Fischer matrices of the affine subgroups of the classical linear groups

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Thesis accepted for the degree Doctor of Philosophy in Mathematics at the North-West University

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Graduation: July 2023

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Abstract

The work of this thesis is motivated by the study and classification of finite groups in Group Theory. Finite simple groups were studied and classified in the recent past. The character tables of these finite simple groups were published in the ATLAS [16] about three decades ago. Character tables contain vital information about a group. After that, a process to study and construct the character tables of maximal subgroups of these finite simple groups was initiated. Currently these character tables are known except for some maximal subgroups of the Monster M and the Baby Monster B. This then brings the challenge to study and classify the finite groups that have normal subgroups. Our contribution in this thesis is the study and the discussion of the computation of the Fischer matrices and eventually the construction of the character tables of the affine subgroups of the classical linear groups. Classical linear groups comprise the general linear groups, symplectic groups, orthogonal groups and unitary groups. Our approach in this thesis is to express these affine subgroups as group extensions of the form $\overline{G} = N.G$, where $N \trianglelefteq \overline{G}$ and G is isomorphic to the complement of N in $\overline{G}$. It turns out that all these affine subgroups are split extensions of the form $\overline{G} = N{:}G$. We construct the character tables of these affine subgroups using the Clifford-Fischer Theory which is due to Fischer [23] and is based on Clifford's Theorem. The whole of Chapter 3 is dedicated to the study of the Clifford-Fischer Theory. A necessary condition for this technique is that the irreducible characters of N are extendable to the inertia groups of $\overline{G}$. Moreover, to use this technique we need to have the Fischer matrices of $\overline{G}$, the conjugacy classes of $\overline{G}$, the character tables of the inertia factor groups and the fusion maps of the inertia factor groups into the group G. When this has been achieved then it becomes relatively easy to compute and construct the character table of a group extension. The coset analysis technique, which was developed by Moori [45], is utilized to compute the conjugacy classes of $\overline{G}$. The affine subgroups that are studied in this thesis are of the form $5^2{:}\mathrm{GL}(2,5)$, $2^9{:}Sp(8,2)$, $2^6{:}Sp(2,4)$, $2^{10}{:}Sp(4,4)$, $3^3{:}GO(3,3)$ and $2_+^{1+4}{:}GU(2,4)$. We prove a few results regarding the centre $Z(\overline{G})$ of the affine subgroup $\overline{G} = 2^{2n-1}{:}Sp(2n-2,2)$ of $Sp(2n,2)$ and show that the quotient group $\overline{G}/Z(\overline{G}) = 2^{2n-2}{:}Sp(2n-2,2)$ is a split extension. We extend these results to the affine subgroup of the form $2^{2(2n-1)}{:}Sp(2n-2,2^2)$ of $Sp(2n,2^2)$. Since the symplectic group is generated by symplectic transvections, we study these transvections in great detail and prove a few results on them. We also have a discussion, in a general context, on the character degrees of the affine subgroups of the classical linear groups.

Preface

The work described in this thesis was carried out under the supervision and direction of Professor Jamshid Moori, Department of Mathematics and Applied Mathematics, School of Mathematical and Statistical Sciences, Faculty of Natural and Agricultural Sciences, North West University, Mahikeng.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other university. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

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S.L.T. Mkiva (Student)

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Prof J. Moori (Supervisor)

Dedication

Dedicated to my late parents

Acknowledgements

Praise and glory be to God through Jesus Christ my Lord.

I am immensely grateful to Prof Moori, my supervisor, for his valuable guidance, counsel and mentorship throughout the duration of my PhD studies. His knowledge, love and passion for Mathematics, in particular Group Theory, inspired and motivated me a great deal. His readiness and willingness to assist also served as a source of encouragement and motivation. His professionalism and work ethic are exemplary. I am blessed, honoured and will forever be thankful.

A word of thanks also goes to Dr Chrisper Chileshe who assisted me with the GAP software. I am very thankful to the university for the financial support through the bursary office. I also acknowledge and am very grateful for the funds from the University Capacity Development Programme (UCDP) that were used to pay my replacement so that I could take study leave from July 2019 to December 2019. My thanks also to Prof Sanne Ter Horst for his numerous assistances with the UCDP application process. Financial support organized by my supervisor and Prof Helen Drummond, the deputy dean of the Faculty of Natural and Agricultural Sciences, is acknowledged and highly appreciated. My gratitude also to the School of Mathematical and Statistical Sciences for the assistance towards my research visit to the University of KwaZulu-Natal (UKZN), Pietermaritzburg Campus. Thank you very much to Prof Sicelo Goqo and Prof Fortune Massamba for the warm welcome and for hosting me at UKZN. Another sincere word of thanks to Mr Leonard Els who helped me with the GAP software at UKZN. I also thank the NWU Department of Mathematics and Applied Mathematics for enabling me to attend the South African Mathematical Society (SAMS) conferences and the African Institute of Mathematical Sciences (AIMS) and UCDP workshops.

Keywords

Affine subgroup

Centre of a group

Character degree

Character table

Clifford-Fischer Theory

Conjugacy classes

Coset analysis

Finite group

Fischer matrices

Fusion of conjugacy classes

General linear group

Group character

Group extension

Group representation

Induced characters

Inertia factor group

Inertia group

Normal subgroup

Orbit

Orthogonal group

Permutation character

Point stabilizer

Quotient group

Restriction of characters

Simple group

Split extension

Symplectic group

Symplectic transvections

Transvections

Unitary group

List of notations

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{F}$	field
$\mathbb{F}^*$	multiplicative group of $\mathbb{F}$
$GF(q)$	Galois field of q elements
$\mathbb{F}_p$	Galois field of p elements
V	vector space
G	group (finite)
$e, 1_G$	identity of G
$\|G\|$	order of G (number of elements in G)
$o(g)$	order of g as an element of G
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$H \cong G$	H is isomorphic to G
$\|G{:}H\|$	index of H in G
$N.G$	extension of G by N

$N{:}G$	split extension of G by N
G/N	quotient group
C_r	cyclic group of order r
$[g]$ or C_g	conjugacy class of g
$C_G(g)$	centralizer of g in G
G_x	stabilizer of $x \in X$ in G ($\mathrm{Stab}_G(x)$)
$\mathrm{Aut}(G)$	automorphism group of G
G'	derived subgroup of G
$Z(G)$	centre of G
D_{2n}	dihedral group of $2n$ elements
S_n	symmetric group on n elements
A_n	alternating group on n elements
$\mathbb{Z}_n$	additive group of integers modulo n
$\mathrm{GL}(n, \mathbb{F})$	general linear group over $\mathbb{F}$
$\mathrm{GL}(n, q)$	general linear group over $GF(q)$
$Sp(2n, q)$	symplectic group over $GF(q)$
$O(n, q)$	orthogonal group over $GF(q)$
$U(n, q^2)$	unitary group over $GF(q^2)$
$\mathrm{SL}(n, \mathbb{F})$	special linear group over $\mathbb{F}$
$\mathrm{PGL}(n, \mathbb{F})$	projective linear group over $\mathbb{F}$
$\mathrm{PSL}(n, \mathbb{F})$	projective special linear group over $\mathbb{F}$
χ	ordinary character of a group
$\overline{\chi}$	complex conjugate of χ
$Irr(G)$	set of ordinary irreducible characters of G
$\chi(1_G)$	degree of a character χ
$\chi{\downarrow}_N$	restriction of χ to N
χ^G	induced character of G from χ
$h \sim g$	h is conjugate to g

H^g	conjugate of H by g in G
$\mathrm{Ker} f$	kernel of a homomorphism f
2_+^{1+2n}	extra special 2-group of order 2^{1+2n} of type "+"
2_-^{1+2n}	extra special 2-group of order 2^{1+2n} of type "-"

Contents

1

Introduction

In this thesis we compute the Clifford-Fischer matrices, commonly known as Fischer matrices, of affine subgroups of classical linear groups, namely: the general linear group, the symplectic group, the general orthogonal group and the general unitary group. Fischer matrices play a crucial role in the construction of character tables of group extensions $\overline{G} = N.G$ through the Clifford-Fischer Theory. This technique of constructing character tables of group extensions is due to Fischer [23]. There are other methods of constructing character tables, namely: the Schreier-Sims algorithm, Todd-Coxeter coset enumeration method, the Burnside-Dixon algorithm and others. It is believed that the Clifford-Fischer Theory is more efficient, powerful and systematic. This theory is discussed in detail in Chapter 3. The Clifford-Fischer Theory entails having, first, the conjugacy classes of a group extension, fusion maps of the inertia factor groups into the group G, Fischer matrices of $\overline{G}$ and the character tables of the inertia factor groups. We employ the coset analysis technique, which is due to Moori [45], to determine all the conjugacy classes of $\overline{G}$. More details about this technique are found in Section 2.3. The interest in the character table of a group is because these tables contain vital information about a group. The above-mentioned affine subgroups are in fact split extensions of the form $\overline{G} = N{:}G$, where $N \trianglelefteq \overline{G}$ and G is isomorphic to the complement of N in $\overline{G}$. The specific affine subgroups we study in this work are of the form

- $5^2{:}\mathrm{GL}(2,5) \leq \mathrm{GL}(3,5)$,
- $2^9{:}Sp(8,2) \leq Sp(10,2)$,
- $2^6{:}Sp(2,4) \leq Sp(4,4)$,
- $2^{10}{:}Sp(4,4) \leq Sp(6,4)$,
- $3^3{:}GO(3,3) \leq GO(5,3)$ and
- $2_+^{1+4}{:}GU(2,4) \leq GU(4,4)$.

Since the symplectic groups are generated by symplectic transvections we look deeper into the theory of symplectic transvections and prove a few new results on them in Section 4.2.1. In Section 4.2.6 we also prove some new results regarding the centre $Z(\overline{G})$ of the affine subgroup $\overline{G} = 2^{2n-1}{:}Sp(2n-2,2)$. We show that the quotient group $\overline{G}/Z(\overline{G}) = 2^{2n-2}{:}Sp(2n-2,2)$ is a split extension. We extend these results to the affine subgroup of the form $2^{2(2n-1)}{:}Sp(2n-2,2^2)$ of $Sp(2n,2^2)$. We further discuss in a general context how to derive the Fischer matrices and the character table of $\overline{G}/Z(\overline{G})$ directly from the Fischer matrices and the character table of $\overline{G}$.

In Chapter 2 we discuss background theory on group extensions, group representations and group characters. We first define, in Section 2.1, group extensions $\overline{G} = N.G$ in terms of short exact sequences, where $N \trianglelefteq \overline{G}$ and $G \cong \overline{G}/N$. Then in Section 2.2 we distinguish split extensions from non-split extensions. We concentrate more on the split extensions since the affine subgroups we are dealing with in this work are split extensions. We relate split extensions and semidirect products. One of the key components of this chapter is the determination of the conjugacy classes of group extensions. In Section 2.3 we discuss in detail a technique referred to as the coset analysis. This technique is used to determine the conjugacy classes of both split and non-split extensions. Thereafter, we discuss a method of determining the orders of elements of the split extensions $\overline{G} = N{:}G$, with N a normal elementary abelian p-subgroup, p a prime. We then consider the representation and the character theory of finite groups in Section 2.4. We discuss the orthogonality relations and the notion of a character table in Section 2.5. Then in Section 2.6 we define and discuss the concept of a permutation character. This character plays an important role in the computation of the conjugacy classes of group extensions of an elementary abelian group by a finite group. We also consider the notion of a rank of a permutation group.

Chapter 3 is about the Theory of Fischer matrices. We follow a method presented by Fischer in [23] on how to construct character tables of group extensions $\overline{G} = N.G$ using Clifford's Theory. This technique entails using the character tables of inertia factor groups together with the Fischer matrices. This method requires that the irreducible characters of N be extendable to the inertia groups. Section 3.1 is about Clifford's Theorem and extension theorems, including Mackey's Theorem. We also have Gallagher's Theorem that employs Clifford's Theory and the extension theorems to describe the irreducible characters of $\overline{G}$. Due to these results the character table of $\overline{G}$ is divided into blocks of rows corresponding to the inertia factor groups. The definition and the construction of a Fischer matrix is dealt with in Section 3.2. For each class representative g of a conjugacy class $[g]$ of G, we construct a Fischer matrix $M(g)$. Section 3.3 deals with the properties and the orthogonality relations of a Fischer matrix. These properties and orthogonality relations are then used to compute the entries of a Fischer matrix. In Section 3.4 we provide details on how to construct a Fischer matrix when N is a normal non-abelian extra special 2-subgroup of a split extension $\overline{G} = N{:}G$. Then in Section 3.5 we look at the application of the Clifford-Fischer Theory in constructing the character tables of group extensions.

General theory of affine subgroups of classical linear groups is dealt with in Chapter 4. This chapter also deals with the description of the irreducible characters of these affine subgroups. The Clifford-Fischer Theory, discussed in Chapter 3, is utilized to describe and determine the irreducible characters of these subgroups. By definition, affine subgroups are subgroups that fix a non-zero vector of an underlying vector space. We prove that these affine subgroups are split extensions. This then means that these subgroups are of the form $\overline{G} = N{:}G$, where G is isomorphic to one of the classical linear groups. Another key consideration of this chapter is the analysis of the actions of $\overline{G}$ on N and on the irreducible characters of N, for each of the classical linear groups. The first action will yield the point stabilizers and the latter will yield the inertia factor groups. The structures of these point stabilizers and inertia factor groups are discussed. In Section 4.1, we consider the affine subgroups of the general linear groups. We discuss the affine subgroups of symplectic groups in Section 4.2. Since symplectic groups are generated by symplectic transvections, we dedicate Section 4.2.1 to the discussion on symplectic transvections. We consider a number of results involving symplectic transvections relevant in this thesis. We provide alternative proofs of the following already established theorems: that an inverse of a transvection is also a transvection, a conjugate of a transvection is also a transvection, and when two transvections are equal. We also prove some new results which are given in Proposition 4.2.10, Corollary 4.2.11, Proposition 4.2.14, Proposition 4.2.15, Proposition 4.2.17, Proposition 4.2.18 and Proposition 4.2.19. For further reading on symplectic groups, readers may consult [11], [14], [21], [49], [55], [62] and [64]. We also determine in this section the centre, $Z(\overline{G})$, of the affine subgroup $\overline{G}$ of $Sp(2n, q)$. We further prove some new results given in Proposition 4.2.37, Corollary 4.2.38, Proposition 4.2.39, Corollary 4.2.40, Proposition 4.2.41 and Proposition 4.2.42. In Remark 4.2.43 we establish a method of determining the Fischer matrices of the quotient group $\overline{G}/Z(\overline{G})$ directly from the Fischer matrices of $\overline{G}$. In Remark 4.2.44 we outline a similar process to determine the character table of $\overline{G}/Z(\overline{G})$ directly from the character table of $\overline{G}$. The latter remark is due to Isaacs [37]. In Section 4.3 we study the affine subgroups of orthogonal groups. Then in Section 4.4 we deal with the affine subgroups of unitary groups. We refer the reader to [1], [18], [19], [30], [35], [48] and [49] for further reading on affine subgroups of classical linear groups.

The affine subgroup $\overline{G} = D(3){:}GL(2,5) = N{:}G$ of the general linear group $\mathrm{GL}(3,5)$ is studied in Chapter 5, where N is an elementary abelian 5-group of order 25. This subgroup is of order 12000 and of index 124 in $\mathrm{GL}(3,5)$. In Section 5.1 we obtain, using GAP, generators of N and G, respectively. We then express these generators as non-singular 3×3 matrices with entries in $\mathbb{F} = GF(5)$ since $\overline{G}$ sits in $\mathrm{GL}(3,5)$. In Section 5.2 we consider the action of G on N. This action fixes the zero and is transitive on the non-zero elements of N. Thus we obtain two orbits of lengths 1 and 24. The corresponding point stabilizers are isomorphic to G and the affine subgroup $5{:}GL(1,5)$ of G, where $GL(1,5) = \mathbb{F}^*$. We also compute the permutation character of G on the holomorph 5:4. We use the coset analysis technique to determine the conjugacy classes of $\overline{G}$ in Section 5.3. From the 24 conjugacy classes of G we obtain 29 conjugacy classes of $\overline{G}$. In Section 5.4 we compute the 24 Fischer matrices of $\overline{G}$, corresponding to each conjugacy class representative of G. We utilize the Clifford-Fischer Theory in Section 5.5 to construct the character table of $\overline{G}$. In Section 5.6 we deal with the fusion of $\overline{G}$ into $GL(3,5)$.

In Chapter 6 we consider the affine subgroup $\overline{G} = 2^9{:}Sp(8,2) = N{:}G$ of the symplectic group $Sp(10,2)$, where N is an elementary abelian 2-group of order 512. The index of $\overline{G}$ in $Sp(10,2)$ is 1023. Section 6.1 deals with the transvections of G. Let $\mathbb{F}$ be the Galois field of two elements. We note that there are 255 transvections in G. The order of each transvection is 2 since $\mathrm{Char}(\mathbb{F}) = 2$. There are 6 conjugacy classes of elements of order 2 in G. Since $|\mathbb{F}| = 2$, G has one conjugacy class of transvections. The class 2A of G has 255 elements and this coincides with number of transvetions in G. Thus we conclude that 2A is the class of transvections. We further note that the centralizer of a transvection is isomorphic to the affine subgroup $2^7{:}Sp(6,2)$ of G. In Section 6.2 we express the generators of N and G in terms of 10×10 symplectic matrices over $\mathbb{F}$ since $\overline{G}$ sits in $Sp(10,2)$. The action of G on N, in Section 6.3, yields 4 orbits of lengths

1, 1, 255 and 255, with the respective corresponding point stabilizers being isomorphic to G, G, $2^7{:}Sp(6,2)$ and $2^7{:}Sp(6,2)$. In the same section we apply the coset analysis technique to determine all the 322 conjugacy classes of $\overline{G}$ from the 81 conjugacy classes of G. The action of G on the $Irr(N)$, in Section 6.4, yields 4 orbits of lengths 1, 120, 136 and 255. We determine 4 corresponding inertia factor groups isomorphic to G, $GO^-(8,2)$, $GO^+(8,2)$ and $2^7{:}Sp(6,2)$, respectively. Then we compute the fusion maps of these inertia factor groups into G. In Section 6.5 we discuss the computation of the Fischer matrices of $\overline{G}$ using the theory outlined in Chapter 3. In Section 6.6 we use the Clifford-Fischer Theory to discuss the construction of the character table of $\overline{G}$. The centre $Z(\overline{G})$, which is isomorphic to $\mathbb{Z}_2$, is determined in Section 6.7. We also determine the quotient group $\overline{G}/Z(\overline{G})$ which is isomorphic to the split extension $2^8{:}Sp(8,2)$. We then show how to derive the Fischer matrices of the split extension $2^8{:}Sp(8,2)$ directly from the Fischer matrices of $\overline{G}$.

In Chapter 7 we work on the affine subgroup $\overline{G} = 2^6{:}Sp(2,4) = N{:}G$ of the symplectic group $Sp(4,4)$. This chapter is in preparation of Chapter 8, where we will consider the affine subgroup $2^{10}{:}Sp(4,4)$ of the symplectic group $Sp(6,4)$. It turns out that the group $\overline{G}$ is a point stabilizer when $Sp(4,4)$ acts on 2^{10} and is also one of the inertia factor groups when the group $Sp(4,4)$ acts on the irreducible characters of 2^{10}. This then means that we will need the character table of $\overline{G}$ in Chapter 8. Let $\mathbb{F} = GF(4)$ be the Galois field of 4 elements. The normal subgroup N is elementary abelian 2-group of order 64. This affine subgroup is a split extension of index 255 in $Sp(4,4)$. In Section 7.1 we deal with the transvections of G. We observe that there are 15 transvections in G. Since $\text{Char}(\mathbb{F}) = 2$, the order of each transvection is 2. There is one conjugacy class of transvections in G since $|\mathbb{F}| = 2^2$. The class 2A in G is therefore the class of transvections. The centralizer of a transvection is isomorphic to the affine subgroup D_4 of G, which is isomorphic to the Klein four-group. In Section 7.2 we derive the generators of N and G as 4×4 symplectic matrices since $\overline{G}$ is a subgroup of $Sp(4,4)$. The generators of G are expressed in terms of transvections since G is generated by transvections. In Section 7.3 we compute the permutation character of G on D_4 using the fusion of conjugacy classes of D_4 into G. This permutation character is then used in Section 7.4, together with the coset analysis technique, to obtain 30 conjugacy classes of $\overline{G}$ from the 5 conjugacy classes of G. The action of G on N yields orbits of lengths 1,1,1,1,15,15,15 and 15. The corresponding point stabilizers are, respectively, G, G, G, G, D_4, D_4, D_4 and D_4. In Section 7.5 we obtain 8 orbits of lengths 1, 15, 6, 6, 6, 10, 10 and 10, respectively, from the action of G on $Irr(N)$. The corresponding inertia factor groups are isomorphic to G, D_4, D_{10}, D_{10}, D_{10}, S_3, S_3 and S_3, respectively. At the end of this section we compute the fusion maps of these inertia factor groups into G. We use these fusion maps and the Clifford-Fischer Theory to compute the 5 Fischer matrices of $\overline{G}$ in Section 7.6. Since $\overline{G}$ is a split extension and N is elementary abelian, then the $Irr(N)$ are extendable to the inertia groups. This then enables us to utilize the Clifford-Fischer Theory to construct the character table of $\overline{G}$ in Section 7.7. The fusion of the conjugacy classes of $\overline{G}$ into the conjugacy classes of $Sp(4,4)$ is done in Section 7.8. We conclude this chapter by considering the quotient group $\overline{G}/Z(\overline{G})$ in Section 7.9. The centre $Z(\overline{G})$ is isomorphic to $\mathbb{Z}_4$ and the quotient group $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^4{:}Sp(2,4)$. We demonstrate how to obtain the Fischer matrices and the character table of $\overline{G}/Z(\overline{G})$ directly from the Fischer matrices and the character table of $\overline{G}$.

We study the affine subgroup $\overline{G} = 2^{10}{:}Sp(4,4) = N{:}G$ of the symplectic group $Sp(6,4)$ in

Chapter 8. This affine subgroup is of order 1002700800 and of index 4095 in $Sp(6,4)$. The normal subgroup N is an elementary abelian 2-group of order 1024. Let $\mathbb{F} = GF(4)$. In Section 8.1 we have that G has 255 transvections. All these are of order 2 since $\text{Char}(\mathbb{F}) = 2$. Because $|\mathbb{F}| = 2^2$, then there is one conjugacy class of transvections in G. According to the character table of G in the ATLAS there are three conjugacy classes of elements of order 2 in G. However, either the class 2A or 2B is the class of transvections. We use GAP to analyse the elements of these two classes. We observe that the elements of 2A satisfy the conditions of the definition of a transvection. We thus conclude that the class 2A is the class of transvections. We note that the centralizer of a transvection is isomorphic to the affine subgroup $2^6{:}Sp(2,4)$ of G. In Section 8.2 we express the generators of N and G as 6×6 symplectic matrices since $\overline{G}$ is a subgroup of $Sp(6,4)$. In Section 8.3 we have that the action of G on N has 8 orbits. The orbit lengths are 1, 1, 1, 1, 255, 255, 255 and 255 and the corresponding point stabilizers are isomorphic to G, G, G, G, $2^6{:}Sp(2,4)$, $2^6{:}Sp(2,4)$, $2^6{:}Sp(2,4)$ and $2^6{:}Sp(2,4)$, respectively. We then determine all the 165 conjugacy classes of $\overline{G}$ using the coset analysis technique. In Section 8.4 we have that the action of G on $Irr(N)$ yields 8 orbits of lengths 1, 255, 136, 136, 136, 120, 120 and 120. The corresponding inertia factor groups are isomorphic to $H_1 = G$, $H_2 \cong 2^6{:}Sp(2,4)$, H_3, H_4, H_5 being isomorphic to the full orthogonal group $GO^+(4,4)$ and H_6, H_7, H_8 being isomorphic to the full orthogonal group $GO^-(4,4)$. The fusion maps of these inertia factor groups are also computed in this section. In Section 8.5 we determine all the 27 Fischer matrices of $\overline{G}$. We discuss the construction of the character table of $\overline{G}$ in Section 8.6 using the Clifford-Fischer Theory. In Section 8.7 we note that the centre $Z(\overline{G})$ is isomorphic to $\mathbb{Z}_4$ and that the quotient group $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^8{:}Sp(4,4)$. We then show how to obtain the Fischer matrices of $2^8{:}Sp(4,4)$ directly from the Fischer matrices of $2^{10}{:}Sp(4,4)$.

The analysis of the affine subgroup $\overline{G} = 3^3{:}GO(3,3) = N{:}G$ of the general orthogonal group $GO(5,3)$ is considered in Chapter 9. The order of this affine subgroup is 1296 and is of index 80 in $GO(5,3)$. The group $\overline{G}$ is a semidirect product of the normal elementary abelian 3-group N of order 27 by the group G of order 48. In Section 9.1 we express the generators of the groups N and G in terms of the 5×5 invertible matrices with entries in the Galois field of 3 elements since $\overline{G}$ is contained in $GO(5,3)$. This process entails analysing in GAP the subgroups of index 80 in $GO(5,3)$. Using the ATLAS we note that $GO(5,3)$ is isomorphic to $2 \times (O(5,3){:}2)$ where $O(5,3)$ is the simple orthogonal group isomorphic to the simple symplectic group $S_4(3)$. The affine subgroup $\overline{G}$ sits maximally in $O(5,3){:}2$. The affine subgroup of the full symplectic group $Sp(4,3)$ is $3^3{:}S_4$. Our $\overline{G}$ sits maximally in $Sp(4,3)$ containing the affine subgroup of $Sp(4,3)$. In Section 9.2 we deal with the action of G on N. This action yields 4 orbits of lengths 1, 8, 12 and 6. The corresponding point stabilizers are isomorphic to G, $3{:}GO(1,3) \cong D_6$, $GO^+(2,3) \cong D_4$ and $GO^-(2,3) \cong D_8$, where $3{:}GO(1,3)$ is the affine subgroup of G. We determine the fusion maps of these point stabilizers into G. Then we compute the respective permutation characters. Thereafter we express these permutation characters in terms of the irreducible characters of G. This process is in aid of the determination of the permutation character $\chi(G|N)$. In Section 9.3 we use the coset analysis to obtain 22 conjugacy classes of $\overline{G}$ from the 10 conjugacy classes of G. The action of G on $Irr(N)$ is dealt with in Section 9.4. This action yields 4 orbits of lengths 1, 8, 12 and 6. The corresponding inertia factor groups coincide with the point stabilizers, and so are isomorphic to G, D_6, D_4 and D_8, respectively. We also list the character tables of these inertia factor groups in this section. In Section 9.5 we employ an alternative method of constructing the Fischer matrices of $\overline{G}$. This technique is due to List and it is briefly discussed towards the end of Section 3.3. It is used among others by Almestady [2], [3], Almestady and Morris [4], Chileshe [15], Darafsheh and Iranmanesh [19], Iranmanesh [35], Mpono [49], Whitley [63] and Zimba [65]. The fusion maps of the inertia factor groups into G together with some properties of character tables are also used to construct these Fischer matrices. The affine subgroup $\overline{G}$ has 10 Fischer matrices corresponding to each class representative of the conjugacy classes of G. These Fischer matrices together with the character tables of the inertia factor groups are then used to construct the character table of $\overline{G}$ in Section 9.6. We conclude this chapter by considering the fusion of the conjugacy classes of $\overline{G}$ into the conjugacy classes of $GO(5,3)$ in Section 9.7. This section is also vital in ensuring the correctness of the Fischer matrices and the character table of $\overline{G}$.

The last chapter, Chapter 10, is about the affine subgroup $\overline{G} = 2_+^{1+4}{:}GU(2,4) = N{:}G$ of the general unitary group $GU(4,4)$. This subgroup is of order 576 and of index 135 in $GU(4,4)$. We show that N is an extra special 2-group of order 32 and that the quadratic form associated with N is of type +. We analyse in Section 10.1, using GAP, the subgroups of index 135 in $GU(4,4)$ to ultimately find the generators of N and G in terms of 4×4 invertible matrices with entries in the Galois field of 4 elements since $\overline{G}$ sits in $GU(4,4)$. In Section 10.2 we look at the structure of the 17 conjugacy classes of N. We note that N has 1 central involution, 9 classes of non-central involutions and 6 classes of elements of order 4. Each of the latter classes is of cardinality of 2. We remark that, unlike in the previous chapters, the normal subgroup N is not abelian. In Section 10.3 we first consider the action $\overline{G}$ on N. This action produces 4 orbits of lengths 1, 1, 18 and 12. The corresponding point stabilizers are isomorphic to $\overline{G}$, $\overline{G}$, $2^4{:}C_2$ and $2^4{:}C_3$, respectively. We then utilize the coset analysis to determine 23 conjugacy classes of $\overline{G}$ from the 9 conjugacy classes of G. In Section 10.4 we consider the action of $\overline{G}$ on $Irr(N)$. By Brauer's Theorem this action must also yield 4 orbits. We show that N has 17 irreducible characters of which 16 are linear characters and one is the unique faithful irreducible character of degree 4. The orbits on the linear characters are of lengths 1, 9 and 6. The corresponding inertia factor groups are $H_1 = G$, $H_2 = C_2$ and $H_3 = C_3$. The inertia factor group corresponding to the faithful character is isomorphic to $H_4 = G$. The full inertia groups are then $N{:}H_1$, $N{:}H_2$, $N{:}H_3$ and $N{:}H_4$. The computations of the 9 Fischer matrices of $\overline{G}$ are done in Section 10.5. Since N is not abelian we use a different approach than in the previous chapters to construct these Fischer matrices. In fact, we use Lemma 3.4.1 and Lemma 3.4.2, due to Pahlings [52], to construct the first two columns and the second row of a Fischer matrix. We then use Remark 3.4.3 (c), due to Basheer [8], and the orthogonality relations of a Fischer matrix to calculate the rest of the entries. In Section 10.6 we construct the character table of $\overline{G}$ using the Clifford-Fischer Theory. We require that the $Irr(N)$ be extendable to the inertia groups. The 16 linear characters of N are extendable by Theorem 3.1.15. Since the Schur multiplier of G is trivial, the faithful character is also extendable. Then we use the Fischer matrices of $\overline{G}$ and the character tables of the inertia factor groups to construct the character table of $\overline{G}$. Lastly, in Section 10.7 we deal with the fusion of the conjugacy classes of $\overline{G}$ into the conjugacy classes of $GU(4,4)$. This section played a critical role in the construction of the character table of $\overline{G}$, in particular on the choices of the entries of the first two columns in the second row of the Fischer matrices of $\overline{G}$.

2
Group extensions and group characters

In this thesis we construct the character tables of affine subgroups of classical linear groups, namely the general linear groups, the symplectic groups, the general orthogonal groups and the general unitary groups. It turns out that these affine subgroups are split extensions. The first part of this section is dedicated to the theory of group extensions, in particular split extensions. Thereafter we deal with the representation and the character theory of finite groups. In Section 2.1 we define group extensions $\overline{G}$ of a group N by a group G in terms of short exact sequences $\{1\} \to N \overset{\beta}{\to} \overline{G} \overset{\pi}{\to} G \to \{1\}$. Then in Section 2.2 we define split extensions, non-split extensions and semidirect products. Section 2.3 is about the determination of the conjugacy classes of group extensions $\overline{G} = N.G$, where N is a normal subgroup of $\overline{G}$. We discuss in detail a technique known as the coset analysis which was developed by Moori [45] to compute these conjugacy classes. After that we discuss ways of determining the orders of elements of split extensions $\overline{G} = N{:}G$ with the normal subgroup N an elementary abelian p-group. We then proceed to discuss some representation and character theories relevant in this work. We will use some of these theories in Chapter 3 when we deal with the theory of Clifford-Fischer matrices and the construction of character tables of finite groups. In Section 2.4 we deal with group representations and group characters. Section 2.5 is about the orthogonality relations of characters and the character table of a finite group G. Lastly in Section 2.6 we discuss the notion of a permutation character. This character plays an important role when computing the conjugacy classes of group extensions of an elementary abelian group by a finite group. We also define the rank of a transitive permutation group to be the number of orbits of a point stabilizer. The reader may consult Humphreys [32], Robinson [54], Rose [56] and Rotman [57] for further reading on group extensions. For further reading on group representations and group characters, the reader may consult Alperin and Bell [5], Aschbacher [6], Gorenstein [29], Huppert [33], Isaacs [37], James and Liebeck [39], Kleidman and Liebeck [42], Lux and Pahlings [44] and Moori [47].

2.1 Group extensions

Let $\{\cdots, B_{n-1}, B_n, B_{n+1}, \cdots\}$ and $\{\cdots, \alpha_{n-1}, \alpha_n, \alpha_{n+1}, \cdots\}$ be sets of groups and homomorphisms respectively. A sequence of groups and homomorphisms of the form

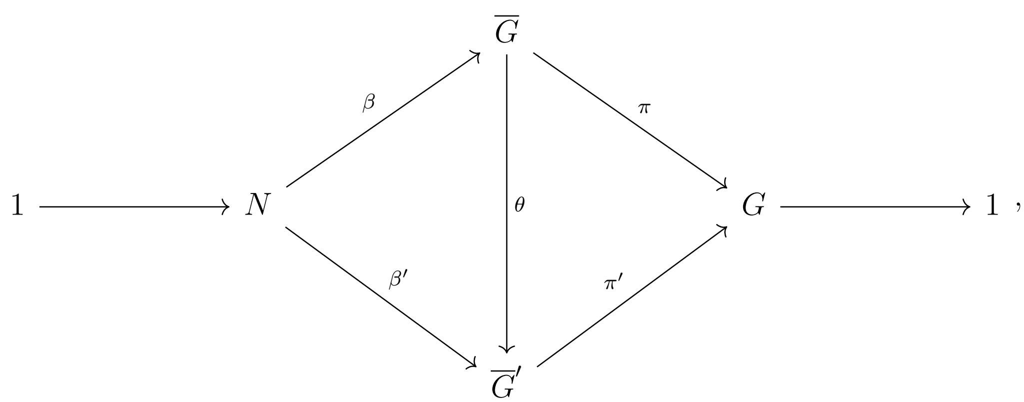
$$\cdots B_{n-1} \overset{\alpha_{n-1}}{\to} B_n \overset{\alpha_n}{\to} B_{n+1} \overset{\alpha_{n+1}}{\to} \cdots$$

is called an **exact sequence** if Ker(α_n)=Im(α_{n-1}) for every successive pair (α_{n-1}, α_n). A **group extension** $\overline{G}$ of N by G is a short exact sequence

$$\{1\} \to N \overset{\beta}{\to} \overline{G} \overset{\pi}{\to} G \to \{1\} \tag{2.1}$$

with Ker(π)=Im(β). In Equation 2.1, β is an inclusion monomorphism and π a projection epimorphism. If N and G are arbitrary groups, then an extension of N by G is a group $\overline{G}$ possessing a normal subgroup N such that $\overline{G}/N \cong G$. An extension of N by G is denoted by $N.G$. The group G is referred to as the complement of N in $\overline{G}$ and the group N the kernel of the extension. We further note that Ker(π)=Im(β) $\cong N$. This isomorphism becomes equality since $N \leq \overline{G}$. We remark that complements of N, if they exist, need not be unique. However, if complements of N in $\overline{G}$ exist then they are unique up to isomorphism since they are all isomorphic to $\overline{G}/N$.

Definition 2.1.1 *Extensions* $\{1\} \to N \overset{\beta}{\to} \overline{G} \overset{\pi}{\to} G \to \{1\}$ *and* $\{1\} \to N \overset{\beta'}{\to} \overline{G}' \overset{\pi'}{\to} G \to \{1\}$ *are said to be* **equivalent** *if there exists a homomorphism* $\theta : \overline{G} \to \overline{G}'$ *such that the following diagram is commutative:*



that is $\theta \circ \beta = \beta'$ *and* $\pi' \circ \theta = \pi$.

The homomorphism θ is in fact an isomorphism by *five lemma*.

2.2 Semidirect products and split extensions

Definition 2.2.1 *Let* $\overline{G}$ *be a group. If* $N \leq \overline{G}$ *and* $G \leq \overline{G}$ *such that* *(i)* $N \trianglelefteq \overline{G}$,

(ii) $\overline{G} = NG$ *and*

(iii) $N \cap G = \{1_G\}$,

then $\overline{G}$ *is referred to as a* **semidirect product** *of* N *by* G. *A semidirect product* $\overline{G}$ *of* N *by* G *is denoted by* $\overline{G} = N{:}G$.

If $N \trianglelefteq \overline{G}$ and $G \trianglelefteq \overline{G}$ then $\overline{G}$ is a direct product of N and G. Let $\mathrm{Aut}(N)$ be the group of automorphisms of N. The function $\theta : G \to \mathrm{Aut}(N)$ defined by $\theta(g) = \theta_g$ is a group homomorphism, where $\theta_g(n) = gng^{-1}\ \forall\ g \in G$ and $n \in N$.

Remark 2.2.2 *If* $\overline{G}$ *is a semidirect product of* N *by* G *then every element in* $\overline{G}$ *can be expressed uniquely in the form* ng, *where* $n \in N$ *and* $g \in G$. *The multiplication of elements of* $\overline{G}$ *is given by*

$$(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2,$$

where $n^g = gng^{-1} = \theta_g(n)$. *This defines an* **action** *of* G *on* N.

Proposition 2.2.3 *A semidirect product* $N{:}G$ *for arbitrary finite groups* N *and* G *exists if and only if there exists a homomorphism* $\theta : G \to Aut(N)$.

Proof See [8]. □

If $\overline{G}$ is a semidirect product of N by G then $\overline{G}$ is isomorphic to the semidirect product $N{:}_\theta G$ realized by $\theta : G \to \mathrm{Aut}(N)$. Hence, most often we omit the homomorphism θ when dealing with a specific semidirect product $N{:}_\theta G$ as we know how G acts on N.

Definition 2.2.4 *Let* $\overline{G} = N.G$ *and* $\{1\} \to N \xrightarrow{\beta} \overline{G} \xrightarrow{\pi} G \to \{1\}$ *be the corresponding short exact sequence. Let* $g \in G$ *and* $\overline{g} \in \overline{G}$ *such that* $\pi(\overline{g}) = g$. *Then* $\overline{g}$ *is called a* **lifting** *of* g *in* $\overline{G}$.

Suppose that $\overline{G}$ is an extension of N by G. Then there exists an epimorphism $\pi : \overline{G} \to G$ such that $\mathrm{Ker}(\pi) = N$ since $\overline{G}/N \cong G$. If we choose a lifting for each $g \in G$ then we obtain the set $\{\overline{g} : g \in G\}$. This set is referred to as the **transversal** for N in $\overline{G}$. From this we obtain a **transversal function** $\eta : G \to \overline{G}$. Now if $\overline{g} \in \overline{G}$, the coset representative of $N\overline{g}$ is $\eta(\pi(\overline{g})) = (\eta \circ \pi)(\overline{g})$. In general η is not a homomorphism. However, η always satisfies the following relation:

$$\eta \circ \pi = 1. \tag{2.2}$$

The converse is that any function, say η, that satisfies the Equation 2.2 defines a transversal of N in $\overline{G}$, namely $\{\eta(g) \mid g \in G\}$.

Definition 2.2.5 *An extension* $\{1\} \to N \xrightarrow{\beta} \overline{G} \xrightarrow{\pi} G \to \{1\}$ *is said to be*

(a) **abelian** *if* $\overline{G}$ *is abelian,*

(b) **central** *if* $\mathrm{Im}(\beta) = \beta(N) \subseteq Z(\overline{G})$,

(c) **cyclic** *if G is cyclic,*

(d) **split** *if there is a monomorphism $\zeta : G \to \overline{G}$ such that $\pi \circ \zeta = 1_G$.*

An extension $\overline{G}$ of N by G is said to be **non-split** if it is not a split extension. Non-split extensions are denoted by $N^{\cdot}G$.

Theorem 2.2.6 *If an extension splits then so does any equivalent extension.*

Proof Suppose the split extension $\{1\} \to N \xrightarrow{\beta} \overline{G} \xrightarrow{\pi} G \to \{1\}$ is equivalent to the extension $\{1\} \to N \xrightarrow{\beta'} \overline{G}' \xrightarrow{\pi'} G \to \{1\}$. Let θ be the homomorphism that yields the equivalence. Then there is a monomorphism $\phi : G \to \overline{G}$ such that $\pi\phi = 1_G$. Let $\phi' = \theta\phi$. Then $\phi' : G \to \overline{G}'$ is a monomorphism such that $\pi'\phi' = \pi'\theta\phi = \pi\phi = 1_G$. □

The following result relates split extensions and semidirect products.

Theorem 2.2.7 *Every split extension $\{1\} \to N \xrightarrow{\beta} \overline{G} \xrightarrow{\pi} G \to \{1\}$ is equivalent to a semidirect product $N{:}G$.*

Proof See [54]. □

Theorem 2.2.8 *(Schur-Zassenhaus) An extension $\{1\} \to N \xrightarrow{\beta} \overline{G} \xrightarrow{\pi} G \to \{1\}$ splits if the orders $|N|$ and $|G|$ are finite and are relatively prime.*

Proof See [29] and [57]. □

Remark 2.2.9 *If $\overline{G}$ is a split extension of N by G then $\overline{G} = NG = \bigcup_{g \in G} Ng$.*

In the following result we show when $\overline{G}$ is a non-split extension of N by G and N is abelian, then G acts on N.

Lemma 2.2.10 *Let $\overline{G}$ be an extension of N by G with N abelian. Then there exists a homomorphism $\theta : G \to \mathrm{Aut}(N)$ such that $\theta_g(n) = \overline{g}n\overline{g}^{-1}$, $n \in N$, and θ is independent of the choice of the liftings $\{\overline{g} \ : \ g \in G\}$.*

Proof See [15] and [57]. □

2.3 Conjugacy classes of group extensions

We consider in this section the determination of conjugacy classes of group extensions $\overline{G} = N.G$, where N is a normal subgroup of $\overline{G}$. The conjugacy classes of elements of a group provide vital information about the structure of the group. We discuss in detail the coset analysis technique which we use to compute the conjugacy classes. We note the crucial role played by the centralizers in the determination of these conjugacy classes. We conclude this section with a method of determining the orders of split extensions $\overline{G} = N{:}G$, where N is an elementary abelian p-group, with p a prime. More on conjugacy classes of group extensions may also be found in [12] and [13].

2.3.1 Coset analysis

We use the coset analysis technique to determine the conjugacy classes of elements of group extensions $\overline{G} = N.G$, where N is a normal abelian subgroup of $\overline{G}$. This technique was designed and developed by Moori [45]. It can be used for both the split and non-split extensions.

For each conjugacy class $[g]$ in G with representative $g \in G$, we analyse the coset $N\overline{g}$ where $\overline{g}$ is a lifting of g in $\overline{G}$ and

$$\overline{G} = \bigcup_{g \in G} N\overline{g}.$$

To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$ we define the centralizer of $N\overline{g}$ in $\overline{G}$ under the action by conjugation of $\overline{G}$ on $N\overline{g}$ as

$$C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\}.$$

It then follows that $C_{\overline{g}} \leq \overline{G}$ and $N \trianglelefteq C_{\overline{g}}$.

Lemma 2.3.1 *$C_{\overline{g}}/N = C_{\overline{G}/N}(N\overline{g})$.*

Proof See [49]. □

Corollary 2.3.2 *If $\overline{G} = N{:}G$, then $C_g = N{:}C_G(g)$.*

Proof See [49]. □

In general, the conjugacy classes of $\overline{G} = N.G$, where N is abelian, will be determined by the action by conjugation of $C_{\overline{g}}$ for each conjugacy class $[g]$ of G on the elements of $N\overline{g}$. To act $C_{\overline{g}}$ on the elements of $N\overline{g}$, we first act N and then act $\{\overline{h} \mid h \in C_G(g)\}$, where $\overline{h}$ is a lifting of h in $\overline{G}$. This process is carried out in the following two steps:

Step 1 - the action of N on $N\overline{g}$: [49] Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$\begin{aligned} x \in C_N(n\overline{g}) &\Leftrightarrow x(n\overline{g})x^{-1} = n\overline{g} \\ &\Leftrightarrow xnx^{-1}x\overline{g}x^{-1} = n\overline{g} \\ &\Leftrightarrow n(x\overline{g}x^{-1}) = n\overline{g} \\ &\Leftrightarrow x\overline{g}x^{-1} = \overline{g} \\ &\Leftrightarrow x \in C_N(\overline{g}). \end{aligned}$$

This means $C_N(\overline{g})$ fixes every element on $N\overline{g}$. So under this action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, \cdots, Q_k$ where

$$|Q_i| = [N{:}C_N(\overline{g})] = \frac{|N|}{k},$$

for $i \in \{1, 2, \cdots, k\}$.

Step 2 - The action of $\{\overline{h} \mid h \in C_G(g)\}$ on $N\overline{g}$: We now need only act $\{\overline{h} \mid h \in C_G(g)\}$ on the orbits $Q_1, Q_2, \cdots, Q_k$ since the elements of $N\overline{g}$ are in these k orbits. Suppose that under this action f_j of these k orbits fuse together to form one orbit Δ_j, then the f_j's obtained in this way must satisfy

$$\sum_j f_j = k$$

and we have

$$|\Delta_j| = f_j \times \frac{|N|}{k}.$$

Thus for $x = d_j\overline{g} \in \Delta_j$ we have that

$$\begin{aligned} |[x]_{\overline{G}}| &= |\Delta_j| \times |[g]_G| \\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\overline{G}|}{k|C_G(g)|} \end{aligned}$$

and therefore

$$|C_{\overline{G}}(x)| = \frac{|\overline{G}|}{|[x]_{\overline{G}}|} = |\overline{G}| \times \frac{k|C_G(g)|}{f_j|\overline{G}|} = \frac{k|C_G(g)|}{f_j}.$$

This means that to compute the conjugacy classes of a group extension $\overline{G} = N.G$, we need to obtain the values k and f_j for each class representative $g \in G$. Mpono [49] developed CAYLEY programmes for the coset analysis to calculate the k and f values. Ali [1] wrote MAGMA versions to compute the k and f values. In this thesis we use GAP programmes that were designed by Chileshe [15].

Remark 2.3.3 *[49] In the case of a split extension $\overline{G} = N{:}G$ we have that $G \leq \overline{G}$ and thus we analyse the coset Ng instead of $N\overline{g}$. Under the action of N on Ng we always assume that $g \in Q_1$. Again we just act $C_G(g)$ on the orbits $Q_1, Q_2, \cdots, Q_k$ instead of acting $\{\overline{h} \mid h \in C_G(g)\}$ on these k orbits. Since $g \in Q_1$ then $C_G(g)$ always fixes Q_1 and thus we will always have $f_1 = 1$. Hence*

$$k = \sum_j f_j = 1 + \sum_m f_m,$$

where the sum is taken over all m such that $g \notin Q_m$.

2.3.2 Orders of the elements of $\overline{G} = N{:}G$

In this subsection we discuss the theory that is used to determine the orders of the elements of $\overline{G} = N{:}G$, where N is an elementary p-group for some prime p.

Theorem 2.3.4 *Let $\overline{G} = N{:}G$ and $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then $m|k$.*

Proof See [49]. □

Theorem 2.3.5 *Let $\overline{G} = N{:}G$ such that N is an elementary abelian p-group where p is prime. Let $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then either $k = m$ or $k = pm$.*

Proof See [49]. □

Remark 2.3.6 *[49] Let $\overline{G} = N{:}G$ such that N is elementary abelian p-group where p is prime. Let $dg \in \overline{G}$ where $d \in N$ and $g \in G$ such that $o(g) = m$ and $o(dg) = k$. Then we observe that*

$$(dg)^m = d \cdot d^g \cdot d^{g^2} \cdots d^{g^{m-1}} g^m.$$

Since $g^m = 1_G$, we obtain that $(dg)^m = w$, where $w \in N$ and it is given by

$$w = d \cdot d^g \cdot d^{g^2} \cdots d^{g^{m-1}}.$$

By Theorem 2.3.5 we have that if $w = 1_N$ then $k = m$ and if $w \neq 1_N$ then $k = pm$.

2.4 Representations and characters

The rest of this chapter is about the representation and the character theory of finite groups. The results that are presented here will prove useful, in particular, in Chapter 3 where we will discuss the computation of Fischer matrices and the construction of character tables of finite groups using the Clifford-Fischer Theory. The reader may consult the following for further reading, Alperin and Bell [5], Aschbacher [6], Gorenstein [29], Huppert [33], Isaacs [37], James and Liebeck [39], Kleidman and Liebeck [42], Lux and Pahlings [44] and Moori [47]. In this section we deal with group representations and group characters.

Definition 2.4.1 *Let $\mathbb{F}_q$ be the Galois field of q elements and $n \in \mathbb{N}$. The* **general linear group** *is the group of all invertible $n \times n$ matrices whose entries are in $\mathbb{F}_q$. This group is denoted by $\mathrm{GL}(n, q)$ and*

$$|\mathrm{GL}(n, q)| = \prod_{i=1}^{n} (q^n - q^{i-1}).$$

Definition 2.4.2 *A* **representation** *of G over a field $\mathbb{F}$ is a homomorphism*

$$\rho : G \to \mathrm{GL}(n, \mathbb{F})$$

for $n \in \mathbb{N}$. In this setup n is referred to as the degree of ρ. In the case $\mathbb{F} = \mathbb{C}$, we say ρ is an ordinary representation of G. Two representations $\rho : G \to \mathrm{GL}(n, \mathbb{F})$ and $\beta : G \to \mathrm{GL}(m, \mathbb{F})$ are said to be equivalent if $n = m$ and $\exists\ T \in \mathrm{GL}(n, \mathbb{F})$ such that $\beta(g) = T^{-1}(\rho(g))\ T$.

Definition 2.4.3 *Let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ for $n \in \mathbb{N}$ be a representation. The function $\chi : G \to \mathbb{F}$ defined by $\chi(g) = \mathrm{Trace}(\rho(g))$ induced by ρ is the* **character** *of G afforded by the representation ρ. The degree of χ is the same as the degree of ρ.*

A representation ρ of G is said to be reducible if it is equivalent to a representation α which is given by

$$\alpha(g) = \begin{pmatrix} \beta(g) & \gamma(g) \\ 0 & \delta(g) \end{pmatrix},$$

for all $g \in G$ and where β, γ and δ are representations of G. If ρ is not reducible then it is called an irreducible representation. The character afforded by an irreducible representation is called an irreducible character. Similar matrices have the same trace.

Theorem 2.4.4 Schur's Lemma *Let $\rho_1 : G \to \mathrm{GL}(n, \mathbb{F})$ and $\rho_2 : G \to \mathrm{GL}(m, \mathbb{F})$ be two irreducible representations of a group G over a field $\mathbb{F}$. Suppose that there exists a matrix P such that $P\rho_1(g) = \rho_2(g)P\ \forall\ g \in G$. Then either P is the zero matrix or P is invertible so that $\rho_1(g) = P^{-1}\rho_2(g)P$.*

Proof See [47]. □

Corollary 2.4.5 *If $\rho : G \to GL(n, \mathbb{F})$ is an irreducible representation of a group G over an algebraically closed field $\mathbb{F}$, then the only matrices which commute with all matrices $\rho(g)$ are scalar matrices αI_n, where $g \in G$, $\alpha \in \mathbb{F}$ and I_n is the $n \times n$ identity matrix,*

Proof See [47]. □

Theorem 2.4.6 Mascheke's theorem *Let G be a finite group. Let ϕ be a representation of G over the field $\mathbb{F}$ such that the characteristic $Char(\mathbb{F})$ is either 0 or is a prime p which does not divide the order of $|G|$. Then the representation ϕ can be expressed as a sum of irreducible representations of G.*

Proof See [47]. □

Definition 2.4.7 *A function $f : G \to \mathbb{F}$ is called a* **class function** *if it is constant on the conjugacy classes of G. This means $f(gxg^{-1}) = f(x)$ for all $g \in G$.*

Any character of a group G is a class function. In this work we will consider ordinary representations and ordinary characters of a finite group G, that is representations and characters over the complex field $\mathbb{C}$. Every class function ϕ of G can be uniquely expressed in the form $\phi = \sum_{\chi \in Irr(G)} b_\chi \chi$, where $b_\chi \in \mathbb{C}$.

Remark 2.4.8 *The set of all ordinary irreducible characters of G is denoted by Irr(G).*

Remark 2.4.9 *The characters of a finite group G have the following properties:*

(i) A character of G is constant on the conjugacy classes of G.

(ii) Equivalent representations afford the same character.

(iii) $\chi(1)$ is the degree of χ for all χ.

(iv) The sum of two characters of G is also a character of G.

(v) Any character of G can be written as a sum of irreducible characters.

(vi) The number of irreducible characters of G is equal to the number of conjugacy classes of G.

If $\chi(1) = 1$ then we say that χ is a linear character.

Definition 2.4.10 *The* **inner product** *of two class functions ψ_1 and ψ_2 is given by*

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_1(g) \overline{\psi_2(g)},$$

where $\overline{\psi_2(g)}$ is a complex conjugate of $\psi_2(g)$

Corollary 2.4.11 *The set $Irr(G)$ forms a basis of the class functions of G. Let ψ be a class function and $\chi_i \in Irr(G)$. Then*

$$\psi = \sum_{i=1}^{k} n_i \chi_i \tag{2.3}$$

where $n_i = \langle \psi, \chi_i \rangle$ and $1 \leq i \leq k$.

Proof See [39]. □

Remark 2.4.12 *We remark that the class function ψ in Equation 2.3 is a character if and only if $n_i > 0$ for every $1 \leq i \leq k$.*

We define addition and multiplication, respectively, of class functions ψ_1 and ψ_2 by

$$(\psi_1 + \psi_2)(g) = \psi_1(g) + \psi_2(g) \ \forall \ g \in G$$

and

$$\psi_1\psi_2(g) = \psi_1(g)\psi_2(g) \ \forall \ g \in G.$$

We note that $\psi_1 + \psi_2$ and $\psi_1\psi_2$ are also class functions. If $\alpha \in \mathbb{C}$ and ψ is a class function then $\alpha\psi$ is again a class function. The set of all class functions forms an **algebra**. This set is denoted by $\zeta(G)$. The set of all characters of G is a **subalgebra** of $\zeta(G)$.

Definition 2.4.13 *Let G be a group, χ be a character of G and $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$ such that $\chi = \sum_{i=1}^{r} n_i\chi_i,\ n_i \in \mathbb{Z}^+ \cup \{0\}$. Then the χ_i,s for which $n_i \neq 0$ are called the* **irreducible constituents** *of χ. In general, if ψ is a character of G such that $\chi - \psi$ is a character or zero then ψ is a constituent of χ.*

Remark 2.4.14 *The inner product in Corollary 2.4.11 is used to compute the coefficients n_i in Definition 2.4.13 and thus determine the constituents of a character of a group G.*

Lemma 2.4.15 *Let χ be a character of a group G. Then $\chi \in Irr(G)$ if and only if $\langle \chi, \chi \rangle = 1$.*

Proof Let G be a group, $\chi_i \in Irr(G)$ and $\chi = \sum_{i=1}^{k} n_i\chi_i$. If $\chi \in Irr(G)$ then $\langle \chi, \chi \rangle = 1$. Conversely, assume that $1 = \langle \chi, \chi \rangle = \sum_{i=1}^{k} n_i^2$. Since $n_i \in \mathbb{N} \cup \{0\}$ then it follows that one of the n_i^s must be one and the rest equal to zero. Thus χ must be irreducible. □

2.5 Character tables

Due to Remark 2.4.9, the $Irr(G)$ are presented in a square matrix referred to as the character table. In this section we focus on the orthogonality relations of the characters and the character table of a group G.

Definition 2.5.1 *Let G be a group, $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_m\}$ be a set of irreducible characters of G. Let $\{g_1, g_2, \cdots, g_m\}$ be the class representatives of the conjugacy classes of G. Let $i, j \in \{1, 2, \cdots, m\}$. Then the* **character table** *of G is an $m \times m$ matrix whose entries are the values $\chi_i(g_j)$. The rows of this table are indexed by χ_i and the columns by g_j.*

The class representative g_1 is always the identity 1_G of G and the character χ_1 is always the trivial character. The character table is an invertible matrix. The character table contains very important details about a group G. This table can be used for instance to

(a) decide the commutativity of G,

(b) determine the normal subgroups, the centre and the commutator subgroups of G,

(c) determine the simplicity of G,

(d) determine the degrees of all the representations of G,

(e) determine the conjugacy class sizes of G.

Proposition 2.5.2 *Some properties of characters.*

(a) $\chi(1_G) \mid |G| \ \forall\ g \in G$,

(b) $\sum_{i=1}^{|Irr(G)|} (\chi_i(1_G))^2 = |G|$,

(c) If $\chi \in Irr(G)$ *then* $\overline{\chi} \in Irr(G)$, *where* $\overline{\chi}(g) = \overline{\chi(g)}\ \forall\ g \in G$,

(d) $\chi(g^{-1}) = \overline{\chi(g)}\ \forall\ g \in G$. *In particular if* $g^{-1} \in [g]$ *then* $\chi(g) \in \mathbb{R}\ \forall\ \chi$.

Proof See [39] and [47]. □

In the next theorem we state the row and column relations of a character table of a group G.

Theorem 2.5.3 *Let $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_m\}$ and $\{g_1, g_2, \cdots, g_m\}$ be a collection of class representatives of the conjugacy classes of G. Let $C_G(g_k)$ be the centralizer of g_k in G for $1 \leq i \leq m$. Then for any $r, s \in \{1, 2, \cdots, m\}$ we have:*

1. **The row orthogonality relations**

$$\sum_{i=1}^{m} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \langle \chi_r, \chi_s \rangle = \delta_{rs}$$

2. **The column orthogonality relations**

$$\sum_{i=1}^{m} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs}|C_G(g_r)|$$

Proof See [8] and [39]. □

2.6 Permutation characters

In this section we deal with the concept of a permutation character. This character plays an important role when computing the conjugacy classes of group extensions, in particular, of an elementary abelian group by a finite group.

If there exists a homomorphism $\phi : G \to S_X$ then we say that G acts on the set X. We assume in this section that X is a finite set and $|X| = n$. If ϕ is a monomorphism then we say that the action is faithful. Then G can be identified with a subgroup of S_X and G is referred to as the permutation group on X.

Definition 2.6.1 *Let $x \in X$ and G be a group acting on the set X. The set $X^G = \{x^g \mid g \in G\}$ is the* **orbit** *of G which contains x.*

Definition 2.6.2 *Let $x \in X$ and G be a group acting on the set X. The set $G_x = \{g \in G \mid x^g = x\}$ is the* **stabilizer** *of x in G.*

Theorem 2.6.3 *Let X be a finite set and G be a group acting on the set X.*

(a) $G_x \leq G$,

(b) $|X^G| = |G{:}G_x|$.

Proof See [47]. □

Definition 2.6.4 *Let G be a group acting on a set X such that for any two k-tuples $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$, the x_i and y_i are pairwise distinct, and there exists $g \in G$ such that*

$$(x_1^g, x_2^g, \cdots, x_k^g) = (y_1, y_2, \cdots, y_k),$$

*then we say that G is k-**transitive** on X.*

If G is 1-transitive on a set X then we say that the action of G on X is transitive. That is, G has only one orbit on the set X.

Definition 2.6.5 *Consider an action of G on a finite set $X = \{x_1, x_2, \cdots, x_n\}$ and the representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$. Suppose for each $g \in G$ the $n \times n$ matrix $\rho_g = (a_{ij})$ has entries*

$$a_{ij} = \begin{cases} 1 & \text{if} \quad x_i^g = x_j \\ 0 & \text{if} \quad x_i^g \neq x_j. \end{cases}$$

*Then the matrix ρ_g is called the **permutation matrix** of the action of g. The representation ρ defined by $\rho(g) = \rho_g$ is the **permutation representation** of G corresponding to the action of G on X.*

The values of the induced character of G from χ, χ^G, can be computed by using Theorem 2.6.6 below.

Theorem 2.6.6 *Let $H \leq G$, χ be a character of H, $[g]_G$ be a conjugacy class of G and $\{x_1, x_2, \cdots, x_r\}$ be a set of representatives of the conjugacy classes of H that fuse to $[g]_G$. Then*

$$\chi^G(g) = \begin{cases} |C_G(g)| \sum_{i=1}^{r} \frac{\chi(x_i)}{|C_K(x_i)|} & \text{if} \quad H \cap [g]_G \neq \emptyset \\ 0 & \text{if} \quad H \cap [g]_G = \emptyset \end{cases}$$

Proof See [46]. □

Definition 2.6.7 *Let G be a group acting on a set X. The character afforded by the permutation representation ρ is called the **permutation character** of G associated with the action of G on X. This character is denoted by $\chi(G|X)$. Then we have that*

$$\chi(G|X) = |\{x \in X \mid x^g = x\}|,$$

which is the number of points of X fixed by g.

The degree of this permutation character is $|X|$. Suppose that G acts transitively on X and that G_x is a stabilizer of $x \in X$. Then the action of G on X is the same as the action of G on the cosets of $H = G_x$. This then means that for every $g \in G$, $\chi(G|H)(g)$ yields the number of cosets of H which are fixed by g. Since for all $g \in G$, the number $\chi(G|X)$ is the same as the number of cosets of $H = G_x$ which are fixed by g then $\chi(G|H) = \chi(G|X)$.

Theorem 2.6.8 *Let G be a group acting transitively on a set X. Let $x \in X$, $H = G_x$ and $\chi(G|H)$ be the permutation character of this action. Then*

$$\chi(G|H) = (1_H)^G.$$

Proof See [37] and [49]. □

The above theorem implies that the permutation character is in fact the trivial character 1_H of H induced to G. Let $\chi(G|H) = \sum \lambda_i \chi_i$ be a permutation character of G, where $\lambda_i \in \mathbb{N} \cup \{0\}$ and $\chi_i \in Irr(G)$. If $\lambda_i \in \{0, 1\}$ then $\chi(G|H)$ is said to be **multiplicity-free**.

Corollary 2.6.9 *If G acts on X and has k orbits on X with the permutation character χ, then $\langle \chi, 1_G \rangle = k$.*

Proof See [37]. □

Corollary 2.6.10 *Let G act transitively on X with the permutation character χ. Suppose that $x \in X$. If G_x has exactly r orbits on X then $\langle \chi, \chi \rangle = r$.*

Proof See [37]. □

Lemma 2.6.11 *Let $x \in X$. If G acts transitively on X then all subgroups G_x of G are conjugate in G.*

Proof See [63]. □

Theorem 2.6.12 below will be very useful for the determination of conjugacy class fusions of subgroups of G.

Theorem 2.6.12 *Let $H \leq G$ with $\chi = (1_H)^G$. Let $g \in G$ and let $x_1, x_2, \cdots, x_m$ be the representatives of the conjugacy classes of H that fuse to $[g]$. Then*

$$\chi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_H(x_i)|}.$$

If $H \cap [g] = \emptyset$ then $\chi(g) = 0$.

Proof Follows from Theorem 2.6.6. □

Below we list a number of necessary conditions that $(1_H)^G$ must satisfy since it is a transitive permutation character.

Theorem 2.6.13 *Let $H \leq G$ and $\chi = (1_H)^G$. Then*

(i) $\chi(1)$ *divides the order of G,*

(ii) $\langle \chi, \psi \rangle \leq \psi(1) \ \forall \ \psi \in Irr(G)$,

(iii) $\langle \chi, 1_G \rangle = 1$,

(iv) $\chi(g) \in \{0\} \cup \mathbb{Z}^+ \ \forall\ g \in G$,

(v) $\chi(g) \leq \chi(g^m)\ \forall\ g \in G$ and $m \in \{0\} \cup \mathbb{Z}^+$,

(vi) $\chi(g) = 0$ *if the order of* g *does not divide* $\frac{|G|}{\chi(1)}$,

(vii) $\chi(g)\frac{|[g]|}{\chi(1)}$ *is an integer for all* $g \in G$.

Proof See [37]. □

Definition 2.6.14 *Let* X *be a set and* $x \in X$*. The number of orbits of a point stabilizer* G_x *is called the* **rank** *of a transitive permutation group.*

In Corollary 2.6.10 we have that when G acts transitively on X with permutation character χ and has rank r then $\langle \chi, \chi \rangle = r$. For a doubly transitive (2-transitive) group G we have that G_x is transitive on $X - \{x\}$ and $\langle \chi, \chi \rangle = 2$.

Corollary 2.6.15 *Let* G *act on* X *with permutation character* χ*. Then the action is 2-transitive if and only if* $\chi = 1_G + \psi$ *where* $\psi \in Irr(G)$ *and* $\psi \neq 1_G$*.*

Proof Using the fact that $\langle \chi, \chi \rangle = 2$ and $\chi = 1_G + \sum \lambda_i \chi_i$, $i \neq 1$, $\chi_i \in Irr(G)$, we get $2 = 1 + \sum \lambda_i^2$. □

Remark 2.6.16 *[61] [6] Let* G *be a transitive permutation group acting on a finite set* X *and let* G_x *be the stabilizer of a point* $x \in X$*. Then*

(a) *G is said to be* **primitive** *on* X *if and only if* G_x *is maximal in* G*,*

(b) *G is said to be* **primitive rank 3** *on* X *if* G_x *is maximal in* G *and* G_x *has exactly three orbits on* X*.*

Remark 2.6.17 *[61] Let* χ *be a permutation character of* G *on* X*. If* G *is primitive rank 3 then* χ *has a decomposition*

$$\chi = 1 + \chi_s + \chi_t$$

where 1 *is a trivial character and* $\chi_s, \chi_t \in Irr(G)$*.*

3

Theory of Clifford-Fischer matrices

In [23] Fischer presents an interesting method of constructing the character tables of group extensions $\overline{G} = N.G$. His method entails using Clifford's Theory, the characters of inertia factor groups and Clifford-Fischer matrices, commonly known as Fischer matrices. A necessary condition for this method is that the irreducible characters of N must be extendable to the inertia groups. We discuss in Section 3.1 Clifford's Theorem 3.1.3 and some extension theorems, including Mackey's Theorem 3.1.12. At the end of this section we have Gallagher's Theorem 3.1.19 that utilizes Clifford's Theory and the extension theorems to describe the characters of $\overline{G}$. It turns out that the characters of $\overline{G}$ are arranged in blocks of rows that correspond to the inertia factor groups. In Section 3.2 we define the notion of a Fischer matrix. As mentioned earlier these matrices are vital in the construction of a character table of a group extension. For each class representative, say g, of a conjugacy class $[g]$ of G, we construct a Fischer matrix $M(g)$. Then in Section 3.3 we discuss the properties and the orthogonality relations of Fischer matrices. These properties and orthogonality relations are then used to compute the entries of a Fischer matrix. In Section 3.4 we briefly discuss a method to construct the Fischer matrices of a group extension where N is a normal extra special 2-subgroup of $\overline{G}$. We then discuss the application of the Clifford-Fischer Theory in Section 3.5 to construct the character table of $\overline{G}$. The technique of using the Clifford-Fischer Theory to construct character tables of group extensions is used, among others, by [1], [2], [4], [8], [15], [19], [23], [35], [43], [48], [49], [52], [53], [58], [63] and [65].

3.1 The Clifford Theory

Definition 3.1.1 *Let $\overline{G} = N.G$ and $\theta \in \mathrm{Irr}(N)$, where $N \trianglelefteq \overline{G}$ with N not necessarily abelian. The function $\theta^{\overline{g}}$ defined by $\theta^{\overline{g}}(n) = \theta(\overline{g}n\overline{g}^{-1})$, $g \in \overline{G}$ and $n \in N$, is a character of N. This character is said to be a* **conjugate character** *of θ.*

Remark 3.1.2 *We note from Definition 3.1.1 that $\theta^{\overline{g}} \in Irr(N)$ if and only if $\theta \in Irr(N)$. We also note that $\overline{G}$ acts on $Irr(N)$ by conjugation and since N acts trivially on $Irr(N)$ then $Irr(N)$ is permuted by $\overline{G}/N$, by $N\overline{g} : \theta \mapsto \theta^{\overline{g}}$.*

Theorem 3.1.3 *[39]* **Clifford's Theorem** *Let $N \trianglelefteq \overline{G}$ and $\chi \in \mathrm{Irr}(\overline{G})$. Let θ be an irreducible constituent of $\chi{\downarrow_N}$ and suppose that $\theta = \theta_1, \theta_2, \cdots, \theta_t$ are distinct conjugates of θ in $\overline{G}$. Then $\chi{\downarrow_N} = e\sum_{i=1}^{t}\theta_i$, where $e =< \chi{\downarrow_N}, \theta >$.*

Proof See [39] and [63]. □

Clifford's Theorem asserts that for $N \trianglelefteq \overline{G}$, $\chi \in Irr(\overline{G})$ and $\theta \in Irr(N)$ an irreducible constituent of $\chi{\downarrow_N}$, then every $\overline{G}$-conjugate of θ will also be an irreducible constituent of $\chi{\downarrow_N}$.

Theorem 3.1.4 *Let G be a group and $K, H \leq G$ such that $K \leq H \leq G$ and χ be a character of K. Then for all $g \in G$ we have*

(i) $(\chi^H)^g = (\chi^g)^{g^{-1}Hg}$,

(ii) $(\chi^g)^G = \chi^G$.

Proof See [40]. □

Theorem 3.1.5 *[29]* **Brauer's Theorem** *Let G be a group and K be a group of automorphisms of G. Then the number of orbits of K as a group of permutations on the $Irr(G)$ is the same as the number of orbits of K as a group of permutations on the conjugacy classes of G.*

Proof See [29] and [49]. □

Remark 3.1.6 *(a) From Brauer's Theorem we have that the action of $\overline{G}$ on the conjugacy classes of N and on the $Irr(N)$ yields the same number of orbits, however the orbit lengths need not be the same.*

(b) If N is non-abelian, then

$$\sum_{k=1}^{t}\left|\theta_k^{\overline{G}}\right| = |Irr(N)| \neq |N| = \sum_{k=1}^{t}\left|[n_k]_N^{\overline{G}}\right|,$$

where

(i) t is the number of orbits of the action of $\overline{G}$ on $Irr(N)$ or on the conjugacy classes of N,

(ii) n_k is a class representative of a conjugacy class of N and θ_k is a representative character,

(iii) $\theta_k^{\overline{G}}$ and $[n_k]_N^{\overline{G}}$ are, respectively, orbits of N containing θ_k and n_k, for the action of $\overline{G}$ on $Irr(N)$ and on the conjugacy classes of N, respectively.

Definition 3.1.7 *Let G be a group and $K \leq G$. Then for a character χ of K, we define*

$$I_G(\chi) = \{g \in N_G(K) \mid \chi^g = \chi\},$$

where χ^g is defined similarly to Definition 3.1.1. We call $I_G(\chi)$ the **inertia group** *of χ in G. If $K \trianglelefteq G$, then*

$$I_G(\chi) = \{g \in G \mid \chi^g = \chi\}.$$

Remark 3.1.8 *Let* $N \trianglelefteq \overline{G}$, $\theta \in \mathrm{Irr}(N)$ *and* $\overline{H} = I_{\overline{G}}(\theta) = \{g \in \overline{G} \mid \theta^g = \theta\}$ *be the inertia group of* θ *in* $\overline{G}$*. It is clear that* $\overline{H}$ *is the stabilizer of* θ *in the action of* $\overline{G}$ *on* $Irr(N)$*. It can be shown that* $N \trianglelefteq \overline{H} \leq \overline{G}$*. We note as well that* $\left[\overline{G}{:}\overline{H}\right]$ *is orbit length of the orbit containing* θ*. Thus in the formula* $\chi\downarrow_N = e\sum_{i=1}^{t}\theta_i$*, in Clifford's Theorem 3.1.3,* $t = \left[\overline{G}{:}\overline{H}\right]$*. The quotient group* $H \cong \overline{H}/N$ *is called the* **inertia factor group***. The group* H *can be regarded as the inertia group of* θ *in the factor group* $\overline{G}/N \cong G$*. That is* $H = \{g \in G \mid \theta^g = \theta\} = I_G(\theta)$*.*

Remark 3.1.9 *If* $\overline{G} = N{:}G$ *and* $\overline{H} = I_{\overline{G}}(\theta)$*, then* $\overline{H} = N{:}H$*.*

Next, we define the notion of an **extendable character**. This notion plays a vital role in the construction of character tables of finite groups using the Clifford Theory. We also discuss the conditions for a character to be extendable. More theory on the extension of characters can be found in the following, Curtis and Reiner [17], Gagola [25], Gallagher [26], Huppert [33], Isaacs [37] and Karpilovsky [41], among other relevant items.

Definition 3.1.10 *Let* $N \leq \overline{G}$, $\theta \in Irr(N)$ *and* $\chi \in Irr(\overline{G})$ *such that* $\chi\downarrow_N = \theta$*. Then* θ *is said to be* **extendable** *to an irreducible character of* $\overline{G}$*.*

Theorem 3.1.11 *[40] Let* $\overline{G} = N.G$*, where* $N \trianglelefteq \overline{G}$ *and* $G \leq \overline{G}$ *such that* $N \cap G \subseteq N'$*. If* θ *is an irreducible* $\overline{G}$*-invariant character of* N *such that* $(\deg(\theta), |G|) = 1$*, then* θ *can be extended to* $\overline{G}$*.*

Proof See [40]. □

Theorem 3.1.12 *[17]* **Mackey's Theorem** *Let* $N \trianglelefteq \overline{G}$ *and* θ *be an irreducible* $\overline{G}$*-invariant character of* N*. If* N *is abelian and* $\overline{G}$ *splits over* N*, then* θ *can be extended to* $\overline{G}$*.*

Proof See [17]. □

Remark 3.1.13 *An alternative proof of Mackey's Theorem is by applying Theorem 3.1.11. Suppose that* $\overline{G} = N{:}G$ *and* $\theta \in Irr(N)$*. If* N *is abelian, then* $N' = \{1\}$ *and* $deg(\theta) = 1$*. Since* $\overline{G}$ *is a split extension then* $N \cap G = \{1\}$*. Thus* $N \cap G \subseteq N'$ *and* $(\deg(\theta), |\overline{G}|) = 1$*. Therefore* θ *is extendable to* $\overline{G}$*.*

Theorem 3.1.14 *Let* $N \trianglelefteq \overline{G}$ *and* $\theta \in Irr(N)$ *which is invariant in* $\overline{G}$*. If* $\left(\left[\overline{G}{:}N\right], \frac{|N|}{\deg(\theta)}\right) = 1$*, then* θ *is extendable to a character of* $\overline{G}$*.*

Proof See [25]. □

Theorem 3.1.15 *Suppose* $\overline{G} = N{:}G$*. If* $\theta \in Irr(N)$ *and* $deg(\theta) = 1$*, then* θ *can be extended to the inertia group* $I_{\overline{G}}(\theta)$ *of* θ *in* $\overline{G}$*.*

Proof [49] Suppose $\overline{G} = N{:}G$, $\theta \in Irr(N)$ and $\deg(\theta) = 1$. Let $\overline{H} = I_{\overline{G}}(\theta)$. Then by Remark 3.1.9, $\overline{H} = N{:}H$, where $H = I_G(\theta)$. Since $\overline{H}$ is a split extension, then $N \cap H = \{1\} \subseteq N'$. We also note that $(\deg(\theta), |H|) = 1$ and that θ is $\overline{H}$-invariant. Therefore θ can be extended to $\overline{H}$ by Theorem 3.1.11. □

Remark 3.1.16 *The above theorem reinforces Mackey's Theorem since for N abelian, all $\theta \in Irr(N)$ are linear and hence extendable to their inertia groups.*

Suppose once more that $\overline{G} = N.G$, $N \trianglelefteq \overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. The latter part of this section is devoted to the description of the $Irr(\overline{G})$ using Clifford's Theorem and the notion of the extension of characters. Theorem 3.1.18 shows that to obtain the $Irr(\overline{G})$ that contain θ in their restriction to N, it is sufficient to find the $Irr(\overline{H})$ that contain θ in their restriction. If θ can be extended to $Irr(\overline{H})$, then the relevant characters of $\overline{H}$ can be determined by using Gallagher's Theorem, Theorem 3.1.19. But first we state the following result from Isaacs [37].

Theorem 3.1.17 *[37] Let $N \trianglelefteq \overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. Suppose*

$$A = \{\psi \in Irr(\overline{H}) \mid \langle \psi\downarrow_N, \theta\rangle \neq 0\}$$

and

$$B = \{\chi \in Irr(\overline{G} \mid \langle \chi\downarrow_N, \theta\rangle \neq 0\}.$$

Then

(a) If $\psi \in A$, then $\psi^{\overline{G}} \in Irr(\overline{G})$.

(b) The function $\psi \mapsto \psi^{\overline{G}}$ is a bijection of A to B.

(c) If $\psi^{\overline{G}} = \chi$ and $\psi \in A$, then ψ is the unique irreducible constituent of $\chi\downarrow_{\overline{H}}$ which sits in A.

(d) If $\psi^{\overline{G}} = \chi$ and $\psi \in A$, then $\langle \psi\downarrow_N, \theta\rangle = \langle \chi\downarrow_N, \theta\rangle$.

Proof See [37]. □

Theorem 3.1.18 *Let $\overline{G} = N.G$, $N \trianglelefteq \overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. Then induction to $\overline{G}$ maps the $Irr(\overline{H})$ that contain θ in their restriction to N faithfully onto the $Irr(\overline{G})$ which contain θ in their restriction to N.*

Proof See [63]. □

Theorem 3.1.19 *[26]* **Gallagher's Theorem** *Let $N \trianglelefteq \overline{G}$, $\theta \in Irr(\theta)$ and $\overline{H} = I_{\overline{G}}(\theta)$. If θ can be extended to $\psi \in Irr(\overline{H})$, then as β ranges over all $Irr(\overline{H})$ which contain N in their kernels, $\beta\psi$ ranges over all the $Irr(\overline{H})$ which contain θ in their restriction to N.*

Proof See [26]. □

Remark 3.1.20 *If $\overline{G} = N.G$ and every $Irr(N)$ can be extended to its inertia group in $\overline{G}$, then the application of Theorem 3.1.18 and Gallagher's Theorem gives the characters of $\overline{G}$ in the following manner. Let $\theta_1, \theta_2 \cdots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$. For each i, let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and $\psi_i \in Irr(\overline{H}_i)$ with $\psi_i\downarrow_N = \theta_i$. By Clifford's Theorem, each $Irr(\overline{G})$ contains some θ_i in its restriction. Then by Gallagher's Theorem and Theorem 3.1.18 we have*

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \Big\{ (\beta\psi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i) \text{ and } N \subset \mathrm{Ker}(\beta) \Big\}.$$

This then means that the characters of $\overline{G}$ are arranged into blocks, with each block corresponding to an inertia group.

3.2 Fischer matrices

In Remark 3.1.20, let $\theta_1 = 1_N$ so that $\overline{H}_1 = \overline{G}$ and $H_1 = G$. Consider a conjugacy class $[g]$ of G with g a class representative. Let $\overline{g} \in \overline{G}$ be a lifting of g under the natural homomorphism $\overline{G} \to G$. Let $c(g)$ be the number of the conjugacy classes of elements of $\overline{G}$ which correspond to the coset $N\overline{g}$. Set $X(g) = \{x_1, x_2, \cdots, x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy classes of elements of the coset $N\overline{g}$ such that the images under the natural homomorphism $\overline{G} \to G$ are in $[g]$. Put $x_1 = \overline{g}$. Set $y_1, y_2, \cdots, y_r$ to be the representatives of the conjugacy classes of elements of H_i which fuse to $[g]$ in G. Let $y_{l_k} \in \overline{H}_i$ such that y_{l_k} ranges over all the representatives of the conjugacy classes of elements $\overline{H}_i$ which map y_k under the homomorphism $\overline{H}_i \to H_i$ whose kernel is N. Let $\beta \in Irr(\overline{H}_i)$ such that $N \subseteq Ker(\beta)$. This means β is a lifting of $\hat{\beta} \in Irr(H_i)$ such that $\beta(y_{l_k}) = \hat{\beta}(y_k)$ for any lifting $y_{l_k} \in \overline{H}_i$ of $y_k \in H_i$. Using the formula for induced characters, Theorem 2.6.6, we have

$$(\beta\psi_i)^{\overline{G}}(x_j) = \sum_{1\leq k\leq r} \sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i\beta(y_{l_k})$$

$$= \sum_{1\leq k\leq r} \sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k})\beta(y_{l_k})$$

$$= \sum_{1\leq k\leq r} \sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k})\overline{\beta}(y_{l_k})$$

where $\sum_l{}'$ is the sum over all l for which y_{l_k} is conjugate to x_j in $\overline{G}$.

Definition 3.2.1 *Let us define a matrix $M_i(g) = (a_{uv}),\ 1 \leq u \leq r$ and $1 \leq v \leq c(g)$, where*

$$a_{uv} = \sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k}).$$

Then we have

$$(\beta\psi_i)^{\overline{G}}(x_j) = \sum_{1\leq k\leq r} a_{uv}\overline{\beta}(y_k).$$

Doing this for all $1 \leq i \leq t$ such that H_i contains an element in $[g]$ to obtain

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix},$$

where $M_i(g)$ is the submatrix corresponding to the inertia group $\overline{H}_i$ and its inertia factor group H_i. Now if $H_i \cap [g] = \emptyset$, then $M_i(g)$ will not exist and $M(g)$ will not have $M_i(g)$. The size of the matrix $M(g)$ is $p \times c(g)$, where p is the number of conjugacy classes of elements of the inertia *factor groups H_i for $1 \leq i \leq t$ which fuse into $[g] \in G$, and $c(g)$ is the number of conjugacy classes of elements of G which correspond to the coset Ng. The matrix $M(g)$ is referred to as a* **Clifford-Fischer** *matrix of $\overline{G}$ corresponding to the coset $N\overline{g}$.*

Most often Clifford-Fischer matrices are simply referred to as Fischer matrices. Let $R(g) = \{(i, y_k) \mid 1 \leq i \leq t,\ H_i \cap [g] \neq \emptyset, 1 \leq k \leq r\}$, where y_k runs over the representatives of the conjugacy classes of elements of H_i which fuse into $[g]$ in G. Now let's write $M(g)$ as $M(g) = (a_j^{(i,y_k)})$, where

$$a^j_{(i,y_k)} = \sum_l{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k})$$

with columns indexed by $X(g)$ and rows by $R(g)$.

3.3 Properties of Fischer matrices

Lemma 3.3.1 *Let $A = \mathrm{Aut}(K)$. Then A acts on $Irr(K)$ and on the conjugacy classes of K with the same number of orbits on each by Brauer's Theorem 3.1.5. Suppose we have the following matrix describing these actions:*

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 = l_1 & l_2 & l_3 & \cdots & l_j & \cdots & l_t \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2t} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{it} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & a_{t3} & \cdots & a_{tj} & \cdots & a_{tt} \end{pmatrix} \end{array},$$

where
$a_{1j} = 1$ for $j \in \{1, 2, \cdots, t\}$,
l_j 's are lengths of orbits of A on the conjugacy classes of K,
s_j 's are lengths of orbits of A on the $Irr(K)$,
a_{ij} is the sum of s_i irreducible characters of K on the element x_j, where x_j is an element of the orbit of length l_j.
Then the following relation holds for $i, i' \in \{1, \cdots, t\}$:

$$\sum_{j=1}^{t} a_{ij}\ \overline{a_{i'j}}\ l_j = |K| s_i \delta_{ii'}.$$

Proof See [63]. □

Remark 3.3.2 *Let $x_j \in X(g)$, $m_j = \left[C_{\overline{g}}{:}C_{\overline{G}}(x_j)\right]$ and $M(g)$ be the Fischer matrix of $\overline{G}$. The m_j are referred to as weights. The columns of $M(g)$ are indexed by $X(g)$. At the top of the columns we have the centralizer orders $|C_{\overline{G}}(x_j)|$ and at the bottom we have the weights m_j for each $x_j \in X(g)$. The rows of $M(g)$ are indexed by $R(g)$. On the left of $M(g)$ we have $|C_{H_i}(y_k)|$ next to each row, where $[y_k]$ is the class of H_i that fuses into $[g] \in G$. If we let $M(g) = \left(a^j_{(i,y)}\right)$ be the Fischer matrix of $\overline{G} = N.G$ at $g \in G$, then the general form of $M(g)$ is presented as follows:*

$$
\begin{array}{r}
\\
|C_G(g)| \\
|C_{H_2}(y_1)| \\
|C_{H_2}(y_2)| \\
\vdots \\
|C_{H_3}(y_1)| \\
\vdots \\
|C_{H_i}(y_1)| \\
\vdots \\
\\
\end{array}
\begin{array}{ccccc}
|C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & |C_{\overline{G}}(x_3)| & \cdots & |C_{\overline{G}}(x_{c(g)})| \\
\hline
1 & 1 & 1 & \cdots & 1 \\
\hline
a^1_{(2,y_1)} & a^2_{(2,y_1)} & a^3_{(2,y_1)} & \cdots & a^{c(g)}_{(2,y_1)} \\
a^1_{(2,y_2)} & a^2_{(2,y_2)} & a^3_{(2,y_2)} & \cdots & a^{c(g)}_{(2,y_2)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\hline
a^1_{(3,y_1)} & a^2_{(3,y_1)} & a^3_{(3,y_1)} & \cdots & a^{c(g)}_{(3,y_1)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\hline
a^1_{(i,y_1)} & a^2_{(i,y_1)} & a^3_{(i,y_1)} & \cdots & a^{c(g)}_{(i,y_1)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\hline
m_1 & m_2 & m_3 & \cdots & m_{c(g)}
\end{array}
$$

The horizontal lines indicate the blocks corresponding to the inertia factor groups H_i. In the coset analysis, m_j is given by $m_j = \left[C_{\overline{g}}{:}C_{\overline{G}}(x_j)\right] = \frac{f|N|}{k}$.

Proposition 3.3.3

$$\sum_{(i,y)} |C_{H_i}(y)| a^i_{(i,y)} \, \overline{a^{j}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|.$$

Proof [63] The partial character table of $\overline{G}$ at classes $x_1, x_2, \cdots, x_{c(g)}$ is

$$\begin{bmatrix} T_1(g) \cdot M_1(g) \\ T_2(g) \cdot M_2(g) \\ \vdots \\ T_t(g) \cdot M_t(g) \end{bmatrix}.$$

where $M_i(g)$ is the submatrix corresponding to H_i and $T_i(g)$ is the partial character table of H_i consisting of columns corresponding to the classes of H_i that fuse into $[g]$ in G. By column orthogonality of the character table of $\overline{G}$, we have

$$
\begin{aligned}
|C_{\overline{G}}(x_j)|\delta_{jj'} &= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \left(\sum_{y:(i,y)\in R(g)} a^{j}_{(i,y)} \beta_i(y) \overline{\left(\sum_{y':(i,y')\in R(g)} a^{j'}_{(i,y')} \beta_i(y') \right)} \right) \\
&= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \left(\sum_{y} a^{j}_{(i,y)} \overline{a^{j'}_{(i,y')}} \beta_i(y) \overline{\beta_i(y)} + \sum_{y} \sum_{y' \neq y} a^{j}_{(i,y)} \overline{a^{j'}_{(i,y')}} \beta_i(y) \overline{\beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \left(\sum_{y} a^{j}_{(i,y)} \overline{a^{j'}_{(i,y')}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y) \overline{\beta_i(y)} + \sum_{y} \sum_{y' \neq y} a^{j}_{(i,y)} \overline{a^{j'}_{(i,y')}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y) \overline{\beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \left(\sum_{y} a^{j}_{(i,y)} \overline{a^{j'}_{(i,y')}} |C_{H_i}(y)| + 0 \right) \\
&= \sum_{(i,y)\in R(g)} a^{j}_{(i,y)} \overline{a^{j'}_{(i,y')}} |C_{H_i}(y)|.
\end{aligned}
$$

□

Theorem 3.3.4

$$a^{j}_{1,g} = 1 \ \forall\ j \in \{1, \cdots, c(g)\}.$$

Proof [49] If $y_{l_k} \sim x_j$, then $|C_{\overline{H}_1}(y_{l_k})| = |C_{\overline{G}}(x_j)|$. It then follows that

$$a^{j}_{(1,g)} = \sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_1}(y_{l_k})|} \psi_1(y_{l_k}) = 1.$$

□

Proposition 3.3.5 *[43] At the identity of G, the matrix $M(1_G)$ is the matrix with rows equal to orbit sums of the action of $\overline{G}$ on $Irr(N)$ with duplicate columns discarded. For this matrix we have $a^{j}_{(i,1_G)}[G{:}H_i]$, and an orthogonality relation for rows:*

$$\sum_{j=1}^{t} a^{j}_{(i,1_G)} a^{j}_{(i',1_G)} |C_{\overline{G}}(x_j)|^{-1} = \delta_{ii'} |C_{H_i}(1_G)|^{-1} = \delta_{ii'} |H_i|^{-1}.$$

Proof [63] The $(i, 1_G), j^{th}$ entry of $M(1_G)$ is

$$a^{j}_{(i,1)} = \sum_{k} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{lk})|} \psi_i(y_{lk})$$

where we sum over representatives of conjugacy classes of $\overline{H}_i$ that fuse to $[x_j]$ in $\overline{G}$. Therefore $a^{j}_{(i,1)} = \psi^{\overline{G}}(x_j)$. By Theorem 3.1.18, $\psi^{\overline{G}} \in Irr(\overline{G})$, and $\langle \psi_i^{\overline{G}}\downarrow_N, \theta_i \rangle = \langle \psi_i\downarrow_N, \theta_i \rangle = 1$. It then follows by Clifford's Theorem 3.1.3, that $\psi_i^{\overline{G}}\downarrow_N = \sum_\alpha \chi_\alpha$, where we sum over all $\chi_\alpha \in Irr(N)$ in the orbit containing θ_i. Then $x_j \in N$ and $a^{j}_{(i,1)} = \sum_\alpha \chi_\alpha(x_j)$. The orthogonality relation follows by Lemma 3.3.1. □

Due to the results we proved in this section and the work of Fischer in [23], we summarize some of the properties of Fischer matrices in the following remark.

Remark 3.3.6 *For a Fischer matrix $M(g) = (a^j_{(i,y)})$ of $\overline{G} = N.G$, the following relations apply:*

(i) $a^j_{(1,g)} = 1 \ \forall \ j \in \{1, \cdots, c(g)\}$,

(ii) $\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \ \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$,

(iii)
$$a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|},$$

(iv)
$$\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} a^j_{(i',y')} = \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|,$$

(v) $|X(g)| = |R(g)|$,

(vi) $M(g)$ *is square and singular,*

(vii) $|a^1_{i,y_k}| \geq |a^j_{i,y_k}|$.

Note that (iii) and (iv) apply if N is elementary abelian.

Remark 3.3.7 *[49]*

(i) Let χ be a character of a group G and $g \in G$. Then we have $|\chi(g)| \leq \chi(1_G)$, where 1_G is the identity element of G.

(ii) Let χ be a character of a group G and g a p-singular element of G, where p is a prime. Then we have $\chi(g) \equiv \chi(g^p) \bmod p$.

The rest of the theory we discuss on Fischer matrices is due to List [43]. This theory can be seen somehow as an alternative way of computing the entries of a Fischer matrix. It is used, among others, by Chileshe [15], Darafsheh and Iranmanesh [19], Iranmanesh [35], Mpono [49], Almestady [2], Almestady and Morris [4], Whitley [63] and Zimba [65].

Proposition 3.3.8 *Let $\overline{G} = N.G$ with N an elementary abelian p-group. Let $\overline{g} \in \overline{G}$. The map $l_{\overline{g}} : N \to N$ defined by $l_{\overline{g}}(n) = n\overline{g}n^{-1}\overline{g}^{-1}$ is an endomorphism of N.*

Proof See [15]. □

Remark 3.3.9 *List in [43] proves that if N is a vector space which is invariant under the action of $\overline{G}$, then $Im(l_{\overline{g}})$ and $Ker(l_{\overline{g}})$ are $C_{\overline{g}}$-modules. In Corollary 2.3.2 we proved that $C_{\overline{g}} = N{:}C_G(g)$ for $\overline{G} = N{:}G$.*

Remark 3.3.10 *Let $Im(l_{\overline{g}}) = M$. The quotient group N/M plays a vital role in the computation of the entries of a Fischer matrix. The group N acts on $N\overline{g}$ by conjugation and M acts on $N\overline{g}$ by left multiplication such that the resulting orbits of the two actions are the same. This then means the action of $C_{\overline{g}}$ on the orbits of N acting on $N\overline{g}$ is the same as the action of $C_{\overline{g}}$ on the module N/M. Therefore the orbits of the action of M on $N\overline{g}$ can be identified with the elements of N/M. Let $\theta_i \in Irr(N)$, $\psi_i \in \overline{H}_i$ and ψ_i be an extension of θ_i in $\overline{H}_i$. Then ψ_i is constant on the orbits of N acting on $N\overline{g}$. We define a class function μ on N/M by $\mu(Mn_j\overline{g}) = \psi_i(n_j\overline{g})$, where $n_j \in N$ and $n_j\overline{g} \in Q_j$ is a representative of the j-th orbit of N acting on $N\overline{g}$, and $n_1 = 1_N$. Then $\mu(M\overline{g}) = \psi_i(\overline{g})$. Let $\hat{\mu}$ be an extension of μ to the inertia group of μ in $C_{\overline{g}}$. Then induction of $\hat{\mu}$ to $\overline{G}$ evaluated on the elements of $N\overline{g}$ is equivalent to the induction of $\hat{\mu}$ to $C_{\overline{g}}/M$ evaluated on the elements of N/M. From [43] we have that if $\overline{G}$ is a split extension, then the Fischer matrix at a non-identity coset of N in $\overline{G}$ is the matrix of orbit sums of $C_{\overline{g}}$ acting on the rows of the character table of N/M with duplicate columns discarded. If $\overline{G}$ is a non-split extension, then it may happen that μ may not be a character of N/M. Then $\xi\mu$ will be a character of N/M, where ξ is an appropriate p-th root of unity. Therefore when $\overline{G}$ is a non-split extension, the Fischer matrix is the matrix of orbit sums of $C_{\overline{g}}$ acting on the rows of the character table of N/M with duplicate columns discarded and with each row multiplied by an appropriate p-th root of unity. It may happen that the p-th root of unity for each row is 1.*

Proposition 3.3.11 *Let N be an elementary abelian subgroup of $\overline{G} = N.G$ such that $\overline{G}/N \cong G$. Let $M = \{n\overline{g}n^{-1}\overline{g}^{-1} \mid n \in N\}$. Then $\left[N{:}M\right] = k$, where k is the number of elements of N fixed by a class representative $g \in G$.*

Proof [49] We have that the orbits $Q_1, Q_2, \cdots, Q_k$ of N acting on $N\overline{g}$ are the same as the orbits $D_1, D_2, \cdots, D_k$ of M acting on $N\overline{g}$ by left multiplication. Also the orbits $D_1, D_2, \cdots, D_k$ can be identified with the elements of N/M. Then it follows that $|N/M| = \left[N{:}M\right] = k$. □

Remark 3.3.12 *Let N be an elementary abelian p-group such that $|N| = p^n$. From the coset analysis of the extension $\overline{G} = N.G$, we have that $k = p^m$ for $0 \leq m \leq n$, where k is the number of elements of N fixed by a class representative $g \in G$. Suppose that $Q_1, Q_2, \cdots, Q_k$ are the orbits from the action of N on $N\overline{g}$ and that the action of $C_{\overline{g}}$ on the listed orbits results into $f_i = 1\ \forall\ i \in \{1, 2, \cdots, k\}$. This then means that the action of $C_{\overline{g}}$ on $Irr(N/M)$ is trivial. In this case the Fischer matrix $M(g)$ coincides with the character table of an abelian group N/M of order k. We also note that for $\overline{G} = N{:}G$, the set stabilizer $C_g = N{:}C_G(g)$ and the action of C_g on the character table of N/M is the same as that of $C_G(g)$.*

3.4 Fischer matrices where N is an extra special 2-group

Let $\overline{G} = N{:}G$, with N a normal non-abelian extra special 2-subgroup of $\overline{G}$. We adopt the notation in Sections 3.2 and 3.3.

Lemma 3.4.1 *For every $c(g_i) \times c(g_i)$ Fischer matrix $M(g_i)$, the sum of the $1^{st}, 3^{rd}, \cdots, c(g_i)^{th}$ rows equal (component wise) to the square of the modulus of the 2^{nd} row.*

Proof See [52]. □

Lemma 3.4.2 *For each Fischer matrix $M(g_i)$, we can order the $x_{i,j}$, $1 \leq j \leq c(g_i)$, so that the second row of $M(g_i)$ is of the form $[q_i, -q_i, 0, \cdots, 0]$ and we may choose $x_{i,2} = zx_{i,1}$, where z is the central involution in H_2. Furthermore,*

$$a^1_{(k,y_m)} = a^2_{(k,y_m)} = \frac{|C_G(g_i)|}{|C_{H_k}(y_m)|},$$

where $k \neq 2$, the indices i, k and m correspond to the number of conjugacy classes of G, the number of inertia factor groups of $\overline{G}$ and the number of classes of the inertia factor group H_k that fuse to $[g_i]$, respectively.

Proof The proof follows from the proof of Lemma 7 by Pahlings [52]. □

Remark 3.4.3 *We then compute the entries of $M(g_i)$ as follows:*

(a) The first row comes from Theorem 3.3.4.

(b) Lemma 3.4.1 and Lemma 3.4.2 provide the second row and the first two columns.

