Nonparametric estimation of the off-pulse interval(s) of a pulsar light curve

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It all starts here ™

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"Lift up your eyes and see.
Who has made these stars?
It is the One Who leads them out by number.
He calls them all by name.
Because of the greatness of His strength,
and because He is strong in power,
not one of them is missing."
(Isaiah 40:26, NLV)

Summary

The main objective of this thesis is the development of a nonparametric sequential estimation technique for the off-pulse interval(s) of a source function originating from a pulsar. It is important to identify the off-pulse interval of each pulsar accurately, since the properties of the off-pulse emissions are further researched by astrophysicists in an attempt to detect potential emissions from the associated pulsar wind nebula (PWN). The identification technique currently used in the literature is subjective in nature, since it is based on the visual inspection of the histogram estimate of the pulsar light curve. The developed nonparametric estimation technique is not only objective in nature, but also accurate in the estimation of the off-pulse interval of a pulsar, as evident from the simulation study and the application of the developed technique to observed pulsar data.

The first two chapters of this thesis are devoted to a literature study that provides background information on the pulsar environment and γ-ray astronomy, together with an explanation of the on-pulse and off-pulse interval of a pulsar and the importance thereof for the present study. This is followed by a discussion on some fundamental circular statistical ideas, as well as an overview of kernel density estimation techniques. These two statistical topics are then united in order to illustrate kernel density estimation techniques applied to circular data, since this concept is the starting point of the developed nonparametric sequential estimation technique.

Once the basic theoretical background of the pulsar environment and circular kernel density estimation has been established, the new sequential off-pulse interval estimator is formulated. The estimation technique will be referred to as 'SOPIE'. A number of tuning parameters form part of SOPIE, and therefore the performed simulation study not only serves as an evaluation of the performance of SOPIE, but also as a mechanism to establish which tuning parameter configurations consistently perform better than some other configurations.

In conclusion, the optimal parameter configurations are utilised in the application of SOPIE to pulsar data. For several pulsars, the sequential off-pulse interval estimators are compared to the off-pulse intervals published in research papers, which were identified with the *subjective* "eye-ball" technique. *It is found that the sequential off-pulse interval estimators are closely related to the off-pulse intervals identified with subjective visual inspection, with the benefit that the estimated intervals are objectively obtained with a nonparametric estimation technique.*

KEY WORDS: Off-pulse interval — Pulsar light curve — Nonparametric sequential estimation — Circular statistics — Circular kernel density estimation — Pulsar Wind Nebulae — FERMI — Gamma Rays.

Opsomming

Die primêre doelwit van hierdie proefskrif is die ontwikkeling van 'n sekwensiële nie-parametriese beramingsmetode vir die af-pulsinterval(le) van 'n bronfunksie afkomstig vanaf 'n pulsar. Dit is belangrik om die af-pulsinterval van elke pulsar akkuraat te identifiseer, aangesien die eienskappe van die af-pulsuitstralings verder deur astrofisici nagevors word in 'n poging om die potensiële uitstraling van die geassosieerde pulsarwindnewel (PWN) waar te neem. Die identifiseringstegniek wat tans in die literatuur gebruik word, is subjektief van aard, aangesien dit gegrond is op visuele ondersoeke van die histogramberamer van die pulsar se ligkromme. Die nie-parametriese beramingsmetode wat ontwikkel word in hierdie proefskrif, is nie net objektief nie, maar ook akkuraat in die beraming van die af-pulsinterval van 'n pulsar, soos blyk uit die simulasiestudie en die toepassing van die ontwikkelde metode op waargenome pulsardata.

Die eerste twee hoofstukke in die proefskrif is 'n literatuuroorsig wat die agtergrondinligting verskaf ten opsigte van die pulsaromgewing en γ-straal astronomie. Hiermee saam word die begrippe van af- en aan-pulsinterval van 'n pulsar verduidelik, asook die belangrikheid daarvan vir die huidige studie. Dit word opgevolg deur 'n bespreking van grondliggende sirkulêre statistiese konsepte, asook 'n oorsig van kerndigtheidsfunksieberaming. Hierdie twee onderwerpe word dan verenig om kerndigtheidsfunksieberaming, toegepas op sirkulêre data, te illustreer. Laasgenoemde konsep is dan ook die vertrekpunt van die ontwikkelde sekwensiële nie-parametriese beramingsmetode.

Sodra die basiese teoretiese agtergrond van die pulsaromgewing, asook dié van sirkulêre kerndigtheidsfunksieberaming gevestig is, word die nuwe sekwensiële beramingsmetode vir die af-pulsinterval(le) geformuleer. Die beramingsmetode sal voortaan 'SOPIE' genoem word. 'n Aantal instellingsparameters vorm deel van SOPIE. Die doel van die uitgevoerde simulasiestudie is eerstens om die werkverrigting van SOPIE te beoordeel, en tweedens om te bepaal watter konfigurasie van die instellingsparameters konsekwent beter werkverrigting as ander konfigurasies lewer.

Ten slotte word die optimale konfigurasie van die instellingsparameters gebruik in die toepassing van SOPIE op waargenome pulsardata. Verskeie waargenome pulsare word gebruik om die sekwensiële beramingsmetode vir die af-pulsinterval(le) te vergelyk met die af-pulsintervalle wat gepubliseer is in navorsingsartikels en geïdentifiseer is met *subjektiewe* visuele ondersoeke van die histogram. *Daar word bevind dat die sekwensiële berамings van die af-pulsintervalle nou verwant is aan die af-pulsintervalle soos geïdentifiseer deur subjektiewe visuele ondersoek, met die voordeel dat die intervalle objektief beraam word met SOPIE.*

SLEUTELWOORDE: Af-pulsinterval — Pulsarligkromme — Sekwensiële nie-parametriese beraming — Sirkulêre statistiek — Sirkulêre kerndigtheidsfunksieberaming — Pulsarwindnewel — FERMI — Gammastraal.

Contents

Chapter 1

Concepts in Astrophysics: Overview, motivation and importance to the present study

1.1 Introduction

Optical astronomy is a field of astronomy (and part of physics) where the focus is primarily on the investigation of celestial events by detecting emissions from cosmic sources in the optical waveband. Up until 1945, astronomers could only study the universe at large in the optical waveband. Since then there has been tremendous expansion of the wavebands available for astronomical study. The advancements in rockets capable of lifting scientific payloads above the atmosphere opened the research-horizon in the field of astrophysics. The reason being that X-ray astronomy can only be carried out at very high altitudes because of the photoelectric absorption of X-rays by the atoms and molecules of the Earth's atmosphere (Longair, 2004). One should take note that X-rays and γ-rays have a much higher frequency, shorter wavelength and higher photons energy compared to normal light (optical region) observed by the human eye.

The new disciplines of radio, infrared, X- and γ-ray astronomies combined with optical astronomy resulted in a renewed interest in astrophysical research. Photons (detectable in γ-ray, X-ray, microwave or Infra-red, depending on their energy) are emitted from various astrophysical sources such as stars, supernova explosions, nuclear reactions and even the decay of radioactive material in space. Other rotating stellar bodies, such as *pulsars* also emit various photons that can be detected by either low-Earth-orbit or ground-based telescopes. Since this thesis is concerned with pulsars and specifically the estimated pulsar light curve, this chapter will provide an introduction to pulsars, pulsar wind nebulae and the pulsar magnetosphere, which can be seen as the environment wherein these pulsars function. A number of other related topics required to understand and motivate this study will also be discussed.

The chapter will firstly review some elementary *pulsar* theory. The discussion will then focus on *pulsar wind nebulae* and the pulsar magnetosphere. The attention will then shift toward the definition of the *pulse* and *off-pulse region* of the pulsar light curve. In conclusion, the provided information will be used to motivate this study and to formulate the objectives.

1.2 Pulsars and Neutron Stars

Pulsating sources of radio emission, or *pulsars*, are highly magnetised, rapidly-rotating neutron stars, with masses more or less that of our sun's mass (2×10^{30} kg, abbreviated with $M_{\odot}$) and radii of approximately 10km. They emit beams of electromagnetic radiation analogous to a lighthouse. The magnetic field consists of *open* and *closed* field lines, with the outermost closed field lines touching the *light cylinder* and defining the *polar cap* region in the surface of the pulsar. Electromagnetic radiation from pulsars is observed in the form of two conical beams directed along the magnetic axis, believed to be due to the pulsar's spin and magnetic axis being misaligned. The radiation can only be observed when the beam of emission is pointing towards the distant observer every time a beam sweeps past them, whether it is a person or a telescope (Chaisson & McMillan, 2008). This is called the lighthouse effect and gives rise to the pulsed nature that ultimately is responsible for the name of these stars, viz. *pulsars*. A simplified pulsar model is illustrated in Figure 1.1.

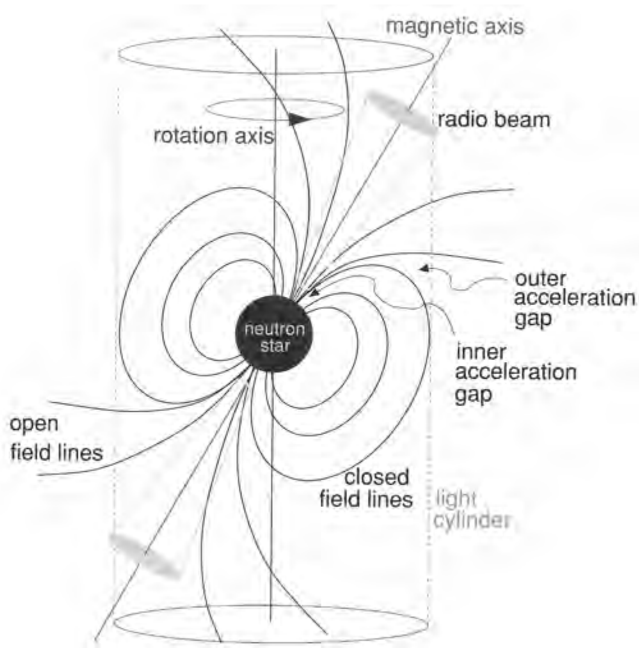

Figure 1.1: *A simplified model for the rotating neutron star (not drawn to scale) illustrating some fundamental concepts (Lorimer & Kramer, 2005, p. 55).*

1.2.1 Historical overview

In the 1930s, two astronomers tentatively proposed the existence of a then, new kind of star, the *neutron star* (Baade & Zwicky, 1934). This neutron star was hypothesised to have a very small radius with extreme density, and was assumed to consist primarily of neutrons, as its name suggests. It was also believed that a supernova represents the transition of an ordinary star into this new form of a neutron star. These stars represent one stellar-evolutionary endpoint and are created by the gravitational collapse of a star with sufficient mass. At that time, the hypothesis seemed of only theoretical importance, since neutron stars would be small and would emit very little light, with the result that their radiation would be nearly impossible to detect (Lyne & Graham-Smith, 1990).

It must be noted, though, that neutron stars represent but one stellar-evolutionary endpoint, as stated. However, there are a few other end-points worth mentioning. When massive stars have exhausted the nuclear fuel in their cores, it will cool and contract or collapse under its own gravity and the overburden of the matter lying on top. A star with the density of normal matter then ends up in one of three possible states, i.e., a white dwarf, a neutron star or a black hole. The extent of the collapse depends on the initial mass of the star before collapse. On the one end of the spectrum, the least massive stars become white dwarfs. The progenitors of white dwarfs normally have a mass of order 1 to 1.4 solar mass ($M_\odot$). On the other hand, the most massive stars become black holes. The remainder become neutron stars, where the progenitors' masses range from 6 to 15 $M_\odot$ (Lyne & Graham-Smith, 2005).

Since a neutron star represents one stellar-evolutionary endpoint, it must be the product of some other stellar phenomenon. A well-known neutron star is believed to be the result of the supernova explosion in 1054 AD, which was observed by astronomers to the Chinese court, Yang Wei-T'e. On 4 July of that year, the astronomers observed what we observe today as the Crab Nebula (NGC 1952) (see Figure 1.2). Duyvendak (1942) and Mayall & Oort (1942) concluded that the Crab Nebula had to be the remnant of the supernova explosion observed by the Chinese, by putting forward several arguments involving the estimated distance, position and other information pertaining to its magnitude. At that time, the neutron star was an interesting concept and various speculations were made to estimate and calculate the properties of such a star. Pacini (1967), for example, predicted that the Crab Nebula was powered by the rapid rotation of a highly magnetised neutron star, emitting electromagnetic radiation. It was not long after this speculation that unusual, rapidly pulsating radio signals were detected at the Mullard Radio Astronomy Observatory (Hewish, Bell, Pilkington, Scott & Collins, 1968). Initial thoughts assigned the signals to man-made space probes or the reflection of terrestrial signals from the moon, but the signals were soon placed far outside the solar system, proving their initial ideas incorrect. Several other such pulsating sources were also discovered, verifying that this was a natural phenomenon.

Figure 1.2: *An optical image of the Crab Nebula. This object represents a significant 'standard candle' in γ-ray Astronomy, against which new telescopes are calibrated.*

An important feature of the signals that had to be accounted for was the absolute regularity of the pulses, suggesting the pulsation of the entire source. Interestingly though, is that previous speculations indicated that radial pulsation of neutron stars and white dwarves could play an important role in the historical evolution of supernovae and supernova remnants (SNRs). This led to the conclusion that the pulsating radio signals could be associated with degenerate stars (Hewish et al., 1968). Gold (1968) then immediately researched this phenomenon and set forward a very clear case for associating pulsars with rotating neutron stars.

One fundamental question that still required clarification was whether the sources were white dwarfs or neutron stars. Theories involving white dwarfs could only account for rotational periods longer than 0.25 s, but the discovery of the Vela pulsar (Large, Vaughan & Mills, 1968) and Crab pulsar (Staelin & Reifenstein, 1968) with periods of 89 ms and 33 ms, respectively, addressed this question. Only a neutron star could rotate as fast as 30 times a second without breaking apart. It was also of great significance that both the Crab and Vela pulsars were located within SNRs, providing clear confirmation of the prediction made by Baade & Zwicky (1934) as highlighted at the beginning of this subsection.

1.2.2 Formation of the pulsar wind

Shortly after the conclusion that the Crab and Vela pulsars were located within SNRs, a classic paper on pulsar theory saw the light. In this paper, Goldreich & Julian (1969) calculated the dynamic properties of an aligned rotator. One of the fundamental conclusions was that the pulsar could not be surrounded by a vacuum, but should instead be surrounded by a dense magnetosphere filled with a plasma of positive and negative particles. Another conclusion was that the particles would be accelerated due to potential differences and would escape from the pulsar magnetosphere along the open field lines (see Figure 1.1). This leads to the notion of a *Pulsar Wind nebulae* that will be discussed in the next section.

1.3 Pulsar Wind Nebulae (PWN)

Pulsar wind nebulae are highly luminous nebulae observable over a large wavelength range (from radio to γ-rays). When the particle wind from a rotating neutron star (as described above) interacts with its surrounding interstellar medium or surrounding SNR, such a complex PWN arises. A PWN can therefore be seen as a bubble of shocked relativistic particles, produced when a pulsar's relativistic wind interacts with its environment (Gaensler & Slane, 2006). In a similar way, Lyne & Graham-Smith (2005) define a *plerion* by mentioning that most SNRs are shells, but some SNRs (like the Crab nebula) are filled with material emitting at all wavelengths, and are then referred to as a plerion. Figure 1.3 is an illustration of this idea. It is therefore a defining characteristic of a PWN that it is centrally filled, meaning that a central source must be continuously supplying energy to the PWN. Weiler & Panagia (1980) probably provided the first real summary of the properties of a filled centre PWN (or plerion).

With the inception of the High Energy Stereoscopic System (abbreviated with HESS hereafter) in 2004, the quality and quantity of γ-ray data have improved tremendously, ultimately leading to advances in the understanding of PWNe. The reader is referred to de Jager & Venter (2005), de Jager & Djannati-Ataï (2009) and Aharonian et al. (2006) who added a number of additional features to the definition of PWNe, but this is not regarded as within the scope of this thesis.

1.4 Introduction to γ-ray astronomy

This study will focus primarily on high-energy radiation as the source of data to develop a statistical technique to identify (and estimate) the pulse- and off-pulse region of a pulsar light curve. Therefore, a broad overview of γ-ray astronomy should be provided. In the case of γ-rays, these high-energy photons are not deflected by magnetic fields present in the space between the source and detector, resulting in the preservation of directional information. High-energy γ-ray sources may therefore be uniquely identified. In order to explain these concepts in slightly more detail, a brief overview

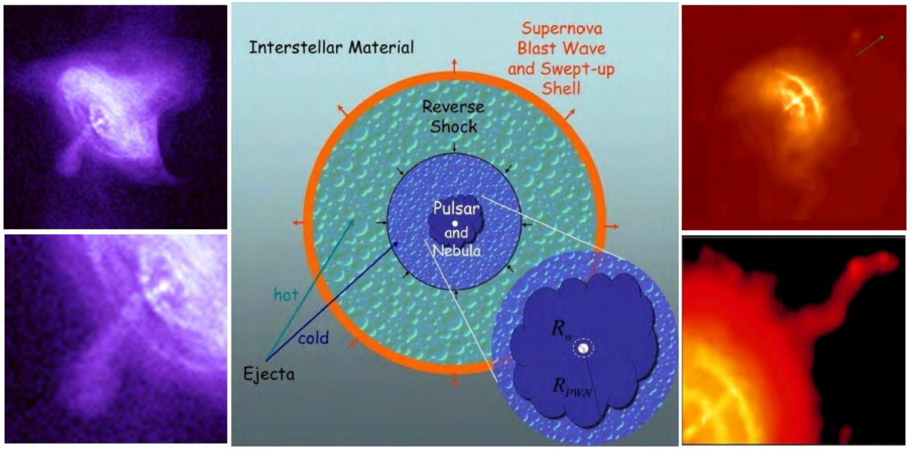

Figure 1.3: *Schematic diagram of a PWN within an SNR. On the left are images of the Crab Nebula, with the jet underneath. On the right are images of Vela, with the jet-like structure visible in the lower left. Source: Slane (2008).*

of γ-ray astronomy will now follow in the next subsection, with some definitions and references to observed γ-ray pulsars.

1.4.1 Definition

Gamma-rays consist of photons with energy higher than 1 MeV. In particular, γ-rays are produced by interactions of particles accelerated to the highest energies by electromagnetic and other shock processes (Thompson, 2004). γ-rays constitute approximately 0.1% to 1% of the total radiation classified as cosmic rays. Cosmic rays mainly consist of charged particles such as protons (about 90%), helium nuclei (less than 10%), ionised heavier elements (less than 1%) and electrons (less than 1%), with the remainder classified as γ-rays (Angelis, Mansutti & Persic, 2008). As γ-radiation represents the most energetic part of the electromagnetic spectrum, it fundamentally provides information about the most energetic processes in the universe, and therefore the study of γ-ray astronomy is so important. The reader is also referred to an excellent review paper by Hinton & Hofmann (2009), where an overview is provided of the principal astrophysical results of high-energy astronomy in the last few years.

1.4.2 Observed γ-ray pulsars

Caraveo (2006) noted that the largest proportion of energy loss from pulsars is converted into high-energy γ-rays, even though such a small number of these pulsars are detected relative to the number of radio pulsars. Based on the data of the Energetic Gamma Ray Experiment Telescope (abbreviated with EGRET hereafter), only seven gamma-ray pulsars have been identified up to 2004 (Thompson, 2004; Roberts, 2004): Crab, Vela, PSR B1706-44, PSR B1951+32, PSR B1055-52, PSR B1509-58 (up to 10 MeV) and the radio-quiet Geminga, as illustrated in Figure 1.4. Kanbach (2002) also summarised a few general characteristics of these gamma-ray pulsars. The most important characteristic relevant for this thesis, is that the light curves are usually double-peaked with a large pulsed fraction or duty cycle (the pulsed emission cover more than 50% of the emission).

Research into pulsars and their surrounding PWN continued to expand since the amount of γ-ray observations increased, and scientists analysed these pulsars with even greater intensity. In order

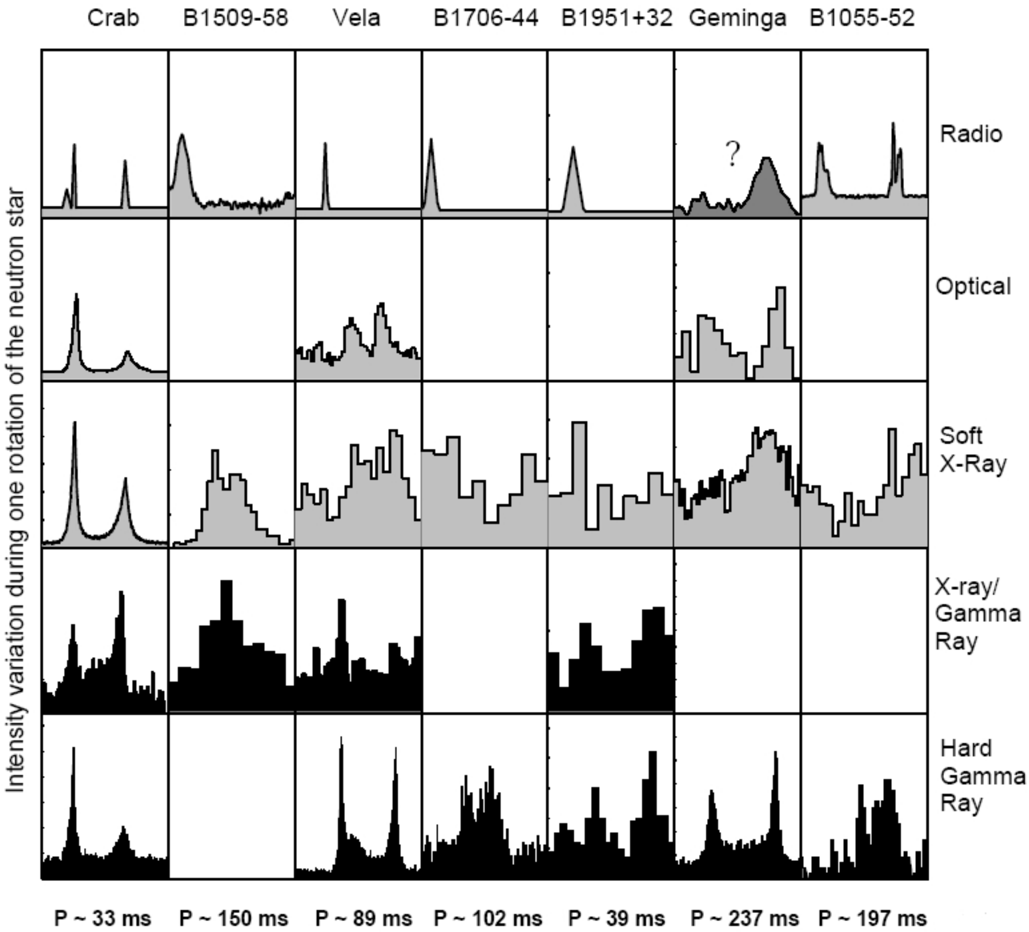

Figure 1.4: *Light curves of seven gamma-ray pulsars in five energy bands, from left to right in order of characteristic age. Adopted from Thompson (2004).*

to understand the pulsar, and the associated PWN, it is first required to define the pulse- and off-pulse window in the next section. The definition of the pulse- and off-pulse window is regarded as the most important section of this chapter, as the defined concepts will be utilised throughout this thesis.

1.5 Defining the pulse- and off-pulse window

As defined in Section 1.2 of this chapter, pulsars produce pulses through the *lighthouse effect* of an emission beam of a neutron star as it sweeps past our (or the observation instrument's) line of sight once per rotation. Pulsars produce weak radio signals, and therefore only the strongest sources are observed. In order to gather and construct a pulse profile or pulse window that is discernible above the background noise of the receiver, most pulsars require the coherent addition of many hundreds of pulses together through a process known as *folding* (Lorimer & Kramer, 2005). This folding-procedure requires the detection of the arrival times of individual photons by large optical telescopes or satellite-borne γ-ray telescopes. The precise timing over long observation periods allows the detection of the periodicity of the pulsar, and by *folding* thousands of data periods over one another, the creation of a pulse profile becomes possible (Lyne & Graham-Smith, 2005). It is important to note that the key quantity of interest is the *time of arrival* (TOA) of pulses at the telescope. The TOA can be defined as the arrival time of the nearest pulse to the mid-point of the observation. As the pulse has a certain width, the TOA refers to some reference point on the profile. It would be ideal if this reference point coincides with the plane defined by the magnetic axis and rotation of the pulsar and the line of sight to the observer (Lorimer & Kramer, 2005).

Because neutron stars are very dense objects, the rotation period and therefore the interval between observed pulses is very regular, with periods ranging from 1 millisecond up to a few hundred seconds. Some of these pulsars radiate γ-rays with the same period as the spin period of the pulsar. It must be noted that not only the emissions of the stellar bodies or events are detected. Typically, aperiodic cosmic rays – normally referred to as background radiation or noise – are also detected. **Therefore, a typical data set consists not only of pulsed radiation from an identifiable source, but also of noise.** Such a data set would therefore contain arrival times t_i, each arrival time representing either noise or pulsed radiation. Usually, the data set is pre-analysed so that the arrival times t_i are folded modulo 1 with pulsar period q:

$$X_i = \frac{t_i}{q} - \left[\frac{t_i}{q}\right], \quad i = 1, 2, \ldots$$

The pulsar's signal period q can be accurately determined from the TOAs. The unknown periodic density function (or light curve) $f(\theta)$ of the folded (modulo 1) arrival times can be presented as

$$f(\theta) = 1 - p + p f_s(\theta), \tag{1.1}$$

where $0 \leq p \leq 1$ is the unknown strength of the periodic (or pulsed) signal and $f_s(\theta)$ is the unknown source function that characterises the radiation pattern of the source. To explain this folding process in some more detail, the reader can inspect Figure 1.5, as well as the discussion in Emadzadeh & Speyer (2010). In order to clarify the notation used in the illustration, the following definitions are provided.

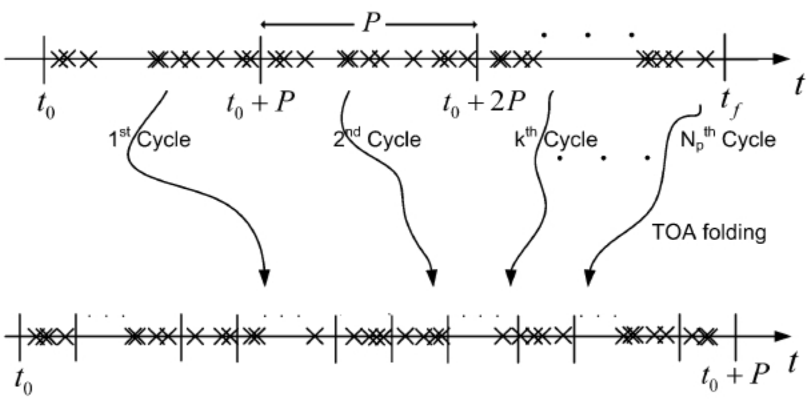

Figure 1.5: *The epoch folding procedure (adopted from Emadzadeh & Speyer (2010)).*

Let (t_0, t_f) be the observation interval, with t_i the time of arrival (TOA) of the ith pulsar photon or γ-ray. All the time tags during the observation interval $T_{obs} = t_f - t_0$ are collected, and it is assumed that there are N_p pulsar cycles in the observation interval. From the observational interval, it is trivial to see that $T_{obs} \approx N_p P$, where P is the observed pulsar period. Furthermore, from (1.1), it is evident that the estimation of p may be of importance. The problem relating to the estimation of the strength of the pulsed signal, p, in a series of high-energy photon arrival times, has been previously researched by Loots (1995) and Swanepoel (1999). In Section 2.5, a brief discussion will be provided on the estimation of the pulsed signal p.

What became recently even more of a burning question is the following: *Suppose one suspects that the pulsar is surrounded by some other stellar structure or body (such as a PWN), what is required to investigate such a source?*

Since the launch of the *Fermi Gamma-Ray Space Telescope* (formerly GLAST), the number of detected pulsars in the γ-ray domain has dramatically increased. Most of the pulsars detected by the *Fermi*-large area telescope (LAT) are bright point sources in the γ-ray sky. In order to study the surroundings of the pulsars, such as their associated PWNe, requires the assignment of phases to the γ-ray photons, and consequently requires the observer to focus only on those γ-ray emissions where the pulsed emission beam is not in the line of sight as described in the first paragraph of this section. It is therefore mandatory to establish the off-pulse window of the pulsar before any research can proceed into the surrounding stellar structure of a pulsar. **This off-pulse window (synonyms: off-peak or off-pulse interval) can therefore be defined as the period of time where the pulsar is "off", implying the time period where the pulsar is not recognised as the source of (or "responsible" for) any γ-rays that are received by the detection instruments.** Differently stated, it is possible to separate the pulse- and off-pulse window by taking the whole phase emission and subtracting the off-pulse window to remain with the pulsed emission only (Abdo et al., 2010b).
The on-peak interval can also be defined as the complement of the off-peak interval, and vice versa.

Figure 1.6 is a visual illustration of the Crab pulsar (pulse window) and nebula (off-pulse window). The pulsed emission dominates in the on-pulse window, while the nebula stands out in the off-pulse interval from the emission of the diffuse background only at high energies.

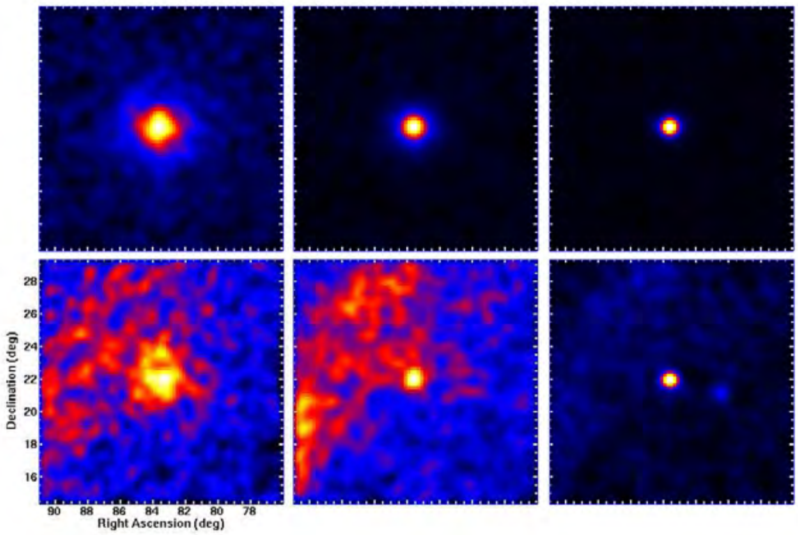

Figure 1.6: *Counts map (arbitrary units) of the pulse window (displaying the Crab pulsar – top row) and the off-pulse window (displaying the nebula – bottom row) for three different energy bands (Left: 100 MeV < E < 300 MeV; middle: 300 MeV < E < 1 GeV; right: E > 1 GeV). Illustration adopted from Abdo et al. (2010b).*

It must be noted that, although the basic characteristic of the pulsar signal that facilitates its recognition is the exact periodicity of the pulsar, a second important characteristic is the frequency dispersion in arrival times due to the ionised interstellar medium. The reader should consult Lyne & Graham-Smith (2005, p. 25) for a detailed discussion on the frequency dispersion in the pulsed arrival times, as it is considered outside the scope of this thesis.

1.6 Motivation

As stated, the launch of the *Fermi Gamma-ray Space Telescope* in 2008 resulted in a dramatic increase in the number of known γ-ray pulsars. The opportunity to study a large number of these high-energy objects suddenly arose, and several recent research papers were published on pulsars and their associated PWNe (Abdo et al., 2009; Abdo et al., 2010a; Abdo et al., 2010b; Abdo et al., 2010c; Abdo et al., 2010f; Abdo et al., 2010e; Ackermann et al., 2011). From the last referenced paper of Ackermann et al. (2011), the key objective of the study was to examine the properties of the off-pulsed emission of each pulsar and to attempt to detect the potential emission associated with its PWN. All of the mentioned research papers utilise the estimated light curves to perform subsequent analyses on these curves in order to understand the pulsar magnetosphere even better, including the PWNe associated with most pulsars. It is evident from these papers that the times of arrival (TOA) of the photons (defined in Section 1.5) are used in the process of generating the light curve. The TOAs are then fitted to a timing model – in most cases, the *fermi*-plugin distributed with the *Tempo2* pulsar timing package – which is available on the *Fermi* Science Support Center (FSSC) website.* The output from this timing model is sufficient statistics to construct the estimated light curve of the pulsar. What is evident from all the above articles, including the articles by Ray et al. (2011) and Parkinson et al. (2010), is the fact that a histogram representation is made of the light curve, based on a choice for the number of bins in this histogram. The number of bins in most of these papers varies from 25 to 50 bins per phase, depending on certain factors. In some articles, the bins are chosen so that no bin contains fewer than a certain threshold number of TOAs, e.g., at least 50 counts per bin are used in Abdo et al. (2010f).

What is also evident is that some analyses performed on these histogram-type light curves are done with the "eye-ball" technique, or visual inspection of the data in order to identify the off-peak phase interval (Parkinson et al., 2010). The identified off-peak window is then used to estimate the unpulsed emission, which in turn is used to calculate other physical quantities of interest. It is therefore crucial that the identification of the off-peak interval be done accurately, since other research is based on this interval. Figure 1.7 is a typical illustration found in research papers where the off-pulse interval is identified by visual inspection of the histogram estimate of the pulsar light curve. In this specific illustration, the off-pulse intervals for two pulsars are identified from their light curves, constructed from the arrival times of photons with energies above 100 MeV in a region of 1° around each pulsar. For both light curves in Figure 1.7, 2 phases are illustrated.

In contrast to this *subjective* approach to identify the off-pulse interval visually, the intention is to develop an *objective* technique to estimate the off-pulse interval nonparametrically. Figure 1.8 is used to illustrate the concept of estimating the off-pulse interval.

The figure shows three scenarios, with each graph representing a simplified hypothetical pulsar light curve. The pulsar light curves are specifically illustrated as "smooth" light curves in contrast to the histogram estimate of the pulsar light curve in Figure 1.7. The reason will become evident in Chapter 3, where the proposed estimation technique is discussed in detail. Each graph contains

*http://fermi.gsfc.nasa.gov/ssc/data/access/lat/ephems/

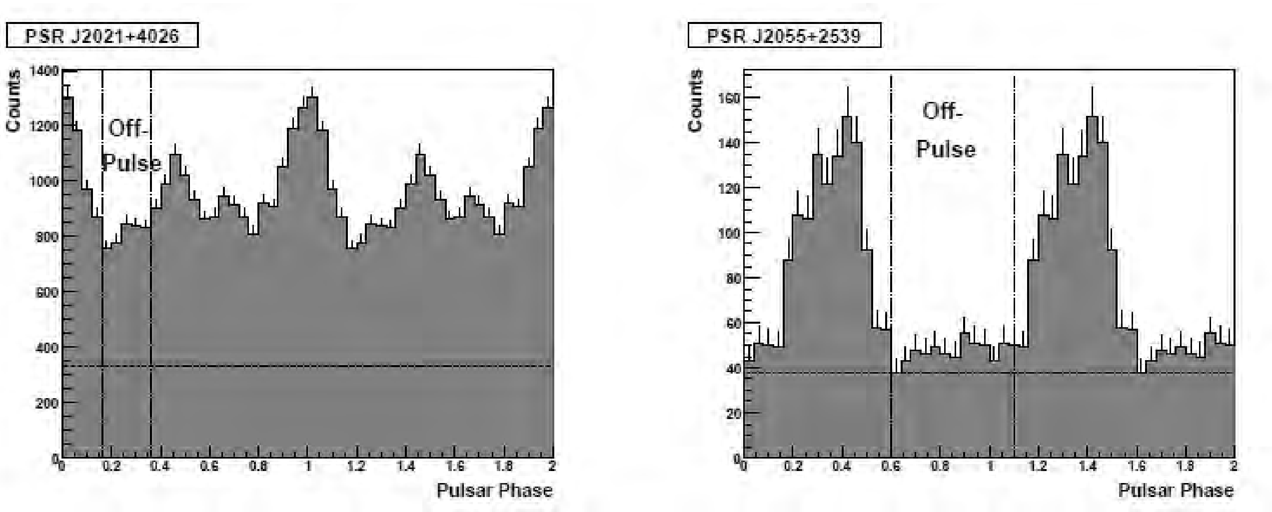

Figure 1.7: *Histogram estimate of the pulsar light curve used to identify the off-pulse interval with visual inspection. Illustration adopted from Ackermann et al. (2011).*

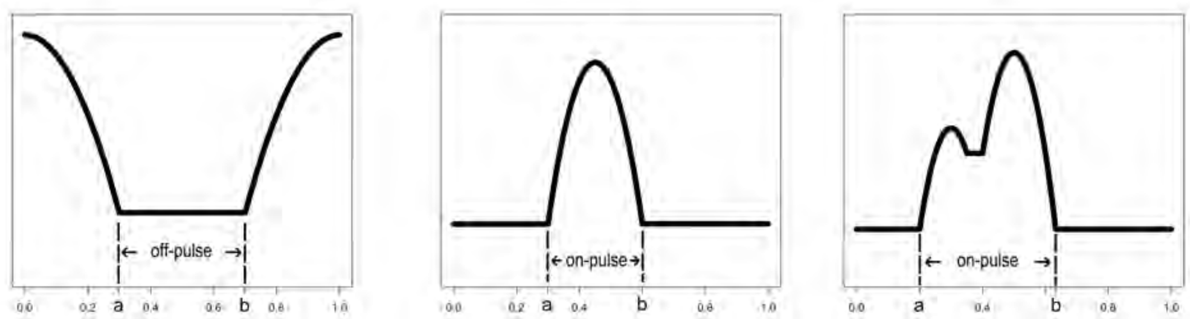

Figure 1.8: *Illustration of estimating the off-pulse interval of a pulsar light curve for three different scenarios.*

an interval where pulsed emissions are primarily observed (on-pulse interval), as well as a disjoint interval where emissions are present that are not related to the pulsar (e.g. noise and possible emissions from the PWN, called the off-pulse interval). It is assumed that there are only these two intervals present in each scenario. Therefore, there are 2 unknown points a and b that need to be estimated from the data.

The aim is therefore to develop an objective procedure that can be applied to the data, resulting in an estimate for the unknown values a and b without using any parametric assumptions.

It is believed that this thesis and future research will contribute towards establishing a methodology and/or procedure that is not subjective in nature, such as the "eye-ball" technique referred to above, but *objective*, with specific reference to the estimation of the off-pulse interval or window of a pulsar.

1.7 Objectives

It is essential to summarise the specific objectives of this thesis from the background and motivation given in the previous sections. The following list highlights the key outcomes that need to be achieved in this study.

- Provide an overview of the existing literature on density estimation, circular data estimation and estimation of the background levels of photon arrival times (addressed in Chapter 2).
- Develop a statistical methodology or technique to estimate the off-pulse window nonparametrically (addressed in Chapter 3).
- The developed technique should be flexible and take light curves with single or multiple off-pulse events into account. The technique should be capable of accepting different classes of light curves, such as:
  - unimodal light curves with a minimum interval, and
  - bimodal light curves with multiple minimum intervals (addressed in Chapter 3).
- Conduct simulation studies to measure the performance of the developed technique to estimate the off-pulse window of a pulsar light curve (addressed in Chapter 4).
- Identify tuning parameter sets that produce optimal results on the simulated data sets (addressed in Section 4.7).
- Apply the optimal tuning parameter sets to astrophysical data to derive estimates for the off-pulse window of actual pulsar light curves (addressed in Chapter 5).
- Compare the nonparametric estimation (from actual pulsar data) to the estimated off-pulse window obtained with the "eye-ball" technique (addressed in Chapter 5).
- Comment on the applicability of the new technique for future research on pulsar data (addressed in Chapter 6).

The following section will briefly describe the framework of the remainder of this thesis to provide the reader with a broad impression of what is to follow, together with some references in terms of the chapter division.

1.8 Thesis outline

Following on this introductory chapter, the thesis commences by looking at a review of directional statistics in Chapter 2. This chapter provides the reader with an improved understanding of the problem and how it relates to circular data. It also provides a brief overview of the literature pertaining to circular data to enable the readers to acquaint themselves with the terminology and notation that are used. This chapter also discusses some fundamental circular statistical ideas followed by an overview of kernel density estimation techniques. Only then the two ideas are combined to illustrate kernel density estimation techniques applied to circular data. The chapter concludes with a section that addresses the problem of estimating the strength of the pulsed signal, p, for the periodic light curve function.

Chapter 3 discusses the proposed estimation technique of the off-pulse interval(s) of a pulsar light curve. This is the most important chapter of the thesis. In this chapter, the proposed technique to estimate the endpoints of the off-pulse interval(s) of an unknown source function (originating from a pulsar) is developed, defined and discussed. A simple proposal to address the problem is firstly discussed for a theoretical situation in the absence of noise. Some justification is then provided why the proposed technique could be applied, before supplying the algorithm of the estimation technique in the presence of noise or background radiation. Broadly speaking, this technique is based in a sequential way on the P-values of goodness-of-fit tests for the uniform distribution. To be more specific, it is the intention to use well-known test statistics for uniformity sequentially on subintervals of the light curve to assess the point at which uniformity is rejected, resulting in an estimator for the off-pulse interval.

Chapter 4 presents the results of the empirical study performed on simulated data to evaluate the performance of the proposed method to estimate the off-pulse interval of a source function originating from a pulsar. This chapter provides a detailed outline of the simulation design that was followed, together with the simulation study results obtained from different study populations. The chapter concludes with a summary of the optimal choice of tuning parameters that consistently performed better than other selections.

In the penultimate chapter of this thesis, Chapter 5, the results of the empirical study performed on pulsar data are presented. The chapter makes use of the optimal tuning parameter configurations to estimate the off-pulse interval of the pulsars using the proposed methodology. These estimated intervals are then compared to the off-pulse intervals published in research papers where the *subjective* “eye-ball” technique was used.

Finally, in Chapter 6, some concluding remarks are presented as a synthesis of the study. This chapter also endeavours to provide suggestions on some intended future research.

Chapter 2

Directional statistics: An overview

2.1 Introduction

The focus of directional statistics is mainly on data that were measured or obtained in the form of angles or two-dimensional orientations, usually referred to as unit vectors in the plane. Circular data can therefore be seen in two ways, according to Mardia & Jupp (2000, p.1):

1. a point on a circle of unit radius, or

2. a unit vector in the plane (therefore a direction).

Each observation on the circle can therefore be measured or classified according to the angle from some chosen initial direction, bearing in mind that the orientation of the circle also plays a role (whether angles are measured clockwise or counter-clockwise from the initial direction). When observations are considered (and represented) as a unit vector in the plane, each point $\boldsymbol{y}$ on the circle can then be represented in terms of the angle θ:

$$\boldsymbol{y} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \tag{2.1}$$

This representation will be discussed in more detail in Section 2.3.4. The reader is also referred to Figure 2.11. Throughout this thesis, angles will be measured in radians without loss of generality, since any angle (in radians) can be converted to degrees by multiplying by $\frac{180}{\pi}$. In some examples, though, angles are represented in degrees for simplicity of calculations.

It is important to note that circular data analysis sits somewhere between the analysis of linear- and spherical data, and linear techniques cannot always be applied directly. Therefore, specific statistical methods and techniques have been developed to handle circular data. During the 1980s and the early 1990s, a brisk development of circular and spherical data analysis techniques took place as evident in published books such as Mardia (1972), Fisher (1993), Mardia (1992) and Mardia & Jupp (2000).

The development in the field of density estimation pertaining to circular- and spherical data also received ample attention in the early 1990s, continuing into the new millennium as evident in Agostinelli (2007), Bai, Rao & Zhao (1988), Fisher (1989), Hall, Watson & Cabrera (1987), Klemelä (2000) and Taylor (2008). Density estimation, and specifically kernel density estimation (abbreviated with KDE) on the straight line is, however, no new topic, and was first introduced by Rosenblatt (1956) and Parzen (1962), although a less known paper by Fix & Hodges (1951) introduced the basic algorithm of nonparametric density estimation. Other excellent references such as Scott (1992), Silverman (1986) and Wand & Jones (1995) also exist.

In order to get a better understanding of the problem, it is required that a brief overview of the literature be provided to enable the reader to understand the terminology and notation that will be used. The next section will discuss some fundamental circular statistical ideas, followed by an overview of kernel density estimation techniques. Only then will the two topics be unified in order to illustrate kernel density estimation techniques applied to circular data. The chapter will conclude with a brief synopsis of the estimation techniques that can be applied when estimating the strength of the pulsed signal p of a pulsar light curve.

2.2 Basic circular data representation

As stated, circular data can be seen as a direction and can be classified according to the angle from some chosen initial direction and orientation of the circle. A first step in exploratory data analysis would be to represent the data in some form of a diagram. The simplest representation of circular data is a raw data plot on a circle, where each data point is represented by a point on the unit circle. Figure 2.1 is such a raw data plot of the departing direction of 76 female turtles after laying their eggs on a beach (Gould, 1959).

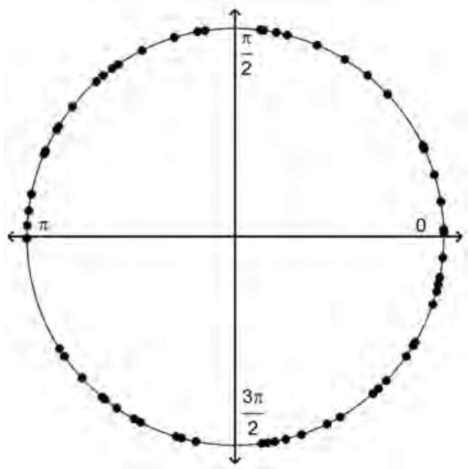

Figure 2.1: *Raw circular data plot of the orientation data of 76 turtles after laying eggs*

When it is possible to group the data, a circular histogram seems to be an improved way of representing the data. With such a circular histogram, the bars are centred at the midpoint of each group of angles, with the area of each bar representative of the frequency of that group, as illustrated in Figure 2.2. This is similar to a histogram on the real line, and care must be taken with the width of each bin in the histogram, as the histogram will be sensitive to the choice of width of the bin.

It is possible to “unwrap” the histogram around the circle into a histogram on the real line (as some statisticians are more familiar with histograms on the real line). The methodology to do this requires the selection of a certain point on the circle as “cutting point”. That is the point where the circle will be “bisected” and be unrolled from. Figure 2.3 represents some circular data, but the data are unrolled onto the real line with the zero angle chosen as the cutting point. The data illustrate the Vela pulsar and were obtained from the *Fermi* Science Support Centre (FSSC)

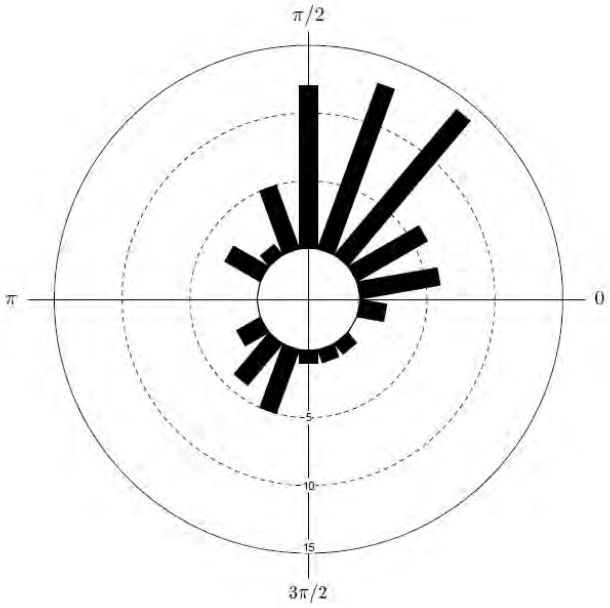

Figure 2.2: *Circular histogram with 18 bins of the orientation data of 76 turtles after laying eggs*

website. *
It is very important to note that the choice of the cutting point will lead to different visual impressions of the histogram. Normally, the cutting point is chosen to be on the opposite side (on the circle) of the mode. Should the cutting point be chosen as the mode, it might result in the interpretation that the data are bimodal. Another way of countering different visual impressions of circular data unwrapped on the real line – based on a certain choice of cutting point – is to present two phases of the data on the same histogram. The resultant histogram will contain two phases and the visual image of the data just before the cutting point will continue (or flow) into the data just after the cutting point. For a graphical representation, the reader is referred to Figure 2.4.

2.3 Circular ground work and notation

Observations on the (unit) circle can be regarded as unit vectors $\boldsymbol{y}$. It is important to choose a zero direction and orientation for the unit circle. Each point $\boldsymbol{y}$ on the circle can then be represented as an angle θ or differently stated by $\boldsymbol{y} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. It is also important to state that all angles will be measured in radians from now on, and for all calculations, two points on the circle, namely θ and $\theta + 2\pi$ will be exactly the same point.

The key characteristic that differentiates circular data from data measured on a linear scale is exactly this wrap-around nature with no maximum or minimum. That is, the start of the measurement interval coincides with the end of the interval, i.e., the angles θ and $\theta + 2\pi$ (radians) are the same point on the circle. In general, the measurement is periodic with θ being the same as $\theta + 2p\pi$ for any integer p. To state it simplistically, the difference between the theory of linear statistics and statistics on the circle can be attributed to the fact that the circle is a closed curve,

*http://fermi.gsfc.nasa.gov/ssc/data/access/lat/ephems/

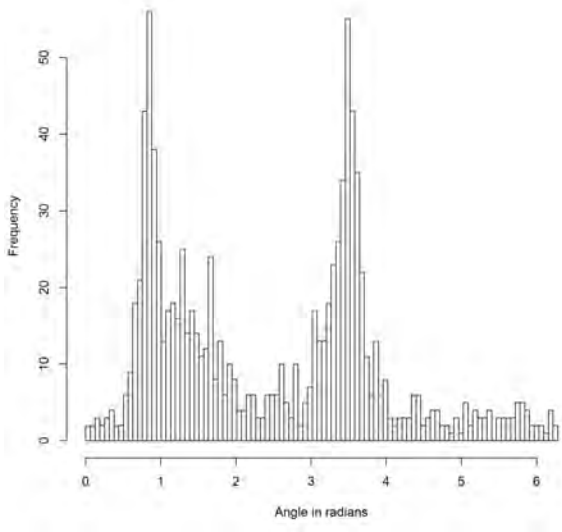

Figure 2.3: *Histogram on the real line (unwrapped around the circle) with 100 bins of the first thousand observations (times of arrival) of the Vela pulsar, represented as angles (in radians).*

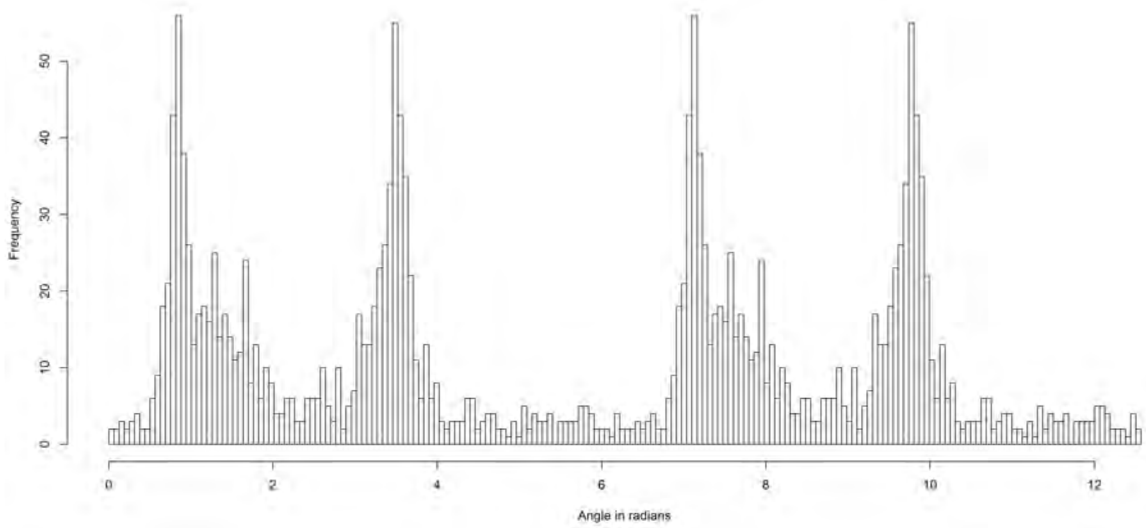

Figure 2.4: *Histogram on the real line of two phases of the first thousand observations of the Vela pulsar (100 bins per phase)*

while the line is not. Therefore, distribution functions, characteristic functions and moments on the circle have to be defined by taking into account the periodicity of observations on the circle. For example, consider the following definition of a *distribution function* on the circle. Suppose that an initial zero direction and an orientation of the unit circle have been chosen. Let Z be a random variable that takes values on the circumference of the unit circle. All possible values of Z can then be identified with random angles θ measured from the initial zero direction and with the correct orientation. The distribution function F is defined as the function on the complete real line, and given by

$$F(x) = Pr(0 < \theta \leq x), \quad 0 \leq x \leq 2\pi,$$

and

$$F(x + 2\pi) - F(x) = 1, \quad -\infty < x < \infty. \tag{2.2}$$

Equation (2.2) states that any arc of length 2π on the unit circle has a probability of 1.

The inherent periodicity of circular data brings with it a distinctive nature that does not occur elsewhere in statistics. This distinctive nature can be seen when calculating some basic (linear) statistical measures.

Consider two angles that are 2 degrees apart. If the interval $[-180\,^\circ, 180\,^\circ]$ is considered, the two angles may be chosen as $-1\,^\circ$ and $1\,^\circ$. If the interval is chosen differently, say $[0\,^\circ, 360\,^\circ]$, the same angles will be $1\,^\circ$ and $359\,^\circ$. If these two scenarios are graphically depicted on a circle, no problem is apparent, as the figures would be equivalent. However, when approaching the problem numerically, potential problems exist. When estimating the mean direction of the latter pair of angles, these observations are clearly centred about $0\,^\circ$. However, when using naive linear methods, the sample mean and standard deviation of these two observations would be $180\,^\circ$ and $253\,^\circ$, respectively. Had the pair of angles been $1\,^\circ$ and $-1\,^\circ$, and by using the same naive linear methods, more sensible values of 0 as the sample mean and $\sqrt{2}$ as the sample standard deviation are obtained. This illustrates the need for different measures of location and scale when dealing with circular data. It must be emphasised that, since the choice of a zero-direction and the sense of rotation is arbitrary, one needs measures that are invariant under such choices. A point estimate $\tilde{\theta}$ is said to be location (translation) invariant if

$$\tilde{\theta}\big(\theta_1 + \nu, \ldots, \theta_n + \nu\big) = \nu + \tilde{\theta}(\theta_1, \ldots, \theta_n), \tag{2.3}$$

for every ν and $(\theta_1, \ldots, \theta_n)$. That is, if the data are shifted by a certain amount ν, the value of the point estimate also changes by the same amount.

Three common choices to summarise the preferred direction of circular data are the mean direction, the median direction and the modal direction (Fisher, 1993). The sample mean direction (when combined with a measure of sample dispersion) is usually preferred for moderate to large samples, because it acts as a summary value of the data, which is suitable for comparison when working with multiple data sets. The mean direction will be discussed in Section 2.3.1. The sample median can be thought of as balancing the number of observations on two halves of the circle and will be discussed in Section 2.3.2. The sample modal direction is the direction corresponding to the maximum concentration of the data. The sample modal direction is less useful because of difficulties in its calculation, in drawing inferences, and in ascertaining its sampling error. What is interesting to note is that all three measures of preferred direction are undefined if the sample data are equally spaced around the circle, i.e., if the data are symmetric around the circle there is no preferred direction. In case of bimodal data, there are two preferred directions, and consequently the three measures are also not that meaningful.

The next subsection will firstly explain the concept of the resultant vector and the significance of the direction and length of it. The subsequent subsection will investigate the concept of a circular median and how to calculate it. The discussion will then focus on circular measures of dispersion, followed by a discussion on distance measures for circular data. *These measures are of importance, since most of them are utilised in the empirical study in Chapters 4 and 5.*

2.3.1 The mean direction and the resultant length

The mean direction is an appropriate and meaningful measure for a unimodal set of circular observations. The mean direction is obtained by treating the data as unit vectors and using the direction of their resultant vector. For a sample of unit vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$, with corresponding angles $\theta_1, \ldots, \theta_n$, the resultant vector $\boldsymbol{E}$ is obtained by adding up the unit vectors component-wise:

$$\boldsymbol{E} = \sum_{i=1}^{n} \boldsymbol{y}_i. \tag{2.4}$$

This is a vector with length between 0 and n, and pointing in the mean direction $\bar{\theta}$ of the sample. The sample mean resultant length (standardised length) is given by:

$$\bar{R} = \frac{\|\boldsymbol{E}\|}{n}, \tag{2.5}$$

with $\bar{R} \in [0, 1]$, and $\| \cdot \|$ the Euclidean norm. If the data are closely clustered around the mean, then $\bar{R}$ is close to 1. However, if the data are evenly spread around the circle, $\bar{R}$ will be near zero. Hence, $\bar{R}$ is a natural measure of dispersion (see Section 2.3.3).

Therefore, the resultant vector $\boldsymbol{E}$ can be decomposed into two components, namely:

1. the mean direction $\bar{\theta}$, and
2. the mean resultant length $\bar{R}$.

This forms a useful starting point for any analysis regarding circular data. Since the Cartesian coordinates of each $\boldsymbol{y}_i$ are $(\cos\theta_i\,, \sin\theta_i)$ for $i = 1, \ldots, n$, the mean resultant length $\bar{R}$ can be written differently:

$$\bar{R} = \sqrt{(\bar{C}^2 + \bar{S}^2)} > 0, \quad \text{with} \quad \bar{C} = \frac{1}{n}\sum_{i=1}^{n} \cos\theta_i \quad \text{and} \quad \bar{S} = \frac{1}{n}\sum_{i=1}^{n} \sin\theta_i. \tag{2.6}$$

As stated above, the direction of the resultant vector $\boldsymbol{E}$ is known as the circular mean direction, and is denoted by $\bar{\theta}$. A "quadrant-specific" inverse of the tangent definition of the circular mean direction (Mardia & Jupp, 2000) is

$$\bar{\theta} = \begin{cases} \arctan(\bar{S}/\bar{C}), & \text{if } \bar{C} > 0 \\ \frac{\pi}{2}, & \text{if } \bar{C} = 0 \text{ and } \bar{S} > 0 \\ \frac{-\pi}{2}, & \text{if } \bar{C} = 0 \text{ and } \bar{S} < 0 \\ \pi + \arctan(\bar{S}/\bar{C}), & \text{otherwise.} \end{cases} \tag{2.7}$$

Note, the inverse tangent function, arctan (or $\tan^{-1}$), takes values in $[\frac{-\pi}{2}, \frac{\pi}{2}]$. The above definition results in the correct unique inverse on $[0, 2\pi)$, which takes into account the signs of $\bar{C}$ and $\bar{S}$. *The reader must note that, within the context of circular statistics, $\bar{\theta}$ does not denote the standard linear average $(\theta_1 + \cdots + \theta_n)/n$.*

Geometrically, the mean direction is equivalently obtained with vector polygons, as shown in Figure 2.5 and Figure 2.6. Furthermore, Jammalamadaka & SenGupta (2001) proved that $\bar{\theta}$ is location invariant, i.e., if the data are shifted by a certain amount, the value of $\bar{\theta}$ also changes by the same amount. Otieno (2002) also proved that $\bar{\theta}$ is invariant with respect to changes in the choice of rotation, i.e., when the measurement direction switches from clockwise to counter-clockwise so that all θ's become $(2\pi - \theta)$'s, then $\bar{\theta}$ becomes $(2\pi - \bar{\theta})$. Therefore, the point estimate does not depend on what direction is taken to be the positive direction.

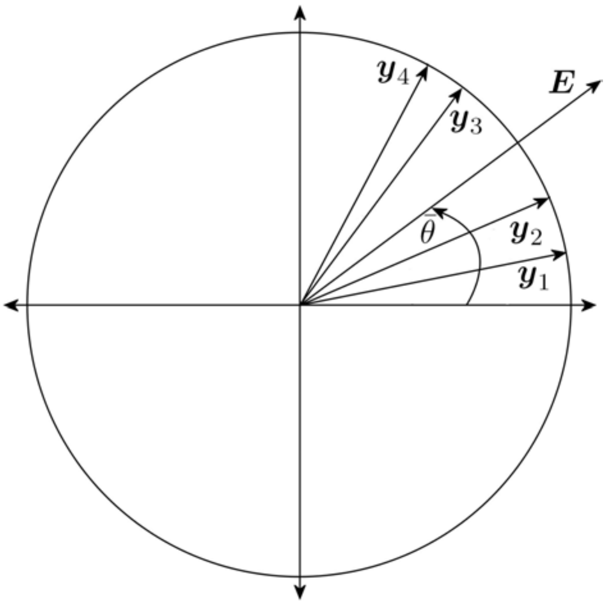

Figure 2.5: *The resultant vector and mean direction as obtained from vector polygons*

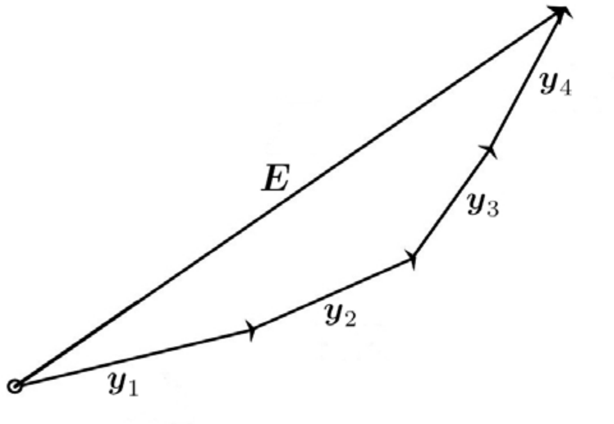

Figure 2.6: *Addition of unit vectors by forming a vector polygon to calculate the mean direction*

2.3.2 The median direction

For the purposes of robust estimation, it is appropriate to have a version of the sample median for circular data. Since the sample median is a robust estimate for the preferred direction of a distribution, it has a different character than the sample mean as illustrated by the different properties. The circular median was defined more formally by Fisher & Powell (1989) as the angle about which the sum of absolute angular deviations is a minimum.

According to Mardia & Jupp (2000), the sample median direction $\tilde{\theta}$ of angles $\theta_1, \ldots, \theta_n$ is the point P on the circumference of the circle that satisfies the following two properties:

1. the diameter PQ through P divides the circle into semi-circles, each with an equal number of observed data points, and

2. the majority of the observed data are closer to P than to the antimedian Q.

Mardia & Jupp (2000) proved that the circular median of a unimodal distribution is unique. It is also rotationally invariant as shown by Ackermann (1997). Similar to the case of the median for linear data, the circular median is defined separately for odd and even number of observations. When n is odd, the sample median is one of the observation in the data set. When n is even, the sample median is taken to be the midpoint of two appropriate adjacent observations. Figure 2.7 and Figure 2.8 depict the circular median for even and odd sample sizes, respectively. Note the balance between the number of observations in both half circles. In both cases, the majority of sample observations are closer to P (median), than to Q (antimedian).

However, the calculation procedure (based on the ranking of observations) for computing the median for linear data cannot be applied directly to circular data. For example, consider the following data set (in degrees): $43°, 45°, 52°, 61°, 75°, 88°, 88°, 279°, 357°$, shown in Figure 2.9 (Ackermann, 1997). If these observations are treated as linear measurements, then the median is $75\,°$. However, when considered as observations on the circle, the median is $52\,°$. Clearly, these answers are not equal.

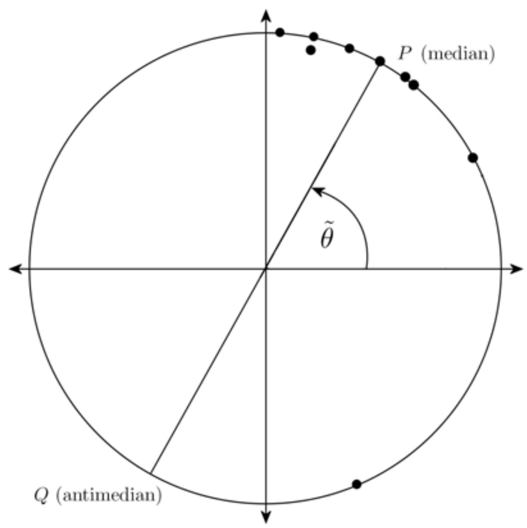

Figure 2.7: *The median direction is the direction of one of the observations when n is odd*

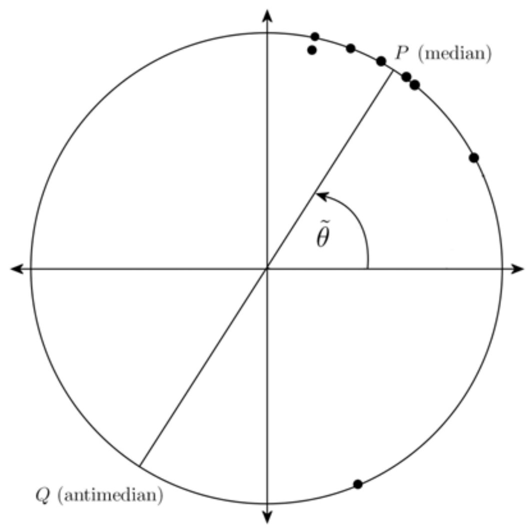

Figure 2.8: *The median direction is the midpoint between two observations when n is even*

In addition, a straight line through 75 ° and the midpoint of the circle, extended to 255 °, will not lead to an equal number of observations on each semi-circle. Furthermore, the mean and median directions typically yield different estimates of preferred direction. Figure 2.10 shows an example where the circular median (denoted by P) is one of the sample values, while the circular mean (denoted by $\bar{\theta}$) is not necessarily one of the sample values.

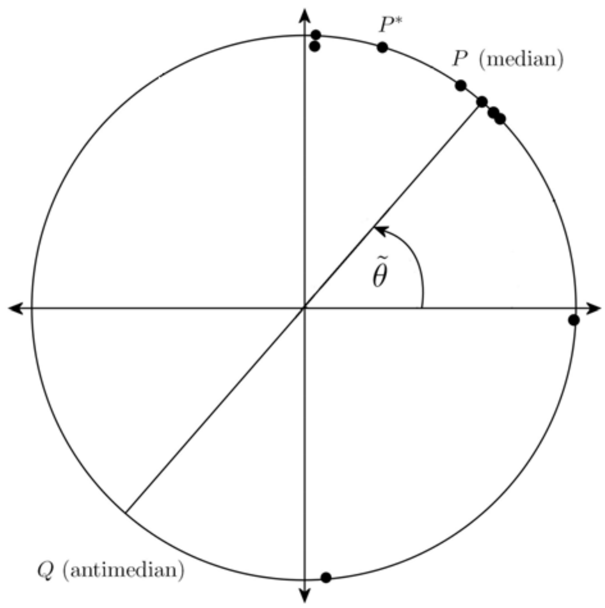

Figure 2.9: *Representation of data on a circle with P^* the linear median, P the circular median and Q the circular antimedian*

It is possible though that the circular mean and circular median can coincide if the underlying distribution is symmetric about the reference direction. Ease of computation and availability of relevant statistical theory (e.g., to calculate confidence regions or to pool independent estimates

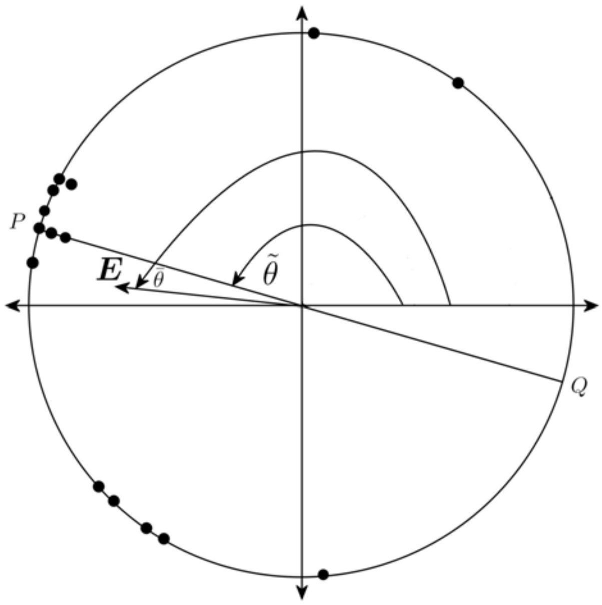

Figure 2.10: *Representation of data on a circle with $\tilde{\theta}$ the circular median, and $\bar{\theta}$ the circular mean direction of the resultant vector* $\boldsymbol{E}$

of the same quantity) make the mean direction the most commonly used measure of preferred direction, particularly for moderate to large samples (Fisher, 1993). However, robust estimation of the preferred direction for the von Mises distribution has been based mainly on the median direction.

2.3.3 Circular measures of dispersion

The measures of spread associated with the circular mean and the circular median directions are, respectively, the circular variance and the circular mean deviation (Mardia & Jupp, 2000). The circular variance V, is a common dispersion statistic defined in terms of the length of the standardised resultant vector:

$$V = 1 - \bar{R}, \tag{2.8}$$

where $0 \leq V \leq 1$ since $0 \leq \bar{R} \leq 1$. Minimum variation occurs when $V = 0\,(\bar{R} = 1)$, and corresponds to the situation where all of the observations in a given sample occur at precisely the same location. A natural upper limit to the possible variation occurs for data uniformly distributed around the circle, and corresponds to $V = 1\,(\bar{R} = 0)$. Calculation of $\bar{R}$, and hence V is straightforward, and the interpretation of results does not depend on assumptions about the original data. It is interesting to note that for *any* data set of the form $\theta_1, \ldots, \theta_n, \theta_1 + \pi, \ldots, \theta_n + \pi$ the mean resultant length $\bar{R}$ will always be zero. Therefore, it is not true to state that, when $\bar{R} \approx 0$, then the directions are spread almost evenly around the circle (Mardia & Jupp, 2000). Some authors such as Jammalamadaka & SenGupta (2001) also refer to the quantity $2(1 - \bar{R})$ as the circular variance. However, the representation in equation (2.8) is preferred.

Based on the circular variance, an appropriate transformed statistic is the sample circular standard deviation given by (Mardia & Jupp, 2000):

$$v = \{-2\log(1 - V)\}^{1/2} = \{-2\log\bar{R}\}^{1/2}. \tag{2.9}$$

Note that v takes values in $[0, \infty]$, whereas V takes values in $[0, 1]$. For small V, (2.9) reduces to

$$v \approx (2V)^{1/2} = \{2(1 - \bar{R})\}^{1/2}. \tag{2.10}$$

The circular mean deviation is a measure of spread associated with any measure of the preferred direction, say α. It is defined about α using

$$d_0(\alpha) = \frac{1}{n}\sum_{i=1}^{n}\{\pi - |\pi - |\theta_i - \alpha||\}. \tag{2.11}$$

That is, the mean distance between the preferred direction α and the data points. Mardia & Jupp (2000) showed that it has a minimum when the sample median $\tilde{\theta}$ is used as the measure of the preferred direction. The circular mean deviation can then also be defined as $d_0(\bar{\theta})$. Two other measures of dispersion then remain to be defined. The first being the circular range and the second being the circular median absolute deviation.

The circular range is the length of the smallest arc that contains all the observations. One way of calculating the circular range is described in Mardia & Jupp (2000). The procedure is to cut the circle at an initial direction and consider $\theta_1, \ldots, \theta_n$ in the range $0 \leq \theta_i \leq 2\pi$. The next step is to obtain the order statistics of $\theta_1, \ldots, \theta_n$, namely $\theta_{(1)} \leq \cdots \leq \theta_{(n)}$. The arc lengths between adjacent observations are then given by:

$$T_i = \theta_{(i+1)} - \theta_{(i)},\ i = 1, \ldots, n-1;\ \ T_n = 2\pi - (\theta_{(n)} - \theta_{(1)}).$$

The circular range w can then be calculated as follows:

$$w = 2\pi - \max(T_1, \ldots, T_n). \tag{2.12}$$

Another useful measure of spread is the quantiles of circular data. Mardia (1972) defined the first and third quartile directions Q_1 and Q_3 when there is prior knowledge about the circular distribution as

$$\int_{\tilde{\theta}-Q_1}^{\tilde{\theta}} f(\theta)d\theta = 0.25 \text{ and } \int_{\tilde{\theta}}^{\tilde{\theta}+Q_3} f(\theta)d\theta = 0.25\,,$$

respectively. In most cases, however, the circular distribution is unknown. To date, only limited literature exists on the nonparametric estimation of Q_1 and Q_3 for circular data. Abuzaid, Mohamed & Hussin (2011) did, however, find it sensible to estimate Q_1 and Q_3 by classifying the sample observations into two groups based on their location with respect to the sample median direction $\tilde{\theta}$. Q_1 can therefore be considered as the median of the first group and Q_3 as the median of the second group. If the value of Q_1 is larger than the value of Q_3, then the labels are simply interchanged.

For simplicity and to avoid confusion caused by the interchanging of the labels of Q_1 and Q_3, it is possible to make use of the rotatable property of circular data to ensure the consistent definition of the quartiles, and the interquartile range. The rotatable property ensures that the estimated mean direction $\bar{\theta}$ can be subtracted from each sample observation, resulting in the situation that the mean is then in the zero direction. This rotation is helpful in identifying Q_1 and Q_3 in a more consistent way so that, following the rotation, it can be assumed that $Q_1 - \bar{\theta} \in [0, \pi]$ and

$Q_3 - \bar{\theta} \in [\pi,\, 2\pi]$.

It is now possible to define the circular interquartile range ($CIQR$) in a consistent way, analogous to the interquartile range of data on the real line. Following the rotation of the sample observations, the circular interquartile range can be defined by:

$$CIQR = 2\pi - (Q_3 - Q_1). \tag{2.13}$$

For highly concentrated data, it is possible to have quartiles and mean directions at the same point so that $CIQR = 0$.

The final measure of dispersion is the circular median absolute deviation from $\tilde{\theta}$, which is defined by

$$\text{median}\left(|\theta_1 - \tilde{\theta}|, \ldots, |\theta_n - \tilde{\theta}|\right). \tag{2.14}$$

The following measures of dispersion will frequently be used, especially when selecting the smoothing parameter for the kernel density estimator:

1. the square root of the circular variance defined in (2.8);
2. the circular mean deviation defined in (2.11) for $\alpha = \bar{\theta}$ and $\alpha = \tilde{\theta}$, and
3. the circular median absolute deviation defined in (2.14).

2.3.4 Circular distance measure

For a random sample of unit vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ with corresponding angles (measured in radians) $\theta_1, \ldots, \theta_n \in [0, 2\pi]$, it is important to define some distance measures between the sample observations on the circle. In order to thoroughly explain the idea of the distance measure between observations on a circle, the following definitions must be made (the reader is also referred to Figure 2.11).

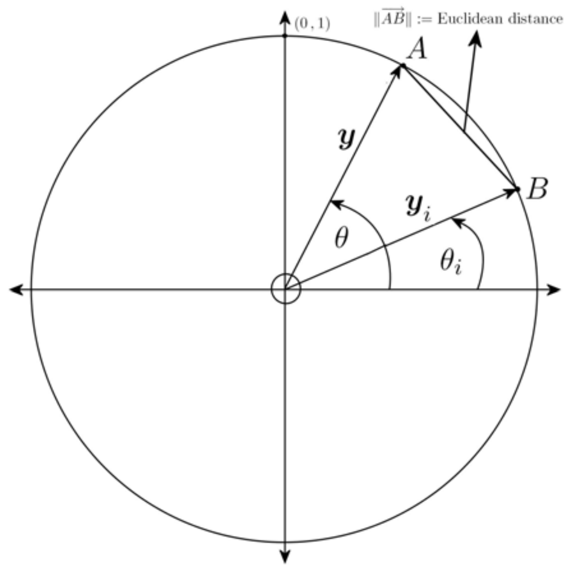

Figure 2.11: *Representation of data on a circle classified as angles and Euclidean distance*

Let $\boldsymbol{y} = \begin{bmatrix} \cos\,\theta \\ \sin\,\theta \end{bmatrix}$ and $\boldsymbol{y}_i = \begin{bmatrix} \cos\,\theta_i \\ \sin\,\theta_i \end{bmatrix}$.

It is then similar to say that point $A = (\cos\,\theta\,, \sin\theta\,)$ and point $B = (\cos\,\theta_i\,, \sin\theta_i\,)$. Also define

$$\|\overrightarrow{AB}\| := \text{Euclidean distance (straight line distance) between points } A \text{ and } B,$$

where $\overrightarrow{AB}$ denotes the vector from A to B.

It is possible to calculate the distance $\|\overrightarrow{AB}\|^2$ with simple vector algebra as follows:

$$\|\overrightarrow{AB}\|^2 = (\cos\theta - \cos\theta_i)^2 + (\sin\theta - \sin\theta_i)^2. \tag{2.15}$$

Note that this representation of the Euclidean distance $\|\overrightarrow{AB}\|^2$ is not written in terms of the angle between the vectors $\boldsymbol{y}$ and $\boldsymbol{y}_i$, i.e. $|\theta - \theta_i|$. However, applying the law of cosines (see, e.g., Johnson, Riess & Arnold (2002)) on the triangle OAB in Figure 2.11 one obtains

$$\begin{aligned} \|\overrightarrow{AB}\|^2 &= \|\boldsymbol{y}\|^2 + \|\boldsymbol{y}_i\|^2 - 2\|\boldsymbol{y}\|\,\|\boldsymbol{y}_i\|\,\cos(\theta - \theta_i) \\ &= 2(1 - \cos(\theta - \theta_i)) \\ &= 2(1 - \cos(|\theta - \theta_i|)). \end{aligned} \tag{2.16}$$

Furthermore, note the following:

$$\begin{aligned} \boldsymbol{y}^T\,\boldsymbol{y}_i &= (\cos\theta\,,\,\sin\theta)\begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} \\ &= \cos\theta\cos\theta_i + \sin\theta\sin\theta_i \\ &= \cos(\theta - \theta_i). \end{aligned}$$

Hence, we also have

$$\|\overrightarrow{AB}\|^2 = 2(1 - \boldsymbol{y}^T\,\boldsymbol{y}_i). \tag{2.17}$$

The following definition is introduced (see, e.g., Taylor (2008)) to define the distance between two angles, θ and θ_i:

$$d_i(\theta) := \min(|\theta - \theta_i|, 2\pi - |\theta - \theta_i|), \tag{2.18}$$

where $|\theta - \theta_i|$ denotes the usual absolute value. This definition follows from the fact that the smallest angle between observations must be used as distance measure, even if the angles seem to be distant from one another on the real line, such as $1\,^\circ$ and $359\,^\circ$. For this example, the angle between the observations is not $358\,^\circ$ (difference on the real line), but rather $2\,^\circ$ from (2.18).

Some properties of $d_i(\theta)$ are:

$$\begin{aligned} d_i(0) &= \min\,(|0 - \theta_i|\,,\,2\pi - |0 - \theta_i|) \\ &= \min\,(\theta_i\,,\,2\pi - \theta_i),\ \text{since}\ \theta_i \geq 0. \end{aligned}$$

$$\begin{aligned} d_i(2\pi) &= \min\,(|2\pi - \theta_i|\,,\,2\pi - |2\pi - \theta_i|) \\ &= \min\,(2\pi - \theta_i\,,\,2\pi - (2\pi - \theta_i)),\ \text{since}\ \theta_i \leq 2\pi \end{aligned}$$

$$= \quad \min\,(2\pi - \theta_i\,,\,\theta_i).$$

Therefore, $d_i(0) = d_i(2\pi)$. Furthermore, it is known that:

1. $\cos|\theta| = \cos\theta \;\forall\; \theta$, and
2. $\cos(2\pi - \theta) = \cos\theta \;\forall\; \theta$.

Consequently, the following identities can be established:

1. $\cos(\theta - \theta_i) = \cos d_i(\theta)$; $1 - \cos(\theta - \theta_i) = 1 - \cos d_i(\theta)$, and
2. $d_i(\theta) = \cos^{-1}(\boldsymbol{y}^T\,\boldsymbol{y}_i) = \arccos(\boldsymbol{y}^T\,\boldsymbol{y}_i)$.

Remark: These distance measures will be used in kernel density estimation on the circle.

2.4 Kernel estimation of density functions: An overview

Considerable literature exists that investigates the non-parametric estimation of a probability density function of a random variable through the use of kernel functions. Frequently, references such as Silverman (1986) and Wand & Jones (1995) apply kernel density estimation to data on the real line. There are some references that apply kernel density estimation to circular data, such as Hall et al. (1987), Bai et al. (1988), Fisher (1989), Klemelä (2000), Taylor (2008), Oliveira, Crujeiras & Rodriquez-Casal (2012) and Garcia-Portugues, Crujeiras & Gonzalez-Manteiga (2013). In order to develop standard notation that will be used throughout the text, a synopsis of the most frequent kernel density estimation techniques will be provided.

2.4.1 Kernel estimation on the real line

According to Silverman (1986, p. 7), the histogram can be seen as the oldest and most widely used density estimator. For a detailed study of histograms, the reader is referred to Scott (1992). It is important to note that even for such a simple estimator as the histogram, a certain bin width h must be chosen, together with some point of origin, say, x_0. Define the bins of the histogram to be the intervals $[x_0 + mh,\, x_0 + (m+1)h)$ for positive and negative integers m. The histogram is then defined in Silverman (1986, p. 9) by

$$\hat{f}_{n,h}(x) = \frac{1}{nh}(\text{number of } X_i \text{ in the same bin as } x),$$

where $X_1, \ldots, X_n$ are the sample observations on the real line.

If one considers the definition of a probability density, and define f as the density of a random variable X, then

$$f_h(x) = \lim_{h\to 0} \frac{1}{2h} P(x - h < X < x + h).$$

For any given h it is possible to estimate $P(x - h < X < x + h)$ by calculating the proportion of the sample observations within the interval $(x - h, x + h)$. An estimator $\hat{f}_{n,h}(x)$ of the density is then given when choosing a small value for h and setting (Silverman, 1986, p. 12)

$$\hat{f}_{n,h}(x) = \frac{1}{2nh}[\text{number of } X_1, \ldots X_n \text{ within the interval } (x - h,\, x + h)]. \tag{2.19}$$

This density estimator can now be defined in a slightly different fashion in order to get closer to the well-known and popular kernel method introduced by Rosenblatt (1956). Firstly, it is required to introduce some weight function w, defined as follows (Silverman, 1986, p. 12):

$$w(x) = \begin{cases} \frac{1}{2} & \text{if } |x| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The so-called *naive* estimator in (2.19) can then be re-written as:

$$\hat{f}_{n,h}(x) = \frac{1}{nh} \sum_{i=1}^{n} w\left(\frac{x - X_i}{h}\right).$$

It is now possible to substitute the weight function $w(x)$ with some general weight function, say $k(x) \geq 0$. The latter is usually called the kernel function. Furthermore, it is interesting to note that Cacoullos (1966) appears to have been the first to call k a kernel function.

The *kernel estimator* based on the kernel function k is then defined by (Silverman, 1986, p. 15)

$$\hat{f}_{n,h}(x) = \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{x - X_i}{h}\right). \tag{2.20}$$

This estimator is, however, only applicable to data on the real line. What should be changed and how should the method be adopted if the data are circular observations? The following subsection will address this question.

2.4.2 Kernel estimation on the circle

Taylor (2008) proposes the use of angular distances when the data are on the circle, rather than using the Euclidean distance $x - X_i$ in (2.20). From the representations of the distance between two angles in (2.18), the kernel density estimator on the circle is written as follows:

$$\hat{f}_{n,h}(\theta) = \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{1 - \cos(d_i(\theta))}{h}\right) \Big/ c_f(h), \tag{2.21}$$

where $d_i(\theta)$ is given in (2.18), and

$$c_f(h) = \int_0^{2\pi} \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{1 - \cos(d_i(\theta))}{h}\right) d\theta\,, \tag{2.22}$$

is known as the normalisation constant to ensure that $\hat{f}_{n,h}(\theta)$ integrates to unity.

This representation of the kernel density estimator in (2.21) is also used by Jammalamadaka & SenGupta (2001, p. 282) and Hall et al. (1987). The latter states that this representation of the kernel density estimator is "closer in spirit to the Euclidean estimator" in (2.20). Furthermore, since the argument in the kernel function k is $1 - \cos(d_i(\theta))$, which is a quadratic distance measure (see (2.16)), this representation can be seen as an L2-norm.

Both Hall et al. (1987) and Mardia & Jupp (2000) indicate furthermore that another generalisation of the kernel density estimator on the circle can be given by:

$$\hat{g}_{n,h}(\theta) = \frac{1}{nh}\sum_{i=1}^{n} k\left(\frac{\cos(d_i(\theta))}{h}\right) \Big/ c_g(h)\,, \tag{2.23}$$

where

$$c_g(h) = \int_0^{2\pi} \frac{1}{nh}\sum_{i=1}^{n} k\left(\frac{\cos(d_i(\theta))}{h}\right) d\theta\,, \tag{2.24}$$

is again referred to as the normalisation constant to ensure that $\hat{g}_{n,h}(\theta)$ integrates to unity.

Hall et al. (1987) calculated the bias and variance of both the estimators in (2.21) and (2.23). These estimators are often equivalent. To show this, take a simple kernel function $k(t) = e^{-t}$ in (2.21), and $k(t) = e^{t}$ in (2.23). It is, however, not always possible to find two kernel functions with these properties. For both these two forms of estimators, various kernel functions k can now be chosen. There is a wide array of kernels that can be selected, as will become evident in the discussion of these kernels in the next subsection.
Both these density estimators represented in (2.21) and (2.23) will be used throughout the text, although the actual calculations will verify that the two forms of the kernel estimators are equivalent.

Furthermore, it is evident from (2.21) and (2.23) that some value of the smoothing parameter h should be used. One of the difficulties in nonparametric density estimation is to make a good data-based choice of the smoothing parameter h. Several excellent references on this topic exist, such as Silverman (1986), Hall, Sheather, Jones & Marron (1991), Sheather & Jones (1991), Jones, Marron & Sheather (1996) and Oliveira et al. (2012), to name but a few. When the data are observed in Euclidean space, there are many approaches to the problem, with one example being the “normal-scale rule”or a “rule of thumb”(Taylor, 2008). The next subsection will firstly expand on the choice of kernel functions k, followed by a subsection that will investigate the data-based choice of the smoothing parameter h for data on a circle.

2.4.3 Choice of kernel function k

It is important to note that various kernel functions can be chosen for the two different representations of the kernel estimators. Popular choices of kernels include the Gaussian-like kernels in circular data such as the von Mises, and wrapped normal distribution with unbounded support. Other popular kernels with compact support will be the so-called “polynomial” kernels of the form:

$$k(x) = \begin{cases} \kappa_{rs}(1-|x|^r)^s, & \text{if } -1 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\kappa_{rs} = \frac{r}{2B\left(s+1, \frac{1}{r}\right)}, \quad r > 0,\, s \ge 0,$$

with $B(\cdot,\cdot)$ denoting the beta-function. The rectangular kernel is obtained if $s = 0\,(\kappa_{r0} = \frac{1}{2})$; the triangular kernel if $r = 1, s = 1\,(\kappa_{11} = 1)$; the Epanechnikov kernel if $r = 2, s = 1\,(\kappa_{21} = \frac{3}{4})$; the bi-weight kernel if $r = 2, s = 2\,(\kappa_{22} = \frac{15}{16})$; and the tri-weight kernel if $r = 2, s = 3\,(\kappa_{23} = \frac{35}{32})$. The simplest form of the Gaussian kernel can be obtained if $r = 2, s = \infty$ after a suitable rescaling (Loots, 1995).

Hall et al. (1987) highlighted the fact that circular kernel estimators with different characteristics require different kernel functions. Therefore, it must be ascertained whether the two representations of kernel density estimators in (2.21) and (2.23) are different in character. It is found that the two forms of estimators exhibit similar characteristics, and similar kernel functions can therefore be applied. The only difference between the two estimators is the support region of the kernel function that will be used, e.g., for (2.21) the co-domain of $1 - \cos(d_i(\theta))$ is $[0; 2]$, in comparison to the co-domain of $\cos(d_i(\theta))$ in (2.23), which is $[-1; 1]$.

It is a well-known fact that, based on the definition of efficiency, the choice of kernel function is not the most important component of kernel density estimation, as there is little to choose between the various kernels on the basis of the mean integrated squared error (Silverman, 1986; Wand & Jones, 1995). Nevertheless, some kernel function must be chosen, and therefore the following list highlights possible choices of kernel functions, equivalent to the kernels proposed by Hall et al. (1987), Swanepoel (1987) and Taylor (2008). It must be stated that this is not a complete list of all kernel functions, and neither will all the kernels in this list be used in the empirical study. It will be clearly stated in Chapter 4 which kernel functions are used.

I) Kernels with compact support on $[0; 1]$

1 Uniform or rectangular kernel

$$k(t) = 1. \tag{2.25}$$

2 Triangular kernel

$$k(t) = 1 - t. \tag{2.26}$$

3 Epanechnikov kernel

$$k(t) = 1 - t^2. \tag{2.27}$$

4 Quartic kernel

$$k(t) = (1 - t^2)^2. \tag{2.28}$$

5 Swanepoel kernel

$$k(t) = c_1(\cos(at) - \sin(a|t|)\exp(a|t|) + c_2(\cos(at) + \sin(a|t|)\exp(-a|t|)\,, \tag{2.29}$$

$$\text{where } a = \frac{\pi}{2}\,,\; c_1 = \frac{a\exp(-a)}{2\exp(a) - \exp(-a)} \text{ and } c_2 = \frac{a\exp(a)}{2\exp(a) - \exp(-a)}.$$

II) Kernels with unbounded support on $[0; \infty)$

1 Exponential kernel

$$k(t) = \exp(-t). \tag{2.30}$$

2 Gaussian kernel

$$k(t) = \exp(-t^2). \tag{2.31}$$

2.4.4 Choice of smoothing parameter h

The aim of this subsection is to discuss some equivalent plug-in rules for density estimation on the circle. When the kernel function is taken as the Gaussian density in Euclidean space, the plug-in selector for h is provided by Silverman (1986):

$$\hat{h} = 1.06\hat{\sigma}n^{-1/5}, \tag{2.32}$$

where $\hat{\sigma}$ is some estimate of the measure of dispersion. Taylor (2008) performed some research on the choice of smoothing parameter h for circular density estimation, and derived a "plug-in rule" for the value of the smoothing parameter based on the concentration parameter of the von Mises reference density. Since this is not in line with our aim of developing a *nonparametric* estimation technique, the proposed choices for the smoothing parameter h are not adopted as such, although there are several outcomes of the research worth mentioning.

Before mentioning some of these outcomes, it is required to provide a quick synopsis of the von Mises distribution, found in Mardia & Jupp (2000). The *von Mises distribution* $vM(\mu, \kappa)$ has probability density function

$$g(\theta;\, \mu,\, \kappa) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa\cos(\theta-\mu)}, \tag{2.33}$$

where I_0 denotes the modified Bessel function of the first kind and order 0, which can be defined by

$$I_0(\kappa) = \frac{1}{2\pi}\int_0^{2\pi} e^{\kappa\cos\theta} d\theta. \tag{2.34}$$

The function $I_0(\kappa)$ has power series expansion

$$I_0(\kappa) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}\left(\frac{\kappa}{2}\right)^{2r}. \tag{2.35}$$

The parameter μ is the mean direction and the parameter κ is known as the *concentration parameter*. When $\kappa = 0$, $vM(\mu, \kappa)$ is the uniform distribution.

Taylor (2008) found that his derived estimate for the smoothing parameter, say $\hat{h}_T$, (based on the concentration parameter κ of the von Mises reference density) tends towards the plug-in selector given in (2.32) when the concentration parameter $\kappa \to \infty$. Furthermore, it is mentioned that the estimator $\hat{h}_T$ will not perform that well for bimodal or multimodal data, since $\hat{h}_T$ is based on the concentration parameter κ of the von Mises reference density, which is not really of any use for bimodal and multimodal data.

In order to remedy the problem stated above, Taylor (2008) then referred to Silverman (1986), who proposed another class of smoothing parameter estimates which uses a more robust measure of dispersion, such as the interquartile range instead of $\hat{\sigma}$ in (2.32), i.e.,

$$\hat{h} = 0.79 IQR_{\circ} n^{-1/5} = 1.06\frac{IQR_{\circ}}{1.349} n^{-1/5}, \tag{2.36}$$

with $IQR_{\circ}$ the estimated circular inter-quartile range defined in (2.13).

The other proposed estimated smoothing parameter is (Silverman, 1986, pp. 48)

$$\hat{h} = 0.9\hat{\sigma}n^{-1/5}. \tag{2.37}$$

In this case, either the square root of the circular variance $s_\circ$, or the estimated circular inter-quartile range $\frac{IQR_\circ}{1.349}$ can be used as the estimated measure of dispersion $\hat{\sigma}$. In Chapter 4, the details will be provided of all the estimated smoothing parameters that are used in the empirical study.

The final section of this chapter briefly highlights some techniques that can be applied when estimating the strength of the pulsed signal p of a pulsar light curve. Since it is not the primary aim of this thesis to develop a new estimation technique for the strength of the pulsed signal p, several known and existing techniques are discussed.

2.5 Nonparametric estimation of the minimum value of a density function

The estimation of the strength of the pulsed signal p is an important topic in Astrophysics, as mentioned in Section 1.5 in Chapter 1. The main aim of this thesis, however, is to identify the global minimum and/or a support region over which the light curve reaches this minimum level $1 - p$. Although the estimation of the off-pulse interval of a pulsar light curve is discussed in the next chapter, a brief overview on the estimation of the strength of the pulsed signal p, is provided. This topic has been studied by Loots (1995) and Swanepoel, de Beer & Loots (1996).

The unknown periodic density function (or light curve) $f(\theta)$ of the folded (modulo 1) arrival times can be presented as

$$f(\theta) = 1 - p + pf_s(\theta), \qquad 0 \leq \theta \leq 1, \tag{2.38}$$

where $0 \leq p \leq 1$ is the unknown *strength* of the periodic (or pulsed) signal and $f_s(\theta)$ is the unknown *source function* that characterises the radiation pattern of the source. Lorimer & Kramer (2005, p. 201) proposed a similar model for the discretely sampled profile $f(\theta)$ when determining the time of arrivals through a process of cross-correlating the observed profile with a so-called high signal to noise *template* profile $(f_s(\theta))$.

Emadzadeh & Speyer (2010) utilised the same model, and referred to $f(\theta)$ as the overall rate function, $1-p$ as the effective background arrival rate, p as the source arrival rate and $f_s(\theta)$ as the detected phase. Another reference to such a model can be found in Ray et al. (2011), where the pulse profile is described as the sum of a constant background component and a small number of Gaussian peaks. It is therefore not an uncommon model to use in the modelling of the light curve. It is also assumed that $f_s(\theta)$ is a periodic function defined on the phase interval $[0, 1)$ (or wrapped to $[0, 1)$). Furthermore, it is usually assumed that

$$\min_{0\leq\theta\leq1} f_s(\theta) = 0.$$

Hence, the estimation of $1 - p$ can be reduced to the estimation of the minimum value of f:

$$1 - \hat{p}_n := \min_{\theta} \widehat{f(\theta)},$$

where $\widehat{f(\theta)}$ is some estimator of f on the circle (histogram or kernel density estimator). However, since $0 \leq p \leq 1$, we suggest to estimate p by

$$\hat{p}_n = \max\{0, 1 - \hat{f}_{n,h}(\theta)\},$$

where $\hat{f}_{n,h}(\theta)$ is the circular kernel density estimator.

The light curves of several γ-ray pulsars illustrated in Figures 2.12 – 2.19 are useful to highlight the above concepts for an improved understanding of the model. The figures show the photon counts (on the y-axis) over two phases of the Pulsar (x-axis) binned in a histogram format. In each of the figures, the estimated light curve $f(\theta)$ is sub-divided into two parts. Below the dashed horizontal line $(1-\hat{p}_n)$, the estimated background or noise is illustrated. Above the dashed line, the estimated source function $\hat{f}_s(\theta)$ is displayed. It is evident from Figures 2.12 – 2.14 that all the light curves are unimodal in nature (only one peak per phase). However, pulsars may also have bimodal light curves, as illustrated in Figures 2.15 – 2.19.

Remark: Another estimator of the strength of the pulsed signal p was introduced and studied by Swanepoel et al. (1996), which contains an extensive discussion on the estimation that is based on the histogram representation of the pulsar light curve.

