Conserved vectors and analytic solutions of potential Kadomtsev-Petviashvili and Korteweg-de Vries-like equations

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Declaration

I MDUDUZI YOLANE THABO LEPHOKO, student number 24860425, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Mr. MDUDUZI YOLANE THABO LEPHOKO

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF CHAUDRY MASOOD KHALIQUE

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 3

C.M. Khalique, M.Y.T. Lephoko, Conserved vectors and solutions of the two-dimensional potential Kadomtsev-Petviashvili equation, has been submitted for publication to Open Physics

Chapter 4

M.Y.T. Lephoko, C.M. Khalique Exact solutions and conserved vectors of the Korteweg-de Vries-like equation, has been submitted for publication in Results in Physics

Dedication

I dedicate this dissertation to the people who have supported and inspired me throughout my academic journey.

To my mother, Nontombi Mabel Lephoko, and my father, Demu Phineas Simelane, I am forever grateful for the sacrifices you have made and the unwavering love and encouragement you have shown me.

To Gogo Elda Lephoko who always prays for my well-being and success.

To the Mahlaku family, who have welcomed me into their home and hearts, I thank you for your kindness, generosity, and the sense of belonging you have given me.

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Abstract

In this work, we study three nonlinear partial differential equations that are used to simulate a variety of physical processes in science. As an example of what will be done in this dissertation, we start off by looking at the second order nonlinear Boltzmann equation. The key equations that we examine in this dissertation are the potential Kadomtsev-Petviashvili and Korteweg-de Vries-like equation. Lie symmetry analysis is employed to obtain commutation relations, one-parameter group of transformations and exact solution for the equations under study. Moreover, for more physical illustration of the extracted solutions, three and two dimensional plots for the solutions are presented. We further explore methods like Noether's approach, multiplier method and the conservation theorem due to Ibragimov to derive their conservation laws where they may apply.

Keywords: Boltzmann equation; potential Kadomtsev-Petviashvili equation; Korteweg-de Vries-like equation; Lie symmetry analysis; Kudryashov's method; conservation laws; Noether's theorem; Ibragimov's theorem; multiplier method.

Introduction

It is a known fact that almost all physical phenomena of physics and other fields obey mathematical laws and hence nonlinear partial differential equations (NLPDEs) can be used to describe such phenomena [1]. This makes the theory of NLPDEs to be one of the most important fields in mathematics. Considering the foregoing, a number of important methods for generating exact solutions to NLPDEs have been developed by noteworthy scientists. These methods include the bifurcation method [2], the Bäcklund transformation [3], Hirota bilinear method [4], Kudryashov's method [5], the (G'/G)-expansion method [6] and Lie group method [7–10].

With the use of Galois' theory, Sophus Lie (1842–1899), a Norwegian mathematician, established the symmetry method and demonstrated that many of the known ad hoc methods of integration of ordinary differential equations (ODEs) could be obtained in a systematic manner [11]. ODEs and partial differential equations (PDEs) have both been studied using Lie group theory. See for example [12–17]. The approach has evolved into a helpful tool for solving differential equations (DEs), classifying them, and preserving the solution set in the DEs.

Furthermore, it has been observed that conservation laws are established and entrenched natural laws that have been studied by many researchers in various scientific fields. Conservation laws that are commonly used in this context include, conservation of linear momentum in an isolated system, conservation of electric charge, conservation of energy, conservation of mechanical energy in the absence of dissipative forces, and many others [12, 17].

One of the fundamental principles in the formulation and investigation of models in mathematics is conservation laws. The existence of a large number of conservation laws of NLPDEs, for example, is sometimes a strong indicator of their integrability [18]. They also help in development of numerical schemes, and theory of non-classical transformations [19]. Hence many methods have been developed to compute conservation laws. These include, Noether's theorem [20], the multiplier approach [8], and conservation theorem due to Ibragimov [21].

The outline of this dissertation can simply be stated as follows:

In Chapter 1 we present preliminaries on Lie symmetry analysis, that is, definitions, theorems and various methods for obtaining conservation laws of PDEs. Also a brief description of Kudryashov's method for finding exact solutions of DEs is given that will be needed in our study.

In Chapter 2 we give an illustrative example to demonstrate how the methods, theorems and definitions stated in Chapter one will be used throughout this dissertation. We investigate the second-order nonlinear Boltzmann equation and using Lie group theory to obtain its group-invariant solutions. Thereafter, we employ the multiplier method, Noether's theorem and the conservation theorem due to Ibragimov to derive its conserved vectors.

In Chapter 3 we compute Lie point symmetries for the potential Kadomtsev-Petviashvili equation. We then determine its commutator table and one-parameter group of transformations. Moreover, we use its Lie point symmetries to perform reductions. Travelling wave solutions are obtained by applying Kudryashov's method. Finally, conservation laws are derived using Ibragimov's approach, multiplier method and Noether's theorem.

Chapter 4 deals with the Korteweg-de Vries-like equation We obtain its Lie point symmetries, present the commutator table and compute the one-parameter group of transformations. Reductions are performed using the obtained symmetries. We then provide conservation laws of the Korteweg-de Vries-like equation using the theorem due to Ibragimov.

In the final chapter, a summary of the results obtained in the dissertation is given and future work is discussed.

Bibliography is given at the end.

Chapter 1

Symmetries of PDEs

In this chapter we give some basic concepts, definitions and theorems of the Lie group theory that will be used throughout this dissertation. Moreover, we provide methods of obtaining conservation laws and closed-form solutions of PDEs.

1.1 Introduction

Marius Sophus Lie, a Norwegian mathematician, born in Nordfjordeid in 1842 realized more than a century ago that through group theory many of the approaches for solving DEs could be unified [12]. Through his study on continuous groups he devised a method for obtaining closed-form solutions to DEs based on symmetry. Nearly all known approaches for solving DEs are particular cases of Lie's theory. Many books and articles have been published demonstrating the power of this method. This includes Ovsiannikov [7], Olver [8], Bluman and Kumei [9], Stephani [10], Ibragimov [22,23], Hydon [24]. The definitions, theorems and conclusions offered in this chapter are based on the aforementioned publications.

1.2 Local one-parameter Lie group

Definition 1.1 Suppose H is a set of transformations

$$\Psi_a : \bar{x}^j = \psi^j(x, a), \;\; \alpha = 1, \ldots, n, \tag{1.1}$$

where a is a real parameter that ranges in values from the neighbourhood $\mathbb{D} \subset \mathbb{R}$ of $a = 0$ and ψ^j are differentiable functions, then the set H is said to be a *continuous one-parameter (local) Lie group of transformations* if it satisfies the properties:

(i) Closure: For $\Psi_a, \Psi_b \in H$ and $a, b \in \mathbb{D}' \subset \mathbb{D}$, then $\Psi_b\Psi_a = \Psi_c \in H, c = \varphi(a, b) \in \mathbb{D}$.

(ii) Identity: $\Psi_0 \in H$ such that $\Psi_0\Psi_a = \Psi_a\Psi_0 = \Psi_a$ for any $a \in \mathbb{D}' \subset \mathbb{D}$ and $\Psi \in H$.

(iii) Inverse: For any $\Psi_a \in H, a \in \mathbb{D}' \subset \mathbb{D}$, there exists $\Psi_a^{-1} = \Psi_{a^{-1}} \in H, a^{-1} \in \mathbb{D}$ such that $\Psi_a\Psi_{a^{-1}} = \Psi_0 = \Psi_{a^{-1}}\Psi_a$.

1.3 Infinitesimal transformations

According to Lie group theory construction of the symmetry group H is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x}^j \approx x^j + a\xi^j(x) \tag{1.2}$$

by the Taylor series expansion in a of (1.1) about $a = 0$ and also taking into account the initial conditions

$$\psi^j\big|_{a=0} = x^j.$$

Consequently,

$$\xi^j(x) = \left.\frac{\partial \psi^j(x, a)}{\partial a}\right|_{a=0} \tag{1.3}$$

elements of the vector field $v = \left(\frac{\partial \psi^1(x,a)}{\partial a}, \frac{\partial \psi^2(x,a)}{\partial a}, \ldots, \frac{\partial \psi^n(x,a)}{\partial a}\right)$ at local coordinates $(x^1, x^2, \ldots, x^n)$. By introducing the operator

$$X = \xi^j(x, u)\frac{\partial}{\partial x^j} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{1.4}$$

we note that equation (1.2) can now be written in the form

$$\bar{x}^j \approx (1 + a\,X)x^j.$$

The operator X is known as the *infinitesimal operator* of the group H.

1.4 Group invariants

Definition 1.2 A function $\mathcal{P}(x, u)$ is the invariant of the group of transformations (1.1) if

$$\mathcal{P}(\bar{x}, \bar{u}) = \mathcal{P}(x, u), \tag{1.5}$$

for all x, u and the parameter $a \in \mathbb{D}' \subset \mathbb{D}$.

Theorem 1.1 (Infinitesimal criterion of invariance) The function $\mathit{\Gamma}(x, u)$ is an invariant of group H if

$$X\,\mathit{\Gamma} \equiv \ \xi^j(x, u)\frac{\partial \mathit{\Gamma}}{\partial x^j} + \eta^\alpha(x, u)\frac{\partial \mathit{\Gamma}}{\partial u^\alpha} = 0. \tag{1.6}$$

The aforementioned theorem suggests that every one-parameter group of point transformations (1.1) has functionally independent invariants. The linear equation (1.6) is solvable by the method of characteristics giving invariant curves that are tangent to the vector (ξ^j, η^α) for $j = 1, \ldots, n$ and $\alpha = 1, \ldots, m$ which can be regarded as the left-hand side of any first integrals

$$J_1(x, u) = c_1, \ldots, J_{m+n-1}(x, u) = c_{m+n-1},$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^m}{\eta^m(x, u)}.$$

Theorem 1.2 If the infinitesimal transformation (1.2) or its generator X is given, the corresponding one-parameter group H is then obtained by solving the Lie equations

$$\frac{d\bar{x}^j}{da_s} = \xi^j_s(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da_s} = \eta^\alpha_s(\bar{x}, \bar{u}), \quad s = 1, \ldots, l, \tag{1.7}$$

subject to the initial conditions

$$\bar{x}^j\big|_{a=0} = x^j, \quad \bar{u}^\alpha\big|_{a=0} = u^\alpha.$$

1.5 Construction of a symmetry group

The algorithm for determining a symmetry group for a given PDE is described below, but first some definitions are provided.

1.5.1 Prolongation of point transformations

Consider a second-order PDE

$$E(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = 0, \tag{1.8}$$

where t and x are two independent variables and u is a dependent variable. Let

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{1.9}$$

be the infinitesimal generator of the one-parameter group G of transformation (1.1). The second prolongation of X, denoted by $X^{[2]}$, is given by

$$X^{[2]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}, \tag{1.10}$$

where,

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi)
\end{aligned}$$

and the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \tag{1.11}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{1.12}$$

Applying the definitions of D_t and D_x given above we obtain the respective values of ζs to be

$$\zeta_t = \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u, \tag{1.13}$$

$$
\begin{aligned}
\zeta_x &= \eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u, &(1.14)\\
\zeta_{tt} &= \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + (u_t)^2\eta_{uu} - 2u_{tt}\tau_t - u_t\tau_{tt} - 2(u_t)^2\tau_{tu}\\
&\quad -3u_tu_{tt}\tau_u - (u_t)^3\tau_{uu} - 2u_{tx}\xi_t - u_x\xi_{tt} - 2u_tu_x\xi_{tu} - (u_t)^2u_x\xi_{uu}\\
&\quad -(u_xu_{tt} + 2u_tu_{tx})\xi_u, &(1.15)\\
\zeta_{tx} &= \eta_{tx} + u_x\eta_{tu} + u_t\eta_{xu} + u_{tx}\eta_u + u_tu_x\eta_{uu} - u_{tx}(\tau_t + \xi_x) - u_t\tau_{tx} - u_{tt}\tau_x\\
&\quad -u_tu_x(\tau_{tu} + \xi_{xu}) - u_t^2\tau_{xu} - (2u_tu_{tx} + u_xu_{tt})\tau_u - (u_t)^2u_x\tau_{uu} - u_x\xi_{tx}\\
&\quad -u_{xx}\xi_t - (u_x)^2\xi_{tu} - (2u_xu_{tx} + u_tu_{xx})\xi_u - u_t(u_x)^2\xi_{uu}, &(1.16)\\
\zeta_{xx} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + (u_x)^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2(u_x)^2\xi_{xu}\\
&\quad -3u_xu_{xx}\xi_u - (u_x)^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} - (u_tu_{xx} + 2u_xu_{tx})\tau_u\\
&\quad -2u_tu_x\tau_{xu} - u_t(u_x)^2\tau_{uu}. &(1.17)
\end{aligned}
$$

1.5.2 Group admitted by a PDE

Consider the PDE of second-order

$$E(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0. \tag{1.18}$$

The vector-field

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{1.19}$$

is an infinitesimal symmetry of (1.18) if the Lie condition

$$X^{[2]}\, E\big|_{E=0} = 0 \tag{1.20}$$

holds, where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Equation (1.20) is said to be the determining equation of (1.18) since it determines all the infinitesimal symmetries of (1.18).

Theorem 1.3 A symmetry of (1.18) transforms any solution of (1.18) into another solution of the same equation.

The aforementioned theorem implies that we can construct more solutions of (1.18) from known ones.

1.6 Lie algebras

Suppose we have the operators, $X_i = \xi_l \frac{\partial}{\partial x_l}$ and $X_j = \eta_l \frac{\partial}{\partial x_l}, i, j = 1, \ldots, r$ and $l = 1, \ldots, n$

Definition 1.3 (Commutator) The commutator of X_i and X_j is defined by

$$[X_i, X_j] = X_i X_j - X_j X_i = \sum_{l}^{n} \sum_{m}^{n} \left(\xi_m \frac{\partial \eta_l}{\partial x_m} - \eta_m \frac{\partial \xi_l}{\partial x_m} \right) \frac{\partial}{\partial x_l}.$$

Definition 1.4 (Lie algebra) A Lie algebra is a vector space L (over the field of real numbers) of operators with the property: For all $X_i, X_j \in L$, the commutator $[X_i, X_j] \in L$.

The dimension of a Lie algebra is the dimension of the vector space L.

Theorem 1.4 The set of all solutions of any determining equation forms a Lie algebra.

1.7 Solution methods for differential equations

A few approaches for finding exact solutions to DEs are presented in this section.

1.7.1 Kudryashov's method to obtain travelling wave solutions

Consider the nonlinear evolution equation (NLEE)

$$F(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xt}, u_{xx}, ...) = 0. \tag{1.21}$$

Here, F is a polynomial in $u(x, y, z, t)$ and its various partial derivatives involving the highest order derivatives and nonlinear terms. The Kudryashov's method can be described by the following fundamental steps [25]:

First step

We transform equation (1.21) from PDE to ODE by using the travelling wave transformation $\xi = k(x + y + z - Vt)$, $u(x, y, z, t) = u(\xi)$ in the form

$$P(u, u', u'', u''', ...) = 0. \tag{1.22}$$

Here, P is a function of $u(\xi)$, prime denotes $\dfrac{d}{d\xi}$ and V is the speed of travelling wave.

Second step

Assume that the solution of equation (1.22) has the following form

$$U(\xi) = \sum_{i=0}^{N} a_i q^i(\xi). \tag{1.23}$$

Here, $a_i (i = 0, 1, 2, ..., N)$ are constants to be investigated afterwards such that $a_N \neq 0$.

Consider the ODE

$$\frac{d}{d\xi} q(\xi) = q^2(\xi) - q(\xi) \tag{1.24}$$

whose solution is

$$q(\xi) = \frac{1}{1 + Ae^{\xi}}, \tag{1.25}$$

where A is an integrating constant.

Third step

The positive integer N in equation (1.23) will be determined by using the homogeneous balance between the highest order linear term and the highest order nonlinear term occurring in equation (1.22).

Fourth step

Using equations (1.23) and (1.24) into equation (1.22), we obtain a polynomial in q^i, where $(i = 0, 1, 2, ...)$. Allocating all terms of same power and equating them to zero, we obtain a system of algebraic equations for this polynomial. These equations can be solved by Maple or Mathematica to find the values of a_i and V and using these values we can obtain the desired solutions. Implementing these solutions then into equations (1.21) and (1.22), we can justify whether the solutions are exact or not.

1.8 Conservation laws

1.8.1 Fundamental operators and their relationship

Consider a pth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, given by

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(p)}) = 0, \quad \alpha = 1, \ldots, m. \tag{1.26}$$

Definition 1.5 (Euler-Lagrange operator) The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2\ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{1.27}$$

Definition 1.6 (Lagrangian) If there exists a function
$\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)})$, $s \leq p$, p being the order of equation (1.26), such that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{1.28}$$

then $\mathcal{L}$ is called a Lagrangian of equation (1.26). Equation (1.28) is known as the Euler-Lagrange equation.

Definition 1.7 (Lie-Bäcklund operator) The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.29}$$

where $\mathcal{A}$ is the space of differential functions. The operator (1.29) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2\ldots i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2\ldots i_s}}, \tag{1.30}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s > 1, \end{aligned} \tag{1.31}$$

in which W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{1.32}$$

One can write the Lie-Bäcklund operator (1.30) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1}\dots D_{i_s}(W^\alpha)\frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}. \tag{1.33}$$

Definition 1.8 (Conservation law) The n-tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a conserved vector of (1.26) if T^i satisfies

$$D_i T^i|_{(1.26)} = 0. \tag{1.34}$$

The equation (1.34) defines a local conservation law of system (1.26).

1.8.2 Multiplier method

The algorithm of finding the conservation laws for DEs is given in [26]. The advantage of this approach is that it does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.35}$$

holds identically.

The right hand side of (1.35) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.36}$$

When the multipliers are found, then the conserved vectors are constructed by invoking the homotopy operator [26].

1.8.3 Noether's theorem

Consider the pth-order system of PDEs of n independent and N dependent variables

$$E_\alpha\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(p)}\right)=0, \quad \alpha=1,2, \ldots N. \tag{1.37}$$

Definition 1.9 If there exists a function $\mathcal{L}\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right) \in \mathcal{A}, s \leq p$, such that (1.37) are equivalent to

$$\frac{\delta \mathcal{L}}{\delta u^\alpha}=0, \quad \alpha=1,2, \ldots, N, \tag{1.38}$$

then $\mathcal{L}$ is called a Lagrangian of (1.37) and (1.38) are the corresponding Euler-Lagrange DEs.

Definition 1.10 A Lie-Bäcklund operator X is a Noether symmetry generator associated with a Lagrangian $\mathcal{L}$ of (1.37) if there exists a vector $B=\left(B^1, B^2, \ldots, B^n\right)$ such that

$$X(\mathcal{L})+\mathcal{L} D_i\left(\xi^i\right)=D_i\left(B^i\right). \tag{1.39}$$

Theorem 1.5 For each Noether symmetry generator X associated with a given Lagrangian $\mathcal{L}$ corresponding to the Euler-Lagrange DEs, there corresponds a vector $T=\left(T^1, T^2, \ldots, T^n\right)$ with T^i defined by

$$\begin{aligned} T^i & =B^i-N^i \mathcal{L} \\ & =B^i-\xi^i \mathcal{L}-W^\alpha \frac{\delta \mathcal{L}}{\delta u_i^\alpha}-\sum_{s \geq 1} D_{i_1 \cdots i_s}\left(W^\alpha\right) \frac{\delta \mathcal{L}}{\delta u_{i_1 \cdots i_s}^\alpha}, \end{aligned} \tag{1.40}$$

which is a conservation law for the Euler-Lagrange DEs (1.38). In Noether's method we find $\mathcal{L}\left(x, u, \ldots, u_{(p-1)}\right)$ and then equation (1.39) is used for the construction of Noether symmetries. Moreover, equation (1.40) yields the corresponding Noether conservation laws. The Lie characteristics W^α of the Noether symmetry generator are the characteristics of the conserved vectors.

1.8.4 A conservation theorem due to Ibragimov

A new conservation theorem due to Ibragimov [21] provides the procedure for computing the conserved vectors associated with all symmetries of the system of pth-order PDEs, that is

$$F_\alpha(x,u,u_{(1)},...,u_{(p)})=0,\ \alpha=1,...,m. \tag{1.41}$$

Definition 1.11 (Adjoint equations) Consider a system of pth-order PDEs given by (1.41). Let

$$F^*_\alpha(x,u,v,...,u_{(p)},v_{(p)})=\frac{\delta(v^\beta F_\beta)}{\delta u^\alpha},\ \alpha=1,...,m, \tag{1.42}$$

where $v=(v^1,...,v^m)$ are new dependent variables, $v=v(x)$, and define the system of adjoint equations to system (1.41) by

$$F^*_\alpha(x,u,v,...,u_{(p)},v_{(p)})=0,\ \alpha=1,...,m. \tag{1.43}$$

Theorem 1.6 Any system of PDEs (1.41) considered together with its adjoint system (1.43) has a Lagrangian

$$\mathcal{L}=v^\beta F^*_\beta(x,u,v,...,u_{(p)}). \tag{1.44}$$

Theorem 1.7 Consider the system of PDEs (1.41). The adjoint system given by (1.43) inherits the symmetries of system (1.41). If system (1.41) admits a point transformation group with a generator (1.29), then the adjoint system (1.43) admits the operator (1.29) extended to the variables v^α by the formula

$$Y=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^i}+\eta^\alpha_*\frac{\partial}{\partial v^i}. \tag{1.45}$$

with

$$\eta^\alpha_*=\eta^\alpha_*(x,u,v,...)=-\left\{\lambda^\alpha_\beta v^\beta+v^\alpha D_i(\xi^i)\right\} \tag{1.46}$$

in which (1.45) and (1.46) are extensions of (1.29) to the variable v^α and λ^α_β and are obtained from

$$X(F_\alpha)=\lambda^\alpha_\beta F_\beta. \tag{1.47}$$

Theorem 1.8 Any infinitesimal symmetry (Lie point, Lie-Bäcklund, nonlocal) given by (1.29) of system (1.41) leads to a conservation law $D_i(T^i)=0$ for the system (1.41) and (1.43). The components of the conserved vector are given by the formula

$$
\begin{aligned}
T^i \;&=\; \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} + D_j D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right] \\
&\quad + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots \right] + D_j D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right], \qquad (1.48)
\end{aligned}
$$

where, W^α is the Lie characteristic function given by (1.32) and $\mathcal{L}$ is the Lagrangian (1.44) [23]. The functions ξ^i and η^α are the coefficient functions of the generator (1.29). The conserved vectors (1.48) involve the arbitrary solutions v of the system of adjoint equations (1.43) and hence one obtains an infinite number of conservation laws for (1.41) by specifying v.

1.9 Conclusion

We discussed some basic concepts and definitions of Lie group analysis in this chapter. Moreover methods to obtain exact solutions and conservation laws of PDEs were presented. These methods were Kudryashov's method (to obtain exact solutions), Noether's theorem, multiplier method and the conservation theorem due to Ibragimov. All the aforementioned methods will be used throughout this dissertation.

Chapter 2

Symmetries and conservation laws of the Boltzmann differential equation: an illustrative example

The generalized $(p+1)$th Boltzmann equation in $(1+1)-$dimensions has the form [27]

$$\left(\frac{\partial}{\partial t}+1\right)\left(-\frac{\partial}{\partial x}\right)^{p} u(t,x)=\frac{\Gamma(2p)}{\Gamma(p)}u^{2}(t,x), \tag{2.1}$$

where p is a positive integer. When $p=1$, equation (2.1) reduces to

$$u_{tx}+u_{x}+u^{2}=0. \tag{2.2}$$

In this chapter we study equation (2.2). We calculate Lie point symmetries, group invariant solutions and its conservation laws. The nonlinear Boltzmann equation was derived in [28] by Krook and Wu, and arises in the formation of Maxwellian tails in gas dynamics [29, 30]. In kinetic theory, Boltzmann equation provides a statistical description of a gas of interacting particles [31].

