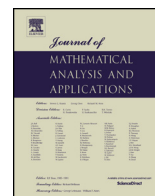




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A Toeplitz-like operator with rational matrix symbol having poles on the unit circle: Invertibility and Riccati equations

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ABSTRACT

This paper is a continuation of the work on unbounded Toeplitz-like operators T_Ω with rational matrix symbol Ω initiated in Groenewald et al. (2021) [11], where a Wiener-Hopf type factorization of Ω is obtained and used to determine when T_Ω is Fredholm and to compute the Fredholm index in case T_Ω is Fredholm. Due to the high level of non-uniqueness and complicated form of the Wiener-Hopf type factorization, it does not appear useful in determining when T_Ω is invertible. In the present paper we use state space methods to characterize invertibility of T_Ω in terms of the existence of a stabilizing solution of an associated nonsymmetric discrete algebraic Riccati equation, which in turn leads to a pseudo-canonical factorization of Ω and concrete formulas of T_Ω^{-1} .

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1. Introduction

In two recent papers [11,12], we explored the matrix analogue of an unbounded Toeplitz-like operator that was investigated in [8–10] for scalar rational symbols with poles on the unit circle $\mathbb{T}$. While many of the classical operator theory topics, like Fredholmness, invertibility and spectrum, are well understood in the scalar case, the case of matrix symbols appears to be more intricate. In [11] a Wiener-Hopf type factorization was obtained, from which the Fredholm index can be determined, in case the symbol has no zeroes on $\mathbb{T}$. However, this Wiener-Hopf type factorization has a high level of non-uniqueness and, unlike in the classical case, generally does not lead to a diagonalization of the symbol. As a result, although some

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further Fredholm characteristics can be determined from the Wiener-Hopf type factorization [12], it does not seem to be an adequate tool to compute the dimensions of the kernel and the cokernel, nor does it seem to give a clear characterization of invertibility. In the present paper we take a different approach to the question of invertibility of this Toeplitz-like operator, using Riccati equations and pseudo-canonical factorization.

Unbounded Toeplitz operators appeared first in a paper of Hartman and Wintner [13] in 1950, but only became an active topic with the seminal paper of Sarason [18] in connection to truncated Toeplitz operators; see [17] for how unbounded Toeplitz operators with matrix symbols come into play. More recently, kernels of unbounded Toeplitz operators appeared in the study of nearly backward shift invariant subspaces and Toeplitz inverses [3,4].

Next we introduce some notation, which is required to define our Toeplitz-like operator and state our main results. We write Rat for the space of rational functions, $\text{Rat}(\mathbb{T})$ for the functions in Rat that only have poles on the unit circle $\mathbb{T}$, $\text{Rat}_0(\mathbb{T})$ for the strictly proper functions in $\text{Rat}(\mathbb{T})$, $\mathcal{P}$ for the space of polynomials and for positive integers k we indicate the polynomials of degree at most k by $\mathcal{P}_k$, extending it to all integers by setting $\mathcal{P}_k = \{0\}$ in case $k \leq 0$. For positive integers m and n , we indicate the spaces of $m \times n$ matrices with entries from these function spaces by $\text{Rat}^{m \times n}$, $\text{Rat}_0(\mathbb{T})^{m \times n}$, etc. In the case of vector functions, when $n = 1$, we will just write m instead of $m \times 1$. For $1 < p < \infty$, L^p and H^p denote the Lebesgue space and Hardy space, respectively, and K^p is the standard complement of H^p in L^p . With L_m^p , H_m^p and K_m^p we indicate the spaces of vectors of length m with entries from L^p , H^p and K^p , respectively.

Let $\Omega \in \text{Rat}^{m \times m}$ with possibly poles on $\mathbb{T}$ and $\det \Omega \not\equiv 0$, and let $1 < p < \infty$. We then define the Toeplitz-like operator $T_\Omega (H_m^p \rightarrow H_m^p)$ by

$$\text{Dom}(T_\Omega) = \left\{ f \in H_m^p : \begin{array}{l} \Omega f = h + \eta \text{ where } h \in L_m^p(\mathbb{T}), \\ \text{and } \eta \in \text{Rat}_0^m(\mathbb{T}) \end{array} \right\}, \quad (1.1)$$

$$T_\Omega f = \mathbb{P}h \text{ with } \mathbb{P} \text{ the Riesz projection of } L_m^p(\mathbb{T}) \text{ onto } H_m^p.$$

By the Riesz projection, $\mathbb{P}$, we mean the projection of L_m^p onto H_m^p , as discussed in [15, pages 149–153].