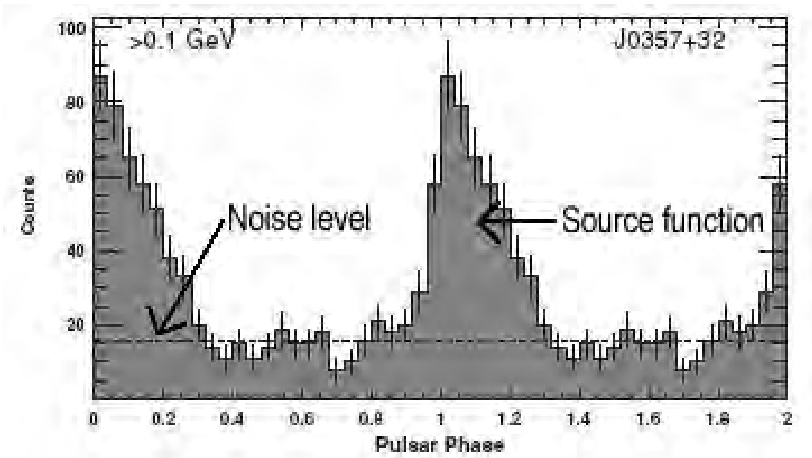

Figure 2.12: *Unimodal light curve of Pulsar J0357+32 as extracted from the FERMI catalogue (Abdo et al., 2010f).*

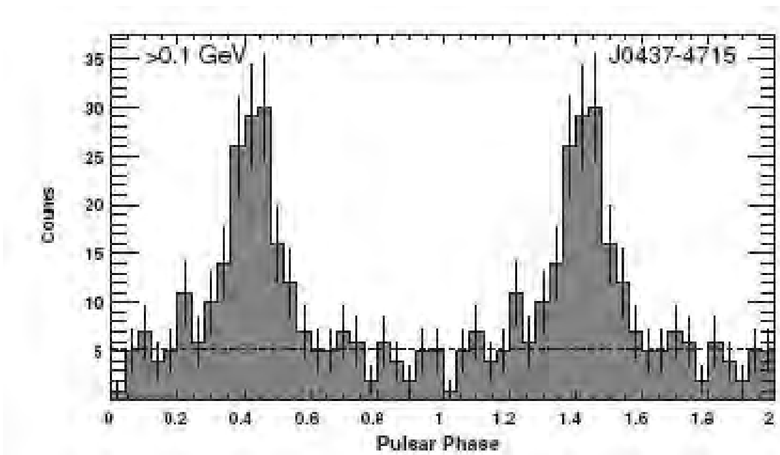

Figure 2.13: *Unimodal light curve of Pulsar J0437-4715 as extracted from the FERMI catalogue (Abdo et al., 2010f).*

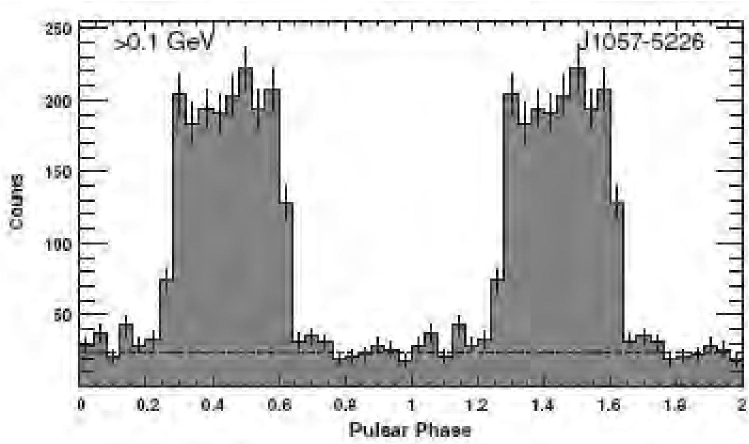

Figure 2.14: *Unimodal light curve of Pulsar J1057-5226 as extracted from the FERMI catalogue (Abdo et al., 2010f).*

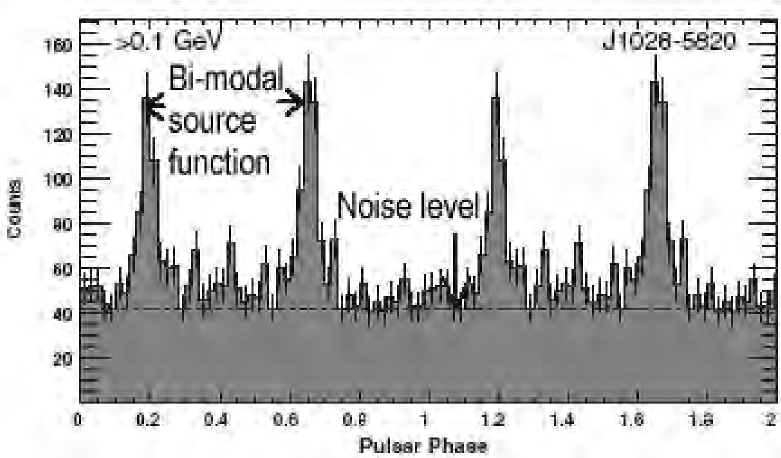

Figure 2.15: *Bimodal light curve of Pulsars J1028-5820 as extracted from the FERMI catalogue (Abdo et al., 2010f).*

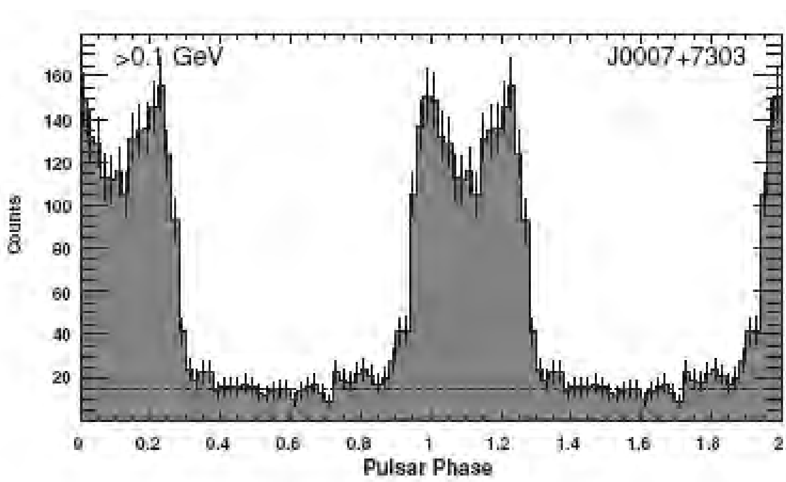

Figure 2.16: *Bimodal light curve of Pulsar J0007+7303 as extracted from the FERMI catalogue (Abdo et al., 2010f).*

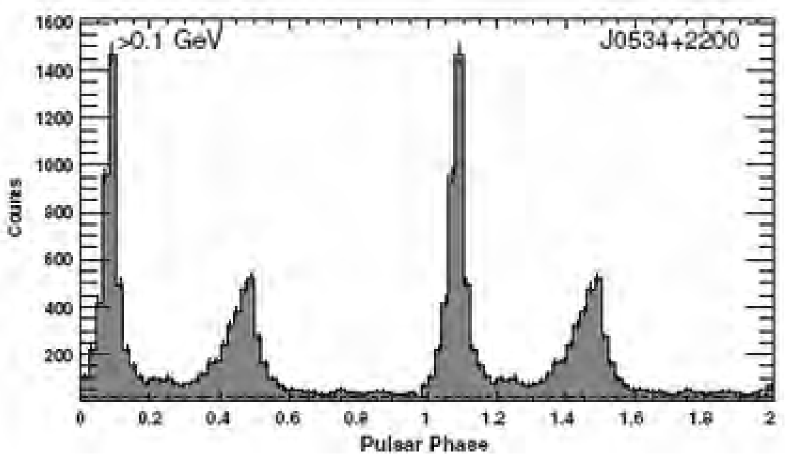

Figure 2.17: *Bimodal light curve of Pulsars J0534+2200 (also known as Crab) as extracted from the FERMI catalogue (Abdo et al., 2010f).*

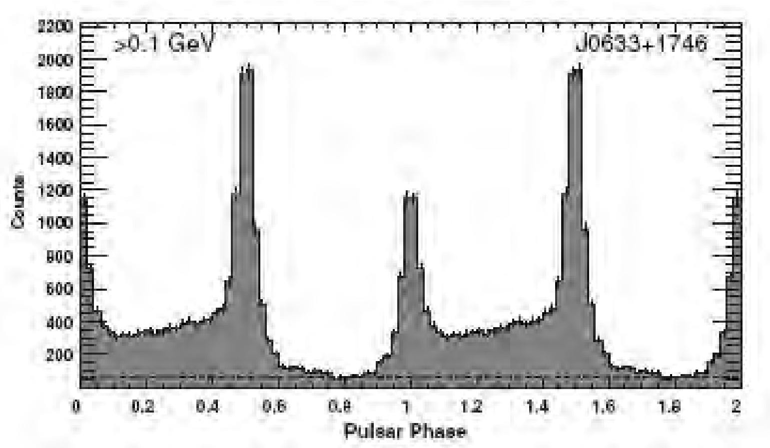

Figure 2.18: *Bimodal light curve of Pulsars J0633+1746 (also known as Geminga) as extracted from the FERMI catalogue (Abdo et al., 2010f).*

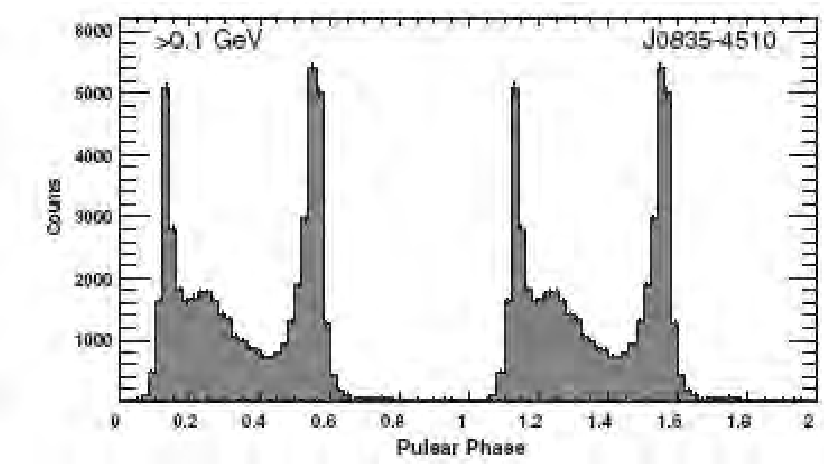

Figure 2.19: *Bimodal light curve of Pulsars J0835-4510 (also known as Vela) as extracted from the FERMI catalogue (Abdo et al., 2010f).*

Chapter 3

Estimation of the off-pulse interval(s)

3.1 Introduction

As mentioned throughout, the main aim is to estimate the endpoints of the interval(s) on which the unknown source function $f_s(\theta)$ is zero. Following from the discussion in the final section of Chapter 2, suppose $p = 1$ in (2.38), implying that no noise component is present in the light curve $f(\theta)$. If this is the case, then the off-pulse interval $[a, b]$ (as described in Section 1.6) can easily be estimated. To illustrate this, consider the following notation.

Let $\theta_1, \theta_2, \ldots, \theta_n$ be a continuous random sample and denote the order statistics by

$$0 \leq \theta_{(1)} \leq \theta_{(2)} \leq \cdots \leq \theta_{(n)} \leq 2\pi.$$

The arc lengths between adjacent observations are defined by Mardia & Jupp (2000) as:

$$T_i = \theta_{(i+1)} - \theta_{(i)},\ 1 \leq i \leq n-1; \qquad T_n = 2\pi - (\theta_{(n)} - \theta_{(1)}).$$

The maximal arc length between two adjacent observations is then defined by

$$T_N = \max_{1 \leq i \leq n} T_i.$$

From this maximal arc length, it is possible to obtain a nonparametric estimator for the interval $[a, b]$:

$$\hat{a} = \theta_{(N)}; \qquad \hat{b} = \theta_{(N+1)}. \tag{3.1}$$

Now, suppose that not only a single off-pulse interval exists, but that the light curve is of bimodal nature. A similar argument is then followed to obtain estimators for the two minimum intervals. In fact, the estimation of the first minimum interval is a replication of the above described technique, but the estimation of the second interval requires some explanation. Following the estimation of the first minimum interval, define the second order maximal arc length by

$$T_M = \max_{\substack{1 \leq i \leq n \\ i \neq N}} T_i.$$

From this second order maximal arc length, it is possible to obtain an estimator for the second off-pulse interval $[c, d]$:

$$\hat{c} = \theta_{(M)}; \qquad \hat{d} = \theta_{(M+1)}. \tag{3.2}$$

Remark: *For typical astrophysical data, noise is always present. Therefore, the sequential estimation technique developed in the next section will take this into account.*

3.2 A new sequential method to estimate the off-pulse interval (SOPIE)

The remainder of this thesis will be concerned with estimating the off-pulse interval $[a, b]$ in the presence of noise. Broadly speaking, this technique is based in a sequential way on the P-values of goodness-of-fit tests for the uniform distribution (see, e.g., D'Agostino & Stephens (1986) to name one reference). To be more specific, the well-known Kolmogorov-Smirnov, Cramér-von Mises, Anderson-Darling and Rayleigh test statistics will be applied sequentially on **subintervals** of $[0, 1]$.

The precise procedure will be provided shortly. Firstly, a result will be stated and proved that provides the theoretical justification for the proposed procedure. For this, some further notation needs to be introduced:

Let $\theta_1, \theta_2, \ldots, \theta_n$ be a sample of independent and identically distributed (i.i.d.) **uniform** random variables on the interval $[0, 1]$ with corresponding order statistics

$$\theta_{(1)} \leq \theta_{(2)} \leq \cdots \leq \theta_{(n)}.$$

Suppose that r and s are integers such that $1 \leq r \leq s-1 \leq n-2$. Denote by $f^*(x_{r+1}, \ldots, x_s | x_r, x_{s+1})$ the joint conditional density function of $\theta_{(r+1)}, \ldots, \theta_{(s)}$ given that $\theta_{(r)} = x_r$ and $\theta_{(s+1)} = x_{s+1}$. Also, let $g^*(x_r, \ldots, x_{s+1})$ denote the joint density function of $\theta_{(r)}, \ldots, \theta_{(s+1)}$ and $h^*(x_r, x_{s+1})$ the joint density function of $\theta_{(r)}$ and $\theta_{(s+1)}$.

Proposition

$$f^*(x_{r+1}, \ldots, x_s | x_r, x_{s+1}) = \begin{cases} \frac{(s-r)!}{(x_{s+1}-x_r)^{s-r}}\ , & \text{for } x_r \leq x_{r+1} \leq \cdots \leq x_s \leq x_{s+1}, \\ 0\,, & \text{elsewhere.} \end{cases}$$

Proof

It is well known from the theory of order statistics (see, e.g., the books by Karlin (1966) and David (1981) and the references therein) that

$$g^*(x_r, \ldots, x_{s+1}) = \frac{n!}{(r-1)!(n-s-1)!}\ x_r^{r-1}(1-x_{s+1})^{n-s-1}\ ,$$

and

$$h^*(x_r, x_{s+1}) = \frac{n!}{(r-1)!(s-r)!(n-s-1)!}\ x_r^{r-1}(x_{s+1}-x_r)^{s-r}(1-x_{s+1})^{n-s-1}\ .$$

The proof now follows directly from these two expressions and the fact that

$$f^*(x_{r+1}, \ldots, x_s | x_r, x_{s+1}) := \frac{g^*(x_r, \ldots, x_{s+1})}{h^*(x_r, x_{s+1})}.$$

Remark
The result stated in the Proposition can be interpreted as follows:
"The joint **conditional** distribution of $\theta_{(r+1)}, \ldots, \theta_{(s)}$, given that $\theta_{(r)} = x_r$ and $\theta_{(s+1)} = x_{s+1}$, is the same as the joint **unconditional** distribution of $s-r$ order statistics corresponding to an i.i.d. sample of size $s-r$ from the uniform distribution on the interval $[x_r, x_{s+1}]$."

The following algorithm is now proposed to estimate the off-pulse interval(s) of a source function originating from a pulsar. This suggested procedure will henceforth be denoted by SOPIE (Sequential Off-Pulse Interval Estimation).

1. Calculate the point where the kernel density estimator $\hat{f}_{n,h}(\theta)$ attains its *global* minimum value:
$$x_1 := \arg\min_{\theta} \hat{f}_{n,h}(\theta).$$
Also, determine the next m *local* minimum points, i.e. for $i = 2, 3, \ldots, m$, let
$$x_i := \arg \min_{\theta \notin \{x_1,\ldots,x_{i-1}\}} \hat{f}_{n,h}(\theta).$$

   Remark about the choice of m

   By utilizing the theory presented in Section 2.5 regarding the nonparametric estimation of the minimum value of a density function, the user must ensure that each of the minima points selected is in close proximity of the estimated background or noise level $(1 - \hat{p}_n)$. Typically (see Section 2.5), $\hat{p}_n$ is calculated as follows:
$$\hat{p}_n = \max\{0, 1 - \min_{0\leq\theta\leq 1} \hat{f}_{n,h}(\theta)\}.$$
In other words, choose m so that, for each $i = 2, 3, \ldots, m$,
$$|\hat{f}_{n,h}(x_i) - (1 - \hat{p}_n)| \leq \epsilon\,, \tag{3.3}$$
for some small value of $\epsilon > 0$.

2. For each of the selected minima points $x_1, x_2, \ldots, x_m$, find the nearest ordered observation to this point:

   Let
$$k_i := \arg \min_{1\leq j\leq n} |\theta_{(j)} - x_i|, \qquad i = 1, 2, \ldots, m\,,$$
then the nearest observation to x_i will be $\theta_{(k_i)}$, $i = 1, 2, \ldots, m$.

3. Duplicate the initial ordered observations between 0 and 1 to the left of 0 and to the right of 1. That is, define

   - $\theta_{(-n+i)} = \theta_{(i)} - 1$ for $i = 1, 2, \ldots, n$.
   - $\theta_{(n+i)} = \theta_{(i)} + 1$ for $i = 1, 2, \ldots, n$.

   The result is the ordered observation $\theta_{(-n+1)}, \theta_{(-n+2)}, \ldots, \theta_{(0)}, \theta_{(1)}, \ldots, \theta_{(n)}, \theta_{(n+1)}, \ldots, \theta_{(2n)} \in [-1; 2]$.

4. For some specified integer g, define
$$n_g := \lfloor\frac{n-1}{g}\rfloor \text{ and } \rho_g := \frac{n-1}{g}.$$
Evaluate ρ_g to determine whether it is an integer or not.

- If ρ_g is integer-valued, firstly consider k_1 (corresponding to x_1) and define for each $\ell = 1, 2, \ldots, n_g$ the following set of observations:

$$\chi_\ell := \{\theta_{(k_1)}, \theta_{(k_1+1)}, \ldots, \theta_{(k_1+\ell g)}, \theta_{(k_1+\ell g+1)}\}.$$

- If ρ_g is not an integer, consider k_1 (corresponding to x_1) and define

$$\chi_\ell^0 := \begin{cases} \chi_\ell, & \ell = 1, 2, \ldots, n_g, \\ \{\theta_{(k_1)}, \theta_{(k_1+1)}, \theta_{(k_1+2)}, \ldots, \theta_{(k_1+n)}\}, & \ell = n_g + 1\,. \end{cases}$$

5. Let $T_n(\chi_\ell)$ (or $T_n(\chi_\ell^0)$) be a given test statistic to test *uniformity* based on the observations in the set χ_ℓ (or χ_ℓ^0), when ρ_g is an integer (or not). Denote by P_ℓ the corresponding P-value. Calculate P_ℓ sequentially for $\ell = 1, ..., N$, where N is a stopping time defined by

$$N := \text{the smallest integer}\, j,\ \text{with}\, 1 \le j \le n_g - r + 1 \text{ for which } \max_{j \le \ell \le j+r-1} P_\ell \le \alpha\,,$$

if ρ_g is an integer, or

$$N := \text{the smallest integer}\, j,\ \text{with}\, 1 \le j \le n_g - r + 2 \text{ for which } \max_{j \le \ell \le j+r-1} P_\ell \le \alpha\,,$$

if ρ_g is not an integer. *Note that it is possible that N does not exist.*

Here α is the so-called significance level which is usually taken as $\alpha = 1\%$, 5% or 10% in the statistical literature when hypothesis testing is conducted. Furthermore, r is an integer representing an applicable tuning parameter.

6. The right endpoint (boundary) of the unknown off-pulse interval $I = [a, b]$ is then estimated (when using x_1) by

$$\hat{b}_1 := \begin{cases} \theta_{(k_1+Ng+1)}, & \text{if } N \text{ exists and } k_1 + Ng + 1 \le n, \\ \theta_{(k_1+Ng-n+1)}, & \text{if } N \text{ exists and } k_1 + Ng + 1 > n\,, \\ \theta_{(k_1)}, & \text{if } N \text{ does not exist.} \end{cases} \tag{3.4}$$

7. Repeat steps 4 - 6 for $x_2, x_3, \ldots, x_m$ to obtain $\hat{b}_2, \hat{b}_3, \ldots, \hat{b}_m$.

8. The final estimate of the right endpoint (boundary) of the unknown off-pulse interval $I = [a, b]$ is then given by

$$\hat{b} := \frac{1}{m} \sum_{j=1}^{m} \hat{b}_j\,.$$

Next, consider the estimation of a of the unknown off-pulse interval $I = [a, b]$. The procedure is similar to the procedure to estimate b, but for completeness, the procedure is provided. Therefore, follow a similar procedure as the above, but replace steps 4 - 8 with the following:

4' For some specified integer g, define

$$n_g := \lfloor \frac{n-1}{g} \rfloor \text{ and } \rho_g := \frac{n-1}{g}.$$

Evaluate ρ_g to determine whether it is an integer or not.

- If ρ_g is integer-valued, we first consider k_1 (corresponding to x_1) and define for each $\ell = 1, 2, \ldots, n_g$ the following set of observations:

$$\mathcal{X}_\ell := \{\theta_{(k_1-\ell g-1)}, \theta_{(k_1-\ell g)}, \ldots, \theta_{(k_1-1)}, \theta_{(k_1)}\}.$$

- If ρ_g is not an integer, consider k_1 (corresponding to x_1) and define

$$\mathcal{X}_\ell^0 := \begin{cases} \mathcal{X}_\ell, & \ell = 1, 2, \ldots, n_g, \\ \{\theta_{(k_1-n)}, \theta_{(k_1-n+1)}, \theta_{(k_1-n+2)}, \ldots, \theta_{(k_1)}\}, & \ell = n_g + 1\,. \end{cases}$$

5' Let $T_n(\mathcal{X}_\ell)$ (or $T_n(\mathcal{X}_\ell^0)$) be a given test statistic to test *uniformity* based on the observations in the set $\mathcal{X}_\ell$ (or $\mathcal{X}_\ell^0$), when ρ_g is an integer (or not). Denote by P_ℓ the corresponding P-value. Calculate P_ℓ sequentially for $\ell = 1, ..., N$, where N is a stopping time defined by

$$N := \text{the smallest integer}\, j,\ \text{with}\, 1 \le j \le n_g - r + 1 \text{ for which } \max_{j \le \ell \le j+r-1} P_\ell \le \alpha\,,$$

if ρ_g is an integer, or

$$N := \text{the smallest integer}\, j,\ \text{with}\, 1 \le j \le n_g - r + 2 \text{ for which } \max_{j \le \ell \le j+r-1} P_\ell \le \alpha\,,$$

if ρ_g is not an integer.

6' The left endpoint (boundary) of the unknown off-pulse interval $I = [a, b]$ is then estimated (when using x_1) by

$$\hat{a}_1 := \begin{cases} \theta_{(k_1-Ng-1)}, & \text{if } N \text{ exists and } k_1 - Ng - 1 \ge 1, \\ \theta_{(k_1-Ng+n-1)}, & \text{if } N \text{ exists and } k_1 - Ng - 1 < 1\,, \\ \theta_{(k_1)}, & \text{if } N \text{ does not exist.} \end{cases} \tag{3.5}$$

7' Repeat Steps 4'- 6' for $x_2, x_3, \ldots, x_m$ to obtain $\hat{a}_2, \hat{a}_3, \ldots, \hat{a}_m$.

8' The final estimate of the left endpoint (boundary) of the unknown off-pulse interval $I = [a, b]$ is then given by

$$\hat{a} := \frac{1}{m} \sum_{j=1}^{m} \hat{a}_j\,.$$

Remark: Estimation of a second off-pulse region

It must be noted that there exists a possibility of more than one off-pulse region or interval. The aim of this thesis is only to provide a thorough methodology for the estimation of one such off-pulse interval of a source function originating from a pulsar. From literature such as the Fermi Large Area Telescope catalogue (Abdo et al., 2010f), it is also evident that only a single off-pulse interval is defined for the LAT-detected Pulsars in the catalogue.

Nevertheless, the method described above can be adapted to accommodate a situation where more than one off-pulse interval exists. In order to make the above method fit for such a purpose, apply the following, rather simple, adjustments.

- In the process of executing step 1, establish whether one can identify two subsets of minima points, with each subset belonging to two **disjoint** intervals, e.g., say several minima points can be grouped in a subset $A \in [z_1; z_2]$, and several other minima point can be grouped in another subset $B \in [z_3; z_4]$, with $[z_1; z_2] \cap [z_3; z_4] = \emptyset$. The user must ascertain that, for both subsets, each minima point is still in close proximity to the estimated noise level $(1 - \hat{p}_n)$, as remarked above. If this is possible, it is an initial indication that two off-pulse intervals may exist.

- The proposed method can now be applied just as presented above, firstly on the one subset of minima points A, and then on the other subset B.

- In conclusion, the user must verify that the individual off-pulse intervals, calculated for each of the subsets of minima points, are disjoint in nature. If this is not the case, only one off-pulse interval exists.

3.3 Goodness-of-fit tests for the uniform distribution

The algorithm specified in Section 3.2 for estimating a and b is based on tests for uniformity. Throughout this thesis, the Kolmogorov-Smirnov, Cramér-von-Mises, Anderson-Darling and Rayleigh goodness-of-fit tests will be applied (see, e.g., D'Agostino & Stephens (1986) and Mardia & Jupp (2000)). Each of these test statistics is briefly discussed in the next subsections.

3.3.1 Kolmogorov-Smirnov test for uniformity

The test statistic for the Kolmogorov-Smirnov goodness-of-fit test is given (Stephens, 1970) for $i = 1, 2, \ldots, m$ by

$$T_{n,i}^{D^+}(\mathcal{X}_\ell) := \max_{1 \leq j \leq \ell g} \left(\frac{j}{\ell g} - \frac{\theta_{(k_i+j)} - \theta_{(k_i)}}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}} \right);$$

$$T_{n,i}^{D^-}(\mathcal{X}_\ell) := \max_{1 \leq j \leq \ell g} \left(\frac{\theta_{(k_i+j)} - \theta_{(k_i)}}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}} - \frac{j-1}{\ell g} \right);$$

$$T_{n,i}^{KS}(\mathcal{X}_\ell) := \max\left(T_{n,i}^{D^+}(\mathcal{X}_\ell), T_{n,i}^{D^-}(\mathcal{X}_\ell) \right),$$

for $\ell = 1, 2, \ldots, n_g$. $T_{n,i}^{KS}(\mathcal{X}_\ell^0)$ is defined similarly for $\ell = 1, 2, \ldots, n_g, n_g + 1$.

For calculating the P-values, the method of Marsaglia, Tsang & Wang (2003) is used. This method provides an accuracy of up to 15 digits for sample sizes ranging from 2 to at least 16000.

3.3.2 Cramér-von Mises test for uniformity

The Cramér-von Mises goodness-of-fit test is a well-known and widely used non-parametric two-sample test. The Cramér-von Mises goodness-of-fit test was introduced by Cramér (1928) and von Mises (1931), while Anderson (1962) studied it as an alternative two-sample goodness-of-fit test. The test statistic is given by Stephens (1970) for $i = 1, 2, \ldots, m$ by

$$T_{n,i}^{CvM}(\mathcal{X}_\ell) := \sum_{j=1}^{\ell g} \left(\frac{\theta_{(k_i+j)} - \theta_{(k_i)}}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}} - \frac{j - \frac{1}{2}}{\ell g} \right)^2 + \frac{1}{12 \ell g},$$

for $\ell = 1, 2, \ldots, n_g$. $T_{n,i}^{CvM}(\mathcal{X}_\ell^0)$ is defined similarly for $\ell = 1, 2, \ldots, n_g, n_g + 1$.

For calculating the P-values, the modified version of the Cramér-von Mises statistic is used, which can be calculated as follows:

$$W^*_{n,i} = (T^{CvM}_{n,i} - \frac{0.4}{\ell g} + \frac{0.6}{(\ell g)^2})(1 + \frac{1}{\ell g})\,.$$

For the calculation of the P-values of the test, the program of Xiao, Gordon & Yakovlev (2007) was investigated. Due to the iterative nature of the proposed procedure and large sample sizes, too much computer time was used to calculate the exact P-values. Discrete P-values are calculated instead for the upper and lower tail according to Stephens (1970), with linear interpolation between discrete P-value levels.

3.3.3 Anderson-Darling test for uniformity

The Anderson-Darling test statistic is given (Stephens, 1970) for $i = 1, 2, \ldots, m$ by

$$T^{AD}_{n,i}(\mathcal{X}_\ell) := -\ell g - \frac{1}{\ell g}\sum_{j=1}^{\ell g}(2j-1)\ln\left(\frac{\theta_{(k_i+j)} - \theta_{(k_i)}}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}}\left(1 - \frac{\theta_{(k_i+\ell g-j+1)} - \theta_{(k_i)}}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}}\right)\right),$$

for $\ell = 1, 2, \ldots, n_g$. $T^{AD}_{n,i}(\mathcal{X}^0_\ell)$ is defined similarly for $\ell = 1, 2, \ldots, n_g, n_g + 1$. For the P-values, the two-term recursion method of Marsaglia & Marsaglia (2004) is used for accuracy to the fourth digit in terms of the approximation of the null distribution.

3.3.4 Rayleigh test for uniformity

The Rayleigh goodness-of-fit test is a test for uniformity on the circle proposed in 1894 by Lord Rayleigh (also known by the name of J. Strutt). The test is based on the sample mean resultant length $\bar{R}$ defined in (2.5). The relevant test statistic is $n\bar{R}^2$ (Mardia & Jupp, 2000) and the P-values are calculated with the approximation proposed by Greenwood & Durand (1955). As mentioned in the first paragraph of Section 3.2, SOPIE is based on the sequential application of goodness-of-fit tests on *subintervals* of $[0, 1]$. By contrast, the Rayleigh goodness-of-fit test was developed to test uniformity of sample observations *on the entire circle.* It is, therefore, essential to *scale* the observations when SOPIE is applied to each subinterval, before calculating the resultant length $\bar{R}$.

Define the sample observations in the subinterval $[\theta_{(k_i)}, \theta_{(k_i+\ell g+1)}]$, for $i = 1, 2, \ldots, m$, by

$$\mathcal{X}_{\ell,i} := \{\theta_{(k_i)}, \theta_{(k_i+1)}, \ldots, \theta_{(k_i+\ell g)}, \theta_{(k_i+\ell g+1)}\},$$

for $\ell = 1, 2, \ldots, n_g$. $\mathcal{X}^0_{\ell,i}$ is defined similarly for $\ell = 1, 2, \ldots, n_g, n_g + 1$. The *scaled* observations used in the calculation of $\bar{R}$, are given by

$$\mathcal{X}^*_{\ell,i} := \left\{\frac{2\pi(\theta_{(k_i)} - \theta_{(k_i)})}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}}, \frac{2\pi(\theta_{(k_i+1)} - \theta_{(k_i)})}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}}, \ldots, \frac{2\pi(\theta_{(k_i+\ell g)} - \theta_{(k_i)})}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}}, \frac{2\pi(\theta_{(k_i+\ell g+1)} - \theta_{(k_i)})}{\theta_{(k_i+\ell g+1)} - \theta_{(k_i)}}\right\}.$$

$\mathcal{X}^{0*}_{\ell,i}$ is defined similarly for $\ell = 1, 2, \ldots, n_g, n_g + 1$.

Remark

Note that, in the notation of the Proposition in Section 3.2 which provides the theoretical justification for SOPIE, $\theta_{(k_i)}$ and $\theta_{(k_i+\ell g+1)}$ play the role of x_r and x_{s+1}, respectively.

Chapter 4

Empirical studies

4.1 Introduction

This chapter reports the results of extended simulation studies performed in order to evaluate the performance of the proposed sequential method (SOPIE) to estimate the off-pulse interval of a source function originating from a pulsar. SOPIE is not only tested on simulated data from different target populations using various tuning parameter configurations, but also on data from the field of Astrophysics (see Chapter 5). All of the results obtained in the simulation study can be found on the DVD-ROM included with the thesis. Section 4.2 starts with a discussion of the different classes of probability density functions used in the simulation study. Section 4.3 then deals with the simulation design and the various tuning parameter configurations that influence the estimated off-pulse interval. The measures used to assess the performance of SOPIE are also discussed. Sections 4.4 – 4.6 contain the simulation study results of both the von Mises and triangular study populations used to evaluate SOPIE. The chapter concludes in Section 4.7 with a discussion of the tuning parameter configurations that consistently yielded accurate estimation, and can therefore be seen as the recommendations for the analysis of the pulsar data from the field of Astrophysics in Chapter 5.

4.2 Target densities

This chapter reports the results of a series of Monte Carlo studies performed to investigate the accuracy and consistency of SOPIE. This section discusses the underlying probability density functions that are used to generate the data for the simulation study. The data are generated from two distinct classes of density functions, namely the von Mises probability density function and the triangular probability density function. For each of these classes, several different study populations are generated based on different parameter configurations for each of the density functions.

4.2.1 Von Mises distribution

The von Mises distribution is the first class of probability density functions considered in the simulation design. The reader can find a detailed exposition of the von Mises distribution in Section 2.4.4. For completeness, some essential details of the von Mises distribution are given.

A random variable X is said to have a von Mises distribution if its probability density function is given by

$$f(\theta) := \frac{1}{2\pi I_o(\kappa)} e^{\kappa\cos(\theta-\mu)}, \quad 0 < \theta \leq 2\pi, \quad \kappa > 0, \quad 0 \leq \mu < 2\pi,$$

where $I_o(\kappa)$ is the modified Bessel function of the first kind and order zero, i.e.,

$$I_o(\kappa) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{1}{2}\kappa\right)^{2r}.$$

The parameters μ and κ are known as the mean direction and concentration parameters, respectively. Also note that $f(0) = f(2\pi)$. For statistical inference on the circle, the von Mises distribution is comparable to the normal distribution on the line and exhibits similar properties, i.e., it is unimodal and symmetric about $\theta = \mu$. Furthermore, the mode is at $\theta = \mu$ and the antimode is at $\theta = \mu - \pi$. Finally, as $\kappa \to 0$, the von Mises distribution converges to the uniform distribution, and as $\kappa \to \infty$, the distribution tends to the point distribution concentrated in the direction μ. For an extended discussion of the von Mises distribution, as well as its relation with other distributions, the reader is referred to Mardia (1972), Fisher (1993) and Mardia & Jupp (2000).

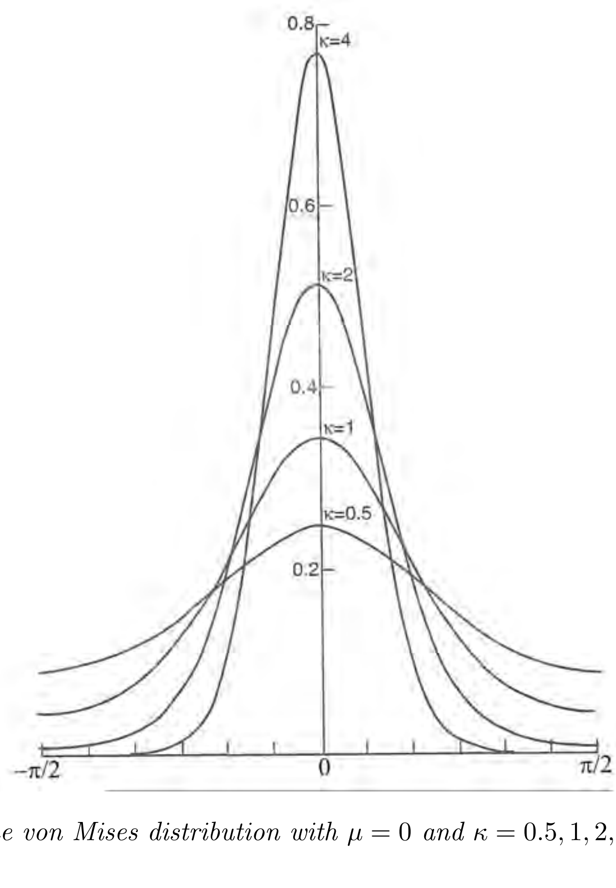

Figure 4.1: *Density of the von Mises distribution with $\mu = 0$ and $\kappa = 0.5, 1, 2, 4$ (Mardia & Jupp, 2000).*

Figure 4.1 illustrates several von Mises distributions with different values of κ. It can be seen that the von Mises densities $f(\theta)$ are never equal to zero, i.e., $f(\theta) \neq 0 \;\forall\; \theta$.

Since SOPIE is developed to find the off-pulse interval, it is essential that random variates are obtained from a distribution where $f(\theta) = 0$ for θ belonging to some finite interval(s). Furthermore, the considered densities are limited to the interval $[0, 1]$, and therefore the following von Mises density is used:

$$f_{vm}(x) := \begin{cases} \frac{1}{I_o(\kappa)} e^{\kappa \cos[2\pi(x-\mu)]}, & 0 \leq x \leq 1, \quad \kappa > 0, \quad 0 \leq \mu < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The process to create such a distribution – where $f(\theta) = 0$ for θ belonging to some interval(s) – from the above-mentioned von Mises distribution, is to remove some density in the tails of the distribution in the intervals $[0, c]$ and $[1 - c, 1]$ for some value c. Furthermore, the distribution must be standardised to ensure that it is a density function. Figure 4.2 is a depiction of this transformation (for $\mu = 0.5$). The coloured part of the density on the left graph must be removed in order to end up with a density similar to the graph on the right.

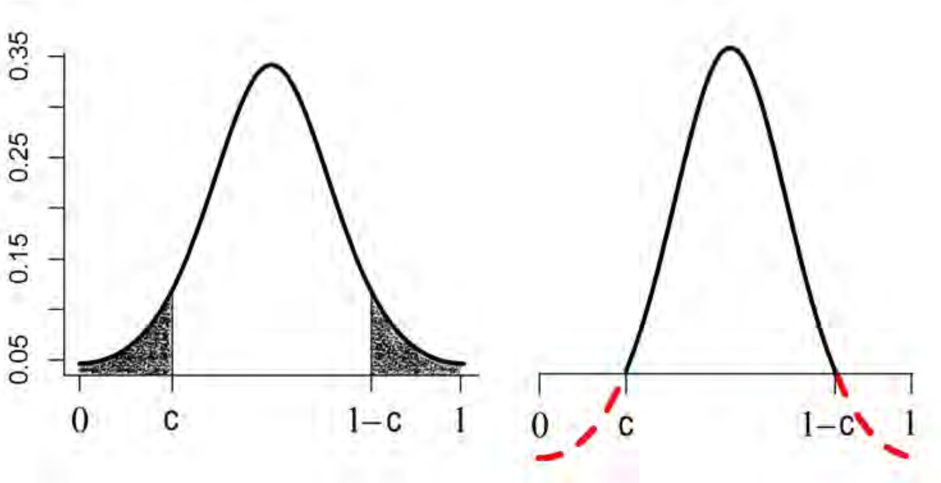

Figure 4.2: *Scaling of the von Mises density function with $\mu = 0.5$ and $\kappa = 1$.*

The dashed lines reflect the part of the original von Mises distribution that is no longer used. Mathematically, the density must be scaled from a support of $[0, 1]$ to support on $[c, 1 - c]$. Thus to construct such a scaled von Mises density function $f_{vms}(x)$ with $\mu = 0.5$, the following holds for $f_{vm}(x) = \frac{1}{I_o(\kappa)} e^{\kappa \cos[2\pi(x-0.5)]}$:

$$\begin{aligned} f_{vms}(x) &= \frac{f_{vm}(x) - f_{vm}(c)}{\int\limits_c^{1-c} (f_{vm}(x) - f_{vm}(c)) \, dx} \\ &= \frac{f_{vm}(x) - \frac{e^{\kappa \cos(2\pi[(c-\mu)]}}{I_o(\kappa)}}{\int\limits_c^{1-c} \frac{1}{I_o(\kappa)} e^{\kappa \cos[2\pi(x-\mu)]} dx - \frac{1-2c}{I_o(\kappa)} e^{\kappa \cos[2\pi(c-\mu)]}} \\ &= \frac{f_{vm}(x) - \frac{e^{\kappa \cos[2\pi(c-\mu)]}}{I_o(\kappa)}}{F_{vm}(1-c) - F_{vm}(c) - \frac{1-2c}{I_o(\kappa)} e^{\kappa \cos[2\pi(c-\mu)]}} \end{aligned}$$

with $F_{vm}(x)$ the von Mises distribution function with $0 \leq x \leq 1$ and $\mu = 0.5$.

In order to generate random variates from the defined scaled von Mises distribution $f_{vms}(x)$, the accept-reject method (Robert & Casella, 2010) is used. This method requires only prior knowledge of the target density (in this case the density function $f_{vms}(x)$). A much simpler density g is then used to generate random variates, where only those variates are accepted that pass some test. The only two constraints imposed on the simpler density g are:

1. g and the target density f_{vms} should have compatible supports (i.e., $g(x) > 0$ when $f_{vms}(x) > 0$), and

2. there must exist a constant M with $\frac{f_{vms}(x)}{g(x)} \leq M \ \forall\ x$.

It is then possible to generate random variates from $f_{vms}(x)$ as follows:

1. Generate Y from $g(x)$ and independently generate U from an uniform density over the interval $[0, 1]$.

2. If $U \leq \frac{f_{vms}(Y)}{Mg(Y)}$, then a random variate X from $f_{vms}(x)$ is Y. If the inequality is not satisfied, discard Y and U and restart from 1.

The simpler density g is taken throughout the study as the uniform density over the interval $[0, 1]$, i.e., $g(x) = 1 \ \forall\ x$. The calculation of M is trivial, since $M \geq \max f_{vms}(x)$. In most cases, $M = \max\ f_{vms}(x)$ would suffice. Furthermore, for this specific instance of the scaled von Mises distribution, $\max f_{vms}(x)$ is always attained at $x = \mu$.

Since it is possible to generate random variates from the scaled von Mises distribution, the reader must also keep the context of the problem in mind. Recall from Section 1.5 of Chapter 1 that the light curve $f(\theta)$ is defined as an unknown density function of the folded (modulo 1) arrival times:

$$f(\theta) = 1 - p + pf_s(\theta), \quad 0 \leq \theta \leq 1\,,$$

where $p\,(0 \leq p \leq 1)$ is the unknown strength of the pulsed signal and $f_s(\theta)$ is the unknown *source function* that characterizes the radiation pattern of the source.

The random variates of the scaled von Mises distribution are thus a simulation of the source function $f_s(\theta)$. In order to satisfy the properties of $f(\theta)$, the random variates must be contaminated with uniform noise proportional to $1 - p$. Thus, if n random variates are required, then one generates $\lfloor pn \rfloor$ values from $f_{vms}(x)$ and $n - \lfloor pn \rfloor$ values from a uniform density over the interval $[0, 1]$, and combine the random variates to produce a simulated form of $f(\theta)$. Figure 4.3 is an illustration of the histogram representation of the random variates generated from the scaled von Mises distribution.

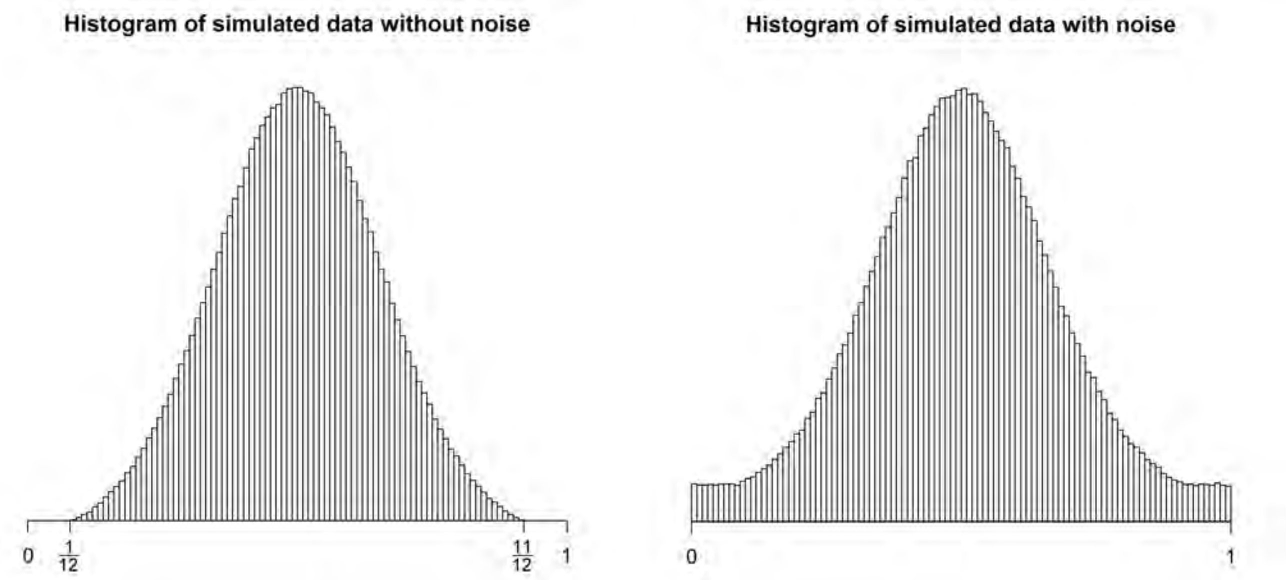

Figure 4.3: *Histograms of scaled von Mises variates ($n = 1000000$, $c = \frac{1}{12}$, $\kappa = 1$) without noise (left) and with noise $1 - p = 0.2$ (right).*

4.2.2 Triangular distribution

The second class of probability density functions used to generate different target populations to assess the performance of SOPIE, is the triangular probability density function. This density function is relatively easy to use and to simulate random variates from.

A random variable X is said to have a general triangular distribution if its probability density function is given by:

$$f(x) := \begin{cases} 0, & x < a, \\ \frac{2(x-a)}{(b-a)(c-a)}, & a \leq x \leq c, \\ \frac{2(b-x)}{(b-a)(b-c)}, & c < x \leq b, \\ 0, & b < x, \end{cases}$$

and $a < b$ and $a \leq c \leq b$. Figure 4.4 is a representation of the general triangular probability density function.

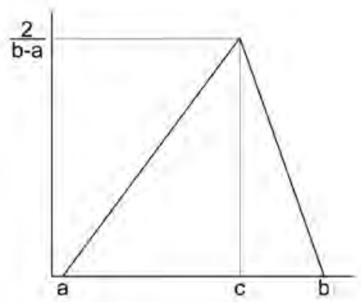

Figure 4.4: *Probability density function of the general triangular distribution.*

Since densities with support $[0,1]$ are required, the general triangular distribution must be scaled to have support on $[0,1]$, similar to the von Mises distribution (see Section 4.2.1). It is also possible to shift the distribution so that the interval over which the density function attains its minimum, is not in the tails of the distribution, but rather to the middle of the distribution. Figure 4.5 graphically represents such a scaled triangular density. Mathematically, a random variable X is said to have this scaled triangular distribution if its probability density function is given by:

$$f_{ts}(x) := \begin{cases} -\frac{\alpha}{a}\,x + \alpha, & 0 \leq x \leq a \leq b, \\ \frac{\alpha}{1-b}\,x - \frac{\alpha b}{1-b}, & b \leq x \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha = \frac{2}{a+1-b}$ when a and b are given.

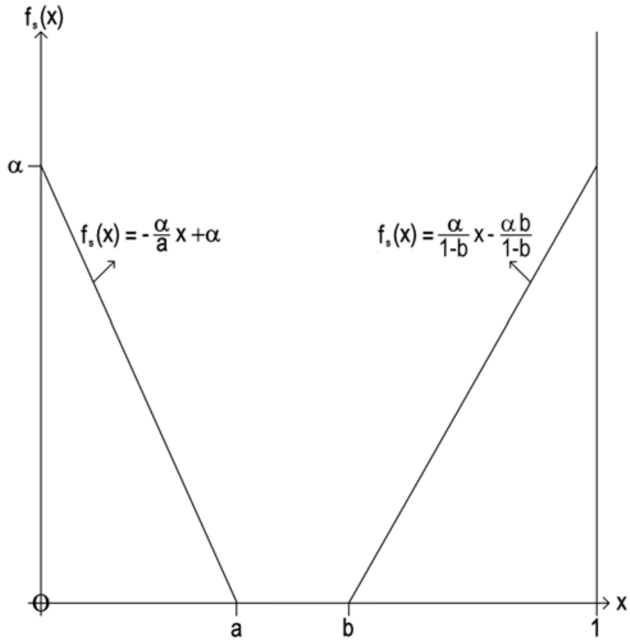

Figure 4.5: *Probability density function of the scaled triangular distribution.*

In order to generate random variates from the scaled triangular density, the inverse transformation method will be used (Robert & Casella, 2010). By applying the probability integral transformation, it is possible to transform a random variable into a uniform random variable and vice versa. The distribution function of the random variable is required though, which will now be derived.

For $0 \leq x \leq a$*:*

$$\begin{aligned} F_{ts}(x) &= \int_0^x \left(-\frac{\alpha}{a}t + \alpha\right) dt \\ &= -\frac{\alpha x^2}{2a} + \alpha x \\ &= \frac{a\alpha}{2} - \frac{\alpha}{2a}(x-a)^2 . \end{aligned}$$

For $a \leq x \leq b$:

$$F_{ts}(x) = \frac{a\alpha}{2}.$$

For $b \leq x \leq 1$:

$$\begin{aligned} F_{ts}(x) &= \frac{a\alpha}{2} + \int_b^x \left(\frac{\alpha}{1-b}(t-b)\right) dt \\ &= \frac{a\alpha}{2} + \frac{\alpha}{2(1-b)}(x-b)^2. \end{aligned}$$

The inverse transformation method then requires the calculation of the inverse of the distribution function, $F_{ts}(x)$. This can easily be done, and the results are as follows:

$$F_{ts}^{-1}(x) := \begin{cases} a - \sqrt{a^2 - \frac{2ax}{\alpha}}, & \text{when } 0 \leq x \leq \frac{a\alpha}{2}, \\ b + \sqrt{\frac{2(1-b)}{\alpha}\left(x - \frac{a\alpha}{2}\right)}, & \text{when } \frac{a\alpha}{2} < x \leq 1. \end{cases}$$

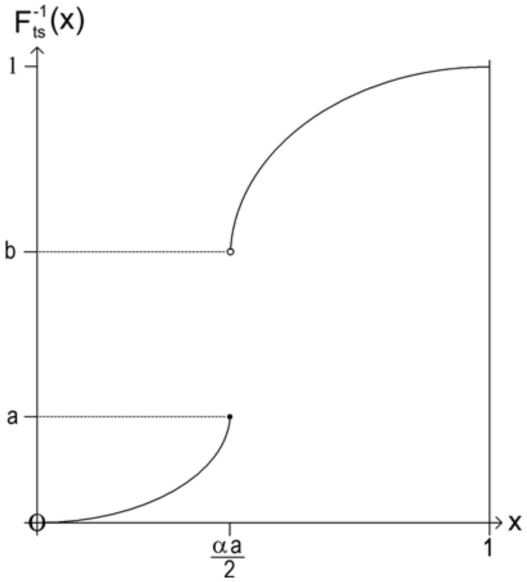

Figure 4.6: *The inverse function of the scaled triangular distribution.*

A representation of $F_{ts}^{-1}(x)$ can be found in Figure 4.6. It is now possible to generate random variates X_i, $i = 1, 2, \ldots$, from the scaled triangular distribution by using a uniform random variable on the interval $[0, 1]$ as follows:

$$X_i := \begin{cases} a - \sqrt{a^2 - \frac{2aU_i}{\alpha}}, & \text{when } 0 \leq U_i \leq \frac{a\alpha}{2}, \\ b + \sqrt{\frac{2(1-b)}{\alpha}\left(U_i - \frac{a\alpha}{2}\right)}, & \text{when } \frac{a\alpha}{2} < U_i \leq 1, \end{cases}$$

where U_i is a random variable from a uniform distribution over the interval $[0, 1]$. Equivalent to the explanation given in the final paragraphs of Section 4.2.1, the generated variates from the above procedure do not contain any random noise. It is again easy to contaminate this random variable with some uniform noise over the interval $[0, 1]$. Figure 4.7 is a histogram representation of the random variates without and with noise.

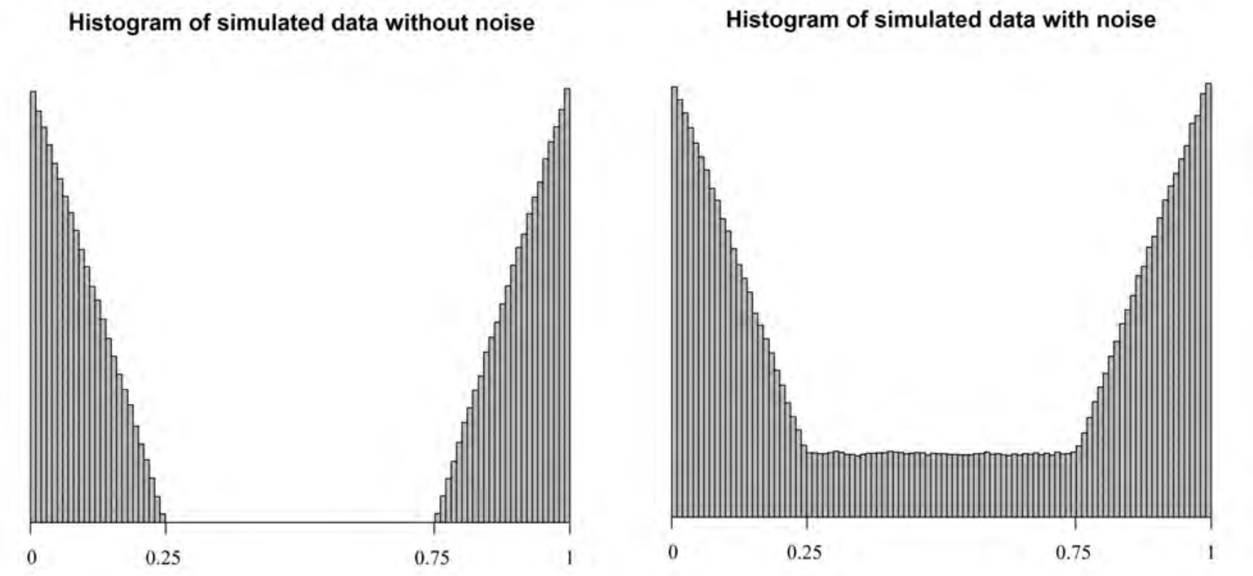

Figure 4.7: *Histograms of the scaled triangular variates ($n = 100000$, $a = \frac{1}{4}$, $b = \frac{3}{4}$) without noise (left) and with noise $1 - p = 0.4$ (right).*

4.3 Simulation design

Before commencing with the simulation study results, it is essential to give a detailed outline of the simulation design that is followed. The aim of such an outline is to facilitate the reader to easily identify and understand the different parameter configurations and variables that are used throughout the simulation study. References to the relevant sections and chapters are also given.

4.3.1 Kernel functions k

Kernel density estimation is discussed in Section 2.4, where two general representations of the kernel density estimator on the circle are given in (2.21) and (2.23). It is also mentioned that these estimators are often equivalent. Furthermore, two distinct sets of kernels are discussed in Section 2.4.3, with the first set being kernels with compact support on $[0, 1]$ and the second set being kernels with unbounded support on $[0, \infty)$. It is also a well-known fact that the choice of kernel function is not the most important aspect in kernel density estimation (Silverman, 1986; Wand & Jones, 1995). Therefore, in the simulation study, both general representations of the kernel density estimator on the circle are used, in conjunction with only 3 different kernel functions k, namely:

1. the Epanechnikov kernel;
2. the Swanepoel kernel, and
3. the Gaussian (normal) kernel.

For the circular kernel density estimator represented in (2.21), only the Epanechnikov and Swanepoel kernels are used, since both these kernel functions have compact support on $[0, 1]$. For the circular kernel density estimator in (2.23), only the Gaussian kernel function (unbounded support on $[0, \infty]$) is used. Only these three circular kernel density estimators are used in the empirical study.

4.3.2 Choice of estimated smoothing parameter $\hat{h}$

It is evident from (2.21) and (2.23) that the smoothing parameter h should be estimated. For a discussion on smoothing parameters, see Section 2.4.4 in Chapter 2. The choice of the value of the smoothing parameter is critically important in kernel density estimation. The smoothing parameter will influence the conclusions that can be drawn from the density estimator, and the reader will not be able to "unsmooth" the estimator following an extreme choice for the smoothing parameter. Silverman (1986) discusses several methods for choosing the smoothing parameter, which vary from subjective techniques, reference to a standard distribution, least squares cross-validation, likelihood cross-validation, test graph methods and internal estimation of the density roughness. What is of importance, though, is the fact that astrophysical data originating from a pulsar may be unimodal, bimodal and/or skew. Therefore, the estimated smoothing parameter should take this fact into consideration. For data on the real line, a popular and simple method known as the plug-in estimate (Silverman, 1986), is given by

$$\hat{h} = 1.06\hat{\sigma}n^{-1/5}\,, \tag{4.1}$$

where $\hat{\sigma}$ is some estimate of the dispersion of the data. Different data-driven choices of $\hat{h}$ are used throughout the simulation study based on linear and circular estimates of the dispersion of the data.

The following list highlights the estimated measures of dispersion that are used in the empirical study:

1. Linear standard deviation, denoted by s;
2. Square root of the circular variance defined in (2.8), denoted by $s_\circ$;
3. Circular mean deviation defined in (2.11) with $\alpha = \bar{\theta}$, denoted by $\bar{D}_\circ$;
4. Circular median deviation defined in (2.11) with $\alpha = \tilde{\theta}$ denoted by $\tilde{D}_\circ$, and
5. Circular median absolute deviation defined in (2.14) denoted by $|D_\circ|$.

In summary, nine different estimated smoothing parameters are used throughout the empirical study. In the discussion that follows, reference will only be made to the choice of smoothing parameter $\hat{h}_i$, $i = 1\ldots 9$, according to Table 4.1.

Table 4.1: *Smoothing parameter references and equations*

Smoothing parameter estimates
$\hat{h}_1 = 1.06sn^{-1/5}$
$\hat{h}_2 = 1.06s_\circ n^{-1/5}$
$\hat{h}_3 = 1.06\bar{D}_\circ n^{-1/5}$
$\hat{h}_4 = 1.06\tilde{D}_\circ n^{-1/5}$
$\hat{h}_5 = 1.06\lvert D_\circ\rvert n^{-1/5}$
$\hat{h}_6 = 1.06IQR_\circ n^{-1/5}$
$\hat{h}_7 = \frac{1.06}{1.349}IQR_\circ n^{-1/5}$
$\hat{h}_8 = 0.9s_\circ n^{-1/5}$
$\hat{h}_9 = \frac{0.9}{1.349}IQR_\circ n^{-1/5}$

4.3.3 Choice of goodness-of-fit test

SOPIE is fundamentally based on sequential goodness-of-fit testing (for uniformity). The type of goodness-of-fit test used to assess uniformity is therefore of importance. Throughout the simulation study, four different goodness-of-fit tests are used, namely the Kolmogorov-Smirnov, Cramér-von-Mises, Anderson-Darling and the Rayleigh goodness-of-fit tests. An exposition of each of these tests can be found in Section 3.3.

4.3.4 Choice of tuning parameters in SOPIE

In Section 3.2, SOPIE is proposed. Several tuning parameters were used in SOPIE and the aim of this section is to inform the reader of the values chosen for each tuning parameter.

Choice of the grid used in the kernel density estimation

The first tuning parameter of importance is the selection of the grid over which the kernel density estimation will be done. Throughout the simulation study, a grid of 1000 equally spaced intervals were used for the kernel density estimation.

Choice of significance level α.

For any situation where a hypothesis is tested, it is required to choose the level of significance α. Since SOPIE sequentially applies goodness-of-fit tests, the value of α must be selected. Throughout the simulation study, several α-values are used, such as 10%, 5% , 1% or 0.1%.

Choice of m, the number of minimum points

The next important tuning parameter is the choice of m, the number of local minimum points that are used in SOPIE. It is evident that the execution time of SOPIE is heavily dependent on the choice of m. The complete procedure is reiterated for every selected minimum point, and therefore the choice of m is critical in the context of computation time. The larger the value of m, the longer the computation time. The remark about the estimation of the second off-pulse region is also of importance when selecting the number of minimum points, as the user must ensure that the selected minimum points do not belong to two disjoint intervals, as described in the last paragraph of Section 3.2 in Chapter 3.

Choice of g, the incremental growth of each interval being tested for uniformity

Another important tuning parameter is g, the value of the incremental growth of each subsequent interval over which uniformity is tested. In SOPIE, uniformity is sequentially tested, with the interval used in the test growing by g observations in every iteration. The selection of g not only influences the computation time of the procedure, but also has an effect on the point where rejection of the hypothesis takes place. For large values of g, the user takes the risk that uniformity is rejected for a certain (larger) interval, while it should have been rejected earlier (for a smaller interval). On the other hand, a very small choice of g results in long execution times. Small values of g may also result in the early rejection of uniformity, e.g. in the situation where a few observations may cause the rejection of uniformity, while uniformity is again confirmed when several more observations are included in the interval. If the user suspects that this situation may occur, the problem can be overcome by selecting a larger value of the integer r. The significance of r will be discussed next.

Choice of r, the number of intervals of rejection

The tuning parameter r represents the number of subsequent intervals that must result in the rejection of uniformity before SOPIE will stop. The choice of r must therefore be linked to the choice of g as explained above. For smaller values of g, it would be safer to select larger values of r, and vice versa. Since small values of g may result in a temporary rejection of uniformity for an interval, a larger value of r would prevent the method from immediately stopping at the first occurrence of rejection. It is very important to note that, for a large value of r, there will be no influence on the value of $\hat{b}_1$ in (3.4) or $\hat{a}_1$ in (3.5) if rejection takes place for each interval after a certain point.

For example, with step 4 of Section 3.2 in mind, suppose that rejection takes place over the following set of observations $\chi_\ell := \{\theta_{(k_1)}, \theta_{(k_1+1)}, \ldots, \theta_{(k_1+\ell g)}, \theta_{(k_1+\ell g+1)}\}$. If rejection continues to take place for the next r sets, e.g., the final set where uniformity is rejected before the method stops is $\chi_r := \{\theta_{(k_1)}, \theta_{(k_1+1)}, \ldots, \theta_{(k_1+rg)}, \theta_{(k_1+rg+1)}\}$, then the values of $\hat{b}_1$ and $\hat{a}_1$ are independent of r. This can also be seen when considering the stopping time defined in step 5 in (3.2), because N is defined as the smallest integer j, while r is only present in the upper bound of j.

4.3.5 Measures of accuracy and consistency of SOPIE

In order to establish which combinations of the above-mentioned parameters are influential in estimating the off-pulse interval of the simulated density functions, two measures are used as evaluation criteria. These measures are used to assess which combinations of parameter values yield optimal estimation. The following two measures are used throughout the simulation study:

Bias

Firstly, the bias is used as measure of how well SOPIE estimates the theoretical (true) off-pulse interval. Since the theoretical off-pulse interval is known for each of the simulated data sets, the bias serves as a measure of the accuracy of the estimation technique. The closer the bias is to zero, the smaller the mean difference between the estimator and the theoretical value.

Mean squared error (MSE)

An overall measure of the accuracy of an estimator is the mean squared error, henceforth abbreviated as MSE. Suppose, for example, an unknown parameter ψ is estimated by $\hat{\psi}$ (some function of the data), then the MSE is defined by

$$\begin{aligned} MSE &= E\left((\hat{\psi} - \psi)^2\right) \\ &= (\text{Variance of } \hat{\psi}) + (\text{Bias of } \hat{\psi})^2. \end{aligned}$$

The MSE criterion, therefore, takes both the bias and the variance of an estimator into account. This is an important measure, since an estimator with good properties should ideally control both the bias and the variance.

4.4 Simulation study

Two broad classes of simulated data are used to assess the performance of SOPIE. Firstly, different study populations are constructed from a scaled Von Mises distribution and then contaminated with noise. Another broad class of study populations are constructed from a scaled triangular distribution and then contaminated with noise. Both types of study populations are used to test the performance of SOPIE.

The reader is reminded that SOPIE is developed to estimate the off-pulse interval of a source function originating from a pulsar. It is therefore essential that the data used in the simulation study must be comparable to data obtained from a typical pulsar light curve. This implies that, for each of the data sets used in the simulation study, one must consider what percentage of the interval $[0, 1]$ consists of pulsed signal $f_s(\theta)$. It is known that the γ-ray light curves of most of the high-energy pulsars contain pulsed emission that covers about 50% of the complete interval $[0, 1]$ (Kanbach, 2002). Also, from Abdo et al. (2010f), it can be concluded that the average pulsed emission coverage of the pulsars listed in that paper is in the order of 60%, with a maximum coverage of 78% for pulsar J1614-2230.

Throughout the simulation study, 1000 Monte Carlo repetitions are used. The standard errors of all the averages of the Monte Carlo estimates of the bias and MSE were found to be negligibly small and are therefore omitted from all the tables. The estimated standard errors are available on the supplied DVD-ROM, together with all of the other results.

For each of these classes, several parameters are varied, resulting in the generation of a vast array of different data sets. The following list represents the different data sets generated from the scaled von Mises distribution:

- Noise level $1 - p = 0.1$, concentration parameter $\kappa = 1$, number of observations $n = 10000$ and $[a, b] = [0.13\,,\,0.87]$.
- Noise level $1 - p = 0.1$, concentration parameter $\kappa = 1$, number of observations $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$.
- Noise level $1 - p = 0.2$, concentration parameter $\kappa = 2$, number of observations $n = 5000$ and $[a, b] = [0.3\,,\,0.7]$.
- Noise level $1 - p = 0.3$, concentration parameter $\kappa = 3$, number of observations $n = 25000$ and $[a, b] = [0.4\,,\,0.6]$.
- Noise level $1 - p = 0.3$, concentration parameter $\kappa = 4$, number of observations $n = 7500$ and $[a, b] = [0.3\,,\,0.7]$.
- Noise level $1 - p = 0.4$, concentration parameter $\kappa = 2$, number of observations $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$.
- Noise level $1 - p = 0.1$, concentration parameter $\kappa = 1$, number of observations $n = 5000$ and $[a, b] = [0.45\,,\,0.55]$.
- Noise level $1 - p = 0.1$, concentration parameter $\kappa = 3$, number of observations $n = 2000$ and $[a, b] = [0.25\,,\,0.75]$.
- Noise level $1 - p = 0.2$, concentration parameter $\kappa = 1$, number of observations $n = 500$ and $[a, b] = [0.2\,,\,0.8]$.

The following list represents the different data sets generated from the scaled triangular distribution:

- Noise level $1 - p = 0.1$, number of observations $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$.

- Noise level $1 - p = 0.1$, number of observations $n = 2000$ and $[a, b] = [0.3\,,\, 0.7]$.
- Noise level $1 - p = 0.2$, number of observations $n = 10000$ and $[a, b] = [0.3\,,\, 0.7]$.
- Noise level $1 - p = 0.2$, number of observations $n = 25000$ and $[a, b] = [0.3\,,\, 0.7]$.
- Noise level $1 - p = 0.4$, number of observations $n = 5000$ and $[a, b] = [0.45\,,\, 0.55]$.
- Noise level $1 - p = 0.2$, number of observations $n = 10000$ and $[a, b] = [0.45\,,\, 0.55]$.
- Noise level $1 - p = 0.4$, number of observations $n = 10000$ and $[a, b] = [0.3\,,\, 0.7]$.
- Noise level $1 - p = 0.4$, number of observations $n = 25000$ and $[a, b] = [0.3\,,\, 0.7]$.
- Noise level $1 - p = 0.1$, number of observations $n = 500$ and $[a, b] = [0.3\,,\, 0.7]$.

The reader is reminded that the on-pulse interval can be defined as the complement of the off-pulse interval and vice versa. The estimation of $[a, b]$ will enable one to identify both the on-pulse and off-pulse interval. Therefore, notice that the meaning of the intervals $[a, b]$ is different for the two classes of simulated data. For the data generated from the scaled von Mises distribution, $[a, b]$ represents the on-pulse interval (or the complement of the off-pulse interval). For the data generated from the scaled triangular distribution, $[a, b]$ represents the off-pulse interval.

Each of the above mentioned study populations is now analysed separately. Each population is inspected on a parameter per parameter basis in order to assess the effect of different tuning parameter choices on the estimated off-pulse interval.

4.5 Simulation study results: Von Mises simulated data

As said above, two broad classes of simulated data are used to assess the performance of SOPIE. Firstly, different study populations are constructed from a scaled Von Mises distribution contaminated with noise. The following subsections present the results of the performance of SOPIE when applied to the simulated data sets generated from the scaled von Mises distribution.

4.5.1 Data set parameters: $1 - p = 0.1$, $\kappa = 1$, $n = 10000$ and $[a, b] = [0.13\,,\, 0.87]$

Figures 4.8 and 4.9 display histogram representations of one Monte Carlo simulation from this target population. Different kernel density estimators (with two choices of the estimated smoothing parameter) are fitted to the data and illustrated by the lines superimposed on each histogram. The histograms of the simulated data contain 100 classes.

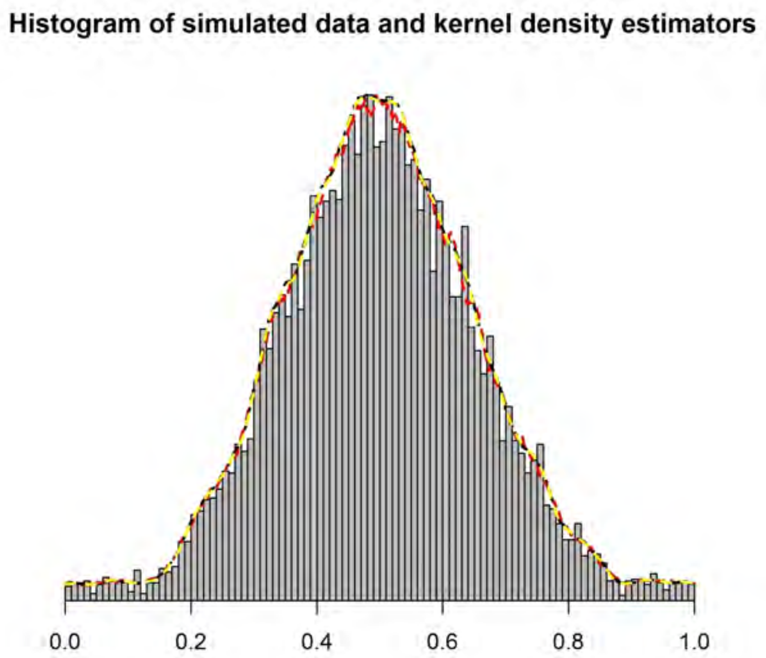

Figure 4.8: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_1 = 0.17$.

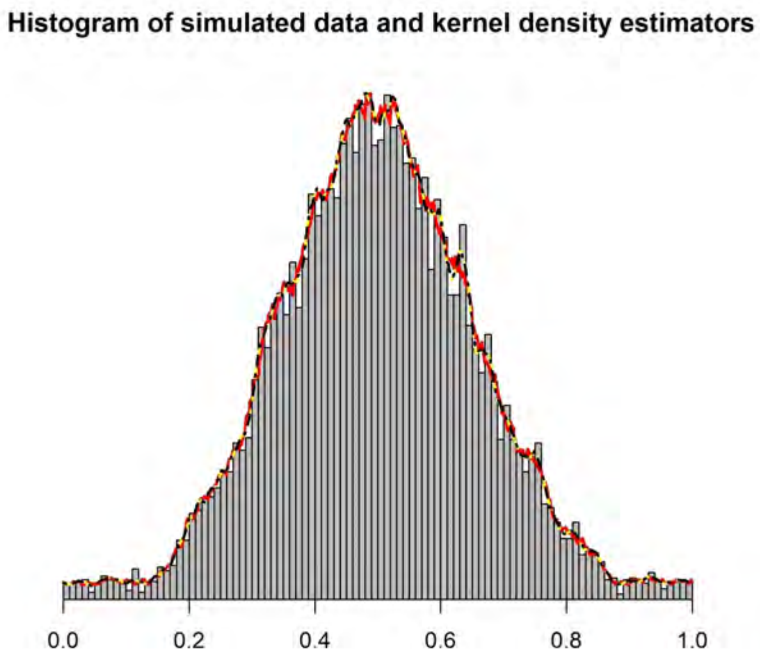

Figure 4.9: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_2 = 0.10$.

It is very important to mention that the population used to simulate the data from, contains a very high percentage (75%) of pulsed signal. It was mentioned earlier that the simulated data from a specific population should mimic pulsar data to a certain degree. *This specific population is therefore an extreme case in terms of the percentage pulsed emission contained in the interval [0,1].*

This study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

During each iteration of SOPIE, the first step is to calculate that point where the circular kernel density estimator attains its global minimum and the next m local minima. For this data set the global minimum point and a maximum of 19 other unique minima are obtained, resulting in 20 minima that can be used as starting point for the sequential method. The reader is referred to step 2 in Section 3.2 of Chapter 3 for the details pertaining to the selection of the minima.

It must be noted that, in several cases, the 20 minimum points $(x_1, x_2, \ldots, x_{20})$ obtained from the kernel density estimator, resulted in less than 20 actual data values $(\theta_{(k_i)}\,,\ i = 1, 2, \ldots, 20)$ used as starting point for SOPIE. The reason is that (in some cases) the obtained minima from the kernel density estimation $(x_1, x_2, \ldots, x_{20})$ resulted in equal values for some of the $\theta_{(k_i)}\,,\ i = 1, 2, \ldots, 20$. Since only unique $\theta_{(k_i)}$ values are used in SOPIE, the number of $\theta_{(k_i)}$ is sometimes less than the selected 20 minimum points obtained from the kernel density estimation.

Table 4.2 compares the minima obtained from each of the different kernel functions fitted to the data for a single Monte Carlo iteration.

Table 4.2: *Minima comparison $(x_1, x_2, \ldots, x_{20})$ for different kernel functions using $\hat{h}_2$.*

	Swanepoel Kernel	**Epanechnikov Kernel**	**Normal Kernel**
1st local min.	0.0355	0.0255	0.0335
2nd local min.	0.0315	0.0265	0.0225
3rd local min.	0.0345	0.0295	0.0235
4th local min.	0.0335	0.0245	0.0245
5th local min.	0.0325	0.0235	0.0265
6th local min.	0.0305	0.0225	0.0295
7th local min.	0.0375	0.0285	0.0185
8th local min.	0.0385	0.0315	0.0205
9th local min.	0.0295	0.0325	0.0215
10th local min.	0.0285	0.0305	0.0285
11th local min.	0.0395	0.0215	0.0305
12th local min.	0.0275	0.0205	0.0315
13th local min.	0.0405	0.0185	0.0325
14th local min.	0.0265	0.0275	0.0175
15th local min.	0.0415	0.0345	0.0195
16th local min.	0.0255	0.8905	0.8905
17th local min.	0.0245	0.0195	0.0275
18th local min.	0.0425	0.0175	0.0345
19th local min.	0.0235	0.0355	0.8985
20th local min.	0.0225	0.0335	0.0355

From Table 4.2 it is evident that similar minima are obtained from the different kernel functions that are used in the kernel density estimator. In most cases, several minima overlap for all the kernel functions, except that the minima are obtained in a different order.

Although Table 4.2 is only an indication of one Monte Carlo iteration, it became evident from the estimation of the off-pulse interval that this situation was true for almost all of the iterations. The reader must keep in mind that, when equal minimum points are obtained from the different kernel functions, the rejection point calculated from a goodness-of-fit test will also be equal. This is true since the resulting estimation from a certain goodness-of-fit test is a function of the choice of starting point and is independent of the choice of the kernel function. This can be seen from Tables 4.3 – 4.6, which compare the bias of the estimators $\hat{a}$ and $\hat{b}$ for the different kernel density functions. Tables 4.7 – 4.10 reflect the same comparison, but display the MSE of the estimators. The estimated bias and MSE-values are displayed for $\hat{h}_6$, for an array of g values, with $m = 2,\ r = 6,\ \alpha = 0.01$, and for all the goodness-of-fit tests. The selected values of g, m, r, α and $\hat{h}$ were chosen merely for illustration purposes. Furthermore, these last mentioned tables, as well as the remainder of the tables, report the results of averages over all of the 1000 Monte Carlo repetitions.

Table 4.3: *Bias (when estimating a and b) for different kernel functions (Anderson-Darling test).*

	Swanepoel		**Epanechnikov**		**Normal**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0386	-0.0400	0.0389	-0.0408	0.0379	-0.0389
Grow=7	0.0380	-0.0388	0.0379	-0.0387	0.0377	-0.0379
Grow=8	0.0378	-0.0384	0.0377	-0.0382	0.0380	-0.0381
Grow=9	0.0381	-0.0386	0.0380	-0.0380	0.0383	-0.0380
Grow=10	0.0388	-0.0389	0.0387	-0.0383	0.0382	-0.0383

Table 4.4: *Bias (when estimating a and b) for different kernel functions (Cramér-von-Mises test).*

	Swanepoel		**Epanechnikov**		**Normal**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0549	-0.0547	0.0551	-0.0545	0.0548	-0.0545
Grow=7	0.0550	-0.0547	0.0552	-0.0546	0.0549	-0.0546
Grow=8	0.0552	-0.0549	0.0554	-0.0548	0.0551	-0.0548
Grow=9	0.0554	-0.0551	0.0556	-0.0549	0.0553	-0.0549
Grow=10	0.0556	-0.0553	0.0557	-0.0552	0.0555	-0.0551

Table 4.5: *Bias (when estimating a and b) for different kernel functions (Kolmogorov-Smirnov test).*

	Swanepoel		**Epanechnikov**		**Normal**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0432	-0.0451	0.0431	-0.0467	0.0422	-0.0426
Grow=7	0.0430	-0.0437	0.0433	-0.0446	0.0422	-0.0432
Grow=8	0.0430	-0.0431	0.0428	-0.0448	0.0422	-0.0432
Grow=9	0.0438	-0.0424	0.0423	-0.0433	0.0422	-0.0435
Grow=10	0.0434	-0.0434	0.0432	-0.0435	0.0426	-0.0435

Table 4.6: *Bias (when estimating a and b) for different kernel functions (Rayleigh test).*

	Swanepoel		Epanechnikov		Normal	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0520	-0.0514	0.0514	-0.0523	0.0514	-0.0512
Grow=7	0.0510	-0.0516	0.0508	-0.0523	0.0522	-0.0500
Grow=8	0.0504	-0.0521	0.0507	-0.0518	0.0523	-0.0502
Grow=9	0.0508	-0.0515	0.0505	-0.0514	0.0514	-0.0501
Grow=10	0.0505	-0.0513	0.0506	-0.0512	0.0498	-0.0501

Table 4.7: *MSE (when estimating a and b) for different kernel functions (Anderson-Darling test).*

	Swanepoel		Epanechnikov		Normal	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0036	0.0056	0.0037	0.0067	0.0026	0.0043
Grow=7	0.0029	0.0038	0.0026	0.0042	0.0023	0.0028
Grow=8	0.0026	0.0031	0.0023	0.0034	0.0023	0.0028
Grow=9	0.0026	0.0031	0.0023	0.0031	0.0023	0.0025
Grow=10	0.0030	0.0031	0.0027	0.0031	0.0020	0.0025

Table 4.8: *MSE (when estimating a and b) for different kernel functions (Cramér-von-Mises test).*

	Swanepoel		Epanechnikov		Normal	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0031	0.0031	0.0031	0.0030	0.0031	0.0030
Grow=7	0.0031	0.0031	0.0031	0.0030	0.0031	0.0031
Grow=8	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031
Grow=9	0.0031	0.0031	0.0032	0.0031	0.0031	0.0031
Grow=10	0.0032	0.0031	0.0032	0.0031	0.0031	0.0031

Table 4.9: *MSE (when estimating a and b) for different kernel functions (Kolmogorov-Smirnov test).*

	Swanepoel		Epanechnikov		Normal	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0050	0.0065	0.0047	0.0082	0.0044	0.0046
Grow=7	0.0047	0.0052	0.0047	0.0062	0.0040	0.0046
Grow=8	0.0044	0.0041	0.0041	0.0058	0.0037	0.0042
Grow=9	0.0048	0.0034	0.0034	0.0045	0.0033	0.0043
Grow=10	0.0040	0.0038	0.0037	0.0042	0.0033	0.0039

Table 4.10: *MSE (when estimating a and b) for different kernel functions (Rayleigh test).*

	Swanepoel		**Epanechnikov**		**Normal**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
Grow=6	0.0048	0.0040	0.0041	0.0048	0.0055	0.0050
Grow=7	0.0037	0.0037	0.0034	0.0044	0.0059	0.0036
Grow=8	0.0030	0.0041	0.0030	0.0037	0.0059	0.0036
Grow=9	0.0031	0.0031	0.0027	0.0031	0.0048	0.0033
Grow=10	0.0027	0.0027	0.0027	0.0027	0.0033	0.0030

It is not conclusive which specific kernel function performs best in terms of estimated bias and MSE. What is interesting, though, is that the Rayleigh goodness-of-fit test frequently performs worse than the other goodness-of-fit tests in terms of bias. In the following paragraphs it will again be shown that the Rayleigh test does not seem to be the optimal test, as it frequently results in larger values of the bias and MSE when compared to the other goodness-of-fit tests. Albeit, the reader will see that the difference is not large in magnitude. In conclusion, it is argued that the kernel function does not have much influence on the estimators $\hat{a}$ and $\hat{b}$. This is in agreement with findings in the literature pertaining to kernel density estimation on the real line (Silverman, 1986; Wand & Jones, 1995).

Choice of the number of minimum points m

The first step of the proposed technique is to select a number of minimum points m . In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus.

Tables 4.11 – 4.19 highlight the values of the bias of $\hat{a}$ and $\hat{b}$ when the normal kernel is used, $g = 6$, $r = 6$, $\alpha = 0.01$ for the various goodness-of-fit tests and for each of the estimated smoothing parameters $\hat{h}_i$, $i = 1, 2, \ldots, 9$. Tables 4.20 – 4.28 report the values of the MSE of $\hat{a}$ and $\hat{b}$.

It can be seen that the choice of m results in only slight variations of the bias and MSE of the estimators $\hat{a}$ and $\hat{b}$. From several of these comparisons that use different values for g, r and α, similar trends are observed. It is evident in some of the comparisons that the choice of $m = 1$ results in a bias and MSE of the estimators for a and b that are closer to zero, compared to other choices of m. This can be explained from the fact that, when only one minimum point is used to estimate the off-pulse interval, averaging over several minima is not present. Furthermore, a small choice of m is preferable, since computing time is reduced.

An important observation is that, as far as the bias and MSE are concerned, all the goodness-of-fit tests, but especially the Cramér-von-Mises test, are insensitive and robust against different choice of m.

