Doob's optional sampling theorem in Riesz spaces

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Dedicated to the memory of Charalambros D Aliprantis

Abstract

The notions of stopping times and stopped processes for continuous stochastic processes are defined and studied in the framework of Riesz spaces. This leads to a formulation and proof of Doob's optional sampling theorem.

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1 Introduction.

An order continuous positive projection $\mathbb{E}$ mapping a Riesz space $\mathfrak{E}$ onto a Riesz subspace $\mathfrak{F}$ that has the property that the band generated by $\mathfrak{F}$ is again $\mathfrak{E}$, is the fundamental notion on which a theory of probability can successfully be developed within the framework of vector lattices (Riesz spaces). This is due to the fact that in classical probability theory the stochastic variables are elements of vector lattices of functions (often the L^p spaces $(1 \leq p \leq \infty)$), and secondly due to the fact that the conditional expectation in probability theory is an operator of the kind mentioned above. So in this abstract theory the notion of conditional expectation rather than that of probability measure, is fundamental.

This paper is a continuation of the paper [5] in which we defined continuous time stochastic processes in Riesz spaces and proved the fundamental Doob-Meyer decomposition theorem for submartingales. We study the notion of a stopping time, that was defined in [5] as an orthomorphism. However, there we needed only to consider stopping times that were step elements (linear combinations of projections), in which case it is easy to define stopped filtrations and processes. It is not outright clear how to do this for general stopping times. We show here that stopped processes that have the expected properties can be defined and then show that Doob's theorem mentioned in the title can be proved. As we remarked in [5], our study exhibits the fundamental role played by positivity in the theory of stochastic processes.

For more background on what has been done in this field we refer the reader to R. DeMarr ([3]) who studied martingales in Banach lattices as early as 1966, followed by Gh. Stoica ([18],[19] [20], [21]), and V.G. Troitsky ([22]). The general theory was considered by W.-C. Kuo, C.C.A. Labuschagne and B.A. Watson (see [8], [9], [10], [11], [12], [13], [14]), who studied countable processes in this setting.

In the next section we set our notation. We choose the notation to make it user friendly to probabilists, hoping that it will not offend specialists in Riesz space theory.

2 Notation and definitions.

As in [5], we recall that a Riesz space $\mathfrak{E}$ is a real vector space endowed with an order relation compatible with the algebraic structure. The standard references for the theory of vector lattices are the textbooks of W.A.J. Luxemburg and A.C. Zaanen ([15]), H.H. Schaefer ([17]), D.H. Fremlin, ([4]), C.D. Aliprantis and O. Burkinshaw ([1, 2]), P. Meyer-Nieberg ([16]) and A.C. Zaanen ([25, 26]). We refer the reader to them for any notion not defined in this paper.

We shall exclusively consider Dedekind complete vector lattices. If $\mathfrak{E}$ is such a lattice and if $\mathfrak{A} \subset \mathfrak{E}$ we denote its disjoint complement in $\mathfrak{E}$ by $\mathfrak{A}^d$. We call the set $\mathfrak{A}$ *weakly order dense in* $\mathfrak{E}$ if $\mathfrak{A}^{dd} = \mathfrak{E}$ (this notion should not be confused with that of quasi-order denseness or order denseness, since the latter notions are only defined for ideals). Of course, $E \in \mathfrak{E}$ is a weak order unit for $\mathfrak{E}$ if the singleton $\{E\}$ is weakly order dense in $\mathfrak{E}$. Note that in our case $\mathfrak{A}^{dd}$ is also the band generated by the set $\mathfrak{A} \subset \mathfrak{E}$ in $\mathfrak{E}$.

A linear operator $\mathbb{T} : \mathfrak{E} \to \mathfrak{F}$ is called *positive* if $\mathbb{T}X \geq 0$ for all $X \geq 0$, i.e., if $\mathbb{T}$ maps the positive cone $\mathfrak{E}^+$ of $\mathfrak{E}$ into the positive cone $\mathfrak{F}^+$ of $\mathfrak{F}$. It is called *strictly positive* if $\mathbb{T}X > 0$ for every $X > 0$. The positive operator $\mathbb{T}$ is called *order continuous* whenever the image of every downwards directed net (X_α) with infimum 0 (write $X_\alpha \downarrow 0$) is again a downwards directed net with infimum 0, i.e., $\mathbb{T}X_\alpha \downarrow 0$.

We denote the range of the operator $\mathbb{T}$ by $\mathcal{R}(\mathbb{T})$ and the set of all order bounded linear maps from $\mathfrak{E}$ into $\mathfrak{F}$ by $\mathcal{L}^b(\mathfrak{E}, \mathfrak{F})$. This space is partially ordered by $\mathbb{T} \leq \mathbb{S}$ if $\mathbb{T}X \leq \mathbb{S}X$ for all $X \in \mathfrak{E}^+$ and it is again a Dedekind complete vector lattice. The order bounded operator $\mathbb{T}$ is called order continuous if it is the difference of two order continuous positive operators.

An operator $\mathbb{S} \in \mathcal{L}(\mathfrak{E})$ is called *band preserving* if $\mathbb{S}$ leaves all bands in $\mathfrak{E}$ invariant, i.e., if $\mathbb{S}\mathfrak{B} \subset \mathfrak{B}$ for all bands $\mathfrak{B}$ in $\mathfrak{E}$. A band preserving order bounded operator is called and *orthomorphism*. The set of orthomorphisms is denoted by $\mathrm{Orth}(\mathfrak{E})$. It is the band generated by the identity operator $\mathbb{I}$ in $\mathcal{L}^b(\mathfrak{E})$.

If E is a weak order unit of $\mathfrak{E}$, if $X \in \mathfrak{E}$ and if $t \in \mathbb{R}$, we define E^ℓ_t to be the component of E in the band generated by $(tE - X)^+ = (X - tE)^-$. The set (E^ℓ_t) is an increasing left continuous system of components of E. We call the system $(E^\ell_t)_{t\in\mathbb{R}}$ the *left continuous spectral system* of X. Also, if $\overline{E}^r_t$ is the component of E in the band generated by $(X - tE)^+ = (tE - X)^-$ and $E^r_t := E - \overline{E}^r_t$, the set (E_t^r) is an increasing right continuous system of components of E (see [15, page 262]). We call the system $(E_t^r)_{t\in\mathbb{R}}$ the *right continuous spectral system* of X and we have

$$E_t^\ell \le E_t^r \le E_s^\ell \text{ for all } t < s.$$

We shall use the fact that the identity operator $\mathbb{I}$ is a weak order unit for $\mathrm{Orth}(\mathfrak{E})$ and that the spectral system of an element $\mathbb{S} \in \mathrm{Orth}(\mathfrak{E})$ consists therefore of components of $\mathbb{I}$ which means that it consists of order projections in $\mathfrak{E}$.

3 Conditional expectations, filtrations and stochastic processes.

As in [5], we use the following definition of conditional expectation that generalizes the one given by Kuo, Labuschagne and Watson [9].

Definition 3.1 The strictly positive order continuous projection $\mathbb{F} : \mathfrak{E} \to \mathfrak{E}$ is called a *conditional expectation* if

(a) $\mathfrak{F} := \mathcal{R}(\mathbb{F})$ is a Dedekind complete Riesz subspace of $\mathfrak{E}$;

(b) $\mathfrak{F}$ is weakly dense in $\mathfrak{E}$.

An important property that we proved in [5, Theorem 4.4] is that if $\mathbb{P}$ is a projection in $\mathfrak{E}$ that maps $\mathfrak{F}$ into itself, then $\mathbb{P}$ commutes with $\mathbb{F}$. It follows easily from this that if $\mathbb{S} \in \mathrm{Orth}(\mathfrak{F})$ then $\mathbb{S}$ commutes with $\mathbb{F}$. This can be interpreted as $\mathbb{F}$ being an "averaging" operator. We shall denote the set of all order projections of $\mathfrak{E}$ that maps $\mathfrak{F}$ into itself by $\mathfrak{P}_{\mathfrak{F}}$ and this set can be identified with the set of all order projections of the vector lattice $\mathfrak{F}$ (we say these projections act on $\mathfrak{F}$, see [5]).

The next notion, due to B.A. Watson [24], is now defined in this more general case.

Definition 3.2 Let $\mathfrak{E}$ be a Dedekind complete Riesz space and let $\mathbb{F}$ be a conditional expectation on $\mathfrak{E}$. We say that $\mathfrak{E}$ is $\mathbb{F}$-universally complete if, whenever $X_\alpha \uparrow$ in $\mathfrak{E}$ and $\mathbb{F}(X_\alpha)$ is bounded in $\mathfrak{E}$, we have that $X_\alpha \uparrow X$ for some $X \in \mathfrak{E}$.

In the terminology of [6] this means that the conditional expectation operator is defined on its natural domain (see also [11]). We need the following existence theorem and therefore have to prove Watson's Radon-Nikodym theorem for the more general case:

Theorem 3.3 *Let $\mathfrak{E}$ be an $\mathbb{F}$-universally complete Riesz space with conditional expectation operator $\mathbb{F}$. Let $\mathfrak{F}$ be a Dedekind complete Riesz subspace of $\mathfrak{E}$ with $\mathcal{R}(\mathbb{F}) \subset \mathfrak{F}$. For each $X \in \mathfrak{E}^+$ there exists a unique $Y \in \mathfrak{F}^+$ such that*

$$\mathbb{F}\mathbb{P}X = \mathbb{F}\mathbb{P}Y, \quad \textit{for all} \;\; \mathbb{P} \in \mathfrak{P}_{\mathfrak{F}}.$$

Proof. The uniqueness part follows exactly as in [24]. We now prove the existence. Let $\{E_\alpha\}$ be a maximal disjoint system in $\mathcal{R}(\mathbb{F})$ (see [15, Theorem 28.5]) and note that since the band generated by the latter space is equal to $\mathfrak{E}$, this is also a maximal disjoint system in $\mathfrak{E}$ (and so also in $\mathfrak{F}$). Denote the band generated by E_α in $\mathfrak{E}$ by $\mathfrak{B}_\alpha$. By Lemma 4.3 in [5], $\mathbb{F}$ maps $\mathbb{B}_\alpha$ into itself and being a projection, it maps the weak order unit E_α of $\mathfrak{B}_\alpha$ onto itself. If we therefore denote the restriction of $\mathbb{F}$ to $\mathfrak{B}_\alpha$ by $\mathbb{F}_\alpha$, we easily see that $\mathbb{F}_\alpha$ is a conditional expectation of $\mathfrak{B}_\alpha$ and it is also easy to see that $\mathfrak{B}_\alpha$ is $\mathbb{F}_\alpha$ universally complete. Put $\mathfrak{F}_\alpha := \mathfrak{F} \cap \mathfrak{B}_\alpha$, then $\mathfrak{F}_\alpha$ is a closed Riesz subspace of $\mathfrak{B}_\alpha$ and $\mathcal{R}(\mathbb{F}_\alpha) \subset \mathfrak{F}_\alpha$. We now apply Watson's result: Let X_α be the component of X in $\mathfrak{B}_\alpha$. Then there exists an element $Y_\alpha \in \mathfrak{F}_\alpha^+$ such that

$$\mathbb{F}_\alpha \mathbb{P} X_\alpha = \mathbb{F}_\alpha \mathbb{P} Y_\alpha \quad \text{for all} \quad \mathbb{P} \in \mathfrak{P}_{\mathfrak{F}_\alpha}.$$

Let $\beta = (\alpha_1, \ldots, \alpha_n)$ be an arbitrary multi-index and let $X_\beta = X_{\alpha_1} \vee X_{\alpha_2} \vee \cdots \vee X_{\alpha_n}$ and similarly for Y_β. Then $X_\beta \uparrow X$ and so $\mathbb{F}X_\beta \uparrow \mathbb{F}X$. Also, by the disjointness of the elements Y_α, we get

$$\mathbb{F}Y_\beta = \mathbb{F}\sum_{k=1}^n Y_{\alpha_k} = \sum_{k=1}^n \mathbb{F}_{\alpha_k} Y_{\alpha_k} = \sum_{k=1}^n \mathbb{F}_{\alpha_k} X_{\alpha_k} = \mathbb{F}X_\beta \uparrow \mathbb{F}X.$$

So, by our assumption that $\mathfrak{E}$ is $\mathbb{F}$-universally complete, we have $Y_\beta \uparrow Y \in \mathfrak{E}$. But each $Y_\beta \in \mathfrak{F}$ (which is order complete) and therefore $Y \in \mathfrak{F}$.

