Entropy in asymmetric topological structures

P Haihambo

orcid.org 0000-0001-6534-4099

Thesis accepted in fulfilment of the requirements for the degree *Doctor of Philosophy in Science with Mathematics* at the North-West University

Promoter: Prof O Olela Otafudu

Graduation December 2023

Abstract

The notion of entropy first appeared in the context of thermodynamics in the first half of the XIX century. The topological entropy and the uniform entropy have been largely studied in the last 60 years.

The main aim of this thesis is to investigate the notion of entropy in asymmetric spaces. More specifically, we are interested in extending the notion of entropy on metric and uniform spaces to quasi-metric and quasi-uniform spaces. In this thesis we managed to generalize most of the results about uniform entropy on metric(uniform) spaces to quasi-metric(uniform) spaces. Indeed a new notion of entropy for a uniformly continuous self-map ψ of quasi-metric(uniform) space has been presented, which we call quasi-uniform entropy in this thesis. We managed to show that the quasi-uniform entropy is less or equal to the uniform entropy of ψ considered as a uniformly continuous self-map of the metric(or uniform) space (X, q^s) (or $(X, \mathcal{U}^s)$), where q^s(or $\mathcal{U}^s$) is the symmetrised metric(or uniformity) of the quasi-metric(or uniformity) q(or $\mathcal{U}$).

We prove that for a join-compact quasi-metric(uniform) space the quasi-uniform entropy of a uniformly continuous self-map coincides with the quasi-uniform entropy of its extension to the bicompletion. Finally, we compared our notion of entropy namely, quasi-uniform entropy, to some well-known notions of topological entropy.

Keywords: quasi-metric(uniform) space, uniformly continuous self-map, quasi-uniform entropy, q-totally bounded, $\mathcal{U}$-totally bounded, join-compact, bicompletion, separated.

Acknowledgements

I wish to convey my sincere gratitude to my supervisor, Professor Olivier Olela Otafudu for his infinite support, patience, stimulating discussions on the subject and advices he has given me throughout my studies.

I express my gratitude to the School of Mathematical and Statistical Sciences at the North West University, and in particular the Topology Research Group for opportunities offered during my studies.

I would like to thank the North West University for providing me with a doctoral bursary. Further financial support was provided by Faculty of Natural and Agricultural Sciences at the North West University. Finally, I acknowledge financial support from the University of Namibia.

My sincere appreciation goes to my friend, Dr. Zechariah Mushaandja for his suggestions and comments he made after going through this thesis.

To my family and friends, thank you for all the support and encouragement.

Dedication

I dedicate this thesis to my late parents, Lucia Nghidengwa-Haihambo and Jesaya Haihambo, to my lovely wife Ingayamwena Amutse-Haihambo and my precious daughter Rhema Perfecta Haihambo, for their patience, motivation and understanding. To my family, friends and Namibians at large, who are willing to carry out any scientific work.

Declaration

I, PAULUS HAIHAMBO

hereby declare that this thesis is my own unaided work and it is hereby being submitted for the degree of Doctor of Philosophy in Science with Mathematics at the North West University. It has not been submitted for any degree or examination in any other university.

Contents

Abstract **i**

Acknowledgements **ii**

Dedication **iii**

Declaration **iv**

Introduction **1**

1 Preliminaries **6**

1.1 Quasi-metric spaces 6

1.2 Quasi-uniform spaces 10

2 The topological entropy **17**

2.1 Topological entropy on compact spaces 17

2.2 Topological entropy on non-compact spaces 20

2.3 Uniform entropy 20

3 Entropy on quasi-metric spaces **25**

3.1 Quasi-uniform entropy on a quasi-metric space 26

3.2 Basic properties of the quasi-uniform entropy on a quasi-metric space 32

3.3 Relationship between the uniform and quasi-uniform entropy on a quasi-metric space 43

3.4 Completion theorem for quasi-uniform entropy on a quasi-metric space 47

4 Entropy on quasi-uniform spaces **51**

4.1 Quasi-uniform entropy on a quasi-uniform space 51

4.2 Basic properties of the quasi-uniform entropy on a quasi-uniform space 60

4.3 Quasi-uniform entropy following Kimura's approach 73

4.4 Completion theorem for quasi-uniform entropy on a quasi-uniform space 75

5 Quasi-uniform entropy vs topological entropy **81**

5.1 Quasi-uniform entropy vs topological entropy on compact quasi-uniform spaces 82

5.2 Quasi-uniform entropy vs topological entropy on non-compact quasi-uniform spaces 86

6 Conclusion **89**

6.1 Summary of the achieved work 90

6.2 Some possible areas for future work 92

Bibliography **93**

Introduction

The notion of entropy first appeared in the context of thermodynamics in 1850 by Rudolf Clausius as a measure of disorder/randomness of a given system. In mathematics, the notion of entropy was first introduced by Shannon in the 1930's in the context of information theory. In the 1950's Kolmogorov and Sinai defined the measure entropy which played a fundamental role in the ergodic theory of dynamical systems. In 1965 Adler, Konheim and McAndrew [1] defined the topological entropy $h(\psi)$ of a continuous self-map ψ of a compact space. Independently in 1970 Dinaburg [6] and in 1971 Bowen [2] defined another notion of entropy for uniformly continuous self-maps of metric spaces. Bowen [2] gave the definition when the metric space is not necessarily compact, which we shall call uniform entropy or Bowen's entropy and denoted by h_U in this thesis. Bowen's entropy coincides with the topological entropy when the metric space is compact. In 1974 Hood [15] and Hofer [14] extended Bowen's entropy to uniform spaces. A shortcoming with the uniform entropy h_U is the fact that it is defined using compact sets, so that in uniform spaces with few compact sets (see for instance Remark 4.1.3) the uniform entropy of each uniformly continuous self-map will be zero regardless of how complicated the map is. This motivated Kimura [16] to propose a modified version h_K of the uniform entropy for uniform spaces using totally bounded sets instead of compact ones and he proved among other things that completing a totally bounded uniform space does not change the entropy of the space, in the sense that if ψ is a uniformly continuous self-map of a totally bounded uniform space $(X,\mathcal{U})$ and $(\widetilde{X},\widetilde{\mathcal{U}})$ is the completion of $(X,\mathcal{U})$ and $\widetilde{\psi}$ is the uniformly continuous extension of ψ, then $h_K(\psi) = h_K(\widetilde{\psi})$ (see [16, Theorem 5.3]). We call this result the completion theorem.

For a good starting point on the concepts of topological entropy in mathematics we recommend Peter Walters book [21]. For more recent developments on the notion of topological entropy we recommend [7, 8, 17], where [7] is concerned with a unifying approach to the study of entropies in mathematics, [8] is a survey paper on uniform entropy for uniform spaces and topological groups, and [17] concerns the connection between uniform entropy and topological entropy.

More recently Sayyari, Molari and Moghayer [20] defined a notion of entropy for a continuous self-map of a quasi-metric space (X, q). It must be noted that their entropy is computed via join-compact subsets for a continuous self-map of the symmetrised metric space (X, q^s) (see the second last sentence of the introduction on page 2 of their paper [20]), i.e., their notion of entropy is computed within the metric space (X, q^s), where q^s is the symmetrised metric on X of the quasi-metric q. This motivated us to study the notion of entropy for a uniformly continuous self-map of a quasi-metric(uniform) space $(X, q)((X,\mathcal{U}))$ using $\tau(q)(\tau(\mathcal{U}))$-compact (not necessarily join-compact) subsets of X.

In this thesis, modifying the notion of entropy by Bowen [2], we introduce a new notion of entropy , which we call quasi-uniform entropy (denoted by $h_{QU}(\psi)$) of a uniformly continuous self-map ψ on a quasi-metric(uniform) space $(X, q)((X,\mathcal{U}))$. With this new notion of quasi-uniform entropy, we manage to extend most of the results about uniform entropy on metric(uniform) spaces to quasi-metric(uniform) spaces. By using Kimura's idea of the completion theorem, we manage to prove the completion theorem in the class of all join-compact quasi-metric(uniform) spaces, that is for join-compact quasi-metric(uniform) spaces the entropy of a uniformly continuous self-map coincides with the entropy of its extension to the bicompletion. Finally we compared our notion of entropy to some of the well-known notions of topological entropy.

A Brief Outline Of The Thesis

This thesis is arranged as stated below.

Chapter 1: This chapter is dedicated to a brief overview of certain well-known results about quasi-metric and quasi-uniform spaces and it follows the outline of [5, 18]. An interesting example of a quasi-metric on $\mathbb{R}$ (see Example 1.1.1) to be used more frequently in the thesis is presented.

Chapter 2: In this chapter we gather all the well-known results about topological entropy which are relevant to this thesis. Indeed we recall the definition of topological entropy $h(\psi)$ given by Adler, Konheim and McAndrew [1] for a continuous self-map ψ of a compact topological space X and in the case of a continuous self-map ψ of a non-compact topological space X, we use the definition of topological entropy $h_f(\psi)$ given by Hofer [14] in terms of finite open covers. We recall that for a continuous self-map ψ of a topological space X, we have that $h_f(\psi) \leq h(\psi)$ with equality if the underlying space X is compact (see Remark 2.2.1). We also recall the notion of topological entropy $h_U(\psi)$ for a uniformly continuous self-map ψ of a metric(or uniform) space X defined in [2] for a metric space and [15] for a uniform space (which we call uniform entropy in this thesis). Most of the materials in this chapter are based on [1, 2, 14, 15, 21].

Our contribution is contained in chapters 3, 4 and 5.

Chapter 3: In this chapter we start our own investigation into the notion of entropy for uniformly continuous self-maps of quasi-metric spaces. In the first section we begin with a scheme which will allow us to eventually extend Bowen's uniform entropy to quasi-metric spaces. In the second section we manage to show that most of the basic results about uniform entropy for metric spaces easily extends to quasi-uniform entropy for quasi-metric spaces. In the third section we managed to establish the relationship between the uniform entropy and the quasi-uniform entropy. Indeed we managed to show that $h_{QU}(\psi, q) \leq h_U(\psi, q^s)$, where ψ is a uniformly continuous self-map of a quasi-metric space (X, q) (see Theorem 3.3.1). The fourth section deals with the completion theorem. Indeed we showed that for a join-compact quasi-metric space the quasi-uniform entropy of a uniformly continuous self-map coincides with the quasi-uniform entropy of its extension to the bicompletion (see Theorem 3.4.1). Most of the materials in this chapter are extracted from our recent paper [11].

Chapter 4: In this chapter we continue with our investigation on the notion of entropy in asymmetric spaces. Indeed we managed to extend our recent results in [11] on the notion of quasi-uniform entropy to quasi-uniform spaces. The first section is devoted to defining quasi-uniform entropy $h_{QU}(\psi)$ of a uniformly continuous self-map ψ of a quasi-uniform space $(X,\mathcal{U})$. In the second section we managed to extend most of our results in section 3.2 to quasi-uniform spaces. We finish off this section by establishing the relationship between the quasi-uniform entropy h_{QU} and the Kimura's entropy h_K (see Proposition 4.2.4). In the third section following Kimura's approach, we introduced a modified version h_{QUK} (which we call *quasi-uniform K-entropy*) of the quasi-uniform entropy for quasi-uniform spaces using $\mathcal{U}$-totally bounded subsets instead of compact ones. We managed to establish the relationships between the quasi-uniform entropy h_{QU} and the quasi-uniform K-entropy h_{QUK} (see Theorems 4.3.1 and 4.3.2). We also managed to extended Theorem 3.3.1 to quasi-uniform spaces (see Proposition 4.3.2). Finally, in the fourth section we managed to prove among other things the completion theorem for quasi-uniform spaces (see Theorem 4.4.2), hence extending Theorem 3.4.1 to quasi-uniform spaces. Most of the materials in this chapter are extracted from our recent paper [12].

Chapter 5: In this chapter we continue our investigations by comparing our notion of quasi-uniform entropy to certain well-known notions of topological entropy, namely the topological entropy defined in [1, 14, 15] and [16]. The first section is devoted to studying the relationship between quasi-uniform entropy and topological entropy on compact quasi-uniform spaces. Indeed we managed to establish the relationship between the quasi-uniform entropy function h_{QU} and the topological entropy function h for (join-)compact quasi-uniform spaces (see Theorem 5.1.1). Finally, in the second section the comparison is carried out on non-compact quasi-uniform spaces. In particular, we managed to establish the relationship between the quasi-uniform entropy function h_{QU} and the topological entropy function h_f for T_2-quasi-uniform spaces (see Theorem 5.2.1).

Chapter 6: In this final chapter we conclude our investigations by reflecting on the main results of the thesis and point out some connections of this current work with the established work in the literature. Furthermore, we present some open problems which can constitute some topics for further research. The study of entropy in asymmetric spaces leads to many open problems. For instance one could investigate the notion of entropy in asymmetrically normed linear spaces, quasi-soft metric spaces, quasi-ultra metric spaces and other asymmetric structures which are not investigated in this thesis.

Chapter 1

Preliminaries

In this chapter we give a brief overview of notation, terminology and some well-known basic concepts from the theory of quasi-metric and quasi-uniform spaces to be used throughout the thesis. We present an interesting example of a quasi-metric space (see Example 1.1.1) which will be used more frequently in the thesis. For more on these basics, the reader is referred to, among others, [5, 9, 18].

1.1 Quasi-metric spaces

In this section we are going to recall some basic facts from the theory of quasi-metric spaces which are relevant to the materials presented in this thesis.

1.1.1 Definition. *Let X be a set and let $q : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of nonnegative reals.*

Then q is called a quasi-metric *on X if for any $x, y, z \in X$,*

(a) $q(x, y) = 0 = q(y, x)$ *implies* $x = y$,

(b) $q(x,y) \leq q(x,z) + q(z,y)$.

The ordered pair (X, q) is called a *quasi-metric space*.

If q is a quasi-metric on X, then the function $q^t : X \times X \to [0, \infty)$ defined by $q^t(x,y) = q(y,x)$ for all $x, y \in X$ is also a quasi-metric on X, often called the *conjugate quasi-metric* of q. As usual a quasi-metric q on X such that $q = q^t$ is called a *metric*.

The symmetrised metric of q is the function $q^s : X \times X \to [0, \infty)$ given by $q^s(x,y) = \max\{q(x,y), q(y,x)\} = q(x,y) \vee q^t(x,y)$ whenever $x, y \in X$.

(Here $\vee$ is used to denote a supremum, i.e., for any $x, y \in X, q(x,y) \vee q^t(x,y) = \sup\{q(x,y), q^t(x,y)\}$). It is easy to see that q^s is indeed a metric on X and $(q^t)^s = q^s$.

Let (X, q) be a quasi-metric space. For each $x \in X$ and $\epsilon > 0$,

$$B_q(x, \epsilon) = \{y \in X : q(x,y) < \epsilon\}$$

denotes the open ϵ-ball at x. The collection of all open ϵ-balls yields a base for a topology $\tau(q)$ on X, (i.e., a subset G of X belongs to $\tau(q)$ if and only if for each $x \in G$ there exists some $\epsilon = \epsilon_x > 0$ such that $B_q(x, \epsilon) \subseteq G$). It is called the *topology induced by q on X*.

Similarly for each $x \in X$ and $\epsilon > 0, C_q(x, \epsilon) = \{y \in X : q(x,y) \leq \epsilon\}$. It must be noted that this set is $\tau(q^t)$-closed, but not $\tau(q)$-closed in general. Also, the following inclusions hold

$$B_{q^s}(x, \epsilon) \subseteq B_q(x, \epsilon) \text{ and } B_{q^s}(x, \epsilon) \subseteq B_{q^t}(x, \epsilon),$$

with similar inclusions for the closed balls.

Hence for any quasi-metric space (X,q), we have that $\tau(q), \tau(q^t) \subseteq \tau(q^s)$. In particular, $\tau(q^s) = \tau(q) \vee \tau(q^t)$.

Throughout this thesis any topological statement about a (quasi-)metric space (X,q) will refer to the topology $\tau(q)$, unless otherwise stated.

If τ is a topology on X, such that $\tau = \tau(q)$ for some quasi-metric q on X, then X is said to be *quasi-metrizable*. Observe that if q is a quasi-metric on X, then $\tau(q)$ is a T_0-topology on X. It is for this reason that recently some authors (see for instance [18]) call quasi-metric spaces T_0-quasi-metric spaces.

For an arbitrary quasi-metric q on X, there is no known direct link between the topologies $\tau(q)$ and $\tau(q^t)$, except the following result.

1.1.1 Remark. *(compare [18, Remark 2.1.5]) Let q be a quasi-metric on a set X. If $\tau(q)$ has a base $\mathcal{B}$ of (infinite) cardinality κ, then $\tau(q^t)$ has a base of cardinality κ. This means that $\tau(q)$ is second countable if and only if $\tau(q^t)$ is second countable.*

A map $\psi : (X,q) \to (Y,\rho)$ between two quasi-metric spaces (X,q) and (Y,ρ) is called *nonexpansive* provided that $\rho(\psi(x), \psi(x')) \leq q(x,x')$ whenever $x, x' \in X$.

A map $\psi : (X,q) \to (Y,\rho)$ between two quasi-metric spaces (X,q) and (Y,ρ) is called an *isometric map* or *isometry* provided that $\rho(\psi(x), \psi(x')) = q(x,x')$ whenever $x, x' \in X$. It is easy to see that every isometric map between quasi-metric spaces is injective.

Two quasi-metric spaces (X,q) and (Y,ρ) will be called *isometric* provided that there exists a bijective isometric map between them.

A map $\psi : (X,q) \to (Y,\rho)$ between two quasi-metric spaces (X,q) and (Y,ρ) is called *uniformly continuous* provided that for each $\epsilon > 0$ there is some $\delta > 0$ such that $\rho(\psi(x), \psi(x')) < \epsilon$ for all $x, x' \in X$ with $q(x,x') < \delta$. By the definition of the topology generated by a quasi-metric, it is clear that a uniformly continuous mapping is continuous with respect to the topologies $\tau(q), \tau(\rho)$.

We are now going to give an example of a quasi-metric on $\mathbb{R}$ which will be used more frequently throughout the thesis.

1.1.1 Example. *(see for instance [10, Example 1.1.1]) For any $x, y \in \mathbb{R}$, define $u(x,y) = \max\{x-y, 0\}$. Then u is a quasi-metric on $\mathbb{R}$, called the* usual quasi-metric *of $\mathbb{R}$. It is clear that $u^s(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, hence $(\mathbb{R}, u^s)$ is the real line.*

1.1.2 Definition. *Let (X,q) be a quasi-metric space. We say that*

1. *(X,q) is* join-compact *provided that the metric space (X,q^s) is compact.*
2. *(X,q) is* bicomplete *if the metric space (X,q^s) is complete. i.e., if every q^s-Cauchy sequence in X is q^s-convergent in X.*
3. *(X,q) is* q-totally bounded *provided that the metric space (X,q^s) is totally bounded. i.e., for each $\epsilon > 0$, there exists a finite subset $F_\epsilon \subseteq X$ such that $X = \bigcup_{x \in F_\epsilon} B_{q^s}(x,\epsilon)$.*

Let (X,q) be a quasi-metric space, $A \subseteq X$ and $\psi : X \to X$ a function. If $q_A = q|_{A\times A}$, then (A, q_A) is a quasi-metric space, called *the subspace of the quasi-metric space* (X,q). Also $(q_A)^t = q_A^t$ and $(q_A)^s = q_A^s$. We say that A is ψ*-invariant* if $\psi(A) \subseteq A$.

1.1.2 Remark. *Let (X,q) be a quasi-metric space. Since $\tau(q), \tau(q^t) \subseteq \tau(q^s)$, where $\tau(q), \tau(q^t)$ and $\tau(q^s)$ is the topology on X induced by q, q^t and q^s, respectively, then every join-compact subset of X is both compact in (X,q) and (X,q^t).*

It must be noted that if $K \subseteq X$ is $\tau(q)$-compact, then K need not be $\tau(q^t)$-compact and vice-versa. For instance if one considers the usual quasi-metric u of $\mathbb{R}$, then *for each $a \in \mathbb{R}$ the set (a, ∞) is $\tau(u)$-compact, but not $\tau(u^t)$-compact. Similarly, the set $(-\infty, a)$ is $\tau(u^t)$-compact, but not $\tau(u)$-compact.*

If (X, q) is a quasi-metric space, by Remark 1.1.2 every join-compact subset of X is compact, but the converse is not true in general. For instance consider $\mathbb{R}$ equipped with its usual quasi-metric u, then for each $a \in \mathbb{R}$, the intervals $[a, \infty)$ and (a, ∞) are both compact, but not join-compact, as they are not compact subsets of the real line.

1.1.2 Example. *Let A be a non-empty bounded and closed subset of the real line and u denotes the usual quasi-metric of $\mathbb{R}$, then (A, u_A) is a join-compact quasi-metric space, as (A, u_A^s) is a compact metric space.*

1.1.1 Lemma. *Let (X, q) be a quasi-metric space. Then*

(i) *$\psi : (X, q) \to (X, q)$ is uniformly continuous if and only if $\psi : (X, q^t) \to (X, q^t)$ is uniformly continuous;*

(ii) *if $\psi : (X, q) \to (X, q)$ is uniformly continuous, then $\psi : (X, q^s) \to (X, q^s)$ is uniformly continuous. The converse does not hold in general (see for instance [19, Example 3.3]). In particular, if (X, q) is a quasi-metric space and ψ is a uniformly continuous self-map of the symmetrised metric space (X, q^s), then ψ need not be a uniformly continuous self-map of (X, q).*

1.2 Quasi-uniform spaces

This section contains the basic definitions and results from the theory of quasi-uniform spaces which will be used in the thesis. For more on this basics, the reader is referred to, among others, [5, 9, 18].

1.2.1 Definition. *A quasi-uniformity $\mathcal{U}$ on a set X is a filter on $X \times X$ such that*

(a) *Every member* $U \in \mathcal{U}$ *contains the diagonal* $\Delta := \{(x,x) : x \in X\}$ *(where if confusion might occur, we specify which set* X *we are referring to by writing* Δ_X*),*

(b) *For each* $U \in \mathcal{U}$ *there is a* $V \in \mathcal{U}$ *such that* $V^2 \subseteq U$ *(Here* $V^2 = V \circ V$ *and* $\circ$ *is the usual composition of binary relations).*

The ordered pair $(X, \mathcal{U})$ is called a *quasi-uniform space.*

The members $U \in \mathcal{U}$ are called the *entourages* or *surroundings* of $\mathcal{U}$. The elements of X are called *points.* Each quasi-uniformity $\mathcal{U}$ on a set X *induces a topology* $\tau(\mathcal{U})$ as follows: For each $x \in X$ and $U \in \mathcal{U}$ set $U(x) = \{y \in X : (x,y) \in U\}$. A subset $G \subseteq X$ belongs to $\tau(\mathcal{U})$ if and only if it satisfies the following condition: For each $x \in G$ there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G$. If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X. (Here U^{-1} is the inverse of the binary relation U on X). The quasi-uniformity $\mathcal{U}^{-1}$ is called the *conjugate* of $\mathcal{U}$. A quasi-uniformity that is equal to its conjugate is called a *uniformity.* If $U \in \mathcal{U}$, we define $U^s = U \cap U^{-1}$. The union of a quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ yields a subbase of the coarsest uniformity finer than both $\mathcal{U}$ and $\mathcal{U}^{-1}$. It will be denoted by $\mathcal{U}^s$ in this thesis and is called the *symmetrization of* $\mathcal{U}$. It must be observed that $U^s \in \mathcal{U}^s$, whenever $U \in \mathcal{U}$.

Throughout this thesis any topological statement about a (quasi-)uniform space $(X, \mathcal{U})$ will refer to the topology $\tau(\mathcal{U})$ and all quasi-uniform spaces are taken to be T_0 (i.e., $\tau(\mathcal{U})$ is a T_0 topology on X). We are interested in T_0 quasi-uniform space, because of the following result which is crucial in establishing Lemma 4.1.3, which is needed in eventually defining the notion of entropy on a quasi-uniform space.

1.2.1 Proposition. *(see [9, Proposition 1.9]) Let* $(X, \mathcal{U})$ *be a quasi-uniform space. Then* $(X, \mathcal{U})$ *is* T_0 *if and only if* $(X, \mathcal{U}^s)$ *is* T_2*. i.e.,* $(X, \tau(\mathcal{U}))$ *is* T_0 *if and* *only if* $(X, \tau(\mathcal{U}^s))$ *is* T_2.

Proof. [9, Proposition 1.9]. $\square$

A map $\psi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called *uniformly continuous* provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(\psi \times \psi)(U) \subseteq V$. Here $\psi \times \psi$ is the product map from $X \times X$ to $Y \times Y$ defined by $(\psi \times \psi)(x, x') = (\psi(x), \psi(x'))$ for all $x, x' \in X$. By the definition of the topology generated by a quasi-uniformity, it is clear that a uniformly continuous mapping is continuous with respect to the topologies $\tau(\mathcal{U}), \tau(\mathcal{V})$.

For a subset Y of a quasi-uniform space $(X,\mathcal{U})$, we set

$$\mathcal{U}_Y = \{U \cap (Y \times Y) : U \in \mathcal{U}\}.$$

Then $(Y,\mathcal{U}_Y)$ is also a quasi-uniform space. The quasi-uniform space $(Y,\mathcal{U}_Y)$ is called a *subspace of the quasi-uniform space* $(X,\mathcal{U})$. Observe that for each $y \in Y$ and $U \in \mathcal{U}$, we have that $\Big(U \cap (Y \times Y)\Big)(y) = U(y) \cap Y$.

For every quasi-uniform space $(X,\mathcal{U})$ and any subset Y of X, the formula $i_Y(y) = y$ defines a uniformly continuous mapping $i_Y : (Y,\mathcal{U}_Y) \to (X,\mathcal{U})$, the mapping i_Y is called the *embedding of the subspace* $(Y,\mathcal{U}_Y)$ *in the space* $(X,\mathcal{U})$.

Let (X, q) be a quasi-metric space. For each $\epsilon > 0$, we define

$$V_\epsilon = \{(x, y) \in X \times X : q(x, y) < \epsilon\}.$$

It is well-known that $\{V_\epsilon : \epsilon > 0\}$ forms a base of a quasi-uniformity on X, called the *quasi-uniformity induced by* q on X or simply the *quasi-metric quasi-uniformity* and denoted by $\mathcal{U}_q$ (see for instance [18]). It can easily be shown that $(\mathcal{U}_q)^{-1} = \mathcal{U}_{q^t}$ and $(\mathcal{U}_q)^s = \mathcal{U}_{q^s}$. In this case $\psi : (X, q) \to (X, q)$ is uniformly continuous if and only if $\psi : (X,\mathcal{U}_q) \to (X,\mathcal{U}_q)$ is uniformly continuous. Also for each $\epsilon > 0$, we have that $V_\epsilon(x) = B_q(x, \epsilon)$ for each $x \in X$. Therefore $\tau(q) = \tau(\mathcal{U}_q)$.