Table 4.11: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0388	-0.0393	0.0550	-0.0546	0.0433	-0.0409	0.0511	-0.0505
m=2	0.0380	-0.0395	0.0550	-0.0547	0.0426	-0.0431	0.0507	-0.0513
m=3	0.0384	-0.0397	0.0550	-0.0546	0.0428	-0.0431	0.0512	-0.0514
m=4	0.0385	-0.0389	0.0550	-0.0546	0.0429	-0.0431	0.0515	-0.0512
m=5	0.0386	-0.0392	0.0550	-0.0546	0.0433	-0.0432	0.0513	-0.0514
m=6	0.0387	-0.0390	0.0550	-0.0546	0.0432	-0.0432	0.0510	-0.0515
m=7	0.0386	-0.0390	0.0550	-0.0547	0.0430	-0.0432	0.0510	-0.0516
m=8	0.0385	-0.0393	0.0550	-0.0547	0.0431	-0.0435	0.0511	-0.0517
m=9	0.0384	-0.0394	0.0550	-0.0547	0.0429	-0.0438	0.0513	-0.0519
m=10	0.0383	-0.0392	0.0550	-0.0548	0.0428	-0.0436	0.0514	-0.0519
m=11	0.0384	-0.0392	0.0550	-0.0548	0.0428	-0.0435	0.0513	-0.0519
m=12	0.0383	-0.0392	0.0550	-0.0548	0.0426	-0.0434	0.0513	-0.0519
m=13	0.0384	-0.0392	0.0550	-0.0548	0.0426	-0.0434	0.0513	-0.0520
m=14	0.0384	-0.0392	0.0551	-0.0548	0.0426	-0.0433	0.0512	-0.0519
m=15	0.0384	-0.0391	0.0551	-0.0549	0.0425	-0.0433	0.0512	-0.0518
m=16	0.0384	-0.0391	0.0551	-0.0549	0.0425	-0.0433	0.0511	-0.0519
m=17	0.0384	-0.0391	0.0551	-0.0549	0.0425	-0.0433	0.0511	-0.0519
m=18	0.0384	-0.0391	0.0551	-0.0549	0.0425	-0.0433	0.0511	-0.0519
m=19	0.0384	-0.0391	0.0551	-0.0549	0.0425	-0.0433	0.0511	-0.0519

Table 4.12: *Bias for different choices of m for $\hat{h}_2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0392	-0.0434	0.0557	-0.0549	0.0444	-0.0454	0.0514	-0.0524
m=2	0.0389	-0.0434	0.0558	-0.0550	0.0440	-0.0468	0.0520	-0.0523
m=3	0.0388	-0.0422	0.0557	-0.0551	0.0440	-0.0461	0.0516	-0.0522
m=4	0.0400	-0.0415	0.0556	-0.0551	0.0451	-0.0454	0.0514	-0.0522
m=5	0.0398	-0.0409	0.0556	-0.0551	0.0446	-0.0451	0.0515	-0.0523
m=6	0.0395	-0.0408	0.0556	-0.0552	0.0443	-0.0450	0.0514	-0.0522
m=7	0.0392	-0.0403	0.0556	-0.0552	0.0440	-0.0446	0.0516	-0.0520
m=8	0.0393	-0.0401	0.0556	-0.0552	0.0440	-0.0447	0.0516	-0.0519
m=9	0.0392	-0.0400	0.0556	-0.0552	0.0438	-0.0444	0.0516	-0.0520
m=10	0.0392	-0.0400	0.0556	-0.0553	0.0437	-0.0445	0.0515	-0.0520
m=11	0.0391	-0.0400	0.0556	-0.0553	0.0435	-0.0445	0.0514	-0.0519
m=12	0.0390	-0.0399	0.0557	-0.0553	0.0434	-0.0442	0.0513	-0.0517
m=13	0.0390	-0.0397	0.0557	-0.0554	0.0433	-0.0443	0.0513	-0.0516
m=14	0.0390	-0.0397	0.0557	-0.0554	0.0433	-0.0442	0.0513	-0.0516
m=15	0.0390	-0.0397	0.0557	-0.0554	0.0433	-0.0441	0.0513	-0.0515
m=16	0.0390	-0.0397	0.0557	-0.0554	0.0433	-0.0441	0.0513	-0.0515
m=17	0.0390	-0.0397	0.0558	-0.0555	0.0433	-0.0441	0.0513	-0.0515
m=18	0.0390	-0.0397	0.0558	-0.0555	0.0433	-0.0441	0.0513	-0.0515
m=19	0.0390	-0.0397	0.0558	-0.0555	0.0433	-0.0441	0.0513	-0.0515
m=20	0.0390	-0.0397	0.0558	-0.0555	0.0433	-0.0441	0.0513	-0.0515

Table 4.13: *Bias for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0387	-0.0401	0.0558	-0.0555	0.0415	-0.0443	0.0493	-0.0497
m=2	0.0389	-0.0406	0.0564	-0.0555	0.0424	-0.0460	0.0489	-0.0511
m=3	0.0402	-0.0411	0.0569	-0.0558	0.0437	-0.0466	0.0495	-0.0516
m=4	0.0409	-0.0418	0.0575	-0.0559	0.0442	-0.0466	0.0504	-0.0526
m=5	0.0411	-0.0417	0.0577	-0.0560	0.0449	-0.0461	0.0506	-0.0529
m=6	0.0413	-0.0412	0.0579	-0.0560	0.0449	-0.0456	0.0511	-0.0529
m=7	0.0410	-0.0413	0.0578	-0.0561	0.0448	-0.0455	0.0515	-0.0529
m=8	0.0412	-0.0414	0.0578	-0.0561	0.0446	-0.0454	0.0513	-0.0529
m=9	0.0409	-0.0415	0.0577	-0.0562	0.0443	-0.0457	0.0510	-0.0527
m=10	0.0410	-0.0412	0.0577	-0.0562	0.0444	-0.0454	0.0512	-0.0526
m=11	0.0409	-0.0410	0.0577	-0.0563	0.0442	-0.0455	0.0511	-0.0526
m=12	0.0408	-0.0409	0.0577	-0.0563	0.0442	-0.0452	0.0511	-0.0524
m=13	0.0407	-0.0410	0.0577	-0.0565	0.0442	-0.0454	0.0510	-0.0525
m=14	0.0408	-0.0409	0.0578	-0.0565	0.0442	-0.0453	0.0510	-0.0524
m=15	0.0409	-0.0410	0.0578	-0.0566	0.0443	-0.0453	0.0510	-0.0524
m=16	0.0409	-0.0410	0.0578	-0.0566	0.0443	-0.0453	0.0510	-0.0524
m=17	0.0409	-0.0410	0.0579	-0.0566	0.0443	-0.0453	0.0510	-0.0524
m=18	0.0409	-0.0410	0.0579	-0.0566	0.0443	-0.0453	0.0510	-0.0524
m=19	0.0409	-0.0410	0.0579	-0.0566	0.0443	-0.0453	0.0510	-0.0524
m=20	0.0409	-0.0410	0.0579	-0.0566	0.0443	-0.0453	0.0510	-0.0524

Table 4.14: *Bias for different choices of m for $\hat{h}_4$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0382	-0.0367	0.0555	-0.0547	0.0427	-0.0423	0.0524	-0.0523
m=2	0.0383	-0.0382	0.0554	-0.0547	0.0435	-0.0433	0.0513	-0.0522
m=3	0.0384	-0.0385	0.0554	-0.0548	0.0429	-0.0440	0.0510	-0.0525
m=4	0.0384	-0.0390	0.0554	-0.0548	0.0429	-0.0436	0.0513	-0.0524
m=5	0.0386	-0.0391	0.0554	-0.0548	0.0436	-0.0436	0.0514	-0.0527
m=6	0.0387	-0.0393	0.0554	-0.0548	0.0438	-0.0439	0.0518	-0.0523
m=7	0.0387	-0.0392	0.0553	-0.0549	0.0437	-0.0437	0.0518	-0.0524
m=8	0.0388	-0.0397	0.0553	-0.0549	0.0438	-0.0440	0.0519	-0.0523
m=9	0.0388	-0.0398	0.0553	-0.0550	0.0438	-0.0443	0.0519	-0.0522
m=10	0.0388	-0.0400	0.0553	-0.0550	0.0439	-0.0443	0.0519	-0.0522
m=11	0.0387	-0.0398	0.0553	-0.0550	0.0437	-0.0443	0.0518	-0.0523
m=12	0.0386	-0.0396	0.0553	-0.0551	0.0435	-0.0441	0.0518	-0.0522
m=13	0.0385	-0.0396	0.0553	-0.0551	0.0434	-0.0441	0.0517	-0.0522
m=14	0.0385	-0.0396	0.0553	-0.0551	0.0433	-0.0440	0.0516	-0.0521
m=15	0.0385	-0.0396	0.0554	-0.0551	0.0433	-0.0440	0.0516	-0.0521
m=16	0.0385	-0.0396	0.0554	-0.0551	0.0433	-0.0440	0.0515	-0.0520
m=17	0.0385	-0.0396	0.0554	-0.0552	0.0433	-0.0440	0.0515	-0.0520
m=18	0.0385	-0.0396	0.0554	-0.0552	0.0433	-0.0439	0.0515	-0.0520
m=19	0.0385	-0.0396	0.0554	-0.0552	0.0433	-0.0439	0.0515	-0.0520
m=20	0.0385	-0.0396	0.0554	-0.0552	0.0433	-0.0439	0.0515	-0.0520

Table 4.15: *Bias for different choices of m for $\hat{h}_5$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0380	-0.0412	0.0556	-0.0550	0.0445	-0.0459	0.0519	-0.0505
m=2	0.0379	-0.0409	0.0555	-0.0550	0.0439	-0.0459	0.0512	-0.0520
m=3	0.0384	-0.0394	0.0555	-0.0551	0.0440	-0.0453	0.0515	-0.0521
m=4	0.0388	-0.0392	0.0554	-0.0551	0.0442	-0.0452	0.0513	-0.0522
m=5	0.0392	-0.0398	0.0554	-0.0551	0.0442	-0.0450	0.0512	-0.0521
m=6	0.0391	-0.0396	0.0554	-0.0551	0.0439	-0.0446	0.0511	-0.0522
m=7	0.0393	-0.0398	0.0555	-0.0551	0.0440	-0.0445	0.0514	-0.0521
m=8	0.0393	-0.0402	0.0555	-0.0551	0.0441	-0.0446	0.0513	-0.0522
m=9	0.0391	-0.0402	0.0555	-0.0552	0.0439	-0.0444	0.0512	-0.0522
m=10	0.0389	-0.0402	0.0555	-0.0552	0.0437	-0.0444	0.0513	-0.0522
m=11	0.0388	-0.0402	0.0555	-0.0552	0.0436	-0.0447	0.0512	-0.0521
m=12	0.0387	-0.0401	0.0556	-0.0552	0.0435	-0.0445	0.0511	-0.0519
m=13	0.0387	-0.0401	0.0556	-0.0553	0.0434	-0.0445	0.0511	-0.0518
m=14	0.0387	-0.0399	0.0556	-0.0553	0.0434	-0.0444	0.0511	-0.0518
m=15	0.0387	-0.0399	0.0556	-0.0553	0.0434	-0.0444	0.0511	-0.0517
m=16	0.0387	-0.0399	0.0557	-0.0553	0.0433	-0.0444	0.0511	-0.0517
m=17	0.0387	-0.0399	0.0557	-0.0553	0.0434	-0.0444	0.0511	-0.0517
m=18	0.0388	-0.0399	0.0557	-0.0553	0.0434	-0.0443	0.0511	-0.0517
m=19	0.0388	-0.0399	0.0557	-0.0553	0.0434	-0.0443	0.0511	-0.0517
m=20	0.0388	-0.0399	0.0557	-0.0553	0.0434	-0.0443	0.0511	-0.0517

Table 4.16: *Bias for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0377	-0.0399	0.0547	-0.0545	0.0428	-0.0425	0.0515	-0.0518
m=2	0.0379	-0.0389	0.0548	-0.0545	0.0422	-0.0426	0.0514	-0.0512
m=3	0.0379	-0.0388	0.0548	-0.0545	0.0419	-0.0423	0.0518	-0.0513
m=4	0.0380	-0.0388	0.0548	-0.0545	0.0417	-0.0421	0.0519	-0.0514
m=5	0.0379	-0.0386	0.0548	-0.0545	0.0417	-0.0420	0.0515	-0.0513
m=6	0.0380	-0.0387	0.0548	-0.0545	0.0418	-0.0417	0.0515	-0.0511
m=7	0.0381	-0.0387	0.0548	-0.0545	0.0418	-0.0420	0.0514	-0.0509
m=8	0.0381	-0.0385	0.0548	-0.0545	0.0418	-0.0421	0.0513	-0.0510
m=9	0.0381	-0.0385	0.0548	-0.0546	0.0419	-0.0423	0.0513	-0.0508
m=10	0.0381	-0.0386	0.0548	-0.0546	0.0419	-0.0424	0.0514	-0.0506
m=11	0.0383	-0.0386	0.0548	-0.0546	0.0420	-0.0424	0.0516	-0.0507
m=12	0.0383	-0.0385	0.0548	-0.0547	0.0419	-0.0423	0.0514	-0.0506
m=13	0.0384	-0.0386	0.0548	-0.0547	0.0419	-0.0425	0.0513	-0.0507
m=14	0.0384	-0.0386	0.0548	-0.0548	0.0419	-0.0426	0.0512	-0.0508
m=15	0.0383	-0.0386	0.0548	-0.0548	0.0419	-0.0426	0.0512	-0.0508
m=16	0.0384	-0.0386	0.0548	-0.0548	0.0419	-0.0426	0.0511	-0.0508
m=17	0.0384	-0.0386	0.0548	-0.0548	0.0419	-0.0426	0.0511	-0.0508
m=18	0.0384	-0.0386	0.0548	-0.0549	0.0419	-0.0426	0.0511	-0.0508
m=19	0.0384	-0.0386	0.0548	-0.0549	0.0419	-0.0426	0.0511	-0.0509

Table 4.17: *Bias for different choices of m for $\hat{h}_7$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0392	-0.0396	0.0551	-0.0546	0.0423	-0.0421	0.0507	-0.0515
m=2	0.0381	-0.0393	0.0550	-0.0547	0.0418	-0.0428	0.0508	-0.0519
m=3	0.0387	-0.0396	0.0551	-0.0547	0.0425	-0.0433	0.0513	-0.0521
m=4	0.0384	-0.0392	0.0551	-0.0547	0.0423	-0.0434	0.0514	-0.0524
m=5	0.0384	-0.0392	0.0551	-0.0547	0.0424	-0.0435	0.0513	-0.0523
m=6	0.0385	-0.0391	0.0551	-0.0547	0.0426	-0.0433	0.0512	-0.0520
m=7	0.0383	-0.0391	0.0551	-0.0547	0.0425	-0.0434	0.0514	-0.0520
m=8	0.0385	-0.0391	0.0551	-0.0547	0.0428	-0.0436	0.0513	-0.0521
m=9	0.0382	-0.0390	0.0551	-0.0548	0.0427	-0.0436	0.0515	-0.0518
m=10	0.0382	-0.0389	0.0551	-0.0548	0.0428	-0.0435	0.0517	-0.0518
m=11	0.0382	-0.0388	0.0551	-0.0548	0.0427	-0.0435	0.0516	-0.0517
m=12	0.0382	-0.0389	0.0551	-0.0548	0.0427	-0.0437	0.0516	-0.0517
m=13	0.0381	-0.0388	0.0551	-0.0549	0.0427	-0.0436	0.0515	-0.0516
m=14	0.0381	-0.0387	0.0551	-0.0549	0.0426	-0.0435	0.0513	-0.0515
m=15	0.0381	-0.0387	0.0551	-0.0549	0.0426	-0.0435	0.0513	-0.0516
m=16	0.0381	-0.0388	0.0551	-0.0549	0.0426	-0.0436	0.0512	-0.0516
m=17	0.0381	-0.0389	0.0552	-0.0549	0.0426	-0.0436	0.0512	-0.0516
m=18	0.0381	-0.0389	0.0552	-0.0549	0.0426	-0.0436	0.0512	-0.0516
m=19	0.0381	-0.0389	0.0552	-0.0549	0.0426	-0.0436	0.0512	-0.0516
m=20	0.0381	-0.0389	0.0552	-0.0549	0.0426	-0.0436	0.0512	-0.0516

Table 4.18: *Bias for different choices of m for $\hat{h}_8$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0390	-0.0442	0.0557	-0.0551	0.0435	-0.0468	0.0513	-0.0533
m=2	0.0394	-0.0424	0.0557	-0.0552	0.0449	-0.0468	0.0518	-0.0527
m=3	0.0391	-0.0410	0.0556	-0.0553	0.0444	-0.0461	0.0516	-0.0523
m=4	0.0393	-0.0408	0.0557	-0.0553	0.0443	-0.0457	0.0512	-0.0521
m=5	0.0395	-0.0405	0.0556	-0.0553	0.0444	-0.0459	0.0512	-0.0522
m=6	0.0395	-0.0406	0.0557	-0.0553	0.0443	-0.0454	0.0511	-0.0521
m=7	0.0394	-0.0403	0.0557	-0.0553	0.0440	-0.0452	0.0510	-0.0521
m=8	0.0394	-0.0402	0.0557	-0.0554	0.0441	-0.0450	0.0511	-0.0519
m=9	0.0393	-0.0400	0.0557	-0.0554	0.0438	-0.0448	0.0511	-0.0518
m=10	0.0394	-0.0401	0.0557	-0.0554	0.0438	-0.0447	0.0511	-0.0517
m=11	0.0392	-0.0400	0.0558	-0.0555	0.0436	-0.0445	0.0510	-0.0516
m=12	0.0391	-0.0399	0.0558	-0.0555	0.0436	-0.0444	0.0510	-0.0515
m=13	0.0391	-0.0398	0.0558	-0.0555	0.0435	-0.0443	0.0511	-0.0514
m=14	0.0391	-0.0398	0.0558	-0.0556	0.0435	-0.0443	0.0510	-0.0515
m=15	0.0391	-0.0398	0.0558	-0.0556	0.0434	-0.0443	0.0510	-0.0514
m=16	0.0391	-0.0397	0.0559	-0.0556	0.0434	-0.0443	0.0510	-0.0514
m=17	0.0391	-0.0397	0.0559	-0.0556	0.0434	-0.0443	0.0510	-0.0514
m=18	0.0391	-0.0397	0.0559	-0.0556	0.0435	-0.0443	0.0510	-0.0514
m=19	0.0391	-0.0397	0.0559	-0.0556	0.0435	-0.0443	0.0510	-0.0514

Table 4.19: *Bias for different choices of m for $\hat{h}_9$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0385	-0.0389	0.0555	-0.0544	0.0428	-0.0411	0.0517	-0.0518
m=2	0.0389	-0.0372	0.0555	-0.0546	0.0434	-0.0423	0.0508	-0.0526
m=3	0.0386	-0.0372	0.0554	-0.0547	0.0430	-0.0431	0.0520	-0.0526
m=4	0.0386	-0.0383	0.0553	-0.0547	0.0430	-0.0439	0.0521	-0.0524
m=5	0.0387	-0.0384	0.0553	-0.0548	0.0433	-0.0437	0.0517	-0.0522
m=6	0.0386	-0.0390	0.0552	-0.0548	0.0431	-0.0439	0.0520	-0.0524
m=7	0.0385	-0.0391	0.0552	-0.0549	0.0433	-0.0442	0.0518	-0.0525
m=8	0.0383	-0.0391	0.0552	-0.0549	0.0432	-0.0439	0.0518	-0.0526
m=9	0.0385	-0.0392	0.0552	-0.0549	0.0432	-0.0437	0.0518	-0.0524
m=10	0.0386	-0.0393	0.0552	-0.0549	0.0433	-0.0436	0.0519	-0.0523
m=11	0.0385	-0.0394	0.0552	-0.0550	0.0430	-0.0436	0.0518	-0.0522
m=12	0.0384	-0.0393	0.0552	-0.0550	0.0429	-0.0435	0.0517	-0.0520
m=13	0.0383	-0.0393	0.0553	-0.0550	0.0428	-0.0436	0.0516	-0.0519
m=14	0.0383	-0.0393	0.0553	-0.0550	0.0427	-0.0435	0.0515	-0.0519
m=15	0.0383	-0.0393	0.0553	-0.0550	0.0427	-0.0435	0.0515	-0.0518
m=16	0.0383	-0.0393	0.0553	-0.0551	0.0427	-0.0434	0.0514	-0.0518
m=17	0.0383	-0.0393	0.0553	-0.0551	0.0427	-0.0435	0.0514	-0.0518
m=18	0.0384	-0.0393	0.0553	-0.0551	0.0427	-0.0435	0.0514	-0.0518
m=19	0.0384	-0.0393	0.0553	-0.0551	0.0427	-0.0435	0.0514	-0.0518

Table 4.20: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0036	0.0050	0.0031	0.0030	0.0046	0.0032	0.0047	0.0040
m=2	0.0029	0.0050	0.0031	0.0031	0.0043	0.0049	0.0044	0.0047
m=3	0.0031	0.0052	0.0031	0.0030	0.0044	0.0050	0.0047	0.0047
m=4	0.0031	0.0045	0.0031	0.0030	0.0045	0.0049	0.0049	0.0046
m=5	0.0032	0.0045	0.0031	0.0030	0.0048	0.0049	0.0047	0.0048
m=6	0.0033	0.0044	0.0031	0.0031	0.0047	0.0049	0.0045	0.0048
m=7	0.0032	0.0043	0.0031	0.0031	0.0045	0.0048	0.0045	0.0049
m=8	0.0032	0.0045	0.0031	0.0031	0.0046	0.0051	0.0046	0.0050
m=9	0.0032	0.0045	0.0031	0.0031	0.0045	0.0053	0.0048	0.0051
m=10	0.0030	0.0043	0.0031	0.0031	0.0044	0.0051	0.0049	0.0051
m=11	0.0030	0.0043	0.0031	0.0031	0.0043	0.0050	0.0048	0.0051
m=12	0.0030	0.0042	0.0031	0.0031	0.0042	0.0048	0.0048	0.0051
m=13	0.0030	0.0042	0.0031	0.0031	0.0041	0.0048	0.0049	0.0052
m=14	0.0030	0.0041	0.0031	0.0031	0.0041	0.0046	0.0048	0.0051
m=15	0.0030	0.0040	0.0031	0.0031	0.0040	0.0046	0.0048	0.0050
m=16	0.0030	0.0039	0.0031	0.0031	0.0040	0.0046	0.0047	0.0051
m=17	0.0029	0.0039	0.0031	0.0031	0.0040	0.0046	0.0047	0.0051
m=18	0.0029	0.0039	0.0031	0.0031	0.0040	0.0046	0.0047	0.0051
m=19	0.0029	0.0039	0.0031	0.0031	0.0040	0.0046	0.0047	0.0051

Table 4.21: *MSE for different choices of m for $\hat{h}_2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0030	0.0087	0.0032	0.0031	0.0048	0.0074	0.0034	0.0040
m=2	0.0026	0.0086	0.0032	0.0031	0.0044	0.0083	0.0038	0.0041
m=3	0.0028	0.0074	0.0032	0.0031	0.0045	0.0075	0.0036	0.0041
m=4	0.0039	0.0068	0.0032	0.0031	0.0055	0.0068	0.0036	0.0041
m=5	0.0038	0.0062	0.0032	0.0031	0.0052	0.0065	0.0038	0.0042
m=6	0.0037	0.0060	0.0032	0.0031	0.0050	0.0063	0.0039	0.0042
m=7	0.0034	0.0054	0.0032	0.0031	0.0047	0.0058	0.0041	0.0040
m=8	0.0034	0.0052	0.0032	0.0031	0.0046	0.0059	0.0041	0.0040
m=9	0.0034	0.0051	0.0032	0.0031	0.0045	0.0056	0.0041	0.0041
m=10	0.0034	0.0051	0.0032	0.0031	0.0045	0.0057	0.0040	0.0041
m=11	0.0033	0.0049	0.0032	0.0031	0.0043	0.0056	0.0040	0.0040
m=12	0.0032	0.0047	0.0032	0.0031	0.0042	0.0054	0.0039	0.0039
m=13	0.0031	0.0046	0.0032	0.0031	0.0040	0.0054	0.0039	0.0039
m=14	0.0030	0.0045	0.0032	0.0031	0.0040	0.0053	0.0038	0.0038
m=15	0.0030	0.0044	0.0032	0.0031	0.0040	0.0052	0.0038	0.0038
m=16	0.0030	0.0043	0.0032	0.0031	0.0039	0.0051	0.0038	0.0038
m=17	0.0030	0.0043	0.0032	0.0031	0.0039	0.0051	0.0038	0.0038
m=18	0.0030	0.0043	0.0032	0.0031	0.0039	0.0051	0.0038	0.0038
m=19	0.0030	0.0043	0.0032	0.0031	0.0039	0.0051	0.0038	0.0038
m=20	0.0030	0.0043	0.0032	0.0031	0.0039	0.0051	0.0038	0.0038

Table 4.22: *MSE for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0016	0.0043	0.0032	0.0031	0.0019	0.0046	0.0038	0.0039
m=2	0.0017	0.0050	0.0033	0.0031	0.0026	0.0062	0.0035	0.0046
m=3	0.0026	0.0055	0.0033	0.0032	0.0035	0.0068	0.0037	0.0047
m=4	0.0029	0.0061	0.0034	0.0032	0.0036	0.0067	0.0040	0.0052
m=5	0.0030	0.0059	0.0034	0.0032	0.0041	0.0064	0.0041	0.0054
m=6	0.0031	0.0055	0.0035	0.0032	0.0041	0.0061	0.0044	0.0054
m=7	0.0031	0.0056	0.0035	0.0033	0.0041	0.0060	0.0048	0.0054
m=8	0.0033	0.0055	0.0036	0.0033	0.0041	0.0059	0.0047	0.0054
m=9	0.0032	0.0055	0.0036	0.0033	0.0039	0.0061	0.0045	0.0053
m=10	0.0033	0.0052	0.0037	0.0033	0.0040	0.0058	0.0046	0.0053
m=11	0.0034	0.0049	0.0036	0.0033	0.0039	0.0058	0.0046	0.0053
m=12	0.0033	0.0047	0.0036	0.0033	0.0039	0.0055	0.0046	0.0052
m=13	0.0032	0.0047	0.0036	0.0034	0.0038	0.0055	0.0045	0.0052
m=14	0.0032	0.0045	0.0036	0.0034	0.0038	0.0054	0.0045	0.0051
m=15	0.0032	0.0045	0.0037	0.0034	0.0039	0.0053	0.0045	0.0051
m=16	0.0032	0.0045	0.0037	0.0034	0.0039	0.0053	0.0044	0.0050
m=17	0.0032	0.0045	0.0037	0.0034	0.0039	0.0053	0.0044	0.0051
m=18	0.0032	0.0045	0.0037	0.0034	0.0039	0.0052	0.0044	0.0050
m=19	0.0032	0.0045	0.0037	0.0034	0.0039	0.0052	0.0044	0.0050
m=20	0.0032	0.0045	0.0037	0.0034	0.0039	0.0052	0.0044	0.0050

Table 4.23: *MSE for different choices of m for $\hat{h}_4$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0023	0.0029	0.0031	0.0031	0.0033	0.0046	0.0048	0.0047
m=2	0.0026	0.0040	0.0031	0.0031	0.0043	0.0052	0.0040	0.0047
m=3	0.0027	0.0041	0.0031	0.0031	0.0039	0.0056	0.0038	0.0049
m=4	0.0027	0.0045	0.0031	0.0031	0.0039	0.0055	0.0041	0.0049
m=5	0.0029	0.0046	0.0031	0.0031	0.0045	0.0054	0.0042	0.0051
m=6	0.0031	0.0048	0.0031	0.0031	0.0048	0.0057	0.0045	0.0048
m=7	0.0031	0.0046	0.0031	0.0031	0.0048	0.0054	0.0047	0.0048
m=8	0.0033	0.0049	0.0031	0.0031	0.0050	0.0056	0.0048	0.0048
m=9	0.0033	0.0049	0.0031	0.0031	0.0049	0.0058	0.0047	0.0048
m=10	0.0033	0.0051	0.0031	0.0031	0.0051	0.0058	0.0048	0.0047
m=11	0.0032	0.0048	0.0031	0.0031	0.0049	0.0057	0.0047	0.0048
m=12	0.0031	0.0046	0.0031	0.0031	0.0047	0.0055	0.0047	0.0047
m=13	0.0030	0.0046	0.0031	0.0031	0.0046	0.0054	0.0046	0.0047
m=14	0.0030	0.0044	0.0031	0.0031	0.0045	0.0053	0.0045	0.0046
m=15	0.0029	0.0044	0.0031	0.0031	0.0044	0.0052	0.0045	0.0046
m=16	0.0029	0.0044	0.0031	0.0031	0.0044	0.0052	0.0045	0.0046
m=17	0.0029	0.0044	0.0031	0.0031	0.0043	0.0051	0.0045	0.0046
m=18	0.0029	0.0044	0.0031	0.0031	0.0043	0.0051	0.0044	0.0046
m=19	0.0029	0.0044	0.0031	0.0031	0.0043	0.0051	0.0044	0.0046
m=20	0.0029	0.0044	0.0031	0.0031	0.0043	0.0051	0.0044	0.0046

Table 4.24: *MSE for different choices of m for $\hat{h}_5$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0023	0.0072	0.0032	0.0031	0.0047	0.0079	0.0041	0.0027
m=2	0.0023	0.0065	0.0031	0.0031	0.0044	0.0076	0.0037	0.0040
m=3	0.0027	0.0050	0.0031	0.0031	0.0047	0.0068	0.0041	0.0041
m=4	0.0032	0.0048	0.0031	0.0031	0.0049	0.0068	0.0039	0.0042
m=5	0.0035	0.0052	0.0031	0.0031	0.0050	0.0065	0.0038	0.0042
m=6	0.0034	0.0049	0.0031	0.0031	0.0047	0.0061	0.0037	0.0043
m=7	0.0036	0.0050	0.0031	0.0031	0.0047	0.0059	0.0040	0.0043
m=8	0.0036	0.0053	0.0031	0.0031	0.0048	0.0059	0.0039	0.0043
m=9	0.0034	0.0053	0.0031	0.0031	0.0047	0.0058	0.0039	0.0044
m=10	0.0032	0.0051	0.0031	0.0031	0.0045	0.0057	0.0039	0.0044
m=11	0.0031	0.0051	0.0032	0.0031	0.0044	0.0058	0.0039	0.0043
m=12	0.0030	0.0049	0.0032	0.0031	0.0043	0.0056	0.0039	0.0042
m=13	0.0030	0.0048	0.0032	0.0031	0.0042	0.0056	0.0038	0.0041
m=14	0.0029	0.0047	0.0032	0.0031	0.0042	0.0055	0.0038	0.0041
m=15	0.0029	0.0047	0.0032	0.0031	0.0041	0.0055	0.0038	0.0040
m=16	0.0029	0.0046	0.0032	0.0031	0.0041	0.0054	0.0038	0.0040
m=17	0.0028	0.0046	0.0032	0.0031	0.0041	0.0054	0.0038	0.0040
m=18	0.0028	0.0046	0.0032	0.0031	0.0041	0.0054	0.0038	0.0040
m=19	0.0028	0.0046	0.0032	0.0031	0.0041	0.0054	0.0038	0.0040
m=20	0.0028	0.0046	0.0032	0.0031	0.0041	0.0054	0.0038	0.0040

Table 4.25: *MSE for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0023	0.0050	0.0031	0.0030	0.0047	0.0046	0.0055	0.0053
m=2	0.0026	0.0043	0.0031	0.0030	0.0044	0.0046	0.0055	0.0050
m=3	0.0025	0.0040	0.0031	0.0030	0.0040	0.0043	0.0057	0.0051
m=4	0.0026	0.0039	0.0031	0.0030	0.0038	0.0041	0.0058	0.0052
m=5	0.0027	0.0039	0.0031	0.0030	0.0039	0.0040	0.0056	0.0051
m=6	0.0028	0.0040	0.0031	0.0030	0.0039	0.0038	0.0056	0.0049
m=7	0.0029	0.0040	0.0031	0.0030	0.0038	0.0040	0.0055	0.0048
m=8	0.0028	0.0038	0.0031	0.0030	0.0037	0.0041	0.0054	0.0048
m=9	0.0029	0.0037	0.0031	0.0030	0.0039	0.0041	0.0054	0.0047
m=10	0.0028	0.0037	0.0031	0.0030	0.0038	0.0042	0.0055	0.0045
m=11	0.0030	0.0036	0.0031	0.0030	0.0038	0.0041	0.0057	0.0045
m=12	0.0029	0.0036	0.0031	0.0031	0.0038	0.0040	0.0054	0.0045
m=13	0.0029	0.0035	0.0031	0.0031	0.0037	0.0041	0.0054	0.0046
m=14	0.0029	0.0035	0.0031	0.0031	0.0036	0.0040	0.0053	0.0047
m=15	0.0028	0.0034	0.0031	0.0031	0.0036	0.0040	0.0053	0.0046
m=16	0.0028	0.0034	0.0031	0.0031	0.0035	0.0039	0.0053	0.0046
m=17	0.0028	0.0034	0.0031	0.0031	0.0035	0.0039	0.0053	0.0046
m=18	0.0028	0.0034	0.0031	0.0031	0.0035	0.0039	0.0053	0.0045
m=19	0.0028	0.0034	0.0031	0.0031	0.0035	0.0039	0.0053	0.0046

Table 4.26: *MSE for different choices of m for $\hat{h}_7$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0036	0.0056	0.0031	0.0030	0.0039	0.0045	0.0040	0.0047
m=2	0.0029	0.0050	0.0031	0.0031	0.0036	0.0049	0.0040	0.0051
m=3	0.0034	0.0052	0.0031	0.0031	0.0042	0.0052	0.0045	0.0052
m=4	0.0031	0.0048	0.0031	0.0031	0.0040	0.0052	0.0046	0.0054
m=5	0.0031	0.0047	0.0031	0.0031	0.0040	0.0052	0.0045	0.0052
m=6	0.0032	0.0045	0.0031	0.0031	0.0042	0.0050	0.0044	0.0050
m=7	0.0029	0.0044	0.0031	0.0031	0.0041	0.0050	0.0046	0.0051
m=8	0.0031	0.0044	0.0031	0.0031	0.0044	0.0051	0.0046	0.0052
m=9	0.0029	0.0042	0.0031	0.0031	0.0042	0.0051	0.0048	0.0050
m=10	0.0030	0.0040	0.0031	0.0031	0.0043	0.0050	0.0050	0.0050
m=11	0.0028	0.0039	0.0031	0.0031	0.0042	0.0050	0.0049	0.0048
m=12	0.0028	0.0039	0.0031	0.0031	0.0042	0.0051	0.0050	0.0048
m=13	0.0027	0.0038	0.0031	0.0031	0.0041	0.0050	0.0048	0.0047
m=14	0.0027	0.0037	0.0031	0.0031	0.0040	0.0049	0.0047	0.0046
m=15	0.0026	0.0037	0.0031	0.0031	0.0039	0.0048	0.0047	0.0047
m=16	0.0026	0.0037	0.0031	0.0031	0.0039	0.0049	0.0046	0.0047
m=17	0.0026	0.0038	0.0031	0.0031	0.0039	0.0049	0.0046	0.0047
m=18	0.0026	0.0038	0.0031	0.0031	0.0039	0.0049	0.0046	0.0047
m=19	0.0026	0.0038	0.0031	0.0031	0.0039	0.0049	0.0046	0.0047
m=20	0.0026	0.0038	0.0031	0.0031	0.0039	0.0049	0.0046	0.0047

Table 4.27: *MSE for different choices of m for* $\hat{h}_8$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0030	0.0094	0.0032	0.0031	0.0041	0.0081	0.0034	0.0048
m=2	0.0033	0.0075	0.0032	0.0031	0.0051	0.0080	0.0038	0.0044
m=3	0.0032	0.0062	0.0032	0.0031	0.0047	0.0073	0.0036	0.0041
m=4	0.0033	0.0061	0.0032	0.0031	0.0047	0.0070	0.0034	0.0039
m=5	0.0035	0.0057	0.0032	0.0031	0.0049	0.0070	0.0034	0.0039
m=6	0.0035	0.0057	0.0032	0.0031	0.0048	0.0066	0.0034	0.0040
m=7	0.0035	0.0053	0.0032	0.0031	0.0046	0.0064	0.0033	0.0041
m=8	0.0035	0.0052	0.0032	0.0031	0.0046	0.0062	0.0035	0.0040
m=9	0.0034	0.0050	0.0032	0.0031	0.0044	0.0059	0.0035	0.0039
m=10	0.0034	0.0050	0.0032	0.0031	0.0044	0.0058	0.0035	0.0038
m=11	0.0033	0.0048	0.0032	0.0031	0.0043	0.0056	0.0035	0.0037
m=12	0.0032	0.0046	0.0032	0.0032	0.0042	0.0054	0.0035	0.0036
m=13	0.0031	0.0045	0.0032	0.0032	0.0041	0.0053	0.0035	0.0036
m=14	0.0030	0.0044	0.0032	0.0032	0.0040	0.0053	0.0035	0.0036
m=15	0.0030	0.0044	0.0032	0.0032	0.0039	0.0053	0.0034	0.0036
m=16	0.0030	0.0043	0.0032	0.0032	0.0039	0.0052	0.0034	0.0036
m=17	0.0030	0.0043	0.0032	0.0032	0.0039	0.0052	0.0034	0.0036
m=18	0.0030	0.0043	0.0032	0.0032	0.0039	0.0052	0.0034	0.0036
m=19	0.0030	0.0043	0.0032	0.0032	0.0039	0.0052	0.0034	0.0036

Table 4.28: *MSE for different choices of m for* $\hat{h}_9$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0029	0.0050	0.0031	0.0030	0.0040	0.0039	0.0047	0.0047
m=2	0.0033	0.0032	0.0031	0.0030	0.0046	0.0045	0.0037	0.0054
m=3	0.0031	0.0031	0.0031	0.0031	0.0044	0.0052	0.0048	0.0054
m=4	0.0031	0.0039	0.0031	0.0031	0.0043	0.0059	0.0049	0.0052
m=5	0.0032	0.0038	0.0031	0.0031	0.0045	0.0055	0.0046	0.0050
m=6	0.0032	0.0043	0.0031	0.0031	0.0044	0.0056	0.0049	0.0052
m=7	0.0031	0.0044	0.0031	0.0031	0.0046	0.0057	0.0047	0.0052
m=8	0.0030	0.0044	0.0031	0.0031	0.0045	0.0055	0.0048	0.0052
m=9	0.0031	0.0044	0.0031	0.0031	0.0045	0.0053	0.0048	0.0050
m=10	0.0032	0.0045	0.0031	0.0031	0.0046	0.0052	0.0048	0.0049
m=11	0.0030	0.0044	0.0031	0.0031	0.0043	0.0051	0.0048	0.0049
m=12	0.0029	0.0043	0.0031	0.0031	0.0042	0.0050	0.0047	0.0047
m=13	0.0029	0.0043	0.0031	0.0031	0.0041	0.0050	0.0046	0.0046
m=14	0.0028	0.0043	0.0031	0.0031	0.0040	0.0049	0.0046	0.0046
m=15	0.0028	0.0042	0.0031	0.0031	0.0040	0.0048	0.0045	0.0045
m=16	0.0028	0.0042	0.0031	0.0031	0.0039	0.0048	0.0045	0.0045
m=17	0.0028	0.0041	0.0031	0.0031	0.0039	0.0048	0.0045	0.0045
m=18	0.0027	0.0041	0.0031	0.0031	0.0039	0.0048	0.0045	0.0045
m=19	0.0027	0.0041	0.0031	0.0031	0.0039	0.0048	0.0045	0.0045

Choice of estimated smoothing parameters

In Section 2.4.4 in Chapter 2, the choices of different smoothing parameters were highlighted. The reader is also referred to Table 4.1 for details pertaining to the calculation of the estimated smoothing parameter $\hat{h}$.

Tables 4.29, 4.30 and 4.31 highlight the values of the bias when estimating a and b for $m = 1$, $m = 5$ and $m = 10$, respectively. In each table, the normal kernel is used, $g = 6$, $r = 6$ and $\alpha = 0.01$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Tables 4.32 – 4.34 present the estimated MSE.

Table 4.29: *Comparison of the bias for different combinations of the estimated smoothing parameter (normal kernel) and $m = 1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0388	-0.0393	0.0550	-0.0546	0.0433	-0.0409	0.0511	-0.0505
$\hat{h}_2$	0.0392	-0.0434	0.0557	-0.0549	0.0444	-0.0454	0.0514	-0.0524
$\hat{h}_3$	0.0387	-0.0401	0.0558	-0.0555	0.0415	-0.0443	0.0493	-0.0497
$\hat{h}_4$	0.0382	-0.0367	0.0555	-0.0547	0.0427	-0.0423	0.0524	-0.0523
$\hat{h}_5$	0.0380	-0.0412	0.0556	-0.0550	0.0445	-0.0459	0.0519	-0.0505
$\hat{h}_6$	0.0377	-0.0399	0.0547	-0.0545	0.0428	-0.0425	0.0515	-0.0518
$\hat{h}_7$	0.0392	-0.0396	0.0551	-0.0546	0.0423	-0.0421	0.0507	-0.0515
$\hat{h}_8$	0.0390	-0.0442	0.0557	-0.0551	0.0435	-0.0468	0.0513	-0.0533
$\hat{h}_9$	0.0385	-0.0389	0.0555	-0.0544	0.0428	-0.0411	0.0517	-0.0518

Table 4.30: *Comparison of the bias for different combinations of the estimated smoothing parameter (normal kernel) and $m = 5$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0386	-0.0392	0.0550	-0.0546	0.0433	-0.0432	0.0513	-0.0514
$\hat{h}_2$	0.0398	-0.0409	0.0556	-0.0551	0.0446	-0.0451	0.0515	-0.0523
$\hat{h}_3$	0.0411	-0.0417	0.0577	-0.0560	0.0449	-0.0461	0.0506	-0.0529
$\hat{h}_4$	0.0386	-0.0391	0.0554	-0.0548	0.0436	-0.0436	0.0514	-0.0527
$\hat{h}_5$	0.0392	-0.0398	0.0554	-0.0551	0.0442	-0.0450	0.0512	-0.0521
$\hat{h}_6$	0.0379	-0.0386	0.0548	-0.0545	0.0417	-0.0420	0.0515	-0.0513
$\hat{h}_7$	0.0384	-0.0392	0.0551	-0.0547	0.0424	-0.0435	0.0513	-0.0523
$\hat{h}_8$	0.0395	-0.0405	0.0556	-0.0553	0.0444	-0.0459	0.0512	-0.0522
$\hat{h}_9$	0.0387	-0.0384	0.0553	-0.0548	0.0433	-0.0437	0.0517	-0.0522

Table 4.31: *Comparison of the bias for different combinations of the estimated smoothing parameter (normal kernel) and $m = 10$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0383	-0.0392	0.0550	-0.0548	0.0428	-0.0436	0.0514	-0.0519
$\hat{h}_2$	0.0392	-0.0400	0.0556	-0.0553	0.0437	-0.0445	0.0515	-0.0520
$\hat{h}_3$	0.0410	-0.0412	0.0577	-0.0562	0.0444	-0.0454	0.0512	-0.0526
$\hat{h}_4$	0.0388	-0.0400	0.0553	-0.0550	0.0439	-0.0443	0.0519	-0.0522
$\hat{h}_5$	0.0389	-0.0402	0.0555	-0.0552	0.0437	-0.0444	0.0513	-0.0522
$\hat{h}_6$	0.0381	-0.0386	0.0548	-0.0546	0.0419	-0.0424	0.0514	-0.0506
$\hat{h}_7$	0.0382	-0.0389	0.0551	-0.0548	0.0428	-0.0435	0.0517	-0.0518
$\hat{h}_8$	0.0394	-0.0401	0.0557	-0.0554	0.0438	-0.0447	0.0511	-0.0517
$\hat{h}_9$	0.0386	-0.0393	0.0552	-0.0549	0.0433	-0.0436	0.0519	-0.0523

Table 4.32: *Comparison of the MSE for different combinations of the estimated smoothing parameter (normal kernel) and $m = 1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0036	0.0050	0.0031	0.0030	0.0046	0.0032	0.0047	0.0040
$\hat{h}_2$	0.0030	0.0087	0.0032	0.0031	0.0048	0.0074	0.0034	0.0040
$\hat{h}_3$	0.0016	0.0043	0.0032	0.0031	0.0019	0.0046	0.0038	0.0039
$\hat{h}_4$	0.0023	0.0029	0.0031	0.0031	0.0033	0.0046	0.0048	0.0047
$\hat{h}_5$	0.0023	0.0072	0.0032	0.0031	0.0047	0.0079	0.0041	0.0027
$\hat{h}_6$	0.0023	0.0050	0.0031	0.0030	0.0047	0.0046	0.0055	0.0053
$\hat{h}_7$	0.0036	0.0056	0.0031	0.0030	0.0039	0.0045	0.0040	0.0047
$\hat{h}_8$	0.0030	0.0094	0.0032	0.0031	0.0041	0.0081	0.0034	0.0048
$\hat{h}_9$	0.0029	0.0050	0.0031	0.0030	0.0040	0.0039	0.0047	0.0047

Table 4.33: *Comparison of the MSE for different combinations of the estimated smoothing parameter (normal kernel) and $m = 5$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0032	0.0045	0.0031	0.0030	0.0048	0.0049	0.0047	0.0048
$\hat{h}_2$	0.0038	0.0062	0.0032	0.0031	0.0052	0.0065	0.0038	0.0042
$\hat{h}_3$	0.0030	0.0059	0.0034	0.0032	0.0041	0.0064	0.0041	0.0054
$\hat{h}_4$	0.0029	0.0046	0.0031	0.0031	0.0045	0.0054	0.0042	0.0051
$\hat{h}_5$	0.0035	0.0052	0.0031	0.0031	0.0050	0.0065	0.0038	0.0042
$\hat{h}_6$	0.0027	0.0039	0.0031	0.0030	0.0039	0.0040	0.0056	0.0051
$\hat{h}_7$	0.0031	0.0047	0.0031	0.0031	0.0040	0.0052	0.0045	0.0052
$\hat{h}_8$	0.0035	0.0057	0.0032	0.0031	0.0049	0.0070	0.0034	0.0039
$\hat{h}_9$	0.0032	0.0038	0.0031	0.0031	0.0045	0.0055	0.0046	0.0050

Table 4.34: *Comparison of the MSE for different combinations of the estimated smoothing parameter (normal kernel) and $m = 10$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0030	0.0043	0.0031	0.0031	0.0044	0.0051	0.0049	0.0051
$\hat{h}_2$	0.0034	0.0051	0.0032	0.0031	0.0045	0.0057	0.0040	0.0041
$\hat{h}_3$	0.0033	0.0052	0.0037	0.0033	0.0040	0.0058	0.0046	0.0053
$\hat{h}_4$	0.0033	0.0051	0.0031	0.0031	0.0051	0.0058	0.0048	0.0047
$\hat{h}_5$	0.0032	0.0051	0.0031	0.0031	0.0045	0.0057	0.0039	0.0044
$\hat{h}_6$	0.0028	0.0037	0.0031	0.0030	0.0038	0.0042	0.0055	0.0045
$\hat{h}_7$	0.0030	0.0040	0.0031	0.0031	0.0043	0.0050	0.0050	0.0050
$\hat{h}_8$	0.0034	0.0050	0.0032	0.0031	0.0044	0.0058	0.0035	0.0038
$\hat{h}_9$	0.0032	0.0045	0.0031	0.0031	0.0046	0.0052	0.0048	0.0049

When inspecting the estimated bias, it is found that $\hat{h}_1$, $\hat{h}_4$ and $\hat{h}_6$ are associated with estimated bias-values closer to zero than some of the other choices of $\hat{h}$. This is especially true for the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests. When Tables 4.30 and 4.31 are viewed, it is even clearer that $\hat{h}_6$ is performing better than the other choices of $\hat{h}$ for $m = 5$ and $m = 10$.

When comparing the MSE of the estimators for a and b, $\hat{h}_3$ and $\hat{h}_4$ result in slightly smaller values of the estimated MSE for $m = 1$. When m is increased, again it is found that $\hat{h}_6$ produces smaller values of the estimated MSE. On the other hand, certain choices of $\hat{h}$ seldom result in small values of the estimated bias and MSE, and would therefore not be recommended, such as $\hat{h}_8$ and $\hat{h}_9$.

In the light of the conclusions just made from the estimated bias and MSE, it is fair to recommend any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) as a good choice for the estimated smoothing parameter. Again, note the good performance of the Cramér-von-Mises goodness-of-fit test, especially as far as MSE is concerned.

Choice of goodness-of-fit test

SOPIE is based in a sequential way on the P-values of goodness-of-fit tests for the uniform distribution. Different goodness-of-fit tests exist and, therefore, it must be evaluated whether there is a superior goodness-of-fit test when compared to other tests. Some of the tables presented earlier, were already used to assess the goodness-of-fit tests. Moreover, the reader can inspect Tables 4.35 – 4.43 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ for each of the estimated smoothing parameters $\hat{h}$. Tables 4.44 – 4.52 are even more important, since the MSE is compared in these tables. Recall that the MSE takes both the bias and variance into account when measuring the performance of an estimator.

Table 4.35: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0388	-0.0393	0.0550	-0.0546	0.0433	-0.0409	0.0511	-0.0505
g=7	0.0392	-0.0373	0.0551	-0.0548	0.0437	-0.0413	0.0516	-0.0505
g=8	0.0393	-0.0368	0.0553	-0.0549	0.0440	-0.0408	0.0503	-0.0505
g=9	0.0395	-0.0372	0.0554	-0.0550	0.0442	-0.0413	0.0496	-0.0506
g=10	0.0400	-0.0376	0.0556	-0.0553	0.0446	-0.0425	0.0497	-0.0507

Table 4.36: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_2$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0392	-0.0434	0.0557	-0.0549	0.0444	-0.0454	0.0514	-0.0524
g=7	0.0387	-0.0421	0.0559	-0.0550	0.0448	-0.0462	0.0517	-0.0527
g=8	0.0389	-0.0414	0.0561	-0.0553	0.0432	-0.0441	0.0520	-0.0529
g=9	0.0390	-0.0413	0.0562	-0.0554	0.0434	-0.0430	0.0521	-0.0524
g=10	0.0391	-0.0406	0.0564	-0.0556	0.0429	-0.0434	0.0524	-0.0518

Table 4.37: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_3$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0387	-0.0401	0.0558	-0.0555	0.0415	-0.0443	0.0493	-0.0497
g=7	0.0388	-0.0396	0.0559	-0.0556	0.0426	-0.0445	0.0488	-0.0501
g=8	0.0390	-0.0389	0.0560	-0.0557	0.0422	-0.0449	0.0491	-0.0503
g=9	0.0392	-0.0402	0.0562	-0.0558	0.0423	-0.0443	0.0485	-0.0507
g=10	0.0395	-0.0379	0.0564	-0.0561	0.0424	-0.0437	0.0488	-0.0493

Table 4.38: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_4$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0382	-0.0367	0.0555	-0.0547	0.0427	-0.0423	0.0524	-0.0523
g=7	0.0385	-0.0363	0.0555	-0.0548	0.0429	-0.0418	0.0501	-0.0520
g=8	0.0386	-0.0358	0.0557	-0.0550	0.0425	-0.0416	0.0504	-0.0522
g=9	0.0388	-0.0371	0.0559	-0.0551	0.0425	-0.0409	0.0508	-0.0527
g=10	0.0392	-0.0364	0.0560	-0.0553	0.0427	-0.0406	0.0510	-0.0514

Table 4.39: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_5$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0380	-0.0412	0.0556	-0.0550	0.0445	-0.0459	0.0519	-0.0505
g=7	0.0385	-0.0399	0.0557	-0.0551	0.0446	-0.0457	0.0521	-0.0509
g=8	0.0386	-0.0405	0.0559	-0.0553	0.0432	-0.0454	0.0516	-0.0518
g=9	0.0389	-0.0401	0.0561	-0.0554	0.0428	-0.0442	0.0519	-0.0514
g=10	0.0391	-0.0384	0.0562	-0.0556	0.0430	-0.0428	0.0522	-0.0519

Table 4.40: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_6$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0377	-0.0399	0.0547	-0.0545	0.0428	-0.0425	0.0515	-0.0518
g=7	0.0379	-0.0380	0.0549	-0.0547	0.0424	-0.0433	0.0517	-0.0507
g=8	0.0383	-0.0380	0.0551	-0.0548	0.0430	-0.0437	0.0519	-0.0508
g=9	0.0385	-0.0383	0.0553	-0.0549	0.0423	-0.0440	0.0515	-0.0502
g=10	0.0389	-0.0386	0.0555	-0.0551	0.0428	-0.0435	0.0497	-0.0500

Table 4.41: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_7$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0392	-0.0396	0.0551	-0.0546	0.0423	-0.0421	0.0507	-0.0515
g=7	0.0393	-0.0382	0.0552	-0.0547	0.0427	-0.0420	0.0510	-0.0511
g=8	0.0389	-0.0368	0.0555	-0.0548	0.0430	-0.0415	0.0496	-0.0515
g=9	0.0390	-0.0380	0.0555	-0.0550	0.0432	-0.0419	0.0499	-0.0511
g=10	0.0392	-0.0383	0.0558	-0.0552	0.0435	-0.0431	0.0501	-0.0507

Table 4.42: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_8$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0390	-0.0442	0.0557	-0.0551	0.0435	-0.0468	0.0513	-0.0533
g=7	0.0385	-0.0404	0.0558	-0.0552	0.0440	-0.0459	0.0518	-0.0529
g=8	0.0388	-0.0410	0.0560	-0.0554	0.0423	-0.0438	0.0520	-0.0532
g=9	0.0390	-0.0395	0.0562	-0.0556	0.0427	-0.0441	0.0520	-0.0527
g=10	0.0392	-0.0391	0.0564	-0.0557	0.0428	-0.0429	0.0525	-0.0522

Table 4.43: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_9$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0385	-0.0389	0.0555	-0.0544	0.0428	-0.0411	0.0517	-0.0518
g=7	0.0382	-0.0368	0.0557	-0.0547	0.0427	-0.0423	0.0496	-0.0514
g=8	0.0383	-0.0372	0.0558	-0.0548	0.0418	-0.0415	0.0498	-0.0518
g=9	0.0384	-0.0376	0.0560	-0.0549	0.0420	-0.0418	0.0501	-0.0508
g=10	0.0388	-0.0378	0.0562	-0.0550	0.0424	-0.0429	0.0504	-0.0510

Table 4.44: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0036	0.0050	0.0031	0.0030	0.0046	0.0032	0.0047	0.0040
g=7	0.0036	0.0029	0.0031	0.0031	0.0047	0.0032	0.0047	0.0033
g=8	0.0036	0.0021	0.0031	0.0031	0.0047	0.0025	0.0034	0.0034
g=9	0.0036	0.0021	0.0031	0.0031	0.0047	0.0025	0.0026	0.0027
g=10	0.0037	0.0021	0.0032	0.0031	0.0047	0.0032	0.0026	0.0027

Table 4.45: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_2$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0030	0.0087	0.0032	0.0031	0.0048	0.0074	0.0034	0.0040
g=7	0.0023	0.0072	0.0032	0.0031	0.0048	0.0075	0.0034	0.0041
g=8	0.0023	0.0065	0.0032	0.0031	0.0033	0.0052	0.0034	0.0041
g=9	0.0023	0.0058	0.0032	0.0031	0.0034	0.0039	0.0034	0.0035
g=10	0.0023	0.0051	0.0032	0.0032	0.0027	0.0039	0.0035	0.0028

Table 4.46: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_3$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0016	0.0043	0.0032	0.0031	0.0019	0.0046	0.0038	0.0039
g=7	0.0016	0.0036	0.0032	0.0032	0.0026	0.0046	0.0032	0.0039
g=8	0.0017	0.0028	0.0032	0.0032	0.0019	0.0046	0.0032	0.0039
g=9	0.0017	0.0036	0.0032	0.0032	0.0019	0.0039	0.0025	0.0039
g=10	0.0017	0.0015	0.0033	0.0032	0.0020	0.0032	0.0025	0.0026

Table 4.47: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_4$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0023	0.0029	0.0031	0.0031	0.0033	0.0046	0.0048	0.0047
g=7	0.0023	0.0022	0.0031	0.0031	0.0033	0.0039	0.0027	0.0040
g=8	0.0023	0.0015	0.0032	0.0031	0.0026	0.0032	0.0027	0.0040
g=9	0.0023	0.0022	0.0032	0.0031	0.0027	0.0025	0.0027	0.0041
g=10	0.0023	0.0015	0.0032	0.0031	0.0027	0.0018	0.0027	0.0027

Table 4.48: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_5$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0023	0.0072	0.0032	0.0031	0.0047	0.0079	0.0041	0.0027
g=7	0.0023	0.0057	0.0032	0.0031	0.0047	0.0074	0.0041	0.0027
g=8	0.0023	0.0057	0.0032	0.0031	0.0034	0.0066	0.0034	0.0034
g=9	0.0023	0.0050	0.0032	0.0031	0.0027	0.0052	0.0034	0.0028
g=10	0.0023	0.0035	0.0032	0.0032	0.0027	0.0038	0.0035	0.0028

Table 4.49: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_6$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0023	0.0050	0.0031	0.0030	0.0047	0.0046	0.0055	0.0053
g=7	0.0023	0.0028	0.0031	0.0030	0.0040	0.0046	0.0055	0.0040
g=8	0.0023	0.0028	0.0031	0.0031	0.0040	0.0046	0.0055	0.0040
g=9	0.0023	0.0028	0.0031	0.0031	0.0033	0.0046	0.0048	0.0033
g=10	0.0023	0.0028	0.0031	0.0031	0.0034	0.0039	0.0033	0.0026

Table 4.50: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_7$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0036	0.0056	0.0031	0.0030	0.0039	0.0045	0.0040	0.0047
g=7	0.0036	0.0035	0.0031	0.0031	0.0040	0.0038	0.0040	0.0040
g=8	0.0030	0.0021	0.0031	0.0031	0.0039	0.0031	0.0026	0.0040
g=9	0.0030	0.0028	0.0031	0.0031	0.0040	0.0031	0.0027	0.0034
g=10	0.0030	0.0028	0.0032	0.0031	0.0040	0.0037	0.0027	0.0027

Table 4.51: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_8$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0030	0.0094	0.0032	0.0031	0.0041	0.0081	0.0034	0.0048
g=7	0.0023	0.0058	0.0032	0.0031	0.0041	0.0068	0.0034	0.0041
g=8	0.0023	0.0057	0.0032	0.0031	0.0026	0.0045	0.0034	0.0041
g=9	0.0023	0.0043	0.0032	0.0032	0.0027	0.0046	0.0034	0.0035
g=10	0.0023	0.0036	0.0032	0.0032	0.0027	0.0032	0.0035	0.0028

Table 4.52: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_9$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0029	0.0050	0.0031	0.0030	0.0040	0.0039	0.0047	0.0047
g=7	0.0023	0.0029	0.0032	0.0031	0.0033	0.0046	0.0027	0.0040
g=8	0.0023	0.0028	0.0032	0.0031	0.0027	0.0031	0.0027	0.0040
g=9	0.0023	0.0028	0.0032	0.0031	0.0027	0.0032	0.0027	0.0027
g=10	0.0023	0.0028	0.0032	0.0031	0.0027	0.0038	0.0027	0.0027

In terms of the estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Cramér-von-Mises test. The Rayleigh goodness-of-fit test performs worst in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs best, followed by the other tests. Due to the importance of the MSE, it is recommended to use the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests.

Choice of the significance level α

The significance level α is another tuning parameter that may influence the point where rejection of uniformity takes place for each goodness-of-fit test, and therefore α may influence the values of $\hat{a}$ and $\hat{b}$. Several different values of α are utilised in the simulation study. Careful consideration must be given when the value of α is selected. Should α be chosen too small, rejection of uniformity may occur too soon, resulting in a large bias. On the contrary, if α is chosen too large, rejection of uniformity will not occur soon enough, also resulting in biased estimation of a and b. Several tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit tests. Tables 4.53 – 4.56 compare the estimated bias and MSE for the four goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_1$. From these tables it is concluded that the estimated bias is not sensitive to the choice of α, but that a slightly smaller estimated MSE results from $\alpha = 0.01$.

Tables 4.57 – 4.60 compare the estimated bias and MSE for the goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_4$. Tables 4.61 – 4.64 present the same comparison, but for $\hat{h}_6$ and Tables 4.65 – 4.68 for $\hat{h}_7$. Furthermore, Tables 4.69 – 4.72 compare the estimated bias and MSE when $\hat{h}_6$ and $m = 5$ are chosen.

Table 4.53: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0388	-0.0393	0.0036	0.0366	0.0366	-0.0447	0.0091	0.0186
g=7	0.0392	-0.0373	0.0036	0.0341	0.0341	-0.0429	0.0062	0.0163
g=8	0.0393	-0.0368	0.0036	0.0342	0.0342	-0.0441	0.0054	0.0164
g=9	0.0395	-0.0372	0.0036	0.0339	0.0339	-0.0434	0.0048	0.0156
g=10	0.0400	-0.0376	0.0037	0.0343	0.0343	-0.0404	0.0048	0.0128

Table 4.54: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0550	-0.0546	0.0031	0.0030	0.0526	-0.0521	0.0028	0.0028
g=7	0.0551	-0.0548	0.0031	0.0031	0.0528	-0.0522	0.0029	0.0028
g=8	0.0553	-0.0549	0.0031	0.0031	0.0530	-0.0524	0.0029	0.0028
g=9	0.0554	-0.0550	0.0031	0.0031	0.0531	-0.0525	0.0029	0.0028
g=10	0.0556	-0.0553	0.0032	0.0031	0.0532	-0.0527	0.0029	0.0028

Table 4.55: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0433	-0.0409	0.0046	0.0032	0.0422	-0.0507	0.0123	0.0215
g=7	0.0437	-0.0413	0.0047	0.0032	0.0403	-0.0493	0.0103	0.0194
g=8	0.0440	-0.0408	0.0047	0.0025	0.0407	-0.0479	0.0096	0.0173
g=9	0.0442	-0.0413	0.0047	0.0025	0.0403	-0.0433	0.0088	0.0128
g=10	0.0446	-0.0425	0.0047	0.0032	0.0401	-0.0433	0.0081	0.0122

Table 4.56: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Rayleigh goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0511	-0.0505	0.0047	0.0040	0.0572	-0.0539	0.0184	0.0153
g=7	0.0516	-0.0505	0.0047	0.0033	0.0522	-0.0511	0.0127	0.0119
g=8	0.0503	-0.0505	0.0034	0.0034	0.0497	-0.0494	0.0099	0.0091
g=9	0.0496	-0.0506	0.0026	0.0027	0.0490	-0.0472	0.0085	0.0070
g=10	0.0497	-0.0507	0.0026	0.0027	0.0486	-0.0477	0.0078	0.0070

Table 4.57: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_4$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0382	-0.0367	0.0023	0.0029	0.0385	-0.0471	0.0105	0.0221
g=7	0.0385	-0.0363	0.0023	0.0022	0.0399	-0.0433	0.0105	0.0172
g=8	0.0386	-0.0358	0.0023	0.0015	0.0396	-0.0418	0.0097	0.0151
g=9	0.0388	-0.0371	0.0023	0.0022	0.0390	-0.0418	0.0090	0.0143
g=10	0.0392	-0.0364	0.0023	0.0015	0.0399	-0.0403	0.0091	0.0130

Table 4.58: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_4$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0555	-0.0547	0.0031	0.0031	0.0530	-0.0521	0.0029	0.0028
g=7	0.0555	-0.0548	0.0031	0.0031	0.0532	-0.0522	0.0029	0.0028
g=8	0.0557	-0.0550	0.0032	0.0031	0.0534	-0.0524	0.0029	0.0028
g=9	0.0559	-0.0551	0.0032	0.0031	0.0535	-0.0525	0.0029	0.0028
g=10	0.0560	-0.0553	0.0032	0.0031	0.0537	-0.0527	0.0029	0.0028

Table 4.59: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_4$ and the normal kernel (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0427	-0.0423	0.0033	0.0046	0.0455	-0.0492	0.0150	0.0205
g=7	0.0429	-0.0418	0.0033	0.0039	0.0457	-0.0512	0.0137	0.0214
g=8	0.0425	-0.0416	0.0026	0.0032	0.0451	-0.0519	0.0130	0.0213
g=9	0.0425	-0.0409	0.0027	0.0025	0.0444	-0.0485	0.0121	0.0172
g=10	0.0427	-0.0406	0.0027	0.0018	0.0434	-0.0477	0.0109	0.0159

Table 4.60: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_4$ and the normal kernel (Rayleigh goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0524	-0.0523	0.0048	0.0047	0.0554	-0.0552	0.0149	0.0153
g=7	0.0501	-0.0520	0.0027	0.0040	0.0535	-0.0517	0.0121	0.0111
g=8	0.0504	-0.0522	0.0027	0.0040	0.0507	-0.0482	0.0092	0.0077
g=9	0.0508	-0.0527	0.0027	0.0041	0.0491	-0.0463	0.0070	0.0056
g=10	0.0510	-0.0514	0.0027	0.0027	0.0476	-0.0469	0.0057	0.0056

Table 4.61: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_6$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0377	-0.0399	0.0023	0.0050	0.0364	-0.0422	0.0093	0.0156
g=7	0.0379	-0.0380	0.0023	0.0028	0.0356	-0.0423	0.0078	0.0149
g=8	0.0383	-0.0380	0.0023	0.0028	0.0353	-0.0392	0.0071	0.0113
g=9	0.0385	-0.0383	0.0023	0.0028	0.0341	-0.0378	0.0057	0.0099
g=10	0.0389	-0.0386	0.0023	0.0028	0.0338	-0.0392	0.0050	0.0105

Table 4.62: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_6$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0547	-0.0545	0.0031	0.0030	0.0525	-0.0520	0.0028	0.0028
g=7	0.0549	-0.0547	0.0031	0.0030	0.0526	-0.0521	0.0028	0.0028
g=8	0.0551	-0.0548	0.0031	0.0031	0.0528	-0.0523	0.0029	0.0028
g=9	0.0553	-0.0549	0.0031	0.0031	0.0530	-0.0525	0.0029	0.0028
g=10	0.0555	-0.0551	0.0031	0.0031	0.0531	-0.0527	0.0029	0.0028

Table 4.63: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_6$ and the normal kernel (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0428	-0.0425	0.0047	0.0046	0.0417	-0.0462	0.0123	0.0165
g=7	0.0424	-0.0433	0.0040	0.0046	0.0423	-0.0454	0.0125	0.0150
g=8	0.0430	-0.0437	0.0040	0.0046	0.0376	-0.0465	0.0081	0.0158
g=9	0.0423	-0.0440	0.0033	0.0046	0.0378	-0.0413	0.0074	0.0101
g=10	0.0428	-0.0435	0.0034	0.0039	0.0391	-0.0427	0.0081	0.0108

Table 4.64: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_6$ and the normal kernel (Rayleigh goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0515	-0.0518	0.0055	0.0053	0.0497	-0.0571	0.0130	0.0193
g=7	0.0517	-0.0507	0.0055	0.0040	0.0453	-0.0507	0.0086	0.0128
g=8	0.0519	-0.0508	0.0055	0.0040	0.0463	-0.0492	0.0087	0.0107
g=9	0.0515	-0.0502	0.0048	0.0033	0.0452	-0.0473	0.0071	0.0085
g=10	0.0497	-0.0500	0.0033	0.0026	0.0457	-0.0460	0.0064	0.0063

Table 4.65: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_7$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0392	-0.0396	0.0036	0.0056	0.0373	-0.0436	0.0091	0.0174
g=7	0.0393	-0.0382	0.0036	0.0035	0.0373	-0.0416	0.0083	0.0153
g=8	0.0389	-0.0368	0.0030	0.0021	0.0372	-0.0405	0.0076	0.0138
g=9	0.0390	-0.0380	0.0030	0.0028	0.0352	-0.0396	0.0055	0.0124
g=10	0.0392	-0.0383	0.0030	0.0028	0.0365	-0.0393	0.0062	0.0116

Table 4.66: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_7$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0551	-0.0546	0.0031	0.0030	0.0527	-0.0520	0.0028	0.0028
g=7	0.0552	-0.0547	0.0031	0.0031	0.0529	-0.0522	0.0029	0.0028
g=8	0.0555	-0.0548	0.0031	0.0031	0.0531	-0.0524	0.0029	0.0028
g=9	0.0555	-0.0550	0.0031	0.0031	0.0532	-0.0525	0.0029	0.0028
g=10	0.0558	-0.0552	0.0032	0.0031	0.0533	-0.0527	0.0029	0.0028

Table 4.67: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_7$ and the normal kernel (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0423	-0.0421	0.0039	0.0045	0.0456	-0.0522	0.0144	0.0217
g=7	0.0427	-0.0420	0.0040	0.0038	0.0417	-0.0489	0.0108	0.0189
g=8	0.0430	-0.0415	0.0039	0.0031	0.0403	-0.0499	0.0087	0.0190
g=9	0.0432	-0.0419	0.0040	0.0031	0.0383	-0.0451	0.0066	0.0145
g=10	0.0435	-0.0431	0.0040	0.0037	0.0398	-0.0436	0.0073	0.0125

Table 4.68: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_7$ and the normal kernel (Rayleigh goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0507	-0.0515	0.0040	0.0047	0.0538	-0.0562	0.0148	0.0165
g=7	0.0510	-0.0511	0.0040	0.0040	0.0512	-0.0562	0.0112	0.0159
g=8	0.0496	-0.0515	0.0026	0.0040	0.0470	-0.0507	0.0072	0.0104
g=9	0.0499	-0.0511	0.0027	0.0034	0.0495	-0.0498	0.0085	0.0091
g=10	0.0501	-0.0507	0.0027	0.0027	0.0466	-0.0494	0.0057	0.0084

Table 4.69: *Comparison of the estimated bias and MSE for different α-values, with $m = 5, r = 6, \hat{h}_6$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0379	-0.0386	0.0027	0.0039	0.0376	-0.0405	0.0103	0.0140
g=7	0.0378	-0.0384	0.0024	0.0031	0.0377	-0.0393	0.0097	0.0122
g=8	0.0382	-0.0381	0.0024	0.0027	0.0361	-0.0387	0.0077	0.0109
g=9	0.0383	-0.0380	0.0023	0.0024	0.0352	-0.0380	0.0064	0.0099
g=10	0.0381	-0.0385	0.0019	0.0026	0.0356	-0.0388	0.0064	0.0102

Table 4.70: *Comparison of the estimated bias and MSE for different α-values, with $m = 5, r = 6, \hat{h}_6$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0548	-0.0545	0.0031	0.0030	0.0525	-0.0521	0.0028	0.0028
g=7	0.0549	-0.0547	0.0031	0.0031	0.0526	-0.0522	0.0028	0.0028
g=8	0.0551	-0.0549	0.0031	0.0031	0.0527	-0.0524	0.0028	0.0028
g=9	0.0553	-0.0550	0.0031	0.0031	0.0529	-0.0525	0.0029	0.0028
g=10	0.0555	-0.0552	0.0031	0.0031	0.0530	-0.0527	0.0029	0.0028

Table 4.71: *Comparison of the estimated bias and MSE for different α-values, with $m = 5, r = 6, \hat{h}_6$ and the normal kernel (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0417	-0.0420	0.0039	0.0040	0.0429	-0.0476	0.0135	0.0178
g=7	0.0421	-0.0426	0.0039	0.0040	0.0406	-0.0464	0.0112	0.0160
g=8	0.0423	-0.0426	0.0036	0.0037	0.0393	-0.0451	0.0094	0.0142
g=9	0.0423	-0.0429	0.0033	0.0036	0.0383	-0.0432	0.0081	0.0118
g=10	0.0422	-0.0428	0.0029	0.0034	0.0390	-0.0430	0.0081	0.0112

Table 4.72: *Comparison of the estimated bias and MSE for different α-values, with $m = 5, r = 6, \hat{h}_6$ and the normal kernel (Rayleigh goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$			
	Bias		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0515	-0.0513	0.0056	0.0051	0.0529	-0.0535	0.0160	0.0164
g=7	0.0519	-0.0502	0.0057	0.0038	0.0486	-0.0485	0.0114	0.0111
g=8	0.0518	-0.0500	0.0055	0.0034	0.0485	-0.0472	0.0104	0.0092
g=9	0.0511	-0.0503	0.0045	0.0034	0.0480	-0.0469	0.0091	0.0081
g=10	0.0504	-0.0504	0.0038	0.0032	0.0479	-0.0460	0.0083	0.0064

From all of these tables, similar conclusions can be made. The bias seems to be insensitive to the choice of α, but a smaller estimated MSE results from $\alpha = 0.01$. The remark is therefore made that the choice of $\alpha = 0.01$ would be slightly better, but any value of α in the range from 0.01 to 0.05 will have only a marginal effect on the results. It is again observed that the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests perform slightly better in terms of bias and MSE.

Choice of incremental growth g and number of intervals of rejection r

The choice of g, the incremental growth of each interval over which rejection is tested, together with the choice of r, the number of intervals before rejection of uniformity takes place, are two important tuning parameters in the estimation process. Together, these two parameters will determine the accuracy of the estimated point where rejection of uniformity takes place. Large values (and very small values) of g may influence the point of rejection negatively, as illustrated in Figure 4.10 and explained with the following example.

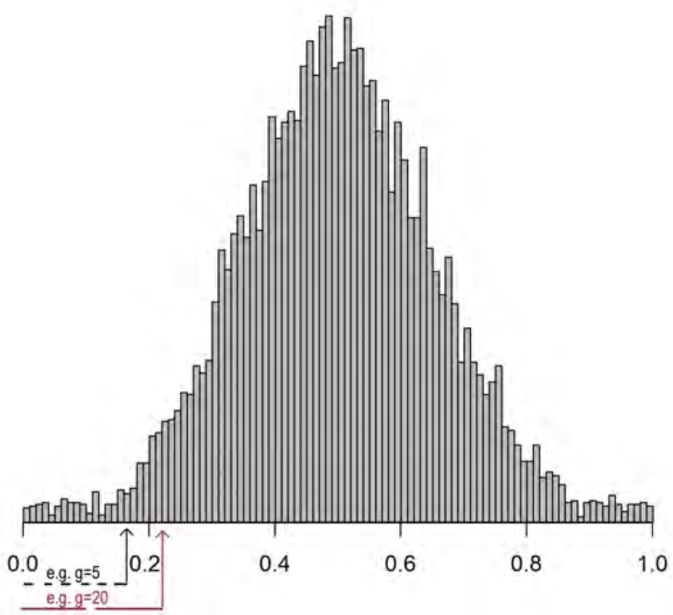

Figure 4.10: *Effect of the choice of g on the point of rejection.*

At the bottom of Figure 4.10, the black line segments indicate the intervals over which rejection is tested, e.g., say an interval of length 5. Furthermore, let $r = 1$. The interval will grow in each subsequent step with 5 observations, until the first rejection of uniformity takes place (in the illustration after 6 iterations). The hypothetical point of rejection is illustrated with the black arrow. If the value of g is chosen to be 20, for example, as illustrated in red, then the estimated point where rejection of uniformity takes place, maybe much larger, as indicated by the red arrow. In this example rejection took place after 2 iterations. On the other hand, if g is chosen too small, rejection may take place too early (due to a possible clustering of noise), causing the estimation to be biased.

For the tuning parameter r, larger values of r will not necessarily have a negative impact on the estimated point where rejection takes place. The reason can be explained with the following example. Suppose $r = 5$ and hypothetically the point of rejection of uniformity is calculated as z, i.e., $\hat{b} = z$. This implies that z was the largest value in the first interval where rejection of uniformity took place, followed by $r-1$ subsequent rejections of uniformity for intervals growing at a rate of g observations. Suppose the same example is used, except that $r = 10$. If rejection of uniformity took place not only for 5 subsequent intervals growing at a rate of g observations per iteration, but also for 5 more iterations, then the same point of rejection of uniformity (i.e., $\hat{b} = z$) would have been identified. Therefore, the parameter r can be seen as a tuning parameter since it serves the purpose of confirming that, after a certain point where rejection of uniformity repeatedly occurred, rejection of uniformity continues to take place for several more intervals.

On close inspection of Tables 4.73 – 4.88, the reader will find some numeric evidence for the explanation just provided. As said above, a too large and too small value of g may result in inaccurate estimation of the value of a and b. For $g < 5$ and $g > 10$, large values of the estimated bias can be clearly seen. For $5 \leq g \leq 10$ the best results are obtained for the estimated bias and the MSE, albeit, the variation in the bias and MSE-values for different combinations of r and g are very small.

The influence of the tuning parameter r can also be investigated from the tables. From $g = 9$ up to the bottom of the table, the bias and MSE reach a certain value for a selected r, and when r is increased even more, the values of the bias and MSE remain constant for a fixed value of g. This shows that after a certain point where rejection of uniformity repeatedly occurred, rejection of uniformity continues to take place for several more intervals.

In conclusion, it is recommended to use values of g in the range from 5 to 10, and r-values from 6 to 8 for moderate to large sample sizes.

Table 4.73: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0672	0.0545	0.0493	0.0472	0.0445					
g=2	0.0651	0.0469	0.0434	0.0430	0.0409					
g=3	0.0557	0.0442	0.0413	0.0402	0.0401					
g=4	0.0537	0.0449	0.0408	0.0402	0.0403					
g=5	0.0554	0.0414	0.0401	0.0406	0.0393					
g=6						0.0388	0.0391	0.0391	0.0391	0.0393
g=7						0.0392	0.0392	0.0393	0.0396	0.0396
g=8						0.0393	0.0396	0.0398	0.0391	0.0384
g=9						0.0395	0.0399	0.0400	0.0393	0.0385
g=10						0.0400	0.0396	0.0388	0.0380	0.0380
g=20		0.0404		0.0408		0.0402		0.0402		0.0402
g=25		0.0433		0.0409		0.0411		0.0411		0.0411
g=30		0.0440		0.0424		0.0424		0.0424		0.0424
g=35		0.0441		0.0431		0.0431		0.0431		0.0431
g=40		0.0455		0.0443		0.0443		0.0443		0.0443
g=45		0.0469		0.0451		0.0451		0.0451		0.0451
g=50		0.0462		0.0459		0.0459		0.0459		0.0459
g=100		0.0540		0.0540		0.0540		0.0540		0.0540
g=200		0.0715		0.0715		0.0715		0.0715		0.0715
g=300		0.0797		0.0797		0.0797		0.0797		0.0797
g=400		0.0989		0.0989		0.0989		0.0989		0.0989
g=500		0.1170		0.1170		0.1170		0.1170		0.1170

Table 4.74: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.0796	-0.0615	-0.0544	-0.0497	-0.0441					
g=2	-0.0674	-0.0555	-0.0485	-0.0456	-0.0440					
g=3	-0.0668	-0.0509	-0.0463	-0.0430	-0.0397					
g=4	-0.0628	-0.0461	-0.0416	-0.0402	-0.0403					
g=5	-0.0607	-0.0474	-0.0407	-0.0400	-0.0403					
g=6						-0.0393	-0.0364	-0.0366	-0.0368	-0.0369
g=7						-0.0373	-0.0375	-0.0370	-0.0371	-0.0364
g=8						-0.0368	-0.0371	-0.0372	-0.0365	-0.0365
g=9						-0.0372	-0.0366	-0.0366	-0.0366	-0.0366
g=10						-0.0376	-0.0369	-0.0369	-0.0369	-0.0369
g=20		-0.0424		-0.0388		-0.0388		-0.0388		-0.0388
g=25		-0.0413		-0.0399		-0.0399		-0.0399		-0.0399
g=30		-0.0427		-0.0412		-0.0412		-0.0412		-0.0412
g=35		-0.0424		-0.0421		-0.0421		-0.0421		-0.0421
g=40		-0.0444		-0.0429		-0.0429		-0.0429		-0.0429
g=45		-0.0436		-0.0438		-0.0438		-0.0438		-0.0438
g=50		-0.0445		-0.0447		-0.0447		-0.0447		-0.0447
g=100		-0.0530		-0.0530		-0.0530		-0.0530		-0.0530
g=200		-0.0707		-0.0707		-0.0707		-0.0707		-0.0707
g=300		-0.0794		-0.0794		-0.0794		-0.0794		-0.0794
g=400		-0.0989		-0.0989		-0.0989		-0.0989		-0.0989
g=500		-0.1171		-0.1171		-0.1171		-0.1171		-0.1171

Table 4.75: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0411	0.0264	0.0204	0.0170	0.0136					
g=2	0.0376	0.0185	0.0129	0.0107	0.0079					
g=3	0.0284	0.0136	0.0093	0.0072	0.0064					
g=4	0.0261	0.0136	0.0079	0.0064	0.0057					
g=5	0.0259	0.0093	0.0064	0.0057	0.0043					
g=6						0.0036	0.0036	0.0036	0.0036	0.0036
g=7						0.0036	0.0036	0.0036	0.0036	0.0036
g=8						0.0036	0.0036	0.0036	0.0028	0.0022
g=9						0.0036	0.0036	0.0036	0.0028	0.0022
g=10						0.0037	0.0029	0.0022	0.0016	0.0016
g=20		0.0031		0.0024		0.0017		0.0017		0.0017
g=25		0.0046		0.0018		0.0018		0.0018		0.0018
g=30		0.0040		0.0019		0.0019		0.0019		0.0019
g=35		0.0033		0.0020		0.0020		0.0020		0.0020
g=40		0.0034		0.0021		0.0021		0.0021		0.0021
g=45		0.0043		0.0021		0.0021		0.0021		0.0021
g=50		0.0030		0.0022		0.0022		0.0022		0.0022
g=100		0.0031		0.0031		0.0031		0.0031		0.0031
g=200		0.0054		0.0054		0.0054		0.0054		0.0054
g=300		0.0068		0.0068		0.0068		0.0068		0.0068
g=400		0.0100		0.0100		0.0100		0.0100		0.0100
g=500		0.0138		0.0138		0.0138		0.0138		0.0138

Table 4.76: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0531	0.0348	0.0271	0.0214	0.0151					
g=2	0.0417	0.0271	0.0187	0.0152	0.0131					
g=3	0.0402	0.0220	0.0159	0.0115	0.0080					
g=4	0.0361	0.0165	0.0101	0.0079	0.0072					
g=5	0.0328	0.0166	0.0095	0.0072	0.0064					
g=6						0.0050	0.0021	0.0021	0.0021	0.0021
g=7						0.0029	0.0028	0.0021	0.0021	0.0014
g=8						0.0021	0.0021	0.0021	0.0014	0.0014
g=9						0.0021	0.0015	0.0015	0.0015	0.0015
g=10						0.0021	0.0015	0.0015	0.0015	0.0015
g=20		0.0059		0.0016		0.0016		0.0016		0.0016
g=25		0.0039		0.0017		0.0017		0.0017		0.0017
g=30		0.0039		0.0018		0.0018		0.0018		0.0018
g=35		0.0026		0.0019		0.0019		0.0019		0.0019
g=40		0.0034		0.0020		0.0020		0.0020		0.0020
g=45		0.0021		0.0021		0.0021		0.0021		0.0021
g=50		0.0022		0.0022		0.0022		0.0022		0.0022
g=100		0.0031		0.0031		0.0031		0.0031		0.0031
g=200		0.0053		0.0053		0.0053		0.0053		0.0053
g=300		0.0067		0.0067		0.0067		0.0067		0.0067
g=400		0.0100		0.0100		0.0100		0.0100		0.0100
g=500		0.0139		0.0139		0.0139		0.0139		0.0139

Table 4.77: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0534	0.0539	0.0540	0.0541	0.0542					
g=2	0.0537	0.0541	0.0542	0.0543	0.0543					
g=3	0.0541	0.0543	0.0544	0.0545	0.0545					
g=4	0.0543	0.0545	0.0546	0.0546	0.0546					
g=5	0.0545	0.0547	0.0547	0.0548	0.0548					
g=6						0.0550	0.0550	0.0550	0.0550	0.0550
g=7						0.0551	0.0551	0.0551	0.0551	0.0551
g=8						0.0553	0.0553	0.0553	0.0553	0.0553
g=9						0.0554	0.0554	0.0554	0.0554	0.0554
g=10						0.0556	0.0556	0.0556	0.0556	0.0556
g=20		0.0572		0.0572		0.0572		0.0572		0.0572
g=25		0.0581		0.0581		0.0581		0.0581		0.0581
g=30		0.0590		0.0590		0.0590		0.0590		0.0590
g=35		0.0596		0.0596		0.0596		0.0596		0.0596
g=40		0.0604		0.0604		0.0604		0.0604		0.0604
g=45		0.0612		0.0612		0.0612		0.0612		0.0612
g=50		0.0621		0.0621		0.0621		0.0621		0.0621
g=100		0.0696		0.0696		0.0696		0.0696		0.0696
g=200		0.0800		0.0800		0.0800		0.0800		0.0800
g=300		0.0904		0.0904		0.0904		0.0904		0.0904
g=400		0.0992		0.0992		0.0992		0.0992		0.0992
g=500		0.1170		0.1170		0.1170		0.1170		0.1170

Table 4.78: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.0530	-0.0535	-0.0536	-0.0537	-0.0538					
g=2	-0.0535	-0.0537	-0.0539	-0.0540	-0.0540					
g=3	-0.0536	-0.0540	-0.0541	-0.0542	-0.0542					
g=4	-0.0540	-0.0542	-0.0543	-0.0543	-0.0544					
g=5	-0.0542	-0.0544	-0.0544	-0.0545	-0.0545					
g=6						-0.0546	-0.0546	-0.0546	-0.0546	-0.0546
g=7						-0.0548	-0.0548	-0.0548	-0.0548	-0.0548
g=8						-0.0549	-0.0549	-0.0549	-0.0549	-0.0549
g=9						-0.0550	-0.0550	-0.0550	-0.0550	-0.0550
g=10						-0.0553	-0.0553	-0.0553	-0.0553	-0.0553
g=20		-0.0571		-0.0571		-0.0571		-0.0571		-0.0571
g=25		-0.0578		-0.0578		-0.0578		-0.0578		-0.0578
g=30		-0.0586		-0.0586		-0.0586		-0.0586		-0.0586
g=35		-0.0594		-0.0594		-0.0594		-0.0594		-0.0594
g=40		-0.0604		-0.0604		-0.0604		-0.0604		-0.0604
g=45		-0.0608		-0.0608		-0.0608		-0.0608		-0.0608
g=50		-0.0621		-0.0621		-0.0621		-0.0621		-0.0621
g=100		-0.0695		-0.0695		-0.0695		-0.0695		-0.0695
g=200		-0.0797		-0.0797		-0.0797		-0.0797		-0.0797
g=300		-0.0907		-0.0907		-0.0907		-0.0907		-0.0907
g=400		-0.0991		-0.0991		-0.0991		-0.0991		-0.0991
g=500		-0.1171		-0.1171		-0.1171		-0.1171		-0.1171

Table 4.79: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0029	0.0030	0.0030	0.0030	0.0030					
g=2	0.0030	0.0030	0.0030	0.0030	0.0030					
g=3	0.0030	0.0030	0.0030	0.0030	0.0030					
g=4	0.0030	0.0030	0.0030	0.0030	0.0030					
g=5	0.0030	0.0031	0.0031	0.0031	0.0031					
g=6						0.0031	0.0031	0.0031	0.0031	0.0031
g=7						0.0031	0.0031	0.0031	0.0031	0.0031
g=8						0.0031	0.0031	0.0031	0.0031	0.0031
g=9						0.0031	0.0031	0.0031	0.0031	0.0031
g=10						0.0032	0.0032	0.0032	0.0032	0.0032
g=20		0.0033		0.0033		0.0033		0.0033		0.0033
g=25		0.0034		0.0034		0.0034		0.0034		0.0034
g=30		0.0035		0.0035		0.0035		0.0035		0.0035
g=35		0.0036		0.0036		0.0036		0.0036		0.0036
g=40		0.0037		0.0037		0.0037		0.0037		0.0037
g=45		0.0038		0.0038		0.0038		0.0038		0.0038
g=50		0.0039		0.0039		0.0039		0.0039		0.0039
g=100		0.0050		0.0050		0.0050		0.0050		0.0050
g=200		0.0066		0.0066		0.0066		0.0066		0.0066
g=300		0.0085		0.0085		0.0085		0.0085		0.0085
g=400		0.0101		0.0101		0.0101		0.0101		0.0101
g=500		0.0138		0.0138		0.0138		0.0138		0.0138

Table 4.80: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0029	0.0029	0.0029	0.0029	0.0030					
g=2	0.0029	0.0030	0.0030	0.0030	0.0030					
g=3	0.0030	0.0030	0.0030	0.0030	0.0030					
g=4	0.0030	0.0030	0.0030	0.0030	0.0030					
g=5	0.0030	0.0030	0.0030	0.0030	0.0030					
g=6						0.0030	0.0030	0.0030	0.0030	0.0030
g=7						0.0031	0.0031	0.0031	0.0031	0.0031
g=8						0.0031	0.0031	0.0031	0.0031	0.0031
g=9						0.0031	0.0031	0.0031	0.0031	0.0031
g=10						0.0031	0.0031	0.0031	0.0031	0.0031
g=20		0.0033		0.0033		0.0033		0.0033		0.0033
g=25		0.0034		0.0034		0.0034		0.0034		0.0034
g=30		0.0035		0.0035		0.0035		0.0035		0.0035
g=35		0.0036		0.0036		0.0036		0.0036		0.0036
g=40		0.0037		0.0037		0.0037		0.0037		0.0037
g=45		0.0038		0.0038		0.0038		0.0038		0.0038
g=50		0.0039		0.0039		0.0039		0.0039		0.0039
g=100		0.0049		0.0049		0.0049		0.0049		0.0049
g=200		0.0066		0.0066		0.0066		0.0066		0.0066
g=300		0.0086		0.0086		0.0086		0.0086		0.0086
g=400		0.0100		0.0100		0.0100		0.0100		0.0100
g=500		0.0139		0.0139		0.0139		0.0139		0.0139

Table 4.81: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0699	0.0589	0.0532	0.0501	0.0476					
g=2	0.0679	0.0545	0.0456	0.0418	0.0418					
g=3	0.0576	0.0464	0.0415	0.0415	0.0397					
g=4	0.0659	0.0449	0.0411	0.0397	0.0389					
g=5	0.0630	0.0450	0.0401	0.0377	0.0378					
g=6						0.0377	0.0378	0.0379	0.0379	0.0382
g=7						0.0379	0.0380	0.0380	0.0383	0.0375
g=8						0.0383	0.0383	0.0386	0.0379	0.0379
g=9						0.0385	0.0386	0.0378	0.0379	0.0379
g=10						0.0389	0.0393	0.0384	0.0384	0.0384
g=20		0.0433		0.0412		0.0403		0.0403		0.0403
g=25		0.0424		0.0412		0.0412		0.0412		0.0412
g=30		0.0413		0.0422		0.0422		0.0422		0.0422
g=35		0.0445		0.0433		0.0433		0.0433		0.0433
g=40		0.0460		0.0445		0.0445		0.0445		0.0445
g=45		0.0455		0.0451		0.0451		0.0451		0.0451
g=50		0.0467		0.0463		0.0463		0.0463		0.0463
g=100		0.0541		0.0541		0.0541		0.0541		0.0541
g=200		0.0707		0.0707		0.0707		0.0707		0.0707
g=300		0.0790		0.0790		0.0790		0.0790		0.0790
g=400		0.0994		0.0994		0.0994		0.0994		0.0994
g=500		0.1174		0.1174		0.1174		0.1174		0.1174

Table 4.82: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.0682	-0.0545	-0.0504	-0.0476	-0.0449					
g=2	-0.0638	-0.0513	-0.0444	-0.0445	-0.0428					
g=3	-0.0588	-0.0480	-0.0445	-0.0426	-0.0416					
g=4	-0.0610	-0.0460	-0.0416	-0.0407	-0.0415					
g=5	-0.0579	-0.0436	-0.0423	-0.0414	-0.0415					
g=6						-0.0399	-0.0379	-0.0379	-0.0379	-0.0379
g=7						-0.0380	-0.0380	-0.0380	-0.0380	-0.0381
g=8						-0.0380	-0.0382	-0.0382	-0.0375	-0.0368
g=9						-0.0383	-0.0383	-0.0385	-0.0370	-0.0370
g=10						-0.0386	-0.0379	-0.0372	-0.0372	-0.0372
g=20		-0.0428		-0.0397		-0.0390		-0.0390		-0.0390
g=25		-0.0422		-0.0404		-0.0404		-0.0404		-0.0404
g=30		-0.0433		-0.0412		-0.0412		-0.0412		-0.0412
g=35		-0.0432		-0.0421		-0.0421		-0.0421		-0.0421
g=40		-0.0448		-0.0432		-0.0432		-0.0432		-0.0432
g=45		-0.0456		-0.0441		-0.0441		-0.0441		-0.0441
g=50		-0.0458		-0.0450		-0.0450		-0.0450		-0.0450
g=100		-0.0532		-0.0532		-0.0532		-0.0532		-0.0532
g=200		-0.0703		-0.0703		-0.0703		-0.0703		-0.0703
g=300		-0.0785		-0.0785		-0.0785		-0.0785		-0.0785
g=400		-0.0986		-0.0986		-0.0986		-0.0986		-0.0986
g=500		-0.1168		-0.1168		-0.1168		-0.1168		-0.1168

Table 4.83: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0436	0.0302	0.0230	0.0187	0.0159					
g=2	0.0402	0.0238	0.0144	0.0095	0.0088					
g=3	0.0301	0.0152	0.0095	0.0080	0.0059					
g=4	0.0360	0.0128	0.0081	0.0059	0.0045					
g=5	0.0329	0.0122	0.0066	0.0037	0.0030					
g=6						0.0023	0.0023	0.0023	0.0023	0.0023
g=7						0.0023	0.0023	0.0023	0.0023	0.0015
g=8						0.0023	0.0023	0.0023	0.0015	0.0015
g=9						0.0023	0.0023	0.0015	0.0015	0.0015
g=10						0.0023	0.0023	0.0016	0.0016	0.0016
g=20		0.0054		0.0025		0.0017		0.0017		0.0017
g=25		0.0033		0.0018		0.0018		0.0018		0.0018
g=30		0.0019		0.0019		0.0019		0.0019		0.0019
g=35		0.0034		0.0020		0.0020		0.0020		0.0020
g=40		0.0036		0.0021		0.0021		0.0021		0.0021
g=45		0.0029		0.0022		0.0022		0.0022		0.0022
g=50		0.0030		0.0023		0.0023		0.0023		0.0023
g=100		0.0031		0.0031		0.0031		0.0031		0.0031
g=200		0.0053		0.0053		0.0053		0.0053		0.0053
g=300		0.0066		0.0066		0.0066		0.0066		0.0066
g=400		0.0101		0.0101		0.0101		0.0101		0.0101
g=500		0.0139		0.0139		0.0139		0.0139		0.0139

Table 4.84: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0440	0.0286	0.0221	0.0179	0.0149					
g=2	0.0385	0.0221	0.0143	0.0130	0.0108					
g=3	0.0328	0.0177	0.0130	0.0102	0.0087					
g=4	0.0335	0.0151	0.0094	0.0079	0.0079					
g=5	0.0300	0.0124	0.0094	0.0079	0.0072					
g=6						0.0050	0.0028	0.0028	0.0028	0.0028
g=7						0.0028	0.0028	0.0028	0.0028	0.0028
g=8						0.0028	0.0028	0.0028	0.0021	0.0014
g=9						0.0028	0.0028	0.0028	0.0015	0.0015
g=10						0.0028	0.0021	0.0015	0.0015	0.0015
g=20		0.0059		0.0023		0.0016		0.0016		0.0016
g=25		0.0038		0.0017		0.0017		0.0017		0.0017
g=30		0.0039		0.0018		0.0018		0.0018		0.0018
g=35		0.0032		0.0019		0.0019		0.0019		0.0019
g=40		0.0033		0.0020		0.0020		0.0020		0.0020
g=45		0.0034		0.0021		0.0021		0.0021		0.0021
g=50		0.0028		0.0022		0.0022		0.0022		0.0022
g=100		0.0030		0.0030		0.0030		0.0030		0.0030
g=200		0.0052		0.0052		0.0052		0.0052		0.0052
g=300		0.0066		0.0066		0.0066		0.0066		0.0066
g=400		0.0099		0.0099		0.0099		0.0099		0.0099
g=500		0.0138		0.0138		0.0138		0.0138		0.0138

Table 4.85: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0531	0.0536	0.0537	0.0539	0.0539					
g=2	0.0534	0.0538	0.0540	0.0541	0.0541					
g=3	0.0538	0.0541	0.0542	0.0543	0.0543					
g=4	0.0540	0.0543	0.0544	0.0544	0.0544					
g=5	0.0543	0.0545	0.0546	0.0546	0.0546					
g=6						0.0547	0.0547	0.0547	0.0547	0.0547
g=7						0.0549	0.0549	0.0549	0.0549	0.0549
g=8						0.0551	0.0551	0.0551	0.0551	0.0551
g=9						0.0553	0.0553	0.0553	0.0553	0.0553
g=10						0.0555	0.0555	0.0555	0.0555	0.0555
g=20		0.0571		0.0571		0.0571		0.0571		0.0571
g=25		0.0578		0.0578		0.0578		0.0578		0.0578
g=30		0.0587		0.0587		0.0587		0.0587		0.0587
g=35		0.0593		0.0593		0.0593		0.0593		0.0593
g=40		0.0603		0.0603		0.0603		0.0603		0.0603
g=45		0.0610		0.0610		0.0610		0.0610		0.0610
g=50		0.0620		0.0620		0.0620		0.0620		0.0620
g=100		0.0696		0.0696		0.0696		0.0696		0.0696
g=200		0.0801		0.0801		0.0801		0.0801		0.0801
g=300		0.0892		0.0892		0.0892		0.0892		0.0892
g=400		0.0996		0.0996		0.0996		0.0996		0.0996
g=500		0.1174		0.1174		0.1174		0.1174		0.1174

Table 4.86: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0531	-0.0534	-0.0535	-0.0536	-0.0537					
g=2	-0.0534	-0.0537	-0.0538	-0.0539	-0.0539					
g=3	-0.0536	-0.0539	-0.0540	-0.0541	-0.0541					
g=4	-0.0538	-0.0540	-0.0542	-0.0542	-0.0542					
g=5	-0.0541	-0.0543	-0.0543	-0.0543	-0.0544					
g=6						-0.0545	-0.0545	-0.0545	-0.0545	-0.0545
g=7						-0.0547	-0.0547	-0.0547	-0.0547	-0.0547
g=8						-0.0548	-0.0548	-0.0548	-0.0548	-0.0548
g=9						-0.0549	-0.0549	-0.0549	-0.0549	-0.0549
g=10						-0.0551	-0.0551	-0.0551	-0.0551	-0.0551
g=20		-0.0568		-0.0568		-0.0568		-0.0568		-0.0568
g=25		-0.0577		-0.0577		-0.0577		-0.0577		-0.0577
g=30		-0.0587		-0.0587		-0.0587		-0.0587		-0.0587
g=35		-0.0595		-0.0595		-0.0595		-0.0595		-0.0595
g=40		-0.0602		-0.0602		-0.0602		-0.0602		-0.0602
g=45		-0.0609		-0.0609		-0.0609		-0.0609		-0.0609
g=50		-0.0617		-0.0617		-0.0617		-0.0617		-0.0617
g=100		-0.0694		-0.0694		-0.0694		-0.0694		-0.0694
g=200		-0.0803		-0.0803		-0.0803		-0.0803		-0.0803
g=300		-0.0900		-0.0900		-0.0900		-0.0900		-0.0900
g=400		-0.0987		-0.0987		-0.0987		-0.0987		-0.0987
g=500		-0.1168		-0.1168		-0.1168		-0.1168		-0.1168

Table 4.87: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0029	0.0030	0.0030	0.0030	0.0030					
g=2	0.0030	0.0030	0.0030	0.0030	0.0030					
g=3	0.0030	0.0030	0.0030	0.0030	0.0030					
g=4	0.0030	0.0030	0.0030	0.0030	0.0030					
g=5	0.0030	0.0030	0.0030	0.0030	0.0030					
g=6						0.0031	0.0031	0.0031	0.0031	0.0031
g=7						0.0031	0.0031	0.0031	0.0031	0.0031
g=8						0.0031	0.0031	0.0031	0.0031	0.0031
g=9						0.0031	0.0031	0.0031	0.0031	0.0031
g=10						0.0031	0.0031	0.0031	0.0031	0.0031
g=20		0.0033		0.0033		0.0033		0.0033		0.0033
g=25		0.0034		0.0034		0.0034		0.0034		0.0034
g=30		0.0035		0.0035		0.0035		0.0035		0.0035
g=35		0.0036		0.0036		0.0036		0.0036		0.0036
g=40		0.0037		0.0037		0.0037		0.0037		0.0037
g=45		0.0038		0.0038		0.0038		0.0038		0.0038
g=50		0.0039		0.0039		0.0039		0.0039		0.0039
g=100		0.0050		0.0050		0.0050		0.0050		0.0050
g=200		0.0066		0.0066		0.0066		0.0066		0.0066
g=300		0.0083		0.0083		0.0083		0.0083		0.0083
g=400		0.0101		0.0101		0.0101		0.0101		0.0101
g=500		0.0139		0.0139		0.0139		0.0139		0.0139

Table 4.88: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, normal kernel density estimator, $\alpha = 0.01$, $\hat{h}_6$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0029	0.0029	0.0029	0.0029	0.0029					
g=2	0.0029	0.0029	0.0030	0.0030	0.0030					
g=3	0.0029	0.0030	0.0030	0.0030	0.0030					
g=4	0.0030	0.0030	0.0030	0.0030	0.0030					
g=5	0.0030	0.0030	0.0030	0.0030	0.0030					
g=6						0.0030	0.0030	0.0030	0.0030	0.0030
g=7						0.0030	0.0030	0.0030	0.0030	0.0030
g=8						0.0031	0.0031	0.0031	0.0031	0.0031
g=9						0.0031	0.0031	0.0031	0.0031	0.0031
g=10						0.0031	0.0031	0.0031	0.0031	0.0031
g=20		0.0033		0.0033		0.0033		0.0033		0.0033
g=25		0.0034		0.0034		0.0034		0.0034		0.0034
g=30		0.0035		0.0035		0.0035		0.0035		0.0035
g=35		0.0036		0.0036		0.0036		0.0036		0.0036
g=40		0.0037		0.0037		0.0037		0.0037		0.0037
g=45		0.0038		0.0038		0.0038		0.0038		0.0038
g=50		0.0039		0.0039		0.0039		0.0039		0.0039
g=100		0.0049		0.0049		0.0049		0.0049		0.0049
g=200		0.0067		0.0067		0.0067		0.0067		0.0067
g=300		0.0085		0.0085		0.0085		0.0085		0.0085
g=400		0.0100		0.0100		0.0100		0.0100		0.0100
g=500		0.0138		0.0138		0.0138		0.0138		0.0138

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.1$, $\kappa = 1$, $n = 10000$ and $[a, b] = [0.13\,, 0.87]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combinations of parameter values will result in the best possible estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions. The normal kernel is used in most of the results that will follow, if not explicitly stated differently.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) is recommended as a good choice for the estimated smoothing parameter.
- Both the Cramér-von-Mises and the Anderson-Darling goodness-of-fit tests are recommended with preference given to the Cramér-von-Mises test.
- For both of these goodness-of-fit tests, α-values of 1% or 5% can be used.
- The choices of $6 \leq r \leq 8$ and $5 \leq g \leq 10$ are recommended for optimal results.

The following four tables (Tables 4.89 – 4.92) provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the interval $[a, b] = [0.13\,,\,0.87]$.

Table 4.89: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Cramér-von-Mises goodness-of-fit test.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.18	0.82	0.18	0.82
g=7	0.18	0.82	0.18	0.82
g=8	0.18	0.82	0.18	0.82
g=9	0.18	0.82	0.18	0.82
g=10	0.18	0.82	0.18	0.82

Table 4.90: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.16	0.84	0.16	0.84
g=7	0.16	0.84	0.16	0.84
g=8	0.16	0.84	0.16	0.84
g=9	0.16	0.84	0.17	0.84
g=10	0.17	0.84	0.16	0.84

Table 4.91: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_6$ for the Cramér-von-Mises goodness-of-fit test.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.18	0.82	0.18	0.82
g=7	0.18	0.82	0.18	0.82
g=8	0.18	0.82	0.18	0.82
g=9	0.18	0.82	0.18	0.82
g=10	0.18	0.82	0.18	0.82

Table 4.92: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_6$ for the Anderson-Darling goodness-of-fit test.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.16	0.84	0.16	0.84
g=7	0.16	0.84	0.16	0.84
g=8	0.16	0.84	0.16	0.84
g=9	0.16	0.84	0.16	0.84
g=10	0.16	0.84	0.16	0.84

Remark: *For this data set, the estimated off-pulse interval is still close to the actual off-pulse interval, even though this is an extreme case in terms of the percentage pulsed emission contained in the interval [0,1].*

4.5.2 Data set parameters: $1-p=0.1$, $\kappa=1$, $n=10000$ and $[a,b]=[0.3\,,\,0.7]$

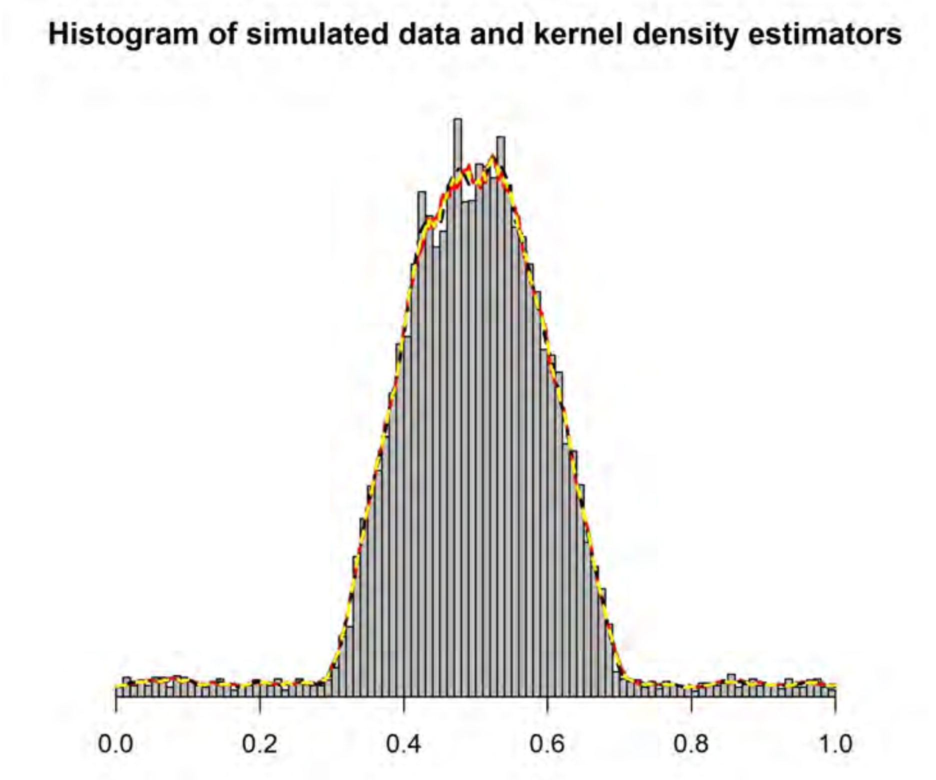

Figure 4.11: *A simulated data set from the above distribution with different circular kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_1=0.13$.

Figure 4.11 provides a histogram representation (with 100 classes) of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

In the analysis of the previous target population, it was found that different choices of the kernel function resulted in almost similar behaviour of the kernel density estimator. The same behaviour is evident when inspecting the different kernel density estimators in Figure 4.11. Therefore, the normal kernel is used in the analysis of this target population.

Choice of the number of minimum points m

The first step of SOPIE is to select a number of minimum points m. In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus. For this study population and most of the populations to follow, the global minimum point and a maximum of 9 other unique minima are obtained, resulting in 10 minima that can be used as starting point for SOPIE.

Tables 4.93 and 4.94 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$, $g = 6$, $r = 6$, $\alpha = 0.05$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_1$.

From the tables it is observed that different values of m influence the estimated bias or MSE very slightly. It can again be concluded that a small value of m is preferable, since computing time is reduced and it seems as if the goodness-of-fit tests are insensitive to the value of m.

Table 4.93: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=2	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=3	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=4	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=5	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=6	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=7	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=8	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=9	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
m=10	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02

Table 4.94: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=2	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=3	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=4	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=5	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=6	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=7	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=8	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=9	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
m=10	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00

Choice of estimated smoothing parameters

Table 4.95 and Table 4.96 highlight the values of the estimated bias and MSE when estimating a and b for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$.

When inspecting the estimated bias and MSE, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero. The recommendation is still that any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ is a good choice for the estimated smoothing parameter.

Table 4.95: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $r = 6$ and $g = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
$\hat{h}_2$	0.02	-0.02	0.02	-0.02	0.02	-0.03	0.02	-0.02
$\hat{h}_3$	0.01	-0.02	0.02	-0.02	0.01	-0.02	0.02	-0.02
$\hat{h}_4$	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
$\hat{h}_5$	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
$\hat{h}_6$	0.01	-0.02	0.02	-0.02	0.01	-0.02	0.02	-0.02
$\hat{h}_7$	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
$\hat{h}_8$	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
$\hat{h}_9$	0.02	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02

Table 4.96: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $r = 6$ and $g = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
$\hat{h}_2$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
$\hat{h}_3$	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00
$\hat{h}_4$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
$\hat{h}_5$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
$\hat{h}_6$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
$\hat{h}_7$	0.00	0.01	0.00	0.00	0.01	0.01	0.00	0.00
$\hat{h}_8$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
$\hat{h}_9$	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00

Choice of goodness-of-fit test

SOPIE is based in a sequential way on the P-values of goodness-of-fit tests for the uniform distribution. Different goodness-of-fit tests exist, and therefore it must be evaluated whether there is a superior goodness-of-fit test when compared to other tests. Some of the tables presented earlier can be used to assess the goodness-of-fit tests. Moreover, the reader can inspect Table 4.97 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ with $\hat{h}_1$. Table 4.98 is even more important, since the MSE is compared in this table.

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g, with the other three tests almost as well. When comparing the estimated MSE, all four tests perform very well.

Table 4.97: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	-0.02	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=7	0.02	-0.01	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=8	0.01	-0.01	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=9	0.01	-0.01	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=10	0.01	-0.01	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=20	0.01	-0.01	0.02	-0.02	0.02	-0.02	0.02	-0.02

Table 4.98: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
g=7	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Choice of the significance level α

Several different values of α are utilised in the simulation study, and therefore, multiple tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit tests. Tables 4.99 - 4.100 compare the estimated bias for two goodness-of-fit tests for $m = 1$, $r = 6$ and for $\hat{h}_1$. From these tables it is obvious that the estimated bias is close to zero when $0.01 \leq \alpha \leq 0.05$.

