

Constraining modified gravity models with cosmological data

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Abstract

In this dissertation, we looked at the cosmological constraints of some $f(R)$ -modified gravity models, such as $f(R) = \beta R^n$ (our first toy model), $f(R) = \alpha R + \beta R^n$ (our second toy model), and more realistic ones like the Starobinsky and Hu-Sawicki models. We used 236 intermediate-redshift and 123 low-redshift Supernovae Type 1A data obtained from the SDSS-II/SNLS3 Joint Light-curve Analysis (JLA), with absolute magnitudes, for the B-filter, found on the NASA Extragalactic Database (NED). We also developed a Markov Chain Monte-Carlo (MCMC) simulation to find the best-fitting luminosity distance function value for each combination of cosmological parameters, namely the matter density distribution Ω_m and the Hubble uncertainty parameter $\bar{h}$ (firstly for the Λ CDM model and then for the $f(R)$ -gravity models). We then used the Λ CDM model results to constrain the priors for the $f(R)$ -gravity models. We assumed a flat universe $\Omega_k = 0$ and a radiation density distribution Ω_r that is negligible to simplify these models. Therefore, the only difference between the Λ CDM model and $f(R)$ -gravity models are the dark energy component and the arbitrary free parameters. This gave us an indication if there exist viable $f(R)$ -gravity models when we compared them statistically to the results of the Λ CDM model. Furthermore, we developed a numerical method to solve the models to which we were not able to find an analytical solution, and incorporated it into the MCMC simulation.

We found 2 viable models, namely the Starobinsky model and a reduced version of the Starobinsky model. These models obtained the cosmological parameters values to be $\Omega_m = 0.268^{+0.027}_{-0.024}$, $\bar{h} = 0.690^{+0.005}_{-0.005}$, and $\Omega_m = 0.266^{+0.026}_{-0.024}$, $\bar{h} = 0.694^{+0.018}_{-0.006}$, respectively. Both were able to predict an accelerating universe. We also found a further three models that were able to fit the data, but were statistically rejected, namely the second toy model where n is fixed to the parameter values of $n = 0$ and $n = 2$, as well as the Hu-Sawicki model. Lastly, we found a further three models that were not able to fit the supernova data and as a consequence were statistically rejected, namely the first toy model, and the second toy model for fixed n -values of $n = \frac{1}{2}$ and $n = 1$. Therefore, we were able to constrain the viability of some of the $f(R)$ -gravity models with cosmological data.

Keywords: general relativity, cosmic acceleration, cosmological parameters, dark energy, modified gravity, $f(R)$, supernova, distance modulus, numerical methods, MCMC simulation

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Introduction to gravitation and cosmology

1.1 History of gravity model

Since the dawn of humanity, there has been a search for the purpose of life itself. Ancient civilisations gazed to the stars for answers. The proof of these is written all over history. For instance, an ancient calendar discovered in Swaziland, Africa dates back to about ~ 35000 BC. This ‘calendar’ consists of 29 notches carved into a bone, which could be an indication of the number of days between two consecutive full moons [7]. In a museum web exhibit held by [8], they showed how some of the ancient calendars based on space events came to be. However, calendars are not the only evidence of humankind’s curiosity with space. Before the invention of the telescope by Galileo Galilei in 1609/10, comets were believed to be signs of God and were recorded [9].

It turns out, this recording of comets proved to be one of the most constructive recordings ever made, as it resulted in Edmund Halley discovering a pattern in the data [10]. The discovery of comets by Halley set him on a path to understand how the orbits of comets work and if they follow similar paths to that of the planets/celestial objects. He met up with Sir Isaac Newton in August 1684 [11]. By the time of the meeting, Sir Isaac Newton had explained the motions of celestial objects, but has misplaced his proof thereof. Halley went on to fund Sir Isaac Newton to write up *Philosophiæ Naturalis Principia Mathematica*, which was published in 1687 [12]. Newton’s gravitational law in combination with his three other famous laws changed the world’s understanding of deterministic actions.

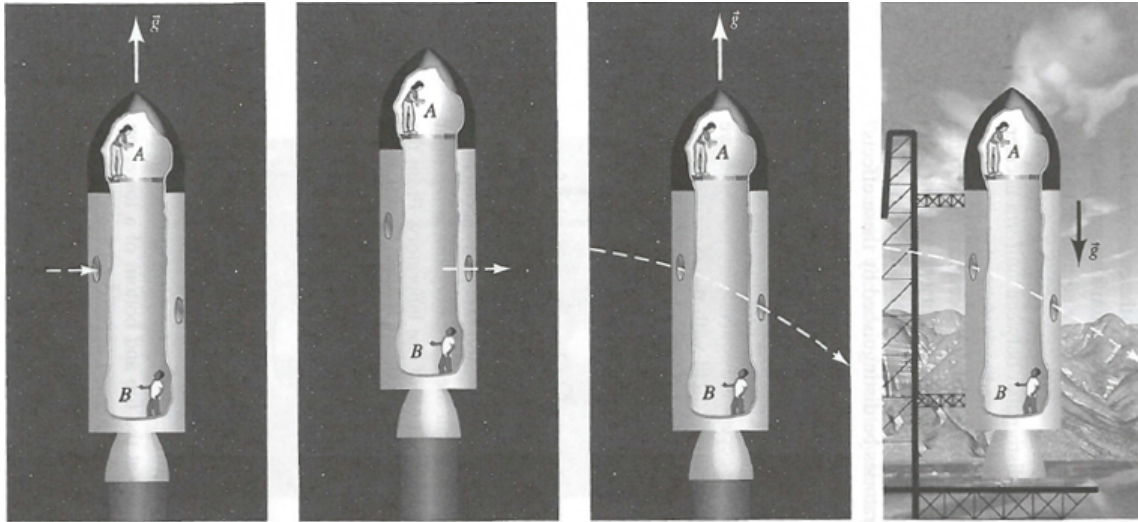


Figure 1.1: The first three panels represent a laboratory accelerating upwards in the vacuum of space, while the last panel contains a stationary laboratory situated in a uniform gravitational field. This image was obtained from [1].

From Newton's laws, physics grew into what is known today as classical mechanics, the study of relative motions. Classical mechanics opened up all types of other study fields in physics, such as electrodynamics, which still has a massive influence in our day-to-day life.

The question that arises now, if Newton's gravitational law explained everything from the fact that we are not falling off the Earth to the motion of celestial objects, why do we still study the effects of gravity? The answer came in the form of what is known as the Ultraviolet Catastrophe, [13]. The Ultraviolet Catastrophe is essentially a disagreement between the classical theory and an experiment. In the textbook by [14], they explain this disagreement with theory by using an everyday practice as an experiment.

It turns out that classical mechanics fail when we proceed to realms that are remote from the common-day experience [15]; for instance, when we are moving at extremely high velocities, or transverse an extremely strong gravitational field, or moving into subatomic space. These problems are being studied within the fields of special relativity, general relativity and quantum mechanics, respectively.

Since we are looking at gravity models, we will be working with the theory of general relativity (GR) and its modifications. The Theory of General Relativity was published by Albert Einstein in 1915 [16]. According to [1], Einstein realised that there exists an experiment that can distinguish between a uniform acceleration from a uniform gravitational field. This is called Einstein's equivalent principle. As an example of this principle, consider the following situation as shown in Figure 1.1.

This example was done by [1]. Consider a light ray that transverses our laboratory. In the first case, our laboratory is accelerating upwards in empty space. The first two panels show the view of an observer in the inertial frame of reference and, according to him, the path of the light ray is

straight. However, since the laboratory moved upwards in the time the light needed to transverse the laboratory, the light exits the laboratory at a lower point than the entry point in our laboratory. A second observer in the accelerating laboratory will observe a different scenario. According to him, the light ray falls with an acceleration equal to a gravitational acceleration (g), as shown in panel three. Using the equivalence principle for the same observation with the laboratory at rest in a uniform gravitational field, the light ray must fall with the same acceleration as other objects, resulting in the conclusion that gravity attracts light. Using the equivalence principle, Einstein considered how gravity will affect the curvature of the Universe.

1.2 Hubble's confirmation of GR?

Now that we understand that Einstein's GR is based on how gravity is affecting the curvature of the Universe, how can we use it to understand the cosmos? Since the discovery of Newton's gravitation law, people were studying Newtonian Cosmology. However, as mentioned, classical mechanics could not explain certain phenomena. This led to other problems being discovered in Newtonian Cosmology and ultimately being concluded that Newtonian gravity lacks a fully satisfactory cosmological formulation, as stated by [17].

This is where GR takes centre stage. When GR was published in 1915, it was used to explain the matter and radiation density distributions of the Universe, by using the density, pressure and energy distribution within the said Universe. This led to a conclusion that the Universe is expanding or contracting, whereas it had previously been thought to be static. In the book written by [18], they actually looked at how this paradigm shift affected the understanding, and thought process of the Universe was changed.

But how do we know that GR gives the correct prediction of an expanding/contracting universe? The answers lies with the discovery made by Edwin Hubble in 1929 [19]. According to [19], astronomers observed that the light of distant galaxies was shifted towards the red end of the electromagnetic spectrum. This shift was the result of the galaxies that are moving away from the Earth. Hubble then compared the redshift of different galaxies at different distances from the Earth, and discovered that the further a galaxy is away from the Earth, the faster it is moving away from the Earth. To give a more formal definition: Hubble's law is the empirically confirmed uniformity that the apparent magnitude of galaxies is linearly correlated with the red-shift in their spectra [20].

Figure 1.2 is an example of Hubble's law for Supernovae Type 1A. The velocity at which the supernovae are moving away from the Earth is linearly proportional to the distance at which the supernovae are situated w.r.t. to the Earth. Furthermore, due to the Doppler effect, the light will then be redshift in proportion to the velocity at which an object is moving. Therefore, the faster the object is moving, the more redshifted the emission from the object is.

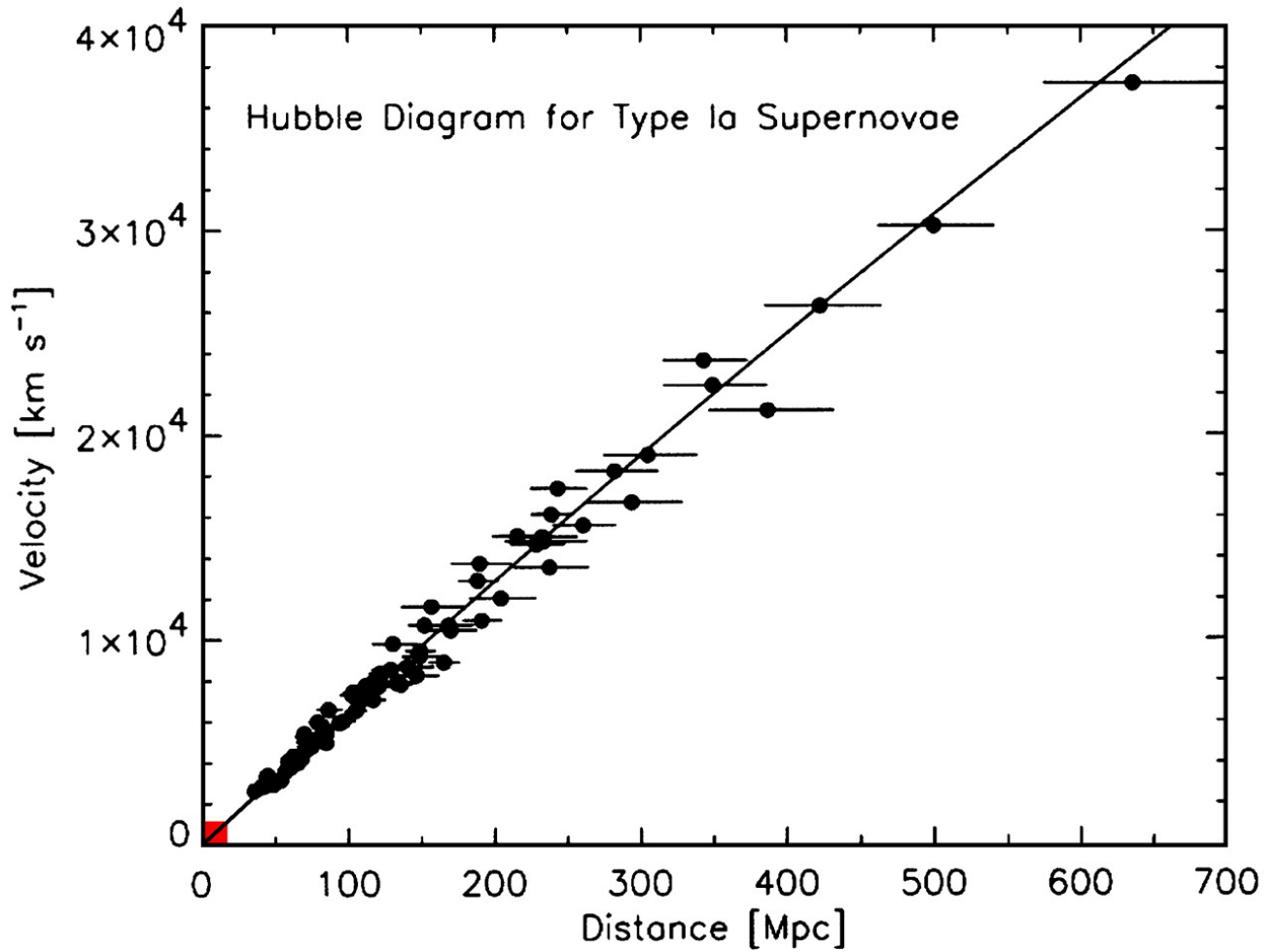


Figure 1.2: An example of Hubble's Law [2]. Velocity can be used to calculate the redshift of an object. DOA: 26 August 2019.

This discovery confirmed the prediction made by GR. Therefore, the Universe had to be expanding. Although GR was Einstein's famous theory, he believed the Universe was static as previously thought. According to [19], Einstein did, however, embrace the idea of an expanding universe, and even referring to his previous belief, which led him to alter GR to accommodate a static universe, to be his greatest blunder.

1.3 General relativity: A mathematics perspective

Now that we have seen that GR explains the expansion of the Universe and without going into too much of the mathematics behind the evolution of GR, we can start with defining the Einstein-Hilbert action. The Einstein-Hilbert action tries to extremize the path between two time-like separated points in space. We include a cosmological constant Λ (that represents a repulsive force working out against gravity, called "dark energy") and a Lagrangian L'_m , which describes the matter contained

in the field:

$$A = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} [R + 2(L'_m - \Lambda)]. \quad (1.1)$$

From the Einstein-Hilbert action, we can derive Einstein's famous field equations. Einstein's field equations in conjunction with the *cosmological principle* are used to derive the Friedmann and Raychaudhuri equations (also simply called the Friedmann equations). The *cosmological principle* states that the Universe is isotropic and homogeneous on large scales. This will allow us to have a reference frame to study the dynamics of the Universe [21]. Furthermore, for the rest of this dissertation, it should be noted that when we use Greek indices, they are part of the main derivation, whereas if we use Latin indices, it will be used to show a certain math construction to aid the main derivation.

1.3.1 Einstein's famous field equations

In this section, we will derive Einstein's field equation following the method shown in [22], but using a similar notation that was used by [23]. Consider the Einstein-Hilbert action shown in Equation 1.1, where R is known as the Ricci scalar, g being the metric tensor, G is Newton's gravitational constant and as usual c being the speed of light. We will also adopt a 4-D spacetime, which includes a temporal component and three spatial components, with a construction of $(-, +, +, +)$ for our 4-vector space. We can then rewrite the Einstein-Hilbert action in the form

$$A = \int d^4x \sqrt{-g} \left(\frac{k}{2} R + L_m - \kappa \Lambda \right), \quad (1.2)$$

where $k = \frac{c^4}{8\pi G}$ with $L_m = \kappa L'_m$. The variation principle (action principle) tells us that the variation of this Einstein-Hilbert action w.r.t. the inverse metric tensor $g^{\mu\nu}$ is zero, meaning $\delta A = 0$. So we have

$$\begin{aligned} \delta A &= \int d^4x \delta g^{\mu\nu} \left(\frac{k}{2} \frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} - \frac{\delta(\sqrt{-g}\kappa\Lambda)}{\delta g^{\mu\nu}} \right), \\ \Rightarrow \delta A &= \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[\frac{k}{2} \left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} - \frac{k}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\Lambda)}{\delta g^{\mu\nu}} \right], \end{aligned} \quad (1.3)$$

where we used the product rule for differentiation and also extracted $\sqrt{-g}$ as a common factor. Furthermore, we know that Equation 1.3 must hold for any variation of $\delta g^{\mu\nu}$, which means that we can simplify Equation 1.3 to obtain

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} - \frac{2\Lambda}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = - \frac{2}{k\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \quad (1.4)$$

We can now calculate each term in Equation 1.4 separately, to find the end result in a similar fashion as was done in [22]. This is shown in the subsequent subsections. After the calculations are done, we can substitute Equations 1.15, 1.19, 1.20, and 1.21 into Equation 1.4 and reverting back to Greek indices, we find

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (1.5)$$

We can simplify Equation 1.5 by using the Einstein tensor ($G_{\mu\nu}$), which is defined as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (1.6)$$

Our final result then reads

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (1.7)$$

Equation 1.7 is known as Einstein's field equations.

1.3.1.1 Variation of the Ricci scalar

In this section, we will examine what happens to the first term in Equation 1.4. To achieve this, we will need to calculate the variation of the Riemann curvature tensor, since the Ricci scalar is a second-generation descendent from the Riemann tensor. By definition, the Riemann curvature tensor is defined as

$$R_{bcd}^a = \partial_c \Gamma_{bd}^a - \partial_d \Gamma_{bc}^a + \Gamma_{bd}^e \Gamma_{ce}^a - \Gamma_{bc}^e \Gamma_{de}^a, \quad (1.8)$$

and remembering that we are using the Einstein summing convention for the index e . We also note for the Riemann tensor definition a new type of symbol. This new symbol is called the Christoffel symbols and is defined as

$$\Gamma_{bd}^a = \frac{1}{2}g^{ae}(\partial_b g_{ed} + \partial_d g_{be} - \partial_e g_{bd}). \quad (1.9)$$

From Equation 1.8, we can see that the Riemann curvature tensor depends only on the Levi-Civita connection (Christoffel symbols). Therefore, the variation of the Riemann tensor is

$$\delta R_{bcd}^a = \partial_c \delta \Gamma_{bd}^a - \partial_d \delta \Gamma_{bc}^a + \delta \Gamma_{bd}^e \Gamma_{ce}^a + \Gamma_{bd}^e \delta \Gamma_{ce}^a - \delta \Gamma_{bc}^e \Gamma_{de}^a - \Gamma_{bc}^e \delta \Gamma_{de}^a. \quad (1.10)$$

We can then calculate the covariant derivative (which is shown in Appendix A). Using Equation A.3 (where we already cancelled the last two terms), we can then simplify Equation 1.10 to obtain

$$\delta R_{bcd}^a = \nabla_c \delta \Gamma_{bd}^a - \nabla_d \delta \Gamma_{bc}^a. \quad (1.11)$$

We can now construct the Ricci tensor by contracting the Riemann tensor's top and bottom (middle) indices, which gives us

$$\delta R_{bcd}^a \equiv \delta R_{bd} = \nabla_a \delta \Gamma_{bd}^a - \nabla_d \delta \Gamma_{ba}^a. \quad (1.12)$$

Furthermore, per definition, the Ricci scalar is given as

$$R = g^{bd} R_{bd}. \quad (1.13)$$

Calculating the variation of the Ricci scalar, we need to use Equation 1.12. So we obtain

$$\begin{aligned} \delta R &= R_{bd} \delta g^{bd} + g^{bd} \delta R_{bd}, \\ \Rightarrow \delta R &= R_{bd} \delta g^{bd} + g^{bd} (\nabla_a \delta \Gamma_{bd}^a - \nabla_d \delta \Gamma_{ba}^a), \\ \Rightarrow \delta R &= R_{bd} \delta g^{bd} + \nabla_a (g^{bd} \delta \Gamma_{bd}^a - g^{ba} \delta \Gamma_{ba}^a), \end{aligned} \quad (1.14)$$

where we used the covariant derivative $\nabla_b g^{cd} = 0$. The second term becomes a total derivative, if multiplied by $\sqrt{-g}$, and then, according to the Stokes theorem, yields only a boundary term when the variation metric $\delta g^{\mu\nu}$ vanishes in a region close to the boundary, thereby resulting in a term that does not contribute, according to [22]. The remaining terms then leave us with

$$\frac{\delta R}{\delta g^{bd}} = R_{bd}. \quad (1.15)$$

1.3.1.2 Variation of the metric tensor

To be able to calculate the the variation of the metric tensor, we will need to use Jacobi's formula for differentiating a determinant. Jacobi's formula is defined as

$$\delta g = \delta \det(g_{bd}) = g g^{bd} \delta g_{bd}. \quad (1.16)$$

Using this formula gives us

$$\begin{aligned} \delta \sqrt{-g} &= -\frac{1}{2\sqrt{-g}} \delta g, \\ \Rightarrow \delta \sqrt{-g} &= \frac{1}{2} \sqrt{-g} (g^{bd} \delta g_{bd}), \\ \Rightarrow \delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} (g_{bd} \delta g^{bd}), \end{aligned} \quad (1.17)$$

where we used the fact that

$$g^{bd}\delta g_{bd} = -g_{bd}\delta g^{bd}. \quad (1.18)$$

Using this result, we can calculate the second and third terms in Equation 1.4, resulting in

$$\frac{R}{\sqrt{-g}} \frac{\left[-\frac{1}{2}\sqrt{-g}(g_{bd}\delta g^{bd}) \right]}{\delta g^{bd}} = -\frac{1}{2}Rg_{bd}, \quad (1.19)$$

for the second term, and

$$-\frac{2\Lambda}{\sqrt{-g}} \frac{\left[-\frac{1}{2}\sqrt{-g}(g_{bd}\delta g^{bd}) \right]}{\delta g^{bd}} = \Lambda g_{bd}, \quad (1.20)$$

for the third term.

1.3.1.3 Hilbert stress-energy tensor

The right hand side (R.H.S.) of Equation 1.4 is per definition known as the Hilbert stress-energy tensor with the exception of the $\frac{1}{k}$. Since the Hilbert stress-energy tensor [24] is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_M}{\delta g^{\mu\nu}}, \quad (1.21)$$

we can use it as is.

1.3.2 Friedmann and Raychaudhuri expansion equations

Since we have derived Einstein's field equations (Eq. 1.7), we can now use it to derive the Friedmann and Raychaudhuri expansion equations. These equations give us information about how the Universe is expanding/contracting depending on a certain density distribution. We will be following a similar fashion for deriving these two equations as done by [21]. First thing to notice from Equation 1.5 is that the field equations depend on the Ricci tensor $R_{\mu\nu}$, as well as the Ricci scalar. As mentioned above, we also have two other tensor dependences in the form of the metric tensor $g_{\mu\nu}$ and the stress-energy tensor (also called the energy-momentum tensor). The energy-momentum (EM) tensor for a perfect fluid is defined as

$$T_{\mu\nu} = \varepsilon(t)u_\mu u_\nu + p(t)(u_\mu u_\nu + g_{\mu\nu}). \quad (1.22)$$

In Equation 1.22, we have used $\varepsilon(t)$ as the density parameter, since we assumed that the *cosmological principle* holds. However, we could only have done this since we assumed that we are working with a homogeneous universe. If that was not the case, we would have had to use $\varepsilon(\vec{x}, t)$. However, since

we assumed a homogeneous universe, the density distribution looks the same in every direction, at every position in the Universe and therefore the density distribution is only dependent on time.

1.3.2.1 The Robertson-Walker metric

To calculate the metric tensor $g_{\mu\nu}$, we will use the Robertson-Walker metric with the 4-vector space. This set-up, in spherical coordinates, is given by [25]

$$ds^2 = -(cdt)^2 + a^2(t) \left[\frac{1}{1 - \kappa r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]. \quad (1.23)$$

We choose the Robertson-Walker metric since it describes an isotropic and homogeneous universe due to symmetry. $a^2(t)$ is called the cosmological scale factor. It tells us the relative expansion rate of the Universe. The variable κ^2 is used to distinguish between the three different space curvatures. In the case that $\kappa^2 = 0$, we have $a^2(t) = 1$, known as Euclidean space (Flat Universe). From Equation 1.23, we can set up a matrix to find the metric tensor ($g_{\mu\nu}$) components. The matrix for Equation 1.23 is defined as

$$\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2 \sin^2 \theta \end{bmatrix} = \begin{bmatrix} g_{tt} & g_{tr} & g_{t\theta} & g_{t\phi} \\ g_{rt} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta t} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi t} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} \quad (1.24)$$

The metric tensor is used to calculate the corresponding Christoffel symbols. We then use the Christoffel symbols to calculate the Riemann tensor, and then find the resulting Ricci-tensor and -scalar.

1.3.2.1.1 The Christoffel symbols

Since the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are confined within the Einstein tensor $G_{\mu\nu}$, we will need to calculate the Christoffel symbols to determine $R_{\mu\nu}$ and R from the Robertson-Walker metric in order to calculate $G_{\mu\nu}$. Using the definition of the Christoffel symbols, as defined in Section 1.3.1.1, and remembering that we are busy using the Einstein's summation convention (sum over all indices *i.e.* $e = t, r, \theta, \phi$). We can then calculate the Christoffel symbols (shown in Appendix B). One will notice that most of the Christoffel symbols are zero. This is due to the

Robertson-Walker metric being diagonal and symmetric. The only non-zero Christoffel symbols are

$$\begin{aligned}
 \bullet \Gamma_{rr}^t &= \frac{a\dot{a}}{c^2(1-\kappa r^2)}, & \bullet \Gamma_{\theta\theta}^t &= \frac{r^2 a\dot{a}}{c^2}, \\
 \bullet \Gamma_{tr}^r &= \Gamma_{rt}^r = \Gamma_{t\theta}^\theta = \Gamma_{\theta t}^\theta = \Gamma_{t\phi}^\phi = \Gamma_{\phi t}^\phi = \frac{\dot{a}}{a}, & \bullet \Gamma_{rr}^r &= \frac{\kappa r}{(1-\kappa r^2)}, \\
 \bullet \Gamma_{\theta\theta}^r &= -r(1-\kappa r^2), & \bullet \Gamma_{\phi\phi}^r &= -r(1-\kappa r^2)\sin^2\theta, \\
 \bullet \Gamma_{r\theta}^\theta &= \Gamma_{\theta r}^\theta = \Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \frac{1}{r}, & \bullet \Gamma_{\phi\phi}^\theta &= -\sin\theta\cos\theta, \\
 \bullet \Gamma_{\phi\theta}^\phi &= \Gamma_{\theta\phi}^\phi = \frac{1}{\tan\theta}, & \bullet \Gamma_{\phi\phi}^t &= \frac{a\dot{a}}{c^2}r^2\sin^2\theta.
 \end{aligned}$$

Noting that $\dot{a} = \frac{da}{dt}$. These results correspond to the general representation presented in [25, 26].

1.3.2.1.2 The Riemann tensor, Ricci tensor and Ricci scalar

After calculating the Christoffel symbols, we can calculate the Ricci tensor by contracting the Riemann tensor given by Equation 1.8. This results in

$$R_{bcd}^a \Rightarrow R_{bmd}^m = R_{bd}. \quad (1.25)$$

Using Equation 1.25, as well as the Einstein summation convention (*i.e.*, $m = t, r, \theta, \phi$), we can calculate the Ricci tensor components (shown in Appendix B). Using these results, we can summarise the Ricci tensor components as temporal and spacial equations, to find

$$\begin{aligned}
 \bullet R_{tt} &= -3\frac{\ddot{a}}{a}, \\
 \bullet R_{ii} &= \frac{g_{ii}}{a^2 c^2} (a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2).
 \end{aligned} \quad (1.26)$$

With $i = r, \theta, \phi$ and $\ddot{a} = \frac{d^2 a}{dt^2}$ ($a = a(t)$ for the entire derivation). Using the Ricci tensor components, we calculate the Ricci scalar by using Equation 1.13 (shown in Appendix B). The Ricci scalar for the Robertson-Walker metric is then

$$R = \frac{6}{a^2 c^2} (a\ddot{a} + \dot{a}^2 + \kappa c^2). \quad (1.27)$$

1.3.2.1.3 The energy-momentum tensor $T_{\mu\nu}$

We also need to calculate the energy-momentum tensor, defined in Equation 1.22. However, a perfect fluid cannot have a privileged direction, so it has only temporal components: $u^\mu = (1, 0, 0, 0)$. Furthermore, we also know that $\|T_\mu^\nu\| = \text{diag}(-\varepsilon(t), p, p, p)$. This leads to component values for

the energy-momentum tensor resulting in:

$$\begin{aligned}
& \bullet \quad T_{tt} = c^2 \varepsilon, \\
& \bullet \quad T_{rr} = \frac{pa^2}{1 - \kappa r^2}, \\
& \bullet \quad T_{\theta\theta} = pr^2 a^2, \\
& \bullet \quad T_{\phi\phi} = pr^2 a^2 \sin^2 \theta,
\end{aligned} \tag{1.28}$$

which can be summarised as

$$\begin{aligned}
& \bullet \quad T_{tt} = -\varepsilon g_{tt}, \\
& \bullet \quad T_{ii} = pg_{ii}.
\end{aligned} \tag{1.29}$$

1.3.2.2 The Friedmann and Raychaudhuri equations

We now have all of the elements needed to calculate Equation 1.5. Let us start with the temporal equation for Equation 1.5. We have

$$\begin{aligned}
& R_{tt} - \frac{1}{2}g_{tt}R + \Lambda g_{tt} = \frac{8\pi G}{c^4}T_{tt}, \\
\Rightarrow & -3\frac{\ddot{a}}{a} - \frac{1}{2}(-c^2)\frac{6}{a^2c^2}(a\ddot{a} + \dot{a}^2 + \kappa c^2) + \Lambda(-c^2) = \frac{8\pi G}{c^4}\varepsilon c^2, \\
\Rightarrow & \left[\frac{\dot{a}(t)}{a(t)}\right]^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{a^2(t)} + \frac{c^2\Lambda}{3}.
\end{aligned} \tag{1.30}$$

For simplicity, we make the substitution $\rho(t) = \frac{\varepsilon}{c^2}$, where $\rho(t)$ is the total energy density parameter. The resulting equation is known as the Friedmann equation

$$\left[\frac{\dot{a}(t)}{a(t)}\right]^2 = \frac{8\pi G}{3}\rho(t) - \frac{\kappa c^2}{a^2(t)} + \frac{c^2\Lambda}{3}. \tag{1.31}$$

We can now calculate for the spacial part:

$$\begin{aligned}
& R_{ii} - \frac{1}{2}g_{ii}R + \Lambda g_{ii} = \frac{8\pi G}{c^4}T_{ii}, \\
\Rightarrow & \frac{g_{ii}}{a^2c^2}(a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2) - \frac{1}{2}g_{ii}\frac{6}{a^2c^2}(a\ddot{a} + \dot{a}^2 + \kappa c^2) + \Lambda g_{ii} = \frac{8\pi G}{c^4}pg_{ii}, \\
\Rightarrow & \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{c^4}p + \frac{c^2\Lambda}{2} - \frac{\kappa c^2}{2a^2}.
\end{aligned} \tag{1.32}$$

After doing a simple mathematical manipulation, we find our second equation as

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\frac{3p}{c^2} + \rho(t)\right) + \frac{c^2\Lambda}{3}. \tag{1.33}$$

This is known as the Raychaudhuri equation.

1.4 Energy conservation: Thermodynamics

We now need to solve the Friedmann equation. Since the Raychaudhuri equation is related to the Friedmann equation, it is unnecessary to solve the Raychaudhuri equation. However, we will not be able to solve Equation 1.31, without knowing how the pressure and the density for a particular system relate. This problem can be solved by using the equation of state (EOS). According to [14], the First Law of Thermodynamics (conservation of energy) is given by

$$\Delta U = Q + W, \quad (1.34)$$

which states that the change in energy (ΔU) is equal to the heat (Q) plus the work (W) done on the system [14]. Since we are working with a system that grows (expansion of the Universe), we have $W = -P\Delta V$ [14]. If we assume a spherical symmetric universe, the expansion of the Universe models is generally modelled as an adiabatic system, meaning that

$$Q = 0. \quad (1.35)$$

A change in the volume within the system should be caused by a change in the internal energy U . Furthermore, we assume that we have a homogeneous universe, which allows us to use $\rho(t)$ as the density. This is possible since the position and the direction of a moving point will be experiencing a uniform density distribution throughout the system, as mentioned in Section 1.3.2. Therefore, the density is only time dependent.

Let us consider a certain volume of the Universe with a unit comoving radius of $r_c = 1$. We also assume that the density is given as $\rho(t)$ within this volume, giving a volume mass ($M = \rho V$) given by

$$M = \frac{4}{3}\pi a^3 r_c^3 \rho. \quad (1.36)$$

Furthermore, the internal energy is given by [27]

$$U = \frac{4}{3}\pi a^3 \rho c^2. \quad (1.37)$$

We assume that we are working with an infinitesimal small volume segment in the Universe [14], meaning that the First Law of Thermodynamics becomes

$$dU = dQ + dW. \quad (1.38)$$

However, since we are working with an adiabatic system ($dQ = 0$ and $dW = -PdV$), we have

$$dU = -PdV. \quad (1.39)$$

We then calculate each side of Equation 1.39 separately. Starting with the L.H.S, we find the change in energy of the expanding universe over time as

$$\begin{aligned} \frac{dU}{dt} &= \frac{d}{dt} \left(\frac{4}{3} \pi a^3 \rho c \right), \\ &= 4\pi a^2 \rho c^2 \dot{a} + \frac{4}{3} \pi a^3 c^2 \dot{\rho}. \end{aligned} \quad (1.40)$$

On the R.H.S, we obtain

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt} \left(\frac{4}{3} \pi a^3 r_c^3 \right), \\ &= 4\pi a^2 \dot{a}, \end{aligned} \quad (1.41)$$

since $r_c = 1$. Now using Equations 1.40 and 1.41 in Equation 1.39, we obtain

$$\begin{aligned} 4\pi a^2 \rho c^2 \dot{a} + \frac{4}{3} \pi a^3 c^2 \dot{\rho} + 4\pi a^2 \dot{a} P &= 0, \\ \Rightarrow 4\pi a^2 \left[(\rho c^2 + P) \dot{a} + \frac{1}{3} a c^2 \dot{\rho} \right] &= 0, \\ \Rightarrow \dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{P}{c^2} \right) &= 0. \end{aligned} \quad (1.42)$$

Equation 1.42 is known as the fluid conservation equation and it describes how the density of the Universe evolves with time.

1.4.1 The equation of state solutions

To solve the Equation 1.42, we will need to know how the pressure term and the density distribution relates to each other. According to [28], we need to assume that the pressure is a single valued function of the energy density $P = P(\rho)$. For convenience, we defined an equation of state (EOS) parameter ω , such that we obtain an EOS as:

$$P = \omega \rho. \quad (1.43)$$

We will use the values of ω , as given in [28], to find the pressure term:

- $\omega_d = 0$ for a pressure-less dust (baryonic and dark matter), which represents the collection of massive non-relativistic particles,
- $\omega_r = \frac{1}{3}$ for a gas of radiation such as photons or ultra-high relativistic particles,
- $\omega_\Lambda = -1$ for a cosmological constant, which represents the repulsive pressure being exerted on the Universe called dark energy, *etc.*

The values of ω also correspond to values given in [23, 29, 30]. Furthermore, we will also assume that the speed of light $c = 1$ for convenience. Therefore, we have

$$\begin{aligned} \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \omega\rho) &= 0, \\ \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho x &= 0. \end{aligned} \tag{1.44}$$

By using the separation of variables method, we can then solve this ordinary differential equation (ODE), by simply integrating both sides of the equation. We then obtain an expression for the density as a function of the cosmological scale factor, resulting in an expression that determines the density of the Universe at a specific time:

$$\rho = Aa^{-3(1+\omega)}, \tag{1.45}$$

where $A = e^c$ is an integration constant. For the three cases, we then have

- For a pressure-less dust $\omega = 0$:

$$\rho_d = Aa^{-3}. \tag{1.46}$$

- For radiation such as photons $\omega = \frac{1}{3}$:

$$\rho_r = Ba^{-4}. \tag{1.47}$$

- For dark energy $\omega = -1$:

$$\rho_\Lambda = C. \tag{1.48}$$

In Figure 1.3¹, an example for how each of the densities, for the different arbitrary constants, will behave is shown. However, it is possible to obtain the values for each of these arbitrary constants. We just chose these values for illustrative purposes. Even though the arbitrary constants were

¹The Python script for this graph can be found on the website <https://drive.google.com/drive/folders/1ag87ouKHZwmuCTBPqsrfLLdRrDmsG4Bi?usp=sharing>.

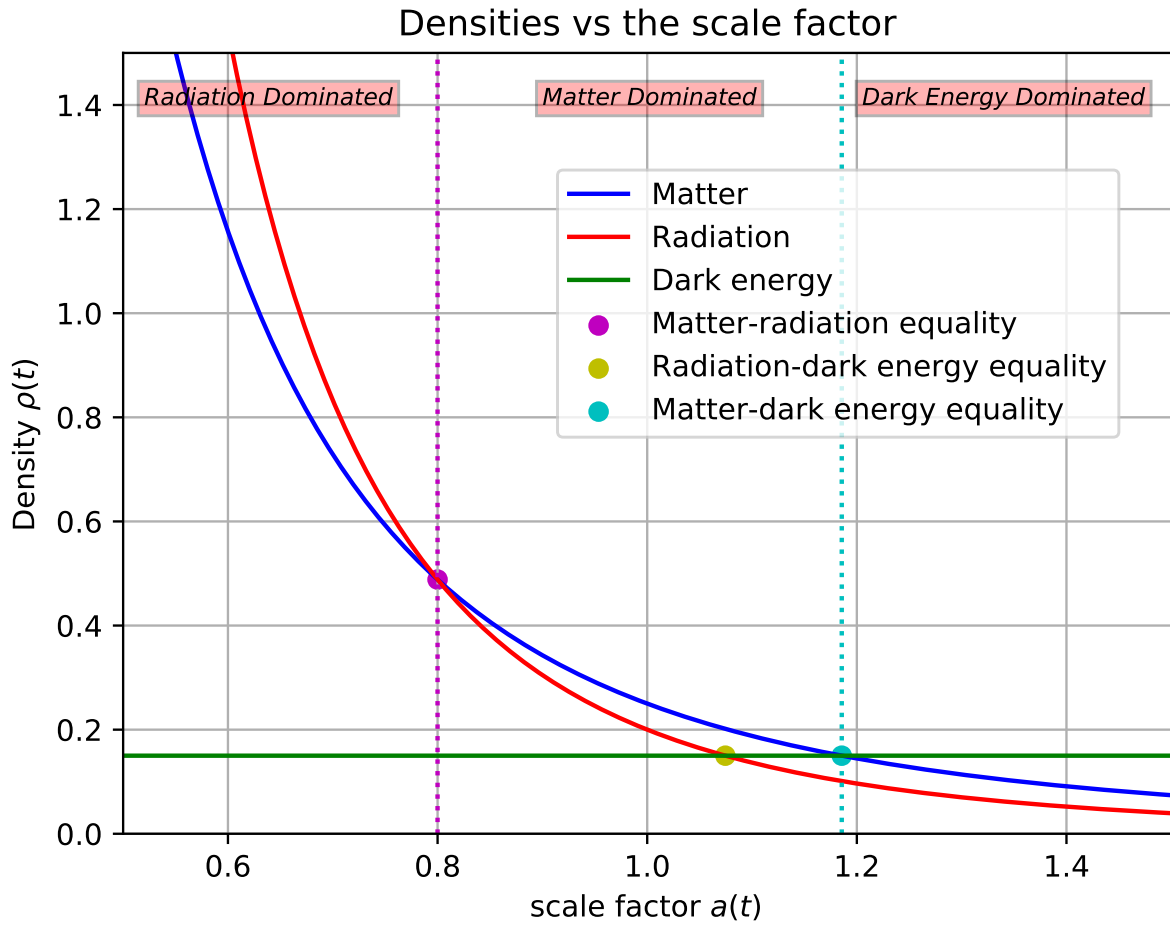


Figure 1.3: The density distribution strength for each of the different densities presented in Equations 1.46, 1.47, and 1.48, using arbitrary constant values ($A = 0.25$, $B = 0.2$, $C = 0.15$) for illustrative purposes.

randomly chosen, the layout of the graph should be similar for the actual values. The labels at the top indicate which density distribution dominated in that period of time. At a scale factor of zero is the beginning of the Universe (the Universe is infinitely small). As the cosmological scale factor increases with time, the size of the Universe also increases. Meaning that for our case, until we reach a scale factor of 0.8, the radiation density distribution will be the dominating density for that period.

We then reach the matter-radiation (MR) equality, where the density caused by matter is equal in strength to that of the radiation density. Following the MR equality, we enter the matter-dominated epoch, after which we will reach the matter-dark energy (MDE) equality. Meaning, the Universe's density distribution caused by dark energy equals the matter density distribution. We then move into the dark-energy-dominated epoch. In this epoch, we are observing that the expansion of the Universe is accelerating, since the density distribution is no longer declining, causing the Universe to accelerate outwards to compensate for this change.

1.4.2 The Friedmann equation solutions

By using the Friedmann equation, as given in Equation 1.31, making the substitution $b = \frac{8\pi G}{3}$, and using the densities distribution Equations 1.46, 1.47, and 1.48, we obtain the general Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = b(\rho_d + \rho_r) + \frac{c^2 \rho_\Lambda}{3} - \frac{\kappa c^2}{a^2}. \quad (1.49)$$

Now, if we assume a certain density, for example “matter density distribution” to be dominating a specific epoch, the density contribution of the other two (radiation and dark energy) will be negligible. The same applies for the other two density distributions dominating their respective epochs. The general Friedmann equation (Eq. 1.49) becomes:

- For a matter-dominated epoch:

$$\dot{a}^2 = bBa^{-2} - \kappa. \quad (1.50)$$

- For a radiation-dominated epoch:

$$\dot{a}^2 = bBa^{-2} - \kappa. \quad (1.51)$$

- For a dark-energy-dominated epoch:

$$\dot{a}^2 = \frac{Ca^2}{3} - \kappa. \quad (1.52)$$

With A, B, C and b as they were previously given. The only “unknown” variable in these equations is the space curvature variable. There exist three possible spaces of constant curvature. These include $\kappa = 0$ for a flat space, $\kappa = -1$ for an open space and lastly $\kappa = 1$ for a closed space, according to [3, 21, 23]. Figure 1.4 shows an illustration of each of these types of curvatures.

1.4.3 The Friedmann equation: A Hubble universe

In Section 1.2, we concluded that we can use Einstein’s gravity theory because Edwin Hubble discovered that the Universe is indeed expanding. Therefore, following this logic we can find a few background information pieces about the Universe. We start with the general Friedmann equation (Eq. 1.49). By defining a matter density distribution, which contains both the dust and radiation

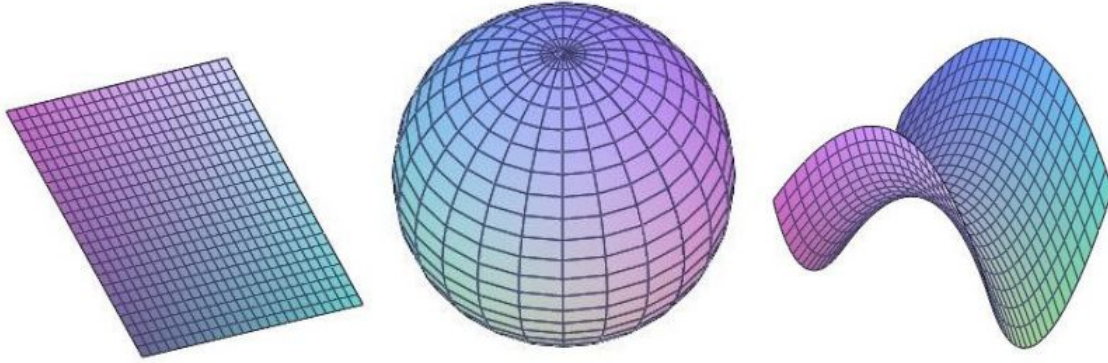


Figure 1.4: Two-dimensional analogues for a flat universe $\kappa = 0$ (left panel), a closed universe $\kappa = 1$ (middle panel), and an open universe $\kappa = -1$ (right panel). This image was obtained on the 28 of February 2018 from [3].

density distributions, and taking the factor $\frac{1}{3}$ out of the constant b^2 , we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{b\rho_m}{3} + \frac{c^2\rho_\Lambda}{3} - \frac{\kappa c^2}{a^2}, \quad (1.53)$$

where $\rho_m = \rho_d + \rho_r$. We can then define the Hubble constant as

$$H = \frac{\dot{a}}{a}. \quad (1.54)$$

This is known as the Hubble parameter or the Hubble rate, according to [23, 31, 32]. Using this definition, we obtain

$$H^2 = \frac{b\rho_m}{3} + \frac{c^2\rho_\Lambda}{3} - \frac{\kappa c^2}{a^2}. \quad (1.55)$$

From the definition of the Hubble constant, we can see that if $H \neq 0$ then we have an expanding/contracting universe, whereas if $H = 0$ we have a static universe. With a current value of $H_0 \simeq 73.24 \pm 1.73$ km/s/Mpc [33–36], we can conclude that the Universe is expanding due to Hubble’s law. Also, it is common to express the Hubble constant as $H_0 = 100\tilde{h}$ km/s/Mpc, where $\tilde{h}$ is to account for the uncertainty in the ever-changing value of H_0 [37]. For example: the current CMB Planck2018 results suggest a Hubble constant value of $H_0 = 67.4 \pm 0.5$ [4]. Normalising Equation 1.55, we find

$$1 = \frac{b\rho_m}{3H^2} + \frac{c^2\rho_\Lambda}{3H^2} - \frac{\kappa c^2}{H^2 a^2}. \quad (1.56)$$

We can then define the fractional energy density distribution parameters as

$$\begin{aligned} \bullet \quad \Omega_m &= \frac{\rho_m}{3H^2}, \\ \bullet \quad \Omega_\Lambda &= \frac{\rho_\Lambda}{3H^2}, \\ \bullet \quad \Omega_k &= -\frac{\kappa}{H^2 a^2}. \end{aligned} \quad (1.57)$$

²So we have $b = 8\pi G$ for this section.

