On a new class of tests for the Poisson distribution based on empirical weight functions

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Abstract

The aim of our study is to introduce a new class of tests for the Poisson distribution. The testing procedure entails fitting a weighted Poisson distribution to observed data. The Poisson distribution is a special case of this more general class of distributions. If the data are realised from a Poisson distribution, then the weight function associated with the weighted Poisson distribution is expected to be a constant (since a weighted Poisson distribution with a constant weight function is simply a Poisson distribution). The proposed test statistics are weighted L_p distances between an empirical version of the weight function and the corresponding constant. Computational forms are derived for test statistics based on weighted L_1, L_2 and L_∞ distances for three choices of weight function in each case. The tests considered reject the assumption that the observed data are Poisson for large values of these distance measures.

A Monte Carlo study is included in which the finite sample performance of the newly proposed class of tests is investigated and compared to a wide range of existing tests for the Poisson distribution. The numerical powers are obtained using a parametric bootstrap approach known as the warp-speed bootstrap. The finite sample results indicate that the newly proposed tests are competitive in terms of their power performance. In fact, these tests often outperform the existing tests for the Poisson distribution considered, especially in the case of overdispersed alternatives (where the variance of the alternative distribution considered exceeds its mean). The study is concluded with the analysis of two observed datasets. Here we demonstrate the use of the newly proposed tests to test the hypothesis that a given dataset is realised from a Poisson distribution.

Key words: *Goodness-of-fit, Monte Carlo simulation, Poisson distribution, Weighted L_p distance.*

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Chapter 1

Introduction and motivation

The Poisson distribution, originally introduced in Poisson (1828), is a useful model for count data and it is used in various fields of study. For a detailed treatment of this distribution, together with its applications, see Haight (1967). Furthermore, the role of the Poisson distribution in biostatistics is explained in Zar (1999). For further discussions of the properties, applications and generalisations of the Poisson distribution, see Patil & Rao (1978) and Johnson, Kemp & Kotz (2005).

The Poisson distribution has been studied extensively and several generalisations have been proposed, for instance the generalised Poisson as well as the zero inflated Poisson. An important generalisation, which plays a central role in this dissertation, is the weighted Poisson distribution introduced in Fisher (1934).

The interest in this study centers on testing the hypothesis that observed data are compatible with the assumption of being realised from the Poisson distribution; this is known as goodness-of-fit testing. Due to the popularity of the Poisson distribution, several goodness-of-fit tests have been proposed for this distribution. A recent review of existing test for the Poisson distribution can be found in Mijburgh & Visagie (2020), while the most recent reviews on this subject prior to the mentioned paper are Gürtler & Henze (2000) as well as Karlis & Xekalaki (2000). A review of goodness-of-fit testing procedures for discrete distributions in general can be found in Horn (1977) as well as Kocherlakota & Kocherlakota (1986).

To highlight the importance of goodness-of-fit testing, consider the following statement from Ebner, Fisher, Henze & Mayer (2022): " $\cdots$ a first step to serious statistical inference involving this family of distributions is to check whether given data are sufficiently compatible with some Weibull law." While this statement references the Weibull distribution, it is no less true for the Poisson distribution. If the Poisson assumption is made for a dataset which is not compatible with this assumption, then the resulting inference may lead to incorrect results. This may result in serious miscalculations in practice. In addition to discussing many of the existing tests for the Poisson distribution, this dissertation proposes a new class of tests for this distribution.

The remainder of the dissertation is structured as follows. Chapter 2 provides an overview of a wide range of existing goodness-of-fit tests for the Poisson distribution while Chapter 3 introduces a new class of tests. A comprehensive Monte Carlo power study is included in Chapter 4 in which the performance of the newly proposed class of tests is compared to that of the existing tests in the literature. It is demonstrated that the newly proposed class of tests compares favourably in terms of powers to the existing tests. This chapter also includes two examples in which observed datasets are analysed and the hypothesis of the Poisson distribution is tested for each dataset. Finally, Chapter 5 provides some conclusions and recommendations.

Chapter 2

A review of existing goodness-of-fit tests for the Poisson distribution

In this chapter we consider various tests for the Poisson distribution. The majority of the tests below are based on characterisations of this distribution. An advantage of characterisation based tests is that they are consistent against a wide range of fixed alternatives. A notable exception is the first test discussed; the Fisher index based test, which is not based on a characterisation of the Poisson distribution. The discussion surrounding this test is included mainly because of the historical importance of the test. The results pertaining to the Fisher index based test are not included in the numerical study shown in Chapter 4.

In order to proceed, we introduce some notation. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with distribution function F. We denote the order statistics of the sample by $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. The sample maximum is used extensively below; let $m = X_{(n)}$. The Poisson distribution function, F_λ, with mean $\lambda > 0$, is given by

$$F_\lambda(x) = \mathrm{e}^{-\lambda} \sum_{k=0}^{x} \frac{\lambda^k}{k!}, \quad \text{for} \quad x \in \{0, 1, \ldots\}.$$

The corresponding probability mass function is

$$f_\lambda(x) = \frac{\mathrm{e}^{-\lambda} \lambda^x}{x!}, \quad \text{for} \quad x \in \{0, 1, \ldots\}.$$

Below we use the notation $P(\lambda)$ to indicate the Poisson distribution with mean λ.

In this dissertation, we are interested in testing the composite goodness-of-fit hypothesis that

$$H_0 : F(x) = F_\lambda(x), \quad \text{for} \quad x \in \{0, 1, ...\} \text{ and for some } \lambda > 0. \tag{2.1}$$

This hypothesis is to be tested against general alternatives.

2.1 Tests based on the Fisher index (FI)

A well-known property of the Poisson distribution is that its expected value and variance are equal Mijburgh (2020). This property is known as equidispersion. The Poisson distribution is not uniquely characterised by this property; for example, the discrete uniform distribution with support $[0, 4]$ is also equidispersed.

Overdispersion occurs when a distribution's variance is greater than its mean, while a distribution with variance smaller than the mean is said to be underdispersed Mijburgh (2020). In Fisher (1950), Fisher conducted experiments to check the deviation between the mean and variance in a random sample drawn from a Poisson distribution. The author then introduced a sample version of the ratio in question, referred to as the Fisher index, given by

$$FI = \frac{S^2}{\overline{X}}, \quad \text{with } \overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k, \ \ S^2 = \frac{1}{n}\sum_{k=1}^{n}\left(X_k - \overline{X}\right)^2.$$

The observed sample value of FI can be used to test if a distribution is under or overdispersed. A large value of FI indicates overdispersion whereas a small value indicates underdispersion.

One of the earliest tests for the Poisson distribution is based on FI. This test rejects the hypothesis of the Poisson distribution if $FI - 1$ is large in absolute terms. However, authors have historically drawn erroneous conclusions regarding over or underdispersion based on the sample values of the Fisher index. This is due to the fact that practitioners of statistics have often failed to take the characteristics of the underlying distribution into account when interpreting the observed sample value of FI; the interested reader is referred to Henze (1996) for further details. The mentioned paper recommends using rescaled versions of FI which measure only deviations from equidispersion and not specifically over or underdispersion. The recommended statistics are

$$S^* = \frac{n^2\overline{X}(FI-1)^2}{\sum_{j=1}^{n}\{(X_j - \overline{X})^2 - X_j\}}$$

and

$$U^2 = \frac{n}{2}\left(FI - 1\right)^2,$$

where U^2 is originally proposed by Rayner & McIntyre (1985). In both cases the null hypothesis of the Poisson distribution is rejected for large values of the test statistic.

The Fisher index based test has great historical importance, as witnessed by the large number of papers studying the properties thereof; see, for example, Darwin (1957), Selby (1965), Potthoff & Whittinghill

(1966), Bartko, Greenhouse & Patlak (1968), Dahiya & Gurland (1969), Kharshikar (1970), Soo Kim & Park (1992) as well as Anderson & Siddiqui (1994). However, tests based on the Fisher index are not consistent. As a concrete example, consider the mentioned discrete uniform distribution with support $[0, 4]$. Since this distribution is equidispersed, a Fisher index based test will typically fail to reject the hypothesis of the Poisson distribution, even in the case of large samples.

2.2 Other tests based on sample moments

In addition to the Fisher index based test described in Section 2.1, which relies on the first two sample moments, goodness-of-fit tests have also been developed using other sample moments. For example, tests based on sample versions of the higher order moments of the Poisson distribution were proposed in Pettigrew & Mohler (1967). This paper found that tests based on lower order moments typically outperform those based on higher order moments. This result is not unexpected as lower order moments can often be estimated more accurately than is the case for higher order moments Pettigrew & Mohler (1967). Other papers in which sample moment based tests for the Poisson distribution are proposed include Gupta, Móri & Székely (1994) as well as Kyriakoussis, Li & Papadopoulos (1998); the latter introduces a test based on the second product moment of a distribution.