A quasi-uniform space $(X,\mathcal{U})$ is said to be *quasi-metrizable* if $\mathcal{U} = \mathcal{U}_q$ for some quasi-metric q on X.

A topological space (X,τ) is said to be *quasi-uniformizable* if $\tau = \tau(\mathcal{U})$ for some quasi-uniformity $\mathcal{U}$ on X.

1.2.1 Remark. *(see for instance [18, page 243])*

It is well known that every topological space is quasi-uniformizable. Indeed if (X,τ) is a topological space, then the set

$$\{[(X-G)\times X]\cup[X\times G] : G\in\tau\}^1$$

yields a subbase of a quasi-uniformity on X. This quasi-uniformity is called the Pervin quasi-uniformity *$\mathcal{P}(\tau)$ of the topological space (X,τ). One checks that $\tau(\mathcal{P}(\tau))=\tau$.*

1.2.1 Example. *Consider $\mathbb{R}$ equipped with its usual quasi-metric u (see Example 1.1.1 for the definition of u). It can be shown that $\mathcal{U}_u$ is a quasi-uniformity on $\mathbb{R}$, which we call the* usual quasi-uniformity *of $\mathbb{R}$ and $\mathcal{U}_{u^s}$ is the usual/standard uniformity of $\mathbb{R}$.*

Just as for quasi-metric spaces, we have the following similar result to Lemma 1.1.1 for quasi-uniform spaces.

1.2.1 Lemma. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then*

(i) *$\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous if and only if $\psi : (X,\mathcal{U}^{-1}) \to (X,\mathcal{U}^{-1})$ is uniformly continuous;*

(ii) *if $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous, then $\psi : (X,\mathcal{U}^s) \to (X,\mathcal{U}^s)$ is uniformly continuous. The converse does not hold in general.*

[1] Let us mention that some authors write equivalently $\{[(X-G)\times X]\cup[G\times G] : G\in\tau\}$.

1.2.2 Definition. *Let $(X,\mathcal{U})$ be a quasi-uniform space, $A \subseteq X$. We say that*

1. *A is* join-compact *provided that A is a compact subset of the uniform space $(X,\mathcal{U}^s)$. A quasi-uniform space $(X,\mathcal{U})$ is* join-compact *provided that the uniform space $(X,\mathcal{U}^s)$ is compact.*

2. *A is* $\mathcal{U}$-totally bounded *provided for each $U \in \mathcal{U}$, the open cover $\{U(x) : x \in X\}$ of A has a finite subcover. i.e., for each $U \in \mathcal{U}$, there exists a finite subset $F_U \subseteq X$ such that $A \subseteq \bigcup_{x \in F_U} U(x)$.*

3. *A is* totally bounded *provided that A is totally bounded with respect to the associated uniformity $\mathcal{U}^s$. i.e., for each $U \in \mathcal{U}$, the open cover $\{U^s(x) : x \in X\}$ of A has a finite subcover.*

4. *$(X,\mathcal{U})$ is called* $\mathcal{U}$-totally bounded *(resp.* totally bounded*) provided that X is $\mathcal{U}$-totally bounded (resp. totally bounded).*

It must be noted that every $\tau(\mathcal{U})$-compact subset of X is $\mathcal{U}$-totally bounded and every totally bounded subset is $\mathcal{U}$-totally bounded.

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : X \to X$ a function. A subset Y of X is called *ψ-invariant* if $\psi(Y) \subseteq Y$.

1.2.3 Definition. *(see [18, Definition 2.6.1]) Let $(X,\mathcal{U})$ be a quasi-uniform space.*

1. *A filter $\mathcal{F}$ on $(X,\mathcal{U})$ is called a* $\mathcal{U}^s$-Cauchy filter *if for each $U \in \mathcal{U}$ there exists some $F \in \mathcal{F}$ such that $F \times F \subseteq U$.*

2. *$(X,\mathcal{U})$ is said to be* bicomplete *if the uniform space $(X,\mathcal{U}^s)$ is complete. i.e., every $\mathcal{U}^s$-Cauchy filter is $\tau(\mathcal{U}^s)$-convergent in X.*

1.2.4 Definition. *(see [18, Definition 2.1.8]) A quasi-uniform space* $(X,\mathcal{U})$ *is said to have the* Lebesgue property *if for every open cover* α *of* $(X,\mathcal{U})$ *there is* $U \in \mathcal{U}$ *such that* $\{U(x) : x \in X\}$ *refines* α*. i.e., every member of* $\{U(x) : x \in X\}$ *is a subset of some member of* α*.*

The following result is well known for compact metric spaces.

1.2.2 Proposition. *(see [18, Proposition 2.1.9]) Every compact quasi-uniform space* $(X,\mathcal{U})$ *has the Lebesgue property.*

Proof. [18, Proposition 2.1.9]. □

1.2.2 Remark. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space.*
If $\tau(\mathcal{U}), \tau(\mathcal{U}^{-1})$ *and* $\tau(\mathcal{U}^s)$ *is the topology on* X *induced by* $\mathcal{U}, \mathcal{U}^{-1}$ *and* $\mathcal{U}^s$*, respectively, then* $\tau(\mathcal{U}), \tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U}^s)$*. Hence every join-compact subset of* X *is both compact in* $(X, \tau(\mathcal{U}))$ *and* $(X, \tau(\mathcal{U}^{-1}))$*. It must be noted that there is no direct relationship between* $\tau(\mathcal{U})$ *and* $\tau(\mathcal{U}^{-1})$*. In particular, if* $K \subseteq X$ *is* $\tau(\mathcal{U})$*-compact, then* K *need not be* $\tau(\mathcal{U}^{-1})$*-compact and vice-versa. For a counter example one can use the usual quasi-uniformity* $\mathcal{U}_u$ *of* $\mathbb{R}$*.*

The following result is well known for uniform spaces.

1.2.3 Proposition. *A quasi-uniform space is join-compact if and only if it is bicomplete and totally bounded.*

Proof. Let $(X,\mathcal{U})$ be a quasi-uniform space. Then

$$\begin{aligned}(X,\mathcal{U}) \text{ is join-compact} &\Leftrightarrow (X,\mathcal{U}^s) \text{ is a compact uniform space}\\ &\Leftrightarrow (X,\mathcal{U}^s) \text{ is complete and totally bounded}\\ &\Leftrightarrow (X,\mathcal{U}) \text{ is bicomplete and totally bounded.}\end{aligned}$$

□

If $(X,\mathcal{U})$ is a quasi-uniform space, by Remark 1.2.2 every join-compact subset of X is $\tau(\mathcal{U})$-compact, but the converse is not true in general. For a counter example one can consider $\mathbb{R}$ equipped with its usual quasi-uniformity $\mathcal{U}_u$.

Chapter 2

The topological entropy

In this chapter we are going to recall the well-known notion of topological entropy given in [1] for compact spaces and the one given in [14] for non-compact spaces. For more on topological entropy, the reader is referred to, among others, [7, 8, 21, 17], where [7] is concerned with a unifying approach to the study of entropies in mathematics, [8] is a survey paper on uniform entropy for uniform spaces and topological groups, [21] is an introductory book on Ergodic Theory, and [17] concerns the connection between uniform entropy and topological entropy.

2.1 Topological entropy on compact spaces

Let us recall the definition of a topological entropy given in [1, 8]. Let X be a compact topological space and let $\mathrm{cov}(X)$ be the family of all open covers of X. For any $\alpha \in \mathrm{cov}(X)$, the *entropy* of α is

$$h(\alpha) = \log N(\alpha),$$

where $N(\alpha) = \min\{|\beta| : \beta$ is a finite subcover of $\alpha\}$. If $\alpha, \beta \in \mathrm{cov}(X)$, the *join* of α and β is the open cover

$$\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}.$$

If $\alpha, \beta \in \mathrm{cov}(X)$, α is said to be a *refinement* of β, written $\alpha < \beta$, if every member of α is a subset of a member of β. Hence $\alpha \vee \beta < \alpha$ and $\alpha \vee \beta < \beta$. Also if α is a subcover of β then $\alpha < \beta$.

If $\psi : X \to X$ is a continuous self-map and $\alpha \in \mathrm{cov}(X)$, then for each $k \in \mathbb{N}_+ \cup \{0\}$,

$$\psi^{-k}(\alpha) = \{\psi^{-k}(A) : A \in \alpha\}$$

is an open cover of X and $\psi^{-1}(\alpha \vee \beta) = \psi^{-1}(\alpha) \vee \psi^{-1}(\beta)$ whenever $\alpha, \beta \in \mathrm{cov}(X)$. If $\alpha, \beta \in \mathrm{cov}(X)$ such that $\alpha < \beta$, then $\psi^{-1}(\alpha) < \psi^{-1}(\beta)$.

For each $n \in \mathbb{N}_+$, and each $\alpha \in \mathrm{cov}(X)$, we define

$$C_n(\alpha, \psi) = \bigvee_{k=0}^{n-1} \psi^{-k}(\alpha).$$

Since ψ is continuous, then $C_n(\alpha, \psi) \in \mathrm{cov}(X)$ and $C_n(\alpha, \psi) < \psi^{-i}(\alpha)$ for all $0 \leq i < n$. The *topological entropy of ψ with respect to α* is defined as

$$h(\alpha, \psi) = \lim_{n \to \infty} \frac{\log N(C_n(\alpha, \psi))}{n}.$$

The *topological entropy* of ψ is

$$h(\psi) = \sup_{\alpha \in \mathrm{cov}(X)} h(\alpha, \psi).$$

2.1.1 Remark. *(compare [1] and [21, Section 7.1]) Let X be a compact topological space and $\psi : X \to X$ a continuous self-map. Let $\alpha, \beta \in cov(X)$. Then*

1. *If $\alpha < \beta$, then*

(i) $N(\beta) \leq N(\alpha)$,

(ii) $C_n(\alpha, \psi) < C_n(\beta, \psi)$ *and* $N(C_n(\alpha, \psi)) \geq N(C_n(\beta, \psi))$ *for each* $n \in \mathbb{N}_+$.

2. If $n \in \mathbb{N}_+$, *then* $N(\psi^{-i}(\alpha)) \leq N(C_n(\alpha, \psi))$ *for each* $0 \leq i < n$.

Proof. Suppose X is a compact topological space and $\psi : X \to X$ is a continuous self-map. Let $\alpha, \beta \in \text{cov}(X)$.

1. Suppose $\alpha < \beta$.

   (i) Let $\{A_1, ..., A_{N(\alpha)}\}$ be a subcover of α with minimal cardinality. Since $\alpha < \beta$, for each $1 \leq i \leq N(\alpha)$ there exists $B_i \in \beta$ such that $A_i \subseteq B_i$. Therefore $\{B_1, ..., B_{N(\alpha)}\}$ covers X and is a subcover of β. Thus $N(\beta) \leq N(\alpha)$ as required.

   (ii) Let $n \in \mathbb{N}_+$. Here it only suffices to show that $\psi^{-1}(\alpha) < \psi^{-1}(\beta)$, then the first part is achieved by using induction on n and the second part is achieved by using (i) and the first part. To this, let $A' \in \psi^{-1}(\alpha)$. Then $A' = \psi^{-1}(A)$ for some $A \in \alpha$. Since $\alpha < \beta$, then $A \subseteq B$ for some $B \in \beta$. Hence $A' = \psi^{-1}(A) \subseteq \psi^{-1}(B)$. Since $A' \subseteq \psi^{-1}(B)$ and $\psi^{-1}(B) \in \psi^{-1}(\beta)$, then $\psi^{-1}(\alpha) < \psi^{-1}(\beta)$.

2. Let $n \in \mathbb{N}_+$ be fixed and let $0 \leq i < n$. Since $C_n(\alpha, \psi) < \psi^{-i}(\alpha)$, then by $1\,(i)$ we get that $N(\psi^{-i}(\alpha)) \leq N(C_n(\alpha, \psi))$.

$\square$

If (X, τ) is a topological space, $\alpha \in \text{cov}(X), n \in \mathbb{N}_+$ and $\psi : X \to X$ is a continuous self-map, we will write $C_n(\alpha, \psi, \tau), h(\alpha, \psi, \tau)$ and $h(\psi, \tau)$ if we need to emphasise on the topology τ used.

2.2 Topological entropy on non-compact spaces

In this section we are going to recall the notion of topological entropy introduced by Hofer [14] by means of finite open covers. Let X be a non-compact topological space and let $\mathrm{cov}_f(X)$ be the family of all finite open covers of X. For any $\alpha \in \mathrm{cov}_f(X)$ the *entropy* of α is

$$h_f(\alpha) = \log N(\alpha),$$

where $N(\alpha) = \min\{|\beta| : \beta \text{ is a subcover of } \alpha\}$ and for each continuous self-map $\psi : X \to X$ the *topological entropy* of ψ is

$$h_f(\psi) = \sup_{\alpha \in \mathrm{cov}_f(X)} h_f(\alpha, \psi),$$

where $h_f(\alpha, \psi)$ is defined exactly in the same way as for the compact spaces. Defining topological entropy in this way has far reaching advantages, as it does not require any specific separation property for the underlying space.

2.2.1 Remark. *Let X be a topological space. Since $cov_f(X) \subseteq cov(X)$ and every open cover has a finite subcover when X is compact, then for each continuous self-map $\psi : X \to X$, we have that*

$$h_f(\psi) \leq h(\psi),$$

with equality if the underlying space is compact.

For further readings on the topological entropy using finite open covers, we recommend, among others, [8, 15, 17].

2.3 Uniform entropy

In 1971 Bowen [2] extended the notion of topological entropy on compact spaces to uniformly continuous self-maps of metric spaces via separating and spanning sets.

Later on in 1974, Hood [15] adapted Bowen's definition to uniformly continuous self-maps on a uniform space. Bowen [2] and Hood [15] gave the definition when the metric(uniform) space is not necessarily compact, which we shall call uniform entropy or Bowen's entropy and denoted by h_U in this thesis. The uniform entropy coincides with the classical topological entropy given in [1] when the metric(uniform) space is compact (see [21, Corollary 7.5.2] for metric spaces and [15, Page 635 line 12 from above]). For basic facts and properties about uniform entropy we refer the readers to, among others, [2, 8, 15] and [21, Section 7.2].

Let (X, d) (or $(X, \mathcal{U})$) be a metric (or uniform) space and $\psi : (X, d) \to (X, d)$ (or $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$) be a uniformly continuous self-map. Let $\mathcal{K}(X)$ be the family of all non-empty compact subsets of X. If $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $\epsilon > 0$ (or $U \in \mathcal{U}$)

1. a subset F of X is said to (n, ϵ) (or (n, U))-*span* K with respect to ψ, if for every $x \in K$ there is $y \in F$ such that

$$d(\psi^k(x), \psi^k(y)) < \epsilon \text{ (or } (\psi^k(x), \psi^k(y)) \in U \text{)}$$

   for each $0 \le k < n$.

2. a subset F of X is said to be (n, ϵ) (or (n, U))-*separated* with respect to ψ, if for each pair of distinct points $x, y \in F$ there exists some $k \in \mathbb{N}_+$ such that $0 \le k < n$ and $d(\psi^k(x), \psi^k(y)) \ge \epsilon$ (or $(\psi^k(x), \psi^k(y)) \notin U$).

For $\epsilon > 0$ (or $U \in \mathcal{U}$), $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}_+$, set

(a) $r_n(\epsilon, K, \psi) = \min\left\{ |F| : F \subseteq X \text{ and } F \ (n, \epsilon)\text{-span } K \right\}$

or

$$r_n(U, K, \psi) = \min\left\{ |F| : F \subseteq X \text{ and } F \ (n, U)\text{-span } K \right\}.$$

(b) $s_n(\epsilon, K, \psi) = \max\{|F| : F \subseteq K$ and F is (n, ϵ)-separated with respect to $\psi\}$

or

$$s_n(U, K, \psi) = \max\{|F| : F \subseteq K \text{ and } F \text{ is } (n, U)\text{-separated with respect to } \psi\}.$$

Compactness of K guarantees that the numbers $r_n(\epsilon, K, \psi), r_n(U, K, \psi), s_n(\epsilon, K, \psi)$ $s_n(U, K, \psi)$ are finite and well defined.

Given $K \in \mathcal{K}(X), n \in \mathbb{N}_+$, and letting $\sigma = r, s$, define

(i) $\sigma(\epsilon, K, \psi) = \lim_{n\to\infty} \sup \frac{\log \sigma_n(\epsilon,K,\psi)}{n}$ for each $\epsilon > 0$. Furthermore, let

$$h_\sigma(K, \psi) = \lim_{\epsilon\to 0} \sigma(\epsilon, K, \psi).$$

(ii) $\sigma(U, K, \psi) = \lim_{n\to\infty} \sup \frac{\log \sigma_n(U,K,\psi)}{n}$ for each $U \in \mathcal{U}$. Furthermore, let

$$h_\sigma(K, \psi) = \sup_{U\in\mathcal{U}} \sigma(U, K, \psi).$$

The following lemma easily follows from [2, Lemma 1] and [15, Lemma 1].

2.3.1 Lemma. *Let (X, d) (or $(X, \mathcal{U})$) be a metric (or uniform) space and let $\psi : (X, d) \to (X, d)$ (or $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$) be a uniformly continuous self-map. If $K \in \mathcal{K}(X)$, then $h_r(K, \psi) = h_s(K, \psi)$.*

With the above Lemma 2.3.1 we get the notion of *uniform entropy* $h_U(\psi)$ of a uniformly continuous self-map ψ of a metric(or uniform) space X.

2.3.1 Definition. *Let X be a metric (or uniform) space and $\psi : X \to X$ a uniformly continuous self-map. The* uniform entropy *of ψ is*

$$h_U(\psi) = \sup_{K\in\mathcal{K}(X)} h_r(K, \psi) = \sup_{K\in\mathcal{K}(X)} h_s(K, \psi).$$

Hence $h_U(\psi)$ can be defined using either spanning sets (in the case of r) or separating sets (in the case of s).

2.3.2 Definition. *Let (X,d) (or $(X,\mathcal{U})$) be a metric (or uniform) space and let $\psi : (X,d) \to (X,d)$ (or $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$) be a uniformly continuous self-map. For $x \in X$, $n \in \mathbb{N}_+$ and $\epsilon > 0$ (or $U \in \mathcal{U}$). Then we define the* Bowen's ball *by*

$$D_n(x,\epsilon,\psi) := \bigcap_{k=0}^{n-1} \psi^{-k}(B(\psi^k(x),\epsilon)),$$

where $B(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$ denotes the open ϵ-ball at x

or

$$D_n(x,U,\psi) := \bigcap_{k=0}^{n-1} \psi^{-k}(U(\psi^k(x))),$$

where

$$U(x) = \{y \in X : (x,y) \in U\}.$$

It must be noted that each Bowen's ball is an open subset of the metric(or uniform) space (X,d) (or $(X,\mathcal{U})$).

With this definition we obtain the following result about spanning and separating sets. The following result will be used to define the notion of entropy for quasi-metric(uniform) spaces (see Section 3.1 and 4.1).

2.3.1 Remark. *Let (X,d) (or $(X,\mathcal{U})$) be a metric (or uniform) space and $\psi : (X,d) \to (X,d)$ (or $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$) be a uniformly continuous self-map. Let $\mathcal{K}(X)$ be the family of all non-empty compact subsets of X.*
If $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $\epsilon > 0$ (or $U \in \mathcal{U}$), then

*(i) a subset F of X is said to (n,ϵ)-*span *K with respect to ψ if and only if*

$$K \subseteq \bigcup_{x \in F} D_n(x,\epsilon,\psi).$$

(ii) a subset F of X is said to (n, U)-span K *with respect to* ψ *if and only if*

$$K \subseteq \bigcup_{x \in F} D_n(x, U, \psi).$$

(iii) a subset F of X is said to be (n, ϵ)-separated *with respect to* ψ *if and only if*

$$D_n(x, \epsilon, \psi) \cap D_n(y, \epsilon, \psi) = \emptyset$$

for any $x, y \in F$ with $x \neq y$.

(iv) a subset F of X is said to be (n, U)-separated *with respect to* ψ *if and only if*

$$D_n(x, U, \psi) \cap D_n(y, U, \psi) = \emptyset$$

for any $x, y \in F$ with $x \neq y$.

It must be noted that the topological entropy given by Bowen [2] for uniformly continuous self-maps of a metric space (X, d) can be obtained from the one given by Hood [15] by using the metric uniformity $\mathcal{U}_d$.

Chapter 3

Entropy on quasi-metric spaces

In this chapter we start our own investigation on the notion of entropy. Modifying the notion of entropy by Bowen [2], we introduce a new notion of entropy, which we call quasi-uniform entropy (denoted by $h_{QU}(\psi)$) of a uniformly continuous self-map ψ of a quasi-metric space (X, q). We decided to give our entropy function a new name, as it is computed in a completely different manner from what is done by Bowen [2]. The only literature we are aware of on the notion of entropy on quasi-metric spaces is [20], but their notion of entropy is computed within the symmetrised metric space, as they are using what we call join-compact subsets.

With this new notion of quasi-uniform entropy, we manage to extend most of the basic results about uniform entropy on metric spaces to quasi-metric spaces. Finally using Kimura's idea of the completion theorem, we manage to prove the completion theorem in the class of all join-compact quasi-metric spaces, that is for join-compact quasi-metric spaces the entropy of a uniformly continuous self-map coincides with the entropy of its extension to the bicompletion. Most of the results in this chapter are contained in our recent paper [11].

3.1 Quasi-uniform entropy on a quasi-metric space

In this section, following step by step the work of Bowen, we extend the notion of uniform entropy for self-maps of quasi-metric spaces which are uniformly continuous with respect to the topology associated with the quasi-metric space (X, q).

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous self-map. For $x \in X$, $n \in \mathbb{N}_+$ and $\epsilon > 0$. Then we define

$$D_n^q(x, \epsilon, \psi) := \bigcap_{k=0}^{n-1} \psi^{-k}(B_q(\psi^k(x), \epsilon))$$

and

$$D_n^{q^s}(x, \epsilon, \psi) := D_n^q(x, \epsilon, \psi) \cap D_n^{q^t}(x, \epsilon, \psi).$$

It must be noted that $D_n^{q^s}(x, \epsilon, \psi)$ is the Bowen's ball and if q is a metric, then

$$D_n^q(x, \epsilon, \psi) = D_n^{q^t}(x, \epsilon, \psi).$$

Let $\mathcal{K}(X)$ denote the collection of all non-empty $\tau(q)$-compact subsets of X. We define

$$r_n(\epsilon, K, \psi) = \min \left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^q(x, \epsilon, \psi) \right\} \tag{3.1}$$

whenever $K \in \mathcal{K}(X)$.

Throughout this thesis we will adopt definition (3.1) for $r_n(\epsilon, K, \psi)$.

A subset F of X is said to be (n, ϵ)-*separated* with respect to ψ if

$$D_n^{q^s}(x, \epsilon, \psi) \cap D_n^{q^s}(y, \epsilon, \psi) = \emptyset$$

for any $x, y \in F$ with $x \neq y$.
For $K \in \mathcal{K}(X)$, we set

$$s_n(\epsilon, K, \psi) = \max\{|F| : F \subseteq K \text{ and } F \text{ is } (n, \epsilon)\text{-separated with respect to } \psi\} \tag{3.2}$$

whenever $K \in \mathcal{K}(X)$.

In light of Lemma 2.3.1 the definitions in equation (3.1) and (3.2) are essentially the same as the one given by Bowen [2] via spanning and separating sets, respectively.

3.1.1 Remark. *1. The choice of defining the notion of separatedness within the symmmetrised metric space (X, q^s) is to enable us to prove Lemma 3.1.1, which is very crucial in proving Proposition 3.1.1 which is very important for this section, as it allow us to eventually define the notion of entropy of a uniformly continuous self-map of a quasi-metric space.*

Even if the Bowen balls for separating sets are taken within the symmetrised metric space (X, q^s), our notion of entropy is purely computed within the quasi-metric space (X, q). This is so, as we are using compact (not necessarily join-compact) subsets of X.

2. Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous self-map. Let $n \in \mathbb{N}_+, \epsilon > 0$ and $x \in X$. Since ψ is continuous, then $D_n^q(x, \epsilon, \psi)$ is a $\tau(q)$-open subset of X.

Since K is $\tau(q)$-compact, then the quantities $r_n(\epsilon, K, \psi)$ and $s_n(\epsilon, K, \psi)$ are finite and well defined.

Furthermore, we define:

$$r(\epsilon, K, \psi) = \lim_{n\to\infty} \sup \frac{\log r_n(\epsilon, K, \psi)}{n}$$

and

$$s(\epsilon, K, \psi) = \lim_{n\to\infty} \sup \frac{\log s_n(\epsilon, K, \psi)}{n}.$$

Then, the quantities $h_r(K,\psi)$ and $h_s(K,\psi)$ are defined by

$$h_r(K,\psi) = \lim_{\epsilon\to 0} r(\epsilon, K, \psi) \text{ and } h_s(K,\psi) = \lim_{\epsilon\to 0} s(\epsilon, K, \psi).$$

We write $r_n(\epsilon, K, \psi, q), s_n(\epsilon, K, \psi, q), r(\epsilon, K, \psi, q), s(\epsilon, K, \psi, q), h_r(K, \psi, q)$ and $h_s(K, \psi, q)$ if we need to emphasise on the quasi-metric q used.

3.1.2 Remark. *Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous self-map. For each $n \in \mathbb{N}_+$, define*

$$q_n(x, y) = \max_{0\leq k<n} q(\psi^k(x), \psi^k(y))$$

for all $x, y \in X$. One easily check that q_n is a quasi-metric on X. Furthermore, for each $n \in \mathbb{N}_+, \epsilon > 0$ and each $x \in X$ we have that $B_{q_n}(x, \epsilon) = D_n^q(x, \epsilon, \psi)$ and $B_{q_n^s}(x, \epsilon) = D_n^{q^s}(x, \epsilon, \psi)$. Hence the reason we have given our definitions of r_n and s_n without mentioning the quasi-metric q_n.

3.1.1 Lemma. *Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous self-map. Let $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $\epsilon > 0$. If $F \subseteq K$ is (n, ϵ)-separated with respect to ψ and $s_n(\epsilon, K, \psi) = |F|$, then $K \subseteq \bigcup_{x\in F} D_n^{q^s}(x, \epsilon, \psi)$.*

Proof. Suppose $F \subseteq K$ is (n, ϵ)-separated with respect to ψ and $s_n(\epsilon, K, \psi) = |F|$, but K is not a subset of $\bigcup_{x\in F} D_n^{q^s}(x, \epsilon, \psi)$. Pick $y \in K \setminus \bigcup_{x\in F} D_n^{q^s}(x, \epsilon, \psi)$. It follows that $y \notin F$. Set $F_0 := F \cup \{y\} \subseteq K$. We claim that F_0 is (n, ϵ)-separated with respect to ψ.

Let $z, w \in F_0$ such that $z \neq w$. We have the following two cases:

Case 1. Suppose $z, w \in F$. Then we are done, as F is (n, ϵ)-separated with respect to ψ.