(c) In Note 10.3.2 in [8] by Basheer, we have that at this point the Fischer matrix $M(g_i)$ has $c(g_i)^2 - 4c(g_i) + 4$ unknowns. These unknowns are calculated using the orthogonality properties of Fischer matrices discussed in Section 3.3.

3.5 Character table of $\overline{G} = N{:}G$

At this stage we are able to construct the character table of a group extension using the Clifford-Fischer Theory. In this thesis we confine our discussion to character tables of split extensions. Character tables of non-split extensions are discussed by Ali [1], Basheer [8] and Pahlings [52], among others. The Clifford-Fischer Theory for character tables of split extensions $\overline{G} = N{:}G$ entails having first the following:

(a) conjugacy classes of $\overline{G}$,

(b) character tables of the inertia factor groups,

(c) fusion maps of the inertia factor groups into G,

(d) the Fischer matrices of $\overline{G}$.

We know that the rows of a Fischer matrix $M(g)$ can be divided into blocks with each block corresponding to the inertia factor group H_i. Let us denote the submatrix corresponding to H_i by $M_i(g)$, and the partial character table of H_i consisting of columns corresponding to the classes of H_i that fuse into $[g]$ in G by $T_i(G)$. The character table of $\overline{G}$ is then obtained by multiplying the relevant characters of H_i by rows of $M(g)$. Thus the partial character table on the classes $\{x_1, x_2, \cdots, x_{c(g)}\}$ is via matrix multiplication as follows:

$$\begin{bmatrix} T_1(g) \cdot M_1(g) \\ T_2(g) \cdot M_2(g) \\ \vdots \\ T_t(g) \cdot M_t(g) \end{bmatrix}.$$

This then means that a character table of $\overline{G}$ will also be divided into blocks corresponding to the inertia factor groups H_i. We note as well that

$$|Irr(\overline{G})| = \sum_i |Irr(H_i)|.$$

Examples of character tables of split extensions are found in Chapters 5 to 10. In the latter chapter our N is non-abelian.

4

Affine subgroups of classical linear groups

This chapter is dedicated to the description of the irreducible characters of affine subgroups of classical linear groups, namely the general linear group, the symplectic group, the orthogonal group and the unitary group. We use the Clifford-Fischer Theory discussed in Chapter 3 to determine the irreducible characters of these affine subgroups. Affine subgroups are subgroups that fix a non-zero vector of an underlying vector space. In each classical linear group, we prove that these affine subgroups are split extensions. We adopt the notation we used in Chapter 2 for split extensions. Therefore these affine subgroups are of the form $\overline{G} = N{:}G$, where $N \trianglelefteq \overline{G}$, $\overline{G}/N \cong G$ and G is one of the classical linear groups. We recall that the description of $Irr(\overline{G})$ via the Clifford-Fischer Theory entails having the conjugacy classes of $\overline{G}$, the character tables of the inertia factor groups, the fusion maps of these inertia factor groups into G and lastly the Fischer matrices of $\overline{G}$. Conjugacy classes of $\overline{G}$ are determined by utilizing the coset analysis technique discussed in Subsection 2.3.1. Therefore, one of the key considerations of this chapter is to analyse the actions of $\overline{G}$ on N and on $Irr(N)$, for each of the above-mentioned classical linear groups. The first action will yield the point stabilizers and the latter will yield the inertia factor groups. The structures of these point stabilizers and inertia factor groups are discussed. In Section 4.1 we deal with the affine subgroups of the general linear groups. In Section 4.2 we discuss affine subgroups of symplectic groups. Symplectic groups, including symplectic forms and symplectic spaces are discussed in detail by Mpono [49] and Rodrigues in [49] and [55], respectively, among others. We first discuss symplectic transvections and prove a number of results involving symplectic transvections relevant in this thesis. We give alternative proofs of the following already established theorems: that an inverse of a transvection is also a transvection, a conjugate of a transvection is also a transvection, and when two transvections are equal. We also prove some new results which are given in Proposition 4.2.10, Corollary 4.2.11, Proposition 4.2.14, Proposition 4.2.15, Proposition 4.2.17, Proposition 4.2.18 and Proposition 4.2.19. Readers may consult the following for further reading on symplectic groups, [14], [21], [49], [55], [62] and [64]. In this section we also determine the centre, $Z(\overline{G})$, of the affine subgroup $\overline{G}$ of $Sp(2n, q)$. We further prove some new results given in Proposition 4.2.37, Corollary 4.2.38, Proposition 4.2.39, Corollay 4.2.40, Proposition 4.2.41 and Proposition 4.2.42. In Remark 4.2.43 we establish a method of determining the Fischer matrices of the quotient group $\overline{G}/Z(\overline{G})$ directly from the Fischer matrices of $\overline{G}$. A similar process is outlined in Remark 4.2.44 to determine the character table of $\overline{G}/Z(\overline{G})$ directly from the character table of $\overline{G}$. The latter remark is due to Isaacs [37]. We continue with the discussion of affine subgroups of orthogonal groups in Section 4.3. Then in Section 4.4 we deal with the affine subgroups of unitary groups. For further reading on affine subgroups of classical groups, we refer the reader to [1], [18], [19], [30], [35], [48] and [49].

4.1 Affine subgroup of the general linear group GL(n, q)

Let $G(n) = \mathrm{GL}(n, q)$ be a general linear group of dimension n over the $GF(q) = \mathbb{F}$, where $q = p^k$, for $k \in \mathbb{N}$ and p a prime. We consider the affine subgroup $A(n)$ of $G(n)$. We use the following lemma due to Gow in [30] to show that $A(n)$ is a split extension.

Lemma 4.1.1 *[30] The affine subgroup $A(n)$ of $G(n)$ is a split extension of an elementary abelian p-group $D(n)$ of order q^{n-1} by a subgroup isomorphic to $G(n-1)$ which acts transitively by conjugation on the non-identity elements of $D(n)$.*

Proof *See [30].* □

Let us denote the normal elementary abelian p-subgroup $D(n)$, the subgroup $G(n-1)$ and the affine subgroup $A(n)$ in Lemma 4.1.1 by N, G and $\overline{G}$, respectively. Then $\overline{G} = N{:}G = q^{n-1}{:}\mathrm{GL}(n-1, q)$. Thus $|\overline{G}| = q^{n-1}|\mathrm{GL}(n-1, q)| = q^{n-1} \cdot \prod_{i=1}^{n-1}(q^{n-1} - q^{i-1})$.

Remark 4.1.2 *We observe that* $\left[G(n){:}\overline{G}\right] = q^n - 1$.

4.1.1 Elements of N and G in terms of $n \times n$ matrices

Since the affine subgroup $\overline{G} = N{:}G$ sits in $\mathrm{GL}(n, q)$, we express the elements of N and G in terms of $n \times n$ matrices with entries in $\mathbb{F}$. We use the discussion by Darafsheh and Iranmanesh in [35] to achieve this. The group $\mathrm{GL}(n, q)$ acts transitively on the non-zero vectors of the n-dimensional vector space $V(n, q)$ over the field $\mathbb{F}$. Let

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

be a vector in $V(n, q)$. The stabilizer of v_0 in $\mathrm{GL}(n, q)$ is the subgroup

$$G_{v_0} = \left\{ \left[\begin{array}{c|cccc} 1 & a & b & \cdots & c \\ \hline 0 & & & & \\ \vdots & & & A & \\ 0 & & & & \end{array}\right] \;\middle|\; A \text{ is an } n \times n \text{ non} - \text{singular matrix over } \mathbb{F} \text{ and } v = (a, b, \cdots, c) \in \mathbb{F}^{n-1} \right\}.$$

Since

$$\left[\begin{array}{c|c} 1 & v \\ \hline 0 & \\ \vdots & \text{A} \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} 1 & vA^{-1} \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right] \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right]$$

for $v = (a, b, \cdots, c) \in \mathbb{F}^{n-1}$, then we obtain that $A(n) = V(n-1, q){:}\mathrm{GL}(n-1, q) = N{:}G$, where

$$N = \left\{ \left[\begin{array}{c|c} 1 & v \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right] \middle| \; v = (a, b, \cdots, c) \in \mathbb{F}^{n-1} \right\}$$

and

$$G = \left\{ \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right] \middle| \; A \text{ is an } n-1 \times n-1 \text{ non} - \text{singular matrix over } \mathbb{F} \right\}.$$

The group $M = \{n\overline{g}n^{-1}\overline{g}^{-1} \mid n \in N, \; \overline{g} \in \overline{G}\}$, defined in Remark 3.3.10, has elements of the form

$$\left[\begin{array}{c|c} 1 & v(I-A) \\ \hline 0 & \\ \vdots & I_{(n-1)\times(n-1)} \\ 0 & \end{array}\right],$$

$M \leq N$.

4.1.2 The actions of G on N and on $Irr(N)$

The action of G on N is by conjugation and is given by

$$\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right]^{-1} \left[\begin{array}{c|c} 1 & v \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right] \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|c} 1 & vA \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right].$$

This action yields two orbits. The orbit lengths are 1 and $q^{n-1} - 1$. Thus we have two point stabilizers, one isomorphic to G and the other to the affine subgroup of G. The affine subgroup of G is the split extension $q^{n-2}{:}\mathrm{GL}(n-2, q)$. By Brauer's Theorem the action of G on $Irr(N)$ will again yield two orbits. The latter action is given by

$$\chi^g\left(\left[\begin{array}{c|c} 1 & v \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right]\right) = \chi\left(\left[\begin{array}{c|c} 1 & vA \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right]\right),$$

where $\chi \in Irr(N)$ and

$$g = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right] \in G.$$

The orbit lengths of this action are also 1 and $q^{n-1} - 1$. This then means that we have two inertia factor groups $H_1 \cong G$ and $H_2 \cong q^{n-2}$:GL$(n-2, q)$.

4.1.3 Holomorph of C_p

In this subsection we discuss the character table of the holomorph of C_p, where p is prime, which we will use in Chapter 5 where we will deal with the affine subgroup 5^2:GL$(2, 5)$ of GL(3,5).

Definition 4.1.3 *The holomorph of a group M is the split extension M:Aut(M), where Aut(M) acts naturally on M.*

Lemma 4.1.4 *If C_p is the cyclic group of order a prime p, then $Aut(C_p) \cong C_{p-1}$.*

Proof See [63]. □

In [63] Whitley shows that the character table of the holomorph of C_p is always of the following form:

Table 4.1: Character Table of C_p:C_{p-1}

Class rep	[1]	[p]	Cl_1	Cl_2	$\cdots$	Cl_{p-2}
|Centralizer|	$p(p-1)$	p	$p-1$	$p-1$	$\cdots$	$p-1$
χ_1	1	1				
χ_2	1	1				
$\vdots$	$\vdots$	$\vdots$		X		
χ_{p-1}	1	1				
χ_p	$p-1$	-1	0	0	$\cdots$	0

where

- $Cl_1, Cl_2, \cdots, Cl_{p-2}$ are class representatives of non-identity elements of C_{p-1},
- X denotes the character table of C_{p-1}.

4.1.4 Character degrees of $\overline{G}$

Let $\overline{G} = q^{n-1}{:}\mathrm{GL}(n-1, q) = N{:}G$ be the affine subgroup of $\mathrm{GL}(n, q)$, where q is a prime power. By Lemma 4.1.1, $\overline{G}$ is a split extension and N is abelian. Then by Mackey's Theorem 3.1.12, the $Irr(N)$ are extendable to the inertia groups of $\overline{G}$. This then means that the Clifford-Fischer Theory can be used to construct the character table of $\overline{G}$. According to Gallagher's Theorem 3.1.19, the characters of $\overline{G}$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \Big\{ (\beta\psi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i) \text{ and } N \subset \mathrm{Ker}(\beta) \Big\}.$$

Lemma 4.1.5 *The degrees of the $Irr(\overline{G})$ are the degrees of the $Irr(G)$ and the degrees of the irreducible characters of the affine subgroup of G multiplied by $q^{n-1} - 1$.*

Proof See [20]. □

Corollary 4.1.6 *$|Irr(\overline{G})|$ is equal to $\sum_{i=0}^{n-1} r_i(q)$, where $r_i(q)$ is the number of the $Irr(\mathrm{GL}_i(q))$ with $r_0(q) = 1$.*

Proof See [20]. □

Remark 4.1.7 *In [19] we have that the number of irreducible characters of an affine subgroup $\overline{G} = A(n)$ of $G(n) = \mathrm{GL}(n, q)$ can be computed by using the following polynomials:*

(i) $q^2 + q - 1$ *for* $n = 3$,

(ii) $q^3 + q^2 - 1$ *for* $n = 4$,

(iii) $q^4 + q^3 + q^2 - q - 1$ *for* $n = 5$.

Theorem 4.1.8 *$\overline{G}$ has $q-1$ irreducible characters of degree $(q^{n-1} - 1)(q^{n-2} - 1) \cdots (q^{n-i} - 1)$ for each $1 \le i \le n-2$ and a unique character of degree $(q^{n-1} - 1)(q^{n-2} - 1) \cdots (q - 1)$.*

Proof See [20]. □

4.1.5 Fischer matrices of $\overline{G}$ generalized

In [19] and [35] Darafsheh and Iranmanesh generalized the Fischer matrices of the affine subgroup of the general linear group $\mathrm{GL}(n, q)$. We take a closer look at the generalizations of 2×2 and 3×3 Fischer matrices.

Theorem 4.1.9 *Let $\overline{G} = q^{n-1}{:}\mathrm{GL}(n-1, q)$ be the affine subgroup of $GL(n, q)$ and let $[g]$ be a conjugacy class of G. Then the 2×2 Fischer matrices of $\overline{G}$ are of the form*

$$M(g) = \begin{bmatrix} 1 & 1 \\ q^m - 1 & -1 \end{bmatrix},$$

where $1 \le m \le n-1$.

Proof See [35]. □

Remark 4.1.10 *From Subsection 4.1.2 we have that the action of G on $Irr(N)$ yields two orbits of lengths 1 and $q^{n-1}-1$. It then follows from Theorem 4.1.9 that the Fischer matrix at the identity class of G is of the form*

$$M(1_G) = \begin{bmatrix} 1 & 1 \\ q^{n-1}-1 & -1 \end{bmatrix}.$$

Theorem 4.1.11 *[35] Let $\overline{G} = q^{n-1}{:}\mathrm{GL}(n-1,q)$ be the affine subgroup of $GL(n,q)$ and let $[g]$ be a conjugacy class of G. Then the 3×3 Fischer matrices of $\overline{G}$ are of the form*

$$M(g) = \begin{bmatrix} 1 & 1 & 1 \\ q-1 & q-1 & -1 \\ q^t-q & -q & 0 \end{bmatrix},$$

where $2 \leq t \leq n-2$.

Proof See [35]. □

4.2 Affine subgroup of the symplectic group $Sp(2n,q)$

We discuss in this section the affine subgroup of the symplectic group $Sp(2n,q)$, where q is a prime power. We start the discussion on symplectic transvections. Thereafter we look at the general theory regarding the affine subgroup of the symplectic group $Sp(2n,q)$.

Theorem 4.2.1 *[21] Let (V,f) be a non-degenerate symplectic space of dimension $2n$ over $GF(q)$. Then*

$$|Sp(2n,q)| = q^{n^2}\prod_{i=1}^{n}(q^{2i}-1).$$

Proof See [21] and [49]. □

4.2.1 Symplectic transvections

Definition 4.2.2 *If V is a vector space of finite dimension n and W is a subspace of V of dimension $n-1$, then W is referred to as a* **hyperplane** *of V. If $\mathbb{F} = \mathrm{GF}(q)$ and W is a hyperplane of V, then $|W| = q^{n-1}$.*

Definition 4.2.3 *Let V be a non-degenerate symplectic space over a field $\mathbb{F}$ and $T \in Sp(2n,\mathbb{F})$ and $T \neq I$, such that for some hyperplane W of V, we have*

(i) $T(w) = w$ for all $w \in W$,

(ii) $T(u) - u \in W$ *for all* $u \in V - W$.

In this case T *is referred as the* **symplectic transvection** *of* V.

Theorem 4.2.4 *[34] Let* T *be a symplectic transvection with hyperplane* W*. Then there is a non-zero* $v \in V$ *such that* $W = \langle v \rangle^{\perp}$ *and for all* $x \in V$ *we have* $T(x) = x + \lambda f(x, v)v$ *for* $\lambda \in \mathbb{F}$*. Conversely, for* $0 \neq v \in V$ *and* $0 \neq \lambda \in \mathbb{F}$ *define* $T : V \to V$ *by* $T(x) = x + \lambda f(x, v)v$ *for all* $x \in V$*. Then* T *is a symplectic transvection with hyperplane* $\langle v \rangle^{\perp}$.

Proof See [34] and [49]. □

Theorem 4.2.5 *[34] The symplectic group* $Sp(2n, \mathbb{F})$ *is generated by the set of all symplectic transvections.*

Proof See [34] and [49]. □

Corollary 4.2.6 *The symplectic group* $Sp(2n, \mathbb{F})$ *is transitive on* V^*.

Proof See [49]. □

Theorem 4.2.7 *Let* q *be a power of an odd prime* p*. Then* $Sp(2n, \mathbb{F})$ *has irreducible characters* ψ_1 *and* ψ_2 *of degrees* $\frac{q^n+1}{2}$ *and* $\frac{q^n-1}{2}$ *respectively. Moreover*

$$|\psi_1(x) + \psi_2(x)|^2 = |C_V(x)|$$

for all $x \in Sp(2n, q)$ *and* $V = V(2n, q)$ *is the natural module of* $Sp(2n, \mathbb{F})$.

Proof See [36]. □

Theorem 4.2.8 *The inverse of a transvection* $T_{v,\lambda}$ *is also a transvection* $T_{v,-\lambda}$.

Proof Claim $T_{v,\lambda}^{-1} = T_{v,-\lambda}$. Let $T_{v,\lambda}^{-1} = T_{w,\beta}$. Then $\forall\ x \in V$

$$(T_{v,\lambda} \circ T_{w,\beta})(x) = x$$

$$\Rightarrow T_{v,\lambda}(T_{w,\beta}(x)) = x$$

$$\Rightarrow T_{v,\lambda}(x + \beta f(x, w)w) = x$$

$$\Rightarrow x + \beta f(x, w)w + \lambda f[x + \beta f(x, w)w, v]v = x$$

$$\Rightarrow \beta f(x, w)w + \lambda f(x, v)v + \lambda\beta f(x, w)f(w, v)v = 0.$$

If $\beta = -\lambda$ and $w = v$, then the sum on the left equals to zero. Hence $T_{v,\lambda}^{-1} = T_{v,-\lambda}$ as claimed.

We show that the two composite functions $T_{v,\lambda} \circ T_{v,-\lambda}$ and $T_{v,-\lambda} \circ T_{v,\lambda}$ are identity functions.

Let $x \in V$, $v \in V^*$, $\lambda \in \mathbb{F}^*$ and $g \in Sp(2n, q)$. Then

$$\begin{aligned} T_{v,\lambda}(T_{v,-\lambda}(x)) &= T_{v,\lambda}(x - \lambda f(x,v)v) \\ &= x - \lambda f(x,v)v + \lambda f(x - \lambda f(x,v)v, v)v \\ &= x - \lambda f(x,v)v + \lambda f(x,v)v - \lambda^2 f(x,v)f(v,v)v \\ &= x. \end{aligned}$$

And

$$\begin{aligned} T_{v,-\lambda}(T_{v,\lambda}(x)) &= T_{v,-\lambda}(x + \lambda f(x,v)v) \\ &= x + \lambda f(x,v)v - \lambda f(x + \lambda f(x,v)v, v)v \\ &= x + \lambda f(x,v)v - \lambda f(x,v)v - \lambda^2 f(x,v)f(v,v)v \\ &= x. \end{aligned}$$

□

Theorem 4.2.9 *The conjugate of a transvection is also a transvection.*

Proof Let $x \in V$, $v \in V^*$, $\lambda \in \mathbb{F}^*$ and $g \in Sp(2n, q)$. Then

$$\begin{aligned} (gT_{v,\lambda}g^{-1})(x) &= (gT_{v,\lambda})(g^{-1}(x)) \\ &= g(g^{-1}(x) + \lambda f(g^{-1}(x), v)v) \\ &= g(g^{-1}(x)) + \lambda f(g^{-1}(x), v)g(v) \\ &= x + \lambda f(g^{-1}(x), g^{-1}(g(v))g(v) \\ &= x + \lambda f(x, g(v))g(v) \\ &= T_{g(v),\lambda}(x). \end{aligned}$$

□

Proposition 4.2.10 *Let p be a prime. If Char$(\mathbb{F}) = p$, then order of the transvection $T_{v,\lambda}$ is p.*

Proof By definition $T_{v,\lambda}(x) = x + \lambda f(x,v)v$. It can be shown that $T^2_{v,\lambda}(x) = x + 2\lambda f(x,v)v$ and $T^3_{v,\lambda}(x) = x + 3\lambda f(x,v)v$, etc. Assume that $T^k_{v,\lambda}(x) = x + k\lambda f(x,v)v$, for $k \in \mathbb{Z}^+$. Now

$$\begin{aligned} T^{k+1}_{v,\lambda}(x) &= (T^k_{v,\lambda} \circ T^1_{v,\lambda})(x) \\ &= (T^k_{v,\lambda}(T_{v,\lambda}(x)) \\ &= T^k_{v,\lambda}(x + \lambda f(x,v)v) \\ &= x + \lambda f(x,v)v + k\lambda f(x + \lambda f(x,v)v, v)v \\ &= x + \lambda f(x,v)v + k\lambda f(x,v)v + k\lambda^2 f(x,v)f(v,v)v \\ &= x + (k+1)\lambda f(x,v)v. \end{aligned}$$

Therefore $T^n_{v,\lambda}(x) = x + n\lambda f(x,v)v$, $\forall\, n \in \mathbb{Z}^+$. If Char$(\mathbb{F}) = p$, then $T^p_{v,\lambda}(x) = x + p\lambda f(x,v)v = x$, $\forall\, x \in V$. Thus $o(T_{v,\lambda}(x)) = p$. □

Corollary 4.2.11 *If the characteristic is 2, then order of a transvection is 2. This implies that these transvections are involutions in* $Sp(2n, q), \quad q = 2^k$.

Remark 4.2.12 *Theorem 4.2.5 and Corollary 4.2.11 imply that the symplectic group* $Sp(2n, 2)$ *is generated by a set of involutions. It can also be shown that* $Sp(2n, 2)$ *is a 3-transposition group. For details on 3-transposition groups see [24] and [7].*

Lemma 4.2.13 *[51]*

(a) Two transvections are equal, $T_{w,\lambda'} = T_{v,\lambda}$*, if and only if* $\exists\ \alpha \in \mathbb{F}^*$ *such that* $w = \alpha v$ *and* $\alpha^2\lambda' = \lambda$.

(b) $T_{\alpha v,\lambda} = T_{v,\alpha^2\lambda}$.

Proof (a)

$$\begin{aligned} T_{v,\lambda} = T_{w,\lambda'} &\Leftrightarrow\ \forall\ x \in V, x + \lambda f(x, v)v = x + \lambda' f(x, w)w \\ &\Leftrightarrow\ \forall\ x \in V, \lambda f(x, v)v = \lambda' f(x, w)w. \end{aligned}$$

Since f is non-degenerate, $\exists\ x \in V^*$ such that $f(x, v) \neq 0$. Then using the above, we must also have $f(x, w) \neq 0$ and hence $w = \frac{\lambda f(x,v)}{\lambda' f(x,w)} v$. Let $\alpha = \frac{\lambda f(x,v)}{\lambda' f(x,w)}$. Then $w = \alpha v$. Now

$$\alpha = \frac{\lambda f(x, v)}{\lambda' f(x, \alpha v)} = \frac{\lambda f(x, v)}{\alpha\lambda' f(x, v)} = \frac{\lambda}{\alpha\lambda'}.$$

Hence $\lambda = \alpha^2\lambda'$.

(b) $T_{\alpha v,\lambda}(x) = x + \lambda f(x, \alpha v)\alpha v = x + \lambda\alpha^2 f(x, v)v = T_{v,\alpha^2\lambda}(x)$. □

Proposition 4.2.14 *If* $|\mathbb{F}| = 2$*, then there is one conjugacy class of transvections in* $Sp(2n, 2)$.

Proof If $|\mathbb{F}| = 2$, then $\lambda = 1$. The group $Sp(2n, 2)$ is transitive on V^* by Corollary 4.2.6. This implies that $\exists\ g \in G$ such that $g(v) = v' \in V^*$. By Theorem 4.2.9, $g(T_{v,1})g^{-1} = T_{g(v),1} = T_{v',1}$. Hence there is only one conjugacy class of transvection in $Sp(2n, 2)$. □

Proposition 4.2.15 *If* $|\mathbb{F}| = 2^k$*, then there is one conjugacy class of transvections in* $Sp(2n, 2^k)$.

Proof Let $q = 2^m$, $m \in \mathbb{Z}^+$. Consider $\mathbb{F}^* = \{1, \alpha, \alpha^2, \cdots, \alpha^{q-2}\}$ with $\alpha^{q-1} = 1$, that is $o(\alpha) = q-1$. By Theorem 4.2.9, there are at most $|\mathbb{F}^*|$ possible conjugacy classes of transvections in $Sp(2n, q)$. The representatives of these possible conjugacy classes are

$$T_{v,1},\ T_{v,\alpha},\ T_{v,\alpha^2}, \cdots, T_{v,\alpha^{q-2}}.$$

We have that $T_{\alpha v,1} = T_{v,\alpha^2}$ by Lemma 4.2.13(b). If we square each element of $\mathbb{F}^*$, then we must have $T_{v,\alpha^{2k}} = T_{\alpha^k v,1}, k \in \mathbb{Z}^+ \cup \{0\}$. This means that $T_{v,\alpha^{2k}} \in [T_{v,1}]$.

Now if $\beta \in \mathbb{F}^*$ such that $\beta = \alpha^{2k+1}$, then $\beta = \alpha^{2k+1} \cdot \alpha^{2^m-1} = \alpha^{2(k+2^{m-1})} = \alpha^{2k'}$, for some k'. That is $T_{v,\alpha^{2k+1}} = T_{v,\alpha^{2k'}} = T_{\alpha^{k'}v,1}$. This implies that $T_{v,\alpha^{2k+1}}$ is also an element of $[T_{v,1}]$. Thus for every $\alpha \in \mathbb{F}^*$, $T_{v,\alpha} \in [T_{v,1}]$. Hence there is one conjugacy class of transvections in $Sp(2n, 2^k)$. □

Proposition 4.2.16 *The cardinality of the set of symplectic transvections in* $Sp(2n, 2^k)$ *is* $|V^*| = 2^{2n} - 1$.

Proof By Proposition 4.2.15 the conjugacy class of transvection is $[T_{v,1}]$ and hence

$$[T_{v,1}] = \{gT_{v,1}g^{-1} \mid g \in Sp(2n, 2^k)\} = \{T_{g(v),1} \mid g \in Sp(2n, 2^k)\} = \{T_{w,1} \mid w \in V^*\},$$

since $Sp(2n, 2^k)$ is transitive on V^*. Hence $|[T_{v,1}]| = |V^*|$. □

Proposition 4.2.17 *For* $0 \neq v \in V$, $G_v \leq C_G(T_{v,\lambda})\ \forall\ \lambda \in \mathbb{F}^*$.

Proof Let $g \in G_v$. Then $g(v) = v$. Then $\forall\ \lambda \in \mathbb{F}^*$, $gT_{v,\lambda}g^{-1} = T_{g(v),\lambda} = T_{v,\lambda}$. Hence $g \in C_G(T_{v,\lambda})$. □

Proposition 4.2.18 *If Char*($\mathbb{F}$) $= 2$, *then* $G_v = C_G(T_{v,\lambda})\ \forall\ \lambda \in \mathbb{F}^*$.

Proof By Proposition 4.2.17 $G_v \leq C_G(T_{v,\lambda})$. Conversely let $g \in C_G(T_{v,\lambda})$. Claim $g(v) = v$. Now

$g \in C_G(T_{v,\lambda}) \Leftrightarrow gT_{v,\lambda}g^{-1} = T_{v,\lambda}$

$\Leftrightarrow T_{g(v),\lambda} = T_{v,\lambda}$

$\Leftrightarrow \exists\ \alpha \in \mathbb{F}^*$ such that $g(v) = \alpha v, \alpha^2\lambda = \lambda$

$\Leftrightarrow \exists\ \alpha \in \mathbb{F}^*$ such that $g(v) = \alpha v, \alpha^2 = 1$.

Since Char($\mathbb{F}$) $= 2$, $1 = -1$, and hence $\alpha = 1$ is the only solution. In this case $g \in C_G(T_{v,\lambda})$ implies $g(v) = v$. Hence $g \in G_v$. □

Proposition 4.2.19 *If Char*($\mathbb{F}$) $\neq 2$, *then* $C_G(T_{v,\lambda}) = \{g \mid g(v) = v \text{ or } g(v) = -v\}$.

Proof From the proof of Proposition 4.2.18 we have

$$C_G(T_{v,\lambda}) = \{g \mid g(v) = \alpha v,\ \alpha^2 = 1\}.$$

If Char($\mathbb{F}$) $\neq 2$, then $\alpha^2 = 1 \Rightarrow \alpha = 1$ or $\alpha = -1$. Thus

$$C_G(T_{v,\lambda}) = \{g \mid g(v) = v \text{ or } g(v) = -v\} \supseteq G_v.$$

□

4.2.2 Affine subgroup of $Sp(2n, q)$

Let (V, f) be a non-degenerate symplectic space over $\mathbb{F} = GF(q)$. Let $B = \{e_1, e_2, \cdots, e_{2n}\}$ be a basis for V and $f : V \times V \to \mathbb{F}$ defined by $f(e_i, e_j) = \delta(i, 2n + m - j)$, where $i \leq j$. Let T be an isometry of (V, f) and

$$G(n) = Sp(2n, q) = \{T \mid f(T(x), T(y)) = f(x, y)\ \forall\ x, y \in V\}.$$

By Corollary 4.2.6, the symplectic group $G(n)$ is transitive on V^*. Let $\alpha \in V^*$. The affine subgroup of a symplectic group is the point stabilizer $A(n) = \{T \in G(n) \mid T(\alpha) = \alpha\}$. Then by Theorem 2.6.3 $A(n) \leq G(n)$.

Remark 4.2.20 *Since $A(n)$ is the stabilizer of $\alpha \in V^*$ in $G(n)$, then*

$$[G(n){:}A(n)] = |V^*| = q^{2n} - 1.$$

Lemma 4.2.21 *Let $a \in \mathbb{F} = GF(q)$, with $Char(\mathbb{F}) = p$ and $(n, p) = 1$, $n \in \mathbb{N}$. Then there exists $b \in \mathbb{F}$ such that $nb = a$.*

Proof See [48]. □

As $G(n)$ acts transitively on V^*, then let $A(n)$ be the stabilizer in $G(n)$ of the non-zero vector $e_1 \in V^*$. That is $A(n) = \{T \in G(n) \mid T(e_1) = e_1\}$.

Remark 4.2.22 *We define $P(n)$ to be the subgroup of $A(n)$ consisting of elements $T \in G(n)$ such that*

$$T(e_1) = e_1$$

$$T(e_i) = \alpha_i e_1 + e_i, \quad 2 \leq i \leq 2n-1$$

and

$$T(e_{2n}) = \sum_{i=1}^{2n} \beta_i e_i$$

with $\beta_{2n} = 1$ and

$$\alpha_j = \begin{cases} -\beta_{2n+1-j} & 2 \leq j \leq n \\ \beta_{2n+1-j} & n < j \leq 2n-1. \end{cases}$$

If $T \in P(n)$, then T is represented by the following matrix with respect to the basis given in introduction of this section:

$$\begin{pmatrix} 1 & -\beta_{2n-1} & -\beta_{2n-2} & \cdots & \beta_2 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_{2n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We describe $P(n)$ as an abstract group P as follows: Let (V, f) be a non-degenerate symplectic space of dimension $2n-2$ over $\mathbb{F}$ and consider the pairs $[v, a]$, where $v \in V$ and $a \in \mathbb{F}$. Define multiplication on such pairs by

$$[v, a][u, b] = [u + v, a + b + f(u, v)].$$

It is clear that $|P| = q^{2n-2} \times q = q^{2n-1}$.

Definition 4.2.23 *Let G be a group. The* **Frattini subgroup** *$\Phi(G)$ of G is the intersection of all maximal subgroups of G.*

Definition 4.2.24 *Let p be a prime and G a p-group. If $Z(G) = G' = \Phi(G)$ are elementary abelian, then G is referred to as a* **special p-group**.

We refer to $P(n)$ and P in Remark 4.2.22 in the following lemma.

Lemma 4.2.25 *If $Char(\mathbb{F}) = p$, where p is an odd prime, then the group P is a non-abelian special p-group of order q^{2n-1} isomorphic to $P(n)$.*

Proof See [48]. □

Remark 4.2.26 *Let $T \in P$, then $T = [v, a]$, where $v \in V$ and $a \in \mathbb{F}$. If $p = 2$, then we have $T^2 = [v, a]^2 = [2v, 2a] = [0_V, 0] = 1_P$. Thus P is an elementary abelian 2-group. Since $P(n) \cong P$, then $P(n)$ is also an elementary abelian 2-group.*

Lemma 4.2.27 *Let $G \leq A(n)$ which fixes e_{2n}. Then G fixes both e_1 and e_{2n} and acts on $W = \langle e_2, e_3, \cdots, e_{2n-1}\rangle$ as $G(n-1)$. Moreover, $G \cong G(n-1) \cong Sp(2n-2, q)$.*

Proof See [55]. □

In [48], Moori and Rodrigues prove that $P(n) \trianglelefteq A(n)$, $P(n) \cap G = \{1_{A(n)}\}$ and $P(n) \cdot G = A(n)$. These results are then used to prove the following result.

Theorem 4.2.28 *Let q be a power of an odd prime p. Then $A(n)$ is a split extension of $P(n)$ by G where $G \cong G(n-1) \cong Sp(2n-2, q)$.*

Proof See [48]. □

That is, $A(n) = q^{2n-1}{:}Sp(2n-2, q)$, hence $|A(n)| = q^{2n-1} \cdot q^{(n-1)^2} \prod_{i=1}^{n-1}(q^{2i} - 1)$.

4.2.3 Elements of $P(n)$ and $Sp(2n-2)$ in terms of $2n \times 2n$ matrices

We express the elements of $P(n)$ and $Sp(2n-2, q)$ in Theorem 4.2.28 as $2n \times 2n$ matrices with entries in $GF(q)$ since $A(n)$ is the subgroup of the symplectic group $Sp(2n, q)$. We use Remark 4.2.22 to express the elements of $P(n)$ in terms of $2n \times 2n$ matrices. An element $g \in Sp(2n-2, q)$ can be expressed in $Sp(2n, q)$ as follows:

$$\left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & & & & 0 \\ \vdots & & g & & \vdots \\ 0 & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right].$$

4.2.4 Character degrees of $A(n)$

Since $P(n)$ is abelian and $A(n)$ is a split extension, then by Mackey's Theorem, the characters of $P(n)$ are extendable to their inertia groups in $A(n)$. Thus the characters of $A(n)$ can be described using Gallagher's Theorem and Remark 3.1.20. Furthermore, the character table of $A(n)$ can be constructed using the Clifford-Fischer Theory. We concentrate in this subsection on the character degrees of $A(n)$.

Theorem 4.2.29 *Let $G(n) = Sp(2n, q)$, q a power of an odd prime. Let $A(n)$ be the stabilizer of a non-zero vector. The degrees of some of the $Irr(A(n))$ are as follows: the degrees of $Irr(Sp(n-2))$, the degrees of the $Irr(A(n-1))$ multiplied by $q^{2n-2}-1$.*

Proof See [18]. □

Remark 4.2.30 *The corollary of Theorem 4.2.29 in [18] is that the affine subgroup $A(n)$ has some irreducible characters of degree $(q^{2n-2}-1)(q^{2n-3}-1)\cdots(q^{2n-k}-1)$, where $k < n$.*

Theorem 4.2.31 *The affine subgroup $A(n)$ has non-faithful irreducible characters of degree $(q^{2n-2}-1)(q^{2n-3}-1)\cdots(q^{2n-2i}-1)$, for $i \in \mathbb{N}$ satisfying $1 \le i \le n-1$, . The kernel of these characters is the centre of $P(n)$. The characters remain irreducible modulo any prime distinct from p.*

Proof See [30]. □

Next we consider the case where q is a power of 2. If the dimension of a vector space V is even, q even, then there are two non-equivalent quadratic forms defined over V which are denoted by Q^+ and Q^-. If $V = V(2n, q)$, then the subgroups of $\mathrm{GL}(2n, q)$, leaving these forms invariant are the orthogonal groups $O^+(2n, q)$ and $O^-(2n, q)$, respectively. Orthogonal groups are discussed in Section 4.3.

Theorem 4.2.32 *[18] Let $G(n) = Sp(2n, q)$, q even. Let $A(n)$ be the stabilizer of a non-zero vector. The degrees of some of the $Irr(A(n))$ are as follows: the degrees of $Irr(Sp(n-2))$, the degrees of the $Irr(A(n-1))$ multiplied by $q^{2n-2}-1$, the degrees of the $Irr(O^+(2n, q))$ multiplied by $\frac{1}{2}q^{n-1}(q^{n-1}+1)$ and the degrees of the $Irr(O^-(2n, q))$ multiplied by $\frac{1}{2}q^{n-1}(q^{n-1}-1)$. Moreover, the number of characters in each of the last two cases is $q-1$.*

Proof [18] Let $A(n) = P(n){:}Sp(2n-2, q)$, where q is a power of 2. By Remark 4.2.26, $P(n)$ is an elementary abelian 2-group and $|P(n)| = q^{2n-1}$. The 2-group $P(n)$ is isomorphic to the group $P = \{[v, a] \mid v \in V \text{ and } a \in GF(q)\}$. The group P and the multiplication on the elements of P are defined and described in Remark 4.2.22, respectively. The action of $A(n)$ on $P(n)$ is defined in the following manner:

$$[v, a]^{[u,b]A} = [A^{-1}v, a],$$

where $A \in Sp(2n-2, q)$ and $[v, a], [u, b] \in P(n)$. Therefore $A(n)$ has $2q$ orbits on $P(n)$. Due to Brauer's Theorem the action of $A(n)$ on $Irr(P(n))$ has $2q$ orbits as well. We describe the latter orbits. For every vector $u \in V(2n-2, q)$, the function $\chi_u : P(n) \to C$ defined by $\chi_u([v, a]) = (-1)^{tr(f(u,v))}$ is an irreducible character of $P(n)$. The tr is the trace map of $GF(q)$

onto $GF(2)$. The group $G(n-1) = Sp(2n-2,q)$ acting on the set of characters of this type produces two orbits of size 1 and $q^{2n-2}-1$, with corresponding inertia factor groups $G(n-1)$ and $A(n-1)$, respectively. Let Q^{ϵ}, where $\epsilon = \pm$, be the representatives of the two classes of quadratic forms with associated bilinear form f. Then for any $b \in GF(q)$, the function $\chi_b : P(n) \to C$ defined by $\chi_b([v,a]) = (-1)^{tr(b(Q^{\epsilon}(v)+a))}$ is a linear character of $P(n)$. If b and c are distinct elements of $GF(q)$, then χ_b and χ_c lie in different orbits under $G(n-1)$. Because if $\chi_b^A = \chi_c$ for some $A \in G(n-1)$, then $\chi_b^A([0,a]) = \chi_c([0,a])$ for all $a \in GF(q)$ which implies that $tr((b-c)a) = 0$ and since $b \neq c$ this implies that the trace function is identically zero. Therefore if b runs through the $q-1$ non-zero elements of $GF(q)$ we have $2(q-1)$ orbit representatives χ_b for the orbits of $G(n-1)$ on $Irr(P(n))$. The orbit sizes are $\frac{1}{2}q^{n-1}(q^{n-1}+\epsilon)$, where $\epsilon = \pm 1$. The stabilizer of χ_b in $G(n-1)$, $0 \neq b \in GF(q)$, is a group isomorphic to $Q^{\epsilon}(2n-2,q)$ if χ_b is taken from an orbit of length $\frac{1}{2}q^{n-1}(q^{n-1}+\epsilon)$. Recall that the character degrees of $A(n)$ are from the partial character table of $A(n)$ corresponding to the Fischer matrix $M(e)$ on the identity class e of $Sp(2n-2,q)$. Since $P(n)$ is abelian, by Remark 3.3.6, the first column of the Fischer matrix $M(e)$ is given by the indices of the respective inertia factor groups in $Sp(2n-2,q)$, which are the orbit lengths of the action of $Sp(2n-2,q)$ on $Irr(P(n))$, as given above. The result now follows from the discussion in Section 3.5 and Remark 2.4.9 (iii). □

Remark 4.2.33 *The corollary of Theorem 4.2.32 in [18] is that the affine subgroup $A(n)$ has $q-1$ irreducible characters of degree $\frac{1}{2}q^{n-1}(q^{n-1}+\epsilon)$ and for each $k \in \mathbb{N}, 1 \leq k \leq n-2$ it has $q-1$ irreducible characters of degree $\frac{1}{2}q^{n-k-1}(q^{n-k-1}+\epsilon)(q^{2n-2}-1)(q^{2n-3}-1)\cdots(q^{2n-2k}-1)$, where $\epsilon = \pm 1$.*

Theorem 4.2.34 *Let q be a power of 2. Let $A(n)$ be the affine subgroup of $Sp(2n,q)$. Then $A(n)$ has non-faithful irreducible characters of degree $(q^{2n-2}-1)(q^{2n-4}-1)\cdots(q^{2n-2i}-1)$, $1 \leq i \leq n-1$, which are realisable in the rational field and remain irreducible as modular characters for any odd prime.*

Proof See [30]. □

Other characters of $A(n)$ are obtained from the Steinberg characters of the groups $O^+(2n-2,q)$ and $O^-(2n-2,q)$ of degree $q^{(n-1)(n-2)}$. The characters of $A(n)$ from the Steinberg characters are of degrees $\frac{1}{2}q^{(n-1)^2}(q^{n-1}\pm 1)$.

4.2.5 The action of $A(n)$ on $P(n)$ and on $Irr(P(n))$, q even

In this subsection we consider the action of $A(n)$ on $P(n)$ and the action of $A(n)$ on $Irr(N)$. We discuss the orbit lengths, point stabilizers and inertia factor groups of these actions.

Remark 4.2.35 *Let $q = 2^k$, $k \in \mathbb{N}$. Let $A(n) = P(n){:}G(n-1) = q^{2n-1}{:}Sp(2n-2,q)$ be the affine subgroup of the symplectic group $Sp(2n,q)$. From the proof of Theorem 4.2.32, we have that the action of $A(n)$ on $P(n)$ has $2q$ orbits $\Delta_1, \Delta_2, \cdots, \Delta_{2q}$. The respective orbit lengths are*

$$|\Delta_1| = |\Delta_2| = \cdots = |\Delta_q| = 1$$

and

$$|\Delta_{q+1}| = |\Delta_{q+2}| = \cdots = |\Delta_{2q}| = q^{2n-2}-1,$$

by Remark 4.4.7 in [49]. The corresponding point stabilizers are of indices 1 and $q^{2n-2}-1$ in $Sp(2n-2,q)$. These are isomorphic to $Sp(2n-2,q)$ and $A(n-1)$, the affine subgroup of $Sp(2n-2)$, respectively.

Remark 4.2.36 *Let $q=2^k$, $k\in\mathbb{N}$. Let $A(n)=P(n){:}G(n-1)=q^{2n-1}{:}Sp(2n-2,q)$ be the affine subgroup of the symplectic group $Sp(2n,q)$. The action of $A(n)$ on $Irr(P(n))$ has $2q$ orbits $\Gamma_1,\Gamma_2,\cdots,\Gamma_{2q}$. The respective orbit lengths are*

$$\begin{aligned}
|\Gamma_1| &= 1,\\
|\Gamma_2| &= q^{2n-2}-1,\\
|\Gamma_3| = |\Gamma_4| = \cdots = |\Gamma_{q+1}| &= \frac{1}{2}q^{n-1}(q^{n-1}+1),\\
|\Gamma_{q+2}| = |\Gamma_{q+3}| = \cdots = |\Gamma_{2q}| &= \frac{1}{2}q^{n-1}(q^{n-1}-1),
\end{aligned}$$

by Theorem 4.2.32. The corresponding point stabilizers are of indices 1, $q^{2n-2}-1$, $\frac{1}{2}q^{n-1}(q^{n-1}+1)$ ($q-1$ copies) and $\frac{1}{2}q^{n-1}(q^{n-1}-1)$ ($q-1$ copies) in $Sp(2n-2,q)$. These are isomorphic to the inertia factor groups $Sp(2n-2,q)$, $A(n-1)$, $O^+(2n-2,q)$ ($q-1$ copies) and $O^-(2n-2,q)$ ($q-1$ copies), respectively.

4.2.6 The centre of $A(n)$ and the quotient group $A(n)/Z(A(n))$, q even

In the following, Proposition 4.2.37 and Corollary 4.2.38 are for any split extension of the form $\overline{G}=N{:}G$. We start this section by showing that $Z(\overline{G})$ is contained in the split extension $M{:}Z(G)$, where $M=\{m\in N \mid gmg^{-1}=m\ \forall\ g\in G\}$. In the case that $Z(G)=\{e\}$ then we show that $Z(\overline{G})=M$. We use these results to show that if $\overline{G}=2^{2n-1}{:}Sp(2n-2,2)$ is an affine subgroup of $Sp(2n,2)$, then $Z(\overline{G})$ is isomorphic to $\mathbb{Z}_2$, and we then consider the quotient of $\overline{G}$ by its centre. We prove that this quotient group is a split extension $2^{2n-2}{:}Sp(2n-2,2)$. We then show how to derive the Fischer matrices of this quotient group directly from the Fischer matrices of $\overline{G}$.

Proposition 4.2.37 *Let $\overline{G}=N{:}G$ and N be elementary abelian. If*

$$Fix_G(N)=Fix_{\overline{G}}(N)=M=\{m\in N \mid gmg^{-1}=m\ \forall\ g\in G\}$$

then $Z(\overline{G})\leq M{:}Z(G)$.

Proof Since $Z(G)\leq G$ and $M\trianglelefteq N$ then $M\cap Z(G)=\{e\}$ and $M\trianglelefteq M.Z(G)$. This means that $M.Z(G)$ is a split extension. Let $\overline{z}\in Z(\overline{G})$. Then $\overline{z}=nz$, $n\in N$ and $z\in G$. For all $\overline{g}\in\overline{G}$ we have $\overline{g}\ \overline{z}\ \overline{g}^{-1}=\overline{z}$. Then $(\overline{g}n\overline{g}^{-1})(\overline{g}z\overline{g}^{-1})=nz$. This means $(gng^{-1})(gzg^{-1})=nz$, since $G\leq\overline{G}$. But then $gng^{-1}=n'\in N$ and $gzg^{-1}=z'\in G$. Thus $n'z'=nz$. Since $N{:}G$ is a split extension we must have $n'=n$ and $z'=z$. Then $gng^{-1}=n$ and hence $n\in M$. Therefore $\overline{z}=nz\in M{:}Z(G)$. Thus $Z(\overline{G})\leq M{:}Z(G)$. □

Corollary 4.2.38 *Let $\overline{G}=N{:}G$ and N be elementary abelian. If $Z(G)=\{e\}$, then $Z(\overline{G})=M$.*

Proof By the definition of M, $M \leq Z(\overline{G})$. From Proposition 4.2.37 we have that $Z(\overline{G}) \leq M{:}Z(G)$. This implies that $Z(\overline{G}) \leq M$. Thus $Z(\overline{G}) = M$. □

Proposition 4.2.39 *If $\overline{G}$ is an affine subgroup of $Sp(2n, 2)$, then $Z(\overline{G}) = \{e, \alpha\} \cong \mathbb{Z}_2$.*

Proof By Theorem 4.2.28, $\overline{G}$ is a split extension with $N = 2^{2n-1}$ and $G = Sp(2n-2, 2)$. By Remark 4.2.35, $\overline{G}$ has 4 orbits on N such that their respective orbit lengths are $|\Delta_1| = |\Delta_2| = 1$ and $|\Delta_3| = |\Delta_4| = 2^{2n-2} - 1$. Let $e \in \Delta_1$ and $\alpha \in \Delta_2$. This then means $M = \{e, \alpha\}$. Now since $Z(G) = \{e\}$ then the result follows by Corollary 4.2.38. □

Corollary 4.2.40 *If $\overline{G} = 2^{2(2n-1)}{:}Sp(2n-2, 2^2)$ is an affine subgroup of $Sp(2n, 2^2)$, then $Z(\overline{G}) = \{e, \alpha, \alpha^2, \alpha^3\} \cong \mathbb{Z}_4$.*

Proof The result follows by Remark 4.2.35 and Corollary 4.2.38. □

Proposition 4.2.41 *The factor group $\overline{G}/Z(\overline{G})$, where $\overline{G}$ is an affine subgroup of $Sp(2n, 2)$, is a split extension $2^{2n-2}{:}Sp(2n-2, 2)$.*

Proof We know that the split extension $\overline{G} \cong 2^{2n-1}{:}Sp(2n-2, 2)$, where $N = 2^{2n-1}$ and $G = Sp(2n-2, 2)$. By Proposition 4.2.39, $\mathbb{Z}_2 \cong \langle\alpha\rangle = Z(\overline{G}) \trianglelefteq \overline{G}$. Consider the epimorphism $\overline{G} \to \overline{G}/\langle\alpha\rangle$. Since $N \trianglelefteq \overline{G}$ then $N/\langle\alpha\rangle \trianglelefteq \overline{G}/\langle\alpha\rangle$. But $N/\langle\alpha\rangle \cong 2^{2n-2}$. Thus $2^{2n-2} \trianglelefteq \overline{G}/\langle\alpha\rangle$. Since $\overline{G}$ is a split extension, we have

$$\overline{G}/\langle\alpha\rangle/N/\langle\alpha\rangle \cong \overline{G}/N \cong Sp(2n-2, 2).$$

We show that $\overline{G}/\langle\alpha\rangle = N/\langle\alpha\rangle{:}K$, where $K \cong Sp(2n-2, 2)$. Now $\overline{G}/\langle\alpha\rangle = \{ng\langle\alpha\rangle \mid n \in N,\ g \in G\} = \{n\langle\alpha\rangle g \mid n \in N,\ g \in G\} = \{\overline{n}g \mid \overline{n} \in N/\langle\alpha\rangle,\ g \in Sp(2n-2, 2)\}$. Then $\overline{G}/\langle\alpha\rangle = N/\langle\alpha\rangle.K$, where $K \cong Sp(2n-2, 2)$. Then we must have $|\overline{G}/\langle\alpha\rangle| = |N/\langle\alpha\rangle.K| = 2^{2n-2} \times |Sp(2n-2, 2)| = |N/\langle\alpha\rangle| \times |K|/|N/\langle\alpha\rangle \cap K|$. This then means $|N/\langle\alpha\rangle \cap K| = 1$ and $N/\langle\alpha\rangle \cap K = \{e\}$. Therefore $2^{2n-2}{:}Sp(2n-2, 2)$ is a split extension. □

Proposition 4.2.42 *If $\overline{G} = 2^{2(2n-1)}{:}Sp(2n-2, 2^2)$ is an affine subgroup of $Sp(2n, 2^2)$, then the factor group $\overline{G}/Z(\overline{G})$ is a split extension of the form $2^{2(2n-2)}{:}Sp(2n-2, 2^2)$.*

Proof The proof is similar to the proof of Proposition 4.2.41. □

Remark 4.2.43 The Fischer matrices of $\overline{G}/Z(\overline{G})$ from the Fischer matrices of $\overline{G}$.

In this remark we refer to the general form of a Fischer matrix $M(g)$ given in Remark 3.3.2. Let $k \in \{1, 2\}$. If $\overline{G}$ is the affine subgroup of $Sp(2n, 2^k)$, then the quotient $\overline{G}/Z(\overline{G})$ is the split extension $2^{k(2n-2)}{:}Sp(2n-2, 2^k)$. The group $Sp(2n-2, 2^k)$ acts transitively on the set V^* of non-zero vectors. Thus the natural action of $Sp(2n-2, 2^k)$ on $2^{k(2n-2)}$ produces two orbits. By Brauer's Theorem, the action of $Sp(2n-2, 2^k)$ on $Irr(2^{k(2n-2)})$ will again produce 2 orbits. This then means that we have two inertia factor groups for $2^{k(2n-2)}{:}Sp(2n-2, 2^k)$ instead of the original $2q = 2^{k+1}$ for $2^{k(2n-1)}{:}Sp(2n-2, 2^k)$. In fact, the inertia factor groups are $H_1 \cong Sp(2n-2, 2^k)$ and the affine subgroup $H_2 \cong 2^{k(2n-3)}{:}Sp(2n-4, 2^k)$ of $Sp(2n-2, 2^k)$. If $M(g)$ is a Fischer matrix of $\overline{G}$ corresponding to g, then to find the corresponding Fischer matrix of $\overline{G}/Z(\overline{G})$ we delete the rows (blocks) corresponding to the inertia factor groups that do not contribute in $2^{k(2n-2)}{:}Sp(2n-2, 2^k)$. The repeated columns are then discarded.

Remark 4.2.44 The character table of $\overline{G}/Z(\overline{G})$ from the character table of $\overline{G}$.

The character table of $\overline{G}/Z(\overline{G})$ can be constructed directly from the character table of $\overline{G}$. To achieve this, we follow the process outlined in Remark 4.2.46.

Lemma 4.2.45 *[37] Let $N \trianglelefteq G$.*

(i) If χ is a character of G and $N \subseteq \mathrm{Ker}(\chi)$, then χ is constant on the cosets of N in G. The mapping $\hat{\chi}$ on G/N defined by $\hat{\chi}(Ng) = \chi(g)$ is a character of G/N.

(ii) If $\hat{\chi}$ is a character of G/N, then the mapping χ defined by $\chi(g) = \hat{\chi}(Ng)$ is a character of G.

(iii) $\chi \in Irr(G) \Leftrightarrow \hat{\chi} \in Irr(G/N)$.

Proof See [37]. □

Remark 4.2.46 *[37] The character table of G/N from the character table of G.*

(i) $\hat{\chi} \in Irr(G/N) = \{\chi \in Irr(G) \mid N \subseteq \mathrm{Ker}(\chi)\}$,

(ii) identify the classes of G that are contained in N and utilize the definition of the $\mathrm{Ker}(\chi)$ to determine the characters of G/N,

(iii) step (ii) will yield a table with repeated columns. But since in G/N, Na and Nb are conjugate if and only if $\chi(a) = \chi(b)$, for $\chi \in Irr(G)$, we can delete the repeated columns. This in turn provides the conjugacy classes of G/N.

4.3 Affine subgroups of the orthogonal groups $O(2n+1, q)$ and $O(2n, q)$

Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let θ be an automorphism of $\mathbb{F}$. A **sesquilinear form** on V with respect to θ is a function $f : V \times V \to F$ such that for all $u, v, w \in V$ and all $a \in \mathbb{F}$:

$$f(u+v, w) = f(u, w) + f(v, w), \qquad f(au, v) = af(u, v),$$

$$f(u, v+w) = f(u, v) + f(u, w), \qquad f(u, av) = a^{\theta} f(u, v).$$

The form is said to be **bilinear** if $\theta = 1$. Assume $o(\theta) \leq 2$.

Definition 4.3.1 *We say f is* **symmetric** *if f is bilinear and $f(u, v) = f(v, u)\ \forall\ u, v \in V$. On the other hand, f is said to be* **skew symmetric** *if f is bilinear and $f(u, v) = -f(v, u)$.*

Definition 4.3.2 *If $f(u, v) = 0$, then we say that u and v are* **orthogonal**, *and we write $u \perp v$.*

Remark 4.3.3 *For $X \subseteq V$ define*

$$X^{\perp} = \{v \in V \mid x \perp v \ \forall\ x \in X\}.$$

Then $V^{\perp}$ is called the **radical** *of V and is denoted by $Rad(V)$. If $Rad(V) = 0$, then f is said to be* **non-degenerate**.

Remark 4.3.4 *The form f is said to be* **orthogonal** *if f is non-degenerate and symmetric, and if in addition, when $Char(\mathbb{F}) = 2$, $f(v, v) = 0 \ \forall\ v \in V$.*

Definition 4.3.5 *Consider a vector $v \in V$. Then v is called as* **isotropic vector** *if $f(v, v) = 0$. That is, by Definition 4.3.2, v is isotropic if v is perpendicular to itself.*

If f is skew symmetric and $Char(\mathbb{F}) \neq 2$, then every vector is isotropic.

Definition 4.3.6 *Suppose that f is symmetric. A* **quadratic form** *on V associated to f is a function $Q : V \to \mathbb{F}$ such that, for all $u, v \in V$ and $a \in \mathbb{F}$, $Q(au) = a^2 Q(u)$ and*

$$Q(u + v) = Q(u) + Q(v) + f(u, v).$$

Remark 4.3.7 *Note that if $Char(\mathbb{F}) \neq 2$, then Definition 4.3.6 forces $Q(u) = \frac{1}{2} f(u, u)$. This then means that the quadratic form is uniquely determined by f, and hence adds no new information. On the other hand, if $Char(\mathbb{F}) = 2$, then there are many quadratic forms associated to f. Note as well that f is uniquely determined by Q, since*

$$f(u, v) = Q(u + v) - Q(u) - Q(v).$$

Definition 4.3.8 *An* **orthogonal space** *is a pair (V, Q) where Q is a quadratic form on V with associated bilinear form f.*

We recall, with regard to symplectic groups in Chapter 4 in [49], that if f is a form on a vector space V, then an **isometry** of f, (or V), is a linear map $g : V \to V$ which preserves the form, in the sense that $f(u^g, v^g) = f(u, v)$ for all $u, v \in V$. Similarly, an isometry of a quadratic form Q is a linear map g such that $Q(v^g) = Q(v)$, for all $v \in V$. We recall as well that the set of all isometries of V, $\text{Isom}(V, f)$, forms a group. Since, in symplectic groups, the alternating bilinear forms are always equivalent, then we had $\text{Isom}(V, f) = Sp(2n, q)$, the symplectic group of dimension $2n$ with entries in $GF(q)$. Similarly, with regard to orthogonal spaces, the group $\text{Isom}(V, Q)$ is an orthogonal group. However, since the symmetric bilinear forms are not equivalent, $\text{Isom}(V, Q)$ yield non-isomorphic orthogonal groups. In odd dimensions over a finite field of odd characteristic, there is only one orthogonal group $O(V)$. However, in even dimensions over finite fields of any characteristic, there are two orthogonal groups, denoted by $O^+(V)$ and $O^-(V)$ respectively. The forms associated with the latter orthogonal groups are denoted, respectively, by f^+ and f^-. The bases of the respective vector spaces will be dealt with when we discuss the affine subgroups of a specific type of the orthogonal group in Subsections 4.3.1, 4.3.2 and 4.3.3. The orthogonal group of odd dimension over a field of even characteristic is not considered, since this group is isomorphic to a symplectic group. Detailed theory on orthogonal groups may be found in [6], [57] and [64].

4.3.1 The affine subgroup of $O(2n+1, q)$, where q is odd

Let $V = V(2n+1, q)$ with the basis $\{e_1, e_2. \cdots, e_{2n+1}\}$. Define the non-degenerate symmetric bilinear form f on V by

$$f(e_i, e_j) = \delta(i, 2n+2-j),$$

$1 \leq i \leq j \leq 2n+1$. The matrix of f relative to this basis is

$$J_{2n+1} = \begin{array}{c} \\ e_1 \\ \vdots \\ e_i \\ e_j \\ \vdots \\ e_{2n+1} \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & \cdots & e_i & e_j & \cdots & e_{2n+1} \end{array} \\ \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ \hline 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right) \end{array}.$$

The order of the orthogonal group $O(m, q)$, m odd, is given by

$$|O(m,q)| = 2q^{\frac{(m-1)^2}{4}} \prod_{i=1}^{\frac{m-1}{2}} (q^{2i} - 1).$$

Let $G(n) = O(2n+1, q)$ be the group of invertible linear transformations of V leaving f invariant. Then $Q(v) = \frac{1}{2} f(v, v)$ defines a quadratic form and the group $G(n)$ acts transitively on the set of all the non-zero isotropic vectors. Denote the affine subgroup of $G(n)$ by $A(n)$. This subgroup is the stabilizer of a non-zero isotropic vector, say, e_1.

Remark 4.3.9 *We observe that* $[G(n){:}A(n)] = q^{2n} - 1$.

Lemma 4.3.10 *[18] The affine subgroup $A(n)$ is the semidirect product of an abelian group of order q^{2n-1} and a group isomorphic to $G(n-1)$.*

Proof See [18]. □

Remark 4.3.11 The action of $A(n)$ on $P(n)$

We observed in the introduction of this section that when q is odd, then $Q(v) = \frac{1}{2} f(v, v)$ defines a quadratic form on $V(2n+1, q)$. Thus $P(n)$ can be expressed as $P(n) = \{[v, -Q(v)] \mid v \in V(2n+1, q)\}$. Define multiplication on the elements of $P(n)$ as follows: $[v, -Q(v)][u, -Q(u)] = [v+u, -Q(v+u)]$. The action of $A(n)$ on $P(n)$ is given by

$$[v, -Q(v)]^{[u,-Q(u)]A} = [A^{-1}v, -Q(v)],$$

where A is identified with the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $u, v \in V(2n-1,q)$. The number of vectors v for which $Q(v) = 0$ is q^{2n-2} and $A(n)$ has two orbits on the set of these vectors. These two orbits are the zero vector and the set of non-zero isotropic vectors with stabilizers isomorphic, respectively, to $P(n) \cdot G(n-1)$ and $P(n) \cdot A(n-1)$. From this action it is also observed that for the vectors v, where $Q(v) = a$ is a fixed non-zero element of $GF(q)$, the set $\{[v, -Q(v)] \mid v \in V(2n-1,q)\}$ is an orbit of $A(n)$. If a runs through the square elements of $GF(q)$, then we obtain $\frac{q-1}{2}$ orbits each of size $q^{2(n-1)} + q^{n-1}$ with stabilizers isomorphic to $P(n) \cdot O^+(2n-2,q)$. If a runs through the non-square elements of $GF(q)$, then we obtain $\frac{q-1}{2}$ orbits each of size $q^{2(n-1)} - q^{n-1}$ with stabilizers isomorphic to $P(n) \cdot O^-(2n-2,q)$. This then means that the action of $A(n)$ on $P(n)$ produces $q+1$ orbits.

Theorem 4.3.12 *[18] Let $G(n) = O(2n+1,q)$, where q is odd. Let $A(n)$ be the affine subgroup of $G(n)$, that is the stabilizer of a non-zero isotropic vector. Then the degrees of $Irr(A(n))$ are as follows: the degrees of the $Irr(G(n-1)) = Irr(O(2n-1,q))$, the degrees of the $Irr(A(n-1))$ multiplied by $q^{2n-2} - 1$, the degrees of the $Irr(O^+(2n-2,q))$ multiplied by $q^{n-1}(q^{n-1}+1)$ and the degrees of the $Irr(O^-(2n-1,q))$ multiplied by $q^{n-1}(q^{n-1}-1)$. Furthermore, there are $\frac{q-1}{2}$ irreducible characters for each of the latter two families.*

Proof See [18]. $\square$

Remark 4.3.13 *The corollary of Theorem 4.3.12 in [18], is that for each $k \in \mathbb{N}$, $1 \leq k \leq n-1$, the subgroup $A(n)$ has characters of degree $(q^{2n-2}-1)(q^{2n-4}-1)\cdots(q^{2n-2k}-1)$. There are also $\frac{q-1}{2}$ character degrees as follows: $q^{n-1}(q^{n-1}+\epsilon)$ and $(q^{2n-2}-1)(q^{2n-4}-1)\cdots(q^{2n-2k}-1)q^{n-k-1}(q^{n-k-1}+\epsilon)$, where $1 \leq k \leq n-2$ and $\epsilon = \pm 1$.*

Remark 4.3.14 The action of $A(n)$ on $Irr(A(n)$

The action of $A(n)$ on $P(n)$ in Remark 4.3.11 produced $q+1$ orbits. Therefore the action of $A(n)$ on $Irr(P(n))$ must also have $q+1$ orbits, $\Gamma_1, \Gamma_2, \cdots, \Gamma_{q+1}$. We deduce the respective orbits lengths from Theorem 4.3.12, to have

$$\begin{aligned} |\Gamma_1| &= 1, \\ |\Gamma_2| &= q^{2n-2} - 1, \\ \frac{q-1}{2} \text{ copies each of length } |\Gamma_i| &= q^{n-1}(q^{n-1}+1), \\ \frac{q-1}{2} \text{ copies each of length } |\Gamma_j| &= q^{n-1}(q^{n-1}-1). \end{aligned}$$

These yield corresponding inertia factor groups isomorphic to $O(2n-1,q)$, $A(n-1)$, $O^+(2n-2,q)$ ($\frac{q-1}{2}$ copies) and $O^-(2n-2,q)$ ($\frac{q-1}{2}$ copies), respectively.

4.3.2 The affine subgroup of $O^{\pm}(2n,q)$, where q is odd

Let q be a power of an odd prime and $V = V(2n,q)$ with basis $\{e_1, e_2, \cdots, e_{2n}\}$. There are two equivalence classes of symmetric bilinear forms f^+ and f^- defined on V. Let $G(n) = O^{\pm}(2n,q)$ be a group of invertible $2n \times 2n$ matrices that leave $f^{\pm}$ invariant. Let $A(n)$ be the affine subgroup of $G(n)$ that stabilizes a non-zero isotropic vector, say e_1. Since $G(n)$ acts transitively on the non-zero isotropic vectors then we have $[G(n){:}A(n)] = (q^n - \epsilon)(q^{n-1} + \epsilon)$, where $\epsilon = \pm 1$. The order of $O^{\pm}(m, q)$, where m is even, is given by

$$|O^{\pm}(m, q)| = 2q^{\frac{m(m-2)}{4}}(q^{\frac{m}{2}} - \epsilon)\prod_{i=1}^{\frac{m-2}{2}}(q^{2i} - 1).$$

Remark 4.3.15 *[18] The affine subgroup $A(n)$ is a semidirect product of an abelian group of order q^{2n-2} by a group isomorphic to $G(n-1)$. That is $A(n) = P(n){:}G(n-1) = q^{2n-2}{:}O^{\pm}(2n-2, q)$. Let $P(n) = \{[v, -Q^{\pm}(v)] \mid v \in V(2n-2, q)\}$, where $Q^{\pm}(v) = \frac{1}{2}f^{\pm}(v, v)$ is the quadratic form associated with $f^{\pm}$. Multiplication of the elements of $P(n)$ is the same as in the odd dimension and q odd case in Subsection 4.3.1.*

Remark 4.3.16 *[18] Let $A(n)$ be the affine subgroup of $G(n) = O^{\pm}(2n, q)$ with q odd. Then the degrees of the $Irr(A(n))$ are as follows: the degrees of $Irr(G(n-1)) = O^{\pm}(2n-2, q)$, the degrees of $A(n-1)$ multiplied by $(q^{n-1} - \epsilon)(q^{n-2} + \epsilon)$, the degrees of the $Irr(O(2n-3, q)$ multiplied by $q^{n-2}(q^{n-1} - \epsilon)$, where $\epsilon = \pm 1$. Furthermore, there are $q - 1$ characters of the mentioned latter degrees.*

Remark 4.3.17 *[18]* **Action of $A(n)$ on $P(n)$ and on $Irr(P(n))$**

The action of $A(n)$ on $P(n)$ is the same as in the odd dimension and q odd case in Subsection 4.3.1. Thus this action and the action on $Irr(P(n))$ produce $q + 1$ orbits. The inertia factor groups, from the latter action, are deduced from Remark 4.3.16. These are isomorphic to $G(n-1)$, $A(n-1)$ and $q - 1$ copies of $O(2n-3, q)$.

4.3.3 Affine subgroup of $O^{\pm}(2n, q)$, where q is even

Let q be even and $V = V(2n, q)$ with a basis $\{e_1, e_2, \cdots, e_{2n}\}$. Let Q be a quadratic form defined on V and f be the bilinear form associated with Q. The form f is symmetric and non-degenerate. There are two equivalence classes of non-degenerate quadratic forms $Q^{\pm}$ defined on V. Let $G(n) = O^{\pm}(2n, q)$ be a group of invertible $2n \times 2n$ matrices that fix $Q^{\pm}$. In this case an isotropic vector v is such that $Q^{\pm}(v) = 0$. Let $A(n)$ be the affine subgroup of $G(n)$ that stabilizes a non-zero isotropic vector, say e_1. Since $G(n)$ acts transitively on the non-zero isotropic vectors then we have $[G(n){:}A(n)] = (q^n - \epsilon)(q^{n-1} + \epsilon)$, where $\epsilon = \pm 1$.

Remark 4.3.18 *[18] Let $q = 2^k, k \in \mathbb{N}$. The affine subgroup $A(n)$ is a semidirect product of an abelian group of order q^{2n-2} by a group isomorphic to $G(n-1)$. That is, $A(n) = P(n){:}G(n-1) = q^{2n-2}{:}O^{\pm}(2n-2, q)$. Let $P(n) = \{[v, Q^{\pm}(v)] \mid v \in V(2n-2, q)\}$. Multiplication of the elements of $P(n)$ is defined as:*

$$[v, Q^{\pm}(v)][u, Q^{\pm}(u)] = [v + u, Q^{\pm}(v + u)].$$

Remark 4.3.19 *[18]* **Action of $A(n)$ on $P(n)$ and on $Irr(P(n))$**

The action of $A(n)$ on $P(n)$ is as follows:

$$[v, Q^{\pm}(v)]^{[u,b]A} = [A^{-1}v, Q^{\pm}(v)],$$

where A is defined in a similar way as in Remark 4.3.11. This action produces $q+1$ orbits, $\{0\}, \{v \in V(2n-2,q) \mid Q^{\pm}(v)=0\}$ and $q-1$ orbits of the form
$\{v \in V(2n-2,q) \mid Q^{\pm}(v) \text{ is a fixed non-zero element of } GF(q)\}$. *The corresponding orbit lengths are 1, $(q^{n-2}+\epsilon)(q^{n-1}-\epsilon)$ and $q-1$ orbits of lengths $q^{2n-2}(q^{n-1}-\epsilon)$, respectively, where $\epsilon=\pm 1$. There are also $q+1$ orbits of the action of $A(n)$ on $Irr(P(n))$. The corresponding inertia factor groups are isomorphic to $G(n-1)$, $A(n-1)$ and $q-1$ groups isomorphic to $O(2n-3,q)$.*

The character degrees of $A(n)$ are determined the same way described in Remark 4.3.16.

4.4 Affine subgroups of the unitary groups $U(2n,q^2)$ and $U(2n+1,q^2)$

Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let θ be an automorphism of $\mathbb{F}$. A **sesquilinear form** on V with respect to θ is a function $f: V \times V \to \mathbb{F}$ such that for all $u,v,w \in V$ and all $a \in \mathbb{F}$:

$$f(u+v,w)=f(u,w)+f(v,w), \qquad f(au,v)=af(u,v),$$

$$f(u,v+w)=f(u,v)+f(u,w), \qquad f(u,av)=a^{\theta}f(u,v).$$

The form is said to be **bilinear** if $\theta=1$. Assume $o(\theta)\leq 2$.

Definition 4.4.1 *A bilinear form f is* **hermitian symmetric** *if θ is an involution and $f(u,v)=(f(v,u))^{\theta}\ \forall\ u,v \in V$.*

Theorem 4.4.2 *If p is a prime, then the group $Aut(GF(p^n))$ of all field automorphisms of $GF(p^n)$ is cyclic of order n.*

Proof See [57]. □

Remark 4.4.3 *If $\mathbb{F}$ is a finite field, then it has an automorphism θ of order 2 if and only if $\mathbb{F}\cong GF(q^2)$ for some prime power q, in which case $a^{\theta}=a^q$. The latter is a consequence of Theorem 4.4.2.*

Definition 4.4.4 *If f is non-degenerate and hermitian symmetric, then it is said to be* **unitary** *and the pair (V,f) is referred to as a* **unitary space***.*

Lemma 4.4.5 *Let (V,f) be a non-degenerate space, let A be the inner product matrix of f relative to an ordered basis $\{e_1,e_2,\cdots,e_n\}$ of V, and let T be a linear transformation on V.*

(i) If f is bilinear, then T is an isometry if and only if its matrix $M=[m_{ij}]$ relative to the ordered basis satisfies

$$M^tAM=A,$$

with det $M=\pm 1$.

(ii) If f is hermitian, then T is an isometry if and only if

$$M^t A M^\theta = A,$$

where $M^\theta = [(m_{ij})^\theta]$. In this case $(\det M)(\det M^\theta) = 1$.

Proof See [57]. □

Remark 4.4.6 *Since all hermitian forms are equivalent, then the group Isom(V, f) is unique and is called the* **unitary group**. *This group is denoted by $U(V)$ or $U(n, \mathbb{F})$ or $U(n, q^2)$, when $\mathbb{F} = GF(q^2)$.*

General theory on unitary groups may be found in [6], [57] and [64], among other relevant materials. Before we deal with the analysis of the affine subgroups of unitary groups, we discuss a few basics regarding extra special p-groups, since these affine subgroups, as it will be shown later, are split extensions of extra special p-subgroups.

4.4.1 Extra special p-groups

In Definition 4.2.24, we defined a special p-group as follows. Let p be a prime and G a p-group. If $Z(G) = G' = \Phi(G)$ are elementary abelian, then G is called a special p-group. We extend this definition to define an extra special p-group.

Definition 4.4.7 *Let p be a prime and G a p-group. If $Z(G) = G' = \Phi(G)$ are elementary abelian, then G is referred to as a special p-group. If moreover, the quotient $G/Z(G)$ is an elementary abelian p-group, then G is said to be an* **extra special** *p-group. This also means $|Z(G)| = p$ and therefore $Z(G)$ is cyclic.*

Remark 4.4.8 *[64] For every g in an extra special p-group G, $g \notin \Phi(G)$, there exists $h \in G$ such that the commutator $[g, h] \neq 1$. Also for any $g \in G$ we have $g^p \in Z(G)$. It then follows that*

$$\begin{aligned}(gh)^p &= g^p [h^{-1}, g^{-1}]^{g^{p-1}} [h^{-2}, g^{-1}]^{q^{p-2}} \cdots [h^{-p+1}, g^{-1}] h^p \\ &= g^p h^p [g, h]^{-1} \cdots [g, h]^{-p+1} \\ &= g^p h^p [h, g]^{\frac{p(p-1)}{2}}.\end{aligned}$$

This is reduced to $(gh)^2 = g^2 h^2 [h, g]$, when $p = 2$, which is just the multiplicative version of the definition of a quadratic form, so the squaring map $G/Z(G) \to Z(G)$ is a quadratic form. However, if p is odd then we have $(gh)^p = g^p h^p$. Then all elements have order p, or the elements of order 1 or p form a characteristic subgroup of index p. The classification of non-singular quadratic forms implies that there are exactly two isomorphism types of extra special p-groups of each order. We denote extra special groups of order 2^{1+2n} whose associated quadratic form is of type ϵ by 2_ϵ^{2n+1}, where $\epsilon = \pm$. If p is odd, then we write p_+^{1+2n} for the extra special group of exponent p, and we have p_-^{1+2n} for the extra special group of exponent p^2.

Remark 4.4.9 *According to Pahlings in [52], the extra special p-group $P(n)$ of $+$ type has $q^{2n} - q^n$ elements of order 4.*

Lemma 4.4.10 *Let G be an extra special p-group of order p^{1+2n}. Then G has*

(i) $p^{2n}+p-1$ *conjugacy classes,*

(ii) $p^{2n}+p-1$ *irreducible characters,*

(iii) p^{2n} *linear characters,*

(iv) $p-1$ *faithful irreducible characters of degree p^n,*

Proof See [6]. □

4.4.2 Affine subgroup of $U(2n, q^2)$

Let $V = V(2n, q^2)$. The set $\{e_1, e_2, \cdots, e_{2n}\}$ is the basis for V satisfying

$$f(e_i, e_j) = \delta(i, 2n+1-j),$$

$i \leq j$, where f is a non-degenerate hermitian form. Let $G(n) = U(2n, q^2)$ be the unitary group defined on V. The order of $G(n)$ is

$$|G(n)| = q^{\frac{2n(2n-1)}{2}} \prod_{i=1}^{2n} (q^i - (-1)^i).$$

Define the affine subgroup $A(n)$ of $G(n)$ to be the stabilizer of a non-zero isotropic vector, say e_1, under the action of $G(n)$. Since $G(n)$ acts transitively on the non-zero isotropic vectors, then $[G(n){:}A(n)] = (q^{2n}-1)(q^{2n-1}+1)$, the number of isotropic vectors.

Remark 4.4.11 *Let $P = \{[v, a] \mid v \in V(2n-2, q^2),\ a \in GF(q^2),\ tr(a) + f(v, v) = 0\}$, where tr is the trace function $GF(q^2) \to GF(q)$ and $|P| = q^{4n-3}$. Multiplication in P is defined as follows:*

$$[v, a][u, b] = [v+u, a+b-f(v, u)].$$

Lemma 4.4.12 *The affine subgroup $A(n)$ is the semidirect product of a special p-group $P(n)$ of order q^{4n-3} and a subgroup isomorphic to $G(n-1)$. That is $A(n) = P(n){:}G(n-1) = q^{4n-3}{:}U(2n-2, q^2)$.*

Proof See [30]. □

Lemma 4.4.13 *Let w be an isotropic non-zero vector of $V(2n-2, q^2)$. If T is the trace map $GF(q^2) \to GF(q)$ and ϵ is a primitive p-th root of unity in $\mathbb{C}$, then the map defined by $\lambda[v, a] = \epsilon^{T(w,v)}$ is a linear character of $P(n)$. The subgroup of $G(n-1)$ that fixes w also fixes λ, and this subgroup is isomorphic to $A(n-1)$.*

Proof See [30]. □

Theorem 4.4.14 *[18] Let $A(n)$ be the affine subgroup of $G(n) = U(2n, q^2)$. Then the degrees of some of the $Irr(A(n))$ are the degrees of the $Irr(U(2n-2, q^2))$, the degrees of the $Irr(A(n-1))$ multiplied by $(q^{2n-2}-1)(q^{2n-3}+1)$, the degrees of the $Irr(U(2n-3, q^2))$ multiplied by $q^{2n-3}(q^{2n-2}-1)$. There are $q-1$ irreducible characters of the latter type.*

Proof From the proof of Lemma 4.4.12, we have that the centre of $P(n)$ is $Z(P(n)) = \{[0, a] \mid tr(a) = 0\}$. This then means that $P(n)$ has p^{4n-4} linear characters. These linear characters are described as in Lemma 4.4.13. If χ_u is fixed by some $A \in G(n-1)$, then $Au = u$. If $u = 0$, then there is one orbit of length one. The corresponding inertia factor group is isomorphic to $G(n-1)$. If $u \neq 0$ and $f(u, u) = 0$, then since $G(n-1)$ is transitive on the set of non-zero isotropic of $V(2n-2, q^2)$, we obtain one orbit of length $(q^{2n-2}-1)(q^{2n-3}+1)$, with the corresponding inertia factor group isomorphic to $A(n-1)$. For $0 \neq c \in GF(q)$, the action of the group $G(n-1)$ on the set of $u \in V(2n-3, q^2)$ is transitive such that $f(u, u) = c$. This then yields $q-1$ orbits each of length $q^{2n-3}(q^{2n-2}-1)$. The inertia factor groups in this case are isomorphic to $GU(2n-3)$. This process yields $q+1$ orbits of $G(n-1)$ on the linear characters of $P(n)$. The other degrees are obtained from the $q-1$ faithful characters of $P(n)$ of degree q^{2n-2} by Lemma 4.4.10 . The application of the Clifford-Fischer Theory completes the proof. □

Remark 4.4.15 *[30] Let $A(n)$ be the affine subgroup of $U(2n, q^2)$. For $i \in \mathbb{Z}$ satisfying $1 \leq i \leq n-1$, $A(n)$ has non-faithful irreducible characters of degree*

$$(q^{2n-2}-1)(q^{2n-3}+1)\cdots(q^{2n-2i}-1)(q^{2n-2i-1}+1)$$

which remain irreducible as modular characters for any prime different from p.

Remark 4.4.16 Action of $A(n)$ on $P(n)$ and on $Irr(P(n))$

The action of $A(n)$ on $P(n)$ is given by

$$[v, a]^{[u,b]A} = [A^{-1}v, a + f(u, v) - \overline{f(u, v)}],$$

where the bar denotes the involuntary automorphism of $GF(q^2)$ sending each element to its q^{th} power and A is identified with $G(n-1)$. This action of $A(n)$ on $P(n)$ therefore produces $2q$ orbits. The orbits of this action are determined through the coset analysis technique. In particular, the analysis of the identity coset which is $P(n)$. Since $P(n)$ is an extra special p-group, it has $p^{2n}+p-1$ conjugacy classes, by Lemma 4.4.10.

The action of $A(n)$ on the $Irr(P(n))$ also has $2q$ orbits. The orbit lengths are deduced from Theorem 4.4.14. We note that there are $q+1$ orbits from the action of $A(n)$ on the linear characters of $P(n)$. By Lemma 4.4.10, $P(n)$ has p^{2n} linear characters. The corresponding orbit lengths on the linear characters are 1, $(q^{2n-2}-1)(q^{2n-3}+1)$ and $q-1$ orbits of lengths $q^{2n-3}(q^{2n-2}-1)$. This process, so far, accounts for $q+1$ orbits of the $2q$ orbits. The other $q-1$ orbits are due to the $p-1$ faithful characters of $P(n)$. These characters are of degree p^n.

4.4.3 Affine subgroup of $U(2n+1, q^2)$

The description and analysis of the affine subgroups of the odd dimensional unitary groups are the same as that of the even dimensional case in Subsection 4.4.2.

5

The affine subgroup 5^2:GL$(2,5)$ of the general linear group GL$(3,5)$

In this chapter we consider the affine subgroup $\overline{G} = D(3){:}GL(2,5)$ of the general linear group $GL(3,5)$. Let $\mathbb{F} = GF(5)$ be the Galois field of 5 elements and $G = GL(2,5)$, the general linear group of 2×2 invertible matrices with entries in $\mathbb{F}$. This affine subgroup fixes a non-zero vector of the underlying space V of dimension 3 with entries in $\mathbb{F}$. It is of order 12000 and index $|GL(3,5){:}\overline{G}| = 5^3 - 1 = 124$ in $GL(3,5)$ by Remark 4.1.2. The subgroup $D(3)$, by Lemma 4.1.1, is an elementary abelian 5-group of order 5^2. Let us denote this subgroup $D(3)$ of $\overline{G}$ by N. In Lemma 4.1.1, it is proved that $\overline{G}$ is a split extension of N by G. Darafsheh and Iranmanesh, [19] and [35], have made a lot of contributions in the study of the characters and Fischer matrices of affine subgroups of general linear groups. The characters of these affine subgroups have also been studied by D.K. Faddeev [22], S.I. Gel'fand [28] and E.A. Siegel [59] among others. In Section 5.1 we express the generators of N and G as non-singular 3×3 matrices with entries in $\mathbb{F}$ since $\overline{G}$ is a subgroup of $GL(3,5)$. We compute the permutation character of G on the holomorph 5:4 in Section 5.2. The conjugacy classes of $\overline{G}$ are determined in Section 5.3 using the coset analysis technique. In Section 5.4 we compute the Fischer matrices of $\overline{G}$ using the method discussed in Subsection 4.1.5. In Section 5.5 we construct the character table of $\overline{G}$ utilizing the Clifford-Fischer Theory. Lastly in Section 5.6 we consider the fusion of the conjugacy classes of $\overline{G}$ into the conjugacy classes of $GL(3,5)$.

5.1 The generators of the groups 5^2 and $GL(2,5)$

We construct the elements of N and G as 3×3 invertible matrices with entries in $\mathbb{F}$ since $\overline{G}$ sits in $GL(3,5)$. Let $n = 3$ and $\mathbb{F}$, N, V and G be as defined in the introduction above. The general structure of $\overline{G} = N{:}G$ is described and discussed in Subsection 4.1.1 in terms of elements of N and G as matrices of the form

$$\left[\begin{array}{c|c} 1 & v \\ \hline 0 & \\ \vdots & I \\ 0 & \end{array}\right]$$

and

$$\left[\begin{array}{c|c} 1 & 0\cdots0 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right]$$

respectively, where $v = (e_1, e_2)$, $e_i \in \mathbb{F}$, I the 2×2 identity matrix and A a non-singular 2×2 matrix over $\mathbb{F}$. We use GAP [27] to obtain all 25 elements, 2 generators of N and 2 generators of G. The generators of N are

$$n_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$n_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The generators of G are

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

with $o(a) = 4$ and $o(b) = 3$.

5.2 The permutation character of G on $5{:}4$

The action of G on N fixes the zero vector and is transitive on the non-zero elements $N^* = N - \{0\}$. This yields two orbits of lengths 1 and 24 due to Subsection 4.1.2. These correspond to the point stabilizers G and a group H of order 20. In fact this group H is an affine subgroup of G. Thus $H = 5{:}GL(1,5)$ and $GL(1,5) = \mathbb{F}^*$, the group of invertible non-zero elements of the field $\mathbb{F}$. This group is isomorphic to a cyclic group of order 4. Let C_m be a cyclic group of order a natural number m. Then H is the split extension $C_5{:}C_4 = 5{:}4$. In Subsection 4.1.3 we define the holomorph of a group M to be the semidirect product $M{:}\mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ acts naturally on M. In Lemma 4.1.4 we have that when m is a prime then $\mathrm{Aut}(C_m) = C_{m-1}$. Thus the split extension $C_5{:}C_4$ is a holomorph of C_5. In Subsection 4.1.3 we further discuss the construction of the character table of the holomorph $C_m{:}C_{m-1}$. We adopt this method to construct the character table of the holomorph $H = 5{:}4$ below.

Table 5.1: Character Table of 5:4

$\lvert C_G \rvert$	20	5	4	4	4
$o(g)$	1a	5a	2a	4a	4b
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	A	$-A$
χ_4	1	1	-1	$-A$	A
χ_5	4	-1	0	0	0

where $A = \sqrt{-1}$.

We use GAP to generate the character table of G.

Table 5.2: Character Table of $GL(2,5)$

$\lvert C_G \rvert$	480	16	480	24	16	16	16	16	16	480	480	20
$o(g)$	1A	2A	2B	3A	4A	4B	4C	4D	4E	4F	4G	5A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	1	-1	-1	1	1	1
χ_3	1	-1	1	1	-A	-A	1	A	A	-1	-1	1
χ_4	1	-1	1	1	A	A	1	-A	-A	-1	-1	1
χ_5	4	0	4	1	0	0	0	0	0	4	4	-1
χ_6	4	0	4	1	0	0	0	0	0	4	4	-1
χ_7	4	0	-4	-2	0	0	0	0	0	E	-E	-1
χ_8	4	0	-4	-2	0	0	0	0	0	-E	E	-1
χ_9	4	0	4	1	0	0	0	0	0	-4	-4	-1
χ_{10}	4	0	4	1	0	0	0	0	0	-4	-4	-1
χ_{11}	4	0	-4	1	0	0	0	0	0	E	-E	-1
χ_{12}	4	0	-4	1	0	0	0	0	0	E	-E	-1
χ_{13}	4	0	-4	1	0	0	0	0	0	-E	E	-1
χ_{14}	4	0	-4	1	0	0	0	0	0	-E	E	-1
χ_{15}	5	1	5	-1	-1	-1	1	-1	-1	5	5	0
χ_{16}	5	1	5	-1	1	1	1	1	1	5	5	0
χ_{17}	5	-1	5	-1	-A	-A	1	A	A	-5	-5	0
χ_{18}	5	-1	5	-1	A	A	1	-A	-A	-5	-5	0
χ_{19}	6	-2	6	0	0	0	-2	0	0	6	6	1
χ_{20}	6	2	6	0	0	0	-2	0	0	-6	-6	1
χ_{21}	6	0	-6	0	C	-C	0	/C	-/C	F	-F	1
χ_{22}	6	0	-6	0	/C	-/C	0	C	-C	-F	F	1
χ_{23}	6	0	-6	0	-/C	/C	0	-C	C	-F	F	1
χ_{24}	6	0	-6	0	-C	C	0	-/C	/C	F	-F	1

Table 5.2: Character Table of $GL(2,5)$ - continued

$\|C_G\|$	24	24	24	20	24	24	20	20	24	24	24	24
$o(g)$	6A	8A	8B	10A	12A	12B	20A	20B	24A	24B	24C	24D
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1	1	1	-1	-1	-1	-1
χ_3	1	-A	A	1	-1	-1	-1	-1	-A	-A	A	A
χ_4	1	A	-A	1	-1	-1	-1	-1	A	A	-A	-A
χ_5	1	-2	-2	-1	1	1	-1	-1	1	1	1	1
χ_6	1	2	2	-1	1	1	-1	-1	-1	-1	-1	-1
χ_7	2	0	0	1	B	-B	A	-A	0	0	0	0
χ_8	2	0	0	1	-B	B	-A	A	0	0	0	0
χ_9	1	B	-B	-1	-1	-1	1	1	-A	-A	A	A
χ_{10}	1	-B	B	-1	-1	-1	1	1	A	A	-A	-A
χ_{11}	-1	0	0	1	-A	A	A	-A	D	-D	-/D	/D
χ_{12}	-1	0	0	1	-A	A	A	-A	-D	D	/D	-/D
χ_{13}	-1	0	0	1	A	-A	-A	A	-/D	/D	D	-D
χ_{14}	-1	0	0	1	A	-A	-A	A	/D	-/D	-D	D
χ_{15}	-1	1	1	0	-1	-1	0	0	1	1	1	1
χ_{16}	-1	-1	-1	0	-1	-1	0	0	-1	-1	-1	-1
χ_{17}	-1	A	-A	0	1	1	0	0	A	A	-A	-A
χ_{18}	-1	-A	A	0	1	1	0	0	-A	-A	A	A
χ_{19}	0	0	0	1	0	0	1	1	0	0	0	0
χ_{20}	0	0	0	1	0	0	-1	-1	0	0	0	0
χ_{21}	0	0	0	-1	0	0	A	-A	0	0	0	0
χ_{22}	0	0	0	-1	0	0	-A	A	0	0	0	0
χ_{23}	0	0	0	-1	0	0	-A	A	0	0	0	0
χ_{24}	0	0	0	-1	0	0	A	-A	0	0	0	0

where

A $= -E(4) = -\sqrt{-1} = -i$, B $= -2 \times E(4) = -2 \times \sqrt{-1} = -2i$, C $= 1 - E(4) = 1 - \sqrt{-1} = 1 - i$, D $= -E(24) + E(24)^{17}$, E $= -4 \times E(4) = -4 \times \sqrt{-1} = -4i$ and F $= 6 \times E(4) = 6 \times \sqrt{-1} = 6i$.

Next we consider the fusion of the conjugacy classes of H into the conjugacy classes of G. Since there is one class of orders 1 and 5 in G then it follows that the classes $[1a]$ and $[5a]$ of H fuse to the classes $[1A]$ and $[5A]$ of G respectively. Now either the class $[2a]_H$ of H fuses to $[2A]_G$ or $[2B]_G$ of G. Let us consider the elements

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix} \in [2a]_H,$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \in [2A]_G \quad \text{and}$$

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in G$$

with $o(g) = 20$. We note that

$$ghg^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} = y.$$

That is h is conjugate to y in G. Thus the class $[2a]_H$ fuses to the class $[2A]_G$ since the conjugacy classes $[2A]_G$ and $[2B]_G$ are disjoint in G. In a similar fashion we note that the classes $[4a]_H$ and $[4b]_H$ fuse respectively to the classes $[4B]_G$ and $[4E]_G$. These fusion maps are given in Table 5.3 below.

Table 5.3: Fusion of 5:4 into $GL(2,5)$

	$\lvert C_G\rvert$	480	16	480	24	16	16	16	16	16	480	480	20
	$o(g)$	1A	2A	2B	3A	4A	4B	4C	4D	4E	4F	4G	5A
$\lvert C_H\rvert$	$o(h)$												
1a	20	24											
2a	4		4										
4a	4						4						
4b	4									4			
5a	5												4
	$\chi(G\lvert H)$	24	4	0	0	0	4	0	0	4	0	0	4

Table 5.3: Fusion of 5:4 into $GL(2,5)$ - continued

	$\lvert C_G\rvert$	24	24	24	20	24	24	20	20	24	24	24	24
	$o(g)$	6A	8A	8B	10A	12A	12B	20A	20B	24A	24B	24C	24D
1a	20												
2a	4												
4a	4												
4b	4												
5a	5												
	$\chi(G\lvert H)$	0	0	0	0	0	0	0	0	0	0	0	0

In Theorem 2.6.12 we have that for $g \in G$ and $y_1, y_2, \cdots, y_m$ representatives of conjugacy classes of H that fuse to a class $[g]$, then a permutation character is given by

$$\chi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_H(y_i)|}.$$

If $H \cap [g] = \emptyset$ then $\chi(g) = 0$. Now in Table 5.3 above the values of the permutation character of G on H, $\chi(G|H)$, are given in the last row. In the latter part of this section we proceed to express this permutation character in terms of the irreducible characters of G. We recall that H is of index 24 in G. Thus the permutation character $\chi(G|H)$ of the action of G on H is of degree 24. We use the values of $\chi_i \in Irr(G)$ together with the above permutation character values to determine $\psi = \chi(G|H)$ in terms of $Irr(G)$. Let $i \in \{1, 2, 3, \cdots, 24\}$, then for each i we compute the inner products $\langle \chi_i, \psi \rangle$. The values of these inner products are listed in Table 5.4 below.

Table 5.4: Values of the inner products $\langle \chi_i, \psi \rangle$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
$\langle \chi_i, \psi \rangle$	1	0	0	0	0	0	0	0	0	0	0	0

Table 5.4: Values of the inner products $\langle \chi_i, \psi \rangle$ - continued

	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
$\langle \chi_i, \psi \rangle$	0	0	0	1	0	0	0	1	0	0	1	1

We now use the output in Table 5.4 to express $\chi(G|H)$ in terms of $Irr(G)$. In cases where $\langle \chi_i, \psi \rangle \neq 0$, the corresponding χ_i is referred to as the constituent of $\chi(G|H)$. Thus

$$\chi(G|H) = \chi_1 + \chi_{16} + \chi_{20} + \chi_{23} + \chi_{24}.$$

The characters of G are given in Table 5.2.

5.3 The conjugacy classes of $\overline{G}$

The character table of $\overline{G}$ will be constructed using the Clifford-Fischer Theory. But before we construct the character table, we need to compute the conjugacy classes of $\overline{G}$. We utilize the coset analysis technique to compute these conjugacy classes. Complete details of this technique are discussed in Subsection 2.3.1. Due to Remark 2.2.9, since $\overline{G}$ is a split extension, we analyse the coset Ng, where g is a class representative of G, and $\overline{G} = \bigcup_{g \in G} Ng$. This also entails computing the centralizer sizes and orders of these new conjugacy classes. We use $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_i}$ to compute the centralizer sizes of the new classes, where k is the number of fixed points when $g \in G$ acts on N. And where f_i of the k blocks of the coset Ng have fuse together to form a new orbit Ω_i. We have that $|\Omega_i| = \frac{f_i|N|}{k}$. This enables us to calculate the f_i-values for each class representative $g \in G$. In Section 5.2 we have that the action of G on N yields 2 orbits of lengths 1 and 24. The corresponding point stabilizers are $G = GL(2,5)$ and $H = 5{:}4$. By Definition 2.6.7, $\chi(G|N)$ is the permutation character of the action of G on N. Then $\chi(G|N) = 1 + \chi(G|H)$. That is

$$\chi(G|N) = \chi_1 + \chi_1 + \chi_{16} + \chi_{20} + \chi_{23} + \chi_{24}$$

$$= 2 \cdot \chi_1 + \chi_{16} + \chi_{20} + \chi_{23} + \chi_{24},$$

where $\chi_i \in Irr(G)$. We use this permutation character together with the character table of G, Table 5.2, to compute the values of k and we list them in Table 5.5 below.