2.3 The chi-squared test

We provide a brief discussion of the well-known chi-squared test and its variations as it relates to the Poisson distribution. Despite this test being historically significant, the empirical powers associated with this test are not competitive against more recently proposed tests for the Poisson distribution Mijburgh (2020). As a result, we do not include the powers associated with this test in the numerical study presented in Chapter 4.

One difficulty associated with this test is that its power is a function of the choice of so-called “bins” or groups used to compute the test statistic. This introduces a measure of subjectivity into the conclusions arrived at using the test. The asymptotic null distribution of the test statistic has been shown to be standard normal where the null hypothesis is rejected in the case of excessively small or large values of the test statistic. The interested reader is referred to Neyman & Pearson (1933), Freeman & Tukey (1950), Nass (1959), Horn (1977) and Read & Cressie (1988).

2.4 Tests based on the empirical distribution function (edf)

We now turn our attention to several classical goodness-of-fit tests based on the edf. Specifically, we consider the Kolmogorov-Smirnov, Cramér-von Mises and Anderson-Darling tests as well as a test proposed in Klar (1999). The test statistics considered are distance measures between the fitted Poisson distribution function, $F_{\widehat{\lambda}}$, and the edf, F_n, of the sample under consideration. Regarding the fitted distribution, the estimated parameter $\widehat{\lambda}$ is set equal to the sample mean; $\widehat{\lambda} = \bar{X}$ (note that $\bar{X}$ is both the maximum likelihood and method of moments estimator of λ). The edf is

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} I(X_k \leq x),$$

where I($\cdot$) denotes the indicator function. Below we consider the various tests mentioned above in turn.

The Kolmogorov-Smirnov test (KS_n)

The Kolmogorov-Smirnov test statistic measures the maximum distance between the fitted and empirical distribution functions;

$$KS_n = \max_{x\in\{0,1,\dots\}} \left|F_{\widehat{\lambda}}(x) - F_n(x)\right|. \tag{2.2}$$

This test admits a simple form for the test statistic as follows;

$$\begin{aligned} KS_n &= \max_{x\in\{0,1,\dots\}} |F_{\hat{\lambda}}(x) - F_n(x)| \\ &= \max\left\{\max_{x\in\{0,1,\dots,m-1\}} |F_{\hat{\lambda}}(x) - F_n(x)|, \max_{x\in\{m,m+1,\dots\}} |F_{\hat{\lambda}}(x) - 1|\right\}. \end{aligned} \tag{2.3}$$

Since $F_{\hat{\lambda}}(m), F_{\hat{\lambda}}(m+1), \dots$ is an increasing sequence, bounded by 1, $|F_{\hat{\lambda}}(m) - 1|, |F_{\hat{\lambda}}(m+1) - 1|, \dots$ is a decreasing sequence. As a result,

$$\max_{x\in\{m,m+1,\dots\}} |F_{\hat{\lambda}}(x) - 1| = |F_{\hat{\lambda}}(m) - 1|, \tag{2.4}$$

without loss of generality. Combining (2.3) and (2.4), we have that

$$\begin{aligned} KS_n &= \max\left\{\max_{x\in\{0,\dots,m-1\}} \{|F_{\hat{\lambda}}(x) - F_n(x)|, |F_{\hat{\lambda}}(m) - 1|\}\right\} \\ &= \max_{x\in\{0,\dots,m\}} |F_{\hat{\lambda}}(x) - F_n(x)|. \end{aligned}$$

The assumption of the Poisson distribution is rejected for large values of KS_n. For more details relating to this test in the current setting, see Conover (1972) as well as Campbell & Oprian (1979).

The Cramér-von Mises test (CM_n)

This test measures the squared distance, weighted by the fitted mass function, between the fitted and empirical distribution functions. The corresponding test statistic is given by

$$CM_n = \frac{1}{n}\sum_{j=0}^{\infty}(F_{\widehat{\lambda}}(j) - F_n(j))^2 f_{\widehat{\lambda}}(j). \tag{2.5}$$

In order to approximately compute the value of this test statistic, we calculate the terms corresponding to j less than or equal to 100. In Karlis & Xekalaki (2000), the value of M is chosen in such a way that the probability of observing a sample observation exceeding M is less than or equal to 10^4. However we choose $j = 100$ to ensue that a significant number of terms are included compared to Karlis & Xekalaki (2000). For large values of CM_n, the hypothesis in (2.1) is rejected. A detailed treatment of this test in the context of the Poisson distribution can be found in Spinelli & Stephens (1997), while Choulakian, Lockhart & Stephens (1994) considers the Cramér-von Mises test for general discrete distributions.

The Anderson-Darling test (AD_n)

This test is a reweighted version of the Cramér-von Mises test statistic. The AD_n statistic is given by

$$AD_n = \frac{1}{n}\sum_{j=0}^{\infty}\frac{[F_{\widehat{\lambda}}(j) - F_n(j)]^2 f_{\widehat{\lambda}}(j)}{F_{\widehat{\lambda}}(j)\left(1 - F_{\widehat{\lambda}}(j)\right)}. \tag{2.6}$$

As is the case with the CM_n, we approximate the value of AD_n by calculating the terms in the summation up to $j = 100$. The inclusion of the weight function $[F_{\widehat{\lambda}}(j)(1 - F_{\widehat{\lambda}}(j))]^{-1}$ results in a higher sensitivity to deviation in the tails of the distribution than is the case for the CM_n test. The test rejects the hypothesis of the Poisson distribution for large values of AD_n.

The Klar test (L_n)

Klar (1999) proposes the sum of the absolute differences between the fitted and empirical distribution functions as a test statistic;

$$L_n = \sqrt{n}\sum_{j=1}^{n}\left|F\left(X_{(j)}\right) - F_n\left(X_{(j)}\right)\right|. \tag{2.7}$$

For large values of L_n, the null hypothesis of the Poisson distribution is rejected.

2.5 Tests based on the empirical probability generating function (pgf)

We start by defining the pgf of a random variable X;

$$g(t) = E\left[t^X\right], \quad t \in [-1,1].$$

The empirical pgf is defined to be

$$g_n(t) = \frac{1}{n}\sum_{j=1}^{n} t^{X_j}, \quad t \in [-1,1].$$

The pgf of a random variable following a $P(\lambda)$ distribution is

$$g_\lambda(t) = e^{\lambda(t-1)}, \quad t \in [-1,1].$$

Below, we briefly consider several goodness-of-fitness tests based on the quantities defined above.

The test of Rueda, O Reilly & Pérez-Abreu (1991)

The first test based on the empirical pgf is the test of Rueda et al. (1991) where the squared difference between the estimated pgf and its empirical counterpart is integrated over $[0,1]$;

$$R_n = \int_0^1 \left(n\left(g_n(t) - g_{\widehat{\lambda}}(t)\right)\right)^2 \mathrm{d}t,$$

which does not allow a closed-from computational formula. The hypothesis that the distribution is Poisson is rejected for large values of the test statistic.

The test of Baringhaus, Gürtler & Henze (2000)

Motivated by the lack of a closed-form for the test statistic R_n, Baringhaus et al. (2000) proposes a modification of the test statistic of Rueda et al. (1991) obtained by the inclusion of a weight function.

The resulting test statistic is given by

$$R_{n,a} = n \int_0^1 \left(g_n\left(t\right) - g_{\widehat{\lambda}}\left(t\right)\right)^2 t^a dt,$$

where a is a positive tuning parameter. A convenient computational form for $R_{n,a}$, provided in Gürtler & Henze (2000), is

$$R_{n,a} = n \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\left(f_n\left(j\right) - f_{\widehat{\lambda}}\left(j\right)\right)\left(f_n\left(k\right) - f_{\widehat{\lambda}}\left(k\right)\right)}{j+k+a+1} \right),$$

where $f_n(x) = n^{-1} \sum_{j=1}^{n} \mathrm{I}(X_j = x)$ is the empirical pmf. Large values of the test statistic, $R_{n,a}$, result in a rejection of the null hypothesis.

The test of Baringhaus & Henze (1992)

Another test based on the pgf is introduced in Baringhaus & Henze (1992). A characterisation of the Poisson distribution is the following partial differential equation:

$$\frac{\partial}{\partial t} g_\lambda\left(t\right) = \lambda g_\lambda\left(t\right). \tag{2.8}$$

Hence, (Baringhaus & Henze 1992) bases a goodness-of-fit test on the integrated squared difference between the empirical versions of the left and right hand sides of (2.8) Mijburgh (2020).

This test is based on a partial differential equation involving the pgf of the Poisson distribution;

$$\begin{aligned} T_n &= n \int_0^1 \left(\widehat{\lambda} g_n\left(t\right) - g_n'\left(t\right)\right)^2 \mathrm{d}t \\ &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\widehat{\lambda}^2}{X_i + X_j + 1} + \frac{X_i X_j}{X_i + X_j - 1} \right) - \widehat{\lambda}\left(n - f_{\widehat{\lambda}}(0)\right). \end{aligned}$$

This test rejects the null hypothesis for large values of T_n.