Tables 4.101 - 4.102 compare the estimated MSE for the same goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_1$. It is evident that the estimated MSE is also the smallest for $0.01 \leq \alpha \leq 0.05$. It seems valid to recommend α-values in the range from 0.01 to 0.05 with limited effect on the estimated values of a and b.

Table 4.99: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	-0.02	0.02	-0.04	0.04	-0.05
g=7	0.02	-0.01	0.03	-0.04	0.04	-0.04
g=8	0.01	-0.01	0.02	-0.03	0.03	-0.05
g=9	0.01	-0.01	0.02	-0.03	0.03	-0.04
g=10	0.01	-0.01	0.02	-0.03	0.03	-0.04
g=20	0.01	-0.01	0.01	-0.02	0.02	-0.02

Table 4.100: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=7	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=8	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=9	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=10	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=20	0.02	-0.02	0.02	-0.02	0.02	-0.02

Table 4.101: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.01	0.02	0.03	0.04	0.05
g=7	0.00	0.01	0.02	0.03	0.04	0.05
g=8	0.00	0.00	0.01	0.03	0.03	0.05
g=9	0.00	0.00	0.01	0.03	0.03	0.04
g=10	0.00	0.00	0.01	0.02	0.03	0.04
g=20	0.00	0.00	0.00	0.01	0.01	0.02

Table 4.102: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.00	0.00	0.00	0.00
g=7	0.00	0.00	0.00	0.00	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

The choice of g, the incremental growth of each interval over which rejection is tested, together with the choice of r, the number of intervals before rejection of uniformity takes place, are two important tuning parameters in the estimation process. Together, these two tuning parameters will determine the accuracy of the estimated point where rejection of uniformity takes place.

In Tables 4.103 – 4.106 the reader will find a comparison of the estimated bias and MSE when different g and r values are used, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$, for only the Anderson-Darling goodness-of-fit test. Similar trends are observed when using the Cramér-von-Mises goodness-of-fit test.

In terms of the estimated bias, $6 \leq g \leq 25$ with $6 \leq r \leq 10$ result in bias-values close to zero when using the Anderson-Darling goodness-of-fit test. When comparing the MSE, a similar trend is found. For the Cramér-von-Mises goodness-of-fit test, the estimated bias and MSE are not influenced by different choices of g and r.

In conclusion, it is recommended to use a value for g in the range from 6 to 25, and for r, values from 6 to 10 are recommended when the sample size n is moderate to large, such as in this case.

Table 4.103: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.06	0.05	0.04	0.04	0.03					
g=2	0.06	0.04	0.04	0.03	0.03					
g=3	0.06	0.03	0.03	0.02	0.02					
g=4	0.05	0.03	0.02	0.02	0.02					
g=5	0.05	0.03	0.02	0.02	0.02					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.02	0.01	0.01	0.01	0.01
g=8						0.01	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.01
g=10						0.01	0.01	0.01	0.01	0.01
g=20		0.02		0.01		0.01		0.01		0.01
g=25		0.02		0.01		0.01		0.01		0.01
g=30		0.02		0.01		0.01		0.01		0.02
g=35		0.02		0.02		0.02		0.02		0.02
g=40		0.02		0.02		0.02		0.02		0.02
g=45		0.02		0.02		0.02		0.02		0.02
g=50		0.02		0.02		0.02		0.02		0.02
g=100		0.02		0.02		0.02		0.02		0.02
g=200		0.03		0.03		0.03		0.03		0.03
g=300		0.03		0.03		0.03		0.03		0.03
g=400		0.04		0.04		0.04		0.04		0.04
g=500		0.05		0.05		0.05		0.05		0.05

Table 4.104: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.08	-0.07	-0.05	-0.05	-0.04					
g=2	-0.08	-0.05	-0.04	-0.04	-0.03					
g=3	-0.07	-0.05	-0.04	-0.03	-0.02					
g=4	-0.07	-0.04	-0.03	-0.03	-0.02					
g=5	-0.07	-0.04	-0.03	-0.02	-0.02					
g=6						-0.02	-0.01	-0.01	-0.01	-0.01
g=7						-0.01	-0.01	-0.01	-0.01	-0.01
g=8						-0.01	-0.01	-0.01	-0.01	-0.01
g=9						-0.01	-0.01	-0.01	-0.01	-0.01
g=10						-0.01	-0.01	-0.01	-0.01	-0.01
g=20		-0.02		-0.01		-0.01		-0.01		-0.01
g=25		-0.02		-0.01		-0.01		-0.01		-0.01
g=30		-0.02		-0.01		-0.01		-0.01		-0.01
g=35		-0.02		-0.01		-0.01		-0.01		-0.01
g=40		-0.02		-0.01		-0.01		-0.01		-0.01
g=45		-0.02		-0.02		-0.02		-0.02		-0.02
g=50		-0.02		-0.02		-0.02		-0.02		-0.02
g=100		-0.02		-0.02		-0.02		-0.02		-0.02
g=200		-0.03		-0.03		-0.03		-0.03		-0.03
g=300		-0.03		-0.03		-0.03		-0.03		-0.03
g=400		-0.04		-0.04		-0.04		-0.04		-0.04
g=500		-0.04		-0.04		-0.04		-0.04		-0.04

Table 4.105: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.05	0.04	0.03	0.03	0.02					
g=2	0.05	0.03	0.02	0.02	0.01					
g=3	0.05	0.02	0.02	0.01	0.01					
g=4	0.04	0.02	0.01	0.01	0.01					
g=5	0.04	0.02	0.01	0.01	0.01					
g=6						0.00	0.00	0.00	0.00	0.00
g=7						0.00	0.00	0.00	0.00	0.00
g=8						0.00	0.00	0.00	0.00	0.00
g=9						0.00	0.00	0.00	0.00	0.00
g=10						0.00	0.00	0.00	0.00	0.00
g=20		0.00		0.00		0.00		0.00		0.00
g=25		0.00		0.00		0.00		0.00		0.00
g=30		0.00		0.00		0.00		0.00		0.00
g=35		0.00		0.00		0.00		0.00		0.00
g=40		0.00		0.00		0.00		0.00		0.00
g=45		0.00		0.00		0.00		0.00		0.00
g=50		0.00		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		0.00

Table 4.106: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.07	0.06	0.05	0.04	0.03					
g=2	0.06	0.04	0.03	0.03	0.02					
g=3	0.06	0.04	0.03	0.02	0.02					
g=4	0.06	0.03	0.02	0.02	0.01					
g=5	0.05	0.03	0.02	0.02	0.01					
g=6						0.01	0.01	0.00	0.00	0.00
g=7						0.01	0.00	0.00	0.00	0.00
g=8						0.00	0.00	0.00	0.00	0.00
g=9						0.00	0.00	0.00	0.00	0.00
g=10						0.00	0.00	0.00	0.00	0.00
g=20		0.01		0.00		0.00		0.00		0.00
g=25		0.01		0.00		0.00		0.00		0.00
g=30		0.01		0.00		0.00		0.00		0.00
g=35		0.00		0.00		0.00		0.00		0.00
g=40		0.00		0.00		0.00		0.00		0.00
g=45		0.00		0.00		0.00		0.00		0.00
g=50		0.00		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		0.00

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.1$, $\kappa = 1$, $n = 10000$ and with $[a, b] = [0.3\,,\,0.7]$

From the detailed analysis of each tuning parameter that may influence the estimation of a and b, it is concluded that the following tuning parameter values will result in the best possible estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions. It was decided to use the normal kernel for most of the results.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) is recommended as a good choice for the estimated smoothing parameter.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended.
- For both of the above-mentioned goodness-of-fit tests, α-values of 1% or 5% can be used.
- The choices of $6 \leq r \leq 10$ and $6 \leq g \leq 25$ are recommended for optimal results, with preference given to larger values of g when the sample size is moderate to large.

The following two tables (Tables 4.107 – 4.108) provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the interval $[a, b] = [0.3\,,\,0.7]$.

Table 4.107: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $\alpha = 0.01$ for the Anderson-Darling goodness-of-fit test with $\hat{h}_1$.*

	r=6		r=8	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.31	0.68	0.31	0.69
g=7	0.32	0.69	0.31	0.69
g=8	0.31	0.69	0.31	0.69
g=9	0.31	0.69	0.31	0.69
g=10	0.31	0.69	0.31	0.69
g=20	0.31	0.69	0.31	0.69
g=25	0.31	0.69	0.31	0.69
g=30	0.31	0.69	0.31	0.69
g=35	0.32	0.69	0.32	0.69
g=40	0.32	0.69	0.32	0.69

Table 4.108: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $\alpha = 0.01$ for the Cramér-von-Mises goodness-of-fit test with $\hat{h}_1$.*

	r=6		r=8	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.32	0.68	0.32	0.68
g=7	0.32	0.68	0.32	0.68
g=8	0.32	0.68	0.32	0.68
g=9	0.32	0.68	0.32	0.68
g=10	0.32	0.68	0.32	0.68
g=20	0.32	0.68	0.32	0.68
g=25	0.32	0.68	0.32	0.68
g=30	0.32	0.68	0.32	0.68
g=35	0.32	0.68	0.32	0.68
g=40	0.32	0.68	0.32	0.68

Remark: *The estimated off-pulse interval for this data set proves to be quite close to the theoretical off-pulse interval, even for different choices of tuning parameters.*

4.5.3 Data set parameters: $1 - p = 0.2$, $\kappa = 2$, $n = 5000$ and $[a, b] = [0.3\,,\,0.7]$

Figure 4.12 provides a histogram representation (with 100 classes) of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The target population will now be analysed on a parameter per parameter basis, ceteris paribus.

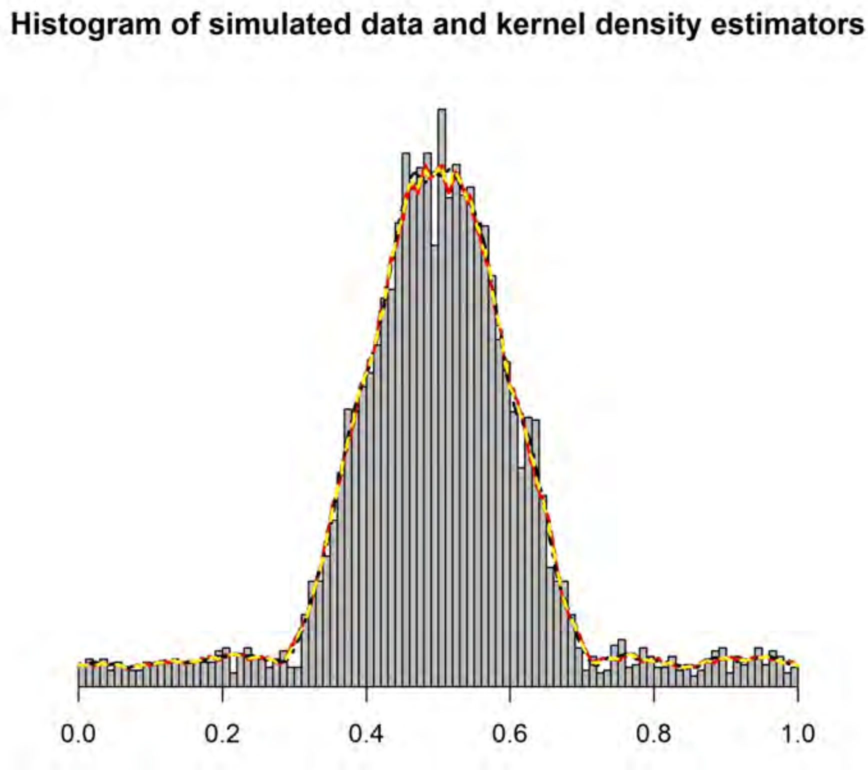

Figure 4.12: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_1 = 0.17$.

Choice of kernel function

In the analyses of all of the previous target populations, it was found that different choices of the kernel function resulted in almost similar behaviour of the kernel density estimator. The same behaviour is evident in this target population when inspecting the different kernel density estimators in Figure 4.12. Therefore, the normal kernel is used in the analysis of this target population.

Choice of the number of minimum points m

The first step of SOPIE is to select a number of minimum points m. In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus.

Tables 4.109 and 4.110 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ when $g = 6$, $r = 6$, $\alpha = 0.05$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_3$. From the tables it is observed that different values of m influence the estimated bias or MSE very slightly. It can again be concluded that a small value of m is preferable, since computing time is reduced and it seems as if the goodness-of-fit tests are insensitive to the choice of m.

Table 4.109: *Bias for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.02	-0.02	0.03	-0.03	0.02	-0.03	0.02	-0.02
m=2	0.02	-0.02	0.03	-0.03	0.02	-0.03	0.02	-0.02
m=3	0.02	-0.03	0.03	-0.03	0.02	-0.04	0.02	-0.02
m=4	0.02	-0.03	0.03	-0.03	0.03	-0.04	0.02	-0.03
m=5	0.02	-0.04	0.03	-0.03	0.03	-0.04	0.03	-0.03
m=6	0.02	-0.04	0.03	-0.03	0.03	-0.04	0.03	-0.03
m=7	0.02	-0.04	0.03	-0.03	0.03	-0.04	0.03	-0.03
m=8	0.02	-0.04	0.03	-0.03	0.03	-0.04	0.03	-0.03
m=9	0.02	-0.04	0.03	-0.03	0.03	-0.04	0.03	-0.03
m=10	0.02	-0.04	0.03	-0.03	0.03	-0.04	0.03	-0.03

Table 4.110: *MSE for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
m=2	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
m=3	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.01
m=4	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02
m=5	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02
m=6	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02
m=7	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02
m=8	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02
m=9	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02
m=10	0.01	0.03	0.00	0.00	0.01	0.03	0.01	0.02

Choice of estimated smoothing parameters

Table 4.111 and Table 4.112 highlight the values of the estimated bias and MSE when estimating a and b for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.05$. When inspecting the estimated bias and MSE, it is found that $\hat{h}_3$ is associated with estimated bias-values closer to zero than some of the other choices of $\hat{h}$. The MSE is also smaller for this choice of $\hat{h}$.

The recommendation is again that any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ is a good choice of the estimated smoothing parameter.

Table 4.111: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $r = 6$ and $g = 6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.03	-0.04	0.03	-0.03	0.04	-0.05	0.04	-0.04
$\hat{h}_2$	0.03	-0.03	0.03	-0.03	0.04	-0.04	0.04	-0.03
$\hat{h}_3$	0.02	-0.02	0.03	-0.03	0.02	-0.03	0.02	-0.02
$\hat{h}_4$	0.03	-0.04	0.03	-0.03	0.04	-0.04	0.04	-0.03
$\hat{h}_5$	0.04	-0.04	0.03	-0.03	0.04	-0.05	0.04	-0.04
$\hat{h}_6$	0.03	-0.04	0.03	-0.03	0.05	-0.05	0.05	-0.04
$\hat{h}_7$	0.03	-0.04	0.03	-0.03	0.04	-0.05	0.04	-0.03
$\hat{h}_8$	0.04	-0.04	0.03	-0.03	0.04	-0.04	0.04	-0.04
$\hat{h}_9$	0.03	-0.04	0.03	-0.03	0.04	-0.04	0.04	-0.03

Table 4.112: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $r = 6$ and $g = 6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.02	0.03	0.00	0.00	0.02	0.03	0.02	0.02
$\hat{h}_2$	0.02	0.03	0.00	0.00	0.02	0.03	0.01	0.01
$\hat{h}_3$	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
$\hat{h}_4$	0.02	0.03	0.00	0.00	0.02	0.03	0.02	0.01
$\hat{h}_5$	0.02	0.04	0.00	0.00	0.02	0.03	0.01	0.01
$\hat{h}_6$	0.02	0.03	0.00	0.00	0.03	0.03	0.02	0.02
$\hat{h}_7$	0.02	0.03	0.00	0.00	0.02	0.03	0.02	0.01
$\hat{h}_8$	0.02	0.04	0.00	0.00	0.02	0.03	0.01	0.01
$\hat{h}_9$	0.02	0.03	0.00	0.00	0.02	0.03	0.01	0.01

Choice of goodness-of-fit test

It is possible to assess the different goodness-of-fit tests from some of the tables provided earlier for this specific data set. Moreover, the reader can inspect Table 4.113 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.05$, $m = 1$ and $r = 6$ with $\hat{h}_3$. Table 4.114 is even more important, since the MSE is compared in this table.

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g, with the Kolmogorov-Smirnov, Rayleigh and Cramér-von-Mises goodness-of-fit tests performing almost as well. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs best. The other three tests have a marginally larger estimated MSE.

Table 4.113: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.05$, $m = 1$, $r = 6, \hat{h}_3$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	-0.02	0.03	-0.03	0.02	-0.03	0.02	-0.02
g=7	0.02	-0.02	0.03	-0.03	0.02	-0.03	0.02	-0.02
g=8	0.02	-0.02	0.03	-0.03	0.02	-0.02	0.02	-0.02
g=9	0.02	-0.02	0.03	-0.03	0.02	-0.02	0.02	-0.02
g=10	0.02	-0.02	0.03	-0.03	0.02	-0.02	0.02	-0.02
g=20	0.02	-0.02	0.04	-0.04	0.02	-0.02	0.03	-0.03
g=25	0.02	-0.02	0.04	-0.04	0.02	-0.02	0.03	-0.03

Table 4.114: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.05$, $m = 1$, $r = 6, \hat{h}_3$ *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
g=7	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
g=8	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
g=9	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=10	0.01	0.01	0.00	0.00	0.01	0.01	0.00	0.00
g=20	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Choice of the significance level α

Several different values of α are utilised in the simulation study, and therefore multiple tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit tests. Tables 4.115 - 4.116 compare the estimated bias for two goodness-of-fit tests for $m = 1$, $r = 6$ and for $\hat{h}_3$. The tables highlight very similar estimated bias-values for all choices of α. Therefore, it is not that easy to choose a level of significance that performs significantly better based on the estimated bias.

Tables 4.117 - 4.118 compare the estimated MSE for the same goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_3$. Slightly larger values of the estimated MSE result when α is increased from 0.01 to 0.10 for the Anderson-Darling goodness-of-fit test. It therefore still seems valid to recommend any value of α from 0.01 up to 0.05, with limited effect on the estimated values of a and b.

Table 4.115: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_3$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	-0.02	0.02	-0.02	0.02	-0.03
g=7	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=8	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=9	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=10	0.02	-0.02	0.02	-0.02	0.02	-0.02
g=20	0.02	-0.03	0.02	-0.02	0.02	-0.02
g=25	0.03	-0.03	0.02	-0.02	0.02	-0.02

Table 4.116: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_3$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.04	-0.04	0.03	-0.03	0.03	-0.03
g=7	0.04	-0.04	0.03	-0.03	0.03	-0.03
g=8	0.04	-0.04	0.03	-0.03	0.03	-0.03
g=9	0.04	-0.04	0.03	-0.03	0.03	-0.03
g=10	0.04	-0.04	0.03	-0.03	0.03	-0.03
g=20	0.04	-0.04	0.04	-0.04	0.03	-0.04
g=25	0.04	-0.04	0.04	-0.04	0.04	-0.04

Table 4.117: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_3$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.01	0.02	0.03	0.05
g=7	0.00	0.00	0.01	0.02	0.02	0.04
g=8	0.00	0.00	0.01	0.02	0.02	0.03
g=9	0.00	0.00	0.01	0.01	0.02	0.03
g=10	0.00	0.00	0.01	0.01	0.01	0.03
g=20	0.00	0.00	0.00	0.01	0.01	0.01
g=25	0.00	0.00	0.00	0.00	0.00	0.01

Table 4.118: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_3$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.00	0.00	0.00	0.00
g=7	0.00	0.00	0.00	0.00	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00
g=25	0.00	0.00	0.00	0.00	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

The choice of g, the incremental growth of each interval over which rejection is tested, together with the choice of r, the number of intervals before rejection of uniformity takes place, are two important tuning parameters in the estimation process. Together, these two tuning parameters will determine the accuracy of the estimated point where rejection of uniformity takes place.

In Tables 4.119 – 4.122 the reader will find a comparison of the estimated bias and MSE when different g and r-values are used, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$, for only the Anderson-Darling goodness-of-fit test. Similar trends are observed when using the Cramér-von-Mises goodness-of-fit test.

In terms of the estimated bias, $6 \leq g \leq 40$ with $4 \leq r \leq 8$ result in bias-values close to zero when using the Anderson-Darling goodness-of-fit test. When comparing the MSE, a similar trend is observed, with slightly larger values of g and r resulting in the smallest estimated MSE. In terms of the estimated bias and MSE for the Cramér-von-Mises goodness-of-fit test, it is evident that different choices of g and r have nearly no effect on the estimated values of a and b.

In conclusion, it is recommended to use a value of g in the range from 6 to 40, and for r, values from 4 to 8.

Table 4.119: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.01	0.02	0.02	0.02	0.02					
g=2	0.01	0.02	0.02	0.02	0.02					
g=3	0.01	0.02	0.02	0.02	0.02					
g=4	0.01	0.02	0.02	0.02	0.02					
g=5	0.02	0.02	0.02	0.02	0.02					
g=6						0.02	0.02	0.02	0.02	0.02
g=7						0.02	0.02	0.02	0.02	0.02
g=8						0.02	0.02	0.02	0.02	0.02
g=9						0.02	0.02	0.02	0.02	0.02
g=10						0.02	0.02	0.02	0.02	0.02
g=20		0.02		0.02		0.02		0.02		0.02
g=25		0.02		0.02		0.02		0.02		0.02
g=30		0.02		0.02		0.02		0.02		0.02
g=35		0.03		0.02		0.02		0.02		0.02
g=40		0.02		0.02		0.02		0.02		0.02
g=45		0.02		0.02		0.02		0.02		0.02
g=50		0.02		0.02		0.02		0.02		0.02
g=100		0.03		0.03		0.03		0.03		0.03
g=200		0.04		0.04		0.04		0.04		0.04
g=300		0.05		0.05		0.05		0.05		0.05
g=400		0.05		0.05		0.05		0.05		0.05
g=500		0.07		0.07		0.07		0.07		

Table 4.120: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.19	-0.15	-0.12	-0.11	-0.10					
g=2	-0.17	-0.12	-0.09	-0.08	-0.06					
g=3	-0.15	-0.10	-0.07	-0.05	-0.04					
g=4	-0.15	-0.09	-0.07	-0.05	-0.04					
g=5	-0.13	-0.08	-0.05	-0.04	-0.03					
g=6						-0.02	-0.02	-0.02	-0.02	-0.02
g=7						-0.02	-0.02	-0.02	-0.02	-0.02
g=8						-0.02	-0.02	-0.02	-0.02	-0.02
g=9						-0.02	-0.02	-0.02	-0.02	-0.02
g=10						-0.02	-0.02	-0.02	-0.02	-0.02
g=20		-0.03		-0.02		-0.02		-0.02		-0.02
g=25		-0.02		-0.02		-0.02		-0.02		-0.02
g=30		-0.03		-0.02		-0.02		-0.02		-0.02
g=35		-0.02		-0.02		-0.02		-0.02		-0.02
g=40		-0.02		-0.02		-0.02		-0.02		-0.02
g=45		-0.02		-0.03		-0.02		-0.02		-0.02
g=50		-0.02		-0.02		-0.02		-0.02		-0.02
g=100		-0.03		-0.03		-0.03		-0.03		-0.03
g=200		-0.03		-0.03		-0.03		-0.03		-0.03
g=300		-0.06		-0.06		-0.06		-0.06		-0.06
g=400		-0.05		-0.05		-0.05		-0.05		-0.05
g=500		-0.06		-0.06		-0.06		-0.06		

Table 4.121: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.08	0.06	0.06	0.05	0.04					
g=2	0.07	0.05	0.04	0.04	0.03					
g=3	0.07	0.05	0.03	0.03	0.03					
g=4	0.06	0.04	0.03	0.02	0.02					
g=5	0.06	0.04	0.03	0.02	0.02					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.01	0.01	0.01	0.01	0.00
g=8						0.01	0.01	0.01	0.00	0.00
g=9						0.01	0.01	0.00	0.00	0.00
g=10						0.01	0.00	0.00	0.00	0.00
g=20		0.02		0.00		0.00		0.00		0.00
g=25		0.01		0.00		0.00		0.00		0.00
g=30		0.01		0.00		0.00		0.00		0.00
g=35		0.01		0.00		0.00		0.00		0.00
g=40		0.01		0.00		0.00		0.00		0.00
g=45		0.01		0.00		0.00		0.00		0.00
g=50		0.00		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		

Table 4.122: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.18	0.15	0.13	0.12	0.11					
g=2	0.17	0.13	0.10	0.09	0.07					
g=3	0.16	0.11	0.08	0.06	0.05					
g=4	0.16	0.10	0.07	0.05	0.04					
g=5	0.14	0.09	0.06	0.04	0.03					
g=6						0.02	0.02	0.01	0.01	0.01
g=7						0.02	0.02	0.01	0.01	0.01
g=8						0.02	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.01
g=10						0.01	0.01	0.01	0.01	0.01
g=20		0.03		0.01		0.01		0.00		0.00
g=25		0.02		0.01		0.00		0.00		0.00
g=30		0.02		0.01		0.00		0.00		0.00
g=35		0.02		0.00		0.00		0.00		0.00
g=40		0.01		0.00		0.00		0.00		0.00
g=45		0.01		0.00		0.00		0.00		0.00
g=50		0.01		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.2$, $\kappa = 2$, $n = 5000$ and with $[a, b] = [0.3\,,\, 0.7]$

From the detailed analysis of each tuning parameter that may influence the estimation of a and b, it is concluded that the following tuning parameter values will result in the best possible estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) is recommended as a good choice for the estimated smoothing parameter.
- Both the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests are recommended.
- For both of these goodness-of-fit tests, α-values of 1% or 5% can be used.
- The choices of $4 \leq r \leq 8$ and $6 \leq g \leq 40$ are recommended for optimal results.

Table 4.123 provides the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in this table should be compared to the interval $[a, b] = [0.3\,,\, 0.7]$.

Table 4.123: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $\alpha = 0.05$ for the Anderson-Darling goodness-of-fit test with $\hat{h}_3$.*

	r=6		r=8	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.32	0.68	0.32	0.68
g=7	0.32	0.68	0.32	0.68
g=8	0.32	0.68	0.32	0.68
g=9	0.32	0.68	0.32	0.68
g=10	0.32	0.68	0.32	0.68
g=20	0.32	0.68	0.32	0.68
g=25	0.32	0.68	0.32	0.68
g=30	0.32	0.68	0.32	0.68
g=35	0.32	0.68	0.32	0.68
g=40	0.32	0.68	0.32	0.68

Remark: *For this data set, the estimation of the off-pulse interval is quite good, even for a wide array of different choices of the tuning parameters.*

4.5.4 Data set parameters: $1 - p = 0.3$, $\kappa = 3$, $n = 25000$ and $[a, b] = [0.4\,,\, 0.6]$

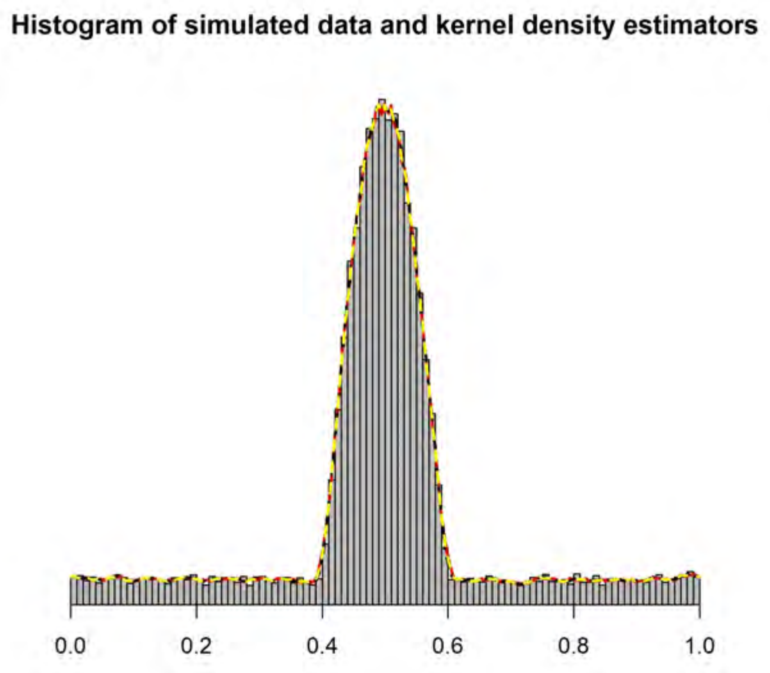

Figure 4.13: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter $\hat{h}_3 = 0.09$.*

Figure 4.13 provides a histogram representation (with 100 classes) of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

From Figure 4.13, and the analyses of all of the previous target populations, it is found that different choices of the kernel function result in almost similar behaviour of the kernel density estimator. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

The first step of SOPIE is to select a number of minimum points m. In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus.

Tables 4.124 and 4.125 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ when $g = 20$, $r = 6$, $\alpha = 0.01$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_1$.

Table 4.124: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=2	0.01	-0.01	0.01	-0.01	0.01	-0.02	0.02	-0.02
m=3	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=4	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=5	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=6	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=7	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=8	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=9	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
m=10	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02

Table 4.125: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=2	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=3	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=4	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=5	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=6	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=7	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=8	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=9	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=10	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01

From the tables it is observed that different values of m have no influence on the estimated bias or MSE. It can again be concluded that a small value of m is preferable, since computing time is reduced and it seems as if the goodness-of-fit tests are insensitive to the choice of m.

Choice of estimated smoothing parameters

Table 4.126 and Table 4.127 highlight the values of the estimated bias and MSE when estimating a and b for $m = 1$, $g = 20$, $r = 6$ and $\alpha = 0.01$.

Table 4.126: *Comparison of the bias for different combinations of the estimated smoothing parameter with* $m = 1$, $r = 6$ *and* $g = 20$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
$\hat{h}_2$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.02	-0.01
$\hat{h}_3$	0.02	-0.01	0.01	-0.01	0.02	-0.01	0.02	-0.02
$\hat{h}_4$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.02	-0.02
$\hat{h}_5$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.01	-0.01
$\hat{h}_6$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.02	-0.01
$\hat{h}_7$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.02	-0.01
$\hat{h}_8$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.01	-0.01
$\hat{h}_9$	0.01	-0.01	0.01	-0.01	0.02	-0.01	0.01	-0.02

Table 4.127: *Comparison of the MSE for different combinations of the estimated smoothing parameter with* $m = 1$, $r = 6$ *and* $g = 20$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_2$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_3$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_4$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_5$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.00
$\hat{h}_6$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_7$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_8$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_9$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01

When inspecting the estimated bias and MSE, it is very difficult to highlight a specific choice of smoothing parameter that performs better than some of the others. Almost all of the choices perform equally well. The recommendation continues to hold that any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ is a good choice for the estimated smoothing parameter.

Choice of goodness-of-fit test

In order to evaluate the different goodness-of-fit tests, the reader may inspect some of the tables already provided for this data set. Moreover, the reader can inspect Tables 4.128 – 4.129 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ with $\hat{h}_1$. In terms of estimated bias and MSE, the Cramér-von-Mises goodness-of-fit test is superior for any choice of $\hat{h}$ and g, with the Kolmogorov-Smirnov, Rayleigh and Cramér-von-Mises goodness-of-fit tests performing almost as well. For large sample sizes, it is recommended to use the Cramér-von-Mises goodness-of-fit test as first choice, followed by the Anderson-Darling goodness-of-fit test.

Table 4.128: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	-0.02	0.01	-0.01	0.02	-0.02	0.02	-0.02
g=7	0.02	-0.02	0.01	-0.01	0.02	-0.02	0.02	-0.02
g=8	0.01	-0.02	0.01	-0.01	0.02	-0.02	0.02	-0.02
g=9	0.01	-0.02	0.01	-0.01	0.02	-0.02	0.02	-0.02
g=10	0.01	-0.02	0.01	-0.01	0.02	-0.02	0.02	-0.02
g=20	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.02	-0.02
g=25	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.01	-0.02
g=30	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.01	-0.01

Table 4.129: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	0.02	0.00	0.00	0.01	0.02	0.02	0.02
g=7	0.02	0.02	0.00	0.00	0.01	0.02	0.02	0.02
g=8	0.02	0.02	0.00	0.00	0.01	0.02	0.01	0.01
g=9	0.02	0.02	0.00	0.00	0.01	0.01	0.01	0.01
g=10	0.02	0.02	0.00	0.00	0.01	0.01	0.01	0.01
g=20	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=25	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=30	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01

Choice of the significance level α

Several different values of α are utilised in the simulation study, and therefore multiple tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit test. Tables 4.130 – 4.133 compare the estimated bias and MSE for two goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_1$. For the Anderson-Darling goodness-of-fit test, it is obvious that the estimated bias and MSE decrease and move closer to zero when α is *decreased*. For the Cramér-von-Mises goodness-of-fit test, though, the bias and MSE are robust against the choice of α.

Table 4.130: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=7	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=8	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=9	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=10	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=20	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=25	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=30	0.01	-0.01	0.01	-0.01	0.01	-0.01
g=35	0.01	-0.01	0.01	-0.01	0.01	-0.01

Table 4.131: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	-0.02	0.03	-0.04	0.05	-0.05
g=7	0.02	-0.02	0.03	-0.03	0.05	-0.04
g=8	0.01	-0.02	0.03	-0.03	0.05	-0.04
g=9	0.01	-0.02	0.03	-0.04	0.04	-0.03
g=10	0.01	-0.02	0.02	-0.03	0.04	-0.04
g=20	0.01	-0.01	0.02	-0.03	0.03	-0.03
g=25	0.01	-0.01	0.02	-0.03	0.03	-0.03
g=30	0.01	-0.01	0.02	-0.03	0.03	-0.03
g=35	0.01	-0.01	0.01	-0.02	0.02	-0.03

Table 4.132: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.00	0.00	0.00	0.00
g=7	0.00	0.00	0.00	0.00	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00
g=25	0.00	0.00	0.00	0.00	0.00	0.00
g=30	0.00	0.00	0.00	0.00	0.00	0.00
g=35	0.00	0.00	0.00	0.00	0.00	0.00

Table 4.133: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.02	0.02	0.06	0.07	0.08	0.09
g=7	0.02	0.02	0.05	0.06	0.08	0.08
g=8	0.02	0.02	0.05	0.06	0.07	0.08
g=9	0.02	0.02	0.05	0.06	0.07	0.08
g=10	0.02	0.02	0.05	0.06	0.07	0.07
g=20	0.01	0.01	0.04	0.04	0.06	0.06
g=25	0.01	0.01	0.03	0.04	0.05	0.06
g=30	0.01	0.01	0.03	0.03	0.05	0.05
g=35	0.01	0.01	0.03	0.03	0.04	0.05

It therefore still seems valid to recommend any value of α in the range from 0.01 to 0.05, with limited effect on the estimated values of a and b, although preference is given to smaller values of α when the sample size is large.

Choice of incremental growth g and number of intervals of rejection r

The choice of g, the incremental growth of each interval over which rejection is tested, together with the choice of r, the number of intervals before rejection of uniformity takes place, are two important tuning parameters in the estimation process. Together, these two tuning parameters will determine the accuracy of the estimated point where rejection of uniformity takes place.

In Tables 4.134 – 4.136 the reader will find a comparison of the estimated bias and MSE when different g and r values are used, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$, for only the Cramér-von-Mises goodness-of-fit test. In terms of the estimated bias and MSE, the Cramér-von-Mises goodness-of-fit test is almost insensitive to different choices of the tuning parameters r and g. Therefore, almost any combinations of r and g-values can be used. For the Anderson-Darling goodness-of-fit test, $20 \leq g \leq 50$ with $4 \leq r \leq 8$ result in the smallest estimated bias and MSE.

In conclusion, it is recommended to use a value of g in the range from 20 to 50, and for r, values from 4 to 8.

Table 4.134: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.01	0.01	0.01	0.01	0.01					
g=2	0.01	0.01	0.01	0.01	0.01					
g=3	0.01	0.01	0.01	0.01	0.01					
g=4	0.01	0.01	0.01	0.01	0.01					
g=5	0.01	0.01	0.01	0.01	0.01					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.01	0.01	0.01	0.01	0.01
g=8						0.01	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.01
g=10						0.01	0.01	0.01	0.01	0.01
g=20		0.01		0.01		0.01		0.01		0.01
g=25		0.01		0.01		0.01		0.01		0.01
g=30		0.01		0.01		0.01		0.01		0.01
g=35		0.01		0.01		0.01		0.01		0.01
g=40		0.01		0.01		0.01		0.01		0.01
g=45		0.01		0.01		0.01		0.01		0.01
g=50		0.01		0.01		0.01		0.01		0.01
g=100		0.01		0.01		0.01		0.01		0.01
g=200		0.01		0.01		0.01		0.01		0.01
g=300		0.02		0.02		0.02		0.02		0.02
g=400		0.02		0.02		0.02		0.02		0.02
g=500		0.02		0.02		0.02		0.02		0.02

Table 4.135: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.01	-0.01	-0.01	-0.01	-0.01					
g=2	-0.01	-0.01	-0.01	-0.01	-0.01					
g=3	-0.01	-0.01	-0.01	-0.01	-0.01					
g=4	-0.01	-0.01	-0.01	-0.01	-0.01					
g=5	-0.01	-0.01	-0.01	-0.01	-0.01					
g=6						-0.01	-0.01	-0.01	-0.01	-0.01
g=7						-0.01	-0.01	-0.01	-0.01	-0.01
g=8						-0.01	-0.01	-0.01	-0.01	-0.01
g=9						-0.01	-0.01	-0.01	-0.01	-0.01
g=10						-0.01	-0.01	-0.01	-0.01	-0.01
g=20		-0.01		-0.01		-0.01		-0.01		-0.01
g=25		-0.01		-0.01		-0.01		-0.01		-0.01
g=30		-0.01		-0.01		-0.01		-0.01		-0.01
g=35		-0.01		-0.01		-0.01		-0.01		-0.01
g=40		-0.01		-0.01		-0.01		-0.01		-0.01
g=45		-0.01		-0.01		-0.01		-0.01		-0.01
g=50		-0.01		-0.01		-0.01		-0.01		-0.01
g=100		-0.01		-0.01		-0.01		-0.01		-0.01
g=200		-0.01		-0.01		-0.01		-0.01		-0.01
g=300		-0.02		-0.02		-0.02		-0.02		-0.02
g=400		-0.02		-0.02		-0.02		-0.02		-0.02
g=500		-0.02		-0.02		-0.02		-0.02		-0.02

Table 4.136: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$. (The MSE values for $\hat{b}$ are equal to the MSE values of $\hat{a}$ $\forall$ g and r-combinations, therefore a separate table is not given.)*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.00	0.00	0.00	0.00	0.00					
g=2	0.00	0.00	0.00	0.00	0.00					
g=3	0.00	0.00	0.00	0.00	0.00					
g=4	0.00	0.00	0.00	0.00	0.00					
g=5	0.00	0.00	0.00	0.00	0.00					
g=6						0.00	0.00	0.00	0.00	0.00
g=7						0.00	0.00	0.00	0.00	0.00
g=8						0.00	0.00	0.00	0.00	0.00
g=9						0.00	0.00	0.00	0.00	0.00
g=10						0.00	0.00	0.00	0.00	0.00
g=20		0.00		0.00		0.00		0.00		0.00
g=25		0.00		0.00		0.00		0.00		0.00
g=30		0.00		0.00		0.00		0.00		0.00
g=35		0.00		0.00		0.00		0.00		0.00
g=40		0.00		0.00		0.00		0.00		0.00
g=45		0.00		0.00		0.00		0.00		0.00
g=50		0.00		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		0.00

Concluding remarks about simulated data from a von Mises distribution with $1-p=0.3$**,** $\kappa=3$**,** $n=25000$ **and with** $[a,b]=[0.4\,,\,0.6]$

From the detailed analysis of each tuning parameter that may influence the estimation of a and b, it is concluded that the following tuning parameter values will result in the best possible estimation of a and b.

- The kernel function does not influence the estimation of a and b substantially, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m=5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1 - \hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) is recommended as a good choice for the estimated smoothing parameter.
- Both the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests are recommended, with preference given to the Cramér-von-Mises goodness-of-fit test when the sample size is large.
- For both of the above-mentioned goodness-of-fit tests, α-values of 1% or 5% can be used, with preference given to smaller α-values when the sample size is large.
- The choices of $4 \leq r \leq 8$ and $20 \leq g \leq 50$ are recommended for optimal results.

Table 4.137 provides the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$ respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in this table should be compared to the interval $[a,b]=[0.4\,,\,0.6]$.

Table 4.137: *Monte Carlo estimates of* $E(\hat{a})$ *and* $E(\hat{b})$ *for* $\alpha=0.01$ *for the Cramér-von-Mises goodness-of-fit test with* $\hat{h}_1$*.*

	r=4		**r=8**	
	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$
g=20	0.41	0.59	0.41	0.59
g=25	0.41	0.59	0.41	0.59
g=30	0.41	0.59	0.41	0.59
g=35	0.41	0.59	0.41	0.59
g=40	0.41	0.59	0.41	0.59
g=45	0.41	0.59	0.41	0.59
g=50	0.41	0.59	0.41	0.59

Remark: *Although this is an extreme data set in terms of the percentage pulsed emission contained in the interval [0,1], the estimation of the off-pulse interval is particularly good.*

4.5.5 Data set parameters: $1-p=0.3$, $\kappa=4$, $n=7500$ and $[a,b]=[0.3\,,\,0.7]$

Figure 4.14 is a histogram representation (with 100 classes) of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

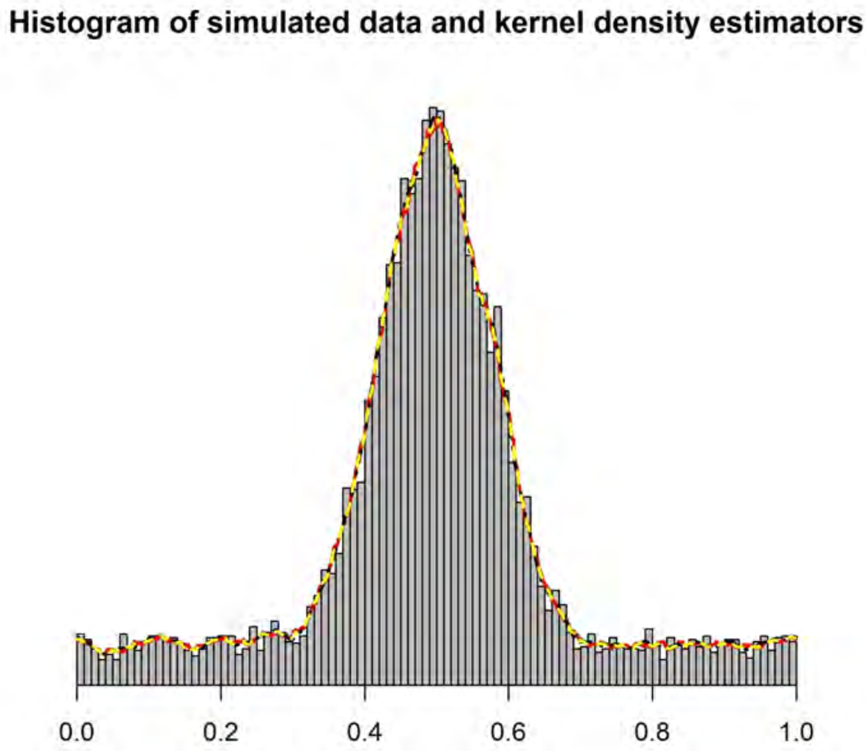

Figure 4.14: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.13$.

Choice of kernel function

In the analyses of all of the previous target populations, it was found that different choices of the kernel function resulted in almost similar behaviour of the kernel density estimator. The same behaviour is evident in this target population when inspecting the different kernel density estimators in Figure 4.14. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus. Tables 4.138 and 4.139 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ for $\hat{h}_1$, $g = 6$, $r = 6$ and $\alpha = 0.01$ for the different goodness-of-fit tests.

From the tables it is observed that different values of m have neither an influence on the estimated bias nor the MSE. It can again be concluded that a small value of m is preferable, since computing time is reduced and it seems as if the goodness-of-fit tests are insensitive to the value of m.

Table 4.138: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=2	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=3	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=4	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=5	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=6	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=7	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=8	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=9	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
m=10	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05

Table 4.139: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=2	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=3	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=4	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=5	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=6	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=7	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=8	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=9	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=10	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01

Choice of estimated smoothing parameters

Table 4.140 and Table 4.141 compare the different estimated smoothing parameters by highlighting the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$.

When inspecting the estimated bias and MSE, it is very difficult to highlight a specific choice of smoothing parameter that performs better than any of the others. Almost all of the choices perform equally well. Therefore, the recommendation is still valid that any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) can be used as a good choice for the estimated smoothing parameter.

Choice of goodness-of-fit test

Since different goodness-of-fit tests exist, it must be evaluated whether there is a superior goodness-of-fit test when compared to other tests. Some of the tables provided earlier can be used to assess the goodness-of-fit tests. Moreover, the reader can inspect Table 4.142 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ with $\hat{h}_1$. Table 4.143 is even more important, since the MSE is compared in this table.

Table 4.140: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$ and $g = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
$\hat{h}_2$	0.04	-0.05	0.05	-0.05	0.04	-0.05	0.05	-0.05
$\hat{h}_3$	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
$\hat{h}_4$	0.04	-0.05	0.05	-0.05	0.04	-0.05	0.05	-0.05
$\hat{h}_5$	0.04	-0.05	0.05	-0.05	0.04	-0.05	0.05	-0.05
$\hat{h}_6$	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
$\hat{h}_7$	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
$\hat{h}_8$	0.04	-0.05	0.05	-0.05	0.04	-0.05	0.05	-0.05
$\hat{h}_9$	0.04	-0.04	0.05	-0.05	0.05	-0.04	0.05	-0.05

Table 4.141: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$ and $g = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_2$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_3$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_4$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_5$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_6$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_7$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_8$	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
$\hat{h}_9$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01

Table 4.142: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.04	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
g=7	0.03	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
g=8	0.03	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
g=9	0.03	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
g=10	0.03	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05
g=20	0.03	-0.04	0.05	-0.05	0.04	-0.04	0.05	-0.05

Table 4.143: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=7	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=8	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=9	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=10	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=20	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.00

In terms of estimated bias, the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests perform better than the other two tests for any choice of $\hat{h}$ and g. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs best, but only with the slightest of margins.

Choice of the significance level α

Several tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit test. Tables 4.144 – 4.145 compare the estimated bias for two goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_1$. From these tables it is obvious that the estimated bias moves closer to zero when α is *decreased*, but only for the Anderson-Darling goodness-of-fit test. For the Cramér-von-Mises goodness-of-fit test, the bias moves slightly away from zero when α is decreased.

Tables 4.146 – 4.147 compare the estimated MSE for the same goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_1$. For smaller values of α, smaller values of the estimated MSE are found (for the Anderson-Darling goodness-of-fit test), while the estimated MSE for the Cramér-von-Mises goodness-of-fit test is not influenced by the choice of α. It therefore seems valid to recommend α-values in the range from 0.01 to 0.05, with limited effect on the estimated values of a and b.

Table 4.144: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\boldsymbol{\alpha = 0.01}$		$\boldsymbol{\alpha = 0.05}$		$\boldsymbol{\alpha = 0.10}$	
	$\hat{\boldsymbol{a}}$	$\hat{\boldsymbol{b}}$	$\hat{\boldsymbol{a}}$	$\hat{\boldsymbol{b}}$	$\hat{\boldsymbol{a}}$	$\hat{\boldsymbol{b}}$
g=6	0.04	-0.04	0.05	-0.06	0.07	-0.08
g=7	0.03	-0.04	0.05	-0.06	0.06	-0.08
g=8	0.03	-0.04	0.05	-0.06	0.06	-0.08
g=9	0.03	-0.04	0.05	-0.06	0.06	-0.07
g=10	0.03	-0.04	0.05	-0.05	0.05	-0.07
g=20	0.03	-0.04	0.04	-0.04	0.04	-0.05
g=25	0.03	-0.04	0.03	-0.04	0.04	-0.05
g=30	0.03	-0.04	0.04	-0.04	0.03	-0.04
g=35	0.03	-0.04	0.03	-0.04	0.04	-0.04

Table 4.145: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\boldsymbol{\alpha = 0.01}$		$\boldsymbol{\alpha = 0.05}$		$\boldsymbol{\alpha = 0.10}$	
	$\hat{\boldsymbol{a}}$	$\hat{\boldsymbol{b}}$	$\hat{\boldsymbol{a}}$	$\hat{\boldsymbol{b}}$	$\hat{\boldsymbol{a}}$	$\hat{\boldsymbol{b}}$
g=6	0.05	-0.05	0.05	-0.05	0.04	-0.04
g=7	0.05	-0.05	0.05	-0.05	0.04	-0.04
g=8	0.05	-0.05	0.05	-0.05	0.04	-0.04
g=9	0.05	-0.05	0.05	-0.05	0.04	-0.04
g=10	0.05	-0.05	0.05	-0.05	0.04	-0.04
g=20	0.05	-0.05	0.05	-0.05	0.05	-0.05
g=25	0.05	-0.05	0.05	-0.05	0.05	-0.05
g=30	0.05	-0.05	0.05	-0.05	0.05	-0.05
g=35	0.05	-0.05	0.05	-0.05	0.05	-0.05

Table 4.146: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	0.01	0.04	0.05	0.06	0.07
g=7	0.01	0.01	0.03	0.04	0.05	0.07
g=8	0.01	0.01	0.03	0.04	0.05	0.07
g=9	0.01	0.01	0.03	0.04	0.05	0.06
g=10	0.01	0.01	0.03	0.04	0.04	0.06
g=20	0.00	0.01	0.01	0.02	0.03	0.04
g=25	0.00	0.01	0.01	0.02	0.02	0.03
g=30	0.00	0.00	0.01	0.02	0.02	0.03
g=35	0.00	0.00	0.01	0.01	0.02	0.02

Table 4.147: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.00	0.00	0.00	0.00
g=7	0.00	0.00	0.00	0.00	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00
g=25	0.00	0.00	0.00	0.00	0.00	0.00
g=30	0.00	0.00	0.00	0.00	0.00	0.00
g=35	0.00	0.00	0.00	0.00	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

On inspection of Tables 4.148 – 4.151, the reader can compare the estimated bias and MSE for different g and r-values, when $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$, only for the Anderson-Darling goodness-of-fit test. Similar trends are observed when using the Cramér-von-Mises goodness-of-fit test. In terms of the estimated bias and MSE, $6 \leq g \leq 40$ with $6 \leq r \leq 8$ result in a smaller MSE (and bias-values closer to zero) when using the Anderson-Darling goodness-of-fit test, compared to other combinations of g and r.

In conclusion, it is recommended to use a value of g in the range from 6 to 40, and for r, values from 6 to 8, although one may use a slightly larger value of g when the sample size n is moderate to large, such as in this case.

Table 4.148: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.08	0.07	0.06	0.06	0.06					
g=2	0.08	0.06	0.06	0.05	0.05					
g=3	0.07	0.06	0.05	0.05	0.05					
g=4	0.07	0.05	0.05	0.04	0.04					
g=5	0.07	0.05	0.05	0.04	0.04					
g=6						0.04	0.03	0.03	0.03	0.03
g=7						0.03	0.03	0.03	0.03	0.03
g=8						0.03	0.03	0.03	0.03	0.03
g=9						0.03	0.03	0.03	0.03	0.03
g=10						0.03	0.03	0.03	0.03	0.03
g=20		0.04		0.03		0.03		0.03		0.03
g=25		0.04		0.04		0.03		0.03		0.03
g=30		0.04		0.03		0.03		0.03		0.03
g=35		0.04		0.03		0.03		0.04		0.03
g=40		0.04		0.03		0.04		0.04		0.04
g=45		0.04		0.04		0.04		0.04		0.04
g=50		0.04		0.04		0.04		0.04		0.04
g=100		0.04		0.04		0.04		0.04		0.04
g=200		0.05		0.05		0.05		0.05		0.05
g=300		0.06		0.06		0.06		0.06		0.06
g=400		0.06		0.06		0.06		0.06		0.06
g=500		0.07		0.07		0.07		0.07		0.07

Table 4.149: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.09	-0.08	-0.07	-0.07	-0.07					
g=2	-0.08	-0.07	-0.07	-0.06	-0.06					
g=3	-0.08	-0.07	-0.06	-0.06	-0.05					
g=4	-0.08	-0.06	-0.06	-0.05	-0.05					
g=5	-0.07	-0.06	-0.05	-0.04	-0.04					
g=6						-0.04	-0.04	-0.04	-0.04	-0.04
g=7						-0.04	-0.04	-0.04	-0.04	-0.03
g=8						-0.04	-0.04	-0.04	-0.03	-0.03
g=9						-0.04	-0.04	-0.03	-0.03	-0.03
g=10						-0.04	-0.04	-0.03	-0.03	-0.03
g=20		-0.04		-0.04		-0.04		-0.04		-0.04
g=25		-0.04		-0.04		-0.04		-0.04		-0.03
g=30		-0.04		-0.04		-0.04		-0.04		-0.03
g=35		-0.04		-0.04		-0.04		-0.04		-0.04
g=40		-0.04		-0.04		-0.04		-0.04		-0.04
g=45		-0.04		-0.04		-0.04		-0.04		-0.04
g=50		-0.04		-0.04		-0.04		-0.04		-0.04
g=100		-0.04		-0.04		-0.04		-0.04		-0.04
g=200		-0.05		-0.05		-0.05		-0.05		-0.05
g=300		-0.06		-0.06		-0.06		-0.06		-0.06
g=400		-0.06		-0.06		-0.06		-0.06		-0.06
g=500		-0.07		-0.07		-0.07		-0.07		-0.07

Table 4.150: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.06	0.04	0.04	0.03	0.03					
g=2	0.05	0.04	0.03	0.03	0.02					
g=3	0.05	0.03	0.03	0.02	0.02					
g=4	0.05	0.03	0.02	0.02	0.01					
g=5	0.04	0.03	0.02	0.02	0.01					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.01	0.01	0.01	0.01	0.00
g=8						0.01	0.01	0.01	0.00	0.00
g=9						0.01	0.01	0.00	0.00	0.00
g=10						0.01	0.00	0.00	0.00	0.00
g=20		0.01		0.00		0.00		0.00		0.00
g=25		0.01		0.00		0.00		0.00		0.00
g=30		0.01		0.00		0.00		0.00		0.00
g=35		0.01		0.00		0.00		0.00		0.00
g=40		0.01		0.00		0.00		0.00		0.00
g=45		0.01		0.00		0.00		0.00		0.00
g=50		0.01		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.01		0.01		0.01		0.01		0.01

Table 4.151: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.07	0.06	0.05	0.05	0.04					
g=2	0.06	0.05	0.04	0.04	0.03					
g=3	0.06	0.04	0.03	0.03	0.03					
g=4	0.06	0.04	0.03	0.03	0.02					
g=5	0.05	0.04	0.03	0.02	0.02					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.01	0.01	0.01	0.01	0.01
g=8						0.01	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.01
g=10						0.01	0.01	0.01	0.01	0.01
g=20		0.02		0.01		0.01		0.00		0.00
g=25		0.01		0.01		0.01		0.00		0.00
g=30		0.01		0.01		0.00		0.00		0.00
g=35		0.01		0.01		0.00		0.00		0.00
g=40		0.01		0.01		0.00		0.00		0.00
g=45		0.01		0.01		0.00		0.00		0.00
g=50		0.01		0.00		0.00		0.00		0.00
g=100		0.01		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		0.00

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.3$, $\kappa = 4$, $n = 7500$ and with $[a, b] = [0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combinations of parameter values may result in the best possible estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$ - $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) is recommended as a good choice for the estimated smoothing parameter.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended.
- For both of the above-mentioned goodness-of-fit tests, α-values of 1% or 5% can be used.
- The choices of $6 \leq r \leq 8$ and $6 \leq g \leq 40$ are recommended for optimal results. An even larger value of g can be used when the sample size is moderate to large.

Table 4.152 provides the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in this table should be compared to the interval $[a, b] = [0.3\,,\,0.7]$.

Table 4.152: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $\alpha = 0.01$ for the Anderson-Darling goodness-of-fit test with $\hat{h}_1$.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.34	0.66	0.33	0.66
g=7	0.33	0.66	0.33	0.66
g=8	0.33	0.66	0.33	0.66
g=9	0.33	0.66	0.33	0.67
g=10	0.33	0.66	0.33	0.67
g=20	0.33	0.66	0.33	0.66
g=25	0.33	0.66	0.33	0.66
g=30	0.33	0.66	0.33	0.66
g=35	0.33	0.66	0.34	0.66
g=40	0.34	0.66	0.34	0.66

Remark: *Even though this data set contains a relatively large percentage of noise, the estimation of the off-pulse interval is still close to the actual off-pulse interval.*

4.5.6 Data set parameters: $1 - p = 0.4$, $\kappa = 2$, $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$

Figure 4.15 provides a histogram representation (with 100 classes) of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

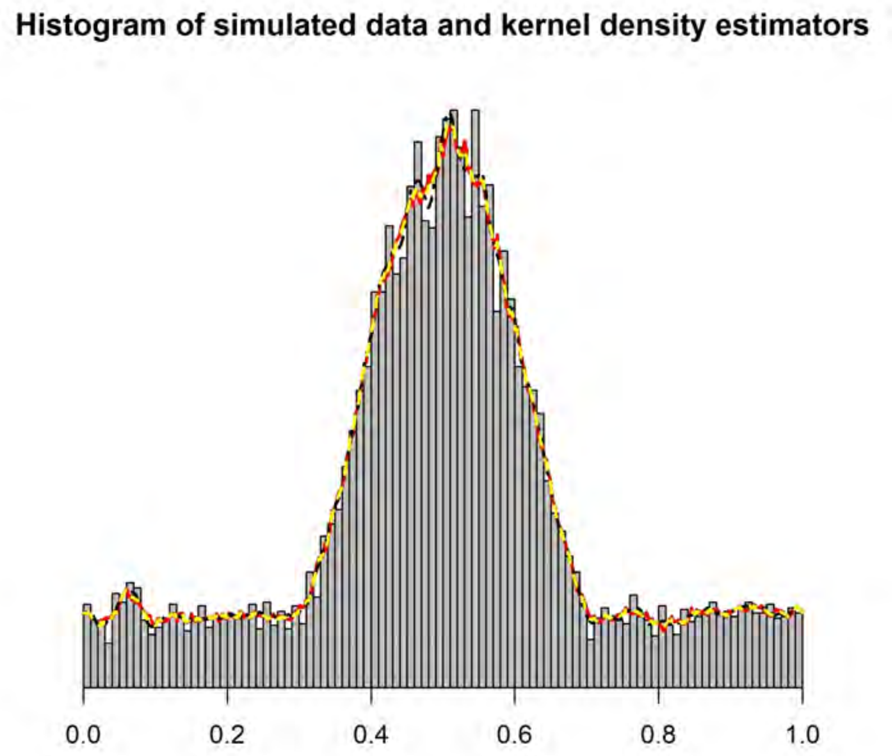

Figure 4.15: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_2 = 0.12$.

Choice of kernel function

From Figure 4.15, and the analyses of all of the previous study populations, it is found that different choices of the kernel function result in almost similar behaviour of the kernel density estimator. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

The first step of SOPIE is to select a number of minimum points m. In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus.

Tables 4.153 and 4.154 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ when $g = 6$, $r = 6$, $\alpha = 0.05$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_2$.

From the tables it is observed that different values of m influence the estimated bias or MSE very slightly. It can again be concluded that a small value of m is preferable, since computing time is reduced and it seems as if the goodness-of-fit tests are insensitive to the value of m.

Table 4.153: *Bias for different choices of m for $\hat{h}_2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=2	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=3	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=4	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=5	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=6	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=7	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=8	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
m=9	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.05
m=10	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04

Table 4.154: *MSE for different choices of m for $\hat{h}_2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=2	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=3	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=4	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=5	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=6	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
m=7	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=8	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=9	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
m=10	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01

Choice of estimated smoothing parameters

Table 4.155 and Table 4.156 highlight the values of the estimated bias and MSE when estimating a and b for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$.

When inspecting the estimated bias and MSE, it is very difficult to highlight a specific choice of smoothing parameter that performs better than any of the others. Almost all of the choices perform equally well. The recommendation continues to hold that any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is a good choice for the estimated smoothing parameter.

Choice of goodness-of-fit test

Table 4.157 compares the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ with $\hat{h}_2$. In Table 4.158 the MSE is compared for the same tuning parameter values. In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g, with the Kolmogorov-Smirnov, Rayleigh and Cramér-von-Mises goodness-of-fit tests performing almost as good. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs best. The other three tests have a marginally larger estimated MSE.

Table 4.155: *Comparison of the bias for different combinations of the estimated smoothing parameter with* $m = 1$, $r = 6$ *and* $g = 6$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.03	-0.04	0.04	-0.04	0.03	-0.04	0.04	-0.05
$\hat{h}_2$	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
$\hat{h}_3$	0.03	-0.04	0.04	-0.04	0.04	-0.05	0.04	-0.04
$\hat{h}_4$	0.03	-0.03	0.04	-0.04	0.04	-0.04	0.04	-0.04
$\hat{h}_5$	0.04	-0.03	0.04	-0.04	0.03	-0.04	0.04	-0.05
$\hat{h}_6$	0.03	-0.03	0.04	-0.04	0.03	-0.04	0.05	-0.05
$\hat{h}_7$	0.03	-0.03	0.04	-0.04	0.04	-0.04	0.04	-0.05
$\hat{h}_8$	0.03	-0.04	0.04	-0.04	0.03	-0.04	0.04	-0.05
$\hat{h}_9$	0.03	-0.03	0.04	-0.04	0.04	-0.04	0.04	-0.04

Table 4.156: *Comparison of the MSE for different combinations of the estimated smoothing parameter with* $m = 1$, $r = 6$ *and* $g = 6$

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_2$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_3$	0.01	0.02	0.00	0.00	0.01	0.02	0.01	0.01
$\hat{h}_4$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_5$	0.02	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_6$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_7$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_8$	0.02	0.02	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_9$	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01

Table 4.157: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_2$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=7	0.03	-0.04	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=8	0.03	-0.03	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=9	0.03	-0.03	0.04	-0.04	0.03	-0.04	0.04	-0.04
g=10	0.03	-0.03	0.04	-0.04	0.04	-0.03	0.04	-0.04
g=20	0.03	-0.03	0.04	-0.04	0.03	-0.03	0.04	-0.04

Table 4.158: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6, \hat{h}_2$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
g=7	0.01	0.02	0.00	0.00	0.01	0.01	0.01	0.01
g=8	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=9	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=10	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
g=20	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01

Choice of the significance level α

Several different values of α are utilised in the simulation study, and therefore multiple tables are constructed to investigate the effect of α, in combination with the effect of the other tuning parameters. Tables 4.159 – 4.162 compare the estimated bias and MSE for two goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_2$.

Table 4.159: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_2$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.03	-0.04	0.07	-0.07	0.10	-0.09
g=7	0.03	-0.04	0.06	-0.07	0.09	-0.08
g=8	0.03	-0.03	0.06	-0.06	0.09	-0.08
g=9	0.03	-0.03	0.06	-0.06	0.08	-0.08
g=10	0.03	-0.03	0.06	-0.06	0.08	-0.08
g=20	0.03	-0.03	0.03	-0.04	0.05	-0.06
g=25	0.03	-0.03	0.03	-0.03	0.05	-0.05
g=30	0.03	-0.03	0.03	-0.04	0.04	-0.04
g=35	0.03	-0.03	0.02	-0.03	0.04	-0.04

Table 4.160: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_2$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=7	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=8	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=9	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=10	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=20	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=25	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=30	0.04	-0.04	0.04	-0.04	0.04	-0.04
g=35	0.04	-0.04	0.04	-0.04	0.04	-0.04

Table 4.161: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_2$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	0.02	0.06	0.06	0.09	0.09
g=7	0.01	0.02	0.05	0.06	0.08	0.08
g=8	0.01	0.01	0.05	0.05	0.08	0.08
g=9	0.01	0.01	0.05	0.05	0.08	0.08
g=10	0.01	0.01	0.04	0.05	0.07	0.07
g=20	0.00	0.01	0.02	0.03	0.05	0.05
g=25	0.00	0.00	0.02	0.02	0.04	0.04
g=30	0.00	0.00	0.02	0.02	0.03	0.04
g=35	0.00	0.00	0.01	0.02	0.03	0.03

Table 4.162: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_2$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.00	0.00	0.00	0.00
g=7	0.00	0.00	0.00	0.00	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00
g=25	0.00	0.00	0.00	0.00	0.00	0.00
g=30	0.00	0.00	0.00	0.00	0.00	0.00
g=35	0.00	0.00	0.00	0.00	0.00	0.00

For the Anderson-Darling goodness-of-fit test, it is obvious that the estimated bias and MSE decrease and move closer to zero when α is *decreased*. For the Cramér-von-Mises goodness-of-fit test, though, the bias and MSE are robust against the choice of α. Therefore, it seems valid to recommend any value of α in the range from 0.01 to 0.05, with limited effect on the estimated values of a and b.

Choice of incremental growth g and number of intervals of rejection r

In Tables 4.163 – 4.166 the reader will find a comparison of the estimated bias and MSE when different g and r-values are used, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$, for only the Anderson-Darling goodness-of-fit test. Similar trends are observed when using the Cramér-von-Mises goodness-of-fit test.

In terms of the estimated bias, $6 \leq g \leq 40$ with $6 \leq r \leq 8$ result in bias-values close to zero when using the Anderson-Darling goodness-of-fit test. When comparing the MSE, a similar trend is observed, with slightly larger values of g resulting in the smallest estimated MSE. In terms of the estimated bias and MSE for the Cramér-von-Mises goodness-of-fit test, it is evident that the choices of g and r have nearly no effect on the estimated values of a and b. In conclusion, it is recommended to use values of g in the range from 6 to 40, and r-values in the range from 6 to 8.

Table 4.163: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.08	0.07	0.06	0.06	0.05					
g=2	0.08	0.06	0.05	0.05	0.05					
g=3	0.07	0.06	0.05	0.05	0.05					
g=4	0.07	0.05	0.05	0.05	0.04					
g=5	0.06	0.05	0.05	0.04	0.04					
g=6						0.03	0.03	0.03	0.03	0.03
g=7						0.03	0.03	0.03	0.03	0.03
g=8						0.03	0.03	0.03	0.03	0.03
g=9						0.03	0.03	0.03	0.03	0.02
g=10						0.03	0.03	0.03	0.03	0.03
g=20		0.03		0.03		0.03		0.03		0.03
g=25		0.03		0.03		0.03		0.03		0.03
g=30		0.03		0.03		0.03		0.03		0.03
g=35		0.03		0.03		0.03		0.03		0.03
g=40		0.03		0.03		0.03		0.03		0.03
g=45		0.03		0.03		0.03		0.03		0.03
g=50		0.03		0.03		0.03		0.03		0.03
g=100		0.03		0.03		0.03		0.03		0.03
g=200		0.04		0.04		0.04		0.04		0.04
g=300		0.04		0.04		0.04		0.04		0.04
g=400		0.05		0.05		0.05		0.05		0.05
g=500		0.05		0.05		0.05		0.05		0.05

Table 4.164: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.08	-0.08	-0.07	-0.06	-0.06					
g=2	-0.08	-0.07	-0.06	-0.05	-0.05					
g=3	-0.08	-0.06	-0.05	-0.05	-0.05					
g=4	-0.08	-0.05	-0.05	-0.05	-0.05					
g=5	-0.07	-0.06	-0.05	-0.05	-0.04					
g=6						-0.04	-0.04	-0.03	-0.03	-0.03
g=7						-0.04	-0.03	-0.03	-0.03	-0.03
g=8						-0.03	-0.03	-0.03	-0.03	-0.03
g=9						-0.03	-0.03	-0.03	-0.03	-0.03
g=10						-0.03	-0.03	-0.03	-0.03	-0.03
g=20		-0.04		-0.03		-0.03		-0.03		-0.02
g=25		-0.04		-0.03		-0.03		-0.02		-0.02
g=30		-0.04		-0.03		-0.03		-0.02		-0.03
g=35		-0.04		-0.03		-0.03		-0.03		-0.03
g=40		-0.03		-0.03		-0.03		-0.03		-0.03
g=45		-0.03		-0.03		-0.03		-0.03		-0.03
g=50		-0.03		-0.03		-0.03		-0.03		-0.03
g=100		-0.03		-0.03		-0.03		-0.03		-0.03
g=200		-0.04		-0.04		-0.04		-0.04		-0.04
g=300		-0.04		-0.04		-0.04		-0.04		-0.04
g=400		-0.04		-0.05		-0.05		-0.05		-0.05
g=500		-0.05		-0.05		-0.05		-0.05		-0.05

Table 4.165: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.07	0.06	0.05	0.04	0.04					
g=2	0.06	0.05	0.04	0.04	0.03					
g=3	0.06	0.04	0.04	0.03	0.03					
g=4	0.06	0.04	0.03	0.03	0.02					
g=5	0.05	0.04	0.03	0.02	0.02					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.01	0.01	0.01	0.01	0.01
g=8						0.01	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.00
g=10						0.01	0.01	0.01	0.01	0.00
g=20		0.02		0.01		0.00		0.00		0.00
g=25		0.01		0.01		0.00		0.00		0.00
g=30		0.01		0.00		0.00		0.00		0.00
g=35		0.01		0.00		0.00		0.00		0.00
g=40		0.01		0.00		0.00		0.00		0.00
g=45		0.01		0.00		0.00		0.00		0.00
g=50		0.01		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		0.00

Table 4.166: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.07	0.06	0.06	0.05	0.05					
g=2	0.07	0.06	0.05	0.04	0.04					
g=3	0.07	0.05	0.04	0.04	0.03					
g=4	0.06	0.05	0.04	0.03	0.03					
g=5	0.06	0.04	0.04	0.03	0.03					
g=6						0.02	0.02	0.01	0.01	0.01
g=7						0.02	0.01	0.01	0.01	0.01
g=8						0.01	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.01
g=10						0.01	0.01	0.01	0.01	0.01
g=20		0.02		0.01		0.01		0.00		0.00
g=25		0.02		0.01		0.00		0.00		0.00
g=30		0.02		0.01		0.00		0.00		0.00
g=35		0.02		0.01		0.00		0.00		0.00
g=40		0.01		0.00		0.00		0.00		0.00
g=45		0.01		0.00		0.00		0.00		0.00
g=50		0.01		0.00		0.00		0.00		0.00
g=100		0.01		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		0.00

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.4$**,** $\kappa = 2$**,** $n = 10000$ **and with** $[a, b] = [0.3\,,\,0.7]$

From the detailed analysis of each tuning parameter that may influence the estimation of a and b, it is concluded that the following tuning parameter values will result in the best possible estimation of a and b.

- The kernel function does not influence the estimation of a and b substantially, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is recommended as a good choice for the estimated smoothing parameter.
- Both the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests are recommended.
- For both of these goodness-of-fit tests, α-values of 1% or 5% can be used.
- The choices of $6 \leq r \leq 8$ and $6 \leq g \leq 40$ are recommended for optimal results.

Table 4.167 provides the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in this table should be compared to the interval $[a, b] = [0.3\,,\,0.7]$.

Table 4.167: *Monte Carlo estimates of* $E(\hat{a})$ *and* $E(\hat{b})$ *for* $\alpha = 0.01$ *for the Anderson-Darling goodness-of-fit test with* $\hat{h}_2$*.*

	r=6		**r=8**	
	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$
g=6	0.33	0.66	0.33	0.67
g=7	0.33	0.66	0.33	0.67
g=8	0.33	0.67	0.33	0.67
g=9	0.33	0.67	0.33	0.67
g=10	0.33	0.67	0.33	0.67
g=20	0.33	0.67	0.33	0.67
g=25	0.33	0.67	0.33	0.68
g=30	0.33	0.67	0.33	0.68
g=35	0.33	0.67	0.33	0.67
g=40	0.33	0.67	0.33	0.67

Remark: *Even though this data set contains* 40% *noise, the estimated off-pulse interval is still quite accurate.*

4.5.7 Data set parameters: $1 - p = 0.1$, $\kappa = 1$, $n = 5000$ and $[a, b] = [0.45\,,\,0.55]$

Figure 4.16 provides a histogram representation (with 100 classes) of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

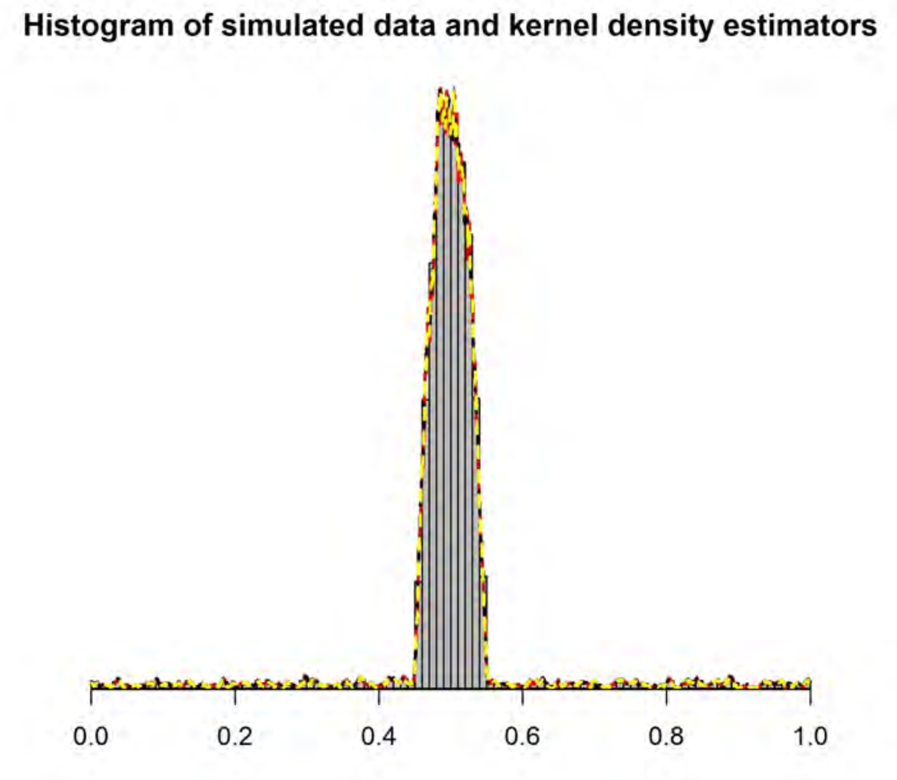

Figure 4.16: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_1 = 0.11$.

Choice of kernel function

From Figure 4.16, and the analyses of all of the previous study populations, it is found that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m. Tables 4.168 and 4.169 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ when $g = 6$, $r = 6$, $\alpha = 0.05$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_1$. It is obvious from the tables that different values of m influence the estimated bias or MSE very slightly. Therefore, a small value for m is preferable, since computing time is reduced and it seems as if the goodness-of-fit tests are insensitive to the choice of m.

Choice of estimated smoothing parameters

Table 4.170 and Table 4.171 highlight the values of the estimated bias and MSE when estimating a and b for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.05$.

Table 4.168: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.00	0.00	-0.01	0.01	-0.00	-0.00	-0.00	-0.00
m=2	-0.00	0.00	-0.01	0.01	-0.00	0.00	-0.00	-0.00
m=3	0.00	0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00
m=4	0.00	0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00
m=5	0.00	0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00
m=6	0.00	-0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00
m=7	0.00	-0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00
m=8	0.00	-0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00
m=9	0.00	-0.00	-0.01	0.01	-0.00	0.00	-0.00	0.00

Table 4.169: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
m=2	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
m=3	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
m=4	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
m=5	0.02	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=6	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
m=7	0.02	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=8	0.02	0.01	0.00	0.00	0.01	0.01	0.01	0.01
m=9	0.02	0.01	0.00	0.00	0.01	0.01	0.01	0.01

Table 4.170: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $r = 6$ and $g = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.00	0.00	-0.01	0.01	-0.00	-0.00	-0.00	-0.00
$\hat{h}_2$	0.00	0.00	-0.01	0.01	0.00	-0.00	0.00	-0.00
$\hat{h}_3$	0.01	-0.01	-0.01	0.01	0.01	-0.01	0.01	-0.02
$\hat{h}_4$	0.00	0.00	-0.01	0.01	-0.01	-0.00	0.00	-0.00
$\hat{h}_5$	0.01	-0.01	-0.01	0.01	0.01	-0.01	0.02	-0.02
$\hat{h}_6$	0.01	0.00	-0.01	0.01	-0.00	-0.00	0.00	-0.00
$\hat{h}_7$	-0.01	-0.00	-0.01	0.01	-0.01	-0.00	0.00	-0.01
$\hat{h}_8$	0.00	0.00	-0.01	0.01	-0.00	-0.00	0.00	-0.01
$\hat{h}_9$	-0.01	-0.01	-0.01	0.01	-0.01	-0.00	0.00	-0.01

When inspecting the estimated bias and MSE, it is found that $\hat{h}_3$ and $\hat{h}_5$ are associated with estimated bias-values closer to zero compared to some of the other choices of $\hat{h}$. The MSE is also smaller for these choices of $\hat{h}$.

The recommendation is still that any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is a good choice for the estimated smoothing parameter.

Table 4.171: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $r = 6$ and $g = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
$\hat{h}_2$	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
$\hat{h}_3$	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01
$\hat{h}_4$	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
$\hat{h}_5$	0.01	0.00	0.00	0.00	0.01	0.00	0.01	0.01
$\hat{h}_6$	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
$\hat{h}_7$	0.03	0.01	0.00	0.00	0.02	0.01	0.01	0.01
$\hat{h}_8$	0.02	0.01	0.00	0.00	0.02	0.01	0.01	0.01
$\hat{h}_9$	0.03	0.01	0.00	0.00	0.02	0.01	0.01	0.01

Choice of goodness-of-fit test

The proposed methodology is based, in a sequential way, on the P-values of goodness-of-fit tests for the uniform distribution. Different goodness-of-fit tests exist, and therefore it must be evaluated whether there is a superior goodness-of-fit test when compared to other tests. Some of the tables provided earlier were already used to assess the goodness-of-fit tests. Moreover, the reader can inspect Table 4.172 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.05$, $m = 1$ and $r = 6$ with $\hat{h}_5$. Table 4.173 is even more important, since the MSE is compared in this table.

Table 4.172: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 6, \hat{h}_5$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	-0.01	-0.01	0.01	0.01	-0.01	0.02	-0.02
g=7	0.01	-0.01	-0.01	0.01	0.01	-0.01	0.02	-0.02
g=8	0.01	-0.01	-0.01	0.01	0.01	-0.01	0.02	-0.01
g=9	0.01	-0.01	-0.01	0.01	0.01	-0.01	0.02	-0.01
g=10	0.01	-0.00	-0.01	0.01	0.01	-0.01	0.01	-0.01
g=20	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00

Table 4.173: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 6, \hat{h}_5$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.01	0.00	0.00	0.00	0.01	0.00	0.01	0.01
g=7	0.01	0.00	0.00	0.00	0.01	0.00	0.01	0.00
g=8	0.01	0.00	0.00	0.00	0.01	0.00	0.01	0.00
g=9	0.01	0.00	0.00	0.00	0.01	0.00	0.01	0.00
g=10	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

In terms of estimated bias, the Cramér-von-Mises goodness-of-fit test is superior for any choice of $\hat{h}$ and g, with the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests performing almost as good. When comparing the estimated MSE, the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests perform marginally better than the Kolmogorov-Smirnov goodness-of-fit test.

Choice of the significance level α

Several different values of α are utilised in the simulation study, and therefore multiple tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit test. Tables 4.174 - 4.175 compare the estimated bias for two goodness-of-fit tests for $m = 1$, $r = 6$ and for $\hat{h}_5$. From this table it is obvious that the estimated bias is closest to zero when $0.01 \leq \alpha \leq 0.05$.

Tables 4.176 - 4.177 compare the estimated MSE for the same goodness-of-fit tests, for $m = 1$, $r = 6$ and for $\hat{h}_5$. A similar trend to the one observed for the estimated bias is found when comparing the estimated MSE. In conclusion, it seems valid to recommend any value of α in the range from 0.01 to 0.05, with limited effect on the estimated values of a and b.

Table 4.174: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_5$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.01	-0.01	0.02	-0.03
g=7	-0.00	0.00	0.01	-0.01	0.02	-0.02
g=8	-0.00	0.00	0.01	-0.01	0.02	-0.02
g=9	-0.00	0.00	0.01	-0.01	0.02	-0.02
g=10	-0.00	0.00	0.01	-0.00	0.01	-0.02
g=20	-0.00	0.00	0.00	0.00	0.01	-0.01

Table 4.175: *Comparison of the estimated bias for different α-values, with $m = 1, r = 6$ and with $\hat{h}_5$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.01	0.01	-0.01	0.01	-0.01	0.01
g=7	-0.01	0.01	-0.01	0.01	-0.01	0.01
g=8	-0.01	0.01	-0.01	0.01	-0.01	0.01
g=9	-0.01	0.01	-0.01	0.01	-0.01	0.01
g=10	-0.01	0.01	-0.01	0.01	-0.01	0.01
g=20	-0.01	0.01	-0.01	0.01	-0.01	0.01

Table 4.176: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_5$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.01	0.00	0.03	0.01
g=7	0.00	0.00	0.01	0.00	0.02	0.01
g=8	0.00	0.00	0.01	0.00	0.02	0.01
g=9	0.00	0.00	0.01	0.00	0.02	0.01
g=10	0.00	0.00	0.01	0.00	0.02	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00

Table 4.177: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 6$ and with $\hat{h}_5$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.00	0.00	0.00	0.00	0.00	0.00
g=7	0.00	0.00	0.00	0.00	0.00	0.00
g=8	0.00	0.00	0.00	0.00	0.00	0.00
g=9	0.00	0.00	0.00	0.00	0.00	0.00
g=10	0.00	0.00	0.00	0.00	0.00	0.00
g=20	0.00	0.00	0.00	0.00	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

In Tables 4.178 – 4.180 the reader will find a comparison of the estimated bias and MSE when different g and r-values are used, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$, for only the Cramér-von-Mises goodness-of-fit test. Similar trends are observed when using the Anderson-Darling goodness-of-fit test. In Table 4.178 only the estimated MSE for $\hat{a}$ is presented, since the estimated MSE-values for $\hat{b}$ are equal to the estimated MSE-values for $\hat{a}$ for all the combinations of g and r.

In terms of the estimated bias, almost any value of g and r can be used, but $1 \leq g \leq 25$ with $1 \leq r \leq 6$ result in a bias closer to zero when using the Cramér-von-Mises goodness-of-fit test. When comparing the MSE, all values are equal for both $\hat{a}$ and $\hat{b}$. For the Anderson-Darling goodness-of-fit test, the smallest values of the MSE (and bias-values close to zero) are obtained when $1 \leq g \leq 4$ with $1 \leq r \leq 4$.

In conclusion, it is recommended to use values of g in the range from 1 to 20 and r-values from 1 to 6, although one may use a slightly smaller value of g when the sample size n is small to moderate, such as in this case.

Table 4.178: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.00	0.00	0.00	0.00	0.01					
g=2	0.00	0.00	0.00	0.01	0.01					
g=3	0.00	0.00	0.01	0.01	0.01					
g=4	0.00	0.00	0.01	0.01	0.01					
g=5	0.01	0.01	0.01	0.01	0.01					
g=6						0.01	0.01	0.01	0.01	0.01
g=7						0.01	0.01	0.01	0.01	0.01
g=8						0.01	0.01	0.01	0.01	0.01
g=9						0.01	0.01	0.01	0.01	0.01
g=10						0.01	0.01	0.01	0.01	0.01
g=20		0.01		0.01		0.01		0.01		0.01
g=25		0.01		0.01		0.01		0.01		0.01
g=30		0.01		0.01		0.01		0.01		0.01
g=35		0.01		0.01		0.01		0.01		0.01
g=40		0.01		0.01		0.01		0.01		0.01
g=45		0.01		0.01		0.01		0.01		0.01
g=50		0.01		0.01		0.01		0.01		0.01
g=100		0.01		0.01		0.01		0.01		0.01
g=200		0.01		0.01		0.01		0.01		0.01
g=300		0.01		0.01		0.01		0.01		0.01
g=400		0.01		0.01		0.01		0.01		0.01
g=500		0.02		0.02		0.02		0.02		

Table 4.179: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.00	-0.00	-0.00	-0.00	-0.00					
g=2	-0.00	-0.00	-0.00	-0.01	-0.01					
g=3	-0.00	-0.00	-0.01	-0.01	-0.01					
g=4	-0.01	-0.00	-0.01	-0.01	-0.01					
g=5	-0.01	-0.01	-0.01	-0.01	-0.01					
g=6						-0.01	-0.01	-0.01	-0.01	-0.01
g=7						-0.01	-0.01	-0.01	-0.01	-0.01
g=8						-0.01	-0.01	-0.01	-0.01	-0.01
g=9						-0.01	-0.01	-0.01	-0.01	-0.01
g=10						-0.01	-0.01	-0.01	-0.01	-0.01
g=20		-0.01		-0.01		-0.01		-0.01		-0.01
g=25		-0.01		-0.01		-0.01		-0.01		-0.01
g=30		-0.01		-0.01		-0.01		-0.01		-0.01
g=35		-0.01		-0.01		-0.01		-0.01		-0.01
g=40		-0.01		-0.01		-0.01		-0.01		-0.01
g=45		-0.01		-0.01		-0.01		-0.01		-0.01
g=50		-0.01		-0.01		-0.01		-0.01		-0.01
g=100		-0.01		-0.01		-0.01		-0.01		-0.01
g=200		-0.01		-0.01		-0.01		-0.01		-0.01
g=300		-0.01		-0.01		-0.01		-0.01		-0.01
g=400		-0.01		-0.01		-0.01		-0.01		-0.01
g=500		-0.01		-0.01		-0.01		-0.01		

Table 4.180: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$. (The MSE values for $\hat{b}$ are equal to the MSE values of $\hat{a}$ $\forall$ g and r-combinations. Therefore, a separate table is not provided.)*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.00	0.00	0.00	0.00	0.00					
g=2	0.00	0.00	0.00	0.00	0.00					
g=3	0.00	0.00	0.00	0.00	0.00					
g=4	0.00	0.00	0.00	0.00	0.00					
g=5	0.00	0.00	0.00	0.00	0.00					
g=6						0.00	0.00	0.00	0.00	0.00
g=7						0.00	0.00	0.00	0.00	0.00
g=8						0.00	0.00	0.00	0.00	0.00
g=9						0.00	0.00	0.00	0.00	0.00
g=10						0.00	0.00	0.00	0.00	0.00
g=20		0.00		0.00		0.00		0.00		0.00
g=25		0.00		0.00		0.00		0.00		0.00
g=30		0.00		0.00		0.00		0.00		0.00
g=35		0.00		0.00		0.00		0.00		0.00
g=40		0.00		0.00		0.00		0.00		0.00
g=45		0.00		0.00		0.00		0.00		0.00
g=50		0.00		0.00		0.00		0.00		0.00
g=100		0.00		0.00		0.00		0.00		0.00
g=200		0.00		0.00		0.00		0.00		0.00
g=300		0.00		0.00		0.00		0.00		0.00
g=400		0.00		0.00		0.00		0.00		0.00
g=500		0.00		0.00		0.00		0.00		

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.1$, $\kappa = 1$, $n = 5000$ and with $[a, b] = [0.45\,,\,0.55]$

From the detailed analysis of each tuning parameter that may influence the estimation of a and b, it is concluded that the following tuning parameter values will result in the best possible estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is recommended as a good choice for the estimated smoothing parameter.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended with preference given to the Cramér-von-Mises test (due to the smaller estimated MSE).
- For both of the above-mentioned goodness-of-fit tests, α-values of 1% or 5% can be used.
- The choices of $1 \leq r \leq 6$ and $1 \leq g \leq 20$ are recommended for optimal results, with preference given to smaller values of g when the sample size is small to moderate.

The following two tables (Tables 4.181 – 4.182) provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the interval $[a, b] = [0.45\,,\,0.55]$.

Table 4.181: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $\alpha = 0.05$ for the Cramér-von-Mises goodness-of-fit test with $\hat{h}_5$.*

	r=2		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.45	0.55	0.45	0.55
g=2	0.45	0.55	0.46	0.54
g=3	0.45	0.55	0.46	0.54
g=4	0.45	0.55	0.46	0.54
g=5	0.46	0.54	0.46	0.54

Table 4.182: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $\alpha = 0.05$ for the Anderson-Darling goodness-of-fit test with $\hat{h}_5$.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.44	0.56	0.44	0.56
g=7	0.44	0.56	0.45	0.56
g=8	0.44	0.56	0.45	0.55
g=9	0.44	0.56	0.45	0.56
g=10	0.45	0.56	0.45	0.56
g=20	0.45	0.55	0.45	0.55
g=25	0.45	0.55	0.45	0.55
g=30	0.45	0.55	0.45	0.55
g=35	0.45	0.55	0.45	0.55
g=40	0.45	0.55	0.45	0.55

Remark: *This data set is an extreme case in terms of the percentage pulsed emission contained in the interval $[0, 1]$, yet the estimated off-pulse interval is exceptionally accurate.*

4.5.8 Data set parameters: $1 - p = 0.1$, $\kappa = 3$, $n = 2000$ and $[a, b] = [0.25\,,\,0.75]$

Figure 4.17 is a histogram representation of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data. The histogram of the simulated data contains 100 classes. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

From Figure 4.17 and the analyses of all the previous study populations, it is found that different choices of the kernel function result in almost similar behaviour of the kernel density estimator. Therefore, the normal kernel is used for the analysis of this study population.

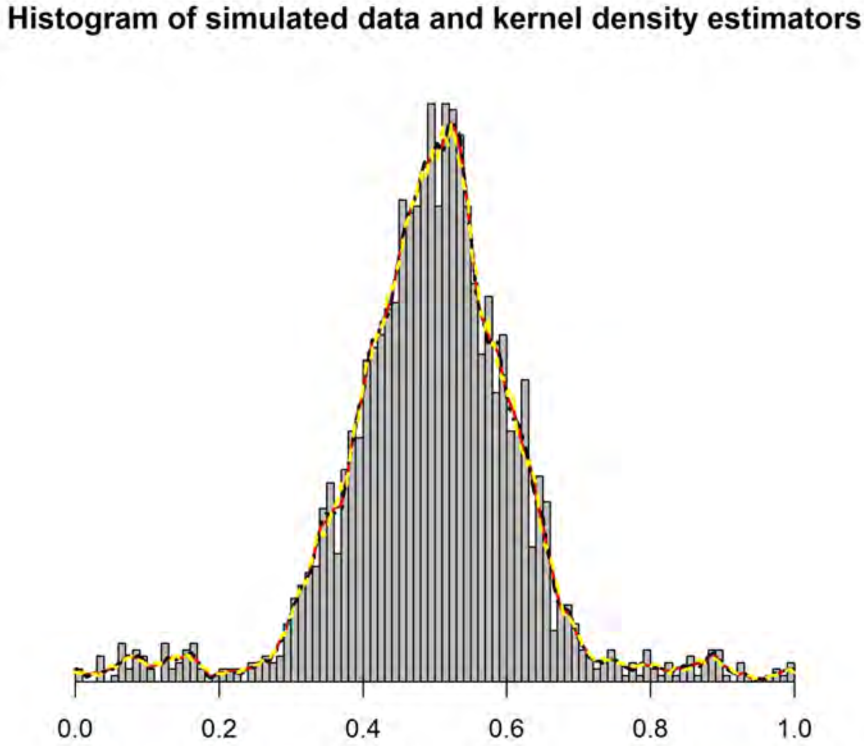

Figure 4.17: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.13$.

Choice of the number of minimum points m

Tables 4.183 and 4.184 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ when the normal kernel is used, $g = 2$, $r = 2$, $\alpha = 0.05$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_3$.

From Table 4.183 it is evident that the estimated bias is robust against different values of m. It can also be seen in Table 4.184 that the value of m has only a very slight impact on the estimated MSE for $\hat{a}$ and $\hat{b}$. Several of these comparisons (by using different values for g, r and α) resulted in the same conclusion that a small value of m is preferable in terms of computing time, without an adverse effect on the estimation.

Choice of estimated smoothing parameters

Tables 4.185 - 4.186 highlight the values of the bias and MSE when estimating a and b for $m = 1$, $g = 2$, $r = 2$ and $\alpha = 0.05$, for all of the goodness-of-fit tests and for each of the nine estimated smoothing parameters $\hat{h}$.

When inspecting the estimated bias and MSE, certain choices of the estimated smoothing parameter perform better than others. In terms of estimated bias and MSE, it seems as if $\hat{h}_3$ results in slightly smaller values of the estimated MSE (and bias-values closer to zero). Therefore, it is recommended to use $\hat{h}_3$ as a good choice of the estimated smoothing parameter, especially when the sample size

Table 4.183: *Bias for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.03	-0.05	0.06	-0.06	0.04	-0.05	0.04	-0.03
m=2	0.03	-0.05	0.06	-0.06	0.04	-0.05	0.04	-0.04
m=3	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=4	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=5	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=6	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=7	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=8	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=9	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
m=10	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04

Table 4.184: *MSE for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.02	0.04	0.00	0.00	0.02	0.05	0.02	0.02
m=2	0.02	0.04	0.00	0.00	0.02	0.04	0.02	0.03
m=3	0.02	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=4	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=5	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=6	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=7	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=8	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=9	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03
m=10	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.03

Table 4.185: *Comparison of the bias for different combinations of the estimated smoothing parameter and $m = 1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.06	-0.08	0.06	-0.06	0.07	-0.09	0.07	-0.08
$\hat{h}_2$	0.05	-0.09	0.06	-0.06	0.06	-0.12	0.07	-0.07
$\hat{h}_3$	0.03	-0.05	0.06	-0.06	0.04	-0.05	0.04	-0.03
$\hat{h}_4$	0.05	-0.09	0.06	-0.06	0.06	-0.10	0.07	-0.08
$\hat{h}_5$	0.05	-0.09	0.06	-0.06	0.07	-0.12	0.07	-0.07
$\hat{h}_6$	0.05	-0.06	0.06	-0.06	0.07	-0.07	0.08	-0.08
$\hat{h}_7$	0.05	-0.08	0.06	-0.06	0.06	-0.10	0.07	-0.08
$\hat{h}_8$	0.05	-0.10	0.06	-0.06	0.07	-0.13	0.06	-0.07
$\hat{h}_9$	0.04	-0.09	0.06	-0.06	0.06	-0.11	0.07	-0.08

Table 4.186: *Comparison of the MSE for different combinations of the estimated smoothing parameter and* $m = 1$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.03	0.06	0.00	0.00	0.04	0.06	0.04	0.04
$\hat{h}_2$	0.02	0.08	0.00	0.00	0.03	0.09	0.03	0.03
$\hat{h}_3$	0.02	0.04	0.00	0.00	0.02	0.05	0.02	0.02
$\hat{h}_4$	0.02	0.07	0.00	0.00	0.03	0.08	0.03	0.04
$\hat{h}_5$	0.02	0.08	0.00	0.00	0.03	0.10	0.03	0.03
$\hat{h}_6$	0.03	0.05	0.00	0.00	0.04	0.06	0.04	0.04
$\hat{h}_7$	0.02	0.07	0.00	0.00	0.03	0.08	0.03	0.04
$\hat{h}_8$	0.02	0.08	0.00	0.00	0.03	0.10	0.03	0.03
$\hat{h}_9$	0.02	0.07	0.00	0.00	0.03	0.09	0.03	0.03

is small.

Choice of goodness-of-fit test

The performance measures of the different goodness-of-fit tests must be investigated to assess whether there is a superior goodness-of-fit test when compared to the other tests. Some of the tables already provided can be used to assess the goodness-of-fit tests. The reader can also inspect Tables 4.187 – 4.188 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.05$, $m = 1$ and $r = 2$ with $\hat{h}_3$.

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best performance is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh test. The Cramér-von-Mises goodness-of-fit test has slightly larger estimated bias-values than the Rayleigh test.

When the estimated MSE is compared, the Cramér-von-Mises goodness-of-fit test performs the best, but only with the slightest of margins. The Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests perform almost as good. For small sample sizes n, the Kolmogorov-Smirnov goodness-of-fit test proves to be quite useful, but preference is still given to the Anderson-Darling goodness-of-fit test.

Table 4.187: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.05$, $m = 1$, $r = 2, \hat{h}_3$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.03	-0.05	0.06	-0.06	0.04	-0.05	0.03	-0.03
g=2	0.03	-0.05	0.06	-0.06	0.04	-0.05	0.04	-0.03
g=3	0.03	-0.05	0.06	-0.06	0.04	-0.06	0.04	-0.04
g=4	0.04	-0.05	0.06	-0.06	0.05	-0.06	0.04	-0.04
g=5	0.04	-0.04	0.07	-0.06	0.05	-0.05	0.04	-0.05
g=20	0.05	-0.04	0.07	-0.07	0.05	-0.05	0.06	-0.06

Table 4.188: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 2, \hat{h}_3$ for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.02	0.05	0.00	0.00	0.03	0.06	0.04	0.04
g=2	0.02	0.04	0.00	0.00	0.02	0.05	0.02	0.02
g=3	0.01	0.03	0.00	0.00	0.02	0.04	0.02	0.02
g=4	0.01	0.03	0.00	0.00	0.02	0.03	0.02	0.01
g=5	0.01	0.02	0.00	0.00	0.01	0.03	0.01	0.01
g=20	0.00	0.01	0.01	0.00	0.00	0.01	0.00	0.00

Choice of the significance level α

The significance level α is another parameter used in goodness-of-fit testing. Several different values of α are utilised in the simulation study and it must be ascertained whether α influences the estimation of a and b.