2.1 Lie point symmetries of (2.2)

Equation (2.2) admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.3}$$

provided

$$X^{[2]}(u_{tx} + u_x + u^2)\big|_{(2.2)} = 0. \tag{2.4}$$

Using the definition of $X^{[2]}$, where $X^{[2]}$ is the second prolongation defined by

$$X^{[2]} = \eta\frac{\partial}{\partial u} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}$$

we obtain

$$\eta(2u) + \zeta_x(1) + \zeta_{tx}(1)\big|_{(2.2)} = 0. \tag{2.5}$$

Substituting the values of ζ_x and ζ_{tx} in equation (2.5) and replacing u_{tx} by $-(u_x + u^2)$ we get

$$\begin{aligned}
&2u\eta + \eta_x + u_x\eta_u - u_t\tau_x - u_tu_x\tau_u - u_x\xi_x - u_x^2\xi_u + \eta_{tx} + u_x\eta_{tu} + u_t\eta_{xu}\\
&-u_x\eta_u - u^2\eta_u + u_tu_x\eta_{uu} + u_x(\tau_t + \xi_x) + u^2(\tau_t + \xi_x) - u_t\tau_{tx} - u_{tt}\tau_x\\
&-u_tu_x(\tau_{tx} + \xi_{xu}) - u_t^2\tau_{xu} + 2(u_tu_x + u_tu^2)\tau_u - u_xu_{tt}\tau_u - u_t^2u_x\tau_{uu}\\
&-u_x\xi_{tx} - u_{xx}\xi_t - u_x^2\xi_{tu} + 2(u_x^2 + u_xu^2)\xi_u - u_tu_{xx}\xi_u - u_tu_x^2\xi_{uu} = 0.
\end{aligned} \tag{2.6}$$

Since τ, ξ and η depend only on t, x and u one can split on the derivatives of u to obtain an over determined system of linear PDEs:

$$u_{tt} \quad : \quad \tau_x = 0, \tag{2.7}$$

$$u_xu_{tt} \quad : \quad \tau_u = 0, \tag{2.8}$$

$$u_tu_{xx} \quad : \quad \xi_u = 0, \tag{2.9}$$

$$u_{xx} \quad : \quad \xi_t = 0, \tag{2.10}$$

$$u_tu_x \quad : \quad \eta_{uu} = 0, \tag{2.11}$$

$$u_t \quad : \quad \eta_{xu} = 0, \tag{2.12}$$

$$\begin{aligned} u_x &: \quad \tau_t + \eta_{tu} = 0, && (2.13) \\ 1 &: \quad 2u\eta + \eta_x + \eta_{tx} - u^2\eta_x + u^2\tau_t - u^2\xi_x = 0. && (2.14) \end{aligned}$$

From equations (2.7) and (2.8) we have

$$\tau = a(t), \tag{2.15}$$

where $a(t)$ is an arbitrary function of t. Equations (2.9) and (2.10) imply that

$$\xi = b(x), \tag{2.16}$$

where $b(x)$ is an arbitrary function of x. Integrating equation (2.11) twice with respect to u, we get

$$\eta = c(t,x)u + d(t,x), \tag{2.17}$$

where $c(t,x)$ and $d(t,x)$ are arbitrary functions of t and x. Substituting this value of η in (2.12), we obtain $c_x(t,x) = 0$, which gives

$$c = c(t). \tag{2.18}$$

Thus, from equation (2.17), we have

$$\eta = c(t)u + d(t,x). \tag{2.19}$$

Substituting the values of τ, ξ and η in equations (2.13) and (2.14), we obtain

$$a'(t) + c'(t) = 0 \tag{2.20}$$

and

$$u^2c(t) + 2ud(t,x) + d_x + d_{tx} + u^2a'(t) + u^2b'(x) = 0,$$

respectively. Separating the above equation with respect to u, we obtain

$$\begin{aligned} u^2 &: \quad c(t) + a'(t) + b'(x) = 0, && (2.21) \\ u &: \quad d(t,x) = 0, && (2.22) \\ \text{rest} &: \quad d_x + d_{tx} = 0. && (2.23) \end{aligned}$$

Using equation (2.22) in (2.19) gives

$$\eta = c(t)u. \tag{2.24}$$

Using (2.20) and replacing $a'(t)$ by $-c'(t)$ in (2.21) we get

$$c'(t) - c(t) = b'(x). \tag{2.25}$$

Since the left-hand side of (2.25) is a function of t only and the right-hand side is a function of x only, both must be equal to a constant, say A_1. Hence

$$b'(x) = A_1, \tag{2.26}$$

$$c'(t) - c(t) = A_1. \tag{2.27}$$

Clearly equation (2.26) gives

$$b(x) = A_1x + A_2, \tag{2.28}$$

where A_2 is an arbitrary constant of integration. Solving equation (2.27) for $c(t)$, we have

$$c(t) = A_3e^t - A_1,$$

where A_3 is an arbitrary constant. We now use equation (2.20) to find the value of $a(t)$.

$$a'(t) = -c'(t) = -A_3e^t. \tag{2.29}$$

Integrating the above equation with respect to t, we get

$$a(t) = -A_3e^t + A_4, \tag{2.30}$$

where A_4 is an arbitrary constant. Thus,

$$\tau = -A_3e^t + A_4, \tag{2.31}$$

$$\xi = A_1x + A_2, \tag{2.32}$$

$$\eta = A_3ue^t - A_1u, \tag{2.33}$$

and so, the Lie point symmetries of (2.2) are

$$X_1 = \frac{\partial}{\partial t}, \tag{2.34}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{2.35}$$

$$X_3 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \tag{2.36}$$

$$X_4 = -e^t\frac{\partial}{\partial t} + e^tu\frac{\partial}{\partial u}. \tag{2.37}$$

2.1.1 Commutator table for the symmetries of (2.2)

We now calculate the commutation relations for all the symmetry generators of (2.2) obtained above. Firstly we compute $[X_2, X_3]$. By the definition of Lie bracket we have

$$\begin{aligned}[X_2, X_3] &= X_2X_3 - X_3X_2 \\ &= \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\right) - \left(x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\right)\frac{\partial}{\partial x} \\ &= \frac{\partial}{\partial x} \\ &= X_2.\end{aligned}$$

In the same way, one can compute all the remaining commutation relations using the above procedure. The table below shows the commutation relations in table form.

Table 2.1: Commutator table of Lie algebra of the differential equation (2.2)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_4
X_2	0	0	X_2	0
X_3	0	$-X_2$	0	0
X_4	$-X_4$	0	0	0

2.1.2 One-parameter groups of (2.2)

We now employ the Lie equations with the initial conditions

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}),\ \bar{t}|_{a=0} = t,\ \ \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}),\ \bar{x}|_{a=0} = x,\ \ \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}),\ \bar{u}|_{a=0} = u$$

to compute the one-parameter group of transformations. For each X_i, let G_{a_i} be the corresponding group. We first compute one-parameter group corresponding to infinitesimal generator X_3, namely

$$X_3 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 0,\ \bar{t}|_{a=0} = t,\ \ \frac{d\bar{x}}{da} = \bar{x},\ \bar{x}|_{a=0} = x,\ \ \frac{d\bar{u}}{da} = -\bar{u},\ \bar{u}|_{a=0} = u.$$

Solving the above equations, we obtain

$$\bar{t} = t,\ \ \bar{x} = xe^{a_3},\ \ \bar{u} = ue^{-a_3}.$$

Thus, the one-parameter group G_{a_3} corresponding to the operator X_3 is given by

$$G_{a_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, xe^{a_3}, ue^{-a_3}).$$

If we continue in the same manner as above, we get the following one-parameter groups for all the operators:

$$\begin{aligned}
G_{a_1} &:\ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u),\\
G_{a_2} &:\ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u),\\
G_{a_3} &:\ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, xe^{a_3}, ue^{-a_3}),\\
G_{a_4} &:\ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow \Big(- \ln\left|a_4 + e^{-t}\right|, x, u(1 + a_4e^{t})\Big).
\end{aligned}$$

Since each group G_{a_i} is a symmetry group, if $u = f(t, x)$ is a solution of (2.2) so are the functions

$$\begin{aligned}
u_1 &= f(t - a, x),\\
u_2 &= f(t, x - a),\\
u_3 &= e^{-a} f(t, e^{-a}x),\\
u_4 &= \left(\frac{1}{1 - ae^{t}}\right) f\left(- \ln\left|e^{-t} - a\right|, x\right).
\end{aligned}$$

2.1.3 Constructing group-invariant solutions of (2.2)

We now utilize the generators obtained in the previous subsection and find symmetry reductions and group invariant solutions for equation (4.1).

Case 1.

We firstly, consider the Lie point symmetry $X_1 = \partial/\partial t$. The characteristic equations associated with the operator X_1 are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which give two invariants $J_1 = x$ and $J_2 = u$. Thus, the group-invariant solution is given by $J_2 = f(J_1)$. This gives $u = f(x)$, where f is an arbitrary function. Substituting this value of u into (2.2), we obtain the first-order nonlinear ordinary differential equation (NLODE)

$$f'(x) + f^2(x) = 0.$$

Integrating with respect to x, we obtain the solution to the ODE to be

$$f(x) = \frac{1}{x+C}, \tag{2.38}$$

where C is an arbitrary constant of integration. Thus, the group-invariant solution of (2.2) under the Lie point symmetry X_1 is

$$u(t,x) = \frac{1}{x+C}. \tag{2.39}$$

Remark. Note that the above invariant solution does not depend on the time variable t, and hence this is the *stationary solution* of (2.2). We now present a dynamical profile picture of solution (2.39) in Figure (2.1) by assigning the value $C = 1$.

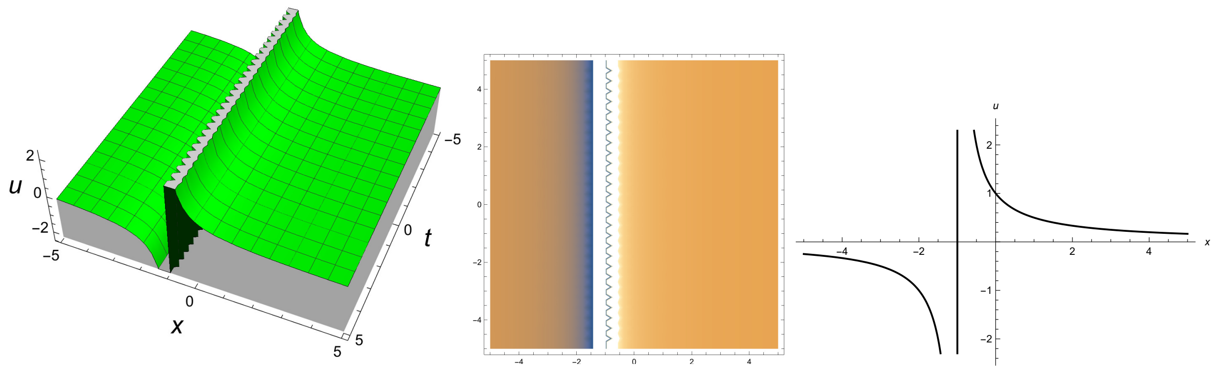

Figure 2.1: 3D and 2D profile of solution of (2.39).

Case 2.

Next, we consider the symmetry operator $X_2 = \partial/\partial x$. The characteristic equations for the operator X_2 are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0},$$

which lead to two invariants $J_1 = t$ and $J_2 = u$. Thus, $u = f(t)$, where f is an arbitrary function of t. Substituting this value of u into (2.2), we obtain

$$f(t) = 0$$

and hence the group-invariant solution of (2.2) under X_2 is just the trivial solution $u(t, x) = 0$.

Case 3.

We now consider the Lie point symmetry $X_3 = x\partial/\partial x - u\partial/\partial u$. The characteristic equations associated with the above operator are

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{-u},$$

that leads to two invariants $J_1 = t$ and $J_2 = ux$. The two invariants imply that $u = (1/x)f(t)$, where f is an arbitrary function of t. Substituting this value of u into (2.2), we obtain the first-order NLODE

$$f'(t) + f(t) - f^2(t) = 0.$$

Solving the above equation, we get

$$f(t) = \frac{1}{1 + C_1 e^t},$$

where C_1 is an arbitrary constant. Hence the group-invariant solution of (2.2) under X_3 is

$$u(t, x) = \frac{1}{x(1 + C_1 e^t)}. \tag{2.40}$$

A dynamical profile picture to the solution (2.40) is shown in Figure (2.2) obtained by assigning the value $C_1 = 1$.

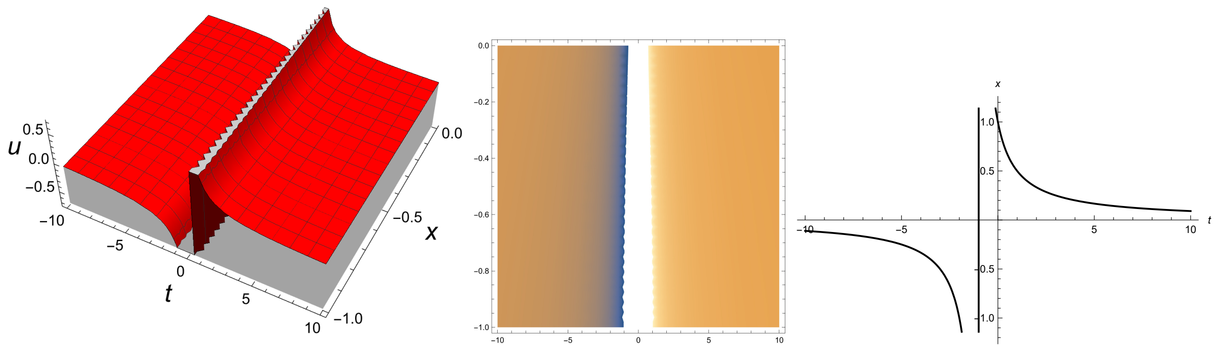

Figure 2.2: 3D and 2D graphics of solution (2.40).

Case 4.

We consider the Lie point symmetry generator $X_4 = -e^t\partial/\partial t + ue^t\partial/\partial u$. The associated characteristic equations to X_4 are

$$\frac{dt}{-e^t} = \frac{dx}{0} = \frac{du}{ue^t}.$$

The characteristic equations yield the two invariants $J_1 = x$ and $J_2 = ue^t$. Hence the group-invariant solution is given by $J_2 = f(J_1)$, where f is an arbitrary function of x. This implies

$$u(t,x) = e^{-t}f(x).$$

Substituting the above value of u into (2.2), we obtain

$$f(x) = 0.$$

Thus, the group-invariant solution of (2.2) under X_4 is also just the trivial solution

$$u(t,x) = 0.$$

Case 5.

We consider the linear combination $X_1 + \mu X_4 = (1 - \mu e^t)\partial/\partial t + \mu ue^t\partial/\partial u$ in order to obtain an exact solution.

The associated characteristic equations to $X_1 + \mu X_4$ are

$$\frac{dt}{1-\mu e^t} = \frac{dx}{0} = \frac{du}{\mu ue^t}.$$

The characteristic equations yield the two invariants $J_1 = x$ and $J_2 = u(1 - \mu e^t)$. Hence the group-invariant solution is given by $J_2 = f(J_1)$, where f is an arbitrary function of x. This implies

$$u(t,x) = \frac{f(x)}{1 - \mu e^t}$$

Substituting the above value of u into (2.2), we obtain the NLODE $f'(x) + f^2(x) = 0$. Solving the NLODE we get

$$f(x) = \frac{1}{x + C_1}, \tag{2.41}$$

where C_1 is an arbitrary constant of integration. Thus, the group-invariant solution of (2.2) under the linear combination $X_1 + \mu X_4$ is

$$u(t,x) = \frac{1}{(C_1 + x)(1 - \mu e^t)}. \tag{2.42}$$

Figure (2.3) depicts a dynamical picture of the above solution by assigning values $C_1 = 1 = \mu = 1$.

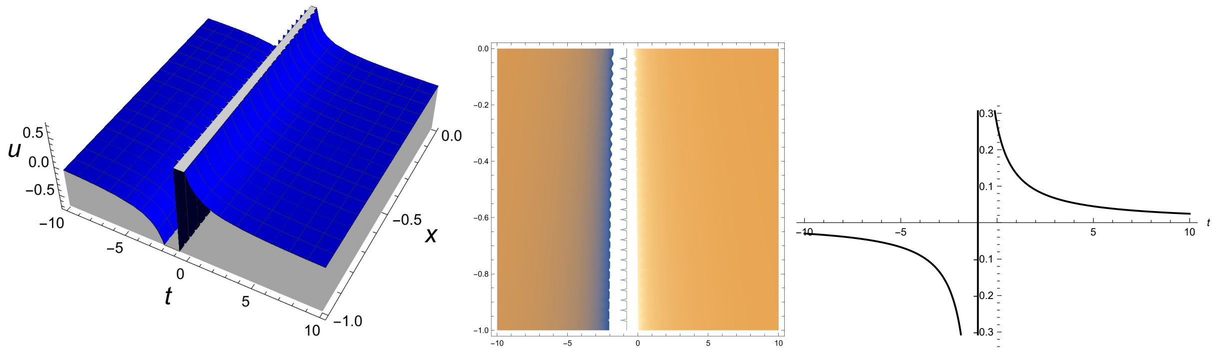

Figure 2.3: 3D and 2D profile solution of (2.42).

Case 6. ***(Traveling wave solutions)*****:** $X_1 + cX_2 = \partial/\partial t + c\partial/\partial x$.

From the above linear combination the characteristic equations are

$$X = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

We consider the linear combination X of the translation operators X_1 and X_2, namely $X = X_1 + cX_2$, where c is an arbitrary constant. Thus,

$$X = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

The associated characteristic equations to X are

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}.$$

Thus, one invariant is $J_1 = u$. The other invariant is obtained from

$$\frac{dt}{1} = \frac{dx}{c}$$

which gives us $J_2 = x - ct$. Hence the group-invariant solution is given by $J_1 = f(J_2)$, where f is an arbitrary function. This implies that

$$u(t,x) = f(\xi),\ \xi = x - ct.$$

Substituting this value of u into (2.2), we obtain

$$-cf''(\xi) + f'(\xi) + f^2(\xi) = 0. \tag{2.43}$$

Thus, the solution of (2.2) under X is $u = f(\xi)$, where f satisfies

$$-cf''(\xi) + f'(\xi) + f^2(\xi) = 0.$$

2.2 Numerical simulations

In this section, we present some three and two dimensional figures for the obtained solutions presented in this chapter. The construction of the figures is carried out with the aid of the mathematical software Mathematica by taking suitable values of the parameters under some limit in order to see the mechanism of the equation (2.2) under investigation. From Figs. 2.1–2.3, one can see that the obtained solutions possess singular solutions.

In Figure 2.1, 3D and 2D graphs of the singular solution (2.39) with parameter $C = 1$ recorded with interval $-5 < t < 5$ and $-5 < x < 5$. Figure 2.2, 3D and 2D graphs of singular solution (2.40) with parameter $C = 1$ recorded on the interval $-5 < t < 5$ and $-5 < x < 5$. Figure 2.3, 3D and 2D graphs of singular solution (2.42) with parameters $C = \mu = 1$ recorded under the limit $-5 < t < 5$ and $-5 < x < 5$.

2.3 Conservation laws for the Boltzmann equation

In this section we derive conservation laws of the Boltzmann equation (2.2) by invoking three approaches.

2.3.1 Conservation laws using the theorem due to Ibragimov

In this subsection we construct conservation laws of (2.2) using Ibragimov's theorem [32]. From (1.30), we get the Euler-Lagrange operator as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_t D_x \frac{\partial}{\partial u_{tx}} + \cdots . \tag{2.44}$$

The adjoint equation for (2.2) from (1.43) is given by

$$F^* \left(t, x, u, v, ..., v_{tx}\right) = \frac{\delta}{\delta u} \left[v(u_{tx} + u_x + u^2)\right] = 0, \tag{2.45}$$

where $v = v(t, x)$ is the new introduced dependent variable. Equation (2.45) yields

$$v_{tx} - v_x + 2uv = 0. \tag{2.46}$$

Since the Boltzmann equation (2.2) is not the same as equation (2.46) we conclude that the Boltzmann equation is not self-adjoint. Consider equation (2.2) and its adjoint equation (2.46) as a system. The Lagrangian for this system from (1.44) is

$$\mathcal{L} = v(u_{tx} + u_x + u^2), \tag{2.47}$$

since

$$\frac{\delta\mathcal{L}}{\delta u}=0,\quad \frac{\delta\mathcal{L}}{\delta v}=0 \tag{2.48}$$

on the system. Now, equation (2.47) admits all the symmetries (2.34), (2.35), (2.36) and (2.37) extended to the new variable $v(t,x)$. That is, the infinitesimal generator becomes

$$Y=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}+\eta^*\frac{\partial}{\partial v} \tag{2.49}$$

with

$$\eta^*=\eta^*(t,x,u,v)=-\left\{\lambda+D_t(\tau)+D_x(\xi)\right\}v. \tag{2.50}$$

We determine the parameter λ by using

$$X^{[2]}(F)=\lambda F, \tag{2.51}$$

where $F=u_{tx}+u_x+u^2$ and $X^{[2]}$ is the second prolongation of X, given by

$$X^{[2]}=X+\zeta_x\frac{\partial}{\partial u_x}+\zeta_{tx}\frac{\partial}{\partial u_{tx}}. \tag{2.52}$$

The conserved vectors of the system of equations (2.2) and (2.46), associated with a symmetry can be obtained from a modified version of (1.48) as

$$T^t=\tau\mathcal{L}+D_x(W)\left(\frac{\partial\mathcal{L}}{\partial u_{tx}}\right), \tag{2.53}$$

$$T^x=\xi\mathcal{L}+W\left[\frac{\partial\mathcal{L}}{\partial u_x}-D_t\left(\frac{\partial\mathcal{L}}{\partial u_{tx}}\right)\right], \tag{2.54}$$

where

$$W=\eta-\tau u_t-\xi u_x, \tag{2.55}$$

since we have one cross term u_{tx} in our equation (2.2). For details see [33].

Case 1. For the translation symmetry $X_1=\partial/\partial t$ we have $\tau=1$, $\xi=\eta=0$. Thus, $\zeta_x=\zeta_{tx}=0$. We then get, $X_1^{[2]}F=0F$, that is, $\lambda=0$. From (2.50) we obtain $\eta^*=0$, whereby the new generator retains the form of X_1. That is, $Y_1=\partial/\partial t$. To compute the conservation laws for X_1 we utilize the formulas (2.53), (2.54) and (2.55). From (2.55) the Lie characteristic function is $W_1=-u_t$. Thus, the conserved vector (T_1^t,T_1^x) from (2.53) and (2.54) is

$$T_1^t=u_xv+u^2v,$$

$$T_1^x = u_t v_t - u_t v.$$

Case 2. For $X_2 = \partial/\partial x$ we note that just like X_1 the generator coefficient is a constant, hence it is easy to see that $\xi = 1$ and $\tau = \eta = \zeta_x = \zeta_{tx} = \eta^* = \lambda = 0$. Similarly, the new generator coincides with X_2. That is, $Y_2 = \partial/\partial x$. The Lie characteristic function is then given by $W_2 = -u_x$. Thus, the conserved vector is

$$T_2^t = -u_{xx}v, \tag{2.56}$$

$$T_2^x = u_x v_t + u^2 v + u_{tx} v. \tag{2.57}$$

Case 3. For the scaling symmetry $X_3 = x\partial/\partial x - u\partial/\partial u$ we have, $\xi = x$, $\tau = 0$, $\eta = -u$, $\zeta_x = -2u_x$, and $\zeta_{tx} = -2u_{tx}$. Then, $X_3^{[2]}F = -2F$, which implies $\lambda = -2$. It follows that $\eta^* = v$, whereby the new generator is $Y_3 = x\partial/\partial x - u\partial/\partial u + v\partial/\partial v$ and the Lie characteristic function is $W_3 = -(u + xu_x)$. Thus, the conserved vector is

$$T_3^t = -2u_x v - x u_{xx} v, \tag{2.58}$$

$$T_3^x = x u_{tx} v + x u^2 v - uv + x u_x v_t + u v_t. \tag{2.59}$$

Case 4. For the symmetry $X_4 = -e^t\partial/\partial t + ue^t\partial/\partial u$, we obtain, $\xi = 0$, $\tau = -e^t$, $\eta = e^t u$, $\zeta_x = -2u_x$, $\zeta_{tx} = -2u_{tx}$, $\lambda = 2e^t$, whereby the new generator is $Y_4 = -e^t\partial/\partial t + e^t u\partial/\partial u - e^t v\partial/\partial v$, and $W_4 = e^t u + e^t u_t$. Hence

$$T_4^t = -e^t u^2 v, \tag{2.60}$$

$$T_4^x = -e^t u_t v_t + e^t u_t v - e^t u v_t + e^t u v. \tag{2.61}$$

Remark. The conserved vectors involve solutions v of the adjoint equation (2.46) and solutions u of the Boltzmann equation (2.2) and hence yields an infinite number of conservation laws. Additionally, the corresponding vectors obtained from case 1, 2, and 3 represent the conservation laws of energy, momentum and mass respectively.