The test of Treutler (1995)

Treutler (1995) proposed a generalisation of the test statistic introduced in Baringhaus & Henze (1992), where the generalisation is achieved via the inclusion of a weight function. The proposed test statistic is

$$T_{n,a} = n \int_0^1 \left(\widehat{\lambda} g_n\left(t\right) - g_n'\left(t\right)\right)^2 t^a dt$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\widehat{\lambda}^2}{X_i+X_j+a+1}-\frac{\widehat{\lambda}\left(X_i+X_j\right)}{X_i+X_j+a}+\frac{X_iX_j}{X_i+X_j+a-1}\right),$$

where $a > 0$ is a tuning parameter. The hypothesis of the Poisson distribution is rejected for large values of $T_{n,a}$.

The test of Nakamura & Pérez-Abreu (1993)

Nakamura & Pérez-Abreu (1993) introduced a test based on a characterisation of the Poisson distribution via a partial differential equation involving the pgf;

$$\frac{d^2}{dt^2}\log\left(g_\lambda\left(t\right)\right) = 0.$$

Based on the squared coefficients of the polynomial

$$g_n^2(t)\frac{d^2}{dt^2}\log(g_n(t)),$$

the authors propose the following test statistic:

$$V_n = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left(X_i\left(X_i-X_j-1\right)X_k\left(X_k-X_l-1\right)\mathrm{I}_{(X_i+X_j=X_k+X_l)}\right).$$

The authors also consider a second test statistic which is the rescaled version of V; $V_n^* = V_n\bar{X}^{-1.45}$. This rescaled version is included as its asymptotic null distribution is approximately independent of the parameter of the underlying Poisson distribution, meaning that asymptotic critical values can be used to obtain power estimates for V_n^*. The proposed test statistics, V_n and V_n^*, reject the null hypothesis for large values.

2.6 A test based on the empirical integrated distribution function (idf)

Klar (1999) introduced a test statistic based on the supremum difference between the idf of the Poisson distribution and the empirical version of this function. The idf is defined to be

$$\Psi\left(t\right) = \int_t^\infty \left(1-F\left(x\right)\right)\mathrm{d}x, \tag{2.9}$$

while the empirical idf is

$$\Psi_n(t) = \frac{1}{n}\sum_{j=1}^{n}(X_j - t)\, I_{(X_j>t)}. \tag{2.10}$$

The associated test statistic is denoted by

$$ID_n = \sup\nolimits_{t\geq 0}\sqrt{n}\left|\Psi_{\widehat{\lambda}}(t) - \Psi_n(t)\right|, \tag{2.11}$$

where $\Psi_{\widehat{\lambda}}$ denotes the idf of the Poisson distribution. This function does not allow substantial simplification. The assumption of the Poisson distribution is rejected for large values of ID_n. Klar (1999) estimates the critical values of ID_n using simulation, and shows that this test outperforms Kolmogorov-Smirnov and Cramér-von Mises type tests against the majority of alternative distributions considered in the paper.

As a result of independent interest, we consider the derivation of the test statistic below. The derivations provided in Klar (1999) are quite compact and we provide more detailed versions of the required calculations below. When considering the empirical idf given in (2.10), it is not immediately clear that this function is the empirical version of the idf given in (2.9). Below we show that the empirical idf is obtained by replacing the distribution function F by F_n in (2.9);

$$\begin{aligned}
\Psi_n(t) &= \int_t^\infty (1 - F_n(x))\,\mathrm{d}x \\
&= \int_t^\infty \left(1 - \frac{1}{n}\sum_{j=1}^{n} I(X_j \leq x)\right)\mathrm{d}x \\
&= \frac{1}{n}\int_t^\infty \left(n - \sum_{j=1}^{n} I(X_j \leq x)\right)\mathrm{d}x \\
&= \frac{1}{n}\int_t^\infty \sum_{j=1}^{n} I(X_j > x)\mathrm{d}x \\
&= \frac{1}{n}\sum_{j=1}^{n}\int_t^\infty I(x < X_j)\mathrm{d}x \\
&= \frac{1}{n}\sum_{j=1}^{n}(X_j - t)\, I_{(X_j>t)},
\end{aligned}$$

where the interchange of summation and the integral in the penultimate step is justified by the fact that a finite number of summands are included in the summation.

In order to evaluate the test statistic in (2.11), we require a computational form for the calculation of the idf. To this end, and recalling that the distribution function is a step function, consider

$$\begin{aligned}\Psi_{\widehat{\lambda}}(t) &= \int_t^{\infty} \left(1 - F_{\widehat{\lambda}}(x)\right) \mathrm{d}x \\ &= \int_t^{\infty} \left(1 - \sum_{j=0}^{\lfloor t \rfloor} F_{\widehat{\lambda}}(j)\right) \mathrm{d}x \\ &= \sum_{j=\lfloor t \rfloor + 1}^{\infty} (j-t) F_{\widehat{\lambda}}(j).\end{aligned}$$

Although this formula allows an explicit formula for the calculation of the idf, we are left with the problem of calculating an infinite summation. However, recalling that

$$\Psi_{\widehat{\lambda}}(0) = \int_t^{\infty} \left(1 - F_{\widehat{\lambda}}(x)\right) \mathrm{d}x = E[X] = \lambda,$$

we may express the idf as a finite summation. Consider the following:

$$\begin{aligned}\Psi_{\widehat{\lambda}}(t) &= \int_0^{\infty} \left(1 - F_{\widehat{\lambda}}(x)\right) \mathrm{d}x - \int_0^{t} \left(1 - F_{\widehat{\lambda}}(x)\right) \mathrm{d}x \\ &= \lambda - \sum_{j=0}^{\lfloor t-1 \rfloor} [1 - F_{\widehat{\lambda}}(j)] + (t - \lfloor t \rfloor) F_{\widehat{\lambda}}(\lfloor t \rfloor).\end{aligned}$$

Using this simple formula, we are able to calculate $\Psi_{\widehat{\lambda}}$, meaning that the calculation of ID_n in (2.11) is a simple matter.

In the next chapter we introduce a new class of tests for the Poisson distribution which is the focus of this study.

Chapter 3

A newly proposed class of tests for the Poisson distribution

In this chapter we introduce a new class of tests for the Poisson distribution. These tests are related to the weighted Poisson distribution; this distribution is discussed below. Thereafter we turn our attention to L_p distances as well as their weighted counterparts. Finally, we discuss the newly proposed class of tests, including the computational form of the proposed test statistics.

3.1 The weighted Poisson distribution

Due to its wide range of applications and its usefulness in the modelling of observed phenomena, many generalisations and extensions of the Poisson distribution have been proposed. An important generalisation is the weighted Poisson distribution introduced in Fisher (1934). In the mentioned paper, Fisher introduces the weighted Poisson distribution via the so-called method of ascertainment. Mijburgh (2020) points out that Rao (1965) is often cited as the first paper to introduce the method of ascertainment, while Fisher (1934) introduces this method in a similar context three decades earlier. The idea underlying this method is based on a random variable with a fixed discrete distribution; meaning that the probability that a given value will realise is fixed. On the other hand, we might not be able to "ascertain" this value in some instances. That is, some of the values may not be observed and, therefore, go unnoticed. As a result, the probability of observing a specified value for the distribution is changed or re-weighted. This can be achieved by introducing a weight function giving more weight to the values which are likely to be "ascertained" and less weight to those which are not. This concept is made precise below.

Let v be some function such that $v(x) \geq 0$ for $x \in \{0, 1, \dots\}$ and let $X \sim P(\lambda)$. $\widetilde{X}$ is said to be a weighted Poisson random variable, with parameter λ and weight function v, if $\widetilde{X}$ has pmf

$$f_{\lambda,v}(x) = \frac{v(x)f_\lambda(x)}{E[v(X)]}, \tag{3.1}$$

for $x \in \{0, 1, \dots\}$, where

$$E[v(X)] = \sum_{x=0}^{\infty} v(x) f_\lambda(x) < \infty.$$

It is a simple matter to show that, if a constant weight function is used, the weighted Poisson distribution simplifies to the (unweighted) Poisson. To demonstrate this, let $v(x) = c$, where $c > 0$ is a constant. In this case, the pmf of $\widetilde{X}$ is

$$f_{\lambda,v}(x) = \frac{v(x) f_\lambda(x)}{E[v(X)]} = \frac{c f_\lambda(x)}{c} = f_\lambda(x).$$

The above demonstrates that, given the value of λ, v does not uniquely define $f_{\lambda,v}$ (since constant multiples of the weight function give rise to the same pmf). In order to ensure that, given λ, the weight function used uniquely determines the pmf, we define $w(x) = v(x)/E[v(X)]$. In this case, the weighted Poisson random variable, $\widetilde{X}$, has pmf

$$f_{\lambda,w}(x) = w(x) f_\lambda(x). \tag{3.2}$$

Note that, for every v such that $E[v(X)] < \infty$, there exists a suitably rescaled weight function w. As a result, we take (3.2) to be the definition of the pmf of a weighted Poisson random variable.