Tables 4.189 – 4.190 compare the estimated bias and MSE for the Anderson-Darling goodness-of-fit tests, for $m = 1$, $r = 2$ and for $\hat{h}_3$. When comparing the estimated bias, values closer to zero are attained when α is *increased*, for all of the goodness-of-fit tests . For the estimated MSE, it seems as if smaller values of α lead to smaller values of the estimated MSE. Therefore, for target populations with small sample sizes n, it is recommended to use any value of α in the range from 0.05 to 0.1.

Table 4.189: *Comparison of the estimated bias for different α-values, with $m = 1, r = 2$ and $\hat{h}_3$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.04	-0.05	0.03	-0.05	0.03	-0.03
g=2	0.05	-0.05	0.03	-0.05	0.03	-0.04
g=3	0.05	-0.05	0.03	-0.05	0.03	-0.04
g=4	0.05	-0.05	0.04	-0.05	0.04	-0.04
g=5	0.05	-0.05	0.04	-0.04	0.04	-0.05
g=20	0.06	-0.05	0.05	-0.04	0.05	-0.04

Table 4.190: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 2$, $\hat{h}_3$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.01	0.02	0.02	0.05	0.04	0.06
g=2	0.01	0.02	0.02	0.04	0.03	0.05
g=3	0.01	0.01	0.01	0.03	0.02	0.04
g=4	0.01	0.01	0.01	0.03	0.02	0.04
g=5	0.00	0.01	0.01	0.02	0.02	0.03
g=20	0.00	0.00	0.00	0.01	0.01	0.01

Choice of incremental growth g and number of intervals of rejection r

On close inspection of Tables 4.191 – 4.194, the reader will find a comparison for different g and r-values for $\hat{h}_3$, $\alpha = 0.05$ and $m = 1$. In terms of the estimated bias and MSE, $2 \leq g \leq 10$ with $2 \leq r \leq 5$ result in the smallest estimated MSE (and bias-values close to zero), and therefore these choices are recommended when the sample size is small.

Table 4.191: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.03	0.03	0.03	0.03	0.03					
g=2	0.03	0.03	0.03	0.04	0.04					
g=3	0.03	0.03	0.04	0.04	0.04					
g=4	0.03	0.04	0.04	0.04	0.04					
g=5	0.03	0.04	0.04	0.04	0.04					
g=6						0.04	0.04	0.04	0.04	0.04
g=7						0.04	0.04	0.04	0.04	0.04
g=8						0.04	0.04	0.04	0.04	0.04
g=9						0.04	0.04	0.04	0.04	0.04
g=10						0.04	0.04	0.04	0.04	0.04
g=20		0.05		0.05		0.05		0.05		0.05
g=25		0.05		0.05		0.05		0.05		0.05
g=30		0.05		0.05		0.05		0.05		0.05
g=35		0.05		0.05		0.05		0.05		0.05
g=40		0.06		0.06		0.06		0.06		0.06
g=45		0.06		0.06		0.06		0.06		0.06
g=50		0.07		0.07		0.07		0.07		0.07
g=100		0.08		0.08		0.08		0.08		0.08

Table 4.192: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.03	-0.05	-0.05	-0.05	-0.05					
g=2	-0.04	-0.05	-0.05	-0.04	-0.04					
g=3	-0.04	-0.05	-0.04	-0.04	-0.04					
g=4	-0.05	-0.05	-0.04	-0.04	-0.04					
g=5	-0.05	-0.04	-0.04	-0.04	-0.04					
g=6						-0.04	-0.04	-0.04	-0.04	-0.04
g=7						-0.04	-0.04	-0.04	-0.04	-0.04
g=8						-0.04	-0.04	-0.04	-0.04	-0.04
g=9						-0.04	-0.04	-0.04	-0.04	-0.04
g=10						-0.04	-0.04	-0.04	-0.04	-0.04
g=20		-0.04		-0.04		-0.04		-0.04		-0.04
g=25		-0.05		-0.05		-0.05		-0.05		-0.05
g=30		-0.05		-0.05		-0.05		-0.05		-0.05
g=35		-0.05		-0.05		-0.05		-0.05		-0.05
g=40		-0.05		-0.05		-0.05		-0.05		-0.05
g=45		-0.06		-0.06		-0.06		-0.06		-0.06
g=50		-0.06		-0.06		-0.06		-0.06		-0.06
g=100		-0.08		-0.08		-0.08		-0.08		-0.08

Table 4.193: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.04	0.02	0.02	0.01	0.01					
g=2	0.03	0.02	0.01	0.01	0.01					
g=3	0.03	0.01	0.01	0.01	0.01					
g=4	0.02	0.01	0.01	0.01	0.00					
g=5	0.02	0.01	0.01	0.01	0.00					
g=6						0.00	0.00	0.00	0.00	0.00
g=7						0.00	0.00	0.00	0.00	0.00
g=8						0.00	0.00	0.00	0.00	0.00
g=9						0.00	0.00	0.00	0.00	0.00
g=10						0.00	0.00	0.00	0.00	0.00
g=20		0.00		0.00		0.00		0.00		0.00
g=25		0.00		0.00		0.00		0.00		0.00
g=30		0.00		0.00		0.00		0.00		0.00
g=35		0.00		0.00		0.00		0.00		0.00
g=40		0.00		0.00		0.00		0.00		0.00
g=45		0.00		0.00		0.00		0.00		0.00
g=50		0.01		0.01		0.01		0.01		0.01
g=100		0.01		0.01		0.01		0.01		0.01

Table 4.194: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_3$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.05	0.05	0.04	0.03	0.03					
g=2	0.05	0.04	0.03	0.02	0.02					
g=3	0.05	0.03	0.02	0.02	0.01					
g=4	0.05	0.03	0.02	0.01	0.01					
g=5	0.04	0.02	0.01	0.01	0.01					
g=6						0.00	0.00	0.00	0.00	0.00
g=7						0.00	0.00	0.00	0.00	0.00
g=8						0.00	0.00	0.00	0.00	0.00
g=9						0.00	0.00	0.00	0.00	0.00
g=10						0.00	0.00	0.00	0.00	0.00
g=20		0.01		0.00		0.00		0.00		0.00
g=25		0.00		0.00		0.00		0.00		0.00
g=30		0.00		0.00		0.00		0.00		0.00
g=35		0.00		0.00		0.00		0.00		0.00
g=40		0.00		0.00		0.00		0.00		0.00
g=45		0.00		0.00		0.00		0.00		0.00
g=50		0.00		0.00		0.00		0.00		0.00
g=100		0.01		0.01		0.01		0.01		0.01

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.1$, $\kappa = 3$, $n = 2000$ and $[a, b] = [0.25\,,\,0.75]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combinations of parameter values may result in the best possible estimation of a and b.

- The kernel function does not influence the estimation of a and b substantially, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$. For smaller sample sizes, a smaller value of m is recommended.
- In terms of the estimated smoothing parameter, $\hat{h}_3$ is recommended as a good choice for small to moderate sample sizes.
- Both the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests are recommended with preference given to the Anderson-Darling test when the sample size is small.
- For both of the recommended goodness-of-fit tests, α-values of 5% or 10% can be used. Larger α-values seem to perform better for small sample sizes.
- The choices of $2 \leq r \leq 5$ and $2 \leq g \leq 10$ are recommended for optimal results.

Table 4.195 provides the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in this table should be compared to the interval $[a, b] = [0.25\,,\,0.75]$.

Table 4.195: *Estimated values of a and b for $m = 1$ for the Anderson-Darling goodness-of-fit test with $\hat{h}_3$ and $\alpha = 0.10$.*

	r=2		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.28	0.70	0.28	0.70
g=2	0.28	0.70	0.29	0.71
g=3	0.28	0.70	0.29	0.71
g=4	0.29	0.70	0.29	0.71
g=5	0.29	0.71	0.29	0.71

Remark: *For this relatively small data set the estimated off-pulse interval is reasonably close to the actual off-pulse interval.*

4.5.9 Data set parameters: $1 - p = 0.2$, $\kappa = 1$, $n = 500$ and $[a, b] = [0.2\,,\,0.8]$

Figure 4.18 is a histogram representation of one Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data. The histogram of the simulated data contains 20 classes. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

From Figure 4.18 it is evident that the different kernel functions result in very similar kernel density estimation, even for this relatively small data set. From the measures used to assess the performance of SOPIE for this study population, it is also evident that the kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used throughout the analysis of this study population.

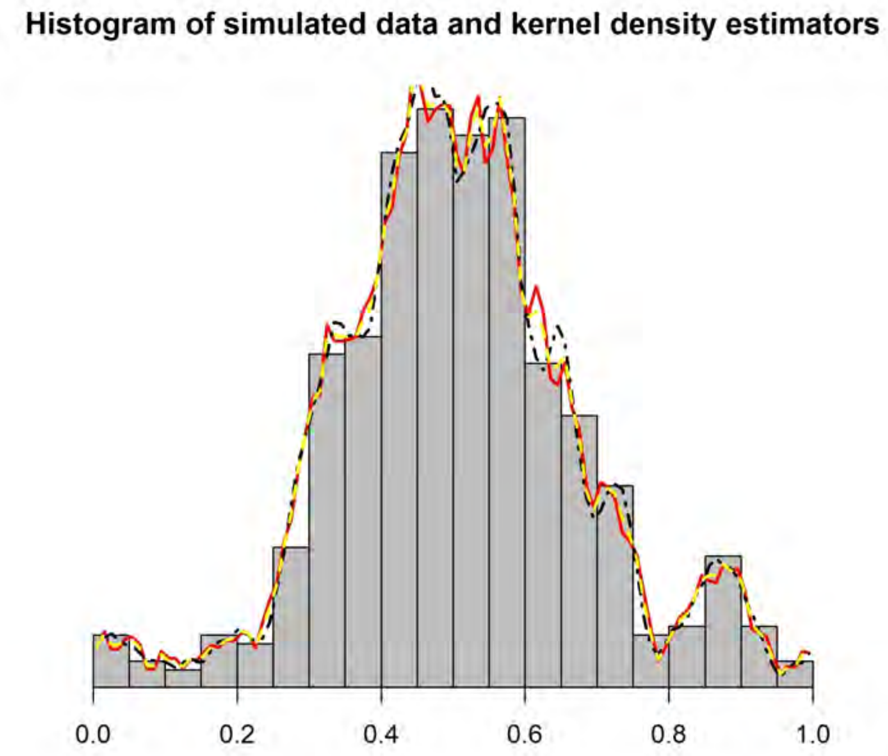

Figure 4.18: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_2 = 0.2$.

Choice of the number of minimum points m

Tables 4.196 and 4.197 highlight the values of the bias and MSE of $\hat{a}$ and $\hat{b}$ when $g = 2$, $r = 2$, $\alpha = 0.10$ for the various goodness-of-fit tests and for the estimated smoothing parameter $\hat{h}_3$. It can be seen that different values of m result in only slight variations of the bias and MSE of the estimators $\hat{a}$ and $\hat{b}$. From several of these comparisons (by using different values for g, r and α) it is observed that different values of m do not dramatically influence the estimated bias or MSE. It can again be concluded that a small value of m is preferable, since computing time is reduced. Furthermore, the estimated MSE is smaller when small m-values are chosen.

Choice of estimated smoothing parameters

Tables 4.198 - 4.199 highlight the values of the bias and MSE when estimating a and b for $m = 1$, $g = 2$, $r = 2$ and $\alpha = 0.10$ for all of the goodness-of-fit tests and for each of the nine estimated smoothing parameters $\hat{h}$.

When inspecting the estimated bias and MSE, it is nearly impossible to identify an estimated smoothing parameter that clearly outperforms any of the other estimated smoothing parameters. In the light of the conclusions from the previous target populations, it is fair to continue to recommend any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ as a good choice of the estimated smoothing parameter.

Table 4.196: *Bias for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.06	-0.07	0.12	-0.11	0.07	-0.09	0.07	-0.07
m=2	0.06	-0.06	0.12	-0.11	0.07	-0.09	0.07	-0.07
m=3	0.07	-0.06	0.12	-0.11	0.07	-0.09	0.08	-0.08
m=4	0.07	-0.07	0.12	-0.12	0.07	-0.10	0.08	-0.09
m=5	0.07	-0.07	0.12	-0.12	0.07	-0.10	0.08	-0.10
m=6	0.07	-0.08	0.12	-0.12	0.08	-0.11	0.08	-0.11
m=7	0.07	-0.08	0.12	-0.12	0.08	-0.11	0.08	-0.12
m=8	0.07	-0.08	0.13	-0.12	0.08	-0.11	0.09	-0.13
m=9	0.08	-0.09	0.13	-0.12	0.08	-0.12	0.09	-0.13
m=10	0.08	-0.09	0.13	-0.12	0.08	-0.12	0.09	-0.14

Table 4.197: *MSE for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.02	0.03	0.02	0.02	0.03	0.06	0.03	0.03
m=2	0.02	0.03	0.02	0.02	0.02	0.06	0.03	0.03
m=3	0.02	0.03	0.02	0.02	0.03	0.06	0.03	0.04
m=4	0.02	0.03	0.02	0.02	0.03	0.06	0.03	0.05
m=5	0.02	0.03	0.02	0.02	0.03	0.06	0.03	0.06
m=6	0.02	0.03	0.02	0.02	0.03	0.06	0.03	0.06
m=7	0.02	0.04	0.02	0.02	0.03	0.06	0.03	0.07
m=8	0.02	0.04	0.02	0.02	0.03	0.06	0.03	0.07
m=9	0.02	0.04	0.02	0.02	0.03	0.06	0.03	0.07
m=10	0.02	0.04	0.02	0.02	0.03	0.06	0.03	0.08

Table 4.198: *Comparison of the bias for different combinations of the estimated smoothing parameter and $m = 1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.06	-0.06	0.11	-0.10	0.07	-0.10	0.09	-0.11
$\hat{h}_2$	0.06	-0.08	0.11	-0.10	0.08	-0.12	0.09	-0.09
$\hat{h}_3$	0.06	-0.07	0.12	-0.11	0.07	-0.09	0.07	-0.07
$\hat{h}_4$	0.06	-0.08	0.11	-0.10	0.08	-0.12	0.08	-0.10
$\hat{h}_5$	0.06	-0.08	0.11	-0.10	0.08	-0.13	0.09	-0.09
$\hat{h}_6$	0.06	-0.06	0.11	-0.10	0.07	-0.09	0.09	-0.10
$\hat{h}_7$	0.06	-0.07	0.11	-0.10	0.08	-0.10	0.08	-0.10
$\hat{h}_8$	0.06	-0.08	0.11	-0.10	0.08	-0.13	0.09	-0.09
$\hat{h}_9$	0.06	-0.08	0.11	-0.10	0.09	-0.12	0.09	-0.09

Table 4.199: *Comparison of the MSE for different combinations of the estimated smoothing parameter and* $m = 1$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.02	0.04	0.01	0.01	0.03	0.06	0.04	0.06
$\hat{h}_2$	0.02	0.05	0.01	0.01	0.03	0.09	0.03	0.04
$\hat{h}_3$	0.02	0.03	0.02	0.02	0.03	0.06	0.03	0.03
$\hat{h}_4$	0.02	0.05	0.01	0.01	0.04	0.09	0.03	0.04
$\hat{h}_5$	0.02	0.05	0.01	0.01	0.03	0.09	0.03	0.04
$\hat{h}_6$	0.01	0.04	0.01	0.01	0.03	0.06	0.04	0.06
$\hat{h}_7$	0.01	0.04	0.01	0.01	0.03	0.07	0.03	0.05
$\hat{h}_8$	0.02	0.05	0.01	0.01	0.03	0.10	0.03	0.04
$\hat{h}_9$	0.02	0.05	0.01	0.01	0.04	0.08	0.04	0.04

Choice of goodness-of-fit test

Tables 4.200 – 4.201 compare the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.10$, $m = 1$ and $r = 2$ for $\hat{h}_1$.

Table 4.200: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.10$, $m = 1$, $r = 2, \hat{h}_1$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.06	-0.07	0.11	-0.10	0.09	-0.11	0.11	-0.14
g=2	0.06	-0.06	0.11	-0.10	0.07	-0.10	0.09	-0.11
g=3	0.06	-0.06	0.11	-0.10	0.07	-0.09	0.09	-0.11
g=4	0.06	-0.06	0.11	-0.10	0.07	-0.09	0.09	-0.10
g=5	0.06	-0.06	0.11	-0.10	0.07	-0.09	0.09	-0.09
g=6	0.07	-0.06	0.11	-0.10	0.07	-0.08	0.09	-0.09
g=7	0.06	-0.06	0.11	-0.11	0.07	-0.08	0.09	-0.09
g=8	0.07	-0.07	0.11	-0.11	0.07	-0.08	0.09	-0.09
g=9	0.07	-0.07	0.12	-0.11	0.07	-0.07	0.09	-0.09
g=10	0.07	-0.07	0.12	-0.11	0.07	-0.07	0.09	-0.09

Table 4.201: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.10$, $m = 1$, $r = 2, \hat{h}_1$ *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.03	0.05	0.01	0.01	0.05	0.08	0.07	0.10
g=2	0.02	0.04	0.01	0.01	0.03	0.06	0.04	0.06
g=3	0.01	0.03	0.01	0.01	0.02	0.06	0.03	0.05
g=4	0.01	0.03	0.01	0.01	0.02	0.05	0.02	0.03
g=5	0.01	0.02	0.01	0.01	0.02	0.04	0.02	0.02
g=6	0.01	0.02	0.01	0.01	0.01	0.04	0.01	0.02
g=7	0.01	0.01	0.01	0.01	0.01	0.03	0.01	0.01
g=8	0.01	0.02	0.01	0.01	0.01	0.03	0.01	0.01
g=9	0.01	0.02	0.01	0.01	0.01	0.03	0.01	0.01
g=10	0.01	0.02	0.01	0.01	0.01	0.02	0.01	0.01

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best performance is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh test. The Cramér-von-Mises goodness-of-fit test has slightly larger values of the estimated bias than the Rayleigh test.

When the estimated MSE is compared, the Cramér-von-Mises goodness-of-fit test performs the best with the slightest of margins, with the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests performing almost as good. For small sample sizes n, the Kolmogorov-Smirnov goodness-of-fit test proves quite useful, but preference is still given to the Anderson-Darling goodness-of-fit test.

Choice of the significance level α

Tables 4.202 – 4.203 compare the estimated bias and MSE for the Anderson-Darling goodness-of-fit tests, for $m = 1$, $r = 2$ and for $\hat{h}_1$. When comparing the estimated bias, values closer to zero are attained when α is *increased* for all of the goodness-of-fit tests. For the estimated MSE, it is not conclusive that larger values of α lead to smaller values of the estimated MSE. Therefore, similar to the previous target populations with small sample sizes n, it is again recommended to use any value of α in the range from 0.05 to 0.1, with larger values of α performing slightly better for small sample sizes.

Table 4.202: *Comparison of the estimated bias for different α-values, with $m = 1, r = 2, \hat{h}_1$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.08	-0.07	0.07	-0.07	0.06	-0.07
g=2	0.08	-0.08	0.07	-0.06	0.06	-0.06
g=3	0.08	-0.08	0.07	-0.06	0.06	-0.06
g=4	0.08	-0.08	0.07	-0.07	0.06	-0.06
g=5	0.08	-0.08	0.07	-0.07	0.06	-0.06
g=6	0.09	-0.07	0.07	-0.06	0.07	-0.06
g=7	0.09	-0.08	0.07	-0.06	0.06	-0.06
g=8	0.09	-0.08	0.07	-0.07	0.07	-0.07
g=9	0.09	-0.08	0.08	-0.07	0.07	-0.07
g=10	0.09	-0.08	0.08	-0.07	0.07	-0.07

Table 4.203: *Comparison of the estimated MSE for different α-values, with $m = 1, r = 2, \hat{h}_1$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.10$	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.01	0.02	0.02	0.04	0.03	0.05
g=2	0.01	0.02	0.01	0.02	0.02	0.04
g=3	0.01	0.01	0.01	0.02	0.01	0.03
g=4	0.01	0.01	0.01	0.02	0.01	0.03
g=5	0.01	0.01	0.01	0.02	0.01	0.02
g=6	0.01	0.01	0.01	0.01	0.01	0.02
g=7	0.01	0.01	0.01	0.01	0.01	0.01
g=8	0.01	0.01	0.01	0.02	0.01	0.02
g=9	0.01	0.01	0.01	0.01	0.01	0.02
g=10	0.01	0.01	0.01	0.01	0.01	0.02

Choice of incremental growth g and number of intervals of rejection r

On close inspection of Tables 4.204 – 4.207, the reader will find a comparison of the estimated bias and MSE over different g and r-values for the normal kernel density estimator with $\hat{h}_1$, $\alpha = 0.10$ and $m = 1$. In terms of the estimated bias and MSE, $2 \leq g \leq 10$ with $2 \leq r \leq 5$ result in the lowest estimated MSE (and bias-values close to zero). When the sample size is small, these combination of g and r-values (and even smaller values) are recommended.

Table 4.204: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
g=2	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
g=3	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
g=4	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
g=5	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
g=6	0.06	0.07	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07
g=7	0.06	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
g=8	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
g=9	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
g=10	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
g=20	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
g=25	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
g=30	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
g=35	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
g=40	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
g=45	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
g=50	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	

Table 4.205: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.07	-0.07	-0.07	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=2	-0.07	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=3	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.05	-0.05	-0.05
g=4	-0.07	-0.06	-0.06	-0.06	-0.06	-0.06	-0.05	-0.05	-0.05	-0.05
g=5	-0.06	-0.06	-0.06	-0.06	-0.06	-0.05	-0.06	-0.05	-0.05	-0.05
g=6	-0.05	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=7	-0.05	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=8	-0.08	-0.07	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=9	-0.07	-0.07	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=10	-0.07	-0.07	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
g=20	-0.06	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07
g=25	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07	-0.07
g=30	-0.08	-0.08	-0.08	-0.08	-0.08	-0.08	-0.08	-0.08	-0.08	-0.08
g=35	-0.09	-0.09	-0.09	-0.09	-0.09	-0.09	-0.09	-0.09	-0.09	-0.09
g=40	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10
g=45	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10	-0.10
g=50	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	

Table 4.206: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.04	0.03	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=2	0.03	0.02	0.01	0.01	0.01	0.01	0.00	0.00	0.00	0.00
g=3	0.03	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=4	0.03	0.01	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00
g=5	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
g=6	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=7	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=8	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=9	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=10	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=20	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=25	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=30	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=35	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=40	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=45	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=50	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	

Table 4.207: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.07	0.05	0.04	0.03	0.03	0.03	0.02	0.02	0.02	0.02
g=2	0.06	0.04	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.01
g=3	0.05	0.03	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0.01
g=4	0.05	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.01
g=5	0.04	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=6	0.03	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=7	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=8	0.05	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=9	0.04	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=10	0.04	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=20	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=25	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=30	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=35	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=40	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=45	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g=50	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	

Concluding remarks about simulated data from a von Mises distribution with $1 - p = 0.2$, $\kappa = 1$, $n = 500$ and $[a, b] = [0.2\,,\,0.8]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combinations of parameter values may result in the best possible estimation of a and b.

- The kernel function influences the estimation of a and b only marginally, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$. For smaller sample sizes, a small value of m is recommended.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$ - $\hat{h}_5$ is recommended as a good choice for the estimated smoothing parameter.
- Both the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests are recommended with preference given to the Anderson-Darling test when the sample size is small.
- For both of the recommended goodness-of-fit tests, α-values of 5% or 10% can be used. Larger α-values seem to perform better for small sample sizes.
- The choices of $2 \leq r \leq 5$ and $2 \leq g \leq 10$ are recommended for optimal results.

The following two tables (Tables 4.208 – 4.209) provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the interval $[a, b] = [0.2\,, \,0.8]$.

Table 4.208: *Estimated values of a and b for $m = 1$ for the Anderson-Darling goodness-of-fit test with $\hat{h}_1$ and $\alpha = 0.10$.*

	r=2		**r=4**	
	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$
g=1	0.26	0.73	0.26	0.74
g=2	0.26	0.74	0.26	0.74
g=3	0.26	0.74	0.26	0.74
g=4	0.26	0.74	0.26	0.74
g=5	0.26	0.74	0.26	0.74
g=6	0.27	0.74	0.27	0.74
g=7	0.26	0.74	0.27	0.74
g=8	0.27	0.73	0.27	0.74
g=9	0.27	0.73	0.27	0.74
g=10	0.27	0.73	0.27	0.74

Table 4.209: *Estimated values of a and b for $m = 1$ for the Kolmogorov-Smirnov goodness-of-fit test with $\hat{h}_1$ and $\alpha = 0.10$.*

	r=2		r=4	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.29	0.69	0.27	0.70
g=2	0.27	0.70	0.27	0.72
g=3	0.27	0.71	0.27	0.72
g=4	0.27	0.71	0.27	0.73
g=5	0.27	0.71	0.27	0.73
g=6	0.27	0.72	0.27	0.73
g=7	0.27	0.72	0.27	0.73
g=8	0.27	0.72	0.27	0.73
g=9	0.27	0.73	0.27	0.74
g=10	0.27	0.73	0.27	0.73

Remark: *For this small data set with a relatively large noise level, the estimated off-pulse interval is still close to the theoretical off-pulse interval.*

4.6 Simulation study results: Scaled triangular simulated data

Two broad classes of simulated data are used to assess the performance of SOPIE. The second class of study populations is constructed from a scaled triangular distribution contaminated with noise. The following subsections present the results of the performance of SOPIE when applied to the simulated data sets generated from the scaled triangular distribution.

4.6.1 Data set parameters: $1 - p = 0.1$, $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$

Figure 4.19 is a histogram representation with 100 classes of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

During each iteration of SOPIE, the first step is to calculate that point where the kernel density estimator attains its global minimum and the next m local minima. For this study population and for several other populations, the global minimum point and 19 other unique minima are obtained, resulting in a maximum of 20 minima that can be used as starting point for the sequential method. The reader is referred to step 2 in Section 3.2 for the details pertaining to the selection of the minima.

Table 4.210 compares the minima obtained from each of the different kernel functions that is fitted to the data for a single Monte Carlo iteration with $\hat{h}_3$. From this table it is evident that similar minima (or minima close to each other) are obtained from the different kernel functions that are used in the kernel density estimation. In most cases, several minima overlap for all of the kernel functions, except that they are obtained in a different order.

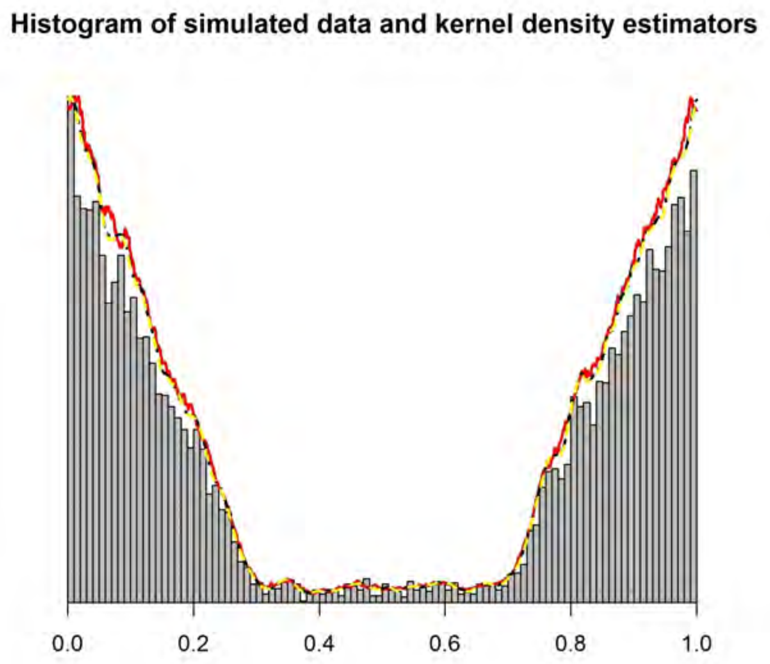

Figure 4.19: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.12$.

Table 4.210: *Minima comparison for different kernel functions.*

	Swanepoel kernel	Epanechnikov kernel	Normal kernel
1st local min.	0.3855	0.3855	0.3885
2nd local min.	0.3865	0.3865	0.3935
3rd local min.	0.3845	0.3845	0.3925
4th local min.	0.3875	0.3875	0.3915
5th local min.	0.3835	0.3835	0.3895
6th local min.	0.3885	0.3825	0.3905
7th local min.	0.3825	0.3805	0.4185
8th local min.	0.3815	0.3815	0.3875
9th local min.	0.3895	0.3885	0.3865
10th local min.	0.3805	0.3795	0.3945
11th local min.	0.3795	0.3785	0.4195
12th local min.	0.3905	0.3775	0.3845
13th local min.	0.3785	0.3895	0.4165
14th local min.	0.3775	0.3765	0.4175
15th local min.	0.3915	0.3905	0.3855
16th local min.	0.3765	0.3755	0.3955
17th local min.	0.3925	0.3915	0.4205
18th local min.	0.6375	0.3745	0.3825
19th local min.	0.6365	0.3925	0.3815
20th local min.	0.3755	0.6365	0.3835

It is therefore still argued that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE.

Choice of the number of minimum points m

The first step of SOPIE is to select a number of minimum points m. In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus. From here on the remainder of the tables report the results of averages over all of the 1000 Monte Carlo repetitions.

Tables 4.211 – 4.216 highlight the values of the bias of $\hat{a}$ and $\hat{b}$ when the normal kernel is used, $g = 6$, $r = 6$, $\alpha = 0.01$ for the various goodness-of-fit tests and for most of the estimated smoothing parameters. The reader can inspect Tables 4.217 – 4.222 for the values of the estimated MSE of $\hat{a}$ and $\hat{b}$. For this study population, it is found that some of the estimated smoothing parameters relate very closely to each other, and can therefore be grouped together. In each of the cases that will be mentioned below, the $\hat{h}$-values are almost identical, resulting in similar minimum points m used as starting point for SOPIE. Consequently, equivalent estimation of $[a, b]$ follows. It is found that the values of $\hat{h}_3$ and $\hat{h}_4$ are very close to each other, and therefore only $\hat{h}_3$ is displayed in the tables. Similar values are also found for $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$, which explains the presentation of only $\hat{h}_6$ in the tables.

Table 4.211: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0166	0.0178	-0.0286	0.0292	-0.0188	0.0199	-0.0232	0.0226
m=2	-0.0164	0.0175	-0.0286	0.0292	-0.0187	0.0200	-0.0238	0.0232
m=3	-0.0161	0.0178	-0.0286	0.0291	-0.0185	0.0199	-0.0235	0.0234
m=4	-0.0161	0.0176	-0.0286	0.0291	-0.0185	0.0197	-0.0234	0.0234
m=5	-0.0162	0.0176	-0.0286	0.0291	-0.0186	0.0197	-0.0235	0.0234
m=6	-0.0162	0.0176	-0.0287	0.0291	-0.0187	0.0198	-0.0234	0.0235
m=7	-0.0163	0.0176	-0.0287	0.0291	-0.0188	0.0198	-0.0234	0.0235
m=8	-0.0164	0.0176	-0.0287	0.0291	-0.0189	0.0198	-0.0234	0.0233
m=9	-0.0164	0.0176	-0.0287	0.0291	-0.0189	0.0198	-0.0234	0.0233
m=10	-0.0164	0.0177	-0.0287	0.0291	-0.0189	0.0198	-0.0234	0.0233
m=11	-0.0165	0.0177	-0.0287	0.0291	-0.0189	0.0198	-0.0233	0.0233
m=12	-0.0165	0.0178	-0.0287	0.0291	-0.0189	0.0198	-0.0233	0.0234
m=13	-0.0165	0.0178	-0.0287	0.0291	-0.0189	0.0198	-0.0233	0.0234
m=14	-0.0165	0.0179	-0.0288	0.0291	-0.0190	0.0198	-0.0234	0.0234
m=15	-0.0165	0.0179	-0.0288	0.0291	-0.0190	0.0199	-0.0233	0.0234
m=16	-0.0166	0.0179	-0.0288	0.0291	-0.0190	0.0199	-0.0233	0.0234
m=17	-0.0166	0.0179	-0.0288	0.0292	-0.0190	0.0199	-0.0233	0.0234
m=18	-0.0166	0.0179	-0.0288	0.0292	-0.0190	0.0199	-0.0233	0.0234
m=19	-0.0166	0.0179	-0.0288	0.0292	-0.0190	0.0199	-0.0233	0.0234

It can be seen that different values of m result in almost equal values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for each of the goodness-of-fit tests. From several of these comparisons using different values for g, r and α, similar results are observed. Therefore it seems fair to recommend the choice of $m = 1$ as a good choice, as it will save on computing time without a sizeable effect on the results.

An important observation is that, as far as bias and MSE are concerned, all of the goodness-of-fit tests are insensitive and robust to the choice of m.

Table 4.212: *Bias for different choices of m for $\hat{h}_2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0124	0.0164	-0.0286	0.0289	-0.0149	0.0194	-0.0265	0.0266
m=2	-0.0125	0.0168	-0.0286	0.0289	-0.0144	0.0195	-0.0259	0.0266
m=3	-0.0127	0.0169	-0.0286	0.0289	-0.0146	0.0195	-0.0258	0.0264
m=4	-0.0132	0.0168	-0.0286	0.0290	-0.0152	0.0194	-0.0259	0.0264
m=5	-0.0133	0.0168	-0.0286	0.0290	-0.0156	0.0193	-0.0259	0.0265
m=6	-0.0136	0.0168	-0.0287	0.0290	-0.0159	0.0192	-0.0261	0.0264
m=7	-0.0139	0.0168	-0.0286	0.0290	-0.0160	0.0192	-0.0260	0.0264
m=8	-0.0141	0.0167	-0.0286	0.0291	-0.0161	0.0191	-0.0260	0.0263
m=9	-0.0142	0.0168	-0.0286	0.0291	-0.0161	0.0192	-0.0260	0.0263
m=10	-0.0145	0.0169	-0.0287	0.0291	-0.0163	0.0191	-0.0260	0.0262
m=11	-0.0144	0.0169	-0.0287	0.0291	-0.0164	0.0191	-0.0259	0.0261
m=12	-0.0146	0.0169	-0.0287	0.0291	-0.0166	0.0192	-0.0260	0.0261
m=13	-0.0147	0.0170	-0.0288	0.0291	-0.0167	0.0192	-0.0260	0.0261
m=14	-0.0148	0.0170	-0.0288	0.0292	-0.0168	0.0193	-0.0260	0.0261
m=15	-0.0149	0.0171	-0.0288	0.0292	-0.0169	0.0194	-0.0261	0.0262
m=16	-0.0149	0.0171	-0.0288	0.0292	-0.0169	0.0194	-0.0261	0.0262
m=17	-0.0149	0.0171	-0.0288	0.0292	-0.0170	0.0194	-0.0261	0.0262
m=18	-0.0149	0.0171	-0.0288	0.0292	-0.0170	0.0194	-0.0261	0.0262
m=19	-0.0149	0.0171	-0.0288	0.0292	-0.0170	0.0194	-0.0261	0.0262

Table 4.213: *Bias for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0125	0.0175	-0.0284	0.0290	-0.0146	0.0197	-0.0260	0.0262
m=2	-0.0130	0.0168	-0.0285	0.0289	-0.0147	0.0189	-0.0262	0.0262
m=3	-0.0129	0.0169	-0.0285	0.0289	-0.0148	0.0190	-0.0259	0.0260
m=4	-0.0132	0.0167	-0.0285	0.0289	-0.0151	0.0186	-0.0258	0.0260
m=5	-0.0132	0.0167	-0.0285	0.0289	-0.0151	0.0187	-0.0257	0.0261
m=6	-0.0133	0.0168	-0.0285	0.0289	-0.0155	0.0188	-0.0256	0.0261
m=7	-0.0137	0.0167	-0.0286	0.0290	-0.0157	0.0189	-0.0255	0.0260
m=8	-0.0137	0.0168	-0.0286	0.0290	-0.0158	0.0189	-0.0255	0.0260
m=9	-0.0140	0.0168	-0.0286	0.0290	-0.0161	0.0191	-0.0256	0.0261
m=10	-0.0142	0.0169	-0.0286	0.0290	-0.0162	0.0190	-0.0256	0.0260
m=11	-0.0144	0.0170	-0.0287	0.0290	-0.0164	0.0191	-0.0255	0.0260
m=12	-0.0145	0.0170	-0.0287	0.0290	-0.0166	0.0192	-0.0256	0.0260
m=13	-0.0146	0.0171	-0.0287	0.0291	-0.0168	0.0193	-0.0256	0.0261
m=14	-0.0147	0.0172	-0.0287	0.0291	-0.0169	0.0194	-0.0256	0.0261
m=15	-0.0147	0.0172	-0.0287	0.0291	-0.0170	0.0194	-0.0256	0.0261
m=16	-0.0147	0.0172	-0.0287	0.0291	-0.0170	0.0194	-0.0256	0.0262
m=17	-0.0148	0.0172	-0.0287	0.0291	-0.0171	0.0194	-0.0256	0.0262
m=18	-0.0148	0.0173	-0.0287	0.0291	-0.0171	0.0195	-0.0256	0.0262
m=19	-0.0148	0.0173	-0.0287	0.0291	-0.0171	0.0195	-0.0256	0.0262
m=20	-0.0148	0.0173	-0.0287	0.0291	-0.0171	0.0195	-0.0256	0.0262

Table 4.214: *Bias for different choices of m for $\hat{h}_5$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0159	0.0183	-0.0288	0.0291	-0.0186	0.0203	-0.0231	0.0237
m=2	-0.0161	0.0181	-0.0289	0.0291	-0.0190	0.0201	-0.0227	0.0234
m=3	-0.0161	0.0182	-0.0289	0.0291	-0.0191	0.0201	-0.0229	0.0236
m=4	-0.0161	0.0181	-0.0289	0.0291	-0.0190	0.0201	-0.0228	0.0236
m=5	-0.0162	0.0180	-0.0289	0.0291	-0.0191	0.0200	-0.0228	0.0235
m=6	-0.0162	0.0179	-0.0289	0.0291	-0.0190	0.0201	-0.0228	0.0234
m=7	-0.0163	0.0179	-0.0289	0.0291	-0.0191	0.0201	-0.0229	0.0233
m=8	-0.0163	0.0180	-0.0289	0.0291	-0.0190	0.0202	-0.0228	0.0233
m=9	-0.0163	0.0180	-0.0289	0.0291	-0.0190	0.0202	-0.0228	0.0233
m=10	-0.0164	0.0180	-0.0289	0.0291	-0.0190	0.0201	-0.0228	0.0233
m=11	-0.0165	0.0180	-0.0289	0.0291	-0.0191	0.0201	-0.0228	0.0232
m=12	-0.0165	0.0180	-0.0289	0.0291	-0.0191	0.0201	-0.0228	0.0233
m=13	-0.0166	0.0180	-0.0289	0.0291	-0.0192	0.0201	-0.0229	0.0233
m=14	-0.0167	0.0181	-0.0289	0.0291	-0.0192	0.0202	-0.0229	0.0233
m=15	-0.0167	0.0181	-0.0289	0.0292	-0.0192	0.0202	-0.0228	0.0233
m=16	-0.0167	0.0181	-0.0289	0.0292	-0.0192	0.0202	-0.0228	0.0234
m=17	-0.0167	0.0181	-0.0289	0.0292	-0.0192	0.0202	-0.0228	0.0234
m=18	-0.0167	0.0181	-0.0289	0.0292	-0.0192	0.0202	-0.0228	0.0234
m=19	-0.0167	0.0181	-0.0289	0.0292	-0.0192	0.0202	-0.0228	0.0234
m=20	-0.0167	0.0181	-0.0289	0.0292	-0.0192	0.0202	-0.0228	0.0234

Table 4.215: *Bias for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0162	0.0183	-0.0281	0.0333	-0.0183	0.0212	-0.0276	0.0242
m=2	-0.0158	0.0182	-0.0269	0.0332	-0.0179	0.0209	-0.0249	0.0248
m=3	-0.0157	0.0186	-0.0268	0.0329	-0.0181	0.0211	-0.0240	0.0243
m=4	-0.0156	0.0187	-0.0270	0.0325	-0.0182	0.0212	-0.0236	0.0242
m=5	-0.0155	0.0188	-0.0273	0.0322	-0.0183	0.0212	-0.0236	0.0243
m=6	-0.0154	0.0189	-0.0277	0.0318	-0.0183	0.0213	-0.0235	0.0244
m=7	-0.0155	0.0188	-0.0280	0.0315	-0.0185	0.0211	-0.0236	0.0244
m=8	-0.0156	0.0188	-0.0284	0.0312	-0.0185	0.0211	-0.0238	0.0243
m=9	-0.0155	0.0188	-0.0287	0.0310	-0.0184	0.0211	-0.0241	0.0242
m=10	-0.0153	0.0189	-0.0289	0.0309	-0.0183	0.0211	-0.0240	0.0242
m=11	-0.0154	0.0189	-0.0291	0.0308	-0.0184	0.0211	-0.0241	0.0241
m=12	-0.0155	0.0190	-0.0292	0.0308	-0.0186	0.0211	-0.0242	0.0241
m=13	-0.0157	0.0190	-0.0293	0.0308	-0.0187	0.0211	-0.0242	0.0242
m=14	-0.0158	0.0190	-0.0294	0.0308	-0.0189	0.0211	-0.0243	0.0242
m=15	-0.0159	0.0190	-0.0295	0.0308	-0.0190	0.0212	-0.0243	0.0243
m=16	-0.0159	0.0190	-0.0295	0.0308	-0.0190	0.0212	-0.0243	0.0243
m=17	-0.0160	0.0191	-0.0295	0.0308	-0.0190	0.0213	-0.0243	0.0243
m=18	-0.0160	0.0191	-0.0295	0.0308	-0.0190	0.0213	-0.0244	0.0243
m=19	-0.0160	0.0191	-0.0295	0.0308	-0.0190	0.0213	-0.0244	0.0243
m=20	-0.0160	0.0191	-0.0295	0.0308	-0.0190	0.0213	-0.0244	0.0243

Table 4.216: *Bias for different choices of m for $\hat{h}_8$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0125	0.0181	-0.0285	0.0292	-0.0147	0.0200	-0.0267	0.0267
m=2	-0.0128	0.0169	-0.0286	0.0291	-0.0141	0.0191	-0.0266	0.0265
m=3	-0.0128	0.0169	-0.0285	0.0292	-0.0146	0.0191	-0.0264	0.0263
m=4	-0.0133	0.0169	-0.0286	0.0292	-0.0152	0.0192	-0.0263	0.0265
m=5	-0.0135	0.0169	-0.0286	0.0292	-0.0154	0.0192	-0.0261	0.0264
m=6	-0.0135	0.0168	-0.0286	0.0292	-0.0154	0.0189	-0.0261	0.0263
m=7	-0.0138	0.0169	-0.0287	0.0292	-0.0157	0.0189	-0.0260	0.0263
m=8	-0.0140	0.0169	-0.0287	0.0292	-0.0160	0.0191	-0.0260	0.0263
m=9	-0.0140	0.0170	-0.0287	0.0292	-0.0162	0.0192	-0.0260	0.0263
m=10	-0.0143	0.0170	-0.0288	0.0292	-0.0164	0.0192	-0.0260	0.0262
m=11	-0.0144	0.0171	-0.0288	0.0293	-0.0165	0.0194	-0.0260	0.0263
m=12	-0.0145	0.0171	-0.0288	0.0293	-0.0166	0.0194	-0.0260	0.0263
m=13	-0.0146	0.0172	-0.0288	0.0293	-0.0167	0.0195	-0.0259	0.0263
m=14	-0.0146	0.0172	-0.0288	0.0293	-0.0168	0.0195	-0.0259	0.0263
m=15	-0.0147	0.0172	-0.0288	0.0293	-0.0168	0.0196	-0.0259	0.0263
m=16	-0.0147	0.0172	-0.0288	0.0293	-0.0168	0.0196	-0.0260	0.0263
m=17	-0.0147	0.0172	-0.0289	0.0293	-0.0169	0.0196	-0.0260	0.0263
m=18	-0.0147	0.0172	-0.0289	0.0293	-0.0169	0.0196	-0.0260	0.0263
m=19	-0.0147	0.0172	-0.0289	0.0293	-0.0169	0.0196	-0.0260	0.0263

Table 4.217: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0006	0.0006	0.0008	0.0009	0.0009	0.0008	0.0013	0.0013
m=2	0.0006	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0013
m=3	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=4	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=5	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=6	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=7	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=8	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=9	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=10	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=11	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=12	0.0007	0.0006	0.0008	0.0009	0.0009	0.0008	0.0012	0.0012
m=13	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012
m=14	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012
m=15	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012
m=16	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012
m=17	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012
m=18	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012
m=19	0.0007	0.0005	0.0009	0.0009	0.0009	0.0008	0.0012	0.0012

Table 4.218: *MSE for different choices of m for $\hat{h}_2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0015	0.0007	0.0008	0.0009	0.0018	0.0008	0.0009	0.0009
m=2	0.0014	0.0007	0.0008	0.0009	0.0020	0.0008	0.0011	0.0009
m=3	0.0014	0.0007	0.0009	0.0009	0.0020	0.0008	0.0011	0.0009
m=4	0.0013	0.0007	0.0008	0.0009	0.0018	0.0009	0.0011	0.0009
m=5	0.0012	0.0007	0.0008	0.0009	0.0017	0.0009	0.0011	0.0009
m=6	0.0012	0.0007	0.0009	0.0009	0.0017	0.0010	0.0010	0.0009
m=7	0.0011	0.0007	0.0008	0.0009	0.0016	0.0009	0.0010	0.0009
m=8	0.0011	0.0007	0.0008	0.0009	0.0016	0.0009	0.0010	0.0009
m=9	0.0011	0.0007	0.0008	0.0009	0.0016	0.0009	0.0010	0.0009
m=10	0.0010	0.0007	0.0009	0.0009	0.0016	0.0010	0.0010	0.0009
m=11	0.0010	0.0007	0.0009	0.0009	0.0016	0.0010	0.0010	0.0009
m=12	0.0010	0.0007	0.0009	0.0009	0.0015	0.0010	0.0010	0.0009
m=13	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=14	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=15	0.0010	0.0006	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=16	0.0010	0.0006	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=17	0.0010	0.0006	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=18	0.0010	0.0006	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=19	0.0010	0.0006	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009

Table 4.219: *MSE for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0014	0.0005	0.0008	0.0009	0.0020	0.0008	0.0011	0.0010
m=2	0.0013	0.0007	0.0008	0.0009	0.0020	0.0009	0.0010	0.0010
m=3	0.0013	0.0006	0.0008	0.0009	0.0019	0.0009	0.0011	0.0010
m=4	0.0013	0.0007	0.0008	0.0009	0.0018	0.0010	0.0011	0.0010
m=5	0.0012	0.0007	0.0008	0.0009	0.0018	0.0010	0.0011	0.0010
m=6	0.0012	0.0007	0.0008	0.0009	0.0017	0.0010	0.0011	0.0010
m=7	0.0011	0.0007	0.0008	0.0009	0.0017	0.0010	0.0012	0.0010
m=8	0.0011	0.0007	0.0008	0.0009	0.0017	0.0010	0.0011	0.0010
m=9	0.0011	0.0006	0.0008	0.0009	0.0016	0.0009	0.0011	0.0009
m=10	0.0010	0.0006	0.0008	0.0009	0.0016	0.0009	0.0011	0.0010
m=11	0.0010	0.0006	0.0008	0.0009	0.0015	0.0009	0.0011	0.0010
m=12	0.0010	0.0006	0.0008	0.0009	0.0015	0.0009	0.0011	0.0010
m=13	0.0010	0.0006	0.0008	0.0009	0.0014	0.0009	0.0011	0.0010
m=14	0.0010	0.0006	0.0008	0.0009	0.0014	0.0009	0.0011	0.0009
m=15	0.0010	0.0006	0.0009	0.0009	0.0014	0.0009	0.0011	0.0009
m=16	0.0010	0.0006	0.0009	0.0009	0.0014	0.0009	0.0011	0.0009
m=17	0.0010	0.0006	0.0009	0.0009	0.0014	0.0009	0.0011	0.0009
m=18	0.0010	0.0006	0.0009	0.0009	0.0014	0.0009	0.0011	0.0009
m=19	0.0010	0.0006	0.0009	0.0009	0.0014	0.0009	0.0011	0.0009
m=20	0.0010	0.0006	0.0009	0.0009	0.0014	0.0009	0.0011	0.0009

Table 4.220: *MSE for different choices of m for $\hat{h}_5$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=2	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=3	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=4	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=5	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=6	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=7	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=8	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=9	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=10	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=11	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=12	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=13	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=14	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=15	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0013	0.0011
m=16	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=17	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=18	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=19	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
m=20	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011

Table 4.221: *MSE for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0003	0.0012	0.0008	0.0013	0.0004	0.0016	0.0009	0.0018
m=2	0.0003	0.0011	0.0008	0.0012	0.0004	0.0014	0.0007	0.0016
m=3	0.0004	0.0009	0.0008	0.0012	0.0005	0.0013	0.0007	0.0016
m=4	0.0004	0.0009	0.0008	0.0011	0.0005	0.0012	0.0007	0.0015
m=5	0.0005	0.0008	0.0008	0.0011	0.0006	0.0012	0.0007	0.0014
m=6	0.0006	0.0008	0.0008	0.0011	0.0007	0.0011	0.0008	0.0013
m=7	0.0006	0.0007	0.0008	0.0010	0.0007	0.0010	0.0009	0.0012
m=8	0.0007	0.0007	0.0008	0.0010	0.0008	0.0010	0.0009	0.0011
m=9	0.0008	0.0007	0.0009	0.0010	0.0009	0.0009	0.0009	0.0011
m=10	0.0009	0.0007	0.0009	0.0010	0.0010	0.0009	0.0009	0.0011
m=11	0.0009	0.0006	0.0009	0.0010	0.0010	0.0009	0.0010	0.0010
m=12	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=13	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=14	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=15	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=16	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=17	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=18	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=19	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010
m=20	0.0009	0.0006	0.0009	0.0010	0.0010	0.0008	0.0010	0.0010

Table 4.222: *MSE for different choices of m for $\hat{h}_8$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0014	0.0004	0.0008	0.0009	0.0020	0.0008	0.0009	0.0008
m=2	0.0014	0.0007	0.0008	0.0009	0.0021	0.0010	0.0009	0.0009
m=3	0.0013	0.0007	0.0008	0.0009	0.0019	0.0010	0.0009	0.0009
m=4	0.0012	0.0007	0.0008	0.0009	0.0018	0.0010	0.0009	0.0009
m=5	0.0012	0.0007	0.0008	0.0009	0.0017	0.0010	0.0010	0.0009
m=6	0.0012	0.0007	0.0008	0.0009	0.0018	0.0010	0.0010	0.0009
m=7	0.0012	0.0007	0.0009	0.0009	0.0017	0.0010	0.0010	0.0009
m=8	0.0011	0.0007	0.0009	0.0009	0.0016	0.0010	0.0010	0.0009
m=9	0.0011	0.0007	0.0009	0.0009	0.0016	0.0010	0.0010	0.0009
m=10	0.0011	0.0007	0.0009	0.0009	0.0015	0.0010	0.0010	0.0009
m=11	0.0010	0.0007	0.0009	0.0009	0.0015	0.0010	0.0010	0.0009
m=12	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=13	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=14	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=15	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=16	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=17	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=18	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009
m=19	0.0010	0.0007	0.0009	0.0009	0.0015	0.0009	0.0010	0.0009

Choice of estimated smoothing parameters

In Section 2.4.4 in Chapter 2, the choices of different smoothing parameters were highlighted. The reader is also referred to Table 4.1 for details pertaining to the calculation of the estimated smoothing parameter $\hat{h}$.

When inspecting the actual values of each $\hat{h}$, it is frequently found that the values of $\hat{h}_3$ and $\hat{h}_4$ are very close to each other. Similarly, the values of $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$ are also close to each other. Therefore, the presentation of the results that follows will take this fact into consideration.

Tables 4.223 and 4.224 highlight the value of the estimated bias and MSE when estimating a and b for $m = 1$, $g = 6$ and $r = 6$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Tables 4.225 – 4.226 present similar results, but for $m = 1$, $g = 3$ and $r = 3$. Finally, Tables 4.227 – 4.228 present the estimated bias and MSE for $m = 1$, $g = 9$ and $r = 8$. In all of the tables, the normal kernel is used and the goodness-of-fit tests are performed at a level of significance $\alpha = 0.01$.

When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero, which make it difficult to choose a specific $\hat{h}$-value that performs better than any of the others. When comparing the MSE of the estimators for a and b, $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_5$ and $\hat{h}_6$ result in marginally smaller values of the estimated MSE. It is again not perfectly clear which $\hat{h}$ would be the best estimated smoothing parameter in terms of the MSE.

Table 4.223: *Comparison of the bias for different combinations of the estimated smoothing parameter with* $m = 1$, $g = 6$ *and* $r = 6$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0166	0.0178	-0.0286	0.0292	-0.0188	0.0199	-0.0232	0.0226
$\hat{h}_2$	-0.0124	0.0164	-0.0286	0.0289	-0.0149	0.0194	-0.0265	0.0266
$\hat{h}_3$	-0.0125	0.0175	-0.0284	0.0290	-0.0146	0.0197	-0.0260	0.0262
$\hat{h}_4$	-0.0125	0.0175	-0.0284	0.0290	-0.0146	0.0197	-0.0260	0.0262
$\hat{h}_5$	-0.0159	0.0183	-0.0288	0.0291	-0.0186	0.0203	-0.0231	0.0237
$\hat{h}_6$	-0.0162	0.0183	-0.0281	0.0333	-0.0183	0.0212	-0.0276	0.0242
$\hat{h}_7$	-0.0162	0.0183	-0.0281	0.0333	-0.0183	0.0212	-0.0276	0.0242
$\hat{h}_8$	-0.0125	0.0181	-0.0285	0.0292	-0.0147	0.0200	-0.0267	0.0267
$\hat{h}_9$	-0.0162	0.0183	-0.0281	0.0333	-0.0183	0.0212	-0.0276	0.0242

Table 4.224: *Comparison of the MSE for different combinations of the estimated smoothing parameter with* $m = 1$, $g = 6$ *and* $r = 6$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0006	0.0006	0.0008	0.0009	0.0009	0.0008	0.0013	0.0013
$\hat{h}_2$	0.0015	0.0007	0.0008	0.0009	0.0018	0.0008	0.0009	0.0009
$\hat{h}_3$	0.0014	0.0005	0.0008	0.0009	0.0020	0.0008	0.0011	0.0010
$\hat{h}_4$	0.0014	0.0005	0.0008	0.0009	0.0020	0.0008	0.0011	0.0010
$\hat{h}_5$	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
$\hat{h}_6$	0.0003	0.0012	0.0008	0.0013	0.0004	0.0016	0.0009	0.0018
$\hat{h}_7$	0.0003	0.0012	0.0008	0.0013	0.0004	0.0016	0.0009	0.0018
$\hat{h}_8$	0.0014	0.0004	0.0008	0.0009	0.0020	0.0008	0.0009	0.0008
$\hat{h}_9$	0.0003	0.0012	0.0008	0.0013	0.0004	0.0016	0.0009	0.0018

Table 4.225: *Comparison of the bias for different combinations of the estimated smoothing parameter with* $m = 1$, $g = 3$ *and* $r = 3$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0059	0.0097	-0.0282	0.0288	-0.0085	0.0135	-0.0158	0.0170
$\hat{h}_2$	0.0108	0.0039	-0.0282	0.0285	0.0019	0.0080	-0.0108	0.0142
$\hat{h}_3$	0.0067	0.0045	-0.0281	0.0286	-0.0011	0.0091	-0.0123	0.0135
$\hat{h}_4$	0.0067	0.0045	-0.0281	0.0286	-0.0011	0.0091	-0.0123	0.0135
$\hat{h}_5$	-0.0071	0.0109	-0.0285	0.0287	-0.0105	0.0148	-0.0150	0.0171
$\hat{h}_6$	-0.0135	0.0048	-0.0274	0.0329	-0.0150	0.0081	-0.0264	0.0065
$\hat{h}_7$	-0.0135	0.0048	-0.0274	0.0329	-0.0150	0.0081	-0.0264	0.0065
$\hat{h}_8$	0.0127	0.0020	-0.0282	0.0288	0.0043	0.0070	-0.0129	0.0170
$\hat{h}_9$	-0.0135	0.0048	-0.0274	0.0329	-0.0150	0.0081	-0.0264	0.0065

Table 4.226: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 3$ and $r = 3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0023	0.0018	0.0008	0.0009	0.0026	0.0016	0.0024	0.0021
$\hat{h}_2$	0.0064	0.0030	0.0008	0.0008	0.0055	0.0030	0.0041	0.0031
$\hat{h}_3$	0.0051	0.0030	0.0008	0.0008	0.0047	0.0029	0.0037	0.0033
$\hat{h}_4$	0.0051	0.0030	0.0008	0.0008	0.0047	0.0029	0.0037	0.0033
$\hat{h}_5$	0.0022	0.0015	0.0008	0.0008	0.0023	0.0014	0.0025	0.0019
$\hat{h}_6$	0.0003	0.0045	0.0008	0.0013	0.0004	0.0051	0.0008	0.0067
$\hat{h}_7$	0.0003	0.0045	0.0008	0.0013	0.0004	0.0051	0.0008	0.0067
$\hat{h}_8$	0.0069	0.0036	0.0008	0.0009	0.0059	0.0036	0.0037	0.0027
$\hat{h}_9$	0.0003	0.0045	0.0008	0.0013	0.0004	0.0051	0.0008	0.0067

Table 4.227: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 9$ and $r = 8$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0183	0.0194	-0.0290	0.0295	-0.0204	0.0216	-0.0258	0.0269
$\hat{h}_2$	-0.0172	0.0189	-0.0290	0.0292	-0.0209	0.0218	-0.0277	0.0275
$\hat{h}_3$	-0.0171	0.0191	-0.0288	0.0293	-0.0204	0.0218	-0.0277	0.0278
$\hat{h}_4$	-0.0171	0.0191	-0.0288	0.0293	-0.0204	0.0218	-0.0277	0.0278
$\hat{h}_5$	-0.0184	0.0194	-0.0292	0.0295	-0.0205	0.0220	-0.0260	0.0255
$\hat{h}_6$	-0.0166	0.0208	-0.0285	0.0337	-0.0190	0.0232	-0.0283	0.0278
$\hat{h}_7$	-0.0166	0.0208	-0.0285	0.0337	-0.0190	0.0232	-0.0283	0.0278
$\hat{h}_8$	-0.0167	0.0193	-0.0289	0.0295	-0.0209	0.0221	-0.0277	0.0274
$\hat{h}_9$	-0.0166	0.0208	-0.0285	0.0337	-0.0190	0.0232	-0.0283	0.0278

Table 4.228: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 9$ and $r = 8$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0005	0.0004	0.0009	0.0009	0.0007	0.0006	0.0010	0.0008
$\hat{h}_2$	0.0005	0.0004	0.0009	0.0009	0.0007	0.0005	0.0008	0.0008
$\hat{h}_3$	0.0006	0.0004	0.0009	0.0009	0.0008	0.0006	0.0008	0.0008
$\hat{h}_4$	0.0006	0.0004	0.0009	0.0009	0.0008	0.0006	0.0008	0.0008
$\hat{h}_5$	0.0005	0.0004	0.0009	0.0009	0.0008	0.0005	0.0009	0.0009
$\hat{h}_6$	0.0003	0.0009	0.0009	0.0013	0.0004	0.0013	0.0009	0.0012
$\hat{h}_7$	0.0003	0.0009	0.0009	0.0013	0.0004	0.0013	0.0009	0.0012
$\hat{h}_8$	0.0007	0.0004	0.0009	0.0009	0.0007	0.0005	0.0008	0.0008
$\hat{h}_9$	0.0003	0.0009	0.0009	0.0013	0.0004	0.0013	0.0009	0.0012

In the light of the conclusions just made from the estimated bias and MSE, an initial recommendation would be to use any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) as a good choice for the estimated smoothing parameter. It must be noted that the next target population will also be inspected to shed some light on the question of the optimal choice of $\hat{h}$. The reader must also note the stable performance of the Cramér-von-Mises goodness-of-fit test especially as far as the MSE is concerned, even for different choices of each parameter.

Choice of goodness-of-fit test

SOPIE is based, in a sequential way, on the P-values of goodness-of-fit tests for the uniform distribution. Different goodness-of-fit tests exist and, therefore, it must be evaluated whether there is a superior goodness-of-fit test, compared to other goodness-of-fit tests. Some of the tables provided earlier can be used to assess the goodness-of-fit tests. Moreover, the reader can inspect Tables 4.229 – 4.233 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ for most of the estimated smoothing parameters $\hat{h}$. Tables 4.234 – 4.238 are even more important, since the MSE is compared in these tables. Recall that the MSE takes both the bias and variance into account when measuring the performance of an estimator.

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Cramér-von-Mises test. The Rayleigh goodness-of-fit test performs the worst in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests perform equally well, followed by the other two tests.

Due to the importance of the MSE, it is recommended to use both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests.

Table 4.229: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_1$*, for different values of* g*.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0166	0.0178	-0.0286	0.0292	-0.0188	0.0199	-0.0232	0.0226
g=7	-0.0165	0.0178	-0.0287	0.0293	-0.0192	0.0205	-0.0242	0.0238
g=8	-0.0169	0.0181	-0.0288	0.0294	-0.0193	0.0206	-0.0246	0.0243
g=9	-0.0176	0.0188	-0.0290	0.0295	-0.0197	0.0212	-0.0247	0.0250
g=10	-0.0180	0.0194	-0.0291	0.0297	-0.0203	0.0210	-0.0251	0.0254
g=20	-0.0202	0.0212	-0.0303	0.0309	-0.0228	0.0240	-0.0282	0.0284

Table 4.230: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_2$*, for different values of* g*.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0124	0.0164	-0.0286	0.0289	-0.0149	0.0194	-0.0265	0.0266
g=7	-0.0134	0.0171	-0.0287	0.0291	-0.0159	0.0203	-0.0270	0.0267
g=8	-0.0147	0.0174	-0.0288	0.0291	-0.0173	0.0208	-0.0274	0.0271
g=9	-0.0154	0.0181	-0.0290	0.0292	-0.0188	0.0212	-0.0274	0.0272
g=10	-0.0160	0.0183	-0.0291	0.0295	-0.0198	0.0211	-0.0274	0.0273
g=20	-0.0197	0.0208	-0.0304	0.0307	-0.0230	0.0235	-0.0292	0.0291

Table 4.231: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_3$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0125	0.0175	-0.0284	0.0290	-0.0146	0.0197	-0.0260	0.0262
g=7	-0.0140	0.0181	-0.0286	0.0291	-0.0157	0.0204	-0.0266	0.0269
g=8	-0.0150	0.0184	-0.0287	0.0292	-0.0167	0.0212	-0.0269	0.0274
g=9	-0.0158	0.0185	-0.0288	0.0293	-0.0184	0.0215	-0.0268	0.0276
g=10	-0.0161	0.0190	-0.0290	0.0294	-0.0187	0.0215	-0.0273	0.0277
g=20	-0.0195	0.0208	-0.0302	0.0307	-0.0230	0.0238	-0.0292	0.0292

Table 4.232: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_5$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0159	0.0183	-0.0288	0.0291	-0.0186	0.0203	-0.0231	0.0237
g=7	-0.0167	0.0183	-0.0290	0.0292	-0.0190	0.0207	-0.0234	0.0233
g=8	-0.0169	0.0186	-0.0291	0.0293	-0.0201	0.0206	-0.0245	0.0244
g=9	-0.0181	0.0190	-0.0292	0.0295	-0.0201	0.0216	-0.0247	0.0244
g=10	-0.0181	0.0194	-0.0294	0.0296	-0.0204	0.0219	-0.0250	0.0251
g=20	-0.0201	0.0211	-0.0307	0.0308	-0.0234	0.0237	-0.0280	0.0281

Table 4.233: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_6$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0162	0.0183	-0.0281	0.0333	-0.0183	0.0212	-0.0276	0.0242
g=7	-0.0165	0.0185	-0.0282	0.0334	-0.0185	0.0216	-0.0278	0.0252
g=8	-0.0165	0.0189	-0.0283	0.0335	-0.0187	0.0224	-0.0280	0.0254
g=9	-0.0166	0.0190	-0.0284	0.0337	-0.0189	0.0225	-0.0282	0.0264
g=10	-0.0168	0.0202	-0.0285	0.0337	-0.0191	0.0229	-0.0284	0.0267
g=20	-0.0183	0.0230	-0.0298	0.0349	-0.0208	0.0264	-0.0297	0.0300

Table 4.234: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_1$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0006	0.0006	0.0008	0.0009	0.0009	0.0008	0.0013	0.0013
g=7	0.0006	0.0006	0.0008	0.0009	0.0009	0.0007	0.0011	0.0011
g=8	0.0006	0.0006	0.0009	0.0009	0.0009	0.0007	0.0011	0.0011
g=9	0.0005	0.0005	0.0009	0.0009	0.0009	0.0007	0.0011	0.0010
g=10	0.0005	0.0004	0.0009	0.0009	0.0008	0.0008	0.0011	0.0010
g=20	0.0005	0.0005	0.0009	0.0010	0.0007	0.0006	0.0009	0.0009

Table 4.235: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_2$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0015	0.0007	0.0008	0.0009	0.0018	0.0008	0.0009	0.0009
g=7	0.0013	0.0007	0.0009	0.0009	0.0017	0.0007	0.0009	0.0009
g=8	0.0011	0.0007	0.0009	0.0009	0.0014	0.0007	0.0008	0.0009
g=9	0.0009	0.0006	0.0009	0.0009	0.0012	0.0007	0.0008	0.0009
g=10	0.0009	0.0006	0.0009	0.0009	0.0010	0.0007	0.0009	0.0009
g=20	0.0005	0.0005	0.0010	0.0010	0.0007	0.0006	0.0009	0.0009

Table 4.236: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_3$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0014	0.0005	0.0008	0.0009	0.0020	0.0008	0.0011	0.0010
g=7	0.0011	0.0005	0.0008	0.0009	0.0018	0.0007	0.0010	0.0009
g=8	0.0009	0.0005	0.0009	0.0009	0.0016	0.0006	0.0010	0.0008
g=9	0.0009	0.0005	0.0009	0.0009	0.0012	0.0006	0.0010	0.0008
g=10	0.0008	0.0004	0.0009	0.0009	0.0013	0.0006	0.0010	0.0008
g=20	0.0005	0.0005	0.0009	0.0010	0.0007	0.0006	0.0009	0.0009

Table 4.237: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_5$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
g=7	0.0007	0.0005	0.0009	0.0009	0.0009	0.0007	0.0012	0.0011
g=8	0.0007	0.0005	0.0009	0.0009	0.0008	0.0007	0.0010	0.0010
g=9	0.0005	0.0005	0.0009	0.0009	0.0008	0.0006	0.0010	0.0010
g=10	0.0006	0.0005	0.0009	0.0009	0.0008	0.0006	0.0010	0.0010
g=20	0.0005	0.0005	0.0010	0.0010	0.0006	0.0006	0.0009	0.0008

Table 4.238: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_6$, *for different values of* g.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0003	0.0012	0.0008	0.0013	0.0004	0.0016	0.0009	0.0018
g=7	0.0003	0.0012	0.0009	0.0012	0.0004	0.0015	0.0009	0.0016
g=8	0.0003	0.0011	0.0009	0.0013	0.0004	0.0014	0.0009	0.0016
g=9	0.0003	0.0011	0.0009	0.0013	0.0004	0.0014	0.0009	0.0014
g=10	0.0003	0.0009	0.0009	0.0013	0.0004	0.0014	0.0009	0.0015
g=20	0.0004	0.0008	0.0009	0.0013	0.0005	0.0009	0.0010	0.0012

Choice of the significance level α

The significance level α is another tuning parameter that may influence the point where rejection of uniformity will take place for each goodness-of-fit test, and therefore α may influence the values of $\hat{a}$ and $\hat{b}$. Several tables are constructed to investigate the effect of α, in combination with the effect of m, $\hat{h}$ and the goodness-of-fit tests. Tables 4.239 – 4.240 compare the estimated bias and MSE for two goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 6$. Tables 4.241 – 4.242 compare the estimated bias and MSE for the same goodness-of-fit tests, but for $\hat{h}_2$. Tables 4.243 – 4.244 present a similar comparison, but for $\hat{h}_5$ and Tables 4.245 – 4.246 for $\hat{h}_6$.

From all of these tables it is concluded that – for the Cramér-von-Mises goodness-of-fit test – the estimated bias and MSE are not sensitive to the choice of α. When investigating the Anderson-Darling goodness-of-fit test, it is found that a smaller estimated MSE results from $\alpha = 0.01$. Furthermore, the estimated bias is the closest to zero when $\alpha = 0.05$. The remark is therefore made that a choice of $\alpha = 0.01$ would be slightly better, but any value of α in the range from 0.01 to 0.05 would marginally influence the results. It is again observed that the Cramér-von-Mises goodness-of-fit test is stable in terms of estimated MSE for different values of the tuning parameters.

Table 4.239: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.01	0.00	0.00
g=7	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.00	0.00	0.00
g=8	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.00	0.00	0.00
g=9	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=10	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=20	-0.02	0.02	0.00	0.00	-0.01	0.02	0.00	0.00	-0.01	0.01	0.00	0.00

Table 4.240: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00

Table 4.241: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_2$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.01	0.02	0.00	0.00	0.01	0.00	0.00	0.00	0.04	-0.01	0.01	0.01
g=7	-0.01	0.02	0.00	0.00	0.01	0.01	0.00	0.00	0.04	-0.01	0.01	0.00
g=8	-0.01	0.02	0.00	0.00	0.01	0.01	0.00	0.00	0.03	-0.01	0.01	0.00
g=9	-0.02	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.02	-0.00	0.01	0.00
g=10	-0.02	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.02	-0.00	0.01	0.00
g=20	-0.02	0.02	0.00	0.00	-0.01	0.02	0.00	0.00	0.00	0.01	0.00	0.00

Table 4.242: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_2$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00

Table 4.243: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_5$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.00	0.00	0.00
g=7	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.00	0.00	0.00
g=8	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=9	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=10	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=20	-0.02	0.02	0.00	0.00	-0.01	0.02	0.00	0.00	-0.01	0.01	0.00	0.00

Table 4.244: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_5$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00

Table 4.245: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_6$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.02	0.02	0.00	0.00	-0.01	-0.00	0.00	0.00	-0.01	-0.03	0.00	0.01
g=7	-0.02	0.02	0.00	0.00	-0.01	-0.00	0.00	0.00	-0.01	-0.02	0.00	0.01
g=8	-0.02	0.02	0.00	0.00	-0.01	0.00	0.00	0.00	-0.01	-0.02	0.00	0.01
g=9	-0.02	0.02	0.00	0.00	-0.01	0.00	0.00	0.00	-0.01	-0.01	0.00	0.01
g=10	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	-0.01	-0.01	0.00	0.01
g=20	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	-0.01	0.00	0.00	0.00

Table 4.246: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_6$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

The choice of g, the incremental growth of each interval over which rejection is tested, together with the choice of r, the number of intervals before rejection of uniformity takes place, are two important parameters in the estimation process. Together, these two parameters will determine the accuracy of the estimated point where rejection of uniformity takes place.

Tables 4.247 – 4.262 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests for two different values of the estimated smoothing parameter $\hat{h}$. In all of the tables the normal kernel is used, with $\alpha = 0.01$ and $m = 1$ kept constant.

For both goodness-of-fit tests, small values of r and g result in estimated bias values that are close to zero, i.e., $2 \leq g \leq 5$ with $2 \leq r \leq 4$. In terms of estimated MSE, the Cramér-von-Mises goodness-of-fit test is relatively robust to choices of r and g. For the Anderson-Darling goodness-of-fit test, the smallest values of the MSE are obtained when $6 \leq g \leq 10$ and $6 \leq r \leq 10$.

In conclusion, it is recommended to use values of g in the range from 5 to 10, and for r, values from 4 to 8.

Table 4.247: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0226	0.0078	0.0026	-0.0012	-0.0029					
g=2	0.0164	0.0015	-0.0030	-0.0060	-0.0077					
g=3	0.0133	-0.0009	-0.0059	-0.0087	-0.0103					
g=4	0.0108	-0.0030	-0.0082	-0.0101	-0.0124					
g=5	0.0053	-0.0059	-0.0100	-0.0117	-0.0144					
g=6						-0.0166	-0.0166	-0.0167	-0.0172	-0.0177
g=7						-0.0165	-0.0170	-0.0176	-0.0179	-0.0179
g=8						-0.0169	-0.0175	-0.0177	-0.0178	-0.0178
g=9						-0.0176	-0.0181	-0.0183	-0.0183	-0.0184
g=10						-0.0180	-0.0182	-0.0182	-0.0185	-0.0186
g=20		-0.0172		-0.0198		-0.0202		-0.0202		-0.0202
g=25		-0.0182		-0.0210		-0.0210		-0.0212		-0.0212
g=30		-0.0204		-0.0219		-0.0219		-0.0221		-0.0221
g=35		-0.0201		-0.0224		-0.0224		-0.0224		-0.0224
g=40		-0.0215		-0.0229		-0.0229		-0.0229		-0.0229
g=45		-0.0233		-0.0241		-0.0241		-0.0241		-0.0241
g=50		-0.0237		-0.0246		-0.0246		-0.0246		-0.0246
g=100		-0.0298		-0.0298		-0.0298		-0.0298		-0.0298
g=200		-0.0378		-0.0378		-0.0378		-0.0378		-0.0378
g=300		-0.0522		-0.0522		-0.0522		-0.0522		-0.0522
g=400		-0.0540		-0.0540		-0.0540		-0.0540		-0.0540
g=500		-0.0678		-0.0678		-0.0678		-0.0678		-0.0678

Table 4.248: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.0125	-0.0016	0.0030	0.0054	0.0079					
g=2	-0.0071	0.0029	0.0075	0.0097	0.0115					
g=3	-0.0033	0.0071	0.0097	0.0124	0.0137					
g=4	-0.0006	0.0084	0.0123	0.0143	0.0158					
g=5	0.0003	0.0100	0.0133	0.0152	0.0161					
g=6						0.0178	0.0180	0.0180	0.0182	0.0185
g=7						0.0178	0.0181	0.0189	0.0189	0.0190
g=8						0.0181	0.0186	0.0191	0.0191	0.0192
g=9						0.0188	0.0193	0.0194	0.0194	0.0194
g=10						0.0194	0.0194	0.0195	0.0195	0.0195
g=20		0.0189		0.0209		0.0212		0.0212		0.0212
g=25		0.0205		0.0220		0.0221		0.0221		0.0221
g=30		0.0214		0.0228		0.0228		0.0228		0.0228
g=35		0.0223		0.0235		0.0235		0.0235		0.0235
g=40		0.0236		0.0239		0.0239		0.0239		0.0239
g=45		0.0242		0.0249		0.0249		0.0249		0.0249
g=50		0.0253		0.0255		0.0255		0.0255		0.0255
g=100		0.0311		0.0311		0.0311		0.0311		0.0311
g=200		0.0390		0.0390		0.0390		0.0390		0.0390
g=300		0.0526		0.0526		0.0526		0.0526		0.0526
g=400		0.0532		0.0532		0.0532		0.0532		0.0532
g=500		0.0670		0.0670		0.0670		0.0670		0.0670

Table 4.249: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0077	0.0047	0.0036	0.0030	0.0027					
g=2	0.0064	0.0035	0.0028	0.0023	0.0020					
g=3	0.0059	0.0031	0.0023	0.0019	0.0016					
g=4	0.0054	0.0029	0.0020	0.0017	0.0013					
g=5	0.0042	0.0024	0.0018	0.0014	0.0009					
g=6						0.0006	0.0006	0.0006	0.0005	0.0004
g=7						0.0006	0.0006	0.0005	0.0004	0.0004
g=8						0.0006	0.0005	0.0005	0.0005	0.0005
g=9						0.0005	0.0005	0.0005	0.0005	0.0004
g=10						0.0005	0.0005	0.0005	0.0004	0.0004
g=20		0.0009		0.0006		0.0005		0.0005		0.0005
g=25		0.0010		0.0005		0.0005		0.0005		0.0005
g=30		0.0008		0.0006		0.0006		0.0006		0.0006
g=35		0.0009		0.0006		0.0006		0.0006		0.0006
g=40		0.0009		0.0006		0.0006		0.0006		0.0006
g=45		0.0008		0.0007		0.0007		0.0007		0.0007
g=50		0.0008		0.0007		0.0007		0.0007		0.0007
g=100		0.0010		0.0010		0.0010		0.0010		0.0010
g=200		0.0016		0.0016		0.0016		0.0016		0.0016
g=300		0.0029		0.0029		0.0029		0.0029		0.0029
g=400		0.0032		0.0032		0.0032		0.0032		0.0032
g=500		0.0048		0.0048		0.0048		0.0048		0.0048

Table 4.250: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0058	0.0036	0.0027	0.0024	0.0021					
g=2	0.0048	0.0029	0.0021	0.0017	0.0015					
g=3	0.0039	0.0022	0.0018	0.0013	0.0011					
g=4	0.0034	0.0020	0.0014	0.0011	0.0008					
g=5	0.0034	0.0017	0.0013	0.0010	0.0008					
g=6						0.0006	0.0006	0.0006	0.0005	0.0005
g=7						0.0006	0.0006	0.0004	0.0004	0.0004
g=8						0.0006	0.0005	0.0004	0.0004	0.0004
g=9						0.0005	0.0004	0.0004	0.0004	0.0004
g=10						0.0004	0.0004	0.0004	0.0004	0.0004
g=20		0.0008		0.0005		0.0005		0.0005		0.0005
g=25		0.0007		0.0005		0.0005		0.0005		0.0005
g=30		0.0008		0.0006		0.0006		0.0006		0.0006
g=35		0.0007		0.0006		0.0006		0.0006		0.0006
g=40		0.0007		0.0006		0.0006		0.0006		0.0006
g=45		0.0007		0.0007		0.0007		0.0007		0.0007
g=50		0.0007		0.0007		0.0007		0.0007		0.0007
g=100		0.0011		0.0011		0.0011		0.0011		0.0011
g=200		0.0017		0.0017		0.0017		0.0017		0.0017
g=300		0.0030		0.0030		0.0030		0.0030		0.0030
g=400		0.0031		0.0031		0.0031		0.0031		0.0031
g=500		0.0047		0.0047		0.0047		0.0047		0.0047

Table 4.251: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0278	-0.0278	-0.0278	-0.0278	-0.0278					
g=2	-0.0279	-0.0279	-0.0279	-0.0281	-0.0281					
g=3	-0.0280	-0.0280	-0.0282	-0.0282	-0.0282					
g=4	-0.0282	-0.0283	-0.0283	-0.0283	-0.0283					
g=5	-0.0283	-0.0285	-0.0285	-0.0285	-0.0285					
g=6						-0.0286	-0.0286	-0.0286	-0.0286	-0.0286
g=7						-0.0287	-0.0287	-0.0287	-0.0287	-0.0287
g=8						-0.0288	-0.0288	-0.0288	-0.0288	-0.0288
g=9						-0.0290	-0.0290	-0.0290	-0.0290	-0.0290
g=10						-0.0291	-0.0291	-0.0291	-0.0291	-0.0291
g=20		-0.0303		-0.0303		-0.0303		-0.0303		-0.0303
g=25		-0.0309		-0.0309		-0.0309		-0.0309		-0.0309
g=30		-0.0316		-0.0316		-0.0316		-0.0316		-0.0316
g=35		-0.0321		-0.0321		-0.0321		-0.0321		-0.0321
g=40		-0.0327		-0.0327		-0.0327		-0.0327		-0.0327
g=45		-0.0333		-0.0333		-0.0333		-0.0333		-0.0333
g=50		-0.0339		-0.0339		-0.0339		-0.0339		-0.0339
g=100		-0.0394		-0.0394		-0.0394		-0.0394		-0.0394
g=200		-0.0459		-0.0459		-0.0459		-0.0459		-0.0459
g=300		-0.0571		-0.0571		-0.0571		-0.0571		-0.0571
g=400		-0.0609		-0.0609		-0.0609		-0.0609		-0.0609
g=500		-0.0678		-0.0678		-0.0678		-0.0678		-0.0678

Table 4.252: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0285	0.0285	0.0285	0.0286	0.0286					
g=2	0.0286	0.0287	0.0287	0.0287	0.0287					
g=3	0.0288	0.0288	0.0288	0.0288	0.0288					
g=4	0.0289	0.0289	0.0289	0.0289	0.0289					
g=5	0.0290	0.0291	0.0291	0.0291	0.0291					
g=6						0.0292	0.0292	0.0292	0.0292	0.0292
g=7						0.0293	0.0293	0.0293	0.0293	0.0293
g=8						0.0294	0.0294	0.0294	0.0294	0.0294
g=9						0.0295	0.0295	0.0295	0.0295	0.0295
g=10						0.0297	0.0297	0.0297	0.0297	0.0297
g=20		0.0309		0.0309		0.0309		0.0309		0.0309
g=25		0.0315		0.0315		0.0315		0.0315		0.0315
g=30		0.0321		0.0321		0.0321		0.0321		0.0321
g=35		0.0327		0.0327		0.0327		0.0327		0.0327
g=40		0.0332		0.0332		0.0332		0.0332		0.0332
g=45		0.0338		0.0338		0.0338		0.0338		0.0338
g=50		0.0344		0.0344		0.0344		0.0344		0.0344
g=100		0.0402		0.0402		0.0402		0.0402		0.0402
g=200		0.0468		0.0468		0.0468		0.0468		0.0468
g=300		0.0579		0.0579		0.0579		0.0579		0.0579
g=400		0.0610		0.0610		0.0610		0.0610		0.0610
g=500		0.0670		0.0670		0.0670		0.0670		0.0670

Table 4.253: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0008	0.0008	0.0008	0.0008	0.0008					
g=2	0.0008	0.0008	0.0008	0.0008	0.0008					
g=3	0.0008	0.0008	0.0008	0.0008	0.0008					
g=4	0.0008	0.0008	0.0008	0.0008	0.0008					
g=5	0.0009	0.0008	0.0008	0.0008	0.0008					
g=6						0.0008	0.0008	0.0008	0.0008	0.0008
g=7						0.0008	0.0008	0.0008	0.0008	0.0008
g=8						0.0009	0.0009	0.0009	0.0009	0.0009
g=9						0.0009	0.0009	0.0009	0.0009	0.0009
g=10						0.0009	0.0009	0.0009	0.0009	0.0009
g=20		0.0009		0.0009		0.0009		0.0009		0.0009
g=25		0.0010		0.0010		0.0010		0.0010		0.0010
g=30		0.0010		0.0010		0.0010		0.0010		0.0010
g=35		0.0011		0.0011		0.0011		0.0011		0.0011
g=40		0.0011		0.0011		0.0011		0.0011		0.0011
g=45		0.0011		0.0011		0.0011		0.0011		0.0011
g=50		0.0012		0.0012		0.0012		0.0012		0.0012
g=100		0.0016		0.0016		0.0016		0.0016		0.0016
g=200		0.0022		0.0022		0.0022		0.0022		0.0022
g=300		0.0034		0.0034		0.0034		0.0034		0.0034
g=400		0.0040		0.0040		0.0040		0.0040		0.0040
g=500		0.0048		0.0048		0.0048		0.0048		0.0048

Table 4.254: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0008	0.0008	0.0008	0.0008	0.0008					
g=2	0.0008	0.0008	0.0008	0.0008	0.0008					
g=3	0.0009	0.0009	0.0009	0.0009	0.0009					
g=4	0.0009	0.0009	0.0009	0.0009	0.0009					
g=5	0.0009	0.0009	0.0009	0.0009	0.0009					
g=6						0.0009	0.0009	0.0009	0.0009	0.0009
g=7						0.0009	0.0009	0.0009	0.0009	0.0009
g=8						0.0009	0.0009	0.0009	0.0009	0.0009
g=9						0.0009	0.0009	0.0009	0.0009	0.0009
g=10						0.0009	0.0009	0.0009	0.0009	0.0009
g=20		0.0010		0.0010		0.0010		0.0010		0.0010
g=25		0.0010		0.0010		0.0010		0.0010		0.0010
g=30		0.0011		0.0011		0.0011		0.0011		0.0011
g=35		0.0011		0.0011		0.0011		0.0011		0.0011
g=40		0.0011		0.0011		0.0011		0.0011		0.0011
g=45		0.0012		0.0012		0.0012		0.0012		0.0012
g=50		0.0012		0.0012		0.0012		0.0012		0.0012
g=100		0.0017		0.0017		0.0017		0.0017		0.0017
g=200		0.0023		0.0023		0.0023		0.0023		0.0023
g=300		0.0035		0.0035		0.0035		0.0035		0.0035
g=400		0.0040		0.0040		0.0040		0.0040		0.0040
g=500		0.0047		0.0047		0.0047		0.0047		0.0047

Table 4.255: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0594	0.0405	0.0303	0.0237	0.0175					
g=2	0.0505	0.0298	0.0179	0.0088	0.0057					
g=3	0.0444	0.0227	0.0108	0.0036	-0.0010					
g=4	0.0404	0.0148	0.0055	-0.0010	-0.0044					
g=5	0.0354	0.0127	0.0015	-0.0041	-0.0071					
g=6						-0.0124	-0.0140	-0.0147	-0.0155	-0.0159
g=7						-0.0134	-0.0150	-0.0155	-0.0162	-0.0169
g=8						-0.0147	-0.0154	-0.0165	-0.0171	-0.0171
g=9						-0.0154	-0.0163	-0.0172	-0.0175	-0.0175
g=10						-0.0160	-0.0167	-0.0174	-0.0180	-0.0181
g=20		-0.0112		-0.0189		-0.0197		-0.0200		-0.0200
g=25		-0.0139		-0.0199		-0.0208		-0.0208		-0.0208
g=30		-0.0172		-0.0207		-0.0212		-0.0215		-0.0215
g=35		-0.0184		-0.0218		-0.0221		-0.0221		-0.0221
g=40		-0.0206		-0.0229		-0.0229		-0.0229		-0.0229
g=45		-0.0213		-0.0233		-0.0233		-0.0233		-0.0233
g=50		-0.0225		-0.0238		-0.0241		-0.0241		-0.0241
g=100		-0.0302		-0.0310		-0.0310		-0.0310		-0.0310
g=200		-0.0405		-0.0405		-0.0405		-0.0405		-0.0405
g=300		-0.0530		-0.0530		-0.0530		-0.0530		-0.0530
g=400		-0.0571		-0.0571		-0.0571		-0.0571		-0.0571
g=500		-0.0662		-0.0662		-0.0662		-0.0662		-0.0662

Table 4.256: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0331	-0.0160	-0.0080	-0.0038	0.0006					
g=2	-0.0263	-0.0083	-0.0000	0.0039	0.0069					
g=3	-0.0209	-0.0034	0.0039	0.0094	0.0114					
g=4	-0.0182	0.0004	0.0078	0.0116	0.0134					
g=5	-0.0129	0.0034	0.0089	0.0130	0.0147					
g=6						0.0164	0.0173	0.0175	0.0177	0.0179
g=7						0.0171	0.0176	0.0179	0.0179	0.0180
g=8						0.0174	0.0180	0.0182	0.0184	0.0188
g=9						0.0181	0.0184	0.0189	0.0190	0.0190
g=10						0.0183	0.0184	0.0188	0.0191	0.0191
g=20		0.0176		0.0206		0.0208		0.0208		0.0208
g=25		0.0192		0.0212		0.0214		0.0214		0.0214
g=30		0.0201		0.0218		0.0221		0.0221		0.0221
g=35		0.0212		0.0232		0.0232		0.0232		0.0232
g=40		0.0232		0.0237		0.0237		0.0237		0.0237
g=45		0.0241		0.0244		0.0244		0.0244		0.0244
g=50		0.0244		0.0250		0.0250		0.0250		0.0250
g=100		0.0316		0.0316		0.0316		0.0316		0.0316
g=200		0.0419		0.0419		0.0419		0.0419		0.0419
g=300		0.0533		0.0533		0.0533		0.0533		0.0533
g=400		0.0590		0.0590		0.0590		0.0590		0.0590
g=500		0.0676		0.0676		0.0676		0.0676		0.0676

Table 4.257: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0170	0.0128	0.0105	0.0090	0.0076					
g=2	0.0150	0.0105	0.0076	0.0056	0.0049					
g=3	0.0137	0.0089	0.0064	0.0047	0.0036					
g=4	0.0127	0.0069	0.0050	0.0037	0.0029					
g=5	0.0116	0.0067	0.0042	0.0030	0.0024					
g=6						0.0015	0.0011	0.0010	0.0008	0.0007
g=7						0.0013	0.0009	0.0008	0.0007	0.0005
g=8						0.0011	0.0009	0.0007	0.0006	0.0006
g=9						0.0009	0.0007	0.0005	0.0005	0.0005
g=10						0.0009	0.0007	0.0006	0.0005	0.0004
g=20		0.0022		0.0007		0.0005		0.0005		0.0005
g=25		0.0019		0.0007		0.0005		0.0005		0.0005
g=30		0.0013		0.0007		0.0006		0.0005		0.0005
g=35		0.0012		0.0006		0.0006		0.0006		0.0006
g=40		0.0010		0.0006		0.0006		0.0006		0.0006
g=45		0.0010		0.0006		0.0006		0.0006		0.0006
g=50		0.0009		0.0007		0.0006		0.0006		0.0006
g=100		0.0012		0.0011		0.0011		0.0011		0.0011
g=200		0.0019		0.0019		0.0019		0.0019		0.0019
g=300		0.0031		0.0031		0.0031		0.0031		0.0031
g=400		0.0036		0.0036		0.0036		0.0036		0.0036
g=500		0.0046		0.0046		0.0046		0.0046		0.0046

Table 4.258: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0106	0.0069	0.0053	0.0044	0.0035					
g=2	0.0091	0.0054	0.0037	0.0029	0.0024					
g=3	0.0082	0.0044	0.0030	0.0019	0.0015					
g=4	0.0073	0.0036	0.0023	0.0015	0.0012					
g=5	0.0063	0.0031	0.0020	0.0013	0.0010					
g=6						0.0007	0.0006	0.0006	0.0005	0.0005
g=7						0.0007	0.0006	0.0006	0.0006	0.0005
g=8						0.0007	0.0005	0.0005	0.0005	0.0004
g=9						0.0006	0.0005	0.0004	0.0004	0.0004
g=10						0.0006	0.0005	0.0004	0.0004	0.0004
g=20		0.0010		0.0005		0.0005		0.0005		0.0005
g=25		0.0009		0.0005		0.0005		0.0005		0.0005
g=30		0.0008		0.0006		0.0005		0.0005		0.0005
g=35		0.0009		0.0006		0.0006		0.0006		0.0006
g=40		0.0007		0.0006		0.0006		0.0006		0.0006
g=45		0.0007		0.0006		0.0006		0.0006		0.0006
g=50		0.0008		0.0007		0.0007		0.0007		0.0007
g=100		0.0011		0.0011		0.0011		0.0011		0.0011
g=200		0.0019		0.0019		0.0019		0.0019		0.0019
g=300		0.0031		0.0031		0.0031		0.0031		0.0031
g=400		0.0038		0.0038		0.0038		0.0038		0.0038
g=500		0.0048		0.0048		0.0048		0.0048		0.0048

Table 4.259: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.0279	-0.0279	-0.0280	-0.0280	-0.0280					
g=2	-0.0280	-0.0280	-0.0281	-0.0281	-0.0281					
g=3	-0.0282	-0.0282	-0.0282	-0.0282	-0.0282					
g=4	-0.0283	-0.0283	-0.0283	-0.0283	-0.0283					
g=5	-0.0284	-0.0284	-0.0284	-0.0284	-0.0284					
g=6						-0.0286	-0.0286	-0.0286	-0.0286	-0.0286
g=7						-0.0287	-0.0287	-0.0287	-0.0287	-0.0287
g=8						-0.0288	-0.0288	-0.0288	-0.0288	-0.0288
g=9						-0.0290	-0.0290	-0.0290	-0.0290	-0.0290
g=10						-0.0291	-0.0291	-0.0291	-0.0291	-0.0291
g=20		-0.0304		-0.0304		-0.0304		-0.0304		-0.0304
g=25		-0.0310		-0.0310		-0.0310		-0.0310		-0.0310
g=30		-0.0316		-0.0316		-0.0316		-0.0316		-0.0316
g=35		-0.0321		-0.0321		-0.0321		-0.0321		-0.0321
g=40		-0.0326		-0.0326		-0.0326		-0.0326		-0.0326
g=45		-0.0332		-0.0332		-0.0332		-0.0332		-0.0332
g=50		-0.0337		-0.0337		-0.0337		-0.0337		-0.0337
g=100		-0.0387		-0.0387		-0.0387		-0.0387		-0.0387
g=200		-0.0477		-0.0477		-0.0477		-0.0477		-0.0477
g=300		-0.0578		-0.0578		-0.0578		-0.0578		-0.0578
g=400		-0.0633		-0.0633		-0.0633		-0.0633		-0.0633
g=500		-0.0669		-0.0669		-0.0669		-0.0669		-0.0669

Table 4.260: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0278	0.0279	0.0279	0.0279	0.0283					
g=2	0.0280	0.0280	0.0284	0.0284	0.0284					
g=3	0.0281	0.0285	0.0285	0.0285	0.0285					
g=4	0.0282	0.0286	0.0286	0.0286	0.0286					
g=5	0.0284	0.0288	0.0288	0.0288	0.0288					
g=6						0.0289	0.0289	0.0289	0.0289	0.0289
g=7						0.0291	0.0291	0.0291	0.0291	0.0291
g=8						0.0291	0.0291	0.0291	0.0291	0.0291
g=9						0.0292	0.0292	0.0292	0.0292	0.0292
g=10						0.0295	0.0295	0.0295	0.0295	0.0295
g=20		0.0307		0.0307		0.0307		0.0307		0.0307
g=25		0.0313		0.0313		0.0313		0.0313		0.0313
g=30		0.0319		0.0319		0.0319		0.0319		0.0319
g=35		0.0325		0.0325		0.0325		0.0325		0.0325
g=40		0.0330		0.0330		0.0330		0.0330		0.0330
g=45		0.0336		0.0336		0.0336		0.0336		0.0336
g=50		0.0342		0.0342		0.0342		0.0342		0.0342
g=100		0.0388		0.0388		0.0388		0.0388		0.0388
g=200		0.0484		0.0484		0.0484		0.0484		0.0484
g=300		0.0572		0.0572		0.0572		0.0572		0.0572
g=400		0.0640		0.0640		0.0640		0.0640		0.0640
g=500		0.0681		0.0681		0.0681		0.0681		0.0681

Table 4.261: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0008	0.0008	0.0008	0.0008	0.0008					
g=2	0.0008	0.0008	0.0008	0.0008	0.0008					
g=3	0.0008	0.0008	0.0008	0.0008	0.0008					
g=4	0.0008	0.0008	0.0008	0.0008	0.0008					
g=5	0.0008	0.0008	0.0008	0.0008	0.0008					
g=6						0.0008	0.0008	0.0008	0.0008	0.0008
g=7						0.0009	0.0009	0.0009	0.0009	0.0009
g=8						0.0009	0.0009	0.0009	0.0009	0.0009
g=9						0.0009	0.0009	0.0009	0.0009	0.0009
g=10						0.0009	0.0009	0.0009	0.0009	0.0009
g=20		0.0010		0.0010		0.0010		0.0010		0.0010
g=25		0.0010		0.0010		0.0010		0.0010		0.0010
g=30		0.0010		0.0010		0.0010		0.0010		0.0010
g=35		0.0011		0.0011		0.0011		0.0011		0.0011
g=40		0.0011		0.0011		0.0011		0.0011		0.0011
g=45		0.0011		0.0011		0.0011		0.0011		0.0011
g=50		0.0012		0.0012		0.0012		0.0012		0.0012
g=100		0.0015		0.0015		0.0015		0.0015		0.0015
g=200		0.0024		0.0024		0.0024		0.0024		0.0024
g=300		0.0035		0.0035		0.0035		0.0035		0.0035
g=400		0.0043		0.0043		0.0043		0.0043		0.0043
g=500		0.0047		0.0047		0.0047		0.0047		0.0047

Table 4.262: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_2$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0009	0.0009	0.0009	0.0009	0.0008					
g=2	0.0009	0.0009	0.0008	0.0008	0.0008					
g=3	0.0009	0.0008	0.0008	0.0008	0.0008					
g=4	0.0009	0.0008	0.0008	0.0008	0.0008					
g=5	0.0009	0.0009	0.0009	0.0009	0.0009					
g=6						0.0009	0.0009	0.0009	0.0009	0.0009
g=7						0.0009	0.0009	0.0009	0.0009	0.0009
g=8						0.0009	0.0009	0.0009	0.0009	0.0009
g=9						0.0009	0.0009	0.0009	0.0009	0.0009
g=10						0.0009	0.0009	0.0009	0.0009	0.0009
g=20		0.0010		0.0010		0.0010		0.0010		0.0010
g=25		0.0010		0.0010		0.0010		0.0010		0.0010
g=30		0.0010		0.0010		0.0010		0.0010		0.0010
g=35		0.0011		0.0011		0.0011		0.0011		0.0011
g=40		0.0011		0.0011		0.0011		0.0011		0.0011
g=45		0.0012		0.0012		0.0012		0.0012		0.0012
g=50		0.0012		0.0012		0.0012		0.0012		0.0012
g=100		0.0016		0.0016		0.0016		0.0016		0.0016
g=200		0.0025		0.0025		0.0025		0.0025		0.0025
g=300		0.0034		0.0034		0.0034		0.0034		0.0034
g=400		0.0044		0.0044		0.0044		0.0044		0.0044
g=500		0.0049		0.0049		0.0049		0.0049		0.0049

Concluding remarks about the simulated data from a scaled triangular distribution with $1-p=0.1$, $n=10000$ and $[a,b]=[0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions. The normal kernel is used in most of the results, if not explicitly stated differently.
- It is recommended to use a single minimum point, but more minima can also be used up to $m=5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ or $\hat{h}_6$ (see the definitions of $\hat{h}$ in Table 4.1) is recommended as a good choice for the estimated smoothing parameter.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended, with preference given to the Cramér-von-Mises test.
- For both of these goodness-of-fit tests, a level of significance of 1% or 5% can be used.
- The choices of $4 \leq r \leq 8$ and $5 \leq g \leq 10$ are recommended for optimal results.