2.3.2 Conservation laws using the multiplier method

In this subsection we employ the multiplier method [8] to construct conservation laws of (2.2). Firstly, we look for the zeroth-order multipliers $\Lambda = \Lambda(t, x, u)$. The determining equation for the multiplier is given by

$$\frac{\delta}{\delta u}\left\{\Lambda(t,x,u)\left(u_{tx}+u_x+u^2\right)\right\}=0. \tag{2.62}$$

Here $\delta/\delta u$ is the Euler operator and is defined as

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_tD_x\frac{\partial}{\partial u_{tx}}+\cdots . \tag{2.63}$$

Expanding equation (2.62) gives

$$\Lambda_u(u_{tx}+u_x+u^2)+D_xD_t(\Lambda)-D_x(\Lambda)+2u\Lambda=0. \tag{2.64}$$

Applying the total derivatives

$$\begin{aligned} D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tx}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}+\cdots , \\ D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{tx}\frac{\partial}{\partial u_t}+\cdots \end{aligned}$$

to equation (2.64) gives

$$\Lambda_u(u_{tx}+u_x+u^2)+\Lambda_{tx}+u_x\Lambda_{tu}+u_t\Lambda_{xu}+u_tu_x\Lambda_{uu}+u_{tx}\Lambda_u-\Lambda_x-u_x\Lambda_u+2u\Lambda=0.$$

Splitting the above equation on the derivatives of u we obtain

$$u_{tx} \quad : \quad \Lambda_u=0, \tag{2.65}$$

$$\text{rest} \quad : \quad \Lambda_{tx}-\Lambda_x+2u\Lambda=0. \tag{2.66}$$

Equations (2.65) gives

$$\Lambda=A(t,x), \tag{2.67}$$

where $A(t,x)$ is an arbitrary function of t and x. Now substituting this value of Λ into equation (2.66) we get

$$A_{tx}-A_x+2uA=0.$$

Since A is independent of u, we can split the above equation on u and get

$$u \quad : \quad A(t,x)=0, \tag{2.68}$$

$$u^0 \quad : \quad A_{tx} - A_x = 0. \tag{2.69}$$

Equation (2.68) implies that the multiplier is given by $\Lambda = 0$. Thus, we conclude that we do not have a zeroth-order multiplier $\Lambda(t, x, u)$. Let us now consider the first-order multiplier $\Lambda(t, x, u, u_t, u_x)$ and likewise the determining equation for the first-order multiplier is

$$\frac{\delta}{\delta u}\left[\Lambda(t, x, u, u_t, u_x)\left\{u_{tx} + u_x + u^2\right\}\right] = 0.$$

Following the same procedure as we did for the zeroth-order multiplier we get the first-order multiplier to be

$$\Lambda = c_3 u e^{2t} + c_3 u_t e^{2t} + c_2 u e^{3t} + c_2 u_t e^{3t} + c_3 x u_x e^{2t} + c_1 u_x e^{2t}, \tag{2.70}$$

where c_1, c_2 and c_3 are arbitrary constants. Thus, we have three first-order multipliers, viz.,

$$\begin{aligned} \Lambda_1 &= u_x e^{2t}, \\ \Lambda_2 &= (u + u_t)\, e^{3t}, \\ \Lambda_3 &= (u + u_t + x u_x)\, e^{2t}. \end{aligned}$$

The conservation laws are now obtained by using the divergence identity

$$D_t T^t + D_x T^x = \left[\Lambda(t, x, u, u_t, u_x)\left\{u_{tx} + u_x + u^2\right\}\right],$$

where T^t is the conserved density, and T^x is the spatial flux.

Case 1.

We first compute conservation law of (2.2) associated with the multiplier $\Lambda_1 = u_x e^{2t}$. Therefore we have

$$D_t T^t + D_x T^x = u_x e^{2t}\left(u_{tx} + u_x + u^2\right), \tag{2.71}$$

where T^t and T^x are defined as $T^t = T^t(t, x, u, u_x)$ and $T^x(t, x, u, u_t)$. Equation (2.71) then becomes

$$T^t_t + u_t T^t_u + u_{tx} T^t_{u_x} + T^x_x + u_x T^x_u = u_x e^{2t}\left(u_{tx} + u_x + u^2\right).$$

By splitting the above equation on second order derivatives of u we get

$$u_{tx} : T^t_{u_x} = u_x e^{2t}, \tag{2.72}$$

$$\text{rest}: T^t_t + u_t T^t_u + T^x_x + u_x T^x_u = u_x e^{2t}\left(u_x + u^2\right). \tag{2.73}$$

Integrating equation (2.72) with respect to u_x yields

$$T^t = \frac{1}{2} e^{2t} u_x^2 + A(t, x, u), \tag{2.74}$$

where A is an arbitrary function of t, x and u. Substituting the value of T^t into (2.73) we obtain

$$A_t + u_t A_u + T^x_x + u_x T^x_u = e^{2t} u^2 u_x.$$

Splitting the above equation on derivatives of u gives

$$u_x : T^x_u = e^{2t} u^2, \tag{2.75}$$

$$\text{rest} : A_t + u_t A_u + T^x_x = 0. \tag{2.76}$$

Integrating equation (2.75) with respect to u we get

$$T^x = \frac{1}{3} u^3 e^{2t} + B(t, x), \tag{2.77}$$

where B is an arbitrary function of t and x. Substituting the new value of T^x into equation (2.76) we get

$$A_t + u_t A_u + B_x = 0.$$

Splitting the above equation on u_t we get

$$u_t : A_u = 0, \tag{2.78}$$

$$\text{rest} : A_t + B_x = 0. \tag{2.79}$$

From equation (2.78) we get $A = A(t, x)$ and hence

$$T^t = \frac{1}{2} e^{2t} u_x^2 + A(t, x). \tag{2.80}$$

We set $A(t, x) = B(t, x) = 0$ since they contribute to the trivial part of the conservation law. Therefore, for the first multiplier $\Lambda_1 = u_x e^{2t}$ the corresponding conserved vector is (T^t_1, T^x_1) whose components are given by

$$T^t_1 = \frac{1}{2} e^{2t} u_x^2,$$
$$T^x_1 = \frac{1}{3} e^{2t} u^3.$$

Likewise, after some calculations, conserved vectors corresponding to the remaining two multipliers Λ_2 and Λ_3 are given below.

Case 2.

For the second multiplier $\Lambda_2 = (u + u_t)\, e^{3t}$ the corresponding conserved vector is (T_2^t, T_2^x), where

$$
\begin{aligned}
T_2^t &= \frac{1}{3} e^{3t} u^3 + e^{3t} u u_x, \\
T_2^x &= \frac{1}{2} e^{3t} u_t^2 - e^{3t} u^2.
\end{aligned}
$$

Case 3.

Finally for the multiplier $\Lambda_3 = (u + u_t + x u_x)\, e^{2t}$ we get the corresponding conserved vector (T_3^t, T_3^x) as

$$
\begin{aligned}
T_3^t &= \frac{1}{3} e^{2t} u^3 + \frac{1}{2} e^{2t} x u_x^2 + e^{2t} u u_x, \\
T_3^x &= \frac{1}{3} e^{2t} x u^3 - \frac{1}{2} e^{2t} u^2 + \frac{1}{2} e^{2t} u_t^2.
\end{aligned}
$$

2.3.3 Conservation laws using Noether's theorem

In this subsection we use Noether's theorem [20] to compute conversation laws for the Boltzmann equation (2.2). It can readily be verified that the first-order Lagrangian for equation (2.2) is given by [30]

$$
\mathcal{L} = \frac{1}{2} e^{2t} \left(u_t u_x + u^2 u_t \right) \tag{2.81}
$$

because $\delta \mathcal{L} / \delta u = 0$ on (2.2). Here $\delta / \delta u$ is the Euler-Lagrange operator defined as

$$
\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + \cdots . \tag{2.82}
$$

Consider the vector field

$$
X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}, \tag{2.83}
$$

where τ, ξ and η depend on t, x and u. The Noether symmetry determining equation in our case is

$$
X^{[1]} \mathcal{L} + \mathcal{L} \left(D_t \tau + D_x \xi \right) = D_t B^t + D_x B^x, \tag{2.84}
$$

where $B^t = B^t(t, x, u)$ and $B^x = B^x(t, x, u)$ are the gauge terms and $X^{[1]}$ is the first prolongation of X defined by

$$X^{[1]} = X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x}$$

with ζ_t and ζ_x given in (1.13) and (1.14), respectively. Equation (2.84) becomes

$$\begin{gathered}\tau\left(e^{2t}u_t u_x + e^{2t}u^2 u_t\right) + \eta\left(e^{2t}u u_t\right) + \zeta_t\left(\frac{1}{2}e^{2t}u_x + \frac{1}{2}e^{2t}u^2\right) + \zeta_x\left(\frac{1}{2}e^{2t}u_t\right) + \\ \left(\frac{1}{2}e^{2t}\left(u_t u_x + u^2 u_t\right)\right)\left(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u\right) = B^t_t + u_t B^t_u + B^x_x + u_x B^x_u,\end{gathered}$$

which gives

$$\begin{aligned}&\left(e^{2t}u_t u_x + e^{2t}u^2 u_t\right)\tau + \left(e^{2t}u u_t\right)\eta + \left(\frac{1}{2}e^{2t}u_x + \frac{1}{2}e^{2t}u^2\right)\eta_t + \left(\frac{1}{2}e^{2t}u_t u_x + \frac{1}{2}e^{2t}u^2 u_t\right)\eta_u \\ &-\left(\frac{1}{2}e^{2t}u_t u_x + \frac{1}{2}e^{2t}u^2 u_t\right)\tau_t - \left(\frac{1}{2}e^{2t}u_t^2 u_x + \frac{1}{2}e^{2t}u^2 u_t^2\right)\tau_u - \left(\frac{1}{2}e^{2t}u_x^2 + \frac{1}{2}e^{2t}u^2 u_x\right)\xi_t \\ &-\left(\frac{1}{2}e^{2t}u_t u_x^2 + \frac{1}{2}e^{2t}u^2 u_t u_x\right)\xi_u + \left(\frac{1}{2}e^{2t}u_t\right)\eta_x + \left(\frac{1}{2}e^{2t}u_t u_x\right)\eta_u - \left(\frac{1}{2}e^{2t}u_t^2\right)\tau_x \\ &-\left(\frac{1}{2}e^{2t}u_t^2 u_x\right)\tau_u - \left(\frac{1}{2}e^{2t}u_t u_x\right)\xi_x - \left(\frac{1}{2}e^{2t}u_t u_x^2\right)\xi_u + \left(\frac{1}{2}e^{2t}\left(u_t u_x + u^2 u_t\right)\right) \\ &\times\left(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u\right) = B^t_t + u_t B^t_u + B^x_x + u_x B^x_u.\end{aligned}$$

Splitting the above equation on derivatives of u, we obtain

$$u_t^2 u_x \;:\; \tau_u = 0, \tag{2.85}$$

$$u_t u_x^2 \;:\; \xi_u = 0, \tag{2.86}$$

$$u_t u_x \;:\; \tau + \eta_u = 0, \tag{2.87}$$

$$u_t^2 \;:\; \tau_x = 0, \tag{2.88}$$

$$u_x^2 \;:\; \xi_t = 0, \tag{2.89}$$

$$u_t \;:\; u^2 e^{2t}\tau + u e^{2t}\eta + \frac{1}{2}u^2 e^{2t}\eta_u + \frac{1}{2}e^{2t}\eta_x + \frac{1}{2}u^2 e^{2t}\xi_x = B^t_u, \tag{2.90}$$

$$u_x \;:\; \frac{1}{2}e^{2t}\eta_t = B^x_u, \tag{2.91}$$

$$rest \;:\; \frac{1}{2}u^2 e^{2t}\eta_t = B^t_t + B^x_x. \tag{2.92}$$

The solution of equations (2.85), (2.88) together with (2.86) and (2.89) yields

$$\tau = A(t),\; \xi = B(x), \tag{2.93}$$

where $A(t)$ and $B(x)$ are arbitrary functions of their arguments. Integrating (2.87) with respect to u we get

$$\eta = -uA(t) + C(t,x), \tag{2.94}$$

where $C(t,x)$ is an arbitrary function of t and x. Equations (2.90) and (2.91) after integrating with respect to u, give

$$B^t = \frac{1}{2}u^2e^{2t}C(t,x) - \frac{1}{6}u^3e^{2t}A(t) + \frac{1}{6}B'(x) + D(t,x), \tag{2.95}$$

$$B^x = \frac{1}{2}ue^{2t}C_t(t,x) - \frac{1}{4}u^2e^{2t}A'(t) + E(t,x). \tag{2.96}$$

Substituting the values of B^t and B^x from equations (2.95) and (2.96) into equation (2.92) gives

$$u^2e^{2t}C(t,x) - \frac{1}{3}u^3e^{2t}A(t) + \frac{1}{3}u^3e^{2t}A'(t) + \frac{1}{3}u^3e^{2t}B'(x) + ue^{2t}C_x(t,x) + ue^{2t}C_{tx}(t,x)$$
$$+ D_t(t,x) + E_x(t,x) = 0.$$

Separating the above equation on u, we obtain

$$u^3 \; : \; A'(t) - A(t) + B'(x) = 0, \tag{2.97}$$

$$u^2 \; : \; C(t,x) = 0, \tag{2.98}$$

$$u \; : \; C_x(t,x) + C_{tx}(t,x) = 0, \tag{2.99}$$

$$\text{rest} \; : \; D_t(t,x) + E_x(t,x) = 0. \tag{2.100}$$

We rewrite equation (2.97) as $A'(t) - A(t) = B'(x)$. Since the left-hand side is a function of t only and the right-hand side is a function of x only, we conclude that both sides must be equal to a constant. Thus,

$$A'(t) - A(t) = C_2,$$
$$B'(x) = C_2,$$

where C_2 is an arbitrary constant. Integrating the above equations we get

$$A(t) = C_1e^t + C_2,$$
$$B(x) = C_2x + C_3,$$

where C_1 and C_3 are arbitrary constants. Therefore

$$\begin{aligned}\tau &= C_1 e^t + C_2,\\ \xi &= C_2 x + C_3,\\ \eta &= -C_1 u e^t - C_2 u.\end{aligned}$$

Thus, from the above we obtain the following Noether symmetry generators:

$$X_1 = \frac{\partial}{\partial x}, \tag{2.101}$$

$$X_2 = e^t \frac{\partial}{\partial t} - u e^t \frac{\partial}{\partial u}, \tag{2.102}$$

$$X_3 = \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} \tag{2.103}$$

and the gauge functions

$$B^t = -\frac{1}{6} u^3 e^{3t} \delta_{i2} + D(t,x), \tag{2.104}$$

$$B^x = -\frac{1}{4} u^2 e^{3t} \delta_{i2} + E(t,x), \tag{2.105}$$

where δ_{ij} is the Knocker delta function defined as

$$\delta_{ij} = \begin{cases} 1, \text{ if } & i = j, \\ 0, \text{ if } & i \neq j. \end{cases}$$

The first prolongation of the Noether symmetries X_1, X_2 and X_3 are

$$X_1 = \frac{\partial}{\partial x}, \tag{2.106}$$

$$X_2 = e^t \frac{\partial}{\partial t} - u e^t \frac{\partial}{\partial u} - (u e^t + 2 e^t u_t) \frac{\partial}{\partial u_t} - e^t u_x \frac{\partial}{\partial u_x}, \tag{2.107}$$

$$X_3 = \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - u_t \frac{\partial}{\partial u_t} - 2 u_x \frac{\partial}{\partial u_x}. \tag{2.108}$$

Now, we use the above results to compute the conserved vectors of the Boltzmann equation (2.2). We use the following formulae [34] for the conserved vector $T = (T^t, T^x)$:

$$T^t = \tau \mathcal{L} + W \left(\frac{\partial \mathcal{L}}{\partial u_t} \right) - B^t, \tag{2.109}$$

$$T^x = \xi \mathcal{L} + W \left(\frac{\partial \mathcal{L}}{\partial u_x} \right) - B^x. \tag{2.110}$$

We choose $D = E = 0$ since they contribute to the trivial part of the conserved vector. Thus, we obtain the following conserved vectors associated with the Noether symmetries X_1, X_2 and X_3:

$$\begin{aligned} T_1^t &= -\frac{1}{2}e^{2t}u_x^2 - \frac{1}{2}e^{2t}u^2u_x, \\ T_1^x &= \frac{1}{2}e^{2t}u^2u_t; \end{aligned}$$

$$\begin{aligned} T_2^t &= -\frac{1}{3}e^{3t}u^3 - \frac{1}{2}e^{3t}uu_x, \\ T_2^x &= -\frac{1}{2}e^{3t}u_t^2 - \frac{1}{2}e^{3t}uu_t + \frac{1}{4}e^{3t}u^2; \end{aligned}$$

$$\begin{aligned} T_3^t &= -\left(u + xu_x\right)\left(\frac{1}{2}e^{2t}u_x + \frac{1}{2}e^{2t}u^2\right), \\ T_3^x &= \frac{1}{2}xe^{2t}u^2u_t - \frac{1}{2}e^{2t}uu_t - \frac{1}{2}e^{2t}u_t^2. \end{aligned}$$

Remark. We note that the conserved vector given by $T = (T_1^t, T_1^x)$ and $T = (T_2^t, T_2^x)$ correspond to the conservation law of momentum and mass respectively.

2.4 Conclusion

In this chapter, we studied the Boltzmann equation (2.2), which is one of the nonlinear partial differential equations that arises in gas dynamics. Lie symmetries were computed and then used to perform symmetry reductions and obtain group-invariant solutions of the Boltzmann equation (2.2). Moreover, we constructed the commutator table and corresponding groups of transformations for these Lie point symmetries. Conservation laws were derived using three methods; Noether's theorem, the conservation theorem due to Ibragimov and the multiplier method. Finally, graphical representations of each solution were presented.

Chapter 3

Conserved vectors and solutions of the two-dimensional potential KP equation

The potential Kadomtsev-Petviashvili (pKP) equation which came as a natural generalization of the celebrated Korteweg-de Vries (KdV) equation was derived by Kadomtsev and Petviashvili in 1970 in the $(2+1)-$dimensional form

$$u_{tx} + \alpha u_x u_{xx} + \beta u_{xxxx} + \gamma u_{yy} = 0,$$

where α, β and γ are non-zero arbitrary constants [35]. The pKP equation has many applications in fields such as plasma physics, adaptive optics, phase imaging and nonlinear mechanics, hence various communities of researchers have employed a variety of effective methods to derive its closed-form solutions. See for example [36–38]. Several forms of the pKP equation have been studied extensively.

Gupta and Bansal [39] in their work investigated the $(2+1)-$dimensional pKP equation with variable coefficients (VCPKP) in the form

$$u_{tx} + \alpha(t) u_x u_{xx} + \beta(t) u_{xxxx} + \mathcal{Z}(t) u_{yy} = 0,$$

where $\alpha(t), \beta(t)$ and $\mathcal{Z}(t)$ are arbitrary functions. In fact, the VCPKP was reduced into a $(1+1)-$dimensional partial differential equation using Lie group methods, and exact solutions were derived using other methods including the extended $(G'/G)-$expansion method.

Moreover, in [40] the authors examined the $(3+1)-$dimensional KP equation

$$u_{tx}+3(u_xu_y)_x+u_{xxxy}+u_{ty}-u_{zz}=0,$$

where several soliton solutions were obtained using the simplified Hirota's technique. Iqbal and Naeem [41] studied the fourth-order nonlinear generalized Kadomtsev-Petviashvili (KP) equation

$$(\alpha u_t+\beta(f(u)^m)_x+\gamma(g(u)^n)_{xxx})_x+\sigma u_{yy}=0,$$

whereby for various choices of $m, n, f(u)$ and $g(u)$ transformed the equation into several forms of KP-like equations. Using the multiplier method they obtained conservation laws for unknown functions $f(u)$ and $g(u)$. The obtained generalized conservation laws were then used to construct conservation laws for certain variants of KP-like equations by choosing values of $f(u)$ and $g(u)$. Moreover, implicit and explicit closed-form solutions were obtained for the various KP-like equations through utilization of the derived conservation laws. Ma et al. [42] studied the $(2+1)-$dimensional combined fourth-order nonlinear equation

$$\begin{aligned}&(6u_xu_{xx}+u_{xxxx})+\alpha\left[3\left(u_xu_t\right)_x+u_{xxxt}\right]+\beta\left[3\left(u_xu_y\right)_x+u_{xxxy}\right]\\&+\gamma_1u_{yt}+\gamma_2u_{xx}+\gamma_3u_{xt}+\gamma_4u_{xy}+\gamma_5u_{yy}+\gamma_6u_{tt}=0,\end{aligned}$$

which possesses diverse lump solutions. For the above equation when $\alpha=\beta=0, \gamma_3=-\gamma_5=1$ and the other γ_s are zero, the potential KP equation in $(2+1)-$dimensional form is obtained, namely

$$6u_xu_{xx}+u_{xxxx}+u_{tx}-u_{yy}=0, \tag{3.1}$$

which also possess lump solutions. Thus, in this chapter we investigate the pKP equation (3.1) which is a nonlinear partial differential equation with two spatial and one temporal coordinate that explains the evolution of nonlinear long waves of small-amplitude with slow transverse coordinate dependence [43].

The contents of this chapter have been submitted for publication [44]

3.1 Exact solutions of (3.1)

In this section we first obtain point symmetries of the pKp equation (3.1) and then use them to construct group invariant solutions, including the travelling wave solutions.