As usual, $\Omega \in \text{Rat}^{m \times m}$ has a pole at $z_0 \in \mathbb{C} \cup \{\infty\}$, if any of its entries has a pole at z_0 . In case $\det \Omega \not\equiv 0$, a zero of Ω is a pole of its inverse $\Omega^{-1}(z) := \Omega(z)^{-1}$. It is not necessarily the case that the zeroes of Ω correspond to the zeroes of $\det \Omega$, and Ω can have both a pole and a zero at the same point $z_0 \in \mathbb{C} \cup \{\infty\}$. It was proved in [11] that T_Ω is Fredholm if and only if Ω has no zeroes on $\mathbb{T}$. In particular, for T_Ω to be invertible, that is, bijective from $\text{Dom}(T_\Omega)$ onto H_m^p , it is necessary that Ω has no zeroes on $\mathbb{T}$.

In the classical case, invertibility of Toeplitz operators with rational symbols can be studied via Riccati equations and canonical factorizations associated with state space realizations of the symbol, cf., [1,2]. In the present paper, we follow the approach of [6]. Let $\Omega \in \text{Rat}^{m \times m}$ and assume Ω is given by a minimal state space realization of the form

$$\Omega(z) = R_0 + zC(I - zA)^{-1}B + \gamma(zI - \alpha)^{-1}\beta, \quad (1.2)$$

with $R_0 \in \mathbb{C}^{m \times m}$ and with A, B, C and α, β, γ matrices of appropriate sizes such that A has all its eigenvalues in the open unit disc $\mathbb{D}$ and α has all its eigenvalues in the closed unit disc $\overline{\mathbb{D}}$, that is, A is stable and α is semi-stable (in [6] α is also stable). Minimality in this setting means that one cannot find a representation for Ω of this form with A and α matrices of smaller size; equivalently, the triples (C, A, B) and (γ, α, β) both provide observable and controllable discrete-time linear systems, cf., [14]. Despite the fact that Ω has poles on $\mathbb{T}$, so that α has eigenvalues on $\mathbb{T}$, there is a fairly direct analogue of the canonical factorization result of [6].

Theorem 1.1. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and no zeroes on $\mathbb{T}$ and assume that Ω is given by the minimal realization (1.2) with A stable and α semi-stable. Then the following are equivalent:*

- (i) *There exists a matrix Q such that $R_0 - \gamma QB$ is invertible, Q satisfies the nonsymmetric discrete algebraic Riccati equation:*

$$Q = \alpha QA + (\beta - \alpha QB)(R_0 - \gamma QB)^{-1}(C - \gamma QA), \quad (1.3)$$

and such that the matrices

$$\begin{aligned} A_o &:= A - B(R_0 - \gamma QB)^{-1}(C - \gamma QA), \\ \alpha_o &:= \alpha - (\beta - \alpha QB)(R_0 - \gamma QB)^{-1}\gamma \end{aligned} \quad (1.4)$$

are both stable.

- (ii) *Ω has a right pseudo-canonical factorization $\Omega(z) = \Psi(z)\Theta(z)$, i.e., Ψ and Θ are $m \times m$ rational matrix functions with $\det \Psi \neq 0$ and $\det \Theta \neq 0$ and such that Θ has poles only outside or on the unit circle $\mathbb{T}$, Θ^{-1} has poles only outside $\mathbb{T}$, Ψ has poles only inside or on the unit circle $\mathbb{T}$, and Ψ^{-1} has poles only inside the unit circle $\mathbb{T}$.*

Moreover, if Q is a solution to (1.3) such that A_o and α_o are stable, then a right pseudo-canonical factorization is obtained as follows: Let δ and D be invertible matrices such that $\delta D = R_0 - \gamma QB$ and set

$$C_o = \delta^{-1}(C - \gamma QA) \quad \text{and} \quad \beta_o = (\beta - \alpha QB)D^{-1}. \quad (1.5)$$