The following two tables (Tables 4.263 – 4.264) provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a,b]=[0.3\,,\,0.7]$.

Table 4.263: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m=1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.28	0.72	0.28	0.72
g=7	0.28	0.72	0.28	0.72
g=8	0.28	0.72	0.28	0.721
g=9	0.28	0.72	0.28	0.72
g=10	0.28	0.72	0.28	0.72

Table 4.264: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m=1$ and $\hat{h}_1$ for the Cramér-von-Mises goodness-of-fit test.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.27	0.73	0.27	0.73
g=7	0.27	0.73	0.27	0.73
g=8	0.27	0.73	0.27	0.73
g=9	0.27	0.73	0.27	0.73
g=10	0.27	0.73	0.27	0.73

Remark: *For this study population the estimated off-pulse interval is fairly close to the actual off-pulse interval for quite a wide array of different values of the tuning parameters.*

4.6.2 Data set parameters: $1-p=0.1$, $n=2000$ and $[a,b]=[0.3\,,\,0.7]$

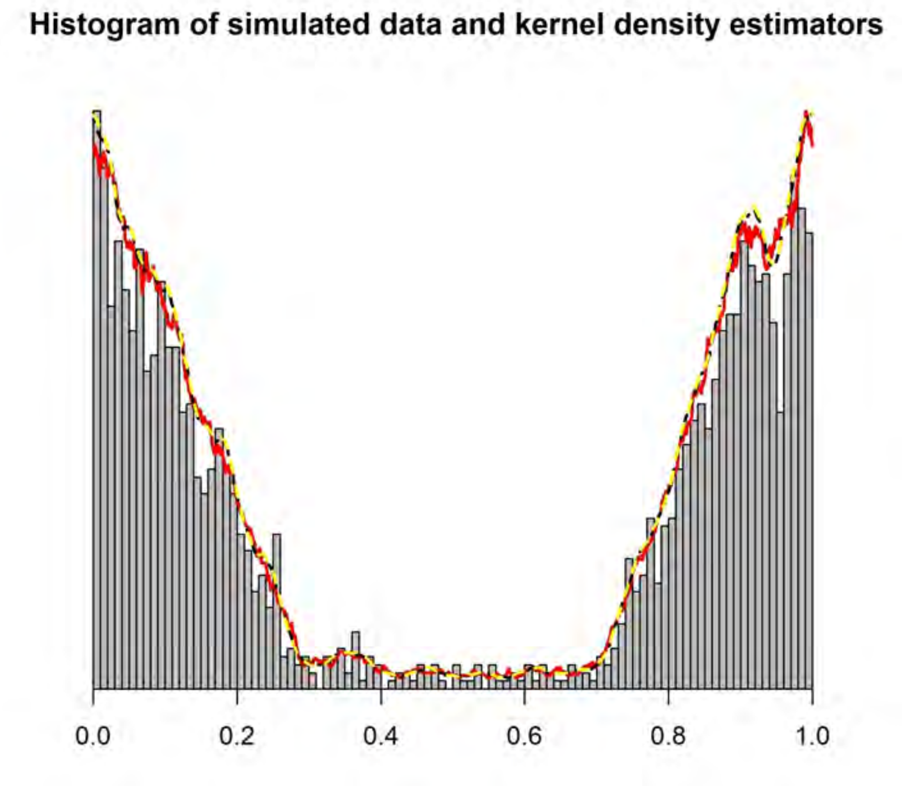

Figure 4.20: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.17$.

Figure 4.20 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

The first step of SOPIE is to calculate that point where the kernel density estimator attains its global minimum and the next m local minima. Table 4.265 compares the minima obtained from each of the different kernel functions that is fitted to the data for a single Monte Carlo iteration with $\hat{h}_3$.

From this table it is again evident that similar minima (or minima close to each other) are obtained from the different kernel functions used in the kernel density estimation. Especially, the Epanechnikov and Swanepoel kernels are quite close to each other. In some cases, several minima overlap for the kernel functions, except that they are obtained in a different order.

It is therefore still argued that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE.

Table 4.265: *Minima comparison for different kernel functions for a single Monte Carlo repetition.*

	Swanepoel kernel	Epanechnikov kernel	Normal kernel
1st local min.	0.5765	0.5755	0.5475
2nd local min.	0.5775	0.5765	0.5485
3rd local min.	0.5755	0.5745	0.5895
4th local min.	0.5785	0.5775	0.5495
5th local min.	0.5045	0.5735	0.5505
6th local min.	0.5745	0.5785	0.5885
7th local min.	0.5055	0.5725	0.5875
8th local min.	0.5795	0.5715	0.4305
9th local min.	0.5035	0.5795	0.5105
10th local min.	0.5735	0.5705	0.4295
11th local min.	0.4245	0.5695	0.4285
12th local min.	0.4235	0.5655	0.4275
13th local min.	0.5725	0.5625	0.5095
14th local min.	0.5065	0.5685	0.4265
15th local min.	0.5605	0.5645	0.5585
16th local min.	0.4255	0.5665	0.4255
17th local min.	0.5615	0.5675	0.5075
18th local min.	0.5805	0.5635	0.4245
19th local min.	0.4225	0.5615	0.5065
20th local min.	0.5025	0.5805	0.5595

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m. Tables 4.266 – 4.267 highlight the values of the estimated bias and MSE for different values of m when $g = 6$, $r = 6$ and $\alpha = 0.01$ for two goodness-of-fit tests and two estimated smoothing parameters. It can be seen that different values for m result in almost equal values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. Therefore, it seems fair to recommend the choice of $m = 1$ as a good choice, since it will save on computing time, without a substantial effect on the results.

Table 4.266: *Bias for different combinations of m for $\hat{h}_1$ and $\hat{h}_3$.*

	$\hat{h}_1$				$\hat{h}_3$			
	Anderson-Darling		**Cramér-von-Mises**		**Anderson-Darling**		**Cramér-von-Mises**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0321	0.0339	-0.0491	0.0503	-0.0306	0.0346	-0.0486	0.0521
m=2	-0.0324	0.0338	-0.0493	0.0502	-0.0306	0.0344	-0.0488	0.0520
m=3	-0.0325	0.0338	-0.0494	0.0503	-0.0311	0.0343	-0.0490	0.0519
m=4	-0.0328	0.0338	-0.0495	0.0503	-0.0314	0.0344	-0.0492	0.0520
m=5	-0.0329	0.0339	-0.0496	0.0503	-0.0318	0.0345	-0.0495	0.0520
m=6	-0.0329	0.0340	-0.0496	0.0504	-0.0319	0.0346	-0.0496	0.0521
m=7	-0.0330	0.0340	-0.0497	0.0505	-0.0321	0.0346	-0.0497	0.0521
m=8	-0.0331	0.0341	-0.0497	0.0505	-0.0322	0.0346	-0.0497	0.0521
m=9	-0.0331	0.0341	-0.0498	0.0506	-0.0322	0.0347	-0.0498	0.0521
m=10	-0.0331	0.0341	-0.0498	0.0506	-0.0322	0.0347	-0.0498	0.0521
m=11	-0.0331	0.0341	-0.0498	0.0506	-0.0323	0.0347	-0.0498	0.0521

Table 4.267: *MSE for different combinations of m for $\hat{h}_1$ and $\hat{h}_3$.*

	$\hat{h}_1$				$\hat{h}_3$			
	Anderson-Darling		**Cramér-von-Mises**		**Anderson-Darling**		**Cramér-von-Mises**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0012	0.0012	0.0025	0.0026	0.0011	0.0013	0.0024	0.0028
m=2	0.0012	0.0012	0.0025	0.0026	0.0011	0.0013	0.0025	0.0028
m=3	0.0012	0.0012	0.0025	0.0026	0.0011	0.0013	0.0025	0.0028
m=4	0.0012	0.0012	0.0025	0.0026	0.0011	0.0013	0.0025	0.0028
m=5	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0025	0.0028
m=6	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0025	0.0028
m=7	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0026	0.0028
m=8	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0026	0.0028
m=9	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0026	0.0028
m=10	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0026	0.0028
m=11	0.0012	0.0012	0.0025	0.0026	0.0012	0.0013	0.0026	0.0028

An important observation is that, as far as the bias and MSE are concerned, all four of the goodness-of-fit tests are insensitive and robust to the choice of m.

Choice of estimated smoothing parameters

Tables 4.268 and 4.269 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$, when the normal kernel is used.

It is found that the values of $\hat{h}_3$ and $\hat{h}_4$ are very close to each other, resulting in near equal values of the estimated bias and MSE. Almost equal values are also found for $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$, which explain why the results for these $\hat{h}$ combinations are almost equal in the tables. When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero, but when $\hat{h}_6$ (related to $\hat{h}_7$ and $\hat{h}_9$) is used, the estimated bias is much larger.

Table 4.268: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 6$ and $r = 6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0321	0.0339	-0.0491	0.0503	-0.0363	0.0373	-0.0427	0.0436
$\hat{h}_2$	-0.0303	0.0353	-0.0486	0.0534	-0.0347	0.0390	-0.0467	0.0476
$\hat{h}_3$	-0.0306	0.0346	-0.0486	0.0521	-0.0349	0.0383	-0.0463	0.0467
$\hat{h}_4$	-0.0306	0.0346	-0.0486	0.0521	-0.0349	0.0383	-0.0463	0.0467
$\hat{h}_5$	-0.0319	0.0340	-0.0492	0.0506	-0.0355	0.0372	-0.0424	0.0434
$\hat{h}_6$	-0.0913	-0.1603	-0.1319	-0.0293	-0.0993	-0.1371	-0.1332	-0.0907
$\hat{h}_7$	-0.0917	-0.1616	-0.1323	-0.0305	-0.0997	-0.1384	-0.1336	-0.0920
$\hat{h}_8$	-0.0303	0.0356	-0.0488	0.0539	-0.0346	0.0393	-0.0469	0.0477
$\hat{h}_9$	-0.0917	-0.1616	-0.1323	-0.0305	-0.0997	-0.1384	-0.1336	-0.0920

Table 4.269: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 6$ and $r = 6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0012	0.0012	0.0025	0.0026	0.0015	0.0015	0.0020	0.0020
$\hat{h}_2$	0.0011	0.0013	0.0024	0.0030	0.0015	0.0016	0.0024	0.0025
$\hat{h}_3$	0.0011	0.0013	0.0024	0.0028	0.0015	0.0016	0.0023	0.0023
$\hat{h}_4$	0.0011	0.0013	0.0024	0.0028	0.0015	0.0016	0.0023	0.0023
$\hat{h}_5$	0.0012	0.0012	0.0025	0.0026	0.0015	0.0015	0.0020	0.0020
$\hat{h}_6$	0.0091	0.0690	0.0191	0.0309	0.0108	0.0628	0.0189	0.0442
$\hat{h}_7$	0.0092	0.0695	0.0192	0.0313	0.0109	0.0632	0.0190	0.0446
$\hat{h}_8$	0.0011	0.0014	0.0025	0.0030	0.0015	0.0016	0.0025	0.0025
$\hat{h}_9$	0.0092	0.0695	0.0192	0.0313	0.0109	0.0632	0.0190	0.0446

When comparing the estimated MSE, $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ and $\hat{h}_5$ result in marginally smaller values of the estimated MSE. The estimated MSE – when using $\hat{h}_6$ – is much larger. This same trend is found for different values of r and g. In the light of this, it is recommended to use any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ (see the definitions of $\hat{h}$ in Table 4.1) as a good choice of the estimated smoothing parameter. Furthermore, for small to moderate sample sizes n, $\hat{h}_6$ is not recommended as first choice for the estimated smoothing parameter.

Choice of goodness-of-fit test

Some of the tables provided earlier can be used to assess the goodness-of-fit tests. For a specific comparison of the goodness-of-fit tests, the reader can inspect Tables 4.270 – 4.271 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ for two of the estimated smoothing parameters $\hat{h}$. Tables 4.272 – 4.273 are even more important, since the MSE is compared in these tables. Recall that the MSE takes both the bias and variance into account when measuring the quality of an estimator.

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs slightly worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests perform equally well for $\hat{h}_1$, followed by the other two tests. For $\hat{h}_3$, the Kolmogorov-Smirnov goodness-of-fit test is equal in performance to the Anderson-Darling goodness-of-fit test.

Due to the importance of the MSE, it is recommended to use both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests. For smaller sample sizes, the Kolmogorov-Smirnov goodness-of-fit test can also be used, as it may lead to improved results.

Table 4.270: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0321	0.0339	-0.0491	0.0503	-0.0363	0.0373	-0.0427	0.0436
g=7	-0.0327	0.0344	-0.0496	0.0507	-0.0369	0.0377	-0.0431	0.0442
g=8	-0.0335	0.0349	-0.0501	0.0511	-0.0375	0.0382	-0.0436	0.0444
g=9	-0.0343	0.0354	-0.0505	0.0515	-0.0382	0.0388	-0.0442	0.0449
g=10	-0.0347	0.0360	-0.0509	0.0518	-0.0385	0.0392	-0.0446	0.0454
g=20	-0.0401	0.0404	-0.0549	0.0556	-0.0436	0.0440	-0.0490	0.0494

Table 4.271: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_3$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0306	0.0346	-0.0486	0.0521	-0.0349	0.0383	-0.0463	0.0467
g=7	-0.0314	0.0353	-0.0491	0.0525	-0.0357	0.0388	-0.0467	0.0470
g=8	-0.0318	0.0353	-0.0493	0.0528	-0.0367	0.0394	-0.0472	0.0475
g=9	-0.0322	0.0362	-0.0499	0.0531	-0.0371	0.0399	-0.0477	0.0479
g=10	-0.0328	0.0365	-0.0503	0.0537	-0.0375	0.0406	-0.0481	0.0486
g=20	-0.0378	0.0411	-0.0542	0.0573	-0.0425	0.0453	-0.0524	0.0524

Table 4.272: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0006	0.0006	0.0008	0.0009	0.0009	0.0008	0.0013	0.0013
g=7	0.0006	0.0006	0.0008	0.0009	0.0009	0.0007	0.0011	0.0011
g=8	0.0006	0.0006	0.0009	0.0009	0.0009	0.0007	0.0011	0.0011
g=9	0.0005	0.0005	0.0009	0.0009	0.0009	0.0007	0.0011	0.0010
g=10	0.0005	0.0004	0.0009	0.0009	0.0008	0.0008	0.0011	0.0010
g=20	0.0005	0.0005	0.0009	0.0010	0.0007	0.0006	0.0009	0.0009

Table 4.273: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_3$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0011	0.0013	0.0024	0.0028	0.0015	0.0016	0.0023	0.0023
g=7	0.0011	0.0013	0.0025	0.0029	0.0015	0.0016	0.0023	0.0024
g=8	0.0012	0.0013	0.0025	0.0029	0.0015	0.0016	0.0024	0.0024
g=9	0.0012	0.0014	0.0026	0.0029	0.0015	0.0017	0.0024	0.0025
g=10	0.0012	0.0014	0.0026	0.0030	0.0016	0.0017	0.0024	0.0025
g=20	0.0016	0.0018	0.0030	0.0034	0.0020	0.0022	0.0029	0.0029

Choice of the significance level α

Tables 4.274 – 4.276 compare the estimated bias and MSE for most of the goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 6$. Several more comparisons were made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

From all of these tables it is concluded that – for the Cramér-von-Mises goodness-of-fit test – the estimated bias and MSE are not sensitive to the choice of α. All of the goodness-of-fit tests produce a estimated bias that tends closer to zero when α is *increased*. This is not a familiar trend compared to all of the previous target populations, but may be explained by the small sample size n. It can be argued that, due to the smaller sample, larger α-values would reject uniformity sooner, implying that the test is more sensitive to changes in the uniform distribution of the underlying data.

Table 4.274: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00	-0.02	0.02	0.00	0.00
g=7	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.02	0.00	0.00
g=8	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.02	0.00	0.00
g=9	-0.03	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00
g=10	-0.03	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00
g=20	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00

Table 4.275: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00
g=7	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00
g=8	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00
g=9	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00
g=10	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00
g=20	-0.05	0.06	0.00	0.00	-0.05	0.05	0.00	0.00	-0.05	0.05	0.00	0.00

Table 4.276: *Comparison of the estimated bias and MSE for different α-values, with $m = 1, r = 6, \hat{h}_1$ and the normal kernel (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.02	0.00	0.00
g=7	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.02	0.00	0.00
g=8	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00
g=9	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00
g=10	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00
g=20	-0.04	0.04	0.00	0.00	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

The choice of g, the incremental growth of each interval over which rejection is tested, together with the choice of r, the number of intervals before rejection of uniformity takes place, are two important tuning parameters in the estimation process. Tables 4.277 – 4.284 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests when $\hat{h}_1$ is chosen. In all of the tables the normal kernel is used, with $\alpha = 0.05$ and $m = 1$ kept constant.

For both goodness-of-fit tests, small values of r and g result in estimated bias-values that are close to zero, i.e., $2 \leq g \leq 5$ with $2 \leq r \leq 4$. In terms of estimated MSE, the Cramér-von-Mises goodness-of-fit test is relatively robust against different choices of r and g. For the Anderson-Darling goodness-of-fit test, the smallest values of the MSE are obtained when $6 \leq g \leq 10$ and $6 \leq r \leq 10$.

In conclusion, it is recommended to use values of g in the range from 5 to 10, and for r, values from 4 to 8. It must be mentioned that, for smaller sample sizes, smaller values of r and g can also be used.

Table 4.277: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0269	0.0093	-0.0002	-0.0051	-0.0102					
g=2	0.0206	-0.0020	-0.0095	-0.0152	-0.0174					
g=3	0.0106	-0.0082	-0.0166	-0.0188	-0.0201					
g=4	0.0099	-0.0126	-0.0189	-0.0211	-0.0220					
g=5	0.0041	-0.0160	-0.0205	-0.0229	-0.0236					
g=6						-0.0247	-0.0252	-0.0255	-0.0255	-0.0257
g=7						-0.0263	-0.0263	-0.0265	-0.0265	-0.0265
g=8						-0.0264	-0.0266	-0.0269	-0.0269	-0.0269
g=9						-0.0273	-0.0278	-0.0278	-0.0278	-0.0278
g=10						-0.0275	-0.0275	-0.0275	-0.0275	-0.0275
g=20		-0.0315		-0.0326		-0.0326		-0.0326		-0.0326
g=25		-0.0337		-0.0339		-0.0339		-0.0339		-0.0339
g=30		-0.0362		-0.0362		-0.0362		-0.0362		-0.0362
g=35		-0.0376		-0.0376		-0.0376		-0.0376		-0.0376
g=40		-0.0398		-0.0398		-0.0398		-0.0398		-0.0398
g=45		-0.0426		-0.0426		-0.0426		-0.0426		-0.0426
g=50		-0.0459		-0.0459		-0.0459		-0.0459		-0.0459
g=100		-0.0681		-0.0681		-0.0681		-0.0681		-0.0681
g=200		-0.1171		-0.1171		-0.1171		-0.1171		
g=300		-0.1516		-0.1516		-0.1516				
g=400		-0.1801		-0.1801						

Table 4.278: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0211	-0.0039	0.0046	0.0100	0.0136					
g=2	-0.0148	0.0070	0.0152	0.0188	0.0204					
g=3	-0.0084	0.0120	0.0190	0.0217	0.0232					
g=4	-0.0065	0.0164	0.0215	0.0235	0.0249					
g=5	0.0025	0.0195	0.0231	0.0259	0.0264					
g=6						0.0276	0.0283	0.0285	0.0285	0.0285
g=7						0.0287	0.0289	0.0289	0.0289	0.0289
g=8						0.0292	0.0295	0.0295	0.0295	0.0295
g=9						0.0299	0.0299	0.0299	0.0299	0.0299
g=10						0.0304	0.0304	0.0304	0.0304	0.0304
g=20		0.0333		0.0350		0.0350		0.0350		0.0350
g=25		0.0368		0.0368		0.0368		0.0368		0.0368
g=30		0.0397		0.0397		0.0397		0.0397		0.0397
g=35		0.0410		0.0410		0.0410		0.0410		0.0410
g=40		0.0424		0.0424		0.0424		0.0424		0.0424
g=45		0.0458		0.0458		0.0458		0.0458		0.0458
g=50		0.0486		0.0486		0.0486		0.0486		0.0486
g=100		0.0677		0.0677		0.0677		0.0677		0.0677
g=200		0.1168		0.1168		0.1168		0.1168		
g=300		0.1514		0.1514		0.1514				
g=400		0.1799		0.1799						

Table 4.279: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0080	0.0053	0.0041	0.0034	0.0026					
g=2	0.0072	0.0039	0.0028	0.0021	0.0018					
g=3	0.0056	0.0031	0.0020	0.0017	0.0015					
g=4	0.0059	0.0026	0.0018	0.0015	0.0014					
g=5	0.0048	0.0022	0.0017	0.0014	0.0013					
g=6						0.0012	0.0011	0.0011	0.0011	0.0011
g=7						0.0011	0.0011	0.0011	0.0011	0.0011
g=8						0.0012	0.0012	0.0011	0.0011	0.0011
g=9						0.0013	0.0012	0.0012	0.0012	0.0012
g=10						0.0013	0.0013	0.0013	0.0013	0.0013
g=20		0.0018		0.0017		0.0017		0.0017		0.0017
g=25		0.0019		0.0018		0.0018		0.0018		0.0018
g=30		0.0019		0.0019		0.0019		0.0019		0.0019
g=35		0.0022		0.0022		0.0022		0.0022		0.0022
g=40		0.0023		0.0023		0.0023		0.0023		0.0023
g=45		0.0024		0.0024		0.0024		0.0024		0.0024
g=50		0.0026		0.0026		0.0026		0.0026		0.0026
g=100		0.0048		0.0048		0.0048		0.0048		0.0048
g=200		0.0138		0.0138		0.0138		0.0138		
g=300		0.0231		0.0231		0.0231				
g=400		0.0325		0.0325						

Table 4.280: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0069	0.0046	0.0035	0.0028	0.0024					
g=2	0.0062	0.0032	0.0023	0.0019	0.0017					
g=3	0.0052	0.0027	0.0019	0.0016	0.0014					
g=4	0.0053	0.0023	0.0017	0.0014	0.0013					
g=5	0.0041	0.0020	0.0016	0.0012	0.0011					
g=6						0.0011	0.0010	0.0010	0.0010	0.0010
g=7						0.0010	0.0010	0.0010	0.0010	0.0010
g=8						0.0011	0.0010	0.0010	0.0010	0.0010
g=9						0.0011	0.0011	0.0011	0.0011	0.0011
g=10						0.0011	0.0011	0.0011	0.0011	0.0011
g=20		0.0018		0.0015		0.0015		0.0015		0.0015
g=25		0.0017		0.0017		0.0017		0.0017		0.0017
g=30		0.0019		0.0019		0.0019		0.0019		0.0019
g=35		0.0021		0.0021		0.0021		0.0021		0.0021
g=40		0.0022		0.0022		0.0022		0.0022		0.0022
g=45		0.0024		0.0024		0.0024		0.0024		0.0024
g=50		0.0027		0.0027		0.0027		0.0027		0.0027
g=100		0.0048		0.0048		0.0048		0.0048		0.0048
g=200		0.0137		0.0137		0.0137		0.0137		
g=300		0.0230		0.0230		0.0230				
g=400		0.0324		0.0324						

Table 4.281: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0444	-0.0448	-0.0449	-0.0449	-0.0449					
g=2	-0.0449	-0.0453	-0.0453	-0.0453	-0.0453					
g=3	-0.0457	-0.0457	-0.0457	-0.0457	-0.0457					
g=4	-0.0460	-0.0462	-0.0462	-0.0462	-0.0462					
g=5	-0.0466	-0.0466	-0.0466	-0.0466	-0.0466					
g=6						-0.0470	-0.0470	-0.0470	-0.0470	-0.0470
g=7						-0.0474	-0.0474	-0.0474	-0.0474	-0.0474
g=8						-0.0480	-0.0480	-0.0480	-0.0480	-0.0480
g=9						-0.0483	-0.0483	-0.0483	-0.0483	-0.0483
g=10						-0.0487	-0.0487	-0.0487	-0.0487	-0.0487
g=20		-0.0527		-0.0527		-0.0527		-0.0527		-0.0527
g=25		-0.0544		-0.0544		-0.0544		-0.0544		-0.0544
g=30		-0.0568		-0.0568		-0.0568		-0.0568		-0.0568
g=35		-0.0586		-0.0586		-0.0586		-0.0586		-0.0586
g=40		-0.0601		-0.0601		-0.0601		-0.0601		-0.0601
g=45		-0.0611		-0.0611		-0.0611		-0.0611		-0.0611
g=50		-0.0617		-0.0617		-0.0617		-0.0617		-0.0617
g=100		-0.0743		-0.0743		-0.0743		-0.0743		-0.0743
g=200		-0.1171		-0.1171		-0.1171		-0.1171		
g=300		-0.1516		-0.1516		-0.1516				
g=400		-0.1801		-0.1801						

Table 4.282: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0456	0.0458	0.0458	0.0459	0.0459					
g=2	0.0460	0.0462	0.0462	0.0462	0.0462					
g=3	0.0466	0.0467	0.0467	0.0467	0.0467					
g=4	0.0470	0.0471	0.0471	0.0471	0.0471					
g=5	0.0475	0.0475	0.0475	0.0475	0.0475					
g=6						0.0479	0.0479	0.0479	0.0479	0.0479
g=7						0.0482	0.0482	0.0482	0.0482	0.0482
g=8						0.0487	0.0487	0.0487	0.0487	0.0487
g=9						0.0493	0.0493	0.0493	0.0493	0.0493
g=10						0.0495	0.0495	0.0495	0.0495	0.0495
g=20		0.0535		0.0535		0.0535		0.0535		0.0535
g=25		0.0552		0.0552		0.0552		0.0552		0.0552
g=30		0.0576		0.0576		0.0576		0.0576		0.0576
g=35		0.0589		0.0589		0.0589		0.0589		0.0589
g=40		0.0609		0.0609		0.0609		0.0609		0.0609
g=45		0.0624		0.0624		0.0624		0.0624		0.0624
g=50		0.0624		0.0624		0.0624		0.0624		0.0624
g=100		0.0744		0.0744		0.0744		0.0744		0.0744
g=200		0.1168		0.1168		0.1168		0.1168		
g=300		0.1514		0.1514		0.1514				
g=400		0.1799		0.1799						

Table 4.283: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0021	0.0021	0.0021	0.0021	0.0021					
g=2	0.0021	0.0021	0.0021	0.0021	0.0021					
g=3	0.0022	0.0022	0.0022	0.0022	0.0022					
g=4	0.0022	0.0022	0.0022	0.0022	0.0022					
g=5	0.0022	0.0022	0.0022	0.0022	0.0022					
g=6						0.0023	0.0023	0.0023	0.0023	0.0023
g=7						0.0023	0.0023	0.0023	0.0023	0.0023
g=8						0.0024	0.0024	0.0024	0.0024	0.0024
g=9						0.0024	0.0024	0.0024	0.0024	0.0024
g=10						0.0024	0.0024	0.0024	0.0024	0.0024
g=20		0.0029		0.0029		0.0029		0.0029		0.0029
g=25		0.0031		0.0031		0.0031		0.0031		0.0031
g=30		0.0033		0.0033		0.0033		0.0033		0.0033
g=35		0.0036		0.0036		0.0036		0.0036		0.0036
g=40		0.0038		0.0038		0.0038		0.0038		0.0038
g=45		0.0039		0.0039		0.0039		0.0039		0.0039
g=50		0.0040		0.0040		0.0040		0.0040		0.0040
g=100		0.0058		0.0058		0.0058		0.0058		0.0058
g=200		0.0138		0.0138		0.0138		0.0138		
g=300		0.0231		0.0231		0.0231				
g=400		0.0325		0.0325						

Table 4.284: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0022	0.0022	0.0022	0.0022	0.0022					
g=2	0.0022	0.0022	0.0022	0.0022	0.0022					
g=3	0.0023	0.0023	0.0023	0.0023	0.0023					
g=4	0.0023	0.0023	0.0023	0.0023	0.0023					
g=5	0.0023	0.0023	0.0023	0.0023	0.0023					
g=6						0.0024	0.0024	0.0024	0.0024	0.0024
g=7						0.0024	0.0024	0.0024	0.0024	0.0024
g=8						0.0025	0.0025	0.0025	0.0025	0.0025
g=9						0.0025	0.0025	0.0025	0.0025	0.0025
g=10						0.0025	0.0025	0.0025	0.0025	0.0025
g=20		0.0030		0.0030		0.0030		0.0030		0.0030
g=25		0.0031		0.0031		0.0031		0.0031		0.0031
g=30		0.0034		0.0034		0.0034		0.0034		0.0034
g=35		0.0036		0.0036		0.0036		0.0036		0.0036
g=40		0.0039		0.0039		0.0039		0.0039		0.0039
g=45		0.0041		0.0041		0.0041		0.0041		0.0041
g=50		0.0041		0.0041		0.0041		0.0041		0.0041
g=100		0.0058		0.0058		0.0058		0.0058		0.0058
g=200		0.0137		0.0137		0.0137		0.0137		
g=300		0.0230		0.0230		0.0230				
g=400		0.0324		0.0324						

Concluding remarks about the simulated data from a scaled triangular distribution with $1 - p = 0.1$, $n = 2000$ and $[a, b] = [0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not significantly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions. The normal kernel is used in most of the results in this study.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is recommended for small to moderate sample sizes n.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended with preference given to the Cramér-von-Mises test.
- For both of the above-mentioned goodness-of-fit tests, a level of significance of 1% or 5% can be used, with preference given to $\alpha = 5\%$ when the sample size is small to moderate.
- The choices of $4 \leq r \leq 8$ and $5 \leq g \leq 10$ are recommended, but smaller values of r and g will perform better for small sample sizes.

Tables 4.285 – 4.286 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a, b] = [0.3\,,\,0.7]$.

Table 4.285: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.05$.*

	r=2		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=2	0.30	0.71	0.28	0.72
g=3	0.29	0.71	0.28	0.72
g=4	0.29	0.72	0.28	0.72
g=5	0.28	0.72	0.28	0.73

Table 4.286: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Cramér-von-Mises goodness-of-fit test with $\alpha = 0.05$.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.25	0.75	0.25	0.75
g=7	0.25	0.75	0.25	0.75
g=8	0.25	0.75	0.25	0.75
g=9	0.25	0.75	0.25	0.75
g=10	0.25	0.75	0.25	0.75

Remark: *For this data set the estimated off-pulse interval is extremely accurate for the Anderson-Darling goodness-of-fit test, even though it is a relatively small data set.*

4.6.3 Data set parameters: $1 - p = 0.2$, $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$

Figure 4.21 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines on top of the histogram. The study population will now be analysed on a parameter per parameter basis.

Choice of kernel function

The first step of SOPIE is to calculate that point where the kernel density estimator attains its global minimum and the next m local minima. Table 4.287 compares the minima obtained from each of the different kernel functions that is fitted to the data for a single Monte Carlo iteration with $\hat{h}_3$. From this table it is again evident that similar minima (or minima close to each other) are obtained from the different kernel functions that are used in the kernel density estimation. Especially, the Epanechnikov and Swanepoel kernels are quite close to each other. In some cases, several minima overlap for the kernel functions, except that they are obtained in a different order.

It is therefore still argued that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel function was used for all results of this study population.

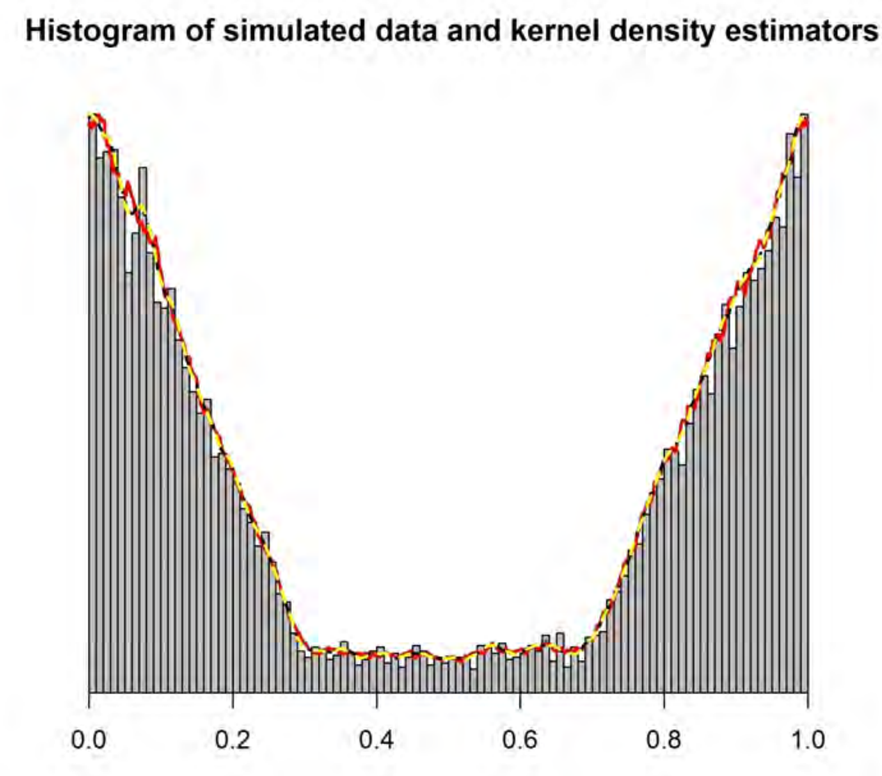

Figure 4.21: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.14$.

Table 4.287: *Minima comparison for different kernel functions for a single Monte Carlo repetition.*

	Swanepoel kernel	Epanechnikov kernel	Normal kernel
1st local min.	0.5195	0.5195	0.5165
2nd local min.	0.5205	0.5205	0.5185
3rd local min.	0.5185	0.5185	0.4905
4th local min.	0.5215	0.5165	0.5175
5th local min.	0.5175	0.5215	0.5115
6th local min.	0.5165	0.5175	0.5105
7th local min.	0.5225	0.5225	0.4925
8th local min.	0.5235	0.5235	0.4915
9th local min.	0.5155	0.5245	0.5125
10th local min.	0.5245	0.4905	0.5135
11th local min.	0.5255	0.4915	0.5145
12th local min.	0.4855	0.5155	0.5195
13th local min.	0.4935	0.5255	0.4875
14th local min.	0.4915	0.4895	0.4885
15th local min.	0.4905	0.4925	0.5095
16th local min.	0.4925	0.4985	0.4865
17th local min.	0.4865	0.4995	0.4935
18th local min.	0.4845	0.5005	0.5155
19th local min.	0.5145	0.4935	0.4895
20th local min.	0.4835	0.4815	0.5205

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus.

Tables 4.288 – 4.289 highlight the values of the estimated bias and MSE for different combinations of m when $g = 6$, $r = 6$ and $\alpha = 0.01$ for two goodness-of-fit tests and two estimated smoothing parameters. It is obvious from the tables that different values of m result in almost equal values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. Therefore, it seems fair to recommend the choice of $m = 1$ as a good choice, as it will save on computing time without affecting the estimation of a and b negatively.

Again it is found that, as far as the bias and MSE are concerned, all four of the goodness-of-fit tests are insensitive and robust to the choice of m, although not all of the tables are shown.

Table 4.288: *Bias for different choices of m for $\hat{h}_1$ and $\hat{h}_3$.*

	$\hat{h}_1$				$\hat{h}_6$			
	Anderson-Darling		**Cramér-von-Mises**		**Anderson-Darling**		**Cramér-von-Mises**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0198	0.0190	-0.0362	0.0363	-0.0099	0.0202	-0.0368	0.0384
m=2	-0.0195	0.0191	-0.0363	0.0363	-0.0111	0.0202	-0.0373	0.0382
m=3	-0.0191	0.0193	-0.0363	0.0363	-0.0130	0.0206	-0.0374	0.0383
m=4	-0.0191	0.0192	-0.0363	0.0363	-0.0129	0.0208	-0.0374	0.0382
m=5	-0.0190	0.0193	-0.0363	0.0363	-0.0132	0.0209	-0.0374	0.0382
m=6	-0.0190	0.0194	-0.0363	0.0363	-0.0139	0.0209	-0.0375	0.0382
m=7	-0.0190	0.0194	-0.0363	0.0363	-0.0142	0.0210	-0.0375	0.0382
m=8	-0.0190	0.0193	-0.0363	0.0363	-0.0145	0.0211	-0.0376	0.0382
m=9	-0.0190	0.0192	-0.0363	0.0363	-0.0149	0.0210	-0.0376	0.0382
m=10	-0.0191	0.0192	-0.0363	0.0363	-0.0152	0.0210	-0.0376	0.0382
m=11	-0.0190	0.0192	-0.0363	0.0363	-0.0156	0.0209	-0.0376	0.0381
m=12	-0.0191	0.0193	-0.0363	0.0363	-0.0157	0.0209	-0.0376	0.0382
m=13	-0.0192	0.0193	-0.0363	0.0363	-0.0158	0.0210	-0.0377	0.0382
m=14	-0.0192	0.0193	-0.0363	0.0363	-0.0161	0.0210	-0.0377	0.0382
m=15	-0.0192	0.0193	-0.0363	0.0363	-0.0163	0.0209	-0.0377	0.0382
m=16	-0.0193	0.0193	-0.0363	0.0363	-0.0164	0.0209	-0.0377	0.0382
m=17	-0.0194	0.0193	-0.0363	0.0363	-0.0166	0.0210	-0.0377	0.0382
m=18	-0.0194	0.0194	-0.0363	0.0363	-0.0167	0.0210	-0.0378	0.0382
m=19	-0.0195	0.0194	-0.0363	0.0364	-0.0167	0.0210	-0.0378	0.0382
m=20	-0.0195	0.0194	-0.0363	0.0364	-0.0167	0.0210	-0.0378	0.0382

Table 4.289: *MSE for different choices of m for $\hat{h}_1$ and $\hat{h}_6$.*

	$\hat{h}_1$				$\hat{h}_6$			
	Anderson-Darling		**Cramér-von-Mises**		**Anderson-Darling**		**Cramér-von-Mises**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0011	0.0013	0.0014	0.0014	0.0029	0.0015	0.0014	0.0015
m=2	0.0011	0.0013	0.0014	0.0014	0.0031	0.0014	0.0014	0.0015
m=3	0.0012	0.0012	0.0014	0.0014	0.0027	0.0014	0.0014	0.0015
m=4	0.0012	0.0012	0.0014	0.0014	0.0027	0.0013	0.0014	0.0015
m=5	0.0012	0.0012	0.0014	0.0014	0.0027	0.0013	0.0014	0.0015
m=6	0.0012	0.0012	0.0014	0.0014	0.0026	0.0013	0.0015	0.0015
m=7	0.0012	0.0012	0.0014	0.0014	0.0025	0.0013	0.0015	0.0015
m=8	0.0012	0.0012	0.0014	0.0014	0.0025	0.0013	0.0015	0.0015
m=9	0.0012	0.0012	0.0014	0.0014	0.0024	0.0013	0.0015	0.0015
m=10	0.0012	0.0012	0.0014	0.0014	0.0023	0.0013	0.0015	0.0015
m=11	0.0012	0.0012	0.0014	0.0014	0.0022	0.0013	0.0015	0.0015
m=12	0.0012	0.0012	0.0014	0.0014	0.0022	0.0013	0.0015	0.0015
m=13	0.0012	0.0012	0.0014	0.0014	0.0022	0.0013	0.0015	0.0015
m=14	0.0012	0.0012	0.0014	0.0014	0.0021	0.0013	0.0015	0.0015
m=15	0.0012	0.0012	0.0014	0.0014	0.0021	0.0013	0.0015	0.0015
m=16	0.0012	0.0012	0.0014	0.0014	0.0021	0.0013	0.0015	0.0015
m=17	0.0012	0.0012	0.0014	0.0014	0.0020	0.0013	0.0015	0.0015
m=18	0.0012	0.0012	0.0014	0.0014	0.0020	0.0013	0.0015	0.0015
m=19	0.0012	0.0012	0.0014	0.0014	0.0020	0.0013	0.0015	0.0015
m=20	0.0012	0.0012	0.0014	0.0014	0.0020	0.0013	0.0015	0.0015

Choice of estimated smoothing parameters

Tables 4.290 and 4.291 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were made for different values of r and g, without showing the results.

Table 4.290: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 6$ and $r = 6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0198	0.0190	-0.0362	0.0363	-0.0224	0.0201	-0.0270	0.0275
$\hat{h}_2$	-0.0122	0.0180	-0.0364	0.0361	-0.0164	0.0178	-0.0288	0.0296
$\hat{h}_3$	-0.0122	0.0179	-0.0361	0.0362	-0.0140	0.0194	-0.0265	0.0278
$\hat{h}_4$	-0.0122	0.0179	-0.0361	0.0362	-0.0140	0.0194	-0.0265	0.0278
$\hat{h}_5$	-0.0202	0.0209	-0.0365	0.0364	-0.0219	0.0223	-0.0273	0.0275
$\hat{h}_6$	-0.0099	0.0202	-0.0368	0.0384	-0.0148	0.0232	-0.0301	0.0285
$\hat{h}_7$	-0.0099	0.0202	-0.0368	0.0384	-0.0148	0.0232	-0.0301	0.0285
$\hat{h}_8$	-0.0111	0.0166	-0.0365	0.0364	-0.0132	0.0194	-0.0300	0.0291
$\hat{h}_9$	-0.0099	0.0202	-0.0368	0.0384	-0.0148	0.0232	-0.0301	0.0285

Table 4.291: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 6$ and $r = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0011	0.0013	0.0014	0.0014	0.0014	0.0018	0.0019	0.0019
$\hat{h}_2$	0.0023	0.0014	0.0014	0.0013	0.0026	0.0024	0.0022	0.0019
$\hat{h}_3$	0.0024	0.0014	0.0013	0.0014	0.0033	0.0020	0.0027	0.0022
$\hat{h}_4$	0.0024	0.0014	0.0013	0.0014	0.0033	0.0020	0.0027	0.0022
$\hat{h}_5$	0.0010	0.0009	0.0014	0.0014	0.0015	0.0014	0.0018	0.0017
$\hat{h}_6$	0.0029	0.0015	0.0014	0.0015	0.0030	0.0017	0.0016	0.0018
$\hat{h}_7$	0.0029	0.0015	0.0014	0.0015	0.0030	0.0017	0.0016	0.0018
$\hat{h}_8$	0.0028	0.0016	0.0014	0.0014	0.0036	0.0020	0.0020	0.0021
$\hat{h}_9$	0.0029	0.0015	0.0014	0.0015	0.0030	0.0017	0.0016	0.0018

It is found that the values of $\hat{h}_3$ and $\hat{h}_4$ are very close to each other, resulting in nearly equal values of the estimated bias and MSE. Almost equal values are also found for $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$, which explains why the results for these $\hat{h}$ combinations are almost equal in the tables. When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero.

When comparing the estimated MSE, $\hat{h}_1$ and $\hat{h}_5$ result in marginally smaller values of the estimated MSE, even for different values of r and g. It must be said that the difference between the best and the worst choice of $\hat{h}$ is very small, i.e., none of the estimated smoothing parameters performs far worse than any of the others. In the light of this, it is recommended to use any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ as a good choice of the estimated smoothing parameter.

Choice of goodness-of-fit test

Some of the tables provided earlier can be used to assess the goodness-of-fit tests. For a specific comparison of the goodness-of-fit tests, the reader can inspect Tables 4.292 – 4.293 for a comparison of the estimated bias of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and $r = 6$ for two of the estimated smoothing parameters $\hat{h}$. Tables 4.294 – 4.295 compare the MSE. Different combinations of g and r were also chosen to investigate the effect of g and r, although the results are not presented.

In terms of the estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs slightly worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests perform equally well for $r = 6$, followed by the other two tests. For $r = 2$, the Cramér-von-Mises goodness-of-fit test is superior in comparison with the other goodness-of-fit tests.

Due to the importance of the MSE, it is recommended to use both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests.

Table 4.292: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0198	0.0190	-0.0362	0.0363	-0.0224	0.0201	-0.0270	0.0275
g=7	-0.0205	0.0196	-0.0363	0.0364	-0.0227	0.0218	-0.0286	0.0287
g=8	-0.0207	0.0202	-0.0364	0.0365	-0.0231	0.0219	-0.0287	0.0290
g=9	-0.0207	0.0209	-0.0365	0.0366	-0.0238	0.0224	-0.0294	0.0292
g=10	-0.0212	0.0217	-0.0366	0.0367	-0.0243	0.0233	-0.0297	0.0298
g=20	-0.0232	0.0237	-0.0376	0.0375	-0.0259	0.0262	-0.0330	0.0325

Table 4.293: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0128	-0.0090	-0.0354	0.0358	0.0051	-0.0022	0.0042	-0.0005
g=2	0.0022	-0.0027	-0.0355	0.0359	-0.0021	0.0031	-0.0066	0.0067
g=3	-0.0018	0.0031	-0.0356	0.0360	-0.0068	0.0068	-0.0122	0.0113
g=4	-0.0050	0.0053	-0.0357	0.0361	-0.0107	0.0086	-0.0139	0.0143
g=5	-0.0068	0.0061	-0.0358	0.0362	-0.0110	0.0106	-0.0157	0.0166
g=20	-0.0198	0.0183	-0.0376	0.0375	-0.0227	0.0213	-0.0276	0.0280

Table 4.294: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0011	0.0013	0.0014	0.0014	0.0014	0.0018	0.0019	0.0019
g=7	0.0010	0.0012	0.0014	0.0014	0.0014	0.0015	0.0017	0.0018
g=8	0.0010	0.0011	0.0014	0.0014	0.0013	0.0015	0.0017	0.0018
g=9	0.0011	0.0010	0.0014	0.0014	0.0013	0.0015	0.0017	0.0018
g=10	0.0010	0.0009	0.0014	0.0014	0.0012	0.0014	0.0017	0.0017
g=20	0.0009	0.0008	0.0014	0.0014	0.0012	0.0012	0.0014	0.0016

Table 4.295: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0069	0.0056	0.0014	0.0013	0.0062	0.0054	0.0076	0.0066
g=2	0.0048	0.0048	0.0014	0.0013	0.0048	0.0044	0.0054	0.0053
g=3	0.0041	0.0039	0.0014	0.0013	0.0040	0.0038	0.0044	0.0045
g=4	0.0035	0.0035	0.0014	0.0013	0.0034	0.0037	0.0042	0.0040
g=5	0.0033	0.0034	0.0014	0.0013	0.0034	0.0033	0.0038	0.0036
g=20	0.0014	0.0017	0.0014	0.0014	0.0017	0.0018	0.0021	0.0021

Choice of the significance level α

Several tables are constructed to investigate the effect of α in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. Tables 4.296 – 4.297 compare the estimated bias and MSE for two of the goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 6$. Tables 4.298 – 4.299 represent a similar comparison with the only change of tuning parameter being $r = 2$. Several more comparisons are made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

From all of these tables it is concluded that – for the Cramér-von-Mises goodness-of-fit test – the estimated MSE is almost insensitive to the choice of α. The estimated bias is slightly more sensitive to the different choices of α. The estimated MSE for the Anderson-Darling goodness-of-fit test is slightly more sensitive to different α-values than the Cramér-von-Mises goodness-of-fit test. The largest estimated MSE-values are obtained when $\alpha = 0.1$. The same is true for the estimated bias, i.e., when $\alpha = 0.1$, the Anderson-Darling goodness-of-fit test has the largest estimated bias.

Table 4.296: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 6$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.02	0.02	0.00	0.00	-0.00	-0.00	0.00	0.00	0.02	-0.02	0.01	0.01
g=7	-0.02	0.02	0.00	0.00	-0.00	0.00	0.00	0.00	0.02	-0.01	0.01	0.01
g=8	-0.02	0.02	0.00	0.00	-0.00	0.00	0.00	0.00	0.01	-0.01	0.01	0.00
g=9	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.01	-0.01	0.00	0.00
g=10	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00	0.01	-0.01	0.00	0.00
g=20	-0.02	0.02	0.00	0.00	-0.01	0.02	0.00	0.00	-0.00	0.00	0.00	0.00

Table 4.297: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 6$ and $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.04	0.04	0.00	0.00	-0.03	0.04	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.04	0.04	0.00	0.00	-0.04	0.04	0.00	0.00	-0.04	0.04	0.00	0.00

Table 4.298: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				**$\alpha = 0.05$**				**$\alpha = 0.10$**			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.01	-0.01	0.01	0.01	0.08	-0.08	0.02	0.02	0.13	-0.13	0.03	0.03
g=2	0.00	-0.00	0.00	0.00	0.07	-0.06	0.02	0.01	0.11	-0.11	0.02	0.02
g=3	-0.00	0.00	0.00	0.00	0.06	-0.05	0.01	0.01	0.09	-0.09	0.02	0.02
g=4	-0.01	0.01	0.00	0.00	0.05	-0.04	0.01	0.01	0.08	-0.08	0.02	0.02
g=5	-0.01	0.01	0.00	0.00	0.04	-0.03	0.01	0.01	0.08	-0.07	0.02	0.02
g=20	-0.02	0.02	0.00	0.00	0.01	-0.00	0.00	0.00	0.03	-0.02	0.01	0.01

Table 4.299: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				**$\alpha = 0.05$**				**$\alpha = 0.10$**			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.04	0.04	0.00	0.00	-0.04	0.04	0.00	0.00	-0.04	0.04	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

Tables 4.300 – 4.307 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests when $\hat{h}_1$ is chosen. In all of the tables the normal kernel is used, with $\alpha = 0.01$ and $m = 1$ kept constant.

In terms of estimated bias and MSE, the Cramér-von-Mises goodness-of-fit test is relatively robust against different choices of r and g, except when g gets very large, i.e., $g > 50$. For both goodness-of-fit tests, small values of r and g result in estimated bias-values that are close to zero, i.e., $2 \leq g \leq 5$ with $2 \leq r \leq 4$. For the Anderson-Darling goodness-of-fit test, the smallest values of the MSE are obtained when $6 \leq g \leq 10$ and $6 \leq r \leq 10$.

In conclusion, it is recommended to use a value of g in the range from 5 to 10, and for r, values from 4 to 8, although smaller values of r and g can also be used.

Table 4.300: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0230	0.0128	0.0047	-0.0002	-0.0040					
g=2	0.0188	0.0022	-0.0044	-0.0070	-0.0109					
g=3	0.0147	-0.0018	-0.0087	-0.0127	-0.0142					
g=4	0.0131	-0.0050	-0.0104	-0.0135	-0.0156					
g=5	0.0095	-0.0068	-0.0129	-0.0153	-0.0171					
g=6						-0.0198	-0.0202	-0.0205	-0.0209	-0.0210
g=7						-0.0205	-0.0206	-0.0211	-0.0212	-0.0212
g=8						-0.0207	-0.0212	-0.0212	-0.0216	-0.0218
g=9						-0.0207	-0.0213	-0.0214	-0.0219	-0.0219
g=10						-0.0212	-0.0214	-0.0218	-0.0219	-0.0219
g=20		-0.0198		-0.0227		-0.0232		-0.0235		-0.0236
g=25		-0.0214		-0.0236		-0.0241		-0.0244		-0.0247
g=30		-0.0214		-0.0241		-0.0246		-0.0253		-0.0254
g=35		-0.0228		-0.0250		-0.0256		-0.0260		-0.0262
g=40		-0.0241		-0.0256		-0.0261		-0.0267		-0.0267
g=45		-0.0246		-0.0260		-0.0263		-0.0269		-0.0271
g=50		-0.0256		-0.0269		-0.0276		-0.0278		-0.0278
g=100		-0.0317		-0.0330		-0.0330		-0.0330		-0.0330
g=200		-0.0400		-0.0400		-0.0400		-0.0400		-0.0400
g=300		-0.0480		-0.0480		-0.0480		-0.0480		-0.0480
g=400		-0.0518		-0.0518		-0.0518		-0.0518		-0.0518
g=500		-0.0652		-0.0652		-0.0652		-0.0652		-0.0652

Table 4.301: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0198	-0.0090	-0.0025	0.0016	0.0049					
g=2	-0.0150	-0.0027	0.0048	0.0086	0.0108					
g=3	-0.0128	0.0031	0.0083	0.0113	0.0136					
g=4	-0.0106	0.0053	0.0110	0.0141	0.0156					
g=5	-0.0063	0.0061	0.0123	0.0158	0.0174					
g=6						0.0190	0.0197	0.0201	0.0210	0.0213
g=7						0.0196	0.0201	0.0209	0.0213	0.0214
g=8						0.0202	0.0208	0.0213	0.0216	0.0218
g=9						0.0209	0.0216	0.0217	0.0218	0.0222
g=10						0.0217	0.0218	0.0218	0.0220	0.0222
g=20		0.0183		0.0229		0.0237		0.0239		0.0241
g=25		0.0203		0.0240		0.0246		0.0252		0.0254
g=30		0.0226		0.0244		0.0254		0.0255		0.0256
g=35		0.0232		0.0255		0.0264		0.0265		0.0265
g=40		0.0243		0.0260		0.0264		0.0266		0.0268
g=45		0.0251		0.0271		0.0272		0.0275		0.0275
g=50		0.0261		0.0276		0.0282		0.0282		0.0282
g=100		0.0320		0.0327		0.0327		0.0327		0.0327
g=200		0.0390		0.0390		0.0390		0.0390		0.0390
g=300		0.0495		0.0495		0.0495		0.0495		0.0495
g=400		0.0518		0.0518		0.0518		0.0518		0.0518
g=500		0.0640		0.0640		0.0640		0.0640		0.0640

Table 4.302: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0086	0.0069	0.0053	0.0044	0.0037					
g=2	0.0079	0.0048	0.0036	0.0032	0.0026					
g=3	0.0072	0.0041	0.0029	0.0022	0.0019					
g=4	0.0067	0.0035	0.0026	0.0021	0.0017					
g=5	0.0064	0.0033	0.0023	0.0018	0.0015					
g=6						0.0011	0.0011	0.0010	0.0010	0.0010
g=7						0.0010	0.0010	0.0010	0.0010	0.0010
g=8						0.0010	0.0010	0.0010	0.0009	0.0009
g=9						0.0011	0.0010	0.0010	0.0009	0.0009
g=10						0.0010	0.0010	0.0009	0.0009	0.0009
g=20		0.0014		0.0010		0.0009		0.0009		0.0009
g=25		0.0012		0.0010		0.0009		0.0009		0.0008
g=30		0.0014		0.0010		0.0009		0.0008		0.0008
g=35		0.0013		0.0010		0.0009		0.0008		0.0008
g=40		0.0012		0.0010		0.0009		0.0008		0.0008
g=45		0.0012		0.0010		0.0010		0.0009		0.0008
g=50		0.0013		0.0010		0.0009		0.0009		0.0009
g=100		0.0014		0.0012		0.0012		0.0012		0.0012
g=200		0.0019		0.0019		0.0019		0.0019		0.0019
g=300		0.0027		0.0027		0.0027		0.0027		0.0027
g=400		0.0031		0.0031		0.0031		0.0031		0.0031
g=500		0.0046		0.0046		0.0046		0.0046		0.0046

Table 4.303: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0073	0.0056	0.0046	0.0040	0.0035					
g=2	0.0065	0.0048	0.0035	0.0029	0.0025					
g=3	0.0062	0.0039	0.0029	0.0024	0.0021					
g=4	0.0060	0.0035	0.0025	0.0020	0.0018					
g=5	0.0053	0.0034	0.0023	0.0018	0.0015					
g=6						0.0013	0.0012	0.0011	0.0009	0.0009
g=7						0.0012	0.0011	0.0010	0.0009	0.0009
g=8						0.0011	0.0010	0.0009	0.0009	0.0008
g=9						0.0010	0.0009	0.0009	0.0009	0.0008
g=10						0.0009	0.0009	0.0009	0.0009	0.0008
g=20		0.0017		0.0009		0.0008		0.0008		0.0008
g=25		0.0015		0.0009		0.0008		0.0007		0.0007
g=30		0.0012		0.0009		0.0008		0.0008		0.0008
g=35		0.0013		0.0009		0.0008		0.0008		0.0008
g=40		0.0012		0.0009		0.0009		0.0008		0.0008
g=45		0.0012		0.0009		0.0009		0.0008		0.0008
g=50		0.0011		0.0009		0.0009		0.0009		0.0009
g=100		0.0013		0.0012		0.0012		0.0012		0.0012
g=200		0.0019		0.0019		0.0019		0.0019		0.0019
g=300		0.0027		0.0027		0.0027		0.0027		0.0027
g=400		0.0031		0.0031		0.0031		0.0031		0.0031
g=500		0.0044		0.0044		0.0044		0.0044		0.0044

Table 4.304: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0353	-0.0354	-0.0355	-0.0355	-0.0355					
g=2	-0.0355	-0.0355	-0.0355	-0.0359	-0.0359					
g=3	-0.0356	-0.0356	-0.0360	-0.0360	-0.0360					
g=4	-0.0357	-0.0357	-0.0360	-0.0360	-0.0360					
g=5	-0.0358	-0.0358	-0.0361	-0.0361	-0.0361					
g=6						-0.0362	-0.0362	-0.0362	-0.0362	-0.0362
g=7						-0.0363	-0.0363	-0.0363	-0.0363	-0.0363
g=8						-0.0364	-0.0364	-0.0364	-0.0364	-0.0364
g=9						-0.0365	-0.0365	-0.0365	-0.0365	-0.0365
g=10						-0.0366	-0.0366	-0.0366	-0.0366	-0.0366
g=20		-0.0376		-0.0376		-0.0376		-0.0376		-0.0376
g=25		-0.0381		-0.0381		-0.0381		-0.0381		-0.0381
g=30		-0.0385		-0.0385		-0.0385		-0.0385		-0.0385
g=35		-0.0391		-0.0391		-0.0391		-0.0391		-0.0391
g=40		-0.0394		-0.0394		-0.0394		-0.0394		-0.0394
g=45		-0.0399		-0.0399		-0.0399		-0.0399		-0.0399
g=50		-0.0403		-0.0403		-0.0403		-0.0403		-0.0403
g=100		-0.0445		-0.0445		-0.0445		-0.0445		-0.0445
g=200		-0.0531		-0.0531		-0.0531		-0.0531		-0.0531
g=300		-0.0571		-0.0571		-0.0571		-0.0571		-0.0571
g=400		-0.0633		-0.0633		-0.0633		-0.0633		-0.0633
g=500		-0.0702		-0.0702		-0.0702		-0.0702		-0.0702

Table 4.305: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0357	0.0358	0.0359	0.0359	0.0359					
g=2	0.0358	0.0359	0.0359	0.0360	0.0360					
g=3	0.0359	0.0360	0.0360	0.0360	0.0360					
g=4	0.0360	0.0361	0.0361	0.0361	0.0361					
g=5	0.0362	0.0362	0.0362	0.0362	0.0362					
g=6						0.0363	0.0363	0.0363	0.0363	0.0363
g=7						0.0364	0.0364	0.0364	0.0364	0.0364
g=8						0.0365	0.0365	0.0365	0.0365	0.0365
g=9						0.0366	0.0366	0.0366	0.0366	0.0366
g=10						0.0367	0.0367	0.0367	0.0367	0.0367
g=20		0.0375		0.0375		0.0375		0.0375		0.0375
g=25		0.0381		0.0381		0.0381		0.0381		0.0381
g=30		0.0385		0.0385		0.0385		0.0385		0.0385
g=35		0.0390		0.0390		0.0390		0.0390		0.0390
g=40		0.0395		0.0395		0.0395		0.0395		0.0395
g=45		0.0398		0.0398		0.0398		0.0398		0.0398
g=50		0.0404		0.0404		0.0404		0.0404		0.0404
g=100		0.0448		0.0448		0.0448		0.0448		0.0448
g=200		0.0535		0.0535		0.0535		0.0535		0.0535
g=300		0.0567		0.0567		0.0567		0.0567		0.0567
g=400		0.0642		0.0642		0.0642		0.0642		0.0642
g=500		0.0698		0.0698		0.0698		0.0698		0.0698

Table 4.306: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0014	0.0014	0.0014	0.0014	0.0014					
g=2	0.0014	0.0014	0.0014	0.0013	0.0013					
g=3	0.0014	0.0014	0.0013	0.0013	0.0013					
g=4	0.0014	0.0014	0.0013	0.0013	0.0013					
g=5	0.0014	0.0014	0.0013	0.0013	0.0013					
g=6						0.0014	0.0014	0.0014	0.0014	0.0014
g=7						0.0014	0.0014	0.0014	0.0014	0.0014
g=8						0.0014	0.0014	0.0014	0.0014	0.0014
g=9						0.0014	0.0014	0.0014	0.0014	0.0014
g=10						0.0014	0.0014	0.0014	0.0014	0.0014
g=20		0.0014		0.0014		0.0014		0.0014		0.0014
g=25		0.0015		0.0015		0.0015		0.0015		0.0015
g=30		0.0015		0.0015		0.0015		0.0015		0.0015
g=35		0.0016		0.0016		0.0016		0.0016		0.0016
g=40		0.0016		0.0016		0.0016		0.0016		0.0016
g=45		0.0016		0.0016		0.0016		0.0016		0.0016
g=50		0.0017		0.0017		0.0017		0.0017		0.0017
g=100		0.0020		0.0020		0.0020		0.0020		0.0020
g=200		0.0029		0.0029		0.0029		0.0029		0.0029
g=300		0.0034		0.0034		0.0034		0.0034		0.0034
g=400		0.0043		0.0043		0.0043		0.0043		0.0043
g=500		0.0052		0.0052		0.0052		0.0052		0.0052

Table 4.307: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0013	0.0013	0.0013	0.0013	0.0013					
g=2	0.0013	0.0013	0.0013	0.0013	0.0013					
g=3	0.0013	0.0013	0.0013	0.0013	0.0013					
g=4	0.0013	0.0013	0.0013	0.0013	0.0013					
g=5	0.0013	0.0013	0.0013	0.0013	0.0013					
g=6						0.0014	0.0014	0.0014	0.0014	0.0014
g=7						0.0014	0.0014	0.0014	0.0014	0.0014
g=8						0.0014	0.0014	0.0014	0.0014	0.0014
g=9						0.0014	0.0014	0.0014	0.0014	0.0014
g=10						0.0014	0.0014	0.0014	0.0014	0.0014
g=20		0.0014		0.0014		0.0014		0.0014		0.0014
g=25		0.0015		0.0015		0.0015		0.0015		0.0015
g=30		0.0015		0.0015		0.0015		0.0015		0.0015
g=35		0.0016		0.0016		0.0016		0.0016		0.0016
g=40		0.0016		0.0016		0.0016		0.0016		0.0016
g=45		0.0016		0.0016		0.0016		0.0016		0.0016
g=50		0.0017		0.0017		0.0017		0.0017		0.0017
g=100		0.0021		0.0021		0.0021		0.0021		0.0021
g=200		0.0030		0.0030		0.0030		0.0030		0.0030
g=300		0.0034		0.0034		0.0034		0.0034		0.0034
g=400		0.0044		0.0044		0.0044		0.0044		0.0044
g=500		0.0051		0.0051		0.0051		0.0051		0.0051

Concluding remarks about the simulated data from a scaled triangular distribution with $1 - p = 0.2$, $n = 10000$ and $[a, b] = [0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is recommended.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended, with preference given to the Cramér-von-Mises test.
- For both of the recommended goodness-of-fit tests, a level of significance of 1% or 5% can be used.
- The choices of $4 \leq r \leq 8$ and $5 \leq g \leq 10$ are recommended, but smaller values of r and g can also be used.

Table 4.308 and Table 4.309 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a, b] = [0.3\,,\,0.7]$.

Table 4.308: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.01$.*

	r=2	
	$E(\hat{a})$	$E(\hat{b})$
g=2	0.30	0.70
g=3	0.30	0.70
g=4	0.30	0.70
g=5	0.29	0.71

Table 4.309: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.01$.*

	r=6	
	$E(\hat{a})$	$E(\hat{b})$
g=6	0.28	0.72
g=7	0.28	0.72
g=8	0.28	0.72
g=9	0.28	0.72
g=10	0.28	0.72

Remark: *The estimated off-pulse interval for this data set is very accurate for a wide array of different values of the tuning parameters.*

4.6.4 Data set parameters: $1-p=0.2$, $n=25000$ and $[a,b]=[0.3\,,\,0.7]$

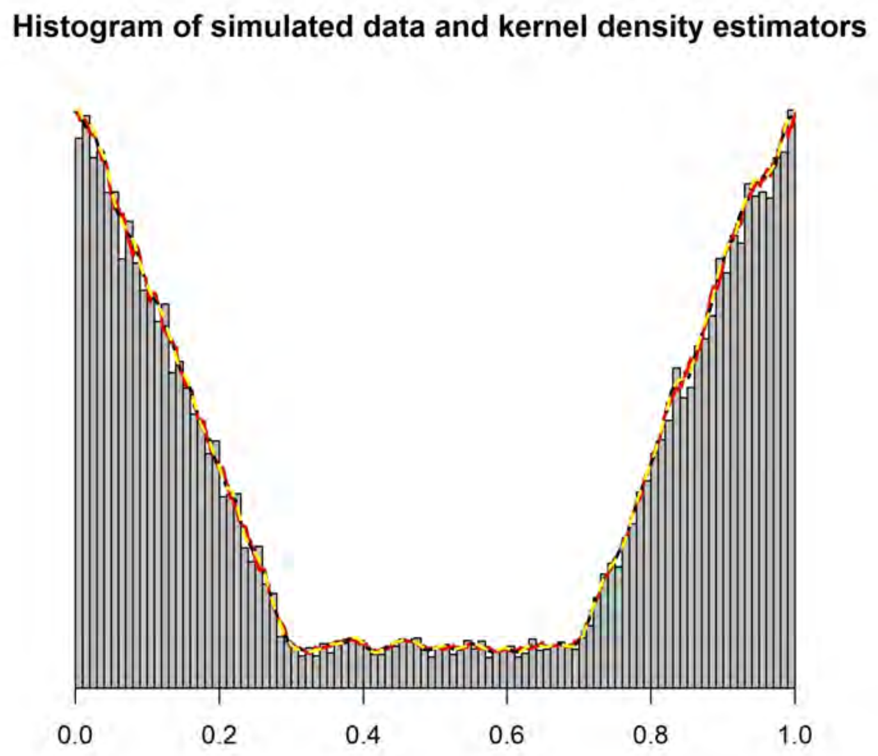

Figure 4.22: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3=0.11$.

Figure 4.22 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis.

Choice of kernel function

The first step of SOPIE is to calculate that point where the kernel density estimator attains its global minimum and the next m local minima. Table 4.310 compares the minima obtained from each of the different kernel functions that is fitted to the data for a single Monte Carlo iteration with $\hat{h}_1$.

From this table it is again evident that similar minima (or minima close to each other) are obtained from the different kernel functions used in the kernel density estimator. Especially, the Epanechnikov and Swanepoel kernels are quite close to each other. In some cases, several minima overlap for the kernel functions, except that they are obtained in a different order.

It is therefore still argued that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel function was

Table 4.310: *Minima comparison for different kernel functions for a single Monte Carlo repetition.*

	Swanepoel kernel	Epanechnikov kernel	Normal kernel
1st local min.	0.6015	0.6025	0.5745
2nd local min.	0.6025	0.6035	0.5715
3rd local min.	0.5965	0.6015	0.5755
4th local min.	0.5955	0.6045	0.5765
5th local min.	0.6005	0.6055	0.5705
6th local min.	0.5975	0.6005	0.5735
7th local min.	0.6035	0.5965	0.5675
8th local min.	0.5995	0.5955	0.5695
9th local min.	0.5945	0.6065	0.5725
10th local min.	0.5985	0.6075	0.5335
11th local min.	0.6045	0.5975	0.5665
12th local min.	0.5935	0.5995	0.5775
13th local min.	0.5925	0.5985	0.5785
14th local min.	0.6055	0.6085	0.5685
15th local min.	0.6065	0.5945	0.5795
16th local min.	0.5915	0.6095	0.5465
17th local min.	0.6075	0.6105	0.5805
18th local min.	0.5905	0.5935	0.5655
19th local min.	0.5895	0.6115	0.6055
20th local min.	0.5885	0.5925	0.6215

used for all results of this study population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m. Tables 4.311 – 4.312 highlight the values of the estimated bias and MSE for different combinations of m when $g = 6$, $r = 6$ and $\alpha = 0.01$ for two goodness-of-fit tests and two estimated smoothing parameters. It can be seen that different values of m result in almost equal values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. It can again be concluded that a small choice of m is preferable, since computing time is reduced.

Choice of estimated smoothing parameters

Tables 4.313 and 4.314 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were made with different values of r and g, although the results are not displayed.

When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero. Also, when comparing the estimated MSE, $\hat{h}_5$ results in marginally smaller values of the estimated MSE, even for different values of r and g. It must be said that the difference between the best and the worst choice of $\hat{h}$ is very small, i.e., none of the estimated smoothing parameters performs far worse than any other one. In the light of this, any one of the previously recommended smoothing parameters may be used, such as $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ (see the definitions of $\hat{h}$ in Table 4.1).

Table 4.311: *Bias for different choices of m for $\hat{h}_1$ and $\hat{h}_6$.*

	$\hat{h}_1$				$\hat{h}_6$			
	Anderson-Darling		**Cramér-von-Mises**		**Anderson-Darling**		**Cramér-von-Mises**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0083	0.0080	-0.0279	0.0274	0.0026	0.0032	-0.0285	0.0281
m=2	-0.0085	0.0081	-0.0279	0.0274	0.0010	0.0038	-0.0285	0.0282
m=3	-0.0084	0.0080	-0.0279	0.0274	0.0003	0.0048	-0.0283	0.0283
m=4	-0.0077	0.0086	-0.0279	0.0274	-0.0008	0.0049	-0.0284	0.0283
m=5	-0.0077	0.0084	-0.0279	0.0274	-0.0016	0.0051	-0.0285	0.0283
m=6	-0.0074	0.0085	-0.0279	0.0274	-0.0014	0.0053	-0.0285	0.0283
m=7	-0.0075	0.0084	-0.0279	0.0274	-0.0017	0.0056	-0.0285	0.0283
m=8	-0.0076	0.0084	-0.0279	0.0274	-0.0023	0.0056	-0.0286	0.0283
m=9	-0.0076	0.0083	-0.0279	0.0274	-0.0025	0.0056	-0.0286	0.0283
m=10	-0.0076	0.0085	-0.0279	0.0274	-0.0026	0.0057	-0.0286	0.0283
m=11	-0.0076	0.0085	-0.0279	0.0274	-0.0031	0.0058	-0.0286	0.0283
m=12	-0.0076	0.0085	-0.0279	0.0274	-0.0030	0.0060	-0.0286	0.0283
m=13	-0.0077	0.0084	-0.0279	0.0274	-0.0032	0.0062	-0.0286	0.0283
m=14	-0.0076	0.0084	-0.0279	0.0274	-0.0035	0.0063	-0.0286	0.0283
m=15	-0.0076	0.0084	-0.0279	0.0274	-0.0036	0.0065	-0.0286	0.0283
m=16	-0.0076	0.0084	-0.0279	0.0275	-0.0038	0.0065	-0.0286	0.0284
m=17	-0.0075	0.0084	-0.0279	0.0275	-0.0039	0.0065	-0.0286	0.0284
m=18	-0.0074	0.0084	-0.0279	0.0275	-0.0041	0.0067	-0.0286	0.0284
m=19	-0.0075	0.0083	-0.0279	0.0275	-0.0043	0.0068	-0.0286	0.0284
m=20	-0.0076	0.0084	-0.0279	0.0275	-0.0044	0.0069	-0.0286	0.0284

Table 4.312: *MSE for different choices of m for $\hat{h}_1$ and $\hat{h}_6$.*

	$\hat{h}_1$				$\hat{h}_6$			
	Anderson-Darling		**Cramér-von-Mises**		**Anderson-Darling**		**Cramér-von-Mises**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0019	0.0019	0.0008	0.0008	0.0049	0.0032	0.0008	0.0008
m=2	0.0019	0.0019	0.0008	0.0008	0.0043	0.0033	0.0008	0.0008
m=3	0.0019	0.0019	0.0008	0.0008	0.0041	0.0030	0.0008	0.0008
m=4	0.0022	0.0018	0.0008	0.0008	0.0039	0.0030	0.0008	0.0008
m=5	0.0022	0.0018	0.0008	0.0008	0.0037	0.0030	0.0008	0.0008
m=6	0.0022	0.0018	0.0008	0.0008	0.0038	0.0029	0.0008	0.0008
m=7	0.0022	0.0018	0.0008	0.0008	0.0037	0.0029	0.0009	0.0008
m=8	0.0022	0.0018	0.0008	0.0008	0.0036	0.0028	0.0009	0.0008
m=9	0.0022	0.0018	0.0008	0.0008	0.0036	0.0028	0.0009	0.0008
m=10	0.0022	0.0018	0.0008	0.0008	0.0036	0.0028	0.0009	0.0008
m=11	0.0022	0.0018	0.0008	0.0008	0.0035	0.0028	0.0009	0.0008
m=12	0.0022	0.0018	0.0008	0.0008	0.0035	0.0028	0.0008	0.0008
m=13	0.0022	0.0018	0.0008	0.0008	0.0035	0.0027	0.0009	0.0008
m=14	0.0022	0.0018	0.0008	0.0008	0.0034	0.0027	0.0009	0.0008
m=15	0.0022	0.0018	0.0008	0.0008	0.0034	0.0027	0.0009	0.0008
m=16	0.0022	0.0018	0.0008	0.0008	0.0034	0.0027	0.0009	0.0008
m=17	0.0022	0.0018	0.0008	0.0008	0.0033	0.0026	0.0009	0.0008
m=18	0.0022	0.0018	0.0008	0.0008	0.0033	0.0026	0.0009	0.0008
m=19	0.0022	0.0018	0.0008	0.0008	0.0032	0.0026	0.0008	0.0008
m=20	0.0022	0.0018	0.0008	0.0008	0.0032	0.0026	0.0008	0.0008

Table 4.313: *Comparison of the bias for different combinations of the estimated smoothing parameter with* $m = 1$*,* $g = 6$ *and* $r = 6$*.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0083	0.0080	-0.0279	0.0274	-0.0096	0.0094	-0.0154	0.0153
$\hat{h}_2$	0.0042	0.0040	-0.0275	0.0275	-0.0047	0.0069	-0.0118	0.0104
$\hat{h}_3$	0.0014	0.0012	-0.0273	0.0274	-0.0070	0.0064	-0.0137	0.0091
$\hat{h}_4$	0.0014	0.0012	-0.0273	0.0274	-0.0070	0.0064	-0.0137	0.0091
$\hat{h}_5$	-0.0093	0.0104	-0.0280	0.0275	-0.0121	0.0119	-0.0170	0.0169
$\hat{h}_6$	0.0026	0.0032	-0.0285	0.0281	-0.0028	0.0080	-0.0177	0.0178
$\hat{h}_7$	0.0026	0.0032	-0.0285	0.0281	-0.0028	0.0080	-0.0177	0.0178
$\hat{h}_8$	0.0043	0.0016	-0.0276	0.0276	-0.0013	0.0063	-0.0101	0.0118
$\hat{h}_9$	0.0026	0.0032	-0.0285	0.0281	-0.0028	0.0080	-0.0177	0.0178

Table 4.314: *Comparison of the MSE for different combinations of the estimated smoothing parameter with* $m = 1$*,* $g = 6$ *and* $r = 6$*.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0019	0.0019	0.0008	0.0008	0.0024	0.0024	0.0027	0.0024
$\hat{h}_2$	0.0048	0.0030	0.0008	0.0008	0.0039	0.0031	0.0037	0.0040
$\hat{h}_3$	0.0044	0.0036	0.0009	0.0008	0.0032	0.0032	0.0034	0.0044
$\hat{h}_4$	0.0044	0.0036	0.0009	0.0008	0.0032	0.0032	0.0034	0.0044
$\hat{h}_5$	0.0018	0.0014	0.0008	0.0008	0.0018	0.0018	0.0022	0.0020
$\hat{h}_6$	0.0049	0.0032	0.0008	0.0008	0.0043	0.0030	0.0022	0.0021
$\hat{h}_7$	0.0049	0.0032	0.0008	0.0008	0.0043	0.0030	0.0022	0.0021
$\hat{h}_8$	0.0050	0.0035	0.0008	0.0008	0.0047	0.0034	0.0042	0.0037
$\hat{h}_9$	0.0049	0.0032	0.0008	0.0008	0.0043	0.0030	0.0022	0.0021

Choice of goodness-of-fit test

Some of the tables provided earlier can be used to assess the goodness-of-fit tests. For a specific comparison of the goodness-of-fit tests, the reader can inspect Tables 4.315 – 4.316 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and with $\hat{h}_1$.

In terms of the estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs slightly worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs slightly better than the Anderson-Darling goodness-of-fit test, followed by the other two tests. Therefore, both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended.

Table 4.315: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_1$, *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.0083	0.0080	-0.0279	0.0274	-0.0096	0.0094	-0.0154	0.0153
g=7	-0.0098	0.0095	-0.0280	0.0275	-0.0102	0.0108	-0.0164	0.0160
g=8	-0.0110	0.0105	-0.0280	0.0275	-0.0115	0.0118	-0.0171	0.0169
g=9	-0.0110	0.0107	-0.0280	0.0276	-0.0139	0.0120	-0.0182	0.0180
g=10	-0.0121	0.0115	-0.0281	0.0276	-0.0135	0.0132	-0.0191	0.0182
g=20	-0.0162	0.0156	-0.0285	0.0280	-0.0170	0.0168	-0.0225	0.0217

Table 4.316: *Comparison of the estimated MSE for different goodness-of-fit tests with* $\alpha = 0.01$, $m = 1$, $r = 6$ *and* $\hat{h}_1$, *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0019	0.0019	0.0008	0.0008	0.0024	0.0024	0.0027	0.0024
g=7	0.0017	0.0016	0.0008	0.0008	0.0023	0.0021	0.0026	0.0022
g=8	0.0015	0.0015	0.0008	0.0008	0.0020	0.0019	0.0025	0.0020
g=9	0.0015	0.0015	0.0008	0.0008	0.0015	0.0019	0.0023	0.0019
g=10	0.0013	0.0013	0.0008	0.0008	0.0017	0.0017	0.0022	0.0019
g=20	0.0007	0.0008	0.0008	0.0008	0.0012	0.0012	0.0015	0.0015

Choice of the significance level α

Several tables are constructed to investigate the effect of α in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. Tables 4.317 – 4.318 compare the estimated bias and MSE for two of the goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 6$. Several more comparisons were made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

From all of these tables it is concluded that – for the Cramér-von-Mises goodness-of-fit test – the estimated bias and MSE are almost insensitive to the choice of α. The estimated MSE for the Anderson-Darling goodness-of-fit test is slightly more sensitive to different α-values than the Cramér-von-Mises goodness-of-fit test. The largest estimated MSE-values are obtained when $\alpha = 0.1$. The same is true for the estimated bias, i.e., when $\alpha = 0.1$, the Anderson-Darling goodness-of-fit test has the largest estimated bias.

Table 4.317: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 6$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.01	0.01	0.00	0.00	0.02	-0.02	0.01	0.01	0.06	-0.05	0.01	0.01
g=7	-0.01	0.01	0.00	0.00	0.02	-0.02	0.01	0.01	0.05	-0.05	0.01	0.01
g=8	-0.01	0.01	0.00	0.00	0.02	-0.02	0.01	0.01	0.05	-0.04	0.01	0.01
g=9	-0.01	0.01	0.00	0.00	0.02	-0.01	0.01	0.00	0.04	-0.04	0.01	0.01
g=10	-0.01	0.01	0.00	0.00	0.01	-0.01	0.01	0.00	0.04	-0.04	0.01	0.01
g=20	-0.02	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.02	-0.01	0.01	0.00

Table 4.318: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 6$ and $\hat{h}_1$ (Cramér-von-Mises goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=7	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=8	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=9	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=10	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00
g=20	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00	-0.03	0.03	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

Tables 4.319 – 4.322 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling goodness-of-fit test when $\hat{h}_1$ is chosen. In all of the tables the normal kernel is used, with $\alpha = 0.01$ and $m = 1$ kept constant.

For the Anderson-Darling goodness-of-fit test, small values of r and g result in estimated bias values that are close to zero, i.e., $2 \leq g \leq 5$ with $2 \leq r \leq 4$. For the Anderson-Darling goodness-of-fit test, the smallest values of the MSE are obtained when $20 \leq g \leq 50$ and $6 \leq r \leq 10$. Compared to the previous study populations, the values of g for which the smallest MSE-values are found, are slightly larger. The reason for this can be explained by the large sample size n of this study population.

In conclusion, it is recommended to use a value of g in the range from 5 to 10, and for r, values from 6 to 8, although larger values of r and g can also be used when the sample size n is large.

Table 4.319: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0467	0.0320	0.0255	0.0206	0.0167					
g=2	0.0410	0.0222	0.0163	0.0097	0.0047					
g=3	0.0369	0.0191	0.0094	0.0035	0.0003					
g=4	0.0329	0.0146	0.0052	-0.0002	-0.0040					
g=5	0.0299	0.0116	0.0020	-0.0030	-0.0057					
g=6						-0.0083	-0.0099	-0.0111	-0.0118	-0.0128
g=7						-0.0098	-0.0112	-0.0120	-0.0125	-0.0137
g=8						-0.0110	-0.0119	-0.0134	-0.0139	-0.0144
g=9						-0.0110	-0.0125	-0.0133	-0.0143	-0.0147
g=10						-0.0121	-0.0132	-0.0142	-0.0151	-0.0154
g=20		-0.0057		-0.0142		-0.0162		-0.0169		-0.0177
g=25		-0.0078		-0.0152		-0.0165		-0.0178		-0.0180
g=30		-0.0102		-0.0163		-0.0177		-0.0182		-0.0182
g=35		-0.0117		-0.0168		-0.0179		-0.0183		-0.0185
g=40		-0.0119		-0.0173		-0.0187		-0.0187		-0.0191
g=45		-0.0144		-0.0180		-0.0187		-0.0192		-0.0195
g=50		-0.0149		-0.0186		-0.0191		-0.0196		-0.0196
g=100		-0.0193		-0.0221		-0.0222		-0.0225		-0.0225
g=200		-0.0256		-0.0266		-0.0266		-0.0266		-0.0266
g=300		-0.0308		-0.0310		-0.0310		-0.0310		-0.0310
g=400		-0.0340		-0.0343		-0.0343		-0.0343		-0.0343
g=500		-0.0361		-0.0367		-0.0367		-0.0367		-0.0367

Table 4.320: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0424	-0.0282	-0.0208	-0.0170	-0.0127					
g=2	-0.0367	-0.0209	-0.0126	-0.0083	-0.0049					
g=3	-0.0319	-0.0149	-0.0077	-0.0033	-0.0004					
g=4	-0.0280	-0.0114	-0.0043	-0.0009	0.0011					
g=5	-0.0266	-0.0089	-0.0017	0.0008	0.0034					
g=6						0.0080	0.0094	0.0107	0.0118	0.0126
g=7						0.0095	0.0105	0.0115	0.0122	0.0126
g=8						0.0105	0.0115	0.0125	0.0130	0.0137
g=9						0.0107	0.0119	0.0131	0.0140	0.0142
g=10						0.0115	0.0130	0.0138	0.0144	0.0146
g=20		0.0054		0.0131		0.0156		0.0163		0.0164
g=25		0.0088		0.0149		0.0165		0.0167		0.0170
g=30		0.0089		0.0156		0.0169		0.0173		0.0173
g=35		0.0105		0.0161		0.0171		0.0175		0.0181
g=40		0.0117		0.0170		0.0176		0.0177		0.0183
g=45		0.0130		0.0174		0.0179		0.0184		0.0189
g=50		0.0144		0.0177		0.0183		0.0191		0.0191
g=100		0.0197		0.0211		0.0217		0.0219		0.0219
g=200		0.0251		0.0261		0.0261		0.0261		0.0261
g=300		0.0308		0.0311		0.0311		0.0311		0.0311
g=400		0.0337		0.0339		0.0339		0.0339		0.0339
g=500		0.0363		0.0366		0.0366		0.0366		0.0366

Table 4.321: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0137	0.0104	0.0091	0.0079	0.0071					
g=2	0.0124	0.0082	0.0070	0.0055	0.0045					
g=3	0.0117	0.0076	0.0054	0.0043	0.0036					
g=4	0.0107	0.0066	0.0046	0.0035	0.0027					
g=5	0.0098	0.0059	0.0039	0.0030	0.0024					
g=6						0.0019	0.0016	0.0014	0.0014	0.0012
g=7						0.0017	0.0015	0.0013	0.0012	0.0010
g=8						0.0015	0.0013	0.0011	0.0010	0.0009
g=9						0.0015	0.0012	0.0011	0.0009	0.0008
g=10						0.0013	0.0011	0.0010	0.0008	0.0007
g=20		0.0027		0.0011		0.0007		0.0006		0.0005
g=25		0.0023		0.0009		0.0007		0.0005		0.0005
g=30		0.0019		0.0008		0.0006		0.0005		0.0005
g=35		0.0017		0.0008		0.0006		0.0006		0.0005
g=40		0.0017		0.0007		0.0005		0.0005		0.0005
g=45		0.0012		0.0007		0.0006		0.0005		0.0005
g=50		0.0012		0.0007		0.0006		0.0005		0.0005
g=100		0.0011		0.0006		0.0006		0.0006		0.0006
g=200		0.0009		0.0008		0.0008		0.0008		0.0008
g=300		0.0011		0.0011		0.0011		0.0011		0.0011
g=400		0.0014		0.0013		0.0013		0.0013		0.0013
g=500		0.0016		0.0015		0.0015		0.0015		0.0015

Table 4.322: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.01$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0137	0.0104	0.0091	0.0079	0.0071					
g=2	0.0124	0.0082	0.0070	0.0055	0.0045					
g=3	0.0117	0.0076	0.0054	0.0043	0.0036					
g=4	0.0107	0.0066	0.0046	0.0035	0.0027					
g=5	0.0098	0.0059	0.0039	0.0030	0.0024					
g=6						0.0019	0.0016	0.0014	0.0014	0.0012
g=7						0.0017	0.0015	0.0013	0.0012	0.0010
g=8						0.0015	0.0013	0.0011	0.0010	0.0009
g=9						0.0015	0.0012	0.0011	0.0009	0.0008
g=10						0.0013	0.0011	0.0010	0.0008	0.0007
g=20		0.0027		0.0011		0.0007		0.0006		0.0005
g=25		0.0023		0.0009		0.0007		0.0005		0.0005
g=30		0.0019		0.0008		0.0006		0.0005		0.0005
g=35		0.0017		0.0008		0.0006		0.0006		0.0005
g=40		0.0017		0.0007		0.0005		0.0005		0.0005
g=45		0.0012		0.0007		0.0006		0.0005		0.0005
g=50		0.0012		0.0007		0.0006		0.0005		0.0005
g=100		0.0011		0.0006		0.0006		0.0006		0.0006
g=200		0.0009		0.0008		0.0008		0.0008		0.0008
g=300		0.0011		0.0011		0.0011		0.0011		0.0011
g=400		0.0014		0.0013		0.0013		0.0013		0.0013
g=500		0.0016		0.0015		0.0015		0.0015		0.0015

Concluding remarks about the simulated data from a scaled triangular distribution with $1 - p = 0.2$, $n = 25000$ and $[a, b] = [0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1 - \hat{h}_5$ is recommended.
- Both the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests are recommended with preference given to the Cramér-von-Mises test.
- For both of these goodness-of-fit tests, a level of significance of 1% or 5% can be used.
- The choices of $6 \leq r \leq 8$ and $5 \leq g \leq 10$ are recommended, but larger values of r and g can also be used, especially if the sample size is large.

Table 4.323 and Table 4.324 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a, b] = [0.3\,,\,0.7]$.

Table 4.323: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.01$.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.29	0.70	0.29	0.71
g=7	0.29	0.70	0.29	0.71
g=8	0.29	0.71	0.29	0.71
g=9	0.29	0.71	0.29	0.71
g=10	0.29	0.71	0.29	0.71
g=20	0.28	0.71	0.28	0.72
g=25	0.28	0.72	0.28	0.72
g=30	0.28	0.72	0.28	0.72
g=35	0.28	0.72	0.28	0.72
g=40	0.28	0.72	0.28	0.72
g=45	0.28	0.72	0.28	0.72
g=50	0.28	0.72	0.28	0.72

Table 4.324: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Cramér-von-Mises goodness-of-fit test with $\alpha = 0.01$.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=6	0.27	0.73	0.27	0.73
g=7	0.27	0.73	0.27	0.73
g=8	0.27	0.73	0.27	0.73
g=9	0.27	0.73	0.27	0.73
g=10	0.27	0.73	0.27	0.73
g=20	0.27	0.73	0.27	0.73
g=25	0.27	0.73	0.27	0.73
g=30	0.27	0.73	0.27	0.73
g=35	0.27	0.73	0.27	0.73
g=40	0.27	0.73	0.27	0.73
g=45	0.27	0.73	0.27	0.73
g=50	0.27	0.73	0.27	0.73

Remark: *For this study population, the estimated off-pulse interval is very accurate for both the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests, and also for a wide array of different values of the tuning parameters.*

4.6.5 Data set parameters: $1 - p = 0.4$, $n = 5000$ and $[a, b] = [0.45\,,\,0.55]$

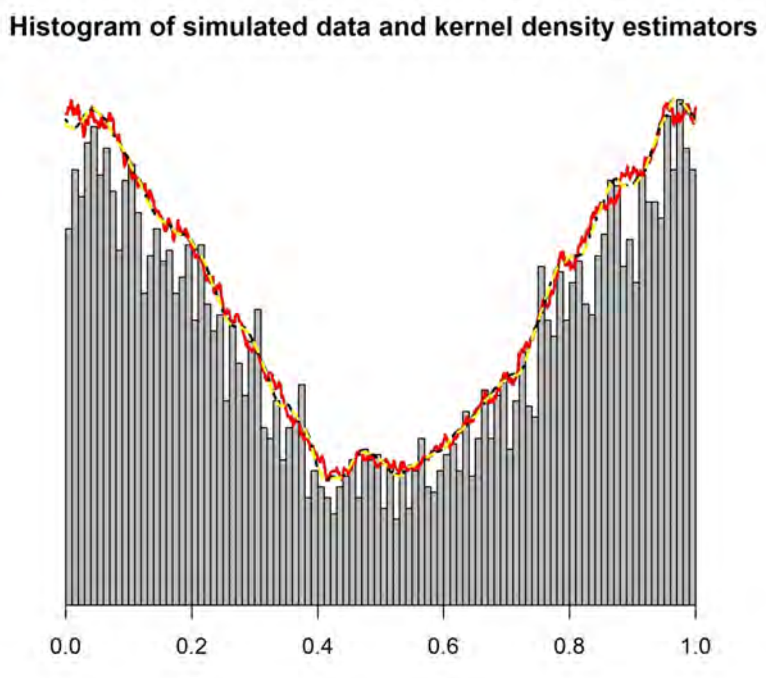

Figure 4.23: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter $\hat{h}_3 = 0.23$.*

Figure 4.23 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis.

Choice of kernel function

From the analyses of all of the previous target populations, it was found that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m. Tables 4.325 – 4.326 highlight the values of the estimated bias and MSE for different combinations of m when $g = 3$, $r = 2$ and $\alpha = 0.05$ for all of the goodness-of-fit tests and for $\hat{h}_6$. It can be seen that different values of m result in small changes in the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. Therefore, it seems fair to recommend the choice of $m = 1$ as a good choice, as it will reduce the computing time.

An important observation is that, as far as the bias and MSE are concerned, all four of the goodness-of-fit tests are relatively insensitive to the choice of m.

Table 4.325: *Bias for different choices of* m *for* $\hat{h}_6$.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0018	0.0330	-0.1108	0.1168	0.0012	0.0369	-0.0480	0.0503
m=2	-0.0020	0.0314	-0.1122	0.1165	-0.0031	0.0357	-0.0498	0.0491
m=3	-0.0057	0.0310	-0.1131	0.1174	-0.0070	0.0354	-0.0499	0.0487
m=4	-0.0078	0.0315	-0.1135	0.1180	-0.0096	0.0357	-0.0496	0.0489
m=5	-0.0106	0.0315	-0.1142	0.1180	-0.0124	0.0355	-0.0499	0.0487
m=6	-0.0126	0.0316	-0.1145	0.1180	-0.0144	0.0356	-0.0500	0.0481
m=7	-0.0138	0.0316	-0.1147	0.1179	-0.0159	0.0357	-0.0498	0.0479
m=8	-0.0149	0.0319	-0.1149	0.1182	-0.0171	0.0359	-0.0491	0.0480
m=9	-0.0160	0.0320	-0.1153	0.1183	-0.0183	0.0361	-0.0492	0.0478
m=10	-0.0170	0.0323	-0.1154	0.1186	-0.0191	0.0363	-0.0489	0.0479
m=11	-0.0178	0.0322	-0.1157	0.1186	-0.0199	0.0363	-0.0488	0.0481
m=12	-0.0182	0.0325	-0.1158	0.1188	-0.0205	0.0367	-0.0487	0.0484
m=13	-0.0187	0.0326	-0.1161	0.1189	-0.0213	0.0367	-0.0489	0.0482
m=14	-0.0194	0.0327	-0.1162	0.1190	-0.0219	0.0367	-0.0489	0.0480
m=15	-0.0199	0.0329	-0.1164	0.1191	-0.0225	0.0368	-0.0489	0.0480
m=16	-0.0202	0.0331	-0.1166	0.1192	-0.0228	0.0372	-0.0488	0.0482
m=17	-0.0205	0.0334	-0.1166	0.1193	-0.0232	0.0375	-0.0489	0.0484
m=18	-0.0207	0.0335	-0.1166	0.1194	-0.0235	0.0377	-0.0489	0.0485
m=19	-0.0208	0.0335	-0.1167	0.1194	-0.0235	0.0378	-0.0489	0.0486
m=20	-0.0208	0.0335	-0.1167	0.1194	-0.0236	0.0378	-0.0490	0.0486

Table 4.326: *MSE for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0040	0.0046	0.0136	0.0149	0.0047	0.0055	0.0075	0.0071
m=2	0.0042	0.0047	0.0141	0.0149	0.0049	0.0056	0.0076	0.0071
m=3	0.0043	0.0048	0.0143	0.0151	0.0050	0.0056	0.0076	0.0072
m=4	0.0044	0.0049	0.0145	0.0154	0.0052	0.0057	0.0076	0.0074
m=5	0.0045	0.0049	0.0147	0.0154	0.0053	0.0058	0.0078	0.0075
m=6	0.0045	0.0049	0.0148	0.0155	0.0054	0.0058	0.0078	0.0075
m=7	0.0046	0.0049	0.0148	0.0155	0.0054	0.0057	0.0078	0.0075
m=8	0.0046	0.0050	0.0148	0.0156	0.0055	0.0058	0.0078	0.0076
m=9	0.0047	0.0050	0.0150	0.0156	0.0055	0.0059	0.0079	0.0076
m=10	0.0047	0.0051	0.0150	0.0157	0.0055	0.0059	0.0079	0.0077
m=11	0.0048	0.0051	0.0151	0.0158	0.0056	0.0059	0.0079	0.0077
m=12	0.0048	0.0051	0.0151	0.0158	0.0056	0.0060	0.0079	0.0078
m=13	0.0049	0.0051	0.0153	0.0159	0.0057	0.0060	0.0080	0.0078
m=14	0.0049	0.0052	0.0153	0.0160	0.0057	0.0060	0.0080	0.0078
m=15	0.0049	0.0052	0.0154	0.0160	0.0057	0.0061	0.0080	0.0079
m=16	0.0050	0.0052	0.0154	0.0161	0.0058	0.0061	0.0081	0.0079
m=17	0.0050	0.0053	0.0155	0.0161	0.0058	0.0061	0.0081	0.0080
m=18	0.0050	0.0053	0.0155	0.0161	0.0058	0.0061	0.0081	0.0080
m=19	0.0050	0.0053	0.0155	0.0161	0.0058	0.0061	0.0081	0.0080
m=20	0.0050	0.0053	0.0155	0.0161	0.0058	0.0061	0.0081	0.0080

Choice of estimated smoothing parameters

Tables 4.327 and 4.328 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 3$, $r = 2$ and $\alpha = 0.05$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were made with different values of r and g, without displaying all of the results.