3.1.1 Lie point symmetries of (3.1)

Equation (3.1) admits the one-parameter Lie group of transformations with infinitesimal generator

$$X=\tau(t,x,y,u)\frac{\partial}{\partial t}+\xi(t,x,y,u)\frac{\partial}{\partial t}+\psi(t,x,y,u)\frac{\partial}{\partial y}+\eta(t,x,y,u)\frac{\partial}{\partial u} \tag{3.2}$$

if and only if

$$X^{[4]}(6u_xu_{xx}+u_{xxxx}+u_{tx}-u_{yy})\big|_{(3.1)}=0. \tag{3.3}$$

Using the definition of $X^{[4]}$, where $X^{[4]}$ is the fourth prolongation defined by

$$X^{[4]}=\zeta_x\frac{\partial}{\partial x}+\zeta_{xx}\frac{\partial}{\partial u_{xx}}+\zeta_{tx}\frac{\partial}{\partial u_{tx}}+\zeta_{yy}\frac{\partial}{\partial u_{yy}}+\zeta_{xxxx}\frac{\partial}{\partial u_{xxxx}}$$

we obtain

$$\zeta_x(6u_{xx})+\zeta_{xx}(6u_x)+\zeta_{tx}(1)+\zeta_{yy}(-1)+\zeta_{xxxx}(1)\big|_{(3.1)}=0, \tag{3.4}$$

where ζ_x, ζ_{xx}, ζ_{tx}, ζ_{yy} and ζ_{xxxx} can be obtained using the prolongation formulas given by

$$\begin{aligned}
\zeta_x &= D_x(\eta)-u_tD_x\left(\xi\right)-u_xD_x\left(\tau\right)-u_yD_x\left(\psi\right),\\
\zeta_y &= D_y(\eta)-u_tD_y\left(\xi\right)-u_xD_y\left(\tau\right)-u_yD_y\left(\psi\right),\\
\zeta_{tx} &= D_x\left(\zeta_t\right)-u_{tt}D_x\left(\xi\right)-u_{tx}D_x\left(\tau\right)-u_{ty}D_x\left(\psi\right),\\
\zeta_{yy} &= D_y\left(\zeta_y\right)-u_{xy}D_y\left(\xi\right)-u_{ty}D_y\left(\tau\right)-u_{yy}D_y\left(\psi\right),\\
\zeta_{xx} &= D_x\left(\zeta_x\right)-u_{tx}D_x\left(\xi\right)-u_{xx}D_x\left(\tau\right)-u_{xy}D_x\left(\psi\right),\\
\zeta_{xxx} &= D_x\left(\zeta_{xx}\right)-u_{txx}D_x\left(\xi\right)-u_{xxx}D_x\left(\tau\right)-u_{xxy}D_z\left(\psi\right),\\
\zeta_{xxxx} &= D_x\left(\zeta_{xxx}\right)-u_{xxxx}D_x\left(\xi\right)-u_{txxx}D_x\left(\tau\right)-u_{xxxy}D_x\left(\psi\right)
\end{aligned} \tag{3.5}$$

and the total differential operators are

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+u_{ty}\frac{\partial}{\partial u_y}+\cdots,\\
D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{tx}\frac{\partial}{\partial u_t}+u_{xy}\frac{\partial}{\partial u_y}+\cdots,\\
D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{ty}\frac{\partial}{\partial u_t}+u_{xy}\frac{\partial}{\partial u_y}+\cdots.
\end{aligned} \tag{3.6}$$

Using equation (3.5) and (3.6) we get ζs in the expanded forms as

$$\begin{aligned}
\zeta_t =& \eta_t + \eta_u u_t - \tau_t u_t - \tau_u u_t^2 - \xi_t u_x - \xi_u u_t u_x - \psi_u u_t u_y - \psi_t u_y,\\
\zeta_x =& \eta_x + \eta_u u_x - \tau_x u_t - \tau_u u_t u_x - \xi_u u_x^2 - \xi_x u_x - \psi_x u_y - \psi_u u_x u_y,\\
\zeta_y =& \eta_y + \eta_u u_y - \tau_y u_t - \tau_u u_t u_y - \xi_u u_x u_y - \xi_y u_x - \psi_y u_y - \psi_u u_y^2,\\
\zeta_{xx} =& \eta_{xx} + \eta_u u_{xx} + 2\eta_{xu} u_x + \eta_{uu} u_x^2 - 2\tau_x u_{tx} - \tau_{xx} u_t - \tau_u u_t u_{xx} - \tau_{uu} u_t u_x^2\\
& - 2\tau_u u_x u_{tx} - 2\tau_{xu} u_t u_x - \xi_{xx} u_x - 2\xi_x u_{xx} - \xi_{uu} u_x^3 - 2\xi_{xu} u_x^2 - 3\xi_u u_x u_{xx}\\
& - 2\psi_x u_{xy} - \psi_{xx} u_y - \psi_u u_y u_{xx} - 2\psi_u u_x u_{xy} - 2\psi_{xu} u_x u_y - \psi_{uu} u_x^2 u_y,\\
\zeta_{tx} =& \eta_{tx} + \eta_{tu} u_x + \eta_u u_{tx} + \eta_{xu} u_t + \eta_{uu} u_t u_x - \tau_t u_{tx} - \tau_x u_{tt} - \tau_{tu} u_t u_x - \tau_{tx} u_t\\
& - \tau_{xu} u_t^2 - \tau_u u_x u_{tt} - 2\tau_u u_t u_{tx} - \tau_{uu} u_t^2 u_x - \xi_t u_{xx} - \xi_{tx} u_x - \xi_x u_{tx} - \xi_{tu} u_x^2\\
& - \xi_{xu} u_t u_x - \xi_u u_t u_{xx} - \xi_{uu} u_t u_x^2 - 2\xi_{xu} u_x u_{tx} - \psi_x u_{ty} - \psi_{tx} u_y - \psi_t u_{xy}\\
& - \psi_u u_y u_{tx} - \psi_u u_x u_{ty} - \psi_u u_t u_{xy} - \psi_{tu} u_x u_y - \psi_{uu} u_t u_x u_y - \psi_{xu} u_t u_y,\\
\zeta_{yy} =& \eta_{yy} + \eta_u u_{yy} + 2\eta_{yu} u_y + \eta_{uu} u_y^2 - 2\tau_y u_{ty} - \tau_u u_t u_{yy} - 2\tau_u u_y u_{ty} - \tau_{yy} u_t\\
& - 2\tau_{yu} u_t u_y - \tau_{uu} u_t u_y^2 - 2\xi_y u_{xy} - \xi_u u_x u_{yy} - 2\xi_u u_y u_{xy} - \xi_{yy} u_x - 2\xi_{yu} u_x u_y\\
& - \xi_{uu} u_x u_y^2 - 2\psi_y u_{yy} - 3\psi_u u_y u_{yy} - \psi_{yy} u_y - 2\psi_{yu} u_y^2 - \psi_{uu} u_y^3,\\
\zeta_{xxxx} =& \eta_{xxxx} + 4\eta_{xu} u_{xxx} + \eta_u u_{xxxx} + 3\eta_{uu} u_{xx}^2 + 4u_x u_{xxx} \eta_{uu} + 4u_x \eta_{xxxu} + 6\eta_{uuu} u_x^2 u_{xx}\\
& + 6\eta_{xxu} u_{xx} + 6\eta_{xxuu} u_x^2 + 12\eta_{xuu} u_x u_{xx} + 4\eta_{xuuu} u_x^3 + \eta_{uuuu} u_x^4 - 12\tau_{xu} u_x u_{txx}\\
& - 4\tau_{xxx} u_{tx} - 4\tau_{xu} u_t u_{xxx} - \tau_{xxxx} u_t - 4\tau_u u_x u_{txxx} - 12u_x \tau_{uu} u_{tx} u_{xx} - 4\tau_{uu} u_t u_x u_{xxx}\\
& - 12\tau_{xuu} u_t u_x u_{xx} - 6\tau_{uuu} u_t u_x^2 u_{xx} - 12\tau_{xu} u_{tx} u_{xx} - 6\tau_u u_{xx} u_{txx} - \tau_u u_t u_{xxxx}\\
& - 6\tau_{xxu} u_t u_{xx} - 6\tau_{xx} u_{txx} - 12\tau_{xxu} u_x u_{tx} - 4\tau_x u_{txxx} - 4\tau_{xxxu} u_t u_x - 4\tau_u u_{tx} u_{xxx}\\
& - 6\tau_{uu} u_x^2 u_{txx} - 4\tau_{uuu} u_x^3 u_{tx} - 12\tau_{xuu} u_x^2 u_{tx} - 3\tau_{uu} u_t u_{xx}^2 - 4\tau_{xuuu} u_t u_x^3 - \tau_{uuuu} u_t u_x^4\\
& - 6\tau_{xxuu} u_t u_x^2 - 4\xi_x u_{xxxx} - 10\xi_{uu} u^2 u_x^2 u_{xxx} - 6\xi_{xx} u_{xxx} - 16\xi_{xu} u_x u_{xxx} - 18\xi_{xxu} u_x u_{xx}\\
& - 5\xi_u u_x u_{xxxx} - 12\xi_{xu} u_{xx}^2 - 10\xi_u u_{xx} u_{xxx} - 4\xi_{xxx} u_{xx} - 10\xi_{uuu} u_x^3 u_{xx} - \xi_{xxxx} u_x\\
& - 4\xi_{xxxu} u_x^2 - 24\xi_{xuu} u_x^2 u_{xx} - 4\xi_{xuuu} u_x^4 - 6\xi_{xxuu} u_x^3 - 15\xi_{uu} u_x u_{xx}^2 - \xi_{uuuu} u_x^5\\
& - 4\psi_u u_x u_{xxxy} - 4\psi_x u_{xxxy} - 4\psi_u u_{xy} u_{xxx} - 12\psi_{xu} u_x u_{xxy} - 6\psi_{xx} u_{xxy}\\
& - 4\psi_{uu} u_x u_y u_{xxx} - 6u_{xx} \psi_u u_{xxy} - 3\psi_{uu} u_y u_{xx}^2 - 12\psi_{xu} u_{xx} u_{xy} - 4\psi_{xu} u_y u_{xxx}\\
& - 6\psi_{uu} u^2 u_x^2 u_{xxy} - 12\psi_{xuu} u_x u_y u_{xx} - 6\psi_{uuu} u_y u_x^2 u_{xx} - 4\psi_{xxxu} u_x u_y - 12\psi_{xxu} u_x u_{xy}\\
& - 6\psi_{xxu} u_y u_{xx} - \psi_{xxxx} u_y - 4\psi_{xxx} u_{xy} - 12\psi_{xuu} u_x^2 u_{xy} - 4\psi_{xuuu} u^3 u_x^3 u_y - 6\psi_{xxuu} u_x^2 u_y
\end{aligned}$$

$$- \psi_{uuuu}u_x^4u_y - 4\psi_{uuu}u_x^3u_{xy} - \psi_u u_y u_{xxxx} - 12\psi_{uu}u_xu_{xx}u_{xy}.$$

Substituting the respective ζs in (3.4) and replacing u_{xxxx} by $u_{yy} - u_{tx} - 6u_xu_{xx}$ we obtain

$$\begin{aligned}
&6\eta_x u_{xx} + 6\eta_u u_x u_{xx} - 6\tau_x u_t u_{xx} - 6\tau_u u_t u_x u_{xx} - 6\xi_u u_x^2 u_{xx} - 6\xi_x u_x u_{xx} - 6\psi_x u_y u_{xx}\\
&- 6\psi_u u_x u_y u_{xx} + 6\eta_{xx} u_x + 6\eta_u u_x u_{xx} + 12\eta_{xu} u_x^2 + 6\eta_{uu} u_x^3 - 12\tau_x u_x u_{tx} - 6\tau_{xx} u_x u_t\\
&- 6\tau_u u_x u_t u_{xx} - 6\tau_{uu} u_t u_x^3 - 12\tau_u u_x^2 u_{tx} - 12\tau_{xu} u_t u_x^2 - 6\xi_{xx} u_x^2 - 12\xi_x u_x u_{xx} - 6\xi_{uu} u_x^4\\
&- 12\xi_{xu} u_x^3 - 18\xi_u u_x^2 u_{xx} - 12\psi_x u_x u_{xy} - 6\psi_{xx} u_x u_y - 6\psi_u u_x u_y u_{xx} - 12\psi_u u_x^2 u_{xy}\\
&- 12\psi_{xu} u_x^2 u_y - \psi_{uu} u_x^2 u_y + \eta_{tx} + \eta_{tu} u_x + \eta_u u_{tx} + \eta_{xu} u_t + \eta_{uu} u_t u_x - \tau_t u_{tx} - \tau_x u_{tt}\\
&- \tau_{tu} u_t u_x - \tau_{tx} u_t - \tau_{xu} u_t^2 - \tau_u u_x u_{tt} - 2\tau_u u_t u_{tx} - \tau_{uu} u_t^2 u_x - \xi_t u_{xx} - \xi_{tx} u_x - \xi_x u_{tx} - \xi_{tu} u_x^2\\
&- \xi_{xu} u_t u_x - \xi_u u_t u_{xx} - \xi_{uu} u_t u_x^2 - 2\xi_{xu} u_x u_{tx} - \psi_x u_{ty} - \psi_{tx} u_y - \psi_t u_{xy} - \psi_u u_y u_{tx} - \psi_u u_x u_{ty}\\
&- \psi_u u_t u_{xy} - \psi_{tu} u_x u_y - \psi_{uu} u_t u_x u_y - \psi_{xu} u_t u_y - \eta_{yy} - \eta_u u_{yy} - 2\eta_{yu} u_y - \eta_{uu} u_y^2 + 2\tau_y u_{ty}\\
&+ \tau_u u_t u_{yy} + 2\tau_u u_y u_{ty} + \tau_{yy} u_t + 2\tau_{yu} u_t u_y + \tau_{uu} u_t u_y^2 + 2\xi_y u_{xy} + \xi_u u_x u_{yy} + 2\xi_u u_y u_{xy} + \xi_{yy} u_x\\
&+ 2\xi_{yu} u_x u_y + \xi_{uu} u_x u_y^2 + 2\psi_y u_{yy} + 3\psi_u u_y u_{yy} + \psi_{yy} u_y + 2\psi_{yu} u_y^2 + \psi_{uu} u_y^3 + \eta_{xxxx} + 4\eta_{xu} u_{xxx}\\
&+ \eta_u u_{yy} - \eta_u u_{tx} - 6\eta_u u u_x u_{xx} + 3\eta_{uu} u_{xx}^2 + 4u_x u_{xxx}\eta_{uu} + 4u_x\eta_{xxxu} + 6\eta_{uuu} u_x^2 u_{xx} + 6\eta_{xxu} u_{xx}\\
&+ 6\eta_{xxuu} u_x^2 + 12\eta_{xuu} u_x u_{xx} + 4\eta_{xuuu} u_x^3 + \eta_{uuuu} u_x^4 - 12\tau_{xu} u_x u_{txx} - 4\tau_{xxx} u_{tx} - 4\tau_{xu} u_t u_{xxx}\\
&- \tau_{xxxx} u_t - 4\tau_u u_x u_{txxx} - 12u_x\tau_{uu} u_{tx} u_{xx} - 4\tau_{uu} u_t u_x u_{xxx} - 12\tau_{xuu} u_t u_x u_{xx} - 6\tau_{uuu} u_t u_x^2 u_{xx}\\
&- 12\tau_{xu} u_{tx} u_{xx} - 6\tau_u u_{xx} u_{txx} - \tau_u u_t u_{yy} + \tau_u u_t u_{tx} + 6\tau_u u_t u_x u_{xx} - 6\tau_{xxu} u_t u_{xx} - 6\tau_{xx} u_{txx}\\
&- 12\tau_{xxu} u_x u_{tx} - 4\tau_x u_{txxx} - 4\tau_{xxxu} u_t u_x - 4\tau_u u_{tx} u_{xxx} - 6\tau_{uu} u_x^2 u_{txx} - 4\tau_{uuu} u_x^3 u_{tx} - 12\tau_{xuu} u_x^2 u_{tx}\\
&- 3\tau_{uu} u_t u_{xx}^2 - 4\tau_{xuuu} u_t u_x^3 - \tau_{uuuu} u_t u_x^4 - 6\tau_{xxuu} u_t u_x^2 - 4\xi_x u_{yy} + 4\xi_x u_{tx} + 24\xi_x u_x u_{xx}\\
&- 10\xi_{uu} u^2 u_x^2 u_{xxx} - 6\xi_{xx} u_{xxx} - 16\xi_{xu} u_x u_{xxx} - 18\xi_{xxu} u_x u_{xx} - 5\xi_u u_x u_{yy} + 5\xi_u u_x u_{tx} + 30\xi_u u_x^2 u_{xx}\\
&- 12\xi_{xu} u_{xx}^2 - 10\xi_u u_{xx} u_{xxx} - 4\xi_{xxx} u_{xx} - 10\xi_{uuu} u_x^3 u_{xx} - \xi_{xxxx} u_x - 4\xi_{xxxu} u_x^2 - 24\xi_{xuu} u_x^2 u_{xx}\\
&- 4\xi_{xuuu} u_x^4 - 6\xi_{xxuu} u_x^3 - 15\xi_{uu} u_x u_{xx}^2 - \xi_{uuuu} u_x^5 - \psi_u u_y u_{yy} + \psi_u u_y u_{tx} + 6\psi_u u_y u_x u_{xx}\\
&- 4\psi_u u_x u_{xxxy} - 4\psi_x u_{xxxy} - 4\psi_u u_{xy} u_{xxx} - 12\psi_{xu} u_x u_{xxy} - 6\psi_{xx} u_{xxy} - 12\psi_{uu} u_x u_{xx} u_{xy}\\
&- 4\psi_{uu} u_x u_y u_{xxx} - 6u_{xx}\psi_u u_{xxy} - 3\psi_{uu} u_y u_{xx}^2 - 12\psi_{xu} u_{xx} u_{xy} - 4\psi_{xu} u_y u_{xxx} - 6\psi_{uu} u^2 u_x^2 u_{xxy}\\
&- 12\psi_{xuu} u_x u_y u_{xx} - 6\psi_{uuu} u_y u_x^2 u_{xx} - 4\psi_{xxxu} u_x u_y - 12\psi_{xxu} u_x u_{xy} - 6\psi_{xxu} u_y u_{xx} - \psi_{xxxx} u_y\\
&- 4\psi_{xxx} u_{xy} - 12\psi_{xuu} u_x^2 u_{xy} - 4\psi_{xuuu} u^3 u_x^3 u_y - 6\psi_{xxuu} u_x^2 u_y - \psi_{uuuu} u_x^4 u_y - 4\psi_{uuu} u_x^3 u_{xy} = 0.
\end{aligned}$$

Since τ, ξ, ψ and η depend only on t, x, y and u we can split the above equation on the derivatives of u and obtain the following linear homogeneous PDEs:

$$u_x u_{txxx} \ : \ \tau_u = 0, \tag{3.7}$$

$$u_{txxx} \ : \ \tau_x = 0, \tag{3.8}$$

$$u_{ty} \ : \ \tau_y = 0, \tag{3.9}$$

$$u_y u_{xy} \ : \ \xi_u = 0, \tag{3.10}$$

$$u_x u_{xxxy} \ : \ \psi_u = 0, \tag{3.11}$$

$$u_y u_{xx} \ : \ \psi_x = 0, \tag{3.12}$$

$$u_y^2 \ : \ 3\eta_u + \tau_t = 0, \tag{3.13}$$

$$u_{xx} \ : \ 2\xi_y - \psi_t = 0, \tag{3.14}$$

$$u_x u_{xx} \ : \ 6\eta_x - \xi_t = 0, \tag{3.15}$$

$$u_x u_{xx} \ : \ 3\xi_x - \tau_t = 0, \tag{3.16}$$

$$u_{yy} \ : \ 3\psi_y - 2\tau_t = 0, \tag{3.17}$$

$$1 \ : \ 6\eta_{yy} - \xi_{tt} = 0. \tag{3.18}$$

From equations (3.7), (3.8) and (3.9) we have

$$\tau = a(t), \tag{3.19}$$

where a is an arbitrary function of t. Equation (3.10) implies that

$$\xi = b(t, x, y), \tag{3.20}$$

where b is an arbitrary function of t, x and y. Similarly, equation (3.11) and equation (3.12) gives

$$\psi = c(t, y), \tag{3.21}$$

where c is an arbitrary function of t and y. Substituting the value of τ into equation (3.13) and integrating once with respect to u, we get

$$\eta = -\frac{1}{3}ua'(t) + d(t, x, y), \tag{3.22}$$

where d is an arbitrary functions of t, x and y. From equation (3.16) we get that

$$3b_x(t, x, y) - a'(t) = 0,$$

which by integrating once with respect to x gives

$$b = \frac{1}{3}xa'(t) + e(t, y)$$

where e is a function of t and x. Moreover, equation (3.17) implies that

$$3c_y(t, y) - 2a'(t) = 0,$$

whereby when we integrate once with respect to y we get

$$c = \frac{2}{3}ya'(t) + f(t),$$

where f is an arbitrary function of t. We now obtain the value of e by substituting the value of b into equation (3.14) such that

$$e_y(t, y) = \frac{1}{3}ya''(t) + \frac{1}{2}f'(t).$$

Integrating the above equation once with respect to y we obtain

$$e = \frac{1}{6}y^2a''(t) + \frac{1}{2}f'(t) + g(t),$$

where g is an arbitrary function of t. The above results imply that

$$b = \frac{1}{3}xa'(t) + \frac{1}{6}y^2a''(t) + \frac{1}{2}f'(t) + g(t).$$

When we substitute the values of b and c into equation (3.15) we obtain

$$d_x(t, x, y) = \frac{1}{18}xa''(t) + \frac{1}{36}y^2a''(t) + \frac{1}{12}yf''(t) + \frac{1}{6}g'(t).$$

Integrating the above equation once with respect to x we get

$$d = \frac{1}{36}x^2a''(t) + \frac{1}{36}xy^2a''(t) + \frac{1}{12}xyf''(t) + \frac{1}{6}xg'(t) + h(t, y),$$

where h is a function of t and y. Finally, equation (3.18) implies that

$$h_{yy}(t, y) = \frac{1}{36}y^2a''''(t) + \frac{1}{12}yf'''(t) + \frac{1}{6}g''(t),$$

which on integrating twice with respect to y leads to

$$h = \frac{1}{432}y^4a''''(t) + \frac{1}{72}y^3f'''(t) + \frac{1}{12}y^2g''(t) + yi(t) + j(t)),$$

where i and j are arbitrary functions of t. Thus,

$$\begin{aligned}
\tau &= a(t),\\
\xi &= \frac{1}{3}xa'(t) + \frac{1}{6}y^2a'''(t) + \frac{1}{2}f'(t) + g(t),\\
\psi &= \frac{2}{3}ya'(t) + f(t),\\
\eta &= \frac{1}{36}x^2a''(t) - \frac{1}{3}ua'(t) + \frac{1}{36}xy^2a'''(t) + \frac{1}{6}xg'(t) + \frac{1}{432}y^4a''''(t)\\
&\quad + \frac{1}{72}y^2f'''(t) + \frac{1}{12}g''(t) + yi(t) + j(t),
\end{aligned}$$

and hence the Lie algebra of point symmetries of equation (3.1) is given as follows:

$$X_1 = j(t)\frac{\partial}{\partial u}, \tag{3.23}$$

$$X_2 = yi(t)\frac{\partial}{\partial u}, \tag{3.24}$$

$$X_3 = 36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y} + \left(6xyf''(t) + y^3f'''(t)\right)\frac{\partial}{\partial u}, \tag{3.25}$$

$$X_4 = 12g(t)\frac{\partial}{\partial x} + \left(2xg'(t) + y^2g''(t)\right)\frac{\partial}{\partial u}, \tag{3.26}$$

$$\begin{aligned}
X_5 &= 432a(t)\frac{\partial}{\partial t} + \left(144xa'(t) + 72y^2a''(t)\right)\frac{\partial}{\partial x} + 288ya'(t)\frac{\partial}{\partial y}\\
&\quad + \left(12x^2a''(t) - 144ua'(t) + 12xy^2a'''(t) + y^4a''''(t)\right)\frac{\partial}{\partial u},
\end{aligned} \tag{3.27}$$

which generates an infinite dimensional Lie algebra.

3.1.2 Commutator table for the symmetries of (3.1)

We now calculate the commutation relations for all the symmetry generators of (3.1) obtained above. Firstly, we compute $[X_2, X_3]$. By the definition of Lie bracket we have

$$\begin{aligned}
[X_2, X_3] =&\ X_2X_3 - X_3X_2\\
=&\ yi(t)\frac{\partial}{\partial u}\left(36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y} + \left(6xyf''(t) + y^3f'''(t)\right)\frac{\partial}{\partial u}\right)\\
&- \left(36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y} + \left(6xyf''(t) + y^3f'''(t)\right)\frac{\partial}{\partial u}\right)yi(t)\frac{\partial}{\partial u}\\
=&\ -72f(t)i(t)\frac{\partial}{\partial u}\\
=&\ X_1
\end{aligned}$$

if we let

$$j(t) = -72f(t)i(t).$$

Similarly, we compute all the remaining commutation relations using the above procedure. The table below shows the commutation relations in table form where for $[X_1, X_5]$, $[X_2, X_5]$, $[X_3, X_5]$, $[X_4, X_5]$, and $[X_1, X_4]$, the following substitutions were made

$$\begin{aligned}
j(t) =& -\left(144j(t)a'(t) + 432j'(t)a(t)\right),\\
i(t) =& -\left(432i(t)a'(t) + i'(t)a(t)\right),\\
f(t) =& \, 288f(t)a'(t) - 432f(t)a(t),\\
g(t) =& \, 144g(t)a'(t) - 432g'(t)a(t),\\
i(t) =& \, 72f'(t)g'(t) + 144f(t)g''(t) - 72f''(t)g(t),
\end{aligned}$$

respectively.