Case 2. Suppose $z \in F$ and $w = y$. Then we claim that

$$D_n^{q^s}(z, \epsilon, \psi) \cap D_n^{q^s}(w, \epsilon, \psi) = \emptyset.$$

Suppose that $D_n^{q^s}(z,\epsilon,\psi)\cap D_n^{q^s}(w,\epsilon,\psi) \neq \emptyset$. Choose $v \in D_n^{q^s}(z,\epsilon,\psi)\cap D_n^{q^s}(w,\epsilon,\psi)$. It follows that for any $0 \leq k < n$ we have

$$q^s(\psi^k(z), \psi^k(v)) < \epsilon \text{ and } q^s(\psi^k(w), \psi^k(v)) < \epsilon, \text{ for all } \epsilon > 0. \tag{3.3}$$

In particular, for $\epsilon = \frac{q^s(z,w)}{2} > 0$ and by taking $k = 0$ in (3.3) we get

$$q^s(z,v) < \epsilon \text{ and } q^s(w,v) < \epsilon.$$

Thus $2\epsilon = q^s(z,w) \leq q^s(z,v) + q^s(v,w) < 2\epsilon$ (contradiction).
Hence F_0 is (n,ϵ)-separated and $F_0 \subseteq K$ with $|F| < |F_0|$. This contradicts the fact that $s_n(\epsilon, K, \psi) = |F|$. Therefore $K \subseteq \bigcup_{x\in F} D_n^{q^s}(x,\epsilon,\psi)$. □

3.1.1 Proposition. *(compare [2, Lemma 1]) Let (X,q) be a quasi-metric space, $K \in \mathcal{K}(X)$ and $\psi : (X,q) \to (X,q)$ be a uniformly continuous self-map. For all $\epsilon, \epsilon' > 0$ and each $n \in \mathbb{N}_+$ we have:*

(i) $r_n(\epsilon,K,\psi) \leq s_n(\epsilon,K,\psi) \leq r_n(\frac{\epsilon}{2},K,\psi)$.

(ii) *If $0 < \epsilon < \epsilon'$, then $r_n(\epsilon,K,\psi) \geq r_n(\epsilon',K,\psi)$ and $s_n(\epsilon,K,\psi) \geq s_n(\epsilon',K,\psi)$.*

Proof. Let $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}_+$.
(i) We first prove that $r_n(\epsilon,K,\psi) \leq s_n(\epsilon,K,\psi)$. Let $E \subseteq K$ such that E is (n,ϵ)-separated with respect to ψ and $s_n(\epsilon,K,\psi) = |E|$. Then by Lemma 3.1.1,

$$K \subseteq \bigcup_{x\in E} D_n^{q^s}(x,\epsilon,\psi) \subseteq \bigcup_{x\in E} D_n^{q}(x,\epsilon,\psi).$$

Then it follows that

$$|E| \in \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x\in F} D_n^{q}(x,\epsilon,\psi)\right\}.$$

Therefore $r_n(\epsilon,K,\psi) \leq s_n(\epsilon,K,\psi)$.

Secondly, we prove that $s_n(\epsilon, K, \psi) \leq r_n(\frac{\epsilon}{2}, K, \psi)$. Let us consider $E \subseteq K$ such that E is (n, ϵ)-separated and $F \subseteq X$ such that $K \subseteq \bigcup_{x \in F} D_n^{q^s}(x, \dfrac{\epsilon}{2}, \psi)$. Define a function $g : E \to F$ by choosing, for each $x \in E$, some point $g(x) \in F$ with $g(x) \in D_n^{q^s}(x, \frac{\epsilon}{2}, \psi)$. We claim that the map g is injective.
Suppose that the map g is not injective. Then there exists $x, y \in E$ with $x \neq y$, but $g(x) = z = g(y)$. Furthermore, we have

$$q^s(\psi^k(x), \psi^k(y)) \leq q^s(\psi^k(x), \psi^k(z)) + q^s(\psi^k(z), \psi^k(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $0 \leq k < n$. Hence

$$y \in D_n^{q^s}(x, \epsilon, \psi) \cap D_n^{q^s}(y, \epsilon, \psi),$$

which is a contradiction, as E is (n, ϵ)-separated. Therefore g is injective. Since the map $g : E \to F$ is injective, then we have $|E| \leq |F|$ and it follows that $s_n(\epsilon, K, \psi) \leq r_n(\frac{\epsilon}{2}, K, \psi)$.

(ii) We now prove that $r_n(\epsilon, K, \psi) \geq r_n(\epsilon', K, \psi)$ whenever $0 < \epsilon < \epsilon'$. If $0 < \epsilon < \epsilon'$, then $D_n^q(x, \epsilon, \psi) \subseteq D_n^q(x, \epsilon', \psi)$ for each $x \in X$. Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^q(x, \epsilon, \psi)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^q(x, \epsilon', \psi)\right\}.$$

Therefore, we have $r_n(\epsilon, K, \psi) \geq r_n(\epsilon', K, \psi)$.

Finally, we prove that $s_n(\epsilon, K, \psi) \geq s_n(\epsilon', K, \psi)$ whenever $0 < \epsilon < \epsilon'$. If $0 < \epsilon < \epsilon'$, then $D_n^{q^s}(x, \epsilon, \psi) \subseteq D_n^{q^s}(x, \epsilon', \psi)$ for each $x \in X$. Indeed, if $F \subseteq X$ is (n, ϵ')-separated, then F is (n, ϵ)-separated, because for any $x, y \in F$ with $x \neq y$, we have that

$$D_n^{q^s}(x, \epsilon, \psi) \cap D_n^{q^s}(y, \epsilon, \psi) \subseteq D_n^{q^s}(x, \epsilon', \psi) \cap D_n^{q^s}(y, \epsilon', \psi) = \emptyset.$$

Therefore, we have $s_n(\epsilon, K, \psi) \geq s_n(\epsilon', K, \psi)$. □

By Proposition 3.1.1, it follows that $h_r(K,\psi) = h_s(K,\psi)$ for each $K \in \mathcal{K}(X)$ and each uniformly continuous self-map $\psi : (X,q) \to (X,q)$. Hence the following definitions make sense:

3.1.1 Definition. *Let (X,q) be a quasi-metric space, $\psi : (X,q) \to (X,q)$ be a uniformly continuous map and $K \in \mathcal{K}(X)$.*

$$h_{QU}(K,\psi) = h_r(K,\psi) = h_s(K,\psi),$$

is the quasi-uniform entropy *of ψ with respect to K. Furthermore, we define the quasi-uniform entropy $h_{QU}(\psi)$ of ψ by*

$$h_{QU}(\psi) = \sup_{K\in\mathcal{K}(X)} h_{QU}(K,\psi).$$

We write $h_{QU}(K,\psi,q)$ and $h_{QU}(\psi,q)$ if we need to emphasise on the quasi-metric q used.

It must be noted that if (X,q) is a metric space, then our quasi-uniform entropy function h_{QU} coincides with that of uniform entropy function h_U given by Bowen.

3.1.2 Proposition. *(compare [21, Theorem 7.5]) Let (X,q) be a quasi-metric space and $\psi : (X,q) \to (X,q)$ a uniformly continuous self-map. If $K, K' \in \mathcal{K}(X)$ with $K \subseteq K'$, then $h_{QU}(K,\psi,q) \leq h_{QU}(K',\psi,q)$. Furthermore $h_{QU}(\psi,q) = h_{QU}(X,\psi,q)$ provided that X is compact.*

Proof. Let $n \in \mathbb{N}_+, \epsilon > 0$ and $K, K' \in \mathcal{K}(X)$ with $K \subseteq K'$. Let $F' \subseteq X$ such that $K' \subseteq \bigcup_{x\in F'} D_n^q(x,\epsilon,\psi)$ and $r_n(\epsilon,K',\psi) = |F'|$.
Since $K \subseteq K' \subseteq \bigcup_{x\in F'} D_n^q(x,\epsilon,\psi)$, then

$$|F'| \in \left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x\in F} D_n^q(x,\epsilon,\psi) \right\}.$$

Therefore $r_n(\epsilon, K, \psi) \leq |F'| = r_n(\epsilon, K', \psi)$. From which we get that

$$h_{QU}(K, \psi, q) \leq h_{QU}(K', \psi, q).$$

If X is compact, then we get that $h_{QU}(K, \psi, q) \leq h_{QU}(X, \psi, q)$ for all $K \in \mathcal{K}(X)$. Thus

$$h_{QU}(X, \psi, q) \leq h_{QU}(\psi, q) = \sup_{K \in \mathcal{K}(X)} h_{QU}(K, \psi, q) \leq h_{QU}(X, \psi, q)$$

and it follows that $h_{QU}(\psi, q) = h_{QU}(X, \psi, q)$. □

3.1.1 Example. *Recall that a map $\psi : (X, q) \to (Y, p)$ between two quasi-metric spaces (X, q) and (Y, p) is called* nonexpansive *provided that $p(\psi(x), \psi(x)) \leq q(x, x)$ for all $x, x' \in X$. Also it is well-known that every nonexpansive map is uniformly continuous.*
If (X, q) is a quasi-metric space and $\psi : (X, q) \to (X, q)$ is a nonexpansive map, then $h_{QU}(\psi) = 0$. We first show that for each $\epsilon > 0$ and each $x \in X$ we have $D_n^q(x, \epsilon, \psi) = B_q(x, \epsilon)$ whenever $n \in \mathbb{N}_+$. Let $y \in D_n^q(x, \epsilon, \psi)$, then $q(\psi^k(x), \psi^k(y)) < \epsilon$ for each $0 \leq k < n$. By taking $k = 0$, we have $y \in B_q(x, \epsilon)$. If $y \in B_q(x, \epsilon)$, then $q(\psi^k(x), \psi^k(y)) \leq q(x, y) < \epsilon$ for each $0 \leq k < n$ (as ψ is nonexpansive). Hence $y \in D_n^q(x, \epsilon, \psi)$. Now we have that $r_n(\epsilon, K, \psi)$ and $s_n(\epsilon, K, \psi)$ do not depend on $n \in \mathbb{N}_+$. Thus, $r(\epsilon, K, \psi) = 0 = s(\epsilon, K, \psi)$. Therefore, $h_{QU}(\psi) = 0$.

3.2 Basic properties of the quasi-uniform entropy on a quasi-metric space

In this section we are going to show that most of the basic results about uniform entropy on metric spaces can easily be extended to quasi-metric spaces.

3.2.1 Definition. *(compare [2, page 403]) Two quasi-metrics q_1 and q_2 on a set X are* uniformly equivalent *if $id_X : (X, q_1) \to (X, q_2)$ and $id_X : (X, q_2) \to$*

(X, q_1) are both uniformly continuous maps of quasi-metric spaces. In this case $\psi : (X, q_1) \to (X, q_1)$ is uniformly continuous if and only if $\psi : (X, q_2) \to (X, q_2)$ is uniformly continuous.

If (X, q) is a quasi-metric space, then (X, q) and (X, q^t) are not uniformly equivalent in general, as the following example illustrates.

3.2.1 Example. *Consider $\mathbb{R}$ equipped with its usual quasi-metric u. Then $id_{\mathbb{R}} : (\mathbb{R}, u) \to (\mathbb{R}, u^t)$ is not uniformly continuous, as it is not continuous. In particular, $id_{\mathbb{R}} : (\mathbb{R}, u) \to (\mathbb{R}, u^t)$ is not continuous on $(-\infty, 0]$. Indeed, for any $\delta > 0$ choose $x < 0$ and set $\epsilon = \frac{|x|}{2} > 0$, then $u(x, 0) = \max\{x, 0\} = 0 < \delta$, but $u^t(id_{\mathbb{R}}(x), id_{\mathbb{R}}(0)) = u^t(x, 0) = u(0, x) = \max\{-x, 0\} = -x = |x| > \epsilon$. Therefore $id_{\mathbb{R}} : (\mathbb{R}, u) \to (\mathbb{R}, u^t)$ is not continuous at 0, and so is not uniformly continuous.*

The following result has been extended to quasi-uniform spaces (see Proposition 4.2.1).

3.2.1 Proposition. *(compare [2, Proposition 3]) If q_1 and q_2 are uniformly equivalent quasi-metrics on X and $\psi : (X, q_1) \to (X, q_1)$ is uniformly continuous, then $h_{QU}(\psi, q_1) = h_{QU}(\psi, q_2)$.*

Proof. Let $\epsilon_1 > 0$. Since $id_X : (X, q_1) \to (X, q_2)$ and $id_X : (X, q_2) \to (X, q_1)$ are both uniformly continuous, we can choose $\epsilon_2 \geq \epsilon_3 > 0$ such that $q_1(x, y) < \epsilon_1$ for all $x, y \in X$ with $q_2(x, y) < \epsilon_2$ and $q_2(x, y) < \epsilon_2$ for all $x, y \in X$ with $q_1(x, y) < \epsilon_3$. Observe that for each $x \in X$, we have

$$D_n^{q_1}(x, \epsilon_3, \psi) \subseteq D_n^{q_2}(x, \epsilon_2, \psi) \subseteq D_n^{q_1}(x, \epsilon_1, \psi)$$

for all $n \in \mathbb{N}_+$.
Let $K \in \mathcal{K}(X)$ and $F \subseteq X$ such that $K \subseteq \bigcup_{x \in F} D_n^{q_2}(x, \epsilon_2, \psi)$, then

$$K \subseteq \bigcup_{x \in F} D_n^{q_1}(x, \epsilon_1, \psi).$$

Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^{q_2}(x, \epsilon_2, \psi)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^{q_1}(x, \epsilon_1, \psi)\right\}.$$

Therefore $r_n(\epsilon_1, K, \psi, q_1) \leq r_n(\epsilon_2, K, \psi, q_2)$. Similarly it can be shown that

$$r_n(\epsilon_2, K, \psi, q_2) \leq r_n(\epsilon_3, K, \psi, q_1).$$

Letting $n \to \infty$, we obtain that

$$r(\epsilon_1, K, \psi, q_1) \leq r(\epsilon_2, K, \psi, q_2) \leq r(\epsilon_3, K, \psi, q_1).$$

Letting $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$, we get $h_{QU}(K, \psi, q_1) = h_{QU}(K, \psi, q_2)$. After taking supremum over $\mathcal{K}(X)$, we get $h_{QU}(\psi, q_1) = h_{QU}(\psi, q_2)$ as required. $\square$

The following result has been extended to quasi-uniform spaces (see Proposition 4.2.2).

3.2.2 Proposition. *(Logarithmic Law.) (compare [2, Proposition 4]) Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous map, then $h_{QU}(\psi^m) = m h_{QU}(\psi)$ for each $m \in \mathbb{N}$.*

Proof. Let $K \in \mathcal{K}(X), \epsilon > 0$ and $m \in \mathbb{N}$.
Case 1. If $m = 0$, then the statement holds, as $\psi^m = \psi^0 = id_X$ and $h_{QU}(id_X) = 0$, as $id_X : (X, q) \to (X, q)$ is nonexpansive (see Example 3.1.1).

Case 2. If $m \in \mathbb{N}_+$, we first show that $r_n(\epsilon, K, \psi^m) \leq r_{mn}(\epsilon, K, \psi)$. To this, let $x \in X$ and $y \in D_{mn}^q(x, \epsilon, \psi)$. Then $\psi^k(y) \in B_q(\psi^k(x), \epsilon)$ for all $0 \leq k < mn$. If we set $k = mi$ for $0 \leq i < n$, then $\psi^{mi}(y) \in B_q(\psi^{mi}(x), \epsilon)$ for all $0 \leq i < n$. Since $\psi^{mi} = (\psi^m)^i$ for all $0 \leq i < n$, then $y \in D_n^q(x, \epsilon, \psi^m)$. Therefore $D_{mn}^q(x, \epsilon, \psi) \subseteq D_n^q(x, \epsilon, \psi^m)$. Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_{mn}^{q}(x, \epsilon, \psi)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^{q}(x, \epsilon, \psi^m)\right\}.$$

Therefore $r_n(\epsilon, K, \psi^m) \leq r_{mn}(\epsilon, K, \psi)$. Taking logarithms on both sides and dividing throughout by n, we get

$$\frac{\log r_n(\epsilon, K, \psi^m)}{n} \leq \frac{m \log r_{mn}(\epsilon, K, \psi)}{mn} = \frac{m \log r_s(\epsilon, K, \psi)}{s},$$

where $s = mn$. Letting $n \to \infty$ and $s \to \infty$ we obtain that

$$r(\epsilon, K, \psi^m) \leq mr(\epsilon, K, \psi).$$

Letting $\epsilon \to 0$, we get $h_{QU}(K, \psi^m) \leq mh_{QU}(K, \psi)$. After taking supremum over $\mathcal{K}(X)$, we get $h_{QU}(\psi^m) \leq mh_{QU}(\psi)$.

Let $\epsilon > 0$. Since $\psi : (X, q) \to (X, q)$ is uniformly continuous, there exists $\delta > 0$ such that if $x, y \in X$ with $y \in B_q(x, \delta)$, then $\psi^j(y) \in B_q(\psi^j(x), \epsilon)$ for all $0 \leq j < m$. We are going to show that $D_n^q(x, \delta, \psi^m) \subseteq D_{mn}^q(x, \epsilon, \psi)$ for each $x \in X$.

For $x \in X$, let $y \in D_n^q(x, \delta, \psi^m)$. Then $(\psi^m)^k(y) \in B_q((\psi^m)^k(x), \delta)$ for all $0 \leq k < n$. Since $\psi^{mk} = (\psi^m)^k$ for all $0 \leq k < n$, then $\psi^{mk}(y) \in B_q(\psi^{mk}(x), \delta)$ for all $0 \leq k < n$. Let $i = mk$, then $0 \leq i < mn$. Thus $\psi^i(y) \in B_q(\psi^i(x), \delta)$ for all $0 \leq i < mn$. Since $\psi^i(y) \in B_q(\psi^i(x), \delta)$, then $\psi^{ij}(y) \in B_q(\psi^{ij}(x), \epsilon)$ for all $0 \leq j < m$ and for all $0 \leq i < mn$. Letting $j = 1$, we get $\psi^i(y) \in B_q(\psi^i(x), \epsilon)$ for all $0 \leq i < mn$. Therefore $y \in D_{mn}^q(x, \epsilon, \psi)$ and $D_n^q(x, \delta, \psi^m) \subseteq D_{mn}^q(x, \epsilon, \psi)$ follows. Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^q(x, \delta, \psi^m)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_{mn}^q(x, \epsilon, \psi)\right\}.$$

Therefore $r_{mn}(\epsilon, K, \psi) \leq r_n(\delta, K, \psi^m)$. Arguing in the same manner as above, we get $mh_{QU}(\psi) \leq h_{QU}(\psi^m)$. Putting everything together we get $h_{QU}(\psi^m) = mh_{QU}(\psi)$ as required. □

The following result has been extended to quasi-uniform spaces (see Theorem 4.2.2).

3.2.3 Proposition. *(Weak Addition Theorem.) (compare [2, Proposition 4]) Let (X_1, q_1) and (X_2, q_2) be quasi-metric spaces. Suppose $\psi_1 : (X_1, q_1) \to (X_1, q_1)$ and $\psi_2 : (X_2, q_2) \to (X_2, q_2)$ are uniformly continuous self-maps. Define a quasi-metric q on $X_1 \times X_2$ by*

$$q((x_1, x_2), (y_1, y_2)) = \max\{q_1(x_1, y_1), q_2(x_2, y_2)\},$$

for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$.
Then $\psi_1 \times \psi_2 : (X_1 \times X_2, q) \to (X_1 \times X_2, q)$ is uniformly continuous and

$$h_{QU}(\psi_1 \times \psi_2, q) \leq h_{QU}(\psi_1, q_1) + h_{QU}(\psi_2, q_2).$$

Furthermore, if X_1 or X_2 is compact, then

$$h_{QU}(\psi_1 \times \psi_2, q) = h_{QU}(\psi_1, q_1) + h_{QU}(\psi_2, q_2).$$

Proof. It is clear that $\psi_1 \times \psi_2 : (X_1 \times X_2, q) \to (X_1 \times X_2, q)$ is uniformly continuous, whenever $\psi_1 : (X_1, q_1) \to (X_1, q_1)$ and $\psi_2 : (X_2, q_2) \to (X_2, q_2)$ are uniformly continuous.
We first show that for any $\epsilon > 0$, we have

$$D_n^q((x_1, x_2), \epsilon, \psi_1 \times \psi_2) = D_n^{q_1}(x_1, \epsilon, \psi_1) \times D_n^{q_2}(x_2, \epsilon, \psi_2)$$

for each pair $(x_1, x_2) \in X_1 \times X_2$ and each $n \in \mathbb{N}_+$. Let $0 \leq k < n$, then we have that

$$\begin{aligned}
(a_1, a_2) \in D_n^q((x_1, x_2), \epsilon, \psi_1 \times \psi_2) &\Leftrightarrow q((\psi_1 \times \psi_2)^k(x_1, x_2), (\psi_1 \times \psi_2)^k(a_1, a_2)) < \epsilon \\
&\Leftrightarrow q((\psi_1^k(x_1), \psi_2^k(x_2)), (\psi_1^k(a_1), \psi_2^k(a_2))) < \epsilon \\
&\Leftrightarrow q_1(\psi_1^k(x_1), \psi_1^k(a_1)) < \epsilon \text{ and } q_2(\psi_2^k(x_2), \psi_2^k(a_2)) < \epsilon \\
&\Leftrightarrow (a_1, a_2) \in D_n^{q_1}(x_1, \epsilon, \psi_1) \times D_n^{q_2}(x_2, \epsilon, \psi_2).
\end{aligned}$$

Therefore $D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2)=D_n^{q_1}(x_1,\epsilon,\psi_1)\times D_n^{q_2}(x_2,\epsilon,\psi_2)$, as required.
Secondly, we show that for each $n\in\mathbb{N}_+$ and each $\epsilon>0$, we have

$$\bigcup_{x_1\in F_1} D_n^{q_1}(x_1,\epsilon,\psi_1)\times\bigcup_{x_2\in F_2} D_n^{q_2}(x_2,\epsilon,\psi_2)\subseteq\bigcup_{(x_1,x_2)\in F_1\times F_2} D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2)$$

for each $F_1\subseteq X_1$ and each $F_2\subseteq X_2$.
Let $(a_1,a_2)\in\bigcup_{x_1\in F_1} D_n^{q_1}(x_1,\epsilon,\psi_1)\times\bigcup_{x_2\in F_2} D_n^{q_2}(x_2,\epsilon,\psi_2)$, then there exists a pair $(x_1,x_2)\in F_1\times F_2$ such that

$$(a_1,a_2)\in D_n^{q_1}(x_1,\epsilon,\psi_1)\times D_n^{q_2}(x_2,\epsilon,\psi_2)=D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2).$$

Since

$$D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2)\subseteq\bigcup_{(x_1,x_2)\in F_1\times F_2} D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2),$$

then

$$(a_1,a_2)\in\bigcup_{(x_1,x_2)\in F_1\times F_2} D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2)$$

as required.
Finally, for each $\epsilon>0$ and each $n\in\mathbb{N}_+$, let $K_i\in\mathcal{K}(X_i)$ and let $F_i\subseteq X_i$ for $i=1,2$ such that

$$K_1\subseteq\bigcup_{x_1\in F_1} D_n^{q_1}(x_1,\epsilon,\psi_1)\text{ and }K_2\subseteq\bigcup_{x_2\in F_2} D_n^{q_2}(x_2,\epsilon,\psi_2).$$

According to what we have shown above, it follows that

$$K_1\times K_2\subseteq\bigcup_{(x_1,x_2)\in F_1\times F_2} D_n^q((x_1,x_2),\epsilon,\psi_1\times\psi_2).$$

Now we have that

$$\text{if } |F_1|\in\left\{|E_1|:E_1\subseteq X_1\text{ and }K_1\subseteq\bigcup_{x_1\in E_1} D_n^{q_1}(x_1,\epsilon,\psi_1)\right\}$$

and

$$|F_2| \in \left\{ |E_2| : E_2 \subseteq X_2 \text{ and } K_2 \subseteq \bigcup_{x_2 \in E_2} D_n^{q_2}(x_2, \epsilon, \psi_2) \right\},$$

then

$$|F_1| \times |F_2| \in \left\{ |E_1 \times E_2| : E_1 \times E_2 \subseteq X_1 \times X_2 \text{ and } K_1 \times K_2 \subseteq \bigcup_{(x_1,x_2) \in E_1 \times E_2} D_n^{q}((x_1, x_2), \epsilon, \psi_1 \times \psi_2) \right\}.$$

Hence $r_n(\epsilon, K_1 \times K_2, \psi_1 \times \psi_2, q) \leq \Big(r_n(\epsilon, K_1, \psi_1, q_1) \Big) \Big(r_n(\epsilon, K_2, \psi_2, q_2) \Big)$. Taking logarithms and dividing throughout by n, we get

$$r(\epsilon, K_1 \times K_2, \psi_1 \times \psi_2, q) \leq r(\epsilon, K_1, \psi_1, q_1) + r(\epsilon, K_2, \psi_2, q_2).$$

Letting $\epsilon \to 0$, we get $h_{QU}(K_1 \times K_2, \psi_1 \times \psi_2, q) \leq h_{QU}(K_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2)$.
For each $\epsilon > 0$ and each $n \in \mathbb{N}_+$, let $K_i \in \mathcal{K}(X_i)$ and $F_i \subseteq K_i$ be (n, ϵ)-separated with respect to ψ_i for each $i = 1, 2$. Then $F_1 \times F_2 \subseteq K_1 \times K_2$. It can easily be seen that for any pairs $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$.

$$D_n^{q^s}((x_1, x_2), \epsilon, \psi_1 \times \psi_2) = D_n^{(q_1)^s}(x_1, \epsilon, \psi_1) \times D_n^{(q_2)^s}(x_2, \epsilon, \psi_2)$$

and the sets

$$D_n^{q^s}((x_1, x_2), \epsilon, \psi_1 \times \psi_2) \cap D_n^{q^s}((y_1, y_2), \epsilon, \psi_1 \times \psi_2)$$

and

$$\Big(D_n^{(q_1)^s}(x_1, \epsilon, \psi_1) \cap D_n^{(q_1)^s}(y_1, \epsilon, \psi_1) \Big) \times \Big(D_n^{(q_2)^s}(x_2, \epsilon, \psi_2) \cap D_n^{(q_2)^s}(y_2, \epsilon, \psi_2) \Big)$$

are equal.
We claim that $F_1 \times F_2$ is (n, ϵ)-separated with respect to $\psi_1 \times \psi_2$. To this, let $(x_1, x_2), (y_1, y_2) \in F_1 \times F_2$ such that $(x_1, x_2) \neq (y_1, y_2)$.
Case 1. If $x_1 \neq y_1$, then $D_n^{(q_1)^s}(x_1, \epsilon, \psi_1) \cap D_n^{(q_1)^s}(y_1, \epsilon, \psi_1) = \emptyset$, as F_1 is (n, ϵ)-separated with respect to ψ_1. Then by what we have done above, it follows that

$$D_n^{q^s}((x_1, x_2), \epsilon, \psi_1 \times \psi_2) \cap D_n^{q^s}((y_1, y_2), \epsilon, \psi_1 \times \psi_2) = \emptyset.$$