We complete the proof by showing that Y has the required property. Let $\mathbb{Q} \in \mathfrak{P}_{\mathfrak{F}}$. Put $\mathbb{Q}_\beta := \mathbb{Q}\mathbb{P}_\beta$, with $\mathbb{P}_\beta$ the projection onto the band $\mathfrak{B}_{\alpha_1} \oplus \cdots \oplus \mathfrak{B}_{\alpha_n}$. Then $\mathbb{Q}_\beta \uparrow \mathbb{Q}$ and

$$\mathbb{F}\mathbb{Q}X = \sup_\beta \mathbb{F}\mathbb{Q}_\beta X = \sup_\beta \sum_{k=1}^n \mathbb{F}_{\alpha_k} \mathbb{Q}\mathbb{P}_{\alpha_k} X =$$

$$\sup_\beta \sum_{k=1}^n \mathbb{F}_{\alpha_k} \mathbb{Q}\mathbb{P}_{\alpha_k} Y = \sup_\beta \mathbb{F}\mathbb{Q}_\beta Y = \mathbb{F}\mathbb{Q}Y.$$

This completes the proof. □

Proposition 3.4 *Let $\mathbb{F}$ be a conditional expectation on the $\mathbb{F}$-universally complete Riesz space $\mathfrak{E}$. Let $\mathfrak{F}$ be a closed Riesz subspace of $\mathfrak{E}$ with $\mathcal{R}(\mathbb{F}) \subset \mathfrak{F}$. Then there exists a unique conditional expectation $\mathbb{F}_{\mathfrak{F}}$ on $\mathfrak{E}$ with $\mathcal{R}(\mathbb{F}_{\mathfrak{F}}) = \mathfrak{F}$ and $\mathbb{F}\mathbb{F}_{\mathfrak{F}} = \mathbb{F}_{\mathfrak{F}}\mathbb{F} = \mathbb{F}$.*

Proof. By Theorem 3.3, the assumptions imply that for every $X \in \mathfrak{E}^+$ there exists a unique $Y \in \mathfrak{F}^+$ such that

$$\mathbb{F}\mathbb{P}X = \mathbb{F}\mathbb{P}Y \text{ for all } \mathbb{P} \in \mathfrak{P}_{\mathbb{F}}. \tag{3.1}$$

Call Y the $\mathbb{F}$-conditional expectation of X given $\mathfrak{F}$. Define $\mathbb{F}_{\mathfrak{F}}X := Y$ for X and Y as above. By the uniqueness part in the Radon-Nikodym theorem, $\mathbb{F}_{\mathfrak{F}}$

is well defined and moreover, $\mathbb{F}_{\mathfrak{F}}$ is positive. We show that $\mathbb{F}_{\mathfrak{F}}$ is additive on $\mathfrak{E}^+$. If $X_1, X_2 \in \mathfrak{E}^+$ with $\mathbb{F}$-conditional expectations $Y_1, Y_2 \in \mathfrak{F}$, we have that $Y_1 + Y_2 \in \mathfrak{F}$ satisfies

$$\mathbb{F}\mathbb{P}(X_1 + X_2) = \mathbb{F}\mathbb{P}X_1 + \mathbb{F}\mathbb{P}X_2 = \mathbb{F}\mathbb{P}Y_1 + \mathbb{F}\mathbb{P}Y_2 = \mathbb{F}\mathbb{P}(Y_1 + Y_2) \text{ for all } \mathbb{P} \in \mathfrak{P}_{\mathfrak{F}},$$

and so $Y_1 + Y_2$ is the $\mathbb{F}$-conditional expectation of $X_1 + X_2$, i.e., $\mathbb{F}_{\mathfrak{F}}(X_1 + X_2) = \mathbb{F}_{\mathfrak{F}}X_1 + \mathbb{F}_{\mathfrak{F}}X_2$. We can therefore extend $\mathbb{F}_{\mathfrak{F}}$ to a positive linear operator on $\mathfrak{E}$ to $\mathfrak{F}$ by defining $\mathbb{F}_{\mathfrak{F}}X := \mathbb{F}_{\mathfrak{F}}X^+ - \mathbb{F}_{\mathfrak{F}}X^-$.

Clearly, $\mathbb{F}_{\mathfrak{F}}Y = Y$ for all $Y \in \mathfrak{F}$ which implies that $\mathbb{F}_{\mathfrak{F}}$ is a projection onto $\mathfrak{F}$. It follows then from $\mathcal{R}(\mathbb{F}) \subset \mathfrak{F}$ that $\mathbb{F}_{\mathfrak{F}}\mathbb{F} = \mathbb{F}$. Also, by taking $\mathbb{P} = \mathbb{I}$ in 3.3 above, it follows from the definition of $\mathbb{F}_{\mathfrak{F}}$ that for all $X \in \mathfrak{E}$ we have $\mathbb{F}X = \mathbb{F}\mathbb{F}_{\mathfrak{F}}X$. Hence we have $\mathbb{F}_{\mathfrak{F}}\mathbb{F} = \mathbb{F}\mathbb{F}_{\mathfrak{F}} = \mathbb{F}$. From this it follows that $\mathbb{F}_{\mathfrak{F}}$ is also strictly positive, for, if $X > 0$ and $\mathbb{F}_{\mathfrak{F}}X = 0$, then $\mathbb{F}X = \mathbb{F}\mathbb{F}_{\mathfrak{F}}X = 0$ contradicting the strict positivity of $\mathbb{F}$.

Let now $X_\alpha \downarrow 0$ in $\mathfrak{E}$, then $\mathbb{F}\mathbb{F}_{\mathfrak{F}}X_\alpha = \mathbb{F}X_\alpha \downarrow 0$. By the positivity of $\mathbb{F}_{\mathfrak{F}}$ we have that $\mathbb{F}_{\mathfrak{F}}X_\alpha \downarrow$ and since $\mathfrak{F}$ is Dedekind complete, $\mathbb{F}_{\mathfrak{F}}X_\alpha \downarrow Y \geq 0$. Hence, $\mathbb{F}Y = 0$ and so $Y = 0$ by the strict positivity of $\mathbb{F}$. It follows that $\mathbb{F}_{\mathfrak{F}}$ is order continuous.

Finally, since the band generated by $\mathcal{R}(\mathbb{F})$ in $\mathfrak{E}$ equals $\mathfrak{E}$ and since $\mathcal{R}(\mathbb{F})$ is contained in $\mathfrak{F} = \mathcal{R}(\mathbb{F}_{\mathfrak{F}})$, the band generated in $\mathfrak{E}$ by the range of $\mathbb{F}_{\mathfrak{F}}$ is also equal to $\mathfrak{E}$. This completes the proof. $\square$

Suppose that the order continuous dual $\mathfrak{E}_n^\sim$ of $\mathfrak{E}$ separates the points of $\mathfrak{E}$ and that $\mathbb{F}$ is a conditional expectation on $\mathfrak{E}$. Then, since $\mathbb{F}$ is order continuous, we have that $\mathbb{F}^\sim$ maps $\mathfrak{E}_n^\sim$ into itself. We proved in [5, Proposition 4.7] that the order adjoint $\mathbb{F}^\sim$ is a conditional expectation on $\mathfrak{E}_n^\sim$, that $\mathcal{R}(\mathbb{F}^\sim) = \mathbb{F}^\sim(\mathfrak{E}_n^\sim)$ separates the points of $\mathfrak{F} = \mathcal{R}(\mathbb{F})$ and that $\mathcal{R}(\mathbb{F}^\sim) = \mathbb{F}^\sim(\mathfrak{E}_n^\sim) = \mathfrak{F}_n^\sim$.

As in [5] we define the following notions that were defined for countable processes in [9]. We refer the reader to [7] for the classical theory.

Definition 3.5 Let $T = [0, \infty)$, let $(\mathbb{F}_t)_{t\in T}$ be a family of conditional expectations on $\mathfrak{E}$ and let $\mathfrak{F}_t := \mathcal{R}(\mathbb{F}_t)$. The family $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ is called a *filtration* on $\mathfrak{E}$ if $\mathbb{F}_s\mathbb{F}_t = \mathbb{F}_t\mathbb{F}_s = \mathbb{F}_s$ for all $s \leq t$.

Remarks: 1. We denote the filtration $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ often simply by either $(\mathbb{F}_t)_{t\in T}$ or $(\mathfrak{F}_t)_{t\in T}$.

2. It follows from $\mathbb{F}_s = \mathbb{F}_t\mathbb{F}_s$ that $\mathfrak{F}_s \subset \mathfrak{F}_t$ for all $s \leq t$. In fact, an alternative way to define a filtration would be to assume that there exists a conditional expectation $\mathbb{F}_0$ on $\mathfrak{E}$ and that $\mathfrak{E}$ is $\mathbb{F}_0$-universally complete. We then define a filtration as a family $(\mathfrak{F}_t)$ of weakly dense order closed Riesz subspaces of $\mathfrak{E}$ with the property that $\mathfrak{F}_s \subset \mathfrak{F}_t$ for all $s \leq t$, with $\mathfrak{F}_0 = \mathcal{R}(\mathbb{F}_0)$. From Proposition 3.4 we then get a family $(\mathbb{F}_t)$ of conditional expectations on $\mathfrak{E}$ with the property used in our definition.

3. We denote the set of all order projections in $\mathfrak{E}$ that act on $\mathfrak{F}_t$ by $\mathfrak{P}_t$ and we recall that $\mathbb{F}_t\mathbb{P} = \mathbb{P}\mathbb{F}_t$ for all $\mathbb{P} \in \mathfrak{P}_t$. For obvious reasons the projections in $\mathfrak{P}_t$ are called the *events* in the process up to time t.

4. $\mathfrak{P}_t$ is a complete Boolean algebra and can be considered to play the rôle of the σ-algebra in the classical case.

5. The family $(\mathbb{F}_t^\sim)_{t\in T}$ is a filtration on $\mathfrak{E}_n^\sim$ and is called the *dual filtration* of the filtration $\mathfrak{F}_t$.

Definition 3.6 Let $(\mathfrak{F}_t)$ be a filtration on $\mathfrak{E}$. We define

1. $\mathfrak{F}_{t+} := \bigcap_{s>t} \mathfrak{F}_s$.
2. $\mathfrak{F}_{t-}$ is defined to be the order closed Riesz subspace of $\mathfrak{E}$ generated by $\{\mathfrak{F}_s : s < t\}$ and $\mathfrak{F}_{0-} := \mathfrak{F}_0$.
3. A filtration is *right-continuous* (resp. *left-continuous*) if for all $t \in T$, $\mathfrak{F}_{t+} = \mathfrak{F}_t$ (resp. $\mathfrak{F}_{t-} = \mathfrak{F}_t$).

The set of projections in $\mathfrak{E}$ that act on $\mathfrak{F}_{t+}$ will be denoted by $\mathfrak{P}_{t+}$ and those acting on $\mathfrak{F}_{t-}$ by $\mathfrak{P}_{t-}$.

Proposition 3.7 *If $(\mathfrak{F}_t)$ is a filtration on $\mathfrak{E}$, we have*

1. $\mathfrak{F}_{t+} = \bigcap_{n=1}^{\infty} \mathfrak{F}_{s_n}$ *for any sequence $s_n \downarrow t$, $s_n > t$.*
2. $\mathfrak{F}_{t-}$ *is the order closed Riesz subspace generated by* $\bigcup_{s<t} \mathfrak{F}_s = \bigcup_{n=1}^{\infty} \mathfrak{F}_{s_n}$ *for any sequence $s_n \uparrow t$, $s_n < t$.*

Proof. 1. Inclusion clearly holds. If X belongs to the countable intersection, and $s > t$ choose $s > s_n > t$. Then $X \in \mathfrak{F}_s$ and so equality holds.