Table 5.5: Fixed points of the action of $GL(2,5)$ on 5^2

$\lvert C_G \rvert$	480	16	480	24	16	16	16	16	16	480	480	20
$o(g)$	1A	2A	2B	3A	4A	4B	4C	4D	4E	4F	4G	5A
k	25	5	1	1	1	5	1	1	5	1	1	5

Table 5.5: Fixed points of the action of $GL(2,5)$ on 5^2 - continued

$\lvert C_G \rvert$	24	24	24	20	24	24	20	20	24	24	24	24
$o(g)$	6A	8A	8B	10A	12A	12B	20A	20B	24A	24B	24C	24D
k	1	1	1	1	1	1	1	1	1	1	1	1

Let us use the class $[2A]$ of G to demonstrate the formation of a conjugacy class of $\overline{G}$. A class representative of $[2A]$ is

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and the respective centralizer is

$$C_G(g) = \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right\rangle$$

with $|C_G(g)| = 16$. The permutation character value when $g \in 2A$ is $k = 5$, in Table 5.5. This means that under the action of g on N we have 5 fixed points. It also means that after the action of N, by conjugation, the coset Ng splits into 5 orbits. Each orbit is of length $\frac{|N|}{k} = \frac{25}{5} = 5$. The action of the centralizer $C_G(g)$ on these orbits yields 2 orbits of lengths $|\Omega_1| = 5$ and $|\Omega_2| = 20$. The corresponding f values are $f_1 = \frac{5 \cdot 5}{25} = 1$ and $f_2 = \frac{5 \cdot 20}{25} = 4$. That is, 4 of the original 5 orbits have fused together to form one orbit Ω_2. These f-values satisfy $\sum_{i=1}^{2} f_i = k$. This means that this class of G produces 2 classes of $\overline{G}$. The centralizer sizes of these new classes are

$$|C_{\overline{G}}(x_1)| = \frac{k|C_G(g)|}{f_1} = \frac{5 \cdot 16}{1} = 80$$

and

$$|C_{\overline{G}}(x_2)| = \frac{k|C_G(g)|}{f_2} = \frac{5 \cdot 16}{4} = 20.$$

Next we calculate the orders of the class representatives of the conjugacy classes of $\overline{G}$. We reconsider the class $2A$ for demonstration. Let $g \in 2A$, then $o(g) = 2 = m$. Recall that $\text{Char}(\mathbb{F}) = 5 = p$. Let $x \in \overline{G}$, $d \in N$ and $w = d \cdot d^g \cdot d^{g^2} \cdots d^{g^{m-1}}$. Since N is elementary abelian we are able to use the method outlined in Subsection 2.3.2. This method implies that $o(x) = m = 2$ if $w = 1_N$, otherwise $o(x) = pm = 5 \cdot 2 = 10$. For x_1 we have $d_1 = (0, 0)$ and $w = (0, 0)$ which means that $o(x_1) = 2$. For x_2, $d_2 = (1, 0)$ and $w = (1, 1)$ and thus $o(x_2) = 10$. The size of each conjugacy class of $\overline{G}$ is determined by $|[x_i]| = \frac{|\overline{G}|}{|C_{\overline{G}}(x_i)|}$. A full list of the conjugacy classes of $\overline{G}$ is contained in Table 5.6 below.

The number of conjugacy classes of $\overline{G}$ turns out to be 29. We accept this number as it coincides with $|Irr(\overline{G})| = \sum_{i=1}^{2} |Irr(\overline{H}_i)| = 24 + 5 = 29$, the number of irreducible characters of $\overline{G}$, according to Section 3.5.

Table 5.6: The conjugacy classes of $5^2{:}GL(2,5)$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
1A	25	1	12000	(0,0)	(0,0)	1A
		24	500	(0,1)	(0,1)	5A
2A	5	1	80	(0,0)	(0,0)	2A
		4	20	(1,0)	(1,1)	10A
2B	1	1	480	(0,0)	(0,0)	2B
3A	1	1	24	(0,0)	(0,0)	3A
4A	1	1	16	(0,0)	(0,0)	4A
4B	5	1	80	(0,0)	(0,0)	4B
		4	20	(1,0)	(1,3)	20A
4C	1	1	16	(0,0)	(0,0)	4C
4D	1	1	16	(0,0)	(0,0)	4D
4E	5	1	80	(0,0)	(0,0)	4E
		4	20	(1,0)	(3,1)	20B
4F	1	1	480	(0,0)	(0,0)	4F
4G	1	1	480	(0,0)	(0,0)	4G
5A	5	1	100	(0,0)	(0,0)	5B
		4	25	(1,0)	(0,0)	5C
6A	1	1	24	(0,0)	(0,0)	6A
8A	1	1	24	(0,0)	(0,0)	8A
8B	1	1	24	(0,0)	(0,0)	8B
10A	1	1	20	(0,0)	(0,0)	10B
12A	1	1	24	(0,0)	(0,0)	12A
12B	1	1	24	(0,0)	(0,0)	12B
20A	1	1	20	(0,0)	(0,0)	20C
20B	1	1	20	(0,0)	(0,0)	20D
24A	1	1	24	(0,0)	(0,0)	24A
24B	1	1	24	(0,0)	(0,0)	24B
24C	1	1	24	(0,0)	(0,0)	24C
24D	1	1	24	(0,0)	(0,0)	24D

5.4 Fischer matrices of $\overline{G}$

We will follow the method used by Fischer in [23] to construct the character table of $\overline{G}$. This method entails utilizing the character tables of the inertia factor groups and Fischer matrices of $\overline{G}$. The Clifford-Fischer Theory requires that the irreducible characters of the normal subgroup $N \trianglelefteq \overline{G}$ be extendable to the inertia groups. Due to Mackey's Theorem, since N is elementary abelian and $\overline{G}$ is a split extension of N by G, then the irreducible characters of N are extendable to its inertia group. The consequence of Brauer's Theorem is that the action of G on $Irr(N)$ will produce 2 orbits since the action of G on N yielded 2 orbits. The orbit lengths are again 1 and 24, and the inertia factor groups are $H_1 = G$ and $H_2 = H = 5{:}4$ as in Section 5.2. In the same section we dealt with the fusion of the inertia factor group H into G.

We can therefore proceed to consider the construction of the Fischer matrices of $\overline{G}$. Since G has 24 conjugacy classes then $\overline{G}$ will have 24 Fischer matrices corresponding to each class of G. Considering the general structure of the Fischer matrix in Remark 3.3.2 and that Fischer matrices are square matrices, by Remark 3.3.6, we are able to deduce the sizes of these matrices from Table 5.6 of the conjugacy classes of $\overline{G}$. Thus the Fischer matrices in this example will be 1×1 and 2×2 square matrices. In Section 4.1.5 the Fischer matrices of $\overline{G}$ are generalized based on a method developed by List in [43]. We demonstrate the computation of the Fischer matrices starting with the identity class of G. We recall that $\overline{G} = q^{n-1}{:}GL(n-1,q) = 5^2{:}GL(2,5)$. Since the action of $\overline{G}$ on N yields 2 orbits then the Fischer matrix corresponding to the identity class is always a 2×2 matrix. The entries of the first column are 1 and $q^{n-1} - 1 = 25 - 1 = 24$, corresponding to the orbit lengths of the action. The remaining entry is determined from the orthogonality relations of a Fischer matrix in Remark 3.3.6. Thus the Fischer matrix from the identity class of G is

$$M(1A) = \begin{bmatrix} 1 & 1 \\ q^{n-1}-1 & a \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 24 & -1 \end{bmatrix}.$$

The following discussion is based on Remarks 3.3.10 and 3.3.12. The Fischer matrices from non-identity classes can be deduced from the character table of the quotient N/M, where $M = \{mgm^{-1}g^{-1} : m \in N\} \leq N$ and $[N{:}M] = k$. Here k has the same meaning as in the coset analysis, that is, k is number of fixed points when $g \in G$ acts on N. These Fischer matrices are square matrices with rows equal to the orbit sums of the action of the centralizer C_G on $Irr(N/M)$. This action is equivalent to the action of C_G on $V_2(5)/M = N/M = V_1(5)$. For the class $[g] = 2A$ we have $|M| = 5$. From the coset analysis, Table 5.6, we note that the Fischer matrix corresponding to $g \in 2A$ is a 2×2 matrix, that is the class $[2A]$ produces 2 classes of $\overline{G}$. We also note from this table that we initially had 5 orbits under the action of N on Ng (by conjugation). We further note that $f_1 = 1$ and $f_2 = 4$, meaning that 4 of these orbits have fused together to form 1 orbit, and ultimately to have 2 orbits. The action of C_G on $Irr(N/M)$ yields

$$\begin{bmatrix} 1 & 1 & 1 & \cdots \\ 4 & -1 & -1 & \cdots \end{bmatrix}$$

Deleting the repeated columns we obtain the Fischer matrix

$$M(2A) = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$$

corresponding to $g \in 2A$. Now let us consider the class $2B$ in G. From the coset analysis, Table 5.6, we note that this class produces 1 class of $\overline{G}$. Following the notation in [60], since the class $2B$ is of type A_1 with the diagonal entries not equal to 1, then $|M| = |N|$ which implies $N/M = 1$. This means after the action of C_G on $Irr(N/M)$ we obtain a 1×1 character table. Therefore the Fischer matrix from this class is

$$M(2B) = \begin{bmatrix} 1 \end{bmatrix}.$$

We remark that in the event that there is no fusion of classes during the coset analysis, that is, C_G fixes the character table of N/M, then the required Fischer matrix is exactly the character table of N/M. Programmes C and D in [15] are utilized to compute all the Fischer matrices of $\overline{G}$ in Table 5.7 below.

Table 5.7: Fischer matrices of $5^2{:}GL(2,5)$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$
$o(x_{1j})$		1	5
$\lvert C_{\overline{G}}(x_{1j})\rvert$		12000	500
(k,m)	$\lvert C_{H_k}(x_{1km})\rvert$		
$(1,1)$	480	1	1
$(2,1)$	20	24	-1
m_{1j}		1	24

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$
$o(x_{2j})$		2	10
$\lvert C_{\overline{G}}(x_{2j})\rvert$		80	20
(k,m)	$\lvert C_{H_k}(x_{2km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	4	4	-1
m_{2j}		5	20

$\mathbb{F}_3 = M(2B)$		$x_{3,1}$
$o(x_{3j})$		2
$\lvert C_{\overline{G}}(x_{3j})\rvert$		480
(k,m)	$\lvert C_{H_k}(x_{3km})\rvert$	
$(1,1)$	480	1
m_{3j}		25

$\mathbb{F}_4 = M(3A)$		$x_{4,1}$
$o(x_{4j})$		3
$\lvert C_{\overline{G}}(x_{4j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{4km})\rvert$	
$(1,1)$	24	1
m_{4j}		25

$\mathbb{F}_5 = M(4A)$		$x_{5,1}$
$o(x_{5j})$		4
$\lvert C_{\overline{G}}(x_{5j})\rvert$		16
(k,m)	$\lvert C_{H_k}(x_{5km})\rvert$	
$(1,1)$	16	1
m_{5j}		25

$\mathbb{F}_6 = M(4B)$		$x_{6,1}$	$x_{6,2}$
$o(x_{6j})$		4	20
$\lvert C_{\overline{G}}(x_{6j})\rvert$		80	20
(k,m)	$\lvert C_{H_k}(x_{6km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	4	4	-1
m_{6j}		5	20

Table 5.7: Fischer matrices of $5^2{:}GL(2,5)$ - continued

$\mathbb{F}_7 = M(4C)$		$x_{7,1}$
$o(x_{7j})$		4
$\lvert C_{\overline{G}}(x_{7j})\rvert$		16
(k,m)	$\lvert C_{H_k}(x_{7km})\rvert$	
$(1,1)$	16	1
m_{7j}		25

$\mathbb{F}_8 = M(4D)$		$x_{8,1}$
$o(x_{8j})$		4
$\lvert C_{\overline{G}}(x_{8j})\rvert$		16
(k,m)	$\lvert C_{H_k}(x_{8km})\rvert$	
$(1,1)$	16	1
m_{8j}		25

$\mathbb{F}_9 = M(4E)$		$x_{9,1}$	$x_{9,2}$
$o(x_{9j})$		4	20
$\lvert C_{\overline{G}}(x_{9j})\rvert$		80	20
(k,m)	$\lvert C_{H_k}(x_{9km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	4	4	-1
m_{9j}		5	20

$\mathbb{F}_{10} = M(4F)$		$x_{10,1}$
$o(x_{10j})$		4
$\lvert C_{\overline{G}}(x_{10j})\rvert$		480
(k,m)	$\lvert C_{H_k}(x_{10km})\rvert$	
$(1,1)$	480	1
m_{10j}		25

$\mathbb{F}_{11} = M(4G)$		$x_{11,1}$
$o(x_{11j})$		4
$\lvert C_{\overline{G}}(x_{11j})\rvert$		480
(k,m)	$\lvert C_{H_k}(x_{11km})\rvert$	
$(1,1)$	480	1
m_{11j}		25

$\mathbb{F}_{12} = M(5A)$		$x_{12,1}$	$x_{12,2}$
$o(x_{12j})$		5	5
$\lvert C_{\overline{G}}(x_{12j})\rvert$		100	25
(k,m)	$\lvert C_{H_k}(x_{12km})\rvert$		
$(1,1)$	20	1	1
$(2,1)$	5	4	-1
m_{12j}		5	20

$\mathbb{F}_{13} = M(6A)$		$x_{13,1}$
$o(x_{13j})$		6
$\lvert C_{\overline{G}}(x_{13j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{13km})\rvert$	
$(1,1)$	24	1
m_{13j}		25

$\mathbb{F}_{14} = M(8A)$		$x_{14,1}$
$o(x_{14j})$		8
$\lvert C_{\overline{G}}(x_{14j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{14km})\rvert$	
$(1,1)$	24	1
m_{14j}		25

$\mathbb{F}_{15} = M(8B)$		$x_{15,1}$
$o(x_{15j})$		8
$\lvert C_{\overline{G}}(x_{15j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{15km})\rvert$	
$(1,1)$	24	1
m_{15j}		25

$\mathbb{F}_{16} = M(10A)$		$x_{16,1}$
$o(x_{16j})$		10
$\lvert C_{\overline{G}}(x_{16j})\rvert$		20
(k,m)	$\lvert C_{H_k}(x_{16km})\rvert$	
$(1,1)$	20	1
m_{16j}		25

Table 5.7: Fischer matrices of $5^2{:}GL(2,5)$ - continued

$\mathbb{F}_{17} = M(12A)$		$x_{17,1}$
$o(x_{17j})$		12
$\lvert C_{\overline{G}}(x_{17j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{17km})\rvert$	
$(1,1)$	24	1
m_{17j}		25

$\mathbb{F}_{18} = M(12B)$		$x_{18,1}$
$o(x_{18j})$		12
$\lvert C_{\overline{G}}(x_{18j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{18km})\rvert$	
$(1,1)$	24	1
m_{18j}		25

$\mathbb{F}_{19} = M(20C)$		$x_{19,1}$
$o(x_{19j})$		20
$\lvert C_{\overline{G}}(x_{19j})\rvert$		20
(k,m)	$\lvert C_{H_k}(x_{19km})\rvert$	
$(1,1)$	20	1
m_{19j}		25

$\mathbb{F}_{20} = M(20D)$		$x_{20,1}$
$o(x_{20j})$		20
$\lvert C_{\overline{G}}(x_{20j})\rvert$		20
(k,m)	$\lvert C_{H_k}(x_{20km})\rvert$	
$(1,1)$	20	1
m_{20j}		25

$\mathbb{F}_{21} = M(24A)$		$x_{21,1}$
$o(x_{21j})$		24
$\lvert C_{\overline{G}}(x_{21j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{21km})\rvert$	
$(1,1)$	24	1
m_{21j}		25

$\mathbb{F}_{22} = M(24B)$		$x_{22,1}$
$o(x_{22j})$		24
$\lvert C_{\overline{G}}(x_{22j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{22km})\rvert$	
$(1,1)$	24	1
m_{22j}		25

$\mathbb{F}_{23} = M(24C)$		$x_{23,1}$
$o(x_{23j})$		24
$\lvert C_{\overline{G}}(x_{23j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{23km})\rvert$	
$(1,1)$	24	1
m_{23j}		25

$\mathbb{F}_{24} = M(24D)$		$x_{24,1}$
$o(x_{24j})$		24
$\lvert C_{\overline{G}}(x_{24j})\rvert$		24
(k,m)	$\lvert C_{H_k}(x_{24km})\rvert$	
$(1,1)$	24	1
m_{24j}		25

The values of $|C_{\overline{G}}(x_{ij})|$ are obtained from Table 5.6 of the conjugacy classes of $\overline{G}$, the values of $|C_{H_k}(x_{ikm})|$ are obtained from the character tables 5.1 and 5.2 of H and G, respectively, and $m_{ij} = \frac{f_i \cdot |N|}{k}$. These values play a crucial role when a Fischer matrix is constructed using the orthogonality relations and the Fischer matrix properties as will be seen in subsequent chapters.

5.5 The character table of $\overline{G}$

We construct the character table of $\overline{G}$ using the Clifford-Fischer Theory. We follow the method used by Fischer. This method entails utilizing the character tables of the inertia factor groups and Fischer matrices of $\overline{G}$. The Clifford-Fischer Theory requires that the irreducible characters of the normal subgroup $N \trianglelefteq \overline{G}$ be extendable to the inertia groups. Due to Mackey's Theorem, since N is elementary abelian and $\overline{G}$ is a split extension of N by $GL(2,5)$, then the irreducible characters of N are extendable to its inertia group. Recall that $|Irr(\overline{G})|$ is the same as the number of the conjugacy classes of $\overline{G}$. According to Table 5.6 $\overline{G}$ has 29 irreducible characters. According to Remark 4.1.7, the polynomial q^2+q-1, which gives $5^2+5-1=29$, can also be used to determine the number of $Irr(\overline{G})$. In Corollary 4.1.6 we have that the number of $Irr(\overline{G})$ can also be computed using $1 + Irr(\mathrm{GL}(1,5)) + Irr(\mathrm{GL}(2,5))$ which gives us $1+4+24=29$. Due to Gallagher's Theorem and Remark 3.1.20 the irreducible characters of $\overline{G}$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\phi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \in ker(\beta)\},$$

where $\overline{H}_i$ is an inertia group and $H_i = \overline{H}_i/N$ is an inertia factor group. This then means that the character table of $\overline{G}$ will be divided into blocks corresponding to the inertia factor groups H_i for $i \in \{1,2\}$. Full details of this process can be seen in Chapter 3. Then $H_1 = GL(2,5)$ and $H_2 = 5{:}4$. Thus the character table of $\overline{G}$ will be of the form

$$\begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \cdots & B_{1,24} \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdots & B_{2,24} \end{bmatrix}$$

where $B_{i,j}$ are blocks corresponding to the inertia factor groups and the 24 conjugacy classes of $GL(2,5)$, $\{1 \le i \le 2\}$ and $\{1 \le j \le 24\}$. The block $B_{i,j}$ is formed by multiplying the relevant columns of the character table of H_i by the rows of the Fischer matrix $M(g)$ corresponding to the classes of H_i that fuse to the class $[g] \in G$. If H_i does not contribute to $M(g)$ then the block $B_{i,j}$ will have zeroes. The character tables and fusion maps of the inertia factor groups are given in Tables 5.1, 5.2 and 5.3 respectively.

The full character table of $\overline{G}$ is given by Table 5.8. First we demonstrate how to compute the entries of this character table. Let us consider the class $1A$ of G. The class $1a$ of H_2 fuses to the class $1A$ of $H_1 = G$. Thus we multiply the column corresponding to $1A$ in Table 5.2 by the first row of matrix $M(1A)$ in Table 5.7. Then we multiply the column corresponding to $1a$ in Table 5.1 by the second row of matrix $M(1A)$ in Table 5.7 corresponding to the inertia factor group H_2 to have the following entries from the class $1A$ of G.

$$B_{1,1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 5 \\ 5 \\ 5 \\ 5 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 5 & 5 \\ 5 & 5 \\ 5 & 5 \\ 5 & 5 \\ 6 & 6 \\ 6 & 6 \\ 6 & 6 \\ 6 & 6 \\ 6 & 6 \\ 6 & 6 \end{bmatrix}$$

$$B_{2,1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 24 & -1 \end{bmatrix} = \begin{bmatrix} 24 & -1 \\ 24 & -1 \\ 24 & -1 \\ 24 & -1 \\ 96 & -4 \end{bmatrix}$$

Table 5.8: Character Table of $5^2{:}GL(2,5)$

$[g]$	1A		2A		2B	3A	4A	4B		4C	4D	4E		4F	4G
$[x]$	1A	5A	2A	10A	2B	3A	4A	4B	20A	4C	4D	4E	20B	4F	4G
$\lvert C_{\overline{G}}\rvert$	12000	500	80	20	480	24	16	80	20	16	16	80	20	480	480
2P	1A	5A	1A	5A	1A	3A	2A	2A	10A	2B	2A	2A	10A	2B	2B
3P	1A	5A	2A	10A	2B	1A	4D	4E	20B	4C	4A	4B	20A	4G	4F
5P	1A	1A	2A	2A	2B	3A	4A	4B	4B	4C	4D	4E	4E	4F	4G
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	1
χ_3	1	1	-1	-1	1	1	A	A	A	1	-A	-A	-A	-1	-1
χ_4	1	1	-1	-1	1	1	-A	-A	-A	1	A	A	A	-1	-1
χ_5	4	4	0	0	4	1	0	0	0	0	0	0	0	4	4
χ_6	4	4	0	0	4	1	0	0	0	0	0	0	0	4	4
χ_7	4	4	0	0	-4	-2	0	0	0	0	0	0	0	-C	C
χ_8	4	4	0	0	-4	-2	0	0	0	0	0	0	0	C	-C
χ_9	4	4	0	0	4	1	0	0	0	0	0	0	0	-4	-4
χ_{10}	4	4	0	0	4	1	0	0	0	0	0	0	0	-4	-4
χ_{11}	4	4	0	0	-4	1	0	0	0	0	0	0	0	-C	C
χ_{12}	4	4	0	0	-4	1	0	0	0	0	0	0	0	-C	C
χ_{13}	4	4	0	0	-4	1	0	0	0	0	0	0	0	C	-C
χ_{14}	4	4	0	0	-4	1	0	0	0	0	0	0	0	C	-C
χ_{15}	5	5	1	1	5	-1	-1	-1	-1	1	-1	-1	-1	5	5
χ_{16}	5	5	1	1	5	-1	1	1	1	1	1	1	1	5	5
χ_{17}	5	5	-1	-1	5	-1	A	A	A	1	-A	-A	-A	-5	-5
χ_{18}	5	5	-1	-1	5	-1	-A	-A	-A	1	A	A	A	-5	-5
χ_{19}	6	6	-2	-2	6	0	0	0	0	-2	0	0	0	6	6
χ_{20}	6	6	2	2	6	0	0	0	0	-2	0	0	0	-6	-6
χ_{21}	6	6	0	0	-6	0	B	-B	-B	0	/B	-/B	-/B	D	-D
χ_{22}	6	6	0	0	-6	0	/B	-/B	-/B	0	B	-B	-B	-D	D
χ_{23}	6	6	0	0	-6	0	-/B	/B	/B	0	-B	B	B	-D	D
χ_{24}	6	6	0	0	-6	0	-B	B	B	0	-/B	/B	/B	D	-D
χ_{25}	24	-1	4	-1	0	0	0	4	-1	0	0	4	-1	0	0
χ_{26}	24	-1	4	-1	0	0	0	-4	1	0	0	-4	1	0	0
χ_{27}	24	-1	-4	1	0	0	0	C	A	0	0	-C	-A	0	0
χ_{28}	24	-1	-4	1	0	0	0	-C	-A	0	0	C	A	0	0
χ_{29}	96	-4	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.8: Character Table of $5^2{:}GL(2,5)$

$[g]$	5A		6A	8A	8B	10B	12A	12B	20C	20D	24A	24B	24C	24D
$[x]$	5B	5C	6A	8A	8B	10B	12A	12B	20C	20D	24A	24B	24C	24D
$\|C_{\overline{G}}\|$	100	25	24	24	24	20	24	24	20	20	24	24	24	24
$2P$	5B	5C	3A	4F	4G	5B	6A	6A	10B	10B	12A	12A	12B	12B
$3P$	5B	5C	2B	8B	8A	10B	4G	4F	20D	20C	8B	8B	8A	8A
$5P$	1A	1A	6A	8A	8B	2B	12A	12B	4F	4G	24A	24B	24C	24D
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	1	A	-A	1	-1	-1	-1	-1	A	A	-A	-A
χ_4	1	1	1	-A	A	1	-1	-1	-1	-1	-A	-A	A	A
χ_5	-1	-1	1	-2	-2	-1	1	1	-1	-1	1	1	1	1
χ_6	-1	-1	1	2	2	-1	1	1	-1	-1	-1	-1	-1	-1
χ_7	-1	-1	2	0	0	1	E	-E	-A	A	0	0	0	0
χ_8	-1	-1	2	0	0	1	-E	E	A	-A	0	0	0	0
χ_9	-1	-1	1	E	-E	-1	-1	-1	1	1	A	A	-A	-A
χ_{10}	-1	-1	1	-E	E	-1	-1	-1	1	1	-A	-A	A	A
χ_{11}	-1	-1	-1	0	0	1	A	-A	-A	A	F	-F	-/F	/F
χ_{12}	-1	-1	-1	0	0	1	A	-A	-A	A	-F	F	/F	-/F
χ_{13}	-1	-1	-1	0	0	1	-A	A	A	-A	-/F	/F	F	-F
χ_{14}	-1	-1	-1	0	0	1	-A	A	A	-A	/F	-/F	-F	F
χ_{15}	0	0	-1	1	1	0	-1	-1	0	0	1	1	1	1
χ_{16}	0	0	-1	-1	-1	0	-1	-1	0	0	-1	-1	-1	-1
χ_{17}	0	0	-1	-A	A	0	1	1	0	0	-A	-A	A	A
χ_{18}	0	0	-1	A	-A	0	1	1	0	0	A	A	-A	-A
χ_{19}	1	1	0	0	0	1	0	0	1	1	0	0	0	0
χ_{20}	1	1	0	0	0	1	0	0	-1	-1	0	0	0	0
χ_{21}	1	1	0	0	0	-1	0	0	-A	A	0	0	0	0
χ_{22}	1	1	0	0	0	-1	0	0	A	-A	0	0	0	0
χ_{23}	1	1	0	0	0	-1	0	0	A	-A	0	0	0	0
χ_{24}	1	1	0	0	0	-1	0	0	-A	A	0	0	0	0
χ_{25}	4	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{26}	4	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	4	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	4	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	-4	1	0	0	0	0	0	0	0	0	0	0	0	0

where

A $= -E(4) = -\sqrt{-1} = -i$, B $= 1 - E(4) = 1 - \sqrt{-1} = 1 - i$, C $= 4 \times E(4) = 4 \times \sqrt{-1} = 4i$, D $= 6 \times E(4) = 6 \times \sqrt{-1} = 6i$, E $= 2 \times E(4) = 2 \times \sqrt{-1} = 2i$ and F $= -E(24) + E(24)^{17}$.

The consistency and accuracy of this character table was checked using Programme E in [15]. The power maps were calculated using Programmes A and B in [15]. These are listed in rows 4, 5 and 6 in Table 5.8 above. We use Lemma 4.1.5 and Theorem 4.1.8 to determine the character degrees of $\overline{G}$. This affine subgroup $\overline{G}$ has 4 irreducible characters of degree 1, 10 irreducible characters of degree 4, 4 irreducible characters of degree 5, 6 irreducible characters of degree 6, 4 irreducible characters of degree 24 and 1 irreducible character of degree 96.

5.6 The fusion of $\overline{G}$ into $GL(3,5)$

Finally in this chapter we consider the fusion of $\overline{G}$ into the general linear group GL$(3,5)$. Let $[y]$ and $[x]$ be the conjugacy classes of GL$(3,5)$ and $\overline{G}$ respectively. We follow the method used in Section 5.2. That is we consider the divisibility of the respective centralizer sizes $\frac{|C(y)|}{|C(x)|}$, where y is the class representative of GL$(3,5)$ and x the class representative of $\overline{G}$ with $o(y) = o(x)$. We also use the permutation character of GL$(3,5)$ on the cosets of $\overline{G}$ in GL$(3,5)$ together with the respective power maps to construct the partial fusion of $\overline{G}$ into GL$(3,5)$. The permutation character $\chi(\mathrm{GL}(3,5)|\overline{G})$ is computed using GAP. We follow a method similar to the one we used in Section 5.2 to express this permutation character in terms of $Irr(\mathrm{GL}(3,5)$. This method yields the following permutaion character

$$\chi(\mathrm{GL}(3,5)|\overline{G}) = \psi_1 + \psi_5 + \psi_{10} + \psi_{13} + \psi_{14},$$

in terms of the irreducible characters of GL$(3,5)$. The values of this permutation character are listed in the last row of Table 5.10 . The partial fusion is Table 5.10 but without considering the square boxes around the entries. With regard to power maps we recall the following. Suppose that $[x_1]$ and $[x_2]$ are conjugacy classes of $\overline{G}$ such that $x_1^p \in [x_2]$ for some prime p. And suppose that $[y_1]$ and $[y_2]$ are conjugacy classes of GL$(3,5)$ such that $y_1^p \in [y_2]$. Now if $[x_2]$ fuses to $[y_2]$ then it must follow that $[x_1]$ fuses to $[y_1]$. The irreducible characters and power maps of GL$(3,5)$ are found in Table A.1 in the Appendix.

In the partial fusion we note that the classes $2A$, $2B$ and $3A$ of $\overline{G}$ fuse into the classes $2b$, $2c$ and $3a$ of GL$(3,5)$ respectively. We also note in the partial fusion that there are instances where there is more than one candidate for a fusion. That is, for an example, there exists more than one class $[y]$ in GL$(3,5)$ such that $o(y) = o(x)$ and the quotient $\frac{|C(y)|}{|C(x)|}$ corresponds to the value of the permutation character. In such instances we utilize the method discussed by Moori in [45], used by Chileshe [15], Mpono [49] and Prince [53] among others. This method is about set intersections for characters which entails restricting the characters of GL$(3,5)$ to the characters of $\overline{G}$. We describe briefly this method.

Let ρ be the character afforded by the regular representation of GL$(2,5)$. It follows that $\rho = \sum_{i=1}^{24} e_i\phi_i$ where $\phi_i \in Irr(\mathrm{GL}(2,5))$ and $e_i = \deg(\phi_i)$. This then means that ρ can be seen as the character of $5^2{:}GL(2,5)$ which contains 5^2 in its kernel such that

$$\rho(g) = \begin{cases} |\mathrm{GL}(2,5)| & g \in 5^2 \\ 0 & \text{otherwise.} \end{cases}$$

Now if ψ is a character of GL(3,5) then

$$\langle \rho, \psi \rangle_{\overline{G}} = \frac{1}{|\overline{G}|}\{\rho(1A)\psi(1A) + 24\rho(5A)\psi(5A)\}$$

$$= \frac{1}{|5^2||\mathrm{GL}(2,5)|}\{|\mathrm{GL}(2,5)|\psi(1A) + 24|\mathrm{GL}(2,5)|\psi(5A)\}$$

$$= \frac{1}{25}\{\psi(1A) + 24\psi(5A)\}$$

$$= \langle \psi\downarrow_N, \tau_1 \rangle$$

where $\psi\downarrow_N$ is the restriction of ψ to 5^2 and τ_1 is the identity character of 5^2. We also note that for ψ we have that

$$\psi\downarrow_N = a_1\theta_1 + a_2\theta_2,$$

where for $i \in \{1,2\}$, θ_i are the sums of the irreducible characters of 5^2 which are in one orbit under the action of GL(2,5) on $Irr(5^2)$, and $a_i \in \{0\} \cup \mathbb{N}$. For $j \in \{1,2,3,\cdots,25\}$, let $\tau_j \in Irr(5^2)$. Recall that under the action of GL(2,5) on $Irr(5^2)$ we have orbits of lengths 1 and 24. Then we have that

$$\theta_1 = \tau_1 \quad \text{and} \quad \deg(\theta_1) = 1,$$

$$\theta_2 = \sum_{j=2}^{25} \tau_j \quad \text{and} \quad \deg(\theta_2) = 24.$$

Then

$$\psi\downarrow_N = a_1\tau_1 + a_2\sum_{j=2}^{25}\tau_j$$

and

$$\langle \psi\downarrow_N, \psi\downarrow_N \rangle = a_1^2 + 24a_2^2.$$

We note firstly that $a_1 = \langle \psi\downarrow_N, \tau_1 \rangle = \langle \rho, \psi \rangle_{\overline{G}}$ and secondly that

$$\langle \psi\downarrow_N, \psi\downarrow_N \rangle = \frac{1}{25}[\psi(1A)\psi(1A) + 24\psi(5A)\psi(5A)].$$

Let us consider the application of this technique. Suppose that $\psi_3 = 1c$, $\psi_6 = 30b$, $\psi_{15} = 31g$ and $\psi_{85} = 124y$, the irreducible characters of GL(3,5) of degrees 1, 30, 31 and 124 respectively. In the case of ψ_3 we have that

$$a_1 = \langle \rho, \psi_3 \rangle_{\overline{G}} = \frac{1}{25}[1 + 24(1)] = 1.$$

Since the degree of ψ_3 is 1 then

$$a_1 + 24a_2 = 1.$$

But since $a_1 = 1$ then it follows that $a_2 = 0$. This means that the restriction $(\psi_3)_{\overline{G}}$ is expressible as a character of degree 1 from the first block of the character table of $\overline{G}$ corresponding to the first inertia factor group H_1 =GL(2,5). Considering the predetermined partial fusion of $\overline{G}$ into $\mathrm{GL}(3,5)$ and the character tables of $\overline{G}$ and GL(3,5), we deduce that

$$(\psi_3)_{\overline{G}} = \chi_3.$$

The values of χ_i are listed in Table 5.8 and the character values of $\mathrm{GL}(3,5)$ can be found in Table A.1 in the Appendix.

On the other hand for ψ_6 we have

$$a_1 = \langle \rho, \psi_6 \rangle_{\overline{G}} = \frac{1}{25}[30 + 24(5)] = 6.$$

Since the degree of ψ_6 is 30 then

$$a_1 + 24a_2 = 30.$$

Because $a_1 = 6$ then we have $a_2 = 1$. This then implies that the restriction $(\psi_6)_{\overline{G}}$ can be expressed as a sum of a character of degree 6 from the first block of the character table of $\overline{G}$ corresponding to the first inertia factor group H_1 and a character of degree 24 from the second block corresponding to the second inertia factor group $H_2 = 5{:}4$. Again considering the predetermined partial fusion of $\overline{G}$ into $\mathrm{GL}(3,5)$ and the character tables of $\overline{G}$ and GL(3,5), we deduce that

$$(\psi_6)_{\overline{G}} = \chi_2 + \chi_{15} + \chi_{26}.$$

For ψ_{15} we have

$$a_1 = \langle \rho, \psi_{15} \rangle_{\overline{G}} = \frac{1}{25}[31 + 24(6)] = 7.$$

Since the degree of ψ_{15} is 31 then

$$a_1 + 24a_2 = 31.$$

This yields $a_2 = 1$ since $a_1 = 7$. This implies that the restriction $(\psi_{15})_{\overline{G}}$ can be expressed as a sum of a character of degree 7 from the first block of the character table of $\overline{G}$ corresponding to the first inertia factor group H_1 and a character of degree 24 from the second block corresponding to the second inertia factor group H_2. Considering the predetermined partial fusion of $\overline{G}$ into $\mathrm{GL}(3,5)$ and the character tables of $\overline{G}$ and GL(3,5), we deduce that

$$(\psi_{15})_{\overline{G}} = \chi_2 + \chi_{21} + \chi_{26}.$$

Lastly for ψ_{85} we have

$$a_1 = \langle \rho, \psi_{85} \rangle_{\overline{G}} = \frac{1}{25}[124 + 24(-1)] = 4.$$

As the degree of ψ_{85} is 124, then

$$a_1 + 24a_2 = 124.$$

This gives $a_2 = 5$. Again this implies that the restriction $(\psi_{85})_{\overline{G}}$ is expressible as a sum of a character of degree 4 from the first block of the character table of $\overline{G}$ corresponding to the first inertia factor group H_1 and a character of degree 120 from the second block corresponding to the second inertia factor group H_2. From the respective character tables and the partial fusion we deduce that

$$(\psi_{85})_{\overline{G}} = \chi_{11} + \chi_{25} + \chi_{29}.$$

In the partial fusion we observe that the class $4A$ of $\overline{G}$ can either fuse to the class $4m$ or $4o$ or $4p$ of GL(3,5). We apply the above method to choose the correct class for $4A$. From the character table of $\mathrm{GL}(3,5)$, ψ_3, ψ_6, ψ_{15} and ψ_{85} yield the following values.

Table 5.9: Values of ψ_i in GL(3,5)

	$[y]$	4m	4o	4p
Degree	ψ_i			
1c	ψ_3	A	$-A$	1
30b	ψ_6	-2	-2	2
31g	ψ_{15}	A	$-A$	1
124y	ψ_{85}	0	0	0

where $A = -\sqrt{-1} = -i$.

In the character table of $\overline{G}$ the values of the restrictions are:

$$(\psi_3)_{\overline{G}}(4A) = A,$$
$$(\psi_6)_{\overline{G}}(4A) = -2,$$
$$(\psi_{15})_{\overline{G}}(4A) = A$$

and

$$(\psi_{85})_{\overline{G}}(4A) = 0.$$

Comparing these values we conclude that the class $4A$ of $\overline{G}$ fuses into the class $4m$ of GL(3,5).

The values of ψ_3, ψ_6, ψ_{15} and ψ_{85} on the classes of $\mathrm{GL}(3,5)$ and the values of the restrictions $(\psi_3)_{\overline{G}}$, $(\psi_6)_{\overline{G}}$, $(\psi_{15})_{\overline{G}}$ and $(\psi_{85})_{\overline{G}}$ on the classes of $\overline{G}$ together with the predetermined fusion enable us to complete the fusion of $\overline{G}$ into $\mathrm{GL}(3,5)$. The complete fusion results are contained in Table 5.10 below.

Table 5.10: The fusion of $5^2{:}GL(2,5)$ into $GL(3,5)$

	$[y]$	$1a$	$2a$	$2b$	$2c$	$3a$	$4a$	$4b$	$4c$	$4d$	$4e$
	$\lvert C_G(y)\rvert$	1488000	1488000	1920	1920	96	1488000	1488000	1920	1920	1920
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$										
$1A$	12000	124									
$2A$	80			24	24						
$2B$	480			4	4						
$3A$	24					4					
	$\chi(GL(3,5)\lvert\overline{G})$	124	0	24	4	4	0	0	0	0	0

Table 5.10: The fusion of $5^2{:}GL(2,5)$ into $GL(3,5)$

	$[y]$	$4f$	$4g$	$4h$	$4i$	$4j$	$4k$	$4l$	$4m$	$4n$	$4o$	$4p$
	$\lvert C_G(y)\rvert$	1920	1920	1920	1920	1920	1920	1920	64	64	64	64
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$											
$4A$	16	120		120	120			120	4		4	4
$4B$	80	24		24	24			24				
$4C$	16	120		120	120			120	4		4	4
$4D$	16	120		120	120			120	4		4	4
$4E$	80	24		24	24			24				
$4F$	480	4		4	4			4				
$4G$	480	4		4	4			4				
	$\chi(GL(3,5)\lvert\overline{G})$	24	0	4	4	0	0	24	4	0	4	4

Table 5.10: The fusion of $5^2{:}GL(2,5)$ into $GL(3,5)$

	$[y]$	$5a$	$5b$	$6a$	$6b$	$6c$	$8a$	$8b$	$8c$	$8d$	$8e$	$8f$	$8g$	$8h$
	$\lvert C_G(y)\rvert$	2000	100	96	96	96	96	96	96	96	96	96	96	96
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$													
$5A$	500	4												
$5B$	100	20	1											
$5C$	25	80	4											
$6A$	24					4								
$8A$	24										4	4		
$8B$	24										4	4		
	$\chi(GL(3,5)\lvert\overline{G})$	24	4	0	0	4	0	0	0	0	4	4	0	0

Table 5.10: The fusion of $5^2{:}GL(2,5)$ into $GL(3,5)$

	$[y]$	$10c$	$10d$	$12f$	$12g$	$12h$	$12i$	$12j$	$12k$	$12l$
	$\lvert C_G(y)\rvert$	80	80	96	96	96	96	96	96	96
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$									
$10A$	20	4	4							
$10B$	20	4	4							
$12A$	24	4			4	4				
$12B$	24	4			4	4				
	$\chi(GL(3,5)\lvert\overline{G})$	4	4	0	4	4	0	0	0	0

Table 5.10: The fusion of $5^2{:}GL(2,5)$ into $GL(3,5)$

	$[y]$	$20h$	$20j$	$20k$	$20n$	$24i$	$24j$	$24k$	$24l$
	$\lvert C_G(y)\rvert$	80	80	80	80	96	96	96	96
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$								
$20A$	20	4	4	4	$\boxed{4}$				
$20B$	20	$\boxed{4}$	4	4	4				
$20C$	20	4	$\boxed{4}$	4	4				
$20D$	20	4	4	$\boxed{4}$	4				
$24A$	24					$\boxed{4}$	4	4	4
$24B$	24					4	$\boxed{4}$	4	4
$24C$	24					4	4	$\boxed{4}$	4
$24D$	24					4	4	4	$\boxed{4}$
	$\chi(GL(3,5)\lvert\overline{G})$	4	4	4	4	4	4	4	4

Table 5.11 below contains the summary of the fusion results of Table 5.10 above.

Table 5.11: The fusion of $5^2{:}GL(2,5)$ into $GL(3,5)$

$5^2{:}GL(2,5)$	$\rightarrow$	$GL(3,5)$	$5^2{:}GL(2,5)$	$\rightarrow$	$GL(3,5)$
$1A$		$1a$	$2A$		$2b$
$2B$		$2c$	$3A$		$3a$
$4A$		$4m$	$4B$		$4l$
$4C$		$4p$	$4D$		$4o$
$4E$		$4f$	$4F$		$4h$
$4G$		$4i$	$5A$ $5B$		$5a$
$5C$		$5b$	$6A$		$6c$
$8A$		$8e$	$8B$		$8f$
$10A$		$10c$	$10B$		$10d$
$12A$		$12h$	$12B$		$12g$
$20A$		$20n$	$20B$		$20h$
$20C$		$20j$	$20D$		$20k$
$24A$		$24i$	$24B$		$24j$
$24C$		$24k$	$24D$		$24l$

6

The affine subgroup $2^9{:}Sp(8,2)$ of the symplectic group $Sp(10,2)$

In this chapter the affine subgroup $A(5) = P(5){:}Sp(8,2)$ of the symplectic group $Sp(10,2)$ is considered. Let us denote this affine subgroup by $\overline{G}$. Let $\mathbb{F} = GF(2)$ be the Galois field of two elements. Let V be a non-degenerate symplectic space of dimension $2n$=10 over the field $\mathbb{F}$. By Theorem 4.2.28 the affine subgroup $\overline{G}$ is a split extension. The normal subgroup $P(5)$ is as defined in Remark 4.2.22. It is an elementary abelian 2-group, since Char($\mathbb{F}$) = 2, by Remark 4.2.26. Throughout this chapter the subgroup $P(5)$ will be denoted by N and the subgroup $Sp(8,2)$ by G. In Section 6.1 we deal with the transvections of G. We express the generators of N and G in Section 6.2 in terms of 10×10 symplectic matrices over $\mathbb{F}$ since $\overline{G}$ sits in $Sp(10,2)$, the symplectic group of 10×10 symplectic matrices with entries in $\mathbb{F}$. We compute the conjugacy classes of $\overline{G}$ using the coset analysis technique in Section 6.3. In Section 6.4 we consider the action of G on the $Irr(N)$ and determine the inertia factor groups. We also deal with the fusion maps of these inertia factor groups into G. In Section 6.5 we discuss the computation of the Fischer matrices of $\overline{G}$. Then in Section 6.6 we discuss the construction of the character table of $\overline{G}$ using the Clifford-Fischer Theory. We use the discussion in Subsection 4.2.6 to determine the centre of $\overline{G}$ in Section 6.7. We note that the quotient $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^8{:}Sp(8,2)$. We demonstrate how to obtain the Fischer matrices of this quotient directly from the Fischer matrices of $\overline{G}$.

6.1 Transvections of $Sp(8,2)$

We recall that the symplectic group G is generated by the set of all symplectic transvections. This section is based on the results we proved in Section 4.2.1 on symplectic transvections. Let $\overline{G}$, $\mathbb{F}$, N and V be as defined in the above introduction. There are $|V^*| = 2^8 - 1 = 255$ transvections in G by Proposition 4.2.16. According to Proposition 4.2.10 since the Char($\mathbb{F}$) = 2, then the order of these transvections is 2. In G there are 6 conjugacy classes of elements of order 2. However, by Proposition 4.2.14 there is one class of transvections in G, since $|\mathbb{F}| = 2$. According to the ATLAS [16] the class $2A$ in G has 255 elements. This number coincides with the cardinality of transvections in G. Therefore we conclude that the class $2A$ is the class of transvections in G. By Proposition 4.2.18 the centralizer of a transvection of $Sp(8,2)$ is isomorphic to the affine subgroup $2^7{:}Sp(6,2)$.

6.2 The generators of the groups N and G

Since $\overline{G}$ is an the affine subgroup of $Sp(10,2)$, we express the elements of N and G in terms of 10×10 symplectic matrices with entries in $\mathbb{F}$. We follow the process outlined in Remark 4.2.22. Since $n=5$, then we have

$$T(e_1) = e_1$$

$$T(e_i) = \alpha_i e_1 + e_i, \quad 2 \leq i \leq 9$$

$$T(e_2) = \alpha_2 e_1 + e_2$$

$$T(e_3) = \alpha_3 e_1 + e_3$$

$$\vdots$$

$$T(e_9) = \alpha_9 e_1 + e_9$$

and

$$T(e_{2n}) = T(e_{10}) = \sum_{i=1}^{10} \beta_i e_i$$

with $\beta_{10} = 1$ and

$$\alpha_j = \begin{cases} -\beta_{2n+1-j} & 2 \leq j \leq 5 \\ \beta_{2n+1-j} & 5 < j \leq 9. \end{cases}$$

This implies $\alpha_2 = -\beta_9$, $\alpha_3 = -\beta_8$, $\alpha_4 = -\beta_7$, $\alpha_5 = -\beta_6$, $\alpha_6 = \beta_5$, $\alpha_7 = \beta_4$, $\alpha_8 = \beta_3$ and $\alpha_9 = \beta_2$. Then the elements T of N are of the form

$$T = \begin{bmatrix} 1 & -\beta_9 & -\beta_8 & -\beta_7 & -\beta_6 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \beta_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \beta_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \beta_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \beta_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \beta_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \beta_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\beta_j \in \mathbb{F}$. The 9 generators of N are:

$$P_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1&0&0&0&0&0&0&1&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1&0&0&0&0&0&1&1&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 1&0&0&0&0&1&1&1&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 1&0&0&0&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 1&0&0&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 1&0&1&1&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$P_9 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using GAP we obtain two generators a and b of G as 8×8 matrices. We employ the method outlined in Subsection 4.2.3 to convert these to be 10×10 symplectic matrices.

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $o(a) = 2$ and $o(b) = 8$.

6.3 Conjugacy classes of $\overline{G}$

When G acts on N we get four orbits of lengths $1, 1, 255$ and 255, by Remark 4.2.35. These correspond with the point stabilizers $Sp(8,2)$, $Sp(8,2)$, $2^7{:}Sp(6,2)$ and $2^7{:}Sp(6,2)$, respectively, also by Remark 4.2.35. In Definition 2.6.14 we define the rank of a group to be the number of orbits of a point stabilizer. We note that G, which is of even order, is a transitive permutation group of rank 3. According to Remark 2.6.16, since $2^7{:}Sp(6,2)$ is maximal in G, then G is primitive of rank 3 and thus the permutation character of the action has a decomposition of the form $1+\chi_s+\chi_t$, where 1 is a trivial character, χ_s and χ_t are irreducible characters of G. From the ATLAS this permutation character is given by $1a+119a+135a$. Let $\chi(G|N)$ be the permuation character when G acts on N. Then

$$\chi(G|N) = 1+1+I^G_{2^7:Sp(6,2)} + I^G_{2^7:Sp(6,2)}$$

$$= 1a+1a+1a+119a+135a+1a+119a+135a$$

$$= 4(1a)+2(119a)+2(135a)$$

where $I^G_{2^7:Sp(6,2)}$ is the identity character of $2^7{:}Sp(6,2)$ induced to G and expressed in terms of the $Irr(G)$. The coset analysis technique is used to determine the conjugacy classes of $\overline{G}$. We use the above permutation character to calculate the values of k in the coset analysis, where k is the number of fixed points of $g \in G$ acting on N. These values are listed in Table 6.1.

Table 6.1: Fixed points of the action of G on N

$o(g)$	1A	2A	2B	2C	2D	2E	2F	3A	3B	3C	3D	4A	4B	4C
k	512	256	128	128	32	64	32	128	2	32	8	64	64	32

Table 6.1: Fixed points of the action of G on N

$o(g)$	4D	4E	4F	4G	4H	4I	4J	4K	4L	5A	5B	6A	6B	6C
k	32	16	16	16	32	16	8	8	8	32	2	64	32	32

Table 6.1: Fixed points of the action of G on N

$o(g)$	6D	6E	6F	6G	6H	6I	6J	6K	6L	6M	6N	6O	6P	7A
k	32	2	4	16	16	8	8	2	8	8	16	4	8	8

Table 6.1: Fixed points of the action of G on N

$o(g)$	8A	8B	8C	8D	8E	8F	9A	9B	10A	10B	10C	10D	12A	12B
k	16	16	8	8	4	4	8	2	16	8	8	2	8	16

Table 6.1: Fixed points of the action of G on N

$o(g)$	12C	12D	12E	12F	12G	12H	12I	12J	12K	12L	12M	14A	15A	15B
k	16	8	2	4	4	8	8	4	4	4	2	4	8	2

Table 6.1: Fixed points of the action of G on N

$o(g)$	15C	17A	17B	18A	20A	20B	21A	24A	24B	30A	30B
k	2	2	2	4	4	4	2	4	4	4	2

As an example, when $g \in 2A$ we note from Table 6.1 that $k = 256$. This means Ng under the action of N produces 256 orbits, each of length $\frac{|N|}{k} = \frac{512}{256} = 2$. After the action of the centralizer $C_G(g)$, using GAP, on these orbits some merge and we end up with 5 orbits, with $|\Omega_1| = 2$, $|\Omega_2| = 2$, $|\Omega_3| = 128$, $|\Omega_4| = 128$ and $|\Omega_5| = 252$. To obtain the corresponding f-values we use $f_i = \frac{k|\Omega_i|}{|N|}$ to get $f_1 = 1$, $f_2 = 1$, $f_3 = 64$, $f_4 = 64$ and $f_5 = 126$. These values satisfy $\sum_{i=1}^{5} f_i = k$. The GAP Programme A in [15] is used to compute the rest of the f-values for each conjugacy class $[g]$ of G. This implies that the conjugacy class $2A$ produces 5 conjugacy classes of $\overline{G}$. We compute the centralizer sizes of each by

$$|C_{\overline{G}}(x_i)| = \frac{k|C_G(g)|}{f_i}.$$

$$|C_{\overline{G}}(x_1)| = \frac{k|C_G(g)|}{f_1} = \frac{256 \times 185794560}{1} = 47563407360,$$

$$|C_{\overline{G}}(x_2)| = \frac{k|C_G(g)|}{f_2} = \frac{256 \times 185794560}{1} = 47563407360,$$

$$|C_{\overline{G}}(x_3)| = \frac{k|C_G(g)|}{f_3} = \frac{256 \times 185794560}{64} = 743178240,$$

$$|C_{\overline{G}}(x_4)| = \frac{k|C_G(g)|}{f_4} = \frac{256 \times 185794560}{64} = 743178240,$$

$$|C_{\overline{G}}(x_5)| = \frac{k|C_G(g)|}{f_5} = \frac{256 \times 185794560}{126} = 377487360.$$

We proceed to compute the orders of class representatives of the new conjugacy classes of $\overline{G}$. We once more use the class 2A for demonstration. Let $g \in 2A$, that is $o(g) = 2 = m$. Let $d \in N$. Recall Char($\mathbb{F}$) $= 2 = p$. Let

$$w = d * d^g * d^{g^2} * \cdots * d^{g^{m-1}}.$$

Now if w is the identity of N, then the order of $x \in \overline{G}$ is $m = 2$ in this case. Otherwise, if w is not the identity of N, then $o(x) = pm = 2 \times 2 = 4$. Since we have represented the elements of N as matrices $[n_{ij}]_{10\times 10}$ with the zero vector represented by the identity matrix $I_{10\times 10}$, we are able to use the multiplication operation to compute our w's. From the class of 2A we have five conjugacy classes of $\overline{G}$. Since g is always in the first conjugacy class, then

$$w = 1_N$$

and hence

$$o(x_1) = 2.$$

For the second class,

$$d = (0,0,0,0,0,0,0,0,1) \text{ and } w = d * d^g = (0,0,0,0,0,0,0,0,0)$$

and therefore

$$o(x_2) = 2.$$

For the third class,

$$d = (0,0,1,1,0,0,1,0,0) \text{ and } w = (0,0,1,1,0,1,1,0,1) \neq 1_N,$$

it then follows that

$$o(x_3) = 4.$$

For the fourth class,

$$d = (0,0,1,1,0,0,1,0,1) \text{ and } w = (0,0,1,1,0,1,1,0,1) \neq 1_N,$$

it then follows that

$$o(x_4) = 4.$$

For the fifth class,

$$d = (0,0,0,0,0,0,0,1,0) \text{ and } w = (0,0,0,0,0,0,0,0,0) = 1_N,$$

thus

$$o(x_5) = 2.$$

Since we already had 3 conjugacy classes of order 2 from 1A and there were no elements of order 4, we label these five conjugacy classes as $[x_1] = 2D,\ [x_2] = 2E,\ [x_5] = 2F,\ [x_3] = 4A$ and $[x_4] = 4B$. The size of each is computed via

$$|[x_i]| = \frac{|\overline{G}|}{|C_{\overline{G}}(x_i)|}.$$

The full conjugacy classes of $\overline{G}$ are listed in Table 6.2 below.

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
1A	512	1	24257337753600	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	1A
		1	24257337753600	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	2A
		255	95126814720	$(0,0,0,0,0,0,0,1,0)$	$(0,0,0,0,0,0,0,1,0)$	2B
		255	95126814720	$(0,0,0,0,0,0,0,1,1)$	$(0,0,0,0,0,0,0,1,1)$	2C
2A	256	1	47563407360	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2D
		1	47563407360	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2E
		64	743178240	$(0,0,1,1,0,0,1,0,0)$	$(0,0,1,1,0,1,1,0,1)$	4A
		64	743178240	$(0,0,1,1,0,0,1,0,1)$	$(0,0,1,1,0,1,1,0,1)$	4B
		126	377487360	$(0,0,0,0,0,0,0,1,0)$	$(0,0,0,0,0,0,0,0,0)$	2F
2B	128	1	1132462080	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2G
		1	1132462080	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2H
		15	75497472	$(0,0,0,0,0,1,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2I
		15	75497472	$(0,0,0,0,0,1,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2J
		96	11796480	$(0,1,0,0,1,0,0,0,0)$	$(0,1,0,0,1,0,0,1,0)$	4C
2C	128	1	377487360	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2K
		1	377487360	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2L
		16	23592960	$(0,0,1,0,0,0,0,0,0)$	$(0,0,1,0,0,0,0,1,0)$	4D
		16	23592960	$(0,0,1,0,0,0,0,0,1)$	$(0,0,1,0,0,0,0,1,0)$	4E
		30	12582912	$(0,0,0,0,0,0,1,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2M
		64	5898240	$(1,0,1,1,1,0,0,0,0)$	$(1,0,1,1,1,1,0,0,1)$	4F
2D	32	1	23592960	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2N
		1	23592960	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2O
		30	786432	$(0,0,0,0,1,0,0,0,0)$	$(0,0,0,0,1,0,0,1,0)$	4G
2E	64	1	9437184	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2P
		1	9437184	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2Q
		4	2359296	$(1,0,1,1,1,0,1,0,0)$	$(1,0,1,1,1,0,1,1,1)$	4H
		4	2359296	$(1,0,1,1,1,0,1,0,1)$	$(1,0,1,1,1,0,1,1,1)$	4I
		6	1572864	$(0,0,0,0,0,1,0,1,0)$	$(0,0,0,0,0,0,0,0,0)$	2R
		24	393216	$(0,0,0,0,1,0,1,1,0)$	$(0,0,0,0,1,0,0,1,0)$	4J
		24	393216	$(1,0,1,1,0,0,0,1,0)$	$(1,0,1,1,0,0,1,0,1)$	4K
2F	32	1	1572864	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	2S
		1	1572864	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	2T
		1	1572864	$(0,0,1,1,1,0,0,0,0)$	$(0,0,1,1,0,1,1,0,0)$	4L
		1	1572864	$(0,0,1,1,1,0,0,0,1)$	$(0,0,1,1,0,1,1,0,0)$	4M
		12	131072	$(0,1,1,0,1,1,1,1,0)$	$(0,1,1,0,1,1,1,0,0)$	4N
		16	98304	$(1,0,1,0,1,1,1,0,0)$	$(1,0,1,0,1,0,1,0,1)$	4O
3A	128	1	557383680	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	3A
		1	557383680	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	6A
		63	8847360	$(1,1,1,1,1,1,0,1,0)$	$(0,0,1,1,1,0,1,0,1)$	6B
		63	8847360	$(1,1,1,1,1,1,0,1,1)$	$(0,0,1,1,1,0,1,0,0)$	6C
3B	2	1	155520	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	3B
		1	155520	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	6D
3C	32	1	414720	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	3C
		1	414720	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	6E
		15	27648	$(0,1,0,0,0,1,0,1,0)$	$(0,0,0,0,1,0,0,0,0)$	6F
		15	27648	$(0,1,0,0,0,1,0,1,1)$	$(0,0,0,0,1,0,0,0,1)$	6G
3D	8	1	31104	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	3D
		1	31104	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	6H
		3	10368	$(1,1,0,1,1,0,0,1,0)$	$(0,1,1,0,0,1,1,1,1)$	6I
		3	10368	$(1,1,0,1,1,0,0,1,1)$	$(0,1,1,0,0,1,1,1,0)$	6J

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
4A	64	1	5898240	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4P
		1	5898240	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4Q
		15	393216	(1,0,1,1,1,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	4R
		15	393216	(1,0,1,1,1,1,0,1,1)	(0,0,0,0,0,0,0,0,0)	4S
		32	184320	(0,1,0,0,1,1,1,0,0)	(0,0,1,0,0,0,0,1,0)	8A
4B	64	1	5898240	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4T
		1	5898240	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4U
		15	393216	(1,1,0,1,0,1,1,1,0)	(0,0,0,0,0,0,0,0,0)	4V
		15	393216	(1,1,0,1,0,1,1,1,1)	(0,0,0,0,0,0,0,0,0)	4W
		32	184320	(1,0,0,0,0,1,0,1,0)	(1,1,0,1,0,1,0,0,0)	8B
4C	32	1	1179648	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4X
		1	1179648	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4Y
		3	393216	(1,1,1,0,0,0,1,1,0)	(0,0,0,0,0,0,0,0,0)	4Z
		3	393216	(1,1,0,1,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AA
		24	49152	(1,1,0,1,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4AB
4D	32	1	393216	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AC
		1	393216	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4AD
		3	131072	(0,0,0,0,1,1,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AE
		3	131072	(1,1,1,1,0,1,1,1,0)	(0,0,0,0,0,0,0,0,0)	4AF
		8	49152	(0,1,0,0,1,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	4AG
		16	24576	(0,1,0,0,1,1,0,1,1)	(0,0,0,0,0,0,0,0,0)	4AH
4E	16	1	98304	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AI
		1	98304	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4AJ
		6	16384	(1,0,0,0,0,1,1,0,0)	(0,0,0,0,0,0,0,0,0)	4AK
		8	12288	(1,1,1,1,0,0,1,0,0)	(1,1,0,1,0,1,0,0,0)	8C
4F	16	1	98304	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AL
		1	98304	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4AM
		6	16384	(0,0,0,1,0,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	4AN
		8	12288	(1,0,0,0,1,1,0,1,0)	(0,1,1,1,1,0,0,1,0)	8D
4G	16	1	49152	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AO
		1	49152	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4AP
		1	49152	(1,1,0,0,0,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	4AQ
		1	49152	(1,0,0,1,1,0,0,1,0)	(0,0,0,0,0,0,0,0,0)	4AR
		6	8192	(0,0,1,1,1,1,1,0,0)	(0,0,0,0,0,0,0,0,0)	4AS
		6	8192	(0,0,1,1,1,1,1,0,1)	(0,0,0,0,0,0,0,0,0)	4AT
4H	32	1	98304	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AU
		1	98304	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4AV
		4	24576	(0,1,1,1,0,0,1,0,0)	(0,0,0,0,0,0,0,0,0)	4AW
		4	24576	(0,1,1,1,0,0,1,0,1)	(0,0,0,0,0,0,0,0,0)	4AX
		6	16384	(1,0,1,1,1,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	4AY
		8	12288	(0,1,0,0,1,1,1,0,0)	(0,0,1,0,0,0,0,1,0)	8E
		8	12288	(0,0,1,1,1,1,0,0,0)	(0,0,1,0,0,0,0,1,0)	8F
4I	16	1	16384	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4AZ
		1	16384	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4BA
		1	16384	(0,1,1,1,1,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	4BB
		1	16384	(0,1,1,1,1,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	4BC
		4	4096	(1,1,0,1,1,1,1,1,0)	(0,1,0,1,1,0,0,0,0)	8G
		4	4096	(1,1,0,1,0,0,1,0,0)	(0,0,0,0,0,0,0,0,0)	4BD
		4	4096	(0,0,0,0,1,1,0,1,0)	(0,1,0,1,1,0,0,0,0)	8H

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
4J	8	1	6144	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	4BE
		1	6144	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	4BF
		6	1024	(1, 1, 1, 1, 1, 1, 0, 1, 0)	(0, 1, 0, 0, 1, 0, 0, 1, 0)	8I
4K	8	1	4096	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	4BG
		1	4096	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	4BH
		2	2048	(0, 0, 0, 1, 1, 1, 1, 1, 0)	(0, 0, 1, 1, 0, 1, 1, 0, 0)	8J
		4	1024	(0, 1, 1, 1, 1, 0, 0, 0, 0)	(0, 1, 1, 0, 1, 1, 1, 0, 0)	8K
4L	8	1	4096	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	4BI
		1	4096	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	4BJ
		2	2048	(0, 1, 0, 1, 0, 1, 0, 0, 0)	(0, 0, 1, 1, 0, 1, 1, 0, 0)	8L
		4	1024	(1, 1, 0, 0, 0, 0, 0, 1, 0)	(0, 1, 1, 0, 1, 1, 1, 0, 0)	8M
5A	32	1	115200	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	5A
		1	115200	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 1)	10A
		15	7680	(1, 0, 0, 1, 0, 1, 0, 0, 0)	(0, 0, 0, 0, 1, 0, 0, 1, 0)	10B
		15	7680	(1, 0, 0, 1, 0, 1, 0, 0, 1)	(0, 0, 0, 0, 1, 0, 0, 1, 1)	10C
5B	2	1	600	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	5B
		1	600	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 1)	10D
6A	64	1	4423680	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6K
		1	4423680	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6L
		16	276480	(0, 0, 1, 1, 1, 1, 1, 0, 0)	(0, 1, 0, 0, 0, 0, 1, 1, 1)	12A
		16	276480	(0, 0, 1, 1, 1, 1, 1, 0, 1)	(0, 1, 0, 0, 0, 0, 1, 1, 1)	12B
		30	147456	(1, 1, 1, 1, 1, 0, 0, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6M
6B	32	1	442368	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6N
		1	442368	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6O
		3	147456	(1, 1, 1, 1, 1, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6P
		3	147456	(1, 1, 1, 1, 1, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6Q
		24	18432	(1, 0, 0, 0, 1, 1, 1, 1, 0)	(0, 1, 0, 0, 1, 0, 0, 1, 0)	12C
6C	32	1	147456	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6R
		1	147456	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6S
		4	36864	(1, 1, 1, 0, 0, 1, 1, 1, 0)	(0, 0, 1, 0, 0, 0, 0, 1, 0)	12D
		4	36864	(1, 1, 1, 0, 0, 1, 1, 1, 1)	(0, 0, 1, 0, 0, 0, 0, 1, 0)	12E
		6	24576	(1, 1, 0, 0, 0, 1, 0, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6T
		16	9216	(0, 1, 0, 0, 0, 1, 0, 0, 0)	(1, 0, 1, 1, 1, 1, 0, 0, 1)	12F
6D	32	1	138240	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6U
		1	138240	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6V
		15	9216	(1, 1, 1, 0, 0, 0, 1, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6W
		15	9216	(1, 1, 1, 0, 0, 0, 1, 1, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6X
6E	2	1	3456	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6Y
		1	3456	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6Z
6F	4	1	5184	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6AA
		1	5184	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6AB
		1	5184	(0, 0, 0, 1, 1, 0, 1, 0, 0)	(0, 0, 1, 1, 0, 1, 1, 0, 1)	12G
		1	5184	(0, 0, 0, 1, 1, 0, 1, 0, 1)	(0, 0, 1, 1, 0, 1, 1, 0, 1)	12H
6G	16	1	18432	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6AC
		1	18432	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	6AD
		1	18432	(1, 1, 0, 1, 1, 0, 0, 1, 0)	(0, 0, 1, 0, 0, 0, 0, 1, 1)	12I
		1	18432	(1, 1, 0, 1, 1, 0, 0, 1, 1)	(0, 0, 1, 0, 0, 0, 0, 1, 1)	12J
		6	3072	(1, 0, 0, 0, 1, 1, 1, 1, 0)	(0, 1, 0, 0, 1, 0, 0, 1, 0)	12K
		6	3072	(0, 1, 0, 1, 0, 1, 1, 0, 0)	(0, 1, 1, 0, 1, 0, 0, 0, 1)	12L

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
6H	16	1	13824	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AE
		1	13824	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AF
		4	3456	$(1,1,1,1,1,1,0,0,0)$	$(0,0,1,0,0,1,0,1,1)$	12M
		4	3456	$(1,1,1,1,1,1,0,0,1)$	$(0,0,1,0,0,1,0,1,1)$	12N
		6	2304	$(1,0,0,0,1,1,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AG
6I	8	1	6912	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AH
		1	6912	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AI
		6	1152	$(0,0,1,1,0,0,0,0,0)$	$(0,1,1,1,1,1,0,0,0)$	12O
6J	8	1	3456	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AJ
		1	3456	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AK
		3	1152	$(1,0,1,1,1,1,1,1,0)$	$(0,0,0,0,0,0,0,0,0)$	6AL
		3	1152	$(1,0,1,1,1,1,1,1,1)$	$(0,0,0,0,0,0,0,0,0)$	6AM
6K	2	1	576	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AN
		1	576	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AO
6L	8	1	2304	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AP
		1	2304	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AQ
		1	2304	$(0,1,0,1,1,1,1,0,0)$	$(1,1,0,1,0,1,0,0,0)$	12P
		1	2304	$(0,1,0,1,1,1,1,0,1)$	$(1,1,0,1,0,1,0,0,0)$	12Q
		4	576	$(0,1,0,0,1,1,0,0,0)$	$(1,1,1,0,1,1,0,0,1)$	12R
6M	8	1	2304	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AR
		1	2304	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AS
		6	384	$(0,1,1,0,1,1,0,1,0)$	$(0,0,1,1,1,0,0,0,0)$	12S
6N	16	1	4608	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AT
		1	4608	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AU
		4	1152	$(1,0,0,1,1,0,0,1,0)$	$(0,0,1,1,0,1,1,0,1)$	12T
		4	1152	$(1,0,0,1,1,0,0,1,1)$	$(0,0,1,1,0,1,1,0,1)$	12U
		6	768	$(1,0,1,0,1,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AV
6O	4	1	576	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AW
		1	576	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AX
		1	576	$(0,0,0,0,0,1,0,0,0)$	$(1,0,1,1,1,0,1,1,1)$	12V
		1	576	$(0,0,0,0,0,1,0,0,1)$	$(1,0,1,1,1,0,1,1,1)$	12W
6P	8	1	768	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	6AY
		1	768	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	6AZ
		1	768	$(1,0,0,1,1,1,1,0,0)$	$(0,0,1,1,0,1,1,0,0)$	12X
		1	768	$(1,0,0,1,1,1,1,0,1)$	$(0,0,1,1,0,1,1,0,0)$	12Y
		4	192	$(0,0,0,1,1,0,0,0,0)$	$(1,0,1,1,0,1,1,1,1)$	12Z
7A	8	1	336	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	7A
		1	336	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	14A
		3	112	$(0,0,1,1,0,0,0,1,0)$	$(0,0,1,0,0,0,0,0,0)$	14B
		3	112	$(0,0,1,1,0,0,0,1,1)$	$(0,0,1,0,0,0,0,0,1)$	14C

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
8A	16	1	6144	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8N
		1	6144	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8O
		3	2048	(1,1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8P
		3	2048	(1,1,1,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8Q
		8	768	(1,0,1,0,0,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	8R
8B	16	1	6144	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8S
		1	6144	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8T
		3	2048	(1,0,1,0,0,1,0,1,0)	(0,0,0,0,0,0,0,0,0)	8U
		3	2048	(1,1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8V
		8	768	(1,1,1,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8W
8C	8	1	1024	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8X
		1	1024	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8Y
		1	1024	(0,1,0,1,1,1,0,0,0)	(0,0,0,0,0,0,0,0,0)	8Z
		1	1024	(0,1,0,1,1,1,0,0,1)	(0,0,0,0,0,0,0,0,0)	8AA
		4	256	(1,1,0,1,0,0,1,1,0)	(0,0,0,0,0,0,0,0,0)	8AB
8D	8	1	1024	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8AC
		1	1024	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8AD
		1	1024	(0,1,1,0,1,0,0,1,0)	(0,0,0,0,0,0,0,0,0)	8AE
		1	1024	(0,1,1,0,1,0,0,1,1)	(0,0,0,0,0,0,0,0,0)	8AF
		4	256	(1,1,1,0,1,0,1,0,0)	(0,0,0,0,0,0,0,0,0)	8AG
8E	4	1	128	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8AH
		1	128	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8AI
		2	64	(0,1,1,0,1,0,0,1,0)	(0,0,1,1,0,1,1,0,0)	16A
8F	4	1	128	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	8AJ
		1	128	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	8AK
		2	64	(1,1,1,0,1,0,0,0,0)	(0,0,1,1,0,1,1,0,0)	16B
9A	8	1	432	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	9A
		1	432	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,1)	18A
		3	144	(1,1,1,0,1,0,0,1,0)	(0,0,0,1,0,1,1,1,1)	18B
		3	144	(1,1,1,0,1,0,0,1,1)	(0,0,0,1,0,1,1,1,0)	18C
9B	2	1	54	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	9B
		1	54	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,1)	18D
10A	16	1	3840	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	10E
		1	3840	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	10F
		4	960	(1,0,0,1,0,1,0,1,0)	(0,1,0,0,0,0,1,1,1)	20A
		4	960	(1,0,0,1,0,1,0,1,1)	(0,1,0,0,0,0,1,1,1)	20B
		6	640	(0,1,1,1,0,0,1,1,0)	(0,0,0,0,0,0,0,0,0)	10G
10B	8	1	1920	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	10H
		1	1920	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	10I
		6	320	(1,1,0,1,0,1,1,1,0)	(0,0,1,0,1,1,0,0,0)	20C
10C	8	1	640	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	10J
		1	640	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	10K
		1	640	(0,0,1,1,1,0,1,0,0)	(1,0,0,0,0,0,1,1,0)	20D
		1	640	(0,0,1,1,1,0,1,0,1)	(1,0,0,0,0,0,1,1,0)	20E
		4	160	(0,0,1,1,0,1,0,1,0)	(1,1,0,0,1,1,1,0,1)	20F
10D	2	1	40	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	10L
		1	40	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	10M
12A	8	1	9216	(0,0,0,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0,0)	12AA
		1	9216	(0,0,0,0,0,0,0,0,1)	(0,0,0,0,0,0,0,0,0)	12AB
		6	1536	(0,1,0,1,0,1,1,0,0)	(0,0,0,0,0,0,0,0,0)	12AC

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
12B	16	1	9216	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AD
		1	9216	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AE
		3	3072	(0, 1, 1, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AF
		3	3072	(0, 1, 1, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AG
		8	1152	(1, 1, 1, 0, 1, 0, 0, 0, 0)	(0, 0, 1, 0, 0, 0, 0, 1, 0)	24A
12C	16	1	9216	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AH
		1	9216	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AI
		3	3072	(0, 1, 1, 1, 1, 1, 1, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AJ
		3	3072	(0, 1, 1, 1, 1, 1, 1, 1, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AK
		8	1152	(1, 1, 0, 0, 1, 1, 0, 0, 0)	(1, 1, 0, 1, 0, 1, 0, 0, 0)	24B
12D	8	1	3072	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AL
		1	3072	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AM
		2	1536	(0, 0, 1, 1, 0, 0, 0, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AN
		4	768	(1, 1, 1, 0, 1, 0, 1, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AO
12E	2	1	288	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AP
		1	288	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AQ
12F	4	1	576	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AR
		1	576	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AS
		2	288	(0, 0, 0, 1, 1, 1, 0, 0, 0)	(1, 1, 0, 1, 0, 1, 0, 0, 0)	24C
12G	4	1	576	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AT
		1	576	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AU
		2	288	(0, 0, 0, 0, 1, 0, 0, 0, 0)	(0, 1, 1, 1, 1, 0, 0, 1, 0)	24D
12H	8	1	768	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AV
		1	768	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AW
		1	768	(1, 0, 1, 1, 0, 1, 0, 1, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AX
		1	768	(1, 0, 1, 1, 0, 1, 0, 1, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AY
		2	384	(0, 0, 1, 0, 1, 1, 1, 1, 0)	(0, 0, 1, 0, 0, 0, 0, 1, 0)	24E
		2	384	(1, 0, 0, 1, 1, 0, 1, 0, 0)	(0, 0, 1, 0, 0, 0, 0, 1, 0)	24F
12I	8	1	576	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12AZ
		1	576	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BA
		3	192	(1, 0, 0, 1, 0, 1, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BB
		3	192	(1, 0, 0, 1, 0, 1, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BC
12J	4	1	192	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BD
		1	192	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BE
		2	96	(1, 1, 1, 1, 1, 1, 1, 0, 0)	(1, 1, 0, 1, 0, 1, 0, 0, 0)	24G
12K	4	1	192	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BF
		1	192	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BG
		2	96	(1, 1, 1, 0, 1, 0, 1, 0, 0)	(1, 1, 0, 1, 0, 1, 0, 0, 0)	24H
12L	4	1	96	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BH
		1	96	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BI
		1	96	(0, 0, 1, 0, 0, 0, 1, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BJ
		1	96	(0, 0, 1, 0, 0, 0, 1, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BK
12M	2	1	48	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BL
		1	48	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	12BM
14A	4	1	56	(0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	14D
		1	56	(0, 0, 0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 0, 0, 0, 0)	14E
		1	56	(1, 0, 0, 0, 0, 1, 1, 1, 0)	(1, 0, 1, 1, 1, 1, 0, 1, 1)	28A
		1	56	(1, 0, 0, 0, 0, 1, 1, 1, 1)	(1, 0, 1, 1, 1, 1, 0, 1, 1)	28B

Table 6.2: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
15A	8	1	720	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	15A
		1	720	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	30A
		3	240	$(1,0,1,0,1,0,0,1,0)$	$(0,0,1,1,1,0,1,1,1)$	30B
		3	240	$(1,0,1,0,1,0,0,1,1)$	$(0,0,1,1,1,0,1,1,0)$	30C
15B	2	1	180	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	15B
		1	180	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	30D
15C	2	1	30	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	15C
		1	30	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	30E
17A	2	1	34	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	17A
		1	34	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	34A
17B	2	1	34	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	17B
		1	34	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	34B
18A	4	1	72	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	18E
		1	72	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	18F
		1	72	$(0,0,0,1,1,0,1,1,0)$	$(0,0,1,1,0,1,1,0,1)$	36A
		1	72	$(0,0,0,1,1,0,1,1,1)$	$(0,0,1,1,0,1,1,0,1)$	36B
20A	4	1	160	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	20G
		1	160	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	20H
		2	80	$(0,1,1,1,1,1,0,1,0)$	$(1,1,0,1,0,1,0,0,0)$	40A
20B	4	1	160	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	20I
		1	160	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	20J
		2	80	$(1,0,0,1,0,1,1,1,0)$	$(1,1,0,0,1,1,1,0,0)$	40B
21A	2	1	42	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	21A
		1	42	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	42A
24A	4	1	192	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	24I
		1	192	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	24J
		2	96	$(1,1,1,0,1,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	24K
24B	4	1	192	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	24L
		1	192	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	24M
		2	96	$(1,1,0,1,1,0,1,0,0)$	$(0,0,0,0,0,0,0,0,0)$	24N
30A	4	1	120	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	30F
		1	120	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	30G
		1	120	$(0,1,1,1,0,1,0,1,0)$	$(0,1,0,0,0,0,1,1,1)$	60A
		1	120	$(0,1,1,1,0,1,0,1,1)$	$(0,1,0,0,0,0,1,1,1)$	60B
30B	2	1	60	$(0,0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0,0)$	30H
		1	60	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,0)$	30I

6.4 Fusion of inertia factor groups into $Sp(8,2)$

The action of G on N yielded 4 orbits. It follows by Brauer's theorem that when G acts on the irreducible characters of N, $Irr(N)$, we again obtain 4 orbits. The orbit lengths of the latter action are 1, 120, 136 and 255 by Remark 4.2.36. From the same result, we determine the inertia factor groups as $H_1 = Sp(8,2)$, $H_2 = O^-(8,2){:}2$, $H_3 = O^+(8,2){:}2$ and $H_4 = 2^7{:}Sp(6,2)$. The dual of a vector space can also be used for the action of G on $Irr(N)$. In the ATLAS we have that $H_2 = GO^-(8,2)$ and $H_3 = GO^+(8,2)$ are respectively the full orthogonal groups. All three are maximal subgroups of G. The character table of H_4 and the fusion of the classes of H_4 into the classes of G were done by Ali in [1]. We do the fusions of H_2 and H_3 into G by using the divisibility of the respective centralizer sizes. That is $\frac{|C_G(a)|}{|C_{H_i}(b)|}$, where a is a class representative of a conjugacy class of G and b a class representative of a conjugacy class of H_i, with $i \in \{2,3\}$ and $o(a) = o(b)$. To ensure correct results we employ for each fusion a permutation character $\chi(G|H_i)$. The degree of $\chi(G|H_4)$ is 255, degree of $\chi(G|H_3)$ is 136 and degree of $\chi(G|H_2)$ is 120. From the ATLAS we have

$$\chi(G|H_2) = 1a + 119a$$

$$\chi(G|H_3) = 1a + 135a$$

and

$$\chi(G|H_4) = 1a + 119a + 135a.$$

In the event that the method outlined above provides more than one candidate for a fusion, we use the following remark.

Remark 6.4.1 *Let $H \leq G$, $[h]$ and $[g]$ be the conjugacy classes of H and G respectively. If there exists more than one class $[g]$ in G such that $o(g) = o(h)$ and the dividend $\frac{|C_G(g)|}{|C_H(h)|}$ corresponds to the permutation character $\chi(G|H)$, then we consider the pairwise intersections $[g_i] \cap [h]$. Since $[g_i] \cap [g_j] = \emptyset$, then only one intersection $[g_i] \cap [h]$ will be non-empty. This intersection provides the suitable candidate for the fusion.*

The fusion maps of $GO^-(8,2)$ into G are given in Table 6.3 and the fusion maps of $GO^+(8,2)$ into G are given in Table 6.4. In Table 6.5 we provide the summary of these fusion maps.

Table 6.3: The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$2F$	$3A$
	$\lvert C_G(g)\rvert$	47377612800	185794560	8847360	2949120	737280	147456	49152	4354560
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$								
$1A$	394813440	$\boxed{120}$							
$2A$	2903040		$\boxed{64}$						
$2B$	368640		504	$\boxed{24}$	8	2			
$2C$	92160		2016	96	$\boxed{32}$	8			
$2D$	9216		20160	960	320	80	$\boxed{16}$		
$2E$	6144		30240	1440	480	120	24	$\boxed{8}$	
$3A$	120960								36
	$\chi(G\lvert H_2)$	120	64	24	32	0	16	8	36

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$3A$	$3B$	$3C$	$3D$	$4A$	$4B$	$4C$	$4D$	$4E$	$4F$	$4G$
	$\lvert C_G(g)\rvert$	4354560	77760	12960	3888	92160	92160	36864	12288	6144	6144	3072
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$											
$3A$	120960	$\boxed{36}$										
$3B$	2160	2016	36	$\boxed{6}$								
$3C$	1296	3360	60	10	$\boxed{3}$							
$4A$	7680					$\boxed{12}$	12					
$4B$	4608					20	$\boxed{20}$	8				
$4C$	3072					30	30	12	$\boxed{4}$	2	2	1
$4D$	3072					30	30	$\boxed{12}$	4	2	2	1
$4E$	768					120	120	48	16	8	8	$\boxed{4}$
$4F$	768					120	120	48	16	$\boxed{8}$	8	4
$4G$	384					240	240	96	32	16	16	8
$4H$	256					360	360	144	48	24	24	12
$4I$	128					720	720	288	96	48	48	24
	$\chi(G\lvert H_2)$	36	0	6	3	12	20	12	4	8	0	4

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$4H$	$4I$	$4J$	$4K$	$4L$	$5A$	$5B$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$
	$\lvert C_G(g)\rvert$	3072	1024	768	512	512	3600	300	69120	13824	4608	4320	1728	1296
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$4A$	7680													
$4B$	4608													
$4C$	3072	1												
$4D$	3072	1												
$4E$	768	4		1										
$4F$	768	4		1										
$4G$	384	$\boxed{8}$		2										
$4H$	256	12	$\boxed{4}$	3	2	2								
$4I$	128	24	8	6	$\boxed{4}$	4								
$5A$	360						10							
$6A$	4320								$\boxed{16}$			1		
$6B$	1296													$\boxed{1}$
$6C$	1152								60	$\boxed{12}$	4			
$6D$	720								96			$\boxed{6}$		
$6E$	576								120	24	$\boxed{8}$		3	
$6F$	288								240	48	16	15	6	
$6G$	216								320	64		20	8	6
	$\chi(G\lvert H_2)$	8	4	0	4	0	10	12	16	12	8	6	0	1

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$6H$	$6I$	$6J$	$6K$	$6L$	$6M$
	$\lvert C_G(g)\rvert$	69120	13824	4608	4320	1728	1296	1152	864	864	432	288	288	288
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$6A$	4320	16			1									
$6B$	1296	1												
$6C$	1152	60	12	4				1						
$6D$	720	96			6									
$6E$	576	120	24	8		3		2						
$6F$	288	240	48	16	15	6		$\boxed{4}$	3	3	0	1	1	1
$6G$	216	320	64		20	8	6		$\boxed{4}$	4	2			
$6H$	144	480	96	32	30	12	9	8	6	6	$\boxed{3}$	2	2	2
$6I$	144	480	96	32	30	12	9	8	6	6	3	2	2	2
$6J$	144	480	96	32	30	12	9	8	6	6	3	2	$\boxed{2}$	2
$6K$	72	960	192	64	60	24	18	16	12	12	6	4	4	4
$6L$	48	1440	288	96	90	36	27	24	18	18	9	6	6	6
	$\chi(G\lvert H_2)$	16	12	8	6	0	1	4	4	0	3	0	2	0

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$6N$	$6O$	$6P$	$7A$	$8A$	$8B$	$8C$	$8D$	$8E$	$8F$	$9A$	$9B$	$10A$
	$\lvert C_G(g)\rvert$	288	144	96	42	384	384	128	128	32	32	54	27	240
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$6A$	4320													
$6B$	1296													
$6C$	1152													
$6D$	720													
$6E$	576													
$6F$	288	1												
$6G$	216													
$6H$	144	2	1											
$6I$	144	2	$\boxed{1}$											
$6J$	144	2	1											
$6K$	72	$\boxed{4}$	2											
$6L$	48	6	3	$\boxed{2}$										
$7A$	42				$\boxed{1}$									
$8A$	192					2	$\boxed{2}$							
$8B$	64					$\boxed{6}$	6	2	2					
$8C$	64					6	6	$\boxed{2}$	2					
$8D$	64					6	6	2	$\boxed{2}$					
$8E$	16					24	24	8	8	$\boxed{2}$	2			
	$\chi(G\lvert H_2)$	4	1	2	1	6	2	2	2	2	0			

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$9A$	$9B$	$10A$	$10B$	$10C$	$10D$	$12A$	$12B$	$12C$	$12D$	$12E$	$12F$	$12G$
	$\lvert C_G(g)\rvert$	54	27	240	240	80	20	1152	576	576	384	144	144	144
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$9A$	18	$\boxed{3}$												
$10A$	60			$\boxed{4}$	4									
$10B$	60			4	$\boxed{4}$									
$10C$	40			6	6	$\boxed{2}$								
$12A$	288							4	2	$\boxed{2}$				
$12B$	96							12	$\boxed{6}$	6	4			
$12C$	96							12	6	6	$\boxed{4}$			
$12D$	72							16	8	8		2	2	$\boxed{2}$
$12E$	48							24	12	12	8	3	3	3
$12F$	24							48	24	24	16	6	6	6
$12G$	24							48	24	24	16	6	6	6
$12H$	24							48	24	24	16	6	6	6
	$\chi(G\lvert H_2)$	3	0	4	4	2	0	0	6	2	4	0	0	2

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$12A$	$12B$	$12C$	$12D$	$12E$	$12F$	$12G$	$12H$	$12I$	$12J$	$12K$	$12L$	$12M$
	$\|C_G(g)\|$	1152	576	576	384	144	144	144	96	72	48	48	24	24
$[x_i]$	$\|C_{\overline{G}}(x)\|$													
$12A$	288													
$12B$	96								1					
$12C$	96								1					
$12D$	72									1				
$12E$	48								$\boxed{2}$		1	1		
$12F$	24								4	$\boxed{3}$	2	2	1	1
$12G$	24								4	3	2	2	$\boxed{1}$	1
$12H$	24								4	3	$\boxed{2}$	2	1	1
	$\chi(G\|H_2)$	0	6	2	4	0	0	2	2	3	2	0	1	0

Table 6.3: Continued - The fusion of $GO^{-}(8,2)$ into $Sp(8,2)$

	$[g]$	$14A$	$15A$	$15B$	$17A$	$17B$	$18A$	$20A$	$20B$	$21A$	$24A$	$24B$	$30A$	$30B$
	$\|C_G(g)\|$	14	90	90	17	17	18	40	40	21	48	48	30	30
$[x_i]$	$\|C_{\overline{G}}(x)\|$													
$14A$	14	$\boxed{1}$												
$15A$	90		$\boxed{1}$	1										
$15B$	90		1	$\boxed{1}$										
$17A$	17				$\boxed{1}$	1								
$17B$	17				1	$\boxed{1}$								
$18A$	18						$\boxed{1}$							
$20A$	20							$\boxed{2}$	2					
$21A$	21									$\boxed{1}$				
$24A$	24										2	$\boxed{2}$		
$30A$	30												$\boxed{1}$	1
$30B$	30												1	$\boxed{1}$
	$\chi(G\|H_2)$	1	1	1	1	1	1	2	0	1	0	2	1	1

Table 6.4: The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$2F$
	$\lvert C_G(g)\rvert$	47377612800	185794560	8847360	2949120	737280	147456	49152
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$							
$1A$	348364800	$\boxed{136}$						
$2A$	2903040		$\boxed{64}$					
$2B$	221184		840	$\boxed{40}$				
$2C$	92160		2016	96	$\boxed{32}$	8		
$2D$	46080		4032	192	64	$\boxed{16}$		
$2E$	9216		20160	960	320	80	$\boxed{16}$	
$2F$	6144		30240	1440	480	120	24	$\boxed{8}$
	$\chi(G\lvert H_3)$	136	64	40	32	16	16	8

Table 6.4: Continued - The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$3A$	$3B$	$3C$	$3D$	$4A$	$4B$	$4C$	$4D$	$4E$	$4F$	$4G$
	$\lvert C_G(g)\rvert$	4354560	77760	12960	3888	92160	92160	36864	12288	6144	6144	3072
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$											
$3A$	155520	$\boxed{28}$										
$3B$	77760	56	$\boxed{1}$									
$3C$	3888	1120	20		$\boxed{1}$							
$3D$	1296	3360	60	$\boxed{10}$	3							
$4A$	9216					10	10	$\boxed{4}$				
$4B$	7680					12	$\boxed{12}$					
$4C$	4608					$\boxed{20}$	20	8				
$4D$	1024					90	90	36	$\boxed{12}$	6	6	3
$4E$	768					120	120	48	16	8	$\boxed{8}$	4
$4F$	768					120	120	48	16	8	8	$\boxed{4}$
$4G$	384					240	240	96	32	16	16	8
$4H$	256					360	360	144	48	24	24	12
$4I$	192					480	480	192	64	32	32	16
$4J$	128					720	720	288	96	48	48	24
	$\chi(G\lvert H_3)$	28	1	10	1	20	12	4	12	0	8	4

Table 6.4: Continued - The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$4H$	$4I$	$4J$	$4K$	$4L$	$5A$	$5B$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$
	$\lvert C_G(g)\rvert$	3072	1024	768	512	512	3600	300	69120	13824	4608	4320	1728	1296
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$4A$	9216													
$4B$	7680													
$4C$	4608													
$4D$	1024	3	1											
$4E$	768	4		1										
$4F$	768	4		1										
$4G$	384	$\boxed{8}$		2										
$4H$	256	12	$\boxed{4}$	3	2	2								
$4I$	192	16		$\boxed{4}$										
$4J$	128	24	8	6	4	$\boxed{4}$								
$5A$	600						$\boxed{6}$							
$5B$	300						12	$\boxed{1}$						
$6A$	4320								$\boxed{16}$			1		
$6B$	3456								20	$\boxed{4}$				
$6C$	1728								40	8			$\boxed{1}$	
$6D$	1296													$\boxed{1}$
$6E$	576								120	24	$\boxed{8}$		3	
$6F$	432								160	32		10	4	3
$6G$	432								160	32		$\boxed{10}$	4	3
	$\chi(G\mid H_3)$	8	4	4	0	4	6	1	16	4	8	10	1	1

Table 6.4: Continued - The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$6H$	$6I$	$6J$	$6K$	$6L$	$6M$
	$\lvert C_G(g)\rvert$	69120	13824	4608	4320	1728	1296	1152	864	864	432	288	288	288
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$6A$	4320													
$6B$	3456													
$6C$	1728													
$6D$	1296													
$6E$	576							2						
$6F$	432								2	2	$\boxed{1}$			
$6G$	432								2	2	1			
$6H$	288							$\boxed{4}$	3	3		1	1	1
$6I$	288							4	3	3		$\boxed{1}$	1	1
$6J$	216								$\boxed{4}$	4	2			
$6K$	216								4	$\boxed{4}$	2			
$6L$	144							8	6	6	3	2	$\boxed{2}$	2
	$\chi(G\mid H_3)$	16	4	8	10	1	1	4	4	4	1	1	2	4

Table 6.4: Continued - The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$6M$	$6N$	$6O$	$6P$	$7A$	$8A$	$8B$	$8C$	$8D$	$8E$	$8F$	$9A$	$9B$
	$\lvert C_G(g)\rvert$	288	288	144	96	42	384	384	128	128	32	32	54	27
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$6H$	288	1	1											
$6I$	288	1	1											
$6J$	216													
$6K$	216													
$6L$	144	2	2	1										
$6M$	144	2	2	$\boxed{1}$										
$6N$	72	$\boxed{4}$	4	2										
$6O$	72	4	$\boxed{4}$	2										
$6P$	48	6	6	3	$\boxed{2}$									
$7A$	14					$\boxed{3}$								
$8A$	192						$\boxed{2}$	2						
$8B$	64						6	$\boxed{6}$	2	2				
$8C$	64						6	6	$\boxed{2}$	2				
$8D$	64						6	6	2	$\boxed{2}$				
$8E$	16						24	24	8	8	2	$\boxed{2}$		
$9A$	54												$\boxed{1}$	
$9B$	27												2	$\boxed{1}$
	$\chi(G\|H_3)$	4	4	1	2	3	2	6	2	2	0	2	1	1

Table 6.4: Continued - The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$10A$	$10B$	$10C$	$10D$	$12A$	$12B$	$12C$	$12D$	$12E$	$12F$	$12G$	$12H$	$12I$
	$\lvert C_G(g)\rvert$	240	240	80	20	1152	576	576	384	144	144	144	96	72
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$10A$	60	$\boxed{4}$	4											
$10B$	40	6	6	$\boxed{2}$										
$10C$	20	12	12	4	$\boxed{1}$									
$12A$	288					$\boxed{4}$	2	2						
$12B$	288					4	$\boxed{2}$	2						
$12C$	144					8	4	4		$\boxed{1}$	1	1		
$12D$	96					12	6	$\boxed{6}$	4				1	
$12E$	72					16	8	8		2	2	2		$\boxed{1}$
$12F$	72					16	8	8		2	$\boxed{2}$	2		1
$12G$	48					24	12	12	8	3	3	3	$\boxed{2}$	
$12H$	24					48	24	24	16	6	6	6	4	3
$12I$	24					48	24	24	16	6	6	6	4	3
$12J$	24					48	24	24	16	6	6	6	4	3
	$\chi(G\|H_3)$	4	0	2	1	4	2	6	0	1	2	0	2	1

Table 6.4: Continued - The fusion of $GO^{+}(8,2)$ into $Sp(8,2)$

	$[g]$	$12J$	$12K$	$12L$	$12M$	$14A$	$15A$	$15B$	$15C$	$18A$	20A	$20B$	24A	30A
	$\lvert C_G(g)\rvert$	48	48	24	24	14	90	90	15	18	40	40	48	30
$[x_i]$	$\lvert C_{\overline{G}}(x)\rvert$													
$12A$	288													
$12B$	288													
$12C$	144													
$12D$	96													
$12E$	72													
$12F$	72													
$12G$	48	1	1											
$12H$	24	2	$\boxed{2}$	1	1									
$12I$	24	2	2	$\boxed{1}$	1									
$12J$	24	2	2	1	$\boxed{1}$									
$14A$	14					$\boxed{1}$								
$15A$	30						$\boxed{3}$	3						
$15B$	15						6	6	$\boxed{1}$					
$18A$	18									$\boxed{1}$				
$20A$	20										2	$\boxed{2}$		
$24A$	24												$\boxed{2}$	
$30A$	30													$\boxed{1}$
	$\chi(G\vert H_3)$	0	2	1	1	1	3	0	1	1	0	2	2	1

Table 6.5: The summary of the fusion maps

$GO^-(8,2)$	$GO^+(8,2)$	$2^7{:}Sp(6,2)$	$\to Sp(8,2)$	$GO^-(8,2)$	$GO^+(8,2)$	$2^7{:}Sp(6,2)$	$\to Sp(8,2)$
1A	1A	1A	1A	2A	2A	2A 2E	2A
2B	2B	2B 2G	2B	2C	2C	2C 2D 2K	2C
	2D	2J	2D	2D	2E	2F 2H 2L	2E
2E	2F	2I 2M 2N	2F	3A	3A	3A	3A
	3B		3B	3B	3D	3C	3C
3C	3C	3B	3D	4A	4C	4B 4S	4A
4B	4B	4A 4O	4B	4D	4A	4D 4K	4C
4C	4D	4C 4E 4V	4D	4F		4G 4P	4E
	4E	4H 4U	4F	4E	4F	4I 4L 4W	4G
4G	4G	4F 4N 4R 4Z	4H	4H	4H	4J 4Q 4T 4AA	4I
	4I	4Y	4J	4I		4M 4AB	4K
	4J	4X 4AC	4L	5A	5A	5A	5A
	5B		5B	6A	6A	6A 6I	6A
6C	6B	6B 6K	6B	6E	6E	6C 6H 6O	6C
6D	6G	6Q	6D		6C		6E
6B	6D	6D	6F	6F	6H	6J 6L 6P	6G
6G	6J	6E 6V	6H		6K	6F	6I
6H	6F	6M	6J		6I		6K
6J	6L	6G 6U	6L		6N	6S	6M
6K	6O	6R 6X	6N	6I	6M	6N	6O
6L	6P	6T 6W	6P	7A	7A	7A	7A

Table 6.5: Summary of fusion maps

$GO^-(8,2)$	$GO^+(8,2)$	$2^7{:}Sp(6,2)$	$\to Sp(8,2)$	$GO^-(8,2)$	$GO^+(8,2)$	$2^7{:}Sp(6,2)$	$\to Sp(8,2)$
8B	8A	8A 8I	8A	8A	8B	8B 8F	8B
8C	8C	8C 8H	8C	8D	8D	8D 8E	8D
8E		8J	8E		8E	8G	8F
9A	9A	9A	9A		9B		9B
10A	10A	10A 10E	10A	10B		10C	10B
10C	10B	10B 10D	10C		10C		10D
	12A	12D	12A	12B	12B	12A 12N	12B
12A	12D	12B 12L	12C	12C		12C 12E	12D
	12C		12E		12F	12H	12F
12D		12G	12G	12E	12G	12F 12K 12M	12H
12F	12E	12O	12I	12H		12I	12J
	12H	12J	12K	12G	12I	12P	12L
	12J		12M	14A	14A	14A	14A
15A	15A	15A	15A	15B			15B
	15B		15C	17A			17A
17B			17B	18A	18A	18A	18A
20A		20A	20A		20A	20B	20B
21A			21A	24A	24A	24A	24A
24A		24B	24B	30A	30A	30A	30A
30B			30B				

6.5 Fischer matrices of $\overline{G}$

The character table of $\overline{G}$ will be constructed using the Clifford-Fischer Theory. This theory entails utilizing the character tables of inertia factor groups together with the Fischer matrices of $\overline{G}$. The Clifford-Fischer Theory requires that the $Irr(N)$ be extendable to the inertia groups. Since $\overline{G} = N{:}G$ is a split extension and N elementary abelian, then by Mackey's Theorem the $Irr(N)$ are extendable. The affine subgroup $\overline{G}$ has 81 Fischer matrices since G has 81 conjugacy classes. The size of each Fischer matrix can be deduced from Table 6.2. Let us consider the construction of the Fischer matrix corresponding to the class $2A$ of G. For this demonstration we consider the general form of the Fischer matrix. The Fischer matrix corresponding to this class is a 5×5 matrix. The fusion maps of the inertia factor groups H_i into G are given in Section 6.4. We note from Table 6.5 that there is one class of H_2, one class of H_3 and two classes of H_4 that fuse into the class $2A$. Thus the Fischer matrix from $2A$ is of the form

Table 6.6: Fischer matrix from $2A$

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$\lvert C_{\overline{G}}(x_{1j})\rvert$		$\lvert C_{\overline{G}}(x_{21})\rvert$	$\lvert C_{\overline{G}}(x_{22})\rvert$	$\lvert C_{\overline{G}}(x_{23})\rvert$	$\lvert C_{\overline{G}}(x_{24})\rvert$	$\lvert C_{\overline{G}}(x_{25})\rvert$
(k, m)	$\lvert C_{H_k}(g_{2km})\rvert$					
$(1, 1)$	$\lvert C_{H_1}(g_{211})\rvert$	1	1	1	1	1
$(2, 1)$	$\lvert C_{H_2}(g_{221})\rvert$	w	a	b	c	d
$(3, 1)$	$\lvert C_{H_3}(g_{231})\rvert$	x	e	f	g	h
$(4, 1)$	$\lvert C_{H_4}(g_{241})\rvert$	y	i	j	k	l
$(4, 2)$	$\lvert C_{H_4}(g_{242})\rvert$	z	m	n	o	p
m_{2j}		m_{21}	m_{22}	m_{23}	m_{24}	m_{25}

The values $|C_{\overline{G}}(x_{1j})|$ are from Table 6.2 and the values $|C_{H_k}(g_{2km})|$ are obtained from the character tables of H_2, H_3 and H_4 or from the respective fusion tables 6.3 and 6.4. From $m_{2j} = \frac{f_i |N|}{k}$, we have $m_{21} = 2 = m_{22}$, $m_{23} = 128 = m_{24}$ and $m_{25} = 252$. Since N is elementary abelian, we have $w = x = 64$, $y = 1$ and $z = 126$. This process yields

Table 6.7: Fischer matrix from $2A$

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$\lvert C_{\overline{G}}(x_{1j})\rvert$		47563407360	47563407360	743178240	743178240	377487360
(k, m)	$\lvert C_{H_k}(g_{2km})\rvert$					
$(1, 1)$	185794560	1	1	1	1	1
$(2, 1)$	2903040	64	a	b	c	d
$(3, 1)$	2903040	64	e	f	g	h
$(4, 1)$	185794560	1	i	j	k	l
$(4, 2)$	1474560	126	m	n	o	p
m_{2j}		2	2	128	128	252

We use the orthogonality relations and properties of a Fischer matrix to compute the rest of entries. Ultimately we obtain the following Fischer matrix from the class $2A$.