Case 2. If $x_2 \neq y_2$, then $D_n^{(q_2)^s}(x_2, \epsilon, \psi_2) \cap D_n^{(q_2)^s}(y_2, \epsilon, \psi_2) = \emptyset$, as F_2 is (n, ϵ)-separated with respect to ψ_2. Similarly, it follows that

$$D_n^{q^s}((x_1, x_2), \epsilon, \psi_1 \times \psi_2) \cap D_n^{q^s}((y_1, y_2), \epsilon, \psi_1 \times \psi_2) = \emptyset.$$

Hence $F_1 \times F_2$ is (n, ϵ)-separated with respect to $\psi_1 \times \psi_2$ as required. Now we have that

$$\text{if } |F_1| \in \Big\{ |E_1| : E_1 \subseteq K_1 \text{ and } E_1 \text{ is } (n, \epsilon)\text{-separated with respect to } \psi_1 \Big\}$$

and

$$|F_2| \in \Big\{ |E_2| : E_2 \subseteq K_2 \text{ and } E_2 \text{ is } (n, \epsilon)\text{-separated with respect to } \psi_2 \Big\},$$

then

$$|F_1| \times |F_2| \in \Big\{ |E_1 \times E_2| : E_1 \times E_2 \subseteq K_1 \times K_2 \text{ and } E_1 \times E_2 \text{ is } (n, \epsilon)\text{-separated with respect to } \psi_1 \times \psi_2 \Big\}.$$

Hence $\Big(s_n(\epsilon, K_1, \psi_1, q_1)\Big)\Big(s_n(\epsilon, K_2, \psi_2, q_2)\Big) \leq s_n(\epsilon, K_1 \times K_2, \psi_1 \times \psi_2, q)$. Taking logarithms and dividing throughout by n, we get

$$s(\epsilon, K_1, \psi_1, q_1) + s(\epsilon, K_2, \psi_2, q_2) \leq s(\epsilon, K_1 \times K_2, \psi_1 \times \psi_2, q).$$

Letting $\epsilon \to 0$, we get

$$h_{QU}(K_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2) \leq h_{QU}(K_1 \times K_2, \psi_1 \times \psi_2, q).$$

Since

$$h_{QU}(K_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2) \leq h_{QU}(K_1 \times K_2, \psi_1 \times \psi_2, q) \leq h_{QU}(K_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2),$$

then

$$h_{QU}(K_1 \times K_2, \psi_1 \times \psi_2, q) = h_{QU}(K_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2). \qquad (3.4)$$

Let $\pi_i : X_1 \times X_2 \to X_i$ be the projection map. If $K \in \mathcal{K}(X_1 \times X_2)$, then $K_i = \pi_i(K) \in \mathcal{K}(X_i)$ for each $i = 1, 2$ and $K \subseteq K_1 \times K_2$. It follows that

$$\begin{aligned} h_{QU}(\psi_1 \times \psi_2, q) &= \sup\{h_{QU}(K, \psi_1 \times \psi_2, q) : K \in \mathcal{K}(X_1 \times X_2)\} \\ &\leq \sup\{h_{QU}(K_1 \times K_2, \psi_1 \times \psi_2, q) : K_1 \in \mathcal{K}(X_1) \text{ and } K_2 \in \mathcal{K}(X_2)\} \\ &= \sup\{h_{QU}(K_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2) : K_1 \in \mathcal{K}(X_1) \text{ and } K_2 \in \mathcal{K}(X_2)\} \\ &= \sup\{h_{QU}(K_1, \psi_1, q_1) : K_1 \in \mathcal{K}(X_1)\} + \sup\{h_{QU}(K_2, \psi_2, q_2) : K_2 \in \mathcal{K}(X_2)\} \\ &= h_{QU}(\psi_1, q_1) + h_{QU}(\psi_2, q_2), \end{aligned}$$

where the third equality from below is obtained by using equation (3.4).
Suppose X_1 or X_2 is compact. Without loss of generality we may assume X_1 is compact. Since every element of $\mathcal{K}(X_1 \times X_2)$ is a subset of $X_1 \times K_2$ for some $K_2 \in \mathcal{K}(X_2)$, we have

$$h_{QU}(\psi_1 \times \psi_2, q) = \sup\{h_{QU}(X_1 \times K_2, \psi_1 \times \psi_2, q) : K_2 \in \mathcal{K}(X_2)\}.$$

Since X_1 is compact, then by Proposition 3.1.2 we get that $h_{QU}(\psi_1, q_1) = h_{QU}(X_1, \psi_1, q_1)$. Replacing K_1 by X_1 in equation (3.4), we get

$$h_{QU}(X_1 \times K_2, \psi_1 \times \psi_2, q) = h_{QU}(X_1, \psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2) = h_{QU}(\psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2).$$

Therefore

$$\begin{aligned} h_{QU}(\psi_1 \times \psi_2, q) &= \sup\{h_{QU}(X_1 \times K_2, \psi_1 \times \psi_2, q) : K_2 \in \mathcal{K}(X_2)\} \\ &= \sup\{h_{QU}(\psi_1, q_1) + h_{QU}(K_2, \psi_2, q_2) : K_2 \in \mathcal{K}(X_2)\} \\ &= h_{QU}(\psi_1, q_1) + \sup\{h_{QU}(K_2, \psi_2, q_2) : K_2 \in \mathcal{K}(X_2)\} \\ &= h_{QU}(\psi_1, q_1) + h_{QU}(\psi_2, q_2). \end{aligned}$$

□

3.2.1 Lemma. *Let (X, q) be a quasi-metric space, $\psi : (X, q) \to (X, q)$ a uniformly continuous self-map and let Y be a ψ-invariant subset of X. For $n \in \mathbb{N}_+$ and $\epsilon > 0$ the following statements holds.*

(i) *If $y \in Y$, then*

$$D_n^{q_Y}(y,\epsilon,\psi|_Y) = D_n^q(y,\epsilon,\psi)\cap Y \text{ and } D_n^{(q_Y)^s}(y,\epsilon,\psi|_Y) = D_n^{q^s}(y,\epsilon,\psi)\cap Y.$$

(ii) $r_n(\epsilon, K, \psi|_Y, q_Y) = r_n(\epsilon, K, \psi, q)$ *for each* $K \in \mathcal{K}(Y)$.

Proof. (i) Let $y \in Y, n \in \mathbb{N}_+$ and $\epsilon > 0$. Since (Y, q_Y) is a subspace of (X, q), then $B_{q_Y}(y,\epsilon) = B_q(y,\epsilon)\cap Y$. Since $\psi(Y) \subseteq Y$, then $Y \subseteq \psi^{-k}(Y)$ for all $k \in \mathbb{N}_+ \cup \{0\}$. Now we get that

$$\begin{aligned} D_n^{q_Y}(y,\epsilon,\psi|_Y) &= \bigcap_{k=0}^{n-1} (\psi|_Y)^{-k}[B_{q_Y}((\psi|_Y)^k(y),\epsilon)] \\ &= \bigcap_{k=0}^{n-1} \psi^{-k}[B_q(\psi^k(y),\epsilon)\cap Y] \\ &= \bigcap_{k=0}^{n-1} \psi^{-k}[B_q(\psi^k(y),\epsilon)] \cap \bigcap_{k=0}^{n-1} \psi^{-k}(Y) \\ &= D_n^q(y,\epsilon,\psi)\cap Y. \end{aligned}$$

Since $D_n^{q_Y}(y,\epsilon,\psi|_Y) = D_n^q(y,\epsilon,\psi)\cap Y$, then

$$D_n^{(q_Y)^s}(y,\epsilon,\psi|_Y) = D_n^{q^s}(y,\epsilon,\psi)\cap Y.$$

(ii) Let $K \in \mathcal{K}(Y)$. By (i) we have that $D_n^{q_Y}(y,\epsilon,\psi|_Y) \subseteq D_n^q(y,\epsilon,\psi)$, thus

$$\left\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y\in F} D_n^{q_Y}(y,\epsilon,\psi|_Y)\right\} \subseteq \left\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y\in F} D_n^q(y,\epsilon,\psi)\right\}.$$

Hence $r_n(\epsilon, K, \psi, q) \le r_n(\epsilon, K, \psi|_Y, q_Y)$.
Let $F \subseteq Y$ such that $K \subseteq \bigcup_{y\in F} D_n^q(y,\epsilon,\psi)$. Since

$$D_n^{q_Y}(y,\epsilon,\psi|_Y) = D_n^q(y,\epsilon,\psi)\cap Y \text{ and } K \subseteq Y,$$

then

$$K = K\cap Y \subseteq \bigcup_{y\in F} D_n^q(y,\epsilon,\psi)\cap Y = \bigcup_{y\in F}\Big(D_n^q(y,\epsilon,\psi)\cap Y\Big) = \bigcup_{y\in F} D_n^{q_Y}(y,\epsilon,\psi|_Y).$$

Therefore

$$\left\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y\in F} D_n^q(y,\epsilon,\psi)\right\} \subseteq \left\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y\in F} D_n^{q_Y}(y,\epsilon,\psi|_Y)\right\}.$$

Hence $r_n(\epsilon, K, \psi|_Y, q_Y) \leq r_n(\epsilon, K, \psi, q)$. Since

$$r_n(\epsilon, K, \psi, q) \leq r_n(\epsilon, K, \psi|_Y, q_Y) \leq r_n(\epsilon, K, \psi, q),$$

then $r_n(\epsilon, K, \psi|_Y, q_Y) = r_n(\epsilon, K, \psi, q)$. □

The following result has been extended to quasi-uniform spaces (see Proposition 4.4.1).

3.2.4 Proposition. *(Monotonicity with respect to subspaces) (compare [16, Lemma 5.1]) Let (X, q) be a quasi-metric space, $\psi : (X, q) \to (X, q)$ a uniformly continuous self-map and let Y be a ψ-invariant subset of X. Then*

$$h_{QU}(\psi|_Y, q_Y) \leq h_{QU}(\psi, q).$$

Proof. Let $K \in \mathcal{K}(Y), n \in \mathbb{N}_+$ and $\epsilon > 0$. Since $\mathcal{K}(Y) \subseteq \mathcal{K}(X)$, then $K \in \mathcal{K}(X)$ and by Lemma 3.2.1 (ii), we have $r_n(\epsilon, K, \psi|_Y, q_Y) = r_n(\epsilon, K, \psi, q)$. Letting $\epsilon \to 0$, we get $h_{QU}(K, \psi|_Y, q_Y) = h_{QU}(K, \psi, q)$. Therefore

$$\begin{aligned} h_{QU}(\psi|_Y, q_Y) &= \sup\{h_{QU}(K, \psi|_Y, q_Y) : K \in \mathcal{K}(Y)\} \\ &= \sup\{h_{QU}(K, \psi, q) : K \in \mathcal{K}(Y)\} \\ &\leq \sup\{h_{QU}(K, \psi, q) : K \in \mathcal{K}(X)\} \\ &= h_{QU}(\psi, q). \end{aligned}$$

□

3.3 Relationship between the uniform and quasi-uniform entropy on a quasi-metric space

Since every quasi-metric space (X, q) has the associated symmetrised metric space (X, q^s) and by Lemma 1.1.1 we have that every uniformly continuous self-map $\psi : (X, q) \to (X, q)$ is also a uniformly continuous self-map of the symmetrised metric space (X, q^s), then it is natural to investigate the relationship between the uniform entropy $h_U(\psi, q^s)$ and the quasi-uniform entropy $h_{QU}(\psi, q)$ for a uniformly continuous self-map $\psi : (X, q) \to (X, q)$. This section is devoted to the investigations of such relationships.

3.3.1 Claim. *If (X, q) is a quasi-metric space and $\psi : (X, q) \to (X, q)$ is a uniformly continuous self-map, then for each $K \in \mathcal{K}(X)$, there exists some non-empty join-compact subset K' of X such that*

$$h_{QU}(K, \psi, q) \leq h_U(K', \psi, q^s).$$

Proof. Let $K \in \mathcal{K}(X)$, $n \in \mathbb{N}_+$ and $\epsilon > 0$. Let $F \subseteq X$ such that $K \subseteq \bigcup_{x \in F} D_n^q(x, \epsilon, \psi)$ and $r_n(\epsilon, K, \psi, q) = |F|$. Since $r_n(\epsilon, K, \psi, q) = |F|$ and $K \neq \emptyset$, then F is a non-empty finite subset of X. Hence $K' = F$ is a non-empty join-compact subset of X. Let $F' \subseteq X$ such that $K' \subseteq \bigcup_{x \in F} D_n^{q^s}(x, \epsilon, \psi)$ and $r_n(\epsilon, K', \psi, q^s) = |F'|$. We first show that

$$K \subseteq \bigcup_{x \in F'} D_n^q(x, 2\epsilon, \psi).$$

For this let $y \in K$. Since $r_n(\epsilon, K, \psi, q) = |F|$ and $r_n(\epsilon, K', \psi, q^s) = |F'|$, then there exists some $x \in F$ and some $z \in F'$ such that $q(\psi^i(x), \psi^i(y)) < \epsilon$ and $q(\psi^i(z), \psi^i(x)) \leq q^s(\psi^i(z), \psi^i(x)) < \epsilon$ for all $0 \leq i < n$. Thus $q(\psi^i(z), \psi^i(y)) \leq q(\psi^i(z), \psi^i(x)) + q(\psi^i(x), \psi^i(y)) < \epsilon + \epsilon = 2\epsilon$ for all $0 \leq i < n$ and it follows that

$$y \in D_n^q(z, 2\epsilon, \psi) \subseteq \bigcup_{x \in F'} D_n^q(x, 2\epsilon, \psi).$$

Therefore $K \subseteq \bigcup_{x \in F'} D_n^q(x, 2\epsilon, \psi)$, as required.

Since $K \subseteq \bigcup_{x \in F'} D_n^q(x, 2\epsilon, \psi)$, then $r_n(2\epsilon, K, \psi, q) \leq |F'| = r_n(\epsilon, K', \psi, q^s)$. Thus $r(2\epsilon, K, \psi, q) \leq r(\epsilon, K', \psi, q^s)$. Letting $\epsilon \to 0$, we get

$$h_{QU}(K, \psi, q) \leq h_U(K', \psi, q^s).$$

□

The following result has been extended to quasi-uniform spaces (see Proposition 4.2.4).

3.3.1 Theorem. *Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous self-map. Then*

(i) $h_{QU}(\psi, q) \leq h_U(\psi, q^s)$*;*

(ii) $\max\{h_{QU}(\psi, q), h_{QU}(\psi, q^t)\} \leq h_U(\psi, q^s)$.

Proof. Suppose $\psi : (X, q) \to (X, q)$ is uniformly continuous, then by Lemma 1.1.1 we have that $\psi : (X, q^s) \to (X, q^s)$ is uniformly continuous. Let K be a non-empty join-compact subset of X. Then by Remark 1.1.2 we have that K is a non-empty compact subset of (X, q).

(i) Let $n \in \mathbb{N}_+$ and $\epsilon > 0$. Since $D_n^{q^s}(x, \epsilon, \psi) \subseteq D_n^q(x, \epsilon, \psi)$, we get that

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n^{q^s}(x, \epsilon, \psi)\right\} \subseteq \left\{|F'| : F' \subseteq X \text{ and } K \subseteq \bigcup_{x \in F'} D_n^q(x, \epsilon, \psi)\right\}.$$

Hence $r_n(\epsilon, K, \psi, q) \leq r_n(\epsilon, K, \psi, q^s)$. Therefore $h_{QU}(K, \psi, q) \leq h_U(K, \psi, q^s)$ for each non-empty join-compact subset K of X. Let K be a non-empty $\tau(q)$-compact subset of X. By Claim 3.3.1, we can choose a non-empty join-compact subset K'

of X such that $h_{QU}(K, \psi, q) \leq h_U(K', \psi, q^s)$. Since this holds for each non-empty compact subset of X, it then follows that

$$\begin{aligned} h_{QU}(\psi, q) &= \sup\{h_{QU}(K, \psi, q) : K \text{ is a compact subset of } X\} \\ &\leq \sup\{h_U(K, \psi, q^s) : K \text{ is a join-compact subset of } X\} \\ &= h_U(\psi, q^s). \end{aligned}$$

Therefore $h_{QU}(\psi, q) \leq h_U(\psi, q^s)$, as required.

(ii) By (i), we get that $h_{QU}(\psi, q^t) \leq h_U(\psi, (q^t)^s) = h_U(\psi, q^s)$, as $(q^t)^s = q^s$. Since $h_{QU}(\psi, q), h_{QU}(\psi, q^t) \leq h_U(\psi, q^s)$, then it follows that

$$\max\{h_{QU}(\psi, q), h_{QU}(\psi, q^t)\} \leq h_U(\psi, q^s).$$

□

It must be noted that if ψ is a nonexpansive self-map of a quasi-metric space (X, q), then by Example 3.1.1 we have that $h_{QU}(\psi, q) = 0$ and it is well known that $h_U(\psi, q^s) = 0$. Hence it is possible for equality in Theorem 3.3.1 to hold. Also by Theorem 3.3.1 we have that our entropy function is less or equal to the entropy function defined by Sayyari, Molari and Moghayer [20].

3.3.1 Example. *Let us consider $\mathbb{R}$ equipped with its usual quasi-metric u. Let $\psi : (\mathbb{R}, u) \to (\mathbb{R}, u)$ be the map defined by $\psi(x) = 2x$ for each $x \in \mathbb{R}$. Then ψ is uniformly continuous and*

$$h_{QU}(\psi, u) < h_U(\psi, u^s).$$

It is well known that $h_U(\psi, u^s) = \log 2$ (see for instance [21, Remark 15 Page 171]). We will proceed to show that $h_{QU}(\psi, u) = 0$. It is easily seen that for each $x \in \mathbb{R}$ and each $k \in \mathbb{N}_+ \cup \{0\}$, we have that $B_u(x, \epsilon) = (x - \epsilon, \infty)$ and $\psi^{-k}(B_u(\psi^k(x), \epsilon)) = B_u(x, \frac{\epsilon}{2^k})$. Hence

$$
\begin{aligned}
D_n^u(x,\epsilon,\psi) &= \bigcap_{k=0}^{n-1} \psi^{-k}(B_u(\psi^k(x),\psi)) \\
&= \bigcap_{k=0}^{n-1} B_u(x,\frac{\epsilon}{2^k}) \\
&= \bigcap_{k=0}^{n-1} (x-\frac{\epsilon}{2^k},\infty) \\
&= (x-\frac{\epsilon}{2^{n-1}},\infty).
\end{aligned}
$$

Since every join-compact subset of $\mathbb{R}$ *is contained in a join-compact subset of the form* $K_m = [-m,m]$ *for some* $m \in \mathbb{N}_+$*, then it suffices to compute the quasi-uniform entropy using join-compact subsets of this form. Let* F *be a finite subset of* $\mathbb{R}$ *of minimal cardinality such that*

$$
K_m \subseteq \bigcup_{x\in F} D_n^u(x,\epsilon,\psi),
$$

then $K_m \subseteq (z,\infty)$*, where* $z = \min\{x_i - \frac{\epsilon}{2^{n-1}} : x_i \in F \quad for \quad i = 1,2,.....,|F|\}$*. This implies that* $K_m \subseteq \bigcup_{x\in\{z\}} (x,\infty)$*. Thus* $r_n(\epsilon,K_m,\psi,u) = 1$ *and it follows that* $h_{QU}(K_m,\psi,u) = 0$*. Thus* $h_{QU}(\psi,u) = 0$ *and it follows that* $h_{QU}(\psi,u) < h_U(\psi,u^s)$ *whenever the quasi-uniform entropy is computed using join-compact subsets of* $\mathbb{R}$*. It only remains to show that the inequality still holds for* $\tau(u)$*-compact subsets of* $\mathbb{R}$ *which are not necessarily join-compact. If we consider* $\tau(u)$*-compact subsets of* $\mathbb{R}$ *of the form* $K_a = [a,\infty)$ *or* $K_a = (a,\infty)$*, for some* $a \in \mathbb{R}$*. It must be noted that* K_a *is not join- compact, as in both cases* K_a *is not bounded. It can also be shown that every* $\tau(u)$*-compact subset of* $\mathbb{R}$ *is contained in a* $\tau(u)$*-compact subset of the form* K_a *for some* $a \in \mathbb{R}$*. Hence in computing* $h_{QU}(\psi,u)$*, it suffices to consider* $\tau(u)$*-compact subsets of this form. Arguing as above, it can be shown that* $r_n(\epsilon,K_a,\psi,u) = 1$ *and it follows that* $h_{QU}(\psi,u) = 0$ *irrespective if we use join-compact subsets or* $\tau(u)$*-compact subsets of* $\mathbb{R}$*. Thus* $h_{QU}(\psi,u) < h_U(\psi,u^s)$*, as required.*

3.4 Completion theorem for quasi-uniform entropy on a quasi-metric space

In [16] Kimura proved the completion theorem for uniform entropy on a uniform space and he pointed out that the completion theorem for Bowen's entropy on a metric space does not hold in general, even if the metric space is totally bounded. We are going to investigate the completion theorem for quasi-uniform entropy on a quasi-metric space. The concept of completing a quasi-metric space has been investigated by several authors, see for instance [4, 5]. It is well known that there are several completion notions for a quasi-metric space and in this section we are going to use the bicompletion.

Recall from definition 1.1.2 that a quasi-metric space (X, q) is said to be *bicomplete* if the metric space (X, q^s) is complete.

3.4.1 Definition. *(see [5, Chapter 1]) Let* (X, q) *be a quasi-metric space. A* bicompletion *of* (X, q) *is a pair* $\left((\widetilde{X}, \widetilde{q}), \varphi\right)$ *consisting of a bicomplete quasi-metric space* $(\widetilde{X}, \widetilde{q})$ *and an isometry* $\varphi : (X, q) \to (\widetilde{X}, \widetilde{q})$ *such that* $\varphi(X)$ *is dense in the metric space* $(\widetilde{X}, (\widetilde{q})^s)$*.*

Since every metric space admits a completion, then it follows that every quasi-metric space admits a bicompletion. The bicompletion $(\widetilde{X}, \widetilde{q})$ of a quasi-metric space (X, q) is taken to be the metric completion $(\widetilde{X}, \widetilde{(q^s)})$ of the metric space (X, q^s) which is constructed as follows:

$\widetilde{X} = \{[(x_n)_{\mathbb{N}}] : (x_n)_{\mathbb{N}}$ is a q^s -Cauchy sequence in $X\}$ and $[(x_n)_{\mathbb{N}}]$ is the equivalence class of the q^s-Cauchy sequence $(x_n)_{\mathbb{N}}$ in X. Two q^s-Cauchy sequences $(x_n)_{\mathbb{N}}$ and $(y_n)_{\mathbb{N}}$ in X are said to be equivalent if and only if $\lim\limits_{n\to\infty} q^s(x_n, y_n) = 0$. The metric $\widetilde{(q^s)}$ on $\widetilde{X}$ is defined by

$$\widetilde{(q^s)}\Big([(x_n)_{\mathbb{N}}],[(y_n)_{\mathbb{N}}]\Big) = \lim_{n\to\infty} q^s(x_n, y_n),$$

for each $[(x_n)_{\mathbb{N}}],[(y_n)_{\mathbb{N}}] \in \widetilde{X}$. The quasi-metric $\widetilde{q}$ on $\widetilde{X}$ is defined by $\widetilde{q} = \widetilde{(q^s)}$. Since $\widetilde{(q^s)}$ is a metric, then we have that $(\widetilde{q})^s = \widetilde{(q^s)}$. The isometric embedding $\varphi : (X,q) \to (\widetilde{X},\widetilde{q})$ is defined by $\varphi(x) = \hat{x}$, where $\hat{x} = [(x)_{\mathbb{N}}]$ is the equivalence class of the constant sequence $(x)_{\mathbb{N}} = \{x, x, ...\}$ for each $x \in X$.

Let (X,q) be a quasi-metric space, $\psi : (X,q) \to (X,q)$ a uniformly continuous function and $(\widetilde{X},\widetilde{q})$ be the bicompletion of (X,q). Let $\widetilde{\psi}$ be the uniformly continuous extension of ψ over $\widetilde{X}$, then $\varphi \circ \psi = \widetilde{\psi} \circ \varphi$, i.e., the following diagram

$$\begin{tikzcd} (X,q) \arrow[r, "\psi"] \arrow[d, "\varphi"] & (X,q) \arrow[d, "\varphi"] \\ (\widetilde{X},\widetilde{q}) \arrow[r, "\widetilde{\psi}"] & (\widetilde{X},\widetilde{q}) \end{tikzcd}$$

commutes. Hence $\widehat{\psi(x)} = \widetilde{\psi}(\hat{x})$ for all $x \in X$. We are interested in the relationship between $h_{QU}(\psi, q)$ and $h_{QU}(\widetilde{\psi},\widetilde{q})$.

The following result is what we call the completion theorem when the underlying space is join-compact.

3.4.1 Theorem. *(compare [16, Theorem 5.2 and 5.3]) Let (X,q) be a quasi-metric space and $\psi : (X,q) \to (X,q)$ a uniformly continuous self-map. If $\Big((\widetilde{X},\widetilde{q}),\varphi\Big)$ is the bicompletion of (X,q) and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$, then*

$$h_{QU}(\psi, q) \le h_{QU}(\widetilde{\psi},\widetilde{q}),$$

where equality holds if the underlying space (X,q) is join-compact.

Proof. Since $q = \widetilde{q}_X = \widetilde{q}|_{X\times X}$ and $\psi = \widetilde{\psi}|_X$, then by Proposition 3.2.4 we have that $h_{QU}(\psi, q) = h_{QU}(\widetilde{\psi}|_X, \widetilde{q}_X) \leq h_{QU}(\widetilde{\psi}, \widetilde{q})$. Suppose (X, q) is join-compact. Then (X, q^s) is totally bounded. Since the completion of a totally bounded metric space is also totally bounded, then $(\widetilde{X}, (\widetilde{q})^s)$ is totally bounded. Hence $(\widetilde{X}, \widetilde{q})$ is $\widetilde{q}$-totally bounded. Since $(\widetilde{X}, \widetilde{q})$ is $\widetilde{q}$- totally bounded and bicomplete, then $(\widetilde{X}, \widetilde{q})$ is join-compact. Let $n \in \mathbb{N}_+, \epsilon > 0$ and $0 \leq i < n$. Since $\left(\widetilde{\psi}\right)^i : (\widetilde{X}, \widetilde{q}) \to (\widetilde{X}, \widetilde{q})$ is uniformly continuous, there exists $\delta_i > 0$ such that $\widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(x), \left(\widetilde{\psi}\right)^i(y)\Big) < \frac{\epsilon}{2}$ for all $x, y \in \widetilde{X}$ with $\widetilde{q}(x, y) < \delta_i$.