2. Since $\mathfrak{F}_s \uparrow$ the union is the Riesz subspace of $\mathfrak{E}$ generated by $\{\mathfrak{F}_s : s < t\}$ (which need not be order closed). If we denote $\mathrm{cl}(S)$ to be the closure of a set $S \subset \mathfrak{E}$ in the order topology, we have $\mathfrak{F}_{t-} = \mathrm{cl}\left(\bigcup_{s<t} \mathfrak{F}_s\right) = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \mathfrak{F}_{s_n}\right)$. □

Proposition 3.8 *Let $\mathfrak{E}$ be a Dedekind complete Riesz space and let $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ be a filtration on $\mathfrak{E}$. Suppose that $\mathfrak{E}$ is $\mathbb{F}_0$-universally complete. Then there exists a conditional expectation $\mathbb{F}_{t+}$ from $\mathfrak{E}$ onto $\mathfrak{F}_{t+}$. Similarly, there exists a conditional expectation $\mathbb{F}_{t-}$ onto $\mathfrak{F}_{t-}$.*

Proof. We note that every $\mathfrak{F}_t$ is not merely Dedekind complete, but even order closed in $\mathfrak{E}$ for if $X_\alpha \to X$ in order in $\mathfrak{F}_s$ then $\mathbb{F}_s(X_\alpha) \to \mathbb{F}_s X \in \mathfrak{F}_s$, but $\mathbb{F}_s(X_\alpha) = X_\alpha \to X$ so $X = \mathbb{F}_s X \in \mathfrak{F}_s$. It follows that $\mathfrak{F}_{t+}$ is an order closed

Riesz subspace of $\mathfrak{E}$ that contains $\mathfrak{F}_0$. By Proposition 3.4 there exists a strictly positive conditional expectation (denote it by $\mathbb{F}_{t+}$) from $\mathfrak{E}$ onto $\mathfrak{F}_{t+}$. For $\mathfrak{F}_{t-}$ it follows by definition that it is an order closed Riesz subspace of $\mathfrak{E}$ and so, for the same reason as above, there exists a strictly positive conditional expectation $\mathbb{F}_{t-}$ from $\mathfrak{E}$ onto $\mathfrak{F}_{t-}$. □

It is also true that for all $s < t$, $\mathbb{F}_s\mathbb{F}_{t-} = \mathbb{F}_{t-}\mathbb{F}_s = \mathbb{F}_s$ and for all $s > t$, $\mathbb{F}_{t+}\mathbb{F}_s = \mathbb{F}_s\mathbb{F}_{t+} = \mathbb{F}_{t+}$. We also note that it follows from the theorem above that if $(\mathbb{F}_t, \mathfrak{F}_t)$ is a filtration on $\mathfrak{E}$ the same is true for $(\mathbb{F}_{t+}, \mathfrak{F}_{t+})$ and this filtration is right continuous. Similarly $(\mathbb{F}_{t-}, \mathfrak{F}_{t-})$ is a left continuous filtration on $\mathfrak{E}$.

Let $\mathfrak{E}$ be a Dedekind complete Riesz space and let $T := [0, \infty)$. As defined in [5], a family $X = (X_t)_{t\in T}$ with $X_t \in \mathfrak{E}$ is called a (continuous time) *stochastic process* in $\mathfrak{E}$. The stochastic process $(X_t)_{t\in T}$ is *right* (resp. *left*) continuous, if

$$o - \lim_{s\downarrow t} X_s = X_t \qquad \text{(resp. } o - \lim_{s\uparrow t} X_s = X_t).$$

A stochastic process $(X_t)_{t\in T}$ is *adapted to the filtration* $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ if $X_t \in \mathfrak{F}_t$ for all $t \in T$. We write $(X_t, \mathfrak{F}_t)_{t\in T}$ to indicate that (X_t) is adapted to $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$. The stochastic process $(X_t, \mathfrak{F}_t)_{t\in T}$ is called a *submartingale* (respectively, *supermartingale*) whenever we have $\mathbb{F}_s(X_t) \geq X_s$ (respectively, $\mathbb{F}_s(X_t) \leq X_s$) for all $s < t < \infty$. If the process is both a sub- and a supermartingale it is called a *martingale* (see also [8, 9, 10, 11, 12, 13, 14] for the case where $T = \mathbb{N}$).

We recall from [5] that a right continuous stochastic process (A_t) is called *increasing* if $A_0 = 0$ and $A_s \leq A_t$ whenever $s \leq t$ and it is called *integrable* if $\sup_{t>0} A_t \in \mathfrak{E}$. If (A_t) is adapted to the filtration $(\mathbb{F}_t)$ and if we have that

$$I_A(\phi_{t^-}) := \lim_{\pi} \sum_{i=1}^{n} \langle \phi_{t_{i-1}}, A_{t_i} - A_{t_{i-1}} \rangle, \quad \pi = \{0 = t_0 < t_1 < \cdots < t_n = t\},$$

exists for every adapted bounded dual martingale $\phi = (\phi_t)$, we call the process (A_t) *tractable on* $[0, t]$ and we call it *tractable* if it is tractable on every interval $[0, t]$. An increasing adapted process is called *predictable on* $[0, t]$ if it is tractable on $[0, t]$ and if $I_A(\phi_{t^-}) = \langle \phi_t, A_t \rangle$. If this holds for all t is is called *predictable.*

4 Optional and stopping times.

In [5] we defined an *optional time* for the filtration $(\mathfrak{F}_t)_{t\in T}$ as a positive orthomorphism $\mathbb{S} \in \mathrm{Orth}(\mathfrak{E})$ such that its *left continuous* spectral system $(\mathbb{S}^\ell_t)$ of projections (with reference to the weak order unit $\mathbb{I}$ of $\mathrm{Orth}(\mathfrak{E})$) satisfies $\mathbb{S}^\ell_t \in \mathfrak{P}_t$ for all $t \in T$. We call it a *stopping time* for the filtration $(\mathfrak{F}_t)_{t\in T}$ whenever its *right continuous* spectral system $(\mathbb{S}^r_t)$ of projections satisfies $\mathbb{S}^r_t \in \mathfrak{P}_t$ for all $t \in T$. As was shown in [5, Proposition 5.5] an easy consequence of these definitions is that every stopping time is optional and the concepts coincide if the filtration is right continuous. We also have

Proposition 4.1 *Let $(\mathfrak{F}_t)_{t\in T}$ be a filtration on $\mathfrak{E}$. Then $\mathbb{S}$ is an optional time for $(\mathfrak{F}_t)_{t\in T}$ if and only if $\mathbb{S}$ is a stopping time for the filtration $(\mathfrak{F}_{t+})_{t\in T}$.*

Proof. Let $\epsilon > 0$ and $t \in T$, be given. Choose $N \in \mathbb{N}$ so that for all $n \geq N$ we have $1/n < \epsilon$. Then, if $\mathbb{S}$ is an optional time, we have $\mathbb{S}^{\ell}_{t+1/n} \in \mathfrak{P}_{t+\epsilon}$ for all $n \geq N$ and $\mathbb{S}^{\ell}_{t+1/n} \downarrow \mathbb{S}^r_t$ by the inequality $\mathbb{S}^{\ell}_t \leq \mathbb{S}^r_t \leq \mathbb{S}^{\ell}_s$ for $t < s$ and the right continuity of $(\mathbb{S}^r_t)$. Hence, $\mathbb{S}^r_t \in \mathfrak{P}_{t+\epsilon}$ for all $\epsilon > 0$ and so $\mathbb{S}^r_t \in \mathfrak{P}_{t+}$. It follows that $\mathbb{S}$ is a stopping time for $(\mathfrak{F}_{t+})_{t\in T}$.

Conversely, if $\mathbb{S}$ is a stopping time for $\mathfrak{F}_{t+}$ it is also an optional time for $\mathfrak{F}_{t+}$. Hence, if $0 < s < t$ we have $\mathbb{S}^{\ell}_s \in \mathfrak{P}_{s+} \subset \mathfrak{P}_t$. But, if $s \uparrow t$, we have by its left continuity that $\mathbb{S}^{\ell}_s \uparrow \mathbb{S}_t$. Since $\mathfrak{P}_t$ is a complete Boolean algebra, $\mathbb{S}^{\ell}_t \in \mathfrak{P}_t$ and so $\mathbb{S}$ is an optional time for $\mathfrak{F}_t$. □

It is easy to see that the constant orthomorphism $\mathbb{S} = a\mathbb{I}$, $a \in \mathbb{R}^+$ is both an optional and a stopping time. For step elements in $\mathrm{Orth}(\mathfrak{E})$ we proved in [5] that if $\mathbb{S}$ is a stopping time (respectively, optional time) of the form $\mathbb{S} := \sum_{k=1}^{n} t_k \mathbb{P}_k$, with $\mathbb{P}_i\mathbb{P}_j = 0$ if $i \neq j$ and $t_i \neq t_j$ if $i \neq j$, (i.e., if $\mathbb{S}$ is written in its canonical form) we have $\mathbb{P}_k \in \mathfrak{P}_{t_k}$ (respectively, $\mathbb{P}_k \in \mathfrak{P}_{t_k+}$) and conversely.

We now prove the following lemma.

Lemma 4.2 *Let $(\mathbb{S}_\alpha)$ be a set of orthomorphisms on $\mathfrak{E}$. Then,*

1. $(\sup \mathbb{S}_\alpha)^r_t = \inf\,[(\mathbb{S}_\alpha)^r_t]$;
2. $(\inf \mathbb{S}_\alpha)^{\ell}_t = \sup\,[(\mathbb{S}_\alpha)^{\ell}_t]$.

Proof. 1. In the Riesz space $\mathrm{Orth}(\mathfrak{E})$ we have

$$\begin{aligned}((\sup \mathbb{S}_\alpha) - t\mathbb{I})^+ &= [(\sup \mathbb{S}_\alpha) + (-t\mathbb{I})] \vee 0 = \sup\,(\mathbb{S}_\alpha - t\mathbb{I}) \vee 0 \\ &= \sup\,[(\mathbb{S}_\alpha - t\mathbb{I}) \vee 0] = \sup\,(\mathbb{S}_\alpha - t\mathbb{I})^+.\end{aligned}$$

It is an easy exercise to show that $\{\sup\,(\mathbb{S}_\alpha - t\mathbb{I})^+\}$, i.e., the band generated in $\mathrm{Orth}(\mathfrak{E})$ by the supremum of the elements $(\mathbb{S}_\alpha - t\mathbb{I})^+$, equals the smallest band containing all the bands $\{(\mathbb{S}_\alpha - t\mathbb{I})^+\}$, i.e., equals the band $\bigvee \{(\mathbb{S}_\alpha - t\mathbb{I})^+\}$. But then the corresponding components of $\mathbb{I}$, i.e., band projections on $\mathfrak{E}$, satisfy the same relation. Thus,

$$\overline{(\sup \mathbb{S}_\alpha)}^r_t = \bigvee_\alpha \overline{(S_\alpha)}^r_l.$$

It follows that

$$(\sup \mathbb{S}_\alpha)^r_t = \mathbb{I} - \overline{(\sup \mathbb{S}_\alpha)}^r_t = \bigwedge_\alpha (\mathbb{I} - \overline{(\mathbb{S}_\alpha)}^r_t) = \bigwedge_\alpha (\mathbb{S}_\alpha)^r_t.$$

2. In this case the conclusion follows immediately from the identity

$$\begin{aligned}(t\mathbb{I} - (\inf \mathbb{S}_\alpha))^+ &= [t\mathbb{I} + \sup\,(-\mathbb{S}_\alpha))] \vee 0 = \sup\,(t\mathbb{I} - \mathbb{S}_\alpha) \vee 0 \\ &= \sup\,[(t\mathbb{I} - \mathbb{S}_\alpha) \vee 0] = \sup\,(t\mathbb{I} - \mathbb{S}_\alpha)^+.\end{aligned}$$

This completes the proof. $\square$

Remark 1. An orthomorphism $\mathbb{T}$ is band preserving and therefore, if $\mathfrak{B}$ is a band in $\mathfrak{E}$, the restriction of $\mathbb{T}$ to $\mathfrak{B}$ is an operator on $\mathfrak{B}$. If this operator has a property $P(\mathbb{T})$, we shall say that $\mathbb{T}$ *has the property* $P(\mathbb{T})$ *on* $\mathfrak{B}$. If $\mathbb{T}$ is strictly positive, i.e., if $\mathbb{T}X > 0$ for every $X > 0$, we write $\mathbb{T} \gg 0$ and if $\mathbb{T} - \mathbb{S} \gg 0$, we write $\mathbb{T} \gg \mathbb{S}$.