Table 6.8: Fischer matrix from $2A$

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$\lvert C_{\overline{G}}(x_{1j})\rvert$		47563407360	47563407360	743178240	743178240	377487360
(k, m)	$\lvert C_{H_k}(g_{2km})\rvert$					
$(1, 1)$	185794560	1	1	1	1	1
$(2, 1)$	2903040	64	-64	8	-8	0
$(3, 1)$	2903040	64	-64	-8	8	0
$(4, 1)$	185794560	1	1	-1	-1	1
$(4, 2)$	1474560	126	126	0	0	-2
m_{2j}		2	2	128	128	252

We used Programmes C and D in [15] to compute the rest of the Fischer matrices of $\overline{G}$. These are listed in Table 6.9 below.

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	2	2	2
$\lvert C_{\overline{G}}(x_{1j})\rvert$		24257337753600	24257337753600	95126814720	95126814720
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$				
$(1,1)$	47377612800	1	1	1	1
$(2,1)$	394813440	120	-120	-8	8
$(3,1)$	348364800	136	-136	8	-8
$(4,1)$	185794560	255	255	-1	-1
m_{1j}		1	1	255	255

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$
$o(x_{2j})$		2	2	4	4	2
$\lvert C_{\overline{G}}(x_{2j})\rvert$		47563407360	47563407360	743178240	743178240	377487360
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$					
$(1,1)$	185794560	1	1	1	1	1
$(2,1)$	2903040	64	-64	8	-8	0
$(3,1)$	2903040	64	-64	-8	8	0
$(4,1)$	185794560	1	1	-1	-1	1
$(4,2)$	1474560	126	126	0	0	-2
m_{2j}		2	2	128	128	252

$\mathbb{F}_3 = M(2B)$		$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$	$x_{3,5}$
$o(x_{3j})$		2	2	2	2	4
$\lvert C_{\overline{G}}(x_{3j})\rvert$		1132462080	1132462080	75497472	75497472	11796480
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$					
$(1,1)$	8847360	1	1	1	1	1
$(2,1)$	368640	24	-24	8	-8	0
$(3,1)$	221184	40	-40	-8	8	0
$(4,1)$	2949120	3	3	3	3	-1
$(4,2)$	147456	60	60	-4	-4	0
m_{3j}		4	4	60	60	384

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_4 = M(2C)$		$x_{4,1}$	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$	$x_{4,5}$	$x_{4,6}$
$o(x_{4j})$		2	2	4	4	2	4
$\|C_{\overline{G}}(x_{4j})\|$		377487360	377487360	23592960	23592960	12582912	5898240
(k,m)	$\|C_{H_k}(g_{4km})\|$						
$(1,1)$	2949120	1	1	1	1	1	1
$(2,1)$	92160	8	-8	2	-2	0	0
$(3,1)$	92160	8	-8	-2	2	0	0
$(4,1)$	2949120	1	1	1	1	1	-1
$(4,2)$	1474560	2	2	-2	-2	2	0
$(4,3)$	49152	12	12	0	0	-1	0
m_{4j}		4	4	64	64	120	256

$\mathbb{F}_5 = M(2D)$		$x_{4,1}$	$x_{4,2}$	$x_{4,3}$
$o(x_{5j})$		2	2	4
$\|C_{\overline{G}}(x_{5j})\|$		23592960	23592960	786432
(k,m)	$\|C_{H_k}(g_{5km})\|$			
$(1,1)$	737280	1	1	1
$(3,1)$	46080	16	-16	0
$(4,1)$	49152	15	15	-1
m_{5j}		16	16	480

$\mathbb{F}_6 = M(2E)$		$x_{6,1}$	$x_{6,2}$	$x_{6,3}$	$x_{6,4}$	$x_{6,5}$	$x_{6,6}$	$x_{6,7}$
$o(x_{6j})$		2	2	4	4	2	4	4
$\|C_{\overline{G}}(x_{6j})\|$		9437184	9437184	2359296	2359296	1572864	393216	393216
(k,m)	$\|C_{H_k}(g_{6km})\|$							
$(1,1)$	147456	1	1	1	1	1	1	1
$(2,1)$	9216	16	-16	8	-8	0	0	0
$(3,1)$	9216	16	-16	-8	8	0	0	0
$(4,1)$	147456	1	1	-1	-1	1	1	-1
$(4,2)$	49152	3	3	3	3	3	-1	-1
$(4,3)$	49152	3	3	-3	-3	3	-1	1
$(4,4)$	6144	24	24	0	0	-8	0	0
m_{6j}		8	8	32	32	48	192	192

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_7 = M(2F)$		$x_{7,1}$	$x_{7,2}$	$x_{7,3}$	$x_{7,4}$	$x_{7,5}$	$x_{7,6}$
$o(x_{7j})$		2	2	4	4	4	4
$\lvert C_{\overline{G}}(x_{7j})\rvert$		1572864	1572864	1572864	1572864	131072	98304
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$						
$(1,1)$	49152	1	1	1	1	1	1
$(2,1)$	6144	8	8	-8	-8	0	0
$(3,1)$	6144	8	-8	-8	8	0	0
$(4,1)$	49152	1	1	1	1	1	-1
$(4,2)$	8192	6	6	6	6	-2	0
$(4,3)$	6144	8	-8	8	-8	0	0
m_{7j}		16	16	16	16	192	256

$\mathbb{F}_8 = M(3A)$		$x_{8,1}$	$x_{8,2}$	$x_{8,3}$	$x_{8,4}$
$o(x_{8j})$		3	6	6	6
$\lvert C_{\overline{G}}(x_{8j})\rvert$		557383680	557383680	8847360	8847360
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$				
$(1,1)$	4354560	1	1	1	1
$(2,1)$	120960	36	-36	4	-4
$(3,1)$	155520	28	-28	-4	4
$(4,1)$	69120	63	63	-1	-1
m_{8j}		4	4	152	152

$\mathbb{F}_9 = M(3B)$		$x_{9,1}$	$x_{9,2}$
$o(x_{9j})$		3	6
$\lvert C_{\overline{G}}(x_{9j})\rvert$		155520	155520
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$		
$(1,1)$	77760	1	1
$(3,1)$	77760	1	-1
m_{9j}		256	256

$\mathbb{F}_{10} = M(3C)$		$x_{10,1}$	$x_{10,2}$	$x_{10,3}$	$x_{10,4}$
$o(x_{10j})$		3	6	6	6
$\lvert C_{\overline{G}}(x_{10j})\rvert$		414720	414720	27648	27648
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$				
$(1,1)$	12960	1	1	1	1
$(2,1)$	2160	6	-6	-2	2
$(3,1)$	1296	10	-10	2	-2
$(4,1)$	864	15	15	-1	-1
m_{10j}		16	16	240	240

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{11} = M(3D)$		$x_{11,1}$	$x_{11,2}$	$x_{11,3}$	$x_{11,4}$
$o(x_{11j})$		3	6	6	6
$\lvert C_{\overline{G}}(x_{11j})\rvert$		31104	31104	10368	10368
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$				
$(1,1)$	3888	1	1	1	1
$(2,1)$	1296	3	3	-1	-1
$(3,1)$	3888	1	-1	-1	1
$(4,1)$	1296	3	-3	1	-1
m_{11j}		64	64	192	192

$\mathbb{F}_{12} = M(4A)$		$x_{12,1}$	$x_{12,2}$	$x_{12,3}$	$x_{12,4}$	$x_{12,5}$
$o(x_{12j})$		4	4	4	4	8
$\lvert C_{\overline{G}}(x_{12j})\rvert$		5898240	5898240	393216	393216	184320
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$					
$(1,1)$	92160	1	1	1	1	1
$(2,1)$	7680	12	-12	4	-4	0
$(3,1)$	4608	20	-20	-4	4	0
$(4,1)$	92160	1	1	1	1	-1
$(4,2)$	3072	30	30	-2	-2	0
m_{12j}		8	8	120	120	256

$\mathbb{F}_{13} = M(4B)$		$x_{13,1}$	$x_{13,2}$	$x_{13,3}$	$x_{13,4}$	$x_{13,5}$
$o(x_{13j})$		4	4	4	4	8
$\lvert C_{\overline{G}}(x_{13j})\rvert$		5898240	5898240	393216	393216	184320
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$					
$(1,1)$	92160	1	1	1	1	1
$(2,1)$	4608	20	-20	-4	4	0
$(3,1)$	7680	12	-12	4	-4	0
$(4,1)$	92160	1	1	1	1	-1
$(4,2)$	3072	30	30	-2	-2	0
m_{13j}		8	8	120	120	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{14} = M(4C)$		$x_{14,1}$	$x_{14,2}$	$x_{14,3}$	$x_{14,4}$	$x_{14,5}$
$o(x_{14j})$		4	4	4	4	4
$\lvert C_{\overline{G}}(x_{14j})\rvert$		1179648	1179648	393216	393216	49152
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$					
$(1,1)$	36864	1	1	1	1	1
$(2,1)$	3072	12	12	-4	-4	0
$(3,1)$	9216	4	-4	4	-4	0
$(4,1)$	12288	3	3	3	3	-1
$(4,2)$	3072	12	-12	-4	4	0
m_{14j}		16	16	48	48	384

$\mathbb{F}_{15} = M(4D)$		$x_{15,1}$	$x_{15,2}$	$x_{15,3}$	$x_{15,4}$	$x_{15,5}$	$x_{15,6}$
$o(x_{15j})$		4	4	4	4	4	4
$\lvert C_{\overline{G}}(x_{15j})\rvert$		393216	393216	131072	131072	49152	24576
(k,m)	$\lvert C_{H_k}(g_{15km})\rvert$						
$(1,1)$	12288	1	1	1	1	1	1
$(2,1)$	3072	8	8	-8	-8	0	0
$(3,1)$	1024	8	-8	-8	8	0	0
$(4,1)$	12288	1	1	1	1	1	-1
$(4,2)$	6144	6	6	6	6	-2	0
$(4,3)$	1024	8	-8	8	-8	0	0
m_{15j}		16	16	48	48	128	256

$\mathbb{F}_{16} = M(4E)$		$x_{16,1}$	$x_{16,2}$	$x_{16,3}$	$x_{16,4}$
$o(x_{16j})$		4	4	4	8
$\lvert C_{\overline{G}}(x_{16j})\rvert$		98304	98304	16384	12288
(k,m)	$\lvert C_{H_k}(g_{16km})\rvert$				
$(1,1)$	6144	1	1	1	1
$(2,1)$	768	8	-8	0	0
$(4,1)$	6144	1	1	1	-1
$(4,2)$	1024	6	6	-2	0
m_{16j}		32	32	192	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{17} = M(4F)$		$x_{17,1}$	$x_{17,2}$	$x_{17,3}$	$x_{17,4}$
$o(x_{17j})$		4	4	4	8
$\lvert C_{\overline{G}}(x_{17j})\rvert$		98304	98304	16384	12288
(k, m)	$\lvert C_{H_k}(g_{17km})\rvert$				
$(1, 1)$	6144	1	1	1	1
$(3, 1)$	768	8	-8	0	0
$(4, 1)$	6144	1	1	1	-1
$(4, 2)$	1024	6	6	-2	0
m_{17j}		32	32	192	256

$\mathbb{F}_{18} = M(4G)$		$x_{18,1}$	$x_{18,2}$	$x_{18,3}$	$x_{18,4}$	$x_{18,5}$	$x_{18,6}$
$o(x_{18j})$		4	4	4	4	4	4
$\lvert C_{\overline{G}}(x_{18j})\rvert$		49152	49152	49152	49152	8192	8192
(k, m)	$\lvert C_{H_k}(g_{18km})\rvert$						
$(1, 1)$	3072	1	1	1	1	1	1
$(2, 1)$	768	4	-4	-4	4	0	0
$(3, 1)$	768	4	-4	4	-4	0	0
$(4, 1)$	3072	1	1	-1	-1	1	-1
$(4, 2)$	1024	3	3	-3	-3	-1	1
$(4, 3)$	1024	3	3	3	3	-1	1
m_{18j}		32	32	32	32	192	192

$\mathbb{F}_{19} = M(4H)$		$x_{19,1}$	$x_{19,2}$	$x_{19,3}$	$x_{19,4}$	$x_{19,5}$	$x_{19,6}$	$x_{19,7}$
$o(x_{19j})$		4	4	4	4	4	8	8
$\lvert C_{\overline{G}}(x_{19j})\rvert$		98304	98304	24576	24576	16384	12288	12288
(k, m)	$\lvert C_{H_k}(g_{19km})\rvert$							
$(1, 1)$	3072	1	1	1	1	1	1	1
$(2, 1)$	384	8	-8	-4	4	0	0	0
$(3, 1)$	384	8	-8	4	-4	0	0	0
$(4, 1)$	3072	1	1	-1	-1	1	-1	1
$(4, 2)$	3072	1	1	1	1	1	-1	-1
$(4, 3)$	3072	1	1	-1	-1	1	1	-1
$(4, 4)$	256	12	12	0	0	-4	0	0
m_{19j}		16	16	64	64	96	128	128

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{20} = M(4I)$		$x_{20,1}$	$x_{20,2}$	$x_{20,3}$	$x_{20,4}$	$x_{20,5}$	$x_{20,6}$	$x_{20,7}$
$o(x_{20j})$		4	4	4	4	8	4	8
$\lvert C_{\overline{G}}(x_{20j})\rvert$		16384	16384	16384	16384	4096	4096	4096
(k,m)	$\lvert C_{H_k}(g_{20km})\rvert$							
$(1,1)$	1024	1	1	1	1	1	1	1
$(2,1)$	256	4	-4	4	-4	0	0	0
$(3,1)$	256	4	-4	-4	4	0	0	0
$(4,1)$	1024	1	1	1	1	-1	-1	1
$(4,2)$	1024	1	1	1	1	-1	1	-1
$(4,3)$	1024	1	1	1	1	1	-1	-1
$(4,4)$	256	4	4	-4	-4	0	0	0
m_{20j}		32	32	32	32	128	128	128

$\mathbb{F}_{21} = M(4J)$		$x_{21,1}$	$x_{21,2}$	$x_{21,3}$
$o(x_{21j})$		4	4	8
$\lvert C_{\overline{G}}(x_{21j})\rvert$		6144	6144	1024
(k,m)	$\lvert C_{H_k}(g_{21km})\rvert$			
$(1,1)$	768	1	1	1
$(3,1)$	192	4	-4	0
$(4,1)$	256	3	3	-1
m_{21j}		64	64	384

$\mathbb{F}_{22} = M(4K)$		$x_{22,1}$	$x_{22,2}$	$x_{22,3}$	$x_{22,4}$
$o(x_{22j})$		4	4	8	8
$\lvert C_{\overline{G}}(x_{22j})\rvert$		4096	4096	2048	1024
(k,m)	$\lvert C_{H_k}(g_{22km})\rvert$				
$(1,1)$	512	1	1	1	1
$(2,1)$	128	4	-4	0	0
$(4,1)$	512	1	1	1	-1
$(4,2)$	256	2	2	-2	0
m_{22j}		64	64	128	256

$\mathbb{F}_{23} = M(4L)$		$x_{23,1}$	$x_{23,2}$	$x_{23,3}$	$x_{23,4}$
$o(x_{23j})$		4	4	8	8
$\lvert C_{\overline{G}}(x_{23j})\rvert$		4096	4096	2048	1024
(k,m)	$\lvert C_{H_k}(g_{23km})\rvert$				
$(1,1)$	512	1	1	1	1
$(3,1)$	128	4	-4	0	0
$(4,1)$	512	1	1	1	-1
$(4,2)$	256	2	2	-2	0
m_{23j}		64	64	128	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{24} = M(5A)$		$x_{24,1}$	$x_{24,2}$	$x_{24,3}$	$x_{24,4}$
$o(x_{24j})$		5	10	10	10
$\lvert C_{\overline{G}}(x_{24j})\rvert$		115200	115200	7680	7680
(k,m)	$\lvert C_{H_k}(g_{24km})\rvert$				
$(1,1)$	3600	1	1	1	1
$(2,1)$	360	10	-10	2	-2
$(3,1)$	600	6	-6	-2	2
$(4,1)$	240	15	15	-1	-1
m_{24j}		16	16	240	240

$\mathbb{F}_{25} = M(5B)$		$x_{25,1}$	$x_{25,2}$
$o(x_{25j})$		5	10
$\lvert C_{\overline{G}}(x_{25j})\rvert$		600	600
(k,m)	$\lvert C_{H_k}(g_{25km})\rvert$		
$(1,1)$	300	1	1
$(3,1)$	300	1	-1
m_{25j}		256	256

$\mathbb{F}_{26} = M(6A)$		$x_{26,1}$	$x_{26,2}$	$x_{26,3}$	$x_{26,4}$	$x_{26,5}$
$o(x_{26j})$		6	6	12	12	6
$\lvert C_{\overline{G}}(x_{26j})\rvert$		4423680	4423680	276480	276480	147456
(k,m)	$\lvert C_{H_k}(g_{26km})\rvert$					
$(1,1)$	69120	1	1	1	1	1
$(2,1)$	4320	16	-16	4	-4	0
$(3,1)$	4320	16	-16	-4	4	0
$(4,1)$	69120	1	1	-1	-1	1
$(4,2)$	2304	30	30	0	0	-2
m_{26j}		8	8	128	128	240

$\mathbb{F}_{27} = M(6B)$		$x_{26,1}$	$x_{26,2}$	$x_{26,3}$	$x_{26,4}$	$x_{26,5}$
$o(x_{27j})$		6	6	6	6	12
$\lvert C_{\overline{G}}(x_{27j})\rvert$		442368	442368	147456	147456	18432
(k,m)	$\lvert C_{H_k}(g_{27km})\rvert$					
$(1,1)$	13824	1	1	1	1	1
$(2,1)$	1152	12	12	-4	-4	0
$(3,1)$	3456	4	-4	4	-4	0
$(4,1)$	4608	3	3	3	3	-1
$(4,2)$	1152	12	-12	-4	4	0
m_{27j}		16	16	48	48	384

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{28} = M(6C)$		$x_{28,1}$	$x_{28,2}$	$x_{28,3}$	$x_{28,4}$	$x_{28,5}$	$x_{28,6}$
$o(x_{28j})$		6	6	12	12	6	12
$\lvert C_{\overline{G}}(x_{28j})\rvert$		147456	147456	36864	36864	24576	9216
(k,m)	$\lvert C_{H_k}(g_{28km})\rvert$						
$(1,1)$	4608	1	1	1	1	1	1
$(2,1)$	576	8	-8	2	-2	0	0
$(3,1)$	576	8	-8	-2	2	0	0
$(4,1)$	4608	1	1	1	1	1	-1
$(4,2)$	2304	2	2	-2	-2	2	0
$(4,3)$	384	12	12	0	0	-1	0
m_{28j}		16	16	64	64	96	256

$\mathbb{F}_{29} = M(6D)$		$x_{29,1}$	$x_{29,2}$	$x_{29,3}$	$x_{29,4}$
$o(x_{29j})$		6	6	6	6
$\lvert C_{\overline{G}}(x_{29j})\rvert$		138240	138240	9216	9216
(k,m)	$\lvert C_{H_k}(g_{29km})\rvert$				
$(1,1)$	4320	1	1	1	1
$(2,1)$	720	6	-6	-2	2
$(3,1)$	432	10	-10	2	-2
$(4,1)$	288	15	15	-1	-1
m_{29j}		16	16	240	240

$\mathbb{F}_{30} = M(6E)$		$x_{30,1}$	$x_{30,2}$
$o(x_{30j})$		6	6
$\lvert C_{\overline{G}}(x_{30j})\rvert$		3456	3456
(k,m)	$\lvert C_{H_k}(g_{30km})\rvert$		
$(1,1)$	1728	1	1
$(3,1)$	1728	1	-1
m_{30j}		256	256

$\mathbb{F}_{31} = M(6F)$		$x_{31,1}$	$x_{31,2}$	$x_{31,3}$	$x_{31,4}$
$o(x_{31j})$		6	6	12	12
$\lvert C_{\overline{G}}(x_{31j})\rvert$		5184	5184	5184	5184
(k,m)	$\lvert C_{H_k}(g_{31km})\rvert$				
$(1,1)$	1296	1	1	1	1
$(2,1)$	1296	1	-1	-1	1
$(3,1)$	1296	1	1	-1	-1
$(4,1)$	1296	1	-1	1	-1
m_{31j}		128	128	128	128

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{32} = M(6G)$		$x_{32,1}$	$x_{32,2}$	$x_{32,3}$	$x_{32,4}$	$x_{32,5}$	$x_{32,6}$
$o(x_{32j})$		6	6	12	12	12	12
$\lvert C_{\overline{G}}(x_{32j})\rvert$		18432	18432	18432	18432	3072	3072
(k,m)	$\lvert C_{H_k}(g_{32km})\rvert$						
$(1,1)$	1152	1	1	1	1	1	1
$(2,1)$	288	4	-4	-4	4	0	0
$(3,1)$	288	4	-4	4	-4	0	0
$(4,1)$	1152	1	1	-1	-1	1	-1
$(4,2)$	384	3	3	-3	-3	-1	1
$(4,3)$	384	3	3	3	3	-1	1
m_{32j}		32	32	32	32	192	192

$\mathbb{F}_{33} = M(6H)$		$x_{33,1}$	$x_{33,2}$	$x_{33,3}$	$x_{33,4}$	$x_{33,5}$
$o(x_{33j})$		6	6	12	12	6
$\lvert C_{\overline{G}}(x_{33j})\rvert$		13824	13824	3456	3456	2304
(k,m)	$\lvert C_{H_k}(g_{33km})\rvert$					
$(1,1)$	864	1	1	1	1	1
$(2,1)$	216	4	-4	2	-2	0
$(3,1)$	216	4	-4	-2	2	0
$(4,1)$	864	1	1	-1	-1	1
$(4,2)$	144	6	6	0	0	-2
m_{33j}		32	32	128	128	192

$\mathbb{F}_{34} = M(6I)$		$x_{34,1}$	$x_{34,2}$	$x_{34,3}$
$o(x_{34j})$		6	6	12
$\lvert C_{\overline{G}}(x_{34j})\rvert$		6912	6912	1152
(k,m)	$\lvert C_{H_k}(g_{34km})\rvert$			
$(1,1)$	864	1	1	1
$(3,1)$	216	4	-4	0
$(4,1)$	288	3	3	-1
m_{34j}		64	64	384

$\mathbb{F}_{35} = M(6J)$		$x_{35,1}$	$x_{35,2}$	$x_{35,3}$	$x_{35,4}$
$o(x_{35j})$		6	6	6	6
$\lvert C_{\overline{G}}(x_{35j})\rvert$		3456	3456	1152	1152
(k,m)	$\lvert C_{H_k}(g_{35km})\rvert$				
$(1,1)$	432	1	1	1	1
$(2,1)$	144	3	-3	1	-1
$(3,1)$	432	1	-1	-1	1
$(4,1)$	144	3	3	-1	-1
m_{35j}		64	64	192	192

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{36} = M(6K)$		$x_{36,1}$	$x_{36,2}$
$o(x_{36j})$		6	6
$\|C_{\overline{G}}(x_{36j})\|$		576	576
(k,m)	$\|C_{H_k}(g_{36km})\|$		
$(1,1)$	288	1	1
$(3,1)$	288	1	-1
m_{36j}		256	256

$\mathbb{F}_{37} = M(6L)$		$x_{37,1}$	$x_{37,2}$	$x_{37,3}$	$x_{37,4}$	$x_{37,5}$
$o(x_{37j})$		6	6	12	12	12
$\|C_{\overline{G}}(x_{37j})\|$		2304	2304	2304	2304	576
(k,m)	$\|C_{H_k}(g_{37km})\|$					
$(1,1)$	288	1	1	1	1	1
$(2,1)$	144	2	-2	-2	2	0
$(3,1)$	144	2	-2	2	-2	0
$(4,1)$	288	1	1	1	1	-1
$(4,2)$	144	2	2	-2	-2	0
m_{37j}		64	64	64	64	256

$\mathbb{F}_{38} = M(6M)$		$x_{38,1}$	$x_{38,2}$	$x_{38,3}$
$o(x_{38j})$		6	6	12
$\|C_{\overline{G}}(x_{38j})\|$		2304	2304	384
(k,m)	$\|C_{H_k}(g_{38km})\|$			
$(1,1)$	288	1	1	1
$(3,1)$	72	4	-4	0
$(4,1)$	96	3	3	-1
m_{38j}		64	64	384

$\mathbb{F}_{39} = M(6N)$		$x_{39,1}$	$x_{39,2}$	$x_{39,3}$	$x_{39,4}$	$x_{39,5}$
$o(x_{39j})$		6	6	12	12	6
$\|C_{\overline{G}}(x_{39j})\|$		4608	4608	1152	1152	768
(k,m)	$\|C_{H_k}(g_{39km})\|$					
$(1,1)$	288	1	1	1	1	1
$(2,1)$	72	4	-4	2	-2	0
$(3,1)$	72	4	-4	-2	2	0
$(4,1)$	288	1	1	-1	-1	1
$(4,2)$	48	6	6	0	0	-2
m_{39j}		32	32	128	128	192

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{40} = M(6O)$		$x_{40,1}$	$x_{40,2}$	$x_{40,3}$	$x_{40,4}$
$o(x_{40j})$		6	6	12	12
$\lvert C_{\overline{G}}(x_{40j})\rvert$		576	576	576	576
(k,m)	$\lvert C_{H_k}(g_{40km})\rvert$				
$(1,1)$	144	1	1	1	1
$(2,1)$	144	1	-1	-1	1
$(3,1)$	144	1	1	-1	-1
$(4,1)$	144	1	-1	1	-1
m_{40j}		128	128	128	128

$\mathbb{F}_{41} = M(6P)$		$x_{41,1}$	$x_{41,2}$	$x_{41,3}$	$x_{41,4}$	$x_{41,5}$
$o(x_{41j})$		6	6	12	12	12
$\lvert C_{\overline{G}}(x_{41j})\rvert$		768	768	768	768	192
(k,m)	$\lvert C_{H_k}(g_{41km})\rvert$					
$(1,1)$	96	1	1	1	1	1
$(2,1)$	48	2	-2	-2	2	0
$(3,1)$	48	2	-2	2	-2	0
$(4,1)$	96	1	1	1	1	-1
$(4,2)$	48	2	2	-2	-2	0
m_{41j}		64	64	64	64	256

$\mathbb{F}_{42} = M(7A)$		$x_{42,1}$	$x_{42,2}$	$x_{42,3}$	$x_{42,4}$
$o(x_{42j})$		7	14	14	14
$\lvert C_{\overline{G}}(x_{42j})\rvert$		336	336	112	112
(k,m)	$\lvert C_{H_k}(g_{42km})\rvert$				
$(1,1)$	42	1	1	1	1
$(2,1)$	42	1	-1	-1	1
$(3,1)$	14	3	3	-1	-1
$(4,1)$	14	3	-3	1	-1
m_{42j}		64	64	192	192

$\mathbb{F}_{43} = M(8A)$		$x_{43,1}$	$x_{43,2}$	$x_{43,3}$	$x_{43,4}$	$x_{43,5}$
$o(x_{43j})$		8	8	8	8	8
$\lvert C_{\overline{G}}(x_{43j})\rvert$		6144	6144	2048	2048	768
(k,m)	$\lvert C_{H_k}(g_{43km})\rvert$					
$(1,1)$	384	1	1	1	1	1
$(2,1)$	64	6	6	-2	-2	0
$(3,1)$	192	2	-2	2	-2	0
$(4,1)$	384	1	1	1	1	-1
$(4,2)$	64	6	-6	-2	2	0
m_{43j}		32	32	96	96	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{44} = M(8B)$		$x_{44,1}$	$x_{44,2}$	$x_{44,3}$	$x_{44,4}$	$x_{44,5}$
$o(x_{44j})$		8	8	8	8	8
$\lvert C_{\overline{G}}(x_{44j})\rvert$		6144	6144	2048	2048	768
(k,m)	$\lvert C_{H_k}(g_{44km})\rvert$					
$(1,1)$	384	1	1	1	1	1
$(2,1)$	192	2	-2	2	-2	0
$(3,1)$	64	6	6	-2	-2	0
$(4,1)$	384	1	1	1	1	-1
$(4,2)$	64	6	-6	-2	2	0
m_{44j}		32	32	96	96	256

$\mathbb{F}_{45} = M(8C)$		$x_{45,1}$	$x_{45,2}$	$x_{45,3}$	$x_{45,4}$	$x_{45,5}$
$o(x_{45j})$		8	8	8	8	8
$\lvert C_{\overline{G}}(x_{45j})\rvert$		1024	1024	1024	1024	256
(k,m)	$\lvert C_{H_k}(g_{45km})\rvert$					
$(1,1)$	128	1	1	1	1	1
$(2,1)$	64	2	-2	-2	2	0
$(3,1)$	64	2	-2	2	-2	0
$(4,1)$	128	1	1	1	1	-1
$(4,2)$	64	2	2	-2	-2	0
m_{45j}		64	64	64	64	256

$\mathbb{F}_{46} = M(8D)$		$x_{46,1}$	$x_{46,2}$	$x_{46,3}$	$x_{46,4}$	$x_{46,5}$
$o(x_{46j})$		8	8	8	8	8
$\lvert C_{\overline{G}}(x_{46j})\rvert$		1024	1024	1024	1024	256
(k,m)	$\lvert C_{H_k}(g_{46km})\rvert$					
$(1,1)$	128	1	1	1	1	1
$(2,1)$	64	2	-2	-2	2	0
$(3,1)$	64	2	-2	2	-2	0
$(4,1)$	128	1	1	1	1	-1
$(4,2)$	64	2	2	-2	-2	0
m_{46j}		64	64	64	64	256

$\mathbb{F}_{47} = M(8E)$		$x_{47,1}$	$x_{47,2}$	$x_{47,3}$
$o(x_{47j})$		8	8	16
$\lvert C_{\overline{G}}(x_{47j})\rvert$		128	128	64
(k,m)	$\lvert C_{H_k}(g_{47km})\rvert$			
$(1,1)$	32	1	1	1
$(2,1)$	16	2	-2	0
$(4,1)$	32	1	1	-1
m_{47j}		128	128	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{48} = M(8F)$		$x_{48,1}$	$x_{48,2}$	$x_{48,3}$
$o(x_{48j})$		8	8	16
$\|C_{\overline{G}}(x_{48j})\|$		128	128	64
(k, m)	$\|C_{H_k}(g_{48km})\|$			
$(1, 1)$	32	1	1	1
$(3, 1)$	16	2	-2	0
$(4, 1)$	32	1	1	-1
m_{48j}		128	128	256

$\mathbb{F}_{49} = M(9A)$		$x_{49,1}$	$x_{49,2}$	$x_{49,3}$	$x_{49,4}$
$o(x_{49j})$		9	18	18	18
$\|C_{\overline{G}}(x_{49j})\|$		432	432	144	144
(k, m)	$\|C_{H_k}(g_{49km})\|$				
$(1, 1)$	54	1	1	1	1
$(2, 1)$	18	3	3	-1	-1
$(3, 1)$	54	1	-1	-1	1
$(4, 1)$	18	3	-3	1	-1
m_{49j}		64	64	192	192

$\mathbb{F}_{50} = M(9B)$		$x_{50,1}$	$x_{50,2}$
$o(x_{50j})$		9	18
$\|C_{\overline{G}}(x_{50j})\|$		54	54
(k, m)	$\|C_{H_k}(g_{50km})\|$		
$(1, 1)$	27	1	1
$(3, 1)$	27	1	-1
m_{50j}		256	256

$\mathbb{F}_{51} = M(10A)$		$x_{51,1}$	$x_{51,2}$	$x_{51,3}$	$x_{51,4}$	$x_{51,5}$
$o(x_{51j})$		10	10	20	20	10
$\|C_{\overline{G}}(x_{51j})\|$		3840	3840	960	960	640
(k, m)	$\|C_{H_k}(g_{51km})\|$					
$(1, 1)$	240	1	1	1	1	1
$(2, 1)$	60	4	-4	2	-2	0
$(3, 1)$	60	4	-4	-2	2	0
$(4, 1)$	240	1	1	-1	-1	1
$(4, 2)$	40	6	6	0	0	-2
m_{51j}		32	32	128	128	192

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{52} = M(10B)$		$x_{52,1}$	$x_{52,2}$	$x_{52,3}$
$o(x_{52j})$		10	10	20
$\lvert C_{\overline{G}}(x_{52j})\rvert$		1920	1920	320
(k,m)	$\lvert C_{H_k}(g_{52km})\rvert$			
$(1,1)$	240	1	1	1
$(2,1)$	60	4	-4	0
$(4,1)$	80	3	3	-1
m_{52j}		64	64	384

$\mathbb{F}_{53} = M(10C)$		$x_{53,1}$	$x_{53,2}$	$x_{53,3}$	$x_{53,4}$	$x_{53,5}$
$o(x_{53j})$		10	10	20	20	20
$\lvert C_{\overline{G}}(x_{53j})\rvert$		640	640	640	640	160
(k,m)	$\lvert C_{H_k}(g_{53km})\rvert$					
$(1,1)$	80	1	1	1	1	1
$(2,1)$	40	2	-2	-2	2	0
$(3,1)$	40	2	-2	2	-2	0
$(4,1)$	80	1	1	1	1	-1
$(4,2)$	40	2	2	-2	-2	0
m_{53j}		64	64	64	64	256

$\mathbb{F}_{54} = M(10D)$		$x_{54,1}$	$x_{54,2}$
$o(x_{54j})$		10	10
$\lvert C_{\overline{G}}(x_{54j})\rvert$		40	40
(k,m)	$\lvert C_{H_k}(g_{54km})\rvert$		
$(1,1)$	20	1	1
$(3,1)$	20	1	-1
m_{54j}		256	256

$\mathbb{F}_{55} = M(12A)$		$x_{55,1}$	$x_{55,2}$	$x_{55,3}$
$o(x_{55j})$		12	12	12
$\lvert C_{\overline{G}}(x_{55j})\rvert$		9216	9216	1536
(k,m)	$\lvert C_{H_k}(g_{55km})\rvert$			
$(1,1)$	1152	1	1	1
$(3,1)$	288	4	-4	0
$(4,1)$	384	3	3	-1
m_{55j}		64	64	384

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{56} = M(12B)$		$x_{56,1}$	$x_{56,2}$	$x_{56,3}$	$x_{56,4}$	$x_{56,5}$
$o(x_{56j})$		12	12	12	12	24
$\lvert C_{\overline{G}}(x_{56j})\rvert$		9216	9216	3072	3072	1152
(k,m)	$\lvert C_{H_k}(g_{56km})\rvert$					
$(1,1)$	576	1	1	1	1	1
$(2,1)$	96	6	6	-2	-2	0
$(3,1)$	288	2	-2	2	-2	0
$(4,1)$	576	1	1	1	1	-1
$(4,2)$	96	6	-6	-2	2	0
m_{56j}		32	32	96	96	256

$\mathbb{F}_{57} = M(12C)$		$x_{57,1}$	$x_{57,2}$	$x_{57,3}$	$x_{57,4}$	$x_{57,5}$
$o(x_{57j})$		12	12	12	12	24
$\lvert C_{\overline{G}}(x_{57j})\rvert$		9216	9216	3072	3072	1152
(k,m)	$\lvert C_{H_k}(g_{57km})\rvert$					
$(1,1)$	576	1	1	1	1	1
$(2,1)$	288	2	-2	2	-2	0
$(3,1)$	96	6	6	-2	-2	0
$(4,1)$	576	1	1	1	1	-1
$(4,2)$	96	6	-6	-2	2	0
m_{57j}		32	32	96	96	256

$\mathbb{F}_{58} = M(12D)$		$x_{58,1}$	$x_{58,2}$	$x_{58,3}$	$x_{58,4}$
$o(x_{58j})$		12	12	12	12
$\lvert C_{\overline{G}}(x_{58j})\rvert$		3072	3072	1536	768
(k,m)	$\lvert C_{H_k}(g_{58km})\rvert$				
$(1,1)$	384	1	1	1	1
$(2,1)$	96	4	-4	0	0
$(4,1)$	384	1	1	1	-1
$(4,2)$	192	2	2	-2	0
m_{58j}		64	64	128	256

$\mathbb{F}_{59} = M(12E)$		$x_{59,1}$	$x_{59,2}$
$o(x_{59j})$		12	12
$\lvert C_{\overline{G}}(x_{59j})\rvert$		288	288
(k,m)	$\lvert C_{H_k}(g_{59km})\rvert$		
$(1,1)$	144	1	1
$(3,1)$	144	1	-1
m_{59j}		256	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{60} = M(12F)$		$x_{60,1}$	$x_{60,2}$	$x_{60,3}$
$o(x_{60j})$		12	12	24
$\lvert C_{\overline{G}}(x_{60j})\rvert$		576	576	288
(k, m)	$\lvert C_{H_k}(g_{60km})\rvert$			
$(1, 1)$	144	1	1	1
$(3, 1)$	72	2	-2	0
$(4, 1)$	144	1	1	-1
m_{60j}		128	128	256

$\mathbb{F}_{61} = M(12G)$		$x_{61,1}$	$x_{61,2}$	$x_{61,3}$
$o(x_{61j})$		12	12	24
$\lvert C_{\overline{G}}(x_{61j})\rvert$		576	576	288
(k, m)	$\lvert C_{H_k}(g_{61km})\rvert$			
$(1, 1)$	144	1	1	1
$(2, 1)$	72	2	-2	0
$(4, 1)$	144	1	1	-1
m_{61j}		128	128	256

$\mathbb{F}_{62} = M(12H)$		$x_{62,1}$	$x_{62,2}$	$x_{62,3}$	$x_{62,4}$	$x_{62,5}$	$x_{62,6}$
$o(x_{62j})$		12	12	12	12	24	24
$\lvert C_{\overline{G}}(x_{62j})\rvert$		768	768	768	768	384	384
(k, m)	$\lvert C_{H_k}(g_{62km})\rvert$						
$(1, 1)$	96	1	1	1	1	1	1
$(2, 1)$	48	2	-2	-2	2	0	0
$(3, 1)$	48	2	-2	2	-2	0	0
$(4, 1)$	96	1	1	-1	-1	1	-1
$(4, 2)$	96	1	1	-1	-1	-1	1
$(4, 3)$	96	1	1	1	1	-1	1
m_{62j}		64	64	64	64	128	128

$\mathbb{F}_{63} = M(12I)$		$x_{63,1}$	$x_{63,2}$	$x_{63,3}$	$x_{63,4}$
$o(x_{63j})$		12	12	12	12
$\lvert C_{\overline{G}}(x_{63j})\rvert$		576	576	192	192
(k, m)	$\lvert C_{H_k}(g_{63km})\rvert$				
$(1, 1)$	72	1	1	1	1
$(2, 1)$	24	3	-3	1	-1
$(3, 1)$	72	1	-1	-1	1
$(4, 1)$	24	3	3	-1	-1
m_{63j}		64	64	192	192

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{64} = M(12J)$		$x_{64,1}$	$x_{64,2}$	$x_{64,3}$
$o(x_{64j})$		12	12	24
$\|C_{\overline{G}}(x_{64j})\|$		192	192	96
(k,m)	$\|C_{H_k}(g_{64km})\|$			
$(1,1)$	48	1	1	1
$(2,1)$	24	2	-2	0
$(4,1)$	48	1	1	-1
m_{64j}		128	128	256

$\mathbb{F}_{65} = M(12K)$		$x_{65,1}$	$x_{65,2}$	$x_{65,3}$
$o(x_{65j})$		12	12	24
$\|C_{\overline{G}}(x_{65j})\|$		192	192	96
(k,m)	$\|C_{H_k}(g_{65km})\|$			
$(1,1)$	48	1	1	1
$(3,1)$	24	2	-2	0
$(4,1)$	48	1	1	-1
m_{65j}		128	128	256

$\mathbb{F}_{66} = M(12L)$		$x_{66,1}$	$x_{66,2}$	$x_{66,3}$	$x_{66,4}$
$o(x_{66j})$		12	12	12	12
$\|C_{\overline{G}}(x_{66j})\|$		96	96	96	96
(k,m)	$\|C_{H_k}(g_{66km})\|$				
$(1,1)$	24	1	1	1	1
$(2,1)$	24	1	-1	-1	1
$(3,1)$	24	1	1	-1	-1
$(4,1)$	24	1	-1	1	-1
m_{66j}		128	128	128	128

$\mathbb{F}_{67} = M(12M)$		$x_{67,1}$	$x_{67,2}$
$o(x_{67j})$		12	12
$\|C_{\overline{G}}(x_{67j})\|$		48	48
(k,m)	$\|C_{H_k}(g_{67km})\|$		
$(1,1)$	24	1	1
$(3,1)$	24	1	-1
m_{67j}		256	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{68} = M(14A)$		$x_{68,1}$	$x_{68,2}$	$x_{68,3}$	$x_{68,4}$
$o(x_{68j})$		14	14	28	28
$\|C_{\overline{G}}(x_{68j})\|$		56	56	56	56
(k, m)	$\|C_{H_k}(g_{68km})\|$				
$(1, 1)$	14	1	1	1	1
$(2, 1)$	14	1	-1	-1	1
$(3, 1)$	14	1	1	-1	-1
$(4, 1)$	14	1	-1	1	-1
m_{68j}		128	128	128	128

$\mathbb{F}_{69} = M(15A)$		$x_{69,1}$	$x_{69,2}$	$x_{69,3}$	$x_{69,4}$
$o(x_{69j})$		15	30	30	30
$\|C_{\overline{G}}(x_{69j})\|$		720	720	240	240
(k, m)	$\|C_{H_k}(g_{69km})\|$				
$(1, 1)$	90	1	1	1	1
$(2, 1)$	90	1	-1	-1	1
$(3, 1)$	30	3	-3	1	-1
$(4, 1)$	30	3	3	-1	-1
m_{69j}		64	64	192	192

$\mathbb{F}_{70} = M(15B)$		$x_{70,1}$	$x_{70,2}$
$o(x_{70j})$		15	30
$\|C_{\overline{G}}(x_{70j})\|$		180	180
(k, m)	$\|C_{H_k}(g_{70km})\|$		
$(1, 1)$	90	1	1
$(2, 1)$	90	1	-1
m_{70j}		256	256

$\mathbb{F}_{71} = M(15C)$		$x_{71,1}$	$x_{71,2}$
$o(x_{71j})$		15	30
$\|C_{\overline{G}}(x_{71j})\|$		30	30
(k, m)	$\|C_{H_k}(g_{71km})\|$		
$(1, 1)$	15	1	1
$(3, 1)$	15	1	-1
m_{71j}		256	256

$\mathbb{F}_{72} = M(17A)$		$x_{72,1}$	$x_{72,2}$
$o(x_{72j})$		17	34
$\|C_{\overline{G}}(x_{72j})\|$		34	34
(k, m)	$\|C_{H_k}(g_{72km})\|$		
$(1, 1)$	17	1	1
$(2, 1)$	17	1	-1
m_{72j}		256	256

$\mathbb{F}_{73} = M(17B)$		$x_{73,1}$	$x_{73,2}$
$o(x_{73j})$		17	34
$\|C_{\overline{G}}(x_{73j})\|$		34	34
(k, m)	$\|C_{H_k}(g_{73km})\|$		
$(1, 1)$	17	1	1
$(2, 1)$	17	1	-1
m_{73j}		256	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{74} = M(18A)$		$x_{74,1}$	$x_{74,2}$	$x_{74,3}$	$x_{74,4}$
$o(x_{74j})$		18	18	36	36
$\|C_{\overline{G}}(x_{74j})\|$		72	72	72	72
(k, m)	$\|C_{H_k}(g_{74km})\|$				
$(1, 1)$	18	1	1	1	1
$(2, 1)$	18	1	-1	-1	1
$(3, 1)$	18	1	1	-1	-1
$(4, 1)$	18	1	-1	1	-1
m_{74j}		128	128	128	128

$\mathbb{F}_{75} = M(20A)$		$x_{75,1}$	$x_{75,2}$	$x_{75,3}$
$o(x_{75j})$		20	20	40
$\|C_{\overline{G}}(x_{75j})\|$		160	160	80
(k, m)	$\|C_{H_k}(g_{75km})\|$			
$(1, 1)$	40	1	1	1
$(2, 1)$	20	2	-2	0
$(4, 1)$	40	1	1	-1
m_{75j}		128	128	256

$\mathbb{F}_{76} = M(20B)$		$x_{76,1}$	$x_{76,2}$	$x_{76,3}$
$o(x_{76j})$		20	20	40
$\|C_{\overline{G}}(x_{76j})\|$		160	160	80
(k, m)	$\|C_{H_k}(g_{76km})\|$			
$(1, 1)$	40	1	1	1
$(3, 1)$	20	2	-2	0
$(4, 1)$	40	1	1	-1
m_{76j}		128	128	256

$\mathbb{F}_{77} = M(21A)$		$x_{77,1}$	$x_{77,2}$
$o(x_{77j})$		21	42
$\|C_{\overline{G}}(x_{77j})\|$		42	42
(k, m)	$\|C_{H_k}(g_{77km})\|$		
$(1, 1)$	21	1	1
$(2, 1)$	21	1	-1
m_{77j}		256	256

Table 6.9: Fischer matrices of $\overline{G}$

$\mathbb{F}_{78} = M(24A)$		$x_{78,1}$	$x_{78,2}$	$x_{78,3}$
$o(x_{78j})$		24	24	24
$\|C_{\overline{G}}(x_{78j})\|$		192	192	96
(k,m)	$\|C_{H_k}(g_{78km})\|$			
$(1,1)$	48	1	1	1
$(3,1)$	24	2	-2	0
$(4,1)$	48	1	1	-1
m_{78j}		128	128	256

$\mathbb{F}_{79} = M(24B)$		$x_{79,1}$	$x_{79,2}$	$x_{79,3}$
$o(x_{79j})$		24	24	24
$\|C_{\overline{G}}(x_{79j})\|$		192	192	96
(k,m)	$\|C_{H_k}(g_{79km})\|$			
$(1,1)$	48	1	1	1
$(2,1)$	24	2	-2	0
$(4,1)$	48	1	1	-1
m_{79j}		128	128	256

$\mathbb{F}_{80} = M(30A)$		$x_{80,1}$	$x_{80,2}$	$x_{80,3}$	$x_{80,4}$
$o(x_{80j})$		30	30	60	60
$\|C_{\overline{G}}(x_{80j})\|$		120	120	120	120
(k,m)	$\|C_{H_k}(g_{80km})\|$				
$(1,1)$	30	1	1	1	1
$(2,1)$	30	1	-1	-1	1
$(3,1)$	30	1	1	-1	-1
$(4,1)$	30	1	-1	1	-1
m_{80j}		128	128	128	128

$\mathbb{F}_{81} = M(30B)$		$x_{81,1}$	$x_{81,2}$
$o(x_{81j})$		30	30
$\|C_{\overline{G}}(x_{81j})\|$		60	60
(k,m)	$\|C_{H_k}(g_{81km})\|$		
$(1,1)$	30	1	1
$(2,1)$	30	1	-1
m_{81j}		256	256

6.6 The character table of $\overline{G}$

Since $\overline{G} = N{:}G$ is a split extension and N elementary abelian, then the $Irr(N)$ are extendable to the inertia groups. This then enables us to construct the character table of $\overline{G}$ using the Clifford-Fischer Theory. According to Gallagher's Theorem, the $Irr(\overline{G})$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\phi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \in ker(\beta)\},$$

where $\overline{H}_i$ is an inertia group and $H_i = \overline{H}_i/N$ is an inertia factor group. This then means that the character table of $\overline{G}$ will be divided into blocks corresponding to the inertia factor groups H_i, for $i \in \{1, 2, 3, 4\}$. Full details of this process are discussed in Chapter 3. Thus the character table of $\overline{G}$ will be of the form

$$\begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \cdots & B_{1,81} \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdots & B_{2,81} \\ B_{3,1} & B_{3,2} & B_{3,3} & \cdots & B_{3,81} \\ B_{4,1} & B_{4,2} & B_{4,3} & \cdots & B_{4,81} \end{bmatrix},$$

where $B_{i,j}$ are blocks corresponding to the 4 inertia factor groups and the 81 conjugacy classes of G, $\{1 \leq i \leq 4\}$ and $\{1 \leq j \leq 81\}$. The block $B_{i,j}$ is formed by multiplying the relevant columns of the character table of H_i by the rows of the Fischer matrix $M(g)$ corresponding to the classes of H_i that fuse to the class $[g] \in G$. If H_i does not contribute to $M(g)$ then the block $B_{i,j}$ will have zeroes. The fusion maps of the inertia factor groups into G are given in Section 6.4. The character table of G is available in the ATLAS. The character tables of H_2 and H_3 can be deduced from the character tables of $O^-(8,2)$ and $O^+(8,2)$ in the ATLAS, respectively. The one for H_4 was computed by Ali in [1]. We demonstrate how to compute the entries of the character table of $\overline{G}$. Let us consider the class $1A$ of G. It is clear that the identity classes of each inertia factor group H_i fuse to the class $1A$ of $H_1 = G$. We multiply the column corresponding to $1A \in G$ by the first row of the Fischer matrix $M(1A)$ in Table 6.9. Then we multiply the column corresponding to $1a \in H_2$ by the second row of $M(1A)$ corresponding to the inertia factor group H_2. Similarly, we multiply the column corresponding to $1a \in H_3$ by the third row of $M(1A)$ corresponding to the inertia factor group H_3. Lastly, we multiply the column corresponding to $1a \in H_4$ by the fourth row of $M(1A)$ corresponding to the inertia factor group H_4.

$$
\begin{bmatrix} 1 \\ 35 \\ 51 \\ 85 \\ 119 \\ 135 \\ 238 \\ 510 \\ 595 \\ 595 \\ 918 \\ 1190 \\ 1275 \\ 1512 \\ 1785 \\ 1785 \\ 2295 \\ 2856 \\ 2856 \\ 2975 \\ 2975 \\ 3213 \\ 3213 \\ 3400 \\ 3570 \\ 3570 \\ 3570 \\ 3808 \\ 4200 \\ 4760 \\ 5712 \\ 5950 \\ 5950 \\ 7140 \\ 8160 \\ 8925 \\ 8925 \\ 8960 \\ 9639 \\ 10200 \\ 11900 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
35 & 35 & 35 & 35 \\
51 & 51 & 51 & 51 \\
85 & 85 & 85 & 85 \\
119 & 119 & 119 & 119 \\
135 & 135 & 135 & 135 \\
238 & 238 & 238 & 238 \\
510 & 510 & 510 & 510 \\
595 & 595 & 595 & 595 \\
595 & 595 & 595 & 595 \\
918 & 918 & 918 & 918 \\
1190 & 1190 & 1190 & 1190 \\
1275 & 1275 & 1275 & 1275 \\
1512 & 1512 & 1512 & 1512 \\
1785 & 1785 & 1785 & 1785 \\
1785 & 1785 & 1785 & 1785 \\
2295 & 2295 & 2295 & 2295 \\
2856 & 2856 & 2856 & 2856 \\
2856 & 2856 & 2856 & 2856 \\
2975 & 2975 & 2975 & 2975 \\
2975 & 2975 & 2975 & 2975 \\
3213 & 3213 & 3213 & 3213 \\
3213 & 3213 & 3213 & 3213 \\
3400 & 3400 & 3400 & 3400 \\
3570 & 3570 & 3570 & 3570 \\
3570 & 3570 & 3570 & 3570 \\
3570 & 3570 & 3570 & 3570 \\
3808 & 3808 & 3808 & 3808 \\
4200 & 4200 & 4200 & 4200 \\
4760 & 4760 & 4760 & 4760 \\
5712 & 5712 & 5712 & 5712 \\
5950 & 5950 & 5950 & 5950 \\
5950 & 5950 & 5950 & 5950 \\
7140 & 7140 & 7140 & 7140 \\
8160 & 8160 & 8160 & 8160 \\
8925 & 8925 & 8925 & 8925 \\
8925 & 8925 & 8925 & 8925 \\
8960 & 8960 & 8960 & 8960 \\
9639 & 9639 & 9639 & 9639 \\
10200 & 10200 & 10200 & 10200 \\
11900 & 11900 & 11900 & 11900
\end{bmatrix}
$$

$$\begin{bmatrix} 11900 \\ 13056 \\ 13600 \\ 14280 \\ 14280 \\ 14688 \\ 16065 \\ 16065 \\ 16065 \\ 16065 \\ 16065 \\ 17850 \\ 17850 \\ 18360 \\ 19040 \\ 23800 \\ 23800 \\ 26775 \\ 26775 \\ 28560 \\ 28917 \\ 30464 \\ 32130 \\ 32130 \\ 32130 \\ 34425 \\ 34425 \\ 34560 \\ 38080 \\ 38556 \\ 42525 \\ 42525 \\ 43520 \\ 47600 \\ 48195 \\ 51408 \\ 53550 \\ 57120 \\ 65536 \\ 68850 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 11900 & 11900 & 11900 & 11900 \\ 13056 & 13056 & 13056 & 13056 \\ 13600 & 13600 & 13600 & 13600 \\ 14280 & 14280 & 14280 & 14280 \\ 14280 & 14280 & 14280 & 14280 \\ 14688 & 14688 & 14688 & 14688 \\ 16065 & 16065 & 16065 & 16065 \\ 16065 & 16065 & 16065 & 16065 \\ 16065 & 16065 & 16065 & 16065 \\ 16065 & 16065 & 16065 & 16065 \\ 16065 & 16065 & 16065 & 16065 \\ 17850 & 17850 & 17850 & 17850 \\ 17850 & 17850 & 17850 & 17850 \\ 18360 & 18360 & 18360 & 18360 \\ 19040 & 19040 & 19040 & 19040 \\ 23800 & 23800 & 23800 & 23800 \\ 23800 & 23800 & 23800 & 23800 \\ 26775 & 26775 & 26775 & 26775 \\ 26775 & 26775 & 26775 & 26775 \\ 28560 & 28560 & 28560 & 28560 \\ 28917 & 28917 & 28917 & 28917 \\ 30464 & 30464 & 30464 & 30464 \\ 32130 & 32130 & 32130 & 32130 \\ 32130 & 32130 & 32130 & 32130 \\ 32130 & 32130 & 32130 & 32130 \\ 34425 & 34425 & 34425 & 34425 \\ 34425 & 34425 & 34425 & 34425 \\ 34560 & 34560 & 34560 & 34560 \\ 38080 & 38080 & 38080 & 38080 \\ 38556 & 38556 & 38556 & 38556 \\ 42525 & 42525 & 42525 & 42525 \\ 42525 & 42525 & 42525 & 42525 \\ 43520 & 43520 & 43520 & 43520 \\ 47600 & 47600 & 47600 & 47600 \\ 48195 & 48195 & 48195 & 48195 \\ 51408 & 51408 & 51408 & 51408 \\ 53550 & 53550 & 53550 & 53550 \\ 57120 & 57120 & 57120 & 57120 \\ 65536 & 65536 & 65536 & 65536 \\ 68850 & 68850 & 68850 & 68850 \end{bmatrix}$$

$$
\begin{bmatrix} 1 \\ 1 \\ 34 \\ 34 \\ 51 \\ 51 \\ 84 \\ 84 \\ 204 \\ 204 \\ 204 \\ 204 \\ 357 \\ 357 \\ 476 \\ 476 \\ 476 \\ 476 \\ 595 \\ 595 \\ 714 \\ 714 \\ 714 \\ 714 \\ 1020 \\ 1020 \\ 1071 \\ 1071 \\ 1071 \\ 1071 \\ 1190 \\ 1190 \\ 1344 \\ 1344 \\ 1428 \\ 1428 \\ 2142 \\ 2176 \\ 2176 \\ 2295 \end{bmatrix}
\begin{bmatrix} 120 & -120 & -8 & 8 \end{bmatrix}
\begin{bmatrix}
120 & -120 & -8 & 8 \\
120 & -120 & -8 & 8 \\
4080 & -4080 & -272 & 272 \\
4080 & -4080 & -272 & 272 \\
6120 & -6120 & -408 & 408 \\
6120 & -6120 & -408 & 408 \\
10080 & -10080 & -672 & 672 \\
10080 & -10080 & -672 & 672 \\
24480 & -24480 & -1632 & 1632 \\
24480 & -24480 & -1632 & 1632 \\
24480 & -24480 & -1632 & 1632 \\
24480 & -24480 & -1632 & 1632 \\
42840 & -42840 & -2856 & 2856 \\
42840 & -42840 & -2856 & 2856 \\
57120 & -57120 & -3808 & 3808 \\
57120 & -57120 & -3808 & 3808 \\
57120 & -57120 & -3808 & 3808 \\
57120 & -57120 & -3808 & 3808 \\
71400 & -71400 & -4760 & 4760 \\
71400 & -71400 & -4760 & 4760 \\
85680 & -85680 & -5712 & 5712 \\
85680 & -85680 & -5712 & 5712 \\
85680 & -85680 & -5712 & 5712 \\
85680 & -85680 & -5712 & 5712 \\
122400 & -122400 & -8160 & 8160 \\
122400 & -122400 & -8160 & 8160 \\
128520 & -128520 & -8568 & 8568 \\
128520 & -128520 & -8568 & 8568 \\
128520 & -128520 & -8568 & 8568 \\
128520 & -128520 & -8568 & 8568 \\
142800 & -142800 & -9520 & 9520 \\
142800 & -142800 & -9520 & 9520 \\
161280 & -161280 & -10752 & 10752 \\
161280 & -161280 & -10752 & 10752 \\
171360 & -171360 & -11424 & 11424 \\
171360 & -171360 & -11424 & 11424 \\
257040 & -257040 & -17136 & 17136 \\
261120 & -261120 & -17408 & 17408 \\
261120 & -261120 & -17408 & 17408 \\
275400 & -275400 & -18360 & 18360
\end{bmatrix}
$$

$$\begin{bmatrix} 2295 \\ 2856 \\ 2856 \\ 2856 \\ 2856 \\ 3264 \\ 3264 \\ 4096 \\ 4096 \\ 4284 \\ 4284 \\ 4284 \\ 4284 \\ 4590 \\ 4760 \\ 4760 \\ 5355 \\ 5355 \\ 5670 \\ 5670 \end{bmatrix} \begin{bmatrix} 120 & -120 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 275400 & -275400 & -18360 & 18360 \\ 342720 & -342720 & -22848 & 22848 \\ 342720 & -342720 & -22848 & 22848 \\ 342720 & -342720 & -22848 & 22848 \\ 342720 & -342720 & -22848 & 22848 \\ 391680 & -391680 & -26112 & 26112 \\ 391680 & -391680 & -26112 & 26112 \\ 491520 & -491520 & -32768 & 32768 \\ 491520 & -491520 & -32768 & 32768 \\ 514080 & -514080 & -34272 & 34272 \\ 514080 & -514080 & -34272 & 34272 \\ 514080 & -514080 & -34272 & 34272 \\ 514080 & -514080 & -34272 & 34272 \\ 550800 & -550800 & -36720 & 36720 \\ 571200 & -571200 & -38080 & 38080 \\ 571200 & -571200 & -38080 & 38080 \\ 642600 & -642600 & -42840 & 42840 \\ 642600 & -642600 & -42840 & 42840 \\ 680400 & -680400 & -45360 & 45360 \\ 680400 & -680400 & -45360 & 45360 \end{bmatrix}$$

$$
\begin{bmatrix} 1 \\ 1 \\ 28 \\ 28 \\ 35 \\ 35 \\ 50 \\ 50 \\ 70 \\ 84 \\ 84 \\ 168 \\ 175 \\ 175 \\ 210 \\ 210 \\ 300 \\ 300 \\ 350 \\ 350 \\ 420 \\ 525 \\ 525 \\ 567 \\ 567 \\ 700 \\ 700 \\ 700 \\ 700 \\ 840 \\ 840 \\ 972 \\ 972 \\ 1050 \\ 1050 \\ 1134 \\ 1344 \\ 1344 \\ 1400 \\ 1400 \end{bmatrix}
\begin{bmatrix} 136 & -136 & 8 & -8 \end{bmatrix} =
\begin{bmatrix}
136 & -136 & 8 & -8 \\
136 & -136 & 8 & -8 \\
3808 & -3808 & 224 & -224 \\
3808 & -3808 & 224 & -224 \\
4760 & -4760 & 280 & -280 \\
4760 & -4760 & 280 & -280 \\
6800 & -6800 & 400 & -400 \\
6800 & -6800 & 400 & -400 \\
9520 & -9520 & 560 & -560 \\
11424 & -11424 & 672 & -672 \\
11424 & -11424 & 672 & -672 \\
22848 & -22848 & 1344 & -1344 \\
23800 & -23800 & 1400 & -1400 \\
23800 & -23800 & 1400 & -1400 \\
28560 & -28560 & 1680 & -1680 \\
28560 & -28560 & 1680 & -1680 \\
40800 & -40800 & 2400 & -2400 \\
40800 & -40800 & 2400 & -2400 \\
47600 & -47600 & 2800 & -2800 \\
47600 & -47600 & 2800 & -2800 \\
57120 & -57120 & 3360 & -3360 \\
71400 & -71400 & 4200 & -4200 \\
71400 & -71400 & 4200 & -4200 \\
77112 & -77112 & 4536 & -4536 \\
77112 & -77112 & 4536 & -4536 \\
95200 & -95200 & 5600 & -5600 \\
95200 & -95200 & 5600 & -5600 \\
95200 & -95200 & 5600 & -5600 \\
95200 & -95200 & 5600 & -5600 \\
114240 & -114240 & 6720 & -6720 \\
114240 & -114240 & 6720 & -6720 \\
132192 & -132192 & 7776 & -7776 \\
132192 & -132192 & 7776 & -7776 \\
142800 & -142800 & 8400 & -8400 \\
142800 & -142800 & 8400 & -8400 \\
154224 & -154224 & 9072 & -9072 \\
182784 & -182784 & 10752 & -10752 \\
182784 & -182784 & 10752 & -10752 \\
190400 & -190400 & 11200 & -11200 \\
190400 & -190400 & 11200 & -11200
\end{bmatrix}
$$

$$\begin{bmatrix} 1400 \\ 1575 \\ 1575 \\ 1680 \\ 2100 \\ 2100 \\ 2100 \\ 2240 \\ 2240 \\ 2268 \\ 2268 \\ 2688 \\ 2835 \\ 2835 \\ 3150 \\ 3200 \\ 3200 \\ 4096 \\ 4096 \\ 4200 \\ 4200 \\ 4200 \\ 4480 \\ 4536 \\ 5670 \\ 6075 \\ 6075 \end{bmatrix} \begin{bmatrix} 136 & -136 & 8 & -8 \end{bmatrix} = \begin{bmatrix} 190400 & -190400 & 11200 & -11200 \\ 214200 & -214200 & 12600 & -12600 \\ 214200 & -214200 & 12600 & -12600 \\ 228480 & -228480 & 13440 & -13440 \\ 285600 & -285600 & 16800 & -16800 \\ 285600 & -285600 & 16800 & -16800 \\ 285600 & -285600 & 16800 & -16800 \\ 304640 & -304640 & 17920 & -17920 \\ 304640 & -304640 & 17920 & -17920 \\ 308448 & -308448 & 18144 & -18144 \\ 308448 & -308448 & 18144 & -18144 \\ 365568 & -365568 & 21504 & -21504 \\ 385560 & -385560 & 22680 & -22680 \\ 385560 & -385560 & 22680 & -22680 \\ 428400 & -428400 & 25200 & -25200 \\ 435200 & -435200 & 25600 & -25600 \\ 435200 & -435200 & 25600 & -25600 \\ 557056 & -557056 & 32768 & -32768 \\ 557056 & -557056 & 32768 & -32768 \\ 571200 & -571200 & 33600 & -33600 \\ 571200 & -571200 & 33600 & -33600 \\ 571200 & -571200 & 33600 & -33600 \\ 609280 & -609280 & 35840 & -35840 \\ 616896 & -616896 & 36288 & -36288 \\ 771120 & -771120 & 45360 & -45360 \\ 826200 & -826200 & 48600 & -48600 \\ 826200 & -826200 & 48600 & -48600 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 7 \\ 15 \\ 21 \\ 21 \\ 27 \\ 28 \\ 28 \\ 35 \\ 35 \\ 36 \\ 36 \\ 56 \\ 63 \\ 63 \\ 70 \\ 84 \\ 105 \\ 105 \\ 105 \\ 120 \\ 168 \\ 168 \\ 168 \\ 189 \\ 189 \\ 189 \\ 210 \\ 210 \\ 216 \\ 252 \\ 252 \\ 280 \\ 280 \\ 280 \\ 315 \\ 315 \\ 315 \\ 315 \\ 315 \end{bmatrix} \begin{bmatrix} 255 & 255 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 255 & 255 & -1 & -1 \\ 1785 & 1785 & -7 & -7 \\ 3825 & 3825 & -15 & -15 \\ 5355 & 5355 & -21 & -21 \\ 5355 & 5355 & -21 & -21 \\ 6885 & 6885 & -27 & -27 \\ 7140 & 7140 & -28 & -28 \\ 7140 & 7140 & -28 & -28 \\ 8925 & 8925 & -35 & -35 \\ 8925 & 8925 & -35 & -35 \\ 9180 & 9180 & -36 & -36 \\ 9180 & 9180 & -36 & -36 \\ 14280 & 14280 & -56 & -56 \\ 16065 & 16065 & -63 & -63 \\ 16065 & 16065 & -63 & -63 \\ 17850 & 17850 & -70 & -70 \\ 21420 & 21420 & -84 & -84 \\ 26775 & 26775 & -105 & -105 \\ 26775 & 26775 & -105 & -105 \\ 26775 & 26775 & -105 & -105 \\ 30600 & 30600 & -120 & -120 \\ 42840 & 42840 & -168 & -168 \\ 42840 & 42840 & -168 & -168 \\ 42840 & 42840 & -168 & -168 \\ 48195 & 48195 & -189 & -189 \\ 48195 & 48195 & -189 & -189 \\ 48195 & 48195 & -189 & -189 \\ 53550 & 53550 & -210 & -210 \\ 53550 & 53550 & -210 & -210 \\ 55080 & 55080 & -216 & -216 \\ 64260 & 64260 & -252 & -252 \\ 64260 & 64260 & -252 & -252 \\ 71400 & 71400 & -280 & -280 \\ 71400 & 71400 & -280 & -280 \\ 71400 & 71400 & -280 & -280 \\ 80325 & 80325 & -315 & -315 \\ 80325 & 80325 & -315 & -315 \\ 80325 & 80325 & -315 & -315 \\ 80325 & 80325 & -315 & -315 \\ 80325 & 80325 & -315 & -315 \end{bmatrix}$$

$$\begin{bmatrix} 336 \\ 378 \\ 378 \\ 378 \\ 405 \\ 420 \\ 420 \\ 420 \\ 420 \\ 420 \\ 504 \\ 504 \\ 512 \\ 560 \\ 560 \\ 560 \\ 567 \\ 567 \\ 630 \\ 630 \\ 630 \\ 630 \\ 630 \\ 630 \\ 672 \\ 672 \\ 720 \\ 720 \\ 756 \\ 756 \\ 840 \\ 840 \\ 945 \\ 945 \\ 945 \\ 945 \\ 1008 \\ 1008 \\ 1008 \\ 1260 \end{bmatrix} \begin{bmatrix} 255 & 255 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 85680 & 85680 & -336 & -336 \\ 96390 & 96390 & -378 & -378 \\ 96390 & 96390 & -378 & -378 \\ 96390 & 96390 & -378 & -378 \\ 103275 & 103275 & -405 & -405 \\ 107100 & 107100 & -420 & -420 \\ 107100 & 107100 & -420 & -420 \\ 107100 & 107100 & -420 & -420 \\ 107100 & 107100 & -420 & -420 \\ 107100 & 107100 & -420 & -420 \\ 128520 & 128520 & -504 & -504 \\ 128520 & 128520 & -504 & -504 \\ 130560 & 130560 & -512 & -512 \\ 142800 & 142800 & -560 & -560 \\ 142800 & 142800 & -560 & -560 \\ 142800 & 142800 & -560 & -560 \\ 144585 & 144585 & -567 & -567 \\ 144585 & 144585 & -567 & -567 \\ 160650 & 160650 & -630 & -630 \\ 160650 & 160650 & -630 & -630 \\ 160650 & 160650 & -630 & -630 \\ 160650 & 160650 & -630 & -630 \\ 160650 & 160650 & -630 & -630 \\ 160650 & 160650 & -630 & -630 \\ 171360 & 171360 & -672 & -672 \\ 171360 & 171360 & -672 & -672 \\ 183600 & 183600 & -720 & -720 \\ 183600 & 183600 & -720 & -720 \\ 192780 & 192780 & -756 & -756 \\ 192780 & 192780 & -756 & -756 \\ 214200 & 214200 & -840 & -840 \\ 214200 & 214200 & -840 & -840 \\ 240975 & 240975 & -945 & -945 \\ 240975 & 240975 & -945 & -945 \\ 240975 & 240975 & -945 & -945 \\ 240975 & 240975 & -945 & -945 \\ 257040 & 257040 & -1008 & -1008 \\ 257040 & 257040 & -1008 & -1008 \\ 257040 & 257040 & -1008 & -1008 \\ 321300 & 321300 & -1260 & -1260 \end{bmatrix}$$

$$\begin{bmatrix} 1260 \\ 1260 \\ 1512 \\ 1512 \\ 1512 \\ 1680 \\ 1680 \\ 1680 \\ 1792 \\ 1792 \\ 1890 \\ 1890 \\ 1890 \\ 1890 \\ 2016 \\ 2016 \\ 2240 \\ 2268 \\ 2268 \\ 2268 \\ 2304 \\ 2304 \\ 2520 \\ 2520 \\ 2520 \\ 2520 \\ 2520 \\ 2520 \\ 2520 \\ 2835 \\ 2835 \\ 2835 \\ 2835 \\ 3240 \end{bmatrix} \begin{bmatrix} 255 & 255 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 321300 & 321300 & -1260 & -1260 \\ 321300 & 321300 & -1260 & -1260 \\ 385560 & 385560 & -1512 & -1512 \\ 385560 & 385560 & -1512 & -1512 \\ 385560 & 385560 & -1512 & -1512 \\ 428400 & 428400 & -1680 & -1680 \\ 428400 & 428400 & -1680 & -1680 \\ 428400 & 428400 & -1680 & -1680 \\ 456960 & 456960 & -1792 & -1792 \\ 456960 & 456960 & -1792 & -1792 \\ 481950 & 481950 & -1890 & -1890 \\ 481950 & 481950 & -1890 & -1890 \\ 481950 & 481950 & -1890 & -1890 \\ 481950 & 481950 & -1890 & -1890 \\ 514080 & 514080 & -2016 & -2016 \\ 514080 & 514080 & -2016 & -2016 \\ 571200 & 571200 & -2240 & -2240 \\ 578340 & 578340 & -2268 & -2268 \\ 578340 & 578340 & -2268 & -2268 \\ 578340 & 578340 & -2268 & -2268 \\ 587520 & 587520 & -2304 & -2304 \\ 587520 & 587520 & -2304 & -2304 \\ 642600 & 642600 & -2520 & -2520 \\ 642600 & 642600 & -2520 & -2520 \\ 642600 & 642600 & -2520 & -2520 \\ 642600 & 642600 & -2520 & -2520 \\ 642600 & 642600 & -2520 & -2520 \\ 642600 & 642600 & -2520 & -2520 \\ 642600 & 642600 & -2520 & -2520 \\ 722925 & 722925 & -2835 & -2835 \\ 722925 & 722925 & -2835 & -2835 \\ 722925 & 722925 & -2835 & -2835 \\ 722925 & 722925 & -2835 & -2835 \\ 826200 & 826200 & -3240 & -3240 \end{bmatrix}$$

This process yields the partial character table of $\overline{G}$ corresponding to the class $1A$ of G and the four inertia factor groups. The full 322×322 character table of $\overline{G}$ is not included in this thesis due to its size. The character degrees of this affine subgroup are given in the first column of this partial character table given by Table 6.10 below. These character degrees can also be computed by using Theorem 4.2.32, Remark 4.2.33 and Theorem 4.2.34 in Section 4.2.4.

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\|C_{\overline{G}}(\overline{g})\|$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_1	1	1	1	1
χ_2	35	35	35	35
χ_3	51	51	51	51
χ_4	85	85	85	85
χ_5	119	119	119	119
χ_6	135	135	135	135
χ_7	238	238	238	238
χ_8	510	510	510	510
χ_9	595	595	595	595
χ_{10}	595	595	595	595
χ_{11}	918	918	918	918
χ_{12}	1190	1190	1190	1190
χ_{13}	1275	1275	1275	1275
χ_{14}	1512	1512	1512	1512
χ_{15}	1785	1785	1785	1785
χ_{16}	1785	1785	1785	1785
χ_{17}	2295	2295	2295	2295
χ_{18}	2856	2856	2856	2856
χ_{19}	2856	2856	2856	2856
χ_{20}	2975	2975	2975	2975
χ_{21}	2975	2975	2975	2975
χ_{22}	3213	3213	3213	3213
χ_{23}	3213	3213	3213	3213
χ_{24}	3400	3400	3400	3400
χ_{25}	3570	3570	3570	3570
χ_{26}	3570	3570	3570	3570
χ_{27}	3570	3570	3570	3570
χ_{28}	3808	3808	3808	3808
χ_{29}	4200	4200	4200	4200
χ_{30}	4760	4760	4760	4760
χ_{31}	5712	5712	5712	5712
χ_{32}	5950	5950	5950	5950
χ_{33}	5950	5950	5950	5950
χ_{34}	7140	7140	7140	7140
χ_{35}	8160	8160	8160	8160
χ_{36}	8925	8925	8925	8925
χ_{37}	8925	8925	8925	8925
χ_{38}	8960	8960	8960	8960
χ_{39}	9639	9639	9639	9639
χ_{40}	10200	10200	10200	10200

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\|C_{\overline{G}}(\overline{g})\|$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{41}	11900	11900	11900	11900
χ_{42}	11900	11900	11900	11900
χ_{43}	13056	13056	13056	13056
χ_{44}	13600	13600	13600	13600
χ_{45}	14280	14280	14280	14280
χ_{46}	14280	14280	14280	14280
χ_{47}	14688	14688	14688	14688
χ_{48}	16065	16065	16065	16065
χ_{49}	16065	16065	16065	16065
χ_{50}	16065	16065	16065	16065
χ_{51}	16065	16065	16065	16065
χ_{52}	16065	16065	16065	16065
χ_{53}	17850	17850	17850	17850
χ_{54}	17850	17850	17850	17850
χ_{55}	18360	18360	18360	18360
χ_{56}	19040	19040	19040	19040
χ_{57}	23800	23800	23800	23800
χ_{58}	23800	23800	23800	23800
χ_{59}	26775	26775	26775	26775
χ_{60}	26775	26775	26775	26775
χ_{61}	28560	28560	28560	28560
χ_{62}	28917	28917	28917	28917
χ_{63}	30464	30464	30464	30464
χ_{64}	32130	32130	32130	32130
χ_{65}	32130	32130	32130	32130
χ_{66}	32130	32130	32130	32130
χ_{67}	34425	34425	34425	34425
χ_{68}	34425	34425	34425	34425
χ_{69}	34560	34560	34560	34560
χ_{70}	38080	38080	38080	38080
χ_{71}	38556	38556	38556	38556
χ_{72}	42525	42525	42525	42525
χ_{73}	42525	42525	42525	42525
χ_{74}	43520	43520	43520	43520
χ_{75}	47600	47600	47600	47600
χ_{76}	48195	48195	48195	48195
χ_{77}	51408	51408	51408	51408
χ_{78}	53550	53550	53550	53550
χ_{79}	57120	57120	57120	57120
χ_{80}	65536	65536	65536	65536

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\lvert C_{\overline{G}}(\overline{g})$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{81}	68850	68850	68850	68850
χ_{82}	120	-120	-8	8
χ_{83}	120	-120	-8	8
χ_{84}	4080	-4080	-272	272
χ_{85}	4080	-4080	-272	272
χ_{86}	6120	-6120	-408	408
χ_{87}	6120	-6120	-408	408
χ_{88}	10080	-10080	-672	672
χ_{89}	10080	-10080	-672	672
χ_{90}	24480	-24480	-1632	1632
χ_{91}	24480	-24480	-1632	1632
χ_{92}	24480	-24480	-1632	1632
χ_{93}	24480	-24480	-1632	1632
χ_{94}	42840	-42840	-2856	2856
χ_{95}	42840	-42840	-2856	2856
χ_{96}	57120	-57120	-3808	3808
χ_{97}	57120	-57120	-3808	3808
χ_{98}	57120	-57120	-3808	3808
χ_{99}	57120	-57120	-3808	3808
χ_{100}	71400	-71400	-4760	4760
χ_{101}	71400	-71400	-4760	4760
χ_{102}	85680	-85680	-5712	5712
χ_{103}	85680	-85680	-5712	5712
χ_{104}	85680	-85680	-5712	5712
χ_{105}	85680	-85680	-5712	5712
χ_{106}	122400	-122400	-8160	8160
χ_{107}	122400	-122400	-8160	8160
χ_{108}	128520	-128520	-8568	8568
χ_{109}	128520	-128520	-8568	8568
χ_{110}	128520	-128520	-8568	8568
χ_{111}	128520	-128520	-8568	8568
χ_{112}	142800	-142800	-9520	9520
χ_{113}	142800	-142800	-9520	9520
χ_{114}	161280	-161280	-10752	10752
χ_{115}	161280	-161280	-10752	10752
χ_{116}	171360	-171360	-11424	11424
χ_{117}	171360	-171360	-11424	11424
χ_{118}	257040	-257040	-17136	17136
χ_{119}	261120	-261120	-17408	17408
χ_{120}	261120	-261120	-17408	17408

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\lvert C_{\overline{G}}(\overline{g})$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{121}	275400	-275400	-18360	18360
χ_{122}	275400	-275400	-18360	18360
χ_{123}	342720	-342720	-22848	22848
χ_{124}	342720	-342720	-22848	22848
χ_{125}	342720	-342720	-22848	22848
χ_{126}	342720	-342720	-22848	22848
χ_{127}	391680	-391680	-26112	26112
χ_{128}	391680	-391680	-26112	26112
χ_{129}	491520	-491520	-32768	32768
χ_{130}	491520	-491520	-32768	32768
χ_{131}	514080	-514080	-34272	34272
χ_{132}	514080	-514080	-34272	34272
χ_{133}	514080	-514080	-34272	34272
χ_{134}	514080	-514080	-34272	34272
χ_{135}	550800	-550800	-36720	36720
χ_{136}	571200	-571200	-38080	38080
χ_{137}	571200	-571200	-38080	38080
χ_{138}	642600	-642600	-42840	42840
χ_{139}	642600	-642600	-42840	42840
χ_{140}	680400	-680400	-45360	45360
χ_{141}	680400	-680400	-45360	45360
χ_{142}	136	-136	8	-8
χ_{143}	136	-136	8	-8
χ_{144}	3808	-3808	224	-224
χ_{145}	3808	-3808	224	-224
χ_{146}	4760	-4760	280	-280
χ_{147}	4760	-4760	280	-280
χ_{148}	6800	-6800	400	-400
χ_{149}	6800	-6800	400	-400
χ_{150}	9520	-9520	560	-560
χ_{151}	11424	-11424	672	-672
χ_{152}	11424	-11424	672	-672
χ_{153}	22848	-22848	1344	-1344
χ_{154}	23800	-23800	1400	-1400
χ_{155}	23800	-23800	1400	-1400
χ_{156}	28560	-28560	1680	-1680
χ_{157}	28560	-28560	1680	-1680
χ_{158}	40800	-40800	2400	-2400
χ_{159}	40800	-40800	2400	-2400
χ_{160}	47600	-47600	2800	-2800

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\|C_{\overline{G}}(\overline{g})\|$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{161}	47600	-47600	2800	-2800
χ_{162}	57120	-57120	3360	-3360
χ_{163}	71400	-71400	4200	-4200
χ_{164}	71400	-71400	4200	-4200
χ_{165}	77112	-77112	4536	-4536
χ_{166}	77112	-77112	4536	-4536
χ_{167}	95200	-95200	5600	-5600
χ_{168}	95200	-95200	5600	-5600
χ_{169}	95200	-95200	5600	-5600
χ_{170}	95200	-95200	5600	-5600
χ_{171}	114240	-114240	6720	-6720
χ_{172}	114240	-114240	6720	-6720
χ_{173}	132192	-132192	7776	-7776
χ_{174}	132192	-132192	7776	-7776
χ_{175}	142800	-142800	8400	-8400
χ_{176}	142800	-142800	8400	-8400
χ_{177}	154224	-154224	9072	-9072
χ_{178}	182784	-182784	10752	-10752
χ_{179}	182784	-182784	10752	-10752
χ_{180}	190400	-190400	11200	-11200
χ_{181}	190400	-190400	11200	-11200
χ_{182}	190400	-190400	11200	-11200
χ_{183}	214200	-214200	12600	-12600
χ_{184}	214200	-214200	12600	-12600
χ_{185}	228480	-228480	13440	-13440
χ_{186}	285600	-285600	16800	-16800
χ_{187}	285600	-285600	16800	-16800
χ_{188}	285600	-285600	16800	-16800
χ_{189}	304640	-304640	17920	-17920
χ_{190}	304640	-304640	17920	-17920
χ_{191}	308448	-308448	18144	-18144
χ_{192}	308448	-308448	18144	-18144
χ_{193}	365568	-365568	21504	-21504
χ_{194}	385560	-385560	22680	-22680
χ_{195}	385560	-385560	22680	-22680
χ_{196}	428400	-428400	25200	-25200
χ_{197}	435200	-435200	25600	-25600
χ_{198}	435200	-435200	25600	-25600
χ_{199}	557056	-557056	32768	-32768
χ_{200}	557056	-557056	32768	-32768

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\lvert C_{\overline{G}}(\overline{g})$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{201}	571200	-571200	33600	-33600
χ_{202}	571200	-571200	33600	-33600
χ_{203}	571200	-571200	33600	-33600
χ_{204}	609280	-609280	35840	-35840
χ_{205}	616896	-616896	36288	-36288
χ_{206}	771120	-771120	45360	-45360
χ_{207}	826200	-826200	48600	-48600
χ_{208}	826200	-826200	48600	-48600
χ_{209}	255	255	-1	-1
χ_{210}	1785	1785	-7	-7
χ_{211}	3825	3825	-15	-15
χ_{212}	5355	5355	-21	-21
χ_{213}	5355	5355	-21	-21
χ_{214}	6885	6885	-27	-27
χ_{215}	7140	7140	-28	-28
χ_{216}	7140	7140	-28	-28
χ_{217}	8925	8925	-35	-35
χ_{218}	8925	8925	-35	-35
χ_{219}	9180	9180	-36	-36
χ_{220}	9180	9180	-36	-36
χ_{221}	14280	14280	-56	-56
χ_{222}	16065	16065	-63	-63
χ_{223}	16065	16065	-63	-63
χ_{224}	17850	17850	-70	-70
χ_{225}	21420	21420	-84	-84
χ_{226}	26775	26775	-105	-105
χ_{227}	26775	26775	-105	-105
χ_{228}	26775	26775	-105	-105
χ_{229}	30600	30600	-120	-120
χ_{230}	42840	42840	-168	-168
χ_{231}	42840	42840	-168	-168
χ_{232}	42840	42840	-168	-168
χ_{233}	48195	48195	-189	-189
χ_{234}	48195	48195	-189	-189
χ_{235}	48195	48195	-189	-189
χ_{236}	53550	53550	-210	-210
χ_{237}	53550	53550	-210	-210
χ_{238}	55080	55080	-216	-216
χ_{239}	64260	64260	-252	-252
χ_{240}	64260	64260	-252	-252

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\lvert C_{\overline{G}}(\overline{g})$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{241}	71400	71400	-280	-280
χ_{242}	71400	71400	-280	-280
χ_{243}	71400	71400	-280	-280
χ_{244}	80325	80325	-315	-315
χ_{245}	80325	80325	-315	-315
χ_{246}	80325	80325	-315	-315
χ_{247}	80325	80325	-315	-315
χ_{248}	80325	80325	-315	-315
χ_{249}	85680	85680	-336	-336
χ_{250}	96390	96390	-378	-378
χ_{251}	96390	96390	-378	-378
χ_{252}	96390	96390	-378	-378
χ_{253}	103275	103275	-405	-405
χ_{254}	107100	107100	-420	-420
χ_{255}	107100	107100	-420	-420
χ_{256}	107100	107100	-420	-420
χ_{257}	107100	107100	-420	-420
χ_{258}	107100	107100	-420	-420
χ_{259}	128520	128520	-504	-504
χ_{260}	128520	128520	-504	-504
χ_{261}	130560	130560	-512	-512
χ_{262}	142800	142800	-560	-560
χ_{263}	142800	142800	-560	-560
χ_{264}	142800	142800	-560	-560
χ_{265}	144585	144585	-567	-567
χ_{266}	144585	144585	-567	-567
χ_{267}	160650	160650	-630	-630
χ_{268}	160650	160650	-630	-630
χ_{269}	160650	160650	-630	-630
χ_{270}	160650	160650	-630	-630
χ_{271}	160650	160650	-630	-630
χ_{272}	160650	160650	-630	-630
χ_{273}	171360	171360	-672	-672
χ_{274}	171360	171360	-672	-672
χ_{275}	183600	183600	-720	-720
χ_{276}	183600	183600	-720	-720
χ_{277}	192780	192780	-756	-756
χ_{278}	192780	192780	-756	-756
χ_{279}	214200	214200	-840	-840
χ_{280}	214200	214200	-840	-840

Table 6.10: Partial character table of $\overline{G}$ corresponding to $1A$

$[g]$	1A			
$\lvert C_{\overline{G}}(\overline{g})$	24257337753600	24257337753600	95126814720	95126814720
$[\overline{g}]$	1A	2A	2B	2C
χ_{281}	240975	240975	-945	-945
χ_{282}	240975	240975	-945	-945
χ_{283}	240975	240975	-945	-945
χ_{284}	240975	240975	-945	-945
χ_{285}	257040	257040	-1008	-1008
χ_{286}	257040	257040	-1008	-1008
χ_{287}	257040	257040	-1008	-1008
χ_{288}	321300	321300	-1260	-1260
χ_{289}	321300	321300	-1260	-1260
χ_{290}	321300	321300	-1260	-1260
χ_{291}	385560	385560	-1512	-1512
χ_{292}	385560	385560	-1512	-1512
χ_{293}	385560	385560	-1512	-1512
χ_{294}	428400	428400	-1680	-1680
χ_{295}	428400	428400	-1680	-1680
χ_{296}	428400	428400	-1680	-1680
χ_{297}	456960	456960	-1792	-1792
χ_{298}	456960	456960	-1792	-1792
χ_{299}	481950	481950	-1890	-1890
χ_{300}	481950	481950	-1890	-1890
χ_{301}	481950	481950	-1890	-1890
χ_{302}	481950	481950	-1890	-1890
χ_{303}	514080	514080	-2016	-2016
χ_{304}	514080	514080	-2016	-2016
χ_{305}	571200	571200	-2240	-2240
χ_{306}	578340	578340	-2268	-2268
χ_{307}	578340	578340	-2268	-2268
χ_{308}	578340	578340	-2268	-2268
χ_{309}	587520	587520	-2304	-2304
χ_{310}	587520	587520	-2304	-2304
χ_{311}	642600	642600	-2520	-2520
χ_{312}	642600	642600	-2520	-2520
χ_{313}	642600	642600	-2520	-2520
χ_{314}	642600	642600	-2520	-2520
χ_{315}	642600	642600	-2520	-2520
χ_{316}	642600	642600	-2520	-2520
χ_{317}	642600	642600	-2520	-2520
χ_{318}	722925	722925	-2835	-2835
χ_{319}	722925	722925	-2835	-2835
χ_{320}	722925	722925	-2835	-2835
χ_{321}	722925	722925	-2835	-2835
χ_{322}	826200	826200	-3240	-3240

6.7 The quotient group $\overline{G}/Z(\overline{G})$

The centre of the affine subgroup $\overline{G}$ of the symplectic group $Sp(10,2)$ is isomorphic to $\mathbb{Z}_2$, by Proposition 4.2.39. According to Proposition 4.2.41 the quotient group $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^8{:}Sp(8,2)$. In Remark 4.2.43 we outlined a method of determining the Fischer matrices of $\overline{G}/Z(\overline{G})$ directly from the Fischer matrices of $\overline{G}$. In Section 6.5 we computed all the 81 Fischer matrices of $\overline{G}$. The action of $Sp(8,2)$ on $Irr(2^8)$ yields two inertia factor groups, namely $Sp(8,2)$ and the affine subgroup of $Sp(8,2)$, which is $2^7{:}Sp(6,2)$. These are labelled H_1 and H_4, respectively, in Section 6.4. This then means that the inertia factor groups $H_2 = GO^-(8,2)$ and $H_3 = GO^+(8,2)$ do not play a role in $\overline{G}/Z(\overline{G})$. Then, to obtain the Fischer matrices of $\overline{G}/Z(\overline{G})$ from those of $\overline{G}$, we delete the rows corresponding to H_2 and H_3 in each Fischer matrix of $\overline{G}$. Thereafter, we discard the repeated columns. In Table 6.9, let us consider the Fischer matrices $M(1A)$ and $M(2A)$ for demonstration. From $M(1A)$ we obtain

$$M(1A) = \begin{bmatrix} 1 & 1 \\ 255 & -1 \end{bmatrix},$$

and from $M(2A)$ we get

$$M(2A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 126 & 0 & -2 \end{bmatrix}.$$

This process will yield 81 Fischer matrices of $\overline{G}/Z(\overline{G})$. These deduced Fischer matrices can then be used to construct the character table of $\overline{G}/Z(\overline{G})$, using the Clifford-Fischer Theory, since $\overline{G}/Z(\overline{G})$ is a split extension. The character table and the Fischer matrices of the non-split extension $2^{8\cdot}Sp(8,2)$ are done by Basheer in [8]. He remarks that the character tables of the non-split extension $2^{8\cdot}Sp(8,2)$ and the split extension $2^8{:}Sp(8,2)$ are the same.

In [1] Ali computed the Fischer matrices and the character table of the affine subgroup $A(4) = 2^7{:}Sp(6,2)$ of the symplectic group $Sp(8,2)$. The centre of $A(4)$ is isomorphic to $\mathbb{Z}_2$. The quotient group $A(4)/Z(A(4))$ is isomorphic to the split extension $2^6{:}Sp(6,2)$. The method discussed in the first paragraph of this section can be used to determine the Fischer matrices of $A(4)/Z(A(4))$. Mpono in [49] dealt with the Fischer matrices and the character table of the split extension $2^6{:}Sp(6,2)$. Basheer in [8] computed the Fischer matrices and the character table of the non-split extension $2^{6\cdot}Sp(6,2)$. It turns out that the character tables of the split extension and the non-split extension coincide.

7

The affine subgroup $2^6{:}Sp(2,4)$ of the symplectic group $Sp(4,4)$

We consider the affine subgroup $\overline{G} = P(2){:}Sp(2,4)$ of the symplectic group $Sp(4,4)$. Let $\mathbb{F} = GF(4)$ be the Galois field of 4 elements and V be a non-degenerate symplectic space of dimension $2n$=4 over the field $\mathbb{F}$. The group $P(2)$ is the subgroup of $\overline{G}$ which satisfies the conditions outlined in Remark 4.2.22. Since the Char($\mathbb{F}$) = 2, the subgroup $P(2)$ is an elementary abelian 2-group of order $4^3 = 2^6$. Recall that this affine subgroup is in fact a split extension. In this section we shall denote this normal subgroup $P(2) = 2^6$ of $\overline{G}$ by N. This chapter is in preparation of the next chapter, Chapter 8, where we will consider the affine subgroup $2^{10}{:}Sp(4,4)$ of the symplectic group $Sp(6,4)$. It turns out that the group $\overline{G}$ is a point stabilizer when $Sp(4,4)$ acts on 2^{10} and is also one of the inertia factor groups when the group $Sp(4,4)$ acts on the irreducible characters of 2^{10}, $Irr(2^{10})$. This then means that we will need the character table of $\overline{G}$ in Chapter 8. In Section 7.1 we look at the transvections of $Sp(2,4)$. In Section 7.2 we derive the generators of N and $G = Sp(2,4)$ as 4×4 symplectic matrices since $\overline{G}$ sits in $Sp(4,4)$, a symplectic group of 4×4 symplectic matrices with entries in the Galois field $\mathbb{F}$. In Section 7.3 we have that the affine subgroup of G is isomorphic to the dihedral group of order 4, D_4. We compute the permutation character of G on D_4 using the fusion of the conjugacy classes of D_4 into G. The computation of the conjugacy classes of $\overline{G}$ is dealt with in Section 7.4. In Section 7.5 we consider the action of G on the $Irr(N)$ and thus the fusion of the respective inertia factor groups into G. This section is in preparation of Section 7.6 where we construct the Fischer matrices of $\overline{G}$. The character table of $\overline{G}$ is then constructed in Section 7.7. In Section 7.8 we consider the fusion of the group $\overline{G}$ into $Sp(4,4)$. Section 7.9 is about the centre, $Z(\overline{G})$, of $\overline{G}$ and the quotient group $\overline{G}/Z(\overline{G})$. We observe that the quotient $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^4{:}Sp(2,4)$. We demonstrate how to obtain the Fischer matrices and the character table of this quotient directly from the Fischer matrices and the character table of $\overline{G}$.

7.1 Transvections of $Sp(2,4)$

In this section we look at the transvections of the symplectic group $Sp(2,4)$ and the results we proved in Section 4.2.1. Background theory on transvections of symplectic groups is dealt with by Mpono, O'Meara, Rodrigues and Wilson in [49], [51], [55] and [64], respectively, among others. Let G, $\overline{G}$, $\mathbb{F}$, N and V be as defined in the above introduction. We know that G is generated by the set of transvections. We have $|V^*| = 2^4 - 1 = 15$ transvections in G. Since the $\text{Char}(\mathbb{F}) = 2$, the order of each of these transvections is 2. By Proposition 4.2.15 we have one conjugacy class of transvections in G since $|\mathbb{F}| = 2^2$. Now there is one conjugacy class of elements of order 2 in G, the class $2A$ with the order of the centralizer equal to $\frac{|G|}{|V^*|} = \frac{60}{15} = 4$. We observe that the centralizer of a transvection of G is isomorphic to the affine subgroup of G, which is isomorphic to the dihedral group D_4 of order 4. We note that D_4 is isomorphic to the Klein four-group.