Then $\Omega(z) = \Psi(z)\Theta(z)$ holds with Ψ and Θ defined as

$$\Theta(z) = D + zC_o(I - zA)^{-1}B \quad \text{and} \quad \Psi(z) = \delta + \gamma(zI - \alpha)^{-1}\beta_o, \quad (1.6)$$

and the inverses of Ψ and Θ are given by

$$\begin{aligned} \Theta^{-1}(z) &= D^{-1} - zD^{-1}C_o(I - zA_o)^{-1}BD^{-1}, \\ \Psi^{-1}(z) &= \delta^{-1} - \delta^{-1}\gamma(zI - \alpha_o)^{-1}\beta_o\delta^{-1}. \end{aligned} \quad (1.7)$$

Furthermore, the solution Q of the Riccati equation (1.3) so that A_o and α_o are stable is unique. Finally, the realizations in (1.6) and (1.7) for Θ , Ψ , Θ^{-1} and Ψ^{-1} are minimal.

The above result is proved in Section 2 and is essentially obtained by specifying Theorem 1.1 of [6] for the function Ω_r defined by

$$\Omega_r(z) = \Omega(rz) \quad (1.8)$$

for $1 < r$ small enough, so that Ω_r does not have poles and zeroes on $\mathbb{T}$. More generally, for any function f , scalar-, vector- or matrix-valued, and scalar $r > 0$ we write f_r for the function $f_r(z) = f(rz)$ defined for $z \in \mathbb{C}$ for which rz is in the domain of f .

In order to characterize invertibility of T_Ω more is required than what is in [6], we want to compare invertibility of the unbounded operator T_Ω with invertibility of the bounded Toeplitz operator T_{Ω_r} . Note that invertibility in both cases means bijectivity on its domain of definition. Since T_Ω is a closed operator, by [11, Proposition 2.1], the inverse of T_Ω will be bounded, cf., Theorem 12.4.1 in [7]. For $r > 1$, we define the annulus

$$\mathfrak{A}_r := \{z \in \mathbb{C} : r^{-1} < |z| < r\}. \quad (1.9)$$

It turns out that T_Ω and T_{Ω_r} can be compared, not only with respect to invertibility, but even with respect to their Fredholm properties, in case they are Fredholm.

Theorem 1.2. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and assume that Ω has no zeroes on $\mathbb{T}$. Define Ω_r by (1.8). Let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$. Then T_Ω is Fredholm and for each $1 < r < r_0$, T_{Ω_r} is bounded and Fredholm and we have*

$$\dim \text{Ker } T_\Omega = \dim \text{Ker } T_{\Omega_r} \quad \text{and} \quad \text{codim Ran } T_\Omega = \text{codim Ran } T_{\Omega_r}.$$

In particular, T_Ω is invertible if and only if T_{Ω_r} is invertible for some (and hence all) $1 < r < r_0$.

We shall prove Theorem 1.2 in Section 3. Our second main result together with [6], shows that invertibility of T_Ω is equivalent to items (i) and (ii) in Theorem 1.1. With some further work we derive, in Section 4 below, formulas for the inverse of T_Ω , as given in the following result.

Theorem 1.3. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and no zeroes on $\mathbb{T}$ and assume that Ω is given by the minimal realization (1.2) with A stable and α semi-stable. Then T_Ω is invertible if and only if the Riccati equation (1.3) has a solution Q such that $A_\circ$ and $\alpha_\circ$ in (1.4) are stable, or, equivalently, if Ω has a pseudo-canonical factorization $\Omega(z) = \Psi(z)\Theta(z)$ as in item (ii) of Theorem 1.1. In that case, the inverse of T_Ω is the bounded operator given by*

$$T_\Omega^{-1} = T_\Theta^{-1} T_\Psi^{-1} = T_{\Theta^{-1}} T_{\Psi^{-1}}. \quad (1.10)$$

Moreover, T_Ω^{-1} has a block matrix representation $[T_\Omega^{-1}]_{i,j}$ with respect to the standard (block) basis of H_m^p that is given by

$$[T_\Omega^{-1}]_{i,j} = \sum_{k=0}^{\min(i,j)} \Theta_{i-k}^\times \Psi_{j-k}^\times, \quad (1.11)$$

where

$$\begin{aligned} \Theta_0^\times &= D^{-1}, & \Theta_j^\times &= -D^{-1}(C_\circ)(A_\circ)^{j-1}BD^{-1}, \quad j = 1, 2, \dots, \\ \Psi_0^\times &= \delta^{-1}, & \Psi_j^\times &= -\delta^{-1}\gamma(\alpha_\circ)^{j-1}(\beta_\circ)\delta^{-1}, \quad j = 1, 2, \dots \end{aligned}$$