In order to proceed, we introduce the empirical pmf;

$$f_n(x) = \frac{1}{n} \sum_{j=0}^{n} I(X_j = x), \tag{3.3}$$

with I denoting the indicator function. In its most general form, w can be chosen such that $f_{\lambda,w}$ takes the form of any specified pmf defined on the non-negative integers. As a result, given an observed dataset, $X_1, X_2, \dots, X_n$, we may estimate λ by $\widehat{\lambda} = \sum_{j=1}^{n} X_j/n$ and then choose w so that $f_{\widehat{\lambda},w}(x) = f_n(x)$ for $x \in \{0, 1, \dots\}$. Denote by w^* the weight function equating $f_{\hat{\lambda},w^*}$ to f_n;

$$w^*(x) = \frac{f_n(x)}{f_{\hat{\lambda}}(x)}. \tag{3.4}$$

We refer to w^* as the empirical weight function.

If the dataset under consideration is realised from a Poisson distribution, then we may expect that $w^*(x)$ would be approximately 1 for all x, at least if the dataset is large. As a result, we can construct a test for the Poisson distribution based on the L_p distance (or some other distance or discrepancy measure) between w^* and the point $\mathbb{1} = \{1, 1, 1, \dots\}$. Below, we propose a class of tests for the Poisson distribution based on this distance. However, before proposing the test, we turn our attention to L_p distances, including weighted versions of these distances in a more general setting.

3.2 Weighted L_p distances

In order to define a distance measure d, consider three points, P, Q and R, in some k-dimensional space (with $k \leq \infty$). The function d is a distance measure if, and only if, it possesses the properties below.

- $d(P, Q) = d(Q, P)$.
- $d(P, Q) > 0$ if $P \neq Q$.
- $d(P, Q) = 0$ if $P = Q$.
- $d(P, Q) \leq d(P, R) + d(R, Q)$.

The last of these properties is known as the triangle inequality. Five of the most well-known types of distance measures are discussed in Hair, Black, Babin, Anderson & Tatham (2006); Euclidean, squared (or absolute) Euclidean, city-block (or Manhattan), Chebychev and Mahalanobis distances. Our focus is on L_p distances, of which the Euclidean and city-block distances are special cases.

In order to be precise, let $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ be two points in $\mathbb{R}^k$, with $k \leq \infty$. In this case, according to Hair et al. (2006), the L_p distance between P and Q is defined to be

$$L_p(P, Q) = \left(\sum_{j=1}^{k} |P_j - Q_j|^p \right)^{\frac{1}{p}}.$$

Special attention is paid to three L_p distances; L_1, L_2 and L_∞.

Consider the distance between w^* and $\mathbb{1}$ in the case where $p < \infty$. The empirical pmf does not assign any probability weight to values larger than the sample maximum; $f_n(x) = 0, \forall\, x > m$. As a consequence, the empirical weight function is zero for all arguments exceeding m; $w^*(x) = 0, \forall\, x > m$. The distance of interest is

$$\begin{aligned} L_p(w^*, \mathbb{1}) &= \left(\sum_{j=0}^{\infty} |w^*(j) - 1|^p \right)^{1/p} \\ &= \left(\sum_{j=0}^{m} |w^*(j) - 1|^p + \sum_{j=m+1}^{\infty} |w^*(j) - 1|^p \right)^{1/p} \end{aligned}$$

$$= \left(\sum_{j=0}^{m} |w^*(j) - 1|^p + \sum_{j=m+1}^{\infty} 1 \right)^{1/p}. \tag{3.5}$$

Since the second summation above is infinite, we have that $L_p(w^*, \{1, 1, \dots\}) = \infty$ for every finite sample. As a result, this distance cannot be employed as a test for the Poisson distribution. However, we may employ a weighted version of the L_p distance between w^* and $\mathbb{1}$; using some weight function g, such that $g(x) > 0$ for all $x \in \{0, 1, \cdots\}$ and $\sum_{j=0}^{\infty} g(x) < \infty$.

Next, we consider the L_∞ distance. In this case, the distance between w^* and $\mathbb{1}$ is

$$\begin{aligned}
L_\infty(w^*, \mathbb{1}) &= \max_{j \in \{0,1,\dots\}} |w^*(j) - 1| \\
&= \max \left\{ \max_{j \in \{0,\dots,m\}} |w^*(j) - 1|, \max_{j \in \{m+1, m+2, \dots\}} |w^*(j) - 1| \right\} \\
&= \max \left\{ \max_{j \in \{0,1,\dots\}} |w^*(j) - 1|, 1 \right\}.
\end{aligned}$$

Although the L_∞ distance between w^* and $\mathbb{1}$ is finite, it is at least 1. Furthermore, we have that

$$\begin{aligned}
\max_{j \in \{0,1,\dots,m\}} \{|w^*(j) - 1|\} > 1 \quad &\Leftrightarrow \quad \max_{j \in \{0,1,\dots,m\}} \left| \frac{f_n(j)}{f_{\hat{\lambda}}(j)}(j) - 1 \right| > 1 \\
&\Leftrightarrow \quad \exists\, j \text{ such that } |f_n(j) - f_{\hat{\lambda}}| > f_{\hat{\lambda}}(j) \\
&\Leftrightarrow \quad \exists j \text{ such that } \left\{ f_n(j) - f_{\hat{\lambda}} > f_{\hat{\lambda}}(j) \right\} \text{ or } \left\{ f_{\hat{\lambda}} - f_n(j) > f_{\hat{\lambda}}(j) \right\} \\
&\Leftrightarrow \quad \exists\, j \text{ such that } f_n(j) > 2 f_{\hat{\lambda}}(j) \\
&\Leftrightarrow \quad \exists\, j \text{ such that } f_n(j) > \frac{2e^{-\hat{\lambda}} \hat{\lambda}^j}{j!} \\
&\Leftrightarrow \quad \exists\, j \text{ such that } \frac{j! f_n(j)}{2} > \frac{\hat{\lambda}^j}{e^{\hat{\lambda}}}.
\end{aligned}$$

As a result, under the assumption of the Poisson distribution, the probability that $L_\infty(w^*, \mathbb{1}) = 1$ depends on the underlying value of λ. For some values of λ, it may be that the probability of realising a distance of exactly 1 is substantial. Again, we consider the use of a weighted distance measure in order to circumvent this potential pitfall.

In the discussion above, it should be noted that we choose to use weighted L_p distances simply because these are convenient and well-known distances. Other distance measures can also be considered in order to develop additional classes of tests for the Poisson distribution.

3.3 The proposed test statistics

Consider weighted versions of the L_p distances between P and Q in some k dimensional space. Let g be some weight function satisfying $g(x) \geq 0$ for all $x \in \{1, \cdots, k\}$ and $\sum_{x=1}^{p} g(x) < \infty$. As mentioned above, we consider three L_p distances, corresponding to $p \in \{1, 2, \infty\}$. As for the choice of g, we again consider three choices; the fitted Poisson pmf, $f_{\widehat{\lambda}}$, the empirical pmf, f_n, and a Laplace-type weight function of the form $g(x) = \mathrm{e}^{-ax}$, where a is a user-defined tuning parameter. Combining the various values of p and the weight functions used we arrive at 9 different test statistics. We discuss these in turn below, starting with the case where p is finite.

The weighted L_p distance between P and Q corresponding to weight function g is

$$L_{p,g}(P, Q) = \left[\sum_{j=1}^{k} |P_j - Q_j|^p g(j)\right]^{1/p}.$$

Note that our notation now includes two separate weight functions; w^* is the empirical weight function associated with the weighted Poisson distribution and g is the weight function associated with the distance measure used. The choice of g does not depend on w^*.

Restricting our attention to the case where $p < \infty$ and considering specifically the distance between w^* and $\mathbb{1}$, the required test statistic can be expressed as

$$\begin{aligned} L_{p,g}(w^*, \mathbb{1}) &= \left(\sum_{j=0}^{\infty} |w^*(j) - 1|^p g(j)\right)^{1/p} \\ &= \left(\sum_{j=0}^{m} |w^*(j) - 1|^p g(j) + \sum_{j=m+1}^{\infty} |w^*(j) - 1|^p g(j)\right)^{1/p} \\ &= \left(\sum_{j=0}^{m} |w^*(j) - 1|^p g(j) + \sum_{j=m+1}^{\infty} g(j)\right)^{1/p}, \end{aligned} \tag{3.6}$$

which is finite since the first summation consists of a finite number of summands while, for the second summation, we have that $\sum_{j=m+1}^{\infty} g(j) \leq \sum_{j=0}^{\infty} g(j)$ since $g(j) \geq 0$, and $\sum_{j=0}^{\infty} g(j) < \infty$ by the definition of g.

The first three test statistic proposed for the Poisson distribution are obtained by setting $p = 1$ in (3.6) together with the various choices of g mentioned. In this case, (3.6) simplifies to

$$L_{1,g}(w^*, \mathbb{1}) = \sum_{j=0}^{m} |w^*(j) - 1| g(j) + \sum_{j=m+1}^{\infty} g(j). \tag{3.7}$$

Below we introduce the test statistics based on the weighted L_1 distances corresponding to the various choices of g considered.