Let $F \subseteq X$ such that $X \subseteq \displaystyle\bigcup_{x \in F} D_n^q(x, \frac{\epsilon}{2}, \psi)$ and $r_n(\frac{\epsilon}{2}, X, \psi, q) = |F|$. We first show that

$$\widetilde{X} \subseteq \bigcup_{x \in F} D_n^{\widetilde{q}}(\hat{x}, \epsilon, \widetilde{\psi}).$$

To this let $z \in \widetilde{X}$. Since $\widetilde{X} = \mathrm{Cl}_{\tau((\widetilde{q})^s)}(\varphi(X))$ (as $\varphi(X)$ is dense in the metric space $(\widetilde{X}, (\widetilde{q})^s)$, there exists a sequence $(w_m)_{\mathbb{N}}$ in X such that the sequence of constant sequences $(\hat{w}_m)_{\mathbb{N}}$ is $(\widetilde{q})^s$-convergent to z. Since $(\hat{w}_m)_{\mathbb{N}}$ is $(\widetilde{q})^s$-convergent to z and $\epsilon > 0$, then there exists some $k \in \mathbb{N}$ such that $(\widetilde{q})^s(\hat{w}_m, z) < \epsilon$ for all $m \geq k$. In particular $(\widetilde{q})^s(\hat{w}_k, z) < \epsilon$. Letting $\epsilon \to 0$, we obtain that $(\widetilde{q})^s(\hat{w}_k, z) = 0$. Since $\widetilde{q}(\hat{w}_k, z) = (\widetilde{q})^s(\hat{w}_k, z) = 0 < \delta_i$, then $\widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(\hat{w}_k), \left(\widetilde{\psi}\right)^i(z)\Big) < \frac{\epsilon}{2}$. Since $w_k \in X$ and $X \subseteq \displaystyle\bigcup_{x \in F} D_n^q(x, \frac{\epsilon}{2}, \psi)$, we can choose $x_k \in F$ such that $q(\psi^i(x_k), \psi^i(w_k)) < \frac{\epsilon}{2}$. Now we have that

$$\begin{aligned}
\widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(\hat{x}_k), \left(\widetilde{\psi}\right)^i(z)\Big) &\leq \widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(\hat{x}_k), \left(\widetilde{\psi}\right)^i(\hat{w}_k)\Big) + \widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(\hat{w}_k), \left(\widetilde{\psi}\right)^i(z)\Big) \\
&= \widetilde{q}\Big(\widehat{\psi^i(x_k)}, \widehat{\psi^i(w_k)}\Big) + \widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(\hat{w}_k), \left(\widetilde{\psi}\right)^i(z)\Big) \\
&= q\Big(\psi^i(x_k), \psi^i(w_k)\Big) + \widetilde{q}\Big(\left(\widetilde{\psi}\right)^i(\hat{w}_k), \left(\widetilde{\psi}\right)^i(z)\Big) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $z \in D_n^{\widetilde{q}}(\hat{x}_k, \epsilon, \widetilde{\psi})$ and it follows that

$$\widetilde{X} \subseteq \bigcup_{x \in F} D_n^{\widetilde{q}}(\hat{x}, \epsilon, \widetilde{\psi}).$$

Therefore

$$|\varphi(F)| \in \left\{ |E| : E \subseteq \widetilde{X} \text{ and } \widetilde{X} \subseteq \bigcup_{y \in E} D_n^{\widetilde{q}}(y, \epsilon, \widetilde{\psi}) \right\},$$

and it follows that $r_n(\epsilon, \widetilde{X}, \widetilde{\psi}, \widetilde{q}) \leq |\varphi(F)|$. Since $\varphi : F \to \widetilde{X}$ is injective (as every isometry is injective), then $|\varphi(F)| = |F|$. Therefore $r_n(\epsilon, \widetilde{X}, \widetilde{\psi}, \widetilde{q}) \leq r_n(\frac{\epsilon}{2}, X, \psi, q)$ and $h_{QU}(\widetilde{X}, \widetilde{\psi}, \widetilde{q}) \leq h_{QU}(X, \psi, q)$ follows.
Since $\widetilde{X}$ and X are both join-compact, then $(\widetilde{X}, \widetilde{q})$ and (X, q) are both compact and applying Proposition 3.1.2, it follows that

$$h_{QU}(\widetilde{\psi}, \widetilde{q}) = h_{QU}(\widetilde{X}, \widetilde{\psi}, \widetilde{q}) \leq h_{QU}(X, \psi, q) = h_{QU}(\psi, q).$$

$\square$

Chapter 4

Entropy on quasi-uniform spaces

In this chapter following our recent work in [11] and the work of Hood [15], we are going to show that the notion of quasi-uniform entropy presented in Chapter 3, can easily be extended to quasi-uniform spaces. In doing so, we also show that certain results about uniform entropy of a uniformly continuous self-map of a uniform space can easily be generalized to a uniformly continuous self-map of a quasi-uniform space. Most of the results in this chapter are contained in our recent article [12].

4.1 Quasi-uniform entropy on a quasi-uniform space

Following a similar approach as for section 3.1, we start this section with a scheme which will eventually allow us to define the notion of quasi-uniform entropy for a uniformly continuous self-map ψ of a quasi-uniform space $(X,\mathcal{U})$.

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. For $V \in \mathcal{U}$, $x \in X$ and $n \in \mathbb{N}_+$, we set

$$D_n(x, V, \psi) = \bigcap_{k=0}^{n-1} \psi^{-k}(V(\psi^k(x)))$$

and

$$D_n(x, V^s, \psi) = D_n(x, V, \psi) \cap D_n(x, V^{-1}, \psi).$$

It follows that

$$D_n(x, V^s, \psi) \subseteq D_n(x, V, \psi) \text{ and } D_n(x, V^s, \psi) \subseteq D_n(x, V^{-1}, \psi).$$

If $\mathcal{U}$ is a uniformity, then $D_n(x, V, \psi) = D_n(x, V^{-1}, \psi)$.
We write $D_n^{\mathcal{U}}(x, V, \psi)$ and $D_n^{\mathcal{U}}(x, V^s, \psi)$ if we need to emphasise on the quasi-uniformity $\mathcal{U}$ used.

Let $\mathcal{K}(X)$ be the collection of all nonempty $\tau(\mathcal{U})$-compact subsets of X. For $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $V \in \mathcal{U}$, we define

$$r_n(V, K, \psi) := \min \left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, V, \psi) \right\}.$$

For $n \in \mathbb{N}_+$ and $V \in \mathcal{U}$, a subset F of X is said to be (n, V)-*separated* with respect to ψ if $D_n(x, V^s, \psi) \cap D_n(y, V^s, \psi) = \emptyset$ for any $x, y \in F$ with $x \neq y$. Given $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $V \in \mathcal{U}$, we set

$$s_n(V, K, \psi) := \max\{|F| : F \subseteq K \text{ and } F \text{ is } (n, V)\text{-separated with respect to } \psi\}.$$

The quantities $r_n(V, K, \psi)$ and $s_n(V, K, \psi)$ are finite and well defined, as K is $\tau(\mathcal{U})$-compact.

Moreover, given $K \in \mathcal{K}(X)$ and $V \in \mathcal{U}$ define:

$$r(V, K, \psi) = \limsup_{n \to \infty} \frac{\log r_n(V, K, \psi)}{n}$$

and

$$s(V, K, \psi) = \limsup_{n \to \infty} \frac{\log s_n(V, K, \psi)}{n}.$$

Then, the quantities $h_r(K,\psi)$ and $h_s(K,\psi)$ are defined by

$$h_r(K,\psi) = \sup\{r(V,K,\psi) : V \in \mathcal{U}\} \text{ and } h_s(K,\psi) = \sup\{s(V,K,\psi) : V \in \mathcal{U}\}.$$

We write $r_n(V,K,\psi,\mathcal{U}), s_n(V,K,\psi,\mathcal{U}), r(V,K,\psi,\mathcal{U}), s(V,K,\psi,\mathcal{U}), h_r(K,\psi,\mathcal{U})$ and $h_s(K,\psi,\mathcal{U})$ if we need to emphasise on the quasi-uniformity $\mathcal{U}$ used.

Since every uniformly continuous map between quasi-uniform spaces is continuous, then it follows that if $(X,\mathcal{U})$ is a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map, then $D_n(x,V,\psi)$ is a $\tau(\mathcal{U})$-open subset of X for each $V \in \mathcal{U}, x \in X$ and $n \in \mathbb{N}_+$.

4.1.1 Remark. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. If $U, V \in \mathcal{U}$ such that $U \subseteq V$, then for each $n \in \mathbb{N}_+$ and each $x \in X$, we have that:*

(i) $D_n(x,U,\psi) \subseteq D_n(x,V,\psi)$, *and*

(ii) $D_n(x,U^s,\psi) \subseteq D_n(x,V^s,\psi)$.

4.1.1 Lemma. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. If $K, K' \in \mathcal{K}(X)$ such that $K \subseteq K'$, then for each $n \in \mathbb{N}_+$ and each $V \in \mathcal{U}$, we have that:*

(i) $r_n(V,K,\psi) \leq r_n(V,K',\psi)$, *and*

(ii) $s_n(V,K,\psi) \leq s_n(V,K',\psi)$.

Proof. Let $n \in \mathbb{N}_+, V \in \mathcal{U}$ and $K, K' \in \mathcal{K}(X)$ with $K \subseteq K'$.

(i) This is similar to the proof of Proposition 3.1.2. One simply replaces ϵ by V.

(ii) Let $F \subseteq K$ such that F is (n, V)-separated with respect to ψ and $s_n(V, K, \psi) = |F|$. Since $K \subseteq K'$, then we have that $F \subseteq K'$ and F is (n, V)-separated with respect to ψ. Therefore

$$|F| \in \{|F'| : F' \subseteq K' \text{ and } F' \text{ is } (n, V)\text{-separated with respect to } \psi\}.$$

It then follows that $s_n(V, K, \psi) = |F| \leq s_n(V, K', \psi)$ as required. $\square$

By Remark 4.1.1 we get the following result which is similar to Proposition 3.1.2.

4.1.1 Proposition. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. If $K, K' \in \mathcal{K}(X)$ with $K \subseteq K'$, then $h_{QU}(K, \psi) \leq h_{QU}(K', \psi)$. Furthermore $h_{QU}(\psi) = h_{QU}(X, \psi)$ provided that X is compact.*

4.1.2 Remark. *Let (X, q) be a quasi-metric space, $\mathcal{U}_q$ the quasi-uniformity induced by q on X and $\psi : (X, q) \to (X, q)$ a uniformly continuous self-map. Let $\epsilon > 0$. If $F \subseteq X$, $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}_+$, we have that*

(i) *F is (n, V_ϵ)-separated with respect to ψ if and only if F is (n, ϵ)-separated with respect to ψ in the sense of [11] (see also section 3);*

(ii) *$K \subseteq \bigcup_{x \in F} D_n^{\mathcal{U}_q}(x, V_\epsilon, \psi)$ if and only if $K \subseteq \bigcup_{x \in F} D_n^q(x, \epsilon, \psi)$ in the sense of [11] (see also section 3).*

4.1.2 Lemma. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then*

(i) *$V \subseteq V^2$ for each $V \in \mathcal{U}$, and*

(ii) $\bigcap_{V \in \mathcal{U}} V^2 = \bigcap_{V \in \mathcal{U}} V$.

Proof. Let $(X,\mathcal{U})$ be a quasi-uniform space. Then

(i) Suppose $V \in \mathcal{U}$. Let $(x, y) \in V$. Since $\Delta \subseteq V$, then $(x, y), (y, y) \in V$. Hence $(x, y) \in V^2$ and therefore $V \subseteq V^2$ follows.

(ii) Since $V \subseteq V^2$ for each $V \in \mathcal{U}$, then $\bigcap_{V \in \mathcal{U}} V \subseteq \bigcap_{V \in \mathcal{U}} V^2$. Let $(x, y) \in \bigcap_{V \in \mathcal{U}} V^2$. Then $(x, y) \in V^2$ for each $V \in \mathcal{U}$. Let $U \in \mathcal{U}$, then there exists $W \in \mathcal{U}$, such that $W^2 \subseteq U$. Since $(x, y) \in V^2$ for each $V \in \mathcal{U}$, then $(x, y) \in W^2$. Thus, $(x, y) \in U$. Therefore $(x, y) \in \bigcap_{V \in \mathcal{U}} V$ and it follows that $\bigcap_{V \in \mathcal{U}} V^2 \subseteq \bigcap_{V \in \mathcal{U}} V$. Now $\bigcap_{V \in \mathcal{U}} V^2 = \bigcap_{V \in \mathcal{U}} V$ follows. $\square$

Since every uniformity on a set X is also a quasi-uniformity on X, then Lemma 4.1.2 also holds for uniformities on X.

4.1.3 Lemma. *(compare [11, Lemma 2]) Let $(X, \mathcal{U})$ be a quasi-uniform space and $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ be a uniformly continuous self-map. Let $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $V \in \mathcal{U}$. If $F \subseteq K$ is (n, V)-separated with respect to ψ and $s_n(V, K, \psi) = |F|$, then $K \subseteq \bigcup_{x \in F} D_n(x, V^s, \psi)$.*

Proof. Let $n \in \mathbb{N}_+$, $K \in \mathcal{K}(X)$ and $V \in \mathcal{U}$. Suppose $F \subseteq K$ is (n, V)-separated with respect to ψ and $s_n(V, K, \psi) = |F|$ and K is not a subset of $\bigcup_{x \in F} D_n(x, V^s, \psi)$. Pick $y \in K \setminus \bigcup_{x \in F} D_n(x, V^s, \psi)$. It follows that $y \notin F$. Set $F_0 := F \cup \{y\} \subseteq K$. We claim that F_0 is (n, V)-separated with respect to ψ.
Let $z, w \in F_0$ such that $z \neq w$. We have the following cases:
Case 1. If $z, w \in F$, then we are done, as F is (n, V)-separated with respect to ψ.
Case 2. If $z \in F$ and $w = y$, then we claim that

$$D_n(z, V^s, \psi) \cap D_n(w, V^s, \psi) = \emptyset.$$

Suppose $D_n(z, V^s, \psi) \cap D_n(w, V^s, \psi) \neq \emptyset$. Choose $t \in D_n(z, V^s, \psi) \cap D_n(w, V^s, \psi)$, then for any $0 \leq k < n$ we have $(\psi^k(z), \psi^k(t)), (\psi^k(w), \psi^k(t)) \in V^s$. Hence

$(\psi^k(z), \psi^k(w)) \in V^s \circ V^s$ for all $0 \leq k < n$. Letting $k = 0$, we get $(z, w) \in V^s \circ V^s$ for all $V \in \mathcal{U}$. Thus by Lemma 4.1.2, we get $(z, w) \in \bigcap_{V \in \mathcal{U}} (V^s \circ V^s) = \bigcap_{V \in \mathcal{U}} V^s$. Since $(X,\mathcal{U})$ is T_0, then by Proposition 1.2.1 $(X,\mathcal{U}^s)$ is T_2 and it follows that $\bigcap_{V \in \mathcal{U}} V^s = \Delta$. Thus, $(z, w) \in \Delta$. Therefore $y = w = z \in F$ (a contradiction, as $y \notin F$). Hence F_0 is (n, V)-separated with respect to ψ and $F_0 \subseteq K$ with $|F| < |F_0|$. This is a contradiction, as $s_n(V, K, \psi) = |F|$. Therefore $K \subseteq \bigcup_{x \in F} D_n(x, V^s, \psi)$, as required. □

The following proposition gives the basic properties of r_n and s_n for $n \in \mathbb{N}_+$, which will be used to define the notion of quasi-uniform entropy for a uniformly continuous self-map of a quasi-uniform space.

4.1.2 Proposition. *(compare [15, Lemma 1] and [11, Proposition 1]) Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. For each $n \in \mathbb{N}_+$ and each $K \in \mathcal{K}(X)$ we have:*

(i) *Let $V, U \in \mathcal{U}$ such that $U^s \circ U^s \subseteq V^s$. Then*

$$r_n(V, K, \psi) \leq s_n(V, K, \psi) \leq r_n(U, K, \psi);$$

(ii) *If $V_1, V_2 \in \mathcal{U}$ such that $V_1 \subseteq V_2$. Then*

$$r_n(V_2, K, \psi) \leq r_n(V_1, K, \psi) \text{ and } s_n(V_2, K, \psi) \leq s_n(V_1, K, \psi).$$

Proof. Let $n \in \mathbb{N}_+$ and $K \in \mathcal{K}(X)$.
(i) Let $V, U \in \mathcal{U}$ such that $U^s \circ U^s \subseteq V^s$. Firstly, we prove that $r_n(V, K, \psi) \leq s_n(V, K, \psi)$. Let $E \subseteq K$ such that E is (n, V)-separated with respect to ψ and $s_n(V, K, \psi) = |E|$. Then by Lemma 4.1.3, we get that

$$K \subseteq \bigcup_{x \in E} D_n(x, V^s, \psi) \subseteq \bigcup_{x \in E} D_n(x, V, \psi),$$

and it follows that

$$|E| \in \left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, V, \psi) \right\}.$$

Therefore $r_n(V, K, \psi) \leq s_n(V, K, \psi)$.
Secondly, we prove that $s_n(V, K, \psi) \leq r_n(U, K, \psi)$. Let $E \subseteq K$ be (n, V)-separated with respect to ψ and $F \subseteq X$ such that $K \subseteq \bigcup_{x \in F} D_n(x, U^s, \psi)$.
Define a map $g : E \to F$ by choosing, for each $x \in E$ a point $g(x) \in F$ with $g(x) \in D_n(x, U^s, \psi)$. Then g is injective. Otherwise, if $x \neq y$ in E, but $g(x) = z = g(y)$, then

$$(\psi^k(x), \psi^k(z)), (\psi^k(y), \psi^k(z)) \in U^s$$

for all $0 \leq k < n$. It follows that

$$(\psi^k(x), \psi^k(y)) \in U^s \circ U^s \subseteq V^s$$

for all $0 \leq k < n$. Therefore $y \in D_n(x, V^s, \psi) \cap D_n(y, V^s, \psi)$, which is a contradiction, as E is (n, V)-separated. Now injectivity of g follows, from which we get that $|E| \leq |F|$. Since $K \subseteq \bigcup_{x \in F} D_n(x, U^s, \psi) \subseteq \bigcup_{x \in F} D_n(x, U, \psi)$, then $s_n(V, K, \psi) \leq r_n(U, K, \psi)$, as required.
(ii) Let $V_1, V_2 \in \mathcal{U}$ such that $V_1 \subseteq V_2$. First we prove that $r_n(V_2, K, \psi) \leq r_n(V_1, K, \psi)$. Since $V_1 \subseteq V_2$, then $D_n(x, V_1, \psi) \subseteq D_n(x, V_2, \psi)$ for each $x \in X$. Thus

$$\left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, V_1, \psi) \right\} \subseteq \left\{ |F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, V_2, \psi) \right\}.$$

Therefore $r_n(V_2, K, \psi) \leq r_n(V_1, K, \psi)$.
Finally, we show that $s_n(V_2, K, \psi) \leq s_n(V_1, K, \psi)$. Since $V_1 \subseteq V_2$, then it follows that $D_n(x, (V_1)^s, \psi) \subseteq D_n(x, (V_2)^s, \psi)$ for each $x \in X$. Now if $F \subseteq X$ is (n, V_2)-separated with respect to ψ, then F is also (n, V_1)-separated with respect to ψ, because for any $x \neq y$ in F, we have that

$$D_n(x, (V_1)^s, \psi) \cap D_n(y, (V_1)^s, \psi) \subseteq D_n(x, (V_2)^s, \psi) \cap D_n(y, (V_2)^s, \psi) = \emptyset.$$

Therefore, we have $s_n(V_2, K, \psi) \leq s_n(V_1, K, \psi)$. $\square$

By Proposition 4.1.2, we get the following results.

4.1.1 Corollary. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. Let $V \in \mathcal{U}$ and K be a non-empty join-compact subset of X. Since $V^s = V \cap V^{-1}$, then $V^s \subseteq V$. Now we have that:*

(1) $r_n(V, K, \psi) \leq r_n(V^s, K, \psi)$ for each $n \in \mathbb{N}_+$;

(2) $s_n(V, K, \psi) \leq s_n(V^s, K, \psi)$ for each $n \in \mathbb{N}_+$.

By Proposition 4.1.2, it follows that $h_r(K, \psi) = h_s(K, \psi)$ for each $K \in \mathcal{K}(X)$ and each uniformly continuous self-map $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$. Hence the following definitions make sense:

4.1.1 Definition. *Let $(X,\mathcal{U})$ be a quasi-uniform space, $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map and $K \in \mathcal{K}(X)$.*

$$h_{QU}(K, \psi) = h_r(K, \psi) = h_s(K, \psi),$$

is the quasi-uniform entropy *of ψ with respect to K. Furthermore, we define the quasi-uniform entropy $h_{QU}(\psi)$ of ψ by*

$$h_{QU}(\psi) = \sup_{K \in \mathcal{K}(X)} h_{QU}(K, \psi).$$

We write $h_{QU}(K, \psi,\mathcal{U})$ and $h_{QU}(\psi,\mathcal{U})$ if we need to emphasise on the quasi-uniformity $\mathcal{U}$ used.

It must be noted that if $(X,\mathcal{U})$ is a uniform space, then our definition of entropy coincides with that of uniform entropy h_U given by Hood [15] and Hofer [14].

In light of Remark 4.1.2 the quasi-uniform entropy defined in Chapter 3 for uniformly continuous self-maps of a quasi-metric space (X, q) can be obtained from the quasi-uniform entropy defined in Section 4.1 by using the quasi-metric quasi-uniformity $\mathcal{U}_q$ (see Proposition 4.1.3).

4.1.3 Remark. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space,* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *be a uniformly continuous self-map and* $K \in \mathcal{K}(X)$*. If* K *is finite, then for each* $U \in \mathcal{U}, n \in \mathbb{N}_+$ *we have* $r_n(U, K, \psi) \leq |K|$ *. Thus,* $r(U, K, \psi) = 0$ *for each* $U \in \mathcal{U}$ *and it follows that* $h_r(K, \psi) = 0$*. Therefore* $h_{QU}(\psi) = 0$*, if* $(X,\mathcal{U})$ *has no infinite compact subsets.*

4.1.1 Example. *If* $(X,\mathcal{U})$ *is a quasi-uniform space and* $id_X : (X,\mathcal{U}) \to (X,\mathcal{U})$ *is the identity map, then* $h_{QU}(id_X) = 0$*.*
Let $n \in \mathbb{N}_+$ *and* $V \in \mathcal{U}$*. If* $x \in X$*, we have that*

$$\begin{aligned} D_n(x, V, id_X) &= \bigcap_{k=0}^{n-1} (id_X)^{-k}(V((id_X)^k(x))) \\ &= \bigcap_{k=0}^{n-1} id_X(V(id_X(x))) \\ &= \bigcap_{k=0}^{n-1} V(x) = V(x), \end{aligned}$$

as $(id_X)^{-k} = id_X = (id_X)^k$ *for all* $k \in \mathbb{N}_+ \cup \{0\}$*. Similarly it can be shown that* $D_n(x, V^s, id_X) = V^s(x)$ *for each* $x \in X$*. Hence* $r_n(V, K, id_X)$ *and* $s_n(V, K, id_X)$ *are independent of* n *for each* $K \in \mathcal{K}(X)$*. Therefore* $r(V, K, id_X) = 0 = s(V, K, id_X)$ *for each* $K \in \mathcal{K}(X)$*. Thus,* $h_{QU}(id_X) = 0$*.*

The following result easily follows by using Remark 4.1.2.

4.1.3 Proposition. *Let* (X, q) *be a quasi-metric space and* $\mathcal{U}_q$ *be the quasi-uniformity induced by* q *on* X*. If* $\psi : (X, q) \to (X, q)$ *is a uniformly continuous self-map, then*

$$h_{QU}(\psi, q) = h_{QU}(\psi, \mathcal{U}_q) \text{ and } h_U(\psi, q^s) = h_U(\psi, \mathcal{U}_{q^s}).$$

In light of Remark 4.1.3 and Proposition 4.1.3, it follows that if $\psi : (X, q) \to (X, q)$ is a uniformly continuous self-map of a quasi-metric space (X, q) and (X, q) has no infinite compact subsets, then $h_{QU}(\psi) = 0$.

4.2 Basic properties of the quasi-uniform entropy on a quasi-uniform space

In this section we are going to show that most of the basic facts about quasi-uniform entropy on quasi-metric spaces presented in Chapter 3 (see also [11]) can easily be extended to quasi-uniform spaces.

4.2.1 Definition. *(compare [11, Definition 2]) Two quasi-uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on a set X are* uniformly equivalent *if $id_X : (X,\mathcal{U}_1) \to (X,\mathcal{U}_2)$ and $id_X : (X,\mathcal{U}_2) \to (X,\mathcal{U}_1)$ are both uniformly continuous maps of quasi-uniform spaces. In this case $\psi : (X,\mathcal{U}_1) \to (X,\mathcal{U}_1)$ is uniformly continuous if and only if $\psi : (X,\mathcal{U}_2) \to (X,\mathcal{U}_2)$ is uniformly continuous.*

It must be noted that, if $(X,\mathcal{U})$ is a quasi-uniform space, then $(X,\mathcal{U})$ and $(X,\mathcal{U}^{-1})$ are not uniformly equivalent in general. See [11, Example 4] by using the usual quasi-uniformity $\mathcal{U}_u$ of $\mathbb{R}$.