For the final result we present in this introduction we restrict to the case where $p = 2$, since it relies on a result from [6], which is proved only for $p = 2$. Let $\Omega \in \text{Rat}^{m \times m}$ be given by the minimal realization with A stable and α semi-stable and assume A and α are of size $s \times s$ and $t \times t$, respectively. We define the observability operator for the pair (C, A) as

$$\mathcal{O}_{C,A} : \mathbb{C}^s \rightarrow H_m^2, \quad \mathcal{O}_{C,A} : x \mapsto C(I - zA)^{-1}x, \quad z \in \mathbb{D}, \quad (1.12)$$

and the controllability operator for the pair (α, β) as

$$\begin{aligned} \mathcal{C}_{\alpha,\beta} : \text{Dom}(\mathcal{C}_{\alpha,\beta}) \rightarrow \mathbb{C}^t, \quad \mathcal{C}_{\alpha,\beta}f &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i\tau}I - \alpha)^{-1}\beta f(e^{i\tau}) \, d\tau, \\ \text{for } f \in \text{Dom}(\mathcal{C}_{\alpha,\beta}) &:= \left\{ f \in H_m^2 : \int_{-\pi}^{\pi} (e^{i\tau}I - \alpha)^{-1}\beta f(e^{i\tau}) \, d\tau \text{ exists} \right\}. \end{aligned} \quad (1.13)$$

Since A is stable, it is clear that $\mathcal{O}_{C,A}$ defines a bounded operator from $\mathbb{C}^s$ into H_m^2 . Due to the semi-stability of α , $\mathcal{C}_{\alpha,\beta}$ need not be bounded, but it is the case that the subspace

$$\mathcal{D}_r := \{f \in H_m^2 : f_r \in H_m^2\},$$

for $r > 1$, is contained in $\text{Dom}(\mathcal{C}_{\alpha,\beta})$. This will be proved in Lemma 4.2 in Section 4 below, where we will also prove the following proposition.

Proposition 1.4. *Consider the case $p = 2$. Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and such that T_Ω is invertible. Assume that Ω is given by the minimal realization (1.2) with A stable and α semi-stable. Then the solution Q of the algebraic Riccati equation (1.3) that makes $A_\circ$ and $\alpha_\circ$ in (1.4) stable is given by*

$$Q = \mathcal{C}_{\alpha,\beta} T_\Omega^{-1} \mathcal{O}_{C,A}. \quad (1.14)$$

The formula for Q is analogous to that in [6], where poles on $\mathbb{T}$ are not allowed, but requires more attention since $\mathcal{C}_{\alpha,\beta}$ is not necessarily bounded. To see that the right-hand side in (1.14) is well defined, we point out that $\mathcal{O}_{C,A}$ maps $\mathbb{C}^s$ into $\mathcal{D}_r$ for $r > 1$ small enough, while T_Ω^{-1} maps $\mathcal{D}_r$ into $\mathcal{D}_r$, again for $r > 1$ small enough, which is contained in the domain of $\mathcal{C}_{\alpha,\beta}$. That T_Ω^{-1} maps $\mathcal{D}_r$ into $\mathcal{D}_r$ follows from Proposition 3.1 below.

We conclude this introduction with a brief overview of the remainder of the paper. In Section 2 we apply the main result of [6] to the function Ω_r in (1.8), and translate back to the state space realization of Ω , leading to a proof of Theorem 1.1. In the next section we investigate the relation between T_Ω and T_{Ω_r} , and give a proof of Theorem 1.2. The work of Sections 2 and 3, is then combined in Section 4 to prove Theorem 1.3 as well as Proposition 1.4.

2. Riccati equation, canonical factorization and inversion for Ω_r

Suppose that $\Omega \in \text{Rat}^{m \times m}$ is given by the minimal realization formula (1.2), that is:

$$\Omega(z) = R_0 + zC(I - zA)^{-1}B + \gamma(zI - \alpha)^{-1}\beta, \quad (2.1)$$

with A being stable and α being semistable. It is then clear that Ω_r defined by (1.8) admits the state space realization

$$\begin{aligned} \Omega_r(z) &= \Omega(rz) = R_0 + z(rC)(I - z(rA))^{-1}B + \gamma\left(zI - \frac{\alpha}{r}\right)^{-1} \frac{\beta}{r} \\ &= R_0 + zC_r(I - zA_r)^{-1}B + \gamma(zI - \alpha_r)^{-1}\beta_r \end{aligned} \quad (2.2)$$

with

$$A_r = rA, \quad C_r = rC, \quad \alpha_r = \frac{\alpha}{r}, \quad \beta_r = \frac{\beta}{r}. \quad (2.3)$$