When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero. Also, when comparing the estimated MSE, none of the choices for the estimated smoothing parameter causes a large change in the MSE, even for different values of r and g. In the light of this, any one of the previously recommended smoothing parameters may be used, such as $\hat{h}_1 - \hat{h}_5$.

Choice of goodness-of-fit test

Some of the tables already provided can be used to assess the goodness-of-fit tests. For a specific comparison of the goodness-of-fit tests, the reader can inspect Tables 4.329 – 4.330 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.05$, $m = 1$ and for $\hat{h}_1$. Several other combinations of g, r and α all led to similar conclusions.

Table 4.327: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 3$ and $r = 2$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0344	0.0398	-0.1044	0.1049	-0.0335	0.0418	-0.0379	0.0414
$\hat{h}_2$	-0.0300	0.0343	-0.1053	0.1058	-0.0337	0.0399	-0.0410	0.0413
$\hat{h}_3$	-0.0340	0.0350	-0.1044	0.1057	-0.0360	0.0407	-0.0397	0.0390
$\hat{h}_4$	-0.0340	0.0350	-0.1044	0.1057	-0.0360	0.0407	-0.0397	0.0390
$\hat{h}_5$	-0.0339	0.0396	-0.1049	0.1058	-0.0352	0.0431	-0.0377	0.0398
$\hat{h}_6$	0.0018	0.0330	-0.1108	0.1168	0.0012	0.0369	-0.0480	0.0503
$\hat{h}_7$	0.0011	0.0327	-0.1108	0.1170	0.0012	0.0364	-0.0483	0.0500
$\hat{h}_8$	-0.0291	0.0328	-0.1063	0.1066	-0.0332	0.0395	-0.0432	0.0418
$\hat{h}_9$	0.0009	0.0321	-0.1110	0.1169	0.0011	0.0361	-0.0484	0.0499

Table 4.328: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 3$ and $r = 2$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0030	0.0033	0.0114	0.0114	0.0034	0.0038	0.0042	0.0044
$\hat{h}_2$	0.0029	0.0031	0.0116	0.0117	0.0035	0.0038	0.0050	0.0050
$\hat{h}_3$	0.0030	0.0031	0.0114	0.0116	0.0035	0.0038	0.0046	0.0047
$\hat{h}_4$	0.0030	0.0031	0.0114	0.0116	0.0035	0.0038	0.0046	0.0047
$\hat{h}_5$	0.0032	0.0034	0.0115	0.0116	0.0037	0.0039	0.0046	0.0046
$\hat{h}_6$	0.0040	0.0046	0.0136	0.0149	0.0047	0.0055	0.0075	0.0071
$\hat{h}_7$	0.0040	0.0046	0.0137	0.0150	0.0047	0.0055	0.0075	0.0070
$\hat{h}_8$	0.0028	0.0030	0.0118	0.0119	0.0036	0.0038	0.0053	0.0052
$\hat{h}_9$	0.0040	0.0046	0.0137	0.0150	0.0047	0.0055	0.0075	0.0070

Table 4.329: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.0185	0.0260	-0.1032	0.1036	-0.0184	0.0275	-0.0176	0.0201
g=2	-0.0286	0.0334	-0.1038	0.1045	-0.0281	0.0368	-0.0314	0.0354
g=3	-0.0344	0.0398	-0.1044	0.1049	-0.0335	0.0418	-0.0379	0.0414
g=4	-0.0377	0.0435	-0.1047	0.1053	-0.0380	0.0456	-0.0424	0.0458
g=5	-0.0411	0.0464	-0.1050	0.1057	-0.0416	0.0477	-0.0455	0.0499
g=20	-0.0581	0.0607	-0.1079	0.1087	-0.0601	0.0618	-0.0652	0.0676

Table 4.330: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0025	0.0028	0.0111	0.0112	0.0030	0.0033	0.0037	0.0039
g=2	0.0028	0.0030	0.0113	0.0114	0.0032	0.0036	0.0040	0.0043
g=3	0.0030	0.0033	0.0114	0.0114	0.0034	0.0038	0.0042	0.0044
g=4	0.0031	0.0035	0.0114	0.0115	0.0036	0.0039	0.0044	0.0045
g=5	0.0033	0.0036	0.0115	0.0116	0.0037	0.0040	0.0045	0.0048
g=20	0.0042	0.0045	0.0121	0.0122	0.0046	0.0047	0.0055	0.0057

In terms of the estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and $g > 1$. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Anderson-Darling goodness-of-fit test performs best, followed by the Kolmogorov-Smirnov and Rayleigh goodness-of-fit tests. The Cramér-von-Mises goodness-of-fit test is, therefore, not the optimal test in this case. Similar observations were made for the other target data sets with small sample sizes n. It seems as if the Cramér-von-Mises goodness-of-fit test is more reliable for larger data sets. For smaller data sets, the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests are recommended.

Choice of the significance level α

Several tables are constructed to investigate the effect of α, in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. Tables 4.331 – 4.332 compare the estimated bias and MSE for two of the goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 2$. Several more comparisons were made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

From all of these tables it is concluded that the estimated bias and MSE are closer to zero when α is *increased*. There is a large change in the estimated bias and MSE-values when α increases from 0.01 to 0.05. The change in the bias and MSE when α increases from 0.05 to 0.10 is not that big. Especially when comparing the estimated MSE, there is only a marginal change when α increases from 0.05 to 0.10. Therefore, $\alpha = 0.05$ is recommended in the case of small data sets, such as for this target population.

Table 4.331: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		MSE		Bias		MSE		Bias		MSE	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.05	0.06	0.00	0.00	-0.02	0.03	0.00	0.00	0.00	0.00	0.00	0.00
g=2	-0.06	0.06	0.00	0.01	-0.03	0.03	0.00	0.00	-0.01	0.01	0.00	0.00
g=3	-0.06	0.07	0.01	0.01	-0.03	0.04	0.00	0.00	-0.01	0.02	0.00	0.00
g=4	-0.06	0.07	0.01	0.01	-0.04	0.04	0.00	0.00	-0.02	0.03	0.00	0.00
g=5	-0.07	0.07	0.01	0.01	-0.04	0.05	0.00	0.00	-0.02	0.03	0.00	0.00
g=20	-0.08	0.08	0.01	0.01	-0.06	0.06	0.00	0.00	-0.05	0.05	0.00	0.00

Table 4.332: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.06	0.07	0.01	0.01	-0.02	0.03	0.00	0.00	0.01	0.00	0.00	0.00
g=2	-0.07	0.07	0.01	0.01	-0.03	0.04	0.00	0.00	-0.01	0.01	0.00	0.00
g=3	-0.07	0.07	0.01	0.01	-0.03	0.04	0.00	0.00	-0.01	0.02	0.00	0.00
g=4	-0.07	0.07	0.01	0.01	-0.04	0.05	0.00	0.00	-0.02	0.03	0.00	0.00
g=5	-0.07	0.07	0.01	0.01	-0.04	0.05	0.00	0.00	-0.02	0.03	0.00	0.00
g=20	-0.08	0.08	0.01	0.01	-0.06	0.06	0.00	0.00	-0.05	0.05	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

Tables 4.333 – 4.336 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling goodness-of-fit test when $\hat{h}_1$ is chosen. In all of the tables the normal kernel is used, with $\alpha = 0.05$ and $m = 1$ kept constant.

Table 4.333: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0047	-0.0185	-0.0274	-0.0329	-0.0374					
g=2	-0.0107	-0.0286	-0.0378	-0.0426	-0.0462					
g=3	-0.0158	-0.0344	-0.0430	-0.0478	-0.0512					
g=4	-0.0184	-0.0377	-0.0466	-0.0513	-0.0541					
g=5	-0.0224	-0.0411	-0.0493	-0.0529	-0.0557					
g=6						-0.0591	-0.0607	-0.0613	-0.0620	-0.0628
g=7						-0.0603	-0.0618	-0.0625	-0.0628	-0.0633
g=8						-0.0610	-0.0622	-0.0627	-0.0630	-0.0634
g=9						-0.0614	-0.0625	-0.0631	-0.0634	-0.0636
g=10						-0.0622	-0.0632	-0.0637	-0.0638	-0.0640
g=20		-0.0581		-0.0641		-0.0649		-0.0651		-0.0651
g=25		-0.0599		-0.0647		-0.0652		-0.0654		-0.0654
g=30		-0.0623		-0.0656		-0.0659		-0.0659		-0.0659
g=35		-0.0630		-0.0662		-0.0663		-0.0663		-0.0663
g=40		-0.0642		-0.0668		-0.0668		-0.0668		-0.0668
g=45		-0.0661		-0.0676		-0.0676		-0.0676		-0.0676
g=50		-0.0663		-0.0682		-0.0682		-0.0682		-0.0682
g=100		-0.0753		-0.0754		-0.0754		-0.0754		-0.0754
g=200		-0.0906		-0.0906		-0.0906		-0.0906		-0.0906
g=300		-0.0959		-0.0959		-0.0959		-0.0959		-0.0959
g=400		-0.1096		-0.1096		-0.1096		-0.1096		-0.1096
g=500		-0.1341		-0.1341		-0.1341		-0.1341		

Table 4.334: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0119	0.0260	0.0329	0.0375	0.0418					
g=2	0.0177	0.0334	0.0423	0.0479	0.0506					
g=3	0.0228	0.0398	0.0484	0.0517	0.0545					
g=4	0.0245	0.0435	0.0506	0.0550	0.0573					
g=5	0.0277	0.0464	0.0527	0.0573	0.0596					
g=6						0.0616	0.0624	0.0635	0.0640	0.0642
g=7						0.0624	0.0634	0.0640	0.0646	0.0648
g=8						0.0632	0.0637	0.0644	0.0645	0.0649
g=9						0.0635	0.0640	0.0644	0.0649	0.0652
g=10						0.0642	0.0645	0.0651	0.0652	0.0657
g=20		0.0607		0.0656		0.0666		0.0669		0.0670
g=25		0.0630		0.0664		0.0669		0.0674		0.0674
g=30		0.0641		0.0670		0.0677		0.0677		0.0677
g=35		0.0653		0.0676		0.0682		0.0682		0.0682
g=40		0.0668		0.0690		0.0691		0.0691		0.0691
g=45		0.0676		0.0694		0.0695		0.0695		0.0695
g=50		0.0691		0.0706		0.0706		0.0706		0.0706
g=100		0.0767		0.0768		0.0768		0.0768		0.0768
g=200		0.0924		0.0924		0.0924		0.0924		0.0924
g=300		0.0978		0.0978		0.0978		0.0978		0.0978
g=400		0.1102		0.1102		0.1102		0.1102		0.1102
g=500		0.1342		0.1342		0.1342		0.1342		

Table 4.335: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0023	0.0025	0.0028	0.0030	0.0031					
g=2	0.0024	0.0028	0.0031	0.0034	0.0035					
g=3	0.0025	0.0030	0.0034	0.0036	0.0037					
g=4	0.0026	0.0031	0.0035	0.0038	0.0039					
g=5	0.0027	0.0033	0.0037	0.0039	0.0040					
g=6						0.0042	0.0043	0.0044	0.0044	0.0045
g=7						0.0043	0.0044	0.0045	0.0045	0.0045
g=8						0.0044	0.0044	0.0045	0.0045	0.0045
g=9						0.0044	0.0045	0.0045	0.0045	0.0046
g=10						0.0045	0.0045	0.0046	0.0046	0.0046
g=20		0.0042		0.0047		0.0047		0.0048		0.0048
g=25		0.0044		0.0047		0.0048		0.0048		0.0048
g=30		0.0046		0.0048		0.0049		0.0049		0.0049
g=35		0.0047		0.0049		0.0049		0.0049		0.0049
g=40		0.0048		0.0050		0.0050		0.0050		0.0050
g=45		0.0050		0.0051		0.0051		0.0051		0.0051
g=50		0.0051		0.0052		0.0052		0.0052		0.0052
g=100		0.0063		0.0063		0.0063		0.0063		0.0063
g=200		0.0091		0.0091		0.0091		0.0091		0.0091
g=300		0.0102		0.0102		0.0102		0.0102		0.0102
g=400		0.0123		0.0123		0.0123		0.0123		0.0123
g=500		0.0181		0.0181		0.0181		0.0181		

Table 4.336: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0024	0.0028	0.0030	0.0032	0.0034					
g=2	0.0026	0.0030	0.0034	0.0037	0.0038					
g=3	0.0027	0.0033	0.0037	0.0039	0.0040					
g=4	0.0027	0.0035	0.0038	0.0040	0.0042					
g=5	0.0028	0.0036	0.0039	0.0042	0.0043					
g=6						0.0045	0.0045	0.0046	0.0047	0.0047
g=7						0.0046	0.0046	0.0047	0.0047	0.0047
g=8						0.0046	0.0046	0.0047	0.0047	0.0047
g=9						0.0046	0.0047	0.0047	0.0047	0.0048
g=10						0.0047	0.0047	0.0048	0.0048	0.0048
g=20		0.0045		0.0049		0.0049		0.0050		0.0050
g=25		0.0047		0.0049		0.0050		0.0050		0.0050
g=30		0.0048		0.0050		0.0051		0.0051		0.0051
g=35		0.0049		0.0051		0.0051		0.0051		0.0051
g=40		0.0051		0.0053		0.0053		0.0053		0.0053
g=45		0.0052		0.0053		0.0053		0.0053		0.0053
g=50		0.0054		0.0055		0.0055		0.0055		0.0055
g=100		0.0065		0.0065		0.0065		0.0065		0.0065
g=200		0.0094		0.0094		0.0094		0.0094		0.0094
g=300		0.0107		0.0107		0.0107		0.0107		0.0107
g=400		0.0124		0.0124		0.0124		0.0124		0.0124
g=500		0.0182		0.0182		0.0182		0.0182		

For the Anderson-Darling goodness-of-fit test, small values of r and g result in estimated bias values that are close to zero, i.e., $1 \leq g \leq 5$ with $1 \leq r \leq 4$. Similar values for g and r also result in the smallest values of the estimated MSE. Compared to some of the previous target data sets, the value of g where the smallest MSE-values are found, is slightly smaller. The explanation for this is the small sample size n of this target data set. In conclusion, it is recommended to use a value of g in the range from 1 to 10, and for r, values from 1 to 10, although smaller values of r and g in the above-mentioned ranges will perform better when the sample size n is small.

Concluding remarks about the simulated data from a scaled triangular distribution with $1 - p = 0.4$, $n = 5000$, and $[a, b] = [0.45\,,\, 0.55]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m = 5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$ – $\hat{h}_5$ is recommended.
- Both the Anderson-Darling and the Kolmogorov-Smirnov goodness-of-fit tests are recommended with preference given to the Anderson-Darling test. For smaller sample sizes, the

Kolmogorov-Smirnov goodness-of-fit test seems to perform better than the Cramér-von-Mises goodness-of-fit test.

- For both of the recommended goodness-of-fit tests, a level of significance of 5% can be used.
- The choices of $1 \leq r \leq 10$ and $1 \leq g \leq 10$ are recommended, but smaller values of r and g can also be used, especially when the sample size is small.

Table 4.337 and Table 4.338 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a, b] = [0.45\,, 0.55]$.

Table 4.337: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.01$.*

	r=1		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.45	0.56	0.42	0.59
g=2	0.44	0.57	0.41	0.60
g=3	0.43	0.57	0.40	0.60
g=4	0.43	0.57	0.40	0.61
g=5	0.43	0.58	0.40	0.61

Table 4.338: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Kolmogorov-Smirnov goodness-of-fit test with $\alpha = 0.01$.*

	r=1		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.45	0.56	0.42	0.59
g=2	0.44	0.57	0.41	0.60
g=3	0.43	0.57	0.40	0.60
g=4	0.43	0.58	0.40	0.61
g=5	0.43	0.58	0.40	0.61

Remark: *Although this is an extreme study population in terms of the percentage pulsed emission contained in the interval [0,1], the estimation of the off-pulse interval is particularly good if one also take the 40% noise level into account.*

4.6.6 Data set parameters: $1 - p = 0.2$, $n = 10000$ and $[a, b] = [0.45\,,\,0.55]$

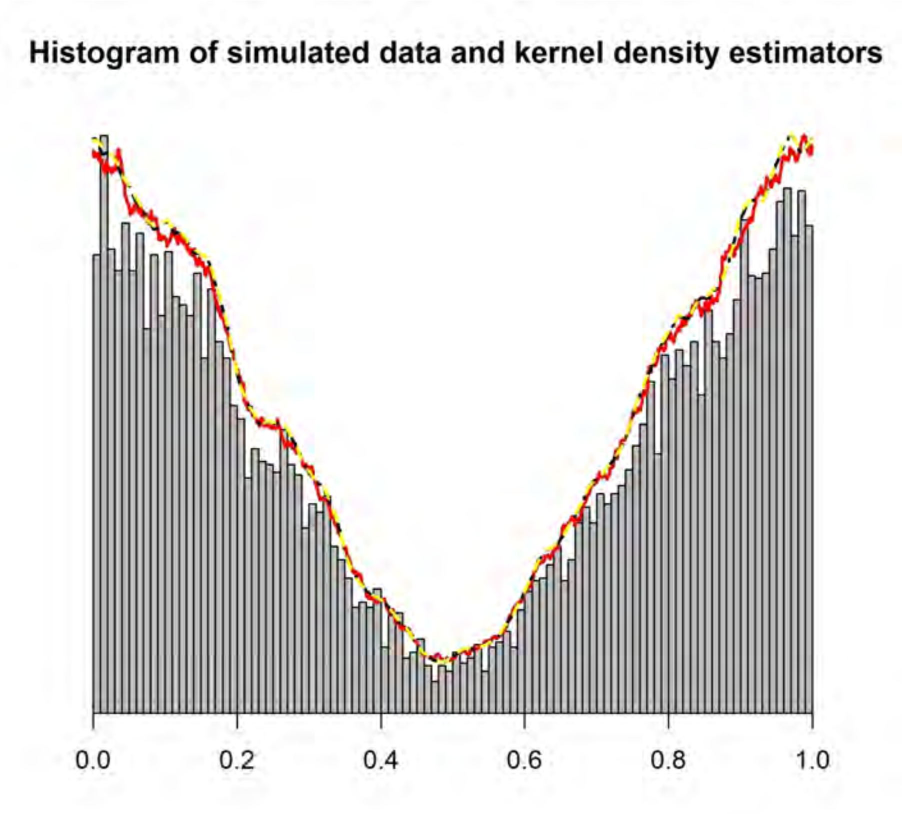

Figure 4.24: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.18$.

Figure 4.24 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis.

Choice of kernel function

From the analyses of all of the previous study populations, it was found that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus. Tables 4.339 – 4.340 highlight the values of the estimated bias and MSE for different combinations of m when $g = 3$, $r = 2$ and $\alpha = 0.01$ for all of the goodness-of-fit tests and for $\hat{h}_6$. It can be seen that different values of m result in small changes in the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. Therefore, it seems fair to recommend $m = 1$ as a good choice.

Table 4.339: *Bias for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0167	0.0365	-0.0589	0.0639	-0.0173	0.0394	-0.0467	0.0473
m=2	-0.0208	0.0342	-0.0596	0.0636	-0.0224	0.0385	-0.0472	0.0465
m=3	-0.0222	0.0344	-0.0598	0.0630	-0.0246	0.0385	-0.0470	0.0464
m=4	-0.0232	0.0349	-0.0596	0.0630	-0.0262	0.0387	-0.0465	0.0463
m=5	-0.0248	0.0344	-0.0601	0.0628	-0.0279	0.0381	-0.0470	0.0458
m=6	-0.0257	0.0345	-0.0602	0.0627	-0.0290	0.0382	-0.0468	0.0457
m=7	-0.0263	0.0345	-0.0604	0.0627	-0.0297	0.0380	-0.0469	0.0457
m=8	-0.0270	0.0343	-0.0604	0.0627	-0.0301	0.0379	-0.0464	0.0456
m=9	-0.0276	0.0342	-0.0605	0.0626	-0.0308	0.0378	-0.0461	0.0454
m=10	-0.0281	0.0342	-0.0606	0.0625	-0.0312	0.0378	-0.0460	0.0454
m=11	-0.0286	0.0346	-0.0606	0.0626	-0.0315	0.0381	-0.0458	0.0456
m=12	-0.0289	0.0349	-0.0607	0.0627	-0.0320	0.0382	-0.0458	0.0456
m=13	-0.0293	0.0348	-0.0608	0.0627	-0.0325	0.0381	-0.0458	0.0456
m=14	-0.0297	0.0350	-0.0609	0.0627	-0.0328	0.0382	-0.0460	0.0456
m=15	-0.0300	0.0351	-0.0610	0.0628	-0.0331	0.0383	-0.0460	0.0457
m=16	-0.0303	0.0352	-0.0611	0.0628	-0.0333	0.0383	-0.0461	0.0458
m=17	-0.0305	0.0352	-0.0612	0.0628	-0.0335	0.0384	-0.0462	0.0458
m=18	-0.0306	0.0353	-0.0612	0.0628	-0.0337	0.0385	-0.0463	0.0459
m=19	-0.0306	0.0353	-0.0612	0.0629	-0.0337	0.0385	-0.0464	0.0459
m=20	-0.0307	0.0353	-0.0612	0.0629	-0.0337	0.0385	-0.0464	0.0459

Table 4.340: *MSE for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0018	0.0020	0.0037	0.0044	0.0022	0.0024	0.0029	0.0031
m=2	0.0017	0.0019	0.0038	0.0043	0.0021	0.0023	0.0029	0.0031
m=3	0.0017	0.0019	0.0039	0.0042	0.0021	0.0023	0.0030	0.0031
m=4	0.0018	0.0019	0.0038	0.0042	0.0021	0.0022	0.0030	0.0031
m=5	0.0017	0.0019	0.0039	0.0042	0.0021	0.0023	0.0030	0.0031
m=6	0.0017	0.0019	0.0039	0.0042	0.0021	0.0023	0.0030	0.0030
m=7	0.0017	0.0019	0.0039	0.0042	0.0021	0.0023	0.0030	0.0031
m=8	0.0017	0.0019	0.0039	0.0042	0.0021	0.0023	0.0030	0.0030
m=9	0.0017	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0030
m=10	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0030
m=11	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0031
m=12	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0031
m=13	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0030
m=14	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0030
m=15	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0031
m=16	0.0018	0.0019	0.0040	0.0042	0.0021	0.0023	0.0030	0.0031
m=17	0.0018	0.0019	0.0041	0.0042	0.0021	0.0023	0.0030	0.0031
m=18	0.0018	0.0019	0.0041	0.0042	0.0021	0.0023	0.0030	0.0031
m=19	0.0018	0.0019	0.0041	0.0043	0.0022	0.0023	0.0030	0.0031
m=20	0.0018	0.0019	0.0041	0.0043	0.0022	0.0023	0.0030	0.0031

Choice of estimated smoothing parameters

Tables 4.341 and 4.342 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 3$, $r = 2$ and $\alpha = 0.01$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were made with different values of r and g (without showing all of the results).

Table 4.341: *Comparison of the bias for different combinations of the estimated smoothing parameter with* $m = 1$*,* $g = 3$ *and* $r = 2$*.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0324	0.0349	-0.0557	0.0574	-0.0337	0.0374	-0.0407	0.0386
$\hat{h}_2$	-0.0286	0.0330	-0.0557	0.0574	-0.0324	0.0372	-0.0405	0.0407
$\hat{h}_3$	-0.0292	0.0324	-0.0552	0.0570	-0.0318	0.0351	-0.0399	0.0390
$\hat{h}_4$	-0.0292	0.0324	-0.0552	0.0570	-0.0318	0.0351	-0.0399	0.0390
$\hat{h}_5$	-0.0321	0.0348	-0.0563	0.0572	-0.0347	0.0365	-0.0406	0.0391
$\hat{h}_6$	-0.0167	0.0365	-0.0589	0.0639	-0.0173	0.0394	-0.0467	0.0473
$\hat{h}_7$	-0.0167	0.0365	-0.0589	0.0639	-0.0173	0.0394	-0.0467	0.0473
$\hat{h}_8$	-0.0284	0.0320	-0.0564	0.0578	-0.0319	0.0359	-0.0431	0.0417
$\hat{h}_9$	-0.0167	0.0365	-0.0589	0.0639	-0.0173	0.0394	-0.0467	0.0473

Table 4.342: *Comparison of the MSE for different combinations of the estimated smoothing parameter with* $m = 1$*,* $g = 3$ *and* $r = 2$*.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0014	0.0015	0.0032	0.0034	0.0016	0.0017	0.0020	0.0020
$\hat{h}_2$	0.0014	0.0015	0.0032	0.0034	0.0016	0.0017	0.0024	0.0024
$\hat{h}_3$	0.0013	0.0015	0.0032	0.0034	0.0016	0.0017	0.0022	0.0022
$\hat{h}_4$	0.0013	0.0015	0.0032	0.0034	0.0016	0.0017	0.0022	0.0022
$\hat{h}_5$	0.0015	0.0016	0.0033	0.0034	0.0016	0.0017	0.0021	0.0021
$\hat{h}_6$	0.0018	0.0020	0.0037	0.0044	0.0022	0.0024	0.0029	0.0031
$\hat{h}_7$	0.0018	0.0020	0.0037	0.0044	0.0022	0.0024	0.0029	0.0031
$\hat{h}_8$	0.0013	0.0015	0.0033	0.0034	0.0016	0.0017	0.0025	0.0025
$\hat{h}_9$	0.0018	0.0020	0.0037	0.0044	0.0022	0.0024	0.0029	0.0031

When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero. Also, when comparing the estimated MSE, none of the choices for the estimated smoothing parameter causes a large change in the MSE, even for different values of r and g. In the light of this, any one of the previously recommended smoothing parameters may be used, such as $\hat{h}_1 - \hat{h}_5$.

Choice of goodness-of-fit test

Some of the tables provided earlier can be used to assess the goodness-of-fit tests. For a specific comparison of the goodness-of-fit tests, the reader can inspect Tables 4.343 – 4.344 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, when $\alpha = 0.01$, $m = 1$ and for $\hat{h}_1$. Several other combinations of g, r and α all led to similar conclusions.

In terms of the estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Anderson-Darling goodness-of-fit test performs best, followed by the Kolmogorov-Smirnov and Rayleigh goodness-of-fit tests. The Cramér-von-Mises goodness-of-fit test is, therefore, not the optimal test in this case. Similar observations were made for the other target data sets with a small to moderate sample size n. It seems as if the Cramér-von-Mises goodness-of-fit test is more reliable for larger data sets. For smaller data sets, the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests are recommended.

Table 4.343: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.0274	0.0291	-0.0551	0.0568	-0.0306	0.0334	-0.0349	0.0333
g=2	-0.0311	0.0331	-0.0554	0.0572	-0.0327	0.0357	-0.0379	0.0361
g=3	-0.0324	0.0349	-0.0557	0.0574	-0.0337	0.0374	-0.0407	0.0386
g=4	-0.0331	0.0362	-0.0558	0.0576	-0.0347	0.0384	-0.0413	0.0391
g=5	-0.0341	0.0368	-0.0561	0.0578	-0.0352	0.0392	-0.0422	0.0404
g=20	-0.0392	0.0408	-0.0579	0.0595	-0.0406	0.0427	-0.0451	0.0453

Table 4.344: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0013	0.0014	0.0031	0.0033	0.0015	0.0017	0.0019	0.0020
g=2	0.0014	0.0015	0.0032	0.0034	0.0016	0.0017	0.0020	0.0020
g=3	0.0014	0.0015	0.0032	0.0034	0.0016	0.0017	0.0020	0.0020
g=4	0.0014	0.0016	0.0032	0.0034	0.0016	0.0017	0.0020	0.0020
g=5	0.0015	0.0016	0.0033	0.0034	0.0016	0.0018	0.0020	0.0021
g=20	0.0017	0.0018	0.0035	0.0036	0.0018	0.0020	0.0022	0.0023

Choice of the significance level α

The significance level α is another parameter that may influence the point where rejection of uniformity takes place, and therefore α may influence the estimated values of a and b. Several tables are constructed to investigate the effect of α, in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. Tables 4.345 – 4.346 compare the estimated bias and MSE for two of the goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 2$. Several more comparisons were made with different combinations of $\hat{h}$ and r resulting in similar conclusions. Not all of the comparative tables are shown.

From all of these tables it is concluded that the estimated bias is close to zero when $\alpha = 0.05$. There is a marginally larger change in the estimated bias and MSE when α increases from 0.01 to 0.05 compared to the change when α increases from 0.05 to 0.10. Especially when comparing the estimated MSE, there is only the slightest variation when α increases from 0.05 to 0.1. Thus,

$\alpha = 0.05$ is recommended in the case of small to moderate data sets, such as for this target population.

Table 4.345: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.03	0.03	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.01	0.00	0.00
g=2	-0.03	0.03	0.00	0.00	-0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
g=3	-0.03	0.03	0.00	0.00	-0.01	0.02	0.00	0.00	-0.00	0.01	0.00	0.00
g=4	-0.03	0.04	0.00	0.00	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00
g=5	-0.03	0.04	0.00	0.00	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00
g=20	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00

Table 4.346: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.03	0.03	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.01	0.00	0.00
g=2	-0.03	0.04	0.00	0.00	-0.01	0.02	0.00	0.00	0.00	-0.00	0.00	0.00
g=3	-0.03	0.04	0.00	0.00	-0.01	0.02	0.00	0.00	-0.00	0.00	0.00	0.00
g=4	-0.03	0.04	0.00	0.00	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00
g=5	-0.04	0.04	0.00	0.00	-0.02	0.02	0.00	0.00	-0.01	0.01	0.00	0.00
g=20	-0.04	0.04	0.00	0.00	-0.03	0.03	0.00	0.00	-0.02	0.03	0.00	0.00

Choice of incremental growth g and number of intervals of rejection r

Tables 4.347 – 4.350 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling goodness-of-fit test when $\hat{h}_1$ is chosen. In all of the tables the normal kernel is used, with $\alpha = 0.05$ and $m = 1$ kept constant.

For the Anderson-Darling goodness-of-fit test, small values of r and g result in estimated bias-values that are close to zero, i.e., $1 \leq g \leq 5$ with $1 \leq r \leq 4$. The same values for g and r also result in the smallest values of the MSE. Compared to some of the previous study populations, the value of g where the smallest MSE-values are found, is slightly smaller. The explanation for this is the moderate sample size n of this study population.

In conclusion, it is recommended to use a value of g in the range from 1 to 10, and for r, values from 1 and 10, although smaller values of r and g can also be used when the sample size n is small to moderate.

Table 4.347: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0066	-0.0038	-0.0100	-0.0137	-0.0163					
g=2	0.0031	-0.0101	-0.0160	-0.0200	-0.0224					
g=3	-0.0019	-0.0138	-0.0207	-0.0238	-0.0254					
g=4	-0.0014	-0.0168	-0.0223	-0.0253	-0.0273					
g=5	-0.0079	-0.0193	-0.0248	-0.0269	-0.0284					
g=6						-0.0298	-0.0303	-0.0307	-0.0310	-0.0316
g=7						-0.0304	-0.0309	-0.0315	-0.0315	-0.0318
g=8						-0.0305	-0.0312	-0.0314	-0.0317	-0.0317
g=9						-0.0311	-0.0313	-0.0317	-0.0318	-0.0318
g=10						-0.0316	-0.0317	-0.0318	-0.0320	-0.0320
g=20		-0.0300		-0.0326		-0.0330		-0.0330		-0.0330
g=25		-0.0315		-0.0334		-0.0335		-0.0335		-0.0335
g=30		-0.0332		-0.0343		-0.0343		-0.0343		-0.0343
g=35		-0.0340		-0.0350		-0.0350		-0.0350		-0.0350
g=40		-0.0347		-0.0358		-0.0358		-0.0358		-0.0358
g=45		-0.0354		-0.0358		-0.0358		-0.0358		-0.0358
g=50		-0.0358		-0.0363		-0.0363		-0.0363		-0.0363
g=100		-0.0423		-0.0423		-0.0423		-0.0423		-0.0423
g=200		-0.0501		-0.0501		-0.0501		-0.0501		-0.0501
g=300		-0.0623		-0.0623		-0.0623		-0.0623		-0.0623
g=400		-0.0827		-0.0827		-0.0827		-0.0827		-0.0827
g=500		-0.1005		-0.1005		-0.1005		-0.1005		-0.1005

Table 4.348: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0013	0.0078	0.0133	0.0170	0.0186					
g=2	0.0017	0.0143	0.0196	0.0240	0.0253					
g=3	0.0061	0.0184	0.0242	0.0257	0.0278					
g=4	0.0073	0.0213	0.0251	0.0275	0.0291					
g=5	0.0111	0.0223	0.0268	0.0292	0.0304					
g=6						0.0315	0.0321	0.0327	0.0328	0.0333
g=7						0.0322	0.0326	0.0333	0.0333	0.0335
g=8						0.0326	0.0331	0.0334	0.0336	0.0340
g=9						0.0328	0.0334	0.0336	0.0341	0.0341
g=10						0.0334	0.0336	0.0338	0.0340	0.0342
g=20		0.0323		0.0349		0.0354		0.0354		0.0354
g=25		0.0334		0.0358		0.0358		0.0358		0.0358
g=30		0.0345		0.0362		0.0362		0.0362		0.0362
g=35		0.0364		0.0370		0.0370		0.0370		0.0370
g=40		0.0365		0.0376		0.0376		0.0376		0.0376
g=45		0.0375		0.0380		0.0380		0.0380		0.0380
g=50		0.0383		0.0390		0.0390		0.0390		0.0390
g=100		0.0440		0.0440		0.0440		0.0440		0.0440
g=200		0.0521		0.0521		0.0521		0.0521		0.0521
g=300		0.0627		0.0627		0.0627		0.0627		0.0627
g=400		0.0832		0.0832		0.0832		0.0832		0.0832
g=500		0.1010		0.1010		0.1010		0.1010		0.1010

Table 4.349: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0012	0.0011	0.0010	0.0010	0.0010					
g=2	0.0012	0.0010	0.0010	0.0010	0.0010					
g=3	0.0011	0.0010	0.0010	0.0010	0.0011					
g=4	0.0011	0.0011	0.0010	0.0011	0.0011					
g=5	0.0010	0.0010	0.0010	0.0011	0.0011					
g=6						0.0011	0.0011	0.0011	0.0011	0.0011
g=7						0.0011	0.0012	0.0012	0.0012	0.0012
g=8						0.0012	0.0011	0.0011	0.0011	0.0011
g=9						0.0012	0.0011	0.0012	0.0012	0.0012
g=10						0.0012	0.0012	0.0012	0.0012	0.0012
g=20		0.0012		0.0012		0.0012		0.0012		0.0012
g=25		0.0013		0.0013		0.0013		0.0013		0.0013
g=30		0.0013		0.0013		0.0013		0.0013		0.0013
g=35		0.0014		0.0014		0.0014		0.0014		0.0014
g=40		0.0014		0.0014		0.0014		0.0014		0.0014
g=45		0.0014		0.0014		0.0014		0.0014		0.0014
g=50		0.0015		0.0015		0.0015		0.0015		0.0015
g=100		0.0020		0.0020		0.0020		0.0020		0.0020
g=200		0.0029		0.0029		0.0029		0.0029		0.0029
g=300		0.0039		0.0039		0.0039		0.0039		0.0039
g=400		0.0069		0.0069		0.0069		0.0069		0.0069
g=500		0.0101		0.0101		0.0101		0.0101		0.0101

Table 4.350: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0012	0.0011	0.0011	0.0011	0.0011					
g=2	0.0012	0.0011	0.0011	0.0011	0.0011					
g=3	0.0011	0.0011	0.0011	0.0011	0.0012					
g=4	0.0011	0.0011	0.0011	0.0012	0.0012					
g=5	0.0011	0.0011	0.0012	0.0012	0.0012					
g=6						0.0012	0.0013	0.0013	0.0013	0.0013
g=7						0.0012	0.0013	0.0013	0.0013	0.0013
g=8						0.0013	0.0013	0.0013	0.0013	0.0013
g=9						0.0013	0.0013	0.0013	0.0013	0.0013
g=10						0.0013	0.0013	0.0013	0.0013	0.0013
g=20		0.0013		0.0014		0.0014		0.0014		0.0014
g=25		0.0014		0.0014		0.0014		0.0014		0.0014
g=30		0.0014		0.0014		0.0014		0.0014		0.0014
g=35		0.0015		0.0015		0.0015		0.0015		0.0015
g=40		0.0015		0.0016		0.0016		0.0016		0.0016
g=45		0.0016		0.0016		0.0016		0.0016		0.0016
g=50		0.0017		0.0017		0.0017		0.0017		0.0017
g=100		0.0021		0.0021		0.0021		0.0021		0.0021
g=200		0.0031		0.0031		0.0031		0.0031		0.0031
g=300		0.0040		0.0040		0.0040		0.0040		0.0040
g=400		0.0070		0.0070		0.0070		0.0070		0.0070
g=500		0.0102		0.0102		0.0102		0.0102		0.0102

Concluding remarks about the simulated data from a scaled triangular distribution with $1-p=0.2$, $n=10000$, and $[a,b]=[0.45\,,\,0.55]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions.
- It is recommended to use a single minimum point, but more minima can also be used up to $m=5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$ – $\hat{h}_5$ is recommended.
- Both the Anderson-Darling and the Kolmogorov-Smirnov goodness-of-fit tests are recommended with preference given to the Anderson-Darling test. For small to moderate sample sizes, the Kolmogorov-Smirnov goodness-of-fit test seems to perform better than the Cramér-von-Mises goodness-of-fit test.
- For both of the recommended goodness-of-fit tests, a level of significance of 5% can be used.
- The choices of $1 \le r \le 10$ and $1 \le g \le 10$ are recommended, but smaller values of r and g can also be used, especially when the sample size is small.

Table 4.351 and Table 4.352 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a,b]=[0.45\,,\,0.55]$.

Table 4.351: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m=1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha=0.05$.*

	r=1		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.46	0.55	0.44	0.57
g=2	0.45	0.55	0.43	0.57
g=3	0.45	0.56	0.43	0.58
g=4	0.45	0.56	0.42	0.58
g=5	0.44	0.56	0.42	0.58

Table 4.352: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Kolmogorov-Smirnov goodness-of-fit test with $\alpha = 0.05$.*

	r=1		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.46	0.54	0.44	0.57
g=2	0.45	0.55	0.43	0.57
g=3	0.45	0.55	0.43	0.58
g=4	0.45	0.56	0.42	0.58
g=5	0.44	0.56	0.42	0.58

Remark: *Although this is an extreme study population in terms of the percentage pulsed emission contained in the interval [0,1], the estimation of the off-pulse interval is very accurate.*

4.6.7 Data set parameters: $1 - p = 0.4$, $n = 10000$ and $[a, b] = [0.3\,,\, 0.7]$

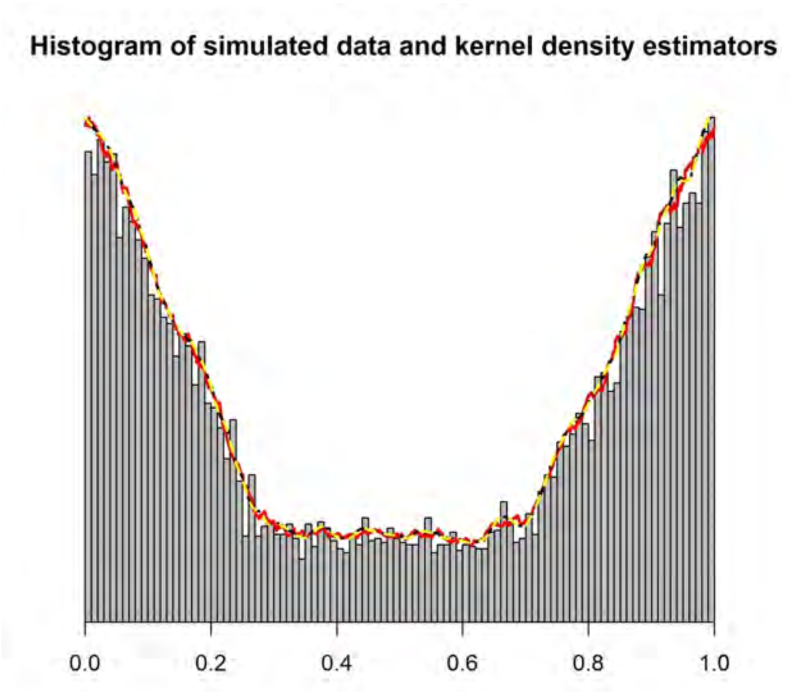

Figure 4.25: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter $\hat{h}_3 = 0.17$.*

Figure 4.25 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

From the analyses of all of the previous target populations, it was found that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used in the analysis of this target population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m. Tables 4.353 – 4.354 highlight the values of the estimated bias and MSE for different combinations of m when $g = 3$, $r = 2$, $\alpha = 0.05$ for all of the goodness-of-fit tests and for $\hat{h}_1$.

It can be seen that different values of m result in small changes in the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. Therefore, it seems fair to recommend the choice of $m = 1$ as a good choice.

Table 4.353: *Bias for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0759	-0.0726	-0.0476	0.0486	0.0635	-0.0680	0.0628	-0.0672
m=2	0.0743	-0.0739	-0.0475	0.0485	0.0637	-0.0665	0.0626	-0.0665
m=3	0.0730	-0.0747	-0.0475	0.0485	0.0634	-0.0658	0.0641	-0.0674
m=4	0.0736	-0.0743	-0.0476	0.0485	0.0634	-0.0650	0.0634	-0.0670
m=5	0.0731	-0.0739	-0.0476	0.0485	0.0635	-0.0641	0.0641	-0.0669
m=6	0.0724	-0.0737	-0.0476	0.0485	0.0627	-0.0635	0.0633	-0.0663
m=7	0.0727	-0.0732	-0.0476	0.0485	0.0624	-0.0637	0.0628	-0.0659
m=8	0.0727	-0.0727	-0.0476	0.0485	0.0621	-0.0630	0.0623	-0.0653
m=9	0.0726	-0.0723	-0.0476	0.0485	0.0617	-0.0625	0.0622	-0.0653
m=10	0.0723	-0.0721	-0.0476	0.0485	0.0614	-0.0623	0.0617	-0.0655
m=11	0.0722	-0.0721	-0.0476	0.0485	0.0612	-0.0622	0.0618	-0.0654
m=12	0.0726	-0.0722	-0.0476	0.0485	0.0613	-0.0623	0.0620	-0.0653
m=13	0.0725	-0.0720	-0.0476	0.0485	0.0612	-0.0621	0.0619	-0.0652
m=14	0.0726	-0.0717	-0.0476	0.0485	0.0617	-0.0620	0.0620	-0.0653
m=15	0.0725	-0.0716	-0.0476	0.0484	0.0616	-0.0618	0.0620	-0.0650
m=16	0.0726	-0.0717	-0.0476	0.0485	0.0619	-0.0618	0.0619	-0.0648
m=17	0.0723	-0.0716	-0.0476	0.0485	0.0618	-0.0620	0.0619	-0.0652
m=18	0.0725	-0.0713	-0.0476	0.0485	0.0621	-0.0616	0.0619	-0.0649
m=19	0.0724	-0.0713	-0.0476	0.0485	0.0620	-0.0615	0.0621	-0.0650
m=20	0.0725	-0.0712	-0.0476	0.0485	0.0619	-0.0613	0.0618	-0.0650

Table 4.354: *MSE for different choices of m for $\hat{h}_6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0195	0.0194	0.0024	0.0024	0.0181	0.0196	0.0192	0.0209
m=2	0.0192	0.0196	0.0024	0.0024	0.0183	0.0192	0.0192	0.0205
m=3	0.0189	0.0198	0.0024	0.0024	0.0182	0.0190	0.0195	0.0207
m=4	0.0190	0.0196	0.0024	0.0024	0.0182	0.0188	0.0192	0.0205
m=5	0.0188	0.0194	0.0024	0.0024	0.0182	0.0186	0.0194	0.0205
m=6	0.0186	0.0194	0.0024	0.0024	0.0180	0.0185	0.0193	0.0203
m=7	0.0187	0.0193	0.0024	0.0024	0.0179	0.0185	0.0191	0.0203
m=8	0.0187	0.0191	0.0024	0.0024	0.0178	0.0184	0.0191	0.0201
m=9	0.0187	0.0190	0.0024	0.0024	0.0177	0.0183	0.0190	0.0201
m=10	0.0186	0.0189	0.0024	0.0024	0.0177	0.0182	0.0190	0.0201
m=11	0.0185	0.0189	0.0024	0.0024	0.0176	0.0182	0.0189	0.0201
m=12	0.0186	0.0189	0.0024	0.0024	0.0176	0.0182	0.0190	0.0201
m=13	0.0186	0.0189	0.0024	0.0024	0.0176	0.0182	0.0191	0.0201
m=14	0.0186	0.0188	0.0024	0.0024	0.0177	0.0181	0.0191	0.0201
m=15	0.0186	0.0188	0.0024	0.0024	0.0177	0.0181	0.0191	0.0200
m=16	0.0186	0.0188	0.0024	0.0024	0.0178	0.0181	0.0191	0.0199
m=17	0.0185	0.0188	0.0024	0.0024	0.0177	0.0181	0.0191	0.0200
m=18	0.0186	0.0188	0.0024	0.0024	0.0178	0.0180	0.0191	0.0200
m=19	0.0186	0.0188	0.0024	0.0024	0.0178	0.0180	0.0191	0.0200
m=20	0.0186	0.0188	0.0024	0.0024	0.0178	0.0180	0.0191	0.0200

Choice of estimated smoothing parameters

Tables 4.355 – 4.356 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 3$, $r = 2$ and $\alpha = 0.05$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were made for different values of r and g.

Table 4.355: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 3$ and $r = 2$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0759	-0.0726	-0.0476	0.0486	0.0635	-0.0680	0.0628	-0.0672
$\hat{h}_2$	0.0983	-0.0888	-0.0476	0.0484	0.0829	-0.0718	0.0697	-0.0733
$\hat{h}_3$	0.0895	-0.0854	-0.0472	0.0482	0.0699	-0.0648	0.0643	-0.0709
$\hat{h}_4$	0.0895	-0.0854	-0.0472	0.0482	0.0699	-0.0648	0.0643	-0.0709
$\hat{h}_5$	0.0685	-0.0593	-0.0486	0.0487	0.0567	-0.0480	0.0524	-0.0620
$\hat{h}_6$	0.1421	-0.1119	-0.0479	0.0506	0.1313	-0.0927	0.0563	-0.0665
$\hat{h}_7$	0.1420	-0.1121	-0.0479	0.0506	0.1313	-0.0929	0.0563	-0.0665
$\hat{h}_8$	0.0960	-0.0925	-0.0480	0.0478	0.0816	-0.0756	0.0711	-0.0723
$\hat{h}_9$	0.1420	-0.1118	-0.0479	0.0506	0.1313	-0.0926	0.0563	-0.0668

When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero. Also, when comparing the estimated MSE, none of the choices for the estimated smoothing parameter causes a large change in the MSE, even for different values of r and g. In the light of this, any one of the previously recommended smoothing parameters may be used, such as $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$.

Table 4.356: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 3$ and $r = 2$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0195	0.0194	0.0024	0.0024	0.0181	0.0196	0.0192	0.0209
$\hat{h}_2$	0.0255	0.0239	0.0024	0.0024	0.0239	0.0217	0.0218	0.0236
$\hat{h}_3$	0.0226	0.0227	0.0025	0.0024	0.0203	0.0197	0.0208	0.0221
$\hat{h}_4$	0.0226	0.0227	0.0025	0.0024	0.0203	0.0197	0.0208	0.0221
$\hat{h}_5$	0.0169	0.0152	0.0024	0.0024	0.0157	0.0141	0.0162	0.0187
$\hat{h}_6$	0.0379	0.0306	0.0030	0.0027	0.0365	0.0276	0.0204	0.0227
$\hat{h}_7$	0.0379	0.0307	0.0030	0.0027	0.0365	0.0277	0.0204	0.0227
$\hat{h}_8$	0.0253	0.0246	0.0024	0.0025	0.0238	0.0222	0.0226	0.0233
$\hat{h}_9$	0.0379	0.0306	0.0030	0.0027	0.0365	0.0276	0.0204	0.0228

Choice of goodness-of-fit test

Some of the tables presented earlier were already used to assess the goodness-of-fit tests. For a specific comparison between the goodness-of-fit tests, the reader can inspect Tables 4.357 – 4.360 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, for two different values of α, $m = 1$ and for $\hat{h}_1$. Several other combinations of g, r and α all led to similar conclusions.

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs best, followed by the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests. The magnitude of the difference between the estimated MSE of the Cramér-von-Mises goodness-of-fit test and the Anderson-Darling (or the Kolmogorov-Smirnov) goodness-of-fit test is not large. Due to the much smaller bias of the Kolmogorov-Smirnov goodness-of-fit test, it can be argued that the Anderson-Darling, Cramér-von-Mises or Kolmogorov-Smirnov goodness-of-fit tests can be used.

Table 4.357: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 6$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0095	-0.0111	-0.0481	0.0489	0.0090	-0.0068	0.0028	-0.0048
g=7	0.0057	-0.0078	-0.0482	0.0489	0.0047	-0.0028	0.0004	-0.0004
g=8	0.0045	-0.0057	-0.0482	0.0490	0.0019	-0.0006	-0.0031	0.0028
g=9	0.0023	-0.0025	-0.0483	0.0490	-0.0003	0.0018	-0.0037	0.0059
g=10	-0.0015	-0.0006	-0.0483	0.0491	-0.0031	0.0033	-0.0067	0.0082
g=20	-0.0121	0.0131	-0.0489	0.0497	-0.0140	0.0142	-0.0210	0.0219

Table 4.358: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.05$, $m = 1$, $r = 6$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	0.0068	0.0072	0.0024	0.0025	0.0076	0.0073	0.0081	0.0085
g=7	0.0062	0.0067	0.0024	0.0025	0.0068	0.0066	0.0078	0.0078
g=8	0.0060	0.0065	0.0024	0.0025	0.0062	0.0063	0.0073	0.0072
g=9	0.0057	0.0059	0.0024	0.0025	0.0059	0.0059	0.0072	0.0066
g=10	0.0052	0.0057	0.0024	0.0025	0.0055	0.0056	0.0066	0.0063
g=20	0.0033	0.0034	0.0025	0.0025	0.0040	0.0040	0.0045	0.0043

Table 4.359: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 2, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0164	-0.0183	-0.0499	0.0506	0.0027	-0.0025	-0.0008	0.0049
g=2	0.0069	-0.0082	-0.0500	0.0508	-0.0042	0.0056	-0.0103	0.0126
g=3	-0.0004	-0.0012	-0.0501	0.0508	-0.0088	0.0099	-0.0170	0.0175
g=4	-0.0042	0.0001	-0.0502	0.0509	-0.0122	0.0118	-0.0191	0.0218
g=5	-0.0058	0.0039	-0.0502	0.0510	-0.0145	0.0146	-0.0210	0.0243
g=20	-0.0208	0.0209	-0.0512	0.0520	-0.0279	0.0278	-0.0354	0.0372

Table 4.360: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 2, \hat{h}_1$ for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0098	0.0100	0.0026	0.0026	0.0087	0.0084	0.0094	0.0093
g=2	0.0077	0.0082	0.0026	0.0026	0.0072	0.0069	0.0077	0.0078
g=3	0.0065	0.0069	0.0026	0.0027	0.0064	0.0061	0.0065	0.0070
g=4	0.0058	0.0068	0.0026	0.0027	0.0056	0.0058	0.0062	0.0061
g=5	0.0057	0.0061	0.0026	0.0027	0.0052	0.0052	0.0059	0.0055
g=20	0.0033	0.0034	0.0027	0.0028	0.0031	0.0032	0.0038	0.0037

Choice of the significance level α

Several tables are constructed to investigate the effect of α in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. Table 4.361 compares the estimated bias and MSE for the Anderson-Darling goodness-of-fit test, with $\hat{h}_1$, $m = 1$ and $r = 2$. Table 4.362 is similar, except that $r = 6$. Several more comparisons were made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

The tables highlight the fact that both $\alpha = 0.01$ and $\alpha = 0.05$ may result in estimated bias-values close to zero, depending on the choice of g and r. For $1 \leq g \leq 5$ with $r = 2$, it is preferable to use $\alpha = 0.01$. When $6 \leq g \leq 10$ with $r = 6$, it is preferable to use $\alpha = 0.05$. For both combination sets, the estimated MSE-values are very close to each other. This is quite interesting, as it illustrates the interdependent nature of the parameters. When a small value of α is used, the user does not have to

Table 4.361: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.02	-0.02	0.01	0.01	0.11	-0.10	0.03	0.03	0.15	-0.15	0.03	0.04
g=2	0.01	-0.01	0.01	0.01	0.09	-0.09	0.02	0.02	0.13	-0.14	0.03	0.03
g=3	-0.00	-0.00	0.01	0.01	0.08	-0.07	0.02	0.02	0.12	-0.12	0.03	0.03
g=4	-0.00	0.00	0.01	0.01	0.07	-0.07	0.02	0.02	0.11	-0.11	0.03	0.03
g=5	-0.01	0.00	0.01	0.01	0.06	-0.06	0.02	0.02	0.10	-0.10	0.02	0.03
g=20	-0.02	0.02	0.00	0.00	0.02	-0.02	0.01	0.01	0.05	-0.05	0.01	0.01

Table 4.362: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 6$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.02	0.02	0.00	0.00	0.01	-0.01	0.01	0.01	0.04	-0.04	0.01	0.01
g=7	-0.02	0.02	0.00	0.00	0.01	-0.01	0.01	0.01	0.03	-0.03	0.01	0.01
g=8	-0.02	0.03	0.00	0.00	0.00	-0.01	0.01	0.01	0.03	-0.03	0.01	0.01
g=9	-0.03	0.03	0.00	0.00	0.00	-0.00	0.01	0.01	0.03	-0.02	0.01	0.01
g=10	-0.03	0.03	0.00	0.00	-0.00	-0.00	0.01	0.01	0.02	-0.02	0.01	0.01
g=20	-0.03	0.03	0.00	0.00	-0.01	0.01	0.00	0.00	0.01	-0.01	0.01	0.01

be too cautious of incorrectly rejecting uniformity by selecting a large value of r. The reason is that the small value of α is already providing protection against the incorrect rejection of uniformity. On the other hand, if $\alpha = 0.05$ and a larger value of r is used, similar protection against the incorrect rejection of uniformity is provided. Therefore, both $\alpha = 0.01$ and $\alpha = 0.05$ are recommended, but the user must then take the choice of r into account, e.g., choose smaller values of r with smaller values of α and vice versa. A similar trend is found for the Kolmogorov-Smirnov and Rayleigh goodness-of-fit tests. The estimated bias and MSE for the Cramér-von-Mises goodness-of-fit test are more robust against different choices of α and r.

Choice of incremental growth g and number of intervals of rejection r

The reader can inspect Tables 4.357 – 4.360 and also Tables 4.361 – 4.362 for a broader understanding of the effect of g, r and α. More specifically, Tables 4.363 – 4.366 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling goodness-of-fit test when $\hat{h}_1$ is chosen. For the last mentioned tables, the normal kernel is used, with $\alpha = 0.05$ and $m = 1$ kept constant.

For the Anderson-Darling goodness-of-fit test, small values of r and g result in estimated bias and MSE-values that are close to zero, i.e., $1 \leq g \leq 5$ with $1 \leq r \leq 4$ if $\alpha = 0.01$. When $\alpha = 0.05$ and $6 \leq g \leq 10$ with $6 \leq r \leq 10$, similar values of the estimated bias and MSE are found.

In conclusion, it is recommended to use a value of g in the range from 1 to 10, and for r, values from 1 to 10, depending on the choice of α, i.e., choose smaller values of r and g with smaller values of α, and vice versa.

Table 4.363: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.1269	0.1052	0.0883	0.0770	0.0696					
g=2	0.1187	0.0869	0.0689	0.0581	0.0487					
g=3	0.1103	0.0759	0.0572	0.0456	0.0365					
g=4	0.1064	0.0673	0.0490	0.0349	0.0253					
g=5	0.1021	0.0613	0.0412	0.0257	0.0184					
g=6						0.0095	0.0057	0.0033	0.0005	-0.0022
g=7						0.0057	0.0024	-0.0002	-0.0031	-0.0052
g=8						0.0045	0.0004	-0.0034	-0.0057	-0.0072
g=9						0.0023	-0.0023	-0.0050	-0.0067	-0.0091
g=10						-0.0015	-0.0047	-0.0063	-0.0079	-0.0098
g=20		0.0169		-0.0043		-0.0121		-0.0177		-0.0192
g=25		0.0117		-0.0082		-0.0157		-0.0192		-0.0201
g=30		0.0085		-0.0115		-0.0183		-0.0202		-0.0215
g=35		0.0048		-0.0132		-0.0199		-0.0225		-0.0239
g=40		0.0004		-0.0162		-0.0206		-0.0226		-0.0244
g=45		-0.0012		-0.0168		-0.0214		-0.0243		-0.0257
g=50		-0.0042		-0.0195		-0.0233		-0.0250		-0.0261
g=100		-0.0189		-0.0275		-0.0307		-0.0310		-0.0310
g=200		-0.0325		-0.0376		-0.0376		-0.0376		-0.0376
g=300		-0.0414		-0.0448		-0.0448		-0.0448		-0.0448
g=400		-0.0468		-0.0473		-0.0473		-0.0473		-0.0473
g=500		-0.0516		-0.0516		-0.0516		-0.0516		-0.0516

Table 4.364: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	-0.1301	-0.1047	-0.0861	-0.0743	-0.0690					
g=2	-0.1185	-0.0854	-0.0686	-0.0578	-0.0493					
g=3	-0.1128	-0.0726	-0.0560	-0.0451	-0.0382					
g=4	-0.1060	-0.0665	-0.0473	-0.0367	-0.0294					
g=5	-0.0984	-0.0607	-0.0422	-0.0302	-0.0203					
g=6						-0.0111	-0.0071	-0.0040	-0.0006	0.0013
g=7						-0.0078	-0.0037	0.0002	0.0020	0.0053
g=8						-0.0057	-0.0023	0.0021	0.0064	0.0088
g=9						-0.0025	0.0017	0.0045	0.0068	0.0087
g=10						-0.0006	0.0034	0.0071	0.0085	0.0104
g=20		-0.0181		0.0032		0.0131		0.0175		0.0203
g=25		-0.0143		0.0070		0.0161		0.0201		0.0222
g=30		-0.0088		0.0117		0.0185		0.0221		0.0240
g=35		-0.0047		0.0155		0.0205		0.0232		0.0247
g=40		-0.0011		0.0164		0.0219		0.0242		0.0259
g=45		0.0011		0.0183		0.0238		0.0259		0.0271
g=50		0.0026		0.0192		0.0240		0.0268		0.0280
g=100		0.0185		0.0284		0.0306		0.0317		0.0319
g=200		0.0326		0.0377		0.0382		0.0385		0.0385
g=300		0.0425		0.0457		0.0457		0.0457		0.0457
g=400		0.0473		0.0487		0.0487		0.0487		0.0487
g=500		0.0540		0.0543		0.0543		0.0543		0.0543

Table 4.365: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0303	0.0256	0.0221	0.0195	0.0182					
g=2	0.0285	0.0218	0.0180	0.0161	0.0141					
g=3	0.0267	0.0195	0.0159	0.0137	0.0119					
g=4	0.0261	0.0178	0.0143	0.0114	0.0096					
g=5	0.0248	0.0167	0.0129	0.0097	0.0083					
g=6						0.0068	0.0062	0.0058	0.0053	0.0050
g=7						0.0062	0.0056	0.0053	0.0049	0.0046
g=8						0.0060	0.0054	0.0048	0.0044	0.0041
g=9						0.0057	0.0050	0.0046	0.0043	0.0038
g=10						0.0052	0.0046	0.0044	0.0041	0.0036
g=20		0.0083		0.0048		0.0033		0.0025		0.0023
g=25		0.0074		0.0041		0.0028		0.0023		0.0022
g=30		0.0069		0.0035		0.0025		0.0023		0.0021
g=35		0.0063		0.0034		0.0023		0.0020		0.0017
g=40		0.0058		0.0030		0.0023		0.0020		0.0017
g=45		0.0056		0.0030		0.0023		0.0018		0.0016
g=50		0.0051		0.0026		0.0020		0.0018		0.0016
g=100		0.0034		0.0021		0.0016		0.0016		0.0016
g=200		0.0029		0.0022		0.0022		0.0022		0.0022
g=300		0.0035		0.0029		0.0029		0.0029		0.0029
g=400		0.0034		0.0034		0.0034		0.0034		0.0034
g=500		0.0041		0.0041		0.0041		0.0041		0.0041

Table 4.366: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_1$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0323	0.0264	0.0223	0.0196	0.0184					
g=2	0.0296	0.0221	0.0185	0.0161	0.0144					
g=3	0.0283	0.0194	0.0156	0.0136	0.0123					
g=4	0.0268	0.0181	0.0141	0.0119	0.0105					
g=5	0.0250	0.0167	0.0130	0.0108	0.0087					
g=6						0.0072	0.0066	0.0061	0.0055	0.0053
g=7						0.0067	0.0061	0.0055	0.0052	0.0046
g=8						0.0065	0.0060	0.0053	0.0043	0.0039
g=9						0.0059	0.0053	0.0048	0.0044	0.0041
g=10						0.0057	0.0050	0.0043	0.0041	0.0037
g=20		0.0087		0.0051		0.0034		0.0027		0.0022
g=25		0.0082		0.0046		0.0031		0.0024		0.0020
g=30		0.0070		0.0038		0.0027		0.0021		0.0017
g=35		0.0066		0.0032		0.0025		0.0020		0.0017
g=40		0.0060		0.0033		0.0023		0.0019		0.0017
g=45		0.0057		0.0030		0.0021		0.0017		0.0015
g=50		0.0056		0.0029		0.0021		0.0016		0.0014
g=100		0.0039		0.0022		0.0018		0.0016		0.0016
g=200		0.0029		0.0021		0.0020		0.0020		0.0020
g=300		0.0034		0.0029		0.0029		0.0029		0.0029
g=400		0.0036		0.0034		0.0034		0.0034		0.0034
g=500		0.0043		0.0042		0.0042		0.0042		0.0042

Concluding remarks about the simulated data from a scaled triangular distribution with $1-p=0.4$, $n=10000$, and $[a,b]=[0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions, although the normal kernel is used for all of the results.
- It is recommended to use a single minimum point, but more minima can also be used up to $m=5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1-\hat{h}_5$ is recommended.
- The Anderson-Darling and Cramér-von-Mises goodness-of-fit tests are recommended without definite exclusion of the Kolmogorov-Smirnov goodness-of-fit test. For smaller sample sizes, the Kolmogorov-Smirnov goodness-of-fit test seems to perform better than the Cramér-von-Mises goodness-of-fit test. The preferential test is still the Anderson-Darling goodness-of-fit test.
- For the goodness-of-fit tests, a level of significance of 1% or 5% can be used.
- The choices of $1\leq r\leq 10$ and $1\leq g\leq 10$ are recommended, with smaller values of r and g preferred when smaller α-values are used, and vice versa.

Tables 4.367 – 4.368 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a,b]=[0.3\,,\,0.7]$.

Table 4.367: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m=1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha=0.01$.*

	r=2		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.32	0.68	0.30	0.70
g=2	0.31	0.69	0.29	0.70
g=3	0.30	0.70	0.29	0.71
g=4	0.30	0.70	0.28	0.71
g=5	0.29	0.70	0.28	0.72

Table 4.368: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.05$.*

	r=6		**r=8**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.31	0.69	0.30	0.70
g=2	0.31	0.69	0.30	0.70
g=3	0.30	0.69	0.30	0.70
g=4	0.30	0.70	0.30	0.70
g=5	0.30	0.70	0.29	0.71
g=20	0.29	0.71	0.28	0.72

Remark: *The estimated off-pulse interval for this study population is extremely good since the noise level* $1 - p = 0.4$.

4.6.8 Data set parameters: $1 - p = 0.4$, $n = 25000$ and $[a, b] = [0.3\,,\,0.7]$

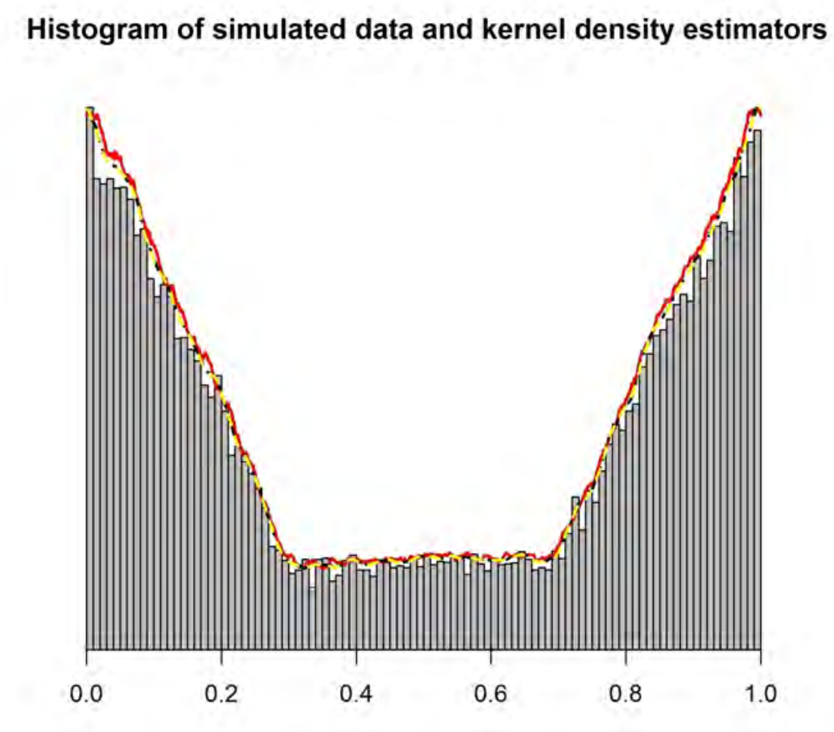

Figure 4.26: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.14$.

Figure 4.26 is a histogram representation (with 100 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis.

Choice of kernel function

From the analyses of all of the previous target populations, it was found that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used in the analysis of this study population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus. Tables 4.369 – 4.370 highlight the values of the estimated bias and MSE for different combinations of m when $g = 6$, $r = 6$, $\alpha = 0.01$ for all of the goodness-of-fit tests and for $\hat{h}_3$. It can be seen that different values of m result in small changes in the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. Therefore, it seems fair to recommend $m = 1$ as a good choice, as it will save computing time, without any reasonable effect on the estimation of a and b.

Table 4.369: *Bias for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0012	0.0027	-0.0384	0.0386	-0.0087	0.0111	-0.0188	0.0174
m=2	-0.0009	0.0038	-0.0385	0.0385	-0.0090	0.0109	-0.0188	0.0176
m=3	-0.0003	0.0038	-0.0385	0.0384	-0.0084	0.0107	-0.0187	0.0183
m=4	0.0002	0.0031	-0.0386	0.0384	-0.0083	0.0101	-0.0186	0.0187
m=5	0.0001	0.0031	-0.0386	0.0384	-0.0089	0.0101	-0.0183	0.0189
m=6	0.0005	0.0032	-0.0386	0.0384	-0.0086	0.0099	-0.0179	0.0193
m=7	0.0003	0.0035	-0.0387	0.0384	-0.0086	0.0102	-0.0179	0.0194
m=8	-0.0001	0.0034	-0.0387	0.0384	-0.0090	0.0103	-0.0181	0.0193
m=9	-0.0002	0.0039	-0.0387	0.0384	-0.0086	0.0106	-0.0180	0.0195
m=10	-0.0004	0.0038	-0.0387	0.0384	-0.0087	0.0107	-0.0178	0.0194
m=11	-0.0005	0.0038	-0.0387	0.0384	-0.0089	0.0106	-0.0179	0.0194
m=12	-0.0008	0.0040	-0.0387	0.0384	-0.0091	0.0106	-0.0181	0.0194
m=13	-0.0007	0.0038	-0.0387	0.0384	-0.0093	0.0106	-0.0182	0.0192
m=14	-0.0009	0.0040	-0.0387	0.0384	-0.0095	0.0108	-0.0183	0.0194
m=15	-0.0009	0.0040	-0.0387	0.0384	-0.0095	0.0108	-0.0183	0.0196
m=16	-0.0009	0.0042	-0.0387	0.0385	-0.0094	0.0110	-0.0184	0.0197
m=17	-0.0010	0.0043	-0.0387	0.0385	-0.0095	0.0110	-0.0185	0.0196
m=18	-0.0011	0.0044	-0.0387	0.0385	-0.0097	0.0111	-0.0186	0.0196
m=19	-0.0011	0.0046	-0.0387	0.0385	-0.0098	0.0113	-0.0185	0.0196
m=20	-0.0012	0.0047	-0.0388	0.0385	-0.0099	0.0113	-0.0186	0.0196

Table 4.370: *MSE for different choices of m for $\hat{h}_3$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0051	0.0049	0.0016	0.0015	0.0046	0.0042	0.0046	0.0049
m=2	0.0050	0.0045	0.0016	0.0015	0.0046	0.0041	0.0046	0.0048
m=3	0.0051	0.0046	0.0016	0.0015	0.0047	0.0042	0.0047	0.0046
m=4	0.0052	0.0047	0.0016	0.0015	0.0047	0.0043	0.0047	0.0045
m=5	0.0052	0.0047	0.0016	0.0015	0.0046	0.0043	0.0047	0.0045
m=6	0.0053	0.0047	0.0016	0.0015	0.0047	0.0044	0.0049	0.0044
m=7	0.0053	0.0046	0.0016	0.0015	0.0047	0.0043	0.0048	0.0044
m=8	0.0052	0.0046	0.0016	0.0015	0.0046	0.0043	0.0048	0.0044
m=9	0.0052	0.0045	0.0016	0.0015	0.0047	0.0042	0.0049	0.0043
m=10	0.0052	0.0046	0.0016	0.0015	0.0047	0.0042	0.0049	0.0043
m=11	0.0051	0.0045	0.0016	0.0015	0.0047	0.0042	0.0049	0.0043
m=12	0.0051	0.0045	0.0016	0.0015	0.0046	0.0042	0.0048	0.0043
m=13	0.0051	0.0045	0.0016	0.0015	0.0046	0.0042	0.0048	0.0044
m=14	0.0050	0.0045	0.0016	0.0015	0.0045	0.0042	0.0047	0.0043
m=15	0.0051	0.0045	0.0016	0.0015	0.0045	0.0042	0.0047	0.0043
m=16	0.0050	0.0045	0.0016	0.0015	0.0045	0.0041	0.0047	0.0042
m=17	0.0050	0.0044	0.0016	0.0015	0.0045	0.0041	0.0047	0.0043
m=18	0.0050	0.0044	0.0016	0.0015	0.0045	0.0041	0.0047	0.0043
m=19	0.0050	0.0044	0.0016	0.0015	0.0044	0.0041	0.0047	0.0043
m=20	0.0050	0.0043	0.0016	0.0015	0.0044	0.0041	0.0047	0.0043

Choice of estimated smoothing parameters

Tables 4.371 and 4.372 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 6$, $r = 6$ and $\alpha = 0.01$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were done with different values of r and g.

Table 4.371: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 6$ and $r = 6$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0072	0.0070	-0.0387	0.0384	-0.0140	0.0117	-0.0214	0.0199
$\hat{h}_2$	0.0023	0.0022	-0.0389	0.0384	-0.0062	0.0102	-0.0186	0.0163
$\hat{h}_3$	-0.0012	0.0027	-0.0384	0.0386	-0.0087	0.0111	-0.0188	0.0174
$\hat{h}_4$	-0.0012	0.0027	-0.0384	0.0386	-0.0087	0.0111	-0.0188	0.0174
$\hat{h}_5$	-0.0116	0.0110	-0.0389	0.0387	-0.0176	0.0154	-0.0241	0.0242
$\hat{h}_6$	0.0075	-0.0114	-0.0402	0.0397	-0.0042	-0.0008	-0.0221	0.0160
$\hat{h}_7$	0.0075	-0.0114	-0.0402	0.0397	-0.0042	-0.0008	-0.0221	0.0160
$\hat{h}_8$	0.0061	0.0040	-0.0388	0.0385	-0.0058	0.0098	-0.0169	0.0183
$\hat{h}_9$	0.0075	-0.0114	-0.0402	0.0397	-0.0042	-0.0008	-0.0221	0.0160

Table 4.372: *Comparison of the MSE for different combinations of the estimated smoothing parameter with $m = 1$, $g = 6$ and $r = 6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0039	0.0037	0.0015	0.0015	0.0036	0.0038	0.0039	0.0039
$\hat{h}_2$	0.0057	0.0050	0.0016	0.0015	0.0053	0.0045	0.0049	0.0052
$\hat{h}_3$	0.0051	0.0049	0.0016	0.0015	0.0046	0.0042	0.0046	0.0049
$\hat{h}_4$	0.0051	0.0049	0.0016	0.0015	0.0046	0.0042	0.0046	0.0049
$\hat{h}_5$	0.0028	0.0030	0.0015	0.0015	0.0025	0.0030	0.0030	0.0028
$\hat{h}_6$	0.0081	0.0084	0.0017	0.0016	0.0066	0.0072	0.0039	0.0049
$\hat{h}_7$	0.0081	0.0084	0.0017	0.0016	0.0066	0.0072	0.0039	0.0049
$\hat{h}_8$	0.0067	0.0046	0.0016	0.0015	0.0054	0.0044	0.0050	0.0046
$\hat{h}_9$	0.0081	0.0084	0.0017	0.0016	0.0066	0.0072	0.0039	0.0049

When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero. Also, when comparing the estimated MSE, only one of the choices for the estimated smoothing parameter results in a slightly smaller value of the MSE, namely $\hat{h}_5$. In the light of this, any one of the previously recommended smoothing parameters may be used, such as $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$.

Choice of goodness-of-fit test

Some of the tables provided earlier can be used to assess the goodness-of-fit tests. The reader can also inspect Tables 4.373 – 4.374 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, for two different values of α, $m = 1$ and for $\hat{h}_5$. Several other combinations of g, r and α all led to similar conclusions.

Table 4.373: *Comparison of the estimated bias for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_5$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.0116	0.0110	-0.0389	0.0387	-0.0176	0.0154	-0.0241	0.0242
g=2	-0.0129	0.0126	-0.0389	0.0387	-0.0181	0.0169	-0.0250	0.0250
g=3	-0.0139	0.0136	-0.0389	0.0387	-0.0193	0.0175	-0.0261	0.0249
g=4	-0.0148	0.0148	-0.0389	0.0388	-0.0196	0.0190	-0.0265	0.0265
g=5	-0.0151	0.0160	-0.0389	0.0388	-0.0205	0.0185	-0.0271	0.0270
g=20	-0.0195	0.0207	-0.0392	0.0391	-0.0235	0.0225	-0.0301	0.0291
g=25	-0.0211	0.0207	-0.0394	0.0392	-0.0243	0.0231	-0.0311	0.0306
g=30	-0.0224	0.0228	-0.0395	0.0393	-0.0252	0.0251	-0.0323	0.0315

Table 4.374: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.01$, $m = 1$, $r = 6$ and $\hat{h}_5$, for different values of g.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0028	0.0030	0.0015	0.0015	0.0025	0.0030	0.0030	0.0028
g=2	0.0026	0.0027	0.0015	0.0015	0.0025	0.0027	0.0029	0.0028
g=3	0.0024	0.0026	0.0016	0.0015	0.0024	0.0026	0.0027	0.0028
g=4	0.0022	0.0024	0.0016	0.0015	0.0024	0.0023	0.0027	0.0026
g=5	0.0022	0.0022	0.0016	0.0015	0.0022	0.0025	0.0026	0.0025
g=20	0.0015	0.0014	0.0016	0.0016	0.0018	0.0018	0.0022	0.0023
g=25	0.0014	0.0014	0.0016	0.0016	0.0017	0.0017	0.0022	0.0020
g=30	0.0012	0.0011	0.0016	0.0016	0.0016	0.0014	0.0020	0.0019

In terms of estimated bias, the Anderson-Darling goodness-of-fit test is superior for any choice of $\hat{h}$ and g. The goodness-of-fit test with the second-best estimated bias is the Kolmogorov-Smirnov goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs worse than the Rayleigh goodness-of-fit test in terms of bias. When comparing the estimated MSE, the Cramér-von-Mises goodness-of-fit test performs best, followed by the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests. Again it seems as if the Cramér-von-Mises and Anderson-Darling goodness-of-fit tests can both be used to produce optimal results when estimating a and b.

Choice of the significance level α

Tables 4.375 – 4.376 are constructed to investigate the effect of α in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. These tables compare the estimated bias and MSE for the Anderson-Darling goodness-of-fit test, with $\hat{h}_5$, $m = 1$ and $r = 2$ (Table 4.375) and $r = 6$ (Table 4.376). Several more comparisons were made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

Table 4.375: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 2$ and $\hat{h}_5$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
	Bias		MSE		Bias		MSE		Bias		MSE	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.03	-0.03	0.01	0.01	0.13	-0.12	0.03	0.03	0.17	-0.16	0.04	0.04
g=2	0.02	-0.02	0.01	0.01	0.11	-0.10	0.03	0.02	0.16	-0.15	0.04	0.03
g=3	0.02	-0.01	0.01	0.01	0.10	-0.09	0.02	0.02	0.15	-0.14	0.03	0.03
g=4	0.01	-0.01	0.01	0.01	0.09	-0.08	0.02	0.02	0.14	-0.13	0.03	0.03
g=5	0.01	-0.00	0.01	0.01	0.09	-0.08	0.02	0.02	0.14	-0.13	0.03	0.03
g=20	-0.01	0.01	0.00	0.00	0.04	-0.04	0.01	0.01	0.09	-0.08	0.02	0.02
g=25	-0.01	0.01	0.00	0.00	0.04	-0.03	0.01	0.01	0.08	-0.07	0.02	0.02
g=30	-0.01	0.01	0.00	0.00	0.03	-0.03	0.01	0.01	0.07	-0.06	0.02	0.02

Table 4.376: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 6$ and $\hat{h}_5$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=6	-0.01	0.01	0.00	0.00	0.03	-0.03	0.01	0.01	0.08	-0.06	0.02	0.02
g=7	-0.01	0.01	0.00	0.00	0.03	-0.03	0.01	0.01	0.07	-0.06	0.02	0.01
g=8	-0.01	0.01	0.00	0.00	0.02	-0.02	0.01	0.01	0.06	-0.06	0.01	0.01
g=9	-0.01	0.01	0.00	0.00	0.02	-0.02	0.01	0.01	0.06	-0.05	0.01	0.01
g=10	-0.02	0.02	0.00	0.00	0.02	-0.02	0.01	0.01	0.05	-0.05	0.01	0.01
g=20	-0.02	0.02	0.00	0.00	0.00	-0.00	0.00	0.00	0.03	-0.02	0.01	0.01
g=25	-0.02	0.02	0.00	0.00	-0.00	0.00	0.00	0.00	0.02	-0.02	0.01	0.01
g=30	-0.02	0.02	0.00	0.00	-0.00	0.01	0.00	0.00	0.01	-0.01	0.01	0.01

The tables highlight the fact that both $\alpha = 0.01$ and $\alpha = 0.05$ may result in estimated bias values close to zero, depending on the choice of g and r. For $1 \leq g \leq 20$ with $r = 2$, it is preferable to use $\alpha = 0.01$. When choosing $20 \leq g \leq 30$ with $r = 6$, it is preferable to use $\alpha = 0.05$. For both combination sets, the estimated MSE-values are very close to each other. This is quite interesting, as it illustrates the interdependent nature of the parameters. When a small value of α is used, the user does not have to be too cautious of incorrectly rejecting uniformity by selecting a large value of r. The reason is that the small value of α is already providing protection against the incorrect rejection of uniformity. On the other hand, if $\alpha = 0.05$ and a larger value of r is used, similar protection against the incorrect rejection of uniformity is provided. Thus, both $\alpha = 0.01$ and $\alpha = 0.05$ are recommended, but the user must then take the choice of r into account, e.g., choose smaller values of r with smaller values of α and vice versa. This same trend is found for the Kolmogorov-Smirnov and Rayleigh goodness-of-fit tests. The estimated bias and MSE for the Cramér-von-Mises goodness-of-fit test are more robust against different choices of α and r.

Choice of incremental growth g and number of intervals of rejection r

The reader can inspect Tables 4.373 – 4.374 and also Tables 4.375 – 4.376 for a broader understanding of the effect of g, r and α. More specifically, Tables 4.377 – 4.380 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Anderson-Darling goodness-of-fit test when $\hat{h}_5$ is chosen. For the last mentioned tables, the normal kernel is used, with $\alpha = 0.05$ and $m = 1$ kept constant.

For the Anderson-Darling goodness-of-fit test, small values of r and g result in estimated bias and MSE-values that are close to zero, i.e., $1 \leq g \leq 20$ with $1 \leq r \leq 4$ if $\alpha = 0.01$. When $\alpha = 0.05$ and $20 \leq g \leq 50$ with $6 \leq r \leq 10$, similar values of the estimated bias and MSE are found. In conclusion, it is recommended to use a value of g in the range from 1 to 40, and for r, values from 1 to 10, depending on the choice of α, i.e., choose smaller values of r and g when using smaller values of α, and vice versa.

Table 4.377: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.1503	0.1299	0.1144	0.1029	0.0958					
g=2	0.1426	0.1115	0.0954	0.0836	0.0747					
g=3	0.1333	0.0990	0.0837	0.0714	0.0634					
g=4	0.1298	0.0948	0.0744	0.0629	0.0538					
g=5	0.1248	0.0865	0.0688	0.0558	0.0472					
g=6						0.0332	0.0261	0.0224	0.0174	0.0146
g=7						0.0291	0.0223	0.0178	0.0153	0.0113
g=8						0.0237	0.0173	0.0141	0.0103	0.0080
g=9						0.0193	0.0152	0.0110	0.0079	0.0053
g=10						0.0182	0.0120	0.0083	0.0052	0.0036
g=20		0.0435		0.0112		0.0025		-0.0029		-0.0076
g=25		0.0371		0.0070		-0.0014		-0.0071		-0.0095
g=30		0.0297		0.0036		-0.0037		-0.0091		-0.0111
g=35		0.0245		0.0026		-0.0075		-0.0104		-0.0128
g=40		0.0191		-0.0019		-0.0090		-0.0117		-0.0135
g=45		0.0183		-0.0039		-0.0100		-0.0131		-0.0141
g=50		0.0150		-0.0055		-0.0114		-0.0139		-0.0155
g=100		-0.0016		-0.0142		-0.0165		-0.0184		-0.0197
g=200		-0.0128		-0.0206		-0.0233		-0.0240		-0.0245
g=300		-0.0210		-0.0254		-0.0274		-0.0284		-0.0287
g=400		-0.0259		-0.0300		-0.0310		-0.0315		-0.0315
g=500		-0.0297		-0.0327		-0.0339		-0.0339		-0.0339

Table 4.378: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.1427	-0.1185	-0.1039	-0.0946	-0.0888					
g=2	-0.1320	-0.1035	-0.0885	-0.0774	-0.0702					
g=3	-0.1227	-0.0944	-0.0764	-0.0682	-0.0588					
g=4	-0.1202	-0.0850	-0.0706	-0.0581	-0.0478					
g=5	-0.1168	-0.0804	-0.0634	-0.0518	-0.0427					
g=6						-0.0302	-0.0247	-0.0192	-0.0165	-0.0135
g=7						-0.0262	-0.0210	-0.0178	-0.0134	-0.0116
g=8						-0.0221	-0.0173	-0.0139	-0.0116	-0.0083
g=9						-0.0193	-0.0139	-0.0114	-0.0081	-0.0056
g=10						-0.0169	-0.0127	-0.0086	-0.0070	-0.0053
g=20		-0.0396		-0.0126		-0.0039		0.0028		0.0080
g=25		-0.0339		-0.0087		0.0017		0.0072		0.0092
g=30		-0.0296		-0.0041		0.0060		0.0087		0.0107
g=35		-0.0253		-0.0016		0.0068		0.0105		0.0131
g=40		-0.0198		0.0008		0.0089		0.0124		0.0140
g=45		-0.0190		0.0045		0.0099		0.0131		0.0151
g=50		-0.0148		0.0047		0.0113		0.0139		0.0157
g=100		-0.0002		0.0151		0.0189		0.0202		0.0207
g=200		0.0147		0.0225		0.0247		0.0250		0.0253
g=300		0.0229		0.0275		0.0282		0.0285		0.0285
g=400		0.0267		0.0310		0.0316		0.0316		0.0316
g=500		0.0313		0.0339		0.0343		0.0343		0.0343

Table 4.379: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0356	0.0310	0.0272	0.0244	0.0228					
g=2	0.0339	0.0265	0.0227	0.0200	0.0181					
g=3	0.0316	0.0234	0.0201	0.0174	0.0157					
g=4	0.0308	0.0226	0.0181	0.0158	0.0138					
g=5	0.0296	0.0207	0.0168	0.0140	0.0125					
g=6						0.0096	0.0082	0.0075	0.0066	0.0062
g=7						0.0089	0.0074	0.0066	0.0064	0.0057
g=8						0.0076	0.0066	0.0062	0.0056	0.0053
g=9						0.0069	0.0064	0.0057	0.0052	0.0048
g=10						0.0069	0.0059	0.0053	0.0047	0.0044
g=20		0.0117		0.0058		0.0042		0.0034		0.0025
g=25		0.0104		0.0051		0.0036		0.0026		0.0022
g=30		0.0091		0.0045		0.0033		0.0023		0.0020
g=35		0.0080		0.0043		0.0026		0.0022		0.0018
g=40		0.0071		0.0036		0.0024		0.0020		0.0018
g=45		0.0071		0.0032		0.0024		0.0018		0.0017
g=50		0.0067		0.0030		0.0021		0.0017		0.0015
g=100		0.0039		0.0020		0.0017		0.0014		0.0012
g=200		0.0028		0.0017		0.0013		0.0012		0.0011
g=300		0.0022		0.0017		0.0013		0.0011		0.0011
g=400		0.0022		0.0016		0.0014		0.0013		0.0013
g=500		0.0022		0.0018		0.0015		0.0015		0.0015

Table 4.380: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Anderson-Darling goodness-of-fit test, $\alpha = 0.05$, $\hat{h}_5$ and $m = 1$.*

	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
g=1	0.0333	0.0278	0.0243	0.0223	0.0210					
g=2	0.0307	0.0243	0.0210	0.0186	0.0171					
g=3	0.0287	0.0223	0.0184	0.0166	0.0146					
g=4	0.0280	0.0201	0.0172	0.0144	0.0124					
g=5	0.0271	0.0192	0.0156	0.0132	0.0115					
g=6						0.0090	0.0081	0.0071	0.0065	0.0059
g=7						0.0083	0.0073	0.0067	0.0059	0.0056
g=8						0.0076	0.0066	0.0060	0.0056	0.0051
g=9						0.0070	0.0060	0.0056	0.0051	0.0047
g=10						0.0065	0.0059	0.0052	0.0050	0.0047
g=20		0.0109		0.0059		0.0046		0.0034		0.0026
g=25		0.0099		0.0052		0.0037		0.0027		0.0024
g=30		0.0091		0.0046		0.0029		0.0025		0.0021
g=35		0.0083		0.0043		0.0028		0.0023		0.0019
g=40		0.0073		0.0037		0.0024		0.0020		0.0018
g=45		0.0074		0.0032		0.0023		0.0020		0.0016
g=50		0.0065		0.0032		0.0022		0.0018		0.0015
g=100		0.0044		0.0019		0.0013		0.0011		0.0011
g=200		0.0026		0.0014		0.0011		0.0011		0.0010
g=300		0.0020		0.0014		0.0013		0.0012		0.0012
g=400		0.0021		0.0016		0.0014		0.0014		0.0014
g=500		0.0020		0.0017		0.0016		0.0016		0.0016

Concluding remarks about the simulated data from a scaled triangular distribution with $1-p=0.4$, $n=25000$, and $[a,b]=[0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions, although the normal kernel is used for all of the results.
- It is recommended to use a single minimum point, but more minima can also be used up to $m=5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$ – $\hat{h}_5$ is recommended.
- The Anderson-Darling and Cramér-von-Mises goodness-of-fit tests are recommended.
- For these goodness-of-fit tests, a level of significance of 1% or 5% can be used.
- The choices of $1 \leq r \leq 10$ and $1 \leq g \leq 40$ are recommended, with smaller values of r and g used with smaller α-values and larger values of r and g used with larger α-values.

Table 4.381 and Table 4.382 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a,b]=[0.3\,,\,0.7]$.

Table 4.381: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m=1$ and $\hat{h}_5$ for the Anderson-Darling goodness-of-fit test with $\alpha=0.01$.*

	r=2		**r=4**	
	$E(\hat{a})$	$E(\hat{b})$	$E(\hat{a})$	$E(\hat{b})$
g=1	0.32	0.68	0.32	0.69
g=2	0.32	0.68	0.31	0.70
g=3	0.32	0.69	0.30	0.70
g=4	0.31	0.69	0.30	0.70
g=5	0.31	0.70	0.30	0.70
g=20	0.30	0.70	0.29	0.71
g=25	0.30	0.71	0.29	0.71
g=30	0.29	0.71	0.28	0.72
g=35	0.29	0.71	0.28	0.72
g=40	0.29	0.71	0.28	0.72
g=45	0.28	0.72	0.28	0.72
g=50	0.28	0.72	0.28	0.72

Table 4.382: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_5$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.05$.*

	r=6		**r=8**	
	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$
g=1	0.33	0.68	0.32	0.68
g=2	0.32	0.68	0.32	0.68
g=3	0.32	0.69	0.31	0.69
g=4	0.31	0.69	0.31	0.69
g=5	0.31	0.69	0.30	0.70
g=20	0.30	0.70	0.30	0.70
g=25	0.30	0.70	0.30	0.70
g=30	0.30	0.70	0.29	0.71
g=35	0.29	0.71	0.29	0.71
g=40	0.29	0.71	0.29	0.71
g=45	0.29	0.71	0.29	0.71
g=50	0.29	0.71	0.29	0.71

Remark: *The estimated off-pulse interval for this study population is extremely good if one takes the noise level of 40% into account.*

4.6.9 Data set parameters: $1 - p = 0.1$, $n = 500$ and $[a, b] = [0.3\,,\,0.7]$

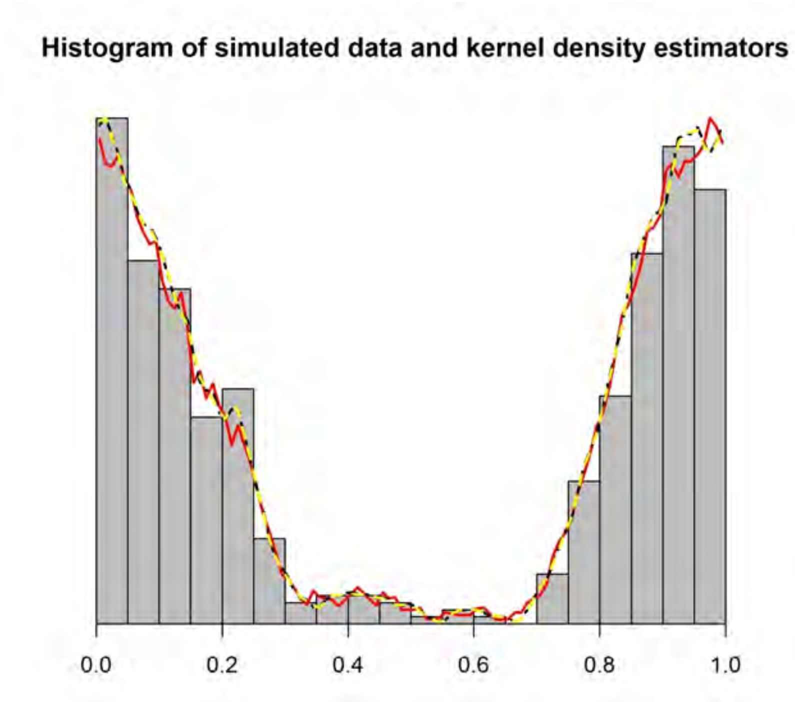

Figure 4.27: *A simulated data set from the above distribution with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter $\hat{h}_3 = 0.23$.*

Figure 4.27 is a histogram representation (with 20 classes) of a single Monte Carlo simulation of the above data set. Different kernel density estimators are fitted to the data and illustrated by the lines superimposed on the histogram. The study population will now be analysed on a parameter per parameter basis, ceteris paribus.

Choice of kernel function

From Figure 4.27 and the analyses of all of the previous target populations, it is found that the choice of kernel function is not the most important aspect of the kernel density estimator in the application of SOPIE. Therefore, the normal kernel is used in the analysis of this target population.

Choice of the number of minimum points m

In order to assess whether the value of m influences the estimators $\hat{a}$ and $\hat{b}$, the bias and MSE of $\hat{a}$ and $\hat{b}$ are compared for different values of m, ceteris paribus.

Tables 4.383 – 4.384 highlight the values of the estimated bias and MSE for different combinations of m when $g = 2$, $r = 2$ and $\alpha = 0.10$ for all of the goodness-of-fit tests and for $\hat{h}_1$. It can be seen that different values of m result in small changes in the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$. From several of these comparisons using different values for g, r, α and $\hat{h}$, similar trends are observed. It can again be concluded that a small choice of m is preferable, since computing time is reduced. Furthermore, as far as bias and MSE are concerned, all four goodness-of-fit tests are relatively insensitive to the choice of m.

Table 4.383: *Bias for different choices of m for* $\hat{h}_1$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	-0.0294	0.0368	-0.0719	0.0783	-0.0169	0.0314	-0.0277	0.0259
m=2	-0.0317	0.0375	-0.0727	0.0775	-0.0208	0.0302	-0.0276	0.0278
m=3	-0.0330	0.0380	-0.0731	0.0776	-0.0225	0.0300	-0.0276	0.0289
m=4	-0.0342	0.0385	-0.0735	0.0779	-0.0243	0.0303	-0.0274	0.0287
m=5	-0.0351	0.0389	-0.0740	0.0781	-0.0254	0.0308	-0.0264	0.0285
m=6	-0.0358	0.0392	-0.0746	0.0783	-0.0260	0.0315	-0.0260	0.0286
m=7	-0.0363	0.0394	-0.0751	0.0787	-0.0267	0.0319	-0.0257	0.0289
m=8	-0.0370	0.0395	-0.0755	0.0790	-0.0272	0.0321	-0.0257	0.0289
m=9	-0.0374	0.0396	-0.0759	0.0792	-0.0275	0.0323	-0.0256	0.0290
m=10	-0.0376	0.0398	-0.0761	0.0795	-0.0278	0.0324	-0.0257	0.0289
m=11	-0.0378	0.0400	-0.0763	0.0796	-0.0280	0.0326	-0.0259	0.0291
m=12	-0.0378	0.0401	-0.0764	0.0797	-0.0281	0.0326	-0.0260	0.0291
m=13	-0.0379	0.0401	-0.0765	0.0798	-0.0281	0.0327	-0.0259	0.0291
m=14	-0.0379	0.0401	-0.0765	0.0798	-0.0282	0.0327	-0.0260	0.0291
m=15	-0.0379	0.0401	-0.0765	0.0798	-0.0282	0.0327	-0.0260	0.0291
m=16	-0.0379	0.0401	-0.0765	0.0798	-0.0282	0.0327	-0.0260	0.0292

Table 4.384: *MSE for different choices of m for $\hat{h}_1$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
m=1	0.0026	0.0025	0.0055	0.0064	0.0048	0.0034	0.0040	0.0045
m=2	0.0025	0.0024	0.0056	0.0063	0.0043	0.0035	0.0039	0.0043
m=3	0.0024	0.0024	0.0056	0.0063	0.0041	0.0035	0.0039	0.0042
m=4	0.0023	0.0024	0.0057	0.0064	0.0040	0.0036	0.0039	0.0042
m=5	0.0023	0.0024	0.0057	0.0064	0.0039	0.0036	0.0041	0.0041
m=6	0.0023	0.0024	0.0058	0.0064	0.0039	0.0035	0.0041	0.0041
m=7	0.0023	0.0024	0.0059	0.0065	0.0039	0.0036	0.0042	0.0041
m=8	0.0023	0.0024	0.0060	0.0065	0.0040	0.0036	0.0042	0.0041
m=9	0.0024	0.0025	0.0060	0.0066	0.0040	0.0036	0.0043	0.0042
m=10	0.0024	0.0025	0.0061	0.0066	0.0040	0.0037	0.0044	0.0042
m=11	0.0024	0.0025	0.0061	0.0066	0.0040	0.0037	0.0044	0.0042
m=12	0.0024	0.0025	0.0061	0.0066	0.0040	0.0037	0.0044	0.0043
m=13	0.0024	0.0025	0.0061	0.0067	0.0040	0.0037	0.0044	0.0043
m=14	0.0024	0.0025	0.0061	0.0067	0.0040	0.0037	0.0044	0.0043
m=15	0.0024	0.0025	0.0061	0.0067	0.0040	0.0037	0.0044	0.0043
m=16	0.0024	0.0025	0.0061	0.0067	0.0040	0.0037	0.0044	0.0043

Choice of estimated smoothing parameters

Tables 4.385 and 4.386 highlight the values of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for $m = 1$, $g = 2$, $r = 2$ and $\alpha = 0.10$ for the various goodness-of-fit tests and for different values of the estimated smoothing parameter $\hat{h}$. Several more comparisons were made with different values of r and g, but not all of the comparisons are shown.