Therefore, Equation 1.56 becomes

$$b\Omega_m + c^2\Omega_\Lambda + c^2\Omega_k = 1, \quad (1.58)$$

where these parameters are called the normalised density parameters at an initial arbitrary time (t) (such as today for the Hubble constant). This is in agreement with results found by [23, 38].

Using cosmological data it was found that the Universe, at the present time, has a matter density distribution of $\Omega_m = 0.315 \pm 0.017$ [4, 39], where the radiation density distribution contributes about $\Omega_{rad} = 2.47 \times 10^{-5} \tilde{h}^{-2}$. This shows that most of the contribution to the matter density comes from the dust and the dark matter density distributions [40]. A cosmological density of $\Omega_\Lambda = 0.679 \pm 0.013$ [4], and a curvature density parameter value of $\Omega_K = 0.001 \pm 0.002$ [4] complete the total density distribution of the Universe. Therefore, we can simplify Equation 1.58 to

$$b\Omega_d + c^2\Omega_\Lambda \approx 1. \quad (1.59)$$

This result is in agreement with [41–43], and gives an interesting piece of information about the Universe in the sense that the Universe is essentially flat. This also raises the question, why is the ratio between matter and dark energy ≈ 1 , with radiation and the curvature densities being negligible, at the present day when we are here to observe it? This is called the Coincidence problem.

This entire model is using GR, solving it to find the Friedmann and Raychaudhuri equations by specifying the different energy densities of the Universe and including the cosmological constant to explain dark energy is called the Lambda Cold Dark Matter (Λ CDM) model. This is the currently accepted cosmological model.

1.5 Size of the Universe for different space curvatures and epochs

In this section, we will only give the solutions to the Friedmann equation for different space curvatures and to give a brief overview of the results. For a more detailed mathematical approach to these results, the mathematical steps for the matter-dominated epoch's solutions are shown in Appendix C. The results are shown in Table 1.1 for a matter-dominating epoch, Table 1.2 for a radiation-dominating epoch, and Table 1.3 for a dark-energy-dominating epoch. For simplicity, we assumed that all the known constants such as $b = \frac{8\pi G}{3}$ and the speed of light c is equal to 1. We are only normalising the equations, not changing the conclusions of the physics.

If we look at these solutions, we can see the expansion of the Universe happening at different rates depending on the different epochs. By defining a deceleration parameter, we can determine the

Matter-dominating epoch		
Space curvature	a	t
Flat space	$a = \left(\frac{3}{2}\right)^{\frac{2}{3}} \sqrt{A} (t - t_0)^{\frac{2}{3}}$	-
Closed space	$a = K(1 - \cos \eta)$	$t = K(\eta - \sin \eta)$
Open space	$a = K(\cosh \eta - 1)$	$t = K(\sinh \eta - \eta)$

Table 1.1: The solutions to the Friedmann equation, with $K = \frac{\rho_0}{2}$, for different space curvatures in an epoch dominated by a matter density distribution.

Radiation-dominating epoch		
Space curvature	a	t
Flat space	$a = \sqrt{2} \sqrt[4]{A} (t - t_0)^{\frac{1}{2}}$	-
Closed space	$a = \sqrt{B} \sin \eta$	$t = -\sqrt{B} \cos \eta$
Open space	$a = \sqrt{B} \sinh \eta$	$t = \sqrt{B} \cosh \eta$

Table 1.2: The solutions to the Friedmann equation, with a new constant related to the old B -value and the new integration constant ($B = B_{old} + IC$), for different space curvatures in an epoch dominated by a radiation density distribution.

different expansion rates [23, 44]

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \quad (1.60)$$

For instance, if we consider a flat matter-dominated universe, the Universe scales as $a \propto t^{\frac{2}{3}}$, which would result in a deceleration parameter value of $q = \frac{3}{4} > 0$. While for the radiation dominating flat universe, it scales as $a \propto t^{\frac{1}{2}}$, resulting in a deceleration parameter value of $q = 1 > 0$. Since both of these are positive and constant, it reveals to us that during these two epochs the expansion of the Universe was slowing down at a constant deceleration, with the radiation-dominated epoch having the most prominent deceleration between the two. Now, if we look at the dark-energy-dominated epoch for a flat universe, we see that our scale factor is $a \propto e^t$, which would give us a deceleration parameter value of $q = -\left(\frac{1}{t^2} + 1\right) < 0$. Since it is negative, the expansion of the Universe must be accelerating during this epoch.

Since we can determine the scale factor for different epochs, as well as determining if they are accelerating or decelerating, we can use this knowledge, combined with the knowledge that the Universe is essentially flat to determine the age of the Universe. We determine this in the following way: We start by taking a desired scale factor. Let us use the scale factor given by a flat universe dominated by a radiation density distribution. Next, we will need to find the derivative w.r.t. time, which in this case is given by $\dot{a} \approx \frac{1}{2}t^{-\frac{1}{2}}$. Substituting these two results into the definition of the

Dark-energy-dominating epoch		
Space curvature	a	t
Flat space	$a = e^{\sqrt{\frac{C}{3}}(t-t_0)}$	-
Closed space	$a = \sqrt{\frac{3}{C}} \cosh \left[\sqrt{\frac{3}{C}}(t - t_0) \right]$	-
Open space	$a = -\sqrt{\frac{3}{C}} \sinh^{-1} \eta$	$t = -\sqrt{\frac{3}{C}} \ln \left \tanh \left(\frac{\eta}{2} \right) \right $

Table 1.3: The solutions to the Friedmann equation, with a new constant related to the old C -value and the new integration constant ($C = C_{old} + IC$) for different space curvatures in an epoch dominated by a dark energy density distribution.

Hubble constant and simplifying, we find

$$H \equiv \frac{1}{2t}. \quad (1.61)$$

By using the current value of the Hubble constant (present day), we can re-write Equation 1.61 to find the age of the Universe as a function of the Hubble constant. The current value of the age of the Universe is estimated to be about $\tau_{age} = 13.73$ Gyr [37]. [45] tried to constrain the data using a computer model and found different values, with the most notable values of $\tau = 13.7 \pm 0.6$ Gyr and $\tau = 13.6 \pm 0.6$ Gyr. This also agrees with the value used by [46, 47]. Therefore, finding an accurate value for the Hubble constant is imperative, if we want to learn more about the cosmic history.

Lastly, it should be mentioned that, in some cases, we had to use the conformal time parametrisation. This was used to simplify the mathematics. According to [37], if we are trying to measure the distances in our co-moving frame in which we are at the origin, we are measuring what is known as the co-moving distance. However, the co-moving distance from us to a galaxy at co-moving coordinates $(\sigma, 0, 0)$ is not observable, because a distant galaxy can only be observed by the light it emitted at an earlier time $t < t_0$. Therefore, meaning a horizon is formed at which we can only observe electromagnetic radiation emitted at a distance smaller than it would take a photon to travel the entire age of the Universe. Anything smaller than that horizon is called the Observable Universe. To parametrise the conformal time we used [30, 48]

$$dt = a d\eta. \quad (1.62)$$

2.1 Problems faced by the Λ CDM model

In Chapter 1, we discussed the need for a gravity model to explain cosmological scales. The first real gravity model was found by Sir Isaac Newton in the 17th century, but it only managed to explain a static universe. Then came Albert Einstein, who improved our understanding of gravity by introducing the theory of general relativity to explain the effects of gravity in extreme gravity fields. Although, at first, he was a firm believer in a static universe, his own theory could not be solved without accepting an expanding universe that had an infinitely small, high density beginning, called the Hot Big Bang [40]. This solution eventually led to the Λ CDM model to explain the accelerated expansion of the Universe. However, we are at another crossroad. With ever improving technology, data has become more accurate with each passing day. This led to a rapid development of observational cosmology [49]. According to the observational data, the Universe has undergone two phases of cosmic acceleration, which introduces new problems for the Λ CDM model.

2.1.1 First phase of cosmic acceleration: Inflation

The first phase of cosmic acceleration is called the inflation epoch, and is believed to have occurred before the radiation epoch [49, 50]. This was also the primary assumption made by [51], where they tried to model the first-order transitions in an expanding universe. The reason why it is believed that an accelerated expansion period happened before the radiation dominated epoch, is due to four

main problems arising in the early-universe cosmology, namely: the Horizon problem, the Flatness problem, the Magnetic monopole problem, and the Anti-matter/Matter ratio problem.

According to [40, 50], the Horizon problem arises from the structure of spacetime. In the standard model, it is assumed that the Universe is homogeneous, but it has been shown to consist of at least $\sim 10^{83}$ separate regions that are causally disconnected because of the great distances between them. These distances exceed the distance that light could have travelled since the Big Bang.

The Flatness problem, as mentioned in Section 1.4.3, is also called the Coincidence problem. Why is the Universe so flat? According to [40], observations had determined that the total density contribution of matter in the Universe is almost equal to the critical density for a flat universe. This means that the contribution to curvature has to be in the same order of magnitude as the contribution from matter throughout the cosmic history. [50] goes even further to show that with a matter density distribution in the same order as the critical density, a universe can only survive $\sim 10^{10}$ years by extreme fine tuning of the initial values. For instance, if our initial thermal equilibrium is taken to be $T_0 = 10^{17}$ GeV, the Hubble constant must be fine-tuned to an accuracy of one part in 10^{55} . Therefore, it again raises the question: Why is the Universe at these particular values, when we are here to observe it?

The Magnetic monopole problem has to do with the fact that the Grand Unified Theory predicts the production of a large number of magnetic monopoles in an early, extremely hot universe [40, 52]. However, not a single magnetic monopole has been observed [53]. This clearly shows that if magnetic monopoles do exist, they are much more rare than the Big Bang theory predicts.

Lastly, we have the Anti-matter/Matter ratio problem. According to [23], at high enough temperatures, there are roughly an equal number of photons (γ), protons (p) and anti-protons ($\bar{p}$) in equilibrium. However, at the present-day value, the ratio between the protons and the photons is $\frac{N_p}{N_\gamma} = (1.5 - 6.3) \times 10^{-10}$, while the ratio between the anti-protons and the protons is $\frac{N_{\bar{p}}}{N_p} \approx 0$ [23, 54]. This disagrees with the prediction made by the Standard Model, which states that due to the conservation of the baryon number, the ratio between the matter particles and their anti-matter counterparts should be roughly equal to 1.

According to [40], the definition of inflation at any period is when the scale factor $a(t)$ of the Universe is accelerating. The basic theory of inflation states that approximately $10^{-37}s$ after the initial Big Bang singularity, there existed a set of highly energetic scalar fields. By definition the total density distribution is driven towards 1 during inflation [40]. [40] goes on to explain that the Universe was dominated during the inflationary phase by a scalar field Φ with a self-interaction potential $V(\Phi)$. Using different forms of this potential, the Horizon problem, the Flatness problem, and the rarity of magnetic monopoles can be explained within the Hot Big Bang model, using the Λ CDM gravity model [40, 50].

2.1.2 Second phase of cosmic acceleration: Dark energy

The second phase of cosmic acceleration is due to the effects that dark energy has on the Universe, as noted in Chapter 1. This acceleration phase was indicated by large-scale observations that occur at late times, unlike the inflation phase [55]. This was not predicted by the standard GR model, since matter without a pressure force acting out against gravity is unable to speed up the expansion of the Universe. This led to the dark energy term, approximated with the cosmological constant, within the standard GR mathematics. According to [55], this new term, dark energy, is typically used as a perfect fluid that counterbalances the action of gravity, providing an effective negative pressure that pushes the Universe outwards. This assumption was used in our derivation of the Friedmann and Raychaudhuri equations in Section 1.3 [21].

The values of the different density distribution contributions in the Universe are $\sim 76\%$ for dark energy, $\sim 20\%$ dark matter and only $\sim 4\%$ ordinary baryonic matter [56]. Baryonic matter is all the different matter particles that emit in the electromagnetic spectrum. This means that $\sim 96\%$ of all of the energy content of the Universe is unknown to us. This value is in a state of ever-improving accuracy due to the more accurate observational tools at our disposal. According to [4], an updated value for the dark energy contribution of the total energy of the Universe is $\sim 68\%$.

By assuming that the cosmological constant is the representation of the dark energy in the Universe, we face a new problem. [56] argues that since the cosmological constant is a constant and the matter density has an exponential decline, once the cosmological constant dominates over matter, there exists no way for matter to dominate again (Fig. 1.3). If you apply this reasoning to the inflation epoch, we can ask: How did it then come to be that the radiation and matter density distributions started to dominate again after the inflation epoch? We know this is important since the Big Bang Nucleosynthesis, which explains the abundance of light baryonic elements formed in the early universe and structure formation, takes place in these epochs [57].

There are two ways of approaching the problem for finding a mechanism that will account for the late-time accelerated expansion. You can either try to find direct solutions to the cosmological constant and coincidence problems, or you can attempt to find alternative ways to explain the late-time acceleration [56]. For this project, we will try to look at alternative ways to explain the acceleration of the expansion of the Universe.

2.2 Modified models to explain the late-time accelerated expansion

In this section, we will briefly mention a few different proposed alternative/modified gravity models. These types of models can be sorted into different categories [23]. To find more intensive reviews on these different types of modified gravity theories, review these sources: [48, 49, 55, 56, 58].

2.2.1 Modified gravity theories with extra fields

If one considers the standard theory of GR, it is assumed that we are working in a 4D-spacetime (with a single rank-2 tensor field) that contains the information about the energy distribution in the Universe, as seen in Section 1.3.2.1. Therefore, it is possible that one can add an extra scalar, vector, tensor or higher-rank field into the Λ CDM model. The additional fields, however, have to be added in such a way that the effects of those fields will be suppressed on scales where GR is highly constrained [23]. Examples of these types of theories are:

- Scalar-Tensor theories
 - Brans-Dicke
 - Chameleon mechanism
- Einstein-/Ether theories
 - Modified Newtonian dynamics
- Bimetric theories
 - Rosen's theory
 - Drummond's theory
- Tensor-Vector-Scalar theories

2.2.2 Higher dimensional theories of gravity

As mentioned, in standard GR, we assume we have a 4D-spacetime. This consists of three spatial dimensions and a temporal dimension. This forms part of the curved 3+1-dimensional Riemannian manifold [23]. In these types of alternative theories, one assumes that gravity exists on a higher-dimensional plane. Examples of these types of theories are:

- Kaluza-Klein theories
- Braneworld paradigm models
- Randall-Sundrum gravity
- Dvali-Gabadadze-Porrati gravity
- Einstein Gauss-Bonnet gravity

2.2.3 Higher derivative theories of gravity

The field equations for GR can have at most a second-order derivative; as seen in Section 1.3, these higher derivative theories of gravity try to allow the field equations to obtain a higher order derivative. However, these types of generalisations tend to have instabilities, but they are outstanding candidates when attempting to renormalise gravity [23]. Examples of these types of theories are:

- General combinations of Ricci and Riemann curvatures
 - Hořava-Lifschitz gravity
 - Galileons
 - $f(R)$ theories
- Generic functions depending on the Ricci scalar in the Einstein-Hilbert action

In this dissertation, we will investigate the $f(R)$ -gravity models.

2.3 Introduction to $f(R)$ -gravity models

$f(R)$ -gravity theories were first proposed by Buchdahl in a publication in 1970 [59]. According to [23], Starobinsky's developments in $f(R)$ gravity theories, such as the Starobinsky model $f(R) = R - 2\Lambda + \alpha R^2$, were crucial in popularising these theories [60]. For a more in-depth review, consult the research papers by [60–63].

In the review paper by Sotiriou [63], they explained that there are two main types of $f(R)$ -gravity theories, mainly the metric formalism and the Palatini formalism. The difference comes from how the variation principle is applied. The metric formalism follows the standard variation principle, while the Palatini formalism assumes that the metric and the connections are independent variables and one has to vary the Einstein-Hilbert action for both of them. Under this assumption, the matter action does not depend on the connection [59, 63].

Furthermore, there exists one more type of $f(R)$ -gravity theory, called the metric-affine $f(R)$ gravity. It follows the Palatini variation, but abandons the assumption that the matter action is independent of the connection [63]. For the rest of this dissertation, we will be following the metric formalism.

2.3.1 Einstein's $f(R)$ -gravity field equations

In this section, we will derive Einstein's field equations for $f(R)$ -gravity theories using the metric formalism. The main derivation follows the same main steps that were used in the derivation of Λ CDM Einstein's field equations (Sec. 1.3.1). We once again consider the Einstein-Hilbert action that includes a Lagrangian L'_m (which describes the matter contained in the field); however, this time we will not include the cosmological constant to ensure that the field equations are not dependent on dark energy. We can then rewrite the Einstein-Hilbert action as

$$A = \int d^4x \sqrt{-g} \left[\frac{k}{2} f(R) + L_m \right], \quad (2.1)$$

where $k = \frac{c^4}{8\pi G}$ with $L_M = kL'_M$. We then use the variation principle, as done in Section 1.3.1. We also remember that since the variation of the Einstein-Hilbert action with regard to the inverse

metric tensor $g^{\mu\nu}$ is zero, this calculation must hold for any variation of $\delta g^{\mu\nu}$. This leads to a simplified equation

$$\frac{\delta f(R)}{\delta g^{\mu\nu}} + \frac{f(R)}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{2}{k\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \quad (2.2)$$

Since we are following the main steps of the original derivation, our next step will be to calculate each term in Equation 2.2. Starting with the first term $\frac{\delta f(R)}{\delta g^{\mu\nu}}$, which we can write as $\delta f(R) = f'(R)\delta R$ [64]. In Section 1.3.1.1, we already calculated the variation of the Ricci scalar, and therefore we can just use Equation 1.14 to calculate $\delta f(R)$, and obtain

$$\delta f(R) = f'(R) \left[R_{bd} \delta g^{bd} + \nabla_e \left(g^{bd} \delta \Gamma_{bd}^e - g^{be} \delta \Gamma_{be}^e \right) \right]. \quad (2.3)$$

As a quick note, the index a in Equation 1.14 has been changed to e for convenience, since this change has no effect on the outcome of this calculation, as indicated in Chapter 1. Furthermore, we will still be using Latin indices as a mathematical tool and Greek indices in the main derivation. In Appendix D, we calculated the variation action on the second term from Equation 2.3. Using the resulting Equation D.12, we have

$$\begin{aligned} \delta f(R) &= f'(R) \left[R_{bd} \delta g^{bd} + \nabla_e \left(g_{gh} \nabla^e \delta g^{gh} - \nabla_f \delta g^{ef} \right) \right], \\ \delta f(R) &= f'(R) \left[R_{bd} \delta g^{bd} + g_{gh} \nabla_e \nabla^e \delta g^{gh} - \nabla_e \nabla_f \delta g^{ef} \right]. \end{aligned} \quad (2.4)$$

But we know that, per definition, the d'Alembert operator is defined as $\square = \nabla_e \nabla^e$ [64]. Using this definition and relabelling some of the indices again, we find

$$\begin{aligned} \delta f(R) &= \delta g^{bd} \left[f'(R) R_{bd} + f'(R) \square g_{bd} - f'(R) \nabla_b \nabla_d \right], \\ \Rightarrow \frac{\delta f(R)}{\delta g^{bd}} &= f'(R) R_{bd} + g_{bd} \square f'(R) - \nabla_b \nabla_d f'(R). \end{aligned} \quad (2.5)$$

We can then take these indices back to their Greek index counterparts, and find

$$\frac{\delta f(R)}{\delta g^{\mu\nu}} = f'(R) R_{\mu\nu} + g_{\mu\nu} \square f'(R) - \nabla_\mu \nabla_\nu f'(R). \quad (2.6)$$

The second term in Equation 2.2 stays mostly unchanged when re-deriving it, with the only change being that it is in terms of $f(R)$ and not R , as shown in Appendix D. While the term on R.H.S. term of Equation 2.2 is by definition known as the Hilbert stress-energy with the exception of $\frac{1}{k}$. Substituting the resulting Equations 2.6 and D.16 into Equation 2.2, we obtain the $f(R)$ -gravity Einstein's field equations as

$$f' R_{\mu\nu} + g_{\mu\nu} \square f' - \nabla_\mu \nabla_\nu f' - \frac{1}{2} g_{\mu\nu} f = \frac{1}{k} T_{\mu\nu}, \quad (2.7)$$

where we set $f'(R) = f'$ and $f(R) = f$. These $f(R)$ -gravity field equations, agrees with the results found in [65–67]

2.3.2 The Friedmann and Raychaudhuri equations for $f(R)$ gravity

Using Einstein's $f(R)$ -gravity field equations (Eq. 2.7) to re-derive the Friedmann and Raychaudhuri equations, we will firstly need to write Equation 2.7 into the same form as Equation 1.6; meaning that we need the Einstein tensor on the L.H.S. and the energy-momentum tensor on the R.H.S. Therefore, we have

$$f'R_{\mu\nu} = \frac{1}{k}T_{\mu\nu} - g_{\mu\nu}\square f' + \nabla_\mu \nabla_\nu f' + \frac{1}{2}g_{\mu\nu}f. \quad (2.8)$$

Now we need to add the term $-\frac{1}{2}g_{\mu\nu}Rf'$ on both sides and simplify to obtain

$$f'\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{1}{k}T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(f - Rf') + \nabla_\mu \nabla_\nu f' - g_{\mu\nu}\square f'. \quad (2.9)$$

If we then divide on both sides with f' , we obtain the Einstein tensor on the L.H.S. as needed

$$G_{\mu\nu} = \frac{1}{f'}\left[\frac{1}{k}T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(f - Rf') + \nabla_\mu \nabla_\nu f' - g_{\mu\nu}\square f'\right]. \quad (2.10)$$

Furthermore, we need to change $T_{\mu\nu}$ to $T_{\mu\nu}^m$, since this energy-momentum tensor, per definition, represents the matter field due to the matter Lagrangian L_M . We can also declare a new energy-momentum tensor representing the remaining terms on the R.H.S. (namely the $f(R)$ terms). We then find

$$G_{\mu\nu} = \frac{1}{k}\tilde{T}_{\mu\nu}^m + T_{\mu\nu}^R = T_{\mu\nu}, \quad (2.11)$$

where

$$\begin{aligned} \bullet \quad \tilde{T}_{\mu\nu}^m &= \frac{T_{\mu\nu}^m}{f'}, \\ \bullet \quad T_{\mu\nu}^R &= \frac{1}{f'}\left[\frac{1}{2}g_{\mu\nu}(f - Rf') + \nabla_\mu \nabla_\nu f' - g_{\mu\nu}\square f'\right]. \end{aligned} \quad (2.12)$$

This result agrees with the work done by [23, 65, 68, 69]. For convenience, we will once again assume that the natural occurring constants within k is equal to one (Sec. 1.5¹). The term on the R.H.S. of Equation 2.11 is called the effective total energy-momentum tensor for a cosmic medium described by $f(R)$ gravity and is given as [23]

$$T_{\mu\nu} = \rho U_\mu U_\nu + Ph_{\mu\nu} + 2q(aU_b) + \pi_{\mu\nu}. \quad (2.13)$$

¹Keep in mind that $k = \frac{c^4}{8\pi G}$, whereas in Section 1.5, we set $b = \frac{8\pi G}{3}$ and the speed of light equal to 1. Thus, the $\frac{1}{3}$ term is not set to be equal to 1 here, since it has not yet been introduced. In fact it will be necessary not to exclude it, as seen in Section 1.4.3 where we had to take $\frac{1}{3}$ out of the equation for b .

2.3.2.1 Thermodynamical quantities

As you might have noticed, Equation 2.13 is the general energy-momentum tensor for a fluid (Sec. 1.3.2). In particular, Equation 2.13 represents a non-perfect fluid. In the case of a perfect fluid, the last two terms vanish due to being equal to zero, since the tensor is a diagonal tensor. As previously stated, ρ is the energy density and P is the isotropic pressure. The two new terms are the heat flux (q) and the anisotropic pressure (π), due to an imperfect fluid distribution. From the effective total energy-momentum tensor, we can define the matter thermodynamical quantities as [23]

$$\begin{aligned}
 \bullet \quad & \rho = \tilde{\rho}_m + \rho_R \quad (\text{Density}), \\
 \bullet \quad & P = \tilde{P}_m + P_R \quad (\text{Pressure}), \\
 \bullet \quad & q_\mu = \tilde{q}_\mu^m + q_\mu^R \quad (\text{Heat flux}), \\
 \bullet \quad & \pi_{\mu\nu} = \tilde{\pi}_{\mu\nu}^m + \pi_{\mu\nu}^R \quad (\text{Anisotropic Pressure}),
 \end{aligned} \tag{2.14}$$

with

$$\tilde{\rho}_m = \frac{\rho_m}{f'}, \quad \tilde{P}_m = \frac{P_m}{f'}, \quad \tilde{q}_\mu^m = \frac{q_\mu^m}{f'}, \quad \tilde{\pi}_{\mu\nu}^m = \frac{\pi_{\mu\nu}^m}{f'}. \tag{2.15}$$

These are the thermodynamical quantities for $f(R)$ gravity. If we consider a perfect fluid, the terms q_μ and $\pi_{\mu\nu}$ vanish as previously mentioned. We also need to remember that since we considered a perfect fluid, we have no preferred direction, resulting in $U^a = (1, 0, 0, 0)$. From this knowledge, we know that for a perfect fluid $U_b^a = -1$. Furthermore, since we have a diagonal matrix (namely $g_{ab} = g^{ab}$), we can define a new tensor, called the projection tensor given as

$$h_{\mu\nu} = U_\mu U_\nu + g_{\mu\nu}. \tag{2.16}$$

2.3.2.2 The $f(R)$ -gravity density and pressure quantities

In this section, we will calculate the thermodynamical density quantity (the calculation for the thermodynamical pressure quantity follows the same path). To calculate ρ , we will start with the definition for density distribution in terms of the energy-momentum tensor, which is defined as

$$\rho = U^a U^b \left(\frac{T_{ab}^m}{f'} + T_{ab}^R \right). \tag{2.17}$$

Next up, we can use Equation 1.22 for T_{ab}^m , with the difference being that we are working with the total matter density distribution (ρ_m) and not the density parameter, to find

$$\begin{aligned}
 \Rightarrow \quad & \rho = U^a U^b \left[\frac{(\rho_m + P_m) U_a U_b + P_m g_{ab}}{f'} + T_{ab}^R \right], \\
 \Rightarrow \quad & \rho = \frac{(\rho_m + P_m)}{f'} U^a U^b U_a U_b + \frac{P_m}{f'} U^a U^b g_{ab} + U^a U^b T_{ab}^R.
 \end{aligned} \tag{2.18}$$

However, we know that $U^a U_a = -1$ and $U^b g_{ab} = U_a$. When using these definitions, we can simplify Equation 2.18 to

$$\rho = \frac{\rho_m}{f'} + U^a U^b T_{ab}^R. \quad (2.19)$$

If we then use the density distribution equation (Eq. 2.15), as well as noting that the second term in Equation 2.19 resembles the definition of the density distribution in terms of the energy-momentum tensor, we obtain

$$\rho = \tilde{\rho}_m + \rho_R. \quad (2.20)$$

By using $\rho_R = U^a U^b T_{ab}^R$, we can determine the density distribution due to the $f(R)$ -gravity terms. This is possible by making use of Equation 2.12 to find

$$\begin{aligned} \Rightarrow \rho_R &= U^a U^b \left[\frac{1}{f'} \left(\frac{1}{2} g_{ab} (f - Rf') + \nabla_a \nabla_b f' - g_{ab} \nabla_c \nabla^c f' \right) \right], \\ \Rightarrow \rho_R &= \frac{1}{f'} \left[\frac{1}{2} (f - Rf') U^a U_a + U^a \nabla_a \dot{f}' - U^a U_a \nabla^2 f' \right], \end{aligned} \quad (2.21)$$

where we used $U^b \nabla_b f' = \dot{f}'$ and $\nabla_c \nabla^c f' = \nabla^2 f'$. We can then use the definitions $\dot{f}' = f'' \dot{R}$ and $\nabla^2 f' = f'' \nabla^2 R$. Furthermore, we can also define the symbol $\Theta = 3 \frac{\dot{a}}{a} = U^a \nabla_a$. When combining all of these definitions, Equation 2.21 becomes

$$\rho_R = \frac{1}{f'} \left[\frac{1}{2} (Rf' - f) - \Theta f'' \dot{R} + f'' \nabla^2 R \right]. \quad (2.22)$$

We can then simplify Equation 2.22, by assuming that we will only be working up to the first-order linear perturbations [68], to find

$$\rho_R = \frac{1}{f'} \left[\frac{1}{2} (Rf' - f) - \Theta f'' \dot{R} \right]. \quad (2.23)$$

If you calculate the pressure due to the $f(R)$ -gravity terms (P_R) in a similar fashion [68], you will find

$$P_R = \frac{1}{f'} \left[\frac{1}{2} (f - Rf') + f'' \ddot{R} + f''' \dot{R}^2 + \frac{2}{3} \Theta f'' \dot{R} \right]. \quad (2.24)$$

2.4 The $f(R)$ -gravity Friedmann equations

Now that we have obtained the density and pressure thermodynamical quantities, we can firstly determine the EOS parameter ω , which we defined in Section 1.4.1, to be

$$\omega = \frac{P_R}{\rho_R} = \frac{3f - 3Rf' + 6f'' \ddot{R} + 6f''' \dot{R}^2 + 4\Theta f'' \dot{R}}{3Rf' - 3f - 6\Theta f'' \dot{R}} = -1 + \frac{6f'' \ddot{R} + 6f''' - 2\Theta f'' \dot{R}}{3Rf' - 3f - 6\Theta f'' \dot{R}}. \quad (2.25)$$

Furthermore, we can also use these quantities to calculate the Friedmann and Raychaudhuri equations. To do this, let us start with the Friedmann equation, as defined in Section 1.3.2.2, with the

exclusion of the dark energy term. Therefore, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{\tilde{\rho}_m + \rho_R}{3} - \frac{\kappa}{a^2}. \quad (2.26)$$

We can then substitute Equation 2.23 into Equation 2.26. We can also use the $\Theta = 3H$ expression to simplify Equation 2.26, to obtain

$$\Theta^2 = \frac{3\rho_m}{f'} + \frac{3}{2}\left(R - \frac{f}{f'}\right) - 3\Theta\dot{R}\frac{f''}{f'} - \frac{9\kappa}{a^2}, \quad (2.27)$$

where $\rho_m = \rho_d + \rho_r$. This is known as the $f(R)$ -gravity Friedmann equation and can also be found in the work done by [49, 70]. Similarly, we can calculate the Raychaudhuri equation to be

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{\rho_m - 3P_m}{2f'} - \frac{1}{2}\left(\frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R}\right] + 3\frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R}\right]\right). \quad (2.28)$$

To simplify Equation 2.28, we will need to use the background trace equation, which is given by [23]

$$3f''\ddot{R} + 3\dot{R}^2 f''' = -9H\dot{R}f'' + \rho_m + Rf' - 2f - 3P_m. \quad (2.29)$$

When we continue to simplify after we used the background equation, we obtain the Raychaudhuri equation as

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{\rho_m}{f'} + \frac{f}{2f'} + \Theta\dot{R}\frac{f''}{f'}. \quad (2.30)$$

Both the Friedmann and the Raychaudhuri equations (Eqs. 2.27 and 2.30) can be found in [23]. However, for understanding supernovae cosmology (see Chapter 3), it is worth writing these two equations in terms of the Hubble parameter. The $f(R)$ -Friedmann equation is given by

$$H^2 = \frac{\rho_m}{3f'} + \frac{1}{6}\left(R - \frac{f}{f'}\right) - H\dot{R}\frac{f''}{f'} - \frac{\kappa}{a^2}, \quad (2.31)$$

while the Raychaudhuri equation is given as

$$\dot{H} = -H^2 - \frac{\rho_m}{3f'} + \frac{f}{6f'} + H\dot{R}\frac{f''}{f'}, \quad (2.32)$$

with R given as

$$R = 6\left(\dot{H} + 2H^2 + \frac{\kappa}{a^2}\right). \quad (2.33)$$

2.5 Analytical conditions for $f(R)$ -gravity validity

Since we will be testing generic $f(R)$ -gravity models, we will need certain analytical conditions that must be satisfied. These criteria have to be added to make sure that the predicted expansion history possesses a matter-dominated epoch followed by a late-time expansion, as currently observed.

Furthermore, it is argued that to match the observations of the diameter distance of acoustic peaks in the cosmic microwave background (CMB) radiation anisotropies, the matter-dominated epoch should be long enough to allow for structure formation and the EOS for the matter-dominated epoch to be $\omega_d \approx 0$. This means, that the scale factor has to expand as $a(t) \sim t^{\frac{2}{3}}$ during the matter-dominated epoch, as concluded by the Planck collaboration that the Universe is close to being flat.

All $f(R)$ -gravity models must satisfy the following conditions [71–75]:

1. We have to ensure that the gravitation energy stays positive and consequently gravity always remains attractive. To do this, the effective Newtonian gravitation constant $G_{eff} = \frac{G}{f'}$, where G is Newton's gravitational constant, must not change sign. Meaning, we have

$$f' > 0 \quad \forall R. \quad (2.34)$$

2. For a stable matter-dominated and high-curvature cosmological regime, we need

$$f'' > 0 \quad \forall R \gg f''. \quad (2.35)$$

3. From observations of the Big Bang Nucleosynthesis (BBN) and the CMB radiation, we obtain constraints that state that the early universe was governed by a GR-like law of gravitation, implying that f' must monotonically asymptote to 1 from below. Meaning, we have

$$\lim_{R \rightarrow \infty} \frac{f(R)}{R} = 1 \Rightarrow f' < 1. \quad (2.36)$$

4. $f' - 1$ must be very small at recent epochs, although it has been argued that this condition is not a necessary condition for ongoing cosmic accelerations, i.e.

$$|f' - 1| \ll 1. \quad (2.37)$$

5. For a late-time de Sitter-type expansion, the ζ parameter (for $\zeta = -2$) is given by

$$\frac{f'}{f''} > R \quad \forall R > 0. \quad (2.38)$$

Supernova cosmology and MCMC simulations

3.1 The Λ CDM model Friedmann equation

In this chapter, we will try to derive an equation that we can use to fit the Λ CDM-Friedmann model to a dataset. For simplicity, we start with the Λ CDM model and expand the same method to the $f(R)$ -gravity models. In this way, we will not only be able to compare how well the $f(R)$ models fair against the Λ CDM model, but it will also grant us the ability to test if the code works, by reproducing known Λ CDM model cosmological values.

To start, we need to change the Friedmann equation into a form in which we have a measurable quantity, called the cosmological redshift. The reason for the redshift is due to the time dependency in the Robertson-Walker geometry. This means that a particle's energy will change as it moves in this geometry. For a photon, whose energy is proportional to its frequency, the frequency will then be redshifted. Per definition [1], redshift is given by

$$1 + z \equiv \frac{\lambda_0}{\lambda_e} = \frac{\omega_e}{\omega_0} = \frac{a(t_0)}{a(t)}, \quad (3.1)$$

where λ represents wavelength and ω the frequency. The subscripts indicate the present time (0) and the time of emission (e). The last remaining variable to declare is z , which is a calculable measurable, called the cosmological redshift. We can rewrite the Friedmann equation into a form in which the equation is dependent on z . Currently, we have the Friedmann equation as

$$H(t)^2 = \frac{\rho_m(t)}{3} - \frac{\kappa}{a^2(t)} + \frac{\Lambda}{3}. \quad (3.2)$$

For simplicity, we once again made all the natural constants equal to 1. This includes $8\pi G = c = \hbar = 1$, where G is Newton's gravitational constant, c the speed of light, and $\hbar$ reduced Planck's constant. If we then divide both sides with the square of Hubble's constant at the present time, and setting $a(t) = 1$ at $t = t_0$, we find

$$\frac{H^2(t)}{H_0^2} = \frac{\rho_m(t)}{3H_0^2} - \frac{\kappa}{a^2(t)H_0^2} + \frac{\Lambda}{3H_0^2}. \quad (3.3)$$

Since we know the dust density distribution scales with $\frac{1}{a^3(t)}$, the radiation density distribution scales with $\frac{1}{a^4(t)}$, and using the dimensionless density distribution parameters as previously defined in Section 1.4.3, we obtain

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_r}{a^4(t)} + \frac{\Omega_d}{a^3(t)} + \frac{\Omega_k}{a^2(t)} + \Omega_\Lambda. \quad (3.4)$$

For completeness, we broke Ω_m up into its components, namely Ω_d and Ω_r . Using the definition of redshift given in Equation 3.1, as well as defining a new constant called the normalized Hubble parameter $h(z) = \frac{H(z)}{H_0}$, we obtain

$$h^2(z) = \Omega_r(1+z)^4 + \Omega_d(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda. \quad (3.5)$$

We can just simplify this equation to find

$$h(z) = \sqrt{\Omega_r(1+z)^4 + \Omega_d(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}. \quad (3.6)$$

This equation can also be found in [23, 76, 77].

3.2 Supernovae Type 1A data

Since we found a normalised dimensionless Friedmann equation in terms of a measurable quantity z (Eq. 3.6), we need a dataset to test this Λ CDM-Friedmann model. We will be using Supernova Type 1A data (as the title of this section suggests). The reason for using this class of star explosions is because they are regarded as standard candles. This means that their absolute magnitudes are relatively the same for all supernovae in this class. A question that may arise is: how can two different star explosions be similar enough to have similar absolute magnitudes?

The reason is due to the source of the explosions. Supernovae Type 1A are the results of white dwarf (WD) stars, accreting low mass main-sequence stars, in a binary system. WD stars have an upper mass limited, called the *Chandrasekhar mass limit*, with a mass value between $\sim 1.2M_\odot$ and $1.44M_\odot$ [78]. It is shown in [79] that only about $\sim 10\%$ of all WD stars have a mass lower than $\sim 0.4M_\odot$, the relative mass difference from a WD to another WD is minimal. Furthermore, most WDs are believed to be composed of carbon and oxygen (with a few traces of other elements) [78].

In the paper [79], they also look at WD stars with other compositions and their evolutionary tracks leading to the WD's explosion. From the above given knowledge about their mass and compositions being similar, it can be concluded that their explosions will be relatively similar. This will result in their light curves, under all aspects, also being similar [80]. This would then result in measured absolute magnitudes of the supernovae being relatively the same. In the research paper by [81], they calculated these absolute magnitudes and found that the mean absolute magnitude for the B-filter for Type 1A Supernovae (with the assumption that $H_0 = 60 \frac{km}{s.Mpc}$) is about ~ -19.46 .

Therefore, we can use the fact that Supernovae Type 1A are considered as standard candles, since the measured luminosity of these supernovae will only be dependent on the distance to the supernovae and not other stellar properties of the explosion. Meaning, an expanding universe (where the distance to the supernovae is changing) can be used to find the best fitting Friedmann equation.

We will be using data obtained from SDSS-II/SNLS3 Joint Light-curve Analysis (JLA). From that particular dataset, we will be using 359 supernovae, where 123 are low-redshift supernovae (with a redshift between $0.01 < z \leq 0.1$) and 236 supernovae with an intermediate redshift (between $0.1 < z \leq 1.1$)¹. A question that may arise is: Why we are not using the full JLA dataset or why we do not even consider the Pantheon data? The reason for this is due to the inefficiency of the MCMC simulation that we developed for this project, as explained in Section 3.4. With more data, more likelihood calculations must be made and with the inefficiency of the code this becomes a time consuming event. Furthermore, we will be using the calculated absolute magnitudes of these supernovae for the B-filter, which can be found in the research papers [82–84]. By using these calculated absolute magnitudes, it also restricts us to the supernovae that already have calculated absolute magnitudes. Therefore, we were not able to use the full dataset. However, for future work we will need to use the larger datasets to improve our statistics.

The reason for using low-redshift data is because we want to test the late-time acceleration of the Universe. It is shown in [85, 86], for two different methods, that the matter-dark energy equality happened at a redshift between ($\sim 0.5 < z < 0.6$), although these are not hard limits, just estimations. Therefore, we need to use supernovae with redshifts between $0 \leq z \leq 0.5$, while it is further suggested by [85] that although the accelerated expansion is most evident in this redshift range, it is not sufficient, and it is necessary to extend the survey to higher redshifts, such as between $1 \leq z \leq 2$.

To add another limit to the upper-redshift limit, it is suggested in [87], that although we need higher redshift measured supernovae, the supernovae become exceedingly faint due to the redshift and have to be measured in near-infrared wavelengths (where the night sky is much brighter than in optical). Furthermore, [87] also suggests to use supernovae with a redshift up to about $z \approx 1.2$,

¹The reduced dataset that we used after calculating the distance modulus from the chosen supernovae can be found on the website <https://drive.google.com/drive/folders/1ag87ouKHZWmuCTBPqsrfLLdRrDmsG4Bi?usp=sharing>.

because in the earlier phases of the expansion, the expansion rate should have decreased, due to the gravitational action being larger during its first few billion (10^9) years. Therefore, we will be fitting the Friedmann equation from deceleration expansion into the late-time accelerated expansion by using supernovae with a redshift between $0 \leq z \leq 1.2$.

For interest sake, at redshift of $z \approx 0.5$, the Universe would be about $\sim 8.6 \times 10^9$ years old, which would mean the late-time acceleration period started at about $\sim 5.2 \times 10^9$ years ago. This age of the Universe is model dependent and uses the assumptions that the cosmological parameters have values of $\Omega_m = 0.315$ and $H_0 = 67.4 \frac{km}{s.Mpc}$ ².