As in Theorem 1.1, let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$, with $\mathfrak{A}_{r_0}$ as defined in (1.9). Then, for $1 < r < r_0$, Ω_r has no poles and no zeroes on $\mathbb{T}$. Therefore, the results of [6] apply to Ω_r and its realization (2.2)–(2.3). Note that the paper [6] only considers the case of Toeplitz operators on H_m^2 . However, since invertibility of Toeplitz operators on H_m^p with rational matrix symbols can be characterized in terms of their Wiener-Hopf factorizations, which are independent of p , invertibility on H_m^p is independent of the value of p . We now specify the main result of [6] to Ω_r , together with some supplementary observations, in the next proposition. This result is subsequently used to prove Theorem 1.1.

Proposition 2.1. Let $\Omega \in \text{Rat}^{m \times m}$ be given by the realization (2.1) with A stable and α semi-stable, so that Ω_r is given by the realization (2.2)–(2.3). Let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$. For $1 < r < r_0$ the following are equivalent:

- (i) T_{Ω_r} is invertible.
- (ii) There exists a matrix Q such that $R_0 - \gamma QB$ is invertible, Q satisfies the nonsymmetric discrete algebraic Riccati equation:

$$Q = \alpha QA + (\beta - \alpha QB)(R_0 - \gamma QB)^{-1}(C - \gamma QA), \quad (2.4)$$

and rA_o and $r^{-1}\alpha_o$ are stable, with A_o and α_o given by (1.4).

- (iii) Ω_r has a canonical factorization $\Omega_r(z) = \Psi^{(r)}(z)\Theta^{(r)}(z)$, i.e., $\Psi^{(r)}$ and $\Theta^{(r)}$ are $m \times m$ rational matrix functions with $\det \Psi^{(r)} \not\equiv 0$ and $\det \Theta^{(r)} \not\equiv 0$ and such that $\Theta^{(r)}$ and $(\Theta^{(r)})^{-1}$ have poles only outside $\mathbb{T}$ and $\Psi^{(r)}$ and $(\Psi^{(r)})^{-1}$ have poles only inside $\mathbb{T}$.

Moreover, the solution Q of the Riccati equation (2.4) such that rA_o and $r^{-1}\alpha_o$ are stable is unique and independent of r , i.e., for each $1 < r < r_0$ one obtains the same solution Q in item (ii). Furthermore, if Q is as in (ii), then a canonical factorization of Ω_r is obtained with $\Theta^{(r)} = \Theta_r$ and $\Psi^{(r)} = \Psi_r$, where Θ and Ψ are defined as in Theorem 1.1, and Θ_r and Ψ_r are defined according to (1.8). In case one of items (i)–(iii) holds, and hence all, we have $T_{\Omega_r}^{-1} = T_{\Theta_r}^{-1}T_{\Psi_r}^{-1} = T_{\Theta_r^{-1}}T_{\Psi_r^{-1}}$.

Proof. Since Ω_r has no poles on $\mathbb{T}$, since $1 < r < r_0$, Theorem 1.1 of [6] applies to Ω_r and its realization (2.2), leading to the equivalence of variations of (i)–(iii) in terms of the matrices in the realizations. Technically, Theorem 1.1 of [6] does not contain an item about the invertibility of T_{Ω_r} , but that invertibility of T_{Ω_r} is equivalent to the two items in the theorem follows by the discussion preceding the theorem, and this is also where the formula for $T_{\Omega_r}^{-1}$ in terms of the canonical factors appears. It thus remains to show that the statements of items (ii) and (iii) in terms of the realization matrices of Ω_r correspond to the statements concerning the Riccati solutions and canonical factorization from Theorem 1.1 of [6], respectively.