4.2.1 Proposition. *(compare [11, Proposition 2]) If $\mathcal{U}_1$ and $\mathcal{U}_2$ are uniformly equivalent quasi-uniformities on X and $\psi : (X,\mathcal{U}_1) \to (X,\mathcal{U}_1)$ is uniformly continuous, then $h_{QU}(\psi,\mathcal{U}_1) = h_{QU}(\psi,\mathcal{U}_2)$.*

Proof. Suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are uniformly equivalent quasi-uniformities on X and $\psi : (X,\mathcal{U}_1) \to (X,\mathcal{U}_1)$ is uniformly continuous. Let $U_1 \in \mathcal{U}_1$. Since $id_X : (X,\mathcal{U}_2) \to (X,\mathcal{U}_1)$ and $id_X : (X,\mathcal{U}_1) \to (X,\mathcal{U}_2)$ are both uniformly continuous, we can choose $U_2 \in \mathcal{U}_2$ and $V_1 \in \mathcal{U}_1$ such that $U_2 = (id_X \times id_X)(U_2) \subseteq U_1$ and $V_1 = (id_X \times id_X)(V_1) \subseteq U_2$. Thus $V_1 \subseteq U_2 \subseteq U_1$. Observe that for each $x \in X$, we have

$$D_n^{\mathcal{U}_1}(x, V_1, \psi) \subseteq D_n^{\mathcal{U}_2}(x, U_2, \psi) \subseteq D_n^{\mathcal{U}_1}(x, U_1, \psi) \text{ for all } n \in \mathbb{N}_+.$$

Let $K \in \mathcal{K}(X)$ and $F \subseteq X$ such that $K \subseteq \bigcup_{x\in F} D_n^{\mathcal{U}_2}(x, U_2, \psi)$. Then

$$K \subseteq \bigcup_{x\in F} D_n^{\mathcal{U}_1}(x, U_1, \psi).$$

Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x\in F} D_n^{\mathcal{U}_2}(x, U_2, \psi)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x\in F} D_n^{\mathcal{U}_1}(x, U_1, \psi)\right\}.$$

Therefore $r_n(U_1, K, \psi, \mathcal{U}_1) \leq r_n(U_2, K, \psi, \mathcal{U}_2)$. Similarly, it can be shown that $r_n(U_2, K, \psi, \mathcal{U}_2) \leq r_n(V_1, K, \psi, \mathcal{U}_1)$. Letting $n \to \infty$, we obtain that

$$r(U_1, K, \psi, \mathcal{U}_1) \leq r(U_2, K, \psi, \mathcal{U}_2) \leq r(V_1, K, \psi, \mathcal{U}_1).$$

Hence

$$\begin{aligned}\sup\{r(U_1, K, \psi, \mathcal{U}_1) : U_1 \in \mathcal{U}_1\} &\leq \sup\{r(U_2, K, \psi, \mathcal{U}_2) : U_2 \in \mathcal{U}_2\} \\ &\leq \sup\{r(V_1, K, \psi, \mathcal{U}_1) : V_1 \in \mathcal{U}_1\},\end{aligned}$$

and it follows that $h_{QU}(K, \psi, \mathcal{U}_1) = h_{QU}(K, \psi, \mathcal{U}_2)$. After taking supremum over $\mathcal{K}(X)$, we get $h_{QU}(\psi, \mathcal{U}_1) = h_{QU}(\psi, \mathcal{U}_2)$ as required. □

4.2.2 Proposition. *(Logarithmic Law.) (compare [11, Proposition 3]) Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map, then*

$$h_{QU}(\psi^m) = m h_{QU}(\psi)$$

for each $m \in \mathbb{N}$.

Proof. Let $U \in \mathcal{U}$ and $m \in \mathbb{N}$.

Case 1. If $m = 0$, then the statement holds, as $\psi^m = id_X$ and $h_{QU}(id_X) = 0$ (see Example 4.1.1).

Case 2. If $m \in \mathbb{N}_+$, then we first show that

$$r_n(U, K, \psi^m) \leq r_{mn}(U, K, \psi) \text{ for each } n \in \mathbb{N}_+ \text{ and each } K \in \mathcal{K}(X).$$

To this end, let $x \in X$ and $y \in D_{mn}(x, U, \psi)$. Then $\psi^k(y) \in U(\psi^k(x))$ for all $0 \leq k < mn$. If we set $k = mi$ for $0 \leq i < n$, then $\psi^{mi}(y) \in U(\psi^{mi}(x))$ for all $0 \leq i < n$. Since $\psi^{mi} = (\psi^m)^i$ for all $0 \leq i < n$, then $y \in D_n(x, U, \psi^m)$. Therefore $D_{mn}(x, U, \psi) \subseteq D_n(x, U, \psi^m)$. Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_{mn}(x, U, \psi)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, U, \psi^m)\right\}.$$

Therefore $r_n(U, K, \psi^m) \leq r_{mn}(U, K, \psi)$. Taking logarithms on both sides and dividing throughout by n, we get

$$\frac{\log r_n(U, K, \psi^m)}{n} \leq \frac{m \log r_{mn}(U, K, \psi)}{mn} = \frac{m \log r_s(U, K, \psi)}{s},$$

where $s = mn$. Letting $n \to \infty$ and $s \to \infty$ we obtain that

$$r(U, K, \psi^m) \leq mr(U, K, \psi).$$

Taking supremum over $\mathcal{U}$, we get $h_{QU}(K, \psi^m) \leq mh_{QU}(K, \psi)$. After taking supremum over $\mathcal{K}(X)$, we get $h_{QU}(\psi^m) \leq mh_{QU}(\psi)$.

Let $U \in \mathcal{U}$. Since $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous, there exists $V \in \mathcal{U}$ such that if $(x, y) \in V$, then $(\psi^j \times \psi^j)(x, y) = (\psi^j(x), \psi^j(y)) \in U$ for all $0 \leq j < m$. We are going to show that $D_n(x, V, \psi^m) \subseteq D_{mn}(x, U, \psi)$ for each $x \in X$ and each $n \in \mathbb{N}_+$.

For $x \in X$ and $n \in \mathbb{N}_+$, let $y \in D_n(x, V, \psi^m)$. Then $(\psi^m)^k(y) \in V((\psi^m)^k(x))$ for all $0 \leq k < n$. Since $\psi^{mk} = (\psi^m)^k$ for all $0 \leq k < n$, then $\psi^{mk}(y) \in V(\psi^{mk}(x))$ for all $0 \leq k < n$. Let $i = mk$, then $0 \leq i < mn$. Thus $\psi^i(y) \in V(\psi^i(x))$ for all $0 \leq i < mn$. Since $\psi^i(y) \in V(\psi^i(x))$, then $\psi^{ij}(y) \in U(\psi^{ij}(x))$ for all $0 \leq j < m$ and for all $0 \leq i < mn$. Letting $j = 1$, we get $\psi^i(y) \in U(\psi^i(x))$ for all $0 \leq i < mn$. Therefore $y \in D_{mn}(x, U, \psi)$ and $D_n(x, V, \psi^m) \subseteq D_{mn}(x, U, \psi)$ follows. Hence

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_n(x, V, \psi^m)\right\} \subseteq \left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x \in F} D_{mn}(x, U, \psi)\right\}.$$

Therefore $r_{mn}(U, K, \psi) \leq r_n(V, K, \psi^m)$. Arguing in a similar manner as above, we get $mh_{QU}(\psi) \leq h_{QU}(\psi^m)$. Putting everything together we get $h_{QU}(\psi^m) = mh_{QU}(\psi)$ as required. □

4.2.2 Definition. *(compare [22, Chapter 9, Definition 37.4]) Let X_1, X_2 be sets and $X = X_1 \times X_2$. For $\alpha = 1, 2$, the α^{th}* biprojection *is the map $P_\alpha : X \times X \to X_\alpha \times X_\alpha$ defined by*

$$P_\alpha(x, y) = (\pi_\alpha(x), \pi_\alpha(y)) \text{ for each } (x, y) \in X \times X,$$

where $\pi_\alpha : X \to X_\alpha$ is the α^{th} projection map. It must be noted that each element x of X has the form $x = (x_1, x_2)$, where $x_1 \in X_1$ and $x_2 \in X_2$.

4.2.1 Theorem. *Let $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ be quasi-uniform spaces. If $X = X_1 \times X_2$ and $P_\alpha : X \times X \to X_\alpha \times X_\alpha$ is the α^{th} biprojection map for $\alpha = 1, 2$, then*

$$\mathcal{U} = \{U \subseteq X \times X : P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U \text{ for some } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}$$

is a quasi-uniformity on X, called the product quasi-uniformity*.*

Proof. We first show that $\mathcal{U}$ is a filter on $X \times X$. To this end, let $\alpha = 1, 2$. Suppose $(X_\alpha, \mathcal{U}_\alpha)$ is a quasi-uniform space and $P_\alpha : X \times X \to X_\alpha \times X_\alpha$ is the α^{th} biprojection map, then $\mathcal{U}_\alpha$ is a filter on $X_\alpha \times X_\alpha$. Thus $X_\alpha \neq \emptyset$ and it follows that $\Delta_{X_\alpha} \neq \emptyset$ and $X \neq \emptyset$. Since $\Delta_{X_\alpha} \subseteq U_\alpha$ for each $U_\alpha \in \mathcal{U}_\alpha$, then $\emptyset \neq \Delta_X \subseteq P_1^{-1}(U_1) \cap P_2^{-1}(U_2)$ for each $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$. Therefore $U \neq \emptyset$ for each $U \in \mathcal{U}$. Since $P_\alpha^{-1}(X_\alpha \times X_\alpha) = X \times X$ and $X_\alpha \times X_\alpha \in \mathcal{U}_\alpha$, then $X \times X \in \mathcal{U}$. Therefore $\mathcal{U}$ is a nonempty collection of nonempty subsets of $X \times X$. Let $U \in \mathcal{U}$ and $V \subseteq X \times X$ such that $U \subseteq V$. Then $P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U$ for some $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$. Hence $P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq V$ and it follows that $V \in \mathcal{U}$. Let $U, V \in \mathcal{U}$, then there exists $U_1, V_1 \in \mathcal{U}_1$ and $U_2, V_2 \in \mathcal{U}_2$ such that $P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U$ and $P_1^{-1}(V_1) \cap P_2^{-1}(V_2) \subseteq V$. Since $U_1 \cap V_1 \in \mathcal{U}_1$ and $U_2 \cap V_2 \in \mathcal{U}_2$, and

$$\begin{aligned} P_1^{-1}(U_1 \cap V_1) \cap P_2^{-1}(U_2 \cap V_2) &= (P_1^{-1}(U_1) \cap P_1^{-1}(V_1)) \cap (P_2^{-1}(U_2) \cap P_2^{-1}(V_2)) \\ &= (P_1^{-1}(U_1) \cap P_2^{-1}(U_2)) \cap (P_1^{-1}(V_1) \cap P_2^{-1}(V_2)) \\ &\subseteq U \cap V, \end{aligned}$$

then $U \cap V \in \mathcal{U}$. Therefore $\mathcal{U}$ is a filter on $X \times X$. Finally we show that $\mathcal{U}$ is a quasi-uniformity on X. Let $U \in \mathcal{U}$. There exists $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ such that $P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U$.
(a) Since $\Delta_X \subseteq P_1^{-1}(U_1) \cap P_2^{-1}(U_2)$ for each $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ (see above), then $\Delta_X \subseteq U$.
(b) Choose $V_1 \in \mathcal{U}_1$ and $V_2 \in \mathcal{U}_2$ such that $V_1 \circ V_1 \subseteq U_1$ and $V_2 \circ V_2 \subseteq U_2$, and let $V = P_1^{-1}(V_1) \cap P_2^{-1}(V_2) \in \mathcal{U}$. Then if $(x, y) \in V \circ V$, we can find $z \in X$ such that $(x, z), (z, y) \in V$. Now $(x_1, z_1), (z_1, y_1) \in V_1$ and $(x_2, z_2), (z_2, y_2) \in V_2$ and hence $(x_1, y_1) \in V_1 \circ V_1 \subseteq U_1$ and $(x_2, y_2) \in V_2 \circ V_2 \subseteq U_2$. Thus $(x, y) \in P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U$, so $V \circ V \subseteq U$. $\square$

4.2.1 Lemma. *Let $(X_1, \mathcal{U}_1), (X_2, \mathcal{U}_2)$ be quasi-uniform spaces and $\mathcal{U}$ be the product quasi-uniformity on $X = X_1 \times X_2$. If $\psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1)$ and $\psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$ are both uniformly continuous self-maps, then $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ is a uniformly continuous self-map, where $\psi = \psi_1 \times \psi_2$ and it is defined by $\psi(x) = (\psi_1(x_1), \psi_2(x_2))$ for each $x \in X$.*

Proof. Suppose $\psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1)$ and $\psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$ are both uniformly continuous. Let $U \in \mathcal{U}$. There exists $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ such that $P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U$. Since $\psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1)$ and $\psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$ are both uniformly continuous, there exists $V_1 \in \mathcal{U}_1$ and $V_2 \in \mathcal{U}_2$ such that $(\psi_1 \times \psi_1)(V_1) \subseteq U_1$ and $(\psi_2 \times \psi_2)(V_2) \subseteq U_2$. Set $V = P_1^{-1}(V_1) \cap P_2^{-1}(V_2)$. Then $V \in \mathcal{U}$ and if $(x, y) \in V$, we have that $(x_1, y_1) \in V_1$ and $(x_2, y_2) \in V_2$. Since $(\psi_1 \times \psi_1)(V_1) \subseteq U_1$ and $(\psi_2 \times \psi_2)(V_2) \subseteq U_2$, then $(\psi_1(x_1), \psi_1(y_1)) \in U_1$ and $(\psi_2(x_2), \psi_2(y_2)) \in U_2$. Since

$$P_\alpha((\psi \times \psi)(x, y)) = P_\alpha(\psi(x), \psi(y)) = (\pi_\alpha(\psi(x)), \pi_\alpha(\psi(y))) = (\psi_\alpha(x_\alpha), \psi_\alpha(y_\alpha)) \in U_\alpha$$

for each $\alpha = 1, 2$, then $(\psi \times \psi)(x, y) \in P_\alpha^{-1}(U_\alpha)$ for each $\alpha = 1, 2$. Thus $(\psi \times \psi)(x, y) \in P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \subseteq U$ and it follows that $(\psi \times \psi)(V) \subseteq U$. Therefore ψ is uniformly continuous. $\square$

4.2.2 Lemma. *Let $(X_1, \mathcal{U}_1), (X_2, \mathcal{U}_2)$ be quasi-uniform spaces and $\mathcal{U}$ be the product quasi-uniformity on $X = X_1 \times X_2$. Let $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$. If $U =$*

$P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \in \mathcal{U}, \psi_1 : (X_1, \mathcal{U}_1) \to (X_1, \mathcal{U}_1)$ *and* $\psi_2 : (X_2, \mathcal{U}_2) \to (X_2, \mathcal{U}_2)$ *are both uniformly continuous self-maps, then for each* $x, y \in X$ *we have that:*

a) $U(x) = U_1(x_1) \times U_2(x_2)$ *and* $U^s(x) = (U_1)^s(x_1) \times (U_2)^s(x_2)$,

b) *If* $\psi = \psi_1 \times \psi_2$ *and* $n \in \mathbb{N}_+$, *then*

(i) $D_n^{\mathcal{U}}(x, U, \psi) = \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, U_\alpha, \psi_\alpha)$,

(ii) $D_n^{\mathcal{U}}(x, U^s, \psi) = \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, (U_\alpha)^s, \psi_\alpha)$, *and*

(iii) $D_n^{\mathcal{U}}(x, U^s, \psi) \cap D_n^{\mathcal{U}}(y, U^s, \psi) = \prod_{\alpha=1}^{2} \Big(D_n^{\mathcal{U}_\alpha}(x_\alpha, (U_\alpha)^s, \psi_\alpha) \cap D_n^{\mathcal{U}_\alpha}(y_\alpha, (U_\alpha)^s, \psi_\alpha) \Big)$.

c) $\bigcup_{x_1 \in F_1} D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \times \bigcup_{x_2 \in F_2} D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2) \subseteq \bigcup_{x \in F_1 \times F_2} D_n^{\mathcal{U}}(x, U, \psi)$

for each $F_1 \subseteq X_1$ *and* $F_2 \subseteq X_2$.

Proof. Let $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2$ and set $U = P_1^{-1}(U_1) \cap P_2^{-1}(U_2) \in \mathcal{U}$. Let $x, y, z \in X$. Then

a) we have that

$$\begin{aligned} z \in U(x) &\Leftrightarrow (x, z) \in U \\ &\Leftrightarrow (x_1, z_1) \in U_1 \text{ and } (x_2, z_2) \in U_2 \\ &\Leftrightarrow z_1 \in U_1(x_1) \text{ and } z_2 \in U_2(x_2) \\ &\Leftrightarrow z \in U_1(x_1) \times U_2(x_2). \end{aligned}$$

Therefore $U(x) = U_1(x_1) \times U_2(x_2)$ as required. Similarly, it can be shown that $U^s(x) = (U_1)^s(x_1) \times (U_2)^s(x_2)$.

b) Suppose $\psi = \psi_1 \times \psi_2$ and $n \in \mathbb{N}_+$. It can be shown via induction that $\psi^k(x) = (\psi_1^k(x_1), \psi_2^k(x_2))$ for each $k \in \mathbb{N}_+ \cup \{0\}$. Let $0 \leq k < n$.

(i) we have that

$$\begin{aligned} z \in D_n^{\mathcal{U}}(x, U, \psi) &\Leftrightarrow \psi^k(z) \in U(\psi^k(x)) \\ &\Leftrightarrow \psi^k(z) \in U_1(\psi_1^k(x_1)) \times U_2(\psi_2^k(x_2)) \\ &\Leftrightarrow \psi_1^k(z_1) \in U_1(\psi_1^k(x_1)) \text{ and } \psi_2^k(z_2) \in U_2(\psi_2^k(x_2)) \\ &\Leftrightarrow z \in D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \times D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2) \\ &\Leftrightarrow z \in \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, U_\alpha, \psi_\alpha). \end{aligned}$$

Therefore $D_n^{\mathcal{U}}(x, U, \psi) = \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, U_\alpha, \psi_\alpha)$ follows.

(ii) Since $U^s(x) = (U_1)^s(x_1) \times (U_2)^s(x_2)$. Then the statement follows, by using a similar argument as in (i) above.

(iii) By applying (ii) above, we get that

$$\begin{aligned} D_n^{\mathcal{U}}(x, U^s, \psi) \cap D_n^{\mathcal{U}}(y, U^s, \psi) &= \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(x_\alpha, (U_\alpha)^s, \psi_\alpha) \cap \prod_{\alpha=1}^{2} D_n^{\mathcal{U}_\alpha}(y_\alpha, (U_\alpha)^s, \psi_\alpha) \\ &= \prod_{\alpha=1}^{2} \Big(D_n^{\mathcal{U}_\alpha}(x_\alpha, (U_\alpha)^s, \psi_\alpha) \cap D_n^{\mathcal{U}_\alpha}(y_\alpha, (U_\alpha)^s, \psi_\alpha) \Big). \end{aligned}$$

c) Let $z \in \bigcup_{x_1 \in F_1} D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \times \bigcup_{x_2 \in F_2} D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2)$, then there exists a pair $(x_1, x_2) \in F_1 \times F_2$ such that $z_1 \in D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1)$ and $z_2 \in D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2)$. Thus $z \in D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \times D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2) = D_n^{\mathcal{U}}(x, U, \psi) \subseteq \bigcup_{x \in F_1 \times F_2} D_n^{\mathcal{U}}(x, U, \psi)$.
Therefore $\bigcup_{x_1 \in F_1} D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \times \bigcup_{x_2 \in F_2} D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2) \subseteq \bigcup_{x \in F_1 \times F_2} D_n^{\mathcal{U}}(x, U, \psi)$ as required. $\square$

4.2.2 Theorem. *(Weak Addition Theorem) (compare [11, Proposition 4]) Let* $(X_1,\mathcal{U}_1)$ *and* $(X_2,\mathcal{U}_2)$ *be quasi-uniform spaces. Suppose*

$$\psi_1 : (X_1,\mathcal{U}_1) \to (X_1,\mathcal{U}_1) \text{ and } \psi_2 : (X_2,\mathcal{U}_2) \to (X_2,\mathcal{U}_2)$$

are uniformly continuous self-maps. If $\mathcal{U}$ *is the product quasi-uniformity on the set* $X = X_1 \times X_2$ *and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *is the uniformly continuous self-map, defined by* $\psi = \psi_1 \times \psi_2$, *then* $h_{QU}(\psi,\mathcal{U}) \leq h_{QU}(\psi_1,\mathcal{U}_1) + h_{QU}(\psi_2,\mathcal{U}_2)$. *Furthermore, if* X_1 *or* X_2 *is compact, then* $h_{QU}(\psi,\mathcal{U}) = h_{QU}(\psi_1,\mathcal{U}_1) + h_{QU}(\psi_2,\mathcal{U}_2)$.

Proof. Let $K_i \in \mathcal{K}(X_i), F_i \subseteq X_i$ and $U_i \in \mathcal{U}_i$ for $i = 1, 2$ such that

$$K_1 \subseteq \bigcup_{x_1 \in F_1} D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \text{ and } K_2 \subseteq \bigcup_{x_2 \in F_2} D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2).$$

Set $U = P_1^{-1}(U_1) \cap P_2^{-1}(U_2)$, then $U \in \mathcal{U}$. Applying Lemma 4.2.2 c), we get that

$$K_1 \times K_2 \subseteq \bigcup_{x \in F_1 \times F_2} D_n^{\mathcal{U}}(x, U, \psi).$$

Now we have that

$$\text{if } |F_1| \in \left\{ |E_1| : E_1 \subseteq X_1 \text{ and } K_1 \subseteq \bigcup_{x_1 \in E_1} D_n^{\mathcal{U}_1}(x_1, U_1, \psi_1) \right\}$$

and

$$|F_2| \in \left\{ |E_2| : E_2 \subseteq X_2 \text{ and } K_2 \subseteq \bigcup_{x_2 \in E_2} D_n^{\mathcal{U}_2}(x_2, U_2, \psi_2) \right\},$$

then

$$|F_1| \times |F_2| \in \left\{ |E_1 \times E_2| : E_1 \times E_2 \subseteq X \text{ and } K_1 \times K_2 \subseteq \bigcup_{x \in E_1 \times E_2} D_n^{\mathcal{U}}(x, U, \psi) \right\}.$$

Hence $r_n(U, K_1 \times K_2, \psi,\mathcal{U}) \leq \Big(r_n(U_1, K_1, \psi_1,\mathcal{U}_1) \Big) \Big(r_n(U_2, K_2, \psi_2,\mathcal{U}_2) \Big)$. Taking logarithms and dividing by n throughout, we get

$$r(U, K_1 \times K_2, \psi,\mathcal{U}) \leq r(U_1, K_1, \psi_1,\mathcal{U}_1) + r(U_2, K_2, \psi_2,\mathcal{U}_2).$$

Taking supremum as U_1 ranges over $\mathcal{U}_1$ and U_2 ranges over $\mathcal{U}_2$, we obtain that

$$h_{QU}(U, K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2). \tag{4.1}$$

Let $U \in \mathcal{U}$. Then $V = P_1^{-1}(V_1) \cap P_2^{-1}(V_2) \subseteq U$ for some $V_1 \in \mathcal{U}_1$ and $V_2 \in \mathcal{U}_2$. Now $V \in \mathcal{U}$ and by equation 4.1, we have that

$$h_{QU}(V, K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2).$$

Since $V, U \in \mathcal{U}$ with $V \subseteq U$, then by Proposition 4.1.2 (ii), we get that

$$h_{QU}(U, K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(V, K_1 \times K_2, \psi, \mathcal{U}).$$

Now

$$h_{QU}(U, K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2)$$

for all $U \in \mathcal{U}$ follows. Taking supremum as U ranges over $\mathcal{U}$, we get that

$$h_{QU}(K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2). \tag{4.2}$$

For each $U_i \in \mathcal{U}_i$ and each $n \in \mathbb{N}_+$, let $K_i \in \mathcal{K}(X_i)$ and $F_i \subseteq K_i$ be (n, U_i)-separated with respect to ψ_i for each $i = 1, 2$. Then $F_1 \times F_2 \subseteq K_1 \times K_2$. Set $U = P_1^{-1}(U_1) \cap P_2^{-1}(U_2)$, then $U \in \mathcal{U}$.
We claim that $F_1 \times F_2$ is (n, U)-separated with respect to ψ. To this, let $x, y \in F_1 \times F_2$ such that $x \neq y$.
Case 1. If $x_1 \neq y_1$, then $D_n^{\mathcal{U}_1}(x_1, (U_1)^s, \psi_1) \cap D_n^{\mathcal{U}_1}(y_1, (U_1)^s, \psi_1) = \emptyset$, as F_1 is (n, U_1)-separated with respect to ψ_1. Then by Lemma 4.2.2 $b)(iii)$, it follows that

$$D_n^{\mathcal{U}}(x, U^s, \psi) \cap D_n^{\mathcal{U}}(y, U^s, \psi) = \emptyset.$$

Case 2. If $x_2 \neq y_2$, then $D_n^{\mathcal{U}_2}(x_2, (U_2)^s, \psi_2) \cap D_n^{\mathcal{U}_2}(y_2, (U_2)^s, \psi_2) = \emptyset$, as F_2 is (n, U_2)-separated with respect to ψ_2. Then by Lemma 4.2.2 $b)(iii)$, it follows that

$$D_n^{\mathcal{U}}(x, U^s, \psi) \cap D_n^{\mathcal{U}}(y, U^s, \psi) = \emptyset.$$

Hence $F_1 \times F_2$ is (n, U)-separated with respect to ψ as required. Now we have that

$$\text{if } |F_1| \in \Big\{|E_1| : E_1 \subseteq K_1 \text{ and } E_1 \text{ is } (n, U_1)\text{-separated with respect to } \psi_1\Big\}$$

and

$$|F_2| \in \Big\{|E_2| : E_2 \subseteq K_2 \text{ and } E_2 \text{ is } (n, U_2)\text{-separated with respect to } \psi_2\Big\},$$

then

$$|F_1| \times |F_2| \in \Big\{|E_1 \times E_2| : E_1 \times E_2 \subseteq K_1 \times K_2 \text{ and } E_1 \times E_2 \text{ is } (n, U)\text{-separated with respect to } \psi\Big\}.$$

Hence $\Big(s_n(U_1, K_1, \psi_1, \mathcal{U}_1)\Big)\Big(s_n(U_2, K_2, \psi_2, \mathcal{U}_2)\Big) \leq s_n(U, K_1 \times K_2, \psi, \mathcal{U})$. Taking logarithms and dividing throughout by n, we get

$$s(U_1, K_1, \psi_1, \mathcal{U}_1) + s(U_2, K_2, \psi_2, \mathcal{U}_2) \leq s(U, K_1 \times K_2, \psi, \mathcal{U}).$$

Taking supremum as U_1 ranges over $\mathcal{U}_1$ and U_2 ranges over $\mathcal{U}_2$, we get

$$h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2) \leq h_{QU}(U, K_1 \times K_2, \psi, \mathcal{U}).$$

Since

$$h_{QU}(U, K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(K_1 \times K_2, \psi, \mathcal{U}),$$

then

$$h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2) \leq h_{QU}(K_1 \times K_2, \psi, \mathcal{U}).$$

By equation 4.2, we get that

$$h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2) \leq h_{QU}(K_1 \times K_2, \psi, \mathcal{U}) \leq h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2).$$

Hence

$$h_{QU}(K_1 \times K_2, \psi, \mathcal{U}) = h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2). \tag{4.3}$$

Let $\pi_i : X_1 \times X_2 \to X_i$ be the i^{th} projection map (for $i = 1, 2$). If $K \in \mathcal{K}(X_1 \times X_2)$, then $K_i = \pi_i(K) \in \mathcal{K}(X_i)$ for each $i = 1, 2$ and $K \subseteq K_1 \times K_2$. It follows that

$$\begin{aligned}
h_{QU}(\psi, \mathcal{U}) &= \sup\{h_{QU}(K, \psi, \mathcal{U}) : K \in \mathcal{K}(X_1 \times X_2)\} \\
&\leq \sup\{h_{QU}(K_1 \times K_2, \psi, \mathcal{U}) : K_1 \in \mathcal{K}(X_1) \text{ and } K_2 \in \mathcal{K}(X_2)\} \\
&= \sup\{h_{QU}(K_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2) : K_1 \in \mathcal{K}(X_1) \text{ and } K_2 \in \mathcal{K}(X_2)\} \\
&= \sup\{h_{QU}(K_1, \psi_1, \mathcal{U}_1) : K_1 \in \mathcal{K}(X_1)\} + \sup\{h_{QU}(K_2, \psi_2, \mathcal{U}_2) : K_2 \in \mathcal{K}(X_2)\} \\
&= h_{QU}(\psi_1, \mathcal{U}_1) + h_{QU}(\psi_2, \mathcal{U}_2),
\end{aligned}$$

where the third equality from below is obtained by using equation (4.3).
Suppose X_1 or X_2 is compact. Without loss of generality we may assume X_1 is compact. Since every element of $\mathcal{K}(X_1 \times X_2)$ is a subset of $X_1 \times K_2$ for some $K_2 \in \mathcal{K}(X_2)$, we have

$$h_{QU}(\psi, \mathcal{U}) = \sup\{h_{QU}(X_1 \times K_2, \psi, \mathcal{U}) : K_2 \in \mathcal{K}(X_2)\}.$$

Since X_1 is compact, then by Proposition 4.1.1 we have that

$$h_{QU}(\psi_1, \mathcal{U}_1) = h_{QU}(X_1, \psi_1, \mathcal{U}_1).$$

Replacing K_1 by X_1 in equation (4.2), we get

$$h_{QU}(X_1 \times K_2, \psi, \mathcal{U}) = h_{QU}(X_1, \psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2) = h_{QU}(\psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2).$$

Therefore

$$\begin{aligned}
h_{QU}(\psi, \mathcal{U}) &= \sup\{h_{QU}(X_1 \times K_2, \psi, \mathcal{U}) : K_2 \in \mathcal{K}(X_2)\} \\
&= \sup\{h_{QU}(\psi_1, \mathcal{U}_1) + h_{QU}(K_2, \psi_2, \mathcal{U}_2) : K_2 \in \mathcal{K}(X_2)\} \\
&= h_{QU}(\psi_1, \mathcal{U}_1) + \sup\{h_{QU}(K_2, \psi_2, \mathcal{U}_2) : K_2 \in \mathcal{K}(X_2)\} \\
&= h_{QU}(\psi_1, \mathcal{U}_1) + h_{QU}(\psi_2, \mathcal{U}_2).
\end{aligned}$$

$\square$

Suppose $(X,\mathcal{U})$ is a uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map. The *Kimura's entropy* h_K of ψ is defined in a similar manner as the uniform entropy h_U, but using totally bounded subsets instead of compact ones. For more details on Kimura's entropy we refer the reader to [16]. We write $h_U(K,\psi,\mathcal{U}), h_K(K,\psi,\mathcal{U})$ and $h_U(\psi,\mathcal{U}), h_K(\psi,\mathcal{U})$ to emphasise the uniformity $\mathcal{U}$ used.