In order to define the first test statistic, let $g(x) = f_{\widehat{\lambda}}(x)$. In this case, the first test statistic is calculated to be

$$\begin{aligned} T^{(1)}_{n,f_{\widehat{\lambda}}} := L_{1,f_{\widehat{\lambda}}}(w^*, \mathbb{1}) &= \sum_{j=0}^{m} |w^*(j) - 1| f_{\widehat{\lambda}}(j) + \sum_{j=m+1}^{\infty} f_{\widehat{\lambda}}(j) \\ &= \sum_{j=0}^{m} |w^*(j) - 1| f_{\widehat{\lambda}}(j) + 1 - F_{\widehat{\lambda}}(m). \end{aligned}$$

The second proposed test statistic is obtained using the empirical pmf, f_n, as a weight function. In this case the test statistic can be expressed as

$$\begin{aligned} T^{(1)}_{n,f_n} := L_{1,f_n}(w^*, \mathbb{1}) &= \sum_{j=0}^{m} |w^*(j) - 1| f_n(j) + \sum_{j=m+1}^{\infty} f_n(j) \\ &= \sum_{j=0}^{m} |w^*(j) - 1| f_n(j), \end{aligned}$$

where the final equality follows since $f_n(j) = 0$ for all j exceeding the sample maximum.

The final weighted L_1 distance based test statistic considered is obtained using the Laplace type weight function $g(j) = \mathrm{e}^{-aj}$ for some $a > 0$. A Monte Carlo study was conducted to test the sensitivity of the powers associated with this test for various values of a. We found that the results are remarkably insensitive to the choice of a. As a result, we use the value of $a = 1$. It should be noted that the same insensitivity to the chosen value of a was observed for the tests based on the weighted L_2 and L_∞ type tests. Therefore, $a = 1$ is used throughout and the weight function used is essentially reduced to $g(j) = \mathrm{e}^{-j}$. The test statistic can, in this case, be expressed as

$$\begin{aligned} T^{(1)}_{n,L} := L_{1,\exp(-j)}(w^*, \mathbb{1}) &= \sum_{j=0}^{m} |w^*(j) - 1| \mathrm{e}^{-j} + \sum_{j=m+1}^{\infty} \mathrm{e}^{-j} \\ &= \sum_{j=0}^{m} |w^*(j) - 1| f_n(j) + \frac{\mathrm{e}^{-(m+1)}}{1 - \mathrm{e}^{-1}}. \end{aligned}$$

Note that the subscript L is used to indicate that the Laplace type weight function is used.

We now turn our attention to the tests based on weighted L_2 type distances. Using notation similar to that used above, we define the test statistics $T^{(2)}_{n,f_{\widehat{\lambda}}}$, $T^{(2)}_{n,f_n}$ and $T^{(2)}_{n,L}$. The required test statistics can be expressed as

$$
\begin{aligned}
T^{(2)}_{n,f_{\widehat{\lambda}}} &:= L_{2,f_{\widehat{\lambda}}}(w^*, \mathbb{1}) = \left(\sum_{j=0}^{m} |w^*(j) - 1|^2 f_{\widehat{\lambda}}(j) + 1 - F_{\widehat{\lambda}}(m) \right)^{1/2}. \\
T^{(2)}_{n,f_n} &:= L_{2,f_n}(w^*, \mathbb{1}) = \left(\sum_{j=0}^{m} |w^*(j) - 1|^2 f_n(j) \right)^{1/2}. \\
T^{(2)}_{n,L} &:= L_{2,L}(w^*, \mathbb{1}) = \left(\sum_{j=0}^{m} |w^*(j) - 1|^2 f_n(j) + \frac{\mathrm{e}^{-2(m+1)}}{1-\mathrm{e}^{-2}} \right)^{1/2}.
\end{aligned}
$$

Next, we consider test statistics based on weighted L_∞ distances. In general, we have that

$$
\begin{aligned}
T^{(\infty)}_{n,g} := L_{\infty,g}(w^*, \mathbb{1}) &= \max_{j \in \{0,1,\dots\}} \{|w^*(j) - 1| g(j)\} \\
&= \max \left\{ \max_{j \in \{0,1,\dots,m\}} \{|w^*(j) - 1| g(j)\}, \max_{j \in \{m+1,m+2,\dots\}} \{|w^*(j) - 1| g(j)\} \right\} \\
&= \max \left\{ \max_{j \in \{0,1,\dots,m\}} \{|w^*(j) - 1| g(j)\}, \max_{j \in \{m+1,m+2,\dots\}} g(j) \right\}, \qquad (3.8)
\end{aligned}
$$

where the final equality follows from the fact that $w^*(j) = 0$ for all j greater than the sample maximum.

When considering the general form of the test statistic given in (3.8), some care is required when calculating

$$
\max_{j \in \{m+1,m+2,\dots\}} g(j). \qquad (3.9)
$$

For two of the weight functions considered, this term is computed with ease. If $g(j) = f_n(j)$, then $g(j) = f_n(j) = 0$ for all $j > m$ and the term given in (3.9) is 0. If $g(j) = \mathrm{e}^{-j}$, then g is a decreasing sequence and the maximum is simply attained at $g(m+1) = \mathrm{e}^{-(m+1)}$. If, on the other hand, $g(j) = f_{\widehat{\lambda}}(j)$, then some subtleties should be addressed before attempting to calculate (3.9); these are discussed below.

The pmf of a $P(\lambda)$ random variable can be shown to be a non-increasing function in the case where $\lambda \leq 1$. On the other hand, if $\lambda > 1$, then the pmf increases initially and decreases after attaining some maximum value. As a result, in order to calculate the term in (3.9), we are required to determine the mode of a $P(\lambda)$ distribution. To this end, consider the following ratio;

$$
\frac{f_\lambda(x)}{f_\lambda(x-1)} = \frac{\lambda^x e^{-\lambda}}{x!} \Big/ \frac{\lambda^{x-1} e^{-\lambda}}{x-1!} = \frac{\lambda}{x},
$$

which shows that

$$
f_\lambda(x) \geq f_\lambda(x-1) \Leftrightarrow \lambda \geq x. \qquad (3.10)
$$

As a result, $f_\lambda(x)$ is non-decreasing in x for $x \le \lambda$ and non-increasing for $x \ge \lambda$. In the case where λ is an integer, the $P(\lambda)$ distribution has two modes; $\lambda - 1$ and λ. Turning our attention to non-integer λ, it follows from (3.10) that the mode is $\lfloor \lambda \rfloor$.

Combining the results obtained for integer and non-integer λ, we know that $\lfloor \lambda \rfloor$ is a mode of the $P(\lambda)$ distribution. However, in the case of non-integer λ, this corresponds to the larger of the two modes. Our interest centers on the smallest of the modes, as we are interested in finding the smallest value of j from which $g(j) = f_{\widehat{\lambda}}(j)$ is non-increasing in j so that we may calculate (3.9). Note that, regardless of whether or not λ is an integer, $\lceil \lambda \rceil - 1$ is a mode. Furthermore, in the case of integer λ, $\lceil \lambda \rceil - 1$ corresponds to the smallest of the two modes. As a result, we have that $g(j) = f_{\widehat{\lambda}}(j)$ is non-increasing in $j \in \{\lceil \widehat{\lambda} \rceil - 1, \lceil \widehat{\lambda} \rceil, \dots\}$.

When computing (3.9), we are interested in determining whether $g(j) = f_{\widehat{\lambda}}(j)$ is non-increasing in $j \in \{m+1, m+2, \dots\}$. From the above, we know that this would be the case if, and only if

$$\begin{aligned} m + 1 \ge \lceil \widehat{\lambda} \rceil - 1 \quad &\Leftrightarrow \quad \widehat{\lambda} \le m + 2 && (3.11) \\ &\Leftrightarrow \quad \sum_{j=1}^{n} X_j \le n(m+2), && (3.12) \end{aligned}$$

and the final event stated above is true for all samples since the maximum of the sum is n times the sample maximum; i.e., $\sum_{j=1}^{n} X_j \le nm$.

Taking the above arguments into account, we have that $g(j) = f_{\widehat{\lambda}}(j)$ is non-increasing in $j \in \{m+1, m+2, \dots\}$. As a result, the term in (3.9) simplifies to

$$\max_{j \in \{m+1, m+2, \dots\}} f_{\widehat{\lambda}}(j) = f_{\widehat{\lambda}}(m+1).$$

Summarising the arguments above, we can express the three test statistics based on weighted L_∞ distances as

$$\begin{aligned} T^{(\infty)}_{n, f_{\widehat{\lambda}}} \quad &:= \quad L_{\infty, f_{\widehat{\lambda}}}(w^*, \mathbb{1}) = \max\left\{ \max_{j \in \{0,1,\dots,m\}} \left\{ |w^*(j) - 1| f_{\widehat{\lambda}}(j) \right\}, f_{\widehat{\lambda}}(m+1) \right\}. \\ T^{(\infty)}_{n, f_n} \quad &:= \quad L_{\infty, f_n}(w^*, \mathbb{1}) = \max\left\{ \max_{j \in \{0,1,\dots,m\}} \left\{ |w^*(j) - 1| f_n(j) \right\} \right\}. \\ T^{(\infty)}_{n, L} \quad &:= \quad L_{\infty, L}(w^*, \mathbb{1}) = \max\left\{ \max_{j \in \{0,1,\dots,m\}} \left\{ |w^*(j) - 1| \mathrm{e}^{-j} \right\}, \mathrm{e}^{-(m+1)} \right\}. \end{aligned}$$

In the next chapter, we consider the power performance of the test statistics proposed above. Additionally, we demonstrate the use of these tests when analysing observed data.