We start with item (ii). From Theorem 1.1 of [6], and the preceding paragraphs, we obtain that invertibility of T_{Ω_r} is equivalent to the existence of a matrix Q_r such that $R_0 - \gamma Q_r B$ is invertible, that satisfies the Riccati equation

$$\begin{aligned} Q_r &= \alpha_r Q_r A_r + (\beta_r - \alpha_r Q_r B)(R_0 - \gamma Q_r B)^{-1}(C_r - \gamma Q_r A_r) \\ &= \left(\frac{\alpha}{r}\right) Q_r(rA) + \left(\frac{\beta}{r} - \frac{\alpha}{r} Q_r B\right)(R_0 - \gamma Q_r B)^{-1}(rC - \gamma Q_r(rA)) \\ &= \alpha Q_r A + (\beta - \alpha Q_r B)(R_0 - \gamma Q_r B)^{-1}(C - \gamma Q_r A) \end{aligned}$$

and such that

$$\begin{aligned} A_{o,r} &= A_r - B(R_0 - \gamma Q_r B)^{-1}(C_r - \gamma Q_r A_r) \\ &= rA - B(R_0 - \gamma Q_r B)^{-1}(rC - \gamma Q_r(rA)) = rA_o, \\ \alpha_{o,r} &= \alpha_r - (\beta_r - \alpha_r Q_r B)(R_0 - \gamma Q_r B)^{-1}\gamma \\ &= \frac{\alpha}{r} - \left(\frac{\beta}{r} - \frac{\alpha}{r} Q_r B\right)(R_0 - \gamma Q_r B)^{-1}\gamma = \frac{\alpha_o}{r} \end{aligned}$$

are both stable, corresponding to the claim of item (ii). Moreover, the matrix Q_r with these properties is unique. It follows that the Riccati equation that Q_r solved is independent of r , but it is less straightforward

that the condition of having $rA_\circ$ and $\frac{\alpha_\circ}{r}$ stable does not introduce a dependency on r ; in particular, $A_\circ$ and $\alpha_\circ$ in the above formulas may depend on r . To see that this is not the case, we note that the matrix Q_r can be obtained as the limit of a Riccati difference equation associated with the finite section method for T_{Ω_r} , as discussed in Section 4 of [6]. Indeed, since Ω_r is continuous on $\mathbb{T}$ and we assume T_{Ω_r} to be invertible, for N large enough the N -th section of T_{Ω_r} , i.e., the Toeplitz block matrix

$$T_{\Omega_r, N} = \begin{bmatrix} R_{0,r} & R_{-1,r} & \cdots & R_{1-N,r} \\ R_{1,r} & R_{0,r} & \cdots & R_{2-N,r} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N-1,r} & R_{N-2,r} & \cdots & R_{0,r} \end{bmatrix}, \quad (2.5)$$

with $R_{j,r}$ the j -th Fourier coefficient of Ω_r , will be invertible, and the matrices

$$Q_{N,r} := \mathcal{C}_{\beta_r, \alpha_r, N} T_{\Omega_r, N}^{-1} \mathcal{O}_{C_r, A_r, N}$$

with $\mathcal{C}_{\beta_r, \alpha_r, N}$ and $\mathcal{O}_{C_r, A_r, N}$ given by

$$\mathcal{C}_{\beta_r, \alpha_r, N} = \text{Row}_{j=0}^{N-1}(\alpha_r^j \beta_r) \quad \text{and} \quad \mathcal{O}_{C_r, A_r, N} = \text{Col}_{j=0}^{N-1}(C_r A_r^j) \quad (2.6)$$

solve the Riccati difference equation

$$\begin{aligned} Q_{N+1,r} &= \alpha_r Q_{N,r} A_r + (\beta_r - \alpha_r Q_{N,r} B)(R_0 - \gamma Q_{N,r} B)^{-1}(C_r - \gamma Q_{N,r} A_r) \\ &= \alpha Q_{N,r} A + (\beta - \alpha Q_{N,r} B)(R_0 - \gamma Q_{N,r} B)^{-1}(C - \gamma Q_{N,r} A) \end{aligned}$$

and $Q_{N,r}$ converges to Q_r as $N \rightarrow \infty$. Hence, in order to see that Q_r is independent of r , it suffices to show that $Q_{N,r}$ is independent of r . Note that the Fourier coefficients of Ω_r are given by

$$R_{n,r} = \begin{cases} C_r A_r^{n-1} B = r^n C A^{n-1} B & \text{for } n > 0, \\ R_0 & \text{for } n = 0, \\ \gamma \alpha_r^{n-1} \beta_r = r^{-n} \gamma \alpha^{n-1} \beta & \text{for } n < 0. \end{cases} \quad (2.7)$$