The following result has been extended to quasi-uniform spaces (see Theorems 4.3.1 and 4.3.2).

4.2.3 Proposition. *Let $(X,\mathcal{U})$ be a uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. Then $h_U(\psi) \leq h_K(\psi)$ with equality if the underlying space is complete.*

Proof. Suppose $(X,\mathcal{U})$ is a uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map. Since every compact subset of a uniform space is totally bounded, then $h_U(\psi) \leq h_K(\psi)$. Suppose $(X,\mathcal{U})$ is complete, it suffices to show that $h_K(\psi) \leq h_U(\psi)$. Let K be a totally bounded subset of X. Since $\mathrm{Cl}_{\tau(\mathcal{U})}(K)$ is totally bounded and complete, then $\mathrm{Cl}_{\tau(\mathcal{U})}(K)$ is compact. Since $K \subseteq \mathrm{Cl}_{\tau(\mathcal{U})}(K)$, then by [16, Basic fact 3.4(b)], we get that

$$h_K(K,\psi) \leq h_U(\mathrm{Cl}_{\tau(\mathcal{U})}(K),\psi) \leq h_U(\psi).$$

This implies that $h_K(\psi) \leq h_U(\psi)$, as required. □

4.2.1 Claim. *(compare [11, Claim 1] and Claim 3.3.1) If $(X,\mathcal{U})$ is a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map, then for each $K \in \mathcal{K}(X)$ there exists some non-empty join-compact subset K' of X such that*

$$h_{QU}(K,\psi,\mathcal{U}) \leq h_U(K',\psi,\mathcal{U}^s).$$

Proof. This proof is easily obtainable by doing appropriate modifications and following similar arguments to the proof of Claim 3.3.1. □

4.2.4 Proposition. *(compare [11, Proposition 6]) Let* $(X,\mathcal{U})$ *be a quasi-uniform space and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *be a uniformly continuous self-map. Then*

$$h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s) \leq h_K(\psi,\mathcal{U}^s).$$

Proof. Suppose $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is uniformly continuous, then by Lemma 1.2.1 we have that $\psi : (X,\mathcal{U}^s) \to (X,\mathcal{U}^s)$ is uniformly continuous. Let K be a non-empty join-compact subset of X. Then by Remark 1.2.2 we have that K is a non-empty compact subset of $(X,\mathcal{U})$. It is well-known that $h_U(\psi,\mathcal{U}^s) \leq h_K(\psi,\mathcal{U}^s)$ (see [16, Theorem 4.1]). Hence we only need to show that $h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s)$.

To this end, let $n \in \mathbb{N}_+$ and $U \in \mathcal{U}$. Since $D_n(x,U^s,\psi) \subseteq D_n(x,U,\psi)$ for each $x \in X$, we get that

$$\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x\in F} D_n(x,U^s,\psi)\right\} \subseteq \left\{|F'| : F' \subseteq X \text{ and } K \subseteq \bigcup_{x\in F'} D_n(x,U,\psi)\right\}.$$

Hence $r_n(U,K,\psi,) \leq r_n(U^s,K,\psi)$. Therefore $h_{QU}(K,\psi,\mathcal{U}) \leq h_U(K,\psi,\mathcal{U}^s)$ for each non-empty join-compact subset K of X. Let K be a non-empty $\tau(\mathcal{U})$-compact subset of X. By Claim 4.2.1, we can choose a non-empty join-compact subset K' of X such that $h_{QU}(K,\psi,\mathcal{U}) \leq h_U(K',\psi,\mathcal{U}^s)$. Since this holds for each non-empty compact subset of X, it then follows that

$$\begin{aligned} h_{QU}(\psi,\mathcal{U}) &= \sup\{h_{QU}(K,\psi,\mathcal{U}) : K \text{ is a compact subset of } X\} \\ &\leq \sup\{h_U(K,\psi,\mathcal{U}^s) : K \text{ is a join-compact subset of } X\} \\ &= h_U(\psi,\mathcal{U}^s). \end{aligned}$$

Therefore $h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s)$, as required. □

4.3 Quasi-uniform entropy following Kimura's approach

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. Let $\mathcal{B}(X)$ denotes the collection of all nonempty $\mathcal{U}$-totally bounded subsets of X. We define the finite number $r_n(V, K, \psi)$ for every $K \in \mathcal{B}(X)$ as we did in section 4.1. Let

$$r(V, K, \psi) = \lim_{n\to\infty} \sup \frac{\log r_n(V, K, \psi)}{n}.$$

Let $h_{QUK}(K, \psi) = \sup\{r(V, K, \psi) : V \in \mathcal{U}\}$. Then the notion of *quasi-uniform K-entropy $h_{QUK}(\psi)$* of ψ is given by

$$h_{QUK}(\psi) = \sup\{h_{QUK}(K, \psi) : K \in \mathcal{B}(X)\}.$$

Since every $\tau(\mathcal{U})$-compact subset of a quasi-uniform space $(X,\mathcal{U})$ is $\mathcal{U}$-totally bounded, we obtain the following theorem.

4.3.1 Theorem. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. Then*

$$h_{QU}(\psi) \leq h_{QUK}(\psi).$$

4.3.1 Lemma. *(compare [16, Basic fact 3.4(b)] and Proposition 4.1.1) Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ be a uniformly continuous self-map. If $K, K' \in \mathcal{B}(X)$ such that $K \subseteq K'$ and $V \in \mathcal{U}$, then*

$$h_{QUK}(K, \psi) \leq h_{QUK}(K', \psi).$$

4.3.1 Remark. *Since every $\tau(\mathcal{U})$-compact subset of a quasi-uniform space $(X,\mathcal{U})$ is $\mathcal{U}$-totally bounded, then by the definition of quasi-uniform entropy, it is clear that for a uniformly continuous self-map ψ on a quasi-uniform space $(X,\mathcal{U})$ we have*

$$h_{QU}(\psi) = \sup\{h_{QUK}(K, \psi) : K \textit{ is a nonempty compact subset of } X\}.$$

4.3.2 Theorem. *Let* $(X,\mathcal{U})$ *be a bicomplete quasi-uniform space and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *be a uniformly continuous self-map. Then*

$$h_{QU}(\psi) = h_{QUK}(\psi).$$

Proof. By Theorem 4.3.1, it suffices to show that $h_{QUK}(\psi) \leq h_{QU}(\psi)$. Let $K \in \mathcal{B}(X)$. Since $\mathrm{Cl}_{\tau(\mathcal{U})}(K)$ is totally bounded and bicomplete, then $\mathrm{Cl}_{\tau(\mathcal{U})}(K)$ is compact (as it is join-compact). Since $K \subseteq \mathrm{Cl}_{\tau(\mathcal{U})}(K)$, then by Lemma 4.3.1 and Remark 4.3.1, we get that

$$h_{QUK}(K,\psi) \leq h_{QUK}(\mathrm{Cl}_{\tau(\mathcal{U})}(K),\psi) \leq h_{QU}(\psi).$$

This implies that $h_{QUK}(\psi) \leq h_{QU}(\psi)$, as required. □

4.3.1 Claim. *(compare [11, Claim 1] and Claims 3.3.1 and 4.2.1) If* $(X,\mathcal{U})$ *is a quasi-uniform space and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *is a uniformly continuous self-map, then for each* $K \in \mathcal{B}(X)$ *there exists some non-empty totally bounded subset* K' *of* X *such that*

$$h_{QUK}(K,\psi,\mathcal{U}) \leq h_K(K',\psi,\mathcal{U}^s).$$

Proof. This proof is easily obtainable by doing appropriate modifications and following similar arguments to the proof of Claim 1 in [11] (see also the proof of Claim 3.3.1). Indeed working in the quasi-uniform space $(X,\mathcal{U})$, one must replace $\mathcal{K}(X)$ by $\mathcal{B}(X)$ and join-compact by totally bounded. □

By Claim 4.3.1 the following result easily follows.

4.3.1 Proposition. *(compare [11, Proposition 6]) Let* $(X,\mathcal{U})$ *be a quasi-uniform space and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *be a uniformly continuous self-map. Then*

$$h_{QUK}(\psi,\mathcal{U}) \leq h_K(\psi,\mathcal{U}^s).$$

By Proposition 4.1.3, Theorem 4.3.1 and Theorem 4.3.2 we get the following results.

4.3.2 Proposition. *Let (X, q) be a quasi-metric space and $\mathcal{U}_q$ be the quasi-uniformity induced by q on X. If $\psi : (X, q) \to (X, q)$ is a uniformly continuous map, then*

a) $h_{QU}(\psi, q) \leq h_{QUK}(\psi, \mathcal{U}_q)$,

b) $h_{QU}(\psi, q) = h_{QUK}(\psi, \mathcal{U}_q)$, *provided that (X, q) is bicomplete.*

4.4 Completion theorem for quasi-uniform entropy on a quasi-uniform space

In [16] Kimura proved the completion theorem for uniform entropy on a uniform space. In this section we are going to investigate the completion theorem for quasi-uniform entropy on a quasi-uniform space.

4.4.1 Lemma. *(compare [11, Lemma 3]) Let $(X, \mathcal{U})$ be a quasi-uniform space, $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ a uniformly continuous self-map and let Y be a ψ-invariant subset of X. For each $n \in \mathbb{N}_+$ and $U \in \mathcal{U}$ the following statements holds.*

(i) *If $y \in Y$, then*

$$D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y) = D_n^{\mathcal{U}}(y, U, \psi) \cap Y$$

and

$$D_n^{\mathcal{U}_Y}(y, (U \cap (Y \times Y))^s, \psi|_Y) = D_n^{\mathcal{U}}(y, U^s, \psi) \cap Y;$$

(ii) $r_n(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y) = r_n(U, K, \psi, \mathcal{U})$ *for each $K \in \mathcal{K}(Y)$.*

Proof. (i) Let $y \in Y, n \in \mathbb{N}_+$ and $U \in \mathcal{U}$. Since $(Y, \mathcal{U}_Y)$ is a subspace of $(X, \mathcal{U})$, then $\Big(U \cap (Y \times Y)\Big)(y) = U(y) \cap Y$. Since $\psi(Y) \subseteq Y$, then $Y \subseteq \psi^{-k}(Y)$ for all $k \in \mathbb{N}_+ \cup \{0\}$. Now we get that

$$\begin{aligned}
D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y) &= \bigcap_{k=0}^{n-1} (\psi|_Y)^{-k}\Big((U \cap (Y \times Y))(\psi|_Y)^k(y)\Big) \\
&= \bigcap_{k=0}^{n-1} \psi^{-k}\Big(U(\psi^k(y)) \cap Y\Big) \\
&= \bigcap_{k=0}^{n-1} \psi^{-k}\Big(U(\psi^k(y))\Big) \cap \bigcap_{k=0}^{n-1} \psi^{-k}(Y) \\
&= D_n^{\mathcal{U}}(y, U, \psi) \cap Y.
\end{aligned}$$

Since

$$D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y) = D_n^{\mathcal{U}}(y, U, \psi) \cap Y$$

and

$$(U \cap (Y \times Y))^{-1} = U^{-1} \cap (Y \times Y),$$

then

$$D_n^{\mathcal{U}_Y}(y, (U \cap (Y \times Y))^s, \psi|_Y) = D_n^{\mathcal{U}}(y, U^s, \psi) \cap Y.$$

(ii) Let $K \in \mathcal{K}(Y)$. By (i) we have that

$$D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y) \subseteq D_n^{\mathcal{U}}(y, U, \psi),$$

thus

$$\Big\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y \in F} D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y)\Big\} \subseteq \Big\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y \in F} D_n^{\mathcal{U}}(y, U, \psi)\Big\}.$$

Hence $r_n(U, K, \psi, \mathcal{U}) \leq r_n(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y)$.
Let $F \subseteq Y$ such that $K \subseteq \bigcup_{y \in F} D_n^{\mathcal{U}}(y, U, \psi)$. Since

$$D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y) = D_n^{\mathcal{U}}(y, U, \psi) \cap Y \text{ and } K \subseteq Y,$$

then

$$K = K \cap Y \subseteq \bigcup_{y\in F} D_n^{\mathcal{U}}(y,U,\psi) \cap Y = \bigcup_{y\in F} \Big(D_n^{\mathcal{U}}(y,U,\psi) \cap Y\Big) = \bigcup_{y\in F} D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y).$$

Therefore

$$\Big\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y\in F} D_n^{\mathcal{U}}(y,U,\psi)\Big\} \subseteq \Big\{|F| : F \subseteq Y \text{ and } K \subseteq \bigcup_{y\in F} D_n^{\mathcal{U}_Y}(y, U \cap (Y \times Y), \psi|_Y)\Big\}.$$

Hence $r_n(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y) \leq r_n(U, K, \psi, \mathcal{U})$. Since

$$r_n(U, K, \psi, \mathcal{U}) \leq r_n(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y) \leq r_n(U, K, \psi, \mathcal{U}),$$

then $r_n(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y) = r_n(U, K, \psi, \mathcal{U})$. □

4.4.1 Proposition. *(Monotonicity with respect to subspaces) (compare [11, Proposition 5]) Let $(X, \mathcal{U})$ be a quasi-uniform space, $\psi : (X, \mathcal{U}) \to (X, \mathcal{U})$ a uniformly continuous self-map and let Y be an ψ-invariant subset of X. Then $h_{QU}(\psi|_Y, \mathcal{U}_Y) \leq h_{QU}(\psi, \mathcal{U})$.*

Proof. Let $K \in \mathcal{K}(Y), n \in \mathbb{N}_+$ and $U \in \mathcal{U}$. After applying Lemma 4.4.1 (ii), we get that

$$r(U \cap (Y \times Y), K, \psi|_Y, \mathcal{U}_Y) = r(U, K, \psi, \mathcal{U}).$$

Thus $h_{QU}(K, \psi|_Y, \mathcal{U}_Y) = h_{QU}(K, \psi, \mathcal{U})$. Since $\mathcal{K}(Y) \subseteq \mathcal{K}(X)$, we get that

$$\begin{aligned} h_{QU}(\psi|_Y, \mathcal{U}_Y) &= \sup\{h_{QU}(K, \psi|_Y, \mathcal{U}_Y) : K \in \mathcal{K}(Y)\} \\ &= \sup\{h_{QU}(K, \psi, \mathcal{U}) : K \in \mathcal{K}(Y)\} \\ &\leq \sup\{h_{QU}(K, \psi, \mathcal{U}) : K \in \mathcal{K}(X)\} \\ &= h_{QU}(\psi, \mathcal{U}). \end{aligned}$$

Therefore $h_{QU}(\psi|_Y, \mathcal{U}_Y) \leq h_{QU}(\psi, \mathcal{U})$, as required. □

4.4.1 Definition. *(See [18, Section 2.6]) Let* $(X,\mathcal{U})$ *be a quasi-uniform space. A* bicompletion *of* $(X,\mathcal{U})$ *is a pair* $\Big((\widetilde{X},\widetilde{\mathcal{U}}),\varphi\Big)$ *consisting of a bicomplete quasi-uniform space* $(\widetilde{X},\widetilde{\mathcal{U}})$ *and a uniform embedding* $\varphi:(X,\mathcal{U})\to(\widetilde{X},\widetilde{\mathcal{U}})$ *such that* $\varphi(X)$ *is dense in the uniform space* $(\widetilde{X},(\widetilde{\mathcal{U}})^s)$*.*

The following outline of the actual construction of the bicompletion of a quasi-uniform space can be found in [18, Proposition 2.7.3] and for the actual construction we refer the reader to [9, Chapter 3].

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi:(X,\mathcal{U})\to(X,\mathcal{U})$ a uniformly continuous self-map. We denote by $(\widetilde{X},\widetilde{\mathcal{U}})$ the bicompletion of $(X,\mathcal{U})$ and by $\widetilde{\psi}$ the uniformly continuous extension of ψ over $\widetilde{X}$. Here $\widetilde{X}$ is the set of all minimal $\mathcal{U}^s$-Cauchy filters on X and $\widetilde{\mathcal{U}}$ is the quasi-uniformity on $\widetilde{X}$ generated by $\widetilde{\mathcal{B}}=\{\widetilde{U}:U\in\mathcal{U}\}$, where

$$\widetilde{U}=\{(\mathcal{F},\mathcal{G})\in\widetilde{X}\times\widetilde{X}:\ \text{there exists some } F\in\mathcal{F} \text{ and } G\in\mathcal{G} \text{ such that } F\times G\subseteq U\}$$

for each $U\in\mathcal{U}$. It must be noted that we can identify $(\widetilde{\mathcal{U}})^s$ with $\widetilde{\mathcal{U}^s}$. For each $x\in X$, set $\mathcal{U}^s(x)=\{U^s(x):U\in\mathcal{U}\}$. One easily verifies that $\mathcal{U}^s(x)\in\widetilde{X}$ for each $x\in X$. The uniform embedding $\varphi:(X,\mathcal{U})\to(\widetilde{X},\widetilde{\mathcal{U}})$ is defined by $\varphi(x)=\mathcal{U}^s(x)$ for each $x\in X$. We are interested in the relationship between $h_{QU}(\psi,\mathcal{U})$ and $h_{QU}(\widetilde{\psi},\widetilde{\mathcal{U}})$.

4.4.1 Theorem. *(Compare [16, Theorem 5.2]) Let* $(X,\mathcal{U})$ *be a quasi-uniform space and* $\psi:(X,\mathcal{U})\to(X,\mathcal{U})$ *a uniformly continuous self-map. If* $(\widetilde{X},\widetilde{\mathcal{U}})$*, is the bicompletion of* $(X,\mathcal{U})$ *and* $\widetilde{\psi}$ *is the uniformly continuous extension of* ψ *over* $\widetilde{X}$*, then*

$$h_{QU}(\psi,\mathcal{U})\leq h_{QU}(\widetilde{\psi},\widetilde{\mathcal{U}}).$$

Proof. Suppose $(\widetilde{X},\widetilde{\mathcal{U}})$ is the bicompletion of $(X,\mathcal{U})$ and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$. Then $\mathcal{U}=\widetilde{\mathcal{U}}_X=\widetilde{\mathcal{U}}|_{X\times X}$ and $\psi=\widetilde{\psi}|_X$ and by

Proposition 4.4.1 we have that $h_{QU}(\psi,\mathcal{U}) = h_{QU}(\widetilde{\psi}|_X,\widetilde{\mathcal{U}}_X) \leq h_{QU}(\widetilde{\psi},\widetilde{\mathcal{U}})$. $\square$

The following result is what we call the completion theorem.

4.4.2 Theorem. *(Compare [16, Theorem 5.3] and [11, Theorem 1]) Let $(X,\mathcal{U})$ be a join-compact quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. If $\Big((\widetilde{X},\widetilde{\mathcal{U}}),\varphi\Big)$ is the bicompletion of $(X,\mathcal{U})$ and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$, then*

$$h_{QU}(\psi,\mathcal{U}) = h_{QU}(\widetilde{\psi},\widetilde{\mathcal{U}}).$$

Proof. Suppose $(X,\mathcal{U})$ is join-compact. Then $(X,\mathcal{U}^s)$ is totally bounded. Since the completion of a totally bounded uniform space is also totally bounded, then $(\widetilde{X},(\widetilde{\mathcal{U}})^s)$ is totally bounded. Hence $(\widetilde{X},\widetilde{\mathcal{U}})$ is totally bounded. Since $(\widetilde{X},\widetilde{\mathcal{U}})$ is totally bounded and bicomplete, then $(\widetilde{X},\widetilde{\mathcal{U}})$ is join-compact. Let $n \in \mathbb{N}_+$ and $V \in \widetilde{\mathcal{U}}$. Since $\{\widetilde{U} : U \in \mathcal{U}\}$ is a base of $\widetilde{\mathcal{U}}$, we can choose $U_1, U_2 \in \mathcal{U}$ such that $\widetilde{U_1} \circ \widetilde{U_1} \subseteq \widetilde{U_2}$ and $\widetilde{U_2} \subseteq V$. Let $F \subseteq X$ such that $X \subseteq \bigcup_{x\in F} D_n^{\mathcal{U}}(x,U_1,\psi)$ and $r_n(U_1,X,\psi,\mathcal{U}) = |F|$. Let us set

$$W = \bigcap_{i=0}^{n-1} \Big((\widetilde{\psi})^{-i} \times (\widetilde{\psi})^{-i}\Big)(\widetilde{U_1}).$$

Since $(\widetilde{\psi})^i : (\widetilde{X},\widetilde{\mathcal{U}}) \to (\widetilde{X},\widetilde{\mathcal{U}})$ is uniformly continuous for all $0 \leq i < n$, then $W \in \widetilde{\mathcal{U}}$. We first show that

$$\widetilde{X} \subseteq \bigcup_{x\in F} D_n^{\widetilde{\mathcal{U}}}(\mathcal{U}^s(x),V,\widetilde{\psi}).$$

To this end, let $z \in \widetilde{X}$. Since $\widetilde{X} = \mathrm{Cl}_{\tau((\widetilde{\mathcal{U}})^s)}(\varphi(X))$, we can choose $z' \in X$ such that $(\mathcal{U}^s(z'),z) \in W$. Then we have that $\Big((\widetilde{\psi})^i(\mathcal{U}^s(z')),(\widetilde{\psi})^i(z)\Big) \in \widetilde{U_1}$ for all $0 \leq i < n$. Since $X \subseteq \bigcup_{x\in F} D_n^{\mathcal{U}}(x,U_1,\psi)$, we can choose $x \in F$ such that $(\psi^i(x),\psi^i(z')) \in U_1$ for all $0 \leq i < n$. Then we have that $\Big((\widetilde{\psi})^i(\mathcal{U}^s(x)),(\widetilde{\psi})^i(\mathcal{U}^s(z'))\Big) \in \widetilde{U_1}$ for all

$0 \leq i < n$. Thus $\Big((\widetilde{\psi})^i(\mathcal{U}^s(x)), (\widetilde{\psi})^i(z)\Big) \in \widetilde{U_1} \circ \widetilde{U_1}$ for all $0 \leq i < n$. Since $\widetilde{U_1} \circ \widetilde{U_1} \subseteq \widetilde{U_2}$, then $\Big((\widetilde{\psi})^i(\mathcal{U}^s(x)), (\widetilde{\psi})^i(z)\Big) \in \widetilde{U_2}$ for all $0 \leq i < n$. Hence $\Big((\widetilde{\psi})^i(\mathcal{U}^s(x)), (\widetilde{\psi})^i(z)\Big) \in V$ for all $0 \leq i < n$, and $z \in D_n^{\widetilde{\mathcal{U}}}(\mathcal{U}^s(x), V, \widetilde{\psi})$ follows. From which we get

$$\widetilde{X} \subseteq \bigcup_{x \in F} D_n^{\widetilde{\mathcal{U}}}(\mathcal{U}^s(x), V, \widetilde{\psi}).$$

Therefore

$$|\varphi(F)| \in \left\{ |E| : E \subseteq \widetilde{X} \text{ and } \widetilde{X} \subseteq \bigcup_{y \in E} D_n^{\widetilde{\mathcal{U}}}(y, V, \widetilde{\psi}) \right\},$$

and it follows that $r_n(V, \widetilde{X}, \widetilde{\psi}, \widetilde{\mathcal{U}}) \leq |\varphi(F)|$. Since $\varphi : F \to \widetilde{X}$ is injective (as every uniform embedding is injective), then $|\varphi(F)| = |F|$. Therefore $r_n(V, \widetilde{X}, \widetilde{\psi}, \widetilde{\mathcal{U}}) \leq r_n(U, X, \psi, \mathcal{U})$ and $h_{QU}(\widetilde{X}, \widetilde{\psi}, \widetilde{\mathcal{U}}) \leq h_{QU}(X, \psi, \mathcal{U})$ follows.
Since $\widetilde{X}$ and X are both join-compact, then $(\widetilde{X}, \widetilde{\mathcal{U}})$ and $(X, \mathcal{U})$ are both compact. Applying Proposition 4.1.1 it follows that

$$h_{QU}(\widetilde{\psi}, \widetilde{\mathcal{U}}) = h_{QU}(\widetilde{X}, \widetilde{\psi}, \widetilde{\mathcal{U}}) \leq h_{QU}(X, \psi, \mathcal{U}) = h_{QU}(\psi, \mathcal{U}).$$

$\square$

Chapter 5

Quasi-uniform entropy vs topological entropy

In this chapter we are going to investigate the relationship between our notion of entropy (quasi-uniform entropy) and some of the well established notions of topological entropy. In particular, we are interested in the relationship between our quasi-uniform entropy function h_{QU} and the following entropy functions h, h_f and h_K, where h, h_f and h_K are the entropy functions defined by [1, 14] and [16], respectively. It is known that the notion of quasi-uniform entropy for a quasi-metric space (X, q) is exactly the same as the notion of quasi-uniform entropy computed on $(X, \mathcal{U}_q)$, where $\mathcal{U}_q$ is the quasi-uniformity of the quasi-metric q (see Proposition 4.1.3). Hence it is sufficient to investigate the relationship between topological entropy and quasi-uniform entropy on quasi-uniform spaces.