3.3 The distance modulus equation

Now that we have the Λ CDM model in terms of a measurable quantity z , and a usable dataset, we need to obtain an equation that can mathematically explain the behaviour of the data by finding the best fit. The most common way of doing this is by finding distance modulus as a function of redshift. In cosmology, the *proper distance* is defined as [23, 77]

$$D_p = \int dt = \int_0^z \frac{dz'}{(1+z')H(z')} = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')h(z')}. \quad (3.7)$$

The *proper distance* equation is derived from Equation 3.1, as shown in [88]. Furthermore, the *line-of-sight comoving distance* is defined as [23]

$$D_c = \int \frac{cdt}{a}. \quad (3.8)$$

This shows us that two nearby objects in the Universe have a constant D_c between them at any epoch if they move with the Hubble flow [77, 88]. We can expand Equation 3.8 by using the Hubble constant. We know that the *Hubble time* is given by $\tau_H = \frac{1}{H_0}$, we can use it to define the *Hubble distance* as

$$D_H = \frac{c}{H_0}, \quad (3.9)$$

where we kept the value of c to have the correct normalisation of the distances. Using Equation 3.9, we find the *line-of-sight comoving distance* as

$$D_c = D_H \int_0^z \frac{dz'}{h(z')} = \frac{c}{H_0} \int_0^z \frac{dz'}{h(z')}. \quad (3.10)$$

We can then define the *proper motion distance* D_M , which is the distance from us to a distant object (such as a distant supernova) [77]. This distance is the ratio of the actual transverse velocity (distance vs. time) of the object to its proper motion (in radians per unit time). The *proper motion*

²These calculations were done on the website <https://home.fnal.gov/~gnedin/cc/>. DOA: 13 August 2019.

distance is given by

$$D_M = \begin{cases} D_H \frac{1}{\sqrt{\Omega_k}} \sinh\left(\frac{\sqrt{\Omega_k} D_c}{D_H}\right) & \text{if } \Omega_k > 0, \\ D_c & \text{if } \Omega_k = 0, \\ D_H \frac{1}{\sqrt{|\Omega_k|}} \sin\left(\frac{\sqrt{|\Omega_k|} D_c}{D_H}\right) & \text{if } \Omega_k < 0. \end{cases} \quad (3.11)$$

Equation 3.11 can be found in [23, 77, 88]. Since we need to calculate the distance modulus equation, we need the *luminosity distance*. This can be obtained by using the total flux received from the source

$$D_L = \sqrt{\frac{L}{4\pi f}}, \quad (3.12)$$

where D_L is the luminosity distance, L the total luminosity of the source and f flux received. From Equation 3.12, we can confirm that the change in flux from one Supernovae Type 1A to another is a direct result of the transverse distance (since the relative luminosity is similar for all supernovae of this type). The change in the received flux changes the apparent magnitude that will be measured. The relation between the *proper motion distance* and the *luminosity distance* is defined as [88]

$$D_L = (1 + z)D_M. \quad (3.13)$$

Since we defined the Hubble constant to be $H_0 = 100\bar{h} \frac{km}{s.Mpc}$ in Section 1.4.3, the calculated Hubble distance is $D_H = \frac{3 \times 10^8 m/s}{100\bar{h} km/s/Mpc} = 3000\bar{h}^{-1} Mpc$. If we then combine all the different distance definitions, we obtain the luminosity distance as

$$D_L = 3000\bar{h}^{-1}(1 + z) \int_0^z \frac{dz'}{\sqrt{\Omega_r(1 + z')^4 + \Omega_d(1 + z')^3 + \Omega_k(1 + z')^2 + \Omega_\Lambda}}. \quad (3.14)$$

Equation 3.14 can be found in [23, 88–90]. Substituting the luminosity distance equation into the distance modulus equation³, given by

$$\mu = m - M = 25 + 5 \times \log_{10}(D_L), \quad (3.15)$$

we obtain [76, 89, 90]

$$\mu = m - M = 25 + 5 \times \log_{10} \left(3000\bar{h}^{-1}(1 + z) \int_0^z \frac{dz'}{\sqrt{\Omega_r(1 + z')^4 + \Omega_d(1 + z')^3 + \Omega_k(1 + z')^2 + \Omega_\Lambda}} \right). \quad (3.16)$$

We are now able to fit the Λ CDM-Friedmann model to cosmological data.

³The distance modulus equation is given in Mpc .

3.4 Markov Chain Monte Carlo simulations

There exist different ways to fit a model to data, starting with the simplest way, called the least squares (LS) method (the LS-method tries to minimise the total distance between the data point's value ($y_{data}(x_i)$) and the model's predicted value ($y_{model}(x_i)$) at an independent position x_i), to more advanced methods, such as the Markov Chain Monte Carlo simulations or MCMC simulations for short. We will be using the latter in our work.

The MCMC simulation method forms part of Bayesian statistics. They are a class of algorithms for sampling probability distributions [91]. The MCMC simulation was invented by Nicholas Metropolis in 1953 and released in a research paper as part of the random process Monte Carlo simulations [92]. In that paper, they simulated a liquid in equilibrium with its gas phase. According to [91], the easiest way to find out about the thermodynamical equilibrium was to simulate the dynamics of the system, and let the simulation run until it reaches equilibrium. They realised that they do not have to simulate the exact dynamics, they only needed to simulate some Markov Chain (MC) having the same equilibrium distribution. The MCMC simulation that follows this scheme is called the Metropolis algorithm. In 1970, further development to the MCMC simulation was done by Wilfred Hastings [93]. We will be using this more developed MCMC simulation, called the Metropolis-Hastings (MH) algorithm. Another notable MCMC simulation method is the Gibbs sampler.

3.4.1 Metropolis-Hastings algorithm

Even though the MH algorithm is the most popular MCMC simulation [94], the basic idea behind the different algorithms is to generate samples $x^{(i)}$ while exploring the state space χ using an MC mechanism. The MC mechanism allows the chain to spend more time in the most important regions by using the probability distributions $p(x)$ to find the highest likelihood of being the best fit in the state space.

According to [94], it is intuitive to introduce MCs on a finite state space, where $x^{(i)}$ can only take n discrete values $x^{(i)} \in \chi = \{x_1, x_2, \dots, x_n\}$. This stochastic process $x^{(i)}$ [94] is defined as

$$p\left(x^{(i)} \middle| x^{(i-1)}, \dots, x^{(1)}\right) = T\left(x^{(i)} \middle| x^{(i-1)}\right). \quad (3.17)$$

This chain is homogeneous if $T \triangleq T(x^{(i)} | x^{(i-1)})$ remains invariant for all i , with $\sum_{x^{(i)}} T(x^{(i)} | x^{(i-1)}) = 1$ [94]⁴, where T is the transition matrix that when multiplied with the probability vector for a few different iterations, would stabilise to find the probability distribution. This means that the

⁴The symbol $\triangleq$ is used as a definition. In our case it would mean that T is defined to be $T(x^{(i)} | x^{(i-1)})$.

evolution of the chain in a space χ depends solely on the current state of the chain and a fixed transition matrix.

The MH algorithm for an invariant probability distribution $p(x)$ and a proposed probability distribution $q(x^*|x)$ follows as:

- Sampling a candidate value x^* given the current value x , depending on $q(x^*|x)$;
- The MC then moves towards x^* and calculates the likelihood of x^* occurring depending on the probability distribution;
- The MH algorithm compares the likelihood of the current position x and the new position x^* ;
- The new position is accepted or rejected based on the outcome of the acceptance probability given by $\alpha(x, x^*) = \min \left\{ 1, \frac{p(x^*)q(x|x^*)}{p(x)q(x^*|x)} \right\}$;
- If the new position x^* is rejected, the MC moves back to the current position x , and sample a new candidate $x^{*'}^5$;
- If the new position is accepted, set the new position as the current position ($x^* = x$), and sample a new candidate position $x^{*'}$ based on the new current position.

The MH algorithm and acceptance probability can be found in [95–97]. Since the MH algorithm finds the most likely position given a probability distribution, we can find the position at which the different gravity models are most likely to fit the data. This includes the Λ CDM model and the $f(R)$ -gravity models, by finding the best fit positions for each free parameter, for the model you will be testing, and its likelihood for occurring.

In a more complex, but more accurate MH algorithm, one will incorporate a prior. This will allow you to set physical constraints on the sampling process. These constraints can be results from previous experiments, therefore “prior” knowledge.

3.4.2 A Gaussian probability distribution

To calculate the likelihood of the new sample candidate, we will need to choose a probability distribution. For simplicity, we will assume a Gaussian distribution, since we assume that we will obtain a maximum probability for some value. We will also be able to find the error approximation on the maximum probability for each free parameter in our sample space, by finding the first standard deviation σ from the maximum. From previous knowledge, we know that the likelihood for a Gaussian distribution is given as

$$\mathcal{L}(\hat{\theta}|data) = \exp \left(-\frac{1}{2} \sum \left[\frac{(\mu_{data} - \mu_{theoretical})^2}{\sigma^2} \right] \right), \quad (3.18)$$

⁵ $x^{*'}$ plays the role of the newly proposed candidate “ x^* ” for the next turn in the algorithm.

where μ_{data} is the distance modulus for each individual data point and $\mu_{theoretical}$ the model's predicted distance modulus value.

You will notice that in the code, nonetheless, we use $\ln\mathcal{L}(\hat{\theta}|data)$ ⁶. This is a requirement for the *EMCEE Hammer Python package* that we will be using for the calculations done by MCMC simulation. Even though we used a package to calculate the MCMC simulation, we did write our own small code to demonstrate how the MCMC simulation (MH algorithm) on a two-parameter situation would work⁷.

3.4.3 The MCMC simulation code

As mentioned, we used the *EMCEE Hammer Python package* to do the calculations for the MCMC simulation. The reason for using this package and not our own code is due to efficiency. Our code's execution time for 1 random walker, is similar to the *EMCEE package's* execution time for 50 random walkers. To put this into perspective, if you have 1 walker you will start the MH algorithm, loop through the algorithm n times (iterations) and return the result. If you have 50 random walkers, you will execute the MH algorithm for each individual random walker (each walker starting at a different initial condition and follows its own path) n times (iterations). Therefore having 50 different walkers you obtain a probability distribution that takes into account 50 different paths, creating on average a better understanding of the results, than the results obtained by 1 walker.

The set-up for the Λ CDM-model code is shown in Appendix F⁸. The important lines to notice are: Firstly, setting the supernovae data's distance modulus (μ_{data}) and redshift (z) to usable variables (line: 40-43). Secondly, calculating a starting point for the *EMCEE package* using *SciPy's optimize* function (line: 121-126). Thirdly, executing the *EMCEE package's* main function (line: 131-135). The *EMCEE package's* main function command uses the "lnprob" function (line: 97-101) to check if the proposed values are valid, by calling the "lnprior" function (line: 87-91) and if proposed values are valid, it also calls the "lnlike" function (line: 69-82) to calculate the likelihood of the proposed values. The "lnprob" function then returns the calculated likelihood for the *EMCEE package's* main function to analyse. The "lnlike" function uses our model (line: 48-63) to calculate the likelihood of our model. The rest of the code is just plotting the results and calculating the code's execution time.

⁶We discuss the MCMC simulation code in Section 3.4.3.

⁷This two-parameter situation code and dataset can be found on the website <https://drive.google.com/drive/folders/1ag87ouKHZWmuCTBPqsrfLLdRrDmsG4Bi?usp=sharing>.

⁸The full Λ CDM model *EMCEE Hammer* code can also be found on the website <https://drive.google.com/drive/folders/1ag87ouKHZWmuCTBPqsrfLLdRrDmsG4Bi?usp=sharing>.

3.5 The Λ CDM model results

In this section, we will calculate the Λ CDM model results and compare them to the known cosmological parameter values to determine if our *EMCEE Hammer* code does indeed give us the expected results, such as those given in [4]. Furthermore, this technique has been done countless other times [89, 90, 98]. The results of this method (finding the best fit parameter values for the Λ CDM model on supernovae data) can even be found in textbooks [1]. Furthermore, by obtaining the Λ CDM model results, we will be able to compare our $f(R)$ -gravity models to the Λ CDM model, to determine if they are viable.

By using Equation 3.16 and assuming flat space $\Omega_k = 0$, as well as using the fact that the density of radiation is negligible at the present epoch $\Omega_R \approx 0$, we obtain a simplified distance modulus for the Λ CDM model as

$$\mu = 25 + 5 \times \log_{10} \left(3000 \bar{h}^{-1} (1+z) \int_0^z \frac{dz'}{\sqrt{\Omega_m (1+z')^3 + 1 - \Omega_m}} \right). \quad (3.19)$$

⁹Since we have made these assumptions about our density distributions (also discussed in Section 1.4.3), we are able to substitute Ω_Λ by $1 - \Omega_m$. Therefore, we only have two free parameters for the Λ CDM model ($\bar{h}$ and Ω_m). The Planck 2018 data [4] for these parameters gave estimate values for the Hubble constant as $H_0 = 67.4 \pm 0.5 \frac{km}{sMpc}$ (resulting in a Hubble uncertainty parameter of $\bar{h} = 0.674 \pm 0.005$), and a matter density distribution parameter value of $\Omega_m = 0.315 \pm 0.007$ ¹⁰. We chose the prior to be

$$\begin{aligned} & \bullet \quad 0.00 < \Omega_m < 1.00, \\ & \bullet \quad 0.00 < \bar{h} < 1.00. \end{aligned} \quad (3.20)$$

The reasoning behind these priors is that either the entire universe is matter-less $\Omega_m = 0$ ¹¹, or the entire universe consists only out of matter $\Omega_m = 1$. The Universe may even consist of a combination of different density distributions (such as: radiation, dust, curvature, and dark energy), resulting in a fraction. For the Hubble uncertainty parameter, only the lower limit is necessary, since we know that the Universe is expanding, the value of the Hubble constant cannot be negative. The upper limit can be chosen as any value, since the original estimate of the Hubble constant, made by Edwin Hubble in 1929, is $H_0 \approx 500 \frac{km}{sMpc}$ [99]. With increasingly better technology, this value has been more precisely calculated to the estimated value given above [4]. So the only reason we chose this prior to have an upper limit of $\bar{h} = 1.00$ is to help us with computational time.

⁹One thing to notice is that we substituted Ω_d for Ω_m , since $\Omega_m \approx \Omega_d$ when Ω_r is negligible. This is just to make it easier to compare to other research, since it is more common to present your finding in terms of Ω_m . Therefore, when comparing we will just have to keep in mind that we neglected Ω_r .

¹⁰Keeping in mind they have not neglected Ω_r . However, it will not have an effect of the 3rd decimal as present here, since we would have had $\Omega_d \approx 0.3149...$

¹¹We did, however, exclude 0 and 1, since it contradicts our daily experience (we are made of matter), so we kept the possibility for a low matter count in the Universe.

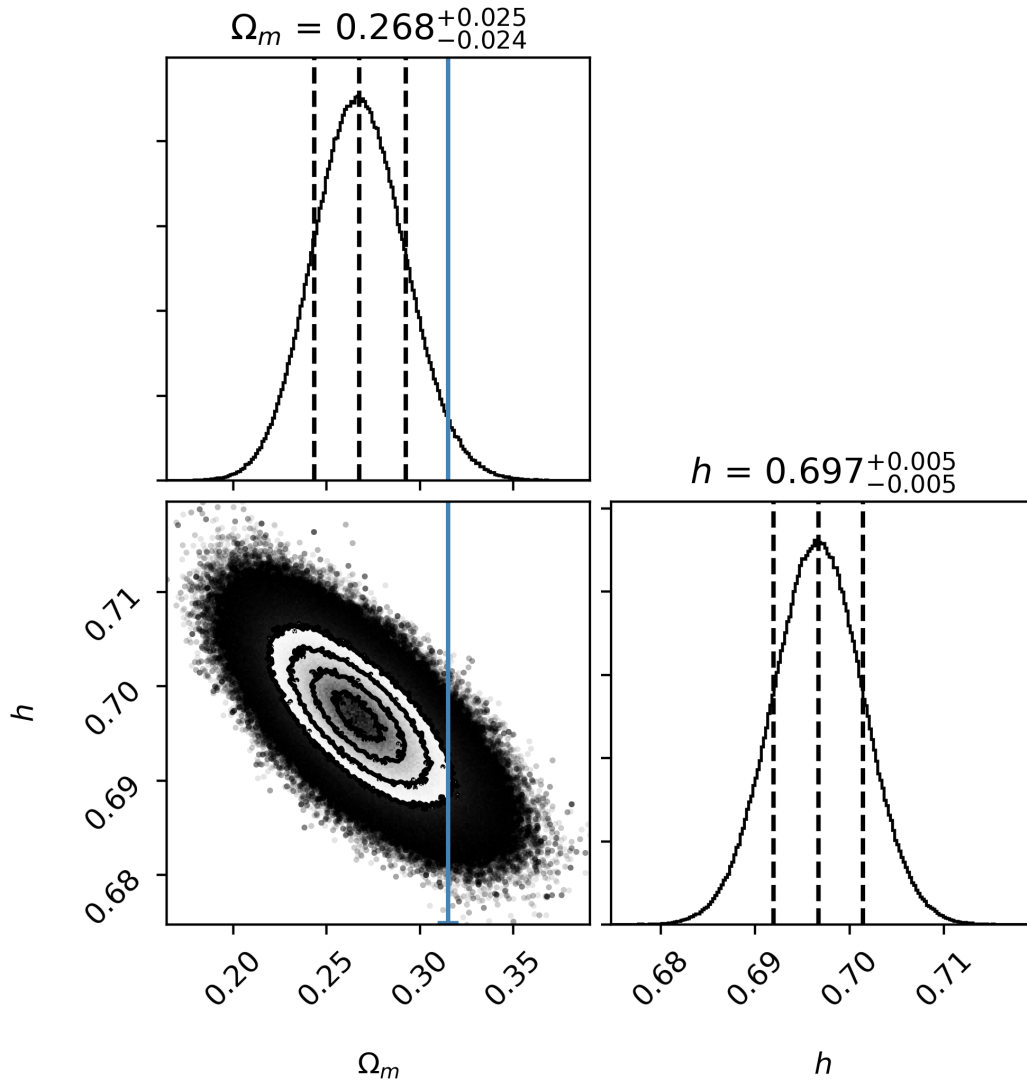


Figure 3.1: MCMC simulation results for the Λ CDM-model's (Eq. 3.19) cosmological free parameters (Ω_m and $\bar{h}$), with “true” values (blue lines: $\Omega_m = 0.315$ and $\bar{h} = 0.674$) provided by the Planck 2018 collaboration data release [4]. We used 100 random walkers, 25 000 iterations and priors provided by Equation 3.20.

When we execute the code, with the above given priors, for a 100 random walkers, with each taking 25000 iterations, we obtain the results shown in Figure 3.1. It must be noted that these chosen amount of random walkers, iterations or step sizes the simulation must execute is not an exact value and can be changed depending on the model and the amount of free parameters you have. We chose these values based on trial and error to determine which resulted in the most complete and accurate results¹².

¹²In Chapter 4 you will notice we decrease the amount of iterations the simulation makes. This is due to having more free parameters so the resulting PDF was overloaded with information and ended up crashing our program. While we lowered it even further for the realistic models due to time constraints, as shown in Appendix E.

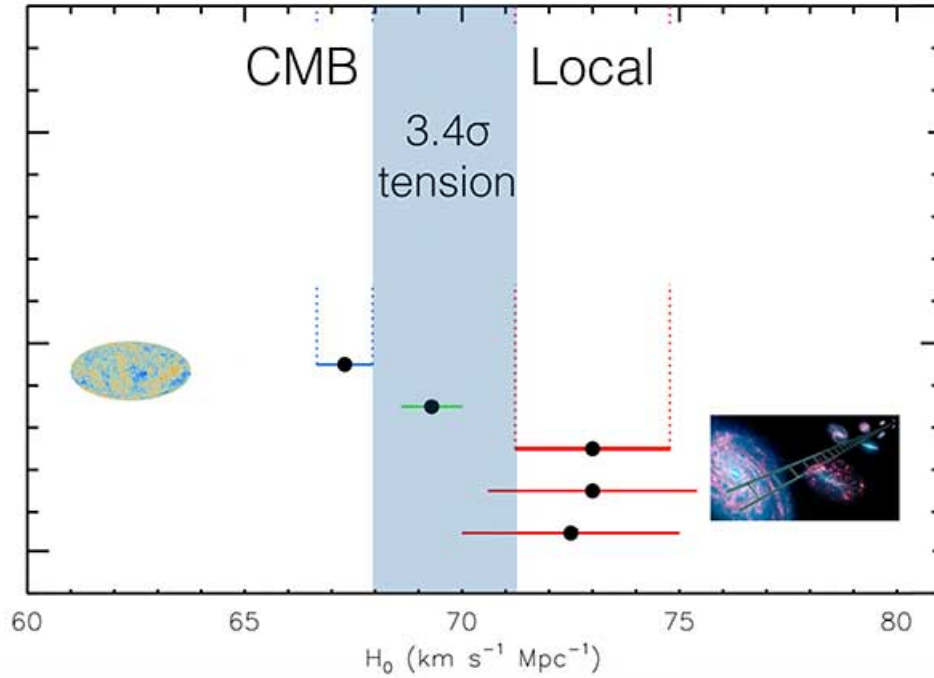


Figure 3.2: Showing the discrepancy between the estimation of the Hubble constant from the CMB radiation and other local determined methods based on a ladder build from “standard candles”, such as Supernovae Type 1A and Cepheid stars. This graph was found on the website of the American Physical Society <https://www.aps.org/publications/apsnews/201805/hubble.cfm> written by [5]. DOA: 19 August 2019.

In Figure 3.1, the middle dashed line in both parameter value cases is the most probable value the MCMC simulation obtained with a Gaussian distribution assumption; while the two dashed lines on either side of the most probable value are the one standard deviation 1σ (error on calculation). In both cases the expected (“true”) value (as presented by the Planck 2018 release [4]), is not within 1σ .

However, before we can conclude that the code is not working, we need to ask why we have this discrepancy. The answer to this question comes in the form of which data we used. The Planck results were determined using the cosmic microwave background (CMB) radiation, while we used supernovae data. It has been shown that this discrepancy between CMB radiation cosmological parameter values and those by supernovae does exist [100]. This discrepancy is not limited to just supernovae and CMB results, as shown in Figure 3.2. The reason for this discrepancy is still being studied.

Since we obtained a higher value for the Hubble uncertainty parameter, which is within the 3.4σ tension introduced in Figure 3.2, we can assume that the lower predicted value for the matter density distribution is also expected. The reason for this is because “less” matter in the Universe (thus “more” dark energy), would result in a faster expanding universe as shown with the higher predicted Hubble constant. We can therefore conclude that our code does indeed work, since we obtain a Hubble constant value within the 3.4σ tension, as well as obtaining a matter density

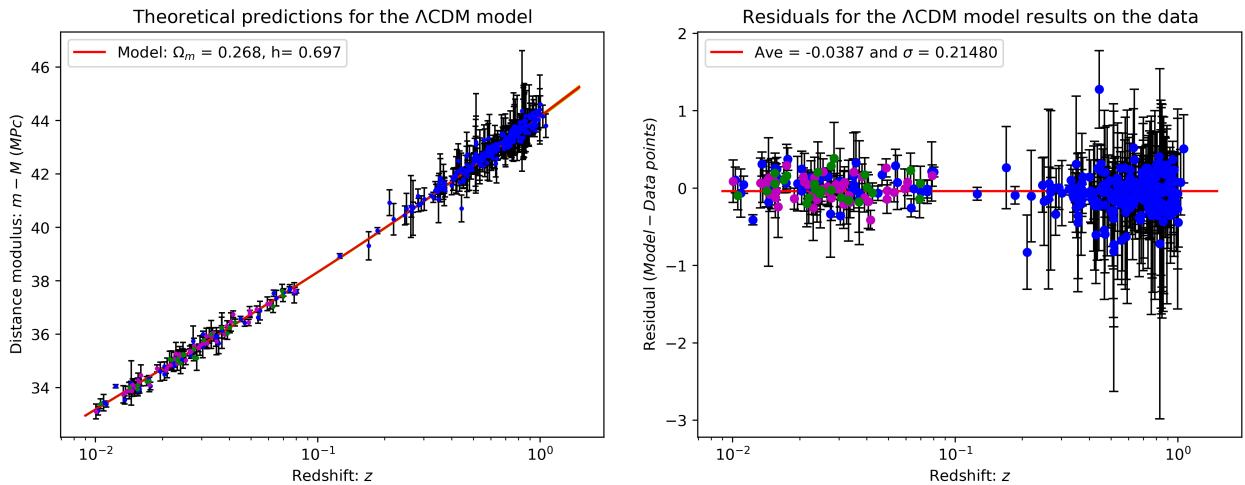


Figure 3.3: The Λ CDM model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.268^{+0.025}_{-0.024}$ and $H_0 = 69.7^{+0.5}_{-0.5} \frac{\text{km}}{\text{s.Mpc}}$, as shown in Figure 3.1.

distribution value that is lower to compensate for the higher Hubble constant value. If we then plot the Λ CDM-model using the results obtained in Figure 3.1 as a function of redshift on the supernovae data, we obtain Figure 3.3.

From Figure 3.3, we can determine that the Λ CDM model, which includes dark energy, can accurately fit to the data. This is made clear in Panel 1 (Red line) of Figure 3.3, which is the predicted model, while the “green lines” on either side of our model show the error on our calculation¹³. To confirm that this is the best fit, we also obtain the residual distance between the predicted model value and supernovae data points at a certain redshift. This is shown in Panel 2 of Figure 3.3. We obtained the average residual distance as $\bar{x}_{res} = -0.0387$ and a standard deviation on this average of $\sigma_{res} = 0.21480$. Since the average distance between the predicted value and the actual data point is close to zero, we can conclude that this is the most probable best fit for the Λ CDM model.

We will use these cosmological parameter results (including their error) to set priors for the $f(R)$ -gravity models. We can do this since we assumed in both cases a flat universe ($\Omega_k = 0$) and that the radiation density is negligible $\Omega_r \approx 0$; meaning, that if these calculated cosmological parameters (Ω_m and $\bar{h}$) and the dataset stay “constant” in the different $f(R)$ -gravity models, the only difference between the Λ CDM model and these modified gravity models will be the dark energy and the arbitrary free parameters, respectively.

¹³It should be noted that the different coloured data points are to show which supernova’s absolute magnitude M was calculated in which research paper with the blue data points determined by [82], the green points by [83], and the magenta points by [84].

3.6 Statistical analysis: AIC and BIC

Since our code is working, we need to discuss how we will use the Λ CDM model results as the standard model, to which we can compare the $f(R)$ -gravity model results, to determine if they are viable. To achieve this, we will use the Akaike information criterion (AIC) and the Bayesian/Schwarz information criterion (BIC, SIC). In the paper by [101], they discuss the AIC and BIC selection for multimodel inference. According to them [101], model selection including variable selection (such as ours) is a biased versus variance trade-off, which is the statistical principle of parsimony. Since the principle of parsimony refers to different “events” being connected or behaving in the simplest/most economical in nature, the number of parameters for the model plays a role. According to [101], inference under models with too few parameters can be biased towards those parameters, such as the Λ CDM model using only the matter density (Ω_m) and the Hubble uncertainty parameter ($\bar{h}$) to explain the expansion of the Universe. While having models with too many parameters, such as the different $f(R)$ -gravity models, may lead to poor precision or identification of the different effects. These considerations call for a balance between under- and over-fitted models.

It must be noted that, in order to use the AIC and BIC selections, we will need a “true” model [101]. However, models are just approximations of reality or certain situations. We will, for that reason, use the Λ CMD model (concordance model) as our “true” model, to which the $f(R)$ -gravity models will be compared. In the papers by [70, 102], they follow the same route, in choosing the Λ CMD model as the “true” model and comparing the $f(R)$ -gravity models against it. The AIC and BIC are defined as:

$$\begin{aligned} \bullet \quad AIC &= -2 \ln (\mathcal{L}(\hat{\theta}|data)) + 2K, \\ \bullet \quad BIC &= -2 \ln (\mathcal{L}(\hat{\theta}|data)) + K \log(n), \end{aligned} \tag{3.21}$$

where K is the number of free parameters (such as 2 for the Λ CDM model), n the number of data points, and $\mathcal{L}(\hat{\theta}|data)$ is the likelihood function of the model at certain parameter value. Our likelihood function is given by Equation 3.18. Taking this into account, the AIC and BIC selections are

$$\begin{aligned} \bullet \quad AIC &= \sum_n \left[\frac{(\mu_{data} - \mu_{theoretical})^2}{\sigma^2} \right] + 2K, \\ \bullet \quad BIC &= \sum_n \left[\frac{(\mu_{data} - \mu_{theoretical})^2}{\sigma^2} \right] + K \log(n). \end{aligned} \tag{3.22}$$

According to [101], the individual AIC and BIC values are not interpretable, since they contain arbitrary constants. The suggestion is to rescale the AIC and BIC selections. For instance, let us say we have a AIC value for the “true” model of 50000 and an alternative model AIC value of 50008. These values mean nothing on their own. However, if you were to find the difference between the alternative model and the “true”, you will be able to interpret the standing of the

alternative model w.r.t. the “true” model. In this case, we have

$$\Delta IC = IC_i - IC_{true}. \quad (3.23)$$

It needs to be noted that this test between the alternative and “true” models will be used for both criteria. So depending on the information criterion you use, you will just add A or B into Equation 3.23. We are now able to define the ranges that will show us whether or not our $f(R)$ -gravity models are viable. First of all, there exist different ranges according to different papers. One such scale is the Jeffreys’s scale shown in [103], but it is also noted in [70, 103], **that this scale is not an exclusive scale and should be handled with care**. The second range can be found in the paper by [104], where they used ranges based on the Jeffreys’s scale, but with shifted range values allowing for bigger ranges. The third range (also the range that we will use), is shown in [101, 102]. These ranges are:

- $\Delta IC \leq 2$ – substantial support,
 - $4 \leq \Delta IC \leq 7$ – less support w.r.t “true” model,
 - $\Delta IC > 10$ – model has no observational support.
- (3.24)

Using this entire method, starting with rewriting the Friedmann equation (Eq. 3.2) into a dimensionless form using the normalised Hubble parameter (Eq. 3.6), all the way down to finding the AIC and BIC values for the different models (Eqs. 3.22) and comparing them to the “true” model (Eq. 3.23), we will be able to test the $f(R)$ -gravity models to determine if they are viable.

Supernovae cosmology in $f(R)$ gravity

4.1 The $f(R)$ -gravity Friedmann model

As we have seen from Section 3.1, we need to find the Friedmann equation in a form in which it is possible to make the transformation to $H(z)$, where z is the measurable quantity called redshift. However, from Section 2.4, it is clear that we are firstly required to simplify our Friedmann equation into a more “usable” form. From Section 2.4, we obtain a $f(R)$ -gravity Friedmann equation given by

$$H(t)^2 = \frac{\rho_m(t)}{3f'(R)} - \frac{\kappa}{a^2(t)} + \frac{1}{6} \left(R - \frac{f(R)}{f'(R)} \right) - H(t) \dot{R} \frac{f''(R)}{f'(R)}. \quad (4.1)$$

Substituting the expressions for the Ricci scalar (Eq. 2.33), and its first-order time derivative $\dot{R} = 6(\ddot{H} + 4H\dot{H})$, into Equation 4.1, we find

$$H^2 = \frac{\rho_m}{3f'} - \frac{\kappa}{a^2} + \frac{1}{6} \left[6(\dot{H} + 2H^2) - \frac{f}{f'} \right] - H \left[6(\ddot{H} + 4H\dot{H}) \right] \frac{f''}{f'}. \quad (4.2)$$

We have once again made the equation “neater” by not writing the function dependencies (for example: $\rho_m(t) \rightarrow \rho_m$). We can define different derivatives “versions” for the Hubble parameter, given as

$$\begin{aligned} \bullet \quad \dot{H} &= -H^2(1+q), \\ \bullet \quad \dot{H} &= \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}, \\ \bullet \quad \ddot{H} &= \frac{\ddot{\ddot{a}}}{a} - 3\frac{\dot{a}\ddot{a}}{a^2} + 2\frac{\dot{a}^3}{a^3}. \end{aligned} \quad (4.3)$$

Substituting these expressions into Equation 4.2 and simplifying, we obtain

$$H^2 = \frac{\rho_m}{3qf'} - \frac{\kappa}{qa^2} - \frac{f}{6qf'} - \frac{6}{q} \left(\frac{\dot{a}\ddot{a}}{a^2} + \frac{\dot{a}^2\ddot{a}}{a^3} - 2H^4 \right) \frac{f''}{f'}. \quad (4.4)$$

By using the Jerk parameter, defined by $j = \frac{\dot{a}\ddot{a}}{\dot{a}^3}$, we can mathematically manipulate the first term in the bracket of Equation 4.4, to find

$$\begin{aligned} \frac{\dot{a}\ddot{a}}{a^2} \frac{H^4}{H^4} &= \frac{\dot{a}\ddot{a}}{a^2} \frac{\dot{a}^4}{a^4} \frac{a^4}{\dot{a}^4}, \\ \Rightarrow \frac{\dot{a}\ddot{a}}{a^2} &= jH^4. \end{aligned} \quad (4.5)$$

Furthermore, we can manipulate the second term in the bracket of Equation 4.4, by using the deceleration parameter given by $q = -\frac{\ddot{a}a}{\dot{a}^2}$, and find

$$\begin{aligned} \frac{\dot{a}^2\ddot{a}}{a^3} \frac{H^4}{H^4} &= \frac{\dot{a}^2\ddot{a}}{a^3} \frac{\dot{a}^4}{a^4} \frac{a^4}{\dot{a}^4}, \\ \Rightarrow \frac{\dot{a}^2\ddot{a}}{a^3} &= -qH^4. \end{aligned} \quad (4.6)$$

Substituting the results of Equations 4.5 and 4.6 into Equation 4.4, we obtain

$$\begin{aligned} \Rightarrow H^2 &= \frac{\rho_m}{3qf'} - \frac{1}{qa^2K^2} - \frac{f}{6qf'} - \frac{6}{q} (jH^4 - qH^4 - 2H^4) \frac{f''}{f'}, \\ \Rightarrow H^2 &= \frac{1}{qf'} \left[\frac{\rho_m}{3} - \frac{f'}{a^2K^2} - \frac{f}{6} + 6H^4(2 + q - j)f'' \right]. \end{aligned} \quad (4.7)$$

For simplicity, we assume a flat universe $\Omega_k = 0$, with a negligible radiation density distribution $\Omega_r \approx 0$ (Sections 1.4.3 and 3.5). Taking these assumptions into account, we obtain our $f(R)$ -gravity model to be

$$H^2 = \frac{1}{qf'} \left[\frac{\rho_d}{3} - \frac{f}{6} + 6H^4(2 + q - j)f'' \right]. \quad (4.8)$$

4.1.1 GR-limit test: Equation 4.8

Before we can continue, we need to test Equation 4.8, to verify whether or not this equation holds in the GR limits. To do this, we need $f(R) = R$, since this was the original change we made in the Einstein-Hilbert action. Our first- and second-order derivatives are then calculated to be $f'(R) = 1$ and $f''(R) = 0$. Equation 4.8 then becomes

$$H^2 = \frac{\rho_d}{3q} - \frac{R}{6q}. \quad (4.9)$$

By making the substitutions $\frac{R}{6} = \dot{H} + 2H^2$ and $qH^2 = -\dot{H} - H^2$ into Equation 4.9 (the second substitution expression comes from the definition $\dot{\Theta} = -\frac{1}{3}\Theta^2(1 + q)$ with $\Theta = 3H$ as defined in

Section 2.4), we obtain

$$\Rightarrow -\dot{H} - H^2 = \frac{\rho_d}{3} - (\dot{H} + 2H^2), \quad (4.10)$$

which can then be simplified to

$$H^2 = \frac{\rho_d}{3}. \quad (4.11)$$

Equation 4.11 is the same equation as derived in Section 1.3.2.2 (Eq. 1.31), namely the GR-Friedmann model. The only difference comes from these assumptions: 1) all the constants, such as the speed of light, are equal to 1; 2) we are using a flat universe $\kappa = 0$; and 3) the radiation density distribution $\rho_r \approx 0$ is negligible. Furthermore, you will also notice that Equation 4.11 does not contain a dark energy density distribution ρ_Λ . This is because the alternative gravity models, such as the Λ CDM model or modified gravity models, try to explain the accelerated expansion of the Universe with either the inclusion of dark energy or modifying the existing GR terms.

Since our $f(R)$ -Friedmann model equation reduces back to GR (a fundamental correctional test), we can continue to test the different $f(R)$ -gravity models against supernovae data. We will be examining four different models. These include two toy models, the general Starobinsky model, and the Hu-Sawicki model.

4.2 First toy model: $f(R) = \beta R^n$

Our first model is defined as $f(R) = \beta R^n$, where $\beta = \beta(n)$ is a running coupling constant, such that for GR we have $\beta = 1$, when $n = 1$. These $f(R)$ -gravity models can then be re-parametrised to be dimensionless in such a way that we obtain

$$f(R) = \beta H_0^{2(1-n)} R^n. \quad (4.12)$$

From Equation 4.8, it is clear that we first need to obtain the first-order and second-order derivatives of Equation 4.12. These are obtained to be:

$$\begin{aligned} \bullet \quad f'(R) &= n\beta H_0^{2(1-n)} R^{n-1}, \\ \bullet \quad f''(R) &= n(n-1)\beta H_0^{2(1-n)} R^{n-2}. \end{aligned} \quad (4.13)$$

We can then substitute Equations 4.12 and 4.13 into our $f(R)$ -Friedmann equation (Eq. 4.8)¹, and obtain

$$H^2 = \frac{\rho_m}{3qn\beta H_0^{2(1-n)} R^{n-1}} - \frac{\beta H_0^{2(1-n)} R^n}{6qn\beta H_0^{2(1-n)} R^{n-1}} + \frac{6H^4}{qn\beta H_0^{2(1-n)} R^{n-1}} (2+q-j)n(n-1)\beta H_0^{2(1-n)} R^{n-2}. \quad (4.14)$$

¹We used ρ_m and not ρ_d as explained in Section 3.5, namely that $\rho_m \approx \rho_d$ when ρ_r is negligible. We will follow this route for all the different $f(R)$ -gravity models.

We simplify this equation, to find

$$H^2 = \frac{\rho_m}{3qn\beta H_0^{2(1-n)} R^{n-1}} - \frac{R}{6qn} + \frac{6H^4(n-1)}{qR}(2+q-j). \quad (4.15)$$

We can then substitute the Ricci scalar (given in terms of the deceleration parameter $R = 6H^2(1-q)$) into our equation and obtain

$$H^2 = \frac{\rho_m}{3qn\beta H_0^{2(1-n)} [6H^2(1-q)]^{n-1}} - \frac{6H^2(1-q)}{6qn} + \frac{6H^4(n-1)}{q6H^2(1-q)}(2+q-j). \quad (4.16)$$

After about a half page of pure algebra in which you multiply both sides of the equation with $H^{2(n-1)}$, and simplifying we obtained

$$\Rightarrow H^2 = \left[\frac{\rho_m}{3qn\beta H_0^{2(1-n)} 6^{n-1} (1-q)^{n-1} \left(1 + \frac{(1-q)}{qn} - \frac{(n-1)}{q(1-q)} (2+q-j) \right)} \right]^{\frac{1}{n}}. \quad (4.17)$$

We can now divide both sides with H_0^2 , as we have done in the case of the Λ CDM model in Section 3.1, to obtain

$$\Rightarrow \frac{H^2}{H_0^2} = \frac{1}{H_0^2} \sqrt[n]{\frac{\rho_m}{3\beta H_0^{2(1-n)}} \left[\frac{1}{qn6^{n-1}(1-q)^{n-1} \left(1 + \frac{(1-q)}{qn} - \frac{(n-1)}{q(1-q)} (2+q-j) \right)} \right]}^{\frac{1}{n}}. \quad (4.18)$$

We can then simplify this equation by using $\Omega_m = \frac{\rho_m}{3H_0^2}$ (as defined in Section 1.4.3), and find

$$\Rightarrow \frac{H^2}{H_0^2} = \sqrt[n]{\frac{\Omega_m}{\beta} \left[\frac{1}{qn6^{n-1}(1-q)^{n-1} \left(1 + \frac{(1-q)}{qn} - \frac{(n-1)}{q(1-q)} (2+q-j) \right)} \right]}^{\frac{1}{n}}. \quad (4.19)$$

We then define the normalised Hubble parameter once again as $h(t) = \frac{H(t)}{H_0}$. Furthermore, we need to remember that the parameters Ω_m , q , and j are all time dependent, while β and n are just arbitrary constants. Therefore, we have

$$h^2(t) = \sqrt[n]{\frac{\Omega_m(t)}{\beta} \left[\frac{1}{q(t)n6^{n-1}(1-q(t))^{n-1} \left(1 + \frac{(1-q(t))}{q(t)n} - \frac{(n-1)}{q(t)(1-q(t))} (2+q(t)-j(t)) \right)} \right]}^{\frac{1}{n}}. \quad (4.20)$$

Before we can complete this derivation to find $h(z)$, we first need to find a way to parametrise the deceleration parameter $q(t)$ and the jerk parameter $j(t)$ in terms of redshift. We have done this in Subsection 4.2.1. We can now make the transition for the normalised Hubble parameter to a function of redshift, by substituting the deceleration and jerk parametrisation Equations 4.24 and

4.26 into Equation 4.20. We find

$$h(z) = \sqrt[2n]{\frac{\Omega_m(1+z)^3}{\beta}} \left[\frac{1}{\left(q_0 + q_1 \frac{z}{z+1}\right) n 6^{n-1} \left(1 - \left(q_0 + q_1 \frac{z}{z+1}\right)\right)^{n-1}} \right. \\ \left. \frac{1}{\left(1 + \frac{1-q_0-q_1 \frac{z}{z+1}}{n \left(q_0 + q_1 \frac{z}{z+1}\right)} - \frac{(n-1)}{q_0 + q_1 \frac{z}{z+1} - \left(q_0 + q_1 \frac{z}{z+1}\right)^2} \left(2 + q_0 + q_1 \frac{z}{z+1} - \left(q_0 + q_1 \frac{z}{z+1}\right) \left(2 \left(q_0 + q_1 \frac{z}{z+1}\right) + 1\right) - \frac{q_1}{1+z}\right)\right)} \right]^{\frac{1}{2n}}. \quad (4.21)$$

We can once again show that this equation does indeed reduce back to GR, confirming its validity. To show this, we have to use the requirement $\beta = 1$ and $n = 1$ (as noted at the start of this section), since these approximations will give $f(R) = \beta H_0^{2(1-n)} R^n = R$. When substituting this requirement into Equation 4.21 and simplifying, we find

$$\Rightarrow h(z) = \sqrt{\Omega_m(1+z)^3}. \quad (4.22)$$

This is the same as the GR model's Friedmann equation, if we assume that the radiation density distribution is negligible $\Omega_r \approx 0$ and that the Universe is flat $\Omega_k = 0$. This confirms that the only difference between the Λ CMD model and this particular $f(R)$ -gravity model is the dark energy term Ω_Λ and the arbitrary free parameters n and β , respectively.

4.2.1 Deceleration and jerk parameter parametrisations

The parametrisations for the deceleration and jerk parameters were found in the research paper by [105]. Starting with the deceleration parameter, we have two possible parametrisations, namely

$$q(z) = q_0 + q_1 z, \quad (4.23)$$

and

$$q(z) = q_0 + q_1 \frac{z}{z+1}. \quad (4.24)$$

We will be using the second parametrisation (Eq. 4.24). The reason for this selection is due to a generalised parametrisation for $q(z)$ that can be found in the paper by [6]. The general form of $q(z)$ is given as

$$q(z) = q_0 - q_1 \left[\frac{(1+z)^{-\alpha} - 1}{\alpha} \right], \quad (4.25)$$

where $\alpha = +1$, resulting in a current deceleration parameter value of $q_0 = -0.537_{-0.11}^{+0.109}$, which is in the vicinity of the observed value [106]. In the case that $\alpha = +1$, we obtain the second parametrisation (Eq. 4.24). It is worth noting that the reason we will not be using the first parametrisation (Eq. 4.23), is because, in the research paper by [6], they explain that this parametrisation becomes

unreliable at high redshift. Even though we will not be using high redshift data (only low and intermediate redshift), some of our higher redshift data points may put our results at risk.

As for the jerk parametrisation, we found that it is possible to find the jerk parameter in terms of the deceleration parameter as [6]

$$j(q) = q(2q + 1) + (1 + z) \frac{dq}{dz}. \quad (4.26)$$

Since we need $\frac{dq}{dz}$, we must first find the derivative of Equation 4.24 with regard to z . This is found to be

$$q'(z) = \frac{q}{(1 + z)^2}. \quad (4.27)$$

We can then substitute this result into Equation 4.26, and obtain

$$\Rightarrow j(q) = q(2q + 1) + \frac{q_1}{1 + z}. \quad (4.28)$$

Substituting Equation 4.24 into this result, we obtain the jerk parameter as a function of the deceleration parameter as

$$\Rightarrow j(q(z)) = \left(q_0 + q_1 \frac{z}{z + 1} \right) \left[2 \left(q_0 + q_1 \frac{z}{z + 1} \right) + 1 \right] + \frac{q_1}{1 + z}. \quad (4.29)$$

The current estimate value for the deceleration parameter is $q_0 = -0.537^{+0.109}_{-0.11}$, with a deceleration correction parameter value of $q_1 = 1.276^{+0.276}_{-0.281}$ within 1σ [6].

4.2.2 $f(R) = \beta R^n$ model results

Since we have obtained the normalised Hubble parameter $h(z)$ for our toy model $f(R) = \beta R^n$, namely Equation 4.21, we can insert Equation 4.21 into the distance modulus equation given by

$$\mu(z) = 25 + 5 \times \log_{10} \left(3000 \bar{h}^{-1} (1 + z) \int_0^z \frac{dz'}{h(z')} \right). \quad (4.30)$$

²It is worth noting that between Equations 4.21 and 4.30, we acquired six free parameters. We will use the following constraints for a flat Λ CDM universe given as:

- $0.1 < \Omega_m < 0.4$,
 - $0.60 < \bar{h} < 0.75$,
 - $-0.8 < q_0 < -0.01$,
 - $-0.75 < q_1 < 1.0$.
- (4.31)

²The distance modulus is once again given in Mpc (as done in Section 3.3). We will continue through-out this chapter using it in terms of Mpc .