Table 3.1: Commutator table of Lie algebra of the pKP equation (3.1)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	X_1
X_2	0	0	X_1	0	X_2
X_3	0	$-X_1$	0	X_2	X_3
X_4	0	0	$-X_2$	0	X_4
X_5	$-X_1$	$-X_2$	$-X_3$	$-X_4$	0

3.1.3 One-parameter groups of transformations of (3.1)

We now employ the Lie equations along with the initial conditions

$$\frac{d\bar{t}}{d\alpha} = \tau(\bar{t}, \bar{x}, \bar{y}, \bar{u}),\ \bar{t}|_{\alpha=0} = t,\ \ \frac{d\bar{x}}{d\alpha} = \xi(\bar{t}, \bar{x}, \bar{y}, \bar{u}),\ \bar{x}|_{\alpha=0} = x,\ \ \frac{d\bar{x}}{d\alpha} = \psi(\bar{t}, \bar{x}, \bar{y}, \bar{u}),\ \bar{y}|_{\alpha=0} = y,$$
$$\frac{d\bar{u}}{d\alpha} = \eta(\bar{t}, \bar{x}, \bar{y}, \bar{u}),\ \bar{u}|_{\alpha=0} = u,$$

to compute the one-parameter group of transformations. For each X_i, let G_{α_i} be the corresponding group. We first compute one-parameter group corresponding to infinitesimal generator X_2, namely

$$X_2 = yi(t)\frac{\partial}{\partial u}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{d\alpha} = 0,\ \bar{t}|_{\alpha=0} = t,\ \ \frac{d\bar{x}}{d\alpha} = 0,\ \bar{x}|_{\alpha=0} = x,\ \ \frac{d\bar{y}}{d\alpha} = 0,\ \bar{y}|_{\alpha=0} = y,\ \ \frac{d\bar{u}}{d\alpha} = \bar{y}i(\bar{t}),\ \bar{u}|_{\alpha=0} = u.$$

Solving the above equations, we obtain

$$\bar{t} = t,\ \ \bar{x} = x,\ \ \bar{y} = y,\ \ \bar{u} = u + \alpha y i(t).$$

Thus, the one-parameter group G_{α_2} corresponding to the operator X_2 is given by

$$G_{\alpha_2} : (\bar{t}, \bar{x}, \bar{y}, \bar{u}) \longrightarrow (t, x, y, u + \alpha y i(t)).$$

If we continue in the same manner as above, we get the following one-parameter groups for all the operators:

$$\begin{aligned}
G_{\alpha_1} :\ & (\bar{t}, \bar{x}, \bar{y}, \bar{u}) \longrightarrow (t, x, y, u + \alpha j(t)),\\
G_{\alpha_2} :\ & (\bar{t}, \bar{x}, \bar{y}, \bar{u}) \longrightarrow (t, x, y, u + \alpha y i(t)),\\
G_{\alpha_3} :\ & (\bar{t}, \bar{x}, \bar{y}, \bar{u}) \longrightarrow \Big(t, x + 36\alpha_3 y f'(t) + 1296\alpha_3^2 f(t)f'(t), u + 6\alpha_3 xyf''(t) + 216\alpha_3^2 xf(t)f''(t)\\
&\quad + 108\alpha_3^2 y^2 f'(t)f''(t) + 7776\alpha_3^3 yf(t)f'(t)f''(t) + 139968\alpha_3^4 f^2(t)f'(t)f''(t) + \alpha_3 y^3 f'''(t)\\
&\quad + 108\alpha_3^2 y^2 f(t)f'''(t) + 5184\alpha_3^3 f(t)f'''(t) + 93312\alpha_3^4 f^3(t)f'''(t)\Big),\\
G_{\alpha_4} :\ & (\bar{t}, \bar{x}, \bar{y}, \bar{u}) \longrightarrow \Big(t, x + 12\alpha_4 g(t), y, u + 2\alpha_4 g'(t) + 12\alpha_4^2 g(t)g'(t) + \alpha_4 y^2 g''(t))\Big).
\end{aligned}$$

Using the aforementioned groups, we may find the corresponding new solutions. Since each group G_{α_i} is a symmetry group, if $u = f(t, x, y)$ is a solution of (3.1) then the corresponding new solutions u_i are obtained as follows:

$$\begin{aligned}
u_1 =& f(t, x, y) + \alpha j(t),\\
u_2 =& f(t, x, y) + \alpha y i(t),\\
u_3 =& f\Big(t, x - 36\alpha y f'(t) + 1296\alpha^2 f(t)f'(t), y - 72\alpha f(t)\Big) + 6\alpha xyf''(t) - 216\alpha^2 xf(t)f'(t)
\end{aligned}$$

$$
\begin{aligned}
&- 108\alpha^2 y^2 f'(t)f''(t) + 7776\alpha^3 y f(t)f'(t)f''(t) - 139968\alpha^4 f^2(t)f'(t)f''(t) + \alpha y^3 f'''(t)\\
&- 108\alpha^2 y^2 f(t)f'''(t) + 5184\alpha^3 y f^2(t)f'''(t) - 93312\alpha^4 f^3(t)f'''(t),\\
u_4 &= f\left(t, x - 12\alpha g(t), y\right) + 2\alpha x g'(t) - 12\alpha^2 g(t)g'(t) + \alpha y^2 g''(t).
\end{aligned}
$$

3.1.4 Constructing group-invariant solutions of (3.1)

We derive multiple group-invariant solutions of (3.1) in this section by applying Lie symmetry reductions via the characteristic equations.

Case 1. Firstly, we consider the Lie point symmetry $X_1 = j(t)\partial/\partial u$. The characteristic equations associated with the operator X_1 are

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{j(t)},$$

which give three invariants $J_1 = t$, $J_2 = x$ and $J_3 = y$. Thus, there is no group-invariant solution for X_1.

Case 2. Next we consider the symmetry operator $X_2 = yi(t)\partial/\partial u$. The characteristic equations for the operator X_2 are

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{yi(t)},$$

which lead to three invariants $J_1 = t$, $J_2 = x$ and $J_3 = y$. Thus, X_2 also does not have a group-invariant solution.

Case 3. We now consider the Lie point symmetry $X_3 = 36yf'(t)\partial/\partial x + 72f(t)\partial/\partial y + \left(6xyf''(t) + y^3 f'''(t)\right)\partial/\partial u$. The characteristic equations associated with X_3 are

$$\frac{dt}{0} = \frac{dx}{36yf'(t)} = \frac{dy}{72f(t)} = \frac{du}{6xyf''(t) + y^3 f'''(t)},$$

that leads to the invariants

$$J_1 = t,\ J_2 = \frac{y^2}{2} - \frac{2xf(t)}{f'(t)},\ J_3 = u - \frac{3x^2 f''(t)}{36f'(t)} + \frac{2x^2 f(t)f'''(t)}{36f'^2(t)} - \frac{xy^2 f'''(t)}{36f'(t)}.$$

The above invariants imply that the group invariant solution is given by

$$u = \Phi\left(t, \varphi\right) + \frac{3x^2 f''(t)}{36f'(t)} - \frac{2x^2 f(t)f'''(t)}{36f'^2(t)} + \frac{xy^2 f'''(t)}{36f'(t)},\ \varphi = \frac{y^2}{2} - \frac{2xf(t)}{f'(t)},$$

where Φ is an arbitrary function of t and φ. Substituting this value of u into (3.1), we obtain the nonlinear PDE

$$\begin{aligned}
&864f^4(t)\Phi_{\varphi\varphi\varphi\varphi} - 2\varphi f(t)f'(t)f'''(t) - 108f(t)f'^3(t)\Phi_{t\varphi} - 2592f^3(t)f'(t)\Phi_{\varphi\varphi}\Phi_{\varphi}\\
&+ 72f^2(t)f'(t)f'''(t)\Phi_{\varphi} + 72\varphi f^2(t)f'''(t)\Phi_{\varphi} + 3\varphi f'^3(t)f''''(t) - 162f'^4(t)\Phi_{\varphi}\\
&- 108\varphi f'^4(t)\Phi_{\varphi\varphi} = 0. \qquad (3.28)
\end{aligned}$$

Consequently, the general group invariant solution of equation (3.1) is

$$u = \Phi\left(t, \varphi\right) + \frac{3x^2 f''(t)}{36f'(t)} - \frac{2x^2 f(t)f'''(t)}{36f'^2(t)} + \frac{xy^2 f'''(t)}{36f'(t)},$$

where Φ is any solution of the nonlinear PDE (3.28).

Particular case $f(t) = t$

We consider the particular case $f(t) = t$ which transforms the nonlinear PDE (3.28) into

$$16t^4\Phi_{\varphi\varphi\varphi\varphi} - 2t\Phi_{t\varphi} - 48t^3\Phi_{\varphi\varphi}\Phi_{\varphi} - 3\Phi_{\varphi} - 2\varphi\Phi_{\varphi\varphi} = 0,\ \varphi = \frac{y^2}{2} - 2tx. \qquad (3.29)$$

The Lie algebra of infinitesimal symmetries of the above equation is spanned by the vector fields

$$\begin{aligned}
\gamma_1 =& t\frac{\partial}{\partial\varphi},\ \gamma_2 = h(t)\frac{\partial}{\partial\Phi},\ \gamma_3 = 3t\frac{\partial}{\partial t} + 4\varphi\frac{\partial}{\partial\varphi} - \phi\frac{\partial}{\partial\phi},\ \gamma_4 = 48t^{3/2}\frac{\partial}{\partial\varphi} + \varphi t^{-3/2}\frac{\partial}{\partial\Phi},\\
\gamma_5 =& 192t^{3/2}\frac{\partial}{\partial t} + 288\sqrt{t}\frac{\partial}{\partial\varphi} + \left(\varphi t^{-5/2} - 96\Phi\sqrt{t}\right)\frac{\partial}{\partial\Phi}.
\end{aligned}$$

We perform reductions using the last Lie point symmetry γ_5. This gives us two invariants $I_1 = \varphi t^{-3/2}$ and $I_2 = \Phi\sqrt{t} - \varphi^2/96t^{5/2}$ and consequently we get the group invariant solution

$$\Phi = \frac{1}{\sqrt{t}}G\left(\frac{\varphi}{t^{3/2}}\right) + \frac{\varphi^2}{96t^3}.$$

Substituting this value of Φ into equation (3.29) we get the fourth-order NLODE

$$G''''(\xi) - 3G'(\xi)G''(\xi) = 0,\ \xi = \frac{\varphi}{t^{3/2}}.$$

Integrating the above equation once with respect to ξ we get

$$G'''(\xi) - 3/2G'^2(\xi) + k_1 = 0,$$

where k_1 is a constant of integration. Multiplying the above equation with the integrating factor $G''(\xi)$ and integrating the resulting equation we get the NLODE

$$\frac{1}{2}G''^2(\xi)-\frac{1}{2}G'^3(\xi)+k_1G'(\xi)+k_2=0$$

with k_2 a constant. Solving the above NLODE with the help of Maple and reverting to the original variables t, x, y and u we get our solution of equation (3.1) as

$$u(t,x,y)=\frac{(y^2-4tx)\left\{\left(27k_2+3\sqrt{81k_2^2-24k_1^3}\right)^{\frac{2}{3}}+k_1\right\}}{6t^2\left(27k_2+3\sqrt{81k_2^2-24k_1^3}\right)^{\frac{1}{3}}}+\frac{(y^2-4tx)^2}{384t^3}+k_3, \tag{3.30}$$

where k_3 is a constant. A dynamical picture of the solution (3.30) with $k_1=0.4, k_2=0.6, k_3=0.8$, at $y=180$, is shown in Figure (3.1)

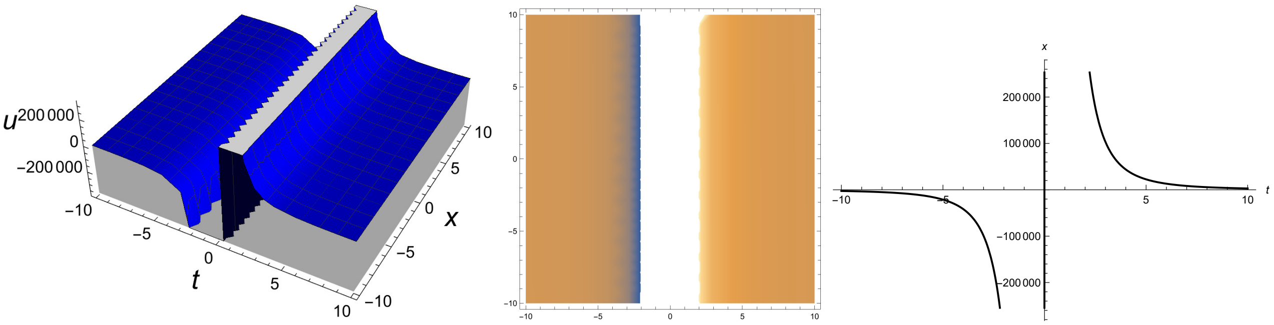

Figure 3.1: 3D and 2D profile of solution (3.30)

Case 4. We now consider the symmetry $X_4=12g(t)\partial/\partial x+(2xg'(t)+y^2g''(t))\,\partial/\partial u$. The associated Lagrange system

$$\frac{dt}{0}=\frac{dx}{12g(t)}=\frac{dy}{0}=\frac{du}{2xg'(t)+y^2g''(t)}.$$

The above system yields the three invariants

$$J_1=t,\;\; J_2=y,\;\; J_3=u-\frac{x^2g'(t)+xy^2g''(t)}{12g(t)}$$

and hence the group-invariant solution is given by

$$u=\Phi(t,y)+\frac{x^2g'(t)+xy^2g''(t)}{12g(t)},$$

where Φ is an arbitrary function of t and y. Substituting the above value of u into (3.1) we obtain

$$12g(t)\Phi_{yy} - y^2 g'''(t) = 0,$$

whose solution is

$$\Phi(t,y) = \frac{y^4 g'''(t)}{144g(t)} + yk(t) + l(t),$$

where k and l are arbitrary functions of t. Hence the group-invariant solution under X_4 is

$$u(t,x,y) = \frac{y^4 g'''(t)}{144g(t)} + \frac{x^2 g'(t)}{12g(t)} + \frac{xy^2 g''(t)}{12g(t)} + yk(t) + l(t). \tag{3.31}$$

A dynamical picture of the solution (3.31) with $g(t) = \sin(t)$, $y = 20$, $k(t) = l(t) = 1$ is shown in Figure (3.2).

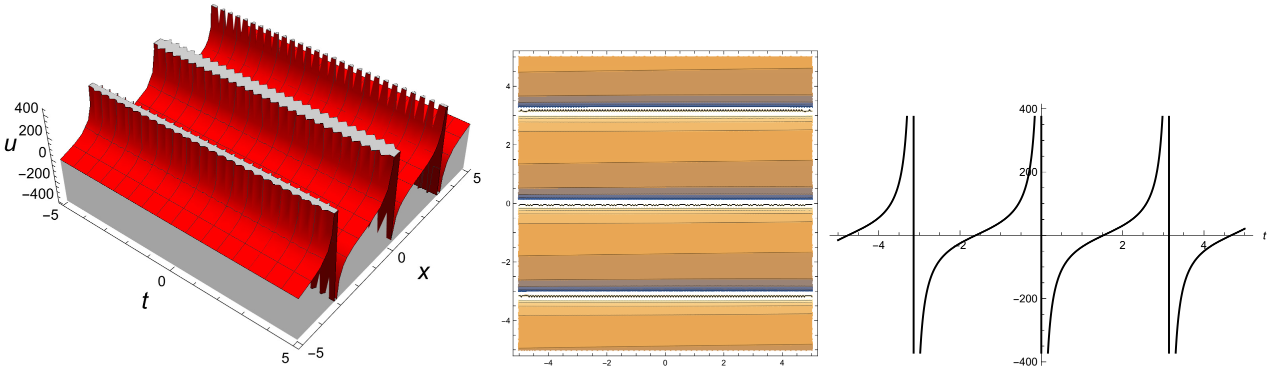

Figure 3.2: 3D and 2D profile of solution (3.31)

Case 5. We consider the symmetry operator

$$\begin{aligned} X_5 =\, & 432a(t)\frac{\partial}{\partial t} + \left(144xa'(t) + 72y^2 a''(t)\right)\frac{\partial}{\partial x} + 288ya'(t)\frac{\partial}{\partial y} \\ & + \left(12x^2 a''(t) - 144ua'(t) + 12xy^2 a'''(t) + y^4 a''''(t)\right)\frac{\partial}{\partial u}. \end{aligned}$$

The associated characteristic equations to X_5 are

$$\begin{aligned} \frac{dt}{432a(t)} = \frac{dx}{144xa'(t) + 72y^2 a''(t)} &= \frac{dy}{288ya'(t)} \\ &= \frac{du}{12x^2 a''(t) - 144ua'(t) + 12xy^2 a'''(t) + y^4 a''''(t)}. \end{aligned}$$

Solving the above equations, we obtain the invariants

$$J_1 = \frac{y}{a(t)^{2/3}}, \quad J_2 = xa(t)^{-1/3} - \frac{y^2 a'(t)}{6a(t)^{4/3}},$$

$$J_3 = ua(t)^{1/3} - \frac{x^2a'(t)}{36a(t)} + \frac{xy^2a'^2(t)}{54a^2(t)} - \frac{xy^2a''(t)}{36a(t)} - \frac{5y^4a'^3(t)}{1944a^3(t)} - \frac{y^4a'''(t)}{432a(t)} + \frac{y^4a'(t)a''(t)}{2116a^2(t)}.$$

Hence the group-invariant solution is given by $J_3 = \Phi(J_1, J_2)$, where Φ is an arbitrary function. This implies

$$u = \frac{1}{a(t)^{1/3}}\Phi(\sigma,\varphi) + \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)} + \frac{5y^4a'^3(t)}{1944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2116a^2(t)}$$

$$\text{with} \quad \sigma = \frac{y}{a(t)^{2/3}},\ \varphi = xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}}.$$

Substituting this value of u into (3.1), we obtain

$$\Phi_{\sigma\sigma} - 6\Phi_\varphi\Phi_{\varphi\varphi} - \Phi_{\varphi\varphi\varphi\varphi} = 0. \tag{3.32}$$

By finding Lie point symmetries of the above equation the PDE can be reduced to an ODE. The symmetry group of equation (3.32) is spanned by

$$X_1 = \frac{\partial}{\partial\sigma},\ X_2 = \frac{\partial}{\partial\varphi},\ X_3 = \frac{\partial}{\partial\Phi},\ X_4 = \sigma\frac{\partial}{\partial\Phi},\ X_5 = \varphi\frac{\partial}{\partial\varphi} + 2\sigma\frac{\partial}{\partial\sigma} - \Phi\frac{\partial}{\partial\Phi}.$$

The linear combination $X = X_1 + cX_2$, where c is a constant yields the two invariants $J_1 = \Phi$ and $J_2 = \varphi - c\sigma$ and consequently we have $\Phi = \chi(\xi)$, $\xi = \varphi - c\sigma$. Substituting the value of Φ into (3.32) we get the nonlinear ODE

$$\chi'''' + 6\chi'\chi'' - c^2\chi'' = 0. \tag{3.33}$$

We can now use Kudryashov's method, which has been fully outlined in Chapter one, to find the exact solution of (3.33). To utilize Kudryashov's method we begin by assuming the solution of equation (3.33) to be of the form

$$\chi(\xi) = \sum_{i=0}^{n} B_i\mathcal{Y}(\xi)^i, \tag{3.34}$$

where $\mathcal{Y}$ satisfies

$$\mathcal{Y}'(\xi) = \mathcal{Y}^2(\xi) - \mathcal{Y}(\xi) \tag{3.35}$$

whose solution is given by

$$\mathcal{Y}(\xi) = \frac{1}{1+\exp(\xi)}. \tag{3.36}$$

Using the balancing procedure, equation (3.33) gives $n = 1$ and hence

$$\chi(\xi) = B_o + B_1\mathcal{Y}(\xi). \tag{3.37}$$

Substituting the above values of $\chi(\xi)$ and its derivatives into equation (3.33) and using (3.35) we get

$$\begin{aligned}&2B_1c^2\mathcal{Y}^3(\xi) - 3\,B_1c^2\mathcal{Y}^2(\xi) + B_1c^2\mathcal{Y}(\xi) - 12\,{B_1}^2\mathcal{Y}^5(\xi) + 30{B_1}^2\mathcal{Y}^4(\xi) - 24\,{B_1}^2\mathcal{Y}^3(\xi)\\ &+ 6{B_1}^2\mathcal{Y}^2(\xi) - 24B_1\mathcal{Y}^5(\xi) + 60B_1\mathcal{Y}^4(\xi) - 50B_1\mathcal{Y}^3(\xi) + 15B_1\mathcal{Y}^2(\xi) - B_1\mathcal{Y}(\xi)\,.\end{aligned} \tag{3.38}$$

Splitting equation (3.38) on powers of $\mathcal{Y}$, we get the

$$\begin{aligned}\mathcal{Y}^5(\xi) &: B_1^2 + 2B_1 = 0,\\ \mathcal{Y}^4(\xi) &: B_1^2 + 2B_1 = 0,\\ \mathcal{Y}^3(\xi) &: c^2B_1 - 12B_1^2 - 25B_1 = 0,\\ \mathcal{Y}^2(\xi) &: c^2B_1 - 2B_1^2 - 5B_1 = 0,\\ \mathcal{Y}(\xi) &: c^2B_1 - B_1 = 0.\end{aligned}$$

Solving the above equations, we get $B_1 = -2$ and $c^2 = 1$ and hence equation (3.37) becomes

$$\chi(\xi) = B_0 - \frac{2}{1+\exp(\xi)}. \tag{3.39}$$

Reverting to the original variables, the exact solutions of (3.1) are

$$\begin{aligned}u(t,x,y) = &\frac{1}{a(t)^{1/3}}\left(K - 2\left[1+\exp\left\{xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}} + \frac{cy}{a(t)^{2/3}}\right\}\right]^{-1}\right)\\ &+ \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)} + \frac{5y^4a'^3(t)}{1944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2116a^2(t)},\end{aligned} \tag{3.40}$$

where $K = B_0$ is an arbitrary constant and $c = \pm 1$. The solution profile of (3.40) for $a(t) = \sin(t), c = 1$ and $K = 1$ at $y = 10$ is presented in Figure (3.3).

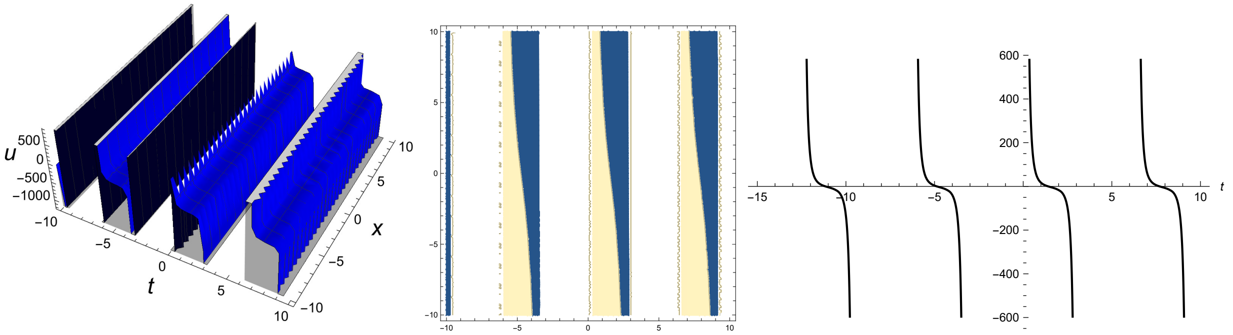

Figure 3.3: 3D and 2D profile of solution (3.40).

3.1.5 Exact solutions of (3.33) using direct integration

Integrating (3.33) with respect to ξ we get

$$\chi''' + 3\chi'^2 - c^2\chi' + k_1 = 0,$$

where k_1 is an arbitrary constant. Moreover, multiplying the above equation by $\chi''(\xi)$ yields

$$\frac{1}{2}\chi''^2 + \chi'^3 - \frac{1}{2}c^2\chi'^2 + k_1\chi' + k_2 = 0, \tag{3.41}$$

where k_2 is a constant of integration. By letting $v = \chi'$ equation (3.41) becomes

$$v'^2 + 2v^3 - c^2v^2 + 2k_1v + 2k_2 = 0,$$

which is a NLODE. If the algebraic equation

$$v^3 - \frac{c^2}{2}v^2 + k_2v + k_2 = 0$$

has the real roots $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 > \alpha_2 > \alpha_3$, then the NLODE becomes

$$v'^2 = 2\left(v - \alpha_1\right)\left(v - \alpha_2\right)\left(v - \alpha_3\right)$$

whose solution is [45, 46]

$$v(\xi) = \alpha_2 - \left(\alpha_1 - \alpha_2\right)\operatorname{cn}^2\left\{\sqrt{\frac{\alpha_1 - \alpha_3}{2}}\xi \,\middle|\, M^2\right\}, M^2 = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}, \tag{3.42}$$

where cn denotes the cosine elliptic function. Since $v = \chi'$ we integrate the above expression with respect to ξ. Thereafter reverting back to the original variables we obtain

$$\begin{aligned} u(t,x,y) = &\frac{1}{a(t)^{1/3}}\left(\mathcal{P}_1\left[\text{ EllipticE }\left\{\operatorname{sn}\left(\mathcal{P}_2\xi \mid M^2\right), M^2\right\}\right] + \left\{\alpha_2 - \left(\alpha_1 - \alpha_2\right)\frac{1 - M^4}{M^4}\right\}\xi + k_3\right) \\ &+ \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)} + \frac{5y^4a'^3(t)}{1944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2116a^2(t)}, \end{aligned} \tag{3.43}$$

where $\xi = xa(t)^{-1/3} - \dfrac{y^2a'(t)}{6a(t)^{4/3}} - \dfrac{cy}{a(t)^{2/3}}$, $\mathcal{P}_1 = \sqrt{\dfrac{2\left(\alpha_1 - \alpha_2\right)^2}{\left(\alpha_1 - \alpha_3\right)M^8}}$, $\mathcal{P}_2 = \sqrt{\dfrac{\alpha_1 - \alpha_3}{2}}$, k_3 a constant and

$$\text{EllipticE }[q, w] = \int_0^w \sqrt{\frac{1 - w^2m^2}{1 - m^2}}dm$$

is the incomplete elliptic integral [47]. Figure (3.4) depicts the wave profile of the periodic solution by assigning the parametric values $\alpha_1 = 10, \alpha_2 = 5, \alpha_3 = 2, k_3 = 0, a(t) = c = 1$ at $y = 180$.