It is well-known that, if $\psi : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a continuous mapping between uniform spaces and $(X, \mathcal{U})$ is compact, then ψ is uniformly continuous. This result is not true in the case of quasi-uniform spaces (see [5] the paragraph above Theorem 1.1.54 on page 33). Hence in studying the relationship between our notion of quasi-uniform entropy and the topological entropies mentioned above, it make sense to start with a self-mapping which is uniformly continuous, as our notion of quasi-uniform entropy is computed for a uniformly continuous self-mapping of a quasi-metric(uniform) space. Most of the materials in this chapter are contained in our recent article [13].

5.1 Quasi-uniform entropy vs topological entropy on compact quasi-uniform spaces

In this section we are going to investigate the relationship between the topological entropy and the quasi-uniform entropy for compact quasi-uniform spaces.

5.1.1 Proposition. *If $(X,\mathcal{U})$ is a join-compact quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map, then*

$$h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s) = h(\psi,\tau(\mathcal{U}^s)) = h_f(\psi,\tau(\mathcal{U}^s)).$$

In particular, we have that $h_{QU}(\psi,\mathcal{U}) \leq h(\psi,\tau(\mathcal{U}^s))$.

Proof. Let $(X,\mathcal{U})$ be a join-compact quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. By Proposition 4.2.4, we have that $h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s)$. Since $(X,\tau(\mathcal{U}^s))$ is compact, then by Remark 2.2.1 we have that $h_f(\psi,\tau(\mathcal{U}^s)) = h(\psi,\tau(\mathcal{U}^s))$. Also compactness of the uniform space $(X,\mathcal{U}^s)$ implies that $h_U(\psi,\mathcal{U}^s) = h(\psi,\tau(\mathcal{U}^s))$ (see for instance [14, 15]). Therefore we get

$$h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s) = h(\psi,\tau(\mathcal{U}^s)) = h_f(\psi,\tau(\mathcal{U}^s))$$

as required. □

Since $(X,\mathcal{U}^s)$ is a compact uniform space, then $h(\psi,\tau(\mathcal{U}^s)) = h_K(\psi,\mathcal{U}^s)$ (see [16, Theorem 4.10]). Then by Proposition 5.1.1 we get that $h_{QU}(\psi,\mathcal{U}) \leq h_K(\psi,\mathcal{U}^s)$.

5.1.2 Proposition. *(compare [21, Theorem 7.7]) Let* $(X,\mathcal{U})$ *be a compact quasi-uniform space and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *a uniformly continuous self-map. Then for each* $\alpha \in cov(X)$*, there exists some* $U \in \mathcal{U}$ *such that*

$$N(C_n(\alpha,\psi)) \le r_n(U,X,\psi) \le s_n(U,X,\psi)$$

for all $n \in \mathbb{N}_+$.

Proof. Let $\alpha \in \mathrm{cov}(X)$ and suppose X is compact. Then by Proposition 1.2.2, $(X,\mathcal{U})$ has the Lebesgue property. Thus there exists some $U \in \mathcal{U}$ such that $\beta = \{U(x) : x \in X\}$ refines α. Let $n \in \mathbb{N}_+$. By Proposition 4.1.2, we have that $r_n(V,K,\psi) \le s_n(V,K,\psi)$ for each $V \in \mathcal{U}$ and each $K \in \mathcal{K}(X)$. Since $X \in \mathcal{K}(X)$, then $r_n(U,X,\psi) \le s_n(U,X,\psi)$. Let $F \subseteq X$ with $X = \bigcup_{x\in F} D_n(x,U,\psi)$ and $r_n(U,X,\psi) = |F|$. Since $\beta < \alpha$, then by 2 (ii) of Remark 2.1.1 we get that $N(C_n(\alpha,\psi)) \le N(C_n(\beta,\psi))$. Since $X = \bigcup_{x\in F} D_n(x,U,\psi)$ and $r_n(U,X,\psi) = |F|$, then

$$\beta' = \left\{ \bigcap_{k=0}^{n-1} \psi^{-k}(U(\psi^k(x))) : x \in F \right\}$$

is a subcover of $C_n(\beta,\psi)$ with $N(\beta') = |F|$. Since $\beta' < C_n(\beta,\psi)$ (as every subcover is a refinement), then by 2 (i) of Remark 2.1.1 we get that $N(C_n(\beta,\psi)) \le N(\beta') = |F|$. Putting everything together we get that

$$N(C_n(\alpha,\psi)) \le N(C_n(\beta,\psi)) \le |F| = r_n(U,X,\psi) \le s_n(U,X,\psi)$$

as required. □

5.1.1 Theorem. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *a uniformly continuous self-map. Then*

(i) *If* $(X,\mathcal{U})$ *is compact, then*

$$h_f(\psi,\tau(\mathcal{U})) = h(\psi,\tau(\mathcal{U})) \le h_{QU}(\psi,\mathcal{U}) \le h_U(\psi,\mathcal{U}^s).$$

In particular, we have that $h(\psi,\tau(\mathcal{U})) \le h_{QU}(\psi,\mathcal{U})$.

(ii) *If* $(X,\mathcal{U})$ *is join-compact, then*

$$h_f(\psi,\tau(\mathcal{U})) = h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s) = h(\psi,\tau(\mathcal{U}^s)) = h_f(\psi,\tau(\mathcal{U}^s)).$$

In particular, we have that $h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U}) \leq h(\psi,\tau(\mathcal{U}^s))$.

Proof. Let $(X,\mathcal{U})$ be a quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map.

(i) Suppose $(X,\mathcal{U})$ is compact. Let $\alpha \in \mathrm{cov}(X)$. By Proposition 5.1.2 there exists some $U \in \mathcal{U}$ with $N(C_n(\alpha,\psi)) \leq r_n(U,X,\psi)$ for all $n \in \mathbb{N}_+$. Thus

$$\begin{aligned} h(\alpha,\psi) &= \lim_{n\to\infty} \frac{\log N(C_n(\alpha,\psi))}{n} \\ &\leq \lim_{n\to\infty} \frac{\log r_n(U,X,\psi)}{n} \\ &\leq \lim_{n\to\infty} \sup \frac{\log r_n(U,X,\psi)}{n} \\ &= h_{QU}(X,\psi). \end{aligned}$$

Since $(X,\mathcal{U})$ is compact, then by Proposition 4.1.1 $h_{QU}(\psi,\mathcal{U}) = h_{QU}(X,\psi)$. Now we get that

$$\begin{aligned} h(\psi,\tau(\mathcal{U})) &= \sup_{\alpha\in\mathrm{cov}(X)} h(\alpha,\psi) \\ &\leq \sup_{\alpha\in\mathrm{cov}(X)} h_{QU}(X,\psi) \\ &= h_{QU}(X,\psi) = h_{QU}(\psi,\mathcal{U}). \end{aligned}$$

Since $(X,\tau(\mathcal{U}))$ is compact (as $(X,\mathcal{U})$ is compact), then by Remark 2.2.1 we get that $h_f(\psi,\tau(\mathcal{U})) = h(\psi,\tau(\mathcal{U}))$. By Proposition 4.2.4 we get $h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s)$. Putting everything together we get

$$h_f(\psi,\tau(\mathcal{U})) = h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s)$$

as required.

(ii) Suppose $(X,\mathcal{U})$ is join-compact. Since every join-compact quasi-uniform space is compact, then $(X,\mathcal{U})$ is compact and by (i) above, it follows that $h_f(\psi,\tau(\mathcal{U})) = h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s)$. Since $(X,\mathcal{U}^s)$ is a compact uniform space, then $h_U(\psi,\mathcal{U}^s) = h(\psi,\tau(\mathcal{U}^s)) = h_f(\psi,\tau(\mathcal{U}^s))$. Therefore

$$h_f(\psi,\tau(\mathcal{U})) = h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s) = h(\psi,\tau(\mathcal{U}^s)) = h_f(\psi,\tau(\mathcal{U}^s))$$

as required.

□

Since $h_{QU}(\psi,\mathcal{U}) \leq h_{QUK}(\psi,\mathcal{U})$ for each uniformly continuous self-map ψ of a quasi-uniform space $(X,\mathcal{U})$ (see Theorem 4.3.1), then by Theorem 5.1.1 (i) above we get that $h(\psi,\tau(\mathcal{U})) \leq h_{QUK}(\psi,\mathcal{U})$ for each uniformly continuous self-map ψ of a compact quasi-uniform space $(X,\mathcal{U})$.

Since $h(\psi,\tau(\mathcal{U})) = h_K(\psi,\mathcal{U})$ for each continuous self-map ψ of a compact uniform space $(X,\mathcal{U})$ (see [16, Theorem 4.10]), then by Theorem 5.1.1 (ii) above we get that

$$h(\psi,\tau(\mathcal{U}) \leq h_{QU}(\psi,\mathcal{U}) \leq h_K(\psi,\mathcal{U}^s)$$

for each uniformly continuous self-map ψ of a join-compact quasi-uniform space $(X,\mathcal{U})$.

It must be noted that in Theorem 5.1.1, the statements in (i) and (ii) are different. In (i), one cannot compare $h_U(\psi,\mathcal{U}^s)$ to $h(\psi,\tau(\mathcal{U}^s))$, as compactness of $(X,\mathcal{U})$ does not imply compactness of $(X,\mathcal{U}^s)$ and it is well-known that the entropy function h is only computed for compact spaces.

5.1.1 Remark. *If (X,τ) is a compact topological space, $\mathcal{U}$ is a quasi-uniformity on X such that $\tau(\mathcal{U}) = \tau$ and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map, then by Theorem 5.1.1 (i) above we get that $h_f(\psi) = h(\psi) \leq h_{QU}(\psi,\mathcal{U})$. In particular, if (X,τ) is a compact topological space and $\mathcal{P}(\tau)$ is the*

Pervin quasi-uniformity of (X,τ) *(see Remark 1.2.1 for the definition of* $\mathcal{P}(\tau)$*), then* $h_f(\psi) = h(\psi) \leq h_{QU}(\psi, \mathcal{P}(\tau))$.

5.2 Quasi-uniform entropy vs topological entropy on non-compact quasi-uniform spaces

In this section we are going to investigate the relationship between the topological entropy and the quasi-uniform entropy for not necessary compact quasi-uniform spaces.

5.2.1 Proposition. *Let* $(X,\mathcal{U})$ *be a* T_2*-quasi-uniform space (i.e.,* $(X,\tau(\mathcal{U}))$ *is a* T_2 *topological space) and* $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ *a uniformly continuous self-map, then for each* $K \in \mathcal{K}(X)$ *there exists some* $\alpha \in cov_f(X)$ *such that*

$$h_{QU}(K, \psi, \mathcal{U}) \leq h_f(\alpha, \psi, \tau(\mathcal{U})).$$

Proof. Suppose $(X,\mathcal{U})$ is a T_2-quasi-uniform space. Let $K \in \mathcal{K}(X), n \in \mathbb{N}_+$ and $U \in \mathcal{U}$. Let $F \subseteq X$ such that $K \subseteq \bigcup_{x \in F} D_n(x, U, \psi)$ and $r_n(U, K, \psi) = |F|$. Since $(X, \tau(\mathcal{U}))$ is T_2 and K is $\tau(\mathcal{U})$-compact, then $X - K$ is $\tau(\mathcal{U})$-open. Since $D_n(x, U, \psi)$ is $\tau(\mathcal{U})$-open for each $x \in X$ and

$$X = K \cup (X - K) \subseteq \left(\bigcup_{x \in F} D_n(x, U, \psi) \right) \cup (X - K),$$

then $\alpha = \{D_n(x, U, \psi) : x \in F\} \cup \{X - K\}$ is a finite open cover of X. i.e., $\alpha \in \mathrm{cov}_f(X)$. Since $r_n(U, K, \psi) = |F|$, then $N(C_n(\alpha, \psi)) = |\alpha| = |F| + 1 > |F| = r_n(U, K, \psi)$. After taking logarithms, dividing throughout by n, then taking supremums and taking limits as $n \to \infty$, we get that $r(U, K, \psi) \leq h_f(\alpha, \psi, \tau(\mathcal{U}))$. Taking supremum as U ranges over $\mathcal{U}$, we get that $h_{QU}(K, \psi, \mathcal{U}) \leq h_f(\alpha, \psi, \tau(\mathcal{U}))$ as required. □

The following result is a consequence of Proposition 5.2.1.

5.2.1 Theorem. *Let $(X,\mathcal{U})$ be a T_2-quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. Then $h_{QU}(\psi,\mathcal{U}) \leq h_f(\psi, \tau(\mathcal{U}))$.*

Proof. Suppose $(X,\mathcal{U})$ is a T_2-quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ is a uniformly continuous self-map. Let $K \in \mathcal{K}(X)$, then by Proposition 5.2.1, there exists some $\alpha \in \mathrm{cov}_f(X)$ with $h_{QU}(K, \psi,\mathcal{U}) \leq h_f(\alpha, \tau(\mathcal{U}))$. Taking supremums as K ranges over $\mathcal{K}(X)$ and α ranges over $\mathrm{cov}_f(X)$, we get that

$$h_{QU}(\psi,\mathcal{U}) \leq h_f(\psi, \tau(\mathcal{U}))$$

as required. $\square$

The following result easily follows from Theorem 5.1.1 and Theorem 5.2.1.

5.2.2 Theorem. *Let $(X,\mathcal{U})$ be a compact T_2-quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. Then*

$$h_{QU}(\psi,\mathcal{U}) = h_f(\psi, \tau(\mathcal{U})) = h(\psi, \tau(\mathcal{U})).$$

In light of Theorem 5.1.1 and Theorem 5.2.2, we get the following corollary.

5.2.1 Corollary. *Let $(X,\mathcal{U})$ be a join-compact T_2-quasi-uniform space and $\psi : (X,\mathcal{U}) \to (X,\mathcal{U})$ a uniformly continuous self-map. Then*

$$h(\psi, \tau(\mathcal{U})) = h_f(\psi, \tau(\mathcal{U})) = h_{QU}(\psi,\mathcal{U}) \leq h_U(\psi,\mathcal{U}^s).$$

By Theorem 4.3.1 we have that $h_{QU}(\psi,\mathcal{U}) \leq h_{QUK}(\psi,\mathcal{U})$ for each uniformly continuous self-map ψ of a quasi-uniform space $(X,\mathcal{U})$. Hence by Theorem 5.2.2 we get that $h(\psi, \tau(\mathcal{U})) = h_f(\psi, \tau(\mathcal{U})) = h_{QU}(\psi,\mathcal{U}) \leq h_{QUK}(\psi,\mathcal{U})$ for each uniformly continuous self-map ψ of a compact T_2-quasi-uniform space $(X,\mathcal{U})$.

If $\psi : (X, \tau) \to (Y, \rho)$ is a continuous map between topological spaces (X, τ) and (Y, ρ), then $\psi : (X, \mathcal{P}(\tau)) \to (Y, \mathcal{P}(\rho))$ is uniformly continuous, where $\mathcal{P}(\sigma)$

denotes the Pervin quasi-uniformity for the topological space (X, σ) for $\sigma = \tau, \rho$ (see [18] page 244, last paragraph). Therefore we get the following result.

5.2.2 Proposition. *Let (X, τ) be a T_2 topological space and $\psi : X \to X$ a continuous self-map. If $\mathcal{P}(\tau)$ denotes the Pervin quasi-uniformity of (X, τ), then*

$$h_{QU}(\psi, \mathcal{P}(\tau)) \leq h_f(\psi, \tau)$$

with equality if (X, τ) is compact.

Proof. Suppose (X, τ) is a T_2 topological space and $\psi : X \to X$ is a continuous self-map. If $\mathcal{P}(\tau)$ is the Pervin quasi-uniformity of (X, τ), then $\tau(\mathcal{P}(\tau)) = \tau$ and $\psi : (X, \mathcal{P}(\tau)) \to (X, \mathcal{P}(\tau))$ is a uniformly continuous self-map. Since $(X, \mathcal{P}(\tau))$ is a T_2-quasi-uniform space, then by Theorem 5.2.1 we get that

$$h_{QU}(\psi, \mathcal{P}(\tau)) \leq h_f(\psi, \tau(\mathcal{P}(\tau))) = h_f(\psi, \tau).$$

If $(X, \tau(\mathcal{P}(\tau))) = (X, \tau)$ is compact and T_2, then $(X, \mathcal{P}(\tau))$ is a compact T_2-quasi-uniform space and by Theorem 5.2.2 we have

$$h_{QU}(\psi, \mathcal{P}(\tau)) = h_f(\psi, \tau(\mathcal{P}(\tau))) = h_f(\psi, \tau)$$

as required. $\square$

Chapter 6

Conclusion

In this final chapter, we make the conclusions of our investigations and point out some open problems encountered throughout the current work that may constitute topics of further research.

Firstly in this thesis, we have achieved the task of establishing some new results from the theory of uniform entropy that was developed by Bowen in [2]. In the second part of the work, we managed to extend our theory of quasi-uniform entropy on quasi-metric spaces to quasi-uniform spaces and in doing so, also extending the notion of uniform entropy that was developed independently by Hood [15] and Hofer [14] to quasi-uniform spaces. In the third part of the thesis, we presented the relationship between our theory of quasi-uniform entropy and some of the notions of topological entropy.

Below we give a summary of the work which we studied in each chapter of the thesis and then suggest some important areas of future research which are related to the study of entropy on quasi-metric(uniform) spaces.

6.1 Summary of the achieved work

The first two chapters recalled well-known results from the literature. In Chapter 1, we presented some preliminaries from the theory of quasi-metric spaces and quasi-uniform spaces that will ease the reading of this work.

In Chapter 2, we recalled some well-known results about the topological entropy which was studied in [1, 2, 14, 15].

In Chapter 3, we presented some of our new results about the notion of entropy (which we call *quasi-uniform entropy* and denoted by h_{QU}) of a uniformly continuous self-map of a quasi-metric space. The properties of this new entropy have been presented. In particular we managed to show that $h_{QU}(\psi, q) \leq h_U(\psi, q^s)$, where ψ is a uniformly continuous self-map of a quasi-metric space (X, q) (see Theorem 3.3.1). We also showed that the completion theorem (in the sense of Kimura [16]) holds in the class of all join-compact quasi-metric spaces. That is for join-compact quasi-metric spaces the entropy of a uniformly continuous self-map coincides with the entropy of its extension to the bicompletion (see Theorem 3.4.1).

In Chapter 4, we continued our investigations of the quasi-uniform entropy to quasi-uniform spaces. We managed to extend most of our results in Chapter 3 to quasi-uniform spaces. Following Kimura's approach, we introduced a modified version h_{QUK} (which we call *quasi-uniform K-entropy*) of the quasi-uniform entropy for quasi-uniform spaces. We managed to show that for a uniformly continuous self-map ψ of a quasi-uniform space $(X,\mathcal{U})$ we have that

1. $h_{QU}(\psi,\mathcal{U}) \leq h_K(\psi,\mathcal{U}^s)$ (see Proposition 4.2.4).

2. $h_{QU}(\psi,\mathcal{U}) \leq h_{QUK}(\psi,\mathcal{U})$ with equality when $(X,\mathcal{U})$ is bicomplete (see Theorems 4.3.1 and 4.3.2).

3. $h_{QUK}(\psi,\mathcal{U}) \leq h_K(\psi,\mathcal{U}^s)$ (see Proposition 4.3.1).

So that 1 is a combination of 2 and 3. We also showed that the completion theorem (in the sense of Kimura [16]) holds in the class of all join-compact quasi-uniform spaces (see Theorem 4.4.2).

In Chapter 5, we established the relationship between quasi-uniform entropy and some of the well-known notions of topological entropy. Indeed we managed to show that for a uniformly continuous self-map ψ of a quasi-uniform space $(X,\mathcal{U})$ we have that

1. $h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U}) \leq h(\psi,\tau(\mathcal{U}^s))$ provided that $(X,\mathcal{U})$ is join-compact (see Proposition 5.1.1 and Theorem 5.1.1).
2. $h(\psi,\tau(\mathcal{U})) \leq h_{QU}(\psi,\mathcal{U})$ provided that $(X,\mathcal{U})$ is compact (see Theorem 5.1.1).
3. $h_{QU}(\psi,\mathcal{U}) \leq h_f(\psi,\tau(\mathcal{U}))$ provided that $(X,\mathcal{U})$ is T_2 and with equality if $(X,\mathcal{U})$ is compact and T_2 (see Theorems 5.2.1 and 5.2.2).

We also managed to show that: If (X,τ) is a T_2 topological space, $\psi : X \to X$ a continuous self-map and $\mathcal{P}(\tau)$ is the Pervin quasi-uniformity of (X,τ), then

$$h_{QU}(\psi,\mathcal{P}(\tau)) \leq h_f(\psi,\tau)$$

with equality if the underlying space is also compact (see Proposition 5.2.2).

In Chapter 6, which is the last chapter of this thesis, we conclude our work by reflecting on the main results of the thesis and pointing out some connections of this current work with what has been studied already in the literature.

Let (X,q) be a quasi-metric space. It is well-known that $\psi : (X,q) \to (X,q)$ is uniformly continuous if and only if $\psi : (X,q^t) \to (X,q^t)$ is uniformly continuous (see Lemma 1.1.1). Also a similar result holds for quasi-uniform spaces (see Lemma 1.2.1). Therefore it is natural to study the following problems, which are related to our work in Chapter 3 and 4.

6.1.1 Problem. *Let (X, q) (or $(X,\mathcal{U})$) be a quasi-metric (or uniform) space and $\psi : X \to X$ a uniformly continuous self-map. What is the relationship between*

1. *$h_{QU}(\psi, q)$ and $h_{QU}(\psi, q^t)$?*
2. *$h_{QU}(\psi,\mathcal{U})$ and $h_{QU}(\psi,\mathcal{U}^{-1})$?*

It is well-known that there are several notions of completeness for quasi-metric(uniform) spaces (see for instance [5, section 1.2]). In this thesis the completion theorem has been studied using the notion of bicompleteness (see Theorems 3.4.1 and 4.4.2). It is natural to ask the following question.

6.1.2 Problem. *Given a quasi-metric(uniform) space X and a uniformly continuous self-map $\psi : X \to X$. Under which notions of completeness (apart from bicompletion) will the completion theorem still hold?*

6.2 Some possible areas for future work

In this section we are going to list some possible areas for future work. We would like to study these problems in future work.

6.2.1 Topological entropy using quasi-uniform entropy

In section 3 of [17] the authors discuss the possibility of defining a new entropy function in the category of Tychonov spaces, using the uniform entropy. Indeed one can define entropy in the category of topological spaces, using uniform entropy. All one needs to do is to connect the category **Top** of Topological spaces and continuous maps with the category **Unif** of uniform spaces and uniformly continuous maps. Along the same line and using the theory of functorial quasi-uniformities in the sense of G. Brümmer [3], one can connect the category **Top** of Topological spaces and continuous maps with the category **QUnif** of quasi-uniform spaces and uniformly continuous maps. Hence we have the following question.

6.2.1 Problem. *Can one define a new entropy function in the category* **Top***, using the quasi-uniform entropy?*

6.2.2 Entropy in other asymmetric structures

In this thesis, our study of entropy was mainly on quasi-metric(uniform) spaces. It make sense for one to explore possibilities of studying the notion of entropy in other asymmetric structures (apart from quasi-metric(uniform) spaces). On that note we pose the following question.

6.2.2 Problem. *Given an asymmetric space, how can one define the notion of entropy and which properties does the entropy function posses?*

Bibliography

[1] R.L. Adler, A. G. Konheim, and M. H. McAndrew, Topological entropy, Trans. Amer. Math. 114 (1965) 309–319.

[2] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc. 153 (1971) 401–414.

[3] G.C.L. Brümmer, Functorial transitive quasi-uniformities, Categorical topology (Toledo, Ohio, 1983), pp.163–184, Sigma Ser. Pure Math.,5, Heldermann, Berlin,1984.

[4] S. Cobzas, Completeness in Quasi-Pseudometric Spaces A Survey, Mathematics 8 (2020) 1279.

[5] S. Cobzas, Functional analysis in Asymmetric Normed spaces, Frontiers in Mathematics, Springers, Basel. (2013).

[6] E. I. Dinaburg, The relation between topological entropy and metric entropy, Soviet Math. Dokl. 11 (1970), 1316.

[7] D. Dikranjan, A. Giordano Bruno, Entropy on normed semigroups (Toward a unifying approach to entropy), Dissertationes Math. 542 (2019) 1–90.

[8] D. Dikranjan, M. Sanchis, S. Virili, New and old facts about entropy in uniform spaces and topological groups, Topology Appl. 159 (2012) 1916–1942.

[9] P. Fletcher and W.F. Lindgren, Quasi-uniform spaces, Lecture Notes in Pure and Appl. Math., vol 77, Marcel Dekker Inc., New York (1982).

[10] P. Haihambo, Hyperconvexity and Endpoints of T_0 quasi-metric spaces, MSc. thesis, University of Cape Town, 2013.

[11] P. Haihambo and O.Olela Otafudu, On entropy on quasi-metric spaces, Topology Appl. 332 (2023) 108512.

[12] P. Haihambo and O.Olela Otafudu, Entropy on quasi-uniform spaces, Acta Math. Hungar. (to appear).

[13] P. Haihambo and O.Olela Otafudu, Quasi-uniform entropy vs topological entropy, Topology Appl. (under review).

[14] J.E. Hofer, Topological entropy for non-compact spaces, Michigan J. Math. 21 (1974) 235–242.

[15] B. M. Hood, Topological entropy and Uniform Spaces, J. London Math. Soc. 8 (1974) 633–641.

[16] T. Kimura, Completion theorem for uniform entropy. Comment. Math. Univ. Carolin. 39 (1998) 389–399.

[17] H.-P.A. Künzi and D. Dikranjan, Uniform entropy vs topological entropy. Topol. Algebra Appl. 3 (2015) 104–114.

[18] H.-P.A. Künzi, An introduction to quasi-uniform spaces, Contemp. Math. 486 (2009) 239–304.

[19] O.Olela Otafudu, Maps that preserve left (right) K-Cauchy sequences. Hacet. J. Math. Stat. 50 (2021) 1466–1476.

[20] Y. Sayyari, M. Molaei and S.M. Moghayer, Entropy of continuous maps on quasi-metric spaces. J. Adv. Res. Dyn. Control Syst. 7 (2015) 1-10.

[21] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag, New-York. (1982).

[22] S. Willard, General Topology, Addison-Wesley Publishing Company, Inc (1970).