The first two constraints are based on the Λ CDM model's results calculated in Section 3.5. However, we have increased the search width on both these parameters, since it was found that the errors are so small that the MCMC simulation struggled with moving around in the parameter space. Therefore, the results would seem like it found the best fit, but in reality all the proposed steps were rejected and simulation got stuck on one value. Even though we increased the search width on both, we tried to restrict the code to search for values that are still in the range of the Λ CDM model. This increased range is close to the chosen priors for Ω_m and $\bar{h}$ that were given in the research paper [102]. Our prior for Ω_m has the same upper-limit, we just extended the lower-limit a bit and the prior for $\bar{h}$ has a similar range. Another prior that might be useful for future work might be the usage of the Riess prior for the Hubble constant. The other two constraints are found in the research paper [107]³, where they determined the constraints for q_0 and q_1 on Type 1a supernovae, assuming a flat Λ CDM universe.

Since all of the above given constraints (including 3 cosmological parameters) are based on a flat Λ CDM universe with a negligible radiation density $\Omega_r \approx 0$, our remaining parameters, namely β and n , are the only difference between the Λ CDM model (using dark energy) and the $f(R)$ -gravity model. Therefore, whatever the outcome for the free parameter is, we will be able to find some conclusion for their viability to explain the accelerated expansion without dark energy.

The requirements for the MCMC simulation are the above given priors and the set-up parameters (6 dimensions, 100 random walkers and 10 000 iterations). Due to the corner plot reaching the upper-limit of the number of information allowed to plot on a PDF/png file using Matplotlib⁴, we had to lower the amount of iterations from the Λ CDM model set-up. This meant that we have a lower accuracy than for the Λ CDM model, but still accurate enough to use. The results for the first toy model's distance modulus equation (Eq. 4.30) are shown in Figure 4.1.

From Figure 4.1, we obtained quite a few different information packets. Firstly, we did obtain plausible matter density distribution (Ω_m) and Hubble uncertainty ($\bar{h}$) parameter values, with the Planck 2018 results [4] well within our error region. However, the search region did not deliver a Gaussian distribution, but more of an average best fit parameter value over the entire search range.

When it comes to the deceleration parameter and its correction term, we found that q_0 is prone towards the upper-bound of the chosen prior, suggesting that this model would most likely fit better when allowed to search for a decelerating universe. The correction term also shows that the parameter space is not large enough (especially towards the negative regions) when restricting q_0 to an accelerating universe. Therefore, a bigger correction term search range is necessary to give more freedom to q_0 to find an appropriate value explaining an accelerating expansion.

³We increased this search width as well to accommodate the larger possible cosmological parameter values.

⁴For instance, we obtained a PDF image for this model's results that reached 7MB in size, which also meant that this dissertation PDF was lagging and in some cases the plotting would fail completely and no results were obtained.

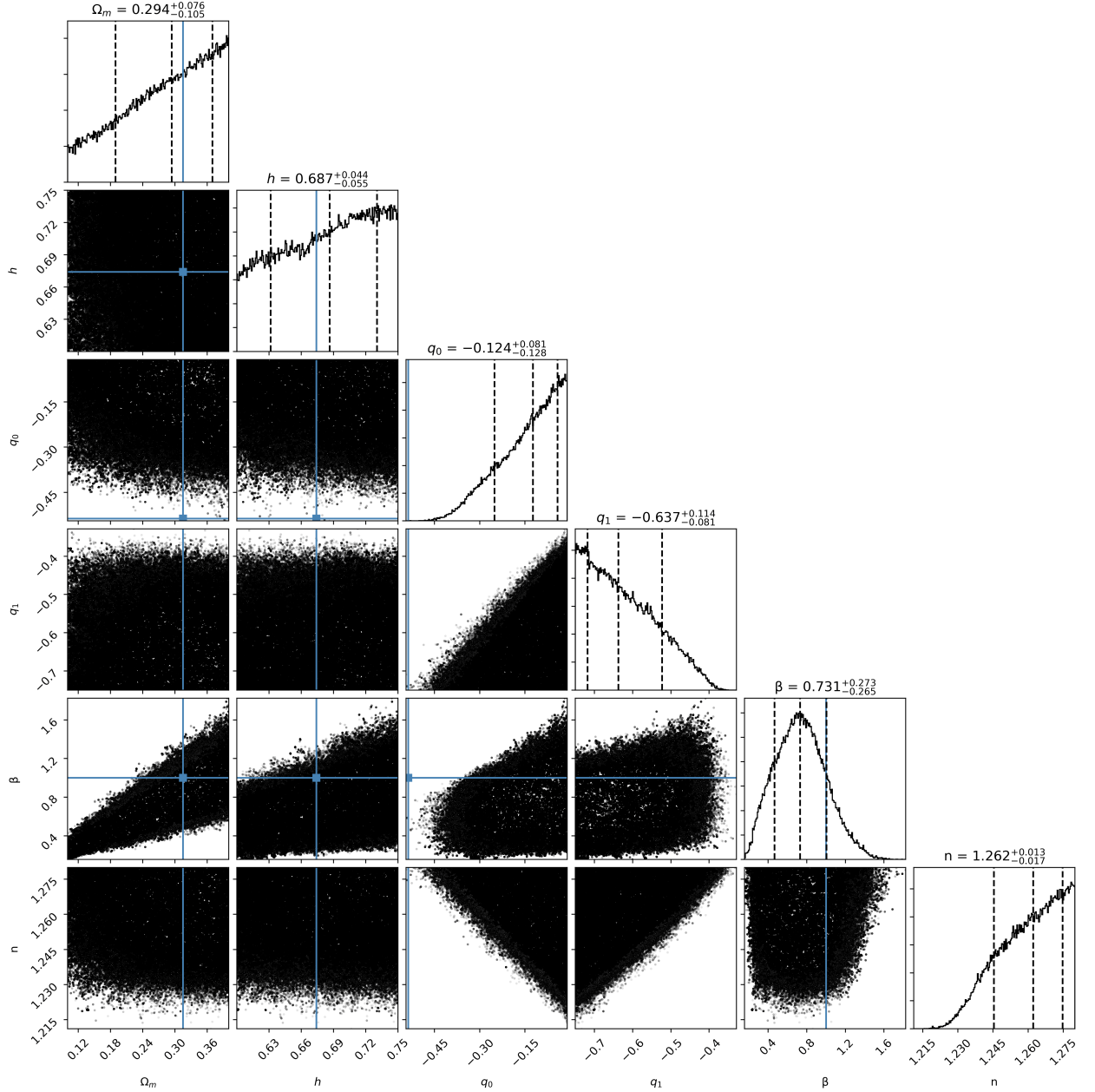


Figure 4.1: MCMC simulation results for the first toy model’s (Eq. 4.30) cosmological parameters (Ω_m , $\bar{h}$, q_0 , and q_1), as well as the model’s arbitrary free parameters (β and n). The “true” values for Ω_m and $\bar{h}$ (blue lines) are provided by Planck 2018 collaboration data release [4], while the “true” values for q_0 and q_1 are obtained from [6]. We used 100 random walkers, 10 000 iterations and priors provided by Equations 4.31 and 4.32. The blue lines for arbitrary free parameter values are to show their initial chosen starting point for the MCMC simulation.

The arbitrary free parameters gave plausible and realistic values (in the sense that the numbers are in proportion to the other free parameters). However, for the arbitrary free parameters, we had to manually constrain them using the priors given in Equation 4.32. This was necessary, since we found that if $n > 1.35$, the integral in Equation 4.30 became infinite. We had to lower the upper-bound even more, due to certain combinations between the different free parameters also resulting in an infinite integral. This is an important piece of information, because this arbitrarily

chosen cut-off below $n \leq 1.35$ could influence the outcome of the other parameters such as q_0 also being stopped by its upper bound. It may be that there lies a combination between $1.28 \leq n < 1.35$ and $q_0 \approx -0.5$, but due to the search limitations, the best-fit combination may have been altered.

Lastly, an important consideration based on this MCMC simulation results must be made due to the nature of the results not all being Gaussian distributions as assumed in Chapter 3. Starting off with the Hubble uncertainty parameter $\bar{h}$, we notice that the MCMC simulation results predict a relatively uniform distribution. This is known as an unconstrained parameter value. Meaning that all parameter values within the prior range have relatively an equal probability of occurring. Therefore, the MCMC simulation was not able to constrain this particular parameter for this particular model in this dataset. It does not mean that any $\bar{h}$ value will always be possible, especially when using other datasets or different models. As for the parameters Ω_m , q_0 , q_1 and n , they all received one-sided tails of the Gaussian distribution. This means that the MCMC simulation did find some possible parameter values that did explain the data better than others, but due to shifting the average and the 1σ deviation with it only being one side of the Gaussian distribution, these results must be used carefully in further studies. These results are known as unreliably constraints. This leaves β as our only truly constrained parameter for this model on this dataset.

The reason this might have happened is due to our chosen priors that is limiting the search regions. For the other MCMC results, I will refer to unconstrained, unreliably unconstrained and constrained when referring to the different parameters. If we do, however, proceed by assuming the results predicted (Fig. 4.1) by the MCMC simulation are indeed the best fit, even if some of the parameters are not constrained, we can plot the model to the supernovae data. This is shown in Figure 4.2:

- $0.01 < \beta < 5$,
 - $0.1 < n < 1.28$.
- (4.32)

From Figure 4.2, even though it is not that clear, the model fits the data at a low redshift, but under-estimates the data at a redshift higher than $z > 0.6$. This can be attributed to the n -value constraint. Furthermore, we obtained an average residual of $\bar{x}_{res} = -0.1405$ (showing MCMC simulation struggles to find a best fit for the model, results in an overall under-estimation) with a standard deviation of $\sigma_{res} = 0.21246$. Therefore, even though it is not clear, the low-redshift part of the model must show some favouritism to an over-estimation. This means that this toy model struggles with the transition phase between a decelerating expansion and an accelerating expansion of the Universe, as is evident with the MCMC simulation prediction for q_0 . The best-fitting model is given by

$$f(R) = 0.731_{-0.265}^{+0.273} R^{1.262_{-0.017}^{+0.013}}. \quad (4.33)$$

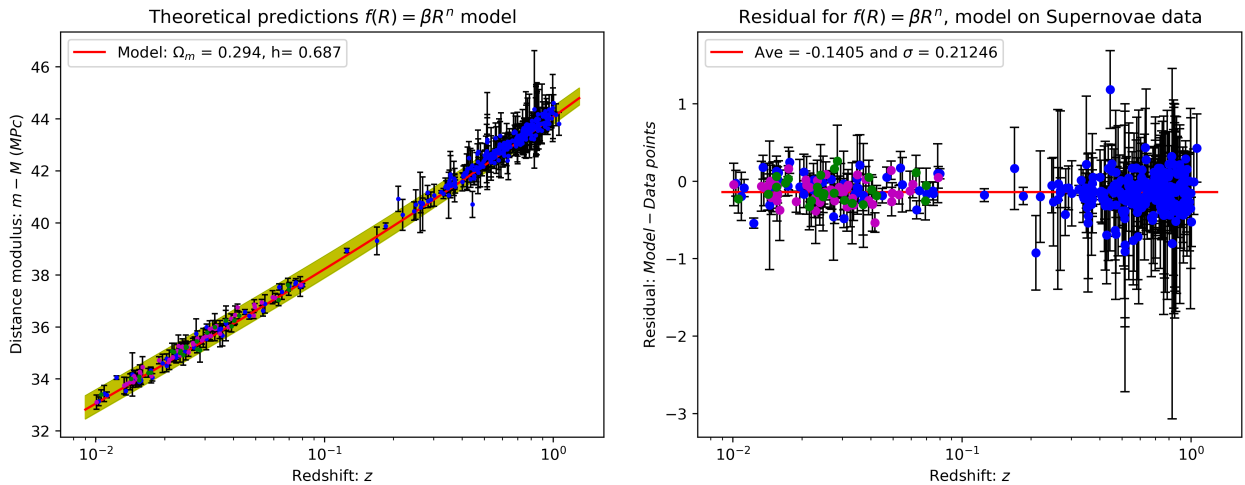


Figure 4.2: The first toy model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.294^{+0.076}_{-0.105}$ (unreliably constrained), $H_0 = 68.7^{+4.4}_{-5.5} \frac{\text{km}}{\text{s.Mpc}}$ (unconstrained), $q_0 = -0.124^{+0.081}_{-0.128}$ (unreliably constrained), and $q_1 = -0.637^{+0.114}_{-0.081}$ (unreliably constrained), as shown in Figure 4.1. The arbitrary free parameters are calculated to be $\beta = 0.731^{+0.273}_{-0.265}$ (constrained) and $n = 1.262^{+0.013}_{-0.017}$ (unreliably constrained).

4.3 Second toy model: $f(R) = \alpha R + \beta R^n$

Our second toy model is defined as $f(R) = \alpha R + \beta R^n$. This $f(R)$ -gravity model is a generalisation of both the GR model and the $f(R) = R^n$ model. If we have $\alpha = 1$ and $\beta = 0$, the model reduces back to the GR model, while having $\alpha = 0$ reduces back to the $f(R) = R^n$ model [23]. We can then once again re-parametrise this $f(R)$ -gravity model to find a dimensionless equation, given by

$$f(R) = \alpha R + H_0^{2(1-n)} \beta R^n, \quad (4.34)$$

where $H_0^{2(1-n)}$ will ensure that we have a dimensionless $f(R)$ -gravity model. We obtain the first-order and second-order derivatives of Equation 4.34 as

$$\begin{aligned} \bullet \quad f'(R) &= \alpha + n H_0^{2(1-n)} \beta R^{n-1}, \\ \bullet \quad f''(R) &= n(n-1) H_0^{2(1-n)} \beta R^{n-2}. \end{aligned} \quad (4.35)$$

If we then substitute Equations 4.34 and 4.35, into our $f(R)$ -Friedmann equation (Eq. 4.8), we obtain

$$H^2 = \frac{\rho_m}{3q(\alpha + n H_0^{2(1-n)} \beta R^{n-1})} - \frac{\alpha R + H_0^{2(1-n)} \beta R^n}{6q(\alpha + n H_0^{2(1-n)} \beta R^{n-1})} + \frac{6H^4(2+q-j)}{q} \left[\frac{n(n-1) H_0^{2(1-n)} \beta R^{n-2}}{\alpha + n H_0^{2(1-n)} \beta R^{n-1}} \right]. \quad (4.36)$$

Since the process for calculating the mathematical steps remains the same, and the calculations becoming longer and more complex, we will be using the Maple mathematics software to do the

calculations. We start by substituting the Ricci scalar into Equation 4.36, and we obtain

$$H^2 = \frac{\rho_m}{3q(\alpha + nH_0^{2(1-n)}\beta 6^{n-1}H^{2(n-1)}(1-q)^{n-1})} - \frac{\alpha 6H^2(1-q) + H_0^{2(1-n)}\beta 6^n H^{2n}(1-q)^n}{6q(\alpha + nH_0^{2(1-n)}\beta 6^{n-1}H^{2(n-1)}(1-q)^{n-1})} + \frac{6H^4(2+q-j)}{q} \left[\frac{n(n-1)H_0^{2(1-n)}\beta 6^{n-2}H^{2(n-2)}(1-q)^{n-2}}{(\alpha + nH_0^{2(1-n)}\beta 6^{n-1}H^{2(n-1)}(1-q)^{n-1})} \right]. \quad (4.37)$$

We can now solve for H^2 , using the Maple program, and we obtain

$$H^2 = \text{RootOF} \left(6(q-1) \left[\beta H_0^2 \left[\alpha H^2(j-2-q)n^2 + \left(\frac{\rho_m(q-1)}{3} - H^2(q^2+j-q-3)\alpha \right)n + \alpha H^2(q-1)^2 \right] \left(-6H^2(q-1) \right)^n + 6H^2\alpha(q-1)^2 \left(\alpha H^2 - \frac{\rho_m}{3} \right) H_0^{2n} \right] - \left[(j-2-q)n^2 + (2q-q^2-j+2)n + (q-1)^2 \right] \beta^2 H_0^{2(2-n)} n \left(-6H^2(q-1) \right)^{2n} \right). \quad (4.38)$$

From the above equation, we see it is not possible to find an exact solution for this particular $f(R)$ -gravity model. However, we can still use this model. We start by testing the equation to see whether it holds in the GR limits. For this model, there are two possible ways of testing this requirement. Either we have $\alpha = 1$ and $\beta = n = 0$, or we have $\alpha = 0$ and $\beta = n = 1$. We will present the first case. If we set $n = 0$ in Equation 4.38, we obtain

$$6(q-1) \left[\beta H_0^2 \alpha H^2(q-1)^2 + 6H^2\alpha(q-1)^2 \left(\alpha H^2 - \frac{\rho_m}{3} \right) \right] = 0. \quad (4.39)$$

We can now set $\alpha = 1$ and $\beta = 0$, and find

$$36(q-1)^3 H^2 \left(H^2 - \frac{\rho_m}{3} \right) = 0. \quad (4.40)$$

When solving for H^2 , we obtain two possible answers, namely

$$\bullet \quad H^2 = 0, \quad \text{or} \quad \bullet \quad H^2 = \frac{\rho_m}{3}. \quad (4.41)$$

$H^2 = 0$ would result in a stationary (non-expanding or -contracting) universe. Since our observations are contradicting this solution, it is not valid. However, the second solution does correspond to a flat universe, with a negligible radiation density distribution in the GR model. Therefore, we can conclude that this model does reduce back to GR in its limits.

4.3.1 Different solutions for Equation 4.38: Fixed n -values

Although it was possible to show that this $f(R)$ -gravity model's Friedmann equation (Eq. 4.38) reduced back to the GR model, we still have the problem that the model does not have an exact solution. We did, however, notice that the substitution of the n -value into Equation 4.38 simplified it to a large extent. So we will attempt to find solutions for H^2 , using discrete values for n , while keeping α and β as arbitrary free parameters. As seen in the first toy model's distance modulus (Eq. 4.30), the model was only able to obtain a finite integral for the values between $n \leq 1.35$ (which we still had to lower to avoid unpleasant error in the MCMC simulation). Since $f(R) = \alpha R + \beta R^n$ is related to this model, we will use some of the allowed values for n with some leniency, in case this model can use higher n -values. The chosen values for n is:

$$n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2. \quad (4.42)$$

The reason for the inclusion of $n = 2$ is because Starobinsky showed that the R^2 corrections to the standard GR gravitational action (i.e. Lagrangians of the form $f(R) = R + \beta R^2$) produce a de Sitter phase of expansion [108]. A de Sitter universe, named after Willem de Sitter, is a model that assumes a spatially flat universe, but neglects any form of ordinary matter, namely $\Omega_m = 0$. This model was created to explain the cosmic inflation model right after the Big Bang, but we can use this model to give us insight into the cosmological constant/dark energy unknown pressure force at a late-time cosmic acceleration, since ordinary matter becomes negligible. The free parameter β in this specific model can then be defined as $\beta \equiv \frac{1}{6M^2}$, where M represents the characteristic mass scale of the Universe [23].

4.3.1.1 Second toy model: $n = 0$

If we insert $n = 0$ into Equation 4.38, as before, we obtain

$$6(q-1) \left[\beta H_0^2 \alpha H^2 (q-1)^2 + 6\alpha H^2 (q-1)^2 \left(\alpha H^2 - \frac{\rho_m}{3} \right) \right] = 0, \quad (4.43)$$

which we can solve for H^2 and once again obtain two possible solutions, given by:

$$\bullet \quad H^2 = 0 \quad \text{or} \quad \bullet \quad H^2 = \frac{\rho_m}{3\alpha} - \frac{\beta H_0^2}{6\alpha}. \quad (4.44)$$

We will use the second solution again (expanding universe). To obtain the normalised Hubble parameter equation for the expansion of the Universe, we divide both sides by H_0^2 , and obtain

$$h(t) = \sqrt{\frac{1}{\alpha} \left(\Omega_m(t) - \frac{\beta}{6} \right)}. \quad (4.45)$$

Changing the time dependency to a function of redshift, we find

$$h(z) = \sqrt{\frac{1}{\alpha} \left(\Omega_m (1+z)^3 - \frac{\beta}{6} \right)}. \quad (4.46)$$

We can then substitute this into the distance modulus model (Eq. 4.30). It is worth mentioning that α cannot be equal to zero⁵, since a value for $\alpha = 0$, in combination with $n = 0$, would result in $f(R) = \beta$. This would mean that the Einstein-Hilbert action will not any more be dependent on the Ricci scalar, which will result in a different field equation (Eq. 2.7). Due to this constraint, as well as the Λ CDM model results constraints, we chose our priors for the MCMC simulation to be:

$$\begin{aligned} & \bullet \quad 0.0 < \Omega_m < 0.4, \\ & \bullet \quad 0.6 < \bar{h} < 0.8, \\ & \bullet \quad -100 < \beta < 0.0, \\ & \bullet \quad 0.1 < \alpha < 100. \end{aligned} \quad (4.47)$$

The question that may arise is, why we only chose negative values for the β parameter, even though there exist combinations where β can be positive, given by this relationship

$$\beta \leq 6\Omega_m(1+z)^3. \quad (4.48)$$

The reason for this is due to the search patterns of the *EMCEE Hammer* package, which will always try the boundaries. Let us assume the matter density to be Ω_m to 0.01, which means that the Universe consist only of 1% matter, this would still result in an upper-bound for the β parameter to be of the order $\beta \leq \times 10^{-2}$, for low redshift supernovae and an upper-limit for β of the order $\times 10^{-1}$ for high redshift supernovae. If the package were to search for an even lower matter density distribution content of the Universe (which it will do), the β parameter upper-limit will also decrease. So for simplification (and to reduce our errors the MCMC simulation can obtain), we will just use negative values for the β parameter.

We once again execute the MCMC simulation for 10 000 iterations, 100 random walkers, and burn the first 500 iterations. The results are shown in Figure 4.3. From these results, we see that the best-fit cosmological parameters Ω_m and $\bar{h}$ are within 1σ from the Planck 2018 CMB results. However, we do notice that their Gaussian distributions are not complete, in the sense that Ω_m obtained an unreliably constrained and $\bar{h}$ was left unconstrained. This is a result of our chosen priors (which prohibits the search for other possible solutions), since we want to constrain the arbitrary free parameters α and β to be valid in the same region as the Λ CMD model. We also noticed that both the arbitrary free parameters α and β have large errors on them, showing that this model is not 100% stable even though they were constrained in this search space.

⁵It will also result in a mathematical error, with a division by zero.

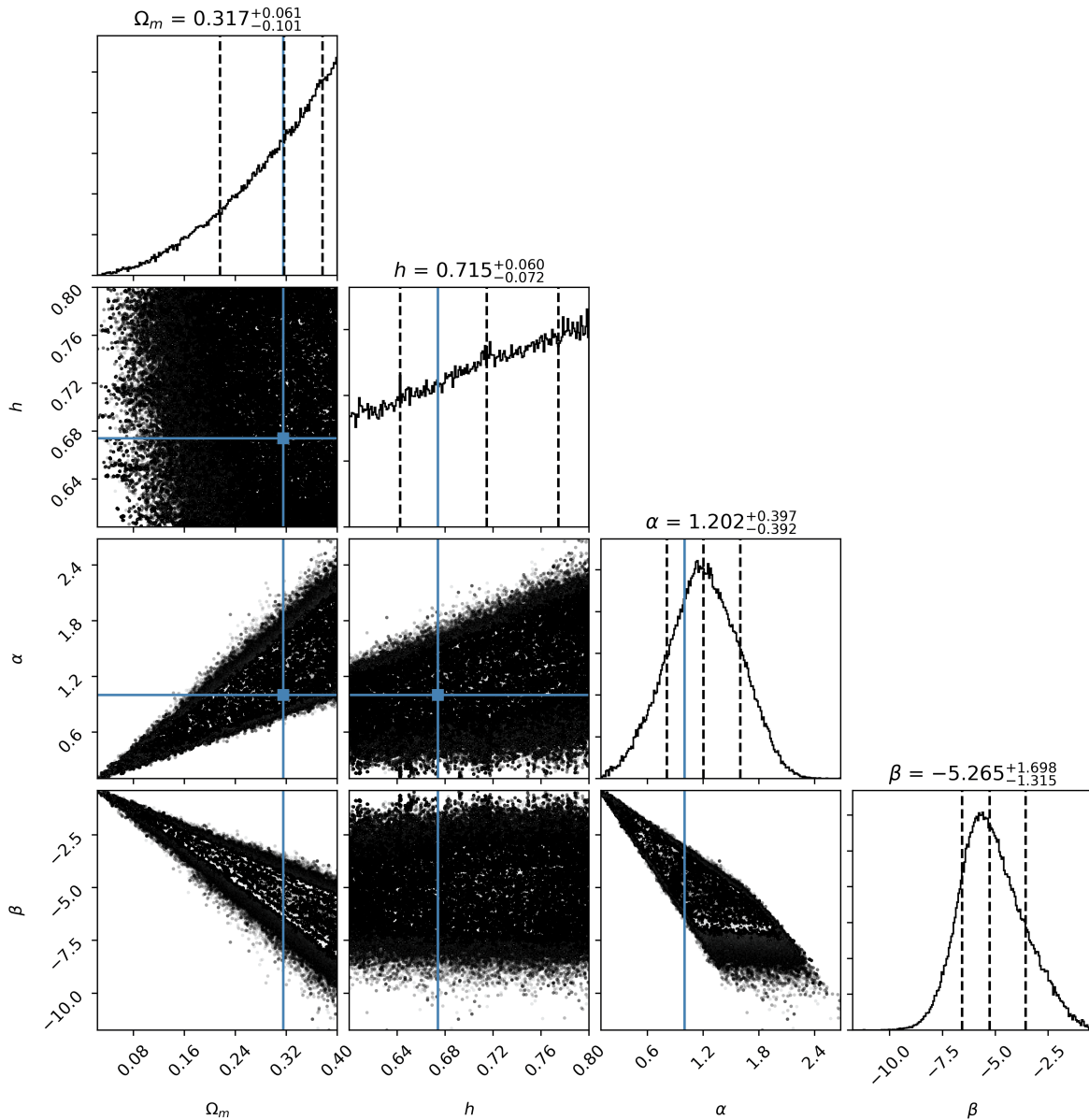


Figure 4.3: MCMC simulation results for the second toy model's (Eq. 4.46 with $n = 0$) cosmological parameters (Ω_m , h), as well as the model's arbitrary free parameters (α and β). The “true” values for Ω_m and h (blue lines) are provided by Planck 2018 collaboration data release [4]. We used 100 random walkers, 10 000 iterations and priors provided by Equation 4.47. The blue lines for arbitrary free parameter values are to show their initial chosen starting point for the MCMC simulation.

The results for the model's cosmological parameters are shown in Figure 4.4, as calculated by the MCMC simulation. We see that the model (red line) fits the Supernovae Type 1A data with an average residual of $\bar{x}_{res} = -0.0856$ and a standard deviation of $\sigma_{res} = 0.21463$. However, the error region for the model in Figure 4.4 is relatively large in comparison to the spread of the data, especially in the low-redshift regime. Even though this model contains a larger error region for the late-time acceleration period, the model still seems to predict the data with good accuracy, since both the cosmological parameters are within 1σ from the Planck 2018 results [4]. This result is

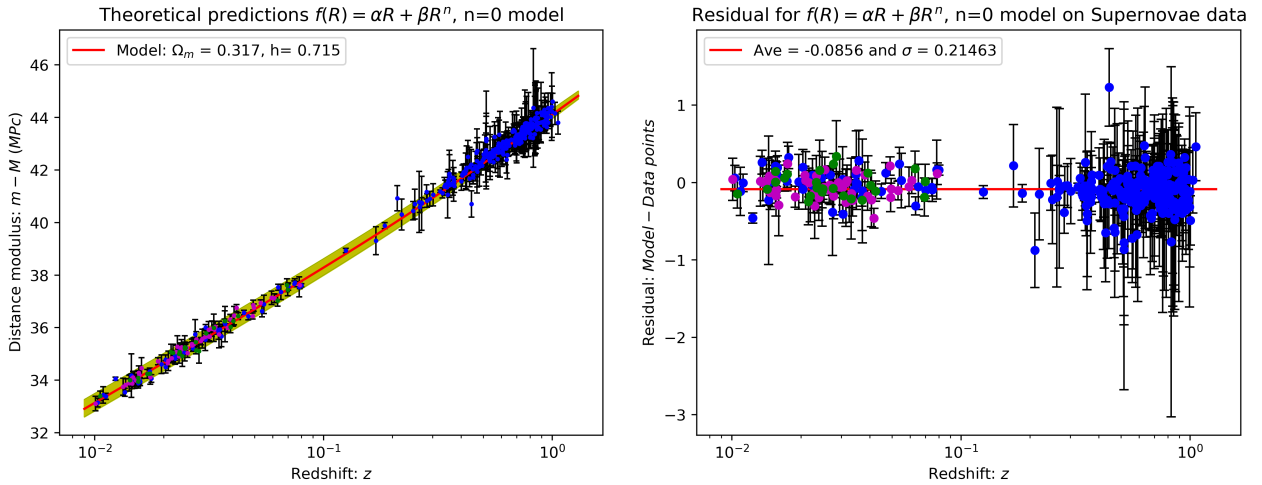


Figure 4.4: The second toy model (with $n = 0$) fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.317^{+0.061}_{-0.101}$ (unreliably constrained) and $H_0 = 71.5^{+6.0}_{-7.2} \frac{\text{km}}{\text{s.Mpc}}$ (unconstrained) as shown in Figure 4.3. The arbitrary free parameters are calculated to be $\alpha = 1.202^{+0.397}_{-0.392}$ (constrained) and $\beta = -5.265^{+1.698}_{-1.315}$ (constrained).

not surprising, since this $f(R)$ -gravity model has a similar form to the Λ CDM model, within the Einstein-Hilbert action, namely $R - 2\Lambda$ (Eq. 1.1). The best-fitting model is given by

$$f(R) = 1.202^{+0.397}_{-0.392}R - 5.265^{+1.698}_{-1.315}. \quad (4.49)$$

If we are to rewrite this result into the form associated with the Λ CDM model, we find

$$f(R) = \alpha(R - 2\Lambda), \quad (4.50)$$

with Λ , the cosmological constant, having a value of ⁶

$$\Lambda = 2.190^{+1.011}_{-0.900}. \quad (4.52)$$

⁶By assuming that the uncertainties from α and β are independent from each other, we can use the calculation

$$\frac{A^{+a_1}_{-a_2}}{B^{+b_1}_{-b_2}} = \left(\frac{A}{B} \right)^{+\frac{A}{B}\sqrt{\left(\frac{a_1}{A}\right)^2 + \left(\frac{b_1}{B}\right)^2}}_{-\frac{A}{B}\sqrt{\left(\frac{a_2}{A}\right)^2 + \left(\frac{b_2}{B}\right)^2}}, \quad (4.51)$$

to calculate the error propagation. This assumption is based on the fact that α and β are independent parameters within the model.

4.3.1.2 Second toy model: $n = \frac{1}{2}$

If we insert $n = \frac{1}{2}$ into Equation 4.38, we obtain

$$3\left(\frac{1}{2} + \frac{3q}{4} - \frac{j}{4} - \frac{q^2}{2} + (q-1)^2\right)\beta^2 H_0^3 H^2 (q-1) + 6(q-1)\left[\beta H_0^2\left(\frac{\alpha H^2(j-2-q)}{4} + \frac{\rho_m(q-1)}{6}\right) - \frac{H^2(q^2 + j - q - 3)\alpha}{2} + \alpha H^2(q-1)^2\right]\sqrt{-6H^2(q-1)} + 6\alpha H^2(q-1)^2\left(\alpha H^2 - \frac{\rho_m}{3}\right)H_0 = 0. \quad (4.53)$$

If we solve for H^2 , we obtain four possible solutions given as:

$$\begin{aligned} \bullet \quad H^2 = 0, \quad \bullet \quad H^2 = -\frac{\beta^2 H_0^2}{24\alpha^2(q-1)}, \\ \bullet \quad H^2 = \pm \frac{H_0\beta(j-2q^2+5q-6)\sqrt{6}H_0^2(j-2q^2+5q-6)^2\beta^2 - 768\alpha\rho_m(q-1)^3}{1152\alpha^2(q-1)^3} \\ + \frac{\left[-6H_0^2(j-2q^2+5q-6)^2\beta^2 + 384\alpha\rho_m(q-1)^3\right]}{1152\alpha^2(q-1)^3}. \end{aligned} \quad (4.54)$$

The first solution gives a stationary universe, while the second solution gives us an $h(t)$ solution that is only dependent on the free parameters (α and β), and the deceleration parameter q (for a negative q -value, we will obtain an expanding universe)⁷. However, we are interested in comparing the solutions to the Λ CDM model, and therefore we will need to use the last two solutions, which are dependent on matter density distribution ρ_m . From these solutions, we obtain the normalised Hubble parameter equation as

$$h(t) = \left[\pm \frac{\beta(j-2q^2+5q-6)\sqrt{(j-2q^2+5q-6)^2\beta^2 - 384\alpha\Omega_m(q-1)^3}}{192\alpha^2(q-1)^3} - \frac{(j-2q^2+5q-6)^2\beta^2}{192\alpha^2(q-1)^3} + \frac{\Omega_m}{\alpha} \right]^{\frac{1}{2}}. \quad (4.55)$$

We can now change $h(t)$ to a function of redshift, but we have to remember that the jerk parameter j and the deceleration parameter q are also time dependent. So we obtain the normalised Hubble parameter equation as a function of redshift as

$$h(z) = \left[\pm \frac{\beta(j(z)-2q^2(z)+5q(z)-6)\sqrt{(j(z)-2q^2(z)+5q(z)-6)^2\beta^2 - 384\alpha\Omega_m(1+z)^3(q(z)-1)^3}}{192\alpha^2(q(z)-1)^3} - \frac{(j(z)-2q^2(z)+5q(z)-6)^2\beta^2}{192\alpha^2(q(z)-1)^3} + \frac{\Omega_m(1+z)^3}{\alpha} \right]^{\frac{1}{2}}, \quad (4.56)$$

⁷We obtain the constraint that $\alpha > 0$ to ensure a decelerating universe, while $\beta = 0$ results in a stationary universe.

with $q(z) = q_0 + q_1 \frac{z}{z+1}$ and $j(z) = q(z)(2q(z) + 1) + \frac{q_1}{z+1}$. We can substitute this normalised Hubble parameter equation, as a function of redshift, into our distance modulus model (Eq. 4.30). We will use the following priors:

$$\begin{aligned}
 & \bullet \quad 0.0 < \Omega_m < 0.4, \\
 & \bullet \quad 0.6 < \bar{h} < 0.8, \\
 & \bullet \quad -0.9 < q_0 < -0.01, \\
 & \bullet \quad -1.0 < q_1 < 1.0, \\
 & \bullet \quad 0.1 < \alpha < 100, \\
 & \bullet \quad -100 < \beta < -0.001.
 \end{aligned} \tag{4.57}$$

The first four priors are similar to those used for the second toy model (with $n = 0$), assuming a flat universe (with slight changes), as discussed in the section for the first $f(R)$ -gravity toy model ($f(R) = \beta R^n$). The α and β ranges were a bit more difficult to obtain, due to the constraint given by

$$(j(z) - 2q^2(z) + 5q(z) - 6)^2 \beta^2 \geq 384\alpha\Omega_m(1+z)^3(q(z) - 1)^3. \tag{4.58}$$

From this constraint, it is clear that the ranges for α and β are dependent on each other, as well as on $q(z)$ and $j(q(z))$ ⁸. From this, we decided to keep α positive (since we used a positive value for α in the case where $n = 0$). Furthermore, this would result in β being the parameter, which will accommodate the changes in accordance with all the other parameters, while suppressing α to allow the R.H.S. of Equation 4.58 to be fulfilled. We also decided to keep β only in a negative value range, even though positive values for β are possible. The reason we kept β negative is due to the end results being a good fit to the data, while a positive β resulted in best-fit cosmological parameter values that were over-estimating the matter density distribution ($\Omega_m \approx 0.4$ -upper bound), or under-estimating the Hubble uncertainty parameter ($\bar{h} \approx 0.6$ -lower boundary).

The obtained results, from the MCMC simulation, for the positive solution of the model $f(R) = \alpha R + \beta\sqrt{R}$, are shown in Figure 4.5. We note that for this particular range for the β -parameter, we obtain a closer value to the expected value than for the first toy model for the deceleration parameter, calculated to be $q_0 = -0.497^{+0.326}_{-0.280}$. Therefore, including the deceleration parameter, all three important cosmological parameters are within 1σ from the Planck 2018 data, although the 1σ deviation on these three cosmological parameters are quite large, with Ω_m being unreliably constrained, while the other two are considered to be unconstrained results. For the other three parameters, only α can be considered to be constrained, and β being very unreliably constrained. Leaving us with the correction on the deceleration parameter also being unconstrained. Although both $\bar{h}$ and q_1 are close to being considered as unreliably constrained⁹. Using these values, we can then also fit the model to the supernovae data and obtain Figure 4.6.

⁸Both $q(z)$ and $j(z)$ can be positive or negative, since q_1 can be both. Furthermore, we also tried to incorporate the constraint shown in Footnote 7.

⁹We will still use them as unconstrained.

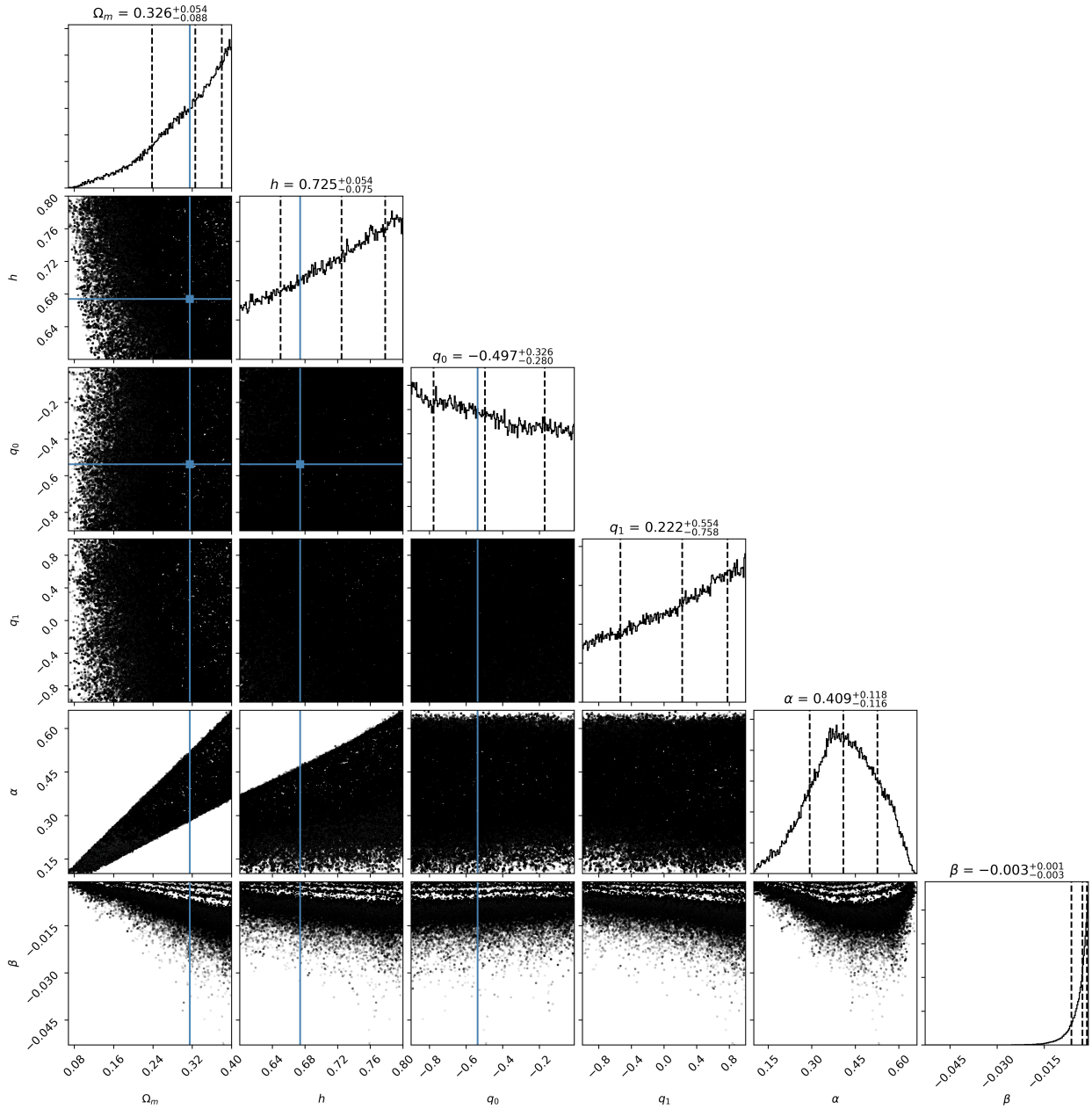


Figure 4.5: MCMC simulation results for the second toy model's (Eq. 4.56 with $n = \frac{1}{2}$ - the positive solution) cosmological parameters (Ω_m , $\bar{h}$, q_0 , and q_1), as well as the model's arbitrary free parameters (α and β). The “true” values for Ω_m and $\bar{h}$ (blue lines) are provided by Planck 2018 collaboration data release [4], while the “true” values for q_0 and q_1 are obtained from [6]. We used 100 random walkers, 10 000 iterations and priors provided by Equation 4.57. The blue lines for arbitrary free parameter values are to show their initial chosen starting point for the MCMC simulation.

From Figure 4.6, we see a few things different from the results shown in Figure 4.4. Firstly, the error regime is smaller, so this model is more stable than the previous one ($n = 0$). However, even though the model obtained its best-fit resulting in an average residual $\bar{x}_{res} = -0.2029$, its standard deviation $\sigma_{res} = 0.33713$ is larger than for the previous model, as mentioned above with each of the cosmological parameters having a large standard deviation. These statistical values, however, are

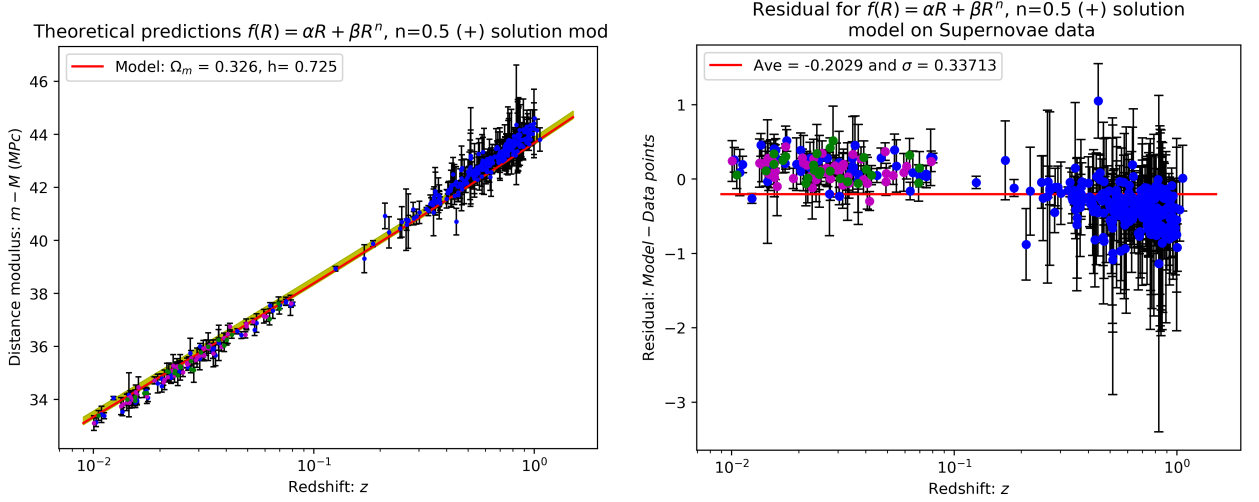


Figure 4.6: The second toy model (with $n = \frac{1}{2}$) fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.326^{+0.054}_{-0.088}$ (unreliably constrained), $H_0 = 72.5^{+5.4}_{-7.5} \frac{\text{km}}{\text{s.Mpc}}$ (unconstrained), $q_0 = -0.497^{+0.326}_{-0.280}$ (unconstrained), and $q_1 = -0.222^{+0.554}_{-0.758}$ (unconstrained), as shown in Figure 4.5. The arbitrary free parameters are calculated to be $\alpha = 0.409^{+0.118}_{-0.116}$ (constrained) and $\beta = -0.003^{+0.001}_{-0.003}$ (unreliably constrained).

not telling the full story. It is evident from the R.H.S. panel (the residuals) for Figure 4.6 that the model over-estimates the distance modulus for the low-redshift supernovae, while under-estimating the high-redshift data. It can be concluded that this model cannot accurately predict the data. However, the best-fitting model is given by

$$f(R) = 0.409^{+0.118}_{-0.116} R - 0.003^{+0.001}_{-0.003} \sqrt{R}. \quad (4.59)$$

For interest's sake, we tried the negative solutions with the same priors as for the positive solution. The MCMC simulation obtained cosmological parameters that are similar/equal to that of the positive solution, as expected, since both are solutions. Since the solutions are the same, the model to be fitted to the distance modulus of the supernovae data will remain the same. Therefore, we are not presenting these results.