Chapter 4

Numerical results

In this chapter we compare the performance of the newly proposed tests to that of the existing tests for the Poisson distribution. This is achieved through a Monte Carlo study in which empirical powers are calculated using a bootstrap approach. Thereafter, we turn our attention to two observed datasets which have been modelled using a Poisson distribution and we demonstrate the use of the proposed techniques in order to test this assumption.

4.1 Monte Carlo setup

A power study is conducted to arrive at empirical powers. In our study, these finite sample powers are calculated using Monte Carlo simulations. A significance level of 5% is used throughout. We include results pertaining to sample sizes of 30, 50, 100 and 200. However, in the main text we include and discuss only the results obtained using a sample size of 50. The remaining results can be found in Appendix A.

We consider the performance of the various tests against a range of alternative distributions. The mass function and notation used for each of the alternatives distributions used can be found in Table 4.1 (this table is similar to Table 1 in Mijburgh & Visagie (2020)). These alternatives are selected since they are commonly used when testing the assumption of the Poisson distribution; see Mijburgh & Visagie (2020), Gürtler & Henze (2000) and Karlis & Xekalaki (2000).

Table 4.1: Alternative distributions considered

Alternative distribution	Notation	Mass function
Discrete uniform	$DU(a,b)$	$(b-a+1)^{-1}$
Binomial	$Bin(m,p)$	$\binom{m}{x}p^x\,(1-p)^x$
Negative binomial	$NB(r,p)$	$\binom{r+x-1}{x}p^r\,(1-p)^x$
Poisson Mixtures	$PM(p,\lambda_1,\lambda_2)$	$(x!)^{-1}\left\{p\lambda_1^x\mathrm{e}^{-\lambda_1}+(1-p)\,\lambda_2^x\mathrm{e}^{-\lambda_2}\right\}$
Zero inflated Poisson	$ZIP(p,\lambda)$	$\left(p\frac{x!}{e^{-\lambda}\lambda^x}\mathrm{I}\,(x=0)+1-p\right)\frac{e^{-\lambda}\lambda^x}{x!}$
Weighted Poisson	$WP(\lambda,a,b)$	$(y!)^{-1}\lambda^y\exp(-\lambda)\frac{ay^2+by+1}{a(\lambda+\lambda^2)+b\lambda+1}$

4.2 Numerical powers

Since λ is an unknown shape parameter, we use a parametric bootstrap procedure in order to approximate the distribution of the test statistics used. In order to speed up the required calculations we use the so-called warp-speed bootstrap, detailed in Giacomini, Politis & White (2013), in order to arrive at the finite sample results shown below. We use the algorithm below to implement the warp-speed bootstrap; this algorithm is an adapted version of the algorithm found in Allison, Betsch, Ebner & Visagie (2022) and Mijburgh & Visagie (2020).

1. Sample $X_1, \ldots, X_n$ from distribution function F and estimate λ by $\widehat{\lambda} = \frac{1}{n}\sum_{j=1}^{n} X_j$.
2. Calculate the value of the test statistic; $S := S(X_1, \ldots, X_n)$.
3. Generate $X_1^*, \ldots, X_n^*$ from a $P(\widehat{\lambda})$ distribution. Calculate the test statistic based on this sample; $S^* = S(X_{n,1}^*, \ldots, X_{n,n}^*)$.
4. Repeat Steps 1 to 3 MC times. Let S_m be the value of the test statistic calculated using the m^{th} dataset generated from F and let S_m^* be the value of the test statistic calculated from the bootstrap sample obtained in the m^{th} iteration of the Monte Carlo simulation. As a result, we obtain $S_1, \ldots, S_{MC}$ and $S_1^*, \ldots, S_{MC}^*$.
5. We reject the hypothesis of the Poisson distribution for the j^{th} sample from F if $S_j > S^*_{(\lfloor MC\cdot(1-\alpha)\rfloor)}$, $j = 1, \ldots, MC$, where $S^*_{(1)} \leq \ldots \leq S^*_{(MC)}$ are the ordered statistics obtained from the bootstrap samples and $\lfloor\cdot\rfloor$ is the floor function.

Table 4.2 shows the empirical powers obtained by the various tests considered for samples of size 50; the table shows the percentage of samples (rounded to the nearest integer) that resulted in a rejection of the null assumption of the Poisson distribution against each of the alternative distributions considered. The alternative distributions considered are similar to those found in the goodness-of-fit literature for the Poisson distribution, see (Mijburgh & Visagie 2020). The results presented are based on 50 000 Monte Carlo replications. Note that this table contains a column indicating FI, the Fisher index of the alternative distribution considered. When discussing the results, our focus is on a comparison between the powers achieved by the newly proposed tests and those obtained by the existing tests. For ease of comparison, the highest power achieved against each alternative distribution considered is printed in bold.

Table 4.2: Empirical powers obtained for samples of size n = 50

Distribution	FI	KS_n	CV_n	AD_n	KL_n	ID_n	$T^{(1)}_{n,f_{\widehat{\lambda}}}$	$T^{(1)}_{n,f_n}$	$T^{(1)}_{n,L}$	$T^{(2)}_{n,f_{\widehat{\lambda}}}$	$T^{(2)}_{n,f_n}$	$T^{(2)}_{n,L}$	$T^{(\infty)}_{n,f_{\widehat{\lambda}}}$	$T^{(\infty)}_{n,f_n}$	$T^{(\infty)}_{n,L}$
$P(0.5)$	1.00	5	5	5	4	4	5	5	5	5	5	5	5	5	4
$P(1)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	4
$P(5)$	1.00	5	5	5	5	5	5	5	4	5	5	4	5	5	4
$P(10)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$DU(4)$	1.00	32	54	**63**	60	17	62	39	34	33	0	28	43	38	28
$B(5, 0.25)$	0.75	6	17	18	18	**23**	13	1	15	1	0	2	12	4	17
$B(5, 0.2)$	0.80	4	13	13	12	**15**	10	2	11	1	0	0	11	4	13
$B(10, 0.2)$	0.80	4	11	11	11	**15**	7	1	7	1	0	5	8	2	8
$B(10, 0.1)$	0.90	3	**7**	6	6	**7**	6	2	6	1	1	2	**7**	3	6
$NB(9, 0.9)$	1.11	10	7	8	8	8	6	**11**	7	**11**	**11**	**11**	5	10	6
$NB(45, 0.9)$	1.11	8	6	8	9	8	6	**11**	7	10	10	7	4	10	7
$PM(0.5, 3, 5)$	1.25	17	13	19	20	**21**	9	20	13	19	16	13	5	16	13
$ZIP(0.9, 3)$	1.30	38	21	36	31	30	28	30	**57**	24	11	56	14	20	56
$PM(0.1, 1, 5)$	1.31	21	19	31	32	32	16	31	**53**	28	15	53	6	24	52
$NB(15, 0.75)$	1.33	21	17	27	28	**30**	11	29	14	28	24	13	5	23	13
$NB(3, 0.75)$	1.33	30	19	25	26	27	17	**32**	24	31	28	28	11	26	21
$DU(6)$	1.33	64	78	**86**	**86**	66	78	71	80	64	5	82	44	54	84
$NB(4, 0.7)$	1.43	36	25	36	37	40	20	**42**	25	40	36	37	11	34	26
$NB(2, 2/3)$	1.50	47	34	42	42	44	30	**48**	40	46	42	43	19	39	35
$NB(3, 2/3)$	1.50	45	32	43	44	47	26	**48**	33	46	41	46	15	40	32
$ZIP(0.8, 3)$	1.60	90	67	84	79	78	78	78	**95**	71	26	94	64	69	**95**
$PM(0.2, 1, 5)$	1.61	63	55	75	75	75	47	67	**81**	63	35	**81**	17	52	79
$NB(1, 0.5)$	2.00	**82**	74	80	77	79	66	**82**	78	79	73	77	51	71	70

The results in Table 4.2 indicate that all tests considered achieve the specified nominal significance level of 5% closely. When considering the Fisher indexes of the alternative distributions used, we note that it is possible to distinguish three classes of distributions; those which are equidispersed as well as under or over dispersed alternatives. We discuss each of these classes in turn.