Table 4.385: *Comparison of the bias for different combinations of the estimated smoothing parameter with $m = 1$, $g = 2$, $r = 2$ and $\alpha = 0.10$.*

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	-0.0294	0.0368	-0.0719	0.0783	-0.0169	0.0314	-0.0277	0.0259
$\hat{h}_2$	-0.0268	0.0410	-0.0700	0.0863	0.0040	0.0304	-0.0492	0.0330
$\hat{h}_3$	-0.0246	0.0423	-0.0699	0.0872	0.0069	0.0298	-0.0454	0.0397
$\hat{h}_4$	-0.0246	0.0423	-0.0699	0.0872	0.0069	0.0298	-0.0454	0.0397
$\hat{h}_5$	-0.0292	0.0390	-0.0715	0.0796	-0.0150	0.0306	-0.0318	0.0312
$\hat{h}_6$	-0.0890	-0.3346	0.1357	-0.1978	-0.0781	-0.3493	-0.0549	-0.3539
$\hat{h}_7$	-0.0939	-0.3843	0.1872	-0.2442	-0.0806	-0.3974	-0.0436	-0.3983
$\hat{h}_8$	-0.0299	0.0377	-0.0702	0.0846	-0.0005	0.0275	-0.0551	0.0253
$\hat{h}_9$	-0.0905	-0.4017	0.2065	-0.2596	-0.0788	-0.4150	-0.0409	-0.4163

It is found that the values of $\hat{h}_3$ and $\hat{h}_4$ are very close to each other, resulting in nearly equal values of the estimated bias and MSE. Almost equal values are also found for $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$, which explain why the results for these $\hat{h}$ combinations are almost equal in the tables. When inspecting the estimated bias, it is found that most of the choices for $\hat{h}$ are associated with estimated bias-values close to zero, but when $\hat{h}_6$ (related to $\hat{h}_7$ and $\hat{h}_9$) is used, the estimated bias is much larger.

Table 4.386: *Comparison of the MSE for different combinations of the estimated smoothing parameter with* $m = 1$, $g = 2$, $r = 2$ *and* $\alpha = 0.10$.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0028	0.0034	0.0065	0.0076	0.0034	0.0039	0.0048	0.0053
$\hat{h}_2$	0.0028	0.0043	0.0065	0.0096	0.0036	0.0048	0.0086	0.0074
$\hat{h}_3$	0.0029	0.0044	0.0064	0.0100	0.0040	0.0050	0.0079	0.0083
$\hat{h}_4$	0.0029	0.0044	0.0064	0.0100	0.0040	0.0050	0.0079	0.0083
$\hat{h}_5$	0.0029	0.0035	0.0064	0.0080	0.0036	0.0040	0.0051	0.0056
$\hat{h}_6$	0.1083	0.1207	0.1576	0.0815	0.1189	0.1223	0.1755	0.1120
$\hat{h}_7$	0.1254	0.1388	0.1806	0.0936	0.1385	0.1408	0.1997	0.1291
$\hat{h}_8$	0.0027	0.0041	0.0065	0.0092	0.0032	0.0046	0.0095	0.0066
$\hat{h}_9$	0.1323	0.1457	0.1872	0.0983	0.1464	0.1476	0.2098	0.1363

When comparing the estimated MSE, $\hat{h}_1$, $\hat{h}_2$ and $\hat{h}_5$ result in marginally smaller values of the estimated MSE, while the estimated MSE when using $\hat{h}_6$ is again much larger. This same trend is found for different values of r and g. In the light of this, it is recommended to use any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ as a good choice of the estimated smoothing parameter. It seems as if – for small to moderate sample sizes n – it is not optimal to choose $\hat{h}_6$.

Choice of goodness-of-fit test

Some of the tables provided earlier can be used to assess the goodness-of-fit tests. For a specific comparison of the goodness-of-fit tests, the reader can inspect Tables 4.387 – 4.388 for a comparison of the estimated bias and MSE of $\hat{a}$ and $\hat{b}$ for the different goodness-of-fit tests, with $m = 1$, $r = 2$ and for $\hat{h}_1$. Several other combinations of g, r and α all led to similar conclusions.

Table 4.387: *Comparison of the estimated bias for different goodness-of-fit tests with* $\alpha = 0.10$, $m = 1$, $r = 2$ *and* $\hat{h}_1$, *for different values of* g.

	Anderson-Darling		Cramér-von-Mises		Kolmogorov-Smirnov		Rayleigh	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.0247	0.0321	-0.0709	0.0772	-0.0033	0.0194	-0.0008	-0.0025
g=2	-0.0294	0.0368	-0.0719	0.0783	-0.0169	0.0314	-0.0277	0.0259
g=3	-0.0346	0.0411	-0.0731	0.0795	-0.0265	0.0384	-0.0309	0.0321
g=4	-0.0346	0.0415	-0.0743	0.0807	-0.0319	0.0417	-0.0420	0.0410
g=5	-0.0395	0.0463	-0.0753	0.0816	-0.0354	0.0458	-0.0462	0.0483
g=6	-0.0442	0.0494	-0.0765	0.0829	-0.0389	0.0469	-0.0511	0.0525
g=7	-0.0482	0.0536	-0.0774	0.0838	-0.0416	0.0490	-0.0528	0.0549
g=8	-0.0405	0.0481	-0.0788	0.0848	-0.0437	0.0513	-0.0549	0.0565
g=9	-0.0426	0.0502	-0.0799	0.0857	-0.0454	0.0525	-0.0561	0.0583
g=10	-0.0444	0.0524	-0.0812	0.0870	-0.0474	0.0542	-0.0585	0.0598

In terms of estimated bias, the Kolmogorov-Smirnov goodness-of-fit test is superior for any choice of $\hat{h}$, g, r and α. The goodness-of-fit test with the second-best estimated bias is the Anderson-Darling goodness-of-fit test, followed by the Rayleigh goodness-of-fit test. The Cramér-von-Mises test performs the worst in terms of estimated bias. When comparing the estimated MSE, the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests perform best, followed by the

Table 4.388: *Comparison of the estimated MSE for different goodness-of-fit tests with $\alpha = 0.10$, $m = 1$, $r = 2$ and $\hat{h}_1$, for different values of g.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	0.0027	0.0026	0.0053	0.0063	0.0063	0.0044	0.0063	0.0068
g=2	0.0026	0.0025	0.0055	0.0064	0.0048	0.0034	0.0040	0.0045
g=3	0.0024	0.0025	0.0056	0.0066	0.0040	0.0031	0.0042	0.0046
g=4	0.0025	0.0026	0.0058	0.0068	0.0035	0.0030	0.0037	0.0041
g=5	0.0026	0.0027	0.0059	0.0070	0.0033	0.0031	0.0036	0.0039
g=6	0.0027	0.0029	0.0061	0.0072	0.0032	0.0032	0.0035	0.0039
g=7	0.0029	0.0032	0.0063	0.0073	0.0032	0.0034	0.0037	0.0040
g=8	0.0030	0.0031	0.0065	0.0075	0.0033	0.0034	0.0038	0.0042
g=9	0.0030	0.0032	0.0066	0.0076	0.0034	0.0035	0.0039	0.0044
g=10	0.0032	0.0033	0.0069	0.0079	0.0034	0.0037	0.0041	0.0045

Rayleigh and Cramér-von-Mises goodness-of-fit tests. Again it seems as if the Kolmogorov-Smirnov test performs quite well when the sample size n is small. Similar to all of the previous simulated target populations, the Anderson-Darling goodness-of-fit test produces accurate and consistent estimates.

Choice of the significance level α

Several tables are constructed to investigate the effect of α in combination with the effect of g, $\hat{h}$ and the goodness-of-fit tests. Tables 4.389 – 4.390 compare the estimated bias and MSE for the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests, with $\hat{h}_1$, $m = 1$ and $r = 1$. Several more comparisons were made with different combinations of $\hat{h}$ and r, resulting in similar conclusions.

Table 4.389: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 1$ and $\hat{h}_1$ (Kolmogorov-Smirnov goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.05	0.06	0.00	0.00	-0.02	0.04	0.00	0.00	-0.00	0.02	0.01	0.00
g=2	-0.05	0.06	0.00	0.00	-0.03	0.04	0.00	0.00	-0.02	0.03	0.00	0.00
g=3	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00	-0.03	0.04	0.00	0.00
g=4	-0.06	0.06	0.00	0.00	-0.04	0.05	0.00	0.00	-0.03	0.04	0.00	0.00
g=5	-0.06	0.06	0.00	0.00	-0.05	0.05	0.00	0.00	-0.04	0.05	0.00	0.00
g=6	-0.06	0.06	0.00	0.00	-0.05	0.05	0.00	0.00	-0.04	0.05	0.00	0.00
g=7	-0.06	0.07	0.00	0.00	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00
g=8	-0.06	0.07	0.00	0.00	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00
g=9	-0.06	0.07	0.00	0.00	-0.05	0.06	0.00	0.00	-0.05	0.05	0.00	0.00
g=10	-0.06	0.07	0.00	0.01	-0.05	0.06	0.00	0.00	-0.05	0.05	0.00	0.00

From the tables (and some other analyses) it is concluded that all of the goodness-of-fit tests produce an estimated bias that tends closer to zero when α is *increased*. These findings are in line with the trends found for some of the previous target populations with a small to moderate sample size n and it is again recommended to use any value of α in the range from 0.05 to 0.1, with larger

Table 4.390: *Comparison of the estimated bias and MSE for different α-values, with $m = 1$, $r = 1$ and $\hat{h}_1$ (Anderson-Darling goodness-of-fit test).*

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	Bias		**MSE**		**Bias**		**MSE**		**Bias**		**MSE**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
g=1	-0.05	0.05	0.00	0.00	-0.03	0.04	0.00	0.00	-0.02	0.03	0.00	0.00
g=2	-0.05	0.05	0.00	0.00	-0.04	0.04	0.00	0.00	-0.03	0.04	0.00	0.00
g=3	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00	-0.03	0.04	0.00	0.00
g=4	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00	-0.03	0.04	0.00	0.00
g=5	-0.05	0.06	0.00	0.00	-0.05	0.05	0.00	0.00	-0.04	0.05	0.00	0.00
g=6	-0.06	0.06	0.00	0.00	-0.05	0.05	0.00	0.00	-0.04	0.05	0.00	0.00
g=7	-0.06	0.06	0.00	0.00	-0.05	0.06	0.00	0.00	-0.05	0.05	0.00	0.00
g=8	-0.06	0.06	0.00	0.00	-0.05	0.05	0.00	0.00	-0.04	0.05	0.00	0.00
g=9	-0.06	0.06	0.00	0.00	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00
g=10	-0.06	0.07	0.00	0.00	-0.05	0.06	0.00	0.00	-0.04	0.05	0.00	0.00

values of α performing better when the sample size is small.

Choice of incremental growth g and number of intervals of rejection r

The reader can inspect Tables 4.387 – 4.388 and also Tables 4.389 – 4.390 for a broader understanding of the effect of g, r and α. More specifically, Tables 4.391 – 4.394 provide the reader with a comparison of the estimated bias and MSE for different combinations of r and g, for the Kolmogorov-Smirnov goodness-of-fit test, when $\hat{h}_1$ is chosen with $\alpha = 0.10$ and $m = 1$.

For the Kolmogorov-Smirnov goodness-of-fit test, small values of r and g result in estimated bias-values that are close to zero, i.e., $1 \leq g \leq 2$ with $1 \leq r \leq 2$ for any value of α. For the estimated MSE, slightly larger values of r result in the lowest estimated MSE-values, i.e., $5 \leq r \leq 10$, although the estimated MSE-values are not vastly different from one another. In conclusion, it is recommended to use a value of g in the range from 1 to 5, and for r, values from 1 to 4, with preference given to smaller values, since this target population has such a small sample size n.

Table 4.391: *Comparison of the bias of $\hat{a}$ for different g and r-combinations for the Kolmogorov-Smirnov goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0219	-0.0033	-0.0127	-0.0199	-0.0233	-0.0261	-0.0284	-0.0296	-0.0302	-0.0306
g=2	0.0031	-0.0169	-0.0234	-0.0298	-0.0309	-0.0311	-0.0318	-0.0320	-0.0323	-0.0323
g=3	-0.0078	-0.0265	-0.0308	-0.0334	-0.0341	-0.0343	-0.0346	-0.0348	-0.0348	-0.0348
g=4	-0.0152	-0.0319	-0.0347	-0.0357	-0.0359	-0.0362	-0.0362	-0.0362	-0.0362	-0.0362
g=5	-0.0234	-0.0354	-0.0372	-0.0380	-0.0382	-0.0382	-0.0382	-0.0382	-0.0382	-0.0382
g=6	-0.0297	-0.0389	-0.0395	-0.0400	-0.0400	-0.0400	-0.0400	-0.0400	-0.0400	-0.0400
g=7	-0.0356	-0.0416	-0.0423	-0.0425	-0.0425	-0.0425	-0.0425	-0.0425	-0.0425	-0.0425
g=8	-0.0374	-0.0437	-0.0441	-0.0441	-0.0441	-0.0441	-0.0441	-0.0441	-0.0441	-0.0441
g=9	-0.0399	-0.0454	-0.0457	-0.0457	-0.0457	-0.0457	-0.0457	-0.0457	-0.0457	-0.0457
g=10	-0.0439	-0.0474	-0.0476	-0.0476	-0.0476	-0.0476	-0.0476	-0.0476	-0.0476	-0.0476

Table 4.392: *Comparison of the bias of $\hat{b}$ for different g and r-combinations for the Kolmogorov-Smirnov goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	-0.0170	0.0194	0.0271	0.0330	0.0359	0.0372	0.0377	0.0383	0.0393	0.0394
g=2	0.0093	0.0314	0.0379	0.0395	0.0408	0.0410	0.0417	0.0417	0.0417	0.0417
g=3	0.0189	0.0384	0.0409	0.0426	0.0431	0.0431	0.0431	0.0431	0.0431	0.0431
g=4	0.0282	0.0417	0.0443	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450
g=5	0.0335	0.0458	0.0474	0.0474	0.0474	0.0474	0.0474	0.0474	0.0474	0.0474
g=6	0.0403	0.0469	0.0483	0.0483	0.0483	0.0483	0.0483	0.0483	0.0483	0.0483
g=7	0.0421	0.0490	0.0499	0.0499	0.0499	0.0499	0.0499	0.0499	0.0499	0.0499
g=8	0.0445	0.0513	0.0515	0.0515	0.0515	0.0515	0.0515	0.0515	0.0515	0.0515
g=9	0.0469	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525
g=10	0.0504	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542

Table 4.393: *Comparison of the MSE of $\hat{a}$ for different g and r-combinations for the Kolmogorov-Smirnov goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0095	0.0063	0.0051	0.0042	0.0038	0.0034	0.0030	0.0029	0.0027	0.0027
g=2	0.0070	0.0048	0.0041	0.0031	0.0030	0.0029	0.0028	0.0028	0.0027	0.0027
g=3	0.0058	0.0040	0.0034	0.0030	0.0029	0.0029	0.0028	0.0028	0.0028	0.0028
g=4	0.0051	0.0035	0.0031	0.0029	0.0029	0.0029	0.0029	0.0029	0.0029	0.0029
g=5	0.0045	0.0033	0.0031	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030
g=6	0.0041	0.0032	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031
g=7	0.0036	0.0032	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031
g=8	0.0037	0.0033	0.0032	0.0032	0.0032	0.0032	0.0032	0.0032	0.0032	0.0032
g=9	0.0037	0.0034	0.0033	0.0033	0.0033	0.0033	0.0033	0.0033	0.0033	0.0033
g=10	0.0036	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034

Table 4.394: *Comparison of the MSE of $\hat{b}$ for different g and r-combinations for the Kolmogorov-Smirnov goodness-of-fit test, $\alpha = 0.10$, $\hat{h}_1$ and $m = 1$.*

	r=1	**r=2**	**r=3**	**r=4**	**r=5**	**r=6**	**r=7**	**r=8**	**r=9**	**r=10**
g=1	0.0087	0.0044	0.0036	0.0030	0.0028	0.0027	0.0026	0.0026	0.0025	0.0025
g=2	0.0056	0.0034	0.0029	0.0028	0.0027	0.0026	0.0025	0.0025	0.0025	0.0025
g=3	0.0049	0.0031	0.0029	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027	0.0027
g=4	0.0039	0.0030	0.0029	0.0028	0.0028	0.0028	0.0028	0.0028	0.0028	0.0028
g=5	0.0038	0.0031	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030
g=6	0.0036	0.0032	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031
g=7	0.0038	0.0034	0.0033	0.0033	0.0033	0.0033	0.0033	0.0033	0.0033	0.0033
g=8	0.0037	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034	0.0034
g=9	0.0038	0.0035	0.0035	0.0035	0.0035	0.0035	0.0035	0.0035	0.0035	0.0035
g=10	0.0038	0.0037	0.0037	0.0037	0.0037	0.0037	0.0037	0.0037	0.0037	0.0037

Concluding remarks about the simulated data from a scaled triangular distribution with $1-p=0.1$, $n=500$, and $[a,b]=[0.3\,,\,0.7]$

From the detailed analysis of each parameter that may influence the estimation of a and b, it is concluded that the following combination of parameter values will result in the optimal estimation of a and b.

- The kernel function does not greatly influence the estimation of a and b, and therefore it is recommended to use any of the kernel functions, although the normal kernel is used for all of the results.
- It is recommended to use a single minimum point, but more minima can also be used up to $m=5$.
- In terms of the estimated smoothing parameter $\hat{h}$, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ is recommended.
- The Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests are recommended, and again it seems as if the Kolmogorov-Smirnov goodness-of-fit test performs quite well when the sample size n is small.
- For both of these goodness-of-fit tests, a level of significance of 5% or 10% can be used, with preference given to larger values of α when the sample size is small.
- The choices of $1 \leq r \leq 4$ and $1 \leq g \leq 5$ are recommended, but smaller values within these ranges are preferred.

Table 4.395 and Table 4.396 provide the Monte Carlo estimates of the expected values of $\hat{a}$ and $\hat{b}$, denoted by $E(\hat{a})$ and $E(\hat{b})$, respectively, for the optimal choice of tuning parameters as recommended above. The results displayed in these tables should be compared to the off-pulse interval $[a,b]=[0.3\,,\,0.7]$.

Table 4.395: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m=1$ and $\hat{h}_1$ for the Kolmogorov-Smirnov goodness-of-fit test with $\alpha=0.10$.*

	r=1		**r=2**	
	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$
g=1	0.32	0.68	0.30	0.70
g=2	0.30	0.70	0.30	0.70
g=3	0.30	0.70	0.29	0.71
g=4	0.29	0.71	0.29	0.71
g=5	0.28	0.72	0.28	0.72
g=6	0.28	0.72	0.28	0.72
g=7	0.27	0.73	0.27	0.73
g=8	0.27	0.73	0.27	0.74
g=9	0.26	0.74	0.26	0.75
g=10	0.26	0.75	0.26	0.75

Table 4.396: *Monte Carlo estimates of $E(\hat{a})$ and $E(\hat{b})$ for $m = 1$ and $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test with $\alpha = 0.10$.*

	r=1		**r=2**	
	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$	$\boldsymbol{E(\hat{a})}$	$\boldsymbol{E(\hat{b})}$
g=1	0.30	0.70	0.29	0.71
g=2	0.29	0.71	0.28	0.72
g=3	0.28	0.72	0.27	0.73
g=4	0.27	0.73	0.27	0.74
g=5	0.27	0.74	0.26	0.74
g=6	0.26	0.74	0.26	0.74
g=7	0.26	0.75	0.25	0.75
g=8	0.26	0.74	0.26	0.74
g=9	0.26	0.74	0.26	0.75
g=10	0.26	0.74	0.26	0.75

Remark: *For this small sample study population, the estimated off-pulse interval is again extremely good, even for a wide array of different values of the tuning parameters.*

4.7 Recommended choice of kernel estimator, goodness-of-fit test and tuning parameters

After careful consideration of the results obtained from the simulation study, it is now essential to give a summary of the choice of kernel estimator, goodness-of-fit test and tuning parameters that consistently performed better than some other selections. This will serve as justification in the next chapter, where these quantities are applied in the analysis of pulsar data.

4.7.1 Kernel density estimator

The reader is once again reminded of the fact that the choice of kernel function is not the most important aspect of kernel density estimation (Silverman, 1986; Wand & Jones, 1995). Nevertheless, SOPIE requires the specification of some kernel function k in the definition of the kernel density estimator. Throughout the simulation study it was found that the choice of kernel function k did not have much impact on the estimated off-pulse interval. Graphically, this can also been seen in the illustration of each of the simulated target populations. Since the kernel density estimators drawn on top of each histogram overlapped in almost all figures, it can be concluded that the choice of kernel function does not influence the derived starting minimum points in the application of SOPIE.

- Hence, it was decided to use the normal kernel function when fitting the kernel density estimator in the implementation of SOPIE.

In kernel density estimation, the estimated smoothing parameter influences the amount of smoothing applied to the kernel density estimator. Therefore, different values of $\hat{h}$ may result in the identification of different minima, resulting in different starting values for SOPIE.

In the simulation study, it was frequently found that most of the choices for $\hat{h}$ can be used. In only a small number of target populations it was found that $\hat{h}_6$, $\hat{h}_8$ and $\hat{h}_9$ were not the best choices for the estimated smoothing parameter. For small data sets, $\hat{h}_6$ performed especially poor, and was usually closely related to $\hat{h}_7$ and $\hat{h}_9$. Otherwise, it did not seem as if specific preference should be given towards any of the remaining choices of the estimated smoothing parameters.

- Therefore, any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ can be used.

When inspecting Table 4.1, it is evident that all of the recommended estimated smoothing parameters relate to the plug-in estimate (Silverman, 1986), defined as

$$\hat{h} = 1.06\hat{\sigma}n^{-1/5} ,$$

where $\hat{\sigma}$ is some estimate of the dispersion of the data.

4.7.2 The goodness-of-fit test

Throughout the simulation study, three of the four goodness-of-fit tests stood out. In terms of estimated bias, the Anderson-Darling goodness-of-fit test performed better than any of the other tests. For small to moderate sample sizes, the Kolmogorov-Smirnov goodness-of-fit test performed second-best, followed by the Cramér-von-Mises goodness-of-fit test. The Rayleigh goodness-of-fit test never appeared in the list of top 3 tests concerning the estimated bias.

In terms of estimated MSE, the Cramér-von-Mises goodness-of-fit test proved to be the best goodness-of-fit test, especially for large sample sizes n. The Anderson-Darling goodness-of-fit test also performed very well based on the MSE, and even better than the Cramér-von-Mises goodness-of-fit test when the sample sizes were small. The Kolmogorov-Smirnov goodness-of-fit test also performed well in terms of MSE, especially when the sample sizes n were small. The recommendation is therefore to use two goodness-of-fit tests when performing SOPIE.

- For *large sample sizes*, the *Anderson-Darling* and *Cramér-von-Mises* goodness-of-fit tests are recommended.
- For *small to moderate sample sizes*, the *Anderson-Darling* and *Kolmogorov-Smirnov* goodness-of-fit tests are recommended, since the Kolmogorov-Smirnov goodness-of-fit test frequently performed better than the Cramér-von-Mises goodness-of-fit test for smaller sample sizes.

4.7.3 Tuning parameters m, α, r and g

The tuning parameter m represents the number of minimum points that will be used in SOPIE. The impact of m is twofold. Firstly, the value of m will heavily influence the computation time when performing SOPIE. Secondly, when the identification of a second off-pulse interval is of importance, the user must ensure that the selected minimum points do not belong to two disjoint intervals, as described in the last paragraph of Section 3.2 in Chapter 3. Thus, be cautious of choosing m-values that are too large.

While performing the simulation study, it became evident that larger values of m influenced the computation time of the procedure. The computation time was, however, not recorded and therefore, it is not valid to base an argument on the computation time. What is important, though, is the impact of m on the estimated off-pulse intervals. For all of the simulated target populations, none of the estimations dramatically changed when $m = 1$ was chosen compared to $m > 1$. Furthermore, for $m > 1$, averaging is used over all of the chosen minimum points m to obtain the estimated off-pulse interval. Therefore, when m is increased from say, $m = 2$ to $m = 20$, the estimated off-pulse interval will only gradually change.

- It is recommended to choose $m = 1$, but one could also choose any value up to $m = 5$ without taking too much risk in terms of the estimation of the off-pulse interval.

SOPIE applies goodness-of-fit tests sequentially, and therefore requires the specification of the significance level α. In the simulation study it was found that the significance level α is also related to the choice of r and g. This relationship will be highlighted below. If only α is considered, the following choices are recommended:

- use $\alpha \leq 0.05$ for *large* sample sizes, and
- use $0.05 \leq \alpha \leq 0.10$ for *small to moderate* sample sizes.

As stated in the previous section, the value of α has a limited effect on the Cramér-von-Mises goodness-of-fit test, but the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests are more sensitive to different choices of α.

The tuning parameter g represents the value of the incremental growth of each subsequent interval over which uniformity is tested. The tuning parameter r represents the number of subsequent intervals that must result in the rejection of uniformity before SOPIE will stop. The relationship between the tuning parameter r and the significance level α can be seen in the following example. Suppose that the observations in a certain subinterval cause the goodness-of-fit test to produce a P-value that is below the level of significance, leading to rejection of uniformity over that subinterval. Choosing a larger value of r would prevent SOPIE to stop after the first rejection of uniformity. The procedure will continue until r consecutive rejections have taken place, which serve as confirmation that a change occurred in the uniform nature of the observations in the subinterval under consideration. For the exact same situation, one could also choose the values of α and r to be slightly smaller to yield a similar estimated off-pulse interval. In summary, and from a logical point of view, larger values of α requires larger values of r to confirm that the P-value remains below α for several subintervals. Similarly, for smaller α-values, uniformity can be rejected if the P-value drops below the significance level, without confirming it in several consecutive intervals (i.e., by selecting a smaller value for the tuning parameter r).

In line with this discussion, it is recommended to use the following values for the tuning parameters g and r in combination with α, when the sample size n is *small to moderate*:

- $1 \leq g \leq 10$ and $1 \leq r \leq 6$ in combination with a small level of significance (i.e., $0.01 \leq \alpha \leq 0.05$), or
- $1 \leq g \leq 10$ and $5 \leq r \leq 10$ in combination with a slightly larger level of significance (i.e., $0.05 \leq \alpha \leq 0.10$).

When the sample sizes n are *large* (which is typical to pulsar data), it is recommended to use a small α-value such as $\alpha = 0.01$, with the following choices for the tuning parameters g and r:

- $1 \leq g \leq 40$ and $1 \leq r \leq 8$.

Chapter 5

Applications to pulsar data

5.1 Introduction

The discussion in the previous chapter dealt with the simulation design and simulation results, which were used to establish the accuracy and consistency of the proposed new methodology to estimate the off-pulse interval(s) of an unknown source function. However, this research originated from an astrophysical context. Therefore, the proposed methodology should be applied to pulsar data in order to assess the performance thereof. This chapter will provide these results and compare the end-point values of the off-pulse interval(s) (estimated with SOPIE) to the *subjective* values obtained with the "eye-ball" technique frequently used in published papers.

It must be emphasised that the values from the published papers are mostly obtained using visual inspection of the *histogram* estimate of the pulsar light curve. For this reason, in the discussion of each pulsar, the pulsar's estimated light curve is illustrated in the beginning of the discussion. This is done so that the reader can verify the off-pulse interval based on his/her own visual inspection of the estimated pulsar light curve. In some cases, the reader may identify an off-pulse interval that is slightly different from the interval quoted in the published papers. This would not be a mistake by the reader, but is based on the fact that the construction of the histogram relies on the choice of origin and bin width. It is also common knowledge that the selection of the bin width primarily controls the amount of smoothing of the histogram. It is therefore possible to reach different conclusions when slightly different bin widths are chosen (Silverman, 1986). This serves as a strong argument and is part of the motivation why an *objective* technique is required to estimate the off-pulse interval of a pulsar light-curve, rather than the *subjective* "eye-ball" technique that relies on the histogram estimator, which cannot be rigorously substantiated.

In Section 5.2 of this chapter, a synopsis will be given of the recommendations pertaining to the optimal parameter configurations found in the simulation study in Chapter 4. This will be followed by a discussion and comparison of the estimated off-pulse intervals of six different pulsars (Sections 5.3 - 5.8). This chapter will then close with some concluding remarks. The pulsar data and all of the results obtained in the analyses thereof can be found on the DVD-ROM included with the thesis.

5.2 Parameter configurations recommended from the simulation study

The aim of Chapter 4 is primarily to establish the accuracy and consistency of the proposed new methodology to estimate the off-pulse interval(s) of an unknown source function. The secondary aim is to determine whether certain parameter configurations consistently result in more accurate estimation of the off-pulse interval. A brief summary of the optimal parameter configurations (discussed in Section 4.7 of Chapter 4) will now be given, since these configurations are applied to the pulsar data throughout this chapter.

- The choice of the kernel function k does not have an impact on the estimated off-pulse interval. The normal kernel is used in the application of SOPIE to pulsar data due to the easy calculation thereof.
- For the number of minimum points m, it is recommended to choose $m = 1$, but one could also choose any value up to $m = 5$ without taking too much risk in terms of the estimation of the off-pulse interval.
- Any one of $\hat{h}_1$, $\hat{h}_2$, $\hat{h}_3$, $\hat{h}_4$ or $\hat{h}_5$ can be used for the value of the estimated smoothing parameter.
- For large sample sizes, the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests should be used, and for smaller sample sizes, the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests are recommended.
- For α, g and r, it is recommended to use $1 \leq g \leq 10$ and $1 \leq r \leq 6$ in combination with a small level of significance (e.g., $\alpha = 0.05$), or $1 \leq g \leq 10$ and $5 \leq r \leq 10$ in combination with a slightly larger level of significance (e.g., $\alpha = 0.10$) for smaller sample sizes n. For larger sample sizes, rather use a small α-value such as $\alpha = 0.01$, together with $1 \leq g \leq 40$ and $1 \leq r \leq 8$.

In the remainder of this chapter, these parameter configurations will be applied to the pulsar data to estimate the off-pulse interval with SOPIE.

5.3 Pulsar PSR J0534+2200 (Crab)

The launch of the *Fermi Gamma-ray Space Telescope* (formerly GLAST) with its Large Area Telescope (LAT) on-board, presented the opportunity to study the high energy behaviour of the Crab pulsar in greater detail. The Crab pulsar is associated with the supernova explosion already reported in 1054 AD (Duyvendak, 1942; Mayall & Oort, 1942). An optical image of the Crab Nebula can be seen in Figure 1.2, while Figure 1.6 is a visual representation of the Crab pulsar (pulsed window) and nebula (off-pulse window).

Figure 5.1 is a histogram representation (with 100 classes) of a single cycle of the estimated γ-ray light curve of the Crab pulsar, together with the different kernel density estimators illustrated by the lines on top of the histogram. This estimated γ-ray light curve above 100 MeV consists of $n = 21145$ times of arrival (TOAs), with an off-pulse interval of $[\hat{a}, \hat{b}] = [0.52, 0.87]$ (Abdo et al., 2010b), obtained with the "eye-ball" technique. The reader must remember that the end-point values of this interval were obtained with the *subjective "eye-ball" technique, and is, therefore, only a "guesstimate."*

Off-pulse interval comparison

Based on the recommendations from the simulation study, related parameter configurations are used to obtain $[\hat{a}, \hat{b}]$ with SOPIE. These estimates are then compared to the published values that were obtained with the subjective "eye-ball" technique.

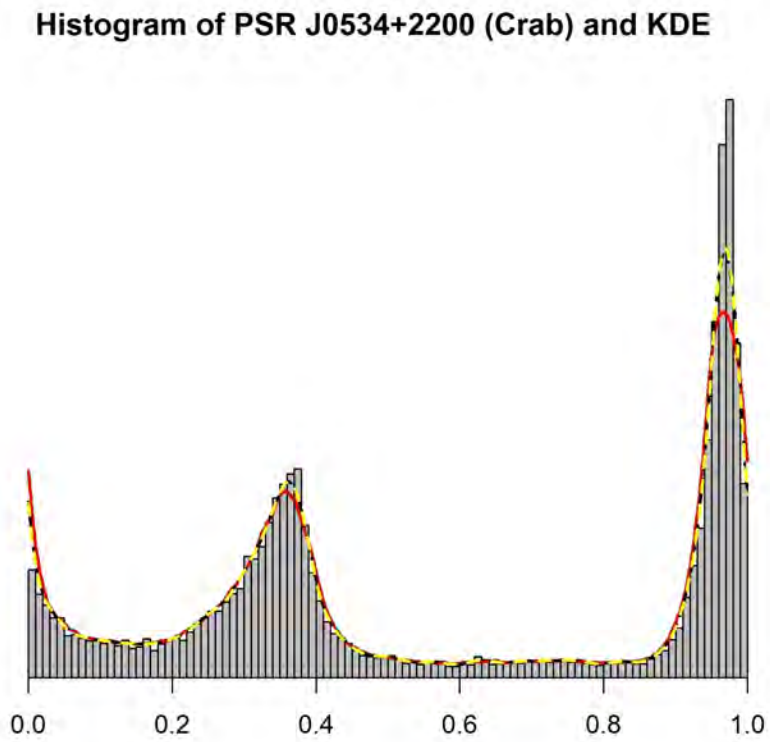

Figure 5.1: *Histogram representation (100 classes) of the estimated γ-ray light curve (obtained from Fermi LAT, energies above 100 MeV) of PSR J0534+2200 with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter $\hat{h}_3 = 0.17$.*

According to the recommendations in Section 5.2, the number of minimum points m does not greatly influence $\hat{a}$ and $\hat{b}$. This is indeed observed for the Crab light curve, and is shown in Tables 5.1 - 5.2. The effect of different choices of m is very marginal, and therefore all estimators for this data set will be based on the choice of $m = 1$.

Pertaining to the estimated smoothing parameters, several choices of $\hat{h}$ are recommended in Section 5.2. For the Crab light curve, it is found that the following estimated smoothing parameters relate very closely to each other, and can therefore be grouped into three distinct sets. In each set listed below, the $\hat{h}$-values are almost identical, resulting in similar minimum points m used as starting points for SOPIE, and consequently equivalent estimation of $[a, b]$ follows. The three distinct sets are:

- $\hat{h}_1$, $\hat{h}_3$, $\hat{h}_4$ and $\hat{h}_5$;
- $\hat{h}_2$ and $\hat{h}_8$, and
- $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$.

Tables 5.1 - 5.2 highlight the comparison of the estimated values $\hat{a}$ and $\hat{b}$ for the above mentioned sets of smoothing parameters. Different choices of the estimated smoothing parameter $\hat{h}$ only have a slight impact on the estimated values. It seems that $\hat{h}_2$ is quite a good choice for most of the goodness-of-fit tests, except for the Rayleigh test. What is more important, is that the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests produce estimates which are closely related to the interval obtained with the subjective "eye-ball" technique. The same cannot be said about the Kolmogorov-Smirnov and Rayleigh goodness-of-fit tests. This trend is in agreement with the findings in the simulation study.

Table 5.1: *Estimated off-pulse intervals for the Crab pulsar with $m = 1$, $r = 10$, $g = 12$ and with $\alpha = 0.05$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.4954	0.8814	0.4566	0.8954	0.5033	0.6301	0.5033	0.6288
$\hat{h}_2$	0.5074	0.8828	0.4765	0.8963	0.5036	0.8872	0.4890	0.6276
$\hat{h}_6$	0.5033	0.6272	0.4580	0.8960	0.5033	0.6288	0.4900	0.6301

Table 5.2: *Estimated off-pulse intervals for the Crab pulsar with $m = 5$, $r = 10$, $g = 12$ and with $\alpha = 0.05$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.4955	0.8289	0.4564	0.8957	0.5017	0.6811	0.5001	0.6290
$\hat{h}_2$	0.5078	0.8325	0.4782	0.8966	0.5049	0.8351	0.4881	0.6282
$\hat{h}_6$	0.4917	0.7786	0.4558	0.8930	0.5115	0.7799	0.5087	0.7840

Table 5.3: *Estimated off-pulse intervals for the Crab pulsar with $m = 1$, $r = 10$, $g = 12$ and with $\hat{h}_2$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
α	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
0.01	0.5055	0.8872	0.4641	0.8971	0.4958	0.8899	0.4771	0.6304
0.05	0.5074	0.8828	0.4765	0.8963	0.5036	0.8872	0.4890	0.6276
0.10	0.5129	0.6276	0.4794	0.8963	0.5074	0.8857	0.4904	0.6276

Pertaining to the level of significance α, the recommendations from the simulation study indicate that α-values between 1% and 5% are preferred. Table 5.3 clearly shows that $\alpha = 0.10$ is not a good choice for the level of significance (see, e.g., the Anderson-Darling and Rayleigh goodness-of-fit tests). The results for α-values of 1% and 5% seem to be similar to each other and the estimates are close to the interval obtained with the subjective "eye-ball" technique.

In terms of the goodness-of-fit tests, the Rayleigh goodness-of-fit test (once again) does not perform as well as the Anderson-Darling and Cramér-von-Mises goodness-of-fit tests. This is in agreement with the simulation study results in Chapter 4. The Cramér-von-Mises goodness-of-fit test produces estimated intervals $[\hat{a}, \hat{b}]$ with great consistency across different values of r, g and α, underlining the robust nature of the test. To obtain a good idea of the estimated values produced by the Cramér-von-Mises goodness-of-fit test, the reader can inspect Table 5.1 and Table 5.3.

From the simulation study it is recommended that larger values for r and g should be used, especially if the sample size n is large. This is not different for the Crab pulsar, and can be seen in Tables 5.4 - 5.5. The choice of r and g has almost no impact on the estimation of a (the left end-point of the off-pulse interval). When b is estimated, it is clear that larger values of $r\,(\geq 8)$ and $g\,(\geq 14)$ are preferred. The results are only given for the Anderson-Darling goodness-of-fit test due to the exceptional consistency of the estimation when the Cramér-von-Mises goodness-of-fit test is used.

Table 5.4: *Estimated values of a (left end-point of the off-pulse interval) for the Crab pulsar with $m = 1$, $\alpha = 0.01$ and with $\hat{h}_2$ for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**	**r=16**
g=1	0.5068	0.5063	0.5063	0.5050	0.5050	0.5050	0.5050	0.5050	0.5050
g=2	0.5068	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062
g=4	0.5068	0.5068	0.5068	0.5068	0.5068	0.5068	0.5068	0.5068	0.5068
g=6	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055
g=8	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062
g=10	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062
g=12	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055	0.5055
g=14	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049
g=15	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048
g=16	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049	0.5049
g=17	0.5061	0.5061	0.5061	0.5061	0.5061	0.5061	0.5061	0.5061	0.5061
g=18	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048	0.5048
g=19	0.5036	0.5036	0.5036	0.5036	0.5036	0.5036	0.5036	0.5036	0.5036
g=20	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062	0.5062

Table 5.5: *Estimated values of b (right end-point of the off-pulse interval) for the Crab pulsar with $m = 1$, $\alpha = 0.01$ and with $\hat{h}_2$ for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**	**r=16**
g=1	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276
g=2	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276
g=4	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276
g=6	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276	0.6276
g=8	0.6281	0.6281	0.6281	0.6281	0.6281	0.6281	0.6281	0.8828	0.8828
g=10	0.6287	0.6287	0.6287	0.6287	0.6287	0.6287	0.8837	0.8837	0.8837
g=12	0.6276	0.6276	0.6276	0.6276	0.6276	0.8828	0.8828	0.8828	0.8828
g=14	0.6287	0.6287	0.6287	0.6287	0.8826	0.8826	0.8826	0.8826	0.8826
g=15	0.6279	0.6279	0.6279	0.6279	0.8842	0.8842	0.8842	0.8842	0.8842
g=16	0.6291	0.6291	0.6291	0.6291	0.8839	0.8839	0.8839	0.8839	0.8839
g=17	0.6281	0.6281	0.6281	0.6281	0.8845	0.8845	0.8845	0.8845	0.8845
g=18	0.6291	0.6291	0.6291	0.6291	0.8838	0.8838	0.8838	0.8838	0.8838
g=19	0.6276	0.6276	0.6276	0.6276	0.8842	0.8842	0.8842	0.8842	0.8842
g=20	0.6287	0.6287	0.6287	0.8837	0.8837	0.8837	0.8837	0.8837	0.8837

In conclusion, it can be seen that the proposed technique provides an objective estimation for the off-pulse interval of a pulsar, closely related to the "eye-ball guesstimate."

5.4 Pulsar PSR J0835-4510 (Vela)

The Vela pulsar is the brightest non-transient source of γ-rays in the sky, and has been intensively studied over the years. Due to its brightness, the Vela pulsar is traditionally the first target object for any new γ-ray observatory. The LAT aboard *Fermi* again contributed towards the gathering of excellent data to study this pulsar in more detail.

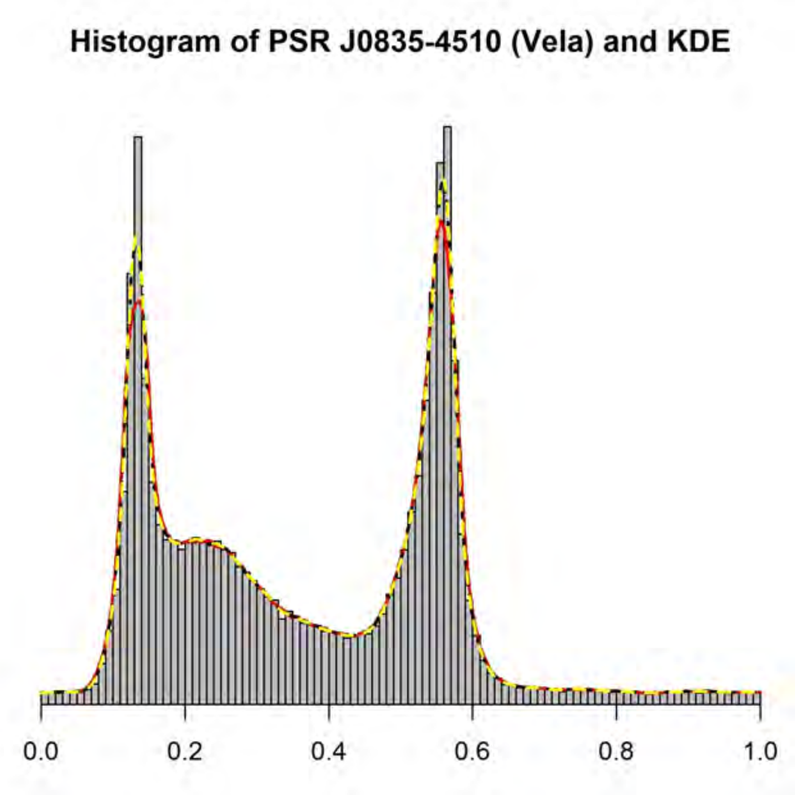

Figure 5.2: *Histogram representation (100 classes) of the estimated γ-ray light curve (obtained from Fermi LAT, energies above* 0.1 *GeV) of PSR J0835-4510 with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.11$.

Figure 5.2 is a histogram representation (with 100 classes) of a single cycle of the estimated γ-ray light curve of the Vela pulsar with different kernel density estimators fitted to the data (illustrated by the lines on top of the histogram). This estimated γ-ray light curve above 0.1 GeV consists of $n = 115657$ times of arrival (TOAs), with an estimated off-pulse interval of $[\hat{a}, \hat{b}] = [0.66, 0.06]$ (Abdo et al., 2010d), obtained with the "eye-ball" technique and by fitting a power law with exponential cut-off to the spectrum.

Off-pulse interval comparison

Based on the optimal tuning parameter configurations in Section 5.2, related configurations are applied to the Vela pulsar in order to obtain $[\hat{a}, \hat{b}]$ with SOPIE. These estimates are then compared to the values obtained with the "eye-ball" technique, viz. $[\hat{a}, \hat{b}] = [0.66, 0.06]$.

For the Vela light curve, it is found that the following smoothing parameters relate very closely to each other, and can be grouped in the following three distinct sets:

- $\hat{h}_1$, $\hat{h}_3$, $\hat{h}_4$, $\hat{h}_5$ and $\hat{h}_9$;
- $\hat{h}_2$ and $\hat{h}_8$, and
- $\hat{h}_6$ and $\hat{h}_7$.

Tables 5.6 - 5.7 compare the estimated values $\hat{a}$ and $\hat{b}$ for one of the $\hat{h}$-values in each of the above-mentioned sets. It is evident that there is only small variations in the estimated values when $m = 1$. The effect when $m = 5$ is substantial when $\hat{h}_2$ is used for the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests, which results in estimated values that are incorrect. This can be explained by the very small value calculated for $\hat{h}_2$, resulting in an under-smoothed kernel density estimator. This in turn results in minimum values that are used as starting points for SOPIE, which are obviously incorrect when inspecting the shape of the light curve. Therefore, different choices of the estimated smoothing parameter, as well as the choice of the goodness-of-fit test, have an impact on the estimated interval. Hence, $\hat{h}_1$ or $\hat{h}_6$ can be used instead of $\hat{h}_2$ for all goodness-of-fit tests. Furthermore, the Cramér-von-Mises goodness-of-fit test produces estimates which are closely related to the interval obtained with the "eye-ball" technique.

Table 5.6: *Estimated off-pulse intervals for the Vela pulsar with $m = 1$, $r = 10$, $g = 10$ and with $\alpha = 0.05$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.7521	0.0713	0.6863	0.0803	0.7521	0.0762	0.7315	0.0792
$\hat{h}_2$	0.7537	0.0714	0.6922	0.0804	0.7522	0.0759	0.7391	0.0790
$\hat{h}_6$	0.7537	0.0695	0.7181	0.0775	0.7531	0.0714	0.7437	0.0714

Table 5.7: *Estimated off-pulse intervals for the Vela pulsar with $m = 5$, $r = 10$, $g = 10$ and with $\alpha = 0.05$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.7634	0.0732	0.7098	0.0806	0.7764	0.0762	0.7481	0.0771
$\hat{h}_2$	0.7567	0.4037	0.7079	0.0795	0.7527	0.3881	0.7376	0.0778
$\hat{h}_6$	0.7538	0.0550	0.7152	0.0772	0.7530	0.0712	0.7429	0.0720

Table 5.8: *Estimated off-pulse intervals for the Vela pulsar with $m = 1$, $r = 10$, $g = 10$ and with $\hat{h}_6$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
α	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
0.01	0.7506	0.0710	0.7031	0.0783	0.7493	0.0758	0.7335	0.0759
0.05	0.7538	0.0695	0.7152	0.0775	0.7530	0.0714	0.7429	0.0714
0.10	0.7611	0.0668	0.7233	0.0775	0.7555	0.0707	0.7685	0.0695

Table 5.8 highlights that the choice of α has a limited impact on the estimated off-pulse interval for all the goodness-of-fit tests. Further analyses confirmed that the Cramér-von-Mises goodness-of-fit

test is the most robust goodness-of-fit test for different values of α, and for different choices of r and g. For the other goodness-of-fit tests, the estimated values tend to fluctuate slightly more when different values of α, g and r are chosen.

The results for the Cramér-von-Mises goodness-of-fit test are shown in Tables 5.9 - 5.10 for $\alpha = 0.01$ and for $\hat{h}_1$. As expected, the Cramér-von-Mises goodness-of-fit test is again almost completely insensitive to different choices of r, g and α, producing consistent results for any choice of these tuning parameters.

Table 5.9: *Estimated values of a (left end-point of the off-pulse interval) for the Vela pulsar with $m = 1$, $\alpha = 0.01$ and with $\hat{h}_1$ for the Cramér-von-Mises goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**
g=1	0.7122	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095
g=2	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095
g=4	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095	0.7095	0.7033
g=6	0.7094	0.7094	0.7094	0.7094	0.7094	0.7033	0.7033	0.7033
g=8	0.7094	0.7094	0.7094	0.7094	0.7033	0.7033	0.7033	0.7033
g=10	0.7092	0.7092	0.7092	0.7040	0.7040	0.7040	0.7040	0.7040
g=20	0.7092	0.7092	0.7040	0.7040	0.7040	0.7040	0.7040	0.7040
g=30		0.7092	0.7033	0.7033	0.7033	0.7033	0.7033	0.7033
g=50		0.7033	0.7033	0.7033	0.7033	0.7033	0.7033	0.7033
g=75		0.7083	0.7083	0.7083	0.7083	0.7083	0.7083	0.7083
g=100		0.7014	0.7014	0.7014	0.7014	0.7014	0.7014	0.7014

Table 5.10: *Estimated values of b (right end-point of the off-pulse interval) for the Vela pulsar with $m = 1$, $\alpha = 0.01$ and with $\hat{h}_1$ for the Cramér-von-Mises goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**
g=1	0.0782	0.0782	0.0782	0.0782	0.0782	0.0782	0.0782	0.0782
g=2	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783
g=4	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783
g=6	0.0784	0.0784	0.0784	0.0784	0.0784	0.0784	0.0784	0.0784
g=8	0.0784	0.0784	0.0784	0.0784	0.0784	0.0784	0.0784	0.0784
g=10	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783
g=20	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783	0.0783
g=30		0.0785	0.0785	0.0785	0.0785	0.0785	0.0785	0.0785
g=50		0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793
g=75		0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793
g=100		0.0793	0.0793	0.0793	0.0793	0.0793	0.0793	0.0793

In conclusion, it can be seen that SOPIE provides an objective estimation for the off-pulse interval of the Vela pulsar, closely related to the interval identified with the "eye-ball" technique.

5.5 Pulsar PSR J2021+3651

Figure 5.3 is a histogram representation (with 100 classes) of a single cycle of the estimated γ-ray light curve of the PSR J2021+3651 pulsar with different kernel density estimators fitted to the data (illustrated by the lines on top of the histogram). This γ-ray light curve above 0.1 GeV consists of $n = 14080$ times of arrival (TOAs), with an estimated off-pulse interval of $[\hat{a}, \hat{b}] = [0.7, 0.04]$ (Abdo et al., 2010f), obtained with the subjective "eye-ball" technique.

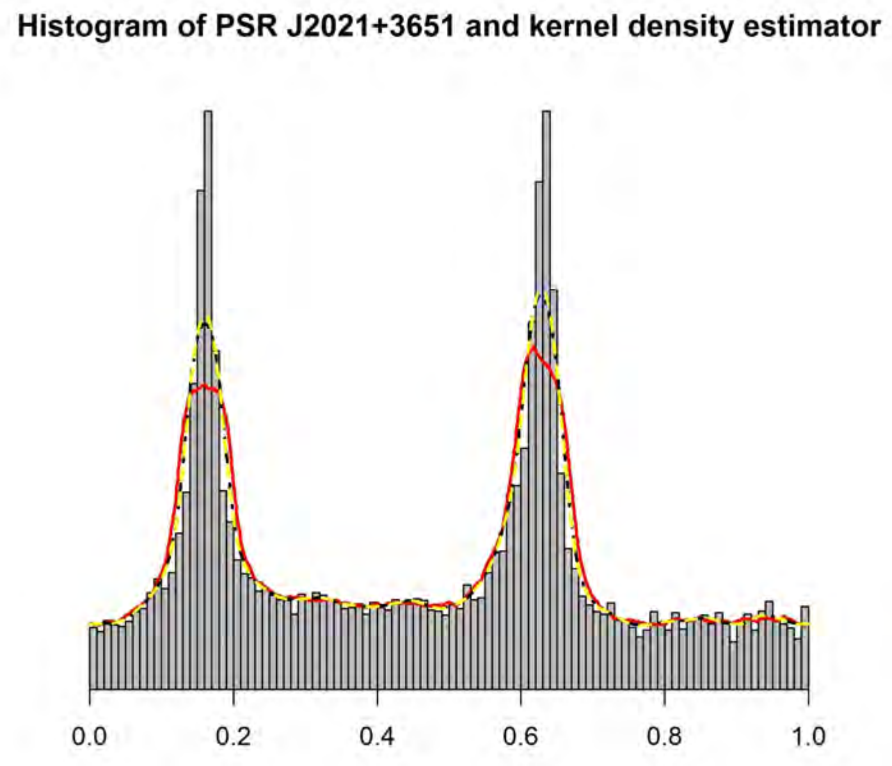

Figure 5.3: *Histogram representation (100 classes) of the estimated γ-ray light curve (obtained from Fermi LAT, energies above* 0.1 *GeV) of PSR J2021+3651 with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.23$.

Off-pulse interval comparison

When utilising the optimal tuning parameters (see Section 5.2) for the PSR J2021+3651 pulsar, it is firstly found that the number of minimum points m does not greatly influence the estimated off-pulse interval (see Tables 5.11 - 5.12). Furthermore, from the estimated PSR J2021+3651 light curve, it is found that the following smoothing parameters relate very closely to each other, and can be grouped into the following three distinct sets:

- $\hat{h}_1$, $\hat{h}_3$, $\hat{h}_4$ and $\hat{h}_5$;
- $\hat{h}_2$ and $\hat{h}_8$, and
- $\hat{h}_6$, $\hat{h}_7$ and $\hat{h}_9$.

In Tables 5.11 - 5.12, when comparing the estimated values $\hat{a}$ and $\hat{b}$ for the above mentioned sets of smoothing parameters, it is found that the off-pulse intervals that result from each set are almost equal in value, with the Rayleigh goodness-of-fit test displaying the largest variation. Therefore, any one of the $\hat{h}$-values can be used.

Table 5.11: *Estimated off-pulse intervals for PSR J2021+3651 with $m = 1$, $r = 8$, $g = 8$ and with $\alpha = 0.01$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.6911	0.0951	0.6753	0.1134	0.6905	0.0967	0.7073	0.1056
$\hat{h}_2$	0.6782	0.0915	0.6706	0.1105	0.6752	0.0964	0.6677	0.1132
$\hat{h}_6$	0.6881	0.0949	0.6751	0.1113	0.6881	0.0955	0.6872	0.0997

Table 5.12: *Estimated off-pulse intervals for PSR J2021+3651 with $m = 5$, $r = 8$, $g = 8$ and with $\alpha = 0.01$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.6904	0.0950	0.6752	0.1127	0.6890	0.0965	0.7134	0.1057
$\hat{h}_2$	0.6844	0.0923	0.6718	0.1103	0.6822	0.0957	0.6811	0.1023
$\hat{h}_6$	0.6879	0.0945	0.6745	0.1110	0.6879	0.0955	0.6887	0.1014

The choice of goodness-of-fit test has a slightly larger impact on the estimated interval. All of the goodness-of-fit tests perform well, but it is still recommended to use the Anderson-Darling, Cramér-von-Mises or the Kolmogorov-Smirnov goodness-of-fit tests. The Rayleigh goodness-of-fit test produces estimates that are slightly more variable. This is in agreement with the simulation study results in Chapter 4. The Cramér-von-Mises goodness-of-fit test produces estimated intervals $[\hat{a}, \hat{b}]$ with great consistency across different values of r, g and α, indicating the robust nature of the test yet again. To obtain a good idea of the estimators produced by the Cramér-von-Mises goodness-of-fit test, the reader can inspect Tables 5.11 – 5.13.

Table 5.13: *Estimated off-pulse intervals for the PSR J2021+3651 with $m = 1$, $r = 8$, $g = 8$ and with $\hat{h}_1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
$\boldsymbol{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
0.01	0.6911	0.0951	0.6753	0.1134	0.6905	0.0967	0.7073	0.1056
0.05	0.7069	0.0910	0.6774	0.1106	0.7069	0.0929	0.7232	0.0993
0.10	0.7210	0.0866	0.6785	0.1102	0.7156	0.0916	0.7284	0.0967

Pertaining to the level of significance α, the recommendations from Section 5.2 indicate that α-values between 1% and 5% are preferred. Table 5.13 contains some evidence that $\alpha = 0.10$ is not the best choice for the level of significance (see, e.g., the estimation of a for all the goodness-of-fit tests). The results for α-values between 1% and 5% seem to be similar to each other and the estimates are in line with the interval identified with the subjective "eye-ball" technique.

From the simulation study it is recommended that larger values for r and g should be used, especially if the sample size n is large. The estimated off-pulse intervals for the Anderson-Darling goodness-of-fit test are shown in Tables 5.14 - 5.15 for $\alpha = 0.05$ and for $\hat{h}_1$, for different g and r-values. It can be seen that the Anderson-Darling goodness-of-fit test is also robust against different choices of r, g and α, producing consistent results for most of the combination-values. It is evident that combinations with slightly larger values for $g(\geq 6)$ and $r(\geq 4)$ are preferred.

Table 5.14: *Estimated values of a (left end-point of the off-pulse interval) for PSR J2021+3651 with $m = 1$, $\alpha = 0.05$ and with $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**	**r=16**
g=1	0.7911	0.7911	0.7911	0.7911	0.7207	0.7207	0.7150	0.7150	0.7110
g=2	0.7911	0.7911	0.7206	0.7206	0.7206	0.7072	0.7072	0.7072	0.7072
g=4	0.7906	0.7204	0.7204	0.7071	0.7071	0.7071	0.7071	0.7071	0.7071
g=6	0.7906	0.7206	0.7070	0.7070	0.7070	0.7070	0.7070	0.7070	0.7070
g=8	0.7204	0.7204	0.7069	0.7069	0.7069	0.7069	0.7069	0.7069	0.7069
g=10	0.7911	0.7204	0.7077	0.7077	0.7077	0.7077	0.7077	0.7077	0.7077
g=15	0.7901	0.7072	0.7072	0.7072	0.7072	0.7072	0.7072	0.7072	0.7072
g=20	0.7204	0.7204	0.7069	0.7069	0.7069	0.7069	0.7069	0.7069	0.7069
g=50		0.7103	0.7103	0.7103	0.7103	0.7103			
g=100		0.7049	0.7049	0.7049	0.7049	0.7049			

Table 5.15: *Estimated values of b (right end-point of the off-pulse interval) for PSR J2021+3651 pulsar with $m = 1$, $\alpha = 0.05$ and with $\hat{h}_1$ for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**	**r=16**
g=1	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907
g=2	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907	0.0907
g=4	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909
g=6	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909
g=8	0.0910	0.0910	0.0910	0.0910	0.0910	0.0910	0.0910	0.0910	0.0910
g=10	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909
g=15	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916
g=20	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916	0.0916
g=50		0.0916	0.0916	0.0916	0.0916	0.0916			
g=100		0.0916	0.0916	0.0916	0.0916	0.0916			

In conclusion, it can be seen that the proposed technique provides an objective estimation for the off-pulse interval of a pulsar light curve, with the estimated value of a ***closely related to the value identified with the "eye-ball" technique.***

5.6 Pulsar PSR J1057-5226

Figure 5.4 is a histogram representation (with 100 classes) of a single cycle of the estimated γ-ray light curve of the PSR J1057-5226 pulsar with different kernel density estimators fitted to the data. This estimated γ-ray light curve above 0.1 GeV consists of $n = 5236$ times of arrival (TOAs), with an off-pulse interval $[\hat{a}, \hat{b}] = [0.2, 0.7]$ (Abdo et al., 2010f), obtained by using the subjective "eye-ball" technique.

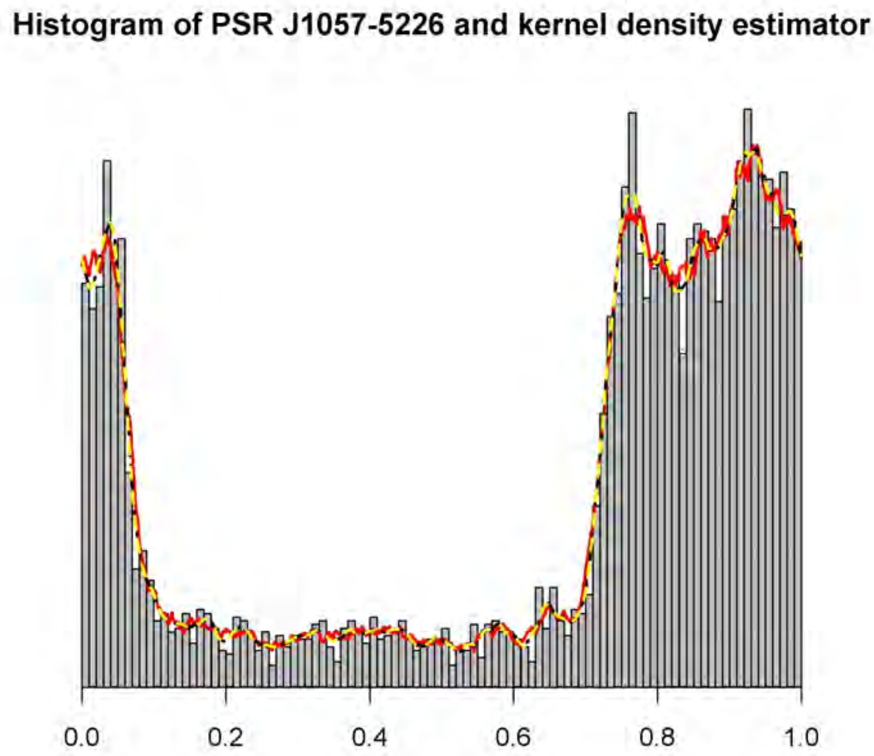

Figure 5.4: *Histogram representation (100 classes) of the estimated γ-ray light curve (obtained from Fermi LAT, energies above* 0.1 *GeV) of PSR J1057-5226 with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.13$.

From Figure 5.4 it is not obvious why visual inspection would result in an off-pulse interval of $[\hat{a}, \hat{b}] = [0.2, 0.7]$, *reported by Abdo et al. (2010f). It is argued that* $[\hat{a}, \hat{b}] = [0.1, 0.7]$ *is a more reasonable off-pulse interval based on visual inspection of the kernel estimators. The reader can also inspect the figure to estimate his/her own off-pulse interval based on visual inspection.*

Off-pulse interval comparison

For the PSR J1057-5226 light curve, it is found that the following smoothing parameters relate very closely to each other, and can be grouped into the following five distinct sets:

- $\hat{h}_1$;
- $\hat{h}_2$, $\hat{h}_3$ and $\hat{h}_4$;
- $\hat{h}_5$ and $\hat{h}_6$;
- $\hat{h}_7$ and $\hat{h}_8$, and
- $\hat{h}_9$.

Tables 5.16 - 5.17 compare the estimated values of $\hat{a}$ and $\hat{b}$ for one of the selected $\hat{h}$-values from the above-mentioned sets of smoothing parameters, and for two different choices of m. The two different choices of m have very little impact on the estimation of $[a, b]$. This trend is in agreement with the recommendations from the simulation study. Furthermore, it seems as if any of the $\hat{h}$-values can be chosen with limited effect on the estimation of a and b. When investigating the effect of the goodness-of-fit tests, it is remarkable that all of the goodness-of-fit tests perform quite well, especially when selecting $\hat{h}_9$ as the estimated smoothing parameter. It is no surprise that the Anderson-Darling, Cramér-von-Mises and Kolmogorov-Smirnov goodness-of-fit tests perform well, but in this case the Rayleigh goodness-of-fit test is also up to par.

Table 5.16: *Estimated off-pulse intervals for PSR J1057-5226 with* $m = 1$, $r = 8$, $g = 8$ *and with* $\alpha = 0.05$.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.0970	0.6428	0.0766	0.7125	0.0970	0.6428	0.0883	0.6985
$\hat{h}_2$	0.1059	0.6565	0.0815	0.7169	0.0982	0.6565	0.0849	0.7142
$\hat{h}_5$	0.1059	0.6564	0.0804	0.7155	0.0978	0.6564	0.0848	0.7109
$\hat{h}_7$	0.1059	0.6534	0.0780	0.7155	0.0978	0.5707	0.0848	0.7141
$\hat{h}_9$	0.1059	0.6874	0.0833	0.7182	0.0978	0.6828	0.0833	0.7092

Table 5.17: *Estimated off-pulse intervals for PSR J1057-5226 with* $m = 5$, $r = 8$, $g = 8$ *and with* $\alpha = 0.05$.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.1045	0.6607	0.0772	0.7145	0.0983	0.6384	0.0868	0.6857
$\hat{h}_2$	0.1120	0.6655	0.0817	0.7166	0.0986	0.6389	0.0837	0.7127
$\hat{h}_5$	0.1104	0.6710	0.0824	0.7169	0.0979	0.6615	0.0837	0.7115
$\hat{h}_7$	0.1075	0.6549	0.0814	0.7163	0.0981	0.6390	0.0841	0.7133
$\hat{h}_9$	0.1077	0.6676	0.0821	0.7168	0.0993	0.6491	0.0837	0.7111

Table 5.18: *Estimated off-pulse intervals for the PSR J1057-5226 with* $m = 1$, $r = 8$, $g = 8$ *and with* $\hat{h}_2$.

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
α	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
0.01	0.0952	0.7028	0.0752	0.7169	0.0896	0.6874	0.0815	0.7185
0.05	0.1059	0.6565	0.0815	0.7169	0.0982	0.6565	0.0849	0.7142
0.10	0.1140	0.6536	0.0815	0.7156	0.1059	0.5709	0.0873	0.7068

Pertaining to the level of significance α, the recommendations from the simulation study indicate that α-values between 1% and 5% are preferred for moderate to larger sample sizes n. Table 5.18 highlights the fact that the smallest choice of α yields estimated values that are close to $[\hat{a}, \hat{b}] = [0.1, 0.7]$, obtained by our own visual inspection of the kernel estimators of the light curve, especially for the Anderson-Darling and the Kolmogorov-Smirnov goodness-of-fit tests. For the Cramér-von-Mises goodness-of-fit test, $\alpha = 0.05$ yields the best results, and for the Rayleigh goodness-of-fit test, an α-value of 0.10 gives the best estimation. The results for the Anderson-Darling goodness-of-fit test are shown in Tables 5.19 - 5.20 for $\alpha = 0.01$ and for $\hat{h}_9$, where it can be seen that the Anderson-Darling goodness-of-fit test is also robust against different choices of r, g and α, producing consistent results for most of these choices.

Table 5.19: *Estimated values of a (left end-point of the off-pulse interval) for PSR J1057-5226 with* $m = 1$, $\alpha = 0.01$ *and with* $\hat{h}_9$ *for the Anderson-Darling goodness-of-fit test.*

	r=1	r=2	r=4	r=6	r=8	r=10	r=12	r=14
g=1	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969
g=2	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969
g=3	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969
g=4	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969
g=5	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952
g=6	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969	0.0969
g=7	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952
g=8	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952
g=9	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952
g=10	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952	0.0952

Table 5.20: *Estimated values of b (right end-point of the off-pulse interval) for PSR J1057-5226 pulsar with* $m = 1$, $\alpha = 0.01$ *and with* $\hat{h}_9$ *for the Anderson-Darling goodness-of-fit test.*

	r=1	r=2	r=4	r=6	r=8	r=10	r=12	r=14
g=1	0.6998	0.7022	0.7022	0.7022	0.7022	0.7022	0.7022	0.7022
g=2	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028
g=3	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028
g=4	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028
g=5	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028
g=6	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028
g=7	0.7022	0.7022	0.7022	0.7022	0.7022	0.7022	0.7022	0.7022
g=8	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028	0.7028
g=9	0.7054	0.7054	0.7054	0.7054	0.7054	0.7054	0.7054	0.7054
g=10	0.7054	0.7054	0.7054	0.7054	0.7054	0.7054	0.7054	0.7054

In conclusion, it can be seen that SOPIE provides an objective estimation for the off-pulse interval of a pulsar light curve, closely related to the results obtained with our visual inspection of the kernel estimators of the light curve in Figure 5.4.

5.7 Pulsar PSR J0034-0534

Figure 5.5 is a histogram representation (with 50 classes) of a single cycle of the estimated γ-ray light curve of the PSR J0034-0534 pulsar with different kernel density estimators fitted to the data (illustrated by the lines on top of the histogram). The estimated γ-ray light curve above 0.1 GeV consists of $n = 425$ times of arrival (TOAs), with an off-pulse interval $[\hat{a}, \hat{b}] = [0.45, 0.85]$ (Abdo et al., 2010f), obtained with the subjective "eye-ball" technique. For this pulsar a relatively small sample is available. In the discussion that follows, the reader will note that the tuning parameter configurations used, are slightly different from the configurations used for the previous pulsars.

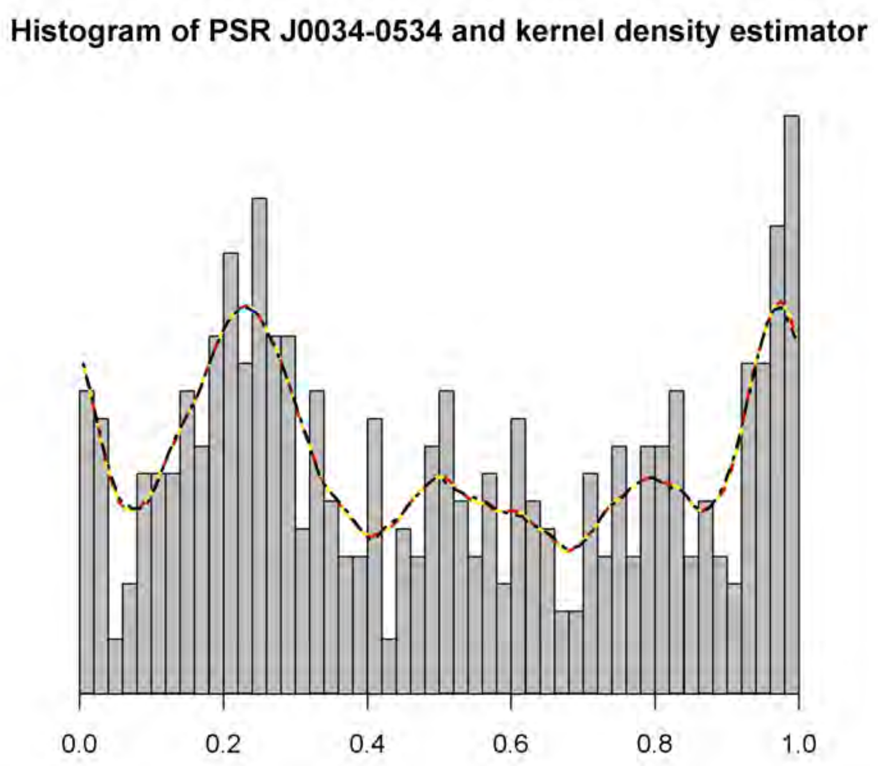

Figure 5.5: *Histogram representation (50 classes) of the estimated γ-ray light curve (obtained from Fermi LAT, energies above* 0.1 *GeV) of PSR J0034-0534 with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.44$.

Off-pulse interval comparison

For the PSR J0034-0534 light curve, it is found that the following smoothing parameters relate very closely to each other, and can be grouped into the following three distinct sets:

- $\hat{h}_1$ and $\hat{h}_5$;
- $\hat{h}_2$, $\hat{h}_6$, $\hat{h}_7$, $\hat{h}_8$ and $\hat{h}_9$, and
- $\hat{h}_3$ and $\hat{h}_4$.

Tables 5.21 – 5.22 compare the estimated interval $[\hat{a}, \hat{b}]$ for different estimated smoothing parameters and for $m = 1$ and $m = 5$. For this light curve, improved estimation followed after selecting $m > 1$. It is also observed that there are noticeable differences in the estimated off-pulse intervals for each different set of the smoothing parameters. When selecting $\hat{h}_2$ as estimated smoothing parameter, the estimated off-pulse intervals are comparable to the interval identified with the subjective "eye-ball" technique.

When investigating the effect of the level of significance α, the recommendations from Section 5.2 indicate that α-values between 5% and 10% are preferred for smaller sample sizes. Table 5.23 clearly shows that $\alpha = 0.10$ is the best choice for the level of significance (see, e.g., the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests). The results for α-values between 1% and 5% fluctuate dramatically.

Table 5.21: *Estimated off-pulse intervals for PSR J0034-0534 with $m = 1$, $r = 2$, $g = 2$ and with $\alpha = 0.1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.2705	0.8313	0.1716	0.9910	0.2591	0.9709	0.5118	0.7074
$\hat{h}_2$	0.6047	0.9671	0.2009	0.9936	0.6022	0.9709	0.2429	0.9898
$\hat{h}_3$	0.2706	0.9485	0.1840	0.9919	0.2642	0.9719	0.5766	0.7145

Table 5.22: *Estimated off-pulse intervals for PSR J0034-0534 with $m = 5$, $r = 2$, $g = 2$ and with $\alpha = 0.1$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.3368	0.9051	0.1914	0.9914	0.2578	0.9717	0.5006	0.7757
$\hat{h}_2$	0.4949	0.8615	0.2082	0.4954	0.4742	0.7574	0.3120	0.7662
$\hat{h}_3$	0.3978	0.8505	0.1962	0.8058	0.3351	0.8622	0.4476	0.8266

Table 5.23: *Estimated off-pulse intervals for the PSR J0034-0534 with $m = 5$, $r = 2$, $g = 2$ and with $\hat{h}_2$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
α	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
0.01	0.2648	0.9851	0.1979	0.2761	0.2446	0.8799	0.2058	0.8850
0.05	0.4940	0.8680	0.2091	0.2532	0.2574	0.7592	0.2319	0.7689
0.10	0.4949	0.8615	0.2082	0.4954	0.4742	0.7574	0.3120	0.7662

In terms of the goodness-of-fit tests, the Rayleigh (once again) and the Cramér-von-Mises goodness-of-fit tests perform poorly, compared to the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests. This trend is in agreement with the simulation study results in Section 5.2 for small sample sizes. For most of the goodness-of-fit tests, the estimated values tend to fluctuate slightly more when different values of α, g and r are chosen, compared to any of the previous pulsar data sets. This is due to the relatively small sample that is available for this pulsar. The results for the Anderson-Darling goodness-of-fit test are shown in Tables 5.24 – 5.25 for $\alpha = 0.1$, $m = 5$ and for $\hat{h}_2$.

From the simulation study it is recommended to use smaller values for r and g, especially if the sample size n is small. It can be seen that the Anderson-Darling goodness-of-fit test gives relatively stable estimated values when small values for $r(\leq 2)$ and $g(\leq 5)$ are chosen. The estimated values are then also in line with the identified off-pulse interval obtained with the subjective "eye-ball" technique.

Table 5.24: *Estimated values of a (left end-point of the off-pulse interval) for PSR J0034-0534 with $m = 5$, $\alpha = 0.1$ and with $\hat{h}_2$ for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**
g=1	0.5062	0.4956	0.4956	0.4806	0.4145	0.3962	0.3300	0.3300
g=2	0.4989	0.4949	0.4158	0.3979	0.3979	0.3315	0.3315	0.3315
g=3	0.4973	0.4815	0.3979	0.3317	0.3317	0.3317	0.2874	0.2874
g=4	0.4938	0.4151	0.3969	0.3300	0.2852	0.2852	0.2852	0.2852
g=5	0.4963	0.4178	0.4002	0.3299	0.2857	0.2857	0.2857	0.2857
g=6	0.4862	0.4122	0.3767	0.2822	0.2822	0.2822	0.2822	0.2822
g=7	0.4882	0.4108	0.3270	0.2804	0.2804	0.2804	0.2804	0.2804
g=8	0.4869	0.3959	0.3300	0.2852	0.2852	0.2852	0.2852	0.2852
g=9	0.4801	0.3431	0.3260	0.2817	0.2817	0.2817	0.2817	0.2817
g=10	0.4109	0.3929	0.2803	0.2803	0.2803	0.2803	0.2803	0.2803
g=11	0.4593	0.3742	0.2790	0.2790	0.2790	0.2790	0.2790	0.2790
g=12	0.4834	0.3877	0.2757	0.2757	0.2757	0.2757	0.2757	0.2757
g=13	0.4901	0.3380	0.2956	0.2956	0.2956	0.2956	0.2956	0.2956
g=14	0.4856	0.3794	0.2823	0.2823	0.2823	0.2823	0.2823	0.2823

Table 5.25: *Estimated values of b (right end-point of the off-pulse interval) for PSR J0034-0534 pulsar with $m = 1$, $\alpha = 0.1$ and with $\hat{h}_2$ for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**
g=1	0.6979	0.7503	0.8615	0.8615	0.8615	0.8615	0.8615	0.9712
g=2	0.6988	0.8615	0.8615	0.8615	0.9713	0.9713	0.9713	0.9713
g=3	0.7535	0.7535	0.8645	0.9718	0.9718	0.9718	0.9718	0.9718
g=4	0.6997	0.7512	0.9722	0.9722	0.9722	0.9722	0.9722	0.9722
g=5	0.8674	0.8674	0.9725	0.9725	0.9725	0.9725	0.9725	0.9725
g=6	0.7645	0.8691	0.9734	0.9734	0.9734	0.9734	0.9734	0.8178
g=7	0.8718	0.9746	0.9746	0.9746	0.9746	0.9746	0.9746	0.8197
g=8	0.6997	0.8644	0.9754	0.9754	0.9754	0.9754	0.8204	0.8204
g=9	0.7568	0.8684	0.9765	0.9765	0.9765	0.8210	0.8210	0.8210
g=10	0.8686	0.9749	0.9749	0.9749	0.9749	0.8201	0.8201	0.8201
g=11	0.7602	0.9745	0.9745	0.9745	0.9745	0.8197	0.8197	0.8197
g=12	0.7678	0.9767	0.9767	0.9767	0.8220	0.8220	0.8220	0.8220
g=13	0.7692	0.9795	0.9795	0.9795	0.8228	0.8228	0.8228	0.8228
g=14	0.8718	0.9761	0.9761	0.9761	0.8212	0.8212	0.8212	0.8212

In conclusion, it can be seen that SOPIE provides an objective estimation for the off-pulse interval of a pulsar light curve, closely related to the interval identified with the "eye-ball" technique, even when the sample size n is small. It must be emphasised that, for smaller sample sizes n, much attention should be given to select the appropriate tuning parameter configurations, due to the varying nature of the estimated values.

5.8 Pulsar PSR J1709-4429

Figure 5.6 is a histogram representation (with 100 classes) of a single cycle of the estimated γ-ray light curve of the PSR J1709-4429 pulsar with different kernel density estimators fitted to the data. This estimated γ-ray light curve above 0.1 GeV consists of $n = 21153$ times of arrival (TOAs), with an off-pulse interval of $[\hat{a}, \hat{b}] = [0.66, 0.14]$ (Abdo et al., 2010f), obtained with the subjective "eye-ball" technique.

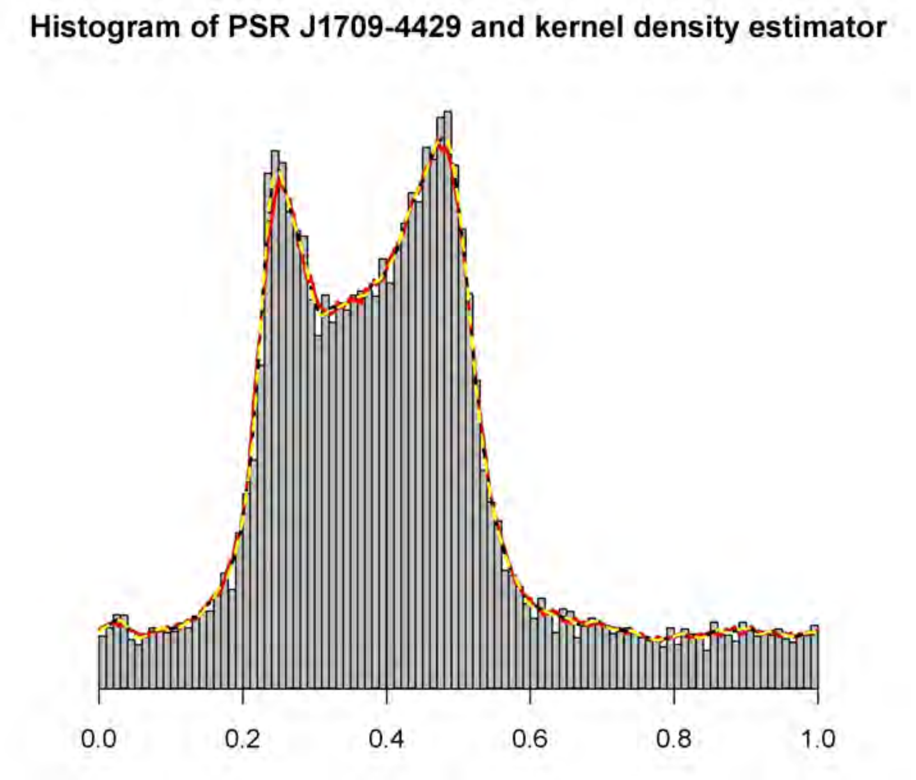

Figure 5.6: *Histogram representation (*100 *classes) of the estimated γ-ray light curve (obtained from Fermi LAT, energies above* 0.1 *GeV) of PSR J1709-4429 with different kernel density estimators (red=normal kernel, yellow=Epanechnikov kernel, and black=Swanepoel kernel) fitted to the data with estimated smoothing parameter* $\hat{h}_3 = 0.13$.

Off-pulse interval comparison

Based on the recommendations in Section 5.2, the number of minimum points m does not greatly influence $\hat{a}$ and $\hat{b}$. A similar situation was observed for the PSR J1709-4429 light curve. Only marginal differences are found in the estimation of $[a, b]$, when $m > 1$ is used, compared to $m = 1$ (see, e.g., the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests in Tables 5.26 – 5.27).

For the estimated smoothing parameters, several choices of $\hat{h}$ are recommended in Section 5.2. For the PSR J1709-4429 light curve, it is found that the following smoothing parameters relate very closely to each other, and can be grouped into three distinct sets:

- $\hat{h}_1$ and $\hat{h}_6$;
- $\hat{h}_2$, $\hat{h}_5$ and $\hat{h}_8$, and

- $\hat{h}_3$, $\hat{h}_4$ and $\hat{h}_9$.

Tables 5.26 - 5.27 compare the estimated off-pulse interval $[\hat{a}, \hat{b}]$ for one of the estimated smoothing parameters in each of the above-mentioned sets. The choice of estimated smoothing parameter does have a slight effect on the estimation, and it seems as if $\hat{h}_2$ performs the best for the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests, with $m = 1$. For this choice of estimated smoothing parameter, the estimated off-pulse interval is closely related to the interval identified with the subjective "eye-ball" technique.

Table 5.26: *Estimated off-pulse intervals for PSR J1709-4429 with $m = 1$, $r = 8$, $g = 8$ and with $\alpha = 0.01$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.6900	0.1440	0.6205	0.1676	0.6821	0.1571	0.6303	0.1676
$\hat{h}_2$	0.6472	0.1399	0.6122	0.1636	0.6451	0.1538	0.6095	0.1706
$\hat{h}_3$	0.6829	0.1413	0.6152	0.1649	0.6472	0.1565	0.6136	0.1691

Table 5.27: *Estimated off-pulse intervals for PSR J1709-4429 with $m = 5$, $r = 8$, $g = 8$ and with $\alpha = 0.01$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
$\hat{h}_1$	0.6801	0.1222	0.6152	0.1662	0.6627	0.1557	0.6295	0.1675
$\hat{h}_2$	0.6466	0.1194	0.6069	0.1636	0.6299	0.1533	0.6079	0.1706
$\hat{h}_3$	0.6683	0.1405	0.6064	0.1639	0.6331	0.1547	0.6046	0.1701

Pertaining to the choice of the goodness-of-fit test, the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests are preferred when comparing the estimates to the results obtained with visual inspection of the kernel estimators of the relevant light curve. The Cramér-von-Mises and Rayleigh goodness-of-fit tests seem to produce estimated values that are slightly different from the off-pulse interval obtained with visual inspection. For the Anderson-Darling goodness-of-fit test, though, the estimated values are comparable.

Table 5.28: *Estimated off-pulse intervals for the PSR J1709-4429 with $m = 1$, $r = 8$, $g = 8$ and with $\hat{h}_2$.*

	Anderson-Darling		**Cramér-von-Mises**		**Kolmogorov-Smirnov**		**Rayleigh**	
$\boldsymbol{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
0.01	0.6472	0.1399	0.6122	0.1636	0.6451	0.1538	0.6095	0.1706
0.05	0.6858	0.0321	0.6145	0.1626	0.6531	0.1409	0.6348	0.1636
0.10	0.6925	0.9005	0.6152	0.1610	0.6829	0.0346	0.6704	0.1610

When inspecting the effect of the level of significance α, Table 5.28 highlights that the choice of α has some impact on the estimated off-pulse interval, especially for the Anderson-Darling goodness-of-fit test when $\alpha = 0.1$. This, however, is no reason for concern, as large α-values were never found to be the best choice when the sample size n is large. Additional analyses again show that the Cramér-von-Mises goodness-of-fit test is the most robust goodness-of-fit test for different values of α and also for different choices of r and g. For the other goodness-of-fit tests, the estimated values tend to fluctuate slightly more when different values of α, g and r are chosen. The results for the Anderson-Darling goodness-of-fit test are shown in Tables 5.29 - 5.30 for $\alpha = 0.01$, $m = 1$ and for $\hat{h}_2$. It can be seen that the Anderson-Darling goodness-of-fit test is also robust against different choices of r, g and α, producing consistent results for different combinations of the tuning parameters. It is evident that combinations with larger values for $g(\geq 15)$ and $r(\geq 4)$ are preferred. This trend is in agreement with the findings from the simulation study.

Table 5.29: *Estimated values of a (left end-point of the off-pulse interval) for PSR J1709-4429 with* $m = 1$, $\alpha = 0.01$ *and with* $\hat{h}_2$ *for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**
g=2	0.6827	0.6809	0.6779	0.6779	0.6779	0.6779	0.6473	0.6473
g=4	0.6827	0.6827	0.6779	0.6472	0.6472	0.6472	0.6472	0.6472
g=6	0.6827	0.6789	0.6789	0.6472	0.6472	0.6472	0.6472	0.6472
g=8	0.6821	0.6791	0.6791	0.6472	0.6472	0.6472	0.6472	0.6472
g=10	0.6805	0.6805	0.6805	0.6463	0.6463	0.6463	0.6463	0.6463
g=12	0.6827	0.6779	0.6472	0.6472	0.6472	0.6472	0.6472	0.6472
g=14	0.6821	0.6821	0.6466	0.6466	0.6466	0.6466	0.6466	0.6466
g=15	0.6805	0.6463	0.6463	0.6463	0.6463	0.6463	0.6463	0.6463
g=16	0.6791	0.6791	0.6493	0.6493	0.6493	0.6493	0.6493	0.6493
g=17	0.6827	0.6827	0.6466	0.6466	0.6466	0.6466	0.6466	0.6466
g=18	0.6827	0.6827	0.6827	0.6495	0.6495	0.6495	0.6495	0.6495
g=19	0.6808	0.6461	0.6461	0.6461	0.6461	0.6461	0.6461	0.6461
g=20	0.6791	0.6791	0.6456	0.6456	0.6456	0.6456	0.6456	0.6456

Table 5.30: *Estimated values of b (right end-point of the off-pulse interval) for PSR J1709-4429 pulsar with* $m = 1$, $\alpha = 0.01$ *and with* $\hat{h}_2$ *for the Anderson-Darling goodness-of-fit test.*

	r=1	**r=2**	**r=4**	**r=6**	**r=8**	**r=10**	**r=12**	**r=14**
g=2	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.1398
g=4	0.0380	0.0380	0.0380	0.0380	0.1399	0.1399	0.1399	0.1399
g=6	0.0382	0.0382	0.0382	0.0382	0.1401	0.1401	0.1401	0.1401
g=8	0.0383	0.0383	0.0383	0.1399	0.1399	0.1399	0.1399	0.1399
g=10	0.0379	0.0379	0.0379	0.1398	0.1398	0.1398	0.1398	0.1398
g=12	0.0391	0.0391	0.1409	0.1409	0.1409	0.1409	0.1409	0.1409
g=14	0.0379	0.0379	0.0379	0.1409	0.1409	0.1409	0.1409	0.1409
g=15	0.0391	0.1409	0.1409	0.1409	0.1409	0.1409	0.1409	0.1409
g=16	0.0383	0.0383	0.1399	0.1399	0.1399	0.1399	0.1399	0.1399
g=17	0.0382	0.0382	0.1406	0.1406	0.1406	0.1406	0.1406	0.1406
g=18	0.0391	0.0391	0.1411	0.1411	0.1411	0.1411	0.1411	0.1411
g=19	0.0382	0.0382	0.1409	0.1409	0.1409	0.1409	0.1409	0.1409
g=20	0.0391	0.0391	0.1409	0.1409	0.1409	0.1409	0.1409	0.1409

In conclusion, it can be seen that SOPIE provides an objective estimation for the off-pulse interval of a pulsar light curve, closely related to the interval obtained with the "eye-ball guesstimate."

Chapter 6

Concluding remarks

6.1 Introduction

The launch of the *Fermi Gamma-ray Space Telescope* in 2008 resulted in a dramatic increase in the number of known γ-ray pulsars, presenting great opportunities to study these high-energy objects. Several recent research papers have focused on examining the properties of the off-pulsed emission of each pulsar and attempted to detect the potential emission associated with its PWN. In these research papers, the estimated light curves were utilised to perform analyses to understand the pulsar magnetosphere even better, including the PWNe associated with most pulsars. It is therefore crucial that the identification of the off-pulse interval be done accurately, since additional research of the pulsar structure is based on this interval. In the Astrophysical literature, the underlying light curve of a pulsar is usually estimated by a histogram, and "eye-ball" techniques are applied to identify the relevant off-pulse interval. This technique is regarded as subjective in nature, and dependent on the parameters used in the construction of the histogram, such as the bin width and the choice of the starting point for the first bin.

In contrast to this *subjective* approach of identifying the off-pulse interval visually, the main objective of this thesis is the development of a nonparametric sequential estimation technique for the off-pulse interval(s) of a source function originating from a pulsar. This technique, called SOPIE, was developed and discussed in Chapter 3. Since SOPIE has several tuning parameters that are essential for the accurate and consistent estimation of the off-pulse interval of a pulsar, a simulation study was performed in Chapter 4. The aim of the simulation study was primarily to establish the accuracy and consistency of SOPIE to estimate the off-pulse interval(s) of an unknown source function. The secondary aim was to determine whether certain parameter configurations consistently resulted in a more accurate estimation of the off-pulse interval. Finally, in Chapter 5, SOPIE was applied to pulsar data, based on the optimal parameter configurations that followed from the simulation study.

In Section 6.2, some of the most important findings of this thesis are highlighted. A brief discussion of future research in this area is then provided in Section 6.3.

6.2 Key findings

As stated several times, the main objective of the research is the development of a nonparametric sequential estimation technique for the off-pulse interval(s) of a source function originating from a pulsar (SOPIE). An essential building block of SOPIE is kernel density estimation on a circle, which is discussed in detail in Chapter 2. The three most important components of a kernel density estimator defined on a circle are:

1. the distance measure;
2. the kernel function, and
3. the smoothing parameter.

From the discussion in Chapter 2, and the simulation study results in Chapter 4, it is clear that the two different choices of the distance measures, defined in (2.21) and (2.23), result in nearly similar behaviour of the kernel density estimator. Furthermore, different choices of the kernel function appearing in the kernel density estimator produce nearly identical density estimation. This is, however, common knowledge when reviewing the literature of kernel density estimation on the real line (Silverman, 1986; Wand & Jones, 1995). Pertaining to the smoothing parameter, nine different estimated smoothing parameters are utilised in the kernel estimation. From the simulation study it is found that the optimal estimated smoothing parameters are those parameters that relate to the plug-in estimates proposed by Silverman (1986), viz. $\hat{h} = 1.06\hat{\sigma}n^{-1/5}$, where $\hat{\sigma}$ is some estimate of the dispersion of the data (see Table 4.1). In Chapter 5, it is again evident that the last-mentioned estimated smoothing parameters perform the best when SOPIE is applied to pulsar data.

Since SOPIE is based in a sequential way on the P-values of goodness-of-fit tests for the *uniform* distribution, the goodness-of-fit test plays a vital role. Throughout the simulation study, three of the four goodness-of-fit tests stand out, namely the Anderson-Darling, Cramér-von-Mises and Kolmogorov-Smirnov goodness-of-fit tests. It is found that the *Anderson-Darling* and *Cramér-von-Mises* goodness-of-fit tests are optimal for *large sample* data sets. The *Anderson-Darling* and *Kolmogorov-Smirnov* goodness-of-fit tests perform best for *smaller* data sets. In the analyses of the pulsars in Chapter 5, exactly the same behaviour is observed.

From the simulation study and the analyses of the pulsars, it is observed that SOPIE is especially effective (both accurate and consistent) for data sets where at most 60% of the interval $[0, 1]$ consists of pulsed signal (this implies that the range of the off-pulse interval is in the order of 40% or more of the interval $[0, 1]$). *Pertaining to pulsar data, this is completely in line with the behaviour of pulsars when reviewing Kanbach (2002) and Abdo et al. (2010f).*

Several tuning parameters also form part of the backbone of SOPIE, namely m, α, r and g. The optimal choices for each tuning parameter are discussed in detail in Section 4.7 of Chapter 4. A summary of the optimal choices is provided again. It is recommended to use $m = 1$ in most cases, but any value of $m \leq 5$ can also be used. For α, g and r, it is recommended to use $1 \leq g \leq 10$ and $1 \leq r \leq 6$ in combination with a small level of significance (e.g., $\alpha = 0.05$), or $1 \leq g \leq 10$ and $5 \leq r \leq 10$ in combination with a slightly larger level of significance (e.g., $\alpha = 0.10$) for smaller sample sizes n. For larger sample sizes, rather use a small α-value such as $\alpha = 0.01$, together with $1 \leq g \leq 40$ and $1 \leq r \leq 8$.

These optimal choices of the tuning parameters are utilised in the application of SOPIE to the pulsar data in Chapter 5. What is especially worth mentioning is that the estimation results from the pulsars verify that these tuning parameter choices are optimal. Therefore, both the simulation study and the analyses of the pulsar data show that SOPIE produces accurate and consistent estimation of the off-pulse interval of a source function, especially when the range of the off-pulse interval is in the order of at most 40%. For the pulsar data, the SOPIE-estimated off-pulse intervals are in close agreement with the off-pulse intervals identified with the subjective "eye-ball" techniques (obtained by visual inspection of the histogram).

6.3 Future research

- The research topic of estimating the off-pulse interval(s) of an unknown source function originated from an astrophysical context. The aim of the research was therefore to develop a specific technique to address this astrophysical problem. Future research relating to this topic will be the investigation of the application of SOPIE (with modifications) to other fields of research. One possible idea that comes to mind is to investigate the modification of SOPIE so that it can be used as an identification technique for *change-points* embedded in circular data. It may also be extended to data on the real line. Since the change-point problem (together with so-called *cusum* procedures) for circular and angular data has been studied before (Lombard, 1986; Jammalamadaka & SenGupta, 2001; Lombard & Maxwell, 2012), these sources will assist in providing some direction for future work.

- A very recent paper by Scargle, Norris, Jackson & Chiang (2013) addresses the problem of detecting and characterising local variability in time series and other forms of sequential data. The aim of the paper is to identify and characterise statistically significant variations in the data using a generalised version of Bayesian Blocks. It is also evident that this approach can be used in Astrophysics, as evident in Abdo et al. (2013), where the Bayesian Block methodology is used to identify the off-pulse interval of pulsars. Future research may compare the off-pulse interval of pulsars in the last-mentioned paper (obtained with Bayesian Blocks) to the results that can be obtained with SOPIE.

- Future research may also investigate the possibility of expressing the optimal choices of tuning parameters as a function of the sample size n, the noise level $1-p$ and/or the expected range of the off-pulse interval. It may even be possible to formulate this as an optimisation problem, where the goal function is some expression for the P-value of different goodness-of-fit tests. The feasible region may then be defined by constraints that contain functions of the tuning parameters.

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