4.3.1.3 Second toy model: $n = 1$

If we insert $n = 1$ into Equation 4.38, we obtain

$$36(q-1)^3(\alpha+\beta)H_0^2 \left[(\alpha+\beta)H^2 - \frac{\rho_m}{3} \right] H^2 = 0. \quad (4.60)$$

We can then solve for H^2 . The first solution once again results in a stationary universe, while the second solution is given by

$$H^2 = \frac{\rho_m}{3(\alpha+\beta)}. \quad (4.61)$$

We then obtain the normalised Hubble parameter equation, in terms of redshift, as

$$h(z) = \sqrt{\frac{\Omega_m(1+z)^3}{\alpha + \beta}}. \quad (4.62)$$

From Equation 4.62, we obtain the constraint $\alpha + \beta > 0$. We can then insert Equation 4.62 into our distance modulus model. We use the priors for the cosmological parameters (Ω_m and $\bar{h}$) previously chosen for the $n = \frac{1}{2}$ model (Eq. 4.57), while using the priors for the arbitrary free parameters as:

$$\begin{aligned} & \bullet \quad 0.01 < \alpha < 1.6, \\ & \bullet \quad 0.01 < \beta < 1.6. \end{aligned} \quad (4.63)$$

We executed the MCMC simulation to obtain the cosmological parameter values for the model $f(R) = (\alpha + \beta)R$. The results are shown in Figure 4.7. The reasoning behind the chosen priors for α and β is even though it is possible to obtain a situation where $x_1 < 0$ and $x_2 > |x_1|$, thereby fulfilling the constraint for $\alpha + \beta > 0$. The resulting parameter values calculated by the MCMC simulation were close to zero. This gave a problem, since we allowed negative values within our priors for α or β , the error on both of these free parameters increased to such a point that if we were to take the lower-bounds on both of them, we would end up with $\alpha_{low} + \beta_{low} < 0$, thereby violating our constraint. We then decided to only use positive values for our priors.

We can then use the calculated cosmological parameter results from Figure 4.7 (all of which are unreliably constrained), to fit the model to the supernovae data, as shown in Figure 4.8. From Figure 4.8, we see that this model over-estimates the low-redshift data, while under-estimating the intermediate redshift. Therefore, it can be concluded that this model cannot accurately predict the data. In any case, the best-fit for this model is given by

$$f(R) = \left(0.183^{+0.160}_{-0.119} + 0.176^{+0.159}_{-0.116}\right)R. \quad (4.64)$$

4.3.1.4 Second toy model: $n = \frac{3}{2}$

If we insert $n = \frac{3}{2}$ into Equation 4.38, we obtain

$$\begin{aligned} & -27H^2 \left\{ \left[\alpha \left(j - \frac{2}{3}q^2 - \frac{11}{3}q + \frac{4}{3} \right) H^2 + \frac{2\rho_m(q-1)}{3} \right] \beta H_0 \sqrt{-6H^2(q-1)} \right. \\ & \left. - 9 \left[\beta^2 \left(j - \frac{2}{3}q^2 - \frac{5}{3}q - \frac{2}{3} \right) H^4 + \frac{4\alpha^2 H^2 H_0^2}{27} - \frac{4\alpha H_0^2 \rho_m}{81} \right] (q-1) \right\} (q-1)^2 H_0 = 0. \end{aligned} \quad (4.65)$$

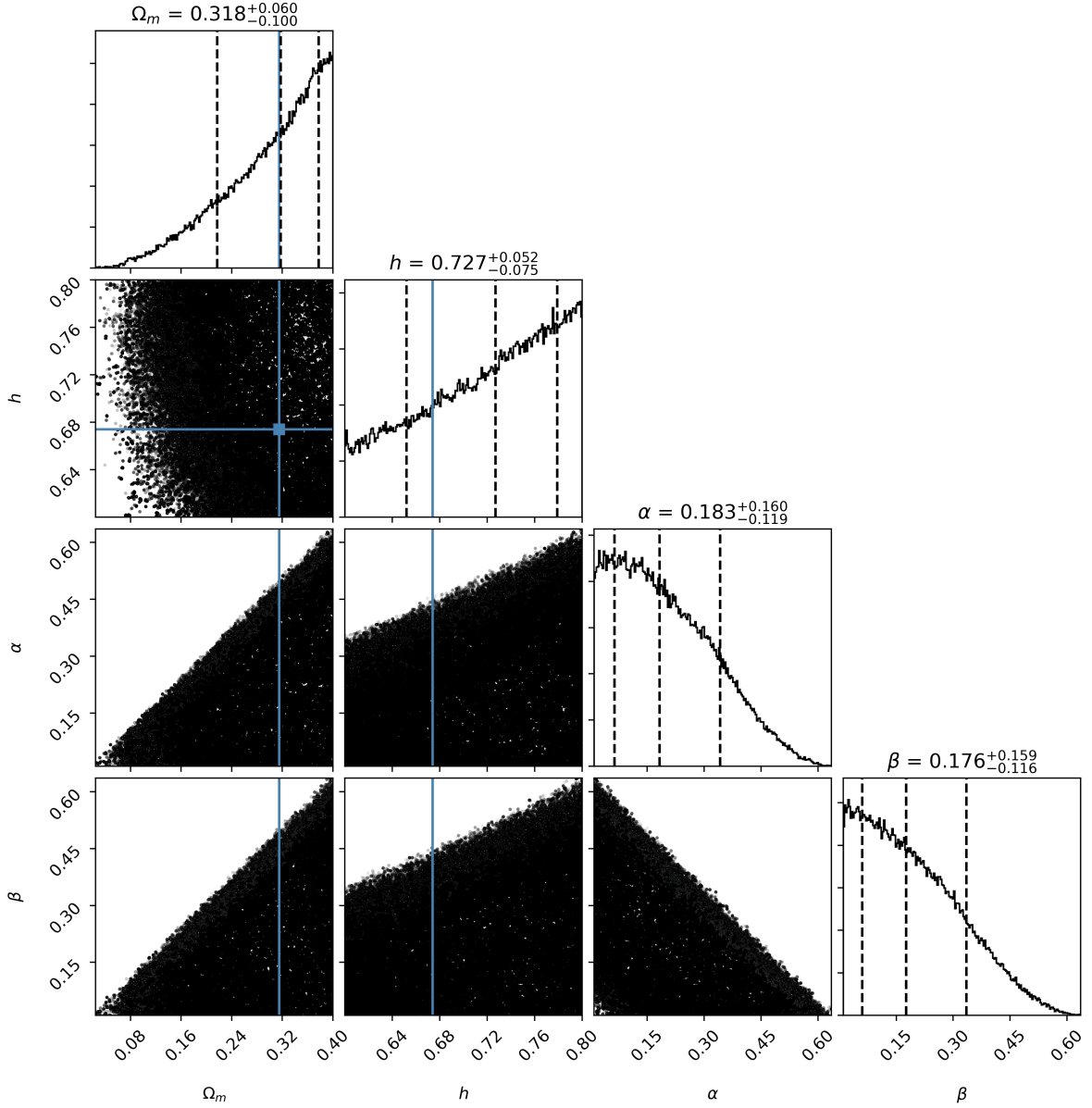


Figure 4.7: MCMC simulation results for the second toy model's (Eq. 4.62 with $n = 1$) cosmological parameters (Ω_m , h), as well as the model's arbitrary free parameters (α and β). The “true” values for Ω_m and h (blue lines) are provided by Planck 2018 collaboration data release [4]. We used 100 random walkers, 10 000 iterations and priors provided by Equations 4.57 and 4.63. The blue lines for arbitrary free parameter values are to show their initial chosen starting point for the MCMC simulation.

Solving this for H^2 , we once again obtain five possible solutions with the first two solutions being

$$\bullet \quad H^2 = 0, \quad \text{or} \quad \bullet \quad H^2 = -\frac{2H_0^2\alpha^2}{27\beta^2(q-1)}. \quad (4.66)$$

These solutions once again result in either a stationary universe or a universe dependent only on the free parameters α and β as well as the deceleration parameter. The only important difference between the second solution for $n = \frac{1}{2}$, as noted in Footnote 7, and the one for $n = \frac{3}{2}$, is that in

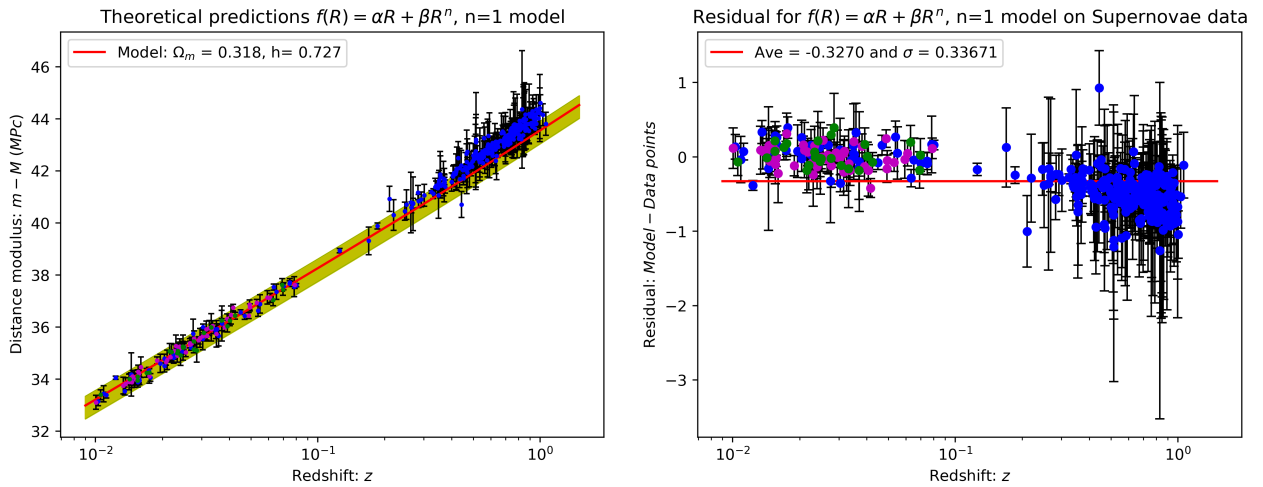


Figure 4.8: The second toy model (with $n = 1$) fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.318^{+0.060}_{-0.100}$ (unreliably constrained) and $H_0 = 72.7^{+5.2}_{-7.5} \frac{\text{km}}{\text{s.Mpc}}$ (unreliably constrained) as shown in Figure 4.7. The arbitrary free parameters are calculated to be $\alpha = 0.183^{+0.160}_{-0.119}$ (unreliably constrained) and $\beta = 0.176^{+0.159}_{-0.116}$ (unreliably constrained).

this case $\beta > 0$, while α may be zero, which would just result in a stationary universe. The three solutions, which have a matter density distribution dependency, become very messy and not really useful. Their solutions, even on a computer program such as Maple, become so long that it will increase our computational time for calculation of the MCMC simulations exponentially to such a degree that it takes too much time to solve. Even just giving their solutions here, but not using them, would result in full pages of mathematics, with no real use.

4.3.1.5 Second toy model: $n = 2$

If we insert $n = 2$ into Equation 4.38, we obtain

$$-5184H^2 \left[\beta \left(j - \frac{1}{2}q^2 - q - \frac{3}{2} \right) H^4 + \frac{\alpha H^2 H_0^2}{12} - \frac{\rho_m H_0^2}{36} \right] (q-1)^3 \left[\beta(q-1)H^2 - \frac{\alpha H_0^2}{12} \right] = 0. \quad (4.67)$$

When solving for H^2 , we obtain again four solutions given by

$$\begin{aligned} & \bullet \quad H^2 = 0, \quad \text{or} \quad \bullet \quad H^2 = \frac{\alpha H_0^2}{12\beta(q-1)}, \\ & \bullet \quad H^2 = \pm \frac{H_0 \sqrt{\alpha^2 H_0^2 - 8\beta\rho_m(q^2 - 2j + 2q + 3)}}{12\beta(2j - q^2 - 2q - 3)} - \frac{H_0^2 \alpha}{12\beta(2j - q^2 - 2q - 3)}. \end{aligned} \quad (4.68)$$

The first solution results, yet again, in a stationary universe, while the second solution gives an expanding universe when the ratio $\frac{\alpha}{\beta} > 0$, but this solution is also independent of ρ_m . Using the

3rd and 4th solutions, we can determine the normalised Hubble parameter equation to be

$$h(t) = \left[\pm \frac{\sqrt{\alpha^2 - 24\beta\Omega_m(q^2 - 2j + 2q + 3)}}{12\beta(2j - q^2 - 2q - 3)} - \frac{\alpha}{12\beta(2j - q^2 - 2q - 3)} \right]^{\frac{1}{2}}. \quad (4.69)$$

Changing $h(t)$ to a function of redshift, we obtain

$$h(z) = \left[\pm \frac{\sqrt{\alpha^2 - 24\beta\Omega_m(1+z)^3(q^2(z) - 2j(z) + 2q(z) + 3)}}{12\beta(2j(z) - q^2(z) - 2q(z) - 3)} - \frac{\alpha}{12\beta(2j(z) - q^2(z) - 2q(z) - 3)} \right]^{\frac{1}{2}}, \quad (4.70)$$

with $q(z)$ and $j(z)$ as previously defined. We can then use this in the supernovae distance modulus model. The chosen cosmological parameter priors for this model are given as:

- $0.10 < \Omega_m < 0.40$,
 - $0.60 < \bar{h} < 0.80$,
 - $-0.70 < q_0 < -0.2$,
 - $-0.70 < q_1 < 0.5$.
- (4.71)

These priors are similar to those chosen for the other $f(R)$ -gravity models. However, these have slight changes due to the difficulty of finding valid ranges for the arbitrary free parameters, which are constrained not only by $\frac{\alpha}{\beta} > 0$, but also by

$$\alpha^2 - 24\beta\Omega_m(1+z)^3(q^2(z) - 2j(z) + 2q(z) + 3) \geq 0. \quad (4.72)$$

We tried to use different ranges for α and β , and finally found for the arbitrary free parameters prior constraints to be

- $17.2 < \alpha < 100$,
 - $0.01 < \beta < 1.0$.
- (4.73)

On the first view, we see it fulfills the first constraint given as $\frac{\alpha}{\beta} > 0$. While the second constraint can be tested by using the lower-bound value for the α parameter, against the upper-bounds of the other parameters used in the second constraint. You will notice that it is possible to find α -parameter values that are lower than this bound. However, certain combinations tend to give errors in the MCMC simulation, as noted in the first toy model's prior selection.

To our surprise, these priors only seemed to work for the negative solution (we searched for these priors using the negative solution). This is unexpected, since we found for the second toy model with $n = \frac{1}{2}$ (Sec. 4.3.1.2), as expected, that the chosen priors worked for both the positive and negative solution. We tried to find a set of priors to which both the solutions can be applied, however, we

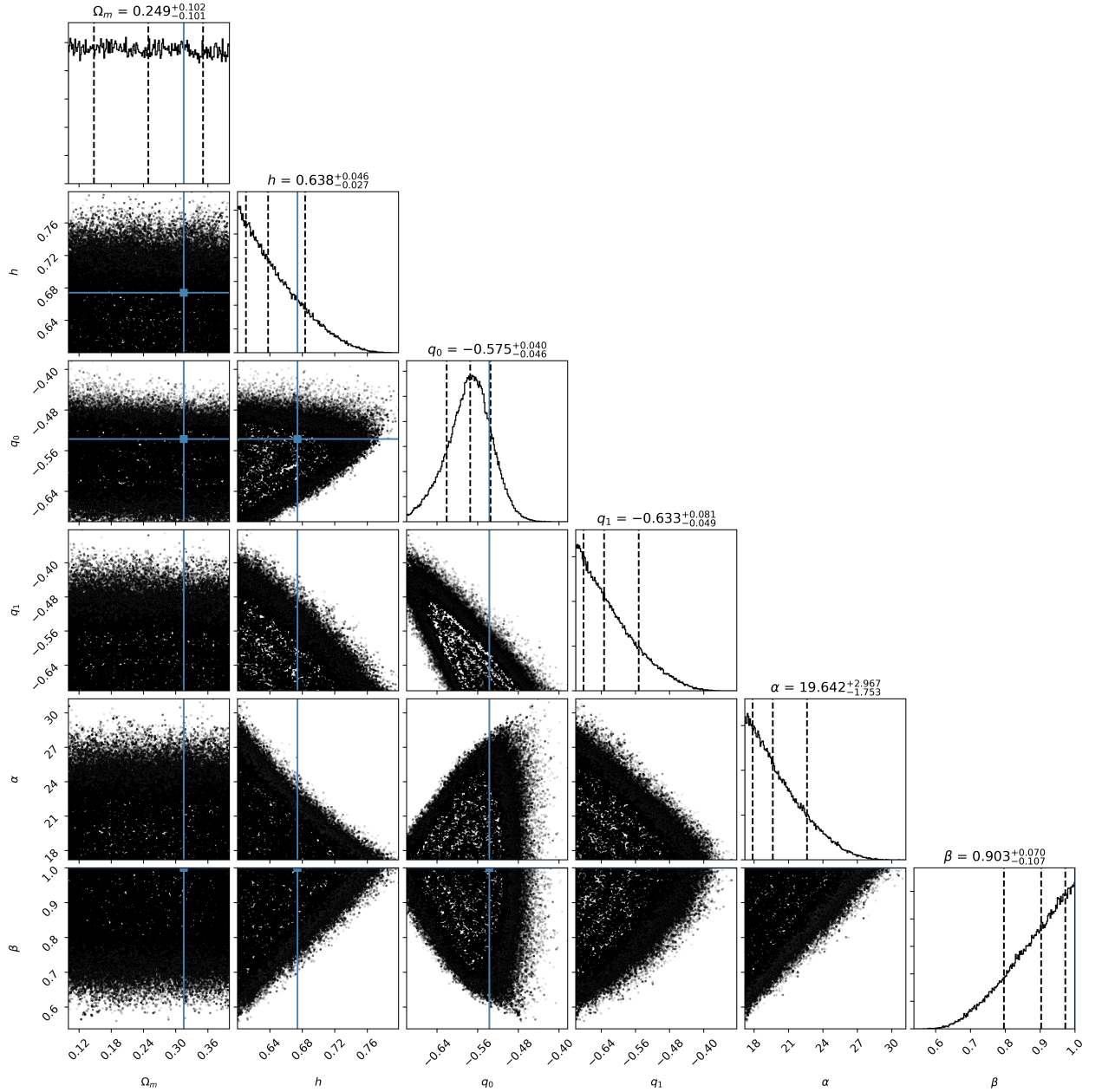


Figure 4.9: MCMC simulation results for the second toy model's (Eq. 4.70 with $n = 2$ - the negative solution) cosmological parameters (Ω_m , h , q_0 , and q_1), as well as the model's arbitrary free parameters (α and β). The “true” values for Ω_m and h (blue lines) are provided by Planck 2018 collaboration data release [4], while the “true” values for q_0 and q_1 are obtained from [6]. We used 100 random walkers, 10 000 iterations and priors provided by Equations 4.71 and 4.73. The blue lines for arbitrary free parameter values are to show their initial chosen starting point for the MCMC simulation.

were unsuccessful in our attempt. Therefore, we will only present the negative solution MCMC simulation results, shown in Figure 4.9.

Using the results calculated by the MCMC simulation, we can fit the best-fit model to the supernovae data, shown in Figure 4.10. From the calculated cosmological parameters and this best-fit model, it is clear that this seems to be a viable model, since our three cosmological parameters

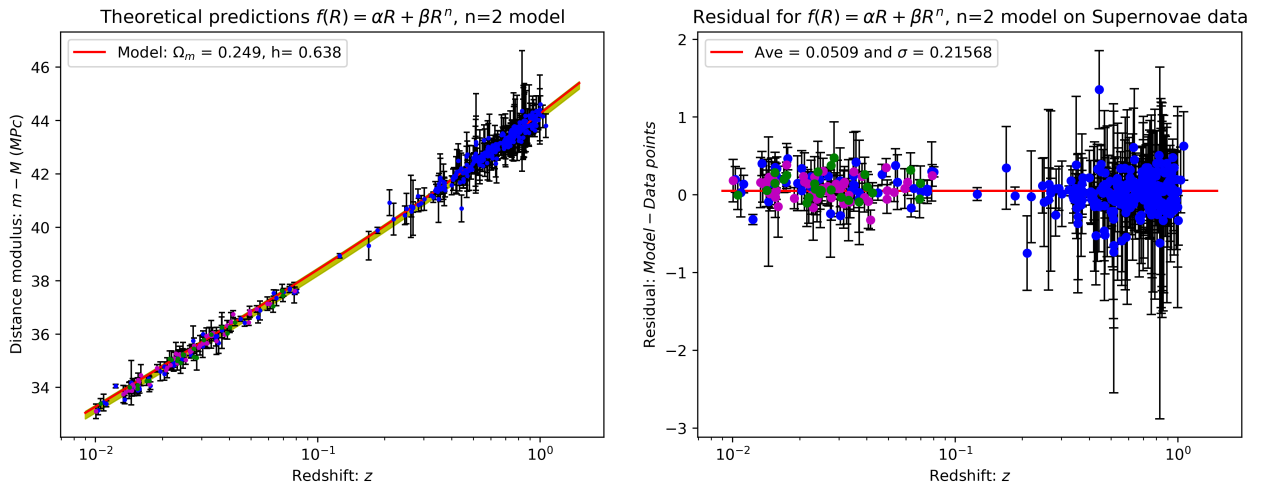


Figure 4.10: The second toy model (with $n = 2$) fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.249^{+0.102}_{-0.101}$ (unconstrained), $H_0 = 63.8^{+4.6}_{-2.7} \frac{\text{km}}{\text{s.Mpc}}$ (unreliably constrained), $q_0 = -0.575^{+0.040}_{-0.046}$ (constrained), and $q_1 = -0.633^{+0.081}_{-0.049}$ (unreliably constrained), as shown in Figure 4.9. The arbitrary free parameters are calculated to be $\alpha = 19.642^{+2.967}_{-1.753}$ (unreliably constrained) and $\beta = 0.903^{+0.070}_{-0.107}$ (unreliably constrained).

are all within a 1σ for the Planck 2018 results. We do, however, see that the Hubble parameter did not find a Gaussian distribution, suggesting that there might be a better fit value outside the lower-bound. Furthermore, it seems as if this model's matter density distribution (Ω_m) does not have an influence on the outcome of the best-fitting model, since it found a uniform distribution (unconstrained). The deceleration parameter (q_0) is the most valuable parameter in this model, since q_0 obtained a true Gaussian distribution (constrained). This means that this model gives useful information about the acceleration of the Universe. All the other parameters gave results that were unreliably constrained.

Furthermore, this model seems to explain the data quite well (with a small error region), suggesting that this model is more stable than the previous ones, and that there are no over- or under-estimations of the distance modulus of the supernovae data. This is evident for average residual distance $\bar{x}_{res} = 0.0509$ (showing that we did obtain a best fit), and a standard deviation on this average residual distance of $\sigma_{res} = 0.21568$ (which is in the range of the better-fitting models we had so far). The best-fitting model is given by

$$f(R) = 19.642^{+2.967}_{-1.753}R + 0.903^{+0.070}_{-0.107}R^2. \quad (4.74)$$

4.4 The Starobinsky models

The Starobinsky $f(R)$ -gravity models are given in the form

$$f(R) = R + \beta R_c \left[\left(1 + \frac{R^2}{R_c^2} \right)^{-n} - 1 \right], \quad (4.75)$$

with the free parameters constrained by $\beta > 0$ and $n > 0$. If either of them would be equal to zero, this $f(R)$ -gravity model would reduce back to the GR model. This form of models can be found in [48, 66, 109, 110]. Furthermore, according to [23, 110], R_c parametrises the curvature scale and has a value in the order of the present-day effective cosmological constant Λ . In the limiting extreme curvature regimes, we have:

$$\begin{aligned} \bullet \quad & \lim_{R/R_c \rightarrow 0} f(R) = R, \\ \bullet \quad & \lim_{R/R_c \rightarrow \infty} f(R) \simeq R - \beta R_c \equiv R - 2\Lambda. \end{aligned} \quad (4.76)$$

From these limiting regimes, we see that in low-curvature regimes, there appears no cosmological constant, while at late-times (high curvature regimes), there is an effective cosmological constant term, which can mimic dark energy, given by $\Lambda \equiv \frac{\beta R_c}{2}$. According to [110], in some limiting cases of n and $\frac{R}{R_c} \gg 1$ (in a de Sitter Universe: $R = \text{constant}$), one can obtain an expansion history that is indistinguishable from the Λ CMD model. If we insert Equation 4.75, with a first-order derivative of

$$f'(R) = 1 - \frac{2n\beta R}{R_c} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+1)}, \quad (4.77)$$

and a second-order one given by

$$f''(R) = \frac{4n(n+1)\beta R^2}{R_c^3} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+2)} - \frac{2n\beta}{R_c} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+1)}, \quad (4.78)$$

into our Friedmann equation (Eq. 4.8), we obtain

$$\begin{aligned} H^2 = & \frac{\rho_m}{3q \left[1 - \frac{2n\beta R}{R_c} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+1)} \right]} - \frac{R + \beta R_c \left[\left(1 + \frac{R^2}{R_c^2} \right)^{-n} - 1 \right]}{6q \left[1 - \frac{2n\beta R}{R_c} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+1)} \right]} \\ & + \frac{6H^4(2+q-j)}{q} \frac{\left[\frac{4n(n+1)\beta R^2}{R_c^3} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+2)} - \frac{2n\beta}{R_c} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+1)} \right]}{\left[1 - \frac{2n\beta R}{R_c} \left(1 + \frac{R^2}{R_c^2} \right)^{-(n+1)} \right]}. \end{aligned} \quad (4.79)$$

We can then substitute the Ricci scalar ($R = 6H^2(1 - q)$) into the above equation, and obtain

$$H^2 = \frac{\rho_m}{3q \left[1 - \frac{12n\beta H^2(1-q)}{R_c} \left(1 + \frac{36H^4(1-q)^2}{R_c^2} \right)^{-(n+1)} \right]} - \frac{6H^2(1-q) + \beta R_c \left[\left(1 + \frac{36H^4(1-q)^2}{R_c^2} \right)^{-n} - 1 \right]}{6q \left[1 - \frac{12n\beta H^2(1-q)}{R_c} \left(1 + \frac{36H^4(1-q)^2}{R_c^2} \right)^{-(n+1)} \right]} + \frac{6H^4(2+q-j)}{q} \frac{\left[\frac{144n(n+1)\beta H^4(1-q)^2}{R_c^3} \left(1 + \frac{36H^4(1-q)^2}{R_c^2} \right)^{-(n+2)} - \frac{2n\beta}{R_c} \left(1 + \frac{36H^4(1-q)^2}{R_c^2} \right)^{-(n+1)} \right]}{\left[1 - \frac{12n\beta H^2(1-q)}{R_c} \left(1 + \frac{36H^4(1-q)^2}{R_c^2} \right)^{-(n+1)} \right]}. \quad (4.80)$$

We can now solve for H^2 . We will once again be using Maple. We find that this model is analogous to the model ($f(R) = \alpha R + \beta R^n$), in the sense that we cannot find an exact solution when solving for H^2 . According to the Maple program, the most simplified solution, is given by

$$H^2 = \text{RootOf} \left\{ R_c \left[5184 \left(\left(\frac{n}{2} + \frac{1}{4} \right) q^2 - \left(n^2 + n + \frac{1}{2} \right) q + \frac{1}{4} + (j-2)n^2 + \left(\frac{j}{2} - 1 \right) n \right) (q-1)^2 H^8 - 72 \left(2q + (j-2)n - (n+1)q^2 - 1 \right) R_c^2 H^4 + R_c^4 \right] \beta \left(\frac{36(q-1)^2 H^4 + R_c^2}{R_c^2} \right)^{-n} - \left(36(q-1)^2 H^4 + R_c^2 \right)^2 \left(R_c \beta - 6H^2 + 2\rho_m \right) \right\}. \quad (4.81)$$

A contrast between the Starobinsky model and the $f(R) = \alpha R + \beta R^n$ toy model is that we were not able to simplify Equation 4.81 sufficiently, by inserting different values for n , for it to become solvable. We can, however, still test this equation to see if it holds in the GR limits. If we introduce these limits (e.g. $n = 0$) and solve for H^2 , we obtain five possible solutions. These solutions are given as

$$\bullet \quad H^2 = \frac{\rho_m}{3}, \quad \text{or} \quad \bullet \quad H^2 = \frac{\pm \frac{i}{6} R_c}{q-1}, \quad (4.82)$$

with the 4th and 5th solutions being exactly the same as the 2nd and 3rd. Furthermore, we noticed that the last four solutions are all complex solutions, leaving us with only one real solution, which is the equation for the GR Friedmann solution. Therefore, this model does reduce back to GR¹⁰.

4.4.1 Numerical method

Since we obtained an unsolvable equation for the Starobinsky Friedmann model (Eq. 4.81), which cannot be simplified enough except in the GR limits, we had to set-up a numerical method capable

¹⁰If you were to test the other constraint, namely $\beta = 0$, then you obtain the same five possible solutions. This also confirms that Equation 4.81 reduces back to the GR Friedmann model.

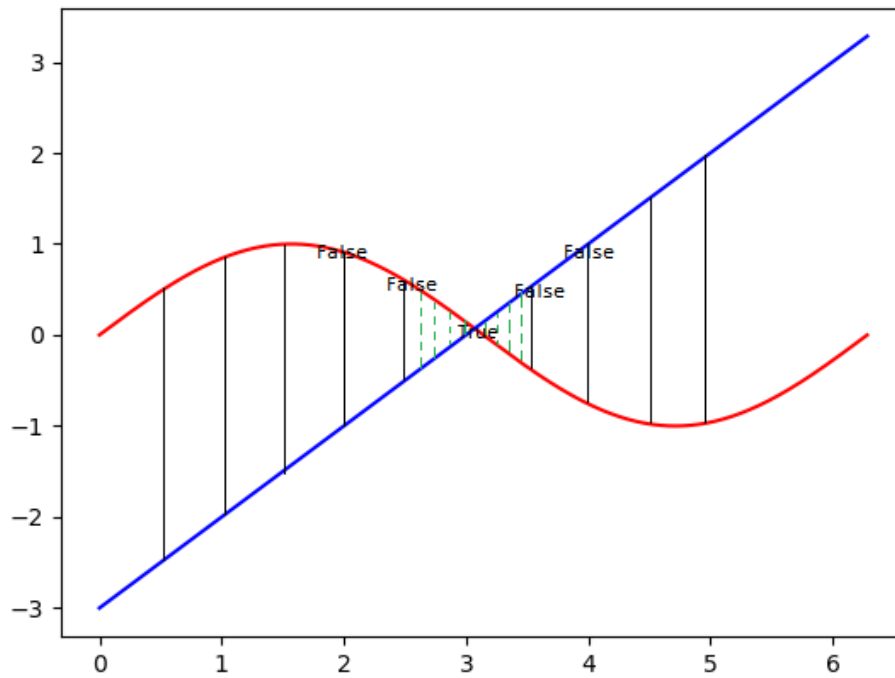


Figure 4.11: Example showing the numerical optimising function to find a solution for the function $f(x) - g(x) = 0$. Red line: $y = \sin(x)$, Blue line: $y = x - 3$, Black lines: Crude calculations, and Green dotted lines: Finer calculations to find local minimum.

of finding the solution. The first problem we faced was finding a way to optimise the function to find $H^2 \approx 0$ numerically. We realised that Equation 4.81 has the form $f(x) - g(x) = 0$, where $x = H^2$. Using this knowledge, we can calculate the difference between the two functions at various points of x , and then search for the position that minimises the difference between the two functions, within out parameter space. In Figure 4.11, we give an example on how to numerically find $f(x) - g(x) \approx 0$.

In Figure 4.11, we calculate the difference between $f(x) = \sin(x)$ (red line) and $g(x) = x - 3$ (blue line) at step intervals of $\Delta x = 0.5$ (shown with the black lines), in a parameter space of $0 \leq x \leq 6$. At each of the black lines, we compare the calculated value for $f(x_i) - g(x_i)$, to its closest neighbours, namely $f(x_{i-1}) - g(x_{i-1})$ and $f(x_{i+1}) - g(x_{i+1})$. If the difference of $f(x_i) - g(x_i)$ is smaller than both its neighbours, we must be close to the local minimum. In Figure 4.11, this is shown with the “False” and “True” statements, with only the one point having a “True” value. However, since it is only a crude searching for the minimum over the entire space, it might be smaller than the neighbours, but is most probably not the exact local minimum solution. Therefore, we have to divide the subspace between its neighbours, namely x_{i-1} and x_{i+1} , also into finer step sizes, say $\delta x = 0.1$ (shown with the green lines).

When calculating the function values at each of these fine steps, we find a numerical approximation for $f(x_i) - g(x_i) \approx 0$, with a numerical error of the order δx . The reason for this is because if x_i is our numerical solution, the ‘true’ solution will be somewhere between $x_i - \delta x$ or $x_i + \delta x$.

In terms of Equation 4.81, we will have to calculate the optimised solutions for H^2 at a certain value for each of our free parameters (Ω_m , $\bar{h}$, q_0 , q_1 , β , and n), at a specific redshift value (z'). Starting at $z = 0$, we can calculate the different $H^2(z')$ solutions for increasing z values. Before we can integrate, however, we will firstly need to divide each $H^2(z')$ solution with H_0^2 , and then take the square-root of each value to find the numerical solutions of $h(z')$ at each z . We can then calculate $\frac{1}{h(z')}$. This creates a solution map for $\frac{1}{h(z')}$ vs. z . We can then use Simpson's integration rule to numerically calculate $\int_0^{z'} \frac{1}{h(z')} dz$. The Simpson integration rule introduces an error of $\frac{h^5}{90} f^{(4)}(\xi)$, which involves the fourth derivative of our function [111]. A full in-depth discussion on both these numerical methods is shown in Appendix E¹¹.

4.4.2 The Starobinsky model results

The first problem we faced, while executing the numerical method for the Starobinsky Friedmann model (Eq. 4.81), was the fact that it is possible to find more than one real solution to H^2 . For this we made graphs for the $f(x) - g(x)$ for various different parameter values. We found an interesting result, that for this model there seems to be always only two possible real solutions. The trivial solution $H^2 = 0$, and one other solution. Since the trivial solution just results in a stationary universe, we can leave this solution and only work with the other solution. The range in which we searched for a solution was that of the Λ CDM model, namely between $H^2 = 0$ and $H^2 = H_0^2 [\Omega_m(1+z)^3 + 1 - \Omega_m]$. We used the following priors for the Starobinsky model as¹²:

- $0.01 < \Omega_m < 0.4$,
 - $0.6 < \bar{h} < 0.9$,
 - $-0.9 < q_0 < -0.01$,
 - $-1.5 < q_1 < 1.5$,
 - $0.001 < \beta < 10$,
 - $0.001 < n < 10$.
- (4.83)

However, from the moment we started to execute our code, we realised that the execution time has been greatly affected by the introduction of the numerical method (as explained in Appendix E). This led us to do a program execution with a lower accuracy, by making the step sizes for the Simpson integration rule and the optimising function larger. We then used the lower accuracy predicted MCMC simulation values as an initial starting point for the MCMC simulation, since

¹¹To test the above given numerical method, we used the Λ CDM model to test the various steps, to which we have expected values to compare our answers to. We also discussed the computational expensiveness of the numerical method. The numerical method code we used is given in Appendix E

¹²The full numerical Starobinsky model and EMCEE code and dataset can also be found on the website: <https://drive.google.com/drive/folders/1ag87ouKHZWmuCTBPqsrflLLdRrDmsG4Bi?usp=sharing>.

it is already close to the possible best-fit value. In doing this, the MCMC simulation needs fewer iterations to find the best fit. We then chose 50 random walkers and 4 000 iteration steps¹³.

We also chose the number of steps necessary for the Simpson integration to be $nsteps_{int} = 101$, which is quite small since the highest redshift value is ≈ 1.2 , which is then divided by 101. For the optimising function, we chose 100 ‘crude’ and ‘finer’ steps, in each of the allowed ranges. The reason we chose these values is due to our testing on the Λ CDM model, which resulted in values that are indistinguishable from the original code’s results. The Starobinsky model’s MCMC simulation results are shown in Figure 4.12.

From Figure 4.12, we obtained cosmological parameter values to be $\Omega_m = 0.268^{+0.027}_{-0.024}$ (constrained), $H_0 = 69.0^{+0.5}_{-0.5} \frac{km}{s.Mpc}$ (constrained), and $q_0 = -0.512^{+0.328}_{-0.265}$ (unconstrained) (with a correction term of $q_1 = 0.037^{+0.991}_{-1.050}$ (unconstrained)). These values correspond extremely well with the cosmological parameter values obtained by the Λ CDM model (Fig. 3.1). We once again obtain the same shift in the Hubble constant being higher than the Planck 2018 results, as previously discussed. This higher Hubble constant value also meant a lower matter density distribution, since gravity seems to have less of an influence, leading to a higher expansion velocity.

Furthermore, the deceleration parameter value predicts an accelerating universe, but did not seem to be a priority in determining the expansion of the Universe (uniform distribution). Lastly, as required [23], we obtained two possible real values for our arbitrary parameters $\beta = 5.284^{+3.191}_{-2.981}$ and $n = 4.567^{+3.346}_{-2.899}$, although both remained unconstrained. This is an interesting result. Not only do they also have a sort of uniform distribution, their error regions are extremely large compared to the other parameters. This means that many different values for these arbitrary free parameters are possible and not one exact value seems to be controlling the outcome of the model.

This may be a result of our accuracy. While obtaining the previous results, we noticed that the MCMC simulation always ends up switching the matter density distribution and the Hubble uncertainty parameter first, since they seemed to control the outcome of the model, more than the other parameters. Which meant that the other parameters were involved in the ‘fine tuning’ of the model. However, we are limited to the accuracy we chose, and therefore it might lead to the arbitrary free parameters obtaining a uniform distribution, therefore leading to the unconstrained nature of their results. Using this argument, we can also explain the random spikes in the different free parameters. The MCMC simulation might have by accident landed on one of these that would indicate a possible best-fit, but due to the accuracy, the MCMC simulation’s movement kept on moving out of the possible best-fit regions.

Using these estimated cosmological parameter values (Fig. 4.12), the best-fit model was fitted to the data, as shown in Figure 4.13. In Figure 4.13, we see that the Starobinsky model fits the supernovae data with quite a high precision. Furthermore, unlike some of the other $f(R)$ -gravity

¹³This is 1 000 iterations more than the number decided on in Appendix E, since the code executed faster than predicted. The 4 000 iterations gave a similar execution time as predicted for the 3 000 iterations.

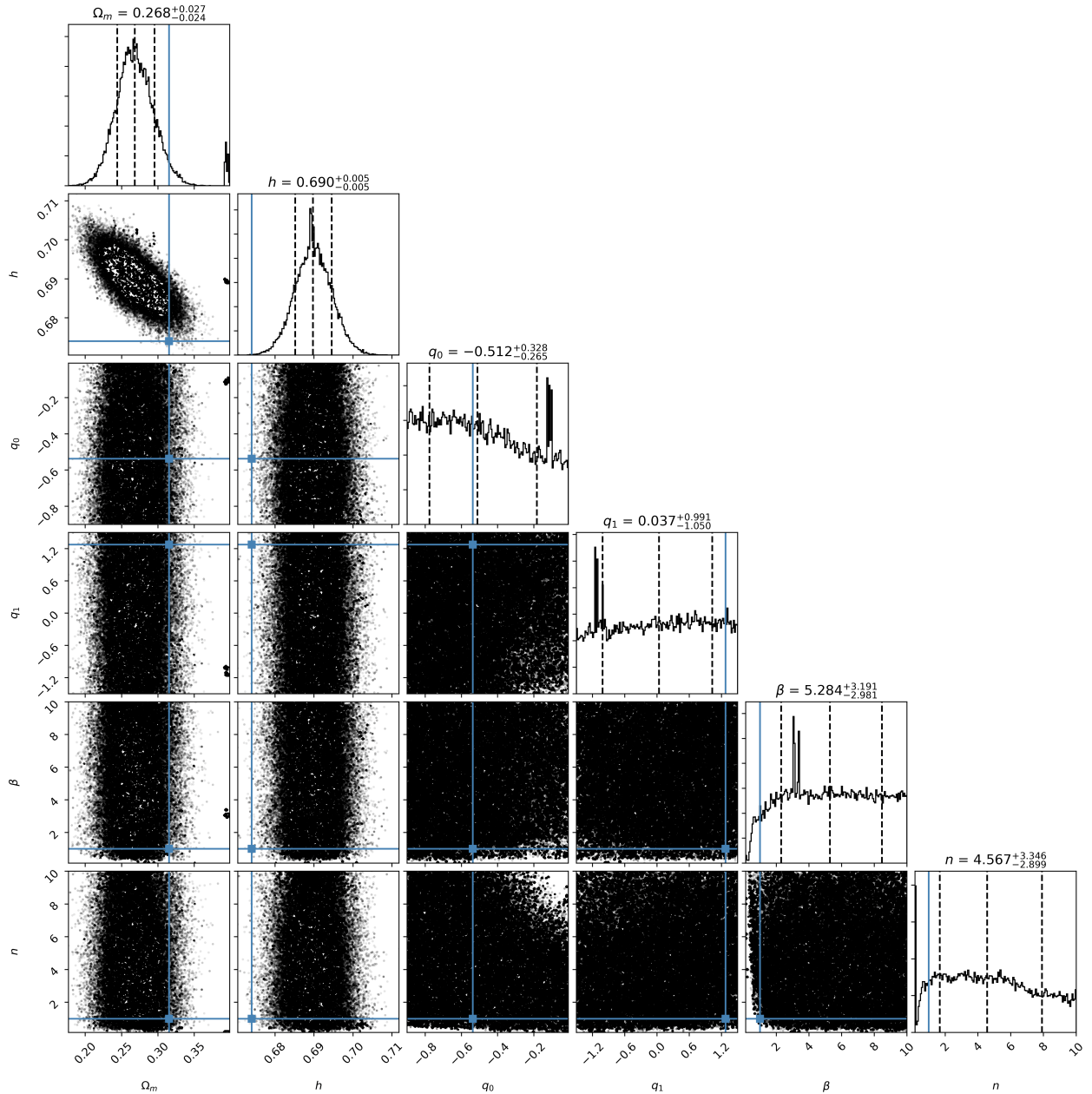


Figure 4.12: The numerical MCMC simulation results for the Starobinsky model's (Eq. 4.81) cosmological parameters (Ω_m , $\bar{h}$, q_0 , and q_1), as well as the model's arbitrary free parameters (β and n). The “true” values for Ω_m and $\bar{h}$ (blue lines) are provided by Planck 2018 collaboration data release [4], while the “true” values for q_0 and q_1 are obtained from [6]. We used 50 random walkers, 4 000 iterations, 101 integration steps, 100 ‘crude’ and ‘finer’ optimisation steps, and priors provided by Equation 4.83. The blue lines for arbitrary free parameter values are to show their original initial chosen starting points for the MCMC simulation.

models, the Starobinsky model is not over-estimating or under-estimating the low- or high-redshift supernovae. This means that the Starobinsky model is predicting an accelerating universe to the same extent as the Λ CDM model. These values are even comparable with the results obtained by [102] for the Starobinsky model and they actually used the full JLA dataset, the BAO data, cosmic chronometer data, as well as H_0 observation data on the state-of-the-art Monte Carlo program,

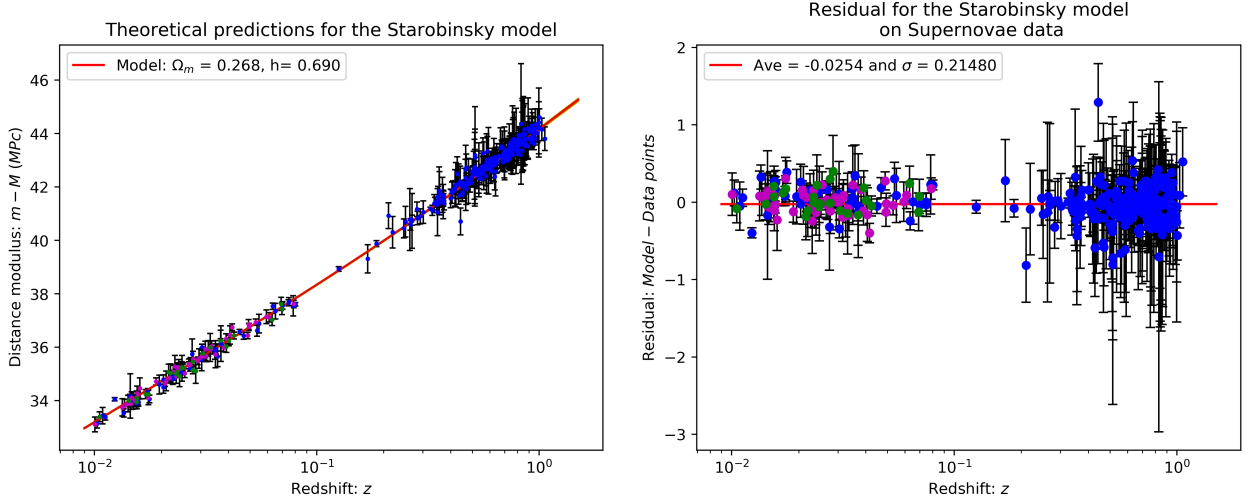


Figure 4.13: The Starobinsky model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.268^{+0.027}_{-0.024}$ (constrained), $H_0 = 69.0^{+0.5}_{-0.5} \frac{\text{km}}{\text{s.Mpc}}$ (constrained), $q_0 = -0.512^{+0.328}_{-0.265}$ (unconstrained), and $q_1 = 0.037^{+0.991}_{-1.050}$ (unconstrained), as shown in Figure 4.12. The arbitrary free parameters are calculated to be $\beta = 5.284^{+3.191}_{-2.981}$ (unconstrained) and $n = 4.567^{+3.346}_{-2.899}$ (unconstrained).

called CLASS, in Python. Their estimation for the matter density parameter is $\Omega_m = 0.269^{+0.050}_{-0.042}$ and the Hubble uncertainty parameter value is given as $\bar{h} = 0.714^{+0.030}_{-0.028}$. Therefore, both our cosmological parameter results are within 1σ from their best-fitting cosmological parameter values. Unfortunately, they use a singular parameter b to represent all the remaining free parameters. Therefore, we cannot compare the free parameters, however, it might help with the reducing of the number of free parameters for statistical purposes in future work. Furthermore, notice that even though our error bars on some of the free parameters are huge, it does not seem to have an effect on the outcome of the model, since the model's error region is almost not even visible. This might mean that those three free parameters really do not even have an effect on the outcome of the model, and that it might not be the result of the accuracy of the numerical method.