Equidispersed alternatives to the Poisson are not particularly prevalent in the statistical literature, as is witnessed by the fact that the first goodness-of-fit test for the Poisson distribution (based on the Fisher index) is a test for equidispersion. As a result, we include a single equidispersed alternative; $D[0,4]$. The AD_n test performs best against this alternative; achieving an empirical power of 63% for a sample of size 50. This performance is closely followed by $T^{(1)}_{n,f_{\widehat{\lambda}}}$ which achieves a power of 62%.

When turning our attention to the underdispersed alternatives, many of the newly proposed tests do not perform well. Notable exceptions are $T^{(1)}_{n,f_{\widehat{\lambda}}}$, $T^{(\infty)}_{n,f_{\widehat{\lambda}}}$ and $T^{(\infty)}_{n,L}$, which exhibit powers comparable to the ID_n test which performs best against underdispersed alternatives. It should be noted that the weighted L_2 type tests do not perform well against the under dispersed alternatives considered. However, the weighted L_1 and L_∞ type tests do not suffer from this disadvantage.

The final class of alternatives considered is the overdispersed distributions. The newly proposed tests notably outperform the existing tests for the Poisson distribution against the majority of the overdispersed alternatives considered. $T^{(1)}_{n,f_n}$ outperforms all other tests in 6 out of 14 instances considered, together with a single instance where this test and the KS_n test jointly produce the highest power. Additionally, the $T^{(1)}_{n,f_n}$ test is outperformed by the ID_n test against both the $PM(0.5,3,5)$ and the $NB(15,0.75)$ distributions. However, in both cases this outperformance is by a single percentage point; in the former case the empirical powers are 21% and 20%, respectively, while powers of 30% and 29% are observed in the latter case. The table further illustrates that $T^{(1)}_{n,L}$ is the second most powerful test against this class of distributions; this test outperforms each of the remaining tests against 4 of the remaining 7 alternatives. In summary, $T^{(1)}_{n,f_n}$ and $T^{(1)}_{n,L}$ either outperformed, or produced empirical powers that were no less than 1% inferior to the highest power achieved by the existing tests in 13 of the 14 cases considered.

The results obtained for the remaining sample sizes, which are n = 30,100 and 200, considered are available in Appendix A. The results for the other sample sizes were also encouraging and generally exhibited a similar pattern to the results obtained for samples of size 50. As expected, the empirical powers increase with sample size.

4.3 Practical applications

We consider two practical examples in this section. The first pertains to the distribution of Sparrows' nests; Zar (1999) records the number of sparrow nests discovered on 40 one hectare plots. This dataset has been analysed in Gürtler & Henze (2000) as well as Mijburgh & Visagie (2020). Both of the mentioned papers test the hypothesis that the number of sparrows nests per hectare are realised from a Poisson distribution. Table 4.3 show the frequencies of the observed number of nests.

Table 4.3: The number of sparrow nests found on 40 one hectare plots

Count	9	22	6	2	1
Frequency	0	1	2	3	4

As a second example, we consider the annual number of deaths due to horse kick in the Prussian army between 1875 and 1894; Table 4.4 shows the observed frequency of the counts in this case. Gürtler & Henze (2000) tested the hypothesis that this dataset is realised from a Poisson distribution.

Table 4.4: The annual number of deaths due to horse kick in the Prussian army

Count	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Frequency	1	1	2	2	1	1	2	1	3	1	0	1	2	0	1	1

We test the hypotheses that the frequency distribution of the Sparrows' nests, as well as that of the deaths due to horse kick are realised from a Poisson distributed. In each case, to estimate the p-values of the tests, a classical parametric bootstrap approach was employed as in Gürtler & Henze (2000). The number of replications used was 100 000. The calculated test statistics and estimated p-values associated with each test are provided in Table 4.5; the two examples are treated separately in the table.

Consider the results pertaining to the first example; the frequency distribution of the sparrows' nests. The p-values reported in Table 4.5 indicate that, at a 10% significance level, the null hypothesis is rejected by all of the tests considered, with the exception of $T^{(2)}_{n,f_{\hat{\lambda}}}$ and $T^{(2)}_{n,f_n}$. As a result, we reject the null hypothesis and we conclude that the dataset is not realised from a Poisson distribution.

Next we consider the p-values obtained for the dataset relating to the deaths by horse kick in the Prussian army. For this dataset, 9 out of the 14 tests considered do not reject the Poisson assumption at a 10% level of significance. In this case, we do not reject the Poisson assumption and we conclude that the observed dataset is sufficiently compatible with this assumption for the Poisson distribution to serve as an accurate model.

Table 4.5:

Tests	Sparrows' nests Statistic	p-value	Horse kicks Statistic	p-value
KS_n	0.480	0.086	0.701	0.049
CV_n	0.000	0.027	0.000	0.102
AD_n	0.001	0.054	0.003	0.017
KL_n	1.364	0.074	5.094	0.016
ID_n	0.682	0.050	2.481	0.013
$T^{(1)}_{n,f_{\widehat{\lambda}}}$	0.377	0.039	0.705	0.265
$T^{(1)}_{n,f_n}$	0.409	0.092	1.432	0.142
$T^{(1)}_{n,L}$	0.574	0.033	1.776	0.116
$T^{(2)}_{n,f_{\widehat{\lambda}}}$	0.155	0.205	1.179	0.176
$T^{(2)}_{n,f_n}$	0.179	0.268	5.282	0.182
$T^{(2)}_{n,L}$	0.223	0.145	2.673	0.119
$T^{(\infty)}_{n,f_{\widehat{\lambda}}}$	0.184	0.017	0.075	0.929
$T^{(\infty)}_{n,f_n}$	0.276	0.064	0.365	0.437
$T^{(\infty)}_{n,L}$	0.324	0.040	1.000	0.055

In conclusion, the Poisson distribution is an appropriate model for the deaths by horse kick in the Prussian army, while it is not an appropriate model for the frequency distribution of the sparrows' nests.

Chapter 5

Conclusions and recommendations

This dissertation discusses a range of existing tests for the Poisson distribution. The majority of the existing tests considered are based on characterisations of the Poisson. These include the distribution function, the probability generating function and the integrated distribution function. The test based on the integrated distribution has been shown to be powerful in previous studies and some interesting derivations are required in order to arrive at the test. We include a detailed account of these derivations as we were unable to find this in the literature. Some existing tests not based on characterisations of the Poisson are also included. Notably, a test based on Fisher's index of dispersion is included due to its historical prominence. Other tests based on sample moments are also discussed as well as the chi-squared test.

The main contribution made in this dissertation is the introduction of a new class of tests for the Poisson distribution. These tests are related to a generalisation of the Poisson distribution known as the weighted Poisson. The probability mass function of a weighted Poisson random variable is obtained by multiplying that of a Poisson random variable with a weight function. In its most general form, the probability mass function of the weighted Poisson distribution can take on any form over the non-negative integers. As a result, we may choose the weight function so that the fitted probability mass function coincides exactly with the empirical mass function. We refer to the weight function for which this is the case as the empirical weight function. If a given dataset is realised from some Poisson distribution, then the empirical weight function is expected to be close to $\mathbb{1} = \{1, 1, \dots\}$. As a result, we may base a test for the Poisson distribution on some distance measure between the empirical weight function and the point $\mathbb{1}$.

We demonstrate that the use of standard L_p distances are problematic, and we rather opt to use weighted versions of these distances. We base tests on weighted versions of the L_1, L_2 and L_∞ distances between the empirical weight function and $\mathbb{1}$. In each of these cases, we use three different weight functions in the calculation of the weighted distances. These are the fitted Poisson mass function, the empirical mass function of the sample as well as a Laplace type kernel. The latter of these contains a tuning parameter, but the tests show remarkable insensitivity to the choice of the tuning parameter, meaning that we restrict our attention to a single choice of this parameter. The three weighted distance measure, each calculated with reference to three different weight functions result in a total of nine new goodness-of-fit tests for the Poisson distribution. It should be stressed that both the use of the specified L_p distances as well as the weight functions used are chosen based on convenience. As a result, a direction for future research is to consider other choices of these functions for the construction of new classes of tests.

A Monte Carlo study is performed in order to compare the empirical powers of the newly proposed tests to that of several of the existing tests discussed. The Monte Carlo study comprises various sample sizes and employs a warp-speed bootstrap methodology in order to calculate empirical powers. We find that the newly proposed tests achieve the specified nominal significance level. Furthermore, these tests are highly competitive, generally outperforming the existing tests against overdispersed alternatives. However, the newly proposed tests are not particularly powerful against underdispersed alternatives.

The dissertation concludes with two practical examples; one relating to the observed numbers of sparrows' nests found on one hectare plots and the other relating to deaths by horse kick in the Prussian army. In both cases, we test whether or not the observed data are compatible with the assumption of being realised from the Poisson distribution. These examples are used to demonstrate the practical implementation of the newly proposed tests as we as the existing tests for the Poisson distribution.

Chapter 6

Additional numerical results

This appendix contains the empirical powers obtained for the sample sizes not discussed in the main text. Table 6.1 contains the results associated with samples size of 30, while Tables 6.2 and 6.3 contain the results associated with samples of size of 100 and 200, respectively. As before, the tables contain the Fisher index of the alternative distributions used and, in order to ease comparison between the tests, the highest power against each alternative distribution is printed in bold.