7.2 The generators of the groups N and $Sp(2,4)$

The affine subgroup $\overline{G}$ sits inside the symplectic group $Sp(4,4)$ of 4×4 symplectic matrices with entries in the Galois field $\mathbb{F}$. For this reason we express the elements of $Sp(2,4)$, which are 2×2 symplectic matrices with entries in $\mathbb{F}$, and elements of N as 4×4 symplectic matrices. We follow the method outlined in Remark 4.2.22. We start with the elements of N. Take $n = 2$, then we have

$$\begin{aligned} T(e_1) &= e_1 \\ T(e_i) &= \alpha_i e_1 + e_i \qquad 2 \le i \le 3 \\ T(e_2) &= \alpha_2 e_1 + e_2 \\ T(e_3) &= \alpha_3 e_1 + e_3 \\ T(e_{2n}) = T(e_4) &= \sum_{i=1}^{4} \beta_i e_i \end{aligned}$$

with $\beta_4 = 1$ and

$$\alpha_j = \begin{cases} -\beta_{2n+1-j}, & \text{if} \quad 2 \le j \le 2 \\ \beta_{2n+1-j}, & \text{if} \quad 2 < j \le 3 \end{cases}.$$

This means $\alpha_2 = -\beta_3$ and $\alpha_3 = \beta_2$ and the elements T of N are of the form

$$T = \begin{bmatrix} 1 & -\beta_3 & \beta_2 & \beta_1 \\ 0 & 1 & 0 & \beta_2 \\ 0 & 0 & 1 & \beta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\beta_j \in \mathbb{F}$. We use GAP to generate all $4^3 = 2^6 = 64$ elements of N and also compute the 6 generators of N. We list these generators:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & z^1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1 & 0 & z^1 & 0 \\ 0 & 1 & 0 & z^1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$P_6 = \begin{bmatrix} 1 & z^1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z^1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $z = Z(4)$ is a primitive root of $\mathbb{F}$. Again from GAP we obtain the generators of G as

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $o(a) = 3$ and $o(b) = 3$. Since symplectic groups are generated by transvections we express a and b as products of transvections:

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

7.3 Permutation character of G on D_4

When G acts on N we have $2 \times q = 2 \times 4 = 8$ orbits of lengths $1, 1, 1, 1, 15, 15, 15$ and 15. These correspond to the point stabilizers G, G, G, G, D_4, D_4, D_4 and D_4 respectively, where D_4 is isomorphic to the affine subgroup of G. In the ATLAS they provide permutation characters of maximal subgroups and so the permutation character of D_4 is not given in the ATLAS. To compute the permutation character of D_4 we use the Theorem 2.6.12 which states that if $H \leq G, g \in G$ with $\chi = (1_H)^G$, then

$$\chi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_H(x_i)|}, \qquad (A)$$

where $x_1, \cdots, x_m$ are representatives of conjugacy classes of H that fuse to $[g]$. If $H \cap [g] = \emptyset$ then $\chi(g) = 0$. We consider the fusion of the conjugacy classes of D_4 in G.

Table 7.1: Conjugacy classes of G

$\lvert C_G\rvert$	60	4	3	5	5
$o(g)$	1A	2A	3A	5A	5B

Table 7.2: Conjugacy classes of D_4

$\lvert C_H\rvert$	4	4	4	4
$o(h)$	1A	2A	2B	2C

We note that $H = D_4$ has three conjugacy classes of order 2 and that there is only one conjugacy class of order 2 in G. This then means all three will fuse to the class $2A$ in G. We use the following table and formula (A) above to compute the permutation character $(1_H)^G$.

Table 7.3: Permutation character of D_4 on G

	$\|C_G\|$	60	4	3	5	5
	$o(g)$	1A	2A	3A	5A	5B
$o(h)$	$\|C_H\|$					
1A	4	15				
2A	4		1			
2B	4		1			
2C	4		1			
	$(1_H)^G$	15	3	0	0	0

Due to the Frobenius reciprocity, we are able to express $(1_H)^G$ in terms of the irreducible characters of G. We use the character table of G in the ATLAS to deduce that $(1_H)^G = 1a + 4a + 2 \cdot 5a$, where $1a$, $4a$ and $5a$ are irreducible characters of G.

7.4 Conjugacy classes of $\overline{G}$

Earlier, in Section 7.3, we noted that when G acts on N we get 8 orbits of lengths $1, 1, 1, 1, 15, 15, 15$ and 15. These corresponding to the point stabilizers G, G, G, G, D_4, D_4, D_4 and D_4 respectively. Let $\chi(G|N)$ be the permutation character when G acts on N. Then

$$\chi(G|N) = 1 + 1 + 1 + 1 + I_{D_4}^G + I_{D_4}^G + I_{D_4}^G + I_{D_4}^G$$
$$= 1a + 1a + 1a + 1a + 4(1a + 4a + 2 \cdot 5a)$$
$$= 8 \cdot 1a + 4 \cdot 4a + 8 \cdot 5a$$

where $I_{D_4}^G$ is the identity character of D_4 induced to G and expressed as the irreducible characters of G. The above permutation character is then used, together with the character table of G in the ATLAS, to compute the values of k, where k is the number of fixed points when $g \in G$ is acting on N. We list these k-values in Table 7.4.

Table 7.4: Fixed points of the action of G on N

$\|C_G\|$	60	4	3	5	5
$o(g)$	1A	2A	3A	5A	5B
k	64	16	4	4	4

Next is to compute the conjugacy classes of $\overline{G}$ by using the coset analysis technique. For each conjugacy class of $\overline{G}$ we compute the centralizer sizes $|C_{\overline{G}}(x_i)| = \frac{k|C_G(g)|}{f_i}$ for $x_i \in \overline{G}$ and conjugacy class $[g] \in G$. Let us use $g \in 3A \in G$ to demonstrate how to obtain the conjugacy classes of $\overline{G}$ from a conjugacy class of G. From Table 7.4 we have that $k = 4$, meaning that Ng under the action of N produces 4 orbits, each of length $\frac{|N|}{k} = \frac{64}{4} = 16$. Using GAP, we note that after the action of the centralizer $C_G(g)$ on these orbits we still have 4 orbits with $|\Omega_1| = 16 = |\Omega_2| = |\Omega_3| = |\Omega_4|$. We compute the f-values f_i using $f_i = \frac{k|\Omega_i|}{|N|}$ to have $f_1 = 1 = f_2 = f_3 = f_4$. The GAP programme A in [15] is used to compute the rest of the f-values for each conjugacy class $[g] \in G$. This then means that this conjugacy class $3A$ produces 4 conjugacy classes of $\overline{G}$. The centralizer sizes of these conjugacy classes is calculated as follows

$$|C_{\overline{G}}(x_i)| = \frac{k|C_G(g)|}{f_i}.$$

$$|C_{\overline{G}}(x_1)| = \frac{k|C_G(g)|}{f_1} = \frac{4 \times 3}{1} = 12,$$

$$|C_{\overline{G}}(x_2)| = \frac{k|C_G(g)|}{f_2} = \frac{4 \times 3}{1} = 12,$$

$$|C_{\overline{G}}(x_3)| = \frac{k|C_G(g)|}{f_3} = \frac{4 \times 3}{1} = 12,$$

$$|C_{\overline{G}}(x_4)| = \frac{k|C_G(g)|}{f_4} = \frac{4 \times 3}{1} = 12.$$

We proceed to calculate the orders of the class representatives of these new conjugacy classes of $\overline{G}$. Since N is an elementary abelian 2-group, we utilize the following method to compute these orders. We again use the class $3A \in G$ for demonstration. Let $g \in 3A$ and let $d \in N$. Then $o(g) = 3 = m$. Recall that $\mathrm{Char}(\mathbb{F}) = 2 = p$. Let

$$w = d * d^g * d^{g^2} * \cdots * d^{g^{m-1}}.$$

Now if w is the identity of N, then the order of $x \in \overline{G}$ is $m = 3$ in this case. Otherwise, if w is not the identity of N, then $o(x) = pm = 2 \times 3 = 6$. Since we have represented the elements of N as matrices $[n_{ij}]_{4\times 4}$ with the zero vector represented by the identity matrix $I_{4\times 4}$, we are able to use the multiplication operation to compute our w's. From the class of 3A we have four conjugacy classes of $\overline{G}$. Since g is always in the first conjugacy class, then

$$w = 1_N$$

and hence

$$o(x_1) = 3.$$

For the second class,

$$d = (0, 0, 1) \text{ and } w = (0, 0, 1) \neq 1_N,$$

it then follows that

$$o(x_2) = pm = 2 \times 3 = 6.$$

For the third class,

$$d = (0, 0, z^1) \text{ and } w = (0, 0, z^1) \neq 1_N,$$

it then follows that

$$o(x_3) = pm = 2 \times 3 = 6.$$

And for the fourth class,

$$d = (0, 0, z^2) \text{ and } w = (0, 0, z^2) \neq 1_N,$$

it then follows that

$$o(x_4) = pm = 2 \times 3 = 6.$$

Note again that z is the primitive root of $GF(4)$. Since these are the first elements of orders 3 and 6 respectively, we label these four conjugacy classes as $[x_1] = 3A$, $[x_2] = 6A$, $[x_3] = 6B$ and $[x_4] = 6C$. The size of each class is

$$|[x_i]| = \frac{|\overline{G}|}{|C_{\overline{G}}(x_i)|}.$$

The full list of conjugacy classes of $\overline{G}$ is given in Table 7.5.

Table 7.5: The conjugacy classes of elements of $\overline{G}$

$[g]_G$	k	f_i	$\|C_{\overline{G}}(x)\|$	d_i	w	$[x]_{\overline{G}}$
1A	64	1	3840	$(0,0,0)$	$(0,0,0)$	1A
		1	3840	$(0,0,1)$	$(0,0,1)$	2A
		1	3840	$(0,0,z^1)$	$(0,0,z^1)$	2B
		1	3840	$(0,0,z^2)$	$(0,0,z^2)$	2C
		15	256	$(0,1,0)$	$(0,1,0)$	2D
		15	256	$(0,1,1)$	$(0,1,1)$	2E
		15	256	$(0,1,z^1)$	$(0,1,z^1)$	2F
		15	256	$(0,1,z^2)$	$(0,1,z^2)$	2G
2A	16	1	64	$(0,0,0)$	$(0,0,0)$	2H
		1	64	$(0,0,1)$	$(0,0,0)$	2I
		1	64	$(0,0,z^1)$	$(0,0,0)$	2J
		1	64	$(0,0,z^2)$	$(0,0,0)$	2K
		2	32	$(1,0,0)$	$(1,1,1)$	4A
		2	32	$(1,0,z^1)$	$(1,1,1)$	4B
		2	32	$(z^1,0,0)$	(z^1,z^1,z^2)	4C
		2	32	$(z^1,0,1)$	(z^1,z^1,z^2)	4D
		2	32	$(z^2,0,0)$	(z^2,z^2,z^1)	4E
		2	32	$(z^2,0,0)$	(z^2,z^2,z^1)	4F
3A	4	1	12	$(0,0,0)$	$(0,0,0)$	3A
		1	12	$(0,0,1)$	$(0,0,1)$	6A
		1	12	$(0,0,z^1)$	$(0,0,z^1)$	6B
		1	12	$(0,0,z^2)$	$(0,0,z^2)$	6C
5A	4	1	20	$(0,0,0)$	$(0,0,0)$	5A
		1	20	$(0,0,1)$	$(0,0,1)$	10A
		1	20	$(0,0,z^1)$	$(0,0,z^1)$	10B
		1	20	$(0,0,z^2)$	$(0,0,z^2)$	10C
5B	4	1	20	$(0,0,0)$	$(0,0,0)$	5B
		1	20	$(0,0,1)$	$(0,0,1)$	10D
		1	20	$(0,0,z^1)$	$(0,0,z^1)$	10E
		1	20	$(0,0,z^2)$	$(0,0,z^2)$	10F

Since $\overline{G}$ has 30 conjugacy classes it then follows that $\overline{G}$ will have 30 irreducible characters.

7.5 Fusion of inertia factor groups into $Sp(2,4)$

In Section 7.3 we had that the action of G on N yielded 8 orbits. By Brauer's Theorem, when G acts on the $Irr(N)$, we will again have 8 orbits. The orbit lengths of the latter action are 1, 15, 6, 6, 6, 10, 10 and 10. We determine the inertia factor groups as $H_1 = G$, $H_2 = D_4$, H_3, H_4, H_5 being isomorphic to D_{10} and H_6, H_7, H_8 isomorphic to S_3. The dual of a vector space can also be used for the action of G on $Irr(N)$. The main aim of this section is to consider the fusion of these inertia factor groups into the group G. The fusion of H_2 into G was dealt with in Section 7.3. We do the fusions of D_{10} and S_3 into G by using the divisibility of the respective centralizer sizes. That is $\frac{|C_G(a)|}{|C_{H_i}(b)|}$, where a is a class representative of a conjugacy class of G and b a class representative of a conjugacy class of H_i, with $i \in \{2,3,4,5,6,7,8\}$ and $o(a) = o(b)$. We also employ for each fusion a respective permutation character $\chi(G|H_i)$. Since the degree of $\chi(G|H_2)$ is 15, the degree of $\chi(G|D_{10})$ is 6 and the degree of $\chi(G|S_3)$ is 10, we have

$$\chi(G|D_4) = 1a + 4a + 2 \cdot 5a$$

$$\chi(G|D_{10}) = 1a + 5a$$

and

$$\chi(G|S_3) = 1a + 4a + 5a,$$

where $1a$, $4a$ and $5a$ are irreducible characters of G.

In the event that the method outlined above provides more than one candidate for a fusion, we use Remark 6.4.1 to obtain the suitable candidate for the fusion.

Table 7.6: The fusion of D_{10} into $Sp(2,4)$

	$[g]$	$1A$	$2A$	$3A$	$5A$	$5B$
	$\lvert C_G(g)\rvert$	60	4	3	5	5
$[x_i]$	$\lvert C_H(h)\rvert$					
$1A$	10	[6]				
$2A$	2		[2]			
$5A$	5				1	[1]
$5B$	5				[1]	1
	$\chi(G\lvert D_{10})$	6	2	0	1	1

Table 7.7: The fusion of S_3 into $Sp(2,4)$

	$[g]$	$1A$	$2A$	$3A$	$5A$	$5B$
	$\|C_G(g)\|$	60	4	3	5	5
$[x_i]$	$\|C_H(h)\|$					
$1A$	6	10				
$2A$	2		2			
$3A$	3			1		
	$\chi(G\|S_3)$	10	2	1	0	0

We summarize the fusions of the inertia factor groups into $Sp(2,4)$ in Table 7.8.

Table 7.8: The fusions of S_3, D_{10} and D_4 into $Sp(2,4)$

S_3	D_{10}	D_4	$\longrightarrow$	$Sp(2,4)$
$1A$	$1A$	$1A$	$\longrightarrow$	$1A$
$2A$	$2A$	$2A$	$\longrightarrow$	$2A$
		$2B$		
		$2C$		
$3A$			$\longrightarrow$	$3A$
	$5B$		$\longrightarrow$	$5A$
	$5A$		$\longrightarrow$	$5B$

7.6 The Fischer matrices of $\overline{G}$

We will construct the character table of $\overline{G}$ using the Clifford-Fischer Theory. We follow the method used by Fischer. This method entails utilizing the character tables of the inertia factor groups and Fischer matrices of $\overline{G}$. The Clifford-Fischer Theory requires that the irreducible characters of the normal subgroup $N \trianglelefteq \overline{G}$ be extendable to the inertia groups. Due to Mackey's Theorem, since N is elementary abelian and $\overline{G}$ is a split extension of N by G, then the irreducible characters of N are extendable to its inertia group. Let us consider the construction of the Fischer matrices of $\overline{G}$. Since G has five conjugacy classes then $\overline{G}$ has five Fischer matrices corresponding to each class of G. From Table 7.5 of the conjugacy classes of $\overline{G}$ we are able to deduce the sizes of these matrices. We use the general form of the Fischer matrix in our demonstration. For demonstration we again consider the class $[g_3] = 3A$. The Fischer matrix corresponding to this class is a 4×4 matrix. From Table 7.8 we note that only the class $3A$ of S_3 that fuses into the class $[g_3] = 3A$ of G. However, according to Remark 4.2.36 and Section 7.5, when G acts on $Irr(N)$ we have 3 copies of the group S_3. This then means that the classes of order 3 in H_6, H_7 and H_8 fuse to the class $3A$ in $Sp(2,4)$. Hence we have the Fischer matrix in Table 7.9 below.

Table 7.9: Fischer matrix from $3A$

$\lvert C_{\overline{G}}(x_{3j})\rvert$	$\lvert C_{\overline{G}}(x_{31})\rvert$	$\lvert C_{\overline{G}}(x_{32})\rvert$	$\lvert C_{\overline{G}}(x_{33})\rvert$	$\lvert C_{\overline{G}}(x_{34})\rvert$
$\lvert C_{H_k}(g_{3km})\rvert$				
$\lvert C_{H_1}(g_{311})\rvert$	1	1	1	1
$\lvert C_{H_6}(g_{361})\rvert$	x	a	b	c
$\lvert C_{H_7}(g_{371})\rvert$	y	d	e	f
$\lvert C_{H_8}(g_{381})\rvert$	z	g	h	i
	m_{31}	m_{32}	m_{33}	m_{34}

We then use the orthogonality relations to compute the entries of this matrix. Referring to Table 7.5 and since $m_i = \frac{f_i|N|}{k}$ then $m_{31} = 16 = m_{32} = m_{33} = m_{34}$. Also according to Table 7.5 we have that $|C_{\overline{G}}(x_{31})| = 3$, $|C_{\overline{G}}(x_{32})| = 6$, $|C_{\overline{G}}(x_{33})| = 6$ and $|C_{\overline{G}}(x_{34})| = 6$. From Table 7.7 we have that $|C_{H_1}(g_{311})| = 3$ and $|C_{H_6}(g_{361})| = 3 = |C_{H_7}(g_{371})| = |C_{H_8}(g_{381})|$. Since N is elementary abelian we have that $x = 1 = y = z$. Thus

Table 7.10: Fischer matrix from $3A$

$\lvert C_{\overline{G}}(x_{3j})\rvert$	3	6	6	6
$\lvert C_{H_k}(g_{3km})\rvert$				
3	1	1	1	1
3	1	a	b	c
3	1	d	e	f
3	1	g	h	i
	16	16	16	16

.

Using the column orthogonality relations we have the following equations:

$$3 + 3a + 3d + 3g = 0, \qquad 3 + 3a^2 + 3d^2 + 3g^2 = 12,$$

$$3 + 3b + 3e + 3h = 0, \qquad 3 + 3b^2 + 3e^2 + 3h^2 = 12,$$

$$3 + 3c + 3f + 3i = 0, \qquad 3 + 3c^2 + 3f^2 + 3i^2 = 12.$$

From row orthogonality relations we have:

$$16 + 16a + 16b + 16c = 0, \qquad 16 + 16a^2 + 16b^2 + 16c^2 = 64,$$

$$16 + 16d + 16e + 16f = 0, \qquad 16 + 16d^2 + 16e^2 + 16f^2 = 64,$$

$$16 + 16g + 16h + 16i = 0, \qquad 16 + 16g^2 + 16h^2 + 16i^2 = 64.$$

The solution for each unknown turns out to be ± 1. Meaning that we have more than one matrix to choose a suitable Fischer matrix. We utilize the additional Fischer matrix properties in Remark 3.3.7 to choose the correct entries for the Fischer matrix for the class $3A$, in particular Remark (ii) which relates the power maps of $\overline{G}$. The power maps of $\overline{G}$ are computed using Programmes A and B in [15]. These power maps are listed in rows 5, 6 and 7 of the character table of $\overline{G}$, Table 7.18. For demonstration, in $\overline{G}$ we have $(3A)^3 = 1A$, $(6A)^3 = 2A$, $(6B)^3 = 2B$ and $(6C)^3 = 2C$. Remark 3.3.7 (ii) implies that the congruency relations $\chi(3A) \equiv \chi(1A) \bmod 3$, $\chi(6A) \equiv \chi(2A) \bmod 3$, $\chi(6B) \equiv \chi(2B) \bmod 3$ and $\chi(6C) \equiv \chi(2C) \bmod 3$ must be satisfied. Eventually, using the orthogonality relations and the guidelines in Remark 3.3.7, we conclude that the Fischer matrix from $3A$ is

Table 7.11: Fischer matrix from $3A$

$\mathbb{F}_3 = M(3A)$		$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$
$o(x_{3j})$		3	6	6	6
$\lvert C_{\overline{G}}(x_{3j})\rvert$		12	12	12	12
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$				
$(1,1)$	3	1	1	1	1
$(6,1)$	3	1	-1	-1	1
$(7,1)$	3	1	-1	1	-1
$(8,1)$	3	1	1	-1	-1
m_{3j}		16	16	16	16

If we had chosen for instance

Table 7.12: wrong matrix from $3A$

$\mathbb{F}_3 = M(3A)$		$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$
$o(x_{3j})$		3	6	6	6
$\lvert C_{\overline{G}}(x_{3j})\rvert$		12	12	12	12
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$				
$(1,1)$	3	1	1	1	1
$(6,1)$	3	1	-1	-1	1
$(7,1)$	3	1	1	-1	-1
$(8,1)$	3	1	-1	1	-1
m_{3j}		16	16	16	16

as our Fischer matrix from $3A$, then the above-mentioned congruency relations are not satisfied. In this case refer to rows of χ_{25}, χ_{26} and χ_{27} and columns of $2A$ and $6A$ in Table 7.18. The rest of the Fischer matrices are constructed using Programmes C and D in [15] and are listed in Table 7.13.

Table 7.13: Fischer matrices of $\overline{G}$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$
$o(x_{1j})$		1	2	2	2	2	2	2	2
$\lvert C_{\overline{G}}(x_{1j})\rvert$		3840	3840	3840	3840	256	256	256	256
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$								
$(1,1)$	60	1	1	1	1	1	1	1	1
$(2,1)$	4	15	15	15	15	-1	-1	-1	-1
$(3,1)$	10	6	-6	-6	6	-2	2	2	-2
$(4,1)$	10	6	-6	6	-6	-2	2	-2	2
$(5,1)$	10	6	6	-6	-6	-2	-2	2	2
$(6,1)$	6	10	-10	-10	10	2	-2	-2	2
$(7,1)$	6	10	-10	10	-10	2	-2	2	-2
$(8,1)$	6	10	10	-10	-10	2	2	-2	-2
m_{1j}		1	1	1	1	15	15	15	15

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$
$o(x_{2j})$		2	2	2	2	4	4	4	4	4	4
$\lvert C_{\overline{G}}(x_{2j})\rvert$		64	64	64	64	32	32	32	32	32	32
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$										
$(1,1)$	4	1	1	1	1	1	1	1	1	1	1
$(2,1)$	4	1	1	1	1	-1	-1	-1	-1	1	1
$(2,2)$	4	1	1	1	1	-1	-1	1	1	-1	-1
$(2,3)$	4	1	1	1	1	1	1	-1	-1	-1	-1
$(3,1)$	2	2	-2	-2	2	0	0	-2	2	0	0
$(4,1)$	2	2	-2	2	-2	0	0	0	0	-2	2
$(5,1)$	2	2	2	-2	-2	-2	2	0	0	0	0
$(6,1)$	2	2	-2	-2	2	0	0	2	-2	0	0
$(7,1)$	2	2	-2	2	-2	0	0	0	0	2	-2
$(8,1)$	2	2	2	-2	-2	2	-2	0	0	0	0
m_{2j}		4	4	4	4	8	8	8	8	8	8

Table 7.13: Fischer matrices of $\overline{G}$

$\mathbb{F}_3 = M(3A)$		$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$
$o(x_{3j})$		3	6	6	6
$\lvert C_{\overline{G}}(x_{3j})\rvert$		12	12	12	12
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$				
$(1,1)$	3	1	1	1	1
$(6,1)$	3	1	-1	-1	1
$(7,1)$	3	1	-1	1	-1
$(8,1)$	3	1	1	-1	-1
m_{3j}		16	16	16	16

$\mathbb{F}_4 = M(5A)$		$x_{4,1}$	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$
$o(x_{4j})$		5	10	10	10
$\lvert C_{\overline{G}}(x_{4j})\rvert$		20	20	20	20
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$				
$(1,1)$	5	1	1	1	1
$(3,1)$	5	1	-1	-1	1
$(4,1)$	5	1	-1	1	-1
$(5,1)$	5	1	1	-1	-1
m_{4j}		16	16	16	16

$\mathbb{F}_5 = M(5B)$		$x_{5,1}$	$x_{5,2}$	$x_{5,3}$	$x_{5,4}$
$o(x_{5j})$		5	10	10	10
$\lvert C_{\overline{G}}(x_{5j})\rvert$		20	20	20	20
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$				
$(1,1)$	5	1	1	1	1
$(3,1)$	5	1	-1	-1	1
$(4,1)$	5	1	-1	1	-1
$(5,1)$	5	1	1	-1	-1
m_{5j}		16	16	16	16

7.7 The character table of $\overline{G}$

Due to Gallagher's Theorem the irreducible characters of $\overline{G}$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\phi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \in ker(\beta)\},$$

where $\overline{H}_i$ is an inertia group and $H_i = \overline{H}_i/N$ is an inertia factor group. This then means that the character table of $\overline{G}$ will be divided into blocks corresponding to the inertia factor groups H_i, for $i \in \{1,2,3,4,5,6,7,8\}$. Details of this process are discussed in Chapter 3. In this case $H_1 = G$, $H_2 = D_4$, $H_3, H_4, H_5 \cong D_{10}$ and $H_6, H_7, H_8 \cong S_3$. Thus the character table of $\overline{G}$ will be of the form

$$\left[\begin{array}{ccccc} B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} \\ \hline B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5} \\ \hline B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5} \\ \hline B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} & B_{4,5} \\ \hline B_{5,1} & B_{5,2} & B_{5,3} & B_{5,4} & B_{5,5} \\ \hline B_{6,1} & B_{6,2} & B_{6,3} & B_{6,4} & B_{6,5} \\ \hline B_{7,1} & B_{7,2} & B_{7,3} & B_{7,4} & B_{7,5} \\ \hline B_{8,1} & B_{8,2} & B_{8,3} & B_{8,4} & B_{8,5} \end{array}\right],$$

where $B_{i,j}$ are blocks corresponding to the inertia factor groups and the five conjugacy classes of G, $\{1 \leq i \leq 8\}$ and $\{1 \leq j \leq 5\}$. The block $B_{i,j}$ is formed by multiplying the relevant columns of the character table of H_i by the rows of the Fischer matrix $M(g)$ corresponding to the classes of H_i that fuse to the class $[g] \in G$. If H_i does not contribute to $M(g)$ then the block $B_{i,j}$ will have zeroes. The fusion maps of the inertia factor groups are given in Table 7.8.

We list below the character tables of the inertia factor groups.

Table 7.14: The character table of $Sp(2,4)$

$C_G(g)$	60	4	3	5	5
$[g]$	1A	2A	3A	5A	5B
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	A^*
χ_3	3	-1	0	A^*	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

where

$$A = \frac{1-\sqrt{5}}{2}.$$

Table 7.15: The character table of D_4

$C(g)$	4	4	4	4
$[g]$	1A	2A	2B	2C
χ_1	1	1	1	1
χ_2	1	-1	-1	1
χ_3	1	-1	1	-1
χ_4	1	1	-1	-1

Table 7.16: The character table of D_{10}

$C(g)$	10	2	5	5
$[g]$	1A	2A	5A	5B
χ_1	1	1	1	1
χ_2	1	-1	1	1
χ_3	2	0	A	A^*
χ_4	2	0	A^*	A

where

$$A = \frac{-1-\sqrt{5}}{2}.$$

Table 7.17: The character table of S_3

$C(g)$	6	2	3
$[g]$	1A	2A	3A
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

We use the class $3A$ of G for demonstration.

$$B_{1,3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

$B_{2,3} = [0]_{4\times 4}$

$B_{3,3} = [0]_{4\times 4}$

$B_{4,3} = [0]_{4\times 4}$

$B_{5,3} = [0]_{4\times 4}$

$$B_{6,3} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$B_{7,3} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$B_{8,3} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

The full character table of $\overline{G}$ is given in Table 7.18. Power maps are computed using Programmes A and B in [15]. These are listed in rows 5, 6 and 7 of Table 7.18. We used Programme E in [15] to check the accuracy and the consistency of this character table.

Table 7.18: The character table of $2^6{:}Sp(2,4)$

$[g]$	1A								2A						
$[\overline{g}]$	1A	2A	2B	2C	2D	2E	2F	2G	2H	2I	2J	2K	4A	4B	4C
Position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert C_{\overline{G}}(\overline{g})\rvert$	3840	3840	3840	3840	256	256	256	256	64	64	64	64	32	32	32
2P	1A	1A	1A	1A	1A	1A	1A	1A	1A	1A	1A	1A	2F	2F	2G
3P	1A	2A	2B	2C	2D	2E	2F	2G	2H	2I	2J	2K	4A	4B	4C
5P	1A	2A	2B	2C	2D	2E	2F	2G	2H	2I	2J	2K	4A	4B	4C
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1
χ_3	3	3	3	3	3	3	3	3	-1	-1	-1	-1	-1	-1	-1
χ_4	4	4	4	4	4	4	4	4	0	0	0	0	0	0	0
χ_5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1
χ_6	15	15	15	15	-1	-1	-1	-1	3	3	3	3	-1	-1	-1
χ_7	15	15	15	15	-1	-1	-1	-1	-1	-1	-1	-1	3	3	-1
χ_8	15	15	15	15	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	3
χ_9	15	15	15	15	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_{10}	6	-6	-6	6	-2	2	2	-2	2	-2	-2	2	0	0	-2
χ_{11}	6	-6	-6	6	-2	2	2	-2	-2	2	2	-2	0	0	2
χ_{12}	12	-12	-12	12	-4	4	4	-4	0	0	0	0	0	0	0
χ_{13}	12	-12	-12	12	-4	4	4	-4	0	0	0	0	0	0	0
χ_{14}	6	-6	6	-6	-2	2	-2	2	2	-2	2	-2	0	0	0
χ_{15}	6	-6	6	-6	-2	2	-2	2	-2	2	-2	2	0	0	0
χ_{16}	12	-12	12	-12	-4	4	-4	4	0	0	0	0	0	0	0
χ_{17}	12	-12	12	-12	-4	4	-4	4	0	0	0	0	0	0	0
χ_{18}	6	6	-6	-6	-2	-2	2	2	2	2	-2	-2	-2	2	0
χ_{19}	6	6	-6	-6	-2	-2	2	2	-2	-2	2	2	2	-2	0
χ_{20}	12	12	-12	-12	-4	-4	4	4	0	0	0	0	0	0	0
χ_{21}	12	12	-12	-12	-4	-4	4	4	0	0	0	0	0	0	0
χ_{22}	10	-10	-10	10	2	-2	-2	2	2	-2	-2	2	0	0	2
χ_{23}	10	-10	-10	10	2	-2	-2	2	-2	2	2	-2	0	0	-2
χ_{24}	20	-20	-20	20	4	-4	-4	4	0	0	0	0	0	0	0
χ_{25}	10	-10	10	-10	2	-2	2	-2	2	-2	2	-2	0	0	0
χ_{26}	10	-10	10	-10	2	-2	2	-2	-2	2	-2	2	0	0	0
χ_{27}	20	-20	20	-20	4	-4	4	-4	0	0	0	0	0	0	0
χ_{28}	10	10	-10	-10	2	2	-2	-2	2	2	-2	-2	2	-2	0
χ_{29}	10	10	-10	-10	2	2	-2	-2	-2	-2	2	2	-2	2	0
χ_{30}	20	20	-20	-20	4	4	-4	-4	0	0	0	0	0	0	0

Table 7.18: The character table of 2^6:$Sp(2,4)$ - continued

$[g]$	2A			3A				5A				5B			
$[\overline{g}]$	4D	4E	4F	3A	6A	6B	6C	5A	10A	10B	10C	5B	10D	10E	10F
Position	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\lvert C_{\overline{G}}(\overline{g})\rvert$	32	32	32	12	12	12	12	20	20	20	20	20	20	20	20
2P	2G	2E	2E	3A	3A	3A	3A	5B	5B	5B	5B	5A	5A	5A	5A
3P	4D	4E	4F	1A	2A	2B	2C	5B	10D	10E	10F	5A	10A	10B	10C
5P	4D	4E	4F	3A	6A	6B	6C	1A	2A	2B	2C	1A	2A	2B	2C
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	0	0	0	0	A	A	A	A	*A	*A	*A	*A
χ_3	-1	-1	-1	0	0	0	0	*A	*A	*A	*A	A	A	A	A
χ_4	0	0	0	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
χ_5	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0
χ_6	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_7	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_8	3	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_9	-1	3	3	0	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	2	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1
χ_{11}	-2	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1
χ_{12}	0	0	0	0	0	0	0	-A	A	A	-A	-*A	*A	*A	-*A
χ_{13}	0	0	0	0	0	0	0	-*A	*A	*A	-*A	-A	A	A	-A
χ_{14}	0	-2	2	0	0	0	0	1	-1	1	-1	1	-1	1	-1
χ_{15}	0	2	-2	0	0	0	0	1	-1	1	-1	1	-1	1	-1
χ_{16}	0	0	0	0	0	0	0	-A	A	-A	A	-*A	*A	-*A	*A
χ_{17}	0	0	0	0	0	0	0	-*A	*A	-*A	*A	-A	A	-A	A
χ_{18}	0	0	0	0	0	0	0	1	1	-1	-1	1	1	-1	-1
χ_{19}	0	0	0	0	0	0	0	1	1	-1	-1	1	1	-1	-1
χ_{20}	0	0	0	0	0	0	0	-A	-A	A	A	-*A	-*A	*A	*A
χ_{21}	0	0	0	0	0	0	0	-*A	-*A	*A	*A	-A	-A	A	A
χ_{22}	-2	0	0	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{23}	2	0	0	1	-1	-1	1	0	0	0	0	0	0	0	0
χ_{24}	0	0	0	-1	1	1	-1	0	0	0	0	0	0	0	0
χ_{25}	0	2	-2	1	-1	1	-1	0	0	0	0	0	0	0	0
χ_{26}	0	-2	2	1	-1	1	-1	0	0	0	0	0	0	0	0
χ_{27}	0	0	0	-1	1	-1	1	0	0	0	0	0	0	0	0
χ_{28}	0	0	0	1	1	-1	-1	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	1	1	-1	-1	0	0	0	0	0	0	0	0
χ_{30}	0	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0

where

$$A = \frac{1-\sqrt{5}}{2}.$$

The degrees of the irreducible characters of the affine subgroup $A(2) = 2^6{:}Sp(2,4)$ of $Sp(4,4)$ are the degrees of $Irr(Sp(2,4))$, the degrees of $Irr(A(1)) = Irr(D_4)$ multiplied by $q^{2n-2} - 1 = 4^2 - 1 = 15$, the degrees of $Irr(D_{10})$ multiplied by $\frac{1}{2}q^{n-1}(q^{n-1} - 1) = \frac{1}{2} \times 4 \times (3) = 6$ and the degrees of $Irr(S_3)$ multiplied by $\frac{1}{2}q^{n-1}(q^{n-1} + 1) = \frac{1}{2} \times 4 \times (5) = 10$, by Theorem 4.2.32, Remark 4.2.33 and Theorem 4.2.34.

7.8 The fusion of $\overline{G}$ into $Sp(4,4)$

The penultimate section of this chapter is about the fusion of $\overline{G}$ into the symplectic group $Sp(4,4)$. Let $[y]$ and $[x]$ be the conjugacy classes of $Sp(4,4)$ and $\overline{G}$ respectively. We follow the method used in Section 7.5 to determine the partial fusion of $\overline{G}$ into $Sp(4,4)$. That is we consider the divisibility of the respective centralizer sizes $\frac{|C(y)|}{|C(x)|}$, where y is the class representative of $Sp(4,4)$ and x the class representative of $\overline{G}$ with $o(y) = o(x)$. We also use the permutation character $\chi(Sp(4,4)|\overline{G})$ of $Sp(4,4)$ on the cosets of $\overline{G}$ in $Sp(4,4)$ together with the respective power maps to construct this partial fusion. This partial fusion is given by Table 7.20 but without considering the squares around the entries. We use GAP to compute the permutation character $\phi = \chi(Sp(4,4)|\overline{G})$. We then express this permutation character in terms of $\psi_i \in Irr(Sp(4,4))$. To determine the constituents of ϕ, we compute the respective inner products $\langle \phi, \psi_i \rangle$. This method yields the following permutation character

$$\chi(Sp(4,4)|\overline{G}) = \psi_1 + \psi_3 + \psi_5 + 2\psi_{10},$$

in terms of the irreducible characters of $Sp(4,4)$. The values of this permutation character are listed in the last row of Table 7.20. With regard to power maps we recall the following. Suppose that $[x_1]$ and $[x_2]$ are conjugacy classes of $\overline{G}$ such that $x_1^p \in [x_2]$ for some prime p. And suppose that $[y_1]$ and $[y_2]$ are conjugacy classes of $Sp(4,4)$ such that $y_1^p \in [y_2]$. Now if $[x_1]$ fuses to $[y_1]$ then it must follow that $[x_2]$ fuses to $[y_2]$. The irreducible characters and power maps of $Sp(4,4)$ are found in the ATLAS.

In the partial fusion we note that the classes $2A$, $2B$ and $2C$ of $\overline{G}$ either fuse into the class $2A$ or the class $2B$ of $Sp(4,4)$. The power maps of $\overline{G}$ in Table 7.18 yield the following: $(6A)^3 = 2A$, $(6B)^3 = 2B$ and $(6C)^3 = 2C$. In the partial fusion we have that the classes $6A$, $6B$ and $6C$ fuse to the class $6A$ of $Sp(4,4)$. From the character table of $Sp(4,4)$ in the ATLAS we have $(6A)^3 = 2A$ and $(6B)^3 = 2B$. This means that the classes $2A$, $2B$ and $2C$ of $\overline{G}$ must fuse to the class $2A$ of $Sp(4,4)$. There are no power maps involving the class $2D$ of $\overline{G}$. In the character table of $\overline{G}$ we have $(4A)^2 = 2F$, $(4B)^2 = 2F$, $(4C)^2 = 2G$, $(4D)^2 = 2G$, $(4E)^2 = 2E$ and $(4F)^2 = 2E$. According to the partial fusion, these classes of order 4 either fuse to $4A$ or $4B$ in $Sp(4,4)$. However in $Sp(4,4)$ we have $(4A)^2 = 2C$ and $(4B)^2 = 2C$. Thus the classes $2E$, $2F$ and $2G$ of $\overline{G}$ fuse to the class $2C$ of $Sp(4,4)$. Going back to the class $2D$, we note from the partial fusion that now the only suitable candidate for this class is $2B$. To complete the fusion we utilize the method of set intersections for characters we used in earlier chapters. This method entails the following.

Let ρ be the character afforded by the regular representation of G. It follows that $\rho = \sum_{i=1}^{5} e_i \phi_i$ where $\phi_i \in Irr(G)$ and $e_i = \deg(\phi_i)$. This then means that ρ can be seen as the character of $\overline{G}$ which contains 2^6 in its kernel such that

$$\rho(g) = \begin{cases} |Sp(2,4)| & g \in 2^6 \\ 0 & \text{otherwise.} \end{cases}$$

Now if ψ is a character of $Sp(4,4)$ then

$$\langle \rho, \psi \rangle_{\overline{G}} = \frac{1}{|\overline{G}|}\{\rho(1A)\psi(1A) + \rho(2A)\psi(2A) + \rho(2B)\psi(2B) + \rho(2C)\psi(2C)$$
$$+15\rho(2D)\psi(2D) + 15\rho(2E)\psi(2E) + 15\rho(2F)\psi(2F) + 15\rho(2G)\psi(2G)\}$$
$$= \frac{1}{|4^3||Sp(2,4)|}\{|Sp(2,4)|\psi(1A) + |Sp(2,4)|\psi(2A) + |Sp(2,4)|\psi(2B) + |Sp(2,4)|\psi(2C)$$
$$+15|Sp(2,4)|\psi(2D) + 15|Sp(2,4)|\psi(2E) + 15|Sp(2,4)|\psi(2F) + 15|Sp(2,4)|\psi(2G)\}$$
$$= \frac{1}{64}\{\psi(1A) + \psi(2A) + \psi(2B) + \psi(2C) + 15\psi(2D) + 15\psi(2E) + 15\psi(2F) + 15\psi(2G)\}.$$

We rearrange the last part of this equation so that it aligns with the discussion in the introduction of this section and the character table of $\overline{G}$, Table 7.18, to have

$$\langle \rho, \psi \rangle_{\overline{G}} = \frac{1}{64}\{\psi(1A) + 15\psi(2D) + \psi(2A) + \psi(2B) + \psi(2C) + 15\psi(2E) + 15\psi(2F) + 15\psi(2G)\}$$
$$= \langle \psi\downarrow_N, \tau_1 \rangle$$

where $\psi\downarrow_N$ is the restriction of ψ to N and τ_1 is the identity character of N. We note that for ψ we have that

$$\psi\downarrow_N = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4 + a_5\theta_5 + a_6\theta_6 + a_7\theta_7 + a_8\theta_8,$$

where for $i \in \{1,2,3,4,5,6,7,8\}$, θ_i are the sums of the irreducible characters of N which are in one orbit under the action of G on $Irr(N)$, and $a_i \in \{0\} \cup \mathbb{N}$. For $j \in \{1,2,3,\cdots,64\}$, let $\tau_j \in Irr(N)$. Recall that under the action of G on $Irr(N)$ we have orbits of lengths 1, 15, 6, 6, 6, 10, 10 and 10. Then we have that

$$\theta_1 = \tau_1 \quad \text{and} \quad \deg(\theta_1) = 1,$$

$$\theta_2 = \sum_{j=2}^{16} \tau_j \quad \text{and} \quad \deg(\theta_2) = 15,$$

$$\theta_3 = \sum_{j=17}^{22} \tau_j \quad \text{and} \quad \deg(\theta_3) = 6,$$

$$\theta_4 = \sum_{j=23}^{28} \tau_j \quad \text{and} \quad \deg(\theta_4) = 6,$$

$$\theta_5 = \sum_{j=29}^{34} \tau_j \quad \text{and} \quad \deg(\theta_5) = 6,$$

$$\theta_6 = \sum_{j=35}^{44} \tau_j \quad \text{and} \quad \deg(\theta_6) = 10$$

$$\theta_7 = \sum_{j=45}^{54} \tau_j \quad \text{and} \quad \deg(\theta_7) = 10,$$

$$\theta_8 = \sum_{j=55}^{64} \tau_j \quad \text{and} \quad \deg(\theta_8) = 10.$$

Then

$$\psi{\downarrow_N} = a_1\tau_1 + a_2\sum_{j=2}^{16}\tau_j + a_3\sum_{j=17}^{22}\tau_j + a_4\sum_{j=23}^{28}\tau_j + a_5\sum_{j=29}^{34}\tau_j + a_6\sum_{j=35}^{44}\tau_j + a_7\sum_{j=45}^{54}\tau_j \quad \text{and} + a_8\sum_{j=55}^{64}\tau_j$$

and

$$\langle \psi{\downarrow_N}, \psi{\downarrow_N}\rangle = a_1^2 + 15a_2^2 + 6a_3^2 + 6a_4^2 + 6a_5^2 + 10a_6^2 + 10a_7^2 + 10a_8^2.$$

We note firstly that $a_1 = \langle \psi{\downarrow_N}, \tau_1\rangle = \langle \rho, \psi\rangle_{\overline{G}}$ and secondly that

$$\langle \psi{\downarrow_N}, \psi{\downarrow_N}\rangle = \frac{1}{64}\{\psi(1A)\psi(1A) + 15\psi(2D)\psi(2D) + \psi(2A)\psi(2A) + \psi(2B)\psi(2B)$$
$$+\psi(2C)\psi(2C) + 15\psi(2E)\psi(2E) + 15\psi(2F)\psi(2F) + 15\psi(2G)\psi(2G)\}.$$

Suppose that $\psi_2 = 18a$ and $\psi_6 = 51a$, the irreducible characters of $Sp(4,4)$ of degrees 18 and 51 respectively. In the case of ψ_2 we have

$$a_1 = \langle \rho, \psi_2\rangle_{\overline{G}} = \frac{1}{64}[18 + 15(-6) + (-6) + (-6) + (-6) + 15(2) + 15(2) + 15(2)] = 0.$$

Since the degree of ψ_2 is 18 then

$$a_1 + 15a_2 + 6a_3 + 6a_4 + 6a_5 + 10a_6 + 10a_7 + 10a_8 = 18.$$

This gives $a_2 = 0 = a_6 = a_7 = a_8$ and $a_3 = a_4 = a_5 = 1$. This means that the restriction $\psi_2{\downarrow_{\overline{G}}}$ is expressible as a sum of characters of degree 6 from the 3rd, 4th and 5th blocks of the character table of $\overline{G}$ corresponding to the inertia factor groups H_3, H_4 and H_5. Considering the predetermined partial fusion and the character tables of $\overline{G}$ and $Sp(4,4)$, we deduce that

$$\psi_2{\downarrow_{\overline{G}}} = \chi_{11} + \chi_{15} + \chi_{19}.$$

The values of χ_i are listed in the character table of $\overline{G}$, Table 7.18. The character table of $Sp(4,4)$ is available in the ATLAS. We denote the characters of $Sp(4,4)$ by ψ_i instead of χ_i as in the ATLAS. This is meant to distinguish the characters of $\overline{G}$ from those of $Sp(4,4)$.

On the other hand for ψ_6 we have

$$a_1 = \langle \rho, \psi_6 \rangle_{\overline{G}} = \frac{1}{64}[51 + 15(3) + (-13) + (-13) + (-13) + 15(3) + 15(3) + 15(3)] = 3.$$

Since the degree of ψ_6 is 51 then

$$a_1 + 15a_2 + 6a_3 + 6a_4 + 6a_5 + 10a_6 + 10a_7 + 10a_8 = 51.$$

One of the suitable solutions is $a_2 = 0$ and $a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 1$. This then implies that the restriction $\psi_6{\downarrow_{\overline{G}}}$ can be expressed as a sum of a character of degree 3 from the 1st block, three characters of degree 6 from the 3rd, 4th and 5th blocks, and three characters of degree 10 from the 6th, 7th and 8th blocks of the character table of $\overline{G}$. Again considering the predetermined partial fusion and the character tables of $\overline{G}$ and $Sp(4,4)$, we deduce that

$$\psi_6{\downarrow_{\overline{G}}} = \chi_3 + \chi_{11} + \chi_{15} + \chi_{19} + \chi_{23} + \chi_{26} + \chi_{29}.$$

In the partial fusion we note that the class $2H$ of $\overline{G}$ can either fuse to the class $2A$ or $2B$ or $2C$ of $Sp(4,4)$. We apply the above technique to choose the right class for $2H$. From the character table of $Sp(4,4)$, ψ_2 and ψ_6 yield the following values.

Table 7.19: Values of ψ_i in Sp(4,4)

	$[y]$	2A	2B	2C
Degree	ψ_i			
18a	ψ_2	-6	-6	2
51a	ψ_6	-13	3	3

In the character table of $\overline{G}$ the values of the restrictions are:

$$\psi_2{\downarrow_{\overline{G}}}(2H) = -6,$$

and

$$\psi_6{\downarrow_{\overline{G}}}(2H) = -13.$$

Comparing these values we conclude that the class $2H$ of $\overline{G}$ fuses into the class $2A$ of $Sp(4,4)$.

The values of ψ_2 and ψ_6 on the classes of $Sp(4,4)$ and the values of the restrictions $\psi_2{\downarrow_{\overline{G}}}$ and $\psi_6{\downarrow_{\overline{G}}}$ on the classes of $\overline{G}$ together with the predetermined fusion enable us to complete the fusion of $\overline{G}$ into $Sp(4,4)$. The complete fusion results are contained in Table 7.20 below.

Table 7.20: The fusion of 2^6:$Sp(2,4)$ into $Sp(4,4)$

	$[g]$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$
	$\|C_{Sp(4,4)}(g)\|$	979200	3840	3840	256	180	180	32	32
$[x_i]$	$\|C_{\overline{G}}(x)\|$								
$1A$	3840	255							
$2A$	3840		1	1					
$2B$	3840		1	1					
$2C$	3840		1	1					
$2D$	256		15	15	1				
$2E$	256		15	15	1				
$2F$	256		15	15	1				
$2G$	256		15	15	1				
$2H$	64		60	60	4				
$2I$	64		60	60	4				
$2J$	64		60	60	4				
$2K$	64		60	60	4				
$3A$	12					15	15		
$4A$	32							1	1
$4B$	32							1	1
$4C$	32							1	1
$4D$	32							1	1
$4E$	32							1	1
$4F$	32							1	1
	$\chi(Sp(4,4)\|\overline{G})$	255	63	15	15	15	0	3	3

Table 7.20: The fusion of 2^6:$Sp(2,4)$ into $Sp(4,4)$

	$[g]$	$5A$	$5B$	$5C$	$5D$	$5E$	$6A$	$6B$	$10A$	$10B$	$10C$	$10D$
	$\|C_{Sp(4,4)}(g)\|$	300	300	300	300	25	12	12	20	20	20	20
$[x_i]$	$\|C_{\overline{G}}(x)\|$											
$5A$	20	15	15	15	15							
$5B$	20	15	15	15	15							
$6A$	12						1	1				
$6B$	12						1	1				
$6C$	12						1	1				
$10A$	20								1	1	1	1
$10B$	20								1	1	1	1
$10C$	20								1	1	1	1
$10D$	20								1	1	1	1
$10E$	20								1	1	1	1
$10F$	20								1	1	1	1
	$\chi(Sp(4,4)\|\overline{G})$	15	15	0	0	0	3	0	3	3	0	0

The fusion results from Table 7.20 are summarized in Table 7.21 below.

Table 7.21: The fusion of $2^6{:}Sp(2,4)$ into $Sp(4,4)$

$2^6{:}Sp(2,4)$	$Sp(4,4)$	$2^6{:}Sp(2,4)$	$Sp(4,4)$
$1A$	$1A$	$2A$	$2A$
		$2B$	
		$2C$	
		$2H$	
$2D$	$2B$	$2E$	$2C$
		$2F$	
		$2G$	
		$2I$	
		$2J$	
		$2K$	
$3A$	$3A$	$4B$	$4A$
		$4D$	
		$4F$	
$4A$	$4B$	$5A$	$5A$
$4C$			
$4E$			
$5B$	$5B$	$6A$	$6A$
		$6B$	
		$6C$	
$10A$	$10A$	$10D$	$10B$
$10B$		$10E$	
$10C$		$10F$	

7.9 The quotient group $\overline{G}/Z(\overline{G})$

The centre of the affine subgroup $\overline{G}$ of the symplectic group $Sp(4,4)$ is isomorphic to $\mathbb{Z}_4$. The quotient group $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^4{:}Sp(2,4)$. The Fischer matrices of $\overline{G}/Z(\overline{G})$ can be determined directly from the Fischer matrices of $\overline{G}$. All the 5 Fischer matrices of $\overline{G}$ were computed in Section 7.6. The action of $Sp(2,4)$ on $Irr(2^4)$ yields two inertia factor groups, namely $Sp(2,4)$ and D_4, the affine subgroup of $Sp(2,4)$. The two are labelled H_1 and H_2, respectively, in Section 7.5. This then means that the inertia factor groups $H_3, H_4, \cdots, H_8$ do not play a role in $\overline{G}/Z(\overline{G})$. Then, to obtain the Fischer matrices of $\overline{G}/Z(\overline{G})$ from those of $\overline{G}$, we delete the rows corresponding to $H_3, H_4, \cdots, H_8$ in each Fischer matrix of $\overline{G}$. Thereafter, we discard the repeated columns. This process yields the following five Fischer matrices.

$$M(1A) = \begin{bmatrix} 1 & 1 \\ 15 & -1 \end{bmatrix},$$

$$M(2A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$

$$M(3A) = \left[\ 1\ \right],$$

$$M(5A) = \left[\ 1\ \right],$$

$$M(5B) = \left[\ 1\ \right].$$

The deduced Fischer matrices can then be used to construct the character table of $\overline{G}/Z(\overline{G})$, using the Clifford-Fischer Theory, since $\overline{G}/Z(\overline{G})$ is a split extension.

On the other hand, the character table of $\overline{G}/Z(\overline{G})$ can be obtained directly from the character table of $\overline{G}$. The character table of $\overline{G}$ is given by Table 7.18. The process, using Remark 4.2.46, entails deleting the blocks corresponding to the inertia factor groups $H_3, H_4, \cdots, H_8$. Thereafter, deleting the repeated columns. The resulting character table is the character table of $\overline{G}/Z(\overline{G})$. This character table is given by Table 7.22 below.

Table 7.22: The character table of $\overline{G}/Z(\overline{G})$

$[g]$	1A		2A				3A	5A	5B
$[\overline{g}]$	1A	2A	2B	2C	2D	2E	3A	5A	5B
χ_1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	-1	-1	-1	0	A	$\overline{A}$
χ_3	3	3	-1	-1	-1	-1	0	$\overline{A}$	A
χ_4	4	4	0	0	0	0	1	-1	-1
χ_5	5	5	1	1	1	1	-1	0	0
χ_6	15	-1	3	-1	-1	-1	0	0	0
χ_7	15	-1	-1	3	-1	-1	0	0	0
χ_8	15	-1	-1	-1	3	-1	0	0	0
χ_9	15	-1	-1	-1	-1	3	0	0	0

where
$A = \frac{1-\sqrt{5}}{2}$.

8

The affine subgroup $2^{10}{:}Sp(4,4)$ of the symplectic group $Sp(6,4)$

In this chapter the affine subgroup $\overline{G} = P(3){:}Sp(4,4)$ of the symplectic group $Sp(6,4)$ is considered. The group $P(3)$ is the subgroup of $\overline{G}$ which satisfies the conditions outlined in Remark 4.2.22. Let $\mathbb{F} = GF(4)$ be the Galois field of 4 elements and V be a non-degenerate symplectic space of dimension $2n = 6$ over the field $\mathbb{F}$. In Section 4.2 we deal with $\overline{G}$ and V in a general context and, more significantly, it is proven that $\overline{G}$ is a split extension. This implies that $P(3)$ is normal in $\overline{G}$. Since the $\mathrm{Char}(\mathbb{F}) = 2$, the subgroup $P(3)$ is an elementary 2-group of order $4^5 = 2^{10}$. Once more, in this chapter we shall denote this normal subgroup $P(3) = 2^{10}$ of $\overline{G}$ by N. In Section 8.1 we deal with the transvections of the symplectic group $Sp(4,4)$. In Section 8.2 we express the generators of N and $Sp(4,4)$ as 6×6 symplectic matrices since $\overline{G}$ sits in $Sp(6,4)$, a symplectic group of 6×6 symplectic matrices with entries in $\mathbb{F}$. We compute the conjugacy classes of $\overline{G}$ in Section 8.3 using the coset analysis technique. We first express the permutation character that was computed in Section 7.8 in terms of the irreducible characters of $Sp(4,4)$. The action of $Sp(4,4)$ on $Irr(N)$ is dealt with in Section 8.4. Thereafter we consider the fusion of the inertia factor groups into the group $Sp(4,4)$. The Fischer matrices of $\overline{G}$ are computed in Section 8.5. We discuss how to construct the character table of $\overline{G}$ in Section 8.6 using the Clifford-Fischer Theory. In Section 8.7, we conclude this chapter by considering the quotient group $\overline{G}/Z(\overline{G})$, where $Z(\overline{G})$ is isomorphic to $\mathbb{Z}_4$. We show that this quotient is isomorphic to the split extension $2^8{:}Sp(4,4)$. We then demonstrate how to obtain the Fischer matrices of $\overline{G}/Z(\overline{G})$ directly from the Fischer matrices of $\overline{G}$.

8.1 Transvections of $Sp(4,4)$

This section is about the transvections of the symplectic group $Sp(4,4)$. Let $G = Sp(4,4)$ and $\overline{G}$, $\mathbb{F}$, N and V be as defined in the above introduction. The symplectic group G is generated by a set of symplectic transvections. There are $|V^*| = 2^8 - 1 = 255$ transvections in G. The order of each of these transvections is 2 as the $\mathrm{Char}(\mathbb{F}) = 2$. Since $|\mathbb{F}| = 2^2$, we have one conjugacy class of transvections in G. There are three conjugacy classes of elements of order 2 in G according to the character table of G in the ATLAS. However, since the order of the centralizer is equal to $\frac{|G|}{|V^*|} = \frac{979200}{255} = 3840$, then either the class 2A or 2B is the class of transvections. GAP is used to analyse the elements of these two classes. We observe that the elements of 2A satisfy the conditions of the definition of a transvection. We therefore conclude that the class 2A is the class of transvections. We note that the centralizer of a transvection is isomorphic to the affine subgroup $2^6{:}Sp(2,4)$ of G.

8.2 The generators of the groups N and $Sp(4,4)$

Since the affine subgroup $\overline{G}$ sits inside the symplectic group $Sp(6,4)$ of 6×6 symplectic matrices with entries in $\mathbb{F}$, we express the elements of $Sp(4,4)$ and N as 6×6 symplectic matrices. We utilize the method outlined in Remark 4.2.22. We start with the elements of N. Take $m = 3$, then we have

$$\begin{aligned} T(e_1) &= e_1 \\ T(e_i) &= \alpha_i e_1 + e_i \qquad 2 \le i \le 5 \\ T(e_2) &= \alpha_2 e_1 + e_2 \\ T(e_3) &= \alpha_3 e_1 + e_3 \\ T(e_4) &= \alpha_4 e_1 + e_4 \\ T(e_5) &= \alpha_5 e_1 + e_5 \\ T(e_{2m}) = T(e_6) &= \sum_{i=1}^{6} \beta_i e_i \end{aligned}$$

with $\beta_6 = 1$ and

$$\alpha_j = \begin{cases} -\beta_{2m+1-j}, & \text{if } 2 \le j \le 3 \\ \beta_{2m+1-j}, & \text{if } 3 < j \le 5. \end{cases}$$

This means $\alpha_2 = -\beta_5$, $\alpha_3 = -\beta_4$, $\alpha_4 = \beta_3$ and $\alpha_5 = \beta_2$ and the elements T of N are of the form

$$T = \begin{bmatrix} 1 & -\beta_5 & -\beta_4 & \beta_3 & \beta_2 & \beta_1 \\ 0 & 1 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 1 & 0 & 0 & \beta_3 \\ 0 & 0 & 0 & 1 & 0 & \beta_4 \\ 0 & 0 & 0 & 0 & 1 & \beta_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\beta_j \in GF(4)$. We compute all $4^5 = 2^{10} = 1024$ elements of N and the 10 generators of N by using GAP. The 10 generators are:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & z & 0 \\ 0 & 1 & 0 & 0 & 0 & z \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_8 = \begin{bmatrix} 1 & 0 & z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_9 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$P_{10} = \begin{bmatrix} 1 & z & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where z is a primitive root of $GF(4)$. We also obtain from GAP the generators of $Sp(4,4)$ as

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with $o(a) = 3$ and $o(b) = 5$. We then express these generators in terms of symplectic transvections since G is generated by symplectic transvections.

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^2 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 1 & 0 & 0 \\ 0 & 0 & z & 0 & 1 & 0 \\ 0 & z & 0 & z & 1 & 0 \\ 0 & 0 & z & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2 & z^2 & 0 & 0 & 0 \\ 0 & 1 & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^2 & z^2 & 0 \\ 0 & 0 & 0 & 1 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2 & 1 & z^2 & 0 & 0 \\ 0 & 0 & z^2 & 0 & z^2 & 0 \\ 0 & 1 & 0 & z^2 & 1 & 0 \\ 0 & 0 & 1 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & z & 0 & 0 \\ 0 & z^2 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

8.3 Conjugacy classes of $\overline{G}$

The action of G on N yields 8 orbits. These are of lengths 1, 1, 1, 1, 255, 255, 255 and 255 and correspond to the point stabilizers G, G, G, G, $2^6{:}Sp(2,4)$, $2^6{:}Sp(2,4)$, $2^6{:}Sp(2,4)$ and $2^6{:}Sp(2,4)$, respectively. The latter is the affine subgroup of G dealt with in Chapter 7. The permutation character of G on $2^6{:}Sp(2,4)$ was computed in Section 7.8. Let us denote this permutation character by $I^{Sp(4,4)}_{2^6:Sp(2,4)}$. We express this permutation character in terms of the $Irr(G)$. We deduce, from the character table of G in the ATLAS, that $I^{Sp(4,4)}_{2^6:Sp(2,4)} = 3\cdot 1a + 3\cdot 34a + 3\cdot 50a$, where $1a$, $34a$ and $50a$ are $Irr(G)$.

Now let $\chi(G|N)$ be the permutation character when G acts on N. Then

$$\chi(G|N) = 1+1+1+1+I^{G}_{2^6:Sp(2,4)} + I^{G}_{2^6:Sp(2,4)} + I^{G}_{2^6:Sp(2,4)} + I^{G}_{2^6:Sp(2,4)}$$

$$= 1a + 1a + 1a + 1a + 4(3\cdot 1a + 3\cdot 34a + 3\cdot 50a)$$

$$= 16\cdot 1a + 12\cdot 34a + 12\cdot 50a$$

where $1a$, $34a$ and $50a$ are irreducible characters of G. This permutation character is then used to compute the number of fixed points when $g \in G$ is acting on N. We refer to these as the k-values and are listed in Table 8.1 below.

Table 8.1: Fixed points of the action of G on N

$\lvert C_G\rvert$	979200	3840	3840	256	180	180	32	32	300	300	300	300	25	12
$o(g)$	1A	2A	2B	2C	3A	3B	4A	4B	5A	5B	5C	5D	5E	6A
k	1024	256	64	64	64	4	16	16	64	64	4	4	4	16

Table 8.1: Fixed points of the action of G on N

$\lvert C_G\rvert$	12	20	20	20	20	15	15	15	15	17	17	17	17
$o(g)$	6B	10A	10B	10C	10D	15A	15B	15C	15D	17A	17B	17C	17D
k	4	16	16	4	4	4	4	4	4	4	4	4	4

The conjugacy classes of $\overline{G}$ are computed using the coset analysis technique. For each conjugacy class of $\overline{G}$ we calculate the centralizer sizes $|C_{\overline{G}}(x_i)| = \frac{k|C_G(g)|}{f_i}$, $x_i \in \overline{G}$ and $g \in G$. We consider the class $[g] = 2A \in G$ to demonstrate the determination of the conjugacy classes of $\overline{G}$ from a conjugacy class of G. The permutation character in Table 8.1 has $k = 256$ for this class. This then means that after the action of N, by conjugation, N_g splits into 256 orbits, each of length $\frac{|N|}{k} = \frac{1024}{256} = 4$. Utilizing GAP, we note that after the action of the centralizer $C_G(g)$ on these orbits that some of them fuse together to ultimately have 11 orbits. We further note that the lengths of these new orbits to have $|\Omega_1| = 4$, $|\Omega_2| = 4$, $|\Omega_3| = 4$, $|\Omega_4| = 4$, $|\Omega_5| = 128$, $|\Omega_6| = 128$, $|\Omega_7| = 128$, $|\Omega_8| = 128$, $|\Omega_9| = 128$, $|\Omega_{10}| = 128$ and $|\Omega_{11}| = 240$. We compute the respective f-values f_i using $f_i = \frac{k|\Omega_i|}{|N|}$ to have $f_1 = 1 = f_2 = f_3 = f_4$, $f_5 = 32 = f_6 = f_7 = f_8 = f_9 = f_{10}$ and $f_{11} = 60$. These f-values satisfy the condition $\sum_{i=1}^{11} f_i = k$. This means that in the analysis of the coset Ng, f_i of the original $k = 256$ orbits have fused together to form a conjugacy class $[x_i]$ of $\overline{G}$. The GAP Programme A in [15] is used to calculate the rest of the f-values for each conjugacy class of $\overline{G}$. This demonstration implies that this class $2A$ produces 11 conjugacy classes of $\overline{G}$. The centralizer sizes of these conjugacy classes are determined by using $|C_{\overline{G}}(x_i)| = \frac{k|C_G(g)|}{f_i}$. Then

$$|C_{\overline{G}}(x_1)| = \frac{k|C_G(g)|}{f_1} = \frac{256 \times 3840}{1} = 983040,$$

$$|C_{\overline{G}}(x_2)| = \frac{k|C_G(g)|}{f_2} = \frac{256 \times 3840}{1} = 983040,$$

$$|C_{\overline{G}}(x_3)| = \frac{k|C_G(g)|}{f_3} = \frac{256 \times 3840}{1} = 983040,$$

$$|C_{\overline{G}}(x_4)| = \frac{k|C_G(g)|}{f_4} = \frac{256 \times 3840}{1} = 983040,$$

$$|C_{\overline{G}}(x_5)| = \frac{k|C_G(g)|}{f_5} = \frac{256 \times 3840}{32} = 30720,$$

$$|C_{\overline{G}}(x_6)| = \frac{k|C_G(g)|}{f_6} = \frac{256 \times 3840}{32} = 30720,$$

$$|C_{\overline{G}}(x_7)| = \frac{k|C_G(g)|}{f_7} = \frac{256 \times 3840}{32} = 30720,$$

$$|C_{\overline{G}}(x_8)| = \frac{k|C_G(g)|}{f_8} = \frac{256 \times 3840}{32} = 30720,$$

$$|C_{\overline{G}}(x_9)| = \frac{k|C_G(g)|}{f_9} = \frac{256 \times 3840}{32} = 30720,$$

$$|C_{\overline{G}}(x_{10})| = \frac{k|C_G(g)|}{f_{10}} = \frac{256 \times 3840}{32} = 30720,$$

and

$$|C_{\overline{G}}(x_{11})| = \frac{k|C_G(g)|}{f_{11}} = \frac{256 \times 3840}{60} = 16384.$$

We proceed to calculate the orders of the class representatives of the conjugacy classes of $\overline{G}$. Let us reconsider the class $2A$ of G for demonstration. Let $g \in 2A$, then $o(g) = 2 = m$. We recall that $\mathrm{Char}(\mathbb{F}) = 2 = p$. Let $d \in N$ and

$$w = d \cdot d^g \cdot d^{g^2} \cdots d^{g^{m-1}}.$$

Now if w is the identity of N, then in this case the order of $x \in \overline{G}$ is $m = 2$. Otherwise, if w is not the identity of N, then $o(x) = pm = 2 \times 2 = 4$. Recall that we have 11 conjugacy classes of $\overline{G}$ from the class $2A$.

Since g is always in the first conjugacy class, then $w = 1_N$ and hence $o(x_1) = 2$. In the second, third and fourth classes we have $d = (0, 0, 0, 0, 1)$, $d = (0, 0, 0, 0, z^1)$ and $d = (0, 0, 0, 0, z^2)$ respectively and in each case $w = (0, 0, 0, 0, 0) = 1_N$ thus $o(x_2) = 2 = o(x_3) = o(x_4)$.

In the fifth class we have $d = (z, z, 1, 0, 0)$ and $w = (z, z, 1, 1, z) \neq 1_N$ and therefore $o(x_5) = 4$.

In the sixth class we have $d = (z, z, 1, 0, 1)$ and $w = (z, z, 1, 1, z)$ and thus $o(x_6) = 4$.

In the seventh class $d = (z^2, z^2, z, 0, 0)$ and $w = (z^2, z^2, z, z, 1)$ and therefore $o(x_7) = 4$.

The eighth class has $d = (z^2, z^2, z, 0, z)$ and $w = (z^2, z^2, z, z, 1)$ and it follows that $o(x_8) = 4$.

The ninth class has $d = (1, 1, z^2, 0, 0)$ and $w = (1, 1, z^2, z^2, z^2)$ and it then follows that $o(x_9) = 4$.

While the tenth class has $d = (1, 1, z^2, 0, 1)$ and $w = (1, 1, z^2, z^2, z^2)$ and hence $o(x_{10}) = 4$.

Lastly the eleventh class has $d = (0, 0, 1, 1, 0)$ and $w = (0, 0, 0, 0, 0)$ and therefore $o(x_{11}) = 2$.

The size of each class is determined by $|[x_i]| = \frac{|\overline{G}|}{|C_{\overline{G}(x_i)}|}$. All 165 conjugacy classes of $\overline{G}$ are listed in Table 8.2 below. It then follows that $\overline{G}$ will have 165 irreducible characters.

Table 8.2: The conjugacy classes of elements of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
1A	1024	1	1002700800	$(0,0,0,0,0)$	$(0,0,0,0,0)$	1A
		1	1002700800	$(0,0,0,0,1)$	$(0,0,0,0,1)$	2A
	1024	1	1002700800	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	2B
		1	1002700800	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	2C
		255	3932160	$(0,0,0,1,0)$	$(0,0,0,1,0)$	2D
		255	3932160	$(0,0,0,1,1)$	$(0,0,0,1,1)$	2E
		255	3932160	$(0,0,0,1,z^1)$	$(0,0,0,1,z^1)$	2F
		255	3932160	$(0,0,0,1,z^2)$	$(0,0,0,1,z^2)$	2G
2A	256	1	983040	$(0,0,0,0,0)$	$(0,0,0,0,0)$	2H
		1	983040	$(0,0,0,0,1)$	$(0,0,0,0,0)$	2I
		1	983040	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	2J
	256	1	983040	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	2K
		32	30720	$(z^1,z^1,1,0,0)$	$(z^1,z^1,1,1,z^1)$	4A
		32	30720	$(z^1,z^1,1,0,1)$	$(z^1,z^1,1,1,z^1)$	4B
		32	30720	$(z^2,z^2,z^1,0,0)$	$(z^2,z^2,z^1,z^1,1)$	4C
		32	30720	$(z^2,z^2,z^1,0,z^1)$	$(z^2,z^2,z^1,z^1,1)$	4D
		32	30720	$(1,1,z^2,0,0)$	$(1,1,z^2,z^2,z^2)$	4E
		32	30720	$(1,1,z^2,0,1)$	$(1,1,z^2,z^2,z^2)$	4F
		60	16384	$(0,0,1,1,0)$	$(0,0,0,0,0)$	2L
2B	64	1	245760	$(0,0,0,0,0)$	$(0,0,0,0,0)$	2M
		1	245760	$(0,0,0,0,1)$	$(0,0,0,0,0)$	2N
		1	245760	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	2O
		1	245760	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	2P
		60	4096	$(0,z^2,z^1,z^1,0)$	$(0,z^2,z^1,z^2,0)$	4G
2C	64	1	16384	$(0,0,0,0,0)$	$(0,0,0,0,0)$	2Q
		1	16384	$(0,0,0,0,1)$	$(0,0,0,0,0)$	2R
		1	16384	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	2S
		1	16384	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	2T
	64	2	8192	$(0,z^2,1,1,0)$	$(0,z^2,1,0,0)$	4H
		2	8192	$(0,z^2,1,1,1)$	$(0,z^2,1,0,0)$	4I
		2	8192	$(0,1,z^1,z^1,0)$	$(0,1,z^1,0,0)$	4J
		2	8192	$(0,1,z^1,z^1,z^1)$	$(0,1,z^1,0,0)$	4K
		2	8192	$(0,z^1,z^2,z^2,0)$	$(0,z^1,z^2,0,0)$	4L
		2	8192	$(0,z^1,z^2,z^2,1)$	$(0,z^1,z^2,0,0)$	4M
		16	1024	$(z^2,1,z^2,z^1,0)$	$(z^2,1,z^1,z^1,1)$	4N
		16	1024	$(1,z^1,1,z^2,0)$	$(1,z^1,z^2,z^2,z^2)$	4O
		16	1024	$(z^1,z^2,z^1,1,0)$	$(z^1,z^2,1,1,z^1)$	4P
3A	64	1	11520	$(0,0,0,0,0)$	$(0,0,0,0,0)$	3A
		1	11520	$(0,0,0,0,1)$	$(0,0,0,0,1)$	6A
		1	11520	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	6B
	64	1	11520	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	6C
		15	768	$(z^2,1,1,z^1,0)$	$(0,z^1,1,1,z^2)$	6D
		15	768	$(z^2,1,1,z^1,1)$	$(0,z^1,1,1,z^1)$	6E
		15	768	$(z^2,1,1,z^1,z^1)$	$(0,z^1,1,1,1)$	6F
		15	768	$(z^2,1,1,z^1,z^2)$	$(0,z^1,1,1,0)$	6G
3B	4	1	720	$(0,0,0,0,0)$	$(0,0,0,0,0)$	3B
		1	720	$(0,0,0,0,1)$	$(0,0,0,0,1)$	6H
		1	720	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	6I
		1	720	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	6J
4A	16	1	512	$(0,0,0,0,0)$	$(0,0,0,0,0)$	4Q
		1	512	$(0,0,0,0,1)$	$(0,0,0,0,0)$	4R
	16	1	512	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	4S
		1	512	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	4T
		4	128	$(z^2,z^2,0,0,0)$	$(0,1,z^1,0,0)$	8A
		4	128	$(1,1,0,0,0,)$	$(0,z^1,z^2,0,0)$	8B
		4	128	$(z^1,z^1,0,0,0,)$	$(0,z^2,1,0,0)$	8C

Table 8.2: The conjugacy classes of elements of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
4B	16	1	512	$(0,0,0,0,0)$	$(0,0,0,0,0)$	4U
		1	512	$(0,0,0,0,1)$	$(0,0,0,0,0)$	4V
		1	512	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	4W
	16	1	512	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	4X
		4	128	$(1,1,z^2,1,0)$	$(0,1,z^1,0,0)$	8D
		4	128	$(z^1,z^1,1,z^1,0)$	$(0,z^1,z^2,0,0)$	8E
		4	128	$(z^2,z^2,z^1,z^2,0)$	$(0,z^2,1,0,0)$	8F
5A	64	1	19200	$(0,0,0,0,0)$	$(0,0,0,0,0)$	5A
		1	19200	$(0,0,0,0,1)$	$(0,0,0,0,1)$	10A
		1	19200	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	10B
		1	19200	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	10C
		15	1280	$(z^1,z^2,z^1,1,0)$	$(0,z^1,1,0,1)$	10D
	64	15	1280	$(z^1,z^2,z^1,1,1)$	$(0,z^1,1,0,0)$	10E
		15	1280	$(z^1,z^2,z^1,1,z^1)$	$(0,z^1,1,0,z^2)$	10F
		15	1280	$(z^1,z^2,z^1,1,z^2)$	$(0,z^1,1,0,z^1)$	10G
5B	64	1	19200	$(0,0,0,0,0)$	$(0,0,0,0,0)$	5B
		1	19200	$(0,0,0,0,1)$	$(0,0,0,0,1)$	10H
		1	19200	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	10I
		1	19200	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	10J
	64	15	1280	$(z^2,z^2,z^1,0,0)$	$(0,z^1,1,0,1)$	10K
		15	1280	$(z^2,z^2,z^1,0,1)$	$(0,z^1,1,0,0)$	10L
		15	1280	$(z^2,z^2,z^1,0,z^1)$	$(0,z^1,1,0,z^2)$	10M
		15	1280	$(z^2,z^2,z^1,0,z^2)$	$(0,z^1,1,0,z^1)$	10N
5C	4	1	1200	$(0,0,0,0,0)$	$(0,0,0,0,0)$	5C
		1	1200	$(0,0,0,0,1)$	$(0,0,0,0,1)$	10O
		1	1200	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	10P
		1	1200	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	10Q
5D	4	1	1200	$(0,0,0,0,0)$	$(0,0,0,0,0)$	5D
		1	1200	$(0,0,0,0,1)$	$(0,0,0,0,1)$	10R
		1	1200	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	10S
		1	1200	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	10T
5E	4	1	100	$(0,0,0,0,0)$	$(0,0,0,0,0)$	5E
		1	100	$(0,0,0,0,1)$	$(0,0,0,0,1)$	10U
		1	100	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	10V
		1	100	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	10W
6A	16	1	192	$(0,0,0,0,0)$	$(0,0,0,0,0)$	6K
		1	192	$(0,0,0,0,1)$	$(0,0,0,0,0)$	6L
		1	192	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	6M
		1	192	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	6N
		2	96	$(1,0,z^2,1,0)$	$(1,0,z^2,1,0)$	12A
	16	2	96	$(1,0,z^2,1,z^1)$	$(1,0,z^2,1,z^1)$	12B
		2	96	$(z^1,0,1,z^1,0)$	$(z^1,0,1,z^1,0)$	12C
		2	96	$(z^1,0,1,z^1,1)$	$(z^1,0,1,z^1,1)$	12D
		2	96	$(z^2,0,z^1,z^2,0)$	$(z^2,0,z^1,z^2,0)$	12E
		2	96	$(z^2,0,z^1,z^2,1)$	$(z^2,0,z^1,z^2,1)$	12F
6B	4	1	48	$(0,0,0,0,0)$	$(0,0,0,0,0)$	6O
		1	48	$(0,0,0,0,1)$	$(0,0,0,0,0)$	6P
		1	48	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	6Q
		1	48	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	6R
10A	16	1	320	$(0,0,0,0,0)$	$(0,0,0,0,0)$	10X
		1	320	$(0,0,0,0,1)$	$(0,0,0,0,0)$	10Y
		1	320	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	10Z
		1	320	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	10AA
	16	2	160	$(0,z^1,0,0,0)$	$(z^1,z^1,1,1,z^1)$	20A
		2	160	$(0,z^1,0,0,1)$	$(z^1,z^1,1,1,z^1)$	20B
		2	160	$(0,z^2,0,0,0)$	$(z^2,z^2,z^1,z^1,1)$	20C
		2	160	$(0,z^2,0,0,z^1)$	$(z^2,z^2,z^1,z^1,1)$	20D
		2	160	$(0,1,0,0,0)$	$(1,1,z^2,z^2,z^2)$	20E
		2	160	$(0,1,0,0,1)$	$(1,1,z^2,z^2,z^2)$	20F

Table 8.2: The conjugacy classes of elements of $\overline{G}$

$[g]_G$	k	f_i	$\lvert C_{\overline{G}}(x)\rvert$	d_i	w	$[x]_{\overline{G}}$
10B	16	1	320	$(0,0,0,0,0)$	$(0,0,0,0,0)$	10AB
		1	320	$(0,0,0,0,1)$	$(0,0,0,0,0)$	10AC
		1	320	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	10AD
		1	320	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	10AE
	16	2	160	$(z^2,z^2,1,0,0)$	$(z^1,z^1,1,1,z^1)$	20G
		2	160	$(z^2,z^2,1,0,1)$	$(z^1,z^1,1,1,z^1)$	20H
		2	160	$(1,1,z^1,0,0)$	$(z^2,z^2,z^1,z^1,1)$	20I
		2	160	$(1,1,z^1,0,z^1)$	$(z^2,z^2,z^1,z^1,1)$	20J
		2	160	$(z^1,z^1,z^2,0,0)$	$(1,1,z^2,z^2,z^2)$	20K
		2	160	$(z^1,z^1,z^2,0,1)$	$(1,1,z^2,z^2,z^2)$	20L
10C	4	1	80	$(0,0,0,0,0)$	$(0,0,0,0,0)$	10AF
		1	80	$(0,0,0,0,1)$	$(0,0,0,0,0)$	10AG
		1	80	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	10AH
		1	80	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	10AI
10D	4	1	80	$(0,0,0,0,0)$	$(0,0,0,0,0)$	10AJ
		1	80	$(0,0,0,0,1)$	$(0,0,0,0,0)$	10AK
		1	80	$(0,0,0,0,z^1)$	$(0,0,0,0,0)$	10AL
		1	80	$(0,0,0,0,z^2)$	$(0,0,0,0,0)$	10AM
15A	4	1	60	$(0,0,0,0,0)$	$(0,0,0,0,0)$	15A
		1	60	$(0,0,0,0,1)$	$(0,0,0,0,1)$	30A
		1	60	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	30B
		1	60	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	30C
15B	4	1	60	$(0,0,0,0,0)$	$(0,0,0,0,0)$	15B
		1	60	$(0,0,0,0,1)$	$(0,0,0,0,1)$	30D
		1	60	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	30E
		1	60	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	30F
15C	4	1	60	$(0,0,0,0,0)$	$(0,0,0,0,0)$	15C
		1	60	$(0,0,0,0,1)$	$(0,0,0,0,1)$	30G
		1	60	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	30H
		1	60	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	30I
15D	4	1	60	$(0,0,0,0,0)$	$(0,0,0,0,0)$	15D
		1	60	$(0,0,0,0,1)$	$(0,0,0,0,1)$	30J
		1	60	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	30K
		1	60	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	30L
17A	4	1	68	$(0,0,0,0,0)$	$(0,0,0,0,0)$	17A
		1	68	$(0,0,0,0,1)$	$(0,0,0,0,1)$	34A
		1	68	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	34B
		1	68	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	34C
17B	4	1	68	$(0,0,0,0,0)$	$(0,0,0,0,0)$	17B
		1	68	$(0,0,0,0,1)$	$(0,0,0,0,1)$	34D
		1	68	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	34E
		1	68	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	34F
17C	4	1	68	$(0,0,0,0,0)$	$(0,0,0,0,0)$	17C
		1	68	$(0,0,0,0,1)$	$(0,0,0,0,1)$	34G
		1	68	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	34H
		1	68	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	34I
17D	4	1	68	$(0,0,0,0,0)$	$(0,0,0,0,0)$	17D
		1	68	$(0,0,0,0,1)$	$(0,0,0,0,1)$	34J
		1	68	$(0,0,0,0,z^1)$	$(0,0,0,0,z^1)$	34K
		1	68	$(0,0,0,0,z^2)$	$(0,0,0,0,z^2)$	34L

8.4 Inertia factor groups into the symplectic group $Sp(4,4)$

We saw in Section 8.3 that the action the action of G on N yielded 8 orbits. Now the action of G on $Irr(N)$ will again yield 8 orbits. The lengths of these new orbits are 1, 255, 136, 136, 136, 120, 120 and 120. We have that the inertia factor groups are $H_1 = G$, $H_2 \cong 2^6{:}Sp(2,4)$, H_3, H_4, H_5 being isomorphic to the full orthogonal group $GO^+(4,4)$ and H_6, H_7, H_8 being isomorphic to the full orthogonal group $GO^-(4,4)$. Mainly in this section we consider the fusion of these inertia factor groups into the symplectic group G. The subgroups $GO^+(4,4)$ and $GO^-(4,4)$ are maximal in G according to the ATLAS and are of orders 7200 and 8160 respectively. We use GAP to compute their generators and eventually to generate their respective character tables. The fusion of $2^6{:}Sp(2,4)$ into G was done in Section 7.8. And so we deal with the fusions of $GO^+(4,4)$ and $GO^-(4,4)$ using the divisibility of the respective centralizer sizes $|C_G(g)|$ and $|C_{H_i}(h_i)|$, where g and h_i are the class representatives of conjugacy classes of G and H_i respectively, with $o(g) = o(h_i)$ and $i \in \{2,3,4,5,6,7,8\}$. Respective permutation characters $\chi(G|H_i)$ are used in aid of each fusion. Since the degree of $\chi(G|H_2)$ is 255, the degree of $\chi(G|GO^+(4,4))$ is 136 and $\chi(G|GO^-(4,4))$ is 120, we have

$$\chi(G|H_2) = 3\cdot 1a + 3\cdot 34a + 3\cdot 50a,$$

$$\chi(G|GO^+(4,4)) = 1a + 50a + 85a$$

and

$$\chi(G|GO^-(4,4)) = 1a + 34a + 85a,$$

where $1a$, $34a$, $50a$ and $85a$ are irreducible characters of G. If there is more than one conjugacy class $[g]$ and / or $[h_i]$ such that $o(g) = o(h)$ and the quotient $\frac{|C_G(g)|}{|C_{H_i}(h_i)|}$ corresponds with the permutation character $\chi(G|H_i)$, then we consider the pairwise intersections $[g] \cap [h_i]$. Since the conjugacy classes are disjoint, then only one intersection will be non-empty. From this intersection we obtain the suitable candidate for the fusion.

Table 8.3: The fusion of $GO^+(4,4)$ into G

	$[g]$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$5B$	$5C$
	$\|C_{Sp(4,4)}(g)\|$	979200	3840	3840	256	180	180	32	32	300	300	300
$[x_i]$	$\|C_{GO^+(4,4)}(x)\|$											
$1A$	7200	$\boxed{136}$										
$2A$	32		120	120	$\boxed{8}$							
$2B$	240		16	$\boxed{16}$								
$2C$	120		$\boxed{32}$	32								
$3A$	18					$\boxed{10}$	10					
$3B$	180					1	$\boxed{1}$					
$4A$	8							4	$\boxed{4}$			
$5A$	50									6	$\boxed{6}$	6
$5B$	50									$\boxed{6}$	6	6
$5C$	300									1	1	$\boxed{1}$
$5D$	300									1	1	1
$5E$	25									12	12	12
	$\chi(Sp(4,4)\|GO^+(4,4))$	136	32	16	8	10	1	0	4	6	6	1

Table 8.3: The fusion of $GO^+(4,4)$ into G

	$[g]$	$5D$	$5E$	$6A$	$6B$	$10A$	$10B$	$10C$	$10D$	$15A$	$15B$	$15C$	$15D$
	$\|C_G(g)\|$	300	25	12	12	20	20	20	20	15	15	15	15
$[x_i]$	$\|C_{GO^+(4,4)}(x)\|$												
$5D$	300	$\boxed{1}$											
$5E$	25	12	$\boxed{1}$										
$6A$	12			1	$\boxed{1}$								
$6B$	6			$\boxed{2}$	2								
$10A$	20					1	1	1	$\boxed{1}$				
$10B$	20					1	1	$\boxed{1}$	1				
$10C$	10					2	$\boxed{2}$	2	2				
$10D$	10					$\boxed{2}$	2	2	2				
$15A$	15									1	1	1	$\boxed{1}$
$15B$	15									1	1	$\boxed{1}$	1
	$\chi(Sp(4,4)\|GO^+(4,4))$	1	1	2	1	2	2	1	1	0	0	1	1

Table 8.4: The fusion of $GO^{-}(4,4)$ into G

	$[g]$	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$4B$	$5A$	$5B$	$5C$
	$\lvert C_{Sp(4,4)}(g)\rvert$	979200	3840	3840	256	180	180	32	32	300	300	300
$[x_i]$	$\lvert C_{GO^{-}(4,4)}(x)\rvert$											
$1A$	8160	120										
$2A$	32		120	120	8							
$2B$	120		32	32								
$3A$	30					6	6					
$4A$	8							4	4			
$5A$	30									10	10	10
$5B$	30									10	10	10
	$\chi(Sp(4,4)\lvert GO^{-}(4,4))$	120	32	0	8	6	0	4	0	10	10	0

Table 8.4: The fusion of $GO^{-}(4,4)$ into G

	$[g]$	$5D$	$5E$	$6A$	$6B$	$10A$	$10B$	$10C$	$10D$	$15A$	$15B$	$15C$	$15D$
	$\lvert C_G(g)\rvert$	300	25	12	12	20	20	20	20	15	15	15	15
$[x_i]$	$\lvert C_{GO^{-}(4,4)}(x)\rvert$												
$6A$	6			2	2								
$10A$	10					2	2	2	2				
$10B$	10					2	2	2	2				
$15A$	15									1	1	1	1
$15B$	15									1	1	1	1
	$\chi(Sp(4,4)\lvert GO^{-}(4,4))$	0	0	2	0	2	2	0	0	1	1	0	0

Table 8.4: The fusion of $GO^{-}(4,4)$ into G

	$[g]$	$17A$	$17B$	$17C$	$17D$
	$\lvert C_G(g)\rvert$	17	17	17	17
$[x_i]$	$\lvert C_{GO^{-}(4,4)}(x)\rvert$				
$17A$	17	1	1	1	1
$17B$	17	1	1	1	1
$17C$	17	1	1	1	1
$17D$	17	1	1	1	1
	$\chi(Sp(4,4)\lvert GO^{-}(4,4))$	1	1	1	1

Table 8.5: The fusion of $2^6{:}Sp(2,4)$, $GO^+(4,4)$ and $GO^-(4,4)$ into G

$2^6{:}Sp(2,4)$	$GO^+(4,4)$	$GO^-(4,4)$	$\longrightarrow\ Sp(4,4)$	$2^6{:}Sp(2,4)$	$GO^+(4,4)$	$GO^-(4,4)$	$\longrightarrow\ Sp(4,4)$
$1A$	$1A$	$1A$	$1A$	$2A$ $2B$ $2C$ $2I$	$2C$	$2B$	$2A$
$2D$	$2B$		$2B$	$2E$ $2F$ $2G$ $2H$ $2J$ $2K$	$2A$	$2A$	$2C$
$3A$	$3A$	$3A$	$3A$		$3B$		$3B$
$4A$ $4D$ $4E$		$4A$	$4A$	$4B$ $4C$ $4F$	$4A$		$4B$
$5A$	$5B$	$5A$	$5A$	$5B$	$5A$	$5B$	$5B$
	$5C$		$5C$		$5D$		$5D$
	$5E$		$5E$	$6A$ $6B$ $6C$	$6B$	$6A$	$6A$
	$6A$		$6B$	$10A$ $10B$ $10C$	$10D$	$10A$	$10A$
$10D$ $10E$ $10F$	$10C$	$10B$	$10B$		$10B$		$10C$
	$10A$		$10D$			$15B$	$15A$
		$15A$	$15B$		$15B$		$15C$
	$15A$		$15D$			$17A$	$17A$
		$17B$	$17B$			$17C$	$17C$
		$17D$	$17D$				

8.5 Fischer matrices of $\overline{G}$

In this section we compute the Fischer matrices of $\overline{G}$. Since N is elementary abelian and $\overline{G}$ is a split extension of N by G, then by Mackey's Theorem the $Irr(N)$ are extendable to the inertia factor groups. Now due to the Clifford-Fischer Theory these Fischer matrices, together with the character tables of the inertia factor groups, will enable us to construct the character table of $\overline{G}$ in Section 8.6. Since G has 27 conjugacy classes, then $\overline{G}$ has 27 Fischer matrices corresponding to each class representative g of G. Each Fischer matrix $M(g)$ is divided into blocks corresponding to an inertia factor group. These Fischer matrices satisfy the column and row orthogonality relations. We use the general form of a Fischer matrix to construct these matrices. Let us consider the class 2A of G to demonstrate the construction and computation of a Fischer matrix. From Table 8.2 we are able to deduce the size of each Fischer matrix. The Fischer matrix corresponding to this class is a 11×11 matrix. The fusion maps computed in

Section 8.4 play a vital role in the construction of a Fischer matrix. We note that the classes 2A, 2B , 2C, 2H of H_2, the class 2C of $GO^+(4,4)$ and the class 2B of $GO^-(4,4)$ fuse to the class 2A of G. We, however, recall that the action of G on $Irr(N)$ yielded 3 copies of $GO^+(4,4)$ and $GO^-(4,4)$ respectively. This then implies that the classes 2C in H_3, H_4, H_5 and the classes 2B in H_6, H_7 and H_8 fuse into the 2A of G, respectively. The Fischer matrix from the class 2A is then of the form:

Table 8.6: Fischer matrix from 2A

$\lvert C_{\overline{G}}(x_{2j})\rvert$	983040	983040	983040	983040	30720	30720	30720	30720	30720	30720	16384
$\lvert C_{H_k}(x_{2km})\rvert$											
3840	1	1	1	1	1	1	1	1	1	1	1
3840	1	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$	$a_{2,8}$	$a_{2,9}$	$a_{2,10}$	$a_{2,11}$
3840	1	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$	$a_{3,8}$	$a_{3,9}$	$a_{3,10}$	$a_{3,11}$
3840	1	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$	$a_{4,8}$	$a_{4,9}$	$a_{4,10}$	$a_{4,11}$
64	60	$a_{5,2}$	$a_{5,3}$	$a_{5,4}$	$a_{5,5}$	$a_{5,6}$	$a_{5,7}$	$a_{5,8}$	$a_{5,9}$	$a_{5,10}$	$a_{5,11}$
120	32	$a_{6,2}$	$a_{6,3}$	$a_{6,4}$	$a_{6,5}$	$a_{6,6}$	$a_{6,7}$	$a_{6,8}$	$a_{6,9}$	$a_{6,10}$	$a_{6,11}$
120	32	$a_{7,2}$	$a_{7,3}$	$a_{7,4}$	$a_{7,5}$	$a_{7,6}$	$a_{7,7}$	$a_{7,8}$	$a_{7,9}$	$a_{7,10}$	$a_{7,11}$
120	32	$a_{8,2}$	$a_{8,3}$	$a_{8,4}$	$a_{8,5}$	$a_{8,6}$	$a_{8,7}$	$a_{8,8}$	$a_{8,9}$	$a_{8,10}$	$a_{8,11}$
120	32	$a_{9,2}$	$a_{9,3}$	$a_{9,4}$	$a_{9,5}$	$a_{9,6}$	$a_{9,7}$	$a_{9,8}$	$a_{9,9}$	$a_{9,10}$	$a_{9,11}$
120	32	$a_{10,2}$	$a_{10,3}$	$a_{10,4}$	$a_{10,5}$	$a_{10,6}$	$a_{10,7}$	$a_{10,8}$	$a_{10,9}$	$a_{10,10}$	$a_{10,11}$
120	32	$a_{11,2}$	$a_{11,3}$	$a_{11,4}$	$a_{11,5}$	$a_{11,6}$	$a_{11,7}$	$a_{11,8}$	$a_{11,9}$	$a_{11,10}$	$a_{11,11}$
m_{2j}	4	4	4	4	128	128	128	128	128	128	240

The centralizer orders in the first row are from the coset analysis process in Section 8.3. The values of the centralizer orders of the first column are from the fusion tables in Section 8.4. Then, since N is elementary abelian, the second column values are the respective indices in $C_{H_1}(2A)$, where $|C_{H_1}(2A)| = 3840$. The weights m_{2j} are obtained from $m_{ij} = \frac{f_i|N|}{k} = \frac{f_i \cdot 1024}{256}$, for $f_i \in \{1,1,1,1,32,32,32,32,32,32,60\}$ from Table 8.2. To compute the entries $a_{i,j}$ we use the orthogonality relations of a Fischer matrix. The column orthogonality relations yield the following 22 equations:

$$\sum_{i=2}^{11} 3840 \cdot a_{i,2} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,2})^2 + 64 \cdot (a_{5,2})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,2})^2 = 983040,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,3} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,3})^2 + 64 \cdot (a_{5,3})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,3})^2 = 983040,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,4} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,4})^2 + 64 \cdot (a_{5,4})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,4})^2 = 983040,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,5} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,5})^2 + 64 \cdot (a_{5,5})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,5})^2 = 30720,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,6} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,6})^2 + 64 \cdot (a_{5,6})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,6})^2 = 30720,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,7} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,7})^2 + 64 \cdot (a_{5,7})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,7})^2 = 30720,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,8} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,8})^2 + 64 \cdot (a_{5,8})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,8})^2 = 30720,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,9} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,9})^2 + 64 \cdot (a_{5,9})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,9})^2 = 30720,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,10} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,10})^2 + 64 \cdot (a_{5,10})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,10})^2 = 30720,$$

$$\sum_{i=2}^{11} 3840 \cdot a_{i,11} = 0,$$

$$3840 + \sum_{i=2}^{4} 3840 \cdot (a_{i,11})^2 + 64 \cdot (a_{5,11})^2 + \sum_{i=6}^{11} 120 \cdot (a_{i,11})^2 = 16384.$$

From the row orthogonality relations we obtain the following 22 equations:

$$4+4\sum_{j=2}^{4}a_{2,j}+128\sum_{j=5}^{10}a_{2,j}+240\cdot a_{2,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{2,j})^2+128\sum_{j=5}^{10}(a_{2,j})^2+240\cdot (a_{2,11})^2=1024,$$

$$4+4\sum_{j=2}^{4}a_{3,j}+128\sum_{j=5}^{10}a_{3,j}+240\cdot a_{3,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{3,j})^2+128\sum_{j=5}^{10}(a_{3,j})^2+240\cdot (a_{3,11})^2=1024,$$

$$4+4\sum_{j=2}^{4}a_{4,j}+128\sum_{j=5}^{10}a_{4,j}+240\cdot a_{4,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{4,j})^2+128\sum_{j=5}^{10}(a_{4,j})^2+240\cdot (a_{4,11})^2=1024,$$

$$4+4\sum_{j=2}^{4}a_{5,j}+128\sum_{j=5}^{10}a_{5,j}+240\cdot a_{5,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{5,j})^2+128\sum_{j=5}^{10}(a_{5,j})^2+240\cdot (a_{5,11})^2=61440,$$

$$4+4\sum_{j=2}^{4}a_{6,j}+128\sum_{j=5}^{10}a_{6,j}+240\cdot a_{6,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{6,j})^2+128\sum_{j=5}^{10}(a_{6,j})^2+240\cdot (a_{6,11})^2=32768,$$

$$4+4\sum_{j=2}^{4}a_{7,j}+128\sum_{j=5}^{10}a_{7,j}+240\cdot a_{7,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{7,j})^2+128\sum_{j=5}^{10}(a_{7,j})^2+240\cdot (a_{7,11})^2=32768,$$

$$4+4\sum_{j=2}^{4}a_{8,j}+128\sum_{j=5}^{10}a_{8,j}+240\cdot a_{8,11}=0,$$

$$4+4\sum_{j=2}^{4}(a_{8,j})^2+128\sum_{j=5}^{10}(a_{8,j})^2+240\cdot (a_{8,11})^2=32768,$$

$$4 + 4\sum_{j=2}^{4} a_{9,j} + 128\sum_{j=5}^{10} a_{9,j} + 240 \cdot a_{9,11} = 0,$$

$$4 + 4\sum_{j=2}^{4}(a_{9,j})^2 + 128\sum_{j=5}^{10}(a_{9,j})^2 + 240 \cdot (a_{9,11})^2 = 32768,$$

$$4 + 4\sum_{j=2}^{4} a_{10,j} + 128\sum_{j=5}^{10} a_{10,j} + 240 \cdot a_{10,11} = 0,$$

$$4 + 4\sum_{j=2}^{4}(a_{10,j})^2 + 128\sum_{j=5}^{10}(a_{10,j})^2 + 240 \cdot (a_{10,11})^2 = 32768,$$

$$4 + 4\sum_{j=2}^{4} a_{11,j} + 128\sum_{j=5}^{10} a_{11,j} + 240 \cdot a_{11,11} = 0,$$

$$4 + 4\sum_{j=2}^{4}(a_{11,j})^2 + 128\sum_{j=5}^{10}(a_{11,j})^2 + 240 \cdot (a_{11,11})^2 = 32768.$$

We utilize GAP or the Maxima Algebraic software to solve these equations. Suitable entries are then chosen together with Remark 3.3.7. This process eventually yields the following Fischer matrix from the class 2A.