In terms of the accuracy of the fitted model, the Starobinsky model obtained an average residual between the predicted model value and the data point of $\bar{x}_{res} = -0.0254$, with a standard deviation on the residuals of $\sigma_{res} = 0.21480$. Both of these statistical values, in combination with the first two cosmological parameters, are comparable to that found by the Λ CDM model. If we just straight up compare these values to the Starobinsky model's, we see that this model is surprisingly in the same order of accuracy as the originals values. Therefore, we can conclude that it might be worth investigating the plausibility that simplified $f(R)$ -gravity models are viable. The best-fitting Starobinsky model is given by

$$f(R) = R + 5.284^{+3.191}_{-2.981} R_c \left[\left(1 + \frac{R^2}{R_c^2} \right)^{-4.567^{+3.346}_{-2.899}} - 1 \right], \quad (4.84)$$

$$\text{with } R_c = \frac{2 \left[3H_0^2 \times (1 - \Omega_m) \right]}{\beta} = \frac{2 \left[69.0^{+0.5}_{-0.5} \left(\frac{km}{s.Mpc} \right)^2 \times (1 - 0.268^{+0.027}_{-0.024}) \right]}{5.284^{+3.191}_{-2.981}}.$$

4.4.3 Testing a reduced Starobinsky model

After coming to the conclusion that three of the free parameters do not really determine the outcome of the model, at least not on the resolution of our code, we decided to test a reduced Starobinsky model, where we fixed those parameter values to certain constants. By fixing these parameters, we are able to determine whether or not these different $f(R)$ -gravity models can be simplified. This is an important conclusion to take note of, since the statistical analysis would favour a more simplified model with a reduced number of free parameters. Without noting this fact earlier, we have actually already been introduced to this concept. For the second toy model, we fixed the n -parameter, thereby changing the outcome of the AIC and BIC information statistical analysis. This led to various different results, with two of the models fitting the data quite well, and might even be viable models, while the others were not able to come close to explaining the data.

To find the reduced Starobinsky model, we had to firstly decide on the fixed values for the q_1 , β , and n parameters. At first, we thought to use their best-fit values, but that meant that we would just end up back where we started, which is not useful. In the end, we decided to use correctional deceleration parameter as $q_1 = 0$, since we wanted to see how the deceleration parameter, on its own, would hold up in this reduced Starobinsky model. Furthermore, this is a valid assumption in terms of Figure 4.12, since we obtained a uniform distribution with $q_1 \approx 0$. Therefore, any chosen value would still be plausible if we were not to choose the exact best-fit value for q_1 . For the arbitrary free parameters, we decided to use $\beta = 1$ and $n = 1$, since it would reduce Equation 4.75 to

$$f(R) = R + \frac{R_c R^2}{R_c^2 + R^2}, \quad (4.85)$$

which is the most simplified version without making extreme assumptions. For the code, we just made it a 3-dimension problem, and then sending these particular three parameters as constants. In this way, we did not need to re-derive Equation 4.81, although in principle it is easy, just simplify the equation. We also kept the priors for the three cosmological parameters the same as for the original (Eq. 4.83). We did, however, reduce the number of iterations to 1 250, due to time constraints and this only being a test to check if it is a viable method. The rest of the conditions, such as the number of optimisation steps, integration steps, and random walkers, were also kept the same. The results are shown in Figure 4.14.

At first look, we notice that we obtained cosmological parameter values for the matter density distribution and the Hubble uncertainty which are very similar to results of the main Starobinsky model, as well as the Λ CDM model. Our deceleration parameter q_0 , however, did have a change from the original result (even though it is still considered to be unconstrained), but it is still

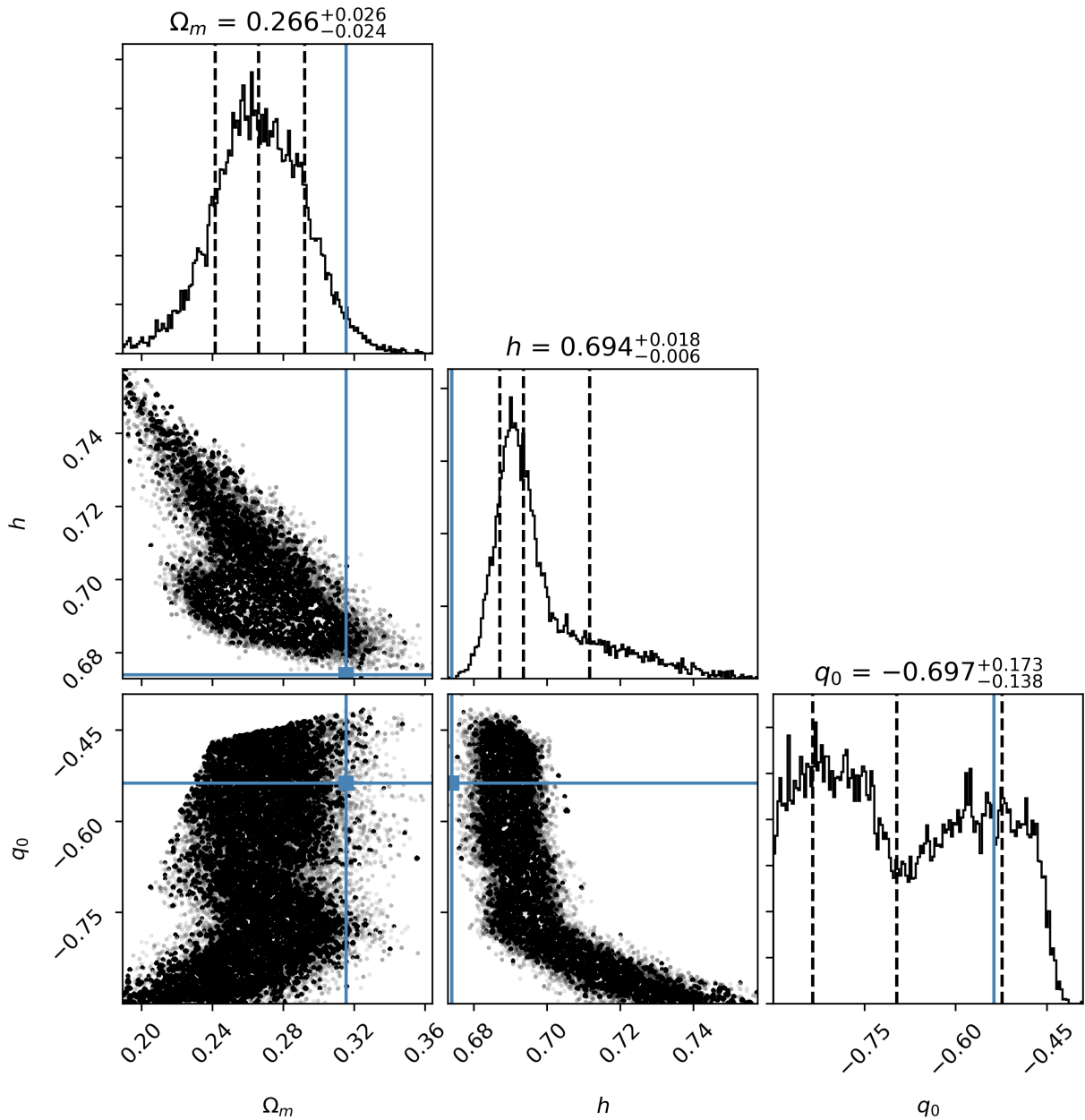


Figure 4.14: The numerical MCMC simulation results for the reduced Starobinsky model’s (simplifying Eq. 4.81) cosmological parameters (Ω_m , $\bar{h}$, and q_0). The “true” values for Ω_m and $\bar{h}$ (blue lines) are provided by Planck 2018 collaboration data release [4], while the “true” values for q_0 are obtained from [6]. We used 50 random walkers, 1 250 iterations, 101 integration steps, 100 ‘crude’ and ‘finer’ optimisation steps, and priors provided by Equation 4.83. We fixed the correctional deceleration parameter to be $q_1 = 0$, and the arbitrary free parameter to be $\beta = n = 1$.

predicting an accelerating universe, as required. There are two different reasons for the change we obtained for q_0 . The first being that it may be a result of the other parameters (q_1 , β , and n) not changing, so all the finer details that had to be made by the MCMC simulation resulted in q_0 being the parameter that receives this change. Secondly, it may be due to the lack of iterations that would be accompanied by a full run. This effect can be seen in the way that the Hubble uncertainty

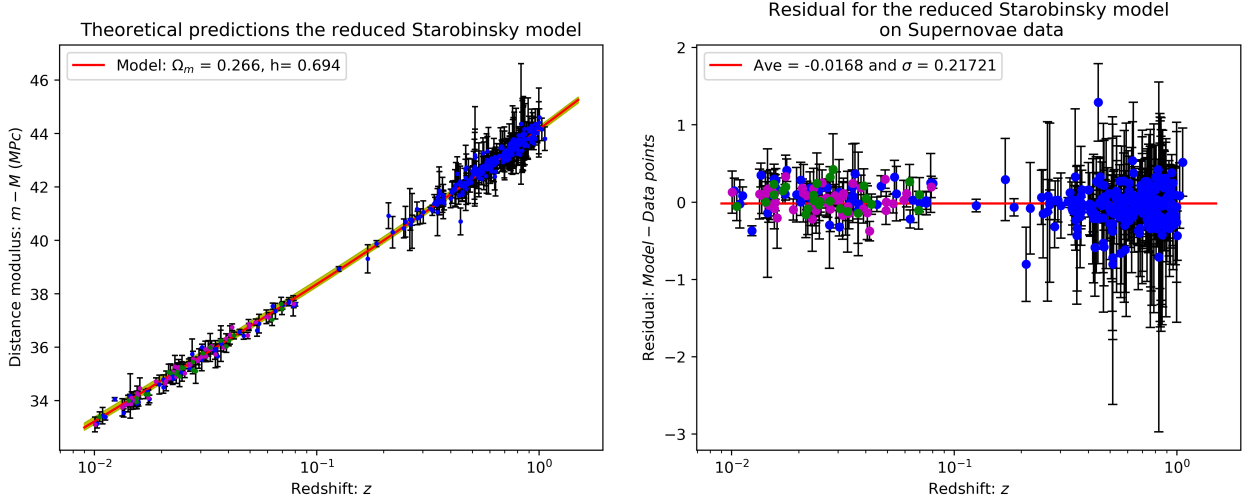


Figure 4.15: The reduced Starobinsky model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.266^{+0.026}_{-0.024}$ (constrained), $H_0 = 69.4^{+1.8}_{-0.6} \frac{km}{s.Mpc}$ (constrained), and $q_0 = -0.697^{+0.173}_{-0.138}$ (unconstrained), as shown in Figure 4.14.

parameter has some sort of tail. The MCMC simulation might still, at this stage, not yet fully settle on a particular spot, so the change in q_0 is a result of the code still making large steps to determine the best possible settling point. We then fitted this best-fit model to the supernovae dataset and obtained Figure 4.15.

From Figure 4.15, we see that the reduced Starobinsky model does indeed fit the Supernovae Type 1A data. This model has a slightly larger error region than the original, but is still small in comparison to some of the other models that we have tested. Furthermore, this model does not seem to be under-estimating or over-estimating the low- or high-redshift data. We can also confirm it is a best-fit, since the average residual between the predicted model value at a certain red-shift and the data point at that redshift is $\bar{x}_{res} = -0.0168$, with a standard deviation on the average residuals of $\sigma_{res} = 0.21721$. The best-fitting parametrised curvature scale is given as

$$R_c = \frac{2[3H_0^2 \times (1 - \Omega_m)]}{\beta} = 2 \left[69.4^{+1.8}_{-0.6} \left(\frac{km}{s.Mpc} \right)^2 \times (1 - 0.266^{+0.026}_{-0.024}) \right]. \quad (4.86)$$

4.5 The Hu-Sawicki model

The Hu-Sawicki model (introduced by Hu and Sawicki), is an $f(R)$ -gravity model of the form [23]

$$f(R) = R - \frac{\beta_1 m^2 \left(\frac{R}{m^2} \right)^n}{1 + \beta_2 \left(\frac{R}{m^2} \right)^n}, \quad (4.87)$$

where β_1 , β_2 and n (where $n > 0$ [67]) are the arbitrary free parameters, and $m^2 = \frac{\rho_m}{3}$ is a curvature scale that depends on the value of the matter-energy density. As we have done with the Starobinsky model, we can determine the influences of the model at a low- and high-curvature regime. For the low-curvature regime, we have

$$\lim_{R/m^2 \rightarrow 0} f(R) = R, \quad (4.88)$$

while a high-curvature regime results in

$$\lim_{R/m^2 \rightarrow \infty} f(R) \simeq R - \frac{\beta_1}{\beta_2} m^2 + \frac{\beta_1}{\beta_2} m^2 \left(\frac{R}{m^2} \right)^{-n}. \quad (4.89)$$

Models of this form can be found in [23, 67, 72, 102, 109]. In [23], they noted that for the low-curvature regime, we once again obtain a model that does not introduce a cosmological constant, similar to the Starobinsky model. Whereas in a high-curvature regime, we can recover an effective cosmological constant at the present epoch. According to [49, 67, 112], there exists another useful form for the Hu-Sawicki model, given by

$$f(R) = R - \alpha R_c \left[\frac{\left(\frac{R}{R_c} \right)^n}{1 + \left(\frac{R}{R_c} \right)^n} \right], \quad (4.90)$$

where we have $\alpha = \beta_1 \beta_2^{\frac{1}{n-1}}$, and $R_c = m^2 \beta_2^{-\frac{1}{n}}$ parametrising the curvature scale as it is presented in Section 4.4.2. It is also required that $\alpha = 0$ for Equation 4.90 to reduce back to the GR model. It was also shown in [49, 67] that we can obtain a relation that holds at the de Sitter point:

$$\alpha = \frac{\left[1 + \left(\frac{R}{R_c} \right)^n \right]^2}{\left(\frac{R}{R_c} \right)^{n-1} \left[2 + 2 \left(\frac{R}{R_c} \right)^n - n \right]}. \quad (4.91)$$

Even though R_c is differently defined for this model, we will be using the definition presented in Section 4.4.2 given by $R_c \equiv \frac{\beta R_c}{2}$. The reason for this is due to m^2 having a dependency on ρ_m , which also has a dependency on the scale factor. Since the Ricci scalar is also a function of the scale factor, this complicates the first- and second-order derivatives with regard to the Ricci scalar. We are allowed to make this change, since both have a value in the order of the cosmological constant, as discussed in [23, 67]. We can find the first- and second-order derivatives as:

$$\begin{aligned} \bullet \quad f'(R) &= \frac{\left(\frac{R}{R_c} \right)^{2n} R + (2R - \alpha R_c n) \left(\frac{R}{R_c} \right)^n + R}{\left[1 + \left(\frac{R}{R_c} \right)^n \right]^2 R}, \\ \bullet \quad f''(R) &= \frac{\alpha R_c n \left[(n+1) \left(\frac{R}{R_c} \right)^{2n} + (1-n) \left(\frac{R}{R_c} \right)^n \right]}{\left[1 + \left(\frac{R}{R_c} \right)^n \right]^3 R}. \end{aligned} \quad (4.92)$$

We can then substitute this into our Friedmann equation (Eq. 4.8), and obtain

$$\Rightarrow H^2 = \frac{\rho_m \left[1 + \left(\frac{R}{R_c}\right)^n\right]^2 R}{3q \left[\left(\frac{R}{R_c}\right)^{2n} R + (2R - \alpha R_c n) \left(\frac{R}{R_c}\right)^n + R\right]} - \frac{\left[R - \alpha R_c \left[\frac{\left(\frac{R}{R_c}\right)^n}{1 + \left(\frac{R}{R_c}\right)^n}\right]\right] \left[1 + \left(\frac{R}{R_c}\right)^n\right]^2 R}{6q \left[\left(\frac{R}{R_c}\right)^{2n} R + (2R - \alpha R_c n) \left(\frac{R}{R_c}\right)^n + R\right]} + \frac{6H^4(2+q-j)}{q} \frac{\alpha R_c n \left[(n+1)\left(\frac{R}{R_c}\right)^{2n} + (1-n)\left(\frac{R}{R_c}\right)^n\right]}{\left[1 + \left(\frac{R}{R_c}\right)^n\right] R \left[\left(\frac{R}{R_c}\right)^{2n} R + (2R - \alpha R_c n) \left(\frac{R}{R_c}\right)^n + R\right]}. \quad (4.93)$$

We must still insert the Ricci scalar defined by $R = 6H^2(1-q)$, but due to space constraints, we only include this in the Maple calculations. Using Maple to solve for H^2 , we obtain

$$H^2 = \text{RootOf} \left\{ \left[\left[\alpha R_c - 6 \left(H^2 - \frac{\rho_m}{3} \right) \right] \left(\frac{-6H^2(q-1)}{R_c} \right)^{3n} - 6H^2 + 2\rho_m \right] (q-1)^2 + \left[\alpha R_c \left[(j-2-q)n^2 - (q^2 + j - 2q - 2)n \right] + (q-1)^2 (6\rho_m - 18H^2 + \alpha R_c) \right] \left(-\frac{6H^2(q-1)}{R_c} \right)^n - \left[\alpha R_c \left[(j-2-q)n^2 + (q^2 + j - 2q - 2)n \right] - 2(q-1)^2 (3\rho_m - 9H^2 + \alpha R_c) \right] \left(-\frac{6H^2(q-1)}{R_c} \right)^{2n} \right\}. \quad (4.94)$$

Without going into much detail, it is possible to show that Equation 4.94 reduces back to the Λ CDM model, when we insert the requirement $\alpha = 0$. When the requirement is fulfilled, Equation 4.94 reduces to two solutions for H^2 , those being the GR solutions and non-real solutions. Since we know that this model does indeed reduce back to the GR model, we can go ahead and find $H^2(z)$. We obtain $H^2(z)$ to be

$$H^2(z) = \text{RootOf} \left\{ \left[\left[\alpha R_c - 6 \left(H^2(z) - \Omega_m(1+z)^3 H_0^2 \right) \right] \left(\frac{-6H^2(z)(q(z)-1)}{R_c} \right)^{3n} - 6H^2(z) + 6\Omega_m(1+z)^3 H_0^2 \right] (q(z)-1)^2 + \left[\alpha R_c \left[(j(z)-2-q(z))n^2 - (q(z)^2 + j(z) - 2q(z) - 2)n \right] + (q(z)-1)^2 [18\Omega_m(1+z)^3 H_0^2 - 18H^2(z) + \alpha R_c] \right] \left(-\frac{6H^2(z)(q(z)-1)}{R_c} \right)^n - \left[\alpha R_c \left[(j(z)-2-q(z))n^2 + (q(z)^2 + j(z) - 2q(z) - 2)n \right] - 2(q(z)-1)^2 (9\Omega_m(1+z)^3 H_0^2 - 9H^2(z) + \alpha R_c) \right] \left(-\frac{6H^2(z)(q(z)-1)}{R_c} \right)^{2n} \right\}. \quad (4.95)$$

We will, once again, use the numerical method introduced in the Starobinsky model section (Sec. 4.4.1). As we have done with the Starobinsky model, we firstly executed the code for a “crude run” to find a good initial starting point close to the real solution. We then executed the code with finer resolutions for the numerical methods, thereby making the result more accurate. Doing it in this manner, we need fewer MCMC iterations to find the best fit. We will be using 50 random walkers with each taking 4 000 steps. We will also be using $nsteps_{int} = 101$ integration steps over the

redshift distance, and 100 optimising steps for both the crude search (first search to find H_i^2) and then finer search (second search between H_{i-1}^2 and H_{i+1}^2) to find H_{ij}^2 (the most accurate optimised numerical solution for $H^2(z)$). We will be using the following priors for the MCMC simulation:

$$\begin{aligned}
 &\bullet \quad 0.01 < \Omega_m < 0.4, \\
 &\bullet \quad 0.6 < \bar{h} < 0.9, \\
 &\bullet \quad -0.9 < q_0 < -0.01, \\
 &\bullet \quad -1.5 < q_1 < 1.5, \\
 &\bullet \quad 0 < \alpha < 10, \\
 &\bullet \quad 0.01 < \beta < 10, \\
 &\bullet \quad 1.2 < n < 3.6.
 \end{aligned} \tag{4.96}$$

The first four priors are the same as the chosen priors for the Starobinsky model, and a slight change in the β -parameter. The range for the n free parameter was changed, but was still kept positive due to the Hu-Sawicki model requirements [67]¹⁴. As for the α free parameter, we chose this range because even though β_1 and β_2 can be positive or negative, this can create a situation where we obtain a non-real square-root [67]. For instance, if $n = 3$ and $\beta_2 < 0$, you will obtain the square-root of a negative value that is non-real¹⁵. The results for the Hu-Sawicki model are shown in Figure 4.16.

From Figure 4.16, we note that we obtained cosmological parameter values comparable to those found by the Λ CDM model (Fig. 3.1), as well as being within 1σ from the results found in [102]. Their cosmological parameter values are $\Omega_m = 0.264_{-0.055}^{+0.059}$ and $\bar{h} = 0.722_{-0.033}^{+0.042}$, respectively. We see that we once again obtain an accelerating universe, with a decelerating parameter value of $q_0 = -0.486_{-0.285}^{+0.300}$. We also note that even though we technically did obtain three parameters that would qualify as constrained, namely Ω_m , $\bar{h}$ and α , we can argue that $\bar{h}$ would rather be classified as unreliably constrained because its peak is at the edge of the prior. Furthermore, both the deceleration parameters were left unconstrained, with the n arbitrary free parameter joining them. Only β received a definite unreliably constrained classification. When we fit this best-fit model to the supernovae data, we obtain Figure 4.17.

From Figure 4.17, we see that the Hu-Sawicki model does fit the supernovae data, although the error region on the best-fit model is still quite large, and this is due to the n -parameter being unstable. We found that it is possible to find n -parameter values that are higher than the best-fit value, but the higher the value the larger the error region. We can also see from the average residual that we did find a best fit, since the average residual distance is $\bar{x}_{res} = 0.0371$, with a standard

¹⁴We did find it is possible to fit the Hu-Sawicki model on the dataset with a higher value for the n -parameter, but it tends to have large errors on this model, so we decided to keep the search range low.

¹⁵The Hu-Sawicki model full EMCEE code and dataset can be found on the website <https://drive.google.com/drive/folders/1ag87ouKHZWmuCTBPqsrfLLdRrDmsG4Bi?usp=sharing>.

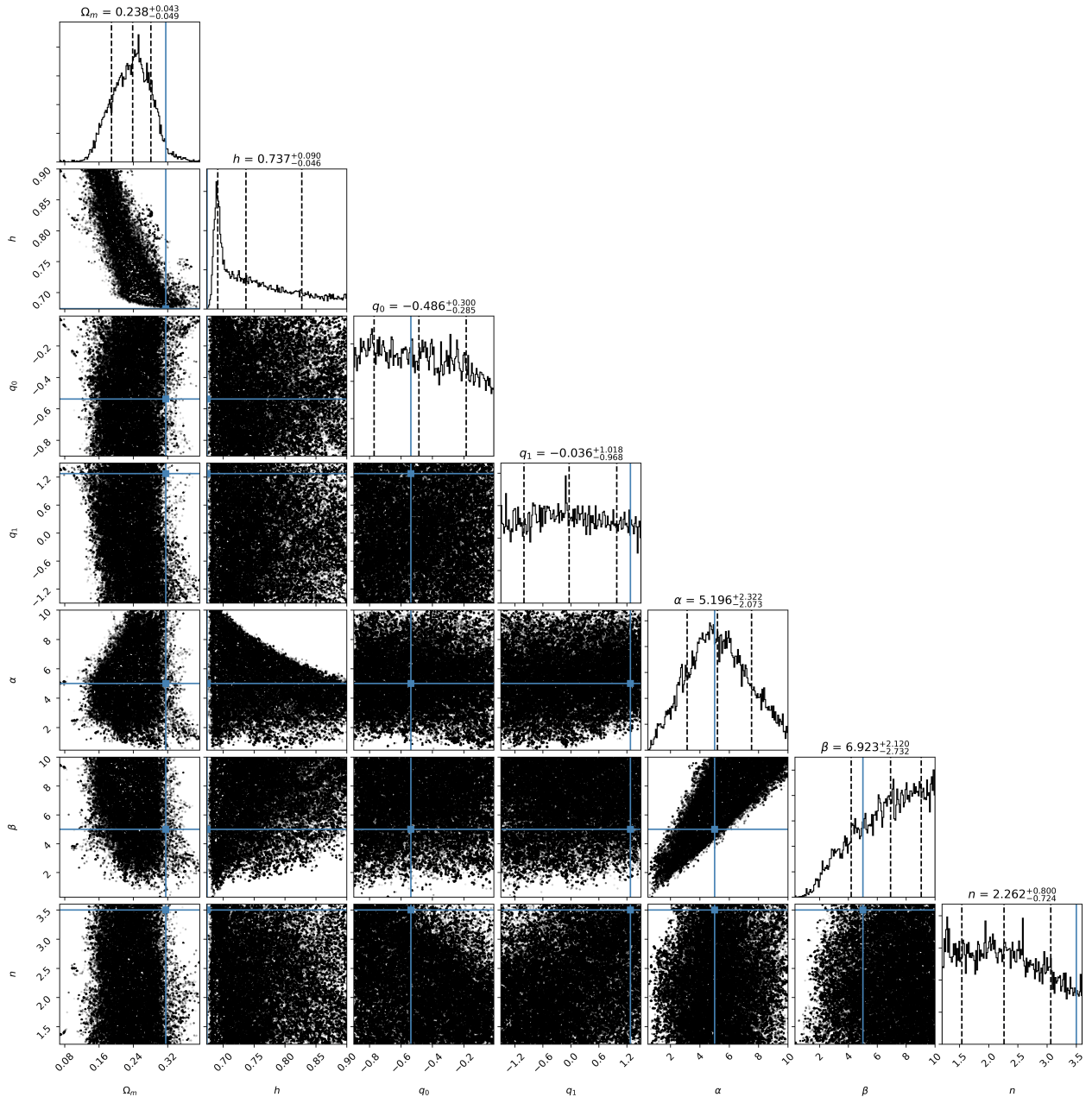


Figure 4.16: The numerical MCMC simulation results for the Hu-Sawicki model’s (Eq. 4.95) cosmological parameters (Ω_m , h , q_0 , and q_1), as well as the model’s arbitrary free parameters (α , β and n). The “true” values for Ω_m and h (blue lines) are provided by Planck 2018 collaboration data release [4], while the “true” values for q_0 and q_1 are obtained from [6]. We used 50 random walkers, 4 000 iterations, 101 integration steps, 100 ‘crude’ and ‘finer’ optimisation steps, and priors provided by Equation 4.96. The blue lines for arbitrary free parameter values are to show their original initial chosen starting points for the MCMC simulation.

deviation on the average residual of $\sigma_{res} = 0.21622$. From the R.H.S. panel, we can conclude that this model does not over- or under-estimate the low- or high-redshift supernovae. The standard deviation on the residual does, however, show us that it is a slightly worse fit than the original

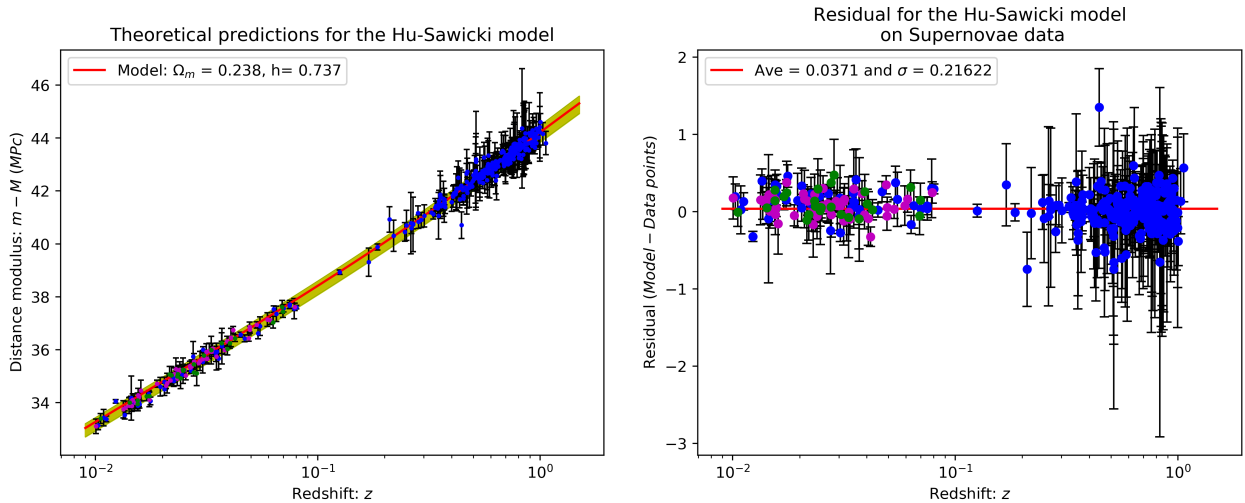


Figure 4.17: The Hu-Sawicki model fitted to the Supernovae Type 1A data. With cosmological parameter values calculated by the MCMC simulation as $\Omega_m = 0.238^{+0.043}_{-0.049}$ (constrained), $H_0 = 73.7^{+9.0}_{-4.6} \frac{km}{s.Mpc}$ (unreliably constrained), $q_0 = -0.486^{+0.300}_{-0.285}$ (unconstrained), and $q_1 = -0.036^{+1.018}_{-0.968}$ (unconstrained), as shown in Figure 4.16. The arbitrary free parameters are calculated to be $\alpha = 5.196^{+2.322}_{-2.073}$ (constrained), $\beta = 6.923^{+2.120}_{-2.732}$ (unreliably constrained), and $n = 2.262^{+0.800}_{-0.724}$ (unconstrained).

Starobinsky model. The best-fitting Hu-Sawicki model is given by

$$f(R) = R - 5.196^{+2.322}_{-2.073} R_c \left[\frac{\left(\frac{R}{R_c}\right)^{2.262^{+0.800}_{-0.724}}}{1 + \left(\frac{R}{R_c}\right)^{2.262^{+0.800}_{-0.724}}} \right], \quad (4.97)$$

$$\text{with } R_c = \frac{2 \left[3H_0^2 \times (1 - \Omega_m) \right]}{\beta} = \frac{2 \left[73.7^{+9.0}_{-4.6} \left(\frac{km}{s.Mpc} \right)^2 \times (1 - 0.238^{+0.043}_{-0.049}) \right]}{6.923^{+2.120}_{-2.732}}.$$

4.6 AIC and BIC results

Using our AIC and BIC Equations 3.22, we can determine the validity of our different $f(R)$ -gravity models. In Table 4.1, we show the $\mathcal{L}(\hat{\theta}|data)$ -value, the χ^2 -value, the reduced χ^2 -value (to determine if the model fits the data), the AIC and BIC values for each of our models, as well as the difference between the Λ CDM model a.k.a our “true” model and the alternative $f(R)$ -gravity models. We used the error on the supernovae data as our standard deviation. It should be noted that the error on the supernovae data, especially at a higher redshift $z > 0.5$, will cause the χ^2 -values to be low, since it is easier to fit a model to data with a large error on it, than on a dataset with smaller errors. This is shown in the χ^2 -value equation, since we divide by the standard deviation (error region).

By rule of thumb, a reduced χ^2 -value between $0.5 < \chi^2 < 3$ is a good fit; however, we expect a lower bound χ^2 -value. Due to this, we will take the reduced χ^2 -value of the Λ CDM model as a

Model	$\mathcal{L}(\hat{\theta} data)$	χ^2	Reduced χ^2	AIC	$ \Delta AIC $	BIC	$ \Delta BIC $
Starobinsky	-120.7052	241.4105	0.6839	253.4105	7.9939	276.7104	23.5272
Λ CDM	-120.7083	241.4166	0.6762	245.4166	0	253.1832	0
Starobinsky reduced	-122.4442	244.8885	0.6879	250.8885	5.4719	262.5385	9.3553
$\alpha R + \beta$	-131.2518	262.5037	0.7394	270.5037	25.0871	286.0370	32.8538
Hu-Sawicki	-140.1668	280.3336	0.7964	294.3336	48.9170	321.5169	68.3336
$\alpha R + \beta R^2$	-155.0369	310.0738	0.8784	322.0738	76.6572	345.3737	92.1905
βR^n	-175.0105	350.0211	0.9916	362.0211	116.6045	385.3210	132.1378
$\alpha R + \beta\sqrt{R}$	-347.0748	694.1496	1.9664	706.1496	460.7330	729.4496	476.2664
$\alpha R + \beta R$	-488.3049	976.6099	2.7510	984.6099	739.1933	1000.1432	746.9600

Table 4.1: The best fit for each test model, including the Λ CDM model. The models are listed in the order from largest likelihood function value $\mathcal{L}(\hat{\theta}|data)$ to the smallest likelihood of being viable. The reduced χ^2 -values are given as an indication of how well a particular model fits the data. The AIC and BIC values are shown, as well as the ΔIC for each information criterion. The Λ CDM model is chosen as the “true” model.

standard description of the best fit, since it is the accepted model. In this way, we can note how well the $f(R)$ -gravity model compares to the Λ CDM model, as well as obtain insight into how good the model fits the data.

From Table 4.1, we notice some interesting packages of information that we have to take note of. Firstly, only the two Starobinsky models are within the criteria for some observational support, although they do not have substantial observational support with regard to the “true” model. Secondly, the original Starobinsky model actually obtained a better fit to the supernovae data than the Λ CDM model, but still struggled with the AIC and BIC statistical analysis, with the latter actually rejecting this model. However, as mentioned in Section 3.6, one should be careful in handling both these criteria. Thirdly, as expected, the more realistic models, such as the Λ CDM model, the Starobinsky model, and the Hu-Sawicki model all favoured reduced χ^2 -values close to the lower bound of what would be considered to be a good fit. This is a result of the larger errors on the higher redshift data.

When we look at the goodness of their fits by comparing the χ^2 - and the reduced χ^2 -values, we note that at least three different $f(R)$ -gravity models (both the Starobinsky models and the $f(R) = \alpha R + \beta$ model) are comparable in accuracy to the Λ CDM model. These three models are in a position where it would be worth it to do further testing on them with a different dataset. There is though an interesting subversion of expectation concerning the $f(R) = \alpha R + \beta$ model, which as mentioned in Section 4.3.1.1, is a particular case of the Λ CDM model. The expectation refers to the fact that this model’s form is the same as the Λ CDM’s model with 2 extra free parameters. Therefore, it is expected that due to the extra freedom this model has, that it would at least fit the data with the same accuracy as the Λ CDM model or even better. Due to this, we firstly did expect that the MCMC simulation is wrong and we went over the code again and re-executed (with fewer steps just to test the simulation) the program, but found the same results.

This led us to rewrite Equation 4.49 to resemble the Λ CDM model and found cosmological parameter to have a value of $\Lambda = 2.190^{+1.011}_{-0.900}$, which is quite higher than the assumed/observed value of $\Lambda \approx 1$. Therefore, this particular model is over-estimating the dark energy content of the Universe in comparison with the Λ CDM model due to its freedom. This can mean that the shape of the best-fitting model is compromised in order to accommodate the extra free parameters. To fix this in future work, we will have to chose different priors to restrict the free parameters to a value for the cosmological constant that is comparable to the observed value, thereby restricting some of the freedom that the arbitrary free parameters have. If this change, however, does not fix the problem it might be due to the fact that the two constrained parameters are the arbitrary free parameters, while we found in Section 4.4.2 that the cosmological parameters are the parameters that determine the shape of the function. Therefore, by having them unconstrained or unreliably constrained the function does not perform to its full capability. This problem can be solved either by using different datasets or more accurate and state-of-the-art software.

We had a further two models (Hu-Sawicki model and $f(R) = \alpha R + \beta R^2$ model) that might give interesting results on other datasets, since they are not as accurate as the Λ CDM model, but were still able to explain the data to a certain extent. We also obtained a further three models (first toy model, and the last two of the second toy model), which seem to fit the supernovae data at all.

If we were to combine the χ^2 -results with the AIC and BIC results, only the two Starobinsky models would be considered viable, since both were in the range of some observational support, as well as fitting the data with the same accuracy as the Λ CDM model. However, since the reduced χ^2 -values of the Hu-Sawicki and the second toy model for fixed $n = 0, 2$ gave an argument for their acceptance, they are also deemed viable even though they were rejected by the AIC and BIC selections. However, this needs to be taken into account when selecting the dataset, as well as what will be tested. Therefore, we constrained the $f(R)$ -gravity models, by finding some viable models, while being able to reject others.

Space exploration and obtaining the knowledge to understand spacetime itself play their part in the search for the purpose of life. The same purpose that has led humankind to build civilisations across the world. From carving notches in a bone, to counting the days between consecutive full moons, to taking a direct image (10 April 2019) of a black hole, to trying to understand the accelerated expansion of the Universe.

In this dissertation, we discussed the background on gravity models (Chapter 1), starting with Newton's law of gravity and its limitations. These limitations led to the discovery of GR. The confirmation of GR's validity came in 1929, when Edwin Hubble confirmed the expansion of the Universe (as predicted by GR) with observational evidence. Due to overwhelming observational support, GR became the accepted theory for gravity. We used the Einstein-Hilbert action as a starting point to derive the Einstein field equations, which explain the matter distribution and dynamics throughout the Universe. From these equations, we derived the Friedmann equations, which can mathematically explain the expansion of the Universe. We had to include an unknown pressure force (dark energy) acting out against gravity in these derivations. This is needed to explain the observational support for an accelerated expansion, which came as a shock to cosmologists, since standard GR was not able to explain it. The GR model with the dark energy term has become the concordance model, called the Λ CDM model.

However, in Chapter 2, we discussed how the concept of dark energy creates problems for the Λ CDM model, since it would mean that $\sim 68\%$ of the Universe consists of a substance that is unknown. This led to the introduction of modified gravity models, which consist of modifying GR's concept of gravity. This is done by introducing higher dimensions, or extra fields, or finding a higher-order

derivative for gravity. We used an alternative gravity model for the latter, classified as $f(R)$ -gravity models. These models change the Einstein-Hilbert action, by changing the dependency on the Ricci scalar (R), to a generic function of the Ricci scalar. Furthermore, these alternative models exclude the dark energy term. Using this knowledge, we re-derived the Einstein's field equations and the Friedmann equations for $f(R)$ gravity in particular.

Up to this point, all the work was either background on the theory of the Λ CMD model or on modified gravity models. We then discussed how we will go forth to test these models (Chapter 3). This led to the introduction of supernovae cosmology, which is a method that involves using the luminosity from Supernovae Type 1A to determine the distance to the exploded star. This distance can be calculated by using the distance modulus. For this method to work, we need to assume that the total luminosity from each supernova remains a constant. In this way, the measured brightness of the supernova will only be dependent on the flux decrease due to the distance. We were allowed to make this assumption, since these types of supernovae are considered to be standard candles. Since the brightness of the supernovae is only dependent on the distance, any change in the distance (e.g. the expansion of the Universe) will then be measurable.

We used 359 (123 low-redshift and 236 intermediate-redshift) supernovae data points from the SDSS-II/SNLS3 Joint Light-curve Analysis (JLA). We did explain why we did not use the full JLA dataset, due to the inefficiency of our MCMC simulation, as well as only using already calculated absolute magnitudes for the supernovae. We also wrote an MCMC simulation program to determine the best-fitting model on this particular dataset. From the best-fitting model, we can determine the Hubble constant (the rate at which the Universe is expanding - H_0) and the matter density distribution (total fraction of the Universe consisting of baryonic matter and dark matter - Ω_m) for each model. We used the Λ CDM model as the “true” model, since we know from other observations, the predicted values for these cosmological parameters. Therefore, we were able to test our MCMC simulation. For the Λ CDM model, we found the cosmological parameters to be $\Omega_m = 0.268^{+0.025}_{-0.024}$ and $H_0 = 69.7^{+0.5}_{-0.5} \frac{km}{s.Mpc}$. Here, we discovered that even though these values are not within 1σ from the CMB results, they do correspond to other supernovae-related results. Therefore, confirming that our code works.

We then looked at how the $f(R)$ -gravity models faired against the supernovae data (Chapter 4). To achieve this, we first had to find the $f(R)$ -Friedmann equation in terms of the Hubble parameter and two new cosmological parameters, namely the deceleration parameter and the jerk parameter. We then used this Friedmann equation for our first $f(R)$ toy model (Sec. 4.2). We obtained the best-fitting cosmological parameters to be $\Omega_m = 0.294^{+0.076}_{-0.105}$, $H_0 = 68.7^{+4.4}_{-5.5} \frac{km}{s.Mpc}$, $q_0 = -0.124^{+0.081}_{-0.128}$, and $q_1 = -0.637^{+0.114}_{-0.081}$, while the arbitrary free parameters were calculated to be $\beta = 0.731^{+0.273}_{-0.265}$ and $n = 1.262^{+0.013}_{-0.017}$. As for our second toy model (Sec. 4.3), we obtained a function that had to be simplified to be solvable. We fixed the n arbitrary free parameter to achieve this. The best-fitting second toy model turned out to be when $n = 0$, for which we obtained the cosmological parameters as $\Omega_m = 0.317^{+0.061}_{-0.101}$ and $H_0 = 71.5^{+6.0}_{-7.2} \frac{km}{s.Mpc}$, while the arbitrary free parameters were calculated

to be $\alpha = 1.202^{+0.397}_{-0.392}$ and $\beta = -5.265^{+1.698}_{-1.315}$. This was an interesting result, since this model has the same form as the Λ CDM model. To improve this result, we will either have to constrain the α and β arbitrary free parameters to obtain a more realistic cosmological constant (current estimation $\Lambda = 2.190^{+1.011}_{-0.900}$) that is not over-estimating the dark energy content of the Universe, or use more accurate simulations with different datasets. This might result in this particular model fitting the data with the same precision as the Λ CDM model, as expected. The only other model worth noting is for $n = 2$ (also known as the original Starobinsky model), for which we obtained the cosmological parameter values as $\Omega_m = 0.249^{+0.102}_{-0.101}$, $H_0 = 63.8^{+4.6}_{-2.7} \frac{km}{s.Mpc}$, $q_0 = -0.575^{+0.040}_{-0.046}$, and $q_1 = -0.633^{+0.081}_{-0.049}$, while the arbitrary free parameters were calculated to be $\alpha = 19.642^{+2.967}_{-1.753}$ and $\beta = 0.903^{+0.070}_{-0.107}$. The models for $n = \frac{1}{2}$ and $n = 1$ did not accurately predict the data.