2. Let $\mathbb{S} \in \mathrm{Orth}(\mathfrak{E})$ and let $\mathbb{P}$ be the component of $\mathbb{I}$ in the band generated by $\mathbb{S}$ in $\mathrm{Orth}(\mathfrak{E})$. We say the projection band $\mathfrak{B} := \mathbb{P}\mathfrak{E}$ is generated by $\mathbb{S}$ in $\mathfrak{E}$.

Proposition 4.3 *Let $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ be a filtration on $\mathfrak{E}$. Then the following assertions hold with reference to this filtration.*

1. *If a is as positive constant and $\mathbb{S}$ is an optional time then $\mathbb{S} + a\mathbb{I}$ is a stopping time.*
2. *If $\mathbb{S}, \mathbb{T}$ are stopping times then the same hold for $\mathbb{S} \wedge \mathbb{T}$, $\mathbb{S} \vee \mathbb{T}$ and $\mathbb{S} + \mathbb{T}$.*
3. *If $(\mathbb{S}_n)$ is a sequence of optional times, then*

$$\sup_{n\geq 1} \mathbb{S}_n, \ \inf_{n\geq 1} \mathbb{S}_n, \ \limsup_n \mathbb{S}_n \ \textit{and} \ \liminf_n \mathbb{S}_n$$

are optional times. If the $\mathbb{S}_n$ are stopping times, then $\sup_{n\geq 1} \mathbb{S}_n$ is a stopping time.

Proof. 1. If $0 \leq t < a$ we have $((\mathbb{S}+a\mathbb{I})-t\mathbb{I})^+ = (\mathbb{S}+(a-t)\mathbb{I})^+ = \mathbb{S}+(a-t)\mathbb{I} \geq (a-t)\mathbb{I}$. Hence, $\overline{(\mathbb{S}+a\mathbb{I})}_t^r = \mathbb{I}$ implying that $(\mathbb{S}+a\mathbb{I})_t^r = 0 \in \mathfrak{F}_t$. If $t \geq a$, we have

$$((\mathbb{S} + a\mathbb{I}) - t\mathbb{I})^+ = (\mathbb{S} - (t-a)\mathbb{I})^+$$

and so by Proposition 4.1 that $(\mathbb{S} + a\mathbb{I})_t^r \in \mathfrak{P}_{(t-a)^+} \subset \mathfrak{P}_t$. This proves 1.

2. The first two assertions follow directly from Lemma 4.2. Assume now that $\mathbb{S}$ and $\mathbb{T}$ are stopping times. Since both $\mathbb{S}$ and $\mathbb{T}$ are orthomorphisms, their null spaces $N_{\mathbb{S}}$ and $N_{\mathbb{T}}$ are bands in $\mathfrak{E}$ that belong to $\mathfrak{F}_0 \subset \mathfrak{F}_t$ for all $t \in T$ since they are stopping times for the filtration. Denote the projections onto these bands by $\mathbb{P}_{\mathbb{S}}$ and $\mathbb{P}_{\mathbb{T}}$ respectively, and the projections $\mathbb{I} - \mathbb{P}_{\mathbb{S}}$ and $\mathbb{I} - \mathbb{P}_{\mathbb{T}}$ onto their carrier bands by $\mathbb{P}_{\mathbb{S}}^c$ and $\mathbb{P}_{\mathbb{T}}^c$ respectively. Then we have the disjoint decomposition

$$\begin{aligned}(\mathbb{T} + \mathbb{S} - t\mathbb{I})^+ &= \mathbb{P}_{\mathbb{T}}(\mathbb{S} - t\mathbb{I})^+ + \mathbb{P}_{\mathbb{T}}^c(\mathbb{T} + \mathbb{S} - t\mathbb{I})^+ \\ &= \mathbb{P}_{\mathbb{T}}(\mathbb{S} - t\mathbb{I})^+ + \mathbb{P}_{\mathbb{S}}\mathbb{P}_{\mathbb{T}}^c(\mathbb{T} - t\mathbb{I})^+ + \mathbb{P}_{\mathbb{S}}^c\mathbb{P}_T^c(\mathbb{T} + \mathbb{S} - t\mathbb{I})^+.\end{aligned}$$

From this the components of $\mathbb{I}$ in the bands generated by each of the terms are

$$\begin{aligned}\overline{(\mathbb{T} + \mathbb{S})}_t^r &= \mathbb{P}_{\mathbb{T}}\overline{\mathbb{S}}_t^r + \mathbb{P}_{\mathbb{S}}\mathbb{P}_{\mathbb{T}}^c\overline{\mathbb{T}}_t^r + \mathbb{P}_{\mathbb{S}}^c\mathbb{P}_T^c\overline{(\mathbb{T} + \mathbb{S})}_t^r \\ &= \mathbb{P}_{\mathbb{T}}\overline{\mathbb{S}}_t^r + \mathbb{P}_{\mathbb{S}}\mathbb{P}_{\mathbb{T}}^c\overline{\mathbb{T}}_t^r + \mathbb{P}_{\mathbb{S}}^c\mathbb{P}_T^c\mathbb{T}_t^\ell\overline{(\mathbb{T} + \mathbb{S})}_t^r + \mathbb{P}_{\mathbb{S}}^c\mathbb{P}_T^c\overline{\mathbb{T}}_t^\ell\overline{(\mathbb{T} + \mathbb{S})}_t^r. \qquad (4.1)\end{aligned}$$

We note also that $\mathbb{P}^c_{\mathbb{S}}\overline{\mathbb{T}}^\ell_t(\mathbb{T}+\mathbb{S}-t\mathbb{I}) \gg 0$ showing that $\mathbb{P}^c_{\mathbb{S}}\overline{\mathbb{T}}^\ell_t \le \overline{(\mathbb{S}+\mathbb{T})}^r_t$. Hence, the last term in 4.1 equals $\mathbb{P}^c_{\mathbb{S}}\mathbb{P}^c_T\overline{\mathbb{T}}^\ell_t$, and so

$$\overline{(\mathbb{T}+\mathbb{S})}^r_t = \mathbb{P}_{\mathbb{T}}\overline{\mathbb{S}}^r_t + \mathbb{P}_{\mathbb{S}}\mathbb{P}^c_{\mathbb{T}}\overline{\mathbb{T}}^r_t + \mathbb{P}^c_{\mathbb{S}}\mathbb{P}^c_T\overline{\mathbb{T}}^\ell_t + \mathbb{P}^c_{\mathbb{S}}\mathbb{P}^c_T\mathbb{T}^\ell_t\overline{(\mathbb{T}+\mathbb{S})}^r_t.$$

The first three terms belong to $\mathfrak{P}_t$ and so we need only consider the last term, which we denote by $\mathbb{P}$. Let $\mathfrak{B} := \mathbb{P}\mathfrak{E}$. We then have that $0 \ll \mathbb{T} \ll t\mathbb{I}$ and $\mathbb{T}+\mathbb{S} \gg t$ and $\mathbb{S} \gg 0$ on $\mathfrak{B}$. For each rational number $0 < r < t$, let $\mathfrak{B}_r$ be the intersection of the bands generated by the elements $(\mathbb{T}-r\mathbb{I})^+$, $(t\mathbb{I}-\mathbb{T})^+$ and $(\mathbb{S}-(t-r))^+$. The restrictions of $\mathbb{S}$ and $\mathbb{T}$ to $\mathfrak{B}_r$ then satisfy $t\mathbb{I} \gg \mathbb{T} \gg r\mathbb{I}$ and $\mathbb{S} \gg (t-r)\mathbb{I}$ which shows that $\mathfrak{B}_r \subset \mathfrak{F}_t$. On $\mathfrak{B}_r$ we have $\mathbb{S}+\mathbb{T} \gg (t-r+r)\mathbb{I} = t\mathbb{I}$ which shows that $\mathfrak{B}_r \subset \mathfrak{B}$ for all $0 < r < t$. We claim that the band generated by the $\mathfrak{B}_r$ equals $\mathfrak{B}$. If not, there exists a non-zero band $\mathfrak{B}_0 \subset \mathfrak{B}$ such that no $\mathfrak{B}_r$ has non-trivial intersection with it. Since $\mathbb{T}$ is strictly positive on $\mathfrak{B}_0$, we can find a sequence of rational step elements $\mathbb{T}_n \uparrow \mathbb{T}$. But then, for each rational coefficient r of such a step element, we have by assumption that it is not true that $\mathbb{S}+r\mathbb{I} \gg t\mathbb{I}$ on any non-zero band contained in $\mathfrak{B}_0$. Hence $\mathbb{S}+r\mathbb{I} \le t\mathbb{I}$ on $\mathfrak{B}_0$. But since $\mathbb{T}_n \uparrow T$, we get from this that on $\mathfrak{B}_0$ we have $\mathbb{S}+\mathbb{T} \le t\mathbb{I}$ which contradicts the fact that $\mathfrak{B}_0 \subset \mathfrak{B}$. Our conclusion is that $\mathfrak{B} \subset \mathfrak{F}_t$ and this completes the proof of 2.

3. Let $(\mathbb{S}_n)$ be a sequence of optional times for $\mathfrak{F}_t$.

(i) By Proposition 4.1 $(\mathbb{S}_n)$ is a sequence of stopping times for the filtration $\mathfrak{F}_{t+}$. Hence, since $\mathfrak{P}_{t+}$ is a complete Boolean algebra, $\inf(\mathbb{S}_n)^r_t \in \mathfrak{P}_{t+}$. By Lemma 4.2, it follows that $(\sup \mathbb{S}_n)^r_t \in \mathfrak{P}_{t+}$ and so $\sup \mathbb{S}_n$ is a stopping time for $\mathfrak{F}_{t+}$. Again by Proposition 4.1, we have that $\sup \mathbb{S}_n$ is an optional time for $\mathfrak{F}_t$.

(ii) By Lemma 4.2 we have $(\inf \mathbb{S}_n)^\ell_t = \sup[(\mathbb{S})^\ell_t] \in \mathfrak{P}_t$. Hence $\inf \mathbb{S}_n$ is an optional time.

The proofs that $\limsup \mathbb{S}_n$ and $\liminf \mathbb{S}_n$ are optional times are now clear.

If all the $\mathbb{S}_n$ are stopping times, the fact that $\sup \mathbb{S}_n$ is a stopping time follows from the identity $(\sup \mathbb{S}_n)^r_t = \inf[(\mathbb{S}_n)^r_t]$ proven in Lemma 4.2. □

5 Stopped filtrations.

If $(\Omega, \mathcal{F}, \mu)$ is a probability space and if $(\mathcal{F}_t)_{t\in T}$ is a filtration of sub-σ-algebras of $\mathcal{F}$, the σ-algebra $\mathcal{F}_{\mathbb{S}}$ of events determined prior to the stopping time $\mathbb{S}$ consists of those events $A \in \mathcal{F}$ satisfying $A \cap (\mathbb{S} \le t) \in \mathcal{F}_t$ for every $t \in T$ (see [7, Definition 2.12]). It is easy to see that

$$\mathcal{F}_{\mathbb{S}} = \{A \in \mathcal{F} : A \cap (\mathbb{S} \le s) \in \mathcal{F}_t,\ s \in [0,t],\ t \in T\}.$$

This shows that $\mathcal{F}_{\mathbb{S}}$ is the smallest σ-algebra such that the map $\omega \mapsto (\omega, \mathbb{S}(\omega))$ from $(\Omega, \mathcal{F}_t)$ into $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}))$ is measurable. This is therefore the smallest

σ-algebra such that the composite map $\omega \mapsto X(\omega, \mathbb{S}(\omega)) = X_{\mathbb{S}(\omega)}(\omega)$ is measurable. The stopped process is then defined to be the process $(X_{\mathbb{S}\wedge t})_{t\in T}$. We proceed to define these notions in the abstract case. The challenge is to define the composite function $X_{\mathbb{S}}$ in general.