Table 8.7: Fischer matrix from 2A

$\lvert C_{\overline{G}}(x_{2j})\rvert$	983040	983040	983040	983040	30720	30720	30720	30720	30720	30720	16384
$\lvert C_{H_k}(x_{2km})\rvert$											
3840	1	1	1	1	1	1	1	1	1	1	1
3840	1	1	1	1	-1	-1	-1	-1	1	1	1
3840	1	1	1	1	-1	-1	1	1	-1	-1	1
3840	1	1	1	1	1	1	-1	-1	-1	-1	1
64	60	60	60	60	0	0	0	0	0	0	-4
120	32	-32	-32	32	0	0	0	0	-8	8	0
120	32	-32	32	-32	-8	8	0	0	0	0	0
120	32	32	-32	-32	0	0	-8	8	0	0	0
120	32	-32	-32	32	0	0	0	0	8	-8	0
120	32	-32	32	-32	8	-8	0	0	0	0	0
120	32	32	-32	-32	0	0	8	-8	0	0	0
m_{2j}	4	4	4	4	128	128	128	128	128	128	240

The rest of the Fischer matrices are computed by using Programmes A and B in [15]. These are listed in Table 8.8 below.

Table 8.8: Fischer matrices of $\overline{G}$

$\mathbb{F}_1 = M(1A)$	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	$x_{1,6}$	$x_{1,7}$	$x_{1,8}$
$o(x_{1j})$	1	2	2	2	2	2	2	2
$\lvert C_{\overline{G}}(x_{1j})\rvert$	1002700800	1002700800	1002700800	1002700800	3932160	3932160	3932160	3932160
$\lvert C_{H_k}(x_{1km})\rvert$								
979200	1	1	1	1	1	1	1	1
3840	255	255	255	255	-1	-1	-1	-1
7200	136	-136	-136	136	8	-8	-8	8
7200	136	-136	136	-136	8	-8	8	-8
7200	136	136	-136	-136	8	8	-8	-8
8160	120	-120	-120	120	-8	8	8	-8
8160	120	-120	120	-120	-8	8	-8	8
8160	120	120	-120	-120	-8	-8	8	8
m_{1j}	1	1	1	1	255	255	255	255

$\mathbb{F}_2 = M(2A)$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$	$x_{2,11}$
$o(x_{2j})$	2	2	2	2	4	4	4	4	4	4	2
$\lvert C_{\overline{G}}(x_{2j})\rvert$	983040	983040	983040	983040	30720	30720	30720	30720	30720	30720	16384
$\lvert C_{H_k}(x_{2km})\rvert$											
3840	1	1	1	1	1	1	1	1	1	1	1
3840	1	1	1	1	-1	-1	-1	-1	1	1	1
3840	1	1	1	1	-1	-1	1	1	-1	-1	1
3840	1	1	1	1	1	1	-1	-1	-1	-1	1
64	60	60	60	60	0	0	0	0	0	0	-4
120	32	-32	-32	32	0	0	0	0	-8	8	0
120	32	-32	32	-32	-8	8	0	0	0	0	0
120	32	32	-32	-32	0	0	-8	8	0	0	0
120	32	-32	-32	32	0	0	0	0	8	-8	0
120	32	-32	32	-32	8	-8	0	0	0	0	0
120	32	32	-32	-32	0	0	8	-8	0	0	0
m_{2j}	4	4	4	4	128	128	128	128	128	128	240

Table 8.8: Fischer matrices of $\overline{G}$

$\mathbb{F}_3 = M(2B)$	$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$	$x_{3,5}$
$o(x_{1j})$	2	2	2	2	4
$\|C_{\overline{G}}(x_{1j})\|$	245760	245760	245760	245760	4096
$\|C_{H_k}(x_{1km})\|$					
3840	1	1	1	1	1
256	15	15	15	15	-1
240	16	-16	-16	16	0
240	16	-16	16	-16	0
240	16	16	-16	-16	0
m_{3j}	16	16	16	16	960

$\mathbb{F}_4 = M(2C)$	$x_{4,1}$	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$	$x_{4,5}$	$x_{4,6}$	$x_{4,7}$	$x_{4,8}$	$x_{4,9}$	$x_{4,10}$	$x_{4,11}$	$x_{4,12}$	$x_{4,13}$
$o(x_{2j})$	2	2	2	2	4	4	4	4	4	4	4	4	4
$\|C_{\overline{G}}(x_{2j})\|$	16384	16384	16384	16384	1024	1024	1024	8192	8192	8192	8192	8192	8192
$\|C_{H_k}(x_{2km})\|$													
256	1	1	1	1	1	1	1	1	1	1	1	1	1
256	1	1	1	1	-1	-1	1	1	1	1	1	1	1
256	1	1	1	1	-1	1	-1	1	1	1	1	1	1
256	1	1	1	1	1	-1	-1	1	1	1	1	1	1
64	4	4	4	4	0	0	0	-4	-4	4	4	-4	-4
64	4	4	4	4	0	0	0	-4	-4	-4	-4	4	4
64	4	4	4	4	0	0	0	4	4	-4	-4	-4	-4
32	8	-8	-8	8	0	0	0	-8	8	0	0	0	0
32	8	-8	8	-8	0	0	0	0	0	-8	8	0	0
32	8	8	-8	-8	0	0	0	0	0	0	0	-8	8
32	8	-8	-8	8	0	0	0	8	-8	0	0	0	0
32	8	-8	8	-8	0	0	0	0	0	8	-8	0	0
32	8	8	-8	-8	0	0	0	0	0	0	0	8	-8
m_{4j}	16	16	16	16	256	256	256	32	32	32	32	32	32

$\mathbb{F}_5 = M(3A)$	$x_{5,1}$	$x_{5,2}$	$x_{5,3}$	$x_{5,4}$	$x_{5,5}$	$x_{5,6}$	$x_{5,7}$	$x_{5,8}$
$o(x_{5j})$	3	6	6	6	6	6	6	6
$\|C_{\overline{G}}(x_{5j})\|$	11520	11520	11520	11520	768	768	768	768
$\|C_{H_k}(x_{5km})\|$								
180	1	1	1	1	1	1	1	1
12	15	15	15	15	-1	-1	-1	-1
18	10	-10	-10	10	2	-2	-2	2
18	10	-10	10	-10	2	-2	2	-2
18	10	10	-10	-10	2	2	-2	-2
30	6	-6	-6	6	-2	2	2	-2
30	6	-6	6	-6	-2	2	-2	2
30	6	6	-6	-6	-2	-2	2	2
m_{5j}	16	16	16	16	240	240	240	240

$\mathbb{F}_6 = M(3B)$	$x_{6,1}$	$x_{6,2}$	$x_{6,3}$	$x_{6,4}$
$o(x_{6j})$	3	6	6	6
$\|C_{\overline{G}}(x_{6j})\|$	720	720	720	720
$\|C_{H_k}(x_{6km})\|$				
180	1	1	1	1
180	1	-1	-1	1
180	1	-1	1	-1
180	1	1	-1	-1
m_{6j}	16	16	16	16

Table 8.8: Fischer matrices of $\overline{G}$

$\mathbb{F}_7 = M(4A)$	$x_{7,1}$	$x_{7,2}$	$x_{7,3}$	$x_{7,4}$	$x_{7,5}$	$x_{7,6}$	$x_{7,7}$
$o(x_{7j})$	4	4	4	4	8	8	8
$\|C_{\overline{G}}(x_{7j})\|$	512	512	512	512	128	128	128
$\|C_{H_k}(x_{7km})\|$							
32	1	1	1	1	1	1	1
32	1	1	1	1	-1	-1	1
32	1	1	1	1	-1	1	-1
32	1	1	1	1	1	-1	-1
8	4	-4	-4	4	0	0	0
8	4	-4	4	-4	0	0	0
8	4	4	-4	-4	0	0	0
m_{7j}	64	64	64	64	256	256	256

$\mathbb{F}_8 = M(4B)$	$x_{8,1}$	$x_{8,2}$	$x_{8,3}$	$x_{8,4}$	$x_{8,5}$	$x_{8,6}$	$x_{8,7}$
$o(x_{8j})$	4	4	4	4	8	8	8
$\|C_{\overline{G}}(x_{8j})\|$	512	512	512	512	128	128	128
$\|C_{H_k}(x_{8km})\|$							
32	1	1	1	1	1	1	1
32	1	1	1	1	-1	-1	1
32	1	1	1	1	-1	1	-1
32	1	1	1	1	1	-1	-1
8	4	-4	-4	4	0	0	0
8	4	-4	4	-4	0	0	0
8	4	4	-4	-4	0	0	0
m_{8j}	64	64	64	64	256	256	256

$\mathbb{F}_9 = M(5A)$	$x_{9,1}$	$x_{9,2}$	$x_{9,3}$	$x_{9,4}$	$x_{9,5}$	$x_{9,6}$	$x_{9,7}$	$x_{9,8}$
$o(x_{9j})$	5	10	10	10	10	10	10	10
$\|C_{\overline{G}}(x_{9j})\|$	19200	19200	19200	19200	1280	1280	1280	1280
$\|C_{H_k}(x_{9km})\|$								
300	1	1	1	1	1	1	1	1
20	15	15	15	15	-1	-1	-1	-1
50	6	-6	-6	6	2	-2	-2	2
50	6	-6	6	-6	-2	2	-2	2
50	6	6	-6	-6	2	2	-2	-2
30	10	-10	-10	10	-2	2	2	-2
30	10	-10	10	-10	2	-2	2	-2
30	10	10	-10	-10	-2	-2	2	2
m_{9j}	16	16	16	16	240	240	240	240

$\mathbb{F}_{10} = M(5B)$	$x_{10,1}$	$x_{10,2}$	$x_{10,3}$	$x_{10,4}$	$x_{10,5}$	$x_{10,6}$	$x_{10,7}$	$x_{10,8}$
$o(x_{10j})$	5	10	10	10	10	10	10	10
$\|C_{\overline{G}}(x_{10j})\|$	19200	19200	19200	19200	1280	1280	1280	1280
$\|C_{H_k}(x_{10km})\|$								
300	1	1	1	1	1	1	1	1
20	15	15	15	15	-1	-1	-1	-1
50	6	-6	-6	6	2	-2	-2	2
50	6	-6	6	-6	-2	2	-2	2
50	6	6	-6	-6	2	2	-2	-2
30	10	-10	-10	10	-2	2	2	-2
30	10	-10	10	-10	2	-2	2	-2
30	10	10	-10	-10	-2	-2	2	2
m_{10j}	16	16	16	16	240	240	240	240

Table 8.8: Fischer matrices of $\overline{G}$

$\mathbb{F}_{11} = M(5C)$	$x_{11,1}$	$x_{11,2}$	$x_{11,3}$	$x_{11,4}$
$o(x_{11j})$	5	10	10	10
$\lvert C_{\overline{G}}(x_{11j})\rvert$	1200	1200	1200	1200
$\lvert C_{H_k}(x_{11km})\rvert$				
300	1	1	1	1
300	1	-1	-1	1
300	1	-1	1	-1
300	1	1	-1	-1
m_{11j}	256	256	256	256

$\mathbb{F}_{12} = M(5D)$	$x_{12,1}$	$x_{12,2}$	$x_{12,3}$	$x_{12,4}$
$o(x_{12j})$	5	10	10	10
$\lvert C_{\overline{G}}(x_{12j})\rvert$	1200	1200	1200	1200
$\lvert C_{H_k}(x_{12km})\rvert$				
300	1	1	1	1
300	1	-1	-1	1
300	1	-1	1	-1
300	1	1	-1	-1
m_{12j}	256	256	256	256

$\mathbb{F}_{13} = M(5E)$	$x_{13,1}$	$x_{13,2}$	$x_{13,3}$	$x_{13,4}$
$o(x_{13j})$	5	10	10	10
$\lvert C_{\overline{G}}(x_{13j})\rvert$	100	100	100	100
$\lvert C_{H_k}(x_{13km})\rvert$				
25	1	1	1	1
25	1	-1	-1	1
25	1	-1	1	-1
25	1	1	-1	-1
m_{13j}	256	256	256	256

$\mathbb{F}_{14} = M(2A)$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$	$x_{2,11}$
$o(x_{2j})$	2	2	2	2	4	4	4	4	4	4	2
$\lvert C_{\overline{G}}(x_{2j})\rvert$	983040	983040	983040	983040	30720	30720	30720	30720	30720	30720	16384
$\lvert C_{H_k}(x_{2km})\rvert$											
3840	1	1	1	1	1	1	1	1	1	1	1
3840	1	1	1	1	-1	-1	-1	-1	1	1	1
3840	1	1	1	1	-1	-1	1	1	-1	-1	1
3840	1	1	1	1	1	1	-1	-1	-1	-1	1
64	60	60	60	60	0	0	0	0	0	0	-4
120	32	-32	-32	32	0	0	0	0	-8	8	0
120	32	-32	32	-32	-8	8	0	0	0	0	0
120	32	32	-32	-32	0	0	-8	8	0	0	0
120	32	-32	-32	32	0	0	0	0	8	-8	0
120	32	-32	32	-32	8	-8	0	0	0	0	0
120	32	32	-32	-32	0	0	8	-8	0	0	0
m_{2j}	4	4	4	4	128	128	128	128	128	128	240

$\mathbb{F}_{15} = M(6B)$	$x_{15,1}$	$x_{15,2}$	$x_{15,3}$	$x_{15,4}$
$o(x_{15j})$	6	6	6	6
$\lvert C_{\overline{G}}(x_{15j})\rvert$	48	48	48	48
$\lvert C_{H_k}(x_{15km})\rvert$				
12	1	1	1	1
12	1	-1	-1	1
12	1	-1	1	-1
12	1	1	-1	-1
m_{15j}	256	256	256	256

Table 8.8: Fischer matrices of $\overline{G}$

$\mathbb{F}_{16} = M(2A)$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$	$x_{2,11}$
$o(x_{2j})$	2	2	2	2	4	4	4	4	4	4	2
$\lvert C_{\overline{G}}(x_{2j})\rvert$	983040	983040	983040	983040	30720	30720	30720	30720	30720	30720	16384
$\lvert C_{H_k}(x_{2km})\rvert$											
3840	1	1	1	1	1	1	1	1	1	1	1
3840	1	1	1	1	-1	-1	-1	-1	1	1	1
3840	1	1	1	1	-1	-1	1	1	-1	-1	1
3840	1	1	1	1	1	1	-1	-1	-1	-1	1
64	60	60	60	60	0	0	0	0	0	0	-4
120	32	-32	-32	32	0	0	0	0	-8	8	0
120	32	-32	32	-32	-8	8	0	0	0	0	0
120	32	32	-32	-32	0	0	-8	8	0	0	0
120	32	-32	-32	32	0	0	0	0	8	-8	0
120	32	-32	32	-32	8	-8	0	0	0	0	0
120	32	32	-32	-32	0	0	8	-8	0	0	0
m_{2j}	4	4	4	4	128	128	128	128	128	128	240

$\mathbb{F}_{17} = M(2A)$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$x_{2,5}$	$x_{2,6}$	$x_{2,7}$	$x_{2,8}$	$x_{2,9}$	$x_{2,10}$	$x_{2,11}$
$o(x_{2j})$	2	2	2	2	4	4	4	4	4	4	2
$\lvert C_{\overline{G}}(x_{2j})\rvert$	983040	983040	983040	983040	30720	30720	30720	30720	30720	30720	16384
$\lvert C_{H_k}(x_{2km})\rvert$											
3840	1	1	1	1	1	1	1	1	1	1	1
3840	1	1	1	1	-1	-1	-1	-1	1	1	1
3840	1	1	1	1	-1	-1	1	1	-1	-1	1
3840	1	1	1	1	1	1	-1	-1	-1	-1	1
64	60	60	60	60	0	0	0	0	0	0	-4
120	32	-32	-32	32	0	0	0	0	-8	8	0
120	32	-32	32	-32	-8	8	0	0	0	0	0
120	32	32	-32	-32	0	0	-8	8	0	0	0
120	32	-32	-32	32	0	0	0	0	8	-8	0
120	32	-32	32	-32	8	-8	0	0	0	0	0
120	32	32	-32	-32	0	0	8	-8	0	0	0
m_{2j}	4	4	4	4	128	128	128	128	128	128	240

$\mathbb{F}_{18} = M(10C)$	$x_{18,1}$	$x_{18,2}$	$x_{18,3}$	$x_{18,4}$
$o(x_{18j})$	10	10	10	10
$\lvert C_{\overline{G}}(x_{18j})\rvert$	80	80	80	80
$\lvert C_{H_k}(x_{18km})\rvert$				
20	1	1	1	1
20	1	-1	-1	1
20	1	-1	1	-1
20	1	1	-1	-1
m_{18j}	256	256	256	256

$\mathbb{F}_{19} = M(10D)$	$x_{19,1}$	$x_{19,2}$	$x_{19,3}$	$x_{19,4}$
$o(x_{19j})$	10	10	10	10
$\lvert C_{\overline{G}}(x_{19j})\rvert$	80	80	80	80
$\lvert C_{H_k}(x_{19km})\rvert$				
20	1	1	1	1
20	1	-1	-1	1
20	1	-1	1	-1
20	1	1	-1	-1
m_{19j}	256	256	256	256

$\mathbb{F}_{20} = M(15A)$	$x_{20,1}$	$x_{20,2}$	$x_{20,3}$	$x_{20,4}$
$o(x_{20j})$	15	30	30	30
$\lvert C_{\overline{G}}(x_{20j})\rvert$	60	60	60	60
$\lvert C_{H_k}(x_{20km})\rvert$				
15	1	1	1	1
15	1	-1	-1	1
15	1	-1	1	-1
15	1	1	-1	-1
m_{20j}	256	256	256	256

$\mathbb{F}_{21} = M(15B)$	$x_{21,1}$	$x_{21,2}$	$x_{21,3}$	$x_{21,4}$
$o(x_{21j})$	15	30	30	30
$\lvert C_{\overline{G}}(x_{21j})\rvert$	60	60	60	60
$\lvert C_{H_k}(x_{21km})\rvert$				
15	1	1	1	1
15	1	-1	-1	1
15	1	-1	1	-1
15	1	1	-1	-1
m_{21j}	256	256	256	256

Table 8.8: Fischer matrices of $\overline{G}$

$\mathbb{F}_{22} = M(15C)$	$x_{22,1}$	$x_{22,2}$	$x_{22,3}$	$x_{22,4}$
$o(x_{22j})$	15	30	30	30
$\lvert C_{\overline{G}}(x_{22j})\rvert$	60	60	60	60
$\lvert C_{H_k}(x_{22km})\rvert$				
15	1	1	1	1
15	1	-1	-1	1
15	1	-1	1	-1
15	1	1	-1	-1
m_{22j}	256	256	256	256

$\mathbb{F}_{23} = M(15D)$	$x_{23,1}$	$x_{23,2}$	$x_{23,3}$	$x_{23,4}$
$o(x_{23j})$	15	30	30	30
$\lvert C_{\overline{G}}(x_{23j})\rvert$	60	60	60	60
$\lvert C_{H_k}(x_{23km})\rvert$				
15	1	1	1	1
15	1	-1	-1	1
15	1	-1	1	-1
15	1	1	-1	-1
m_{23j}	256	256	256	256

$\mathbb{F}_{24} = M(17A)$	$x_{24,1}$	$x_{24,2}$	$x_{24,3}$	$x_{24,4}$
$o(x_{24j})$	17	34	34	34
$\lvert C_{\overline{G}}(x_{24j})\rvert$	68	68	68	68
$\lvert C_{H_k}(x_{24km})\rvert$				
17	1	1	1	1
17	1	-1	-1	1
17	1	-1	1	-1
17	1	1	-1	-1
m_{24j}	256	256	256	256

$\mathbb{F}_{25} = M(17B)$	$x_{25,1}$	$x_{25,2}$	$x_{25,3}$	$x_{25,4}$
$o(x_{25j})$	17	34	34	34
$\lvert C_{\overline{G}}(x_{25j})\rvert$	68	68	68	68
$\lvert C_{H_k}(x_{25km})\rvert$				
17	1	1	1	1
17	1	-1	-1	1
17	1	-1	1	-1
17	1	1	-1	-1
m_{25j}	256	256	256	256

$\mathbb{F}_{26} = M(17C)$	$x_{26,1}$	$x_{26,2}$	$x_{26,3}$	$x_{26,4}$
$o(x_{26j})$	17	34	34	34
$\lvert C_{\overline{G}}(x_{26j})\rvert$	68	68	68	68
$\lvert C_{H_k}(x_{26km})\rvert$				
17	1	1	1	1
17	1	-1	-1	1
17	1	-1	1	-1
17	1	1	-1	-1
m_{26j}	256	256	256	256

$\mathbb{F}_{27} = M(17D)$	$x_{27,1}$	$x_{27,2}$	$x_{27,3}$	$x_{27,4}$
$o(x_{27j})$	17	34	34	34
$\lvert C_{\overline{G}}(x_{27j})\rvert$	68	68	68	68
$\lvert C_{H_k}(x_{27km})\rvert$				
17	1	1	1	1
17	1	-1	-1	1
17	1	-1	1	-1
17	1	1	-1	-1
m_{27j}	256	256	256	256

8.6 The character table of $\overline{G}$

In this section we discuss the construction of the character table of $\overline{G} = N{:}G$ using the Clifford-Fischer Theory. This theory requires that the $Irr(N)$ be extendable to the inertia groups. Since N is elementary abelian and that $\overline{G}$ is a split extension then, by Mackey's Theorem, the $Irr(N)$ are extendable to the inertia groups. Due to Gallagher's Theorem the irreducible characters of $\overline{G}$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\phi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \in ker(\beta)\},$$

where $\overline{H}_i$ is an inertia group and $H_i = \overline{H}_i/N$ is an inertia factor group. This then means that the character table of $\overline{G}$ will be divided into blocks corresponding to the inertia factor groups H_i, for $i \in \{1,2,3,4,5,6,7,8\}$. In this affine subgroup $\overline{G}$ we have that $H_1 \cong G$, $H_2 \cong 2^6{:}Sp(2,4)$, $H_3, H_4, H_5 \cong GO^+(4,4)$ and $H_6, H_7, H_8 \cong GO^-(4,4)$. Therefore the character table of $\overline{G}$ will be of the form

$$\left[\begin{array}{ccccc}
B_{1,1} & B_{1,2} & B_{1,3} & \cdots & B_{1,27} \\ \hline
B_{2,1} & B_{2,2} & B_{2,3} & \cdots & B_{2,27} \\ \hline
B_{3,1} & B_{3,2} & B_{3,3} & \cdots & B_{3,27} \\ \hline
B_{4,1} & B_{4,2} & B_{4,3} & \cdots & B_{4,27} \\ \hline
B_{5,1} & B_{5,2} & B_{5,3} & \cdots & B_{5,27} \\ \hline
B_{6,1} & B_{6,2} & B_{6,3} & \cdots & B_{6,27} \\ \hline
B_{7,1} & B_{7,2} & B_{7,3} & \cdots & B_{7,27} \\ \hline
B_{8,1} & B_{8,2} & B_{8,3} & \cdots & B_{8,27}
\end{array}\right],$$

where $B_{i,j}$ are blocks corresponding to the inertia factor groups and the 27 conjugacy classes of G, $\{1 \le i \le 8\}$ and $\{1 \le j \le 27\}$. The block $B_{i,j}$ is formed by multiplying the relevant columns of the character table of H_i by the rows of the Fischer matrix $M(g)$ corresponding to the classes of H_i that fuse to the class $[g] \in G$. If H_i does not contribute to $M(g)$ then the block $B_{i,j}$ will have zeroes. The fusion maps of the inertia factor groups into G are given in Table 8.5 in Section 8.4.

We list below the character tables of the inertia factor groups. The character table of H_2 is given by Table 7.18 in Section 7.7 of Chapter 7.

Table 8.9: The character table of $Sp(4,4)$

GAP $[g]$	1a	2a	2b	2c	3a	3b	4a	4b	5a	5b	5c	5d	5e
ATLAS $[g]$	1A	2A	2B	2C	3A	3B	4A	4B	5A	5B	5C	5D	5E
$\|C_G(g)\|$	979200	3840	3840	256	180	180	32	32	300	300	300	300	25
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	18	-6	-6	2	0	0	-2	2	3	3	3	3	-2
χ_3	34	10	-6	2	1	4	2	-2	4	4	-1	-1	-1
χ_4	34	-6	10	2	4	1	2	-2	-1	-1	4	4	-1
χ_5	50	10	10	2	5	5	-2	2	0	0	0	0	0
χ_6	51	-13	3	3	3	0	-1	-1	A	$\overline{A}$	$\overline{B}$	B	1
χ_7	51	-13	3	3	3	0	-1	-1	$\overline{A}$	A	B	$\overline{B}$	1
χ_8	51	3	-13	3	0	3	-1	-1	B	$\overline{B}$	A	$\overline{A}$	1
χ_9	51	3	-13	3	0	3	-1	-1	$\overline{B}$	B	$\overline{A}$	A	1
χ_{10}	85	21	5	5	4	-5	1	1	5	5	0	0	0
χ_{11}	85	5	21	5	-5	4	1	1	0	0	5	5	0
χ_{12}	153	9	9	-7	0	0	1	1	3	3	3	3	3
χ_{13}	204	-4	12	-4	3	0	0	0	C	$\overline{C}$	$-\overline{B}$	$-B$	-1
χ_{14}	204	-4	12	-4	3	0	0	0	$\overline{C}$	C	$-B$	$-\overline{B}$	-1
χ_{15}	204	12	-4	-4	0	3	0	0	$-\overline{B}$	$-B$	$\overline{C}$	C	-1
χ_{16}	204	12	-4	-4	0	3	0	0	$-B$	$-\overline{B}$	C	$\overline{C}$	-1
χ_{17}	225	-15	-15	1	0	0	1	1	0	0	0	0	0
χ_{18}	225	-15	-15	1	0	0	1	1	0	0	0	0	0
χ_{19}	225	-15	-15	1	0	0	1	1	0	0	0	0	0
χ_{20}	225	-15	-15	1	0	0	1	1	0	0	0	0	0
χ_{21}	255	-17	15	-1	-3	0	-1	-1	D	$\overline{D}$	0	0	0
χ_{22}	255	-17	15	-1	-3	0	-1	-1	$\overline{D}$	D	0	0	0
χ_{23}	255	15	-17	-1	0	-3	-1	-1	0	0	D	$\overline{D}$	0
χ_{24}	255	15	-17	-1	0	-3	-1	-1	0	0	$\overline{D}$	D	0
χ_{25}	256	0	0	0	4	4	0	0	-4	-4	-4	-4	1
χ_{26}	340	4	20	4	1	-5	0	0	-5	-5	0	0	0
χ_{27}	340	20	4	4	-5	1	0	0	0	0	-5	-5	0

Table 8.9: The character table of $Sp(4,4)$

GAP $[g]$	6a	6b	10a	10b	10c	10d	15a	15b	15c	15d	17a	17b	17c	17d
ATLAS $[g]$	6A	6B	10A	10B	10D	10C	15C	15D	15B	15A	17A	17B	17C	17D
$\lvert C_G(g)\rvert$	12	12	20	20	20	20	15	15	15	15	17	17	17	17
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	0	0	-1	-1	-1	-1	0	0	0	0	1	1	1	1
χ_3	1	0	0	0	-1	-1	-1	-1	1	1	0	0	0	0
χ_4	0	1	-1	-1	0	0	1	1	-1	-1	0	0	0	0
χ_5	1	1	0	0	0	0	0	0	0	0	-1	-1	-1	-1
χ_6	-1	0	E	$\overline{E}$	$-E$	$-\overline{E}$	0	0	$-E$	$-\overline{E}$	0	0	0	0
χ_7	-1	0	$\overline{E}$	E	$-\overline{E}$	$-E$	0	0	$-\overline{E}$	$-E$	0	0	0	0
χ_8	0	-1	$-E$	$-\overline{E}$	$\overline{E}$	E	$-\overline{E}$	$-E$	0	0	0	0	0	0
χ_9	0	-1	$-\overline{E}$	$-E$	E	$\overline{E}$	$-E$	$-\overline{E}$	0	0	0	0	0	0
χ_{10}	0	-1	1	1	0	0	0	0	-1	-1	0	0	0	0
χ_{11}	-1	0	0	0	1	1	-1	-1	0	0	0	0	0	0
χ_{12}	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0
χ_{13}	-1	0	1	1	E	$\overline{E}$	0	0	$-E$	$-\overline{E}$	0	0	0	0
χ_{14}	-1	0	1	1	$\overline{E}$	E	0	0	$-\overline{E}$	$-E$	0	0	0	0
χ_{15}	0	-1	$\overline{E}$	E	1	1	$-E$	$-\overline{E}$	0	0	0	0	0	0
χ_{16}	0	-1	E	$\overline{E}$	1	1	$-\overline{E}$	$-E$	0	0	0	0	0	0
χ_{17}	0	0	0	0	0	0	0	0	0	0	F	G	I	H
χ_{18}	0	0	0	0	0	0	0	0	0	0	G	F	H	I
χ_{19}	0	0	0	0	0	0	0	0	0	0	H	I	F	G
χ_{20}	0	0	0	0	0	0	0	0	0	0	I	H	G	F
χ_{21}	1	0	$-\overline{E}$	$-E$	0	0	0	0	E	$\overline{E}$	0	0	0	0
χ_{22}	1	0	$-E$	$-\overline{E}$	0	0	0	0	$\overline{E}$	E	0	0	0	0
χ_{23}	0	1	0	0	$-E$	$-\overline{E}$	$\overline{E}$	E	0	0	0	0	0	0
χ_{24}	0	1	0	0	$-\overline{E}$	$-E$	E	$\overline{E}$	0	0	0	0	0	0
χ_{25}	0	0	0	0	0	0	-1	-1	-1	-1	1	1	1	1
χ_{26}	1	-1	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{27}	-1	1	0	0	-1	-1	1	1	0	0	0	0	0	0

$$A = \frac{7-\sqrt{5}}{2} \qquad B = \frac{-3-3\sqrt{5}}{2}$$

$$C = -1-2\sqrt{5} \qquad D = \frac{5-5\sqrt{5}}{2}$$

$$E = \frac{-1+\sqrt{5}}{2}$$

$$F = E(17)^6 + E(17)^7 + E(17)^{10} + E(17)^{11}$$

$$G = E(17)^3 + E(17)^5 + E(17)^{12} + E(17)^{14}$$

$$H = E(17) + E(17)^4 + E(17)^{13} + E(17)^{16}$$

$$I = E(17)^2 + E(17)^8 + E(17)^9 + E(17)^{15}$$

Table 8.10: The character table of $GO^+(4,4)$

$[g]$	1A	2A	2B	2C	3A	3B	4A	5A	5B	5C
$\lvert C_G(g)\rvert$	7200	32	240	120	18	180	8	50	50	300
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	1	-1	1	1	1
χ_3	6	-2	2	0	0	3	0	A	$\overline{A}$	C
χ_4	6	-2	2	0	0	3	0	$\overline{A}$	A	$\overline{C}$
χ_5	8	0	4	0	2	5	0	-2	-2	3
χ_6	9	1	-3	-3	0	0	1	B	$\overline{B}$	D
χ_7	9	1	-3	-3	0	0	1	$\overline{B}$	B	$\overline{D}$
χ_8	9	1	-3	3	0	0	-1	B	$\overline{B}$	D
χ_9	9	1	-3	3	0	0	-1	$\overline{B}$	B	$\overline{D}$
χ_{10}	10	2	6	0	-2	4	0	0	0	5
χ_{11}	16	0	0	-4	1	4	0	1	1	-4
χ_{12}	16	0	0	4	1	4	0	1	1	-4
χ_{13}	18	2	-6	0	0	0	0	-2	-2	3
χ_{14}	24	0	-4	0	0	3	0	$-\overline{A}$	$-A$	E
χ_{15}	24	0	-4	0	0	3	0	$-A$	$-\overline{A}$	$\overline{E}$
χ_{16}	25	1	5	-5	1	-5	-1	0	0	0
χ_{17}	25	1	5	5	1	-5	1	0	0	0
χ_{18}	30	-2	-2	0	0	-3	0	0	0	F
χ_{19}	30	-2	-2	0	0	-3	0	0	0	$\overline{F}$
χ_{20}	40	0	4	0	-2	1	0	0	0	-5

Table 8.10: The character table of $GO^+(4,4)$

$[g]$	5D	5E	6A	6B	10A	10B	10C	10D	15A	15B
$\lvert C_G(g)\rvert$	300	25	12	6	20	20	10	10	15	15
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	1	-1	-1	1	1
χ_3	$\overline{C}$	1	-1	0	G	$\overline{G}$	0	0	$-\overline{G}$	$-G$
χ_4	C	1	-1	0	$\overline{G}$	G	0	0	$-G$	$-\overline{G}$
χ_5	3	-2	1	0	-1	-1	0	0	0	0
χ_6	$\overline{D}$	-1	0	0	$\overline{G}$	G	$\overline{G}$	G	0	0
χ_7	D	-1	0	0	G	$\overline{G}$	G	$\overline{G}$	0	0
χ_8	$\overline{D}$	-1	0	0	$\overline{G}$	G	$-\overline{G}$	$-G$	0	0
χ_9	D	-1	0	0	G	$\overline{G}$	$-G$	$-\overline{G}$	0	0
χ_{10}	5	0	0	0	1	1	0	0	-1	-1
χ_{11}	-4	1	0	-1	0	0	1	1	-1	-1
χ_{12}	-4	1	0	1	0	0	-1	-1	-1	-1
χ_{13}	3	3	0	0	-1	-1	0	0	0	0
χ_{14}	$\overline{E}$	-1	-1	0	1	1	0	0	$-G$	$-\overline{G}$
χ_{15}	E	-1	-1	0	1	1	0	0	$-\overline{G}$	$-G$
χ_{16}	0	0	-1	1	0	0	0	0	0	0
χ_{17}	0	0	-1	-1	0	0	0	0	0	0
χ_{18}	$\overline{F}$	0	1	0	$-\overline{G}$	$-G$	0	0	$\overline{G}$	G
χ_{19}	F	0	1	0	$-G$	$-\overline{G}$	0	0	G	$\overline{G}$
χ_{20}	-5	0	1	0	-1	-1	0	0	1	1

$$A = 1-\sqrt{5}, \quad B = \frac{3-\sqrt{5}}{2}, \quad C = \frac{7-\sqrt{5}}{2},$$

$$D = \frac{3-3\sqrt{5}}{2}, \quad E = -1+2\sqrt{5}, \quad F = \frac{5-5\sqrt{5}}{2}, \quad G = \frac{-1-\sqrt{5}}{2}.$$

Table 8.11: The character table of $GO^{-}(4,4)$

$[g]$	1A	2A	2B	3A	4A	5A	5B	6A
$\|C_G(g)\|$	8160	32	120	30	8	30	30	6
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	-1	1	1	-1
χ_3	16	0	-4	1	0	1	1	-1
χ_4	16	0	4	1	0	1	1	1
χ_5	17	1	5	-1	1	2	2	-1
χ_6	17	1	-5	-1	-1	2	2	1
χ_7	17	1	3	2	-1	A	$\overline{A}$	0
χ_8	17	1	3	2	-1	$\overline{A}$	A	0
χ_9	17	1	-3	2	1	A	$\overline{A}$	0
χ_{10}	17	1	-3	2	1	$\overline{A}$	A	0
χ_{11}	30	-2	0	0	0	0	0	0
χ_{12}	30	-2	0	0	0	0	0	0
χ_{13}	30	-2	0	0	0	0	0	0
χ_{14}	30	-2	0	0	0	0	0	0
χ_{15}	34	2	0	-2	0	B	$\overline{B}$	0
χ_{16}	34	2	0	-2	0	$\overline{B}$	B	0

Table 8.11: The character table of $GO^{-}(4,4)$

$[g]$	10A	10B	15A	15B	17A	17B	17C	17D
$\|C_G(g)\|$	10	10	15	15	17	17	17	17
χ_1	1	1	1	1	1	1	1	1
χ_2	-1	-1	1	1	1	1	1	1
χ_3	1	1	1	1	-1	-1	-1	-1
χ_4	-1	-1	1	1	-1	-1	-1	-1
χ_5	0	0	-1	-1	0	0	0	0
χ_6	0	0	-1	-1	0	0	0	0
χ_7	$-\overline{A}$	$-A$	A	$\overline{A}$	0	0	0	0
χ_8	$-A$	$-\overline{A}$	$\overline{A}$	A	0	0	0	0
χ_9	$\overline{A}$	A	A	$\overline{A}$	0	0	0	0
χ_{10}	A	$\overline{A}$	$\overline{A}$	A	0	0	0	0
χ_{11}	0	0	0	0	C	D	F	E
χ_{12}	0	0	0	0	D	C	E	F
χ_{13}	0	0	0	0	E	F	C	D
χ_{14}	0	0	0	0	F	E	D	C
χ_{15}	0	0	$-A$	$-\overline{A}$	0	0	0	0
χ_{16}	0	0	$-\overline{A}$	$-A$	0	0	0	0

$$A = \tfrac{-1-\sqrt{5}}{2}, \qquad B = -1-\sqrt{5},$$

$$C = -E(17)^6 - E(17)^7 - E(17)^{10} - E(17)^{11},$$

$$D = -E(17)^3 - E(17)^5 - E(17)^{12} - E(17)^{14},$$

$$E = -E(17) - E(17)^4 - E(17)^{13} - E(17)^{16},$$

$$F = -E(17)^2 - E(17)^8 - E(17)^9 - E(17)^{15}.$$

We use the class 1A of G to demonstrate the construction of the character table of $\overline{G}$.

$$B_{1,1} = \begin{bmatrix} 1 \\ 18 \\ 34 \\ 34 \\ 50 \\ 51 \\ 51 \\ 51 \\ 51 \\ 85 \\ 85 \\ 153 \\ 204 \\ 204 \\ 204 \\ 204 \\ 225 \\ 225 \\ 225 \\ 225 \\ 255 \\ 255 \\ 255 \\ 255 \\ 256 \\ 340 \\ 340 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 18 & 18 & 18 & 18 & 18 & 18 & 18 & 18 \\ 34 & 34 & 34 & 34 & 34 & 34 & 34 & 34 \\ 34 & 34 & 34 & 34 & 34 & 34 & 34 & 34 \\ 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 \\ 51 & 51 & 51 & 51 & 51 & 51 & 51 & 51 \\ 51 & 51 & 51 & 51 & 51 & 51 & 51 & 51 \\ 51 & 51 & 51 & 51 & 51 & 51 & 51 & 51 \\ 51 & 51 & 51 & 51 & 51 & 51 & 51 & 51 \\ 85 & 85 & 85 & 85 & 85 & 85 & 85 & 85 \\ 85 & 85 & 85 & 85 & 85 & 85 & 85 & 85 \\ 153 & 153 & 153 & 153 & 153 & 153 & 153 & 153 \\ 204 & 204 & 204 & 204 & 204 & 204 & 204 & 204 \\ 204 & 204 & 204 & 204 & 204 & 204 & 204 & 204 \\ 204 & 204 & 204 & 204 & 204 & 204 & 204 & 204 \\ 204 & 204 & 204 & 204 & 204 & 204 & 204 & 204 \\ 225 & 225 & 225 & 225 & 225 & 225 & 225 & 225 \\ 225 & 225 & 225 & 225 & 225 & 225 & 225 & 225 \\ 225 & 225 & 225 & 225 & 225 & 225 & 225 & 225 \\ 225 & 225 & 225 & 225 & 225 & 225 & 225 & 225 \\ 255 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \\ 255 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \\ 255 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \\ 255 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \\ 256 & 256 & 256 & 256 & 256 & 256 & 256 & 256 \\ 340 & 340 & 340 & 340 & 340 & 340 & 340 & 340 \\ 340 & 340 & 340 & 340 & 340 & 340 & 340 & 340 \end{bmatrix}$$

$$B_{2,1} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 4 \\ 5 \\ 15 \\ 15 \\ 15 \\ 15 \\ 6 \\ 6 \\ 12 \\ 12 \\ 6 \\ 6 \\ 12 \\ 12 \\ 6 \\ 6 \\ 12 \\ 12 \\ 10 \\ 10 \\ 20 \\ 10 \\ 10 \\ 20 \\ 10 \\ 10 \\ 20 \end{bmatrix} \begin{bmatrix} 255 & 255 & 255 & 255 & -1 & -1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 255 & 255 & 255 & 255 & -1 & -1 & -1 & -1 \\ 765 & 765 & 765 & 765 & -3 & -3 & -3 & -3 \\ 765 & 765 & 765 & 765 & -3 & -3 & -3 & -3 \\ 1020 & 1020 & 1020 & 1020 & -4 & -4 & -4 & -4 \\ 1275 & 1275 & 1275 & 1275 & -5 & -5 & -5 & -5 \\ 3825 & 3825 & 3825 & 3825 & -15 & -15 & -15 & -15 \\ 3825 & 3825 & 3825 & 3825 & -15 & -15 & -15 & -15 \\ 3825 & 3825 & 3825 & 3825 & -15 & -15 & -15 & -15 \\ 3825 & 3825 & 3825 & 3825 & -15 & -15 & -15 & -15 \\ 1530 & 1530 & 1530 & 1530 & -6 & -6 & -6 & -6 \\ 1530 & 1530 & 1530 & 1530 & -6 & -6 & -6 & -6 \\ 3060 & 3060 & 3060 & 3060 & -12 & -12 & -12 & -12 \\ 3060 & 3060 & 3060 & 3060 & -12 & -12 & -12 & -12 \\ 1530 & 1530 & 1530 & 1530 & -6 & -6 & -6 & -6 \\ 1530 & 1530 & 1530 & 1530 & -6 & -6 & -6 & -6 \\ 3060 & 3060 & 3060 & 3060 & -12 & -12 & -12 & -12 \\ 3060 & 3060 & 3060 & 3060 & -12 & -12 & -12 & -12 \\ 1530 & 1530 & 1530 & 1530 & -6 & -6 & -6 & -6 \\ 1530 & 1530 & 1530 & 1530 & -6 & -6 & -6 & -6 \\ 3060 & 3060 & 3060 & 3060 & -12 & -12 & -12 & -12 \\ 3060 & 3060 & 3060 & 3060 & -12 & -12 & -12 & -12 \\ 2550 & 2550 & 2550 & 2550 & -10 & -10 & -10 & -10 \\ 2550 & 2550 & 2550 & 2550 & -10 & -10 & -10 & -10 \\ 5100 & 5100 & 5100 & 5100 & -20 & -20 & -20 & -20 \\ 2550 & 2550 & 2550 & 2550 & -10 & -10 & -10 & -10 \\ 2550 & 2550 & 2550 & 2550 & -10 & -10 & -10 & -10 \\ 5100 & 5100 & 5100 & 5100 & -20 & -20 & -20 & -20 \\ 2550 & 2550 & 2550 & 2550 & -10 & -10 & -10 & -10 \\ 2550 & 2550 & 2550 & 2550 & -10 & -10 & -10 & -10 \\ 5100 & 5100 & 5100 & 5100 & -20 & -20 & -20 & -20 \end{bmatrix}$$

$$B_{3,1} = \begin{bmatrix} 1 \\ 1 \\ 6 \\ 6 \\ 8 \\ 9 \\ 9 \\ 9 \\ 9 \\ 10 \\ 16 \\ 16 \\ 18 \\ 24 \\ 24 \\ 25 \\ 25 \\ 30 \\ 30 \\ 40 \end{bmatrix} \begin{bmatrix} 136 & -136 & -136 & 136 & 8 & -8 & -8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 136 & -136 & -136 & 136 & 8 & -8 & -8 & 8 \\ 136 & -136 & -136 & 136 & 8 & -8 & -8 & 8 \\ 816 & -816 & -816 & 816 & 48 & -48 & -48 & 48 \\ 816 & -816 & -816 & 816 & 48 & -48 & -48 & 48 \\ 1088 & -1088 & -1088 & 1088 & 64 & -64 & -64 & 64 \\ 1224 & -1224 & -1224 & 1224 & 72 & -72 & -72 & 72 \\ 1224 & -1224 & -1224 & 1224 & 72 & -72 & -72 & 72 \\ 1224 & -1224 & -1224 & 1224 & 72 & -72 & -72 & 72 \\ 1224 & -1224 & -1224 & 1224 & 72 & -72 & -72 & 72 \\ 1360 & -1360 & -1360 & 1360 & 80 & -80 & -80 & 80 \\ 2176 & -2176 & -2176 & 2176 & 128 & -128 & -128 & 128 \\ 2176 & -2176 & -2176 & 2176 & 128 & -128 & -128 & 128 \\ 2448 & -2448 & -2448 & 2448 & 144 & -144 & -144 & 144 \\ 3264 & -3264 & -3264 & 3264 & 192 & -192 & -192 & 192 \\ 3264 & -3264 & -3264 & 3264 & 192 & -192 & -192 & 192 \\ 3400 & -3400 & -3400 & 3400 & 200 & -200 & -200 & 200 \\ 3400 & -3400 & -3400 & 3400 & 200 & -200 & -200 & 200 \\ 4080 & -4080 & -4080 & 4080 & 240 & -240 & -240 & 240 \\ 4080 & -4080 & -4080 & 4080 & 240 & -240 & -240 & 240 \\ 5440 & -5440 & -5440 & 5440 & 320 & -320 & -320 & 320 \end{bmatrix}$$

$$B_{4,1} = \begin{bmatrix} 1 \\ 1 \\ 6 \\ 6 \\ 8 \\ 9 \\ 9 \\ 9 \\ 9 \\ 10 \\ 16 \\ 16 \\ 18 \\ 24 \\ 24 \\ 25 \\ 25 \\ 30 \\ 30 \\ 40 \end{bmatrix} \begin{bmatrix} 136 & -136 & 136 & -136 & 8 & -8 & 8 & -8 \end{bmatrix}$$

$$= \begin{bmatrix}
136 & -136 & 136 & -136 & 8 & -8 & 8 & -8 \\
136 & -136 & 136 & -136 & 8 & -8 & 8 & -8 \\
816 & -816 & 816 & -816 & 48 & -48 & 48 & -48 \\
816 & -816 & 816 & -816 & 48 & -48 & 48 & -48 \\
1088 & -1088 & 1088 & -1088 & 64 & -64 & 64 & -64 \\
1224 & -1224 & 1224 & -1224 & 72 & -72 & 72 & -72 \\
1224 & -1224 & 1224 & -1224 & 72 & -72 & 72 & -72 \\
1224 & -1224 & 1224 & -1224 & 72 & -72 & 72 & -72 \\
1224 & -1224 & 1224 & -1224 & 72 & -72 & 72 & -72 \\
1360 & -1360 & 1360 & -1360 & 80 & -80 & 80 & -80 \\
2176 & -2176 & 2176 & -2176 & 128 & -128 & 128 & -128 \\
2176 & -2176 & 2176 & -2176 & 128 & -128 & 128 & -128 \\
2448 & -2448 & 2448 & -2448 & 144 & -144 & 144 & -144 \\
3264 & -3264 & 3264 & -3264 & 192 & -192 & 192 & -192 \\
3264 & -3264 & 3264 & -3264 & 192 & -192 & 192 & -192 \\
3400 & -3400 & 3400 & -3400 & 200 & -200 & 200 & -200 \\
3400 & -3400 & 3400 & -3400 & 200 & -200 & 200 & -200 \\
4080 & -4080 & 4080 & -4080 & 240 & -240 & 240 & -240 \\
4080 & -4080 & 4080 & -4080 & 240 & -240 & 240 & -240 \\
5440 & -5440 & 5440 & -5440 & 320 & -320 & 320 & -320
\end{bmatrix}$$

$$B_{5,1} = \begin{bmatrix} 1 \\ 1 \\ 6 \\ 6 \\ 8 \\ 9 \\ 9 \\ 9 \\ 9 \\ 10 \\ 16 \\ 16 \\ 18 \\ 24 \\ 24 \\ 25 \\ 25 \\ 30 \\ 30 \\ 40 \end{bmatrix} \begin{bmatrix} 136 & 136 & -136 & -136 & 8 & 8 & -8 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 136 & 136 & -136 & -136 & 8 & 8 & -8 & -8 \\ 136 & 136 & -136 & -136 & 8 & 8 & -8 & -8 \\ 816 & 816 & -816 & -816 & 48 & 48 & -48 & -48 \\ 816 & 816 & -816 & -816 & 48 & 48 & -48 & -48 \\ 1088 & 1088 & -1088 & -1088 & 64 & 64 & -64 & -64 \\ 1224 & 1224 & -1224 & -1224 & 72 & 72 & -72 & -72 \\ 1224 & 1224 & -1224 & -1224 & 72 & 72 & -72 & -72 \\ 1224 & 1224 & -1224 & -1224 & 72 & 72 & -72 & -72 \\ 1224 & 1224 & -1224 & -1224 & 72 & 72 & -72 & -72 \\ 1360 & 1360 & -1360 & -1360 & 80 & 80 & -80 & -80 \\ 2176 & 2176 & -2176 & -2176 & 128 & 128 & -128 & -128 \\ 2176 & 2176 & -2176 & -2176 & 128 & 128 & -128 & -128 \\ 2448 & 2448 & -2448 & -2448 & 144 & 144 & -144 & -144 \\ 3264 & 3264 & -3264 & -3264 & 192 & 192 & -192 & -192 \\ 3264 & 3264 & -3264 & -3264 & 192 & 192 & -192 & -192 \\ 3400 & 3400 & -3400 & -3400 & 200 & 200 & -200 & -200 \\ 3400 & 3400 & -3400 & -3400 & 200 & 200 & -200 & -200 \\ 4080 & 4080 & -4080 & -4080 & 240 & 240 & -240 & -240 \\ 4080 & 4080 & -4080 & -4080 & 240 & 240 & -240 & -240 \\ 5440 & 5440 & -5440 & -5440 & 320 & 320 & -320 & -320 \end{bmatrix}$$

$$B_{6,1} = \begin{bmatrix} 1 \\ 1 \\ 16 \\ 16 \\ 17 \\ 17 \\ 17 \\ 17 \\ 17 \\ 17 \\ 30 \\ 30 \\ 30 \\ 30 \\ 34 \\ 34 \end{bmatrix} \begin{bmatrix} 120 & -120 & -120 & 120 & -8 & 8 & 8 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 120 & -120 & -120 & 120 & -8 & 8 & 8 & -8 \\ 120 & -120 & -120 & 120 & -8 & 8 & 8 & -8 \\ 1920 & -1920 & -1920 & 1920 & -128 & 128 & 128 & -128 \\ 1920 & -1920 & -1920 & 1920 & -128 & 128 & 128 & -128 \\ 2040 & -2040 & -2040 & 2040 & -136 & 136 & 136 & -136 \\ 2040 & -2040 & -2040 & 2040 & -136 & 136 & 136 & -136 \\ 2040 & -2040 & -2040 & 2040 & -136 & 136 & 136 & -136 \\ 2040 & -2040 & -2040 & 2040 & -136 & 136 & 136 & -136 \\ 2040 & -2040 & -2040 & 2040 & -136 & 136 & 136 & -136 \\ 2040 & -2040 & -2040 & 2040 & -136 & 136 & 136 & -136 \\ 3600 & -3600 & -3600 & 3600 & -240 & 240 & 240 & -240 \\ 3600 & -3600 & -3600 & 3600 & -240 & 240 & 240 & -240 \\ 3600 & -3600 & -3600 & 3600 & -240 & 240 & 240 & -240 \\ 3600 & -3600 & -3600 & 3600 & -240 & 240 & 240 & -240 \\ 4080 & -4080 & -4080 & 4080 & -272 & 272 & 272 & -272 \\ 4080 & -4080 & -4080 & 4080 & -272 & 272 & 272 & -272 \end{bmatrix}$$

$$B_{7,1} = \begin{bmatrix} 1 \\ 1 \\ 16 \\ 16 \\ 17 \\ 17 \\ 17 \\ 17 \\ 17 \\ 17 \\ 30 \\ 30 \\ 30 \\ 30 \\ 34 \\ 34 \end{bmatrix} \begin{bmatrix} 120 & -120 & 120 & -120 & -8 & 8 & -8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 120 & -120 & 120 & -120 & -8 & 8 & -8 & 8 \\ 120 & -120 & 120 & -120 & -8 & 8 & -8 & 8 \\ 1920 & -1920 & 1920 & -1920 & -128 & 128 & -128 & 128 \\ 1920 & -1920 & 1920 & -1920 & -128 & 128 & -128 & 128 \\ 2040 & -2040 & 2040 & -2040 & -136 & 136 & -136 & 136 \\ 2040 & -2040 & 2040 & -2040 & -136 & 136 & -136 & 136 \\ 2040 & -2040 & 2040 & -2040 & -136 & 136 & -136 & 136 \\ 2040 & -2040 & 2040 & -2040 & -136 & 136 & -136 & 136 \\ 2040 & -2040 & 2040 & -2040 & -136 & 136 & -136 & 136 \\ 2040 & -2040 & 2040 & -2040 & -136 & 136 & -136 & 136 \\ 3600 & -3600 & 3600 & -3600 & -240 & 240 & -240 & 240 \\ 3600 & -3600 & 3600 & -3600 & -240 & 240 & -240 & 240 \\ 3600 & -3600 & 3600 & -3600 & -240 & 240 & -240 & 240 \\ 3600 & -3600 & 3600 & -3600 & -240 & 240 & -240 & 240 \\ 4080 & -4080 & 4080 & -4080 & -272 & 272 & -272 & 272 \\ 4080 & -4080 & 4080 & -4080 & -272 & 272 & -272 & 272 \end{bmatrix}$$

$$B_{8,1} = \begin{bmatrix} 1 \\ 1 \\ 16 \\ 16 \\ 17 \\ 17 \\ 17 \\ 17 \\ 17 \\ 17 \\ 30 \\ 30 \\ 30 \\ 30 \\ 34 \\ 34 \end{bmatrix} \begin{bmatrix} 120 & 120 & -120 & -120 & -8 & -8 & 8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 120 & 120 & -120 & -120 & -8 & -8 & 8 & 8 \\ 120 & 120 & -120 & -120 & -8 & -8 & 8 & 8 \\ 1920 & 1920 & -1920 & -1920 & -128 & -128 & 128 & 128 \\ 1920 & 1920 & -1920 & -1920 & -128 & -128 & 128 & 128 \\ 2040 & 2040 & -2040 & -2040 & -136 & -136 & 136 & 136 \\ 2040 & 2040 & -2040 & -2040 & -136 & -136 & 136 & 136 \\ 2040 & 2040 & -2040 & -2040 & -136 & -136 & 136 & 136 \\ 2040 & 2040 & -2040 & -2040 & -136 & -136 & 136 & 136 \\ 2040 & 2040 & -2040 & -2040 & -136 & -136 & 136 & 136 \\ 2040 & 2040 & -2040 & -2040 & -136 & -136 & 136 & 136 \\ 3600 & 3600 & -3600 & -3600 & -240 & -240 & 240 & 240 \\ 3600 & 3600 & -3600 & -3600 & -240 & -240 & 240 & 240 \\ 3600 & 3600 & -3600 & -3600 & -240 & -240 & 240 & 240 \\ 3600 & 3600 & -3600 & -3600 & -240 & -240 & 240 & 240 \\ 4080 & 4080 & -4080 & -4080 & -272 & -272 & 272 & 272 \\ 4080 & 4080 & -4080 & -4080 & -272 & -272 & 272 & 272 \end{bmatrix}$$

The above process yields the partial character table of $\overline{G}$ corresponding to the class 1A of G and the 8 inertia factor groups. The full 165×165 character table of $\overline{G}$ was completed using the above-mentioned process but it is not included in this thesis due to its size. The character degrees of this affine subgroup are given in the first column of this partial character table given by Table 8.12 below. These character degrees can also be computed by using Theorem 4.2.32, Remark 4.2.33 and Theorem 4.2.34 in Section 4.2.

Table 8.12: Partial character table of $\overline{G}$ corresponding to 1A

$[g]$	1A							
$\lvert C_{\overline{G}}(\overline{g})\rvert$	1002700800	1002700800	1002700800	1002700800	3932160	3932160	3932160	3932160
$[\overline{g}]$	1A	2A	2B	2C	2D	2E	2F	2G
χ_1	1	1	1	1	1	1	1	1
χ_2	18	18	18	18	18	18	18	18
χ_3	34	34	34	34	34	34	34	34
χ_4	34	34	34	34	34	34	34	34
χ_5	50	50	50	50	50	50	50	50
χ_6	51	51	51	51	51	51	51	51
χ_7	51	51	51	51	51	51	51	51
χ_8	51	51	51	51	51	51	51	51
χ_9	51	51	51	51	51	51	51	51
χ_{10}	85	85	85	85	85	85	85	85
χ_{11}	85	85	85	85	85	85	85	85
χ_{12}	153	153	153	153	153	153	153	153
χ_{13}	204	204	204	204	204	204	204	204
χ_{14}	204	204	204	204	204	204	204	204
χ_{15}	204	204	204	204	204	204	204	204
χ_{16}	204	204	204	204	204	204	204	204
χ_{17}	225	225	225	225	225	225	225	225
χ_{18}	225	225	225	225	225	225	225	225
χ_{19}	225	225	225	225	225	225	225	225
χ_{20}	225	225	225	225	225	225	225	225
χ_{21}	255	255	255	255	255	255	255	255
χ_{22}	255	255	255	255	255	255	255	255
χ_{23}	255	255	255	255	255	255	255	255
χ_{24}	255	255	255	255	255	255	255	255
χ_{25}	256	256	256	256	256	256	256	256
χ_{26}	340	340	340	340	340	340	340	340
χ_{27}	340	340	340	340	340	340	340	340
χ_{28}	255	255	255	255	-1	-1	-1	-1
χ_{29}	765	765	765	765	-3	-3	-3	-3
χ_{30}	765	765	765	765	-3	-3	-3	-3
χ_{31}	1020	1020	1020	1020	-4	-4	-4	-4
χ_{32}	1275	1275	1275	1275	-5	-5	-5	-5
χ_{33}	3825	3825	3825	3825	-15	-15	-15	-15
χ_{34}	3825	3825	3825	3825	-15	-15	-15	-15
χ_{35}	3825	3825	3825	3825	-15	-15	-15	-15
χ_{36}	3825	3825	3825	3825	-15	-15	-15	-15
χ_{37}	1530	1530	1530	1530	-6	-6	-6	-6
χ_{38}	1530	1530	1530	1530	-6	-6	-6	-6
χ_{39}	3060	3060	3060	3060	-12	-12	-12	-12
χ_{40}	3060	3060	3060	3060	-12	-12	-12	-12
χ_{41}	1530	1530	1530	1530	-6	-6	-6	-6

Table 8.12: Partial character table of $\overline{G}$ corresponding to 1A

$[g]$	1A							
$\|C_{\overline{G}}(\overline{g}\|$	1002700800	1002700800	1002700800	1002700800	3932160	3932160	3932160	3932160
$[\overline{g}]$	1A	2A	2B	2C	2D	2E	2F	2G
χ_{42}	1530	1530	1530	1530	-6	-6	-6	-6
χ_{43}	3060	3060	3060	3060	-12	-12	-12	-12
χ_{44}	3060	3060	3060	3060	-12	-12	-12	-12
χ_{45}	1530	1530	1530	1530	-6	-6	-6	-6
χ_{46}	1530	1530	1530	1530	-6	-6	-6	-6
χ_{47}	3060	3060	3060	3060	-12	-12	-12	-12
χ_{48}	3060	3060	3060	3060	-12	-12	-12	-12
χ_{49}	2550	2550	2550	2550	-10	-10	-10	-10
χ_{50}	2550	2550	2550	2550	-10	-10	-10	-10
χ_{51}	5100	5100	5100	5100	-20	-20	-20	-20
χ_{52}	2550	2550	2550	2550	-10	-10	-10	-10
χ_{53}	2550	2550	2550	2550	-10	-10	-10	-10
χ_{54}	5100	5100	5100	5100	-20	-20	-20	-20
χ_{55}	2550	2550	2550	2550	-10	-10	-10	-10
χ_{56}	2550	2550	2550	2550	-10	-10	-10	-10
χ_{57}	5100	5100	5100	5100	-20	-20	-20	-20
χ_{58}	136	-136	-136	136	8	-8	-8	8
χ_{59}	136	-136	-136	136	8	-8	-8	8
χ_{60}	816	-816	-816	816	48	-48	-48	48
χ_{61}	816	-816	-816	816	48	-48	-48	48
χ_{62}	1088	-1088	-1088	1088	64	-64	-64	64
χ_{63}	1224	-1224	-1224	1224	72	-72	-72	72
χ_{64}	1224	-1224	-1224	1224	72	-72	-72	72
χ_{65}	1224	-1224	-1224	1224	72	-72	-72	72
χ_{66}	1224	-1224	-1224	1224	72	-72	-72	72
χ_{67}	1360	-1360	-1360	1360	80	-80	-80	80
χ_{68}	2176	-2176	-2176	2176	128	-128	-128	128
χ_{69}	2176	-2176	-2176	2176	128	-128	-128	128
χ_{70}	2448	-2448	-2448	2448	144	-144	-144	144
χ_{71}	3264	-3264	-3264	3264	192	-192	-192	192
χ_{72}	3264	-3264	-3264	3264	192	-192	-192	192
χ_{73}	3400	-3400	-3400	3400	200	-200	-200	200
χ_{74}	3400	-3400	-3400	3400	200	-200	-200	200
χ_{75}	4080	-4080	-4080	4080	240	-240	-240	240
χ_{76}	4080	-4080	-4080	4080	240	-240	-240	240
χ_{77}	5440	-5440	-5440	5440	320	-320	-320	320
χ_{78}	136	-136	136	-136	8	-8	8	-8
χ_{79}	136	-136	136	-136	8	-8	8	-8
χ_{80}	816	-816	816	-816	48	-48	48	-48
χ_{81}	816	-816	816	-816	48	-48	48	-48
χ_{82}	1088	-1088	1088	-1088	64	-64	64	-64

Table 8.12: Partial character table of $\overline{G}$ corresponding to 1A

$[g]$	1A							
$\lvert C_{\overline{G}}(\overline{g})\rvert$	1002700800	1002700800	1002700800	1002700800	3932160	3932160	3932160	3932160
$[\overline{g}]$	1A	2A	2B	2C	2D	2E	2F	2G
χ_{83}	1224	-1224	1224	-1224	72	-72	72	-72
χ_{84}	1224	-1224	1224	-1224	72	-72	72	-72
χ_{85}	1224	-1224	1224	-1224	72	-72	72	-72
χ_{86}	1224	-1224	1224	-1224	72	-72	72	-72
χ_{87}	1360	-1360	1360	-1360	80	-80	80	-80
χ_{88}	2176	-2176	2176	-2176	128	-128	128	-128
χ_{89}	2176	-2176	2176	-2176	128	-128	128	-128
χ_{90}	2448	-2448	2448	-2448	144	-144	144	-144
χ_{91}	3264	-3264	3264	-3264	192	-192	192	-192
χ_{92}	3264	-3264	3264	-3264	192	-192	192	-192
χ_{93}	3400	-3400	3400	-3400	200	-200	200	-200
χ_{94}	3400	-3400	3400	-3400	200	-200	200	-200
χ_{95}	4080	-4080	4080	-4080	240	-240	240	-240
χ_{96}	4080	-4080	4080	-4080	240	-240	240	-240
χ_{97}	5440	-5440	5440	-5440	320	-320	320	-320
χ_{98}	136	136	-136	-136	8	8	-8	-8
χ_{99}	136	136	-136	-136	8	8	-8	-8
χ_{100}	816	816	-816	-816	48	48	-48	-48
χ_{101}	816	816	-816	-816	48	48	-48	-48
χ_{102}	1088	1088	-1088	-1088	64	64	-64	-64
χ_{103}	1224	1224	-1224	-1224	72	72	-72	-72
χ_{104}	1224	1224	-1224	-1224	72	72	-72	-72
χ_{105}	1224	1224	-1224	-1224	72	72	-72	-72
χ_{106}	1224	1224	-1224	-1224	72	72	-72	-72
χ_{107}	1360	1360	-1360	-1360	80	80	-80	-80
χ_{108}	2176	2176	-2176	-2176	128	128	-128	-128
χ_{109}	2176	2176	-2176	-2176	128	128	-128	-128
χ_{110}	2448	2448	-2448	-2448	144	144	-144	-144
χ_{111}	3264	3264	-3264	-3264	192	192	-192	-192
χ_{112}	3264	3264	-3264	-3264	192	192	-192	-192
χ_{113}	3400	3400	-3400	-3400	200	200	-200	-200
χ_{114}	3400	3400	-3400	-3400	200	200	-200	-200
χ_{115}	4080	4080	-4080	-4080	240	240	-240	-240
χ_{116}	4080	4080	-4080	-4080	240	240	-240	-240
χ_{117}	5440	5440	-5440	-5440	320	320	-320	-320
χ_{118}	120	-120	-120	120	-8	8	8	-8
χ_{119}	120	-120	-120	120	-8	8	8	-8
χ_{120}	1920	-1920	-1920	1920	-128	128	128	-128
χ_{121}	1920	-1920	-1920	1920	-128	128	128	-128
χ_{122}	2040	-2040	-2040	2040	-136	136	136	-136
χ_{123}	2040	-2040	-2040	2040	-136	136	136	-136

Table 8.12: Partial character table of $\overline{G}$ corresponding to 1A

$[g]$	1A							
$\lvert C_{\overline{G}}(\overline{g})\rvert$	1002700800	1002700800	1002700800	1002700800	3932160	3932160	3932160	3932160
$[\overline{g}]$	1A	2A	2B	2C	2D	2E	2F	2G
χ_{124}	2040	-2040	-2040	2040	-136	136	136	-136
χ_{125}	2040	-2040	-2040	2040	-136	136	136	-136
χ_{126}	2040	-2040	-2040	2040	-136	136	136	-136
χ_{127}	2040	-2040	-2040	2040	-136	136	136	-136
χ_{128}	3600	-3600	-3600	3600	-240	240	240	-240
χ_{129}	3600	-3600	-3600	3600	-240	240	240	-240
χ_{130}	3600	-3600	-3600	3600	-240	240	240	-240
χ_{131}	3600	-3600	-3600	3600	-240	240	240	-240
χ_{132}	4080	-4080	-4080	4080	-272	272	272	-272
χ_{133}	4080	-4080	-4080	4080	-272	272	272	-272
χ_{134}	120	-120	120	-120	-8	8	-8	8
χ_{135}	120	-120	120	-120	-8	8	-8	8
χ_{136}	1920	-1920	1920	-1920	-128	128	-128	128
χ_{137}	1920	-1920	1920	-1920	-128	128	-128	128
χ_{138}	2040	-2040	2040	-2040	-136	136	-136	136
χ_{139}	2040	-2040	2040	-2040	-136	136	-136	136
χ_{140}	2040	-2040	2040	-2040	-136	136	-136	136
χ_{141}	2040	-2040	2040	-2040	-136	136	-136	136
χ_{142}	2040	-2040	2040	-2040	-136	136	-136	136
χ_{143}	2040	-2040	2040	-2040	-136	136	-136	136
χ_{144}	3600	-3600	3600	-3600	-240	240	-240	240
χ_{145}	3600	-3600	3600	-3600	-240	240	-240	240
χ_{146}	3600	-3600	3600	-3600	-240	240	-240	240
χ_{147}	3600	-3600	3600	-3600	-240	240	-240	240
χ_{148}	4080	-4080	4080	-4080	-272	272	-272	272
χ_{149}	4080	-4080	4080	-4080	-272	272	-272	272
χ_{150}	120	120	-120	-120	-8	-8	8	8
χ_{151}	120	120	-120	-120	-8	-8	8	8
χ_{152}	1920	1920	-1920	-1920	-128	-128	128	128
χ_{153}	1920	1920	-1920	-1920	-128	-128	128	128
χ_{154}	2040	2040	-2040	-2040	-136	-136	136	136
χ_{155}	2040	2040	-2040	-2040	-136	-136	136	136
χ_{156}	2040	2040	-2040	-2040	-136	-136	136	136
χ_{157}	2040	2040	-2040	-2040	-136	-136	136	136
χ_{158}	2040	2040	-2040	-2040	-136	-136	136	136
χ_{159}	2040	2040	-2040	-2040	-136	-136	136	136
χ_{160}	3600	3600	-3600	-3600	-240	-240	240	240
χ_{161}	3600	3600	-3600	-3600	-240	-240	240	240
χ_{162}	3600	3600	-3600	-3600	-240	-240	240	240
χ_{163}	3600	3600	-3600	-3600	-240	-240	240	240
χ_{164}	4080	4080	-4080	-4080	-272	-272	272	272
χ_{165}	4080	4080	-4080	-4080	-272	-272	272	272

8.7 The quotient group $\overline{G}/Z(\overline{G})$

The centre of the affine subgroup $\overline{G}$ of the symplectic group $Sp(6,4)$ is isomorphic to $\mathbb{Z}_4$. The quotient group $\overline{G}/Z(\overline{G})$ is isomorphic to the split extension $2^8{:}Sp(4,4)$. The Fischer matrices of $\overline{G}/Z(\overline{G})$ can be determined directly from the Fischer matrices of $\overline{G}$. In Section 8.5 we computed all the 27 Fischer matrices of $\overline{G}$. The action of $Sp(4,4)$ on $Irr(2^8)$ yields two inertia factor groups, namely $Sp(4,4)$ and $2^6{:}Sp(2,4)$, the affine subgroup of $Sp(4,4)$. The latter was dealt with in the previous chapter, Chapter 7. In Section 8.4 we labelled these two inertia factor groups by H_1 and H_2, respectively. This then means that the inertia factor groups $H_3, H_4, \cdots, H_8$ do not play a role in $\overline{G}/Z(\overline{G})$. Then, to obtain the Fischer matrices of $\overline{G}/Z(\overline{G})$ from those of $\overline{G}$, we delete the rows corresponding to $H_3, H_4, \cdots, H_8$ in each Fischer matrix of $\overline{G}$. Thereafter, we discard the repeated columns. We use the Fischer matrices corresponding to the identity and the class $2A$ of $Sp(4,4)$, from Table 8.8, to demonstrate this process. From $M(1A)$ we get

$$M(1A) = \begin{bmatrix} 1 & 1 \\ 255 & -1 \end{bmatrix},$$

and from $M(2A)$ we get

$$M(2A) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 60 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

The above-mentioned process is used to determine all the 27 Fischer matrices of $\overline{G}/Z(\overline{G})$. These deduced Fischer matrices can then be used to construct the character table of $\overline{G}/Z(\overline{G})$, using the Clifford-Fischer Theory, since $\overline{G}/Z(\overline{G})$ is a split extension.

9

The affine subgroup $3^3{:}GO(3,3)$ of the general orthogonal group $GO(5,3)$

This chapter is about the affine subgroup $\overline{G} = q^{2n-1}{:}GO(2n-1,q) = 3^3{:}GO(3,3)$ of the full orthogonal group $GO(5,3)$. Let $\mathbb{F}$ be the Galois field, $GF(3)$, of three elements and $G = GO(3,3)$ be the full orthogonal group of invertible 3×3 matrices with entries in $\mathbb{F}$. Let V be a 5-dimensional vector space over $\mathbb{F}$ with basis $\{e_1, e_2, \cdots, e_5\}$. This affine subgroup $A(2)$ is a stabilizer of a non-zero isotropic vector in V. The order of $A(2)$ is 1296 and is of index $|GO(5,3){:}A(2)| = 3^{2\cdot 2} - 1 = 80$ in $GO(5,3)$ by Remark 4.3.9. In Lemma 4.3.10 it is proved that the affine subgroup $A(2) = 3^3{:}GO(3,3)$ is a semidirect product and that the group 3^3 is abelian. Let us denote this normal abelian subgroup 3^3 of $\overline{G}$ by N. In Section 9.1 we construct the groups N and G inside $\overline{G}$ and eventually determine their respective generators as 5×5 matrices. We determine the point stabilizers of the action of G on N in Section 9.2. We then determine the fusion maps of these point stabilizers into G. Their respective permutation characters are also computed. At the end of this section we express these permutation characters in terms of $Irr(G)$. In Section 9.3 we compute the conjugacy classes of $\overline{G}$ using the coset analysis technique. The permutation characters determined in Section 9.2 are used to compute the permutation character $\chi(G|N)$. Section 9.4 is about the inertia factor groups of the action of G on $Irr(N)$. We make a note here that these inertia factor groups are in fact isomorphic to the point stabilizers determined in Section 9.2. The Fischer matrices of $\overline{G}$ are computed in Section 9.5. We then construct the character table of $\overline{G}$ in Section 9.6. Lastly, in Section 9.7, we consider the fusion of the affine subgroup $\overline{G}$ into the full orthogonal group $GO(5,3)$.

9.1 The construction of 3^3 and $GO(3,3)$ in $GO(5,3)$

Since the affine subgroup $\overline{G} = N{:}G$ sits in $GO(5,3)$, we construct the groups N and G inside $GO(5,3)$. The order of $\overline{G}$ is 1296. Using GAP, we analyze subgroups of order 1296 in $GO(5,3)$. We note that there are 13 of these subgroups. Next we obtain that there are 10 that have normal subgroups of order 27. Seven of these normal subgroups are abelian and 3 are non-abelian. We exclude the latter 3 since N must be abelian by Lemma 4.3.10. We note further that the abelian normal subgroups are in fact elementary abelian. We then proceed to construct a complement of N in $\overline{G}$. We require that this complement be isomorphic to G. According to GAP, the structure description of G is $C_2 \times S_4$ and has 10 conjugacy classes. The complement of one of the elementary abelian groups is of order 48 but has 16 conjugacy classes and its structure description is $C_2 \times C_2 \times A_4$. Therefore it is not isomorphic to G. According to Remark 4.3.14, $\overline{G}$ has 4 inertia factor groups, $H_1 = G$, $H_2 = 3{:}GO(1,3)$, $H_3 = GO^+(2,3)$ and $H_4 = GO^-(2,3)$, where H_2 is the affine subgroup of $GO(3,3)$. Also we have $|Irr(H_1)| = 10$, $|Irr(H_2)| = 3$, $|Irr(H_3)| = 4$ and $|Irr(H_4)| = 5$. Thus $\overline{G}$ has 22 conjugacy classes. The complements of the last 6 candidates for N are isomorphic to G. However, two of these produce semidirect products $N{:}G$ have 28 and 34 conjugacy classes, respectively. The number of conjugacy classes of the semidirect products from the other 4 are 22 as required. We further note that from the latter 4 candidates that the respective semidirect products are in fact isomorphic. We then select the normal abelian group N and its complement G from one of these 4. The generators of N are

$$e_1 = \begin{bmatrix} 2 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 2 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix}$$

and

$$e_3 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The generators of G are

$$a = \begin{bmatrix} 2 & 1 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 & 2 \end{bmatrix},$$

$$d = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 2 & 2 & 1 & 2 \end{bmatrix}$$

and

$$e = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

Using the ATLAS we can see that $GO(5,3)$ is isomorphic to $2 \times (O(5,3){:}2)$ where $O(5,3)$ is the simple orthogonal group isomorphic to the simple symplectic group $S_4(3)$. Our group $\overline{G}$ sits maximally in $O(5,3){:}2$. The affine subgroup of the full symplectic group $Sp(4,3)$ is $3^3{:}S_4$. Our $\overline{G}$ sits maximally in $Sp(4,3)$ containing the affine subgroup of $Sp(4,3)$.

9.2 The point stabilizers of the action of G on N

According to Remark 4.3.11, the action of G on N yields $q+1 = 3+1 = 4$ orbits. The orbit lengths are 1, $3^{2(2-1)} - 3^{2-1} = 6$, $3^{2\cdot 2-2} - 1 = 8$ and $3^{2(2-1)} + 3^{2-1} = 12$. We note that these orbit lengths coincide with the indices in G of the inertia factor groups G, $GO^+(2,3)$, $GO^-(2,3)$ and $3{:}GO(1,3)$, the affine subgroup of G. Thus we conclude that the point stabilizers are isomorphic to G, $GO^+(2,3)$, $GO^-(2,3)$ and $3{:}GO(1,3)$. We note further that $GO^+(2,3) \cong D_4$, $GO^-(2,3) \cong D_8$ and $3{:}GO(1,3) \cong D_6$. We use GAP to compute the permutation characters and fusion maps of these point stabilizers into G. The permutation character values are given in the last rows of the fusion tables below.

Table 9.1: Fusion of D_6 into G

	$\|C_G\|$	48	8	16	16	8	48	6	8	8	6
	$o(g)$	1A	2A	2B	2C	2D	2E	3A	4A	4B	6A
$o(h)$	$\|C_H\|$										
$1a$	6	$\boxed{8}$									
$2a$	2		$\boxed{4}$	8	8	4	24				
$3a$	3							$\boxed{2}$			
	$\chi(G\|D_6)$	8	4	0	0	0	0	2	0	0	0

Table 9.2: Fusion of D_4 into G

	$\|C_G\|$	48	8	16	16	8	48	6	8	8	6
	$o(g)$	1A	2A	2B	2C	2D	2E	3A	4A	4B	6A
$o(h)$	$\|C_H\|$										
$1a$	4	$\boxed{12}$									
$2a$	4		$\boxed{2}$	4	4	2	12				
$2b$	4		2	4	$\boxed{4}$	2	12				
$2c$	4		2	4	4	$\boxed{2}$	12				
	$\chi(G\|D_4)$	12	2	0	4	2	0	0	0	0	0

Table 9.3: Fusion of D_8 into G

	$\|C_G\|$	48	8	16	16	8	48	6	8	8	6
	$o(g)$	1A	2A	2B	2C	2D	2E	3A	4A	4B	6A
$o(h)$	$\|C_H\|$										
$1a$	8	$\boxed{6}$									
$2a$	4		$\boxed{2}$	4	4	2	12				
$2b$	8		1	$\boxed{2}$	2	1	6				
$2c$	4		2	4	$\boxed{4}$	2	12				
$4a$	4								2	$\boxed{2}$	
	$\chi(G\|D_8)$	6	2	2	4	2	0	0	0	2	0

We proceed to express these permutation characters in terms of the irreducible characters of G. Let $\chi_i \in Irr(G)$. We restrict the characters of G to the characters of D_n and then compute the respective inner products. The values of these inner products will in turn enable us to determine the constituents of the respective permutation characters.

Table 9.4: Values of the inner product $\langle \chi_i, Irr(D_n) \rangle$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
$\langle \chi_i, Irr(D_6) \rangle$	1	1	0	0	0	0	0	1	1	0
$\langle \chi_i, Irr(D_4) \rangle$	1	0	0	0	0	1	0	1	1	1
$\langle \chi_i, Irr(D_8) \rangle$	1	0	0	0	0	1	0	0	1	0
$\langle \chi_i, \chi(G\|N) \rangle$	3	1	0	0	0	2	0	2	3	1

Thus the respective permutation characters in terms of $Irr(G)$ are

$$\chi(G|D_6) = \chi_1 + \chi_2 + \chi_8 + \chi_9,$$

$$\chi(G|D_4) = \chi_1 + \chi_6 + \chi_8 + \chi_9 + \chi_{10}$$

and

$$\chi(G|D_8) = \chi_1 + \chi_6 + \chi_9.$$

The character tables of D_4, D_6, D_8 and G are in Section 9.4.

9.3 The conjugacy classes of $\overline{G}$

In Section 9.2 we had that the action of G on N yields orbits of lengths 1, 8, 12 and 6. These corresponding to the point stabilizers G, D_6, D_4 and D_8 respectively. Let $I_{D_n}^G$ be the identity character of D_n induced to G and expressed in terms of Irr(G). Let χ(G$|N$) be the permutation character when G acts on N. Then

$$\chi(G|N) = 1 + I_{D_6}^G + I_{D_4}^G + I_{D_8}^G$$

$$= \chi_1 + (\chi_1 + \chi_2 + \chi_8 + \chi_9) + (\chi_1 + \chi_6 + \chi_8 + \chi_9 + \chi_{10}) + (\chi_1 + \chi_6 + \chi_9)$$

$$= 4\chi_1 + \chi_2 + 2\chi_6 + 2\chi_8 + 3\chi_9 + \chi_{10}$$

where $\chi_i \in Irr(G)$. Alternatively, we can obtain χ(G$|N$) directly from the last row of Table 9.4 above. This permutation character is then used to compute the number of fixed points, the k values, when $g \in G$ is acting on N.

Table 9.5: Fixed points of the action of G on N

$\lvert C_G \rvert$	48	8	16	16	8	48	6	8	8	6
$o(g)$	1A	2A	2B	2C	2D	2E	3A	4A	4B	6A
k	27	9	3	9	3	1	3	1	3	1

We proceed to compute the conjugacy classes of $\overline{G}$ using the coset analysis. Since $\overline{G}$ is a semidirect product, we analyse the coset Ng, where g is a class representative of G and $\overline{G} = \bigcup_{g \in G} Ng$. This also entails computing the centralizer sizes and orders of the new classes. We use $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_i}$ for the centralizer sizes, where f_i of the k blocks of the coset Ng have fused together to form a new orbit Ω_i. The orbit lengths are given by $|\Omega_i| = \frac{f_i|N|}{k}$. This enables us to calculate the f_i-values for each class representative $g \in G$. Let us consider the class $2A \in G$. From Table 9.5 above, the permutation character value for this class is $k = 9$. Meaning that under the action of g on N we have 9 fixed points. It also means that after the action of N, by conjugation, the coset Ng splits into 9 orbits. Each orbit length is $\frac{|N|}{k} = \frac{27}{9} = 3$. We note, using GAP, that the action of the centralizer $C_G(g)$ on these orbits yields 4 orbits of lengths $|\Omega_1| = 3$, $|\Omega_2| = 12$, $|\Omega_3| = 6$ and $|\Omega_4| = 6$. The corresponding f-values are $f_1 = 1$, $f_2 = 4$, $f_3 = 2$ and $f_4 = 2$. This means that 4, 2 and 2 of the original 9 orbits have fused together to form the orbits Ω_2, Ω_3 and Ω_4, respectively. These f-values satisfy $\sum_{i=1}^{4} f_i = k$. So this class produces 4 classes of $\overline{G}$. Programme A in [15] is used to calculate the rest of the f-values and the orders of the new classes of $\overline{G}$. For this class we have

$$|C_{\overline{G}}(x_1)| = \frac{k|C_G(g)|}{f_1} = \frac{9 \cdot 8}{1} = 72 \qquad [x_1] = 2A$$

$$|C_{\overline{G}}(x_2)| = \frac{k|C_G(g)|}{f_2} = \frac{9 \cdot 8}{4} = 18 \qquad [x_2] = 6A$$

$$|C_{\overline{G}}(x_3)| = \frac{k|C_G(g)|}{f_3} = \frac{9 \cdot 8}{2} = 36 \qquad [x_3] = 6B$$

$$|C_{\overline{G}}(x_4)| = \frac{k|C_G(g)|}{f_4} = \frac{9 \cdot 8}{2} = 36 \qquad [x_4] = 6C$$

The complete list of the conjugacy classes of $\overline{G}$ is contained in Table 9.6 below. This table also contains the $m_i = \frac{f_i|N|}{k}$ values to aid with the construction of the Fischer matrices in Section 9.5.

Table 9.6: The conjugacy classes of $\overline{G}$

$[g]_G$	k	f_i	m_i	$\lvert C_G(g)\rvert$	$\lvert C_{\overline{G}}(x)\rvert$	$\lvert[x]\rvert$	$[x]_{\overline{G}}$
1A	27	1	1	48	1296	1	1A
		8	8		162	8	3A
		12	12		108	12	3B
		6	6		216	6	3C
2A	9	1	3	8	72	18	2A
		4	6		18	72	6A
		2	12		36	36	6B
		2	6		36	36	6C
2B	3	1	9	16	48	27	2B
		2	18		24	54	6D
2C	9	1	3	16	144	9	2C
		4	12		36	36	6E
		4	12		36	36	6F
2D	3	1	9	8	24	54	2D
		2	18		12	108	6G
2E	1	1	27	48	48	27	2E
3A	3	1	9	6	18	72	3D
		2	18		9	144	9A
4A	1	1	27	8	8	162	4A
4B	3	1	9	8	24	54	4B
		2	18		12	108	12A
6A	1	1	27	6	6	216	6H

9.4 Inertia factor groups of $\overline{G}$

In Section 9.2 we had that the action of G on N yielded 4 orbits. In turn, due to Brauer's Theorem, when G acts on $Irr(N)$ we again have 4 orbits. From Remark 4.3.14 we have that the inertia factor groups of the latter action are isomorphic to $H_1 = G$, $H_2 = 3{:}GO(1,3)$, $H_3 = GO^+(2,3)$ and $H_4 = GO^-(2,3)$. We mentioned in Section 9.2 that these inertia factor groups are isomorphic to the point stabilizers of the action of G on N. In that section we also dealt with the fusions of these inertia factor groups into G and their respective permutation characters. These fusion maps will play a crucial role in the computation of the Fischer matrices in Section 9.5 and the construction of the character table of $\overline{G}$ in Section 9.6.

We list below the character tables of the inertia factor groups.

Table 9.7: Character Table of D_6

$\lvert C_{D_6}\rvert$	6	2	3
$o(j)$	1a	2a	3a
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 9.8: Character Table of D_4

$\lvert C_{D_4}\rvert$	4	4	4	4
$o(k)$	1a	2a	2b	2c
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

Table 9.9: Character Table of D_8

$\lvert C_{D_8}\rvert$	8	4	8	4	4
$o(l)$	1a	2a	2b	2c	4a
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	1	-1
χ_5	2	0	-2	0	0

Table 9.10: Character Table of G

$\|C_G\|$	48	8	16	16	8	48	6	8	8	6
$o(g)$	1a	2a	2b	2c	2d	2e	3a	4a	4b	6a
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1	1	-1	1	-1
χ_4	1	-1	1	1	-1	1	1	-1	-1	1
χ_5	2	0	2	-2	0	-2	-1	0	0	1
χ_6	2	0	2	2	0	2	-1	0	0	-1
χ_7	3	-1	-1	-1	-1	3	0	1	1	0
χ_8	3	1	-1	-1	1	3	0	-1	-1	0
χ_9	3	1	-1	1	-1	-3	0	-1	1	0
χ_{10}	3	-1	-1	1	1	-3	0	1	-1	0

9.5 Fischer matrices of $\overline{G}$

In this section we compute the Fischer matrices of $\overline{G}$. We recall that the Fischer matrix at the identity class $1A$ is the matrix with rows equal to orbit sums of the action of $\overline{G}$ on $Irr(N)$ with duplicate columns discarded. This action yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 12 & 3 & 3 & -3 & -3 & -3 & 0 & 0 & 0 & 3 & 3 & 3 & \cdots \\ 8 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & -4 & -1 & -1 & -1 & \cdots \\ 6 & -3 & -3 & 0 & 0 & 0 & 3 & 3 & 3 & -3 & -3 & -3 & \cdots \end{bmatrix}_{4\times 27}.$$

After discarding the repeated columns we rearrange the columns such that they align with the f-values and the orders of centralizers in Table 9.6 above. We also arrange the rows in blocks such that they coincide with the fusions of the classes of the inertia factor groups into the classes of G. According to the fusion maps in Section 9.2, the Fischer matrix $M(1A)$ will have 4 blocks and 4 rows. Ultimately we have the following Fischer matrix at the identity class.

Table 9.11: Fischer matrix at 1A

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	3	3	3
$\|C_{\overline{G}}(x_{1j})\|$		1296	168	108	216
(k,m)	$\|C_{H_k}(x_{1km})\|$				
$(1,1)$	48	1	1	1	1
$(2,1)$	6	8	-1	2	-4
$(3,1)$	4	12	3	-3	0
$(4,1)$	8	6	-3	0	3
m_{1j}		1	8	12	6

The Fischer matrices from non-identity classes can be deduced from the character table of the quotient N/M, where $M = \{mgm^{-1}g^{-1} : m \in N\}$ and $[N{:}M] = k$. Here k has the same meaning as in the coset analysis. That is k is the number of fixed points when $g \in G$ acts on N. These Fischer matrices are square matrices with rows equal to the orbit sums of the action of the centralizer C_G on $Irr(N/M)$. This action is equivalent to the action of C_G on the orbits of the action of N on Ng. Let us consider the class $2A$ for demonstration. For this class $k = 9$ and thus $|M| = 3$. The class representatives in the character table of N/M are

$$1a, 3a, 3b, 3c, 3d, 3e, 3f, 3g, 3h.$$

From the coset analysis process, Table 9.6, we note that this class produces 4 classes of $\overline{G}$. In the same table we note that the action of N on Ng produced 9 orbits. Since $f_1 = 1$, $f_2 = 4$, $f_3 = 2$ and $f_4 = 2$, this means that after the action of C_G on these orbits some fused together to ultimately have 4 orbits. This then means that the size of the Fischer matrix corresponding to this class will be a 4×4 matrix. The action of C_G on $Irr(N/M)$ yields the following table with entries which are orbit sums of the irreducible characters,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 4 & 1 & -2 & 4 & 1 & 1 & -2 & -2 & -2 & 1 & -2 & -2 & \cdots \\ 2 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & 2 & 2 & \cdots \\ 2 & -1 & -1 & 2 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & \cdots \end{bmatrix}_{4\times 27}.$$

We then delete repeated columns to have the following table,

$$M' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & -2 & -2 \\ 2 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 \end{bmatrix}.$$

Thereafter we rearrange the remaining columns such that they align with the f-values and centralizer orders in Table 9.6 above. According to the properties of Fischer matrices, the entries of the first column must satisfy $|C_{H_1}(2A){:}C_{H_i}(2a)|$, $i \in \{1, 2, 3, 4\}$, since N is elementary abelian. This means the first column of M' above will be the first column of the desired Fischer matrix $M(2A)$ from this class. It is also clear that the second column of M' corresponds to the class $6A$ and thus forms the second column of $M(2A)$. There is a dispute regarding columns 3 and 4 of M'. To resolve this dispute we utilize the congruency relation $\chi(g) \equiv \chi(g^p) \bmod p$ and the power maps of $\overline{G}$. The power maps of $\overline{G}$ are listed in Table 9.14, the character table of $\overline{G}$. In that table we have $(6B)^2 = 3C$ meaning $\chi(3C) \equiv \chi(6B) \bmod 2$. Now if we suppose that column 3 of M' corresponds to the class $6C$ then the above congruency relation yields $0 \equiv \pm 1 \bmod 2$ and $3 \equiv \pm 2 \bmod 2$ in the blocks corresponding to the inertia factor groups H_3 and H_4 respectively, in Table 9.14. This is a contradiction. Thus we conclude that columns 3 and 4 correspond to the classes $6B$ and $6C$ respectively. Lastly we arrange the rows in blocks such that they coincide with the fusions of the classes of the inertia factor groups into the classes of G. According to the fusion maps in Section 9.2, the Fischer matrix $M(2A)$ will have 4 blocks and 4 rows. Ultimately for this class we have the following Fischer matrix.

Table 9.12: Fischer matrix at 2A

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$
$o(x_{2j})$		2	6	6	6
$\lvert C_{\overline{G}}(x_{2j})\rvert$		72	18	36	36
(k, m)	$\lvert C_{H_k}(x_{2km})\rvert$				
$(1, 1)$	8	1	1	1	1
$(2, 1)$	2	4	1	-2	-2
$(3, 1)$	4	2	-1	2	-1
$(4, 1)$	4	2	-1	-1	2
m_{2j}		3	12	6	6

We note that for this class the Fischer matrix $M(2A)$ coincides with M' above. In general this is not always the case. We remark that in cases where there is no fusion of classes during the coset analysis, that is, C_G fixes the character table of N/M, then the resulting Fischer matrix will be exactly the character table of N/M. In the event that $|M| = |N|$ then the corresponding Fischer matrix will be the 1×1 matrix [1]. We use Programmes C and D in [15] to compute all the Fischer matrices of $\overline{G}$ in Table 9.13 below.

Table 9.13: Fischer matrices of $\overline{G}$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	3	3	3
$\lvert C_{\overline{G}}(x_{1j})\rvert$		1296	168	108	216
(k, m)	$\lvert C_{H_k}(x_{1km})\rvert$				
$(1, 1)$	48	1	1	1	1
$(2, 1)$	6	8	-1	2	-4
$(3, 1)$	4	12	3	-3	0
$(4, 1)$	8	6	-3	0	3
m_{1j}		1	8	12	6

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$
$o(x_{2j})$		2	6	6	6
$\lvert C_{\overline{G}}(x_{2j})\rvert$		72	18	36	36
(k, m)	$\lvert C_{H_k}(x_{2km})\rvert$				
$(1, 1)$	8	1	1	1	1
$(2, 1)$	2	4	1	-2	-2
$(3, 1)$	4	2	-1	2	-1
$(4, 1)$	4	2	-1	-1	2
m_{2j}		3	12	6	6

Table 9.13: Fischer matrices of $\overline{G}$

$\mathbb{F}_3 = M(2B)$		$x_{3,1}$	$x_{3,2}$
$o(x_{3j})$		2	6
$\|C_{\overline{G}}(x_{3j})\|$		48	24
(k,m)	$\|C_{H_k}(x_{3km})\|$		
$(1,1)$	16	1	1
$(4,1)$	4	4	-1
m_{3j}		9	18

$\mathbb{F}_4 = M(2C)$		$x_{4,1}$	$x_{4,2}$	$x_{4,3}$
$o(x_{4j})$		2	6	6
$\|C_{\overline{G}}(x_{4j})\|$		144	36	36
(k,m)	$\|C_{H_k}(x_{4km})\|$			
$(1,1)$	16	1	1	1
$(3,1)$	4	4	1	-2
$(4,1)$	4	4	-2	1
m_{4j}		3	12	12

$\mathbb{F}_5 = M(2D)$		$x_{5,1}$	$x_{5,2}$
$o(x_{5j})$		2	6
$\|C_{\overline{G}}(x_{5j})\|$		24	12
(k,m)	$\|C_{H_k}(x_{5km})\|$		
$(1,1)$	8	1	1
$(3,1)$	4	2	-1
m_{5j}		9	18

$\mathbb{F}_6 = M(2E)$		$x_{6,1}$
$o(x_{6j})$		2
$\|C_{\overline{G}}(x_{6j})\|$		48
(k,m)	$\|C_{H_k}(x_{6km})\|$	
$(1,1)$	48	1
m_{6j}		27

$\mathbb{F}_7 = M(3A)$		$x_{7,1}$	$x_{7,2}$
$o(x_{7j})$		3	9
$\|C_{\overline{G}}(x_{7j})\|$		18	9
(k,m)	$\|C_{H_k}(x_{7km})\|$		
$(1,1)$	6	1	1
$(2,1)$	3	2	-1
m_{7j}		9	18

$\mathbb{F}_8 = M(4A)$		$x_{8,1}$
$o(x_{8j})$		4
$\|C_{\overline{G}}(x_{8j})\|$		8
(k,m)	$\|C_{H_k}(x_{8km})\|$	
$(1,1)$	8	1
m_{8j}		27

$\mathbb{F}_9 = M(4B)$		$x_{9,1}$	$x_{9,2}$
$o(x_{9j})$		4	12
$\|C_{\overline{G}}(x_{9j})\|$		24	12
(k,m)	$\|C_{H_k}(x_{9km})\|$		
$(1,1)$	8	1	1
$(4,1)$	4	2	-1
m_{9j}		9	18

$\mathbb{F}_{10} = M(6A)$		$x_{10,1}$
$o(x_{10j})$		6
$\|C_{\overline{G}}(x_{10j})\|$		6
(k,m)	$\|C_{H_k}(x_{10km})\|$	
$(1,1)$	6	1
m_{10j}		27

9.6 The character table of $\overline{G}$

The character table of $\overline{G}$ is constructed using the Clifford-Fischer Theory. This technique entails utilizing the Fischer matrices of $\overline{G}$ together with the character tables of the inertia factor groups. This technique also requires that the irreducible characters of $N \trianglelefteq \overline{G}$ be extendable to the inertia groups. Now since N is elementary abelian and $\overline{G}$ is a semidirect product of N by G, then the $Irr(N)$ are extendable to the inertia groups. We also know that the $Irr(\overline{G})$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\phi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \in ker(\beta)\},$$

where $\overline{H}_i$ is an inertia group and $H_i = \overline{H}_i/N$ is an inertia factor group. This implies that the character table of $\overline{G}$ will be divided into blocks $B_{i,j}$ corresponding to the 4 inertia factor groups H_i and the 10 conjugacy classes of G. That is $1 \leq i \leq 4$ and $1 \leq j \leq 10$. The block $B_{i,j}$ is formed by multiplying the relevant columns of the character table of H_i by the rows of the Fischer matrix $M(g)$ corresponding to the classes of H_i that fuse to the class $[g] \in G$. If H_i does not contribute to $M(g)$ then the block $B_{i,j}$ will have zeroes. The character tables of the inertia factor groups are given in Section 9.4. The fusion maps are discussed in Section 9.2.