Table 6.1: Empirical powers obtained for samples of size n = 30

Distribution	FI	KS_n	CV_n	AD_n	KL_n	ID_n	$T^{(1)}_{n,f_{\widehat{\lambda}}}$	$T^{(1)}_{n,f_n}$	$T^{(1)}_{n,L}$	$T^{(2)}_{n,f_{\widehat{\lambda}}}$	$T^{(2)}_{n,f_n}$	$T^{(2)}_{n,L}$	$T^{(\infty)}_{n,f_{\widehat{\lambda}}}$	$T^{(\infty)}_{n,f_n}$	$T^{(\infty)}_{n,L}$
$P(0.5)$	1.00	5	5	5	4	5	5	5	4	5	5	5	5	5	4
$P(1)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	4
$P(5)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$P(10)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$DU(4)$	1.00	19	32	37	34	10	**38**	18	20	14	0	14	24	17	18
$B(5, 0.25)$	0.75	4	12	12	11	**14**	8	1	10	1	0	2	10	3	11
$B(5, 0.2)$	0.80	3	**10**	9	8	**10**	7	1	8	1	0	1	9	3	9
$B(10, 0.2)$	0.80	4	9	8	8	**10**	6	1	5	1	1	4	8	2	5
$B(10, 0.1)$	0.90	3	**6**	**6**	5	**6**	**6**	2	5	2	2	2	**6**	3	5
$NB(9, 0.9)$	1.11	9	6	7	7	7	6	**10**	6	**10**	**10**	**10**	5	9	6
$NB(45, 0.9)$	1.11	8	6	8	8	8	6	**10**	7	9	9	7	4	9	7
$PM(0.5, 3, 5)$	1.25	13	9	14	15	15	8	**16**	10	15	14	10	4	14	10
$ZIP(0.9, 3)$	1.30	26	14	24	21	20	19	21	43	18	10	45	9	15	**48**
$PM(0.1, 1, 5)$	1.31	14	13	22	22	21	12	23	**37**	22	15	**37**	4	18	36
$NB(15, 0.75)$	1.33	16	12	19	20	21	10	**23**	13	**23**	21	13	4	20	13
$NB(3, 0.75)$	1.33	22	13	18	17	19	12	**25**	16	24	23	23	7	21	14
$DU(6)$	1.33	44	53	63	62	43	52	42	63	37	3	64	23	30	**65**
$NB(4, 0.7)$	1.43	26	17	25	25	27	14	**31**	17	30	28	27	8	27	17
$NB(2, 2/3)$	1.50	32	22	29	28	30	20	**37**	27	36	34	34	12	32	23
$NB(3, 2/3)$	1.50	31	20	29	29	32	17	**36**	21	35	32	32	10	31	20
$ZIP(0.8, 3)$	1.60	70	45	63	56	55	55	54	78	48	21	**79**	36	40	**79**
$PM(0.2, 1, 5)$	1.61	42	35	53	53	53	30	47	**61**	45	28	60	7	36	60
$NB(1, 0.5)$	2.00	64	53	60	57	59	45	**65**	58	63	59	62	31	57	48

Table 6.2: Empirical powers obtained for samples of size n = 100

Distribution	FI	KS_n	CV_n	AD_n	KL_n	ID_n	$T^{(1)}_{n,f_{\widehat{\lambda}}}$	$T^{(1)}_{n,f_n}$	$T^{(1)}_{n,L}$	$T^{(2)}_{n,f_{\widehat{\lambda}}}$	$T^{(2)}_{n,f_n}$	$T^{(2)}_{n,L}$	$T^{(\infty)}_{n,f_{\widehat{\lambda}}}$	$T^{(\infty)}_{n,f_n}$	$T^{(\infty)}_{n,L}$
$P(0.5)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$P(1)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$P(5)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$P(10)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$DU(4)$	1.00	68	89	**96**	94	45	93	84	65	81	0	58	79	79	57
$B(5, 0.25)$	0.75	10	32	37	37	**44**	24	6	29	2	0	2	20	10	33
$B(5, 0.2)$	0.80	7	22	24	23	**27**	18	5	20	1	0	0	15	9	24
$B(10, 0.2)$	0.80	7	19	22	23	**29**	10	1	12	1	0	9	12	5	15
$B(10, 0.1)$	0.90	3	8	8	8	**9**	7	2	7	1	1	1	7	4	8
$NB(9, 0.9)$	1.11	14	9	10	11	11	8	14	10	**15**	14	13	7	12	9
$NB(45, 0.9)$	1.11	10	8	11	11	12	7	**13**	9	12	11	9	5	11	9
$PM(0.5, 3, 5)$	1.25	26	21	31	32	**35**	13	26	19	24	17	19	7	18	19
$ZIP(0.9, 3)$	1.30	68	39	62	55	54	52	55	**86**	44	12	**86**	34	45	**86**
$PM(0.1, 1, 5)$	1.31	39	34	53	54	53	28	51	**77**	46	18	76	10	38	75
$NB(15, 0.75)$	1.33	34	29	44	46	**50**	17	41	21	39	30	21	7	30	20
$NB(3, 0.75)$	1.33	**48**	35	42	43	46	30	**48**	41	44	38	38	20	35	38
$DU(6)$	1.33	93	98	**100**	99	94	99	98	97	97	16	97	83	91	97
$NB(4, 0.7)$	1.43	58	46	58	60	**63**	35	61	43	56	45	57	22	46	45
$NB(2, 2/3)$	1.50	**72**	61	68	68	70	54	71	67	66	56	58	39	55	63
$NB(3, 2/3)$	1.50	69	58	69	69	**73**	47	70	57	66	54	67	32	54	57
$ZIP(0.8, 3)$	1.60	**100**	94	99	98	98	98	98	**100**	96	35	**100**	95	97	**100**
$PM(0.2, 1, 5)$	1.61	91	85	96	96	96	78	92	**98**	88	46	97	38	79	97
$NB(1, 0.5)$	2.00	**97**	96	**97**	96	**97**	92	**97**	**97**	95	89	92	84	90	94

Table 6.3: Empirical powers obtained for samples of size n = 200

Distribution	FI	KS_n	CV_n	AD_n	KL_n	ID_n	$T^{(1)}_{n,f_{\widehat{\lambda}}}$	$T^{(1)}_{n,f_n}$	$T^{(1)}_{n,L}$	$T^{(2)}_{n,f_{\widehat{\lambda}}}$	$T^{(2)}_{n,f_n}$	$T^{(2)}_{n,L}$	$T^{(\infty)}_{n,f_{\widehat{\lambda}}}$	$T^{(\infty)}_{n,f_n}$	$T^{(\infty)}_{n,L}$
$P(0.5)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$P(1)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	4
$P(5)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$P(10)$	1.00	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$DU(4)$	1.00	99	**100**	**100**	**100**	93	**100**	**100**	93	**100**	1	91	99	99	88
$B(5, 0.25)$	0.75	22	60	67	68	**75**	47	20	53	6	0	2	35	24	59
$B(5, 0.2)$	0.80	12	41	45	45	**52**	34	16	38	4	0	0	26	21	44
$B(10, 0.2)$	0.80	13	36	43	45	**52**	20	4	24	1	0	19	19	11	28
$B(10, 0.1)$	0.90	4	12	13	12	**15**	10	4	11	1	1	1	10	7	13
$NB(9, 0.9)$	1.11	**20**	12	15	15	17	11	18	15	17	15	15	9	11	15
$NB(45, 0.9)$	1.11	14	11	15	16	**17**	8	16	10	15	12	10	5	12	10
$PM(0.5, 3, 5)$	1.25	44	41	54	56	**60**	23	40	23	34	22	23	12	21	21
$ZIP(0.9, 3)$	1.30	94	71	90	85	84	84	88	**99**	79	15	**99**	73	85	**99**
$PM(0.1, 1, 5)$	1.31	70	60	82	82	83	52	78	**95**	72	23	**95**	19	62	94
$NB(15, 0.75)$	1.33	57	54	71	74	**77**	31	62	30	57	38	29	13	39	28
$NB(3, 0.75)$	1.33	**73**	61	68	69	**73**	54	71	67	65	50	52	38	48	64
$DU(6)$	1.33	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	49	**100**	**100**	**100**	**100**
$NB(4, 0.7)$	1.43	84	77	86	87	**90**	63	83	72	79	60	80	47	63	75
$NB(2, 2/3)$	1.50	93	89	92	92	**94**	84	93	92	89	74	76	69	77	89
$NB(3, 2/3)$	1.50	92	88	93	93	**95**	79	92	85	88	71	88	64	78	86
$ZIP(0.8, 3)$	1.60	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	68	**100**	**100**	**100**	**100**
$PM(0.2, 1, 5)$	1.61	**100**	99	**100**	**100**	**100**	98	**100**	**100**	**100**	75	**100**	78	98	**100**
$NB(1, 0.5)$	2.00	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	99	99	99	99	**100**

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