Let $\mathbb{S}$ be a stopping time for the filtration $(\mathbb{F}_t)_{t\in T}$ and let $\mathfrak{P} = \mathfrak{P}_{\mathfrak{E}}$ be the Boolean algebra of all band projections on $\mathfrak{E}$.

Definition 5.1 The *set of all events determined prior to the stopping time* $\mathbb{S}$ is defined as the family of projections

$$\mathfrak{P}_{\mathbb{S}} := \{\mathbb{P} \in \mathfrak{P} \,:\, \mathbb{P}\mathbb{S}^r_t\mathbb{F}_t = \mathbb{F}_t\mathbb{P}\mathbb{S}^r_t \text{ for all } t\}.$$

It clearly follows from the definition that the following characterization holds.

Proposition 5.2 *Let $\mathbb{S}$ be a stopping time for the filtration $(\mathbb{F}_t)_{t\in T}$ and let $\mathfrak{P}_{\mathbb{S}}$ be defined as above. Then $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}}$ if and only if $\mathbb{P}\mathbb{S}^r_t \in \mathfrak{P}_t$ for every $t \in T$.*

Proposition 5.3 *Let $\mathbb{S}$ be a stopping time for the filtration $(\mathbb{F}_t)_{t\in T}$ and let $\mathfrak{P}_{\mathbb{S}}$ be defined as above. Then $\mathfrak{P}_{\mathbb{S}}$ is a complete Boolean sub-algebra of $\mathfrak{P}$.*

Proof. Since $\mathbb{S}$ is a stopping time, it follows that $\mathbb{I} \in \mathfrak{P}_{\mathbb{S}}$ and trivially, $0 \in \mathfrak{P}_{\mathbb{S}}$. If $\mathbb{P}_1, \mathbb{P}_2 \in \mathfrak{P}_{\mathbb{S}}$, we have $\mathbb{P}_1 \wedge \mathbb{P}_2 = \mathbb{P}_1\mathbb{P}_2$ and since band projections commute, we get

$$\mathbb{P}_1\mathbb{P}_2\mathbb{S}^r_t\mathbb{F}_t = \mathbb{P}_1\mathbb{S}^r_t\mathbb{P}_2\mathbb{S}^r_t\mathbb{F}_t = \mathbb{P}_1\mathbb{S}^r_t\mathbb{F}_t\mathbb{P}_2\mathbb{S}^r_t = \mathbb{F}_t\mathbb{P}_1\mathbb{S}^r_t\mathbb{P}_2\mathbb{S}^r_2 = \mathbb{F}_t\mathbb{P}_1\mathbb{P}_2\mathbb{S}^r_2.$$

Hence, $\mathbb{P}_1 \wedge \mathbb{P}_2 = \mathbb{P}_1\mathbb{P}_2 \in \mathfrak{P}_{\mathbb{S}}$. Also,

$$(\mathbb{P}_1 + \mathbb{P}_2)\mathbb{S}^r_t\mathbb{F}_t = \mathbb{P}_1\mathbb{S}^r_t\mathbb{F}_t + \mathbb{P}_2\mathbb{S}^r_t\mathbb{F}_t = \mathbb{F}_t\mathbb{P}_1\mathbb{S}^r_t + \mathbb{F}_t\mathbb{P}_2\mathbb{S}^r_t = \mathbb{F}_t(\mathbb{P}_1 + \mathbb{P}_2)\mathbb{S}^r_t$$

and so it follows that $\mathbb{P}_1 \vee \mathbb{P}_2 = \mathbb{P}_1 + \mathbb{P}_2 - \mathbb{P}_1\mathbb{P}_2 \in \mathfrak{P}_{\mathbb{S}}$. Furthermore, if $\mathbb{P} \in \mathfrak{P}_S$ then, since both $\mathbb{I}\mathbb{S}^r_t$ and $\mathbb{P}\mathbb{S}^r_t$ commute with $\mathbb{F}_t$, so does $(\mathbb{I}-\mathbb{P})\mathbb{S}^r_t$. It follows that $\mathbb{I} - \mathbb{P} \in \mathfrak{P}_{\mathbb{S}}$. Hence $\mathfrak{P}_{\mathbb{S}}$ is a Boolean sub-algebra of $\mathfrak{P}$.

If $\mathbb{P}_\alpha \in \mathfrak{P}_S$ and $\mathbb{P}_\alpha \uparrow \mathbb{P}$, then $\mathbb{P} \in \mathfrak{P}_S$ because $\mathbb{P}_\alpha\mathbb{S}^r_t\mathbb{F}_t \uparrow \mathbb{P}\mathbb{S}^r_t\mathbb{F}_t$ by the definition of the supremum for operators and $\mathbb{F}_t\mathbb{P}_\alpha\mathbb{S}^r_t \uparrow \mathbb{F}_t\mathbb{P}\mathbb{S}^r_t$ since $\mathbb{F}_t$ is order continuous. Therefore,

$$\mathbb{P}\mathbb{S}^r_t\mathbb{F}_t = \sup \mathbb{P}_\alpha\mathbb{S}^r_t\mathbb{F}_t = \sup \mathbb{F}_t\mathbb{P}_\alpha\mathbb{S}^r_t = \mathbb{F}_t\mathbb{P}\mathbb{S}^r_t.$$

This shows that $\mathfrak{P}_S$ is an order complete Boolean sub-algebra of $\mathfrak{P}$. □

Let $\{E_\alpha\}$ be a maximal disjoint system in $\mathfrak{F}_0$. With each projection $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}}$ we can associate a set $\{\mathbb{P}E_\alpha\}$ of disjoint elements of $\mathfrak{E}$. Let

$$\mathfrak{C}_{\mathbb{S}} := \{\mathbb{P}E_\alpha \,:\, \mathbb{P} \in \mathfrak{P}_{\mathbb{S}},\ \alpha \in \{\alpha\}\}.$$

Then $\mathfrak{C}_{\mathbb{S}}$ is a Boolean algebra of elements of $\mathfrak{E}$ and we denote the order closed Riesz subspace of $\mathfrak{E}$ generated by these elements by $\mathfrak{F}_{\mathbb{S}}$.

Proposition 5.4 *Let $(\mathfrak{F}_t, \mathbb{F}_t)$ be a filtration on $\mathfrak{E}$ and assume that $\mathfrak{E}$ is $\mathbb{F}_0$-universally complete. If $\mathbb{S}$ is a stopping time for the filtration $(\mathfrak{F}_t, \mathbb{F}_t)$ then there exists a unique conditional expectation $\mathbb{F}_{\mathbb{S}}$ mapping $\mathfrak{E}$ onto $\mathfrak{F}_{\mathbb{S}}$.*

Proof. We have to prove that the conditional expectation of $\mathfrak{E}$ onto $\mathfrak{F}_{\mathbb{S}}$ exists. By Proposition 3.4 we need to prove that $\mathcal{R}(\mathbb{T}_0) \subset \mathfrak{F}_{\mathbb{S}}$. Now if $\mathbb{P} \in \mathfrak{P}_0$, then $\mathbb{P} \in \mathfrak{P}_t$ for every t. It follows from the commutativity of projections and the fact that $\mathbb{S}^r_t$ is a stopping time that

$$\mathbb{P}\mathbb{S}^r_t\mathbb{F}_t = \mathbb{S}^r_t\mathbb{P}\mathbb{F}_t = \mathbb{S}^r_t\mathbb{F}_t\mathbb{P} = \mathbb{F}_t\mathbb{S}^r_t\mathbb{P} = \mathbb{F}_t\mathbb{P}\mathbb{S}^r_t.$$

It follows that $\mathfrak{P}_0 \subset \mathfrak{P}_S$. This implies that $\mathfrak{F}_0 \subset \mathfrak{F}_S$. □

Definition 5.5 For any stopping time relative to the filtration $(\mathfrak{F}_t, \mathbb{F}_t)$, we define the *stopped filtration* to be the pair $(\mathfrak{F}_{\mathbb{S}}, \mathbb{F}_{\mathbb{S}})$.

Example 5.6 Let $\mathfrak{E}$ be a Dedekind complete Riesz space with weak order unit E and let $(\mathfrak{F}_t)$ be a filtration on $\mathfrak{E}$. Let $\mathbb{S} = \sum_{k=1}^n t_k\mathbb{P}_k$ be a stopping time, with $\{\mathbb{P}_k\}$ a partition of $\mathbb{I}$. By our remark preceding Lemma 4.2 we have $\mathbb{P}_k \in \mathfrak{P}_{t_k}$.

Let $\mathfrak{Q}_k := \{\mathbb{P} \in \mathfrak{P}_{t_k} : \mathbb{P} \le \mathbb{P}_k\}$, $k = 1, 2 \ldots, n$ and let $\mathfrak{Q}$ be the Boolean algebra of projections generated by the $\mathfrak{Q}_k$. Note that each $\mathfrak{Q}_k$ is a Boolean complete algebra with largest element $\mathbb{P}_k$. Since they are also disjoint, their union is a σ-algebra with largest element $\mathbb{I}$. We claim that $\mathfrak{Q} = \mathfrak{P}_{\mathbb{S}}$. It is clear that every element in $\mathfrak{Q}$ belongs to $\mathfrak{P}_{\mathbb{S}}$. Conversely, suppose that $\mathbb{P} \in \mathfrak{P}$ is such that $\mathbb{P}\mathbb{S}^r_t \in \mathfrak{P}_t$ for all $t \ge 0$. Assume without loss of generality that the t_k are increasing. We then have, by assumption that $\mathbb{P}\mathbb{P}_1 = \mathbb{P}\mathbb{S}^r_{t_1} \in \mathfrak{P}_{t_1}$, $\mathbb{P}(\mathbb{P}_1 + \mathbb{P}_2) = \mathbb{P}\mathbb{S}^r_{t_2} \in \mathfrak{P}_{t_2}$ and so since $\mathfrak{P}_{t_1} \subset \mathfrak{P}_{t_2}$, $\mathbb{P}\mathbb{P}_2 \in \mathfrak{P}_{t_2}$. By induction, $\mathbb{P}\mathbb{P}_k \in \mathfrak{P}_{t_k}$ for $k = 1, 2, \ldots, n$. Since the $\mathbb{P}_k$ is a partition, it follows that $\mathbb{P} = \sum_{k=1}^n \mathbb{P}\mathbb{P}_k$ belongs to $\mathfrak{Q}$.

It follows that if $C_k := \mathbb{P}_kE$, and if $\mathfrak{C}_k = \{C \in \mathfrak{C}_{t_k} : C \le C_k\}$ then $\mathfrak{C}_{\mathbb{S}}$ is the Boolean algebra generated by the $\mathfrak{C}_k$. Moreover, $\mathfrak{F}_{\mathbb{S}}$ is the order closed Riesz space generated by these sets of components.

We claim that $\mathbb{F}_{\mathbb{S}} = \sum_{k=1}^n \mathbb{P}_k\mathbb{F}_{t_k}$. Clearly, this sum defines a conditional expectation operator on $\mathfrak{E}$. If $X \in \mathfrak{E}$ we have $\mathbb{F}_{\mathbb{S}}X = \sum_{k=1}^n \mathbb{P}_k\mathbb{F}_{t_k}X = \sum_{k=1}^n \mathbb{P}_kX_{t_k}$ where $X_{t_k} \in \mathfrak{F}_{t_k}$. It is clear that each element in the right continuous spectral system of the latter element is a component of E belonging to $\mathfrak{C}_{\mathbb{S}}$ and so $\mathbb{F}_{\mathbb{S}}X \in \mathfrak{F}_{\mathbb{S}}$.

If (X_t) is a stochastic process adapted to the filtration $(\mathfrak{F}_t, \mathbb{F}_t)$ we shall define in the next section the element $X_{\mathbb{S}} := \sum_{k=1}^n \mathbb{P}_kX_{t_k}$. Note then that $\mathbb{F}_{\mathbb{S}}X_{\mathbb{S}} = X_{\mathbb{S}}$ and so $X_{\mathbb{S}} \in \mathfrak{F}_{\mathbb{S}}$.