Table 9.14 below is the full character table of $\overline{G}$. Its consistency and accuracy were tested using Programme E. This table also contains power maps of $\overline{G}$ which were computed using Programmes A and B. The above-mentioned programmes can be found in [15].

Table 9.14: Character Table of $\overline{G}$

$[g]$	1A				2A				2B	
$[x]$	1A	3A	3B	3C	2A	6A	6B	6C	2B	6D
$\lvert C_{\overline{G}}(x)\rvert$	1296	162	108	216	72	18	36	36	48	24
$2P$	1A	3A	3B	3C	1A	3A	3C	3B	1A	3C
$3P$	1A	1A	1A	1A	2A	2A	2A	2A	2B	2B
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1
χ_3	1	1	1	1	-1	-1	-1	-1	1	1
χ_4	1	1	1	1	-1	-1	-1	-1	1	1
χ_5	2	2	2	2	0	0	0	0	2	2
χ_6	2	2	2	2	0	0	0	0	2	2
χ_7	3	3	3	3	-1	-1	-1	-1	-1	-1
χ_8	3	3	3	3	1	1	1	1	-1	-1
χ_9	3	3	3	3	1	1	1	1	-1	-1
χ_{10}	3	3	3	3	-1	-1	-1	-1	-1	-1
χ_{11}	8	-1	2	-4	4	1	-2	-2	0	0
χ_{12}	8	-1	2	-4	-4	-1	2	2	0	0
χ_{13}	16	-2	4	-8	0	0	0	0	0	0
χ_{14}	12	3	-3	0	2	-1	2	-1	0	0
χ_{15}	12	3	-3	0	-2	1	-2	1	0	0
χ_{16}	12	3	-3	0	2	-1	2	-1	0	0
χ_{17}	12	3	-3	0	-2	1	-2	1	0	0
χ_{18}	6	-3	0	3	2	-1	-1	2	2	-1
χ_{19}	6	-3	0	3	-2	1	1	-2	2	-1
χ_{20}	6	-3	0	3	2	-1	-1	2	2	-1
χ_{21}	6	-3	0	3	-2	1	1	-2	2	-1
χ_{22}	12	-6	0	6	0	0	0	0	-4	2

Table 9.14: Character Table of $\overline{G}$ - continued

$[g]$	2C			2D		2E	3A		4A	4B		6A
$[x]$	2C	6E	6F	2D	6G	2E	3D	9A	4A	4B	12A	6H
$\lvert C_{\overline{G}}(x)\rvert$	144	36	36	24	12	48	18	9	8	24	12	6
$2P$	1A	3B	3C	1A	3B	1A	3D	9A	2B	2B	6D	3D
$3P$	2C	2C	2C	2D	2D	2E	1A	3A	4A	4B	4B	2E
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1
χ_3	-1	-1	-1	1	1	-1	1	1	-1	1	1	-1
χ_4	1	1	1	-1	-1	1	1	1	-1	-1	-1	1
χ_5	-2	-2	-2	0	0	-2	-1	-1	0	0	0	1
χ_6	2	2	2	0	0	2	-1	-1	0	0	0	-1
χ_7	-1	-1	-1	-1	-1	3	0	0	1	1	1	0
χ_8	-1	-1	-1	1	1	3	0	0	-1	-1	-1	0
χ_9	1	1	1	-1	-1	-3	0	0	-1	1	1	0
χ_{10}	1	1	1	1	1	-3	0	0	1	-1	-1	0
χ_{11}	0	0	0	0	0	0	2	-1	0	0	0	0
χ_{12}	0	0	0	0	0	0	2	-1	0	0	0	0
χ_{13}	0	0	0	0	0	0	-2	1	0	0	0	0
χ_{14}	4	1	-2	2	-1	0	0	0	0	0	0	0
χ_{15}	4	1	-2	-2	1	0	0	0	0	0	0	0
χ_{16}	-4	-1	2	-2	1	0	0	0	0	0	0	0
χ_{17}	-4	-1	2	2	-1	0	0	0	0	0	0	0
χ_{18}	4	-2	1	0	0	0	0	0	0	2	-1	0
χ_{19}	-4	2	-1	0	0	0	0	0	0	2	-1	0
χ_{20}	-4	2	-1	0	0	0	0	0	0	-2	1	0
χ_{21}	4	-2	1	0	0	0	0	0	0	-2	1	0
χ_{22}	0	0	0	0	0	0	0	0	0	0	0	0

According to Theorem 4.3.12, the degrees of the $Irr(\overline{G})$ are the degrees of $Irr(G)$, the degrees of $Irr(H_2)$ multiplied by $3^{2\cdot 2-2}-1=8$, the degrees of $Irr(H_3)$ multiplied by $3^{2-1}(3^{2-1}+1)=12$ and the degrees of $Irr(H_4)$ multiplied by $3^{2-1}(3^{2-1}-1)=6$.

9.7 Fusion of $\overline{G}$ into $GO(5,3)$

We conclude this chapter with the fusion of the affine subgroup $\overline{G}$ into the full orthogonal group $GO(5,3)$. Let $[x]$ and $[y]$ be the conjugacy classes of $\overline{G}$ and $GO(5,3)$ respectively. We construct the partial fusion table by considering the divisibility of the respective centralizer sizes $\frac{|C(y)|}{|C(x)|}$ with $o(x)=o(y)$. In this process we also employ the permutation character of $GO(5,3)$ on the cosets of $\overline{G}$ in $GO(5,3)$ together with the respective power maps. We use GAP to compute the permutation character $\gamma=\chi(GO(5,3)|\overline{G})$. The values of γ are listed in the last row of the partial fusion table. This is Table 9.16 but without considering the boxes around the entries. We use the values of γ and $\psi_i \in Irr(GO(5,3))$ to express γ in terms of $Irr(GO(5,3))$. We compute the inner products $\langle\psi_i,\gamma\rangle$, for $1\leq i\leq 50$, to determine the constituents of γ. This method yields

$$\chi(GO(5,3)|\overline{G})=\psi_1+\psi_{10}+\psi_{14}+\psi_{28}+\psi_{32},$$

in terms of the $Irr(GO(5,3))$.

The power maps aid in the fusion process in the following manner. Suppose that $[x_1]$ and $[x_2]$ are conjugacy classes of $\overline{G}$ such that $x_1^p \in [x_2]$ for some prime p. And suppose that $[y_1]$ and $[y_2]$ are conjugacy classes of $GO(5,3)$ such that $y_1^p \in [y_2]$. Now $[x_2]$ fuses to $[y_2]$ if and only if $[x_1]$ fuses to $[y_1]$. The irreducible characters and power maps of $GO(5,3)$ are found in Table A.2, the character table of $GO(5,3)$, in the appendix.

It is clear from the partial fusion that the classes $3A$, $3B$, $3C$ and $3D$ of $\overline{G}$ fuse, respectively, to the classes $3c$, $3b$, $3a$ and $3b$ in $GO(5,3)$. We note that the class $2B$, and likewise the class $2E$, of $\overline{G}$ fuse either into the class $2b$ or $2f$ of $GO(5,3)$. In Table 9.14, the character table of $\overline{G}$, the power maps yield the following: $(4B)^2=2B$, $(6D)^3=2B$ and $(6H)^3=2E$. In the partial fusion we note that the classes $4B$, $6D$ and $6H$ fuse to the classes $4a$, $6c$ and $6h$ in $GO(5,3)$ respectively. We further note that the power maps in the character table of $GO(5,3)$, yield the following: $(4a)^2=2b$, $(6c)^3=2b$ and $(6h)^3=2f$. Thus the classes $2B$ and $2E$ fuse to the classes $2b$ and $2f$ respectively in $GO(5,3)$. However, this method of using indices, the permutation character and power maps does not give us the suitable candidate for the class $6B$ in $\overline{G}$. According to the partial fusion, the class $6B$ either fuses to the class $6f$ or $6l$ or $6o$ or $6p$ in $GO(5,3)$. Therefore to complete the fusion we employ the method of set intersections for characters which we outline below.

Let ρ be the character afforded by the regular representation of G. It then follows $\rho=\sum_{i=1}^{10} e_i\phi_i$ where $\phi_i \in Irr(G)$ and $e_i=\deg(\phi_i)$. That is ρ can be viewed as the character of $\overline{G}$ which contains 3^3 in its kernel such that

$$\rho(g)=\begin{cases} |G| & g\in 3^3 \\ 0 & \text{otherwise}\end{cases}$$

If ψ is a character of $GO(5,3)$ then

$$\langle \rho, \psi \rangle = \frac{1}{|\overline{G}|}\{\rho(1A)\psi(1A) + 8\rho(3A)\psi(3A) + 12\rho(3B)\psi(3B) + 6\rho(3C)\psi(3C)\}$$

$$= \frac{1}{|\overline{G}|}\{|G|\psi(1A) + 8|G|\psi(3A) + 12|G|\psi(3B) + 6|G|\psi(3C)\}$$

$$= \frac{1}{27}\{\psi(1A) + 8\psi(3A) + 12\psi(3B) + 6\psi(3C)\}$$

$$= \langle \psi\downarrow_N, \tau_1 \rangle$$

where $\psi\downarrow_N$ is the restriction of ψ to 3^3 and τ_1 is the identity character of 3^3. We note that for ψ we have that

$$\psi\downarrow_N = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4$$

where for $i \in \{1,2,3,4\}$, θ_i are sums of the $Irr(3^3)$ which are in one orbit under the action of G on $Irr(3^3)$, and $a_i \in \{0\} \cup \mathbb{N}$. Let $\tau_j \in Irr(3^3)$ for $j \in \{1,2,3,\cdots,27\}$. We know that under the action of G on $Irr(3^3)$ we have orbits of lengths 1, 8, 12 and 6. We have that

$$\theta_1 = \tau_1 \quad \text{and} \quad \deg(\theta_1) = 1,$$

$$\theta_2 = \sum_{j=2}^{9} \tau_j \quad \text{and} \quad \deg(\theta_2) = 8,$$

$$\theta_3 = \sum_{j=10}^{21} \tau_j \quad \text{and} \quad \deg(\theta_3) = 12,$$

$$\theta_4 = \sum_{j=22}^{27} \tau_j \quad \text{and} \quad \deg(\theta_4) = 6.$$

Then

$$\psi\downarrow_N = a_1\tau_1 + a_2\sum_{j=2}^{9}\tau_j + a_3\sum_{j=10}^{21}\tau_j + a_4\sum_{j=22}^{27}\tau_j$$

and

$$\langle \psi\downarrow_N, \psi\downarrow_N \rangle = a_1^2 + 8a_2^2 + 12a_3^2 + 6a_4^2.$$

It follows that $a_1 = \langle \psi\downarrow_N, \tau_1 \rangle = \langle \rho, \psi \rangle_{\overline{G}}$ and that

$$\langle \psi\downarrow_N, \psi\downarrow_N \rangle = \frac{1}{27}\Big[\psi(1A)\psi(1A) + 8\psi(3A)\psi(3A) + 12\psi(3B)\psi(3B) + 6\psi(3C)\psi(3C)\Big].$$

Now consider ψ_2, ψ_5 and ψ_9, the $Irr(GO(5,3))$ of degrees 1, 6 and 10 respectively. The values of ψ_i and χ_i are in Table A.2 in the appendix and Table 9.14, respectively. For ψ_2 we have

$$a_1 = \langle \rho, \psi_2 \rangle_{\overline{G}} = \frac{1}{27}\Big[1 + 8(1) + 12(1) + 6(1)\Big] = 1.$$

As the degree of ψ_2 is 1 we have

$$a_1 + 8a_2 + 12a_3 + 6a_4 = 1.$$

This implies $a_2 = 0 = a_3 = a_4$. Thus the restriction $(\psi_2)_{\overline{G}}$ is expressible as a character of degree 1 from the first block of the character table of $\overline{G}$ corresponding to the first inertia factor group H_1. Considering the predetermined partial fusion of $\overline{G}$ into $GO(5,3)$ and the character tables of $\overline{G}$ and $GO(5,3)$, we deduce that

$$(\psi_2)_{\overline{G}} = \chi_4.$$

For ψ_5 we have

$$a_1 = \langle \rho, \psi_5 \rangle_{\overline{G}} = \frac{1}{27}\Big[6 + 8(-3) + 12(0) + 6(3)\Big] = 0.$$

As the degree of ψ_5 is 6 we have

$$a_1 + 8a_2 + 12a_3 + 6a_4 = 6.$$

This implies $a_2 = 0 = a_3$ and $a_4 = 1$. Thus the restriction $(\psi_5)_{\overline{G}}$ is expressible as a character of degree 6 from the fourth block of the character table of $\overline{G}$ corresponding to the fourth inertia factor group H_4. Considering the predetermined partial fusion of $\overline{G}$ into $GO(5,3)$ and the character tables of $\overline{G}$ and $GO(5,3)$, we deduce that

$$(\psi_5)_{\overline{G}} = \chi_{19}.$$

For ψ_9 we have

$$a_1 = \langle \rho, \psi_9 \rangle_{\overline{G}} = \frac{1}{27}\Big[10 + 8(1) + 12(4) + 6(-2)\Big] = 2.$$

Since the degree of ψ_9 is 10 we have

$$a_1 + 8a_2 + 12a_3 + 6a_4 = 10.$$

This implies $a_2 = 1$ and $a_3 = a_4 = 0$. Thus the restriction $(\psi_9)_{\overline{G}}$ is expressible as a sum of characters of degree 2 from the first block and of degree 8 from the second block of the character table of $\overline{G}$ corresponding to the first and second inertia factor groups H_1 and H_2. Once more, considering the predetermined partial fusion of $\overline{G}$ into $GO(5,3)$ and the character tables of $\overline{G}$ and $GO(5,3)$, we deduce that

$$(\psi_9)_{\overline{G}} = \chi_3 + \chi_4 + \chi_{12}.$$

We mentioned earlier that the class $6B$ of $\overline{G}$ either fuses to the class $6f$ or $6l$ or $6o$ or $6p$ in $GO(5,3)$. We now apply the above method to choose the suitable candidate for $6B$. In Table 9.15 below we list the values of ψ_2, ψ_5 and ψ_9 from the character table of $GO(5,3)$.

Table 9.15: Values of ψ_i in $GO(5,3)$

		6f	6l	6o	6p
Degree	ψ_i				
1	ψ_2	-1	1	1	-1
6	ψ_5	-2	-1	2	1
10	ψ_9	0	0	0	0

In the character table of $\overline{G}$ we obtain the following values for the restrictions $(\psi_i)_{\overline{G}}$:

$$(\psi_2)_{\overline{G}}(6B) = -1,$$

$$(\psi_5)_{\overline{G}}(6B) = 1$$

and

$$(\psi_9)_{\overline{G}}(6B) = 0.$$

Comparing the values of ψ_i and $(\psi_i)_{\overline{G}}$ above, we conclude that the class $6B$ of $\overline{G}$ fuses into the class $6p$ of $GO(5,3)$.

The values of ψ_2, ψ_5 and ψ_9 on the classes of $GO(5,3)$ and the values of the restrictions $(\psi_2)_{\overline{G}}$, $(\psi_5)_{\overline{G}}$ and $(\psi_9)_{\overline{G}}$ on the classes of $\overline{G}$ together with the predetermined fusion enables us to complete the fusion of $\overline{G}$ into $GO(5,3)$. The complete fusion results are contained in Table 9.16 below. In Table 9.17 we provide the summary of these fusion results.

Table 9.16: The fusion of $\overline{G}$ into $GO(5,3)$

	$[y]$	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$	$2i$
	$\lvert C_{GO(5,3)}(y)\rvert$	103680	103680	384	384	2304	2304	192	192	2880	2880
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$										
$1A$	1296	$\boxed{80}$									
$2A$	72		1440			$\boxed{32}$	32			40	40
$2B$	48		2160	$\boxed{8}$	8	48	48	4	4	60	60
$2C$	144		720			16	16			$\boxed{20}$	20
$2D$	24		4320	16	16	96	96	8	$\boxed{8}$	120	120
$2E$	48		2160	8	8	48	48	$\boxed{4}$	4	60	60
	$\chi(GO(5,3)\lvert\overline{G})$	80	0	8	0	32	0	4	8	20	0

Table 9.16: The fusion of $\overline{G}$ into $GO(5,3)$ - continued

	$[y]$	$3a$	$3b$	$3c$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$4h$
	$\lvert C_{GO(5,3)}(y)\rvert$	432	216	1296	192	192	192	192	32	32	64	64
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$											
$3A$	162			$\boxed{8}$								
$3B$	108	4	$\boxed{2}$	12								
$3C$	216	$\boxed{2}$	1	6								
$3D$	18	24	$\boxed{12}$	72								
$4A$	8				24				$\boxed{4}$			
$4B$	24				$\boxed{8}$							
	$\chi(GO(5,3)\vert\overline{G})$	2	14	8	8	0	0	0	4	0	0	0

Table 9.16: The fusion of $\overline{G}$ into $GO(5,3)$ - continued

	$[y]$	$6c$	$6f$	$6g$	$6h$	$6k$	$6l$	$6o$	$6p$	$9a$	$12a$
	$\lvert C_{GO(5,3)}(y)\rvert$	48	72	24	24	144	72	72	72	18	24
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$										
$6A$	18		4			$\boxed{8}$	4	4	4		
$6B$	36		2			4	2	2	$\boxed{2}$		
$6C$	36		$\boxed{2}$			4	2	2	2		
$6D$	24	$\boxed{2}$	3	1	1	6	3	3	3		
$6E$	36		2			4	2	$\boxed{2}$	2		
$6F$	36		2			4	$\boxed{2}$	2	2		
$6G$	12	4	6	$\boxed{2}$	2	12	6	6	6		
$6H$	6	8	12	4	$\boxed{4}$	24	12	12	12		
$9A$	9									$\boxed{2}$	
$12A$	12										$\boxed{2}$
	$\chi(GO(5,3)\vert\overline{G})$	2	2	2	4	8	2	2	2	2	2

Table 9.17: The fusion of $\overline{G}$ into $GO(5,3)$

$\overline{G}$	$\longrightarrow$	$GO(5,3)$	$\overline{G}$	$\longrightarrow$	$GO(5,3)$
$1A$		$1a$	$4B$		$4a$
$2A$		$2d$	$6A$		$6k$
$2B$		$2b$	$6B$		$6p$
$2C$		$2h$	$6C$		$6f$
$2D$		$2g$	$6D$		$6c$
$2E$		$2f$	$6E$		$6o$
$3A$		$3c$	$6F$		$6l$
$3B$		$3B$	$6G$		$6g$
$3C$		$3A$	$6H$		$6h$
$3D$		$3b$	$9A$		$9a$
$4A$		$4e$	$12A$		$12a$

10

The affine subgroup $2_+^{1+4}{:}GU(2,4)$ of the general unitary group $GU(4,4)$

The affine subgroup of the general unitary group $GU(2n, q^2)$ has the form $N{:}GU(2n-2, q^2)$, where N is a group of order q^{4n-3}. In this chapter we consider the case $n{=}q{=}2$ with $\overline{G} = N{:}GU(2,4)$ as the affine subgroup of $GU(4,4)$, where N is a non-abelian group of order 2^5. Let V be the 4-dimensional unitary space over $\mathbb{F}$ with basis $\{e_1, e_2, e_3, e_4\}$ which satisfies $(e_i, e_j) = \delta(i, 2n+1-j)$ for $i \leq j$. The group $GU(4,4)$ acts transitively on the set of non-zero isotropic vectors of V. The order of $GU(4,4)$ is 77760. The affine subgroup $\overline{G}$ is a stabilizer of a non-zero isotropic vector in V. The order of $\overline{G}$ is 576 and is of index $|GU(4,4){:}\overline{G}| = (2^{2\cdot 2}-1)(2^{2\cdot 2-1}+1) =$ 135 in $GU(4,4)$. That is there are 135 non-zero isotropic vectors in V. The non-abelian group $N = 2^{1+4}$ is a special 2-group since the centre $Z(N)$, the derived subgroup N' and the Frattini subgroup $\Phi(N)$ coincide and are elementary abelian by Definition 4.4.7. In Lemma 4.4.12 it is proved that $\overline{G}$ is a semidirect product of a special 2-group N and a group isomorphic to $G = GU(2,4)$. We note that $|Z(N)| = 2$. Moreover, the quotient $N/Z(N)$ is an elementary abelian 2-group of order $2^4 = 16$. This quotient can also be regarded as a vector space over $\mathbb{F}$. This implies that N is an extra special 2-group, by Definition 4.4.7. We further determine, by Remark 4.4.8 and using GAP, that the quadratic form associated with N is of type $+$. Thus $N = 2_+^{1+4}$ and therefore $\overline{G} = 2_+^{1+4}{:}GU(2,4)$. In Section 10.1 we use GAP to construct the groups N and G inside $\overline{G}$ and then determine their respective generators in terms of 4×4 matrices with entries in $\mathbb{F}$. We discuss the structure of the 17 conjugacy classes of N in Section 10.2. The coset analysis technique is used in Section 10.3 to determine the conjugacy classes of $\overline{G}$. In Section 10.4 we consider the action of G on $Irr(N)$. This action yields 4 orbits of lengths 1, 1, 9 and 6 with corresponding inertia factor groups isomorphic to G, G, C_2 and C_3. We include also the full inertia groups. The Fischer matrices of $\overline{G}$ are computed in Section 10.5. A slightly different method from previous groups is used since in this group N is not elementary abelian. We then construct the character table of $\overline{G}$ in Section 10.6. Again here a slightly different approach is used, based on Theorem 3.1.15 and the fact that the Schur multiplier of G is trivial, to show that the $Irr(N)$ are extendable to the inertia groups. This is a necessary condition to employ the Clifford-Fischer Theory to construct the character tables of group extensions. We conclude this chapter with the fusion of the affine subgroup $\overline{G}$ into the general unitary group $GU(4,4)$ in Section 10.7.

10.1 The construction of N and G in $GU(4,4)$

We construct the groups N and G in terms of 4×4 matrices with entries in $\mathbb{F}$ inside $GU(4,4)$ since the affine subgroup $\overline{G}=N{:}G$ sits in $GU(4,4)$. In the introduction we have that $|GU(4,4){:}\overline{G}| = 135$. We use GAP to analyse the subgroups of index 135 in $GU(4,4)$. We observe that there are 5 of these subgroups. Each of these has one normal subgroup of order 32. We further note that all these normal subgroups are non-abelian. We require by Lemma 4.4.12 that the normal subgroup N be a special 2-group. Our investigations show that all the above normal subgroups are in fact extra special 2-groups. We then proceed to compute their respective complements in $\overline{G}$. We require that the complement be isomorphic to G. We know that G has 9 conjugacy classes, $|G| = 18$ and that the structure description of G is $C_3 \times S_3$. One of the 5 candidates has a complement that has 18 conjugacy classes and a structure description $C_6 \times C_3$ and thus not isomorphic to G. The complements of the other 4 candidates are isomorphic to G. In Section 10.4 we have that $\overline{G}$ has 4 inertia factor groups, $H_1 = G$, $H_2 = G$, $H_3 = C_2$ and $H_4 = C_3$. And also in Section 10.6 we have that $|Irr(H_1)| = 9$, $|Irr(H_2)| = 9$, $|Irr(H_3)| = 2$ and $|Irr(H_4)| = 3$. This implies that $\overline{G}$ has 23 conjugacy classes. We note that one of the last 4 candidates yields a semidirect product that has 48 conjugacy classes. The last three candidates yield semidirect products that have 23 conjugacy classes as required. We further note that latter semidirect products are isomorphic. We therefore choose the groups N and G from one of these semidirect products. Let z be a generator of $\mathbb{F}^*$. The generators of N are

$$d_1 = \begin{bmatrix} 1 & z & z & 0 \\ z^2 & 0 & 1 & z^2 \\ z^2 & 1 & 0 & z^2 \\ 0 & z & z & 1 \end{bmatrix},$$

$$d_2 = \begin{bmatrix} z & z^2 & 1 & 1 \\ z^2 & z & 1 & 1 \\ 0 & 0 & z^2 & z \\ 0 & 0 & z & z^2 \end{bmatrix},$$

$$d_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$d_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$d_5 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The generators of G are

$$a = \begin{bmatrix} 1 & z^2 & z^2 & 0 \\ 1 & z & 0 & z^2 \\ 0 & 0 & z & z^2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$c = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $o(a) = 3$, $o(b) = 2$ and $o(c) = 3$.

10.2 Conjugacy classes of N

The group N has $2^4 + 2 - 1 = 17$ conjugacy classes by Lemma 4.4.10. It has elements of orders 1,2 and 4 since its exponent is 4. It has 1 central involution δ and several non-central involutions. For $n \in N$, the conjugacy class $[n]$ is contained in the coset nN'. This implies $|[n]| \leq |nN'| = |N'| = 2$. It then follows that $|[n]| = 1$ or $|[n]| = 2$ for all $n \in N$. We note that $\delta \in Z(N)$ and therefore the order of its centralizer $|C_N(\delta)| = |N| = 32$. This then means that $|[\delta]| = 1$. Let c and d be class representatives of a conjugacy class of a non-central involution and an element of order 4 in N, respectively. We note that $|C_N(c)| = 2^4 = |C_N(d)|$. It then follows that $|[c]| = 2 = |[d]|$. Since N is of the type 2_+^{1+4}, it has $2^4 - 2^2 = 12$ elements of order 4 by Remark 4.4.9. This means N has 6 conjugacy classes of elements of order 4. Therefore N must have 9 conjugacy classes of non-central involutions. We summarize these details in Table 10.1 below. Thereafter we provide the character table of N in Table 10.2.

Table 10.1: Conjugacy Classes of N

$[n]$	$o(n)$	$\|[n]\|$	Number of conjugacy classes $[n]$	Number of elements
$[1_N]$	1	1	1	1
$[\delta]$	2	1	1	1
$[c]$	2	2	9	18
$[d]$	4	2	6	12
Total			17	32

Table 10.2: Character Table of N

$\lvert C_N(n)\rvert$	32	16	16	32	16	16	16	16	16	16	16	16	16	16	16	16	16
$o(n)$	1a	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	4a	4b	4c	4d	4e	4f
$2P$	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	2c	2c	2c	2c	2c	2c
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1	1	1	-1	1	-1	-1	1	-1	1	-1	-1
χ_3	1	-1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	1
χ_4	1	1	-1	1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	1	-1
χ_5	1	1	-1	1	1	-1	-1	1	1	1	-1	1	-1	-1	-1	-1	1
χ_6	1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	1
χ_7	1	-1	-1	1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	-1
χ_8	1	1	-1	1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	-1	1
χ_9	1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1
χ_{10}	1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	-1	1
χ_{11}	1	-1	1	1	1	-1	-1	-1	1	-1	-1	1	1	-1	1	1	-1
χ_{12}	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
χ_{13}	1	1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	-1
χ_{14}	1	-1	1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	-1
χ_{15}	1	-1	1	1	-1	1	-1	-1	1	1	1	-1	-1	-1	1	-1	1
χ_{16}	1	1	1	1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	-1	-1
χ_{17}	4	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0	0

10.3 Conjugacy classes of $\overline{G}$

We employ the coset analysis technique to determine the conjugacy classes of $\overline{G}$. We first analyse the identity coset $N1_G$. The action of N on this coset produces 17 conjugacy classes of N. The conjugacy classes of N are discussed in Section 10.2 above. We then act the centralizer $C_G(1_G)$ on these classes. This action fuses the non-central involutions into a single orbit of length 18. Likewise, the elements of order 4 fuse together to form a single orbit of length 12. The identity and the central involution classes are left invariant, respectively. This means the action of $\overline{G}$ on N yields 4 orbits. That is, the class 1_G produces 4 classes of $\overline{G}$ of sizes 1, 1, 18 and 12. The point stabilizers in $\overline{G}$ are $\overline{G}$, $\overline{G}$, $2^4{:}C_2$ and $2^4{:}C_3$, respectively. We apply this process to the rest of the 8 class representatives of G to obtain 23 conjugacy classes of $\overline{G}$.

Table 10.3: The conjugacy classes of $\overline{G}$

$[g]_G$	$\lvert C_G(g)\rvert$	$\lvert[x]\rvert$	$\lvert C_{\overline{G}}(x)\rvert$	$[x]_{\overline{G}}$
$1A$	18	1	576	$1A$
		1	576	$2A$
		18	32	$2B$
		12	48	$4A$
$2A$	6	12	48	$2C$
		12	48	$2D$
		72	8	$4B$
$3A$	9	32	18	$3A$
		32	18	$6A$
$3B$	9	8	72	$3B$
		8	72	$6B$
		48	12	$12A$
$3C$	9	8	72	$3C$
		8	72	$6C$
		48	12	$12B$
$3D$	18	16	36	$3D$
		16	36	$6D$
$3E$	18	16	36	$3E$
		16	36	$6E$
$6A$	6	48	12	$6F$
		48	12	$6G$
$6B$	6	48	12	$6H$
		48	12	$6I$

10.4 Inertia factor groups of $\overline{G}$

In Section 10.3 we have that the action of $\overline{G}$ on N has 4 orbits. Thus due to Brauer's Theorem the action of $\overline{G}$ on $Irr(N)$ will also have 4 orbits. By Lemma 4.4.10 the normal subgroup N has $2^4+2-1=17$ irreducible characters. It has $2^4=16$ linear characters and $2-1=1$ unique faithful irreducible character of degree $2^2=4$. From Remark 4.4.16 we have that the orbits on the set of linear characters of N are of lengths 1, 9 and 6. There is $2-1=1$ orbit of length 6. These orbits correspond to the inertia factor groups G, C_2 and $GU(1,4)\cong C_3$, where C_2 is isomorphic to the affine subgroup of G. Let θ be the unique faithful character. In Table 10.2, $\theta=\chi_{17}$. We note that the values of θ are $\theta(1_N)=4$, $\theta(\delta)=\theta(2c)=-4$ and $\theta(r)=0$ for all $r\in N-\{1_N,\delta\}$. The action of G on $Irr(N)$ leaves the faithful character θ invariant. That is the resulting orbit is of length 1. Hence the corresponding inertia factor group is isomorphic to G. Therefore $\overline{G}$ has 4 inertia factor groups $H_1=G$, $H_2=G$, $H_3=C_2$ and $H_4=C_3$. We provide below the fusions of these inertia factor groups into G and thereafter their character tables. Note that the full inertia groups are $N{:}H_1$, $N{:}H_2$, $N{:}H_3$ and $N{:}H_4$, respectively.

Table 10.4: Fusion of C_2 into G

	$\vert C_G\vert$	18	6	9	9	9	18	18	6	6
	$o(g)$	1A	2A	3A	3B	3C	3D	3E	6A	6B
$o(h)$	$\vert C_H\vert$									
$1a$	2	9								
$2a$	2		3							
	$\chi(G\vert C_2)$	9	3	0	0	0	0	0	0	0

Table 10.5: Fusion of C_3 into G

	$\vert C_G\vert$	18	6	9	9	9	18	18	6	6
	$o(g)$	1A	2A	3A	3B	3C	3D	3E	6A	6B
$o(h)$	$\vert C_H\vert$									
$1a$	3	6								
$3a$	3			3	3	3	6	6		
$3b$	3			3	3	3	6	6		
	$\chi(G\vert C_3)$	6	0	0	3	3	0	0	0	0

Table 10.6: Character table of C_2

$\lvert C_G \rvert$	2	2
$o(g)$	1a	2a
$2P$	1a	1a
χ_1	1	1
χ_2	1	-1

Table 10.7: Character table of C_3

$\lvert C_G \rvert$	3	3	3
$o(g)$	1a	3a	3b
$3P$	1a	1a	1a
χ_1	1	1	1
χ_2	1	A	$\overline{A}$
χ_3	1	$\overline{A}$	A

where $A = \frac{-1+\sqrt{-3}}{2}$.

Table 10.8: Character table of G

$\lvert C_G(g) \rvert$	18	6	9	9	9	18	18	6	6
$o(g)$	1a	2a	3a	3b	3c	3d	3e	6a	6b
$2P$	1a	1a	3a	3c	3b	3e	3d	3e	3d
$3P$	1a	2a	1a	1a	1a	1a	1a	2a	2a
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	-1	-1
χ_3	1	-1	1	$-A$	$-\overline{A}$	$-A$	$-\overline{A}$	A	$\overline{A}$
χ_4	1	-1	1	$-\overline{A}$	$-A$	$-\overline{A}$	$-A$	$\overline{A}$	A
χ_5	1	1	1	$-A$	$-\overline{A}$	$-A$	$-\overline{A}$	$-A$	$-\overline{A}$
χ_6	1	1	1	$-\overline{A}$	$-A$	$-\overline{A}$	$-A$	$-\overline{A}$	$-A$
χ_7	2	0	-1	-1	-1	2	2	0	0
χ_8	2	0	-1	$\overline{A}$	A	B	$\overline{B}$	0	0
χ_9	2	0	-1	A	$\overline{A}$	$\overline{B}$	B	0	0

where $A = \frac{1-\sqrt{-3}}{2}$
and $B = -1 - \sqrt{-3}$.

10.5 Fischer matrices of $\overline{G}$

In this section we consider the computation of the Fischer matrices of $\overline{G}$. Since G has 9 conjugacy classes then $\overline{G}$ will have 9 Fischer matrices corresponding to each class of G. The size of each matrix can be deduced from the table of conjugacy classes of $\overline{G}$, Table 10.3. The class 1A produces 4 classes of $\overline{G}$. Thus the Fischer matrix corresponding to this class is a 4×4 matrix. The identity classes of the inertia factor groups H_2, H_3 and H_4 obviously fuse to the class 1A of G. The fusion maps of these inertia factor groups are given in Section 10.4. Then the general form of the Fischer matrix yields the following matrix:

Table 10.9: Fischer matrix from $1A$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(x_{1j})\rvert$		576	576	32	48
(k,m)	$\lvert C_{H_k}(x_{1km})\rvert$				
$(1,1)$	18	1	1	1	1
$(2,1)$	18	a	d	g	j
$(3,1)$	2	b	e	h	k
$(4,1)$	3	c	f	i	l

We remark that, since N is not elementary abelian, not all the properties and the orthogonality relations are applicable in the computation of the Fischer matrices of $\overline{G}$. For instance, the first column does not come directly from the indices of the inertia factor groups in G, which is the case when N is elementary abelian. Instead we employ Lemma 3.4.1, Lemma 3.4.2 and Remark 3.4.3 to compute the first two columns and the second row of a Fischer matrix. By Lemma 3.4.2 the Fischer matrix from $1A$ will have $b = e = 9$ and $c = f = 6$. According to Lemma 3.4.1, a is the square root of the modulus of the sum 1+9+6=16. That is, $a \in \{\pm 4, \pm 4i\}$. We know from Table 10.3 that this class produces 4 classes $1A$, $2A$, $2B$ and $4A$ of $\overline{G}$. Since the second and the third are classes of involutions, the inverses of the elements of $2A$ and $2B$ are contained in $2A$ and $2B$, respectively. We also note that the inverses of the elements of $4A$ lie in $4A$. Then by Proposition 2.5.2, the entries of the portion of the character table corresponding to this class will be real numbers. The character table of $\overline{G}$ is Table 10.13. This then means that $a \in \{-4, 4\}$. The final choice $a = 4$ is chosen such that the fusion of the conjugacy classes of $\overline{G}$ into $GU(4,4)$ works. This fusion is discussed in Section 10.7. Now by Lemma 3.4.2 the second row is given by $[4 \quad -4 \quad 0 \quad 0]$, that is, $d = -4$ and $g = j = 0$. This process yields the following matrix:

Table 10.10: Fischer matrix from $1A$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(x_{1j})\rvert$		576	576	32	48
(k,m)	$\lvert C_{H_k}(x_{1km})\rvert$				
$(1,1)$	18	1	1	1	1
$(2,1)$	18	4	-4	0	0
$(3,1)$	2	9	9	h	k
$(4,1)$	3	6	6	i	l

At this stage, according to Remark 3.4.3, the number of unknowns of every $m \times m$ Fischer matrix is reduced to $m^2 - 4m + 4 = 4^2 - 4 \cdot 4 + 4 = 4$ unknowns. We then use the column orthogonality relations to compute the rest of the entries. The column orthogonality relations of columns 1 and 3 yield:

$$18 \cdot 1 \cdot 1 + 18 \cdot 4 \cdot 0 + 2 \cdot 9 \cdot h + 3 \cdot 6 \cdot i = 0$$

and

$$18 \cdot 1^2 + 0 + 2 \cdot h^2 + 3 \cdot i^2 = 32$$

that is

$$18 + 0 + 18h + 18i = 0$$

and

$$18 + 0 + 2h^2 + 3i^2 = 32.$$

This gives $h = 1$ and $i = -2$.

Similarly, for columns 1 and 4 we have

$$18 + 0 + 18k + 18l = 0$$

and

$$18 + 0 + 2k^2 + 3l^2 = 48$$

which give $k = -3$ and $l = 2$. Therefore the Fischer matrix from 1A is

Table 10.11: Fischer matrix from $1A$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(x_{1j})\rvert$		576	576	32	48
(k, m)	$\lvert C_{H_k}(x_{1km})\rvert$				
$(1, 1)$	18	1	1	1	1
$(2, 1)$	18	4	-4	0	0
$(3, 1)$	2	9	9	1	-3
$(4, 1)$	3	6	6	-2	2

We apply this technique to compute the rest of the Fischer matrices of $\overline{G}$. These Fischer matrices are listed in Table 10.12 below.

Table 10.12: Fischer matrices of $\overline{G}$

$\mathbb{F}_1 = M(1A)$		$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$
$o(x_{1j})$		1	2	2	4
$\lvert C_{\overline{G}}(x_{1j})\rvert$		576	576	32	48
(k, m)	$\lvert C_{H_k}(x_{1km})\rvert$				
$(1, 1)$	18	1	1	1	1
$(2, 1)$	18	4	-4	0	0
$(3, 1)$	2	9	9	1	-3
$(4, 1)$	3	6	6	-2	2

$\mathbb{F}_2 = M(2A)$		$x_{2,1}$	$x_{2,2}$	$x_{2,3}$
$o(x_{2j})$		2	2	4
$\lvert C_{\overline{G}}(x_{2j})\rvert$		48	48	8
(k, m)	$\lvert C_{H_k}(x_{2km})\rvert$			
$(1, 1)$	6	1	1	1
$(2, 1)$	6	-2	2	0
$(3, 1)$	2	3	3	-1

$\mathbb{F}_3 = M(3A)$		$x_{3,1}$	$x_{3,2}$
$o(x_{3j})$		3	6
$\lvert C_{\overline{G}}(x_{3j})\rvert$		18	18
(k, m)	$\lvert C_{H_k}(x_{3km})\rvert$		
$(1, 1)$	9	1	1
$(2, 1)$	9	1	-1

Table 10.12: Fischer matrices of $\overline{G}$

$\mathbb{F}_4 = M(3B)$		$x_{4,1}$	$x_{4,2}$	$x_{4,3}$
$o(x_{4j})$		3	6	12
$\lvert C_{\overline{G}}(x_{4j})\rvert$		72	72	12
(k,m)	$\lvert C_{H_k}(x_{4km})\rvert$			
$(1,1)$	9	1	1	1
$(2,1)$	9	-2	2	0
$(4,1)$	3	3	3	-1

$\mathbb{F}_5 = M(3C)$		$x_{5,1}$	$x_{5,2}$	$x_{5,3}$
$o(x_{5j})$		3	6	12
$\lvert C_{\overline{G}}(x_{5j})\rvert$		72	72	12
(k,m)	$\lvert C_{H_k}(x_{5km})\rvert$			
$(1,1)$	9	1	1	1
$(2,1)$	9	-2	2	0
$(4,1)$	3	3	3	-1

$\mathbb{F}_6 = M(3D)$		$x_{6,1}$	$x_{6,2}$
$o(x_{6j})$		3	6
$\lvert C_{\overline{G}}(x_{6j})\rvert$		36	36
(k,m)	$\lvert C_{H_k}(x_{6km})\rvert$		
$(1,1)$	18	1	1
$(2,1)$	18	1	-1

$\mathbb{F}_7 = M(3E)$		$x_{7,1}$	$x_{7,2}$
$o(x_{7j})$		3	6
$\lvert C_{\overline{G}}(x_{7j})\rvert$		36	36
(k,m)	$\lvert C_{H_k}(x_{7km})\rvert$		
$(1,1)$	18	1	1
$(2,1)$	18	1	-1

$\mathbb{F}_8 = M(6A)$		$x_{8,1}$	$x_{8,2}$
$o(x_{8j})$		6	6
$\lvert C_{\overline{G}}(x_{8j})\rvert$		12	12
(k,m)	$\lvert C_{H_k}(x_{8km})\rvert$		
$(1,1)$	6	1	1
$(2,1)$	6	-1	1

$\mathbb{F}_9 = M(6B)$		$x_{9,1}$	$x_{9,2}$
$o(x_{9j})$		6	6
$\lvert C_{\overline{G}}(x_{9j})\rvert$		12	12
(k,m)	$\lvert C_{H_k}(x_{9km})\rvert$		
$(1,1)$	6	1	1
$(2,1)$	6	-1	1

10.6 Character table of $\overline{G}$

We construct the character table of $\overline{G}$ using the Clifford-Fischer Theory. This technique requires that the $Irr(N)$ be extendable to the inertia groups. In Section 10.4 we determined that N has 16 linear irreducible characters and 1 faithful irreducible character θ of degree 4. We know that the identity character of N is extendable. By Theorem 3.1.15, the linear characters of N are extendable to the ordinary characters of the inertia groups. We use GAP to determine that the Schur multiplier of G is trivial. This then means that the character θ is extendable as well. Thus all $Irr(N)$ are extendable to the inertia groups. The character table of $\overline{G}$ will be divided into 4 blocks that correspond to the 4 inertia factor groups of $\overline{G}$ since, due to Gallagher's theorem, the irreducible characters of $\overline{G}$ are given by

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\phi_i)^{\overline{G}} \mid \beta \in Irr(\overline{H}_i),\ N \in ker(\beta)\},$$

where $\overline{H} = N{:}H$ is an inertia group and $H = \overline{H}/N$ is an inertia factor group. The number of irreducible characters of $\overline{G}$ is $|Irr(\overline{G})| = \sum_{i=1}^{4} |Irr(H_i)| = 9+9+2+3 = 23$. For $1 \leq i \leq 4$ and $1 \leq j \leq 9$, let $B_{i,j}$ be the i,j-th block, in the character table of $\overline{G}$, corresponding to the i-th inertia factor group and the j-th conjugacy class of G. The Clifford-Fischer Theory technique entails utilizing the character tables of the inertia factor groups and the Fischer matrices of $\overline{G}$. The block $B_{i,j}$ is formed by multiplying the relevant columns of the inertia factor group H_i by the rows of the Fischer matrix $M(g)$ corresponding to the classes of H_i that fuse to the class $[g] \in G$. If H_i does not contribute to $M(g)$ then the block $B_{i,j}$ will have zeroes. The character tables of the inertia factor groups H_i together with their respective fusion maps into G are given in Section 10.4.

The full character table of $\overline{G}$ is given by Table 10.13 below. The consistency and accuracy of this table were checked by using Programme E in [15]. The power maps of $\overline{G}$ were computed using GAP and are included in this character table.

Table 10.13: Character Table of $\overline{G}$

$[g]$	1A				2A			3A		3B		
$[x]$	1A	2A	2B	4A	2C	2D	4B	3A	6A	3B	6B	12A
$C_{\overline{G}}(x)$	576	576	32	48	48	48	8	18	18	72	72	12
$2P$	1A	1A	1A	2A	1A	1A	2B	3A	3A	3C	3C	6C
$3P$	1A	2A	2B	4A	2C	2D	4B	1A	2A	1A	2A	4A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	1	1	1	1	1
χ_3	1	1	1	1	-1	-1	-1	1	1	A	A	A
χ_4	1	1	1	1	-1	-1	-1	1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$
χ_5	1	1	1	1	1	1	1	1	1	A	A	A
χ_6	1	1	1	1	1	1	1	1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$
χ_7	2	2	2	2	0	0	0	-1	-1	-1	-1	-1
χ_8	2	2	2	2	0	0	0	-1	-1	-$\overline{A}$	-$\overline{A}$	-$\overline{A}$
χ_9	2	2	2	2	0	0	0	-1	-1	-A	-A	-A
χ_{10}	4	-4	0	0	-2	2	0	1	-1	-2	2	0
χ_{11}	4	-4	0	0	2	-2	0	1	-1	-2	2	0
χ_{12}	4	-4	0	0	2	-2	0	1	-1	B	-B	0
χ_{13}	4	-4	0	0	2	-2	0	1	-1	$\overline{B}$	-$\overline{B}$	0
χ_{14}	4	-4	0	0	-2	2	0	1	-1	B	-B	0
χ_{15}	4	-4	0	0	-2	2	0	1	-1	$\overline{B}$	-$\overline{B}$	0
χ_{16}	8	-8	0	0	0	0	0	-1	1	2	-2	0
χ_{17}	8	-8	0	0	0	0	0	-1	1	-$\overline{B}$	$\overline{B}$	0
χ_{18}	8	-8	0	0	0	0	0	-1	1	-B	B	0
χ_{19}	9	9	1	-3	3	3	-1	0	0	0	0	0
χ_{20}	9	9	1	-3	-3	-3	1	0	0	0	0	0
χ_{21}	6	6	-2	2	0	0	0	0	0	3	3	-1
χ_{22}	6	6	-2	2	0	0	0	0	0	C	C	-A
χ_{23}	6	6	-2	2	0	0	0	0	0	$\overline{C}$	$\overline{C}$	-$\overline{A}$

Table 10.13: Character Table of $\overline{G}$

$[g]$	3C			3D		3E		6A		6B	
$[x]$	3C	6C	12B	3D	6D	3E	6E	6F	6G	6H	6I
$C_{\overline{G}}(x)$	72	72	12	36	36	36	36	12	12	12	12
$2P$	3B	3B	6B	3E	3E	3D	3D	3E	3E	3D	3D
$3P$	1A	2A	4A	1A	2A	1A	2A	2C	2D	2C	2D
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_3	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	-A	-A	-$\overline{A}$	-$\overline{A}$
χ_4	A	A	A	$\overline{A}$	$\overline{A}$	A	A	-$\overline{A}$	-$\overline{A}$	-A	-A
χ_5	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$
χ_6	A	A	A	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	A	A
χ_7	-1	-1	-1	2	2	2	2	0	0	0	0
χ_8	-A	-A	-A	-$\overline{B}$	-$\overline{B}$	-B	-B	0	0	0	0
χ_9	-$\overline{A}$	-$\overline{A}$	-$\overline{A}$	-B	-B	-$\overline{B}$	-$\overline{B}$	0	0	0	0
χ_{10}	-2	2	0	1	-1	1	-1	-1	1	-1	1
χ_{11}	-2	2	0	1	-1	1	-1	1	-1	1	-1
χ_{12}	$\overline{B}$	-$\overline{B}$	0	A	-A	$\overline{A}$	-$\overline{A}$	A	-A	$\overline{A}$	-$\overline{A}$
χ_{13}	B	-B	0	$\overline{A}$	-$\overline{A}$	A	-A	$\overline{A}$	-$\overline{A}$	A	-A
χ_{14}	$\overline{B}$	-$\overline{B}$	0	A	-A	$\overline{A}$	-$\overline{A}$	-A	A	-$\overline{A}$	$\overline{A}$
χ_{15}	B	-B	0	$\overline{A}$	-$\overline{A}$	A	-A	-$\overline{A}$	$\overline{A}$	-A	A
χ_{16}	2	-2	0	2	-2	2	-2	0	0	0	0
χ_{17}	-B	B	0	-$\overline{B}$	$\overline{B}$	-B	B	0	0	0	0
χ_{18}	-$\overline{B}$	$\overline{B}$	0	-B	B	-$\overline{B}$	$\overline{B}$	0	0	0	0
χ_{19}	0	0	0	0	0	0	0	0	0	0	0
χ_{20}	0	0	0	0	0	0	0	0	0	0	0
χ_{21}	3	3	-1	0	0	0	0	0	0	0	0
χ_{22}	$\overline{C}$	$\overline{C}$	-$\overline{A}$	0	0	0	0	0	0	0	0
χ_{23}	C	C	-A	0	0	0	0	0	0	0	0

where

$$A = \frac{-1+\sqrt{-3}}{2}$$

$$B = -2 \times A$$

$$C = 3 \times A$$

We use Theorem 4.4.14 to compute the character degrees of $Irr(\overline{G})$. The degrees of the $Irr(\overline{G})$ are the degrees of $Irr(H_1)$, the degrees of $Irr(H_3)$ multiplied by $(2^{4-2}-1)(2^{4-3}+1) = 9$, the degrees of $Irr(H_4)$ multiplied by $2^{4-3}(2^{4-2}-(-1)^4) = 6$. We observe, following the above pattern, that the other degrees are obtained by multiplying the degrees of $Irr(H_1)$ by the degree of the faithful character θ. In Section 10.4, we determined that the degree of θ is 4.

10.7 Fusion of $\overline{G}$ into $GU(4,4)$

This chapter concludes with the fusion of the affine subgroup $\overline{G}$ into the general unitary group $GU(4,4)$. Let $[x]$ and $[y]$ be the conjugacy classes of $\overline{G}$ and $GU(4,4)$ respectively. We first construct the partial fusion table by considering the divisibility of the respective centralizer sizes $\frac{|C(y)|}{|C(x)|}$ with $o(x) = o(y)$. This is Table 10.15 but without considering the boxes around the entries. This process entails employing the permutation character of $GU(4,4)$ on the cosets of $\overline{G}$ in $GU(4,4)$ together with the respective power maps. The permutation character $\gamma = \chi(GU(4,4)|\overline{G})$ is computed using GAP. The values of γ are listed in the last row of the partial fusion table. We use the values of γ and $\psi_i \in Irr(GU(4,4))$ to express γ in terms of $Irr(GU(4,4))$. We compute the inner products $\langle \psi_i, \gamma \rangle$, for $1 \leq i \leq 60$, to determine the constituents of γ. This method yields

$$\gamma = \psi_1 + \psi_{23} + \psi_{24} + \psi_{25} + \psi_{28} + \psi_{36} + \psi_{37},$$

in terms of the $Irr(GU(4,4))$.

The power maps also play a critical role in the fusion process in the following manner. Suppose that $[x_1]$ and $[x_2]$ are conjugacy classes of $\overline{G}$ such that $x_1^p \in [x_2]$ for some prime p. Suppose also that $[y_1]$ and $[y_2]$ are conjugacy classes of $GU(4,4)$ such that $y_1^p \in [y_2]$. Now $[x_2]$ fuses to $[y_2]$ if and only if $[x_1]$ fuses to $[y_1]$. The irreducible characters and power maps of $GU(4,4)$ are found in the character table of $GU(4,4)$, Table A.3, in the appendix.

We note in the partial fusion that the classes $2A$, $2B$ and $4A$ fuse, respectively, into the classes $2a$, $2b$ and $4a$ of $GU(4,4)$. The class $2C$, likewise $2D$, may fuse either to $2a$ or $2b$. The power maps of $\overline{G}$ in Table 10.13 yield $(6F)^3 = 2C = (6H)^3$ and $(6G)^3 = 2D = (6I)^3$. From the character table of $GU(4,4)$ we have $(6j)^3 = 2a = (6m)^3$ and $(6r)^3 = 2b = (6s)^3$. Now since $6F$ and $6H$ fuse into $6j$ and $6m$ respectively, we conclude that $2C$ fuses to $2a$. Similarly, since $6G$ and $6I$ fuse respectively to $6r$ and $6s$, then the class $2D$ fuses into the class $2b$. This method, however, does not adequately complete the fusion. For instance, for the class $6D$ we cannot choose a suitable fusion candidate using the power maps and the permutation character. According to the partial fusion, the class $6D$ either fuses into the class $6k$ or $6l$ in $GU(4,4)$. We therefore employ the method of set intersections for characters to complete the fusion. We give a brief outline of this technique below.

Let ρ be the character afforded by the regular representation of G. It then follows $\rho = \sum_{i=1}^{9} e_i\phi_i$ where $\phi_i \in Irr(G)$ and $e_i = \deg(\phi_i)$. That is ρ can be viewed as the character of $\overline{G}$ which contains N in its kernel such that

$$\rho(g) = \begin{cases} |G| & g \in N \\ 0 & \text{otherwise} \end{cases}$$

If ψ is a character of $GU(4,4)$ then

$$\langle \rho, \psi \rangle = \frac{1}{|\overline{G}|}\{\rho(1A)\psi(1A) + \rho(2A)\psi(2A) + 18\rho(2B)\psi(2B) + 12\rho(4A)\psi(4A)\}$$

$$= \frac{1}{|\overline{G}|}\{|G|\psi(1A) + |G|\psi(2A) + 18|G|\psi(2B) + 12|G|\psi(4A)\}$$

$$= \frac{1}{32}\{\psi(1A) + \psi(2A) + 18\psi(2B) + 12\psi(4A)\}$$

$$= \langle \psi{\downarrow_N}, \tau_1 \rangle$$

where $\psi{\downarrow_N}$ is the restriction of ψ to N and τ_1 is the identity character of N. We note that for ψ we have that

$$\psi{\downarrow_N} = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4$$

where for $i \in \{1,2,3,4\}$, θ_i are sums of the $Irr(N)$ which are in one orbit under the action of G on $Irr(N)$, and $a_i \in \{0\} \cup \mathbb{N}$. Let $\tau_j \in Irr(N)$ for $j \in \{1,2,3,\cdots,17\}$. In Section 10.4 we have that the action of G on $Irr(N)$ yields orbits of lengths 1, 1, 9 and 6. We have

$$\theta_1 = \tau_1 \quad \text{and} \quad \deg(\theta_1) = 1,$$

$$\theta_2 = \tau_2 \quad \text{and} \quad \deg(\theta_2) = 1,$$

$$\theta_3 = \sum_{j=3}^{11} \tau_j \quad \text{and} \quad \deg(\theta_3) = 9,$$

$$\theta_4 = \sum_{j=12}^{17} \tau_j \quad \text{and} \quad \deg(\theta_4) = 6.$$

Then

$$\psi{\downarrow_N} = a_1\tau_1 + a_2\tau_2 + a_3\sum_{j=3}^{11}\tau_j + a_4\sum_{j=12}^{17}\tau_j$$

and

$$\langle \psi{\downarrow_N}, \psi{\downarrow_N} \rangle = a_1^2 + a_2^2 + 9a_3^2 + 6a_4^2.$$

It follows that $a_1 = \langle \psi{\downarrow_N}, \tau_1 \rangle = \langle \rho, \psi \rangle_{\overline{G}}$ and that

$$\langle \psi{\downarrow_N}, \psi{\downarrow_N} \rangle = \frac{1}{32}\Big[\psi(1A)\psi(1A) + \psi(2A)\psi(2A) + 9\psi(2B)\psi(2B) + 6\psi(4A)\psi(4A)\Big].$$

Now consider ψ_2, ψ_4, ψ_{10}, ψ_{13} and ψ_{19}, the $Irr(GU(4,4))$ of degrees 1, 5, 6, 10 and 15 respectively. We find the values of ψ_i and χ_i in Table A.3 in the appendix and the character table of $\overline{G}$, Table 10.13, respectively. For ψ_2 we have

$$a_1 = \langle \rho, \psi_2 \rangle_{\overline{G}} = \frac{1}{32}\Big[1 + (1) + 18(1) + 12(1)\Big] = 1.$$

Since the degree of ψ_2 is 1 we have

$$a_1 + a_2 + 9a_3 + 6a_4 = 1.$$

This implies $a_2 = 0 = a_3 = a_4$. Thus the restriction $\psi_2{\downarrow_{\overline{G}}}$ is expressible as a character of degree 1 from the first block of the character table of $\overline{G}$ corresponding to the first inertia factor group H_1. Considering the predetermined partial fusion of $\overline{G}$ into $GU(4,4)$ and the character tables of $\overline{G}$ and $GU(4,4)$, we deduce that

$$\psi_2{\downarrow_{\overline{G}}} = \chi_6.$$

For ψ_4 we have

$$a_1 = \langle \rho, \psi_4 \rangle_{\overline{G}} = \frac{1}{32}\Big[5 + (-3) + 18(1) + 12(1)\Big] = 1.$$

Since the degree of ψ_4 is 5 we have

$$a_1 + a_2 + 9a_3 + 6a_4 = 5.$$

This implies $a_2 = 4$ and $a_3 = a_4 = 0$. Thus the restriction $\psi_4{\downarrow_{\overline{G}}}$ is expressible as a sum of characters of degree 1 from the first block and of degree 4 from the second block of the character table of $\overline{G}$ corresponding to the first and second inertia factor groups H_1 and H_2. Considering the predetermined partial fusion of $\overline{G}$ into $GU(4,4)$ and the character tables of $\overline{G}$ and $GU(4,4)$, we deduce that

$$\psi_4{\downarrow_{\overline{G}}} = \chi_3 + \chi_{15}.$$

For ψ_{10} we have

$$a_1 = \langle \rho, \psi_{10} \rangle_{\overline{G}} = \frac{1}{32}\Big[6 + (-2) + 18(2) + 12(2)\Big] = 2.$$

Since the degree of ψ_{10} is 6 we have

$$a_1 + a_2 + 9a_3 + 6a_4 = 6.$$

This implies $a_2 = 4$ and $a_3 = a_4 = 0$. Therefore the restriction $\psi_{10}{\downarrow_{\overline{G}}}$ is expressible as a sum of characters of degree 2 from the first block and of degree 4 from the second block of the character table of $\overline{G}$ corresponding to the inertia factor groups H_1 and H_2. Considering the predetermined partial fusion of $\overline{G}$ into $GU(4,4)$ and the character tables of $\overline{G}$ and $GU(4,4)$, we deduce that

$$\psi_{10}{\downarrow_{\overline{G}}} = \chi_7 + \chi_{10}.$$

For ψ_{13} we have

$$a_1 = \langle \rho, \psi_{13} \rangle_{\overline{G}} = \frac{1}{32}\Big[10 + (2) + 18(-2) + 12(2)\Big] = 0.$$

Since the degree of ψ_{13} is 10 we have

$$a_1 + a_2 + 9a_3 + 6a_4 = 10.$$

This implies $a_2 = 4$, $a_3 = 0$ and $a_4 = 1$. The choice of $a_2 = 1$, $a_3 = 1$ and $a_4 = 0$ we exclude since there is no character of degree 1 of $\overline{G}$ corresponding to H_2. It then follows that the restriction $\psi_{13}\downarrow_{\overline{G}}$ is expressible as a sum of characters of degree 4 from the second block and of degree 6 from the fourth block of the character table of $\overline{G}$ corresponding to the inertia factor groups H_2 and H_4. Considering the predetermined partial fusion of $\overline{G}$ into $GU(4,4)$ and the character tables of $\overline{G}$ and $GU(4,4)$, we deduce that

$$\psi_{13}\downarrow_{\overline{G}} = \chi_{11} + \chi_{23}.$$

For ψ_{19} we have

$$a_1 = \langle \rho, \psi_{19} \rangle_{\overline{G}} = \frac{1}{32}\Big[15 + (-1) + 18(-1) + 12(3)\Big] = 1.$$

Since the degree of ψ_{19} is 15 we have

$$a_1 + a_2 + 9a_3 + 6a_4 = 15.$$

Either $a_2 = 8$, $a_3 = 0$ and $a_4 = 1$ or $a_2 = 5$, $a_3 = 1$ and $a_4 = 0$. We exclude the latter since there is no character of degree 5 of $\overline{G}$ corresponding to H_2. It then follows that the restriction $\psi_{19}\downarrow_{\overline{G}}$ is expressible as a sum of characters of degree 1 from the first block, of degree 8 from the second block and of degree 6 from the fourth block of the character table of $\overline{G}$ corresponding to the inertia factor groups H_1, H_2 and H_4. Once more considering the predetermined partial fusion of $\overline{G}$ into $GU(4,4)$ and the character tables of $\overline{G}$ and $GU(4,4)$, we deduce that

$$\psi_{19}\downarrow_{\overline{G}} = \chi_2 + \chi_{16} + \chi_{21}.$$

Earlier we mentioned that the class $6D$ of $\overline{G}$ may fuse into either the class $6k$ or $6l$ in $GU(4,4)$. We use the above technique to choose a suitable fusion candidate for $6D$. We list the values of ψ_2, ψ_4, ψ_{10}, ψ_{13} and ψ_{19} from the character table of $GU(4,4)$ below.

Table 10.14: The values of ψ_i in $GU(4,4)$

	$[y]$	$6k$	$6l$
Degree	ψ_i		
1	ψ_2	$\frac{-1+\sqrt{-3}}{2}$	$\frac{-1-\sqrt{-3}}{2}$
5	ψ_4	$-\sqrt{-3}$	$\sqrt{-3}$
6	ψ_{10}	1	1
10	ψ_{13}	-1	-1
15	ψ_{19}	-1	-1

The values of the restrictions $\psi_i{\downarrow_{\overline{G}}}$ from the character table of $\overline{G}$ are as follows:

$$\psi_2{\downarrow_{\overline{G}}}(6D) = \frac{-1-\sqrt{-3}}{2}$$

$$\psi_4{\downarrow_{\overline{G}}}(6D) = \sqrt{-3}$$

$$\psi_{10}{\downarrow_{\overline{G}}}(6D) = 1$$

$$\psi_{13}{\downarrow_{\overline{G}}}(6D) = -1$$

$$\psi_{19}{\downarrow_{\overline{G}}}(6D) = -1$$

Comparing the values of ψ_i and $\psi_i{\downarrow_{\overline{G}}}$ above, we conclude that the class $6D$ of $\overline{G}$ fuses into the class $6l$ of $GU(4,4)$.

The values of ψ_2, ψ_4, ψ_{10}, ψ_{13} and ψ_{19} on the classes of $GU(4,4)$ and the values of the restrictions $\psi_2{\downarrow_{\overline{G}}}$, $\psi_4{\downarrow_{\overline{G}}}$, $\psi_{10}{\downarrow_{\overline{G}}}$, $\psi_{13}{\downarrow_{\overline{G}}}$ and $\psi_{19}{\downarrow_{\overline{G}}}$ on the classes of $\overline{G}$ together with the predetermined fusion enable us to complete the fusion of $\overline{G}$ into $GU(4,4)$. The complete fusion results are contained in Table 10.15 below. In Table 10.16 we summarize these fusion results.

Table 10.15: The fusion of $\overline{G}$ into $GU(4,4)$

	$[y]$	$1a$	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$3f$	$3i$	$3k$	$3m$	$4a$	$4b$
	$\lvert C_{GU(4,4)}(y)\rvert$	77760	1728	288	77760	77760	1944	1944	1944	324	324	162	144	24
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$													
$1A$	576	$\boxed{135}$												
$2A$	576		$\boxed{3}$											
$2B$	32		54	$\boxed{9}$										
$2C$	48		$\boxed{36}$	6										
$2D$	48		36	$\boxed{6}$										
$3A$	18						108		108	18	18	$\boxed{9}$		
$3B$	72						27		$\boxed{27}$					
$3C$	72						$\boxed{27}$		27					
$3D$	36						54		54	9	$\boxed{9}$			
$3E$	36						54		54	$\boxed{9}$	9			
$4A$	48												$\boxed{3}$	
$4B$	8												18	$\boxed{3}$
	$\chi(GU(4,4)\lvert\overline{G})$	135	39	15	0	0	27	0	27	9	9	9	3	3

Table 10.15: The fusion of $\overline{G}$ into $GU(4,4)$

	$[y]$	$6e$	$6f$	$6j$	$6k$	$6l$	$6m$	$6r$	$6s$	$6v$	$12d$	$12f$
	$\lvert C_{GU(4,4)}(y)\rvert$	216	216	108	108	108	108	36	36	54	36	36
$[x]$	$\lvert C_{\overline{G}}(x)\rvert$											
$6A$	18	12	12	6	6	6	6	2	2	$\boxed{3}$		
$6B$	72	$\boxed{3}$	3									
$6C$	72	3	$\boxed{3}$									
$6D$	36	6	6	3	3	$\boxed{3}$	3	1	1			
$6E$	36	6	6	3	$\boxed{3}$	3	3	1	1			
$6F$	12	18	18	$\boxed{9}$	9	9	9	3	3			
$6G$	12	18	18	9	9	9	9	$\boxed{3}$	3			
$6H$	12	18	18	9	9	9	$\boxed{9}$	3	3			
$6I$	12	18	18	9	9	9	9	3	$\boxed{3}$			
$12A$	12										3	$\boxed{3}$
$12B$	12										$\boxed{3}$	3
	$\chi(GU(4,4)\mid\overline{G})$	3	3	9	3	3	9	3	3	3	3	3

Table 10.16: The fusion of $\overline{G}$ into $GU(4,4)$

$\overline{G}$	$\longrightarrow$	$GU(4,4)$	$\overline{G}$	$\longrightarrow$	$GU(4,4)$
$1A$		$1a$	$6A$		$6v$
$2A$		$2a$	$6B$		$6e$
$2B$		$2b$	$6C$		$6f$
$2C$		$2a$	$6D$		$6l$
$2D$		$2b$	$6E$		$6k$
$3A$		$3m$	$6F$		$6j$
$3B$		$3f$	$6G$		$6r$
$3C$		$3c$	$6H$		$6m$
$3D$		$3k$	$6I$		$6s$
$3E$		$3i$	$12A$		$12f$
$4A$		$4a$	$12B$		$12d$
$4B$		$4b$			

A

Appendix

A.1 Character table of GL(3, 5)

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	1488000	1488000	1920	1920	96	1488000	1488000	1920	1920	1920	1920	1920	1920
$o(y)$	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	4g	4h
$2P$	1a	1a	1a	1a	3a	2a	2a	2c	2c	2b	2b	2a	2c
$3P$	1a	2a	2b	2c	1a	4b	4a	4d	4c	4j	4l	4k	4i
$5P$	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	4g	4h
$31P$	1a	2a	2b	2c	3a	4b	4a	4d	4c	4j	4l	4k	4i
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	1
ψ_3	1	-1	-1	1	1	A	-A	1	1	-A	-A	A	-1
ψ_4	1	-1	-1	1	1	-A	A	1	1	A	A	-A	-1
ψ_5	30	30	6	6	0	30	30	6	6	6	6	6	6
ψ_6	30	30	6	6	0	-30	-30	6	6	-6	-6	-6	6
ψ_7	30	-30	-6	6	0	B	-B	6	6	-J	-J	J	-6
ψ_8	30	-30	-6	6	0	-B	B	6	6	J	J	-J	-6
ψ_9	31	31	7	7	1	31	31	-5	-5	-5	-5	7	-5
ψ_{10}	31	31	7	7	1	-31	-31	-5	-5	5	5	-7	-5
ψ_{11}	31	-31	-7	7	1	C	-C	-5	-5	I	I	AB	5
ψ_{12}	31	-31	-7	7	1	-C	C	-5	-5	-I	-I	-AB	5
ψ_{13}	31	-31	5	-5	1	C	-C	W	/W	Z	-/Z	-I	-/W
ψ_{14}	31	-31	5	-5	1	-C	C	/W	W	/Z	-Z	I	-W
ψ_{15}	31	-31	5	-5	1	C	-C	/W	W	-/Z	Z	-I	-W
ψ_{16}	31	-31	5	-5	1	-C	C	W	/W	-Z	/Z	I	-/W
ψ_{17}	31	31	-5	-5	1	-31	-31	W	/W	-/W	-W	5	/W
ψ_{18}	31	31	-5	-5	1	-31	-31	/W	W	-W	-/W	5	W
ψ_{19}	31	31	-5	-5	1	31	31	/W	W	W	/W	-5	W
ψ_{20}	31	31	-5	-5	1	31	31	W	/W	/W	W	-5	/W
ψ_{21}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{22}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{23}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{24}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{25}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{26}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{27}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{28}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{29}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{30}	96	96	0	0	0	96	96	0	0	0	0	0	0
ψ_{31}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{32}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{33}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{34}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{35}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{36}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{37}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{38}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{39}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{40}	96	96	0	0	0	-96	-96	0	0	0	0	0	0
ψ_{41}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{42}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{43}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{44}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{45}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{46}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{47}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{48}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{49}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{50}	96	-96	0	0	0	D	-D	0	0	0	0	0	0
ψ_{51}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{52}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{53}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{54}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{55}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{56}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{57}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{58}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{59}	96	-96	0	0	0	-D	D	0	0	0	0	0	0
ψ_{60}	96	-96	0	0	0	-D	D	0	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	1920	1920	1920	1920	64	64	64	64	2000	100	96	96	96	96	96
$o(y)$	4i	4j	4k	4l	4m	4n	4o	4p	5a	5b	6a	6b	6c	8a	8b
$2P$	2c	2b	2a	2b	2b	2c	2b	2c	5a	5b	3a	3a	3a	4h	4i
$3P$	4h	4e	4g	4f	4o	4n	4m	4p	5a	5b	2b	2a	2c	8b	8a
$5P$	4i	4j	4k	4l	4m	4n	4o	4p	1a	1a	6a	6b	6c	8a	8b
$31P$	4h	4e	4g	4f	4o	4n	4m	4p	5a	5b	6a	6b	6c	8b	8a
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	-1	-1	-1	-1	1	-1	1	1	1	1	1	1	-1	-1
ψ_3	-1	A	-A	A	A	-1	-A	1	1	1	-1	-1	1	-A	A
ψ_4	-1	-A	A	-A	-A	-1	A	1	1	1	-1	-1	1	A	-A
ψ_5	6	6	6	6	2	2	2	2	5	0	0	0	0	0	0
ψ_6	6	-6	-6	-6	-2	2	-2	2	5	0	0	0	0	0	0
ψ_7	-6	J	-J	J	AC	-2	-AC	2	5	0	0	0	0	0	0
ψ_8	-6	-J	J	-J	-AC	-2	AC	2	5	0	0	0	0	0	0
ψ_9	-5	-5	7	-5	-1	-1	-1	-1	6	1	1	1	1	-1	-1
ψ_{10}	-5	5	-7	5	1	-1	1	-1	6	1	1	1	1	1	1
ψ_{11}	5	-I	-AB	-I	-A	1	A	-1	6	1	-1	-1	1	A	-A
ψ_{12}	5	I	AB	I	A	1	-A	-1	6	1	-1	-1	1	-A	A
ψ_{13}	-W	/Z	I	-Z	A	-1	-A	1	6	1	-1	-1	1	-1	-1
ψ_{14}	-/W	Z	-I	-/Z	-A	-1	A	1	6	1	-1	-1	1	-1	-1
ψ_{15}	-/W	-Z	I	/Z	A	-1	-A	1	6	1	-1	-1	1	1	1
ψ_{16}	-W	-/Z	-I	Z	-A	-1	A	1	6	1	-1	-1	1	1	1
ψ_{17}	W	-W	5	-/W	-1	1	-1	1	6	1	1	1	1	A	-A
ψ_{18}	/W	-/W	5	-W	-1	1	-1	1	6	1	1	1	1	-A	A
ψ_{19}	/W	/W	-5	W	1	1	1	1	6	1	1	1	1	A	-A
ψ_{20}	W	W	-5	/W	1	1	1	1	6	1	1	1	1	-A	A
ψ_{21}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{22}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{23}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{24}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{25}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{26}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{27}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{28}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{29}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{30}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{31}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{32}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{33}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{34}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{35}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{36}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{37}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{38}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{39}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{40}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{41}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{42}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{43}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{44}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{45}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{46}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{47}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{48}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{49}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{50}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{51}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{52}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{53}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{54}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{55}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{56}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{57}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{58}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{59}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0
ψ_{60}	0	0	0	0	0	0	0	0	-4	1	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	96	96	96	96	96	96	2000	100	80	80	96	96	96	96	96
$o(y)$	8c	8d	8e	8f	8g	8h	10a	10b	10c	10d	12a	12b	12c	12d	12e
$2P$	4c	4d	4h	4i	4c	4d	5a	5b	5a	5a	6c	6c	6a	6a	6b
$3P$	8h	8g	8f	8e	8d	8c	10a	10b	10c	10d	4c	4d	4l	4j	4k
$5P$	8c	8d	8e	8f	8g	8h	2a	2a	2b	2c	12a	12b	12c	12d	12e
$31P$	8h	8g	8f	8e	8d	8c	10a	10b	10c	10d	12b	12a	12i	12j	12l
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1
ψ_3	1	-1	A	-A	-1	1	-1	-1	-1	1	1	1	-A	-A	A
ψ_4	1	-1	-A	A	-1	1	-1	-1	-1	1	1	1	A	A	-A
ψ_5	0	0	0	0	0	0	5	0	1	1	0	0	0	0	0
ψ_6	0	0	0	0	0	0	5	0	1	1	0	0	0	0	0
ψ_7	0	0	0	0	0	0	-5	0	-1	1	0	0	0	0	0
ψ_8	0	0	0	0	0	0	-5	0	-1	1	0	0	0	0	0
ψ_9	-1	-1	-1	-1	-1	-1	6	1	2	2	1	1	1	1	1
ψ_{10}	-1	-1	1	1	-1	-1	6	1	2	2	1	1	-1	-1	-1
ψ_{11}	-1	1	-A	A	1	-1	-6	-1	-2	2	1	1	-A	-A	A
ψ_{12}	-1	1	A	-A	1	-1	-6	-1	-2	2	1	1	A	A	-A
ψ_{13}	A	A	1	1	-A	-A	-6	-1	0	0	-1	-1	A	A	A
ψ_{14}	-A	-A	1	1	A	A	-6	-1	0	0	-1	-1	-A	-A	-A
ψ_{15}	-A	-A	-1	-1	A	A	-6	-1	0	0	-1	-1	A	A	A
ψ_{16}	A	A	-1	-1	-A	-A	-6	-1	0	0	-1	-1	-A	-A	-A
ψ_{17}	A	-A	A	-A	A	-A	6	1	0	0	-1	-1	1	1	-1
ψ_{18}	-A	A	-A	A	-A	A	6	1	0	0	-1	-1	1	1	-1
ψ_{19}	-A	A	A	-A	-A	A	6	1	0	0	-1	-1	-1	-1	1
ψ_{20}	A	-A	-A	A	A	-A	6	1	0	0	-1	-1	-1	-1	1
ψ_{21}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{22}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{23}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{24}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{25}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{26}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{27}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{28}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{29}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{30}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{31}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{32}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{33}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{34}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{35}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{36}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{37}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{38}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{39}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{40}	0	0	0	0	0	0	-4	1	0	0	0	0	0	0	0
ψ_{41}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{42}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{43}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{44}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{45}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{46}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{47}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{48}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{49}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{50}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{51}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{52}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{53}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{54}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{55}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{56}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{57}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{58}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{59}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0
ψ_{60}	0	0	0	0	0	0	4	-1	0	0	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	96	96	96	96	96	96	96	2000	100	2000	100	80	80	80	80
$o(y)$	12f	12g	12h	12i	12j	12k	12l	20a	20b	20c	20d	20e	20f	20g	20h
$2P$	6b	6c	6c	6a	6a	6b	6b	10a	10b	10a	10b	10d	10d	10c	10c
$3P$	4b	4h	4i	4f	4e	4a	4g	20c	20d	20a	20b	20f	20e	20l	20n
$5P$	12f	12g	12h	12i	12j	12k	12l	4a	4a	4b	4b	4c	4d	4e	4f
$31P$	12k	12h	12g	12c	12d	12f	12e	20c	20d	20a	20b	20f	20e	20l	20n
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	-1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	-1
ψ_3	A	-1	-1	A	A	-A	-A	A	A	-A	-A	1	1	-A	-A
ψ_4	-A	-1	-1	-A	-A	A	A	-A	-A	A	A	1	1	A	A
ψ_5	0	0	0	0	0	0	0	5	0	5	0	1	1	1	1
ψ_6	0	0	0	0	0	0	0	-5	0	-5	0	1	1	-1	-1
ψ_7	0	0	0	0	0	0	0	I	0	-I	0	1	1	-A	-A
ψ_8	0	0	0	0	0	0	0	-I	0	I	0	1	1	A	A
ψ_9	1	1	1	1	1	1	1	6	1	6	1	0	0	0	0
ψ_{10}	-1	1	1	-1	-1	-1	-1	-6	-1	-6	-1	0	0	0	0
ψ_{11}	A	-1	-1	A	A	-A	-A	J	A	-J	-A	0	0	0	0
ψ_{12}	-A	-1	-1	-A	-A	A	A	-J	-A	J	A	0	0	0	0
ψ_{13}	A	1	1	-A	-A	-A	-A	J	A	-J	-A	Y	/Y	Y	-/Y
ψ_{14}	-A	1	1	A	A	A	A	-J	-A	J	A	/Y	Y	/Y	-Y
ψ_{15}	A	1	1	-A	-A	-A	-A	J	A	-J	-A	/Y	Y	-/Y	Y
ψ_{16}	-A	1	1	A	A	A	A	-J	-A	J	A	Y	/Y	-Y	/Y
ψ_{17}	-1	-1	-1	1	1	-1	-1	-6	-1	-6	-1	Y	/Y	-/Y	-Y
ψ_{18}	-1	-1	-1	1	1	-1	-1	-6	-1	-6	-1	/Y	Y	-Y	-/Y
ψ_{19}	1	-1	-1	-1	-1	1	1	6	1	6	1	/Y	Y	Y	/Y
ψ_{20}	1	-1	-1	-1	-1	1	1	6	1	6	1	Y	/Y	/Y	Y
ψ_{21}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{22}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{23}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{24}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{25}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{26}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{27}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{28}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{29}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{30}	0	0	0	0	0	0	0	-4	1	-4	1	0	0	0	0
ψ_{31}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{32}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{33}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{34}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{35}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{36}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{37}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{38}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{39}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{40}	0	0	0	0	0	0	0	4	-1	4	-1	0	0	0	0
ψ_{41}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{42}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{43}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{44}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{45}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{46}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{47}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{48}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{49}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{50}	0	0	0	0	0	0	0	K	A	-K	-A	0	0	0	0
ψ_{51}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{52}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{53}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{54}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{55}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{56}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{57}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{58}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{59}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0
ψ_{60}	0	0	0	0	0	0	0	-K	-A	K	A	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	80	80	80	80	80	80	96	96	96	96	96	96	96	96	96
$o(y)$	20i	20j	20k	20l	20m	20n	24a	24b	24c	24d	24e	24f	24g	24h	24i
$2P$	10a	10d	10d	10c	10a	10c	12h	12h	12g	12g	12b	12b	12a	12a	12h
$3P$	20m	20k	20j	20g	20i	20h	8b	8b	8a	8a	8h	8h	8g	8g	8f
$5P$	4g	4h	4i	4j	4k	4l	24a	24b	24c	24d	24e	24f	24g	24h	24i
$31P$	20m	20k	20j	20g	20i	20h	24c	24d	24a	24b	24o	24p	24m	24n	24k
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1
ψ_3	A	-1	-1	A	-A	A	-A	-A	A	A	1	1	-1	-1	A
ψ_4	-A	-1	-1	-A	A	-A	A	A	-A	-A	1	1	-1	-1	-A
ψ_5	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
ψ_6	-1	1	1	-1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_7	A	-1	-1	A	-A	A	0	0	0	0	0	0	0	0	0
ψ_8	-A	-1	-1	-A	A	-A	0	0	0	0	0	0	0	0	0
ψ_9	2	0	0	0	2	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{10}	-2	0	0	0	-2	0	1	1	1	1	-1	-1	-1	-1	1
ψ_{11}	AC	0	0	0	-AC	0	A	A	-A	-A	-1	-1	1	1	-A
ψ_{12}	-AC	0	0	0	AC	0	-A	-A	A	A	-1	-1	1	1	A
ψ_{13}	0	-/Y	-Y	/Y	0	-Y	-1	-1	-1	-1	A	A	A	A	1
ψ_{14}	0	-Y	-/Y	Y	0	-/Y	-1	-1	-1	-1	-A	-A	-A	-A	1
ψ_{15}	0	-Y	-/Y	-Y	0	/Y	1	1	1	1	-A	-A	-A	-A	-1
ψ_{16}	0	-/Y	-Y	-/Y	0	Y	1	1	1	1	A	A	A	A	-1
ψ_{17}	0	/Y	Y	-Y	0	-/Y	A	A	-A	-A	A	A	-A	-A	A
ψ_{18}	0	Y	/Y	-/Y	0	-Y	-A	-A	A	A	-A	-A	A	A	-A
ψ_{19}	0	Y	/Y	/Y	0	Y	A	A	-A	-A	-A	-A	A	A	A
ψ_{20}	0	/Y	Y	Y	0	/Y	-A	-A	A	A	A	A	-A	-A	-A
ψ_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{22}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{23}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{24}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{25}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{26}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{27}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{28}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{29}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{31}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{32}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{33}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{34}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{35}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{36}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{37}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{38}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{39}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{40}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{41}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{42}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{43}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{44}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{45}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{47}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{48}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{49}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{50}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{51}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{52}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{54}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{55}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{56}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{57}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	96	96	96	96	96	96	96	124	124	124	124	124	124	124	124	124
$o(y)$	24j	24k	24l	24m	24n	24o	24p	31a	31b	31c	31d	31e	31f	31g	31h	31i
$2P$	12h	12g	12g	12b	12b	12a	12a	31j	31d	31e	31i	31b	31h	31a	31g	31c
$3P$	8f	8e	8e	8d	8d	8c	8c	31i	31j	31g	31f	31a	31e	31d	31b	31h
$5P$	24j	24k	24l	24m	24n	24o	24p	31a	31b	31c	31d	31e	31f	31g	31h	31i
$31P$	24l	24i	24j	24g	24h	24e	24f	1a	1a	1a	1a	1a	1a	1a	1a	1a
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_3	A	-A	-A	-1	-1	1	1	1	1	1	1	1	1	1	1	1
ψ_4	-A	A	A	-1	-1	1	1	1	1	1	1	1	1	1	1	1
ψ_5	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_6	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_7	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_8	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_9	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{10}	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{11}	-A	A	A	1	1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{12}	A	-A	-A	1	1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{13}	1	1	1	-A	-A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{14}	1	1	1	A	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{15}	-1	-1	-1	A	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{16}	-1	-1	-1	-A	-A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{17}	A	-A	-A	A	A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{18}	-A	A	A	-A	-A	A	A	0	0	0	0	0	0	0	0	0
ψ_{19}	A	-A	-A	-A	-A	A	A	0	0	0	0	0	0	0	0	0
ψ_{20}	-A	A	A	A	A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{21}	0	0	0	0	0	0	0	M	/O	/M	P	N	O	Q	/P	/Q
ψ_{22}	0	0	0	0	0	0	0	N	/P	/N	Q	O	P	/M	/Q	M
ψ_{23}	0	0	0	0	0	0	0	O	/Q	/O	/M	P	Q	/N	M	N
ψ_{24}	0	0	0	0	0	0	0	/M	O	M	/P	/N	/O	/Q	P	Q
ψ_{25}	0	0	0	0	0	0	0	P	M	/P	/N	Q	/M	/O	N	O
ψ_{26}	0	0	0	0	0	0	0	/O	Q	O	M	/P	/Q	N	/M	/N
ψ_{27}	0	0	0	0	0	0	0	Q	N	/Q	/O	/M	/N	/P	O	P
ψ_{28}	0	0	0	0	0	0	0	/P	/M	P	N	/Q	M	O	/N	/O
ψ_{29}	0	0	0	0	0	0	0	/Q	/N	Q	O	M	N	P	/O	/P
ψ_{30}	0	0	0	0	0	0	0	/N	P	N	/Q	/O	/P	M	Q	/M
ψ_{31}	0	0	0	0	0	0	0	M	/O	/M	P	N	O	Q	/P	/Q
ψ_{32}	0	0	0	0	0	0	0	N	/P	/N	Q	O	P	/M	/Q	M
ψ_{33}	0	0	0	0	0	0	0	O	/Q	/O	/M	P	Q	/N	M	N
ψ_{34}	0	0	0	0	0	0	0	/M	O	M	/P	/N	/O	/Q	P	Q
ψ_{35}	0	0	0	0	0	0	0	P	M	/P	/N	Q	/M	/O	N	O
ψ_{36}	0	0	0	0	0	0	0	/O	Q	O	M	/P	/Q	N	/M	/N
ψ_{37}	0	0	0	0	0	0	0	Q	N	/Q	/O	/M	/N	/P	O	P
ψ_{38}	0	0	0	0	0	0	0	/P	/M	P	N	/Q	M	O	/N	/O
ψ_{39}	0	0	0	0	0	0	0	/Q	/N	Q	O	M	N	P	/O	/P
ψ_{40}	0	0	0	0	0	0	0	/N	P	N	/Q	/O	/P	M	Q	/M
ψ_{41}	0	0	0	0	0	0	0	M	/O	/M	P	N	O	Q	/P	/Q
ψ_{42}	0	0	0	0	0	0	0	N	/P	/N	Q	O	P	/M	/Q	M
ψ_{43}	0	0	0	0	0	0	0	O	/Q	/O	/M	P	Q	/N	M	N
ψ_{44}	0	0	0	0	0	0	0	/M	O	M	/P	/N	/O	/Q	P	Q
ψ_{45}	0	0	0	0	0	0	0	P	M	/P	/N	Q	/M	/O	N	O
ψ_{46}	0	0	0	0	0	0	0	/O	Q	O	M	/P	/Q	N	/M	/N
ψ_{47}	0	0	0	0	0	0	0	Q	N	/Q	/O	/M	/N	/P	O	P
ψ_{48}	0	0	0	0	0	0	0	/P	/M	P	N	/Q	M	O	/N	/O
ψ_{49}	0	0	0	0	0	0	0	/Q	/N	Q	O	M	N	P	/O	/P
ψ_{50}	0	0	0	0	0	0	0	/N	P	N	/Q	/O	/P	M	Q	/M
ψ_{51}	0	0	0	0	0	0	0	M	/O	/M	P	N	O	Q	/P	/Q
ψ_{52}	0	0	0	0	0	0	0	N	/P	/N	Q	O	P	/M	/Q	M
ψ_{53}	0	0	0	0	0	0	0	O	/Q	/O	/M	P	Q	/N	M	N
ψ_{54}	0	0	0	0	0	0	0	/M	O	M	/P	/N	/O	/Q	P	Q
ψ_{55}	0	0	0	0	0	0	0	P	M	/P	/N	Q	/M	/O	N	O
ψ_{56}	0	0	0	0	0	0	0	/O	Q	O	M	/P	/Q	N	/M	/N
ψ_{57}	0	0	0	0	0	0	0	Q	N	/Q	/O	/M	/N	/P	O	P
ψ_{58}	0	0	0	0	0	0	0	/P	/M	P	N	/Q	M	O	/N	/O
ψ_{59}	0	0	0	0	0	0	0	/Q	/N	Q	O	M	N	P	/O	/P
ψ_{60}	0	0	0	0	0	0	0	/N	P	N	/Q	/O	/P	M	Q	/M

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	124	124	124	124	124	124	124	124	124	124	124	124	124	124	124	124
$o(y)$	31j	62a	62b	62c	62d	62e	62f	62g	62h	62i	62j	124a	124b	124c	124d	124e
$2P$	31f	31j	31d	31e	31i	31b	31h	31g	31a	31f	31c	62d	62i	62a	62h	62c
$3P$	31c	62j	62i	62h	62f	62a	62e	62b	62d	62c	62g	124o	124p	124s	124l	124n
$5P$	31j	62a	62b	62c	62d	62e	62f	62g	62h	62i	62j	124a	124b	124c	124d	124e
$31P$	1a	2a	2a	2a	2a	2a	2a	2a	2a	2a	2a	4a	4a	4a	4a	4a
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
ψ_3	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-A	-A	-A	-A	-A
ψ_4	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	A	A	A	A	A
ψ_5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_6	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1
ψ_7	-1	1	1	1	1	1	1	1	1	1	1	A	A	A	A	A
ψ_8	-1	1	1	1	1	1	1	1	1	1	1	-A	-A	-A	-A	-A
ψ_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{13}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{17}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{18}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{19}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{21}	/N	M	/O	/M	P	N	O	/P	Q	/N	/Q	/O	M	Q	/P	/Q
ψ_{22}	/O	N	/P	/N	Q	O	P	/Q	/M	/O	M	/P	N	/M	/Q	M
ψ_{23}	/P	O	/Q	/O	/M	P	Q	M	/N	/P	N	/Q	O	/N	M	N
ψ_{24}	N	/M	O	M	/P	/N	/O	P	/Q	N	Q	O	/M	/Q	P	Q
ψ_{25}	/Q	P	M	/P	/N	Q	/M	N	/O	/Q	O	M	P	/O	N	O
ψ_{26}	P	/O	Q	O	M	/P	/Q	/M	N	P	/N	Q	/O	N	/M	/N
ψ_{27}	M	Q	N	/Q	/O	/M	/N	O	/P	M	P	N	Q	/P	O	P
ψ_{28}	Q	/P	/M	P	N	/Q	M	/N	O	Q	/O	/M	/P	O	/N	/O
ψ_{29}	/M	/Q	/N	Q	O	M	N	/O	P	/M	/P	/N	/Q	P	/O	/P
ψ_{30}	O	/N	P	N	/Q	/O	/P	Q	M	O	/M	P	/N	M	Q	/M
ψ_{31}	/N	M	/O	/M	P	N	O	/P	Q	/N	/Q	-/O	-M	-Q	-/P	-/Q
ψ_{32}	/O	N	/P	/N	Q	O	P	/Q	/M	/O	M	-/P	-N	-/M	-/Q	-M
ψ_{33}	/P	O	/Q	/O	/M	P	Q	M	/N	/P	N	-/Q	-O	-/N	-M	-N
ψ_{34}	N	/M	O	M	/P	/N	/O	P	/Q	N	Q	-O	-/M	-/Q	-P	-Q
ψ_{35}	/Q	P	M	/P	/N	Q	/M	N	/O	/Q	O	-M	-P	-/O	-N	-O
ψ_{36}	P	/O	Q	O	M	/P	/Q	/M	N	P	/N	-Q	-/O	-N	-/M	-/N
ψ_{37}	M	Q	N	/Q	/O	/M	/N	O	/P	M	P	-N	-Q	-/P	-O	-P
ψ_{38}	Q	/P	/M	P	N	/Q	M	/N	O	Q	/O	-/M	-/P	-O	-/N	-/O
ψ_{39}	/M	/Q	/N	Q	O	M	N	/O	P	/M	/P	-/N	-/Q	-P	-/O	-/P
ψ_{40}	O	/N	P	N	/Q	/O	/P	Q	M	O	/M	-P	-/N	-M	-Q	-/M
ψ_{41}	/N	-M	-/O	-/M	-P	-N	-O	-/P	-Q	-/N	-/Q	R	U	-/T	S	T
ψ_{42}	/O	-N	-/P	-/N	-Q	-O	-P	-/Q	-/M	-/O	-M	S	V	-/U	T	U
ψ_{43}	/P	-O	-/Q	-/O	-/M	-P	-Q	-M	-/N	-/P	-N	T	-/R	-/V	U	V
ψ_{44}	N	-/M	-O	-M	-/P	-/N	-/O	-P	-/Q	-N	-Q	-/R	-/U	T	-/S	-/T
ψ_{45}	/Q	-P	-M	-/P	-/N	-Q	-/M	-N	-/O	-/Q	-O	U	-/S	R	V	-/R
ψ_{46}	P	-/O	-Q	-O	-M	-/P	-/Q	-/M	-N	-P	-/N	-/T	R	V	-/U	-/V
ψ_{47}	M	-Q	-N	-/Q	-/O	-/M	-/N	-O	-/P	-M	-P	V	-/T	S	-/R	-/S
ψ_{48}	Q	-/P	-/M	-P	-N	-/Q	-M	-/N	-O	-Q	-/O	-/U	S	-/R	-/V	R
ψ_{49}	/M	-/Q	-/N	-Q	-O	-M	-N	-/O	-P	-/M	-/P	-/V	T	-/S	R	S
ψ_{50}	O	-/N	-P	-N	-/Q	-/O	-/P	-Q	-M	-O	-/M	-/S	-/V	U	-/T	-/U
ψ_{51}	/N	-M	-/O	-/M	-P	-N	-O	-/P	-Q	-/N	-/Q	-R	-U	/T	-S	-T
ψ_{52}	/O	-N	-/P	-/N	-Q	-O	-P	-/Q	-/M	-/O	-M	-S	-V	/U	-T	-U
ψ_{53}	/P	-O	-/Q	-/O	-/M	-P	-Q	-M	-/N	-/P	-N	-T	/R	/V	-U	-V
ψ_{54}	N	-/M	-O	-M	-/P	-/N	-/O	-P	-/Q	-N	-Q	/R	/U	-T	/S	/T
ψ_{55}	/Q	-P	-M	-/P	-/N	-Q	-/M	-N	-/O	-/Q	-O	-U	/S	-R	-V	/R
ψ_{56}	P	-/O	-Q	-O	-M	-/P	-/Q	-/M	-N	-P	-/N	/T	-R	-V	/U	/V
ψ_{57}	M	-Q	-N	-/Q	-/O	-/M	-/N	-O	-/P	-M	-P	-V	/T	-S	/R	/S
ψ_{58}	Q	-/P	-/M	-P	-N	-/Q	-M	-/N	-O	-Q	-/O	/U	-S	/R	/V	-R
ψ_{59}	/M	-/Q	-/N	-Q	-O	-M	-N	-/O	-P	-/M	-/P	/V	-T	/S	-R	-S
ψ_{60}	O	-/N	-P	-N	-/Q	-/O	-/P	-Q	-M	-O	-/M	/S	/V	-U	/T	/U

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	124	124	124	124	124	124	124	124	124	124	124	124	124	124	124
$o(y)$	124f	124g	124h	124i	124j	124k	124l	124m	124n	124o	124p	124q	124r	124s	124t
$2P$	62f	62e	62j	62b	62g	62i	62d	62a	62h	62f	62c	62e	62b	62j	62g
$3P$	124q	124m	124t	124k	124r	124e	124f	124h	124a	124g	124d	124c	124b	124j	124i
$5P$	124f	124g	124h	124i	124j	124k	124l	124m	124n	124o	124p	124q	124r	124s	124t
$31P$	4a	4a	4a	4a	4a	4b	4b	4b	4b	4b	4b	4b	4b	4b	4b
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_3	-A	-A	-A	-A	-A	A	A	A	A	A	A	A	A	A	A
ψ_4	A	A	A	A	A	-A	-A	-A	-A	-A	-A	-A	-A	-A	-A
ψ_5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_7	A	A	A	A	A	-A	-A	-A	-A	-A	-A	-A	-A	-A	-A
ψ_8	-A	-A	-A	-A	-A	A	A	A	A	A	A	A	A	A	A
ψ_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{10}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{13}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{16}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{17}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{18}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{19}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{21}	/N	/M	P	N	O	M	/O	Q	/P	/N	/Q	/M	N	P	O
ψ_{22}	/O	/N	Q	O	P	N	/P	/M	/Q	/O	M	/N	O	Q	P
ψ_{23}	/P	/O	/M	P	Q	O	/Q	/N	M	/P	N	/O	P	/M	Q
ψ_{24}	N	M	/P	/N	/O	/M	O	/Q	P	N	Q	M	/N	/P	/O
ψ_{25}	/Q	/P	/N	Q	/M	P	M	/O	N	/Q	O	/P	Q	/N	/M
ψ_{26}	P	O	M	/P	/Q	/O	Q	N	/M	P	/N	O	/P	M	/Q
ψ_{27}	M	/Q	/O	/M	/N	Q	N	/P	O	M	P	/Q	/M	/O	/N
ψ_{28}	Q	P	N	/Q	M	/P	/M	O	/N	Q	/O	P	/Q	N	M
ψ_{29}	/M	Q	O	M	N	/Q	/N	P	/O	/M	/P	Q	M	O	N
ψ_{30}	O	N	/Q	/O	/P	/N	P	M	Q	O	/M	N	/O	/Q	/P
ψ_{31}	-/N	-/M	-P	-N	-O	-M	-/O	-Q	-/P	-/N	-/Q	-/M	-N	-P	-O
ψ_{32}	-/O	-/N	-Q	-O	-P	-N	-/P	-/M	-/Q	-/O	-M	-/N	-O	-Q	-P
ψ_{33}	-/P	-/O	-/M	-P	-Q	-O	-/Q	-/N	-M	-/P	-N	-/O	-P	-/M	-Q
ψ_{34}	-N	-M	-/P	-/N	-/O	-/M	-O	-/Q	-P	-N	-Q	-M	-/N	-/P	-/O
ψ_{35}	-/Q	-/P	-/N	-Q	-/M	-P	-M	-/O	-N	-/Q	-O	-/P	-Q	-/N	-/M
ψ_{36}	-P	-O	-M	-/P	-/Q	-/O	-Q	-N	-/M	-P	-/N	-O	-/P	-M	-/Q
ψ_{37}	-M	-/Q	-/O	-/M	-/N	-Q	-N	-/P	-O	-M	-P	-/Q	-/M	-/O	-/N
ψ_{38}	-Q	-P	-N	-/Q	-M	-/P	-/M	-O	-/N	-Q	-/O	-P	-/Q	-N	-M
ψ_{39}	-/M	-Q	-O	-M	-N	-/Q	-/N	-P	-/O	-/M	-/P	-Q	-M	-O	-N
ψ_{40}	-O	-N	-/Q	-/O	-/P	-/N	-P	-M	-Q	-O	-/M	-N	-/O	-/Q	-/P
ψ_{41}	-/V	-/U	-/S	V	-/R	-U	-R	/T	-S	/V	-T	/U	-V	/S	/R
ψ_{42}	R	-/V	-/T	-/R	-/S	-V	-S	/U	-T	-R	-U	/V	/R	/T	/S
ψ_{43}	S	R	-/U	-/S	-/T	/R	-T	/V	-U	-S	-V	-R	/S	/U	/T
ψ_{44}	V	U	S	-/V	R	/U	/R	-T	/S	-V	/T	-U	/V	-S	-R
ψ_{45}	T	S	-/V	-/T	-/U	/S	-U	-R	-V	-T	/R	-S	/T	/V	/U
ψ_{46}	-/S	-/R	U	S	T	-R	/T	-V	/U	/S	/V	/R	-S	-U	-T
ψ_{47}	U	T	R	-/U	-/V	/T	-V	-S	/R	-U	/S	-T	/U	-R	/V
ψ_{48}	-/T	-/S	V	T	U	-S	/U	/R	/V	/T	-R	/S	-T	-V	-U
ψ_{49}	-/U	-/T	-/R	U	V	-T	/V	/S	-R	/U	-S	/T	-U	/R	-V
ψ_{50}	-/R	V	T	R	S	/V	/S	-U	/T	/R	/U	-V	-R	-T	-S
ψ_{51}	/V	/U	/S	-V	/R	U	R	-/T	S	-/V	T	-/U	V	-/S	-/R
ψ_{52}	-R	/V	/T	/R	/S	V	S	-/U	T	R	U	-/V	-/R	-/T	-/S
ψ_{53}	-S	-R	/U	/S	/T	-/R	T	-/V	U	S	V	R	-/S	-/U	-/T
ψ_{54}	-V	-U	-S	/V	-R	-/U	-/R	T	-/S	V	-/T	U	-/V	S	R
ψ_{55}	-T	-S	/V	/T	/U	-/S	U	R	V	T	-/R	S	-/T	-/V	-/U
ψ_{56}	/S	/R	-U	-S	-T	R	-/T	V	-/U	-/S	-/V	-/R	S	U	T
ψ_{57}	-U	-T	-R	/U	/V	-/T	V	S	-/R	U	-/S	T	-/U	R	-/V
ψ_{58}	/T	/S	-V	-T	-U	S	-/U	-/R	-/V	-/T	R	-/S	T	V	U
ψ_{59}	/U	/T	/R	-U	-V	T	-/V	-/S	R	-/U	S	-/T	U	-/R	V
ψ_{60}	/R	-V	-T	-R	-S	-/V	-/S	U	-/T	-/R	-/U	V	R	T	S