We then looked at the more realistic models, namely the Starobinsky and Hu-Sawicki models. This proved to be difficult, since these models were not solvable (even in their simplest possible forms). We then developed a numerical method that firstly finds an optimised numerical solution for the Friedmann equation, and then to numerically integrate over the redshift of the supernova. This method worked in conjunction with the MCMC simulation to determine the best-fitting model. We again used the Λ CDM model to test the numerical method. With the newly updated program, we found that the Starobinsky model (Sec. 4.4.2) fitted the data the best compared to all the different models, with cosmological parameter values of $\Omega_m = 0.268^{+0.027}_{-0.024}$, $H_0 = 69.0^{+0.5}_{-0.5} \frac{km}{s.Mpc}$, $q_0 = -0.512^{+0.328}_{-0.265}$, and $q_1 = 0.037^{+0.991}_{-1.050}$, while arbitrary free parameters were calculated to be $\beta = 5.284^{+3.191}_{-2.981}$ and $n = 4.567^{+3.346}_{-2.899}$. The Hu-Sawicki model (Sec. 4.5), which on its own also fitted the data with quite a high precision, gave the cosmological parameter values as $\Omega_m = 0.238^{+0.043}_{-0.049}$, $H_0 = 73.7^{+9.0}_{-4.6} \frac{km}{s.Mpc}$, $q_0 = -0.486^{+0.300}_{-0.285}$, and $q_1 = -0.036^{+1.018}_{-0.968}$, while the arbitrary free parameters were calculated to be $\alpha = 5.196^{+2.322}_{-2.073}$, $\beta = 6.923^{+2.120}_{-2.732}$, and $n = 2.262^{+0.800}_{-0.724}$. We also found that both the Starobinsky and Hu-Sawicki models are within 1σ from the results obtained by [102]. This is quite significant since they used different datasets, as well as using the full JLA dataset, on a more efficient software package

In terms of which parameters we were able to constrain for the different models, we found that top 5 models in Table 4.1, including the Λ CDM model, all obtained 2 constrained parameters. It was further found that the model that constrained the cosmological parameters (Ω_m and $\bar{h}$) tended to have better fits than the model that constrained the other parameters. For instance, the two Starobinsky models and the Λ CDM model constrained both the cosmological parameters and were found to be the three best fitting models and all obtaining a likelihood probability that is relatively similar. The only exception to this was between the next two best fitting models (Hu-Sawicki and second toy model for $n = 0$). Both of these also found two constrained parameters with the Hu-Sawicki model obtaining one cosmological parameter Ω_m that was constrained and one of the arbitrary free parameters, while the second toy model for $n = 0$ found both the constrained parameter to be the arbitrary free parameters. In this case it might be the close connection the latter model has with the Λ CDM model that might have explained why it fared better than the Hu-Sawicki

model. The same rule can be applied to the following three models in Table 4.1 that obtained 1 constrained parameter each. The model that fared the best between these three models was the Second toy model for $n = 2$ with the constrained parameter being the deceleration parameter which is a measurable quantity and is considered to be part of the cosmological parameters, while the other two models found the constrained parameter to be one of the arbitrary free parameters. That leaves us with only one model left and it was telling that this model did not received one fully constrained parameter. Therefore, we can conclude that the cosmological parameters, especially Ω_m , $\bar{h}$ and to an extent the deceleration parameter, mostly control the main structure of the function with the correction parameter and the arbitrary free parameters being used for the fine-tuning of the function, as suggested in Section 4.4.2

We can conclude that we constrained some $f(R)$ -gravity model parameters with cosmological data, since we obtained, from a statistical analysis point of view, at least three different $f(R)$ -gravity models that fitted the data (the 2 Starobinsky models and the $f(R) = \alpha R + \beta$ model) with a precision comparable to the Λ CDM model, while a further two models (the Hu-Sawicki and the $f(R) = \alpha R + \beta R^2$ model) might give interesting results in future work. The remaining three models could not accurately predict the supernovae dataset. However, it is worth noting that all the models, except the two Starobinsky models, gave information criteria values that would suggest that they are statistically not probable. This may be a result of the large number of free parameters. For instance, the Starobinsky model is the best-fitting model, but was still rejected by the BIC. This led us to simplify the Starobinsky model (Sec. 4.4.3), to which we found the cosmological parameters to be $\Omega_m = 0.266^{+0.026}_{-0.024}$, $H_0 = 69.4^{+1.8}_{-0.6} \frac{km}{s.Mpc}$, and $q_0 = -0.697^{+0.173}_{-0.138}$, with the assumption that the other three parameters have fixed values of $q_0 = 0$ and $\beta = n = 1$. Even with this assumption, the reduced Starobinsky model was the third best-fitting model, and was the only model to be accepted by both the AIC and BIC suggestions.

For future work, we can use these models (top five best-fitting models), which seem to be able to predict the accelerated expansion of the Universe, on other datasets such as the CMB, large-scale structure formation, the newest and fullest Supernovae Type 1A dataset, namely JLA and Pantheon (to improve our statistics) or the growth rate data to further constrain them. It might also be interesting to see how these $f(R)$ -gravity models hold up against other tests that might not necessarily be testing the rate of expansion. Furthermore, since we understand the basic MCMC simulations now and how it can be used to constrain some of the parameters, we can look at the possibility of using state-of-the-art and more efficient computer software to improve our statistics (on the five possible viable models) even further and be able to constrain all the free parameters to a greater extent. This will also allow us to use a more useful/realistic prior for the Hubble uncertainty parameter, such as the Riess prior. It is also worth checking the plausibility of simplifying the other $f(R)$ -gravity models, as we have done with the Starobinsky model, which ended up being our only statistically viable model. Another simplification can be in the form of $q(z) = q_0$ or using one free parameter to represent all parameters excluding the cosmological parameters.

Appendices

Covariant and contra-variant derivative calculation

To calculate the covariant derivative of the variation of the connection, we will use the general form as follows:

$$\begin{aligned} \bullet \quad \nabla_l A^a &= \partial_l A^a + \Gamma_{lk}^a A^k, \\ \bullet \quad \nabla_l A_a &= \partial_l A_a - \Gamma_{al}^k A_k. \end{aligned} \tag{A.1}$$

In our case we are using $A = \delta\Gamma$, and the index k as e , so we have

$$\begin{aligned} \bullet \quad \nabla_c \delta\Gamma_{bd}^a &= \partial_c \delta\Gamma_{bd}^a + \Gamma_{ce}^a \delta\Gamma_{bd}^e - \Gamma_{bc}^e \delta\Gamma_{de}^a - \Gamma_{dc}^e \delta\Gamma_{be}^a, \\ \bullet \quad \nabla_d \delta\Gamma_{bc}^a &= \partial_d \delta\Gamma_{bc}^a + \Gamma_{de}^a \delta\Gamma_{bc}^e - \Gamma_{bd}^e \delta\Gamma_{ce}^a - \Gamma_{cd}^e \delta\Gamma_{be}^a. \end{aligned} \tag{A.2}$$

If we then subtract these two equations from each other, we find

$$\nabla_c \delta\Gamma_{bd}^a - \nabla_d \delta\Gamma_{bc}^a = \partial_c \delta\Gamma_{bd}^a + \delta\Gamma_{bd}^e \Gamma_{ce}^a - \Gamma_{bc}^e \delta\Gamma_{de}^a - \Gamma_{dc}^e \delta\Gamma_{be}^a - \partial_d \delta\Gamma_{bc}^a - \delta\Gamma_{bc}^e \Gamma_{de}^a + \Gamma_{bd}^e \delta\Gamma_{ce}^a + \Gamma_{cd}^e \delta\Gamma_{be}^a. \tag{A.3}$$

Since c and d are neighbouring connections, the last terms in Equation A.2 are equal [22]. In that case, they just cancel each other out in Equation A.3.

Christoffel symbols, Ricci tensor, Ricci scalar

Christoffel symbol

In this appendix, we will only show an example of each of the different tensor calculations, starting with the Christoffel symbol. Let us calculate $\Gamma_{\phi\phi}^t$:

$$\begin{aligned}\Gamma_{\phi\phi}^t = & \frac{1}{2}g^{tt}\left(\partial_\phi g_{t\phi} + \partial_\phi g_{\phi t} - \partial_t g_{\phi\phi}\right) + \frac{1}{2}g^{tr}\left(\partial_\phi g_{r\phi} + \partial_\phi g_{\phi r} - \partial_r g_{\phi\phi}\right) \\ & + \frac{1}{2}g^{t\theta}\left(\partial_\phi g_{\theta\phi} + \partial_\phi g_{\phi\theta} - \partial_\theta g_{\phi\phi}\right) + \frac{1}{2}g^{t\phi}\left(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}\right).\end{aligned}\tag{B.1}$$

Using metric $g_{\mu\nu}$ given in Equation 1.24, we can see that only the first big term (the $\frac{1}{2}g^{tt}(\dots)$ term) survives in Equation B.1, due to off-diagonal zeros. As for that specific term, we see that only the last term of the three terms (the $-\frac{1}{2}g^{tt}\partial_t g_{\phi\phi}$ term) survives. This results in

$$\begin{aligned}\Gamma_{\phi\phi}^t &= -\frac{1}{2}g^{tt}(\partial_t g_{\phi\phi}), \\ &= -\frac{1}{2}(-c^{-2})\partial_t(a^2(t)r^2\sin^2\theta), \\ &= \frac{a\dot{a}}{c^2}r^2\sin^2\theta,\end{aligned}\tag{B.2}$$

which corresponds to Equation 6 in [25], although they wrote it in a general equation. Next up, we will show an example of how to calculate Ricci tensor components.

Ricci tensor

We will calculate the first component of R_{tt} and just calculate the other two components in a similar fashion of R_{tt} :

$$\begin{aligned}
R_{trt}^r &= \partial_r \Gamma_{tt}^r - \partial_t \Gamma_{tr}^r + \Gamma_{tt}^r \Gamma_{rr}^r + \Gamma_{tt}^\theta \Gamma_{r\theta}^r + \Gamma_{tt}^\phi \Gamma_{r\phi}^r - \Gamma_{tr}^r \Gamma_{tr}^r - \Gamma_{tr}^\theta \Gamma_{t\theta}^r - \Gamma_{tr}^\phi \Gamma_{t\phi}^r, \\
&= \partial_r(0) - \partial_t \left(\frac{\dot{a}}{a} \right) + (0) \left[\frac{r}{K^2(1 - \frac{r^2}{K^2})} \right] + (0)(0) + (0)(0) - \left(\frac{\dot{a}}{a} \right) \left(\frac{\dot{a}}{a} \right) - (0)(0) - (0)(0), \\
&= -\partial_t \left(\frac{\dot{a}}{a} \right) - \left(\frac{\dot{a}}{a} \right)^2 = -\left(\frac{\ddot{a}a - \dot{a}\dot{a}}{a^2} \right) - \left(\frac{\dot{a}}{a} \right)^2 = -\frac{\ddot{a}}{a}, \\
R_{t\theta t}^\theta &= -\frac{\ddot{a}}{a}, \\
R_{t\phi t}^\phi &= -\frac{\ddot{a}}{a}.
\end{aligned} \tag{B.3}$$

Combining the three components for R_{tt} , we find the temporal Ricci tensor component as

$$\begin{aligned}
R_{tt} &= R_{tmt}^m = R_{trt}^r + R_{t\theta t}^\theta + R_{t\phi t}^\phi, \\
&= -\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} = -3\frac{\ddot{a}}{a}.
\end{aligned} \tag{B.4}$$

The spacial Ricci tensor components can be calculated in a similar fashion. The results are:

$$\begin{aligned}
\bullet \quad R_{rr} &= \frac{a\ddot{a}}{c^2(1 - \kappa r^2)} + \frac{2\dot{a}^2}{c^2(1 - \kappa r^2)} + \frac{2\kappa}{(1 - \kappa r^2)} = \frac{1}{c^2(1 - \kappa r^2)}(a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2), \\
\bullet \quad R_{\theta\theta} &= \frac{a\ddot{a}}{c^2}r^2 + \frac{2\dot{a}^2}{c^2}r^2 + 2\kappa r^2 = \frac{r^2}{c^2}(a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2), \\
\bullet \quad R_{\phi\phi} &= \frac{a\ddot{a}}{c^2}r^2 \sin^2 \theta + \frac{2\dot{a}^2}{c^2}r^2 \sin^2 \theta + 2\kappa r^2 \sin^2 \theta = \frac{r^2}{c^2} \sin^2 \theta (a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2).
\end{aligned} \tag{B.5}$$

Ricci scalar

Lastly, using the Ricci tensor, we can calculate the Ricci scalar by following the definition of the Ricci scalar:

$$\begin{aligned}
R &= g^{ab} R_{ab}, \\
&= g^{tt} R_{tt} + g^{tr} R_{tr} + g^{t\theta} R_{t\theta} + g^{t\phi} R_{t\phi} + g^{rt} R_{rt} + g^{rr} R_{rr} + g^{r\theta} R_{r\theta} + g^{r\phi} R_{r\phi} \\
&\quad + g^{\theta t} R_{\theta t} + g^{\theta r} R_{\theta r} + g^{\theta\theta} R_{\theta\theta} + g^{\theta\phi} R_{\theta\phi} + g^{\phi t} R_{\phi t} + g^{\phi r} R_{\phi r} + g^{\phi\theta} R_{\phi\theta} + g^{\phi\phi} R_{\phi\phi}, \\
&= g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}, \\
&= \frac{3\ddot{a}}{ac^2} + \frac{3}{a^2 c^2} (a\ddot{a} + 2\dot{a}^2 + 2\kappa c^2), \\
&= \frac{6}{a^2 c^2} (a\ddot{a} + \dot{a}^2 + \kappa c^2).
\end{aligned} \tag{B.6}$$

Matter-dominated epoch space curvature solutions

Flat universe

We can now solve Equation 1.50, for the three different curvatures of space, starting with a flat universe ($\kappa = 0$). Therefore, we have

$$\begin{aligned}
 \dot{a}^2 &= Aa^{-1} - 0, \\
 \Rightarrow a\dot{a}^2 &= A, \\
 \Rightarrow \sqrt{a}da &= \sqrt{A}dt, \\
 \Rightarrow \int \sqrt{a}da &= \int_{t_0}^t \sqrt{A}dt, \\
 \Rightarrow a^{\frac{3}{2}} &= \frac{3}{2}\sqrt{A}(t - t_0), \\
 \Rightarrow a &= \left(\frac{3}{2}\right)^{\frac{2}{3}}\sqrt{A}(t - t_0)^{\frac{2}{3}},
 \end{aligned} \tag{C.1}$$

where t_0 is the initial time parameter value.

Closed universe

Next up, we have a closed universe ($\kappa = 1$). From Equation 1.50, we have

$$\begin{aligned}
 \dot{a}^2 &= Aa^{-1} - 1, \\
 \Rightarrow a\dot{a}^2 &= A - a, \\
 \Rightarrow \frac{a}{A-a}\dot{a}^2 &= 1, \\
 \Rightarrow \int \sqrt{\frac{a}{A-a}} da &= \int dt.
 \end{aligned} \tag{C.2}$$

We can then use the definition of conformal time $dt = ad\eta$ to parametrise the integral. This leaves us with

$$\begin{aligned}
 \int \sqrt{\frac{a}{A-a}} da &= \int ad\eta, \\
 \Rightarrow \int \sqrt{\frac{a}{a^2(A-a)}} da &= \int d\eta, \\
 \Rightarrow \int \frac{1}{\sqrt{aA-a^2}} da &= \eta - \eta_0.
 \end{aligned} \tag{C.3}$$

Since $A = \rho_0$ for some initial density, we can define a new constant $K = \frac{\rho_0}{2}$, with a boundary set-up 0 for the lower limit and a for the upper limit. On the left-hand side, we then have

$$\begin{aligned}
 \int_0^a \frac{1}{\sqrt{2Ka-a^2}} da &= \cos^{-1} \left(\frac{K-a}{K} \right) \Big|_0^a, \\
 &= \cos^{-1} \left(\frac{K-a}{K} \right) - \cos^{-1} \left(\frac{K}{K} \right), \\
 &= \cos^{-1} \left(\frac{K-a}{K} \right).
 \end{aligned} \tag{C.4}$$

This standard integral was found in [113]. Furthermore, we know $\eta = 0$ at $a = 0$, which means $\eta_0 = 0$, so we have

$$\begin{aligned}
 \cos^{-1} \left(\frac{K-a}{K} \right) &= \eta, \\
 \Rightarrow \frac{K-a}{K} &= \cos \eta, \\
 \Rightarrow a &= K(1 - \cos \eta).
 \end{aligned} \tag{C.5}$$

We can then calculate the time coordinate using the conformal time, resulting in

$$\begin{aligned}
 t &= \int a d\eta, \\
 &= \int K(1 - \cos \eta) d\eta, \\
 &= K(\eta - \sin \eta),
 \end{aligned} \tag{C.6}$$

where we used the fact that $t = 0$ at $\eta = 0$, which sets $t_0 = 0$.

Open universe

Lastly, we have an open universe ($\kappa = -1$). From Equation 1.50, we have

$$\begin{aligned}
 \dot{a}^2 &= Aa^{-1} + 1, \\
 \Rightarrow a\dot{a}^2 &= A + a, \\
 \Rightarrow \frac{a\dot{a}^2}{A + a} &= 1, \\
 \Rightarrow \int \sqrt{\frac{a}{A + a}} da &= \int dt.
 \end{aligned} \tag{C.7}$$

We then once again use the definition of conformal time, and using the same mathematical reshuffling, we find

$$\int \frac{1}{\sqrt{2Ka + a^2}} da = \eta - \eta_0. \tag{C.8}$$

On the left-hand side, we have

$$\begin{aligned}
 \int \frac{1}{\sqrt{2Ka + a^2}} da &= \int \frac{1}{\sqrt{(a^2 + 2Ka + K^2 - K^2)}} da, \\
 &= \int \frac{1}{\sqrt{(a + K)^2 - K^2}} da,
 \end{aligned} \tag{C.9}$$

by completing the square. Let $u = (a + K)$, which has a derivative of $\frac{du}{da} = 1$ and setting the limits again, the same as previously done, we have

$$\begin{aligned}
\int_0^a \frac{1}{\sqrt{u^2 - K^2}} du &= \ln \left| u + \sqrt{u^2 - K^2} \right| \Big|_0^a \\
&= \ln \left| a + K + \sqrt{(a + K)^2 - K^2} \right| \Big|_0^a, \\
&= \ln \left| a + K + \sqrt{a^2 + 2Ka} \right| \Big|_0^a, \\
&= \ln \left| a + K + \sqrt{a^2 + 2Ka} \right| - \ln \left| 0 + K + \sqrt{(0)^2 + 2K0} \right|, \\
&= \ln \left| \frac{a + K + \sqrt{a^2 + 2Ka}}{K} \right|, \\
&= \ln \left| \frac{a}{K} + 1 + \sqrt{\frac{a^2}{K^2} + \frac{2a}{K}} \right|, \\
&= \ln \left| \frac{a}{K} + 1 + \sqrt{\frac{a^2}{K^2} + \frac{2a}{K} + 1 - 1} \right|, \\
&= \ln \left| \frac{a}{K} + 1 + \sqrt{\left(\frac{a}{K} + 1 \right)^2 - 1} \right|.
\end{aligned} \tag{C.10}$$

This standard integral was found in [113]. Let $x = \frac{a}{K} + 1$, so we have

$$\begin{aligned}
\ln \left| x + \sqrt{x^2 - 1} \right| &= \cosh^{-1} x, \\
&= \cosh^{-1} \left(\frac{a}{K} + 1 \right).
\end{aligned} \tag{C.11}$$

The definition for the $\cosh x$ was found in [113]. However, once again, $\eta = 0$ at $a = 0$ means that $\eta_0 = 0$, which leaves us with

$$\begin{aligned}
\eta &= \cosh^{-1} \left(\frac{a}{K} + 1 \right), \\
\Rightarrow \frac{a}{K} + 1 &= \cosh \eta, \\
\Rightarrow a &= K (\cosh \eta - 1).
\end{aligned} \tag{C.12}$$

We can then calculate the time coordinate using the conformal time, resulting in

$$\begin{aligned}
t &= \int a d\eta, \\
&= \int K (\cosh \eta - 1) d\eta, \\
&= K (\sinh \eta - \eta).
\end{aligned} \tag{C.13}$$

Variation of the action

D.1 Varying the action on the second term in Equation 2.3

In this section, we will calculate the variation of the second term in Equation 2.3. This will be done in a similar fashion, as was done in [64], starting with Equation 2.3:

$$\delta R = R_{bd}\delta g^{bd} + \nabla_e(g^{bd}\delta\Gamma_{bd}^e - g^{be}\delta\Gamma_{be}^e). \quad (\text{D.1})$$

For the first step, we will calculate the variation of the Christoffel symbol, namely $\delta\Gamma_{bd}^e$. This results in

$$\delta\Gamma_{bd}^e = \frac{1}{2}\delta g^{ef}(\partial_b g_{fd} + \partial_d g_{fb} - \partial_f g_{bd}) + \frac{1}{2}g^{ef}(\partial_b \delta g_{fd} + \partial_d \delta g_{fb} - \partial_f \delta g_{bd}), \quad (\text{D.2})$$

where f is just an index that was introduced through the definition of the Levi-Civita connection. Using the definitions of the covariant- and contra-variant derivatives, which were defined in Appendix A, we find

$$\begin{aligned} \bullet \quad \nabla_b \delta g_{fd} &= \partial_b \delta g_{fd} - \Gamma_{bf}^e \delta g_{ed} - \Gamma_{bd}^e \delta g_{ef}, \\ \bullet \quad \nabla_d \delta g_{fb} &= \partial_d \delta g_{fb} - \Gamma_{df}^e \delta g_{eb} - \Gamma_{db}^e \delta g_{ef}, \\ \bullet \quad \nabla_f \delta g_{bd} &= \partial_f \delta g_{bd} - \Gamma_{fb}^e \delta g_{ed} - \Gamma_{fd}^e \delta g_{eb}. \end{aligned} \quad (\text{D.3})$$

Substituting these covariant derivatives into Equation D.2, and using the fact that the Levi-Civita connection is symmetric ($\Gamma_{lm}^k = \Gamma_{ml}^k$, where k, l and m are arbitrary indices), we obtain

$$\begin{aligned} \delta\Gamma_{bd}^e &= \frac{1}{2}\delta g^{ef}(\partial_b g_{fd} + \partial_d g_{fb} - \partial_f g_{bd}) + \frac{1}{2}g^{ef}(\nabla_b \delta g_{fd} + \Gamma_{bf}^e \delta g_{ed} + \Gamma_{bd}^e \delta g_{ef} \\ &\quad + \nabla_d \delta g_{fb} + \Gamma_{df}^e \delta g_{eb} + \Gamma_{db}^e \delta g_{ef} - \nabla_f \delta g_{bd} - \Gamma_{fb}^e \delta g_{ed} - \Gamma_{fd}^e \delta g_{eb}), \\ \Rightarrow \delta\Gamma_{bd}^e &= \frac{1}{2}\delta g^{ef}(\partial_b g_{fd} + \partial_d g_{fb} - \partial_f g_{bd}) + g^{ef}\Gamma_{bd}^e \delta g_{ef} \\ &\quad + \frac{1}{2}g^{ef}(\nabla_b \delta g_{fd} + \nabla_d \delta g_{fb} - \nabla_f \delta g_{bd}). \end{aligned} \quad (D.4)$$

We can then use the definition [?]]

$$\delta g_{ef} = -g_{fg}g_{eh}\delta g^{gh} \quad (D.5)$$

to simplify the second term in Equation D.4, where g and h are newly introduced indices. Therefore, we have

$$\begin{aligned} \delta\Gamma_{bd}^e &= \frac{1}{2}\delta g^{ef}(\partial_b g_{fd} + \partial_d g_{fb} - \partial_f g_{bd}) - g^{ef}\Gamma_{bd}^e g_{fg}g_{eh}\delta g^{gh} \\ &\quad + \frac{1}{2}g^{ef}(\nabla_b \delta g_{fd} + \nabla_d \delta g_{fb} - \nabla_f \delta g_{bd}). \end{aligned} \quad (D.6)$$

We can then simplify Equation D.6, through the use of contraction in the second term and mathematically manipulating the first term, to be able to rewrite the first term in terms of the Levi-Civita connection. We then obtain

$$\begin{aligned} \delta\Gamma_{bd}^e &= \delta g^{ef} \left[\frac{1}{2}(\partial_b g_{fd} + \partial_d g_{fb} - \partial_f g_{bd}) \frac{g^{ef}}{g^{ef}} \right] - \delta g^{gh} \delta_g^e g_{eh} \Gamma_{bd}^e \\ &\quad + \frac{1}{2}g^{ef}(\nabla_b \delta g_{fd} + \nabla_d \delta g_{fb} - \nabla_f \delta g_{bd}), \end{aligned} \quad (D.7)$$

$$\Rightarrow \delta\Gamma_{bd}^e = \delta g^{ef} g_{ef} \Gamma_{bd}^e - \delta g^{he} g_{eh} \Gamma_{bd}^e + \frac{1}{2}g^{ef}(\nabla_b \delta g_{fd} + \nabla_d \delta g_{fb} - \nabla_f \delta g_{bd}),$$

where we used the fact that $g_{eh} = \frac{1}{g^{eh}}$ (since the metric tensor is also symmetric), as well as another contraction, namely $\delta g^{gh} \delta_g^e = \delta g^{he}$. We can see that, due to symmetry, and the fact that f and h are just indices introduced through definitions, the first two terms are equal resulting in them cancelling each other out. Therefore, we have

$$\delta\Gamma_{bd}^e = \frac{1}{2}g^{ef}(\nabla_b \delta g_{fd} + \nabla_d \delta g_{fb} - \nabla_f \delta g_{bd}). \quad (D.8)$$

Similarly, we can show that

$$\delta\Gamma_{bf}^f = \frac{1}{2}g^{ef}(\nabla_b \delta g_{ef}). \quad (D.9)$$

Now, we can once again use the definition given by Equation D.5 to calculate Equation D.8 and obtain

$$\begin{aligned}
\delta\Gamma_{bd}^e &= -\frac{1}{2}g^{ef}\left(g_{fg}g_{dh}\nabla_b\delta g^{gh} + g_{fg}g_{bh}\nabla_d\delta g^{gh} - g_{bg}g_{dh}\nabla_f\delta g^{gh}\right), \\
\Rightarrow \delta\Gamma_{bd}^e &= -\frac{1}{2}\left(\delta_g^e g_{dh}\nabla_b\delta g^{gh} + \delta_g^e g_{bh}\nabla_d\delta g^{gh} - g^{ef}g_{bg}g_{dh}\nabla_f\delta g^{gh}\right), \\
\Rightarrow \delta\Gamma_{bd}^e &= -\frac{1}{2}\left(g_{dh}\nabla_b\delta g^{eh} + g_{bh}\nabla_d\delta g^{eh} - g_{bg}g_{dh}\nabla^e\delta g^{gh}\right), \\
\Rightarrow \delta\Gamma_{bd}^e &= -\frac{1}{2}\left(g_{df}\nabla_b\delta g^{ef} + g_{bf}\nabla_d\delta g^{ef} - g_{bg}g_{dh}\nabla^e\delta g^{gh}\right),
\end{aligned} \tag{D.10}$$

where we again used the two contractions mentioned above. We also used the fact that $\nabla^e = g^{ef}\nabla_f$ and relabeled some of the indices from h to f since both are just repeating Einstein's summation convention indices over the same indices. Similarly, we can calculate Equation D.9 as

$$\delta\Gamma_{bf}^f = -\frac{1}{2}g_{gh}\nabla_b\delta g^{gh}. \tag{D.11}$$

We now have all of the terms that are necessary to calculate the term inside the bracket of Equation D.1. Using Equations D.10 and D.11, we have

$$\begin{aligned}
g^{bd}\delta\Gamma_{bd}^e - g^{be}\delta\Gamma_{be}^e &= -\frac{1}{2}\left[\left(g^{bd}g_{df}\nabla_b\delta g^{ef} + g^{bd}g_{bf}\nabla_d\delta g^{ef} - g^{bd}g_{bg}g_{dh}\nabla^e\delta g^{gh}\right) - \left(g^{be}g_{gh}\nabla_b\delta g^{gh}\right)\right], \\
\Rightarrow g^{bd}\delta\Gamma_{bd}^e - g^{be}\delta\Gamma_{be}^e &= -\frac{1}{2}\left[\left(\delta_f^b\nabla_b\delta g^{ef} + \delta_f^d\nabla_d\delta g^{ef} - \delta_g^d g_{dh}\nabla^e\delta g^{gh}\right) - \left(g_{gh}g^{be}\nabla_b\delta g^{gh}\right)\right], \\
\Rightarrow g^{bd}\delta\Gamma_{bd}^e - g^{be}\delta\Gamma_{be}^e &= -\frac{1}{2}\left[\left(\nabla_f\delta g^{ef} + \nabla_f\delta g^{ef} - g_{gh}\nabla^e\delta g^{gh}\right) - \left(g_{gh}\nabla^e\delta g^{gh}\right)\right], \\
\Rightarrow g^{bd}\delta\Gamma_{bd}^e - g^{be}\delta\Gamma_{be}^e &= -\frac{1}{2}\left(2\nabla_f\delta g^{ef} - 2g_{gh}\nabla^e\delta g^{gh}\right), \\
\Rightarrow g^{bd}\delta\Gamma_{bd}^e - g^{be}\delta\Gamma_{be}^e &= g_{gh}\nabla^e\delta g^{gh} - \nabla_f\delta g^{ef}.
\end{aligned} \tag{D.12}$$

D.2 Calculation of the second term in Equation 2.2

To calculate the second term in Equation 2.2, we will need to use Jacobi's formula for differentiating a determinant. Jacobi's formula is defined as

$$\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{D.13}$$

If we then use this formula, we obtain

$$\begin{aligned}
\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g, \\
\Rightarrow \delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\delta g_{\mu\nu}), \\
\Rightarrow \delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu}),
\end{aligned} \tag{D.14}$$

where we used the fact that

$$g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}. \tag{D.15}$$

If we then substitute the result of Equation D.14 into the second term of Equation 2.2, we find

$$\frac{f(R)}{\sqrt{-g}} \frac{\left[-\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu})\right]}{\delta g^{\mu\nu}} = -\frac{1}{2}f(R)g_{\mu\nu}. \tag{D.16}$$

E.1 Numerical optimisation

In this section, we will try to explain the optimisation function we used for the Starobinsky and Hu-Sawick models. We will illustrate the workings of the code by using the Λ CDM model, since we know more about the Λ CDM model than the $f(R)$ -gravity models. Furthermore, it is possible to solve the Λ CDM model’s distance modulus integral function using Scipy, the Python package. In this manner, we can compare our numerical method’s integration value to the value of a trusted Python package. The full numerical methods code is given in [Appendix G](#).

To show how the optimising function works, we will use realistic initial conditions. For instance, we will use a “hypothetical supernovae data point” that happened at a redshift of $z = 1.2$ and a universe that is governed by the cosmological parameters with values of $\Omega_m = 0.315$ for the matter density distribution and a Hubble constant of $H_0 \approx 67.4 \frac{km}{s.Mpc}$. We will also, for explanatory reasons, choose the number of steps needed (to take into account the total range) as *count* = 10 for the coarse and finer searches. Furthermore, in the numerical methods code you will notice we supplied three versions for the Λ CDM model (multiple different uses). For the optimising function, we had to use the Λ CDM model in the following way:

$$H^2(z) - H_0^2 \sqrt{\Omega_m (1+z)^3 + 1 - \Omega_m} = 0. \quad (E.1)$$

When we execute the program (see optimizing function output below), you will firstly see our chosen initial conditions and then the search range. The range we chose is the upper GR limit, therefore

between 0 and $H_0^2 \sqrt{\Omega_m(1+z)^3 + 1 - \Omega_m}$. We also used these boundaries for the Starobinsky and Hu-Sawicki models to restrict us from obtaining solutions not valid under the GR conditions. Furthermore, for the purposes of this test, we already know that the solution should be at the upper bound ($H^2(1.2) = 1.92484 \times 10^{-35}$ by plugging all of the numbers into Equation E.1, since we are using the GR limit to constrain the GR function.

Next up, you will see the calculated H^2 value for the Λ CDM model at each step in our range. As you will notice, these steps are large, since this is the coarse searching step sizes that run over the entire range. You will also see 10 calculations in our range, since we chose 10 equally spaced steps. You will then see the program printing out whether or not a particular step is smaller than its neighbours. Our return shows that all of them are False, except for the last step as predicted. The code then calculates a new search range between the minimum's neighbours. In this case, since the minimum is a boundary, it will make the search range not between the neighbours, but between the existing neighbour and the minimum itself.

It will then calculate the Λ CDM model function values at each of these finer step values again. You will notice these steps are 1/10 smaller than the ones used for the coarse search. This makes the optimised minimum more accurate. You will notice that these function values, even though they are not the minimum, have smaller function values compared to the values found in the coarse search, due to their close proximity to the minimum. We then test each step for the minimum and once again find it to be the final step, as predicted. This minimum value's position is then saved and returned.

```
Redshift = 1.2, Matter density = 0.315, H0^{2} = 4.7654889999999993e-36
H0^2 range between 0 and 1.924838192968e-35
```

```
Coarse H^{2} value= 0.0,          f(H^{2}) = 1.924838192968e-35
Coarse H^{2} value= 2.1387091032977775e-36, f(H^{2}) = 1.710967282638222e-35
Coarse H^{2} value= 4.277418206595555e-36, f(H^{2}) = 1.4970963723084444e-35
Coarse H^{2} value= 6.416127309893332e-36, f(H^{2}) = 1.2832254619786668e-35
Coarse H^{2} value= 8.55483641319111e-36, f(H^{2}) = 1.0693545516488889e-35
Coarse H^{2} value= 1.0693545516488887e-35, f(H^{2}) = 8.55483641319111e-36
Coarse H^{2} value= 1.2832254619786665e-35, f(H^{2}) = 6.416127309893334e-36
Coarse H^{2} value= 1.497096372308444e-35, f(H^{2}) = 4.2774182065955577e-36
Coarse H^{2} value= 1.710967282638222e-35, f(H^{2}) = 2.1387091032977788e-36
Coarse H^{2} value= 1.924838192968e-35, f(H^{2}) = 0.0
```

```
Test each step for a minimum [f(H^{2})= smaller than both neighbours]:
[False False False False False False False False True]
```

```
Minimum H^{2} value = 1.924838192968e-35, f(H^{2}) = 0.0
New search range for H^{2} between 1.710967282638222e-35 and 1.924838192968e-35
```

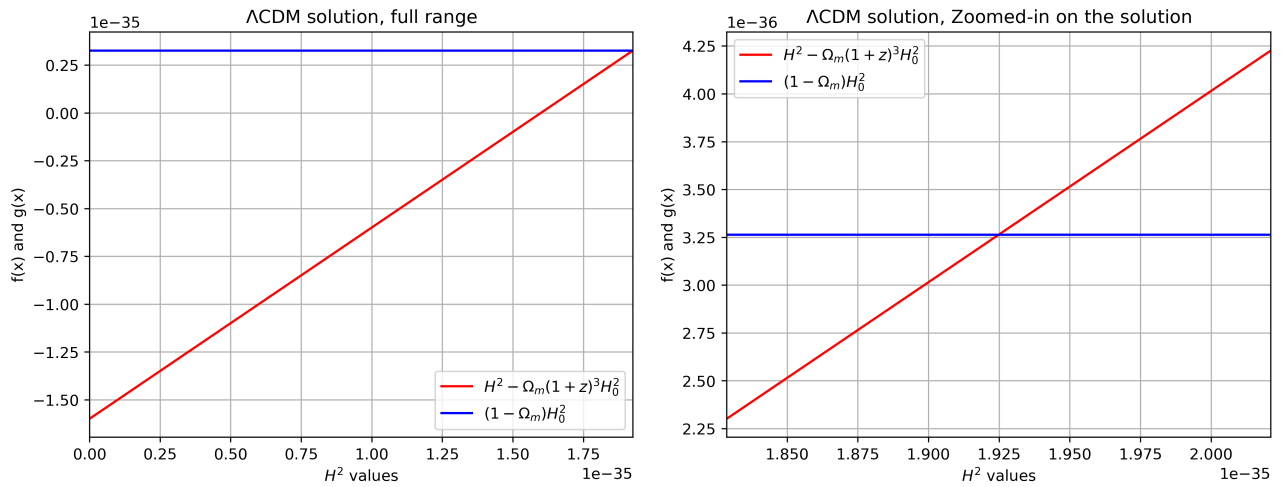


Figure E.1: Optimization solutions to the Λ CDM model

```
Finer H^{2} value= 1.710967282638222e-35, f(H^{2})_new = 2.1387091032977788e-36
Finer H^{2} value= 1.7347307171193083e-35, f(H^{2})_new = 1.9010747584869157e-36
Finer H^{2} value= 1.758494151600395e-35, f(H^{2})_new = 1.66344041367605e-36
Finer H^{2} value= 1.7822575860814812e-35, f(H^{2})_new = 1.4258060688651868e-36
Finer H^{2} value= 1.8060210205625678e-35, f(H^{2})_new = 1.188171724054321e-36
Finer H^{2} value= 1.829784455043654e-35, f(H^{2})_new = 9.505373792434579e-37
Finer H^{2} value= 1.8535478895247407e-35, f(H^{2})_new = 7.1290303443259205e-37
Finer H^{2} value= 1.877311324005827e-35, f(H^{2})_new = 4.752686896217289e-37
Finer H^{2} value= 1.9010747584869136e-35, f(H^{2})_new = 2.3763434481086313e-37
Finer H^{2} value= 1.924838192968e-35, f(H^{2})_new = 0.0
```

```
Test each new step for a minimum [f(H^{2})= smaller than both neighbours]:
[array([False, False, False, False, False, False, False, False, True])]
```

```
Minimum H^{2} value= 1.924838192968e-35, f(H^{2}) = 0.0
Optimized H^{2}(1.2) minimum value= 1.924838192968e-35, f(H^{2}(1.2))_new = 0.0
```

You might wonder, how do we know this calculated value is close to the exact value? For this, we will use two functions plotted on two different graphs, namely $f(H^2) = H^2 - \Omega_m(1+z)^3 H_0^2$ and $g(H^2) = (1 - \Omega_m)H_0^2$. These graphs are shown in Figure E.1. On the L.H.S. of Figure E.1, you will see that the two functions intersect on the boundary, as predicted from the theory and the optimisation function. Furthermore, when we zoom-in on the solution where the intersection happened (R.H.S.), we can visually see this intersection is happening at $H^2(z = 1.2) \approx 1.925$ approximately.

E.2 Numerical integration

We will be using Simpson's rule for integration, since we need to calculate $\int_0^{z'} \frac{1}{h(z')} dz$. We are allowed to do this, since we are able to find an approximated value for H^2 at a certain redshift (z). From here on out, we can just calculate $h(z) = \left(\sqrt{\frac{H^2(z)}{H_0^2}}\right)^{-1}$ at that particular z -value. Using this method, we can then set up a solution map for different values of z . For our hypothetical supernovae data point, we can determine equally spaced step-sizes between $z = 0$ and $z = z' = 1.2$. Let us assume that we have $nsteps_{int} = 11$. Therefore, 10 equally spaced steps in the range $0 \leq z \leq 1.2$. We will then call the optimised function to calculate an approximation for each redshift in our range. You will notice that at a redshift $z = 1.2$, we do once again obtain the approximated value we previously found while testing the optimising function, thereby confirming that the code can calculate multiple different optimisations at different redshifts.

```

Redshift:  0.0,  H^{2} =  [4.765488999999993e-36],  h(z)^{-1} =  1.0
Redshift:  0.12, H^{2} =  [5.37333817788448e-36],   h(z)^{-1} =  0.9417413688792438
Redshift:  0.24, H^{2} =  [6.126448610227839e-36],   h(z)^{-1} =  0.8819609127547395
Redshift:  0.36, H^{2} =  [7.040384002864959e-36],   h(z)^{-1} =  0.8227266436557629
Redshift:  0.48, H^{2} =  [8.130708061630719e-36],   h(z)^{-1} =  0.7655781885977229
Redshift:  0.6,  H^{2} =  [9.412984492360001e-36],   h(z)^{-1} =  0.7115248363893207
Redshift:  0.72, H^{2} =  [1.090277700088768e-35],   h(z)^{-1} =  0.6611274219755168
Redshift:  0.84, H^{2} =  [1.2615649293048637e-35],   h(z)^{-1} =  0.6146090233216716
Redshift:  0.96, H^{2} =  [1.4567165074677757e-35],   h(z)^{-1} =  0.5719607340738297
Redshift:  1.08, H^{2} =  [1.6772888051609917e-35],   h(z)^{-1} =  0.5330277402802114
Redshift:  1.2,  H^{2} =  [1.924838192968e-35],      h(z)^{-1} =  0.4975727891521645

```

My integration value is: 0.7910307546931135

Scipy's integration value is: (0.8701357761300014, 9.660447732096082e-15)

We can then calculate the $\left(\sqrt{\frac{H^2(z')}{H_0^2}}\right)^{-1}$, to which the answers are displayed in the third column. When we make a scatter plot for $\frac{1}{h(z')}$ as a function of redshift, we obtain Figure E.2 (*blue-dots*). The *red-dotted line* represents the total curve if we were to use $nsteps_{int} \rightarrow \infty$. We will effectively be doing an integration of the area under this curve. The Simpson integration rule fits a parabola between three adjacent data points. To illustrate this, we shaded the areas under each parabola with different shades of yellow. This is, however, only for illustrative reasons. The numerical code will execute the Simpson rule function, which is given by

$$\int_{z_0}^{z_2} f(z) dz = \frac{h}{3} \left(f(z_0) + 4f(z_1) + f(z_2) \right) - \frac{h^5}{90} f^{(4)}(\zeta). \quad (\text{E.2})$$

This particular function then finds the integration under the first parabola. The same calculation then must be done for each of the parabolas. The full proof of the numerical method can be found on the website [114], and descriptions on error estimations can be found in the textbook [111]. You might wonder based on the comparison between our integration value and that of Scipy, why do

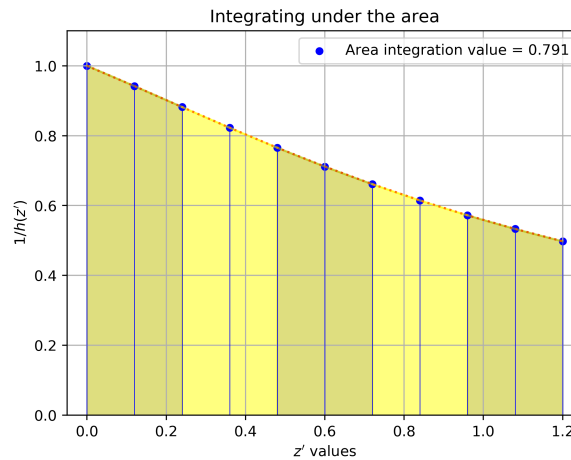


Figure E.2: Visually illustrating the Simpson's integration rule for calculating the area under the curve. Each shade of yellow represents the area under a specific parabola between 3 adjacent H^2 -optimized values. The red-dotted line represents exact curve under which the area must be calculated.

we obtain an integration value that is different from the one found by Scipy? Firstly, Scipy's value is also only an approximation, since it also uses a numerical method to calculate the integration value. Secondly, our step sizes on both the optimisation and integration methods are large, thereby leading to an inaccurate value. If we were to use, for instance, `count = 2000` for the number of steps for both the crude and finer optimisation method and `nstepsint = 2001` for the number of steps for the integration method, we obtain an integration value of

```
My integration value is: 0.869700925667166,
Scipy's integration value is: (0.8701357761300014, 9.660447732096082e-15).
```

As you can see, the difference between the expected value using Scipy and the calculate value becomes much smaller the more steps you use.