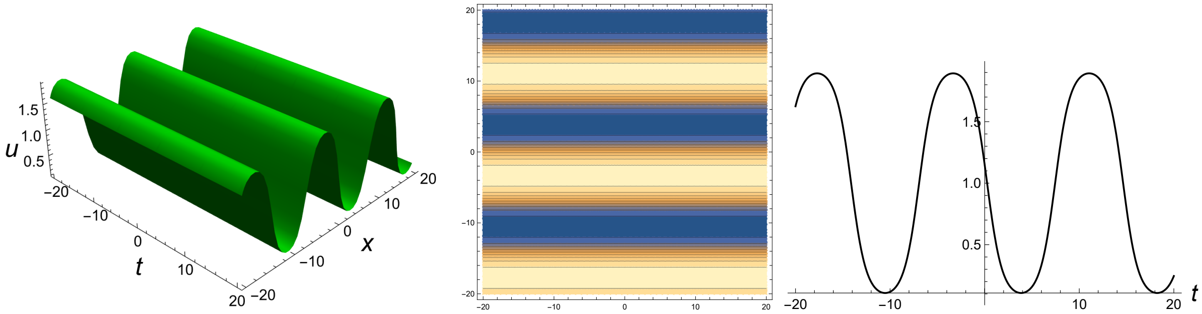

Figure 3.4: 3D and 2D profile of solution (3.43)

Special Case $k_1 = k_2 = 0$

We consider the special case of (3.41) where $k_1 = k_2 = 0$, which upon solving yields the solution

$$\chi(\xi) = c\tanh\left\{\frac{1}{2}c\left(\xi + A_1\right)\right\} + A_2,$$

where A_1 and A_2 are constants of integration. Reverting to the original variables we get

$$\begin{aligned} u(t,x,y) = {} & \frac{1}{a(t)^{1/3}}\left(c\tanh\left\{\frac{1}{2}c\left(xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}} - \frac{cy}{a(t)^{2/3}} + A_1\right)\right\} + A_2\right) \\ & + \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)} + \frac{5y^4a'^3(t)}{1944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2116a^2(t)}. \end{aligned} \quad (3.44)$$

The solution profile of (3.44) for $a(t) = \sinh(t)$, $c = 1$, $A_1 = A_2 = 1$ at $y = 180$ is presented in Figure (3.5).

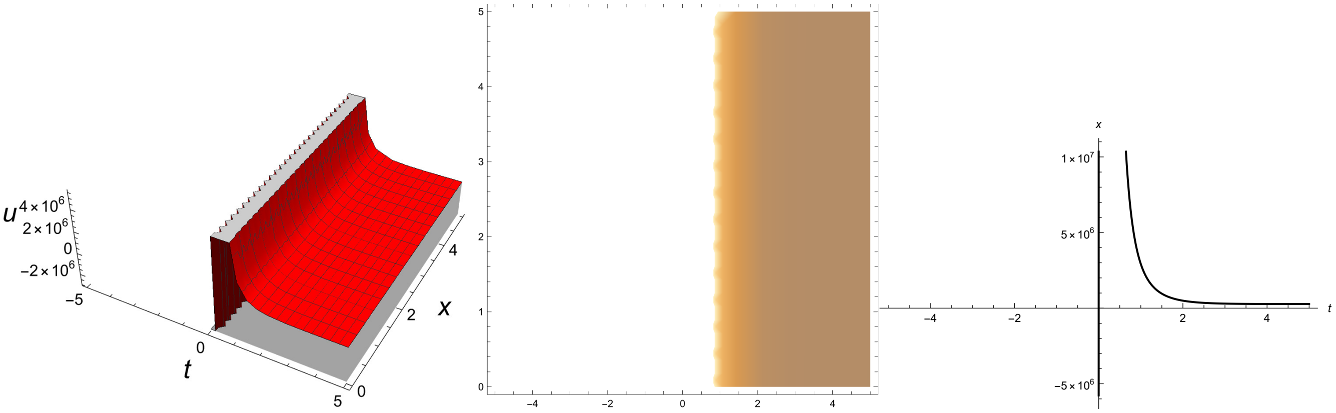

Figure 3.5: 3D and 2D profile of solution (3.44)

3.1.6 Travelling wave solution

In this section we construct travelling wave solutions of (3.1) by considering special values of the functions $f(t)$, $g(t)$ and $a(t)$ in the symmetries X_3, X_4 and X_5, respectively. By taking $f(t) = 1/72$, $g(t) = 1/12$ and $a(t) = 1/432$ in (3.25), (3.26) and (3.27) we obtain

$$X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial x},\ X_5 = \frac{\partial}{\partial t}.$$

We now take the linear combination

$$X_5 + aX_4 + bX_3 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

whose associated Lagrange system is

$$\frac{dt}{1} = \frac{dx}{a} = \frac{dy}{b} = \frac{du}{0}.$$

This gives invariants

$$p = x - at,\ q = y - bt, u = \Theta(p, q) \tag{3.45}$$

and using these invariants equation (3.1) transforms into the following nonlinear partial differential equation in two independent variables:

$$\Theta_{pppp} + 6\Theta_p\Theta_{pp} - a\Theta_{pp} - b\Theta_{pq} - \Theta_{qq} = 0. \tag{3.46}$$

The above equation has five point symmetries, namely

$$\begin{aligned}
&\mathcal{S}_1 = \frac{\partial}{\partial p},\ \mathcal{S}_2 = \frac{\partial}{\partial q},\ \mathcal{S}_3 = \frac{\partial}{\partial \Theta},\ \mathcal{S}_4 = p\frac{\partial}{\partial \Theta},\\
&\mathcal{S}_5 = (6bq + 12p)\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} - (pb^2 - 4pa + 12\Theta)\frac{\partial}{\partial \Theta}.
\end{aligned}$$

The symmetry $\mathcal{S} = \mathcal{S}_1 + c\mathcal{S}_2$ gives two invariants $I_1 = q - cp$ and $I_2 = \Theta$ and consequently the invariant solution is $\Theta = F(q - cp)$. Substituting this value of Θ into (3.46), we get the fourth-order NLODE

$$(bc - ac^2 - 1)F''(z) - 6c^3F'(z)F''(z) + c^4F''''(z) = 0, \tag{3.47}$$

which we rewrite as

$$AF''(z) - BF'(z)F''(z) + CF''''(z) = 0 \tag{3.48}$$

with $A = bc - ac^2 - 1$, $B = 6c^3$, $C = c^4$ and $z = (ac - b)t - cx + y$.

Integrating equation (3.48) with respect to z once gives a third-order ODE

$$AF' - \frac{1}{2}BF'^2 + CF''' + k_1 = 0, \tag{3.49}$$

where k_1 is a constant of integration. Multiplying equation (3.49) by F'', integrating and simplifying the resulting equation, we have the second-order NLODE

$$\frac{1}{2}A(F')^2 - \frac{1}{6}B(F')^3 + \frac{1}{2}C(F'')^2 + k_1F' + k_2 = 0 \tag{3.50}$$

with k_2 an integration constant. Equation (3.50) can be rewritten as

$$(F'')^2 = \frac{B}{3C}(F')^3 - \frac{A}{C}(F')^2 - \frac{2k_1}{C}F' - \frac{2k_2}{C}. \tag{3.51}$$

Let $F' = G$. Equation (3.51) becomes

$$G'^2 = \frac{B}{3C}G^3 - \frac{A}{C}G^2 - \frac{2k_1}{C}G - \frac{2k_2}{C}. \tag{3.52}$$

We consider the cubic equation

$$G^3 - \frac{3A}{B}G^2 - \frac{6k_1}{B}G - \frac{6k_2}{B} = 0 \tag{3.53}$$

and assume that the roots of this equation are r_1, r_2 and r_3 such that $r_3 < r_2 < r_1$. Then equation (3.52) becomes

$$G'^2 = \frac{B}{3C}(G - r_1)(G - r_2)(G - r_3) \tag{3.54}$$

and solution to (3.52) can be expressed in terms of Jacobi elliptic function

$$G(z) = r_2 + (r_1 - r_2)\mathrm{cn}^2\left\{\sqrt{\frac{B(r_1 - r_2)}{12C}}z, S^2\right\},\ S^2 = \frac{r_1 - r_2}{r_1 - r_3} \tag{3.55}$$

with cn denoting the cosine elliptic function. Integrating equation (3.55) and then returning to the original variables t, x, y and u, the pKP equation (3.1) possesses a periodic solution

$$\begin{aligned} u(t,x,y) = {} & \sqrt{\frac{12C(r_1 - r_2)^2}{B(r_1 - r_3)S^8}}\left\{\mathrm{EllipticE}\left[\mathrm{sn}\left(\frac{B(r_1 - r_3)}{12C}z, S^2\right), S^2\right]\right\} \\ & + \left\{r_2 - (r_1 - r_2)\frac{1 - S^4}{S^4}\right\}z + k_3, \end{aligned} \tag{3.56}$$

where $z = (ac - b)t - cx + y$, k_3 is an integration constant and $\text{EllipticE}[q, v]$ denotes the incomplete elliptic integral given by [47]

$$\text{EllipticE}[q, v] = \int_0^v \sqrt{\frac{1 - v^2n^2}{1 - n^2}}dn.$$

Figure (3.6) depicts the wave profile of the periodic solution by assigning the values $r_1 = 10, r_2 = 5, r_3 = 3, k_3 = c = 1$ at $t = 1$ to view the dynamics of solution (3.56) graphically.

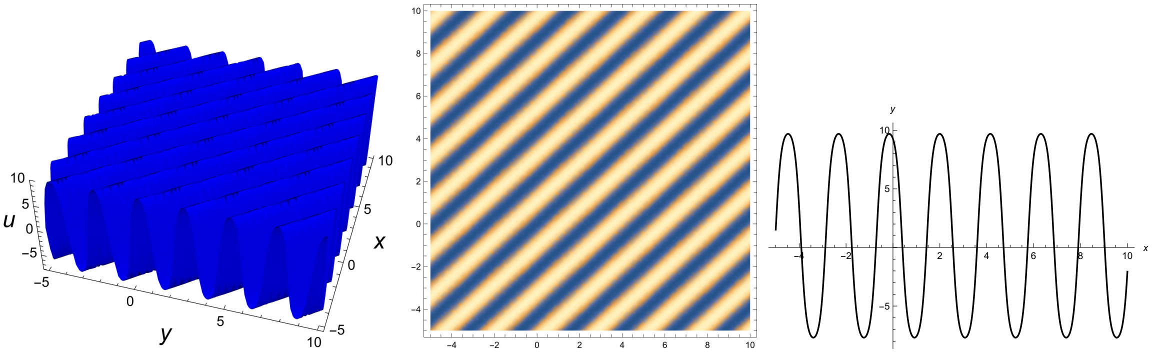

Figure 3.6: 3D and 2D profile of solution (3.56)

3.2 Graphical and physical explanation of the obtained solutions

In this section, we give more details on the obtained group-invariant solutions to the pKP equation (3.1) by discussing their geometrical representation. 3D, 2D and corresponding contour plots in Figs. $3.1 - 3.6$ are constructed by utilizing the mathematical software tool Mathematica. This involves taking acceptable values of the parameters under certain limits in order to visualize the mechanism of the equation under study. We note that from Figs. $3.1 - 3.6$ the achieved solutions of the pKP equation comprise of various solutions such as singular, singular periodic, singular periodic soliton, periodic soliton, singular soliton and periodic soliton, respectively.

In Figure 3.1, 3D and 2D graphs of the singular solution (3.30) with parameters $k_1 = 0.4, k_2 = 0.6, k_3 = 0.8$, at $y = 180$, recorded with interval $-10 < t < 10$ and $-10 < x < 10$. Figure

3.2, 3D and 2D graphs of singular periodic solution (3.31) with parameters $k(t) = l(t) = 1$ when $g(t) = \sin(t)$, $y = 20$, recorded on the interval $-5 < t < 5$ and $-5 < x < 5$. Figure 3.3, 3D and 2D graphs of singular periodic soliton solution (3.40) corresponding to the values $K = c = 1$ when $a(t) = \sin(t)$, $y = 10$ recorded under the limit $-5 < t < 5$ and $-5 < x < 5$. Figure 3.4, 3D and 2D graphs of periodic soliton solution (3.43) assigned the values $\alpha_1 = 10, \alpha_2 = 5, \alpha_3 = 2, k_3 = 0$, when $a(t) = c = 1$, $y = 180$ recorded under the limit $-20 < t < 20$ and $-20 < x < 20$. Figure 3.5, 3D and 2D graphs of singular soliton solution (3.44) for the parameters $c = A_1 = A_2 = 1$, when $a(t) = \sinh(t)$, $y = 180$ on the interval $-5 < t < 5$ and $0 < x < 5$. Figure 3.6, 3D and 2D graphs of periodic soliton solution (3.56) for the values $r_1 = 10, r_2 = 5, r_3 = 3$, when $k_3 = c = 1$ and $t = 1$ recorded with interval $-5 < y < 10$ and $-5 < x < 10$.

3.3 Conservation laws for the pKP equation (3.1)

In this section, we derive the conserved vectors of the pKP equation (3.1) by using three approaches: the conservation theorem due to Ibragimov, the multiplier method and Noether's theorem.

3.3.1 Conservation laws for (3.1) using Ibragimov's theorem

In this subsection, we invoke the conservation theorem of Ibragimov, which has been outlined in chapter one, to find conserved vectors for the pKP equation (3.1). In this case the Euler-Lagrange operator is defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_y^2 \frac{\partial}{\partial u_{yy}} + D_x^4 \frac{\partial}{\partial u_{xxxx}} + \cdots , \tag{3.57}$$

where D_t, D_x and D_y are given in (3.6). The adjoint equation for (3.1) can be obtained from the formula

$$F^* \left(t, x, u, v, ..., v_{xxxx}\right) = \frac{\delta}{\delta u} \left[v(u_{tx} + 6u_x u_{xx} + u_{xxxx} - u_{yy})\right] = 0, \tag{3.58}$$

where $v = v(t, x, y)$. Equation (3.58) yields

$$F^* = v_{tx} + 6v_x u_{xx} + 6u_x v_{xx} + v_{xxxx} - v_{yy} = 0. \tag{3.59}$$

If we consider equation (3.1) and the adjoint equation (3.59) as a system, the Lagrangian for this system from (1.44) is

$$\mathcal{L} = v(u_{tx} + 6u_x u_{xx} + u_{xxxx} - u_{yy}). \tag{3.60}$$

Now, extending all the symmetries of (3.1) to the new variable $v(t, x, y)$ yields the generator

$$Y = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta^*\frac{\partial}{\partial v} \tag{3.61}$$

with

$$\eta^* = \eta^*(t, x, y, u, v) = -\left\{\lambda + D_t(\tau) + D_x(\xi) + D_y(\psi)\right\} v \tag{3.62}$$

and the parameter λ is determined by using

$$X^{[4]}(F) = \lambda F, \tag{3.63}$$

where $F = u_{tx} + 6u_x u_{xx} + u_{xxxx} - u_{yy}$ and $X^{[4]}$ is the fourth prolongation of X, given by

$$X^{[4]} = X + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{yy}\frac{\partial}{\partial u_{yy}} + \zeta_{xxxx}\frac{\partial}{\partial u_{xxxx}}. \tag{3.64}$$

The conservation laws of the system of equations (3.1) and (3.59), associated with a symmetry can be obtained from [33]

$$\begin{aligned}
T^t &= \tau\mathcal{L} + W\left\{\frac{\partial\mathcal{L}}{\partial u_t} - D_x\left(\frac{\partial L}{\partial u_{xt}}\right)\right\},\\
T^x &= \xi\mathcal{L} + W\left\{\frac{\partial\mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right) + D_{xx}\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right) - D_{xxx}\left(\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\right)\right\}\\
&\quad + W_x\left\{\frac{\partial\mathcal{L}}{\partial u_{xx}} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right) + D_{xx}\left(\frac{\partial L}{\partial u_{xxxx}}\right)\right\} + W_t\frac{\partial\mathcal{L}}{\partial u_{xt}}\\
&\quad + W_{xx}\left\{\frac{\partial\mathcal{L}}{\partial u_{xxx}} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\right)\right\} + W_{xxx}\frac{\partial\mathcal{L}}{\partial u_{xxxx}},\\
T^y &= \psi\mathcal{L} + W\left\{\frac{\partial\mathcal{L}}{\partial u_y} - D_y\left(\frac{\partial\mathcal{L}}{\partial u_{yy}}\right)\right\} + W_y\frac{\partial\mathcal{L}}{\partial u_{yy}},
\end{aligned} \tag{3.65}$$

where $W = \eta - \tau u_t - \xi u_x - \psi u_y$.

Case 1. For the symmetry $X_1 = j(t)\partial/\partial u$ we have $\eta = j(t)$, $\xi = \eta = \psi = 0$. Thus, $\zeta_x = \zeta_{tx} = \zeta_{xx} = \zeta_{yy} = \zeta_{xxxx} = 0$. We then get $X_1^{[4]}F = 0F$, that is, $\lambda = 0$. From (3.62) we obtain $\eta^* = 0$, whereby the new generator retains the form of X_1. That is, $Y_1 = j(t)\partial/\partial u$. The

Lie characteristic function is $W_1 = j(t)$. Thus, the conserved vector (T_1^t, T_1^x, T_1^y) using (3.65) is

$$\begin{aligned}
T_1^t =& -j(t)v_x,\\
T_1^x =& j'(t)v - 6j(t)v_x u_x - j(t)v_{xxx},\\
T_1^y =& j(t)v_y.
\end{aligned}$$

Following the above procedure, we obtain the rest of the conserved vectors associated with other symmetries.

Case 2. For $X_2 = yi(t)\partial/\partial u$ we get $Y_2 = yi(t)\partial/\partial u$. The Lie characteristic function is then given by $W_2 = yi(t)$. Thus, the conserved vector is

$$\begin{aligned}
T_2^t =& -yi(t)v_x,\\
T_2^x =& yi'(t)v - 6yi(t)v_x u_x - yi(t)v_{xxx},\\
T_2^y =& yi(t)v_y - i(t)v.
\end{aligned}$$

Case 3. For the symmetry

$$X_3 = 36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y} + \left(6xyf''(t) + y^3 f'''(t)\right)\frac{\partial}{\partial u}$$

the adjoint operator is

$$Y_3 = 36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y} + \left(6xyf''(t) + y^3 f'''(t)\right)\frac{\partial}{\partial u}$$

and the Lie characteristic function is

$$W_3 = 6xyf''(t) + y^3 f'''(t) - 36yf'(t)u_x - 72f(t)u_y.$$

Thus, the conserved vector is

$$\begin{aligned}
T_3^t =& 36yf'(t)u_x v_x - y^3 f'''(t)v_x - 6xyf''(t)v_x + 72f(t)u_y v_x,\\
T_3^x =& y^3 f''''(t)v - 6y^3 f'''(t)u_x v_x - y^3 f'''(t)v_{xxx} + 6xyf'''(t)v - 36xyf''(t)u_x v_x\\
&+ 6yf''(t)v_{xx} - 6xyf''(t)v_{xxx} + 216yf'(t)u_x^2 v_x - 36yf'(t)u_{xx}v_{xx} + 36yf'(t)u_{xxx}v_x\\
&+ 36yf'(t)u_x v_{xxx} - 36yf'(t)u_{yy}v - 72f'(t)u_y v + 432f(t)u_y u_x v_x - 72f(t)u_{ty}v_{xx}\\
&+ 72f(t)u_{xxy}v_x + 72f(t)u_y v_{xxx} - 432f(t)u_x u_{ty}v - 72f(t)u_{xxxy}v - 72f(t)u_{ty}v,
\end{aligned}$$

$$\begin{aligned}T_3^y =& 72f(t)vu_{tx}+36f'(t)vu_x-72f(t)v_yu_y-6xf''(t)v-3y^2f'''(t)v+6xyf''(t)v_y\\&+y^3f'''(t)v_y-36yf'(t)v_yu_x+36yf'(t)vu_{xy}+432f(t)vu_xu_{xx}+72f(t)vu_{xxxx}.\end{aligned}$$

Case 4. For the symmetry

$$X_4=12g(t)\frac{\partial}{\partial x}+\left(2xg'(t)+y^2g''(t)\right)\frac{\partial}{\partial u}$$

the new generator Y_4 is

$$Y_4=12g(t)\frac{\partial}{\partial x}+\left(2xg'(t)+y^2g''(t)\right)\frac{\partial}{\partial u},$$

and

$$W_4=2xg'(t)+y^2g''(t)-12g(t)u_x.$$

Hence

$$\begin{aligned}T_4^t =& 12g(t)v_xu_x+g'(t)v_x-y^2g''(t)v_x-2xg'(t)v_x,\\T_4^x =& y^2g'''(t)v-6y^2g''(t)u_xv_x-y^2g''(t)v_{xxx}+2xg''(t)v-12xg'(t)u_xv_x+2g'(t)v_{xx}\\&-2xg'(t)v_{xxx}+72g(t)u_x^2v_x-12g(t)u_{xx}v_{xx}+12g(t)u_{xxx}v_x+12g(t)u_xv_{xxx}-12g(t)u_{yy}v,\\T_4^y =& 12g(t)vu_{xy}-2yg''(t)v+2xg'(t)v_y-12g(t)v_yu_x.\end{aligned}$$

Case 5. Finally for the symmetry

$$\begin{aligned}X_5 =& 432a(t)\frac{\partial}{\partial t}+\left(144xa'(t)+72y^2a''(t)\right)\frac{\partial}{\partial x}+288ya'(t)\frac{\partial}{\partial y}\\&+\left(12x^2a''(t)-144ua'(t)+12xy^2a'''(t)+y^4a''''(t)\right)\frac{\partial}{\partial u}\end{aligned}$$

the generator with the new variable v is

$$\begin{aligned}Y_5 = 432a(t)\frac{\partial}{\partial t}&+\left(144xa'(t)+72y^2a''(t)\right)\frac{\partial}{\partial x}+288ya'(t)\frac{\partial}{\partial y}\\&+\left(12x^2a''(t)-144ua'(t)+12xy^2a'''(t)+y^4a''''(t)\right)\frac{\partial}{\partial u}-144va'(t)\frac{\partial}{\partial v}\end{aligned}$$

and the Lie characteristic function is

$$\begin{aligned}W_5 = 12x^2a''(t)&-144ua'(t)+12xy^2a'''(t)+y^4a''''(t)432a(t)u_t\\&-\left(144xa'(t)+72y^2a''(t)\right)u_x-288ya'(t)u_y.\end{aligned}$$

Consequently, the associated conserved vector is

$$\begin{aligned}
T_5^t = {} & 72y^2a''(t)u_xv_x - y^4a''''(t)v_x - 12xy^2a'''(t)v_x - 12x^2a''(t)v_x + 288ya'(t)u_yv_x \\
& + 144xa'(t)u_xv_x + 144a'(t)uv_x + 432a(t)u_tv_x - 432a(t)u_{yy}v + 2592a(t)u_xu_{xx}v \\
& + 432a(t)u_{xxxx}v + 432a(t)u_{tx}v, \\
T_5^x = {} & y^4va^{(5)}(t) - 6y^4a''''(t)u_xv_x - y^4a''''(t)v_{xxx} + 12xy^2va''''(t) - 72y^2va''(t)u_{yy} \\
& + 432y^2a''(t)u_x^2v_x - 72xy^2a'''(t)u_xv_x + 12y^2a'''(t)v_{xx} - 72y^2a''(t)u_{xx}v_{xx} \\
& + 72y^2a''(t)v_xu_{xxx} - 12xy^2a'''(t)v_{xxx} + 72y^2a''(t)u_xv_{xxx} - 288yva''(t)u_y \\
& + 1728ya'(t)u_yu_xv_x - 1728yva'(t)u_xu_{xy} - 288ya'(t)u_{xy}v_{xx} + 288ya'(t)v_xu_{xxy} \\
& + 288ya'(t)u_yv_{xxx} - 288yva'(t)u_{xxxy} - 288yva'(t)u_{ty} - 1728va'(t)u_x^2 - 144uva''(t) \\
& + 12x^2va'''(t) - 144xva'(t)u_{yy} + 864xa'(t)u_x^2v_x - 24a''(t)v_x + 864ua'(t)u_xv_x \\
& - 72x^2a''(t)u_xv_x + 432a'(t)v_xu_{xx} + 24xa''(t)v_{xx} - 288a'(t)u_xv_{xx} - 144xa'(t)u_{xx}v_{xx} \\
& - 576va'(t)u_{xxx} + 144xa'(t)v_xu_{xxx} + 144ua'(t)v_{xxx} - 12x^2a''(t)v_{xxx} + 144xa'(t)u_xv_{xxx} \\
& - 576va'(t)u_t + 2592a(t)u_xv_xu_t + 432a(t)v_{xxx}u_t - 2592a(t)vu_xu_{tx} - 432a(t)v_{xx}u_{tx} \\
& + 432a(t)v_xu_{txx} - 432a(t)vu_{txxx} - 432a(t)vu_{tt}, \\
T_5^y = {} & 12x^2a''(t)v_y + 432a'(t)vu_y - 24xya'''(t)v - 4y^3a''''(t)v - 144a'(t)v_yu + 12xy^2a''(t)v_y \\
& + y^4a''''(t)v_y - 288ya'(t)v_yu_y + 144ya''(t)vu_x - 144xa'(t)v_yu_x - 72y^2a''(t)v_yu_x \\
& + 144xa'(t)vu_{xy} + 72y^2a''(t)vu_{xy} + 1728ya'(t)vu_xu_{xx} - 432a(t)v_yu_t + 432a(t)vu_{ty} \\
& + 288ya'(t)vu_{ty} + 288ya'(t)vu_{xxxx}.
\end{aligned}$$

Remark. The conserved vectors T involve solutions u of the pKP equation and solutions v of the adjoint equation (3.59) and hence give an infinite number of conservation laws. Moreover, we observe that when $f(t) = g(t) = 1$ the conserved vectors from case 3 and 4 represent the conservation law momentum, while when $a(t) = 1$, case 5 represents conservation law of energy.