It follows that

$$T_{\Omega_r, N} = \text{Diag}(I_m, rI_m, \dots, r^{N-1}I_m) T_{\Omega, N} \text{Diag}(I_m, rI_m, \dots, r^{N-1}I_m)^{-1},$$

where $T_{\Omega, N}$ is as in (2.5) with $R_{n,r} = R_{n,1}$, where $R_{n,1}$ is defined according to (2.7) with $r = 1$. This shows that for large N , also $T_{\Omega, N}$ invertible and

$$T_{\Omega_r, N}^{-1} = \text{Diag}(I_m, rI_m, \dots, r^{N-1}I_m) T_{\Omega, N}^{-1} \text{Diag}(I_m, rI_m, \dots, r^{N-1}I_m)^{-1}. \quad (2.8)$$

Define $\mathcal{C}_{\beta, \alpha, N}$ and $\mathcal{O}_{C, A, N}$ analogous to $\mathcal{C}_{\beta_r, \alpha_r, N}$ and $\mathcal{O}_{C_r, A_r, N}$, with $\beta_r, \alpha_r, C_r, A_r$ replaced by β, α, C, A , respectively. It is then easy to see that

$$\begin{aligned} \mathcal{C}_{\beta_r, \alpha_r, N} &= \mathcal{C}_{\beta, \alpha, N} \text{Diag}(I_m, rI_m, \dots, r^{N-1}I_m)^{-1}, \\ \mathcal{O}_{C_r, A_r, N} &= \text{Diag}(I_m, rI_m, \dots, r^{N-1}I_m) \mathcal{O}_{C, A, N}. \end{aligned}$$

Combining these identities with (2.8), it follows that

$$Q_{N,r} = \mathcal{C}_{\beta_r, \alpha_r, N} T_{\Omega_r, N}^{-1} \mathcal{O}_{C_r, A_r, N} = \mathcal{C}_{\beta, \alpha, N} T_{\Omega, N}^{-1} \mathcal{O}_{C, A, N},$$

is indeed independent of r , and consequently, Q_r is also independent of r .

It remains to prove the equivalence of (ii) (or (i)) and (iii), and that (iii) can be achieved as described in the proposition, and the formulas for $T_{\Omega_r}^{-1}$. The equivalence of (ii) and (iii), in fact, follows directly from Theorem 1.1 in [6]. We now show that the formulas for the canonical factors from [6] lead to the factorization of Ω_r using Θ and Ψ from Theorem 1.1. By the formulas in [6], the canonical factorization of Ω_r is given by $\Omega_r(z) = \Psi^{(r)}(z)\Theta^{(r)}(z)$ where we factor $R_0 - \gamma Q_r B = \delta D$, as claimed, and set

$$\begin{aligned}\beta_{o,r} &= (\beta_r - \alpha_r Q_r B)D^{-1} = r^{-1}(\beta - \alpha Q_r B)D^{-1} = r^{-1}\beta_o, \\ C_{o,r} &= \delta^{-1}(C_r - \gamma Q_r A) = r\delta^{-1}(C - \gamma Q A) = rC_o,\end{aligned}$$

to arrive at

$$\begin{aligned}\Psi^{(r)}(z) &= \delta + \gamma(zI - \alpha_r)^{-1}\beta_{o,r} = \delta + r^{-1}\gamma(zI - r^{-1}\alpha)^{-1}\beta_o \\ &= \delta + \gamma(rzI - \alpha)^{-1}\beta_o = \Psi_r(z)\end{aligned}$$

with inverse

$$\begin{aligned}\Psi^{(r)}(z)^{-1} &= \delta^{-1} - \delta^{-1}\gamma(zI - \alpha_{o,r})^{-1}\beta_{o,r}\delta^{-1} \\ &= \delta^{-1} - r^{-1}\delta^{-1}\gamma(zI - r^{-1}\alpha_o)^{-1}\beta_o\delta^{-1} = \delta^{-1} - \delta^{-1}\gamma(rzI - \alpha_o)^{-1}\beta_o\delta^{-1}\end{aligned}$$

and

$$\Theta^{(r)}(z) = D + zC_{o,r}(I - zA_r)^{-1}B = D + rzC_o(I - rzA)^{-1}B = \Theta_r(z)$$

with inverse

$$\begin{aligned}\Theta^{(r)}(z)^{-1} &= D^{-1} - zD^{-1}C_{o,r}(I - zA_{o,r})^{-1}BD^{-1} \\ &= D^{-1} - rzD^{-1}C_o(I - rzA_o)^{-1}BD^{-1}.\end{aligned}$$

The formula for $T_{\Omega_r}^{-1}$ now follows simply from the text preceding Theorem 1.1 in [6]. $\square$

Using the equivalence of (ii) and (iii) in Proposition 2.1 it is easy to prove our second main result.