The next separation lemma can be found in [8]. We provide another way to construct a proof for it using Freudenthal's theorem.

Lemma 5.7 *Let $S, T \in \mathfrak{E}$ with $T > S$, and let $\mathfrak{E}$ be a Riesz space satisfying the principal projection property and having a weak order unit E. Then there exist a pair $(s,t) \in \mathbb{R} \times \mathbb{R}$ such that $s < t$ and*

$$(T - tE)^+ \wedge (sE - S)^+ > 0.$$

Proof. If not then for every pair $(s,t) \in \mathbb{R}\times\mathbb{R}$, we have $(T-tE)^+ \wedge (sE-S)^+ > 0$ implies $t \le s$. Freudenthal's theorem then implies the contradiction that $T \le S$, as we will show now.

We may assume that T and S are bounded, i.e., contained in the ideal generated by E. Let (P_α) be a left continuous spectral system for S and let

$$\Sigma(S) = \sum_{i=1}^{k} \alpha_i (P_{\alpha_i} - P_{\alpha_{i-1}}) \ge S \tag{5.1}$$

be an upper sum that approximates S from above. The elements $\Delta P_i = P_{\alpha_i} - P_{\alpha_{i-1}}$ then form a partition of E in disjoint components. Similarly, if (Q_β) is a right continuous spectral system for T (see [15, page 262]) let

$$\sigma(T) = \sum_{j=1}^{l} \beta_{j-1} (Q_{\beta_j} - Q_{\beta_{j-1}}) \le T \tag{5.2}$$

be a lower sum which approximates T from below. Again the elements $\Delta Q_j = Q_{\beta_j} - Q_{\beta_{j-1}}$ form a partition of E in disjoint components. Consider the partition of E consisting of the disjoint components $\Delta R_{i,j} := \Delta Q_i \wedge \Delta Q_j$

We can then write

$$\Sigma(S) = \sum_{i=1}^{k} \sum_{j=1}^{l} \gamma_{i,j} \Delta R_{i,j}, \quad \gamma_{i,j} = \alpha_i$$

$$\sigma(T) = \sum_{j=1}^{l} \sum_{i=1}^{k} \delta_{i,j} \Delta R_{i,j}, \quad \delta_{i,j} = \beta_{j-1}.$$

For every non-zero term in these sums, $\Delta R_{i,j} > 0$. Now,

$$\Delta R_{i,j} \le P_i - P_{i-1} \le P_i$$

with P_i the projection of E onto the band generated by $(\alpha_i E - S)^+$.

Also $Q_j = E - R_j$ with R_j equal to the projection of E onto the band generated by $(T - \beta_j E)^+$. Hence,

$$\Delta R_{i,j} \le Q_j - Q_{j-1} = E - R_j - E + R_{j-1} = R_{j-1} - R_j \le R_{j-1}$$

with R_{j-1} the projection of E onto the band generated by $(T - \beta_{j-1}E)^+$. Thus, it follows from $\Delta R_{i,j} > 0$ that $(\alpha_i E - S)^+ \wedge (T - \beta_{j-1}E)^+ > 0$ which implies by our assumption that $\delta_{i,j} = \beta_{j-1} \le \alpha_i = \gamma_{i,j}$. Since this holds for every pair (i,j) for which the component $\Delta R_{i,j} > 0$, we have that $\sigma(T) \le \Sigma(S)$. It is then an easy consequence of Freudenthal's theorem that $T \le S$, and the proof is complete. $\square$

Corollary 5.8 *Let $\mathbb{S}$ and $\mathbb{T}$ be stopping times for the filtration $(\mathfrak{F}_t, \mathbb{F}_t)$. Let $\mathbb{P}_{(\mathbb{S}-\mathbb{T})^+}$ be the component of $\mathbb{I}$ in the band generated by $(\mathbb{S}-\mathbb{T})^+$. Then*

$$\mathbb{P}_{(\mathbb{S}-\mathbb{T})^+} = \bigvee_{\tau \in \mathbb{Q}} \overline{\mathbb{S}}^r_\tau \mathbb{T}^\ell_\tau.$$

Proof. Consider the restriction of $\mathbb{S}$ and $\mathbb{T}$ to the projection band $\mathfrak{B}$ in $\mathfrak{E}$ corresponding to the projection $\mathbb{P}_{(\mathbb{S}-\mathbb{T})^+}$. As remarked earlier, since $\mathbb{S}$ and $\mathbb{T}$ are orthomorphisms, they map $\mathfrak{B}$ into itself and on $\mathfrak{B}$ we have $\mathbb{S} > \mathbb{T}$. Applying Lemma 5.7 above, there exists a rational number τ such that $\overline{\mathbb{S}}^r_\tau \mathbb{T}^\ell_\tau > 0$. Consider the band $\mathfrak{C}$ generated by the union of all projection bands corresponding to the projection $\overline{\mathbb{S}}^r_\tau \mathbb{T}^\ell_\tau > 0$, equivalently, the supremum of all projections of the form $\overline{\mathbb{S}}^r_\tau \mathbb{T}^\ell_\tau > 0$. Since we have on each of these bands that $\mathbb{S} > \tau\mathbb{I} > \mathbb{T}$, they are all contained in $\mathfrak{B}$ and so $\mathfrak{C} \subset \mathfrak{B}$ and it is clear that since $\mathbb{S} > \mathbb{T}$ on the band $\mathfrak{B} \cap \mathfrak{C}^d$, it follows from the lemma that indeed $\mathfrak{C} = \mathfrak{B}$. This proves the corollary. □

We also use the notation $\mathbb{P}_{(\mathbb{T}<\mathbb{S})}$ for $\mathbb{P}_{(\mathbb{S}-\mathbb{T})^+}$ and $\mathbb{P}_{(\mathbb{S}\leq\mathbb{T})}$ for $\mathbb{I} - \mathbb{P}_{(\mathbb{S}-\mathbb{T})^+}$. It then follows from Corollary 5.8 above that

$$\mathbb{P}_{(\mathbb{S}\leq\mathbb{T})} = \bigwedge_{\tau \in \mathbb{Q}} \mathbb{S}^r_\tau \vee \overline{\mathbb{T}}^\ell_\tau. \tag{5.3}$$

Proposition 5.9 *Let $\mathbb{S}$ and $\mathbb{T}$ be stopping times for the filtration $(\mathfrak{F}_t, \mathbb{P}_t)$. We then have for any $\mathbb{P} \in \mathfrak{P}_\mathbb{S}$ that $\mathbb{P}\mathbb{P}_{(\mathbb{S}\leq\mathbb{T})} \in \mathfrak{P}_\mathbb{T}$. In particular, if $\mathbb{S} \leq \mathbb{T}$ we have $\mathfrak{P}_\mathbb{S} \subset \mathfrak{P}_\mathbb{T}$ and consequently, $\mathfrak{F}_\mathbb{S} \subset \mathfrak{F}_\mathbb{T}$.*

Proof. We observe firstly that if $\mathbb{S} \leq \lambda\mathbb{I}$ and $\mathbb{T} \leq \lambda\mathbb{I}$ then by (5.3) we have that

$$\mathbb{P}_{(\mathbb{S}\leq\mathbb{T})} = \bigwedge_{\substack{\tau \in \mathbb{Q} \\ \tau \leq \lambda}} \mathbb{S}^r_\tau \vee \overline{\mathbb{T}}^\ell_\tau \in \mathfrak{P}_\tau \subset \mathfrak{P}_\lambda.$$

So, in particular, for any $t \in T$ we have that $\mathbb{P}_{(\mathbb{S}\wedge t\mathbb{I}\leq\mathbb{T}\wedge t\mathbb{I})} \in \mathfrak{P}_t$. Thus for any $\mathbb{P} \in \mathfrak{P}_S$ we have

$$\mathbb{P}\mathbb{P}_{(\mathbb{S}\leq\mathbb{T})}\mathbb{T}^r_t = \mathbb{P}(\mathbb{P}_{(\mathbb{S}\leq t\mathbb{I})}\mathbb{P}_{(t\leq t\mathbb{I})}\mathbb{P}_{(\mathbb{S}\wedge t\mathbb{I}\leq\mathbb{T}\wedge t\mathbb{I})}) = (\mathbb{P}\mathbb{P}_{(\mathbb{S}\leq t\mathbb{I})})\mathbb{P}_{(t\leq t\mathbb{I})}\mathbb{P}_{(\mathbb{S}\wedge t\mathbb{I}\leq\mathbb{T}\wedge t\mathbb{I})} \in \mathfrak{P}_t$$

because on the projection band corresponding to the projection $\mathbb{P}_{(\mathbb{S}\leq\mathbb{T})}\mathbb{T}^r_t$ we have $\mathbb{T} \leq t\mathbb{I}$ and $\mathbb{S} \leq \mathbb{T}$ which is equivalent to $\mathbb{S} \leq t\mathbb{I}$, $\mathbb{T} \leq t\mathbb{I}$ and $\mathbb{S} \wedge t\mathbb{I} \leq \mathbb{T} \wedge t\mathbb{I}$. This proves the proposition. □

Following the exposition in [7], we also prove

Proposition 5.10 *Let $\mathbb{S}$ and $\mathbb{T}$ be stopping times for the filtration $(\mathfrak{F}_t, \mathbb{F}_t)$. Then the following assertions hold:*

1. $\mathfrak{P}_{\mathbb{S}\wedge\mathbb{T}} = \mathfrak{P}_\mathbb{S} \cap \mathfrak{P}_\mathbb{T}$.

2. *The following projections are in* $\mathfrak{P}_{\mathbb{S}} \cap \mathfrak{P}_{\mathbb{T}} : \mathbb{P}_{(\mathbb{S}<\mathbb{T})}$, $\mathbb{P}_{(\mathbb{T}<\mathbb{S})}$, $\mathbb{P}_{(\mathbb{T}\le\mathbb{S})}$, $\mathbb{P}_{(\mathbb{S}\le\mathbb{T})}$, $\mathbb{P}_{(\mathbb{S}=\mathbb{T})}$.

Proof. 1. By Proposition 5.9 we have immediately that $\mathfrak{P}_{\mathbb{S}\wedge\mathbb{T}} \subset \mathfrak{P}_{\mathbb{S}} \wedge \mathfrak{P}_{\mathbb{T}}$. For the converse, take $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}} \cap \mathfrak{P}_{\mathbb{T}}$. Then (see the proof of Lemma 4.2)

$$\mathbb{P} \wedge (\mathbb{S} \wedge \mathbb{T})^r_t = \mathbb{P} \wedge (\mathbb{S}^r_t \vee \mathbb{T}^r_t) = \mathbb{P}\mathbb{S}^r_t \vee \mathbb{P}\mathbb{T}^r_t \in \mathfrak{P}_t.$$

Hence, $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}\wedge\mathbb{T}}$.