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	1488000	1488000	1920	1920	96	1488000	1488000	1920	1920	1920	1920	1920	1920
$o(y)$	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	4g	4h
$2P$	1a	1a	1a	1a	3a	2a	2a	2c	2c	2b	2b	2a	2c
$3P$	1a	2a	2b	2c	1a	4b	4a	4d	4c	4j	4l	4k	4i
$5P$	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	4g	4h
$31P$	1a	2a	2b	2c	3a	4b	4a	4d	4c	4j	4l	4k	4i
ψ_{61}	124	124	4	4	1	-124	-124	4	4	-4	-4	-4	4
ψ_{62}	124	124	4	4	1	-124	-124	4	4	-4	-4	-4	4
ψ_{63}	124	124	4	4	1	124	124	4	4	4	4	4	4
ψ_{64}	124	124	4	4	1	124	124	4	4	4	4	4	4
ψ_{65}	124	-124	-4	4	1	E	-E	-4	-4	-K	-K	-K	4
ψ_{66}	124	-124	-4	4	1	-E	E	-4	-4	K	K	K	4
ψ_{67}	124	-124	-4	4	1	E	-E	-4	-4	-K	-K	-K	4
ψ_{68}	124	-124	-4	4	1	-E	E	-4	-4	K	K	K	4
ψ_{69}	124	-124	-4	4	1	E	-E	4	4	K	K	-K	-4
ψ_{70}	124	-124	-4	4	1	-E	E	4	4	-K	-K	K	-4
ψ_{71}	124	-124	-4	4	1	E	-E	4	4	K	K	-K	-4
ψ_{72}	124	-124	-4	4	1	-E	E	4	4	-K	-K	K	-4
ψ_{73}	124	124	4	4	1	-124	-124	-4	-4	4	4	-4	-4
ψ_{74}	124	124	4	4	1	-124	-124	-4	-4	4	4	-4	-4
ψ_{75}	124	124	4	4	1	124	124	-4	-4	-4	-4	4	-4
ψ_{76}	124	124	4	4	1	124	124	-4	-4	-4	-4	4	-4
ψ_{77}	124	124	-4	-4	-2	124	124	-K	K	K	-K	-4	K
ψ_{78}	124	124	-4	-4	-2	124	124	K	-K	-K	K	-4	-K
ψ_{79}	124	124	-4	-4	-2	-124	-124	K	-K	K	-K	4	-K
ψ_{80}	124	124	-4	-4	-2	-124	-124	-K	K	-K	K	4	K
ψ_{81}	124	-124	4	-4	-2	E	-E	-K	K	-4	4	K	-K
ψ_{82}	124	-124	4	-4	-2	-E	E	K	-K	-4	4	-K	K
ψ_{83}	124	-124	4	-4	-2	E	-E	K	-K	4	-4	K	K
ψ_{84}	124	-124	4	-4	-2	-E	E	-K	K	4	-4	-K	-K
ψ_{85}	124	-124	4	-4	1	E	-E	-K	K	-4	4	K	-K
ψ_{86}	124	-124	4	-4	1	E	-E	-K	K	-4	4	K	-K
ψ_{87}	124	-124	4	-4	1	-E	E	K	-K	-4	4	-K	K
ψ_{88}	124	-124	4	-4	1	-E	E	K	-K	-4	4	-K	K
ψ_{89}	124	-124	4	-4	1	E	-E	K	-K	4	-4	K	K
ψ_{90}	124	-124	4	-4	1	E	-E	K	-K	4	-4	K	K
ψ_{91}	124	-124	4	-4	1	-E	E	-K	K	4	-4	-K	-K
ψ_{92}	124	-124	4	-4	1	-E	E	-K	K	4	-4	-K	-K
ψ_{93}	124	124	-4	-4	1	-124	-124	-K	K	-K	K	4	K
ψ_{94}	124	124	-4	-4	1	-124	-124	-K	K	-K	K	4	K
ψ_{95}	124	124	-4	-4	1	-124	-124	K	-K	K	-K	4	-K
ψ_{96}	124	124	-4	-4	1	-124	-124	K	-K	K	-K	4	-K
ψ_{97}	124	124	-4	-4	1	124	124	K	-K	-K	K	-4	-K
ψ_{98}	124	124	-4	-4	1	124	124	K	-K	-K	K	-4	-K
ψ_{99}	124	124	-4	-4	1	124	124	-K	K	K	-K	-4	K
ψ_{100}	124	124	-4	-4	1	124	124	-K	K	K	-K	-4	K
ψ_{101}	125	125	5	5	-1	125	125	5	5	5	5	5	5
ψ_{102}	125	125	5	5	-1	-125	-125	5	5	-5	-5	-5	5
ψ_{103}	125	-125	-5	5	-1	F	-F	5	5	-I	-I	I	-5
ψ_{104}	125	-125	-5	5	-1	-F	F	5	5	I	I	-I	-5
ψ_{105}	155	155	11	11	-1	155	155	-1	-1	-1	-1	11	-1
ψ_{106}	155	155	11	11	-1	-155	-155	-1	-1	1	1	-11	-1
ψ_{107}	155	-155	-11	11	-1	G	-G	-1	-1	A	A	L	1
ψ_{108}	155	-155	-11	11	-1	-G	G	-1	-1	-A	-A	-L	1
ψ_{109}	155	-155	1	-1	-1	G	-G	X	/X	AA	-/AA	-A	-/X
ψ_{110}	155	-155	1	-1	-1	-G	G	/X	X	/AA	-AA	A	-X
ψ_{111}	155	-155	1	-1	-1	G	-G	/X	X	-/AA	AA	-A	-X
ψ_{112}	155	-155	1	-1	-1	-G	G	X	/X	-AA	/AA	A	-/X
ψ_{113}	155	155	-1	-1	-1	-155	-155	X	/X	-/X	-X	1	/X
ψ_{114}	155	155	-1	-1	-1	-155	-155	/X	X	-X	-/X	1	X
ψ_{115}	155	155	-1	-1	-1	155	155	/X	X	X	/X	-1	X
ψ_{116}	155	155	-1	-1	-1	155	155	X	/X	/X	X	-1	/X
ψ_{117}	186	186	-6	-6	0	186	186	6	6	6	6	-6	6
ψ_{118}	186	186	-6	-6	0	-186	-186	6	6	-6	-6	6	6
ψ_{119}	186	-186	6	-6	0	H	-H	6	6	-J	-J	-J	-6
ψ_{120}	186	-186	6	-6	0	-H	H	6	6	J	J	J	-6

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	1920	1920	1920	1920	64	64	64	64	2000	100	96	96	96	96	96
$o(y)$	4i	4j	4k	4l	4m	4n	4o	4p	5a	5b	6a	6b	6c	8a	8b
$2P$	2c	2b	2a	2b	2b	2c	2b	2c	5a	5b	3a	3a	3a	4h	4i
$3P$	4h	4e	4g	4f	4o	4n	4m	4p	5a	5b	2b	2a	2c	8b	8a
$5P$	4i	4j	4k	4l	4m	4n	4o	4p	1a	1a	6a	6b	6c	8a	8b
$31P$	4h	4e	4g	4f	4o	4n	4m	4p	5a	5b	6a	6b	6c	8b	8a
ψ_{61}	4	-4	-4	-4	0	0	0	0	-1	-1	1	1	1	-2	-2
ψ_{62}	4	-4	-4	-4	0	0	0	0	-1	-1	1	1	1	2	2
ψ_{63}	4	4	4	4	0	0	0	0	-1	-1	1	1	1	-2	-2
ψ_{64}	4	4	4	4	0	0	0	0	-1	-1	1	1	1	2	2
ψ_{65}	4	K	K	K	0	0	0	0	-1	-1	-1	-1	1	-2	-2
ψ_{66}	4	-K	-K	-K	0	0	0	0	-1	-1	-1	-1	1	-2	-2
ψ_{67}	4	K	K	K	0	0	0	0	-1	-1	-1	-1	1	2	2
ψ_{68}	4	-K	-K	-K	0	0	0	0	-1	-1	-1	-1	1	2	2
ψ_{69}	-4	-K	K	-K	0	0	0	0	-1	-1	-1	-1	1	AC	-AC
ψ_{70}	-4	K	-K	K	0	0	0	0	-1	-1	-1	-1	1	-AC	AC
ψ_{71}	-4	-K	K	-K	0	0	0	0	-1	-1	-1	-1	1	-AC	AC
ψ_{72}	-4	K	-K	K	0	0	0	0	-1	-1	-1	-1	1	AC	-AC
ψ_{73}	-4	4	-4	4	0	0	0	0	-1	-1	1	1	1	AC	-AC
ψ_{74}	-4	4	-4	4	0	0	0	0	-1	-1	1	1	1	-AC	AC
ψ_{75}	-4	-4	4	-4	0	0	0	0	-1	-1	1	1	1	AC	-AC
ψ_{76}	-4	-4	4	-4	0	0	0	0	-1	-1	1	1	1	-AC	AC
ψ_{77}	-K	-K	-4	K	0	0	0	0	-1	-1	2	-2	2	0	0
ψ_{78}	K	K	-4	-K	0	0	0	0	-1	-1	2	-2	2	0	0
ψ_{79}	K	-K	4	K	0	0	0	0	-1	-1	2	-2	2	0	0
ψ_{80}	-K	K	4	-K	0	0	0	0	-1	-1	2	-2	2	0	0
ψ_{81}	K	-4	-K	4	0	0	0	0	-1	-1	-2	2	2	0	0
ψ_{82}	-K	-4	K	4	0	0	0	0	-1	-1	-2	2	2	0	0
ψ_{83}	-K	4	-K	-4	0	0	0	0	-1	-1	-2	2	2	0	0
ψ_{84}	K	4	K	-4	0	0	0	0	-1	-1	-2	2	2	0	0
ψ_{85}	K	-4	-K	4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{86}	K	-4	-K	4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{87}	-K	-4	K	4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{88}	-K	-4	K	4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{89}	-K	4	-K	-4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{90}	-K	4	-K	-4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{91}	K	4	K	-4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{92}	K	4	K	-4	0	0	0	0	-1	-1	1	-1	-1	0	0
ψ_{93}	-K	K	4	-K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{94}	-K	K	4	-K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{95}	K	-K	4	K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{96}	K	-K	4	K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{97}	K	K	-4	-K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{98}	K	K	-4	-K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{99}	-K	-K	-4	K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{100}	-K	-K	-4	K	0	0	0	0	-1	-1	-1	1	-1	0	0
ψ_{101}	5	5	5	5	1	1	1	1	0	0	-1	-1	-1	-1	-1
ψ_{102}	5	-5	-5	-5	-1	1	-1	1	0	0	-1	-1	-1	1	1
ψ_{103}	-5	I	-I	I	A	-1	-A	1	0	0	1	1	-1	A	-A
ψ_{104}	-5	-I	I	-I	-A	-1	A	1	0	0	1	1	-1	-A	A
ψ_{105}	-1	-1	11	-1	-1	-1	-1	-1	5	0	-1	-1	-1	1	1
ψ_{106}	-1	1	-11	1	1	-1	1	-1	5	0	-1	-1	-1	-1	-1
ψ_{107}	1	-A	-L	-A	-A	1	A	-1	5	0	1	1	-1	-A	A
ψ_{108}	1	A	L	A	A	1	-A	-1	5	0	1	1	-1	A	-A
ψ_{109}	-X	/AA	A	-AA	A	-1	-A	1	5	0	1	1	-1	1	1
ψ_{110}	-/X	AA	-A	-/AA	-A	-1	A	1	5	0	1	1	-1	1	1
ψ_{111}	-/X	-AA	A	/AA	A	-1	-A	1	5	0	1	1	-1	-1	-1
ψ_{112}	-X	-/AA	-A	AA	-A	-1	A	1	5	0	1	1	-1	-1	-1
ψ_{113}	X	-X	1	-/X	-1	1	-1	1	5	0	-1	-1	-1	-A	A
ψ_{114}	/X	-/X	1	-X	-1	1	-1	1	5	0	-1	-1	-1	A	-A
ψ_{115}	/X	/X	-1	X	1	1	1	1	5	0	-1	-1	-1	-A	A
ψ_{116}	X	X	-1	/X	1	1	1	1	5	0	-1	-1	-1	A	-A
ψ_{117}	6	6	-6	6	-2	-2	-2	-2	11	1	0	0	0	0	0
ψ_{118}	6	-6	6	-6	2	-2	2	-2	11	1	0	0	0	0	0
ψ_{119}	-6	J	J	J	-AC	2	AC	-2	11	1	0	0	0	0	0
ψ_{120}	-6	-J	-J	-J	AC	2	-AC	-2	11	1	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	96	96	96	96	96	96	2000	100	80	80	96	96	96	96	96
$o(y)$	8c	8d	8e	8f	8g	8h	10a	10b	10c	10d	12a	12b	12c	12d	12e
$2P$	4c	4d	4h	4i	4c	4d	5a	5b	5a	5a	6c	6c	6a	6a	6b
$3P$	8h	8g	8f	8e	8d	8c	10a	10b	10c	10d	4c	4d	4l	4j	4k
$5P$	8c	8d	8e	8f	8g	8h	2a	2a	2b	2c	12a	12b	12c	12d	12e
$31P$	8h	8g	8f	8e	8d	8c	10a	10b	10c	10d	12b	12a	12i	12j	12l
ψ_{61}	2	2	-2	-2	2	2	-1	-1	-1	-1	1	1	-1	-1	-1
ψ_{62}	-2	-2	2	2	-2	-2	-1	-1	-1	-1	1	1	-1	-1	-1
ψ_{63}	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	1	1	1	1	1
ψ_{64}	2	2	2	2	2	2	-1	-1	-1	-1	1	1	1	1	1
ψ_{65}	AC	AC	2	2	-AC	-AC	1	1	1	-1	-1	-1	A	A	A
ψ_{66}	-AC	-AC	2	2	AC	AC	1	1	1	-1	-1	-1	-A	-A	-A
ψ_{67}	-AC	-AC	-2	-2	AC	AC	1	1	1	-1	-1	-1	A	A	A
ψ_{68}	AC	AC	-2	-2	-AC	-AC	1	1	1	-1	-1	-1	-A	-A	-A
ψ_{69}	-2	2	-AC	AC	2	-2	1	1	1	-1	1	1	-A	-A	A
ψ_{70}	-2	2	AC	-AC	2	-2	1	1	1	-1	1	1	A	A	-A
ψ_{71}	2	-2	AC	-AC	-2	2	1	1	1	-1	1	1	-A	-A	A
ψ_{72}	2	-2	-AC	AC	-2	2	1	1	1	-1	1	1	A	A	-A
ψ_{73}	AC	-AC	AC	-AC	AC	-AC	-1	-1	-1	-1	-1	-1	1	1	-1
ψ_{74}	-AC	AC	-AC	AC	-AC	AC	-1	-1	-1	-1	-1	-1	1	1	-1
ψ_{75}	-AC	AC	AC	-AC	-AC	AC	-1	-1	-1	-1	-1	-1	-1	-1	1
ψ_{76}	AC	-AC	-AC	AC	AC	-AC	-1	-1	-1	-1	-1	-1	-1	-1	1
ψ_{77}	0	0	0	0	0	0	-1	-1	1	1	AC	-AC	-AC	AC	2
ψ_{78}	0	0	0	0	0	0	-1	-1	1	1	-AC	AC	AC	-AC	2
ψ_{79}	0	0	0	0	0	0	-1	-1	1	1	-AC	AC	-AC	AC	-2
ψ_{80}	0	0	0	0	0	0	-1	-1	1	1	AC	-AC	AC	-AC	-2
ψ_{81}	0	0	0	0	0	0	1	1	-1	1	AC	-AC	-2	2	AC
ψ_{82}	0	0	0	0	0	0	1	1	-1	1	-AC	AC	-2	2	-AC
ψ_{83}	0	0	0	0	0	0	1	1	-1	1	-AC	AC	2	-2	AC
ψ_{84}	0	0	0	0	0	0	1	1	-1	1	AC	-AC	2	-2	-AC
ψ_{85}	0	0	0	0	0	0	1	1	-1	1	-A	A	1	-1	-A
ψ_{86}	0	0	0	0	0	0	1	1	-1	1	-A	A	1	-1	-A
ψ_{87}	0	0	0	0	0	0	1	1	-1	1	A	-A	1	-1	A
ψ_{88}	0	0	0	0	0	0	1	1	-1	1	A	-A	1	-1	A
ψ_{89}	0	0	0	0	0	0	1	1	-1	1	A	-A	-1	1	-A
ψ_{90}	0	0	0	0	0	0	1	1	-1	1	A	-A	-1	1	-A
ψ_{91}	0	0	0	0	0	0	1	1	-1	1	-A	A	-1	1	A
ψ_{92}	0	0	0	0	0	0	1	1	-1	1	-A	A	-1	1	A
ψ_{93}	0	0	0	0	0	0	-1	-1	1	1	-A	A	-A	A	1
ψ_{94}	0	0	0	0	0	0	-1	-1	1	1	-A	A	-A	A	1
ψ_{95}	0	0	0	0	0	0	-1	-1	1	1	A	-A	A	-A	1
ψ_{96}	0	0	0	0	0	0	-1	-1	1	1	A	-A	A	-A	1
ψ_{97}	0	0	0	0	0	0	-1	-1	1	1	A	-A	-A	A	-1
ψ_{98}	0	0	0	0	0	0	-1	-1	1	1	A	-A	-A	A	-1
ψ_{99}	0	0	0	0	0	0	-1	-1	1	1	-A	A	A	-A	-1
ψ_{100}	0	0	0	0	0	0	-1	-1	1	1	-A	A	A	-A	-1
ψ_{101}	-1	-1	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	-1
ψ_{102}	-1	-1	1	1	-1	-1	0	0	0	0	-1	-1	1	1	1
ψ_{103}	-1	1	-A	A	1	-1	0	0	0	0	-1	-1	A	A	-A
ψ_{104}	-1	1	A	-A	1	-1	0	0	0	0	-1	-1	-A	-A	A
ψ_{105}	1	1	1	1	1	1	5	0	1	1	-1	-1	-1	-1	-1
ψ_{106}	1	1	-1	-1	1	1	5	0	1	1	-1	-1	1	1	1
ψ_{107}	1	-1	A	-A	-1	1	-5	0	-1	1	-1	-1	A	A	-A
ψ_{108}	1	-1	-A	A	-1	1	-5	0	-1	1	-1	-1	-A	-A	A
ψ_{109}	-A	-A	-1	-1	A	A	-5	0	1	-1	1	1	-A	-A	-A
ψ_{110}	A	A	-1	-1	-A	-A	-5	0	1	-1	1	1	A	A	A
ψ_{111}	A	A	1	1	-A	-A	-5	0	1	-1	1	1	-A	-A	-A
ψ_{112}	-A	-A	1	1	A	A	-5	0	1	-1	1	1	A	A	A
ψ_{113}	-A	A	-A	A	-A	A	5	0	-1	-1	1	1	-1	-1	1
ψ_{114}	A	-A	A	-A	A	-A	5	0	-1	-1	1	1	-1	-1	1
ψ_{115}	A	-A	-A	A	A	-A	5	0	-1	-1	1	1	1	1	-1
ψ_{116}	-A	A	A	-A	-A	A	5	0	-1	-1	1	1	1	1	-1
ψ_{117}	0	0	0	0	0	0	11	1	-1	-1	0	0	0	0	0
ψ_{118}	0	0	0	0	0	0	11	1	-1	-1	0	0	0	0	0
ψ_{119}	0	0	0	0	0	0	-11	-1	1	-1	0	0	0	0	0
ψ_{120}	0	0	0	0	0	0	-11	-1	1	-1	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	96	96	96	96	96	96	96	2000	100	2000	100	80	80	80	80
$o(y)$	12f	12g	12h	12i	12j	12k	12l	20a	20b	20c	20d	20e	20f	20g	20h
$2P$	6b	6c	6c	6a	6a	6b	6b	10a	10b	10a	10b	10d	10d	10c	10c
$3P$	4b	4h	4i	4f	4e	4a	4g	20c	20d	20a	20b	20f	20e	20l	20n
$5P$	12f	12g	12h	12i	12j	12k	12l	4a	4a	4b	4b	4c	4d	4e	4f
$31P$	12k	12h	12g	12c	12d	12f	12e	20c	20d	20a	20b	20f	20e	20l	20n
ψ_{61}	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1
ψ_{62}	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1
ψ_{63}	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{64}	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{65}	A	1	1	-A	-A	-A	-A	-A	-A	A	A	1	1	-A	-A
ψ_{66}	-A	1	1	A	A	A	A	A	A	-A	-A	1	1	A	A
ψ_{67}	A	1	1	-A	-A	-A	-A	-A	-A	A	A	1	1	-A	-A
ψ_{68}	-A	1	1	A	A	A	A	A	A	-A	-A	1	1	A	A
ψ_{69}	A	-1	-1	A	A	-A	-A	-A	-A	A	A	-1	-1	A	A
ψ_{70}	-A	-1	-1	-A	-A	A	A	A	A	-A	-A	-1	-1	-A	-A
ψ_{71}	A	-1	-1	A	A	-A	-A	-A	-A	A	A	-1	-1	A	A
ψ_{72}	-A	-1	-1	-A	-A	A	A	A	A	-A	-A	-1	-1	-A	-A
ψ_{73}	-1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	-1	-1
ψ_{74}	-1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	-1	-1
ψ_{75}	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1
ψ_{76}	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1
ψ_{77}	-2	-AC	AC	AC	-AC	-2	2	-1	-1	-1	-1	-A	A	A	-A
ψ_{78}	-2	AC	-AC	-AC	AC	-2	2	-1	-1	-1	-1	A	-A	-A	A
ψ_{79}	2	AC	-AC	AC	-AC	2	-2	1	1	1	1	A	-A	A	-A
ψ_{80}	2	-AC	AC	-AC	AC	2	-2	1	1	1	1	-A	A	-A	A
ψ_{81}	-AC	AC	-AC	-2	2	AC	-AC	-A	-A	A	A	-A	A	1	-1
ψ_{82}	AC	-AC	AC	-2	2	-AC	AC	A	A	-A	-A	A	-A	1	-1
ψ_{83}	-AC	-AC	AC	2	-2	AC	-AC	-A	-A	A	A	A	-A	-1	1
ψ_{84}	AC	AC	-AC	2	-2	-AC	AC	A	A	-A	-A	-A	A	-1	1
ψ_{85}	A	-A	A	1	-1	-A	A	-A	-A	A	A	-A	A	1	-1
ψ_{86}	A	-A	A	1	-1	-A	A	-A	-A	A	A	-A	A	1	-1
ψ_{87}	-A	A	-A	1	-1	A	-A	A	A	-A	-A	A	-A	1	-1
ψ_{88}	-A	A	-A	1	-1	A	-A	A	A	-A	-A	A	-A	1	-1
ψ_{89}	A	A	-A	-1	1	-A	A	-A	-A	A	A	A	-A	-1	1
ψ_{90}	A	A	-A	-1	1	-A	A	-A	-A	A	A	A	-A	-1	1
ψ_{91}	-A	-A	A	-1	1	A	-A	A	A	-A	-A	-A	A	-1	1
ψ_{92}	-A	-A	A	-1	1	A	-A	A	A	-A	-A	-A	A	-1	1
ψ_{93}	-1	A	-A	A	-A	-1	1	1	1	1	1	-A	A	-A	A
ψ_{94}	-1	A	-A	A	-A	-1	1	1	1	1	1	-A	A	-A	A
ψ_{95}	-1	-A	A	-A	A	-1	1	1	1	1	1	A	-A	A	-A
ψ_{96}	-1	-A	A	-A	A	-1	1	1	1	1	1	A	-A	A	-A
ψ_{97}	1	-A	A	A	-A	1	-1	-1	-1	-1	-1	A	-A	-A	A
ψ_{98}	1	-A	A	A	-A	1	-1	-1	-1	-1	-1	A	-A	-A	A
ψ_{99}	1	A	-A	-A	A	1	-1	-1	-1	-1	-1	-A	A	A	-A
ψ_{100}	1	A	-A	-A	A	1	-1	-1	-1	-1	-1	-A	A	A	-A
ψ_{101}	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0
ψ_{102}	1	-1	-1	1	1	1	1	0	0	0	0	0	0	0	0
ψ_{103}	-A	1	1	-A	-A	A	A	0	0	0	0	0	0	0	0
ψ_{104}	A	1	1	A	A	-A	-A	0	0	0	0	0	0	0	0
ψ_{105}	-1	-1	-1	-1	-1	-1	-1	5	0	5	0	-1	-1	-1	-1
ψ_{106}	1	-1	-1	1	1	1	1	-5	0	-5	0	-1	-1	1	1
ψ_{107}	-A	1	1	-A	-A	A	A	I	0	-I	0	-1	-1	A	A
ψ_{108}	A	1	1	A	A	-A	-A	-I	0	I	0	-1	-1	-A	-A
ψ_{109}	-A	-1	-1	A	A	A	A	I	0	-I	0	A	-A	-1	1
ψ_{110}	A	-1	-1	-A	-A	-A	-A	-I	0	I	0	-A	A	-1	1
ψ_{111}	-A	-1	-1	A	A	A	A	I	0	-I	0	-A	A	1	-1
ψ_{112}	A	-1	-1	-A	-A	-A	-A	-I	0	I	0	A	-A	1	-1
ψ_{113}	1	1	1	-1	-1	1	1	-5	0	-5	0	A	-A	A	-A
ψ_{114}	1	1	1	-1	-1	1	1	-5	0	-5	0	-A	A	-A	A
ψ_{115}	-1	1	1	1	1	-1	-1	5	0	5	0	-A	A	A	-A
ψ_{116}	-1	1	1	1	1	-1	-1	5	0	5	0	A	-A	-A	A
ψ_{117}	0	0	0	0	0	0	0	11	1	11	1	1	1	1	1
ψ_{118}	0	0	0	0	0	0	0	-11	-1	-11	-1	1	1	-1	-1
ψ_{119}	0	0	0	0	0	0	0	L	A	-L	-A	1	1	-A	-A
ψ_{120}	0	0	0	0	0	0	0	-L	-A	L	A	1	1	A	A

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	80	80	80	80	80	80	96	96	96	96	96	96	96	96	96
$o(y)$	20i	20j	20k	20l	20m	20n	24a	24b	24c	24d	24e	24f	24g	24h	24i
$2P$	10a	10d	10d	10c	10a	10c	12h	12h	12g	12g	12b	12b	12a	12a	12h
$3P$	20m	20k	20j	20g	20i	20h	8b	8b	8a	8a	8h	8h	8g	8g	8f
$5P$	4g	4h	4i	4j	4k	4l	24a	24b	24c	24d	24e	24f	24g	24h	24i
$31P$	20m	20k	20j	20g	20i	20h	24c	24d	24a	24b	24o	24p	24m	24n	24k
ψ_{61}	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1
ψ_{62}	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1
ψ_{63}	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1
ψ_{64}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{65}	-A	-1	-1	A	A	A	1	1	1	1	-A	-A	-A	-A	-1
ψ_{66}	A	-1	-1	-A	-A	-A	1	1	1	1	A	A	A	A	-1
ψ_{67}	-A	-1	-1	A	A	A	-1	-1	-1	-1	A	A	A	A	1
ψ_{68}	A	-1	-1	-A	-A	-A	-1	-1	-1	-1	-A	-A	-A	-A	1
ψ_{69}	-A	1	1	-A	A	-A	-A	-A	A	A	1	1	-1	-1	A
ψ_{70}	A	1	1	A	-A	A	A	A	-A	-A	1	1	-1	-1	-A
ψ_{71}	-A	1	1	-A	A	-A	A	A	-A	-A	-1	-1	1	1	-A
ψ_{72}	A	1	1	A	-A	A	-A	-A	A	A	-1	-1	1	1	A
ψ_{73}	1	1	1	-1	1	-1	-A	-A	A	A	-A	-A	A	A	-A
ψ_{74}	1	1	1	-1	1	-1	A	A	-A	-A	A	A	-A	-A	A
ψ_{75}	-1	1	1	1	-1	1	-A	-A	A	A	A	A	-A	-A	-A
ψ_{76}	-1	1	1	1	-1	1	A	A	-A	-A	-A	-A	A	A	A
ψ_{77}	1	A	-A	-A	1	A	0	0	0	0	0	0	0	0	0
ψ_{78}	1	-A	A	A	1	-A	0	0	0	0	0	0	0	0	0
ψ_{79}	-1	-A	A	-A	-1	A	0	0	0	0	0	0	0	0	0
ψ_{80}	-1	A	-A	A	-1	-A	0	0	0	0	0	0	0	0	0
ψ_{81}	A	-A	A	1	-A	-1	0	0	0	0	0	0	0	0	0
ψ_{82}	-A	A	-A	1	A	-1	0	0	0	0	0	0	0	0	0
ψ_{83}	A	A	-A	-1	-A	1	0	0	0	0	0	0	0	0	0
ψ_{84}	-A	-A	A	-1	A	1	0	0	0	0	0	0	0	0	0
ψ_{85}	A	-A	A	1	-A	-1	AD	-AD	-/AD	/AD	AD	-AD	-/AD	/AD	AD
ψ_{86}	A	-A	A	1	-A	-1	-AD	AD	/AD	-/AD	-AD	AD	/AD	-/AD	-AD
ψ_{87}	-A	A	-A	1	A	-1	-/AD	/AD	AD	-AD	-/AD	/AD	AD	-AD	-/AD
ψ_{88}	-A	A	-A	1	A	-1	/AD	-/AD	-AD	AD	/AD	-/AD	-AD	AD	/AD
ψ_{89}	A	A	-A	-1	-A	1	/AD	-/AD	-AD	AD	-/AD	/AD	AD	-AD	/AD
ψ_{90}	A	A	-A	-1	-A	1	-/AD	/AD	AD	-AD	/AD	-/AD	-AD	AD	-/AD
ψ_{91}	-A	-A	A	-1	A	1	-AD	AD	/AD	-/AD	AD	-AD	-/AD	/AD	-AD
ψ_{92}	-A	-A	A	-1	A	1	AD	-AD	-/AD	/AD	-AD	AD	/AD	-/AD	AD
ψ_{93}	-1	A	-A	A	-1	-A	/AD	-/AD	-AD	AD	-AD	AD	-/AD	/AD	-/AD
ψ_{94}	-1	A	-A	A	-1	-A	-/AD	/AD	AD	-AD	AD	-AD	/AD	-/AD	/AD
ψ_{95}	-1	-A	A	-A	-1	A	-AD	AD	/AD	-/AD	/AD	-/AD	AD	-AD	AD
ψ_{96}	-1	-A	A	-A	-1	A	AD	-AD	-/AD	/AD	-/AD	/AD	-AD	AD	-AD
ψ_{97}	1	-A	A	A	1	-A	AD	-AD	-/AD	/AD	/AD	-/AD	AD	-AD	-AD
ψ_{98}	1	-A	A	A	1	-A	-AD	AD	/AD	-/AD	-/AD	/AD	-AD	AD	AD
ψ_{99}	1	A	-A	-A	1	A	-/AD	/AD	AD	-AD	-AD	AD	-/AD	/AD	/AD
ψ_{100}	1	A	-A	-A	1	A	/AD	-/AD	-AD	AD	AD	-AD	/AD	-/AD	-/AD
ψ_{101}	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{102}	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1	1
ψ_{103}	0	0	0	0	0	0	A	A	-A	-A	-1	-1	1	1	-A
ψ_{104}	0	0	0	0	0	0	-A	-A	A	A	-1	-1	1	1	A
ψ_{105}	1	-1	-1	-1	1	-1	1	1	1	1	1	1	1	1	1
ψ_{106}	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	1	1	1	-1
ψ_{107}	A	1	1	-A	-A	-A	-A	-A	A	A	1	1	-1	-1	A
ψ_{108}	-A	1	1	A	A	A	A	A	-A	-A	1	1	-1	-1	-A
ψ_{109}	-A	A	-A	-1	A	1	1	1	1	1	-A	-A	-A	-A	-1
ψ_{110}	A	-A	A	-1	-A	1	1	1	1	1	A	A	A	A	-1
ψ_{111}	-A	-A	A	1	A	-1	-1	-1	-1	-1	A	A	A	A	1
ψ_{112}	A	A	-A	1	-A	-1	-1	-1	-1	-1	-A	-A	-A	-A	1
ψ_{113}	1	-A	A	-A	1	A	-A	-A	A	A	-A	-A	A	A	-A
ψ_{114}	1	A	-A	A	1	-A	A	A	-A	-A	A	A	-A	-A	A
ψ_{115}	-1	A	-A	-A	-1	A	-A	-A	A	A	A	A	-A	-A	-A
ψ_{116}	-1	-A	A	A	-1	-A	A	A	-A	-A	-A	-A	A	A	A
ψ_{117}	-1	1	1	1	-1	1	0	0	0	0	0	0	0	0	0
ψ_{118}	1	1	1	-1	1	-1	0	0	0	0	0	0	0	0	0
ψ_{119}	-A	-1	-1	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{120}	A	-1	-1	-A	-A	-A	0	0	0	0	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	96	96	96	96	96	96	96	124	124	124	124	124	124	124	124	124
$o(y)$	24j	24k	24l	24m	24n	24o	24p	31a	31b	31c	31d	31e	31f	31g	31h	31i
$2P$	12h	12g	12g	12b	12b	12a	12a	31j	31d	31e	31i	31b	31h	31a	31g	31c
$3P$	8f	8e	8e	8d	8d	8c	8c	31i	31j	31g	31f	31a	31e	31d	31b	31h
$5P$	24j	24k	24l	24m	24n	24o	24p	31a	31b	31c	31d	31e	31f	31g	31h	31i
$31P$	24l	24i	24j	24g	24h	24e	24f	1a	1a	1a	1a	1a	1a	1a	1a	1a
ψ_{61}	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{62}	-1	-1	-1	1	1	1	1	0	0	0	0	0	0	0	0	0
ψ_{63}	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
ψ_{64}	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{65}	-1	-1	-1	A	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{66}	-1	-1	-1	-A	-A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{67}	1	1	1	-A	-A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{68}	1	1	1	A	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{69}	A	-A	-A	-1	-1	1	1	0	0	0	0	0	0	0	0	0
ψ_{70}	-A	A	A	-1	-1	1	1	0	0	0	0	0	0	0	0	0
ψ_{71}	-A	A	A	1	1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{72}	A	-A	-A	1	1	-1	-1	0	0	0	0	0	0	0	0	0
ψ_{73}	-A	A	A	-A	-A	A	A	0	0	0	0	0	0	0	0	0
ψ_{74}	A	-A	-A	A	A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{75}	-A	A	A	A	A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{76}	A	-A	-A	-A	-A	A	A	0	0	0	0	0	0	0	0	0
ψ_{77}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{78}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{79}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{80}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{81}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{82}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{83}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{84}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{85}	-AD	-/AD	/AD	AD	-AD	-/AD	/AD	0	0	0	0	0	0	0	0	0
ψ_{86}	AD	/AD	-/AD	-AD	AD	/AD	-/AD	0	0	0	0	0	0	0	0	0
ψ_{87}	/AD	AD	-AD	-/AD	/AD	AD	-AD	0	0	0	0	0	0	0	0	0
ψ_{88}	-/AD	-AD	AD	/AD	-/AD	-AD	AD	0	0	0	0	0	0	0	0	0
ψ_{89}	-/AD	-AD	AD	-/AD	/AD	AD	-AD	0	0	0	0	0	0	0	0	0
ψ_{90}	/AD	AD	-AD	/AD	-/AD	-AD	AD	0	0	0	0	0	0	0	0	0
ψ_{91}	AD	/AD	-/AD	AD	-AD	-/AD	/AD	0	0	0	0	0	0	0	0	0
ψ_{92}	-AD	-/AD	/AD	-AD	AD	/AD	-/AD	0	0	0	0	0	0	0	0	0
ψ_{93}	/AD	AD	-AD	AD	-AD	/AD	-/AD	0	0	0	0	0	0	0	0	0
ψ_{94}	-/AD	-AD	AD	-AD	AD	-/AD	/AD	0	0	0	0	0	0	0	0	0
ψ_{95}	-AD	-/AD	/AD	-/AD	/AD	-AD	AD	0	0	0	0	0	0	0	0	0
ψ_{96}	AD	/AD	-/AD	/AD	-/AD	AD	-AD	0	0	0	0	0	0	0	0	0
ψ_{97}	AD	/AD	-/AD	-/AD	/AD	-AD	AD	0	0	0	0	0	0	0	0	0
ψ_{98}	-AD	-/AD	/AD	/AD	-/AD	AD	-AD	0	0	0	0	0	0	0	0	0
ψ_{99}	-/AD	-AD	AD	AD	-AD	/AD	-/AD	0	0	0	0	0	0	0	0	0
ψ_{100}	/AD	AD	-AD	-AD	AD	-/AD	/AD	0	0	0	0	0	0	0	0	0
ψ_{101}	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1
ψ_{102}	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1
ψ_{103}	-A	A	A	1	1	-1	-1	1	1	1	1	1	1	1	1	1
ψ_{104}	A	-A	-A	1	1	-1	-1	1	1	1	1	1	1	1	1	1
ψ_{105}	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
ψ_{106}	-1	-1	-1	1	1	1	1	0	0	0	0	0	0	0	0	0
ψ_{107}	A	-A	-A	-1	-1	1	1	0	0	0	0	0	0	0	0	0
ψ_{108}	-A	A	A	-1	-1	1	1	0	0	0	0	0	0	0	0	0
ψ_{109}	-1	-1	-1	A	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{110}	-1	-1	-1	-A	-A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{111}	1	1	1	-A	-A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{112}	1	1	1	A	A	A	A	0	0	0	0	0	0	0	0	0
ψ_{113}	-A	A	A	-A	-A	A	A	0	0	0	0	0	0	0	0	0
ψ_{114}	A	-A	-A	A	A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{115}	-A	A	A	A	A	-A	-A	0	0	0	0	0	0	0	0	0
ψ_{116}	A	-A	-A	-A	-A	A	A	0	0	0	0	0	0	0	0	0
ψ_{117}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{118}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{119}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{120}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	124	124	124	124	124	124	124	124	124	124	124	124	124	124	124	124
$o(y)$	31j	62a	62b	62c	62d	62e	62f	62g	62h	62i	62j	124a	124b	124c	124d	124e
$2P$	31f	31j	31d	31e	31i	31b	31h	31g	31a	31f	31c	62d	62i	62a	62h	62c
$3P$	31c	62j	62i	62h	62f	62a	62e	62b	62d	62c	62g	124o	124p	124s	124l	124n
$5P$	31j	62a	62b	62c	62d	62e	62f	62g	62h	62i	62j	124a	124b	124c	124d	124e
$31P$	1a	2a	2a	2a	2a	2a	2a	2a	2a	2a	2a	4a	4a	4a	4a	4a
ψ_{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{62}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{66}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{67}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{68}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{71}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{72}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{73}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{74}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{75}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{76}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{77}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{78}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{79}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{80}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{81}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{82}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{83}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{84}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{85}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{86}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{87}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{88}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{89}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{90}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{91}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{92}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{93}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{94}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{95}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{96}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{97}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{98}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{99}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{100}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{101}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_{102}	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
ψ_{103}	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-A	-A	-A	-A	-A
ψ_{104}	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	A	A	A	A	A
ψ_{105}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{106}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{107}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{108}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{109}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{110}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{111}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{112}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{113}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{114}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{115}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{116}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{117}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{118}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{119}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{120}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table A.1: Character Table of $GL(3,5)$

$C_G(y)$	124	124	124	124	124	124	124	124	124	124	124	124	124	124	124
$o(y)$	124f	124g	124h	124i	124j	124k	124l	124m	124n	124o	124p	124q	124r	124s	124t
$2P$	62f	62e	62j	62b	62g	62i	62d	62a	62h	62f	62c	62e	62b	62j	62g
$3P$	124q	124m	124t	124k	124r	124e	124f	124h	124a	124g	124d	124c	124b	124j	124i
$5P$	124f	124g	124h	124i	124j	124k	124l	124m	124n	124o	124p	124q	124r	124s	124t
$31P$	4a	4a	4a	4a	4a	4b	4b	4b	4b	4b	4b	4b	4b	4b	4b
ψ_{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{62}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{66}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{67}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{68}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{71}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{72}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{73}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{74}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{75}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{76}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{77}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{78}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{79}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{80}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{81}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{82}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{83}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{84}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{85}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{86}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{87}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{88}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{89}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{90}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{91}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{92}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{93}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{94}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{95}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{96}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{97}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{98}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{99}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{100}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{101}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_{102}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{103}	-A	-A	-A	-A	-A	A	A	A	A	A	A	A	A	A	A
ψ_{104}	A	A	A	A	A	-A	-A	-A	-A	-A	-A	-A	-A	-A	-A
ψ_{105}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{106}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{107}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{108}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{109}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{110}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{111}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{112}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{113}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{114}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{115}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{116}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{117}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{118}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{119}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{120}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where

$$A=-E(4)=-\sqrt{-1}=-i,\ B=-30\times E(4)=-30\times\sqrt{-1}=-30i,$$
$$C=-31\times E(4)=-31\times\sqrt{-1}=-31i,\ D=-96\times E(4)=-96\times\sqrt{-1}=-96i,$$
$$E=-124\times E(4)=-124\times\sqrt{-1}=-124i,\ F=-125\times E(4)=-125\times\sqrt{-1}=-125i\ ,$$
$$G=-155\times E(4)=-155\times\sqrt{-1}=-155i\ ,\ H=-186\times E(4)=-186\times\sqrt{-1}=-186i,$$
$$I=-5\times E(4)=-5\times\sqrt{-1}=-5i,\ J=-6\times E(4)=-6\times\sqrt{-1}=-6i,$$
$$K=4\times E(4)=4\times\sqrt{-1}=4i,\ L=-11\times E(4)=-11\times\sqrt{-1}=-11i,$$
$$M=E(31)+E(31)^5+E(31)^{25},\ N=E(31)^{12}+E(31)^{21}+E(31)^{29},$$
$$O=E(31)^4+E(31)^7+E(31)^{20},\ P=E(31)^{17}+E(31)^{22}+E(31)^{23},$$
$$Q=E(31)^{16}+E(31)^{18}+E(31)^{28},\ R=E(124)^3+E(124)^{15}+E(124)^{75},$$
$$S=E(124)^{63}+E(124)^{67}+E(124)^{87},\ T=E(124)^{43}+E(124)^{83}+E(124)^{91},$$
$$U=E(124)^7+E(124)^{35}+E(124)^{51},\ V=E(124)^{23}+E(124)^{79}+E(124)^{115},$$
$$W=-1-6\times E(4)=-1-6\times\sqrt{-1}=-1-6i,$$
$$X=-5-6\times E(4)=-5-6\times\sqrt{-1}=-5-6i,$$
$$Y=-1-E(4)=-1-\sqrt{-1}=-1-i,\ Z=-6-E(4)=-6-\sqrt{-1}=-6-i,$$
$$AA=-6-5\times E(4)=-6-5\times\sqrt{-1}=-6-5i,\ AB=-7\times E(4)=-7\times\sqrt{-1}=-7i,$$
$$AC=-2\times E(4)=-2\times\sqrt{-1}=-2i \text{ and } AD=-E(24)+E(24)^{17}.$$

A.2 Character table of $GO(5,3)$

Table A.2: Character Table of $GO(5,3)$

$\lvert C_G(y)\rvert$	103680	103680	384	384	2304	2304	192	192	2880	2880	432
$o(y)$	1a	2a	2b	2c	2d	2e	2f	2g	2h	2i	3a
$2P$	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	3a
$3P$	1a	2a	2b	2c	2d	2e	2f	2g	2h	2i	1a
$5P$	1a	2a	2b	2c	2d	2e	2f	2g	2h	2i	3a
ψ_1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	-1	1	-1	-1	1	1	-1	1	-1	1
ψ_3	1	-1	1	-1	-1	1	-1	1	-1	1	1
ψ_4	1	1	1	1	1	1	-1	-1	-1	-1	1
ψ_5	6	6	2	2	-2	-2	0	0	-4	-4	3
ψ_6	6	6	2	2	-2	-2	0	0	4	4	3
ψ_7	6	-6	2	-2	2	-2	0	0	-4	4	3
ψ_8	6	-6	2	-2	2	-2	0	0	4	-4	3
ψ_9	10	10	2	2	-6	-6	0	0	0	0	-2
ψ_{10}	10	-10	2	-2	6	-6	0	0	0	0	-2
ψ_{11}	15	15	-1	-1	-1	-1	3	3	-5	-5	3
ψ_{12}	15	15	3	3	7	7	-1	-1	-5	-5	0
ψ_{13}	15	15	-1	-1	-1	-1	-3	-3	5	5	3
ψ_{14}	15	15	3	3	7	7	1	1	5	5	0
ψ_{15}	15	-15	3	-3	-7	7	-1	1	-5	5	0
ψ_{16}	15	-15	-1	1	1	-1	3	-3	-5	5	3
ψ_{17}	15	-15	3	-3	-7	7	1	-1	5	-5	0
ψ_{18}	15	-15	-1	1	1	-1	-3	3	5	-5	3
ψ_{19}	20	20	4	4	4	4	-2	-2	-10	-10	5
ψ_{20}	20	20	4	4	4	4	2	2	10	10	5
ψ_{21}	20	-20	4	-4	-4	4	-2	2	-10	10	5
ψ_{22}	20	-20	4	-4	-4	4	2	-2	10	-10	5
ψ_{23}	20	-20	-4	4	-4	4	0	0	0	0	2
ψ_{24}	20	20	-4	-4	4	4	0	0	0	0	2
ψ_{25}	24	-24	0	0	-8	8	-4	4	-4	4	0
ψ_{26}	24	-24	0	0	-8	8	4	-4	4	-4	0
ψ_{27}	24	24	0	0	8	8	-4	-4	-4	-4	0
ψ_{28}	24	24	0	0	8	8	4	4	4	4	0
ψ_{29}	30	30	2	2	-10	-10	2	2	-10	-10	3
ψ_{30}	30	30	2	2	-10	-10	-2	-2	10	10	3
ψ_{31}	30	-30	2	-2	10	-10	2	-2	-10	10	3
ψ_{32}	30	-30	2	-2	10	-10	-2	2	10	-10	3
ψ_{33}	60	-60	4	-4	4	-4	-2	2	-10	10	-3
ψ_{34}	60	-60	4	-4	4	-4	2	-2	10	-10	-3
ψ_{35}	60	60	4	4	-4	-4	2	2	10	10	-3
ψ_{36}	60	60	4	4	-4	-4	-2	-2	-10	-10	-3
ψ_{37}	60	-60	4	-4	-12	12	0	0	0	0	-6
ψ_{38}	60	60	4	4	12	12	0	0	0	0	-6
ψ_{39}	64	64	0	0	0	0	0	0	-16	-16	4
ψ_{40}	64	64	0	0	0	0	0	0	16	16	4
ψ_{41}	64	-64	0	0	0	0	0	0	-16	16	4
ψ_{42}	64	-64	0	0	0	0	0	0	16	-16	4
ψ_{43}	80	80	0	0	-16	-16	0	0	0	0	-4
ψ_{44}	80	-80	0	0	16	-16	0	0	0	0	-4
ψ_{45}	81	81	-3	-3	9	9	-3	-3	9	9	0
ψ_{46}	81	81	-3	-3	9	9	3	3	-9	-9	0
ψ_{47}	81	-81	-3	3	-9	9	-3	3	9	-9	0
ψ_{48}	81	-81	-3	3	-9	9	3	-3	-9	9	0
ψ_{49}	90	90	-6	-6	-6	-6	0	0	0	0	0
ψ_{50}	90	-90	-6	6	6	-6	0	0	0	0	0

Table A.2: Character Table of $GO(5,3)$ - continued

$\|C_G(y)\|$	216	1296	192	192	192	192	32	32	64	64	20	432	48
$o(y)$	3b	3c	4a	4b	4c	4d	4e	4f	4g	4h	5a	6a	6b
$2P$	3b	3c	2b	2b	2e	2e	2b	2b	2b	2b	5a	3a	3a
$3P$	1a	1a	4a	4b	4c	4d	4e	4f	4g	4h	5a	2a	2c
$5P$	3b	3c	4a	4b	4c	4d	4e	4f	4g	4h	1a	6a	6b
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1
ψ_3	1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1
ψ_4	1	1	-1	-1	1	1	1	1	-1	-1	1	1	1
ψ_5	0	-3	2	2	2	2	0	0	-2	-2	1	3	-1
ψ_6	0	-3	-2	-2	2	2	0	0	2	2	1	3	-1
ψ_7	0	-3	-2	2	2	-2	0	0	-2	2	1	-3	1
ψ_8	0	-3	2	-2	2	-2	0	0	2	-2	1	-3	1
ψ_9	4	1	0	0	2	2	-2	-2	0	0	0	-2	2
ψ_{10}	4	1	0	0	2	-2	2	-2	0	0	0	2	-2
ψ_{11}	0	6	-1	-1	3	3	-1	-1	-1	-1	0	3	-1
ψ_{12}	3	-3	-3	-3	-1	-1	1	1	1	1	0	0	0
ψ_{13}	0	6	1	1	3	3	-1	-1	1	1	0	3	-1
ψ_{14}	3	-3	3	3	-1	-1	1	1	-1	-1	0	0	0
ψ_{15}	3	-3	3	-3	-1	1	-1	1	1	-1	0	0	0
ψ_{16}	0	6	1	-1	3	-3	1	-1	-1	1	0	-3	1
ψ_{17}	3	-3	-3	3	-1	1	-1	1	-1	1	0	0	0
ψ_{18}	0	6	-1	1	3	-3	1	-1	1	-1	0	-3	1
ψ_{19}	-1	2	-2	-2	0	0	0	0	-2	-2	0	5	1
ψ_{20}	-1	2	2	2	0	0	0	0	2	2	0	5	1
ψ_{21}	-1	2	2	-2	0	0	0	0	-2	2	0	-5	-1
ψ_{22}	-1	2	-2	2	0	0	0	0	2	-2	0	-5	-1
ψ_{23}	2	-7	0	0	4	-4	0	0	0	0	0	-2	-2
ψ_{24}	2	-7	0	0	4	4	0	0	0	0	0	2	2
ψ_{25}	3	6	0	0	0	0	0	0	0	0	-1	0	0
ψ_{26}	3	6	0	0	0	0	0	0	0	0	-1	0	0
ψ_{27}	3	6	0	0	0	0	0	0	0	0	-1	0	0
ψ_{28}	3	6	0	0	0	0	0	0	0	0	-1	0	0
ψ_{29}	3	3	4	4	-2	-2	0	0	0	0	0	3	-1
ψ_{30}	3	3	-4	-4	-2	-2	0	0	0	0	0	3	-1
ψ_{31}	3	3	-4	4	-2	2	0	0	0	0	0	-3	1
ψ_{32}	3	3	4	-4	-2	2	0	0	0	0	0	-3	1
ψ_{33}	-3	6	-2	2	0	0	0	0	2	-2	0	3	-1
ψ_{34}	-3	6	2	-2	0	0	0	0	-2	2	0	3	-1
ψ_{35}	-3	6	-2	-2	0	0	0	0	-2	-2	0	-3	1
ψ_{36}	-3	6	2	2	0	0	0	0	2	2	0	-3	1
ψ_{37}	0	-3	0	0	4	-4	0	0	0	0	0	6	2
ψ_{38}	0	-3	0	0	4	4	0	0	0	0	0	-6	-2
ψ_{39}	-2	-8	0	0	0	0	0	0	0	0	-1	4	0
ψ_{40}	-2	-8	0	0	0	0	0	0	0	0	-1	4	0
ψ_{41}	-2	-8	0	0	0	0	0	0	0	0	-1	-4	0
ψ_{42}	-2	-8	0	0	0	0	0	0	0	0	-1	-4	0
ψ_{43}	2	-10	0	0	0	0	0	0	0	0	0	-4	0
ψ_{44}	2	-10	0	0	0	0	0	0	0	0	0	4	0
ψ_{45}	0	0	3	3	-3	-3	-1	-1	-1	-1	1	0	0
ψ_{46}	0	0	-3	-3	-3	-3	-1	-1	1	1	1	0	0
ψ_{47}	0	0	-3	3	-3	3	1	-1	-1	1	1	0	0
ψ_{48}	0	0	3	-3	-3	3	1	-1	1	-1	1	0	0
ψ_{49}	0	9	0	0	2	2	2	2	0	0	0	0	0
ψ_{50}	0	9	0	0	2	-2	-2	2	0	0	0	0	0

Table A.2: Character Table of $GO(5,3)$ - continued

$\lvert C_G(y)\rvert$	48	216	72	72	24	24	1296	144	144	72	72	72	72
$o(y)$	6c	6d	6e	6f	6g	6h	6i	6j	6k	6l	6m	6n	6o
$2P$	3a	3b	3b	3b	3b	3b	3c	3c	3c	3a	3a	3b	3b
$3P$	2b	2a	2e	2d	2g	2f	2a	2e	2d	2h	2i	2i	2h
$5P$	6c	6d	6e	6f	6g	6h	6i	6j	6k	6l	6m	6n	6o
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1
ψ_3	1	-1	1	-1	1	-1	-1	1	-1	-1	1	1	-1
ψ_4	1	1	1	1	-1	-1	1	1	1	-1	-1	-1	-1
ψ_5	-1	0	-2	-2	0	0	-3	1	1	-1	-1	2	2
ψ_6	-1	0	-2	-2	0	0	-3	1	1	1	1	-2	-2
ψ_7	-1	0	-2	2	0	0	3	1	-1	-1	1	-2	2
ψ_8	-1	0	-2	2	0	0	3	1	-1	1	-1	2	-2
ψ_9	2	4	0	0	0	0	1	-3	-3	0	0	0	0
ψ_{10}	2	-4	0	0	0	0	-1	-3	3	0	0	0	0
ψ_{11}	-1	0	2	2	0	0	6	2	2	1	1	-2	-2
ψ_{12}	0	3	1	1	-1	-1	-3	1	1	-2	-2	1	1
ψ_{13}	-1	0	2	2	0	0	6	2	2	-1	-1	2	2
ψ_{14}	0	3	1	1	1	1	-3	1	1	2	2	-1	-1
ψ_{15}	0	-3	1	-1	1	-1	3	1	-1	-2	2	-1	1
ψ_{16}	-1	0	2	-2	0	0	-6	2	-2	1	-1	2	-2
ψ_{17}	0	-3	1	-1	-1	1	3	1	-1	2	-2	1	-1
ψ_{18}	-1	0	2	-2	0	0	-6	2	-2	-1	1	-2	2
ψ_{19}	1	-1	1	1	1	1	2	-2	-2	-1	-1	-1	-1
ψ_{20}	1	-1	1	1	-1	-1	2	-2	-2	1	1	1	1
ψ_{21}	1	1	1	-1	-1	1	-2	-2	2	-1	1	1	-1
ψ_{22}	1	1	1	-1	1	-1	-2	-2	2	1	-1	-1	1
ψ_{23}	2	-2	-2	2	0	0	7	1	-1	0	0	0	0
ψ_{24}	2	2	-2	-2	0	0	-7	1	1	0	0	0	0
ψ_{25}	0	-3	-1	1	1	-1	-6	2	-2	2	-2	1	-1
ψ_{26}	0	-3	-1	1	-1	1	-6	2	-2	-2	2	-1	1
ψ_{27}	0	3	-1	-1	-1	-1	6	2	2	2	2	-1	-1
ψ_{28}	0	3	-1	-1	1	1	6	2	2	-2	-2	1	1
ψ_{29}	-1	3	-1	-1	-1	-1	3	-1	-1	-1	-1	-1	-1
ψ_{30}	-1	3	-1	-1	1	1	3	-1	-1	1	1	1	1
ψ_{31}	-1	-3	-1	1	1	-1	-3	-1	1	-1	1	1	-1
ψ_{32}	-1	-3	-1	1	-1	1	-3	-1	1	1	-1	-1	1
ψ_{33}	1	3	-1	1	-1	1	-6	2	-2	-1	1	1	-1
ψ_{34}	1	3	-1	1	1	-1	-6	2	-2	1	-1	-1	1
ψ_{35}	1	-3	-1	-1	-1	-1	6	2	2	1	1	1	1
ψ_{36}	1	-3	-1	-1	1	1	6	2	2	-1	-1	-1	-1
ψ_{37}	-2	0	0	0	0	0	3	-3	3	0	0	0	0
ψ_{38}	-2	0	0	0	0	0	-3	-3	-3	0	0	0	0
ψ_{39}	0	-2	0	0	0	0	-8	0	0	2	2	2	2
ψ_{40}	0	-2	0	0	0	0	-8	0	0	-2	-2	-2	-2
ψ_{41}	0	2	0	0	0	0	8	0	0	2	-2	-2	2
ψ_{42}	0	2	0	0	0	0	8	0	0	-2	2	2	-2
ψ_{43}	0	2	2	2	0	0	-10	2	2	0	0	0	0
ψ_{44}	0	-2	2	-2	0	0	10	2	-2	0	0	0	0
ψ_{45}	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{47}	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{48}	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ_{49}	0	0	0	0	0	0	9	-3	-3	0	0	0	0
ψ_{50}	0	0	0	0	0	0	-9	-3	3	0	0	0	0

Table A.2: Character Table of $GO(5,3)$ - continued

$\lvert C_G(y) \rvert$	72	72	16	16	18	20	20	20	24	24	24	24	18
$o(y)$	6p	6q	8a	8b	9a	10a	10b	10c	12a	12b	12c	12d	18a
$2P$	3a	3a	4c	4c	9a	5a	5a	5a	6c	6c	6j	6j	9a
$3P$	2d	2e	8a	8b	3c	10a	10b	10c	4a	4b	4d	4c	6i
$5P$	6p	6q	8a	8b	9a	2a	2i	2h	12a	12b	12c	12d	18a
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	-1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1
ψ_3	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1
ψ_4	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1
ψ_5	1	1	0	0	0	1	1	1	-1	-1	-1	-1	0
ψ_6	1	1	0	0	0	1	-1	-1	1	1	-1	-1	0
ψ_7	-1	1	0	0	0	-1	-1	1	1	-1	1	-1	0
ψ_8	-1	1	0	0	0	-1	1	-1	-1	1	1	-1	0
ψ_9	0	0	0	0	1	0	0	0	0	0	-1	-1	1
ψ_{10}	0	0	0	0	1	0	0	0	0	0	1	-1	-1
ψ_{11}	-1	-1	1	1	0	0	0	0	-1	-1	0	0	0
ψ_{12}	-2	-2	1	1	0	0	0	0	0	0	-1	-1	0
ψ_{13}	-1	-1	-1	-1	0	0	0	0	1	1	0	0	0
ψ_{14}	-2	-2	-1	-1	0	0	0	0	0	0	-1	-1	0
ψ_{15}	2	-2	-1	1	0	0	0	0	0	0	1	-1	0
ψ_{16}	1	-1	-1	1	0	0	0	0	1	-1	0	0	0
ψ_{17}	2	-2	1	-1	0	0	0	0	0	0	1	-1	0
ψ_{18}	1	-1	1	-1	0	0	0	0	-1	1	0	0	0
ψ_{19}	1	1	0	0	-1	0	0	0	1	1	0	0	-1
ψ_{20}	1	1	0	0	-1	0	0	0	-1	-1	0	0	-1
ψ_{21}	-1	1	0	0	-1	0	0	0	-1	1	0	0	1
ψ_{22}	-1	1	0	0	-1	0	0	0	1	-1	0	0	1
ψ_{23}	2	-2	0	0	-1	0	0	0	0	0	-1	1	1
ψ_{24}	-2	-2	0	0	-1	0	0	0	0	0	1	1	-1
ψ_{25}	-2	2	0	0	0	1	-1	1	0	0	0	0	0
ψ_{26}	-2	2	0	0	0	1	1	-1	0	0	0	0	0
ψ_{27}	2	2	0	0	0	-1	1	1	0	0	0	0	0
ψ_{28}	2	2	0	0	0	-1	-1	-1	0	0	0	0	0
ψ_{29}	-1	-1	0	0	0	0	0	0	1	1	1	1	0
ψ_{30}	-1	-1	0	0	0	0	0	0	-1	-1	1	1	0
ψ_{31}	1	-1	0	0	0	0	0	0	-1	1	-1	1	0
ψ_{32}	1	-1	0	0	0	0	0	0	1	-1	-1	1	0
ψ_{33}	1	-1	0	0	0	0	0	0	1	-1	0	0	0
ψ_{34}	1	-1	0	0	0	0	0	0	-1	1	0	0	0
ψ_{35}	-1	-1	0	0	0	0	0	0	1	1	0	0	0
ψ_{36}	-1	-1	0	0	0	0	0	0	-1	-1	0	0	0
ψ_{37}	0	0	0	0	0	0	0	0	0	0	-1	1	0
ψ_{38}	0	0	0	0	0	0	0	0	0	0	1	1	0
ψ_{39}	0	0	0	0	1	-1	-1	-1	0	0	0	0	1
ψ_{40}	0	0	0	0	1	-1	1	1	0	0	0	0	1
ψ_{41}	0	0	0	0	1	1	1	-1	0	0	0	0	-1
ψ_{42}	0	0	0	0	1	1	-1	1	0	0	0	0	-1
ψ_{43}	2	2	0	0	-1	0	0	0	0	0	0	0	-1
ψ_{44}	-2	2	0	0	-1	0	0	0	0	0	0	0	1
ψ_{45}	0	0	1	1	0	1	-1	-1	0	0	0	0	0
ψ_{46}	0	0	-1	-1	0	1	1	1	0	0	0	0	0
ψ_{47}	0	0	-1	1	0	-1	1	-1	0	0	0	0	0
ψ_{48}	0	0	1	-1	0	-1	-1	1	0	0	0	0	0
ψ_{49}	0	0	0	0	0	0	0	0	0	0	-1	-1	0
ψ_{50}	0	0	0	0	0	0	0	0	0	0	1	-1	0

A.3 Character table of $GU(4,4)$

Table A.3: Character table of $GU(4,4)$

$\|C_{GU(4,4)}(y)\|$	77760	1728	288	77760	77760	1944	1944	1944	1944	1944
$o(y)$	1a	2a	2b	3a	3b	3c	3d	3e	3f	3g
$2P$	1a	1a	1a	3b	3a	3f	3h	3g	3c	3e
$3P$	1a	2a	2b	1a	1a	1a	1a	1a	1a	1a
$5P$	1a	2a	2b	3b	3a	3f	3h	3g	3c	3e
ψ_1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	1	A	/A	/A	1	A	A	/A
ψ_3	1	1	1	/A	A	A	1	/A	/A	A
ψ_4	5	-3	1	5	5	T	T	T	/T	/T
ψ_5	5	-3	1	5	5	/T	/T	/T	T	T
ψ_6	5	-3	1	B	/B	U	T	/V	/U	V
ψ_7	5	-3	1	/B	B	/U	/T	V	U	/V
ψ_8	5	-3	1	B	/B	V	/T	/U	/V	U
ψ_9	5	-3	1	/B	B	/V	T	U	V	/U
ψ_{10}	6	-2	2	6	6	-3	-3	-3	-3	-3
ψ_{11}	6	-2	2	C	/C	N	-3	/N	/N	N
ψ_{12}	6	-2	2	/C	C	/N	-3	N	N	/N
ψ_{13}	10	2	-2	10	10	W	W	W	/W	/W
ψ_{14}	10	2	-2	10	10	/W	/W	/W	W	W
ψ_{15}	10	2	-2	D	/D	X	W	/Y	/X	Y
ψ_{16}	10	2	-2	/D	D	/X	/W	Y	X	/Y
ψ_{17}	10	2	-2	D	/D	Y	/W	/X	/Y	X
ψ_{18}	10	2	-2	/D	D	/Y	W	X	Y	/X
ψ_{19}	15	-1	-1	15	15	6	6	6	6	6
ψ_{20}	15	7	3	15	15	-3	-3	-3	-3	-3
ψ_{21}	15	-1	-1	E	/E	/C	6	C	C	/C
ψ_{22}	15	-1	-1	/E	E	C	6	/C	/C	C
ψ_{23}	15	7	3	E	/E	N	-3	/N	/N	N
ψ_{24}	15	7	3	/E	E	/N	-3	N	N	/N
ψ_{25}	20	4	4	20	20	2	2	2	2	2
ψ_{26}	20	4	4	F	/F	-O	2	-/O	-/O	-O
ψ_{27}	20	4	4	/F	F	-/O	2	-O	-O	-/O
ψ_{28}	24	8	0	24	24	6	6	6	6	6
ψ_{29}	24	8	0	G	/G	/C	6	C	C	/C
ψ_{30}	24	8	0	/G	G	C	6	/C	/C	C
ψ_{31}	30	-10	2	30	30	3	3	3	3	3
ψ_{32}	30	6	2	30	30	Z	Z	Z	/Z	/Z
ψ_{33}	30	6	2	30	30	/Z	/Z	/Z	Z	Z
ψ_{34}	30	6	2	H	/H	AA	Z	/AB	/AA	AB
ψ_{35}	30	6	2	/H	H	/AA	/Z	AB	AA	/AB
ψ_{36}	30	6	2	H	/H	AB	/Z	/AA	/AB	AA
ψ_{37}	30	6	2	/H	H	/AB	Z	AA	AB	/AA
ψ_{38}	30	-10	2	H	/H	-N	3	-/N	-/N	-N
ψ_{39}	30	-10	2	/H	H	-/N	3	-N	-N	-/N
ψ_{40}	40	-8	0	40	40	AC	AC	AC	/AC	/AC
ψ_{41}	40	-8	0	40	40	/AC	/AC	/AC	AC	AC
ψ_{42}	40	-8	0	I	/I	AD	/AC	/AE	/AD	AE
ψ_{43}	40	-8	0	/I	I	/AD	AC	AE	AD	/AE
ψ_{44}	40	-8	0	I	/I	AE	AC	/AD	/AE	AD
ψ_{45}	40	-8	0	/I	I	/AE	/AC	AD	AE	/AD
ψ_{46}	45	-3	-3	45	45	-/S	-/S	-/S	-S	-S
ψ_{47}	45	-3	-3	45	45	-S	-S	-S	-/S	-/S
ψ_{48}	45	-3	-3	J	/J	-/S	-S	-9	-S	-9
ψ_{49}	45	-3	-3	/J	J	-S	-/S	-9	-/S	-9
ψ_{50}	45	-3	-3	J	/J	-9	-/S	-S	-9	-/S
ψ_{51}	45	-3	-3	/J	J	-9	-S	-/S	-9	-S
ψ_{52}	60	-4	4	60	60	6	6	6	6	6
ψ_{53}	60	-4	4	K	/K	/C	6	C	C	/C
ψ_{54}	60	-4	4	/K	K	C	6	/C	/C	C
ψ_{55}	64	0	0	64	64	-8	-8	-8	-8	-8
ψ_{56}	64	0	0	L	/L	-R	-8	-/R	-/R	-R
ψ_{57}	64	0	0	/L	L	-/R	-8	-R	-R	-/R
ψ_{58}	81	9	-3	81	81	0	0	0	0	0
ψ_{59}	81	9	-3	M	/M	0	0	0	0	0
ψ_{60}	81	9	-3	/M	M	0	0	0	0	0

Table A.3: Character table of $GU(4,4)$

$\lvert C_{GU(4,4)}(y) \rvert$	1944	324	324	324	162	162	162	144	24	15
$o(y)$	3h	3i	3j	3k	3l	3m	3n	4a	4b	5a
$2P$	3d	3k	3j	3i	3n	3m	3l	2a	2b	5a
$3P$	1a	1a	1a	1a	1a	1a	1a	4a	4b	5a
$5P$	3d	3k	3j	3i	3n	3m	3l	4a	4b	1a
ψ_1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	/A	1	A	/A	1	A	1	1	1
ψ_3	1	A	1	/A	A	1	/A	1	1	1
ψ_4	/T	-1	-1	-1	2	2	2	1	-1	0
ψ_5	T	-1	-1	-1	2	2	2	1	-1	0
ψ_6	/T	-/A	-1	-A	-O	2	-/O	1	-1	0
ψ_7	T	-A	-1	-/A	-/O	2	-O	1	-1	0
ψ_8	T	-/A	-1	-A	-O	2	-/O	1	-1	0
ψ_9	/T	-A	-1	-/A	-/O	2	-O	1	-1	0
ψ_{10}	-3	3	3	3	0	0	0	2	0	1
ψ_{11}	-3	-N	3	-/N	0	0	0	2	0	1
ψ_{12}	-3	-/N	3	-N	0	0	0	2	0	1
ψ_{13}	/W	1	1	1	1	1	1	2	0	0
ψ_{14}	W	1	1	1	1	1	1	2	0	0
ψ_{15}	/W	/A	1	A	/A	1	A	2	0	0
ψ_{16}	W	A	1	/A	A	1	/A	2	0	0
ψ_{17}	W	/A	1	A	/A	1	A	2	0	0
ψ_{18}	/W	A	1	/A	A	1	/A	2	0	0
ψ_{19}	6	3	3	3	0	0	0	3	-1	0
ψ_{20}	-3	0	0	0	3	3	3	-1	1	0
ψ_{21}	6	-N	3	-/N	0	0	0	3	-1	0
ψ_{22}	6	-/N	3	-N	0	0	0	3	-1	0
ψ_{23}	-3	0	0	0	-N	3	-/N	-1	1	0
ψ_{24}	-3	0	0	0	-/N	3	-N	-1	1	0
ψ_{25}	2	5	5	5	-1	-1	-1	0	0	0
ψ_{26}	2	/B	5	B	-/A	-1	-A	0	0	0
ψ_{27}	2	B	5	/B	-A	-1	-/A	0	0	0
ψ_{28}	6	0	0	0	3	3	3	0	0	-1
ψ_{29}	6	0	0	0	-N	3	-/N	0	0	-1
ψ_{30}	6	0	0	0	-/N	3	-N	0	0	-1
ψ_{31}	3	3	3	3	3	3	3	-2	0	0
ψ_{32}	/Z	-3	-3	-3	0	0	0	2	0	0
ψ_{33}	Z	-3	-3	-3	0	0	0	2	0	0
ψ_{34}	/Z	N	-3	/N	0	0	0	2	0	0
ψ_{35}	Z	/N	-3	N	0	0	0	2	0	0
ψ_{36}	Z	N	-3	/N	0	0	0	2	0	0
ψ_{37}	/Z	/N	-3	N	0	0	0	2	0	0
ψ_{38}	3	-N	3	-/N	-N	3	-/N	-2	0	0
ψ_{39}	3	-/N	3	-N	-/N	3	-N	-2	0	0
ψ_{40}	/AC	-2	-2	-2	1	1	1	0	0	0
ψ_{41}	AC	-2	-2	-2	1	1	1	0	0	0
ψ_{42}	AC	O	-2	/O	/A	1	A	0	0	0
ψ_{43}	/AC	/O	-2	O	A	1	/A	0	0	0
ψ_{44}	/AC	O	-2	/O	/A	1	A	0	0	0
ψ_{45}	AC	/O	-2	O	A	1	/A	0	0	0
ψ_{46}	-S	0	0	0	0	0	0	1	1	0
ψ_{47}	-/S	0	0	0	0	0	0	1	1	0
ψ_{48}	-/S	0	0	0	0	0	0	1	1	0
ψ_{49}	-S	0	0	0	0	0	0	1	1	0
ψ_{50}	-S	0	0	0	0	0	0	1	1	0
ψ_{51}	-/S	0	0	0	0	0	0	1	1	0
ψ_{52}	6	-3	-3	-3	-3	-3	-3	0	0	0
ψ_{53}	6	N	-3	/N	N	-3	/N	0	0	0
ψ_{54}	6	/N	-3	N	/N	-3	N	0	0	0
ψ_{55}	-8	4	4	4	-2	-2	-2	0	0	-1
ψ_{56}	-8	Q	4	/Q	O	-2	/O	0	0	-1
ψ_{57}	-8	/Q	4	Q	/O	-2	O	0	0	-1
ψ_{58}	0	0	0	0	0	0	0	-3	-1	1
ψ_{59}	0	0	0	0	0	0	0	-3	-1	1
ψ_{60}	0	0	0	0	0	0	0	-3	-1	1

Table A.3: Character table of $GU(4,4)$

$\|C_{GU(4,4)}(y)\|$	1728	1728	216	216	216	216	216	216	108	108
$o(y)$	6a	6b	6c	6d	6e	6f	6g	6h	6i	6j
$2P$	3a	3b	3e	3d	3c	3f	3h	3g	3j	3i
$3P$	2a	2a	2a	2a	2a	2a	2a	2a	2a	2a
$5P$	6b	6a	6h	6g	6f	6e	6d	6c	6n	6m
ψ_{1}	1	1	1	1	1	1	1	1	1	1
ψ_{2}	/A	A	/A	1	A	/A	1	A	1	A
ψ_{3}	A	/A	A	1	/A	A	1	/A	1	/A
ψ_{4}	-3	-3	AF	AF	AF	/AF	/AF	/AF	AG	AG
ψ_{5}	-3	-3	/AF	/AF	/AF	AF	AF	AF	-AG	-AG
ψ_{6}	N	/N	-/AF	AF	-AG	AG	/AF	-AF	AG	/AF
ψ_{7}	/N	N	-AF	/AF	AG	-AG	AF	-/AF	-AG	AF
ψ_{8}	N	/N	AG	/AF	-AF	-/AF	AF	-AG	-AG	-/AF
ψ_{9}	/N	N	-AG	AF	-/AF	-AF	/AF	AG	AG	-AF
ψ_{10}	-2	-2	1	1	1	1	1	1	1	1
ψ_{11}	O	/O	/A	1	A	/A	1	A	1	A
ψ_{12}	/O	O	A	1	/A	A	1	/A	1	/A
ψ_{13}	2	2	T	T	T	/T	/T	/T	-1	-1
ψ_{14}	2	2	/T	/T	/T	T	T	T	-1	-1
ψ_{15}	-O	-/O	U	T	/V	V	/T	/U	-1	-A
ψ_{16}	-/O	-O	/U	/T	V	/V	T	U	-1	-/A
ψ_{17}	-O	-/O	V	/T	/U	U	T	/V	-1	-A
ψ_{18}	-/O	-O	/V	T	U	/U	/T	V	-1	-/A
ψ_{19}	-1	-1	2	2	2	2	2	2	-1	-1
ψ_{20}	7	7	1	1	1	1	1	1	-2	-2
ψ_{21}	-/A	-A	-O	2	-/O	-O	2	-/O	-1	-A
ψ_{22}	-A	-/A	-/O	2	-O	-/O	2	-O	-1	-/A
ψ_{23}	P	/P	/A	1	A	/A	1	A	-2	/O
ψ_{24}	/P	P	A	1	/A	A	1	/A	-2	O
ψ_{25}	4	4	-2	-2	-2	-2	-2	-2	1	1
ψ_{26}	Q	/Q	O	-2	/O	O	-2	/O	1	A
ψ_{27}	/Q	Q	/O	-2	O	/O	-2	O	1	/A
ψ_{28}	8	8	2	2	2	2	2	2	2	2
ψ_{29}	R	/R	-O	2	-/O	-O	2	-/O	2	-/O
ψ_{30}	/R	R	-/O	2	-O	-/O	2	-O	2	-O
ψ_{31}	-10	-10	-1	-1	-1	-1	-1	-1	-1	-1
ψ_{32}	6	6	AF	AF	AF	/AF	/AF	/AF	AG	AG
ψ_{33}	6	6	/AF	/AF	/AF	AF	AF	AF	-AG	-AG
ψ_{34}	/C	C	-/AF	AF	-AG	AG	/AF	-AF	AG	/AF
ψ_{35}	C	/C	-AF	/AF	AG	-AG	AF	-/AF	-AG	AF
ψ_{36}	/C	C	AG	/AF	-AF	-/AF	AF	-AG	-AG	-/AF
ψ_{37}	C	/C	-AG	AF	-/AF	-AF	/AF	AG	AG	-AF
ψ_{38}	-/D	-D	-/A	-1	-A	-/A	-1	-A	-1	-A
ψ_{39}	-D	-/D	-A	-1	-/A	-A	-1	-/A	-1	-/A
ψ_{40}	-8	-8	/O	/O	/O	O	O	O	O	O
ψ_{41}	-8	-8	O	O	O	/O	/O	/O	/O	/O
ψ_{42}	-R	-/R	/O	O	-2	-2	/O	O	/O	O
ψ_{43}	-/R	-R	O	/O	-2	-2	O	/O	O	/O
ψ_{44}	-R	-/R	-2	/O	O	/O	O	-2	O	-2
ψ_{45}	-/R	-R	-2	O	/O	O	/O	-2	/O	-2
ψ_{46}	-3	-3	-N	-N	-N	-/N	-/N	-/N	0	0
ψ_{47}	-3	-3	-/N	-/N	-/N	-N	-N	-N	0	0
ψ_{48}	N	/N	3	-/N	-N	-/N	-N	3	0	0
ψ_{49}	/N	N	3	-N	-/N	-N	-/N	3	0	0
ψ_{50}	N	/N	-/N	-N	3	3	-/N	-N	0	0
ψ_{51}	/N	N	-N	-/N	3	3	-N	-/N	0	0
ψ_{52}	-4	-4	2	2	2	2	2	2	-1	-1
ψ_{53}	-Q	-/Q	-O	2	-/O	-O	2	-/O	-1	-A
ψ_{54}	-/Q	-Q	-/O	2	-O	-/O	2	-O	-1	-/A
ψ_{55}	0	0	0	0	0	0	0	0	0	0
ψ_{56}	0	0	0	0	0	0	0	0	0	0
ψ_{57}	0	0	0	0	0	0	0	0	0	0
ψ_{58}	9	9	0	0	0	0	0	0	0	0
ψ_{59}	S	/S	0	0	0	0	0	0	0	0
ψ_{60}	/S	S	0	0	0	0	0	0	0	0

Table A.3: Character table of $GU(4,4)$

$\lvert C_{GU(4,4)}(y)\rvert$	108	108	108	108	288	288	36	36	36	54
$o(y)$	6k	6l	6m	6n	6o	6p	6q	6r	6s	6t
$2P$	3k	3i	3k	3j	3b	3a	3j	3i	3k	3l
$3P$	2a	2a	2a	2a	2b	2b	2b	2b	2b	2a
$5P$	6l	6k	6j	6i	6p	6o	6q	6s	6r	6u
ψ_{1}	1	1	1	1	1	1	1	1	1	1
ψ_{2}	/A	A	/A	1	A	/A	1	A	/A	A
ψ_{3}	A	/A	A	1	/A	A	1	/A	A	/A
ψ_{4}	AG	-AG	-AG	-AG	1	1	1	1	1	0
ψ_{5}	-AG	AG	AG	AG	1	1	1	1	1	0
ψ_{6}	-AF	-/AF	AF	-AG	A	/A	1	A	/A	0
ψ_{7}	-/AF	-AF	/AF	AG	/A	A	1	/A	A	0
ψ_{8}	AF	/AF	-AF	AG	A	/A	1	A	/A	0
ψ_{9}	/AF	AF	-/AF	-AG	/A	A	1	/A	A	0
ψ_{10}	1	1	1	1	2	2	-1	-1	-1	-2
ψ_{11}	/A	A	/A	1	-/O	-O	-1	-A	-/A	/O
ψ_{12}	A	/A	A	1	-O	-/O	-1	-/A	-A	O
ψ_{13}	-1	-1	-1	-1	-2	-2	1	1	1	-1
ψ_{14}	-1	-1	-1	-1	-2	-2	1	1	1	-1
ψ_{15}	-/A	-A	-/A	-1	/O	O	1	A	/A	-A
ψ_{16}	-A	-/A	-A	-1	O	/O	1	/A	A	-/A
ψ_{17}	-/A	-A	-/A	-1	/O	O	1	A	/A	-A
ψ_{18}	-A	-/A	-A	-1	O	/O	1	/A	A	-/A
ψ_{19}	-1	-1	-1	-1	-1	-1	-1	-1	-1	2
ψ_{20}	-2	-2	-2	-2	3	3	0	0	0	1
ψ_{21}	-/A	-A	-/A	-1	-A	-/A	-1	-A	-/A	-/O
ψ_{22}	-A	-/A	-A	-1	-/A	-A	-1	-/A	-A	-O
ψ_{23}	O	/O	O	-2	-/N	-N	0	0	0	A
ψ_{24}	/O	O	/O	-2	-N	-/N	0	0	0	/A
ψ_{25}	1	1	1	1	4	4	1	1	1	1
ψ_{26}	/A	A	/A	1	/Q	Q	1	A	/A	A
ψ_{27}	A	/A	A	1	Q	/Q	1	/A	A	/A
ψ_{28}	2	2	2	2	0	0	0	0	0	-1
ψ_{29}	-O	-/O	-O	2	0	0	0	0	0	-A
ψ_{30}	-/O	-O	-/O	2	0	0	0	0	0	-/A
ψ_{31}	-1	-1	-1	-1	2	2	-1	-1	-1	-1
ψ_{32}	AG	-AG	-AG	-AG	2	2	-1	-1	-1	0
ψ_{33}	-AG	AG	AG	AG	2	2	-1	-1	-1	0
ψ_{34}	-AF	-/AF	AF	-AG	-/O	-O	-1	-A	-/A	0
ψ_{35}	-/AF	-AF	/AF	AG	-O	-/O	-1	-/A	-A	0
ψ_{36}	AF	/AF	-AF	AG	-/O	-O	-1	-A	-/A	0
ψ_{37}	/AF	AF	-/AF	-AG	-O	-/O	-1	-/A	-A	0
ψ_{38}	-/A	-A	-/A	-1	-/O	-O	-1	-A	-/A	-A
ψ_{39}	-A	-/A	-A	-1	-O	-/O	-1	-/A	-A	-/A
ψ_{40}	O	/O	/O	/O	0	0	0	0	0	1
ψ_{41}	/O	O	O	O	0	0	0	0	0	1
ψ_{42}	-2	-2	/O	O	0	0	0	0	0	A
ψ_{43}	-2	-2	O	/O	0	0	0	0	0	/A
ψ_{44}	/O	O	-2	/O	0	0	0	0	0	A
ψ_{45}	O	/O	-2	O	0	0	0	0	0	/A
ψ_{46}	0	0	0	0	-3	-3	0	0	0	0
ψ_{47}	0	0	0	0	-3	-3	0	0	0	0
ψ_{48}	0	0	0	0	/N	N	0	0	0	0
ψ_{49}	0	0	0	0	N	/N	0	0	0	0
ψ_{50}	0	0	0	0	/N	N	0	0	0	0
ψ_{51}	0	0	0	0	N	/N	0	0	0	0
ψ_{52}	-1	-1	-1	-1	4	4	1	1	1	-1
ψ_{53}	-/A	-A	-/A	-1	/Q	Q	1	A	/A	-A
ψ_{54}	-A	-/A	-A	-1	Q	/Q	1	/A	A	-/A
ψ_{55}	0	0	0	0	0	0	0	0	0	0
ψ_{56}	0	0	0	0	0	0	0	0	0	0
ψ_{57}	0	0	0	0	0	0	0	0	0	0
ψ_{58}	0	0	0	0	-3	-3	0	0	0	0
ψ_{59}	0	0	0	0	/N	N	0	0	0	0
ψ_{60}	0	0	0	0	N	/N	0	0	0	0

Table A.3: Character table of $GU(4,4)$

$\lvert C_{GU(4,4)}(y)\rvert$	54	54	27	27	27	27	27	27	144	144
$o(y)$	6u	6v	9a	9b	9c	9d	9e	9f	12a	12b
$2P$	3n	3m	9e	9d	9f	9b	9a	9c	6a	6b
$3P$	2a	2a	3d	3d	3d	3h	3h	3h	4a	4a
$5P$	6t	6v	9e	9d	9f	9b	9a	9c	12b	12a
ψ_{1}	1	1	1	1	1	1	1	1	1	1
ψ_{2}	/A	1	/A	1	A	1	A	/A	A	/A
ψ_{3}	A	1	A	1	/A	1	/A	A	/A	A
ψ_{4}	0	0	-A	-A	-A	-/A	-/A	-/A	1	1
ψ_{5}	0	0	-/A	-/A	-/A	-A	-A	-A	1	1
ψ_{6}	0	0	-1	-A	-/A	-/A	-1	-A	A	/A
ψ_{7}	0	0	-1	-/A	-A	-A	-1	-/A	/A	A
ψ_{8}	0	0	-A	-/A	-1	-A	-/A	-1	A	/A
ψ_{9}	0	0	-/A	-A	-1	-/A	-A	-1	/A	A
ψ_{10}	-2	-2	0	0	0	0	0	0	2	2
ψ_{11}	O	-2	0	0	0	0	0	0	-/O	-O
ψ_{12}	/O	-2	0	0	0	0	0	0	-O	-/O
ψ_{13}	-1	-1	A	A	A	/A	/A	/A	2	2
ψ_{14}	-1	-1	/A	/A	/A	A	A	A	2	2
ψ_{15}	-/A	-1	1	A	/A	/A	1	A	-/O	-O
ψ_{16}	-A	-1	1	/A	A	A	1	/A	-O	-/O
ψ_{17}	-/A	-1	A	/A	1	A	/A	1	-/O	-O
ψ_{18}	-A	-1	/A	A	1	/A	A	1	-O	-/O
ψ_{19}	2	2	0	0	0	0	0	0	3	3
ψ_{20}	1	1	0	0	0	0	0	0	-1	-1
ψ_{21}	-O	2	0	0	0	0	0	0	-/N	-N
ψ_{22}	-/O	2	0	0	0	0	0	0	-N	-/N
ψ_{23}	/A	1	0	0	0	0	0	0	-A	-/A
ψ_{24}	A	1	0	0	0	0	0	0	-/A	-A
ψ_{25}	1	1	-1	-1	-1	-1	-1	-1	0	0
ψ_{26}	/A	1	-/A	-1	-A	-1	-A	-/A	0	0
ψ_{27}	A	1	-A	-1	-/A	-1	-/A	-A	0	0
ψ_{28}	-1	-1	0	0	0	0	0	0	0	0
ψ_{29}	-/A	-1	0	0	0	0	0	0	0	0
ψ_{30}	-A	-1	0	0	0	0	0	0	0	0
ψ_{31}	-1	-1	0	0	0	0	0	0	-2	-2
ψ_{32}	0	0	0	0	0	0	0	0	2	2
ψ_{33}	0	0	0	0	0	0	0	0	2	2
ψ_{34}	0	0	0	0	0	0	0	0	-/O	-O
ψ_{35}	0	0	0	0	0	0	0	0	-O	-/O
ψ_{36}	0	0	0	0	0	0	0	0	-/O	-O
ψ_{37}	0	0	0	0	0	0	0	0	-O	-/O
ψ_{38}	-/A	-1	0	0	0	0	0	0	/O	O
ψ_{39}	-A	-1	0	0	0	0	0	0	O	/O
ψ_{40}	1	1	A	A	A	/A	/A	/A	0	0
ψ_{41}	1	1	/A	/A	/A	A	A	A	0	0
ψ_{42}	/A	1	A	/A	1	A	/A	1	0	0
ψ_{43}	A	1	/A	A	1	/A	A	1	0	0
ψ_{44}	/A	1	1	A	/A	/A	1	A	0	0
ψ_{45}	A	1	1	/A	A	A	1	/A	0	0
ψ_{46}	0	0	0	0	0	0	0	0	1	1
ψ_{47}	0	0	0	0	0	0	0	0	1	1
ψ_{48}	0	0	0	0	0	0	0	0	A	/A
ψ_{49}	0	0	0	0	0	0	0	0	/A	A
ψ_{50}	0	0	0	0	0	0	0	0	A	/A
ψ_{51}	0	0	0	0	0	0	0	0	/A	A
ψ_{52}	-1	-1	0	0	0	0	0	0	0	0
ψ_{53}	-/A	-1	0	0	0	0	0	0	0	0
ψ_{54}	-A	-1	0	0	0	0	0	0	0	0
ψ_{55}	0	0	1	1	1	1	1	1	0	0
ψ_{56}	0	0	/A	1	A	1	A	/A	0	0
ψ_{57}	0	0	A	1	/A	1	/A	A	0	0
ψ_{58}	0	0	0	0	0	0	0	0	-3	-3
ψ_{59}	0	0	0	0	0	0	0	0	/N	N
ψ_{60}	0	0	0	0	0	0	0	0	N	/N

Table A.3: Character table of $GU(4,4)$

$\lvert C_{GU(4,4)}(y)\rvert$	36	36	36	36	36	36	24	24	15	15
$o(y)$	12c	12d	12e	12f	12g	12h	12i	12j	15a	15b
$2P$	6c	6e	6d	6f	6h	6g	6p	6o	15b	15a
$3P$	4a	4a	4a	4a	4a	4a	4b	4b	5a	5a
$5P$	12g	12f	12h	12d	12c	12e	12j	12i	3b	3a
ψ_1	1	1	1	1	1	1	1	1	1	1
ψ_2	A	/A	1	A	/A	1	A	/A	A	/A
ψ_3	/A	A	1	/A	A	1	/A	A	/A	A
ψ_4	A	A	A	/A	/A	/A	-1	-1	0	0
ψ_5	/A	/A	/A	A	A	A	-1	-1	0	0
ψ_6	/A	1	A	1	A	/A	-A	-/A	0	0
ψ_7	A	1	/A	1	/A	A	-/A	-A	0	0
ψ_8	1	A	/A	/A	1	A	-A	-/A	0	0
ψ_9	1	/A	A	A	1	/A	-/A	-A	0	0
ψ_{10}	-1	-1	-1	-1	-1	-1	0	0	1	1
ψ_{11}	-A	-/A	-1	-A	-/A	-1	0	0	A	/A
ψ_{12}	-/A	-A	-1	-/A	-A	-1	0	0	/A	A
ψ_{13}	-/A	-/A	-/A	-A	-A	-A	0	0	0	0
ψ_{14}	-A	-A	-A	-/A	-/A	-/A	0	0	0	0
ψ_{15}	-1	-A	-/A	-/A	-1	-A	0	0	0	0
ψ_{16}	-1	-/A	-A	-A	-1	-/A	0	0	0	0
ψ_{17}	-/A	-1	-A	-1	-A	-/A	0	0	0	0
ψ_{18}	-A	-1	-/A	-1	-/A	-A	0	0	0	0
ψ_{19}	0	0	0	0	0	0	-1	-1	0	0
ψ_{20}	-1	-1	-1	-1	-1	-1	1	1	0	0
ψ_{21}	0	0	0	0	0	0	-A	-/A	0	0
ψ_{22}	0	0	0	0	0	0	-/A	-A	0	0
ψ_{23}	-A	-/A	-1	-A	-/A	-1	A	/A	0	0
ψ_{24}	-/A	-A	-1	-/A	-A	-1	/A	A	0	0
ψ_{25}	0	0	0	0	0	0	0	0	0	0
ψ_{26}	0	0	0	0	0	0	0	0	0	0
ψ_{27}	0	0	0	0	0	0	0	0	0	0
ψ_{28}	0	0	0	0	0	0	0	0	-1	-1
ψ_{29}	0	0	0	0	0	0	0	0	-A	-/A
ψ_{30}	0	0	0	0	0	0	0	0	-/A	-A
ψ_{31}	1	1	1	1	1	1	0	0	0	0
ψ_{32}	-A	-A	-A	-/A	-/A	-/A	0	0	0	0
ψ_{33}	-/A	-/A	-/A	-A	-A	-A	0	0	0	0
ψ_{34}	-/A	-1	-A	-1	-A	-/A	0	0	0	0
ψ_{35}	-A	-1	-/A	-1	-/A	-A	0	0	0	0
ψ_{36}	-1	-A	-/A	-/A	-1	-A	0	0	0	0
ψ_{37}	-1	-/A	-A	-A	-1	-/A	0	0	0	0
ψ_{38}	A	/A	1	A	/A	1	0	0	0	0
ψ_{39}	/A	A	1	/A	A	1	0	0	0	0
ψ_{40}	0	0	0	0	0	0	0	0	0	0
ψ_{41}	0	0	0	0	0	0	0	0	0	0
ψ_{42}	0	0	0	0	0	0	0	0	0	0
ψ_{43}	0	0	0	0	0	0	0	0	0	0
ψ_{44}	0	0	0	0	0	0	0	0	0	0
ψ_{45}	0	0	0	0	0	0	0	0	0	0
ψ_{46}	A	A	A	/A	/A	/A	1	1	0	0
ψ_{47}	/A	/A	/A	A	A	A	1	1	0	0
ψ_{48}	1	A	/A	/A	1	A	A	/A	0	0
ψ_{49}	1	/A	A	A	1	/A	/A	A	0	0
ψ_{50}	/A	1	A	1	A	/A	A	/A	0	0
ψ_{51}	A	1	/A	1	/A	A	/A	A	0	0
ψ_{52}	0	0	0	0	0	0	0	0	0	0
ψ_{53}	0	0	0	0	0	0	0	0	0	0
ψ_{54}	0	0	0	0	0	0	0	0	0	0
ψ_{55}	0	0	0	0	0	0	0	0	-1	-1
ψ_{56}	0	0	0	0	0	0	0	0	-A	-/A
ψ_{57}	0	0	0	0	0	0	0	0	-/A	-A
ψ_{58}	0	0	0	0	0	0	-1	-1	1	1
ψ_{59}	0	0	0	0	0	0	-A	-/A	A	/A
ψ_{60}	0	0	0	0	0	0	-/A	-A	/A	A

A= $\frac{-1-\sqrt{-3}}{2}$, B $\frac{-5-5\sqrt{-3}}{2}$, C= $-3-3\sqrt{-3}$, D= $-5-5\sqrt{-3}$, E= $\frac{-15-15\sqrt{-3}}{2}$, F= $-10-10\sqrt{-3}$, G= $-12-12\sqrt{-3}$, H= $-15-15\sqrt{-3}$, I= $-20-20\sqrt{-3}$, J= $\frac{-45-45\sqrt{-3}}{2}$, K= $-30-30\sqrt{-3}$, L= $-32-32\sqrt{-3}$, M= $\frac{-81-81\sqrt{-3}}{2}$, N= $\frac{3-3\sqrt{-3}}{2}$, O= $1-\sqrt{-3}$, P= $\frac{-7+7\sqrt{-3}}{2}$, Q= $-2+2\sqrt{-3}$, R= $-4+4\sqrt{-3}$, S= $\frac{-9+9\sqrt{-3}}{2}$, T= $\frac{1-3\sqrt{-3}}{2}$, U= $2+\sqrt{-3}$, V= $\frac{-5-\sqrt{-3}}{2}$, W= $\frac{-7+3\sqrt{-3}}{2}$, X= $\frac{-1-5\sqrt{-3}}{2}$, Y= $4-\sqrt{-3}$, Z= $\frac{-3+9\sqrt{-3}}{2}$, AA= $-6-3\sqrt{-3}$, AB= $\frac{15+3\sqrt{-3}}{2}$, AC= $-5-3\sqrt{-3}$, AD= $-2-4\sqrt{-3}$, AE= $7-\sqrt{-3}$, AF= $\frac{-3-\sqrt{-3}}{2}$, AG= $-\sqrt{-3}$.

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