3.3.2 Conservation laws for (3.1) using the Multiplier method

We now derive the conserved vectors of the pKP equation (3.1) by employing the multiplier method. We seek zeroth-order multipliers $\mathcal{Q} = \mathcal{Q}(t, x, y, u)$. The multipliers are determined by using the determining equation

$$\frac{\delta}{\delta u}\left\{\mathcal{Q}\left(u_{tx}+6u_xu_{xx}+u_{xxxx}-u_{yy}\right)\right\}=0, \tag{3.66}$$

where $\delta/\delta u$ is the Euler operator, given in our case as

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_tD_x\frac{\partial}{\partial u_{tx}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}+D_x^4\frac{\partial}{\partial u_{xxxx}},$$

with D_t, D_x, D_y as total differential operators. Expanding (3.66) and splitting on derivatives of u yields the system of multiplier determining equations:

$$\mathcal{Q}_{yy}=0,\ \mathcal{Q}_x=0,\ \mathcal{Q}_u=0.$$

The solution of the above PDEs gives

$$\mathcal{Q}=f(t)+yg(t),$$

where f and g are arbitrary functions of t. The conservation laws of equation (3.1) are then derived by invoking the divergence identity

$$D_tT^t+D_xT^x+D_yT^y=\mathcal{Q}\left(u_{tx}+6u_xu_{xx}+u_{xxxx}-u_{yy}\right) \tag{3.67}$$

with T^t being conserved density and T^x, T^y spatial fluxes. Thus, using (3.67) we obtain the following conserved vectors corresponding to the two multipliers:

Case 1. For $\mathcal{Q}_1=f(t)$ we have the corresponding conserved vector as

$$\begin{aligned}T_1^t=&f(t)u_x,\\ T_1^x=&3f(t)u_x^2+f(t)u_{xxx}-f'(t)u,\\ T_1^y=&-f(t)u_y.\end{aligned}$$

Case 2. For $\mathcal{Q}_2=yg(t)$, we have the conserved vector given by

$$\begin{aligned}T_2^t=&g(t)yu_x,\\ T_2^x=&3g(t)yu_x^2+yg'(t)u_{xxx}-g(t)yu,\\ T_2^y=&ug(t)-yg(t)u_y.\end{aligned}$$

Remark. Due to the presence of the arbitrary functions $f(t)$ and $g(t)$ in the multiplier, one can obtain infinitely many conservation laws. In fact, if we consider the special cases, $f(t)=t$ and $g(t)=t$, the conserved vectors represent conservation laws of energy and angular momentum respectively.

3.3.3 Conservation laws for (3.1) using Noether's theorem

In this subsection, we utilize the classical Noether's theorem to construct conservation laws for the pKP equation (3.1). It can be easily verified that equation (3.1) has the second-order Lagrangian

$$\mathcal{L} = \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_t u_x + \frac{1}{2}u_y^2 - u_x^3. \tag{3.68}$$

Employing the Lagrangian (3.68) on the determining equation

$$X^{[2]}\mathcal{L} + \mathcal{L}\left\{D_t(\tau) + D_x(\xi) + D_y(\psi)\right\} - D_t(B^t) - D_x(B^x) - D_y(B^y) = 0, \tag{3.69}$$

where $B^t = B^t(t, x, y, u)$, $B^x = B^x(t, x, y, u)$ and $B^y = B^y(t, x, y, u)$ are gauge functions and $X^{[2]}$ is the second-order prolongation of the generator

$$X = \tau(t, x, y, u)\frac{\partial}{\partial t} + \xi(t, x, y, u)\frac{\partial}{\partial x} + \psi(t, x, y, u)\frac{\partial}{\partial y} + \eta(t, x, y, u)\frac{\partial}{\partial u}$$

given by

$$X^{[2]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_y\frac{\partial}{\partial u_y} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}. \tag{3.70}$$

Expanding equation (3.69) we get

$$\begin{aligned}
&- u_{xx}u_t\tau_{ux}u_x - u_{xx}u_y\psi_{ux}u_x - u_{xx}u_{tx}\tau_u u_x + \frac{1}{2}u_t u_x\psi_u u_y - u_{xx}u_{xy}\psi_u u_x - u_{xx}\xi_{xu}u_x^2\\
&- u_y u_x\xi_y - \frac{1}{2}u_t u_x\psi_y - \eta_u u_x u_t - u_y u_t\tau_y - u_{xx}u_{xy}\psi_x + \frac{1}{2}u_x u_y\tau + 3u_t\tau_x u_x^2 - u_{xx}u_{tx}\tau_x\\
&+ \frac{1}{2}u_y\psi_x u_t + 3u_y\psi_x u_x^2 - \frac{1}{2}u_y^2 u_x\xi_u + \frac{1}{2}u_x^2\xi_u u_t - \frac{1}{2}u_{xx}^2\xi_u u_x + \frac{1}{2}u_y^2\xi_x - \frac{1}{2}u_{xx}^2\xi_x + 2u_x^3\xi_x\\
&+ \frac{1}{2}u_t^2\tau_x + \frac{1}{2}u_y^2\tau_t + \frac{1}{2}u_{xx}^2\tau_t + \frac{1}{2}u_x^2\xi_t - u_x^3\tau_t + u_{xx}\eta_{xx} - \frac{1}{2}u_y^3\psi_u + 2u_x^4\xi_u - 3\eta_x u_x^2\\
&+ u_y\eta_y - \frac{1}{2}\eta_x u_t + u_y^2\eta_u - \frac{1}{2}u_x\eta_t - 3\eta_u u_x^3 - u_x^3\psi_y - B_u^y u_y - B_u^x u_x - \frac{1}{2}u_y^2\psi_y + \frac{1}{2}u_{xx}^2\psi_y\\
&- B_u^t u_t - B_y^y - B_x^x - B_t^t = 0
\end{aligned}$$

and splitting on different derivatives of u, we obtain the system of partial differential equations

$$\tau_x = 0, \tag{3.71}$$

$$\tau_y = 0, \tag{3.72}$$

$$\tau_u = 0, \tag{3.73}$$

$$\xi_u = 0, \tag{3.74}$$

$$\psi_x = 0, \tag{3.75}$$

$$\psi_u = 0, \tag{3.76}$$

$$\eta_{xx} = 0, \tag{3.77}$$

$$\psi_y + 2\eta_u = 0, \tag{3.78}$$

$$\xi_{xx} - \eta_{xu} = 0, \tag{3.79}$$

$$\psi_t - 2\xi_y = 0, \tag{3.80}$$

$$\xi_t - 6\eta_x = 0, \tag{3.81}$$

$$\tau_t - \xi_x + \psi_y = 0, \tag{3.82}$$

$$\tau_t + \xi_x - \psi_y + 2\eta_u = 0, \tag{3.83}$$

$$2\xi_x - \tau_t - \psi_y - 3\eta_u = 0, \tag{3.84}$$

$$\eta_y - B^y_u = 0, \tag{3.85}$$

$$\eta_t + 2B^x_u = 0, \tag{3.86}$$

$$\eta_x + 2B^t_u = 0, \tag{3.87}$$

$$B^t_t + B^x_x + B^y_y = 0. \tag{3.88}$$

Now we can obtain the Noether symmetries and guage functions by solving the above system of PDEs. We start by solving for τ, ξ, ψ and η by considering equations (3.71) – (3.77) to get $\tau = a(t), \xi = b(t,x,y), \psi = c(t,y)$ and $\eta = xd(t,y,u) + e(t,y,u)$, where a, b, c, d and e are arbitrary functions of their arguments. Substituting the above values of ψ and η in equation (3.78) we get

$$c_y(t,y) + 2xd_u(t,y,u) + 2e_u(t,y,u) = 0.$$

Since c, d and e does not depend on x, we split the above equation on x and get

$$x : d_u(t,y,u) = 0,$$

$$\text{rest} : c_y(t,y) + 2e_u(t,y,u) = 0.$$

Solving the above equations we get $d = d(t,y)$ and $e = -\frac{1}{2}uc_y(t,y) + f(t,y)$, where f is an arbitrary function of t and y. Equation (3.79) gives $b = xg(t,y) + h(t,y)$, where g and

h are arbitrary functions of t and y. These results imply that $\xi = xg(t,y) + h(t,y)$ and $\eta = xd(t,y) - \frac{1}{2}uc_y(t,y) + f(t,y)$. Substituting these values of ψ and ξ in equation (3.80) we get $c_t(t,y) - 2xg_y(t,y) - 2h_y(t,y) = 0$. Because c, g and h do not depend on x, we can split the equation on x so that

$$x : g_y(t,y) = 0, \tag{3.89}$$

$$\text{rest} : c_t(t,y) - 2h_y(t,y) = 0. \tag{3.90}$$

By equation (3.89) we see that $g = g(t)$ and so $\xi = xg(t) + h(t,y)$. Taking the new value of ξ and η and substituting them in equation (3.81) gives $xg'(t) + h_t(t,y) - 6d(t,y) = 0$. We split this equation on x since g, h and d do not depend on x and obtain

$$x : g'(t) = 0, \tag{3.91}$$

$$\text{rest} : h_t(t,y) - 6d(t,y) = 0. \tag{3.92}$$

Integrating equation (3.91) with respect to t we get $g = C_1$, where C_1 is a constant of integration. Moreover, from equation (3.82) we have $c_y(t,y) = C_1 - a'(t)$. Integrating this equation once with respect to y gives $c(t,y) = C_1y - ya'(t) + i(t)$, where i is an arbitrary function of t. Therefore by equation (3.90) we see that $h_y(t,y) = -\frac{1}{2}ya''(t) + \frac{1}{2}i'(t)$. To obtain the value of h we integrate the equation once with respect to y to get

$$h(t,y) = -\frac{1}{4}y^2a''(t) + \frac{1}{2}yi'(t) + j(t),$$

where j is an arbitrary function of t. We can then find the value of d from equation (3.92), which is

$$d(t,y) = -\frac{1}{24}y^2a'''(t) + \frac{1}{12}yi''(t) + \frac{1}{6}j'(t).$$

This implies that

$$\begin{aligned}\xi &= C_1x - \frac{1}{4}y^2a''(t) + \frac{1}{2}yi'(t) + j(t),\\ \eta &= -\frac{1}{24}xy^2a'''(t) + \frac{1}{12}xyi''(t) + \frac{1}{6}xj'(t) - \frac{1}{2}C_1u + \frac{1}{2}ua'(t) + f(t,y).\end{aligned}$$

From equation (3.83) we get $a(t) = \frac{1}{3}C_1t + C_2$, where C_2 is a arbitrary constant of integration. This means

$$\tau = \frac{1}{3}C_1t + C_2,$$

$$\begin{aligned}\xi &= C_1 x + \frac{1}{2} y i'(t) + j(t),\\ \psi &= \frac{2}{3} C_1 y + i(t),\\ \eta &= \frac{1}{12} x y i''(t) + \frac{1}{6} x j'(t) - \frac{1}{3} C_1 u + f(t,y).\end{aligned}$$

Substituting the above values in equation (3.84) we obtain $C_1 = 0$. Therefore

$$\begin{aligned}\tau &= C_2,\\ \xi &= \frac{1}{2} y i'(t) + j(t),\\ \psi &= i(t),\\ \eta &= \frac{1}{12} x y i''(t) + \frac{1}{6} x j'(t) + f(t,y).\end{aligned}$$

We can now obtain the gauge functions by solving equations (3.85) – (3.87) and obtain

$$\begin{aligned}B^t &= -\frac{1}{24} y u i''(t) - \frac{1}{12} u j'(t) + k(t,x,y),\\ B^x &= -\frac{1}{24} x y u i'''(t) - \frac{1}{12} x u j''(t) - \frac{1}{2} u f_t(t,y) + l(t,x,y),\\ B^y &= \frac{1}{12} x u i''(t) + u f_y(t,y) + m(t,x,y),\end{aligned}$$

where k, l and m are arbitrary functions of t, x and y. Equation (3.88) becomes

$$-\frac{1}{12} y u i'''(t) - \frac{1}{6} u j''(t) + u f_{yy}(t,y) + k_t(t,x,y) + l_x(t,x,y) + m_y(t,x,y) = 0.$$

Splitting the above equation on u since the variables i, j, f, k, l and m do not depend on it we obtain

$$u : -\frac{1}{12} y i'''(t) - \frac{1}{6} j''(t) + f_{yy}(t,y) = 0, \tag{3.93}$$

$$\text{rest} : k_t(t,x,y) + l_x(t,x,y) + m_y(t,x,y) = 0. \tag{3.94}$$

Integrating equation (3.93) twice with respect to y we obtain

$$f(t,y) = \frac{1}{72} y^3 i'''(t) + \frac{1}{12} y^2 j''(t) + y n(t) + o(t),$$

where n and o are arbitrary functions of t. Thus,

$$\tau = C_2,$$

$$
\begin{aligned}
\xi =& \frac{1}{2} y i'(t) + j(t),\\
\psi =& i(t),\\
\eta =& \frac{1}{12} x y i''(t) + \frac{1}{6} x j'(t) + \frac{1}{72} y^3 i'''(t) + \frac{1}{12} y^2 j''(t) + y n(t) + o(t),\\
B^t =& -\frac{1}{24} y u i''(t) - \frac{1}{12} u j'(t) + k(t,x,y),\\
B^x =& -\frac{1}{24} x y u i'''(t) - \frac{1}{12} x u j''(t) - \frac{1}{144} u y^3 i''''(t) - \frac{1}{24} u y^2 j'''(t) - \frac{1}{2} u y n'(t)\\
& - \frac{1}{2} u o'(t) + l(t,x,y),\\
B^y =& \frac{1}{12} x u i''(t) + \frac{1}{24} u y^2 i'''(t) + \frac{1}{6} u y j''(t) + u n(t) + m(t,x,y).
\end{aligned}
$$

Using all the above values and letting $k = l = m = 0$ since they contribute to the trivial part of the conservation laws, we obtain the following Noether symmetries and their respective gauge functions as

$$
\begin{aligned}
& X_1 = \frac{\partial}{\partial t}, \quad B_1^t = 0,\ B_1^x = 0,\ B_1^y = 0,\\
& X_2 = \frac{1}{2} y i'(t) \frac{\partial}{\partial x} + i(t) \frac{\partial}{\partial y} + \left(\frac{1}{12} x y i''(t) + \frac{1}{72} y^3 i'''(t) \right) \frac{\partial}{\partial u},\\
& B_2^t = -\frac{1}{24} y u i''(t),\ B_2^x = -\frac{1}{24} x y u i'''(t) - \frac{1}{144} u y^3 i''''(t),\ B_2^y = \frac{1}{12} x u i''(t) + \frac{1}{24} u y^2 i'''(t),\\
& X_3 = j(t) \frac{\partial}{\partial x} + \left(\frac{1}{6} x j'(t) + \frac{1}{12} y^2 j''(t) \right) \frac{\partial}{\partial u},\\
& B_3^t = -\frac{1}{12} u j'(t),\ \ B_3^x = -\frac{1}{12} x u j''(t) - \frac{1}{24} u y^2 j'''(t),\ \ B_3^y = \frac{1}{6} u y j''(t),\\
& X_4 = y n(t) \frac{\partial}{\partial u}, \quad B_4^t = 0,\ B_4^x = -\frac{1}{2} u y n'(t),\ B_4^y = u n(t),\\
& X_5 = o(t) \frac{\partial}{\partial u}, \quad B_5^t = 0,\ B_5^x = -\frac{1}{2} u o'(t),\ B_5^y = 0.
\end{aligned}
$$

The conserved vectors corresponding to the above obtained Noether point symmetries, using the formulae [34]

$$
\begin{aligned}
T^t =& \tau \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial u_t} - B^t,\\
T^x =& \xi \mathcal{L} + W \left(\frac{\partial \mathcal{L}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right) + W_x \frac{\partial}{\partial u_{xx}} - B^x,\\
T^y =& \psi \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial u_y} - B^y,
\end{aligned}
$$

are given, respectively, by (with $W = \eta - u_t \tau - u_x \xi - u_y \psi$)

$$
T_1^t = \frac{1}{2} u_y^2 + \frac{1}{2} u_{xx}^2 - u_x^3,
$$

$$
\begin{aligned}
T_1^x =& 3u_x^2u_t + \frac{1}{2}u_t^2 + u_{xxx}u_t - u_{xx}u_{tx},\\
T_1^y =& -u_tu_y;
\end{aligned}
$$

$$
\begin{aligned}
T_2^t =& -\frac{1}{144}y^3i'''(t)u_x - \frac{1}{24}xyi''(t)u_x + \frac{1}{24}yi''(t)u + \frac{1}{4}yi'(t)u_x^2 + \frac{1}{2}i(t)u_yu_x,\\
T_2^x =& \frac{1}{144}y^3i''''(t)u - \frac{1}{24}y^3i'''(t)u_x^2 - \frac{1}{72}y^3i'''(t)u_{xxx} - \frac{1}{144}y^3i'''(t)u_t + \frac{1}{24}xyi'''(t)u\\
& - \frac{1}{4}xyi''(t)u_x^2 + \frac{1}{12}yi''(t)u_{xx} - \frac{1}{12}xyi''(t)u_{xxx} - \frac{1}{24}xyi''(t)u_t + yi'(t)u_x^3\\
& + \frac{1}{4}yi'(t)u_y^2 - \frac{1}{4}yi'(t)u_{xx}^2 + \frac{1}{2}yi'(t)u_xu_{xxx} + 3i(t)u_yu_x^2 - i(t)u_{xy}u_{xx} + i(t)u_yu_{xxx}\\
& + \frac{1}{2}i(t)u_yu_t,\\
T_2^y =& \frac{1}{72}y^3i'''(t)u_y - \frac{1}{24}y^2i'''(t)u + \frac{1}{12}xyi''(t)u_y - \frac{1}{12}xi''(t)u - \frac{1}{2}yi'(t)u_yu_x - i(t)u_x^3\\
& - \frac{1}{2}i(t)u_y^2 + \frac{1}{2}i(t)u_{xx}^2 - \frac{1}{2}i(t)u_xu_t;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t =& -\frac{1}{24}y^2j''(t)u_x - \frac{1}{12}xj'(t)u_x + \frac{1}{12}j'(t)u + \frac{1}{2}j(t)u_x^2,\\
T_3^x =& \frac{1}{24}y^2j'''(t)u - \frac{1}{4}y^2j''(t)u_x^2 - \frac{1}{12}y^2j''(t)u_{xxx} - \frac{1}{24}y^2j''(t)u_t + \frac{1}{12}xj''(t)u\\
& - \frac{1}{2}xj'(t)u_x^2 + \frac{1}{6}j'(t)u_{xx} - \frac{1}{6}xj'(t)u_{xxx} - \frac{1}{12}xj'(t)u_t + 2j(t)u_x^3 + j(t)u_{xxx}u_x\\
& + \frac{1}{2}j(t)u_y^2 - \frac{1}{2}j(t)u_{xx}^2,\\
T_3^y =& \frac{1}{12}y^2j''(t)u_y - \frac{1}{6}yj''(t)u + \frac{1}{6}xj'(t)u_y - j(t)u_yu_x;
\end{aligned}
$$

$$
\begin{aligned}
T_4^t =& -\frac{1}{2}yn(t)u_x,\\
T_4^x =& \frac{1}{2}yn'(t)u - 3yn(t)u_x^2 - yn(t)u_{xxx} - \frac{1}{2}yn(t)u_t,\\
T_4^y =& yn(t)u_y - n(t)u;
\end{aligned}
$$

$$
\begin{aligned}
T_5^t =& -\frac{1}{2}o(t)u_x,\\
T_5^x =& \frac{1}{2}o'(t)u - 3o(t)u_x^2 - o(t)u_{xxx} - \frac{1}{2}o(t)u_t,\\
T_5^y =& o(t)u_y.
\end{aligned}
$$

Remark. Due to arbitrary functions appearing in the conserved vectors, infinitely many conservation currents can be obtained. Moreover, in terms of physical representation of the obtained conserved vectors, we note that $T = (T_1^t, T_1^x, T_1^y)$ represent the conservation law of energy.

3.4 Conclusion

In this chapter, we studied the potential Kadomtsev-Petviashvili equation (3.1). Using Lie symmetry methods, its Lie point symmetries were computed and then used to obtain exact solutions through symmetry reductions and with the aid of Kudryashov's method. Moreover, its one-parameter groups of transformations and commutator table were given. We later presented 3D, 2D and contour plots to help analyze the diverse nature of each obtained solution. Finally, since the pKP equation has a Lagrangian and zeroth multiplier, we derived its conservation laws using the conservation theorem due to Ibragimov, multiplier method and Noether's theorem. The results obtained in this study are very helpful for characterizing real-world events that the pKP equation models in engineering and applied sciences.

Chapter 4

A study of the Korteweg-de Vries-like equation

In this paper, we study the Korteweg-de Vries-like (KdV-like) nonlinear differential equation of the form

$$2uu_{tx} - 2u_t u_x + 6u_{xx}^2 = 0 \tag{4.1}$$

taken from [48]. Zhang and Ma [48] considered the nonlinear differential equation (NLDE)

$$u_t + \frac{3}{2}u_x^2 + \frac{3}{2}u^2 u_x + \frac{3}{4}u^4 = 0, \tag{4.2}$$

induced from the bilinear KdV-like equation (4.1). By symbolic computation with Maple, they found 29 classes of polynomial solutions to the equation (4.1) and thus constructed two classes of rational solutions of the NLDE (4.2). Liu et al. [49] derived a new NLDE by also considering the generalized bilinear derivative with prime number $p = 3$. In their case [49] the bilinear KdV-like equation is of the form

$$2uu_{tx} - 2u_t u_x + c(2uu_{xx} - 2u_x^2) + 6u_{xx}^2 = 0 \tag{4.3}$$

and is transformed to the NLDE

$$u_t + cu_x + \frac{3}{2}u_x^2 + \frac{3}{2}u^2 u_x + \frac{3}{8}u^4 = 0. \tag{4.4}$$

Similarly, the authors found rational polynomial solutions to the bilinear KdV-like equation (4.3) and as a result obtained three classes of unequal rational solutions to the new derived

NLDE (4.4). The nonlinear KdV-like equation is a significant mathematical model that has numerous applications in quantum mechanics and nonlinear optics and is employed to model traveling waves in shallow water and harmonic crystal. We compute the KdV-like equation's Lie point symmetries first. The commutator table for the Lie point symmetries is then generated, and we use Lie equations to produce one-parameter groups of transformations. Moreover, group-invariant solutions are obtained under each symmetry. Following that, we use the Ibragimov's theorem to derive the conservation laws for the KdV-like equation (4.1).