Since $\mathbb{I} \in \mathfrak{P}_{\mathbb{S}}$, it follows from Proposition 5.9 that $\mathbb{P}_{(\mathbb{S}\le\mathbb{T})} \in \mathfrak{P}_{\mathbb{T}}$. It follows that $\mathbb{P}_{(\mathbb{S}>\mathbb{T})} \in \mathfrak{P}_T$. Consider $\mathbb{R} := \mathbb{S} \wedge \mathbb{T}$ which is a stopping time according to Proposition 4.2, satisfying $\mathbb{R} \le \mathbb{S}$. By what we have just proved we have $\mathbb{P}_{(\mathbb{T}\le\mathbb{R})} \in \mathfrak{P}_{\mathbb{R}}$ and again $\mathbb{P}_{(\mathbb{T}>\mathbb{R})} \in \mathfrak{P}_{\mathbb{R}} = \mathfrak{P}_{\mathbb{S}} \cap \mathfrak{P}_{\mathbb{T}}$. Thus, $\mathbb{P}_{(\mathbb{S}<\mathbb{T})} = \mathbb{P}_{(\mathbb{R}<\mathbb{T})} \in \mathfrak{P}_{\mathbb{T}}$, and consequently we also have $\mathbb{P}_{(\mathbb{S}\ge\mathbb{T})} \in \mathfrak{P}_{\mathbb{T}}$. Interchanging the roles of $\mathbb{S}$ and $\mathbb{T}$ in the argument above yields the result that all these projections also belong to $\mathfrak{P}_S$. This completes the proof. Finally $\mathbb{P}_{(\mathbb{S}=\mathbb{T})} = \mathbb{P}_{(\mathbb{S}\le\mathbb{T})}\mathbb{P}_{(\mathbb{T}\le\mathbb{S})}$ which therefore also belongs to $\mathfrak{P}_{\mathbb{S}} \cap \mathfrak{P}_{\mathbb{T}}$. □

Proposition 5.11 *Let* $(\mathfrak{F}_t, \mathbb{F}_t)$ *be a filtration on* $\mathfrak{E}$ *and assume that* $\mathfrak{E}$ *is* $\mathbb{F}_0$-*universally complete. If* $\mathbb{S}$ *and* $\mathbb{T}$ *are stopping times for the filtration* $(\mathfrak{F}_t, \mathbb{F}_t)$ *with* $\mathbb{S} \le \mathbb{T}$, *then*

$$\mathbb{F}_{\mathbb{S}} = \mathbb{F}_{\mathbb{S}}\mathbb{F}_{\mathbb{T}} = \mathbb{F}_{\mathbb{T}}\mathbb{F}_{\mathbb{S}}.$$

Proof. It follows from Proposition 5.9 that $\mathfrak{F}_{\mathbb{S}} \subset \mathfrak{F}_{\mathbb{T}}$ and by Proposition 5.6 there exists a unique conditional expectation $\mathbb{F}_{\mathbb{S}}$ onto $\mathfrak{F}_{\mathbb{S}}$. But both $\mathbb{F}_{\mathbb{S}}\mathbb{F}_{\mathbb{T}}$ and $\mathbb{F}_{\mathbb{T}}\mathbb{F}_{\mathbb{S}}$ are conditional expectations mapping $\mathfrak{E}$ onto $\mathfrak{F}_{\mathbb{S}}$. They are therefore equal to $\mathbb{F}_{\mathbb{S}}$ and the proof is complete. □

Thus far in this section all definitions were given for stopping times. We now turn our attention to optional times for similar results.

Definition 5.12 Let $\mathbb{S}$ be an optional time for the filtration $(\mathfrak{F}_t)$. We define the Boolean algebra $\mathfrak{P}_{\mathbb{S}+}$ of *events determined immediately after* the optional time $\mathbb{S}$ as

$$\mathfrak{P}_{\mathbb{S}+} := \{\mathbb{P} \in \mathfrak{P} : \mathbb{P}\mathbb{S}^r_t\mathbb{F}_{t+} = \mathbb{F}_{t+}\mathbb{P}\mathbb{S}^r_t \text{ for all } t\}.$$

Since $\mathbb{S}$ is an optional time for $(\mathfrak{F}_t)$, if and only $\mathbb{S}$ is a stopping time for the right continuous filtration $(\mathfrak{F}_{t+})$, we derive from what has already been proved that $\mathfrak{P}_{\mathbb{S}+}$ is indeed an order complete Boolean sub algebra of $\mathfrak{P}$ and that $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}+}$ if and only if $\mathbb{P}\mathbb{S}^r_t \in \mathfrak{P}_{t+}$ for every $t \in T$ (see 5.1 and 5.2). It is also clear that if the filtration is right continuous then $\mathfrak{P}_{\mathbb{S}+} = \mathfrak{P}_{\mathbb{S}}$. Also, if $\mathbb{S}$ is a stopping time (so that both $\mathfrak{P}_{\mathbb{S}}$ and $\mathfrak{P}_{\mathbb{S}+}$ are defined) we have $\mathfrak{P}_{\mathbb{S}} \subset \mathfrak{P}_{\mathbb{S}+}$, since $\mathfrak{P}_t \subset \mathfrak{P}_{t+}$.

Proposition 5.13 *Let* $\mathbb{S}$ *be an optional time for* $(\mathfrak{F}_t)$. *Then*

$$\mathfrak{P}_{\mathbb{S}+} = \{\mathbb{P} \in \mathfrak{P} : \mathbb{P}\mathbb{S}^\ell_t\mathbb{F}_t = \mathbb{F}_t\mathbb{P}\mathbb{S}^\ell_t \textit{ for all } t\}.$$

Proof. We know that $\mathbb{S}_t^\ell = \bigvee_{n=1}^{\infty} \mathbb{S}^r_{t-1/n}$. Hence, if $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}+}$ we have $\mathbb{P}\mathbb{S}_t^\ell = \bigvee_{n=1}^{\infty} \mathbb{P}\mathbb{S}^r_{(t-1/n)} \in \bigvee_{n=1}^{\infty} \mathfrak{P}_{(t-1/n)+} \subset \mathfrak{P}_t$. It follows that $\mathbb{P}\mathbb{S}_t^\ell \mathbb{F}_t = \mathbb{F}_t \mathbb{P}\mathbb{S}_t^\ell$. Conversely, if the latter relation holds for all t, we have $\mathbb{P}\mathbb{S}_t^r = \bigwedge_{n=1}^{\infty} \mathbb{P}\mathbb{S}^\ell_{t+1/n} \in \bigcap_{n=1}^{\infty} \mathfrak{P}_{t+1/n} = \mathfrak{P}_{t+}$. Hence, $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}+}$. $\square$

Propositions 5.9 and 5.10 remains true if $\mathbb{T}$ and $\mathbb{S}$ are assumed to be optional times and $\mathfrak{P}_{\mathbb{T}}$, $\mathfrak{P}_{\mathbb{S}}$ and $\mathfrak{P}_{\mathbb{T}\wedge\mathbb{S}}$ are replaced by $\mathfrak{P}_{\mathbb{T}+}$, $\mathfrak{P}_{\mathbb{S}+}$ and $\mathfrak{P}_{(\mathbb{T}\wedge\mathbb{S})+}$ respectively.

Proposition 5.14 *If $\mathbb{S}$ is an optional time and $\mathbb{T}$ is a stopping time with $\mathbb{S} \ll \mathbb{T}$ then $\mathfrak{P}_{\mathbb{S}+} \subset \mathfrak{P}_{\mathbb{T}}$.*

Proof. Let us first note that since $S \ll \mathbb{T}$ we have that $\bigvee_{q\in\mathbb{Q}} \mathbb{P}_{(\mathbb{S}<q\mathbb{I}<\mathbb{T})} = \mathbb{I}$ with $\mathbb{Q}$ the set of (positive) rational numbers. It follows that for any $\mathbb{P}$ we have $\mathbb{P} = \bigvee_{q\in\mathbb{Q}} \mathbb{P}\mathbb{P}_{(\mathbb{S}<q\mathbb{I}<\mathbb{T})}$. For any $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}+}$ we moreover have, since $\mathbb{S}$ is an optional and $\mathbb{T}$ is a stopping time.

$$\mathbb{P}\mathbb{P}_{(\mathbb{S}<q\mathbb{I}<\mathbb{T})} = (\mathbb{P}\mathbb{S}_q^\ell)(\mathbb{I} - \mathbb{T}_q^r) \in \mathfrak{P}_q.$$

Now, let $t \in T$ and let $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}+}$ then

$$\mathbb{P}\mathbb{T}_t^r = \bigvee_{q\in\mathbb{Q}} \mathbb{P}\mathbb{P}_{(\mathbb{S}<q\mathbb{I}<\mathbb{T})}\mathbb{P}_{(\mathbb{T}\le t\mathbb{I})} \in \mathfrak{P}_t,$$

since the supremum is taken over $q \in \mathbb{Q}$ such that $q < t$ which implies that $\mathfrak{P}_q \subset \mathfrak{P}_t$. It follows that $\mathbb{P} \in \mathfrak{P}_{\mathbb{T}}$ and the proof is complete. $\square$

Proposition 5.15 *If $(\mathbb{S}_n)$ is a sequence of optional times and $\mathbb{S} = \inf \mathbb{S}_n$, then*

$$\mathfrak{P}_{\mathbb{S}+} = \bigcap_{n=1}^{\infty} \mathfrak{P}_{\mathbb{S}_n+}.$$

Moreover, if each $\mathbb{S}_n$ is a stopping time, and $\mathbb{S} \ll \mathbb{S}_n$, then

$$\mathfrak{P}_{\mathbb{S}+} = \bigcap_{n=1}^{\infty} \mathfrak{P}_{\mathbb{S}_n}.$$

If $(\mathbb{S}_n)$ is a sequence of optional times, then, for $\mathbb{S} = \sup \mathbb{S}_n$ (which is optional), we have

$$\bigcup \mathfrak{P}_{\mathbb{S}_n+} \subset \mathfrak{P}_{\mathbb{S}+}.$$

Proof. It follows from Proposition 4.3(3) that $\mathbb{S}$ is an optional time and then from Proposition 5.10 and our remark preceding Proposition 5.14 that $\mathfrak{P}_{\mathbb{S}+} \subset \mathfrak{P}_{\mathbb{S}_n+}$ for every n. Hence, $\mathfrak{P}_{\mathbb{S}+} \subset \bigcap_{n=1}^{\infty} \mathfrak{P}_{\mathbb{S}_n+}$. For the converse inclusion let $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}_n+}$ for every n, i.e., let $\mathbb{P}(\mathbb{S}_n)_t^\ell \in \mathfrak{P}_t$ for every n and for every $t \in T$. Then

$$\mathbb{P}\mathbb{S}_t^\ell = \bigvee_{n=1}^{\infty} \mathbb{P}(\mathbb{S}_n)_t^\ell \in \mathfrak{P}_t,$$

and so $\mathbb{P} \in \mathfrak{P}_{\mathbb{S}+}$. This proves the first assertion.

For the second assertion we apply Proposition 5.14 to conclude that $\mathfrak{P}_{\mathbb{S}+} \subset \bigcap_{n=1}^{\infty} \mathfrak{P}_{\mathbb{S}_n}$. From our remark preceding Proposition 5.13 we have $\mathfrak{P}_{\mathbb{S}_n} \subset \mathfrak{P}_{\mathbb{S}_n+}$ for every n. Therefore, by the first part equality follows.

The last assertion follows trivially from Proposition 5.9. $\square$

6 Stopped processes and Doob's optional sampling theorem.

We will prove Doob's optional sampling theorem in this section for the class of submartingales that have a Doob-Meyer decomposition.

Definition 6.1 We say the submartingale $(X_t, \mathfrak{F}_t)$ has the Doob-Meyer decomposition property if it can be uniquely decomposed as

$$X_t = M_t + A_t$$

with $(A_t, \mathfrak{F}_t)$ an increasing right continuous predictable process.

We refer the reader to [5, Theorem 7.5] where conditions are given on $\mathfrak{E}$ and on a submartingale (X_t) in order for the latter to have the Doob-Meyer decomposition property.

Let (X_t) be a stochastic process in $\mathfrak{E}$. We start the discussion by considering an arbitrary positive orthomorphism $\mathbb{S}$ and define $X_{\mathbb{S}}$ in several steps. As mentioned earlier, in a function space with a stochastic process $(X_t(\omega)) = (X(\omega, t))$ and with $\mathbb{S}$ represented by a positive real valued measurable function $\mathbb{S}(\omega)$, we have that $X_{\mathbb{S}}(\omega) = X_{\mathbb{S}(\omega)}(\omega) = X(\omega, \mathbb{S}(\omega))$.

Definition 6.2 Let $\mathbb{S} := \sum_{k=1}^{n} t_k \mathbb{P}_k$ be an elementary element of $\mathrm{Orth}(\mathfrak{E})$. We define the element $X_{\mathbb{S}}$ by

$$X_{\mathbb{S}} := \sum_{k=1}^{n} \mathbb{P}_k X_{t_k}.$$

It is a standard exercise to see that $X_{\mathbb{S}}$ is well defined and the definition makes sense for an arbitrary stochastic process (X_t). If $\mathbb{T}$ is also elementary, if $\mathbb{S} \leq \mathbb{T}$ and if the process (X_t) is increasing, it follows immediately that $X_{\mathbb{S}} \leq X_{\mathbb{T}}$.

In order to define elements $X_{\mathbb{S}}$ for more general orthomorphisms $\mathbb{S}$, we firstly extend this definition to right continuous increasing processes.