E.3 Computational expensiveness

Before we can start to look at the computational expensiveness of the numerical methods, we firstly have to mention that the Λ CDM model will obtain a smaller error due to being less complicated (2nd-order derivative) than the Starobinsky and Hu-Sawicki models, which are fourth-order derivatives. As explained in [111], the Simpson rule can give an exact answer for third and lower-order derivatives. The question is now, to which order of numerical errors will we be accepting?

This is a bit harder to find a definite answer to, since the type of computer you are using can also influence this answer. Using a supercomputer, you can choose a much higher accuracy than running the code on a home PC. We will be using an office PC and these execution times are based on this PC. While executing the MCMC simulations for the Λ CDM model using Scipy, we found it takes

the MCMC simulation between 22 and 24 hours to complete the code, for 100 random walkers with each walker taking 25000 steps.

To show the influence to the total execution time, we will compare the execution time for the numerical methods to the execution time for the Scipy program. The execution time for Scipy is on average 0.000109s. Firstly, we will need to find a combination of $count$ and $nsteps_{int}$ steps, which have a similar execution time than the one from Scipy (a reference point). The closest execution time we found, was for $count = 2$ and $nsteps_{int} = 3$, which gave an integration value of $int_{val} = 0.5792$. This is not accurate at all. We then used $count = 10$ and $nsteps_{int} = 11$ (same as in previous section), which resulted in an execution time of 0.00139s (an entire order of magnitude longer than Scipy's execution time). We then used $count = 100$ and $nsteps_{int} = 101$, which resulted in an execution time of 0.02707s (another order of magnitude increase) and an integration value of $int_{val} = 0.8615$ (much closer to the exact value). We then used $count = 250$ and $nsteps_{int} = 101$ and obtained an execution time of 0.05243s, with an integration value of $int_{val} = 0.8615$. We saw that the $count$ steps are not influencing the total accuracy any more, it is just increasing the execution time. We will now keep the $count$ steps constant. Using $nsteps_{int} = 151$, we obtain an integration value of $int_{val} = 0.8644$, with an execution time of 0.039976s. Increasing this to $nsteps_{int} = 201$, we obtain an execution time of 0.05554s (similar to the one found previously), with an integration value of $int_{val} = 0.8658$. Lastly, we used $nsteps_{int} = 251$, which resulted in an execution time of 0.071593s, with an integration value of $int_{val} = 0.8667$.

We will not go any higher than this, even though we still have a difference of about 0.0034 between Scipy's prediction and the numerical methods estimation. Reason for this is because we are already at an increase of execution of about $650\times$ longer than the execution time for the Scipy program. With a total execution for the MCMC simulation being in the range of ± 650 days. The best execution time we will have is one found by using $count = 100$ and $nsteps_{int} = 101$, which has an integration value difference of 0.0086, while having an execution time that is $250\times$ longer than Scipy's execution time. Still, we do not have ± 250 days to execute the MCMC simulation, but going any lower will make the integration too inaccurate. We can, however, change the MCMC simulation inputs. For example, we can make the number of random walkers 50 and the number of iterations each walker takes 5000. This would result in a predicted execution time of ± 25 days, which is still a long time, but plausible.

This, though, is only for the Λ CDM model, which is not that complex. We found that a model such as $f(R) = \beta R^n$ (which has the same number of free parameters as the Starobinsky model), has an execution time of about ± 24 hours for 100 random walkers and 10000 steps for each walker. This would mean that we will have relatively the same execution time for the Starobinsky model using Scipy and 100 random walkers and 10000 steps. Therefore, once again, a total predicted execution time of about ± 250 days for completion. We can then change the input parameters to 50 random walkers and 3000 steps taken by each walker and find a predicted execution time of 37.5 days.

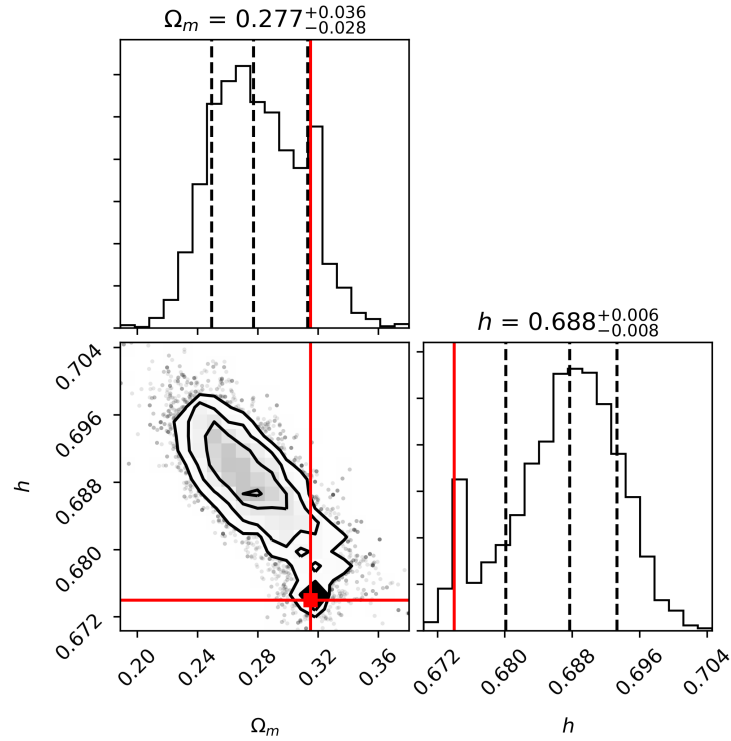


Figure E.3: Visually confirming that the numerical method works and that the errors introduced in the numerical method propagate into the standard deviation calculated by the MCMC simulation.

You might wonder about the error that we are neglecting. This is due to the way the MCMC simulation is structured (finding an unknown answer). It allows for error. Such as starting with a wrong initial condition, or allowing a combination of parameters with a lower likelihood of success to be accepted (since it might have skipped the best combination), or having more random walkers to approach the ‘best fit’ combination from different initial starting points. At the very end, “averaging” the errors out by taking the Gaussian distribution from the results of each walkers and finding the standard deviation σ . Therefore, all the errors are taking into account the standard deviation from the best fit. The larger the error, the larger the calculated standard deviation.

To illustrate this, we executed the Λ CDM model using the numerical method. Our initial conditions were, 40 random walkers, each taking 250 steps, and optimising function steps of $count = 100$ and integration steps of $nsteps_{int} = 101$. The results are shown in Figure E.3. In Figure E.3, we found that the results are almost the same as the results shown in Figure 3.1 (where we used Scipy for our calculation). The small changes may be a result of this being an extremely short and inaccurate run. Therefore, we will be able to use this numerical method with this particular number of steps for the optimising function and the integrating function.

APPENDIX F

Λ CDM-model code

```
1  #importing necessary packages
2  import numpy as np
3  import matplotlib.pyplot as plt
4  import emcee
5  import math
6  from scipy import integrate
7  import corner
8  import scipy.optimize as op
9  from matplotlib.ticker import MaxNLocator
10 import time
11
12 t1=time.clock()                                #Program starting time
13
14 #####
15 ##### True values for cosmological parameters: Planck 2018 and initial conditions
16 #####
17 omega_m_true = 0.315
18 h_true = 0.674
19 ndim =2                                         #number of free parameters
20 nwalker = 100                                 #number of random walkers
21 nsteps=25000                                  #number of steps each walker takes
22 burn =1000                                    #Burning the first N steps
23
24 #####
25 ##### Prior limits: Knowledge from previous experiments
26 #####
27 om_low=0.00
28 om_high=1.00
29 h_low= 0.00
30 h_high=1.00
31
32 #####
```

```

33 ##### Importing the Supernova data set
34 #####
35 supernova_data = np.loadtxt('final_dataset.txt')
36 redshift = supernova_data[:,0]
37 distance_modulus_data=supernova_data[:,5]
38 distance_modulus_data_error = supernova_data[:,6]
39
40 #####
41 ##### The model we are using
42 #####
43 def func(z,omega_m):
44     bottom = np.sqrt(omega_m*(1+z)**3+1-omega_m)
45     return 1/bottom
46
47 #####
48 ##### Calculating the distance modulus for the model
49 #####
50 def model_run(redshift,y_dl,distance_modulus,log2,h,omega_m):
51     for i in range(0,len(redshift)):
52         #integrating over our model and calculating the luminosity distance
53         int_val = integrate.quad(func,0,redshift[i],args=(omega_m))
54         y_dl[i] = (3000/h)*(1+redshift[i])*int_val[0]
55         log2[i] = math.log(y_dl[i],10)
56         distance_modulus[i] = 25+ 5*log2[i]
57     return distance_modulus #return the calculated distance modulus
58
59 #####
60 ##### Calculating the likelihood of the model at a certain current sample
61 ##### value for the free parameters
62 #####
63 def lnlike(theta,redshift,distance_modulus_data):
64     sigma=1.0 #step size that must be taken
65     omega_m,h = theta #the free parameters
66     y_dl = np.zeros(len(redshift))
67     log2 = np.zeros(len(redshift))
68     distance_modulus = np.zeros(len(redshift))
69
70     #running the model to find the likelihood
71     model = model_run(redshift,y_dl,distance_modulus,log2,h,omega_m)
72
73     #Likelihood function plus corrections
74     exponent = -0.5*(np.sum(((distance_modulus_data-model)**2)
75                             /(distance_modulus_data_error**(2))))
76     return exponent #return the calculated likelihood
77
78 #####
79 ##### Check is a sample value is within the physical constrains
80 #####
81 def lnprior(theta):
82     omega_m,h = theta
83     if om_low < omega_m < om_high and h_low < h < h_high:
84         return 0.0
85     return -np.inf
86
87 #####
88 ##### Calling the prior and likelihood functions to calculate the likelihood
89 ##### at a certain sample value and checking if that sample is valid

```

```

90 #####
91 def lnprob(theta,redshift,distance_modulus_data):
92     lp = lnprior(theta)
93     if not np.isfinite(lp):
94         return -np.inf
95     return lp + lnlike(theta,redshift,distance_modulus_data)
96
97 #####
98 ##### Calculate the time the program took to run
99 #####
100 def times(t):
101     time = t/60
102     time_h=0
103     while time>60:
104         time_h = time_h +1
105         time =time -60
106     time_m = int(time)
107     times = time-time_m
108     time_s = (times/100)*6000
109     return time_h,time_m,time_s
110
111 #####
112 ##### Use scipy's opimization to find an initial starting point for the Emcee
113 ##### package
114 #####
115 bnds = ((om_low,om_high),(h_low,h_high))
116 theta = [omega_m_true,h_true]
117 nll = lambda *args: -lnlike(*args)
118 result = op.minimize(nll, theta, args=(redshift,distance_modulus_data)
119                     ,bounds =bnds)
120 print (result['x'])
121
122 #####
123 ##### Running the Emcee package which is a MCMC simulation
124 #####
125 pos =[result["x"]+1e-4*np.random.randn(ndim) for i in range(nwalker)]
126 sampler = emcee.EnsembleSampler(nwalker,ndim,lnprob,args =
127                               (redshift,distance_modulus_data))
128 sampler.run_mcmc(pos,nsteps)
129 samples = sampler.chain[:,burn:,:].reshape((-1,ndim))
130
131 #####
132 ##### Plotting the tracks to see if the MCMC was random enough
133 #####
134 plt.clf()
135 fig, axes = plt.subplots(2, 1, sharex=True, figsize=(8, 9))
136 plt.figure(figsize=(16, 16),dpi=600)
137 axes[0].plot(sampler.chain[:, :, 0].T, color="k", alpha=0.2)
138 axes[0].yaxis.set_major_locator(MaxNLocator(5))
139 axes[0].set_ylabel( "omega_m " )
140 axes[1].set_xlabel("step number")
141
142 axes[1].plot(sampler.chain[:, :, 1].T, color="k", alpha=0.2)
143 axes[1].yaxis.set_major_locator(MaxNLocator(5))
144 axes[1].set_ylabel( "h" )
145 axes[1].set_xlabel("step number")
146

```

```

147 fig.tight_layout(h_pad=0.0)
148 fig.savefig("RVM_line-time.pdf")
149
150 #####
151 ##### Plotting the Emcee results on a corner plot to show the calculated values
152 ##### and compare them to the true value and to each other
153 #####
154 fig = plt.figure()
155 corner.corner(samples, labels=["$\Omega_m$", "$h$"], truths=[omega_m_true, h_true]
156                 , quantiles=[0.16, 0.5, 0.84] , color = "k", truth_color='r'
157                 , show_titles=True, title_fmt = '.3f', bins=200)
158 plt.savefig("Polynomial MCMC results GR1.pdf")
159 plt.show()
160
161 #####
162 ##### Program ending time
163 #####
164 t2 = time.clock()
165 t=t2-t1
166 time_h,time_m,time_s = times(t)
167 print ('Elapsed time:  '+str(time_h)+'h '+str(time_m)+'m  '+str(time_s)+'s')

```

Listing 1: The MCMC simulation, using the EMCEE Hammer Python package, for the Λ CMD, model fitted to Supernovae Type 1A data.

Numerical method code

```

1  ##### Important packages
2  import numpy as np
3  import matplotlib.pyplot as plt
4  from scipy import integrate
5
6  ##### Initial conditions
7  omega_m =0.315
8  H0 = 2.1830*10**(-18)
9  H02 = (H0**(2))
10 z= 1.2
11 count=10
12 nsteps = 1
13
14 #####
15 ##### Concordance model  $H^2 = H0^2 * \sqrt{\omega_m * (1+z)^3 + 1 - \omega_m}$ 
16 #####
17 def function(x, omega_m, H02, z):
18     fx = np.abs(x - H02 * (omega_m * ((1+z)**(3)) + 1 - omega_m))
19     return fx
20
21 ##### Function called by Scipy
22 def func(z, omega_m):
23     bottom = np.sqrt((omega_m * (1+z)**(3)) + 1 - omega_m)
24     function = 1/bottom
25     return function
26
27 ##### Function used for illustrative purposes
28 def function2(x, omega_m, H02, z):
29     gx = np.zeros(len(x))
30     fx = x - (omega_m * ((1+z)**(3)) * H02)
31     for i in range(len(x)):
32         gx[i] = (1 - omega_m) * H02
33     return fx, gx
34
35 ##### Optimizing Function
36 def optimize(z, omega_m, H02, count):

```

```

37     print ("Redshift = "+str(z)+", Matter density = "+str(omega_m)+
38           ", H0^{2} = " +str(H02)+"")
39     print ('')
40     b= H02*(omega_m*((1+z)**(3))+1-omega_m) #Calculate the upper search limit
41     x = np.linspace(0.0,H02*(omega_m*((1+z)**(3))+1-omega_m),count)
42     print ("H0^2 range between 0 and ", b)
43     print ("")
44     #Call function that must be calculate at different H^2 values
45     f = function(x,omega_m,H02,z)
46     for i in range(len(x)):
47         print ("Coarse H^{2} value= "+ str(x[i])+", f(H^{2}) = " +str(f[i]))
48     ans = np.r_[True, f[1:] < f[:-1]] & np.r_[f[:-1] < f[1:], True]
49     #Find all the minimumsIf False then its not a minimum,
50     #if True then we have a minimum
51     print ("")
52     print ("Test each step for a minimum"
53           "[f(H^{2})= smaller than both neighbours]: ")
54     print (ans)
55     print ("")
56
57
58     i = []
59     j=0
60     while j<(len(ans)):
61         if ans[j]==True : #Saving the coarse minimum's position
62             print ("Minimum H^{2} value = "+ str(x[j])+
63                   ", f(H^{2}) = " +str(f[j]))
64             i.append(j)
65             j +=1
66
67     #Calculate the new search range between the two neighbours to both
68     #sides of the minimum. In minimum is the the boundary make the
69     #search range only between the open neighbours and the minimum
70     print ("")
71     x_new =[]
72     f_new =[]
73     if i[0] ==0:
74         x_new.append(np.linspace(x[0],x[1],int(count)))
75         f_new.append(function(x_new[0],omega_m,H02,z))
76         print ("New search range for H^{2} between "+str(x[0])+" and ",x[1])
77     elif i[0] != (len(ans)-1):
78         for a in range(len(i)):
79             x_new.append(np.linspace(x[i[a]-1],x[i[a]+1],int(count)))
80             f_new.append(function(x_new[a],omega_m,H02,z))
81             print ("New search range for H^{2} between "+str(x[i[a]-1])
82                   +" and ",x[i[a]+1])
83     else:
84         for a in range(len(i)):
85             x_new.append(np.linspace(x[i[a]-1],x[i[a]],int(count)))
86             f_new.append(function(x_new[a],omega_m,H02,z))
87             print ("New search range for H^{2} between "+str(x[i[a]-1])
88                   +" and ",x[i[a]])
89     print ("")
90     for j in range(len(x_new)):
91         for i in range(len(x_new[j])):
92             print ("Finer H^{2} value= "+ str(x_new[j][i])+
93                   ", f(H^{2})_new = " +str(f_new[j][i]))

```

```

94     print ("")
95     print("")
96     ans_new =[]
97     for j in range(len(x_new)): #Find all the new minimums
98         ans_new.append(np.r_[True, f_new[j][1:] < f_new[j][:1]]
99                         & np.r_[f_new[j][:1] < f_new[j][1:], True])
100
101     print ("Test each new step for a minimum"
102           "[f(H^{2})= smaller than both neighbours]: ")
103     print (ans_new)
104     print("")
105
106     k2 =[]
107     for j in range(len(ans_new)):
108         for i in range(len(ans_new[j])):
109             if ans_new[j][i]==True: #Saving the finer minimum's position
110                 print ("Minimum H^{2} value= " + str(x_new[j][i])+
111                       ", f(H^{2}) = " +str(f_new[j][i]))
112                 k2.append(i)
113     #Check if more than one solution was found (e.g trivial and another soltuion).
114     print ("")
115     k2_new=[]
116     if len(k2)==2:
117         k2_new.append(k2[1])
118     else:
119         k2_new.append(k2[0])
120
121     x_sol =[] #Save the Optimized functions value
122     for j in range(len(k2)):
123         print ("Optimized H^{2} minimum value= " + str(x_new[j][k2_new[j]])+
124               ", f(H^{2})_new = " +str(f_new[j][k2_new[j]]))
125         x_sol.append(x_new[j][k2_new[j]])
126     return x_sol
127
128     ##### Numerical integration function
129     def simpson_integration(x_sol_norm,stepsize):
130         i=1
131         total=x_sol_norm[0]+x_sol_norm[-1]
132         for x in x_sol_norm[1:-1]:
133             if i%2 == 0:
134                 total+=2*x
135             else:
136                 total+=4*x
137             i+=1
138         return total*(stepsize/(3.0))
139
140     #####
141     ##### Total numerical method, setting up optimized value and calling the
142     ##### integration function to numerically integrate
143     #####
144     def integration(a,b,nsteps,H02,count,omega_m):
145         z=np.linspace(a,b,nsteps)
146         stepsize = (a+b)/nsteps
147         x_sol = []
148         for i in range(len(z)):
149             x_sol.append(optimize(z[i],omega_m, H02,count))
150

```

```

151     x_sol_norm = []
152     for x in x_sol:
153         x_sol_norm.append((1/np.sqrt(x[0]/H02)))
154
155     int_val = simpson_integration(x_sol_norm,stepsize)
156     return int_val
157
158     ##### Optimizing example
159 x_sol=optimize(z,omega_m,H02,count)
160 a=H02*(omega_m*((1+z)**(3))+1-omega_m)
161 x=np.linspace(0,a,1000)
162 f,g = function2(x, omega_m,H02,z)
163
164     ##### Illustrative purposes
165 fig = plt.figure()
166 plt.plot(x,f, color = "r")
167 plt.plot(x,g, color = "b")
168 plt.xlim(0,a)
169 plt.title("GR solution n=0 full range")
170 plt.xlabel("H^2 values")
171 plt.ylabel("f(x) and g(x)")
172 plt.grid(True)
173 plt.savefig("Optimization_solution.png",dpi=400)
174 plt.show()
175
176     ##### Zooming in closer to the solution
177 b=x_sol[0]+((x_sol[0])/10)
178 c=x_sol[0]-((x_sol[0])/10)
179 x=np.linspace(c,b,1000)
180 f,g = function2(x, omega_m,H02,z)
181
182 fig = plt.figure()
183 plt.plot(x,f, color = "r")
184 plt.plot(x,g, color = "b")
185 plt.xlim(c,b)
186 plt.title("GR solution n=0, Zoomed-in on the solution")
187 plt.xlabel("H^2 values")
188 plt.ylabel("f(x) and g(x)")
189 plt.grid(True)
190 plt.savefig("Optimization_solution_zoomed.png",dpi=400)
191 plt.show()

```

Listing 2: The numerical method we used to solve H^2 , and to use the numerical solutions to calculate the numerical integration needed for the distance modulus function. This code was used to explain the choices involving the step sizes, the numerical accuracy, and the computational expensiveness that had to be considered for solving the Starobinsky and Hu-Sawicki models.

Bibliography

- [1] J. B. Hartle. *Gravity: An introduction to Einstein's general relativity*. Pearson Education, 2003.
- [2] R. P. Kirshner. Hubbles diagram and cosmic expansion. *Proceedings of the National Academy of Sciences*, 101(1):8–13, 2004. URL <https://www.pnas.org/content/101/1/8>.
- [3] S. S. Komissarov. Cosmology, 2012. URL <https://www1.maths.leeds.ac.uk/~serguei/teaching/cosmology.pdf> DOA: 28 February 2018 (link for newer version).
- [4] N. Aghanim, Y. Akrami, M. Ashdown, et al. Planck 2018 results. VI. Cosmological parameters. *arXiv:1807.06209*, 2018.
- [5] S. Chen. Hubble trouble: A crisis in cosmology? *American Physical Society news*, 27(5), 2018. URL <https://www.aps.org/publications/apsnews/201805/hubble.cfm>.
- [6] A. Al Mamon and K. Bamba. Observational constraints on the jerk parameter with the data of the Hubble parameter. *European Physical Journal C*, 78:862, 2018. URL <https://doi.org/10.1140/epjc/s10052-018-6355-2>.
- [7] S. Geoff. *The Mayan and other ancient calendars*. Bloomsbury Publishing USA, 2007.
- [8] M. Douma. Calendars through the ages, 2008. URL <http://www.webexhibits.org/calendars>.
- [9] R. J. M. Olson and J. M. Pasachoff. *Fire in the sky: comets and meteors, the decisive centuries, in British art and science*. Cambridge University Press, 1999.
- [10] J. R. Hind. On the past history of the comet of Halley. *Monthly Notices of the Royal Astronomical Society*, 10(3):51–58, 1850. URL <https://doi.org/10.1093/mnras/10.3.51>.

- [11] T. Tilghman. Halley met Newton and then, 2017. URL <https://cadpilot.com/home/jump-on-c3d/jump-on-c3d/2017/08/03/halley-met-newton-and-then/>.
- [12] M. Nauenberg. Visiting Newton’s atelier before the Principia, 1679–1684. *Annals of Science*, 76(1):1–16, 2019. URL <https://doi.org/10.1080/00033790.2019.1566497>.
- [13] R. H. Evans. Blackbody radiation and the “ultraviolet catastrophe”, 2013. URL <https://thecuriousastronomer.wordpress.com/2013/10/21/blackbody-radiation-and-the-ultraviolet-catastrophe/>.
- [14] D. V. Schroeder. *An introduction to thermal physics*. American Association of Physics Teachers, 1999.
- [15] J. D. Norton. *Einstein for everyone*. Nullarbor Press, 10th edition, 2018. URL https://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/index.html.
- [16] N. T. Redd. Einstein’s Theory of General Relativity, 2017. URL <https://www.space.com/17661-theory-general-relativity.html>.
- [17] D. Wallace. More problems for Newtonian cosmology. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 57:35–40, 2017. URL <https://doi.org/10.1016/j.shpsb.2017.01.003>.
- [18] F. Hoyle, G. Burbidge, and J. V. Narlikar. *A different approach to cosmology: from a static universe through the big bang towards reality*. Cambridge University Press, 2000.
- [19] D. Byrd. Today in science: Edwin Hubble and the expanding universe, 2017. URL <http://earthsky.org/space/this-date-in-science-edwin-hubble-and-the-expanding-universe>.
- [20] M. K. Munitz. The logic of cosmology. *The British Journal for the Philosophy of Science*, 13(49):34–50, 1962. URL <http://www.jstor.org/stable/685222>.
- [21] J. Arnau Romeu. Derivation of Friedman equations. 2014. URL <http://hdl.handle.net/2445/59759>.
- [22] D. Sweetser. Einstein field equation derivation in about a dozen steps, 2012. URL http://www.science20.com/standup_physicist/blog/einstein_field_equation_derivation_about_dozen_steps-90263.
- [23] A. A. Gidelew. *Beyond Concordance Cosmology*. PhD thesis, University of Cape Town, 2013.
- [24] B. Jones. Hilberts Stress-Energy Tensor. 2017. Supplement to “Precision Cosmology”.
- [25] W. F. Kao and U. Pen. Generalized Friedmann-Robertson-Walker metric and redundancy in the generalized Einstein equations. *Physical Review D*, 44(12):3974, 1991. URL <https://doi.org/10.1103/PhysRevD.44.3974>.

- [26] D. Baumann. Cosmology: Part III Mathematical Tripos. URL <http://theory.uchicago.edu/~liantaow/my-teaching/dark-matter-472/lectures.pdf>.
- [27] J. Dunning-Davies. Some thoughts on the fluid equation of cosmology. Technical Report arXiv:0805.3183, 2008. URL <http://cds.cern.ch/record/1105573>.
- [28] M. Trodden and S. M. Carroll. TASI lectures: Introduction to cosmology. *ArXiv Astrophysics astro-ph/0401547*, 2004.
- [29] S. M. Carroll. The cosmological constant. *Living Reviews in Relativity*, 4(1):1, 2001. URL <https://doi.org/10.12942/lrr-2001-1>.
- [30] O.F. Piattella. Lecture notes in cosmology. *arXiv:1803.00070*, 2018. URL <https://link.springer.com/book/10.1007%2F978-3-319-95570-4>.
- [31] E. Babichev, D. Gorbunov, and S. Ramazanov. Dark matter and baryon asymmetry from the very dawn of the Universe. *Physical Review D*, 97(12):123543, 2018. URL <https://doi.org/10.1103/PhysRevD.97.123543>.
- [32] H. J. Macpherson, P. D. Lasky, and D. J. Price. The trouble with Hubble: Local versus global expansion rates in inhomogeneous cosmological simulations with numerical relativity. *The Astrophysical Journal Letters*, 865(1):L4, 2018. URL <https://doi.org/10.3847/2041-8213/aadf8c>.
- [33] S. R. Choudhury and S. Choubey. Updated bounds on sum of neutrino masses in various cosmological scenarios. *Journal of Cosmology and Astroparticle Physics*, 2018(09):017, 2018. URL <https://doi.org/10.1088/1475-7516/2018/09/017>.
- [34] S. R. Choudhury and A. Naskar. Strong bounds on sum of neutrino masses in a 12 parameter extended scenario with non-phantom dynamical dark energy ($w(z) \geq -1$). *arXiv:1807.02860*, 2019.
- [35] J. J. Geng, R. Y. Guo, A. Wang, et al. Prospect for cosmological parameter estimation using future Hubble parameter measurements. *Communications in Theoretical Physics*, 70(4):445, 2018. URL <https://doi.org/10.1088/0253-6102/70/4/445>.
- [36] G. Gubitosi, R. Ooijer, C. Ripken, et al. Consistent early and late time cosmology from the RG flow of gravity. *Journal of Cosmology and Astroparticle Physics*, 2018(12):004, 2018. URL <https://doi.org/10.1088/1475-7516/2018/12/004>.
- [37] M. Roos. *Introduction to cosmology*. John Wiley & Sons, 2015.
- [38] G. F. R. Ellis. Issues in the philosophy of cosmology. *ArXiv Astrophysics astro-ph/0602280*, 2006.

- [39] P.A.R. Ade, N. Aghanim, M. Arnaud, et al. Planck 2015 results. XIII. Cosmological parameters. *Astronomy & Astrophysics*, 594:A13, 2016. doi: 10.1051/0004-6361/201525830.
- [40] I. Debono and G. F. Smoot. General relativity and cosmology: Unsolved questions and future directions. *Universe*, 2:23, 2016. URL <https://doi.org/10.3390/universe2040023>.
- [41] C. L. Bennett, D. Larson, J. L. Weiland, et al. The 1% concordance Hubble constant. *The Astrophysical Journal*, 794(2):135, 2014. URL <https://doi.org/10.1088/0004-637X/794/2/135>.
- [42] C. J. A. P. Martins and M. P. Colomer. Constraining late-time transitions in the dark energy equation of state. *Astronomy & Astrophysics*, 616:A32, 2018. URL <https://doi.org/10.1051/0004-6361/201833313>.
- [43] A. Zitrin and D. Eichler. Observing cosmological processes in real time with repeating fast radio bursts. *The Astrophysical Journal*, 866(2):101, 2018. URL <https://doi.org/10.3847/1538-4357/aad6a2>.
- [44] M. S. Berman and F. de Mello Gomide. Cosmological models with constant deceleration parameter. *General Relativity and Gravitation*, 20(2):191–198, 1988. URL <https://doi.org/10.1007/BF00759327>.
- [45] L. Knox, N. Christensen, and C. Skordis. The age of the Universe and the cosmological constant determined from Cosmic Microwave Background anisotropy measurements. *The Astrophysical Journal Letters*, 563(2):L95, 2001. URL <https://doi.org/10.1086/338655>.
- [46] R. Cai. A dark energy model characterized by the age of the Universe. *Physics Letters B*, 657(4-5):228–231, 2007. URL <https://doi.org/10.1016/j.physletb.2007.09.061>.
- [47] A. Hernández-Almada, J. Magaña, M. A. García-Aspeitia, et al. Cosmological constraints on alternative model to Chaplygin fluid revisited. *The European Physical Journal C*, 79(1):12, 2019. URL <https://doi.org/10.1140/epjc/s10052-018-6521-6>.
- [48] T. Clifton, P. G. Ferreira, A. Padilla, et al. Modified gravity and cosmology. *Physics reports*, 513(1-3):1–189, 2012. URL <https://doi.org/10.1016/j.physrep.2012.01.001>.
- [49] A. De Felice and S. Tsujikawa. $f(R)$ Theories. *Living Reviews in Relativity*, 13(1):3, 2010. URL <https://doi.org/10.12942/lrr-2010-3>.
- [50] A. H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23:347–356, 1981. URL <https://link.aps.org/doi/10.1103/PhysRevD.23.347>.
- [51] K. Sato. First-order phase transition of a vacuum and the expansion of the Universe. *Monthly Notices of the Royal Astronomical Society*, 195(3):467–479, 1981. URL <http://dx.doi.org/10.1093/mnras/195.3.467>.

- [52] G. 'tHooft. Magnetic monopoles in unified theories. *Nuclear Physics B*, 79(CERN-TH-1876): 276–284, 1974.
- [53] M. Hasegawa. The catalytic effects of monopoles in QCD. *arXiv:1807.04808*, 2018.
- [54] W. Buchmüller. Matter-antimatter asymmetry of the universe. *Annalen der Physik*, 10(1-2):95–108, 2001. URL [https://doi.org/10.1002/1521-3889\(200102\)10:1/2<95::AID-ANDP95>3.0.CO;2-N](https://doi.org/10.1002/1521-3889(200102)10:1/2<95::AID-ANDP95>3.0.CO;2-N).
- [55] S. Capozziello, O. Luongo, R. Pincak, et al. Cosmic acceleration in non-flat $f(T)$ cosmology. *General Relativity and Gravitation*, 50:53, 2018. URL <https://doi.org/10.1007/s10714-018-2374-4>.
- [56] T. P. Sotiriou. *Modified actions for gravity: Theory and phenomenology*. PhD thesis, International School for Advance studies, 2007.
- [57] R. P. L. Azevedo and P. P. Avelino. Big-bang nucleosynthesis and cosmic microwave background constraints on nonminimally coupled theories of gravity. *Physical Review D*, 98(6): 064045, 2018. URL <https://doi.org/10.1103/PhysRevD.98.064045>.
- [58] S. Carloni and P. K. Dunsby. A dynamical system approach to higher order gravity. *Journal of Physics A: Mathematical and Theoretical*, 40(25):6919, 2007. URL <https://doi.org/10.1088/1751-8113/40/25/S40>.
- [59] H. A. Buchdahl. Non-linear Lagrangians and cosmological theory. *Monthly Notices of the Royal Astronomical Society*, 150(1):1–8, 1970. URL <https://doi.org/10.1093/mnras/150.1.1>.
- [60] O. Bertolami, C. G. Böhmer, T. Harko, et al. Extra force in $f(R)$ modified theories of gravity. *Physical Review D*, 75(10):104016, 2007. URL <https://doi.org/10.1103/PhysRevD.75.104016>.
- [61] S. Nojiri and S. D. Odintsov. Modified $f(R)$ gravity consistent with realistic cosmology: From a matter dominated epoch to a dark energy universe. *Physical Review D*, 74(8):086005, 2006. URL <https://doi.org/10.1103/PhysRevD.74.086005>.
- [62] J. Santos, J. S. Alcaniz, M. J. Reboucas, et al. Energy conditions in $f(R)$ gravity. *Physical Review D*, 76(8):083513, 2007. URL <https://doi.org/10.1103/PhysRevD.76.083513>.
- [63] T. P. Sotiriou and V. Faraoni. $f(R)$ theories of gravity. *Reviews of Modern Physics*, 82(1): 451, 2010. URL <https://doi.org/10.1103/RevModPhys.82.451>.
- [64] A. Guarnizo, L. Castañeda, and J. M. Tejeiro. Boundary term in metric $f(R)$ gravity: field equations in the metric formalism. *General Relativity and Gravitation*, 42:2713–2728, 2010. URL <https://doi.org/10.1007/s10714-010-1012-6>.

- [65] M. Akbar and R. Cai. Thermodynamic behavior of field equations for $f(R)$ gravity. *Physics Letters B*, 648(2):243 – 248, 2007. URL <https://doi.org/10.1016/j.physletb.2007.03.005>.
- [66] S. Tsujikawa, K. Uddin, and R. Tavakol. Density perturbations in $f(R)$ gravity theories in metric and Palatini formalisms. *Physical Review D*, 77(4):043007, 2008. URL <https://doi.org/10.1103/PhysRevD.77.043007>.
- [67] T. Liu, X. Zhang, and W. Zhao. Constraining $f(R)$ gravity in solar system, cosmology and binary pulsar systems. *Physics Letters B*, 777:286–293, 2018. URL <https://doi.org/10.1016/j.physletb.2017.12.051>.
- [68] N. Namane, H. Sami, and A. Abebe. Oscillating cosmological correlations in $f(R)$ gravity. *arXiv:1807.11330*, 2018.
- [69] P. Bari, K. Bhattacharya, and S. Chakraborty. Early universe cosmological solutions in exponential gravity. *arXiv:1805.06673*, 2018.
- [70] J. Pérez-Romero and S. Nesseris. Cosmological constraints and comparison of viable $f(R)$ models. *Physical Review D*, 97(2):023525, 2018. URL <https://link.aps.org/doi/10.1103/PhysRevD.97.023525>.
- [71] L. Pogosian and A. Silvestri. Pattern of growth in viable $f(R)$ cosmologies. *Physical review D*, 77(2):023503, 2008. URL <https://link.aps.org/doi/10.1103/PhysRevD.77.023503>.
- [72] W. Hu and I. Sawicki. Models of $f(R)$ cosmic acceleration that evade solar system tests. *Physical review D*, 76(6):064004, 2007. URL <https://link.aps.org/doi/10.1103/PhysRevD.76.064004>.
- [73] V. Faraoni. de Sitter space and the equivalence between $f(R)$ and scalar-tensor gravity. *Physical review D*, 75(6):067302, 2007. URL <https://link.aps.org/doi/10.1103/PhysRevD.75.067302>.
- [74] V. Faraoni. Matter instability in modified gravity. *Physical review D*, 74(10):104017, 2006. URL <https://link.aps.org/doi/10.1103/PhysRevD.74.104017>.
- [75] A. de la Cruz-Dombriz. Some cosmological and astrophysical aspects of modified gravity theories. *arXiv:1004.5052*, 2010.
- [76] A. Heavens. ICIC Data Analysis Workshop: Supernova MCMC and HMC exercise, 2018. URL <https://www.imperial.ac.uk/media/imperial-college/research-centres-and-groups/astrophysics/public/icic/data-analysis-workshop/2018/SN-MCMC-project-2018.pdf>.

- [77] M. M. Deza and E. Deza. Encyclopedia of distances. In *Encyclopedia of Distances*, pages 1–583. Springer, Berlin, Heidelberg, 2009. URL https://doi.org/10.1007/978-3-642-00234-2_1.
- [78] R. J. Tayler. *The stars: their structure and evolution*. Cambridge University Press, 2nd edition, 1994.
- [79] L. G. Althaus, A. H. Córscico, J. Isern, et al. Evolutionary and pulsational properties of white dwarf stars. *The Astronomy and Astrophysics Review*, 18(4):471–566, 2010. URL <https://doi.org/10.1007/s00159-010-0033-1>.
- [80] R. Barbon, F. Ciatti, and L. Rosino. On the light curve and properties of Type I Supernovae. *Astronomy and Astrophysics*, 25:241–248, 1973.
- [81] D. Richardson, D. Branch, D. Casebeer, et al. A comparative study of the absolute magnitude distributions of supernovae. *The Astronomical Journal*, 123(2):745, 2002. URL <https://doi.org/10.1086/338318>.
- [82] A. Conley, J. Guy, M. Sullivan, et al. Supernova constraints and systematic uncertainties from the first three years of the Supernova Legacy Survey. *The Astrophysical Journal, Supplement series*, 192(1):1, 2010. URL <https://doi.org/10.1088/0067-0049/192/1/1>.
- [83] J. D Neill, M. Sullivan, D. A. Howell, et al. The local hosts of Type Ia Supernovae. *The Astrophysical Journal*, 707(2):1449–1465, 2009. doi: 10.1088/0004-637X/707/2/1449.
- [84] M. Hicken, P. Challis, S. Jha, et al. CfA3: 185 Type Ia Supernova light curves from the CfA. *The Astrophysical Journal*, 700:331–357, 2009. URL <https://doi.org/10.1088/0004-637X/700/1/331>.
- [85] E. V. Linder. Understanding the optimal redshift range for the supernovae Hubble diagram. *arXiv Astrophysics astro-ph/0108280*, 2001.
- [86] S. Capozziello, Ruchika, and A. A. Sen. Model-independent constraints on dark energy evolution from low-redshift observations. *Monthly Notices of the Royal Astronomical Society*, 484(4):4484–4494, 2019. URL <https://doi.org/10.1093/mnras/stz176>.
- [87] B. Leibundgut and J. Sollerman. A cosmological surprise: the universe accelerates. *Euro-physics News*, 32(4):121–125, 2001. URL <https://doi.org/10.1051/epn:2001401>.
- [88] D. W. Hogg. Distance measures in cosmology. *arXiv Astrophysics astro-ph/9905116*, art. astro-ph/9905116, 1999.
- [89] A. G. Riess, A. V. Filippenko, P. Challis, et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *The Astronomical Journal*, 116(3): 1009, 1998. URL <https://doi.org/10.1086/300499>.

- [90] A. G. Riess, L. Strolger, J. Tonry, et al. Type Ia Supernova discoveries at $z > 1$ from the *Hubble Space Telescope*: Evidence for past deceleration and constraints on dark energy evolution. *The Astrophysical Journal*, 607(2):665, 2004. URL <https://doi.org/10.1086/383612>.
- [91] S. Brooks, A. Gelman, G. Jones, et al. *Handbook of Markov Chain Monte Carlo*. CRC press, 2007. URL <https://doi.org/10.1201/b10905>.
- [92] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, et al. Equation of state calculations by fast computing machines. *The journal of Chemical Physics*, 21(6):1087–1092, 1953. URL <https://doi.org/10.1063/1.1699114>.
- [93] W. K. Hastings. Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57:97–109, 1970. URL <https://doi.org/10.1093/biomet/57.1.97>.
- [94] C. Andrieu, N. De Freitas, A. Doucet, et al. An introduction to MCMC for machine learning. *Machine learning*, 50(1-2):5–43, 2003. URL <https://doi.org/10.1023/A:1020281327116>.
- [95] G. O. Roberts and A. F. M. Smith. Simple conditions for the convergence of the Gibbs sampler and Metropolis-Hastings algorithms. *Stochastic Processes and their Applications*, 49(2):207–216, 1994. URL [https://doi.org/10.1016/0304-4149\(94\)90134-1](https://doi.org/10.1016/0304-4149(94)90134-1).
- [96] G. O. Roberts and J. S. Rosenthal. Optimal scaling for various Metropolis-Hastings algorithms. *Statistical Science*, 16(4):351–367, 2001. URL <https://projecteuclid.org/euclid.ss/1015346320>.
- [97] S. Chib and I. Jeliazkov. Marginal likelihood from the Metropolis–Hastings output. *Journal of the American Statistical Association*, 96(453):270–281, 2001. URL <https://doi.org/10.1198/016214501750332848>.
- [98] B. Leibundgut. Supernovae and cosmology. *General Relativity and Gravitation*, 40(2-3):221–248, 2008. URL <https://doi.org/10.1007/s10714-007-0545-9>.
- [99] J. P. Huchra. The hubble constant, 2008. URL <https://www.cfa.harvard.edu/~dfabricant/huchra/hubble/>.
- [100] A. G. Riess, L. M. Macri, S. L. Hoffmann, et al. A 2.4% determination of the local value of the Hubble constant. *The Astrophysical Journal*, 826(1):56, 2016. URL <https://doi.org/10.3847/0004-637X/826/1/56>.
- [101] K. P. Burnham and D. R. Anderson. Multimodel inference: Understanding AIC and BIC in model selection. *Sociological Methods & Research*, 33(2):261–304, 2004. URL <https://doi.org/10.1177/0049124104268644>.

-
- [102] R. C. Nunes, S. Pan, E. N. Saridakis, et al. New observational constraints on $f(R)$ gravity from cosmic chronometers. *Journal of Cosmology and Astroparticle Physics*, 2017(1):005, 2017. URL <https://doi.org/10.1088/1475-7516/2017/01/005>.
- [103] S. Nesseris and J. García-Bellido. Is the Jeffreys' scale a reliable tool for Bayesian model comparison in cosmology? *Journal of Cosmology and Astroparticle Physics*, 2013(8):036, 2013. URL <https://doi.org/10.1088/1475-7516/2013/08/036>.
- [104] A. R. Liddle. Information criteria for astrophysical model selection. *Monthly notices of the Royal Astronomical Society*, 377(1):L74–L78, 2007. URL <https://doi.org/10.1111/j.1745-3933.2007.00306.x>.
- [105] J. V. Cunha and J. A. S. Lima. Transition redshift: new kinematic constraints from supernovae. *Monthly notices of the Royal Astronomical Society*, 390:210–217, 2008. URL <https://doi.org/10.1111/j.1365-2966.2008.13640.x>.
- [106] A. Gómez-Valent. Quantifying the evidence for the current speed-up of the Universe with low and intermediate-redshift data. A more model-independent approach. *Journal of Cosmology and Astroparticle Physics*, 2019. URL <https://doi.org/10.1088/1475-7516/2019/05/026>.
- [107] A. Al Mamon. Constraints on a generalized deceleration parameter from cosmic chronometers. *Modern Physics Letters A*, 33:1850056, 2018. URL <https://doi.org/10.1142/S0217732318500566>.
- [108] A. A. Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91:99–102, 1980. URL [https://doi.org/10.1016/0370-2693\(80\)90670-X](https://doi.org/10.1016/0370-2693(80)90670-X).
- [109] V. F. Cardone, S. Camera, and A. Diaferio. An updated analysis of two classes of $f(R)$ theories of gravity. *Journal of Cosmology and Astroparticle Physics*, 2012(02):30, 2012. URL <https://doi.org/10.1088/1475-7516/2012/02/030>.
- [110] A. A. Starobinsky. Disappearing cosmological constant in $f(R)$ gravity. *Journal of Experimental and Theoretical Physics Letters*, 86:157–163, 2007. URL <https://doi.org/10.1134/S0021364007150027>.
- [111] R. L. Burden and J. D. Faires. *Numerical analysis*. Brooks/Cole CENGAGE Learning, 9th edition, 2010.
- [112] S. Tsujikawa. Observational signatures of $f(R)$ dark energy models that satisfy cosmological and local gravity constraints. *Physical Review D*, 77:023507, 2008. URL <https://link.aps.org/doi/10.1103/PhysRevD.77.023507>.
- [113] J. Stewart. *Calculus*. Brooks/Cole CENGAGE Learning, 7th edition, 2008.
- [114] M. Bourne. Simpson's rule, 2018. URL <https://www.intmath.com/integration/6-simpsons-rule.php>.