The contents of this chapter have been prepared as a paper which has been submitted for publication [50]

4.1 Lie point symmetries of (4.1)

The vector field

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{4.5}$$

is a Lie point symmetry of (4.1) whenever

$$X^{[2]}(2uu_{tx} - 2u_t u_x + 6u_{xx}^2)\big|_{(4.1)} = 0, \tag{4.6}$$

where $X^{[2]}$ is the second prolongation defined as

$$X^{[2]} = \eta\frac{\partial}{\partial u} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}.$$

Expanding (4.6) we obtain

$$\eta(2u_{tx}) + \zeta_x(-2u_t) + \zeta_t(-2u_x) + \zeta_{xx}(12u_{xx}) + \zeta_{tx}(2u)\big|_{(4.1)} = 0. \tag{4.7}$$

Substituting the values of ζ_x, ζ_t, ζ_{xx} and ζ_{tx} in equation (4.7) and replacing u_{tx} by $(u_t u_x - 3u_{xx}^2)/u$ we obtain

$$\begin{aligned}
&\frac{2}{u}u_t u_x\eta - \frac{6}{u}u_{xx}^2\eta - 2u_t u_x\eta_u + 2u_t u_x\tau_t + 2u_t^2 u_x\tau_u + 2u_x^2\xi_x + 4u_t u_x^2\xi_u \\
&-2u_t\eta_x - 2u_t u_x\eta_u + 2u_t^2\tau_x + 2u_t u_x\tau_u + 2u_t u_x\xi_x + 12u_{xx}\eta_{xx} + 24u_x u_{xx}\eta_{xu} \\
&+12u_{xx}^2\eta_u + 12u_x^2 u_{xx}\eta_{uu} - 24u_{xx}^2\xi_{xx} - 12u_x u_{xx}\xi_{xx} - 24u_x^2 u_{xx}\xi_{xu}
\end{aligned}$$

$$\begin{aligned}
&-36u_xu_{xx}^2\xi_u-12u_x^3u_{xx}\xi_{uu}-\frac{24}{u}u_tu_xu_{xx}\tau_x+\frac{72}{u}u_{xx}^3\tau_x-12u_tu_{xx}\tau_{xx}\\
&-24u_tu_xu_{xx}\tau_{xu}-12u_tu_{xx}^2\tau_u-\frac{24}{u}u_tu_x^2u_{xx}\tau_u+\frac{72}{u}u_xu_{xx}^3\tau_u-12u_tu_x^2u_{xx}\tau_{uu}\\
&+2u\eta_{tx}+2u\eta_{tu}+2uu_t\eta_{xu}+2u_tu_x\eta_u-6u_{xx}^2\eta_u+2uu_tu_x\eta_{uu}-2u_tu_x\tau_x\\
&+6u_{xx}^2\tau_x-2u_tu_x\xi_x+6u_{xx}^2\xi_x-2uu_t\tau_{tx}-2uu_{tt}tau_x-2uu_tu_x\tau_{tu}\\
&-2uu_tu_x\xi_{xu}-2uu_t^2\tau_{xu}-4u_t^2u_x\tau_u+12u_tu_{xx}\tau_u-2uu_xu_{tt}\tau_u-2uu_t^2u_x\tau_{uu}\\
&-2uu_x\xi_x-2uu_{xx}\xi_t-2uu_x^2\xi_{tu}-4u_tu_x^2\xi_u+12u_xu_{xx}^2\xi_u-2uu_tu_{xx}\xi_u\\
&-2uu_tu_x^2\xi_{uu}=0.
\end{aligned}\tag{4.8}$$

Since τ, ξ and η depend only on t, x and u we split on the derivatives of u to obtain a system of linear homogeneous PDEs:

$$u_{tt} \ : \ \tau_x = 0, \tag{4.9}$$

$$u_xu_{tt} \ : \ \tau_u = 0, \tag{4.10}$$

$$u_tu_{xx} \ : \ \xi_u = 0, \tag{4.11}$$

$$u_x^2 \ : \ \xi_t = 0, \tag{4.12}$$

$$u_t \ : \ \eta_x = 0, \tag{4.13}$$

$$u_tu_x \ : \ 2u\tau_t - 3\eta = 0, \tag{4.14}$$

$$u_{xx}^2 \ : \ 2u\xi_x - \eta = 0, \tag{4.15}$$

$$1 \ : \ u\eta_u - \eta = 0. \tag{4.16}$$

The above system is now solved for τ, ξ and η. From equations (4.9) and (4.10) we have

$$\tau = a(t), \tag{4.17}$$

where a is an arbitrary function of t. Equation (4.11) and (4.12) implies that

$$\xi = b(x), \tag{4.18}$$

where b is an arbitrary function of x. Substituting the values of τ and ξ in equation (4.14) and (4.15) we get, respectively

$$2ua'(t) - 3\eta = 0,$$

$$2ub'(x) - \eta = 0.$$

The above equations imply that

$$3b'(x) = a'(t)$$

and since the left-hand side of the above equation is a function of x only and right-hand side is a function of t only, then both must be equal to some constant, say C_1. Thus,

$$\begin{aligned} 3b'(x) &= C_1, \\ a'(t) &= C_1. \end{aligned}$$

Solving the above equations we obtain

$$\begin{aligned} b(x) &= \frac{1}{3}C_1 x + C_2, \\ a(t) &= C_1 t + C_3, \end{aligned}$$

where C_2 and C_3 are arbitrary constants of integration. Now from equation (4.16) we get

$$\eta = uc(t, x),$$

where c is an arbitrary function of t and x. Finally, substituting the value of η into equation (4.13) leads to $c_x = 0$ which implies that $c = c(t)$. Thus,

$$\tau = C_1 t + C_3, \tag{4.19}$$

$$\xi = \frac{1}{3}C_1 x + C_2, \tag{4.20}$$

$$\eta = uc(t), \tag{4.21}$$

and so the Lie point symmetries of the KdV-like equation (4.1) are

$$X_1 = \frac{\partial}{\partial t}, \tag{4.22}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{4.23}$$

$$X_3 = uc(t)\frac{\partial}{\partial u}, \tag{4.24}$$

$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{4.25}$$

which form an infinite dimensional Lie algebra. For simplicity we let $c(t) = 1$.

4.1.1 Commutator table for the symmetries of (4.1)

We now calculate the commutation relations for all the symmetry generators of (2.2) obtained above. Firstly, we compute $[X_3, X_4]$. By the definition of Lie bracket we have

$$\begin{aligned}
[X_3, X_4] &= X_3X_4 - X_4X_3 \\
&= ut\frac{\partial}{\partial u}\left(x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}\right) - \left(x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}\right)ut\frac{\partial}{\partial u} \\
&= -3ut\frac{\partial}{\partial u} \\
&= -3X_3,
\end{aligned}$$

Likewise, we obtain the commutation relations between other Lie point symmetries.

Table 4.1: Commutator table of Lie algebra of the KdV-like equation (4.1)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	$3X_1$
X_2	0	0	X_3	X_2
X_3	0	0	0	$-3X_3$
X_4	$-3X_1$	$-X_2$	$3X_3$	0

4.1.2 One-parameter groups of transformations

We now employ the Lie equations together with initial conditions

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}),\ \bar{t}|_{a=0} = t,\ \ \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}),\ \bar{x}|_{a=0} = x,\ \ \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}),\ \bar{u}|_{a=0} = u$$

to compute the one-parameter group of transformations. For each X_i, let G_{a_i} be the corresponding group. We first compute one-parameter group corresponding to infinitesimal generator X_3, namely

$$X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 3\bar{t},\ \bar{t}|_{a=0} = t,\ \ \frac{d\bar{x}}{da} = \bar{x},\ \bar{x}|_{a=0} = x,\ \ \frac{d\bar{u}}{da} = -\bar{u},\ \bar{u}|_{a=0} = u.$$

Solving the above equations, we obtain

$$\bar{t} = t,\ \ \bar{x} = xe^{a_3},\ \ \bar{u} = ue^{-a_3}.$$

Thus, the one-parameter group G_{a_4} corresponding to the operator X_4 is given by

$$G_{a_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, xe^{a_3}, ue^{-a_3}).$$

If we continue in the same manner as above, we get the following one-parameter groups for all the operators:

$$\begin{aligned} G_{a_1} &: \ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u), \\ G_{a_2} &: \ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u), \\ G_{a_3} &: \ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, ue^{a_3}), \\ G_{a_4} &: \ (\bar{t}, \bar{x}, \bar{u}) \longrightarrow \Big(te^{3a_4}, xe^{a_4}, u\Big). \end{aligned}$$

Since each group G_i is a symmetry group, if $u = f(t, x)$ is a solution of (4.1) so are the functions

$$\begin{aligned} u_1 &= f(t, x - a), \\ u_2 &= f(t - a, x), \\ u_3 &= e^{a} f(t, x), \\ u_4 &= f\left(te^{-3a}, xe^{-a}\right). \end{aligned}$$

Constructing group-invariant solutions of (4.1)

We now utilize the optimal system of one-dimensional subalgebras obtained in the previous subsection and find symmetry reductions and group invariant solutions for equation (4.1).

Case 1. $X_1 = \partial/\partial t$

Firstly, we consider the symmetry operator $X_1 = \partial/\partial t$. The characteristic equations for the operator X_1 are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which lead to two invariants $J_1 = x$ and $J_2 = u$. Thus, $u = f(x)$, where f is the invariant solution. Substituting this value of u into (4.1), we obtain the ODE

$$f''(x) = 0$$

Integrating the above equation twice with respect to x we get

$$f(x) = C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants of integration. Thus, the group-invariant solution under X_1 is

$$u(t,x) = C_1 x + C_2. \tag{4.26}$$

We present the solution profile of (4.26) for the parameters $C_1 = C_2 = 1$, in Figure (4.1).

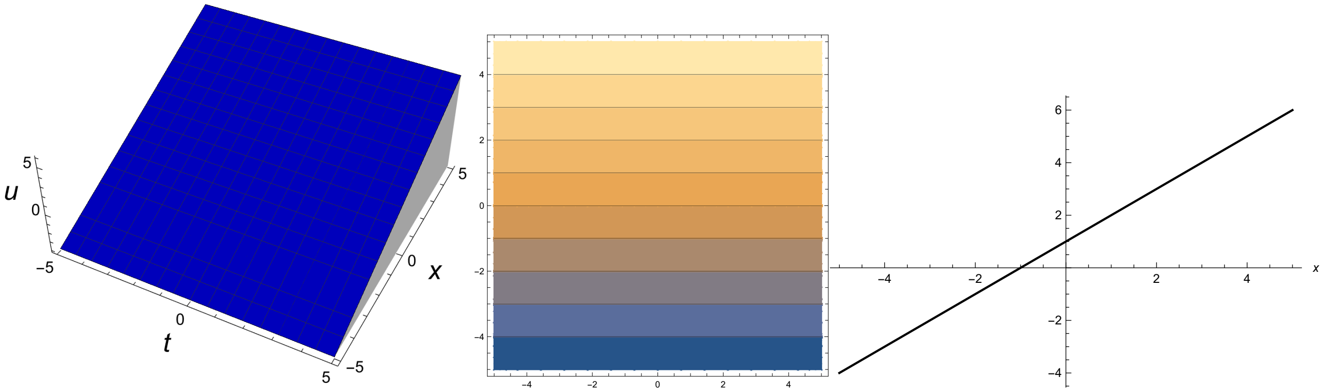

Figure 4.1: 3D and 2D profile of solution (4.26)

Case 2. $X_2 = \partial/\partial x$

Next, we consider the Lie point symmetry $X_2 = \partial/\partial x$. The characteristic equations associated with the operator X_2 are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0},$$

which give two invariants $J_1 = x$ and $J_2 = u$. Thus, the group-invariant solution is given by $J_2 = f(J_1)$. This gives $u = f(t)$, where f is an arbitrary function. Substituting this value of u into (4.1), leads to the conclusion that the group-invariant solution of (4.1) under X_2 is

$$u(t,x) = C.$$

Case 3. We now consider the Lie point symmetry $X_3 = ut\partial/\partial u$. The characteristic equations associated with the above operator are

$$\frac{dt}{0} = \frac{dx}{0} = \frac{du}{ut},$$

that leads to two invariants $J_1 = t$ and $J_2 = x$. Thus, under X_3 (4.1) does not have an invariant solution.

Case 4. $X_4 = 3t\partial/\partial t + x\partial/\partial x$**.**

We consider the Lie point symmetry generator $X_4 = 3t\partial/\partial t + x\partial/\partial x$. The associated characteristic equations to X_4 are

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{du}{0}.$$

The characteristic equations yield the two invariants $J_1 = x/t^{1/3}$ and $J_2 = u$. Hence the group-invariant solution is given by $J_2 = f(J_1)$, where f is an arbitrary function. This implies

$$u(t,x) = f(\Psi),\ \Psi = \frac{x}{t^{1/3}}.$$

Substituting this value of u into (2.2), we obtain

$$9f''^2 - ff' - \Psi ff'' + \Psi f'^2 = 0. \tag{4.27}$$

Consequently, the solution of (4.1) under X is $u = f(\Psi)$, where f satisfies

$$9f''^2 - ff' - \Psi ff'' + \Psi f'^2 = 0.$$

Case 5. ***(Traveling wave solutions)*****:** $X_1 + cX_2 = \partial/\partial t + c\partial/\partial x$

We consider the linear combination $\Gamma = X_1 + cX_2$, to obtain traveling wave solutions i.e.,

$$X = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

The associated characteristic equations to X are

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}.$$

Thus, one invariant is $J_1 = u$. The other invariant is obtained by taking

$$\frac{dt}{1} = \frac{dx}{c}$$

which gives us $J_2 = x - ct$. Hence the group-invariant solution is given by $J_1 = f(J_2)$, where f is an arbitrary function. This implies

$$u(t,x) = f(\xi),\ \xi = x - ct.$$

Substituting this value of u into (4.1), we obtain

$$3f''^2 + cf'^2 - cff'' = 0. \tag{4.28}$$

Thus, the solution of (4.1) under X is $u = f(\xi)$, where f satisfies

$$3f''^2 + cf'^2 - cff'' = 0.$$

Case 6. $X_1 + \varepsilon X_3 = \partial/\partial t + \varepsilon\partial/\partial u$.

From the obtained Lie point symmetries, we take the following linear combination $\Gamma = X_1 + \varepsilon X_3$ to obtain an exact solution for (4.1). The above linear combination gives the invariants

$$\begin{aligned} J_1 &= x, \\ J_2 &= ue^{-\varepsilon t}, \end{aligned}$$

and thus, the invariant solution

$$u = e^{\varepsilon t}F(x).$$

Substituting this value of u into (4.1), we obtain

$$F'' = 0.$$

Solving the above equation, we get

$$F(x) = C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants of integration. Thus, the invariant solution of (4.1) under Γ is

$$u = e^{\varepsilon t}(C_1 x + C_2), \tag{4.29}$$

The solution profile of (4.29) under the parameters $\varepsilon = C_1 = C_2 = 1$, is presented in Figure (4.2).

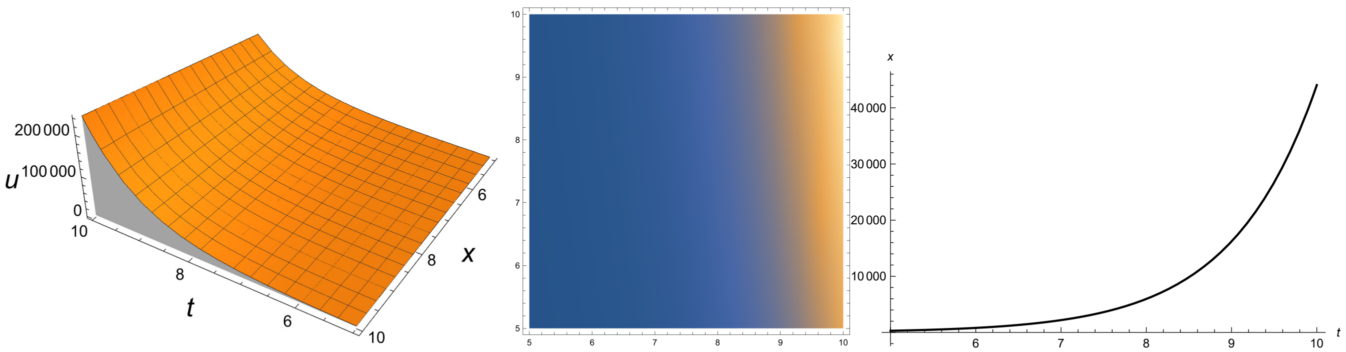

Figure 4.2: 3D and 2D profile of solution (4.29)

4.2 Conservation laws for (4.1)

In this section, we compute the conserved vectors of the KdV-like equation (4.1). Unlike the pKP equation (3.1), the KdV-like equation does not have a Lagrangian and multiplier, hence we apply the conservation theorem due to Ibragimov which has been outlined in Chapter 1.

For the KdV-like equation (4.1) the Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_t \frac{\partial}{\partial u_t} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + \cdots . \tag{4.30}$$

The adjoint equation for (4.1) can be obtained from the formula

$$F^* = \frac{\delta}{\delta u} \left[v(2uu_{tx} - 2u_t u_x + 6u_{xx}^2) \right] = 0, \tag{4.31}$$

where $v = v(t, x)$, and this yields

$$F^* = 12v_{xx}u_{xx} + 24v_x u_{xxx} + 12vu_{xxxx} + 4v_x u_t + 4v_t u_x + 8vu_{tx} + 2v_{tx} = 0. \tag{4.32}$$

Consider equation (4.1) and the adjoint equation (4.32) as a system, the Lagrangian for this system from (1.44) is,

$$\mathcal{L} = v(2uu_{tx} - 2u_t u_x + 6u_{xx}^2). \tag{4.33}$$

Now, equation (4.33) admits all the symmetries of (4.1) extended to the new variable $v(t, x)$. That is, the generators become

$$Y = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \eta^* \frac{\partial}{\partial v} \tag{4.34}$$

with

$$\eta^* = \eta^*(t, x, u, v) = -\left\{\lambda + D_t(\tau) + D_x(\xi)\right\} v. \tag{4.35}$$

For this particular problem we determine the parameter λ by using

$$X^{[2]}(F) = \lambda F, \tag{4.36}$$

where $F = 2uu_{tx} - 2u_t u_x + 6u_{xx}^2$ and $X^{[2]}$ is the second prolongation of X, given by

$$X^{[2]} = X + \zeta_x \frac{\partial}{\partial u_x} + \zeta_t \frac{\partial}{\partial u_t} \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}}. \tag{4.37}$$

The conservation laws of the system of equations (4.1) and (4.32), associated with a symmetry can be obtained by applying the following rules given by [33] due to the existence of the cross term u_{tx}:

$$\begin{aligned} T^t &= \tau \mathcal{L} + W \left\{ \frac{\partial \mathcal{L}}{\partial u_t} - D_x \left(\frac{\partial L}{\partial u_{xt}} \right) \right\}, \\ T^x &= \xi \mathcal{L} + W \left\{ \frac{\partial \mathcal{L}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right\} + W_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) + W_t \frac{\partial \mathcal{L}}{\partial u_{xt}}, \end{aligned} \tag{4.38}$$

where

$$W = \eta - \tau u_t - \xi u_x. \tag{4.39}$$

Case 1. For $X_1 = \partial/\partial t$ the adjoint generator similarly retains the same form $Y_1 = \partial/\partial t$. The Lie characteristic function is then given by $W_1 = -u_t$. The conserved vector is then given by

$$\begin{aligned} T_1^t &= 2uu_t v_x + 6u_{xx}^2 v + 2u_x u_t v + 2uu_{tx} v, \\ T_1^x &= 12u_{xx} u_t v_x + 2u_t^2 v + 12u_{xxx} u_t v - 12u_{xx} u_{tx} v - 2uu_{tt} v. \end{aligned}$$

Case 2. For the symmetry $X_2 = \partial/\partial x$ we have $\xi = \eta = 0$. Thus, $\zeta_x = \zeta_t = \zeta_{xx} = \zeta_{tx} = 0$. We then get, $X_2^{[2]} F = 0F$, that is, $\lambda = 0$. From (4.35) we obtain $\eta^* = 0$, whereby the new generator retains the form of X_2, that is, $Y_2 = \partial/\partial x$. The Lie characteristic function is $W_2 = -u_x$. Thus, the conserved vector from (4.38) is

$$\begin{aligned} T_2^t &= 2uu_x v_x + 4u_x^2 v, \\ T_2^x &= 12u_x u_{xx} v_x - 6u_{xx}^2 v + 12u_x u_{xxx} v. \end{aligned}$$

We obtain the rest of the conserved vectors in similar fashion.

Case 3. For symmetry $X_3 = u\partial/\partial u$ we get the adjoint operator $Y_3 = 2v\partial/\partial v$ and Lie characteristic function $W_3 = u$. Thus, the conserved vector is

$$\begin{aligned} T_3^t &= -4uu_xv - 2u^2v_x, \\ T_3^x &= 12u_xu_{xx}v - 12uu_{xx}v_x - 12uu_{xxx}v. \end{aligned}$$

Case 4. Finally for the symmetry $X_4 = 3t\partial/\partial t + x\partial/\partial x$ the new generator is $Y_4 = 3t\partial/\partial t + x\partial/\partial x$ and $W_4 = -(xu_x + 3tu_t)$. Hence

$$\begin{aligned} T_4^t &= 2xuu_xv_x + 6tuu_tv_x + 4xu_x^2v + 6tu_tu_xv + 18tu_{xx}^2v + 6tuu_{tx}v, \\ T_4^x &= 12xu_xu_{xx}v_x + 36tu_tu_{xx}v_x - 6xu_{xx}^2v - 12u_xu_{xx}v - 36tu_{tx}u_{xx}v \\ &\quad + 6tu_t^2v + 12xu_xu_{xxx}v - 6uu_tv + 36tu_{xxx}u_tv - 6tuu_{tt}v. \end{aligned}$$

Remark. The conserved vectors involve solutions u of the KdV-like equation and solutions v of the adjoint equation (4.32) and hence yield an infinite number of conservation laws. In terms of physical interpretation of the obtained conserved vectors we observe that for case 1, 2 and 4 we have conservation of energy, linear momentum and mass respectively.

4.3 Conclusion

In this chapter, the Korteweg-de Vries-like equation (4.1) was investigated from the Lie symmetry point of view. Infinitesimal generators and commutator table for the KdV-like equation are well presented. Utilising each symmetry, we perform symmetry reductions and group-invariant solutions of the KdV-like equation are obtained. In addition, using Lie equations, we constructed the one-parameter group of transformations. Finally, conserved vectors were derived using the conservation theorem due to Ibragimov.

Chapter 5

Concluding remarks and future work

The purpose of the work done in this dissertation was to apply Lie symmetry methods to obtain analytic solutions and conserved vectors of the potential Kadomtsev-Petviashvili and Korteweg-de Vries-like equations.

In Chapter 1 theoretical justification of principles relevant to this dissertation were given. This included definitions and theorems for various concepts in Lie symmetry analysis, conservation laws, and certain methods for deriving analytic solutions of NLPDEs.

In Chapter 2 we gave an illustrative example on how some of the methods defined in Chapter one were going to be used in this dissertation through the study of the Boltzmann equation. Lie point symmetries were determined and used to compute the commutator table. Under each infinitesimal generator, group-invariant solutions for the Boltzmann equation were obtained. A graphical representation of each solution was given. The multiplier method, Noether's theorem and Ibragimov's method were employed to derive conservation laws of this equation.

In Chapter 3 we examined the potential Kadomtsev-Petviashvili equation by first computing the infinitesimal symmetries of the equation. We proceeded into using the obtained symmetries to construct the commutator table. Thereafter we performed similarity reductions and obtained exact solutions with the aid of Kudryashov's method. The type of solutions obtained were discussed and shown graphically. Moreover, conservation laws of this equation were derived by applying Ibragimov's theorem, multiplier approach and Noether's theorem.

In the fourth Chapter, we studied the Korteweg-de Vries-like equation from the point of view of Lie symmetry analysis. Similarity reductions and exact solutions were obtained. Finally, local conservation laws for the KdV-like equation were derived by employing the conservation theorem by Ibragimov.

In future work, we plan to use the conserved vectors obtained here to construct exact solutions for the potential Kadomtsev-Petviashvili and Korteweg-de Vries-type equations which were studied in this dissertation via the double reduction theory.

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