Definition 6.3 Let (A_t) be a right continuous increasing process, i.e., $0 \leq A_s \leq A_t$ if $s \leq t$ and $A_t \downarrow A_s$ if $t \downarrow s$. Consider a bounded orthomorphism $\mathbb{S}$, say $\mathbb{S} \leq a\mathbb{I}$. Let $\pi = \{t_k\}$, with $0 = t_0 < t_1 < \cdots < t_n = a$ a partition of the interval $[0, a]$ with mesh $|\pi|$ and put $\Delta\mathbb{S}_k := \mathbb{S}^\ell_{t_k} - \mathbb{S}^\ell_{t_{k-1}}$. Consider an upper sum of $\mathbb{S}$ with respect to this partition, i.e., a sum

$$\mathbb{S}_\pi = \sum_{k=1}^{n} t_k \Delta\mathbb{S}_k.$$

Note that $\mathbb{S} \ll \mathbb{S}_\pi$ and that, by Freudenthal's spectral theorem $\mathbb{S}_\pi \downarrow \mathbb{S}$ holds $\mathbb{I}$-uniformly. Now $A_{\mathbb{S}_\pi} \downarrow$ since A_t is an increasing process (this is readily seen by considering a refinement of π by the addition of one point; induction then yields the result). We define

$$A_{\mathbb{S}} := \inf_{\pi} A_{\mathbb{S}_\pi}.$$

Again it follows from the remark in (1) that if $\mathbb{S} \leq \mathbb{T}$ with $\mathbb{T}$ a bounded orthomorphism, we have $A_{\mathbb{S}} \leq A_{\mathbb{T}}$.

Let (A_t) be an integrable right continuous increasing process (i.e., $A_\infty := \sup_{t\in T} A_t \in \mathfrak{E}$). Then we have that $A_{\mathbb{S}\wedge n\mathbb{I}} \leq A_\infty$ and $A_{\mathbb{S}\wedge n\mathbb{I}} \uparrow$. We define

$$A_{\mathbb{S}} := \sup_{n\geq 1} A_{\mathbb{S}\wedge n\mathbb{I}}.$$

Due to the analogue with the definition of an integral, and the analogue with the functional calculus for scalar valued measurable functions (see [23, Chapter XI]), $A_{\mathbb{S}}$ can be written as the (forward) integral of the vector function A_t with respect to the spectral measure $d\mathbb{S}_t$, i.e.,

$$A_{\mathbb{S}} = \int_0^a A_t \, d\mathbb{S}_t.$$

We have thus defined $A_{\mathbb{S}}$ for a right continuous increasing process (A_t) in the case that the orthomorphism $\mathbb{S}$ is bounded and if it is not bounded, we have defined it for the case that (A_t) is integrable.

We now prove the following lemma.

Lemma 6.4 *Let $(A_t, \mathfrak{F}_t)_{t\in T}$ be a positive increasing right continuous stochastic process adapted to the filtration $(\mathbb{F}_t, \mathfrak{F}_t)$ and suppose that $\mathfrak{E}$ is $\mathbb{F}_0$ universally complete. If $\mathbb{S}$ is a bounded optional time for the filtration or an integrable optional time, then $A_{\mathbb{S}} \in \mathfrak{F}_{\mathbb{S}+}$.*

Proof. Let $\mathbb{S}$ be a bounded optional time for the filtration and let $\mathbb{S}_\pi = \sum_{k=1}^{n} t_k \Delta\mathbb{S}_k$ with $\Delta\mathbb{S}_k := \mathbb{S}^\ell_{t_k} - \mathbb{S}^\ell_{t_{k-1}}$. Then, since $\Delta\mathbb{S}_k \in \mathfrak{P}_{t_k+}$ we have that $\mathbb{S}_\pi$ is an optional time (see the remark before Lemma 4.2) and we note that $\mathbb{S} \ll \mathbb{S}_\pi$. Using Freudenthal's theorem, we can choose a sequence of optional times $\mathbb{S}_{\pi_n} \downarrow \mathbb{S}$ uniformly (see [15] Theorem 40.2). It follows as in Example 5.6 that for each $\mathbb{S}_{\pi_n}$ we have $A_{\mathbb{S}_{\pi_n}} \in \mathfrak{F}_{\mathbb{S}_{\pi_n}+}$. Since $A_{\mathbb{S}_{\pi_n}} \downarrow A_{\mathbb{S}}$ by definition, we have that $A_{\mathbb{S}} \in \mathfrak{F}_{\mathbb{S}_{\pi_n}+}$ for every n. But that implies that $A_{\mathbb{S}} \in \mathfrak{F}_{\mathbb{S}+}$ since $\mathfrak{P}_{\mathbb{S}+} = \bigcap_{n=1}^{\infty} \mathfrak{P}_{\mathbb{S}_{\pi_n}+}$ by Proposition 5.15.

If (A_t) is integrable we note that since $(\mathbb{S} \wedge n\mathbb{I}) \uparrow \mathbb{S}$ is upwards directed, $\mathfrak{P}_{(\mathbb{S}\wedge n\mathbb{I})+} \subset \mathfrak{P}_{\mathbb{S}+}$. Hence, also in this case $A_{\mathbb{S}} = \sup A_{\mathbb{S}\wedge n\mathbb{I}} \in \mathfrak{F}_{\mathbb{S}+}$. □

Proposition 6.5 *Let $\mathbb{S}$ be a bounded optional time for the filtration $(\mathfrak{F}_t)$ and suppose (M_t) is a right continuous increasing martingale. If $\mathbb{S} \leq t\mathbb{I}$, then*

$$\mathbb{F}_{\mathbb{S}+} M_t = M_{\mathbb{S}}.$$

If the filtration $(\mathfrak{F}_t)$ is right continuous, then

$$\mathbb{F}_{\mathbb{S}} M_t = M_{\mathbb{S}}.$$

Proof. Let π be a partition of $[0, t]$ and define the step elements $\mathbb{S}_\pi$ as above, with $\mathbb{S}_\pi \downarrow \mathbb{S}$ uniformly. We then have

$$\begin{aligned} \mathbb{F}_{\mathbb{S}+} M_t &= \mathbb{F}_{\mathbb{S}+}\mathbb{F}_{\mathbb{S}_\pi+} M_t \quad (\text{since } \mathfrak{F}_{\mathbb{S}+} \subset \mathfrak{F}_{\mathbb{S}_\pi+}) \\ &= \mathbb{F}_{\mathbb{S}+} M_{\mathbb{S}_\pi+} \quad (\text{since } (M_t) \text{ is a martingale}) \\ &\downarrow \mathbb{F}_{\mathbb{S}+} M_{\mathbb{S}+} = M_{\mathbb{S}} \quad (\text{since } M_{\mathbb{S}} \in \mathfrak{F}_{\mathbb{S}+} \text{ by Lemma 6.4}). \end{aligned}$$

Of course, if the filtration is right continuous, then $\mathfrak{F}_{\mathbb{S}+} = \mathfrak{F}_{\mathbb{S}}$. □

This result serves as a motivation for the definition of $X_{\mathbb{S}}$ in the case that (X_t) is a martingale.

Definition 6.6 1. Let $\mathbb{S}$ be a bounded optional time for the filtration $(\mathfrak{F}_t)$ and suppose (M_t) is a right continuous martingale. We then define

$$M_{\mathbb{S}} := \mathbb{F}_{\mathbb{S}+} M_t \text{ where } t \in T \text{ is such that } \mathbb{S} \leq t\mathbb{I}.$$

2. If $\mathbb{S}$ is an optional time and if (M_t) has a last element M_∞ we define

$$M_{\mathbb{S}} := \mathbb{F}_{\mathbb{S}+} M_\infty.$$

If the filtration $(\mathfrak{F}_t)$ is right continuous, we can replace $\mathbb{F}_{\mathbb{S}+}$ by $\mathbb{F}_{\mathbb{S}}$.

We note that since (M_t) is a martingale, the choice of t does not play a role, for if $\mathbb{S} \leq t\mathbb{I} < s\mathbb{I}$ we have

$$\mathbb{F}_{\mathbb{S}+} M_s = \mathbb{F}_{\mathbb{S}+}\mathbb{F}_t M_s = \mathbb{F}_{\mathbb{S}+} M_t,$$

because $\mathfrak{F}_{\mathbb{S}+} \subset \mathfrak{F}_t \subset \mathfrak{F}_s$.

We can now finally define the stopped process for a submartingale that has the Doob-Meyer decomposition. As mentioned before, conditions on the submartingale and the space $\mathfrak{E}$ for this to hold can be found in [5].

Definition 6.7 Let $\mathbb{S}$ be an optional time for the filtration $(\mathfrak{F}_t)$, let (X_t) be a right continuous submartingale with the Doob-Meyer decomposition property and let

$$X_t = M_t + A_t,\ t \in T$$

be its unique decomposition with (M_t) a martingale and (A_t) an increasing right continuous process. Then

1. if $\mathbb{S}$ is bounded or
2. if (X_t) has a last element X_∞,

we define

$$X_{\mathbb{S}} := M_{\mathbb{S}} + A_{\mathbb{S}}$$

with $M_{\mathbb{S}}$ and $A_{\mathbb{S}}$ as we defined them above.

We conclude with the optional sampling theorem due to Doob (see [7] for a discussion of this result in the concrete case).

Theorem 6.8 *Let $(X_t, \mathfrak{F}_t)$ be a right continuous submartingale with the Doob-Meyer decomposition property and let $\mathbb{S} \leq \mathbb{T}$ be two optional times of the filtration $(\mathfrak{F}_t, \mathbb{F}_t)$. Then, if either*

1. *$\mathbb{T}$ is bounded or*
2. *(X_t) has a last element X_∞,*

we have

$$\mathbb{F}_{\mathbb{S}+} X_{\mathbb{T}} \geq X_{\mathbb{S}}.$$

Proof. We provide a proof for the second case. The first case follows in exactly the same manner. Let $X_t = M_t + A_t$ be the Doob-Meyer decomposition of (X_t). Since $\mathbb{S} \leq \mathbb{T}$ we have $\mathfrak{P}_{\mathbb{S}+} \subset \mathfrak{P}_{\mathbb{T}+}$ and it follows that $\mathbb{F}_{\mathbb{S}+} = \mathbb{F}_{\mathbb{S}+}\mathbb{F}_{\mathbb{T}+} = \mathbb{F}_{\mathbb{T}+}\mathbb{F}_{\mathbb{S}+}$ by the analogue of Proposition 5.11 for optional times. Hence, since M_t is a martingale with a last element M_∞ we have $\mathbb{F}_{\mathbb{S}+} M_{\mathbb{T}} = \mathbb{F}_{\mathbb{S}+}\mathbb{F}_{\mathbb{T}+} M_\infty = \mathbb{F}_{\mathbb{S}+} M_\infty = M_{\mathbb{S}}$.

Furthermore, since (A_t) is an increasing process, we derive from $\mathbb{S} \leq \mathbb{T}$ that $A_{\mathbb{S}} \leq A_{\mathbb{T}}$ (this was remarked when defining $A_{\mathbb{S}}$). Hence we have, since $\mathbb{F}_{\mathbb{S}+}$ is positive and since $A_{\mathbb{S}} \in \mathfrak{F}_{\mathbb{S}+}$ by Lemma 6.4, that

$$\mathbb{F}_{\mathbb{S}+} X_{\mathbb{T}} = \mathbb{F}_{\mathbb{S}+} M_{\mathbb{T}} + \mathbb{F}_{\mathbb{S}+} A_{\mathbb{T}} \geq M_{\mathbb{S}} + \mathbb{F}_{\mathbb{S}+} A_{\mathbb{S}} = M_{\mathbb{S}} + A_{\mathbb{S}} = X_{\mathbb{S}}.$$

This completes our proof. □

We note that the condition that (X_t) should be Doob-Meyer decomposable is the only condition added to the classical conditions found in [7]. As the reader observed, this condition is needed in order to define the object $X_{\mathbb{S}}$ for a stochastic process (X_t) and an orthomorphism $\mathbb{S}$. It would be interesting to know how to remove this extra assumption.

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