Proof of Theorem 1.1. Since Ω_r , Ψ_r and Θ_r are rational matrix functions, the factorization $\Omega_r(z) = \Psi_r(z)\Theta_r(z)$ for some $1 < r < r_0$ implies that also $\Omega(z) = \Psi(z)\Theta(z)$ as well as the formulas for the inverses of Ψ and Θ . Hence (iii) in Proposition 2.1 is equivalent to (ii) in Theorem 1.1.

Next we show that the realizations of Θ and Ψ in (1.6) and of Θ^{-1} and Ψ^{-1} in (1.7) are minimal. Note that since the realization of Ω is minimal and A is stable and α is semi-stable, the McMillan degree, $\deg(\Omega)$, of Ω is equal to the sum of the sizes of A and α , say s and t , respectively. From the formulas of Θ and Ψ it is clear that $\deg(\Theta) \leq s$ and $\deg(\Psi) \leq t$. On the other hand, since the McMillan degree is sublogarithmic, we have

$$s + t = \deg(\Omega) = \deg(\Psi\Theta) \leq \deg(\Psi) + \deg(\Theta) \leq s + t.$$

Hence we have equality in each step, which implies $\deg(\Theta) = s$ and $\deg(\Psi) = t$, in other words, the realizations of Θ and Ψ are minimal. By the observation right after Proposition 7.2 in [1], it follows that the realization for Θ (respectively Ψ) is minimal if and only if the realization of Θ^{-1} (respectively Ψ^{-1}) is minimal. Hence, also the realizations of Θ^{-1} and Ψ^{-1} are minimal.

Since the solution $Q = Q_r$ of (2.4) used to construct $rA_\circ$ and $r^{-1}\alpha_\circ$ is independent of r , it follows that the solution Q in item (ii) in Proposition 2.1 is such that $rA_\circ$ and $r^{-1}\alpha_\circ$ are stable for all $1 < r < r_0$, so that $A_\circ$ is stable and $\alpha_\circ$ is semi-stable.

Thus, from the equivalence of (ii) and (iii) in Proposition 2.1 it follows that we get the equivalence of (i) and (ii) in Theorem 1.1, except that at this stage we only get $\alpha_\circ$ to be semi-stable. To see that $\alpha_\circ$ is in fact stable, note that $\Omega(z) = \Psi(z)\Theta(z)$ implies that $\Psi(z)^{-1} = \Theta(z)\Omega(z)^{-1}$. Since A is stable, Θ has no poles on $\mathbb{T}$, and Ω^{-1} has no poles on $\mathbb{T}$ because Ω is assumed to have no zeroes on $\mathbb{T}$. Therefore, Ψ^{-1} has no poles on $\mathbb{T}$ either, which implies, by minimality of the realization of Ψ^{-1} , that $\alpha_\circ$ has no eigenvalues on $\mathbb{T}$. Hence $\alpha_\circ$ is stable. $\square$

3. Fredholmness of T_Ω versus Fredholmness of T_{Ω_r}

In this section we prove Theorem 1.2. The proof relies heavily on the connection between T_Ω and T_{Ω_r} , with Ω_r as in (1.8), as explained in the next result.

Proposition 3.1. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and assume that Ω has no zeroes on $\mathbb{T}$. Let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$. Define*

$$\mathcal{D}_r := \{f \in H_m^p : f_r \in H_m^p\}. \quad (3.1)$$

Then for each $1 < r < r_0$ we have

$$\text{Ker } T_\Omega \subset \mathcal{D}_r \subset H_m(\overline{\mathbb{D}}) \subset \text{Dom}(T_\Omega). \quad (3.2)$$

Moreover, T_Ω maps $\mathcal{D}_r$ into $\mathcal{D}_r$, the inverse image $T_\Omega^{-1}(\mathcal{D}_r)$ of $\mathcal{D}_r$ under T_Ω lies in $\mathcal{D}_r$, and we have

$$(T_\Omega f)_r = T_{\Omega_r} f_r, \quad f \in \mathcal{D}_r. \quad (3.3)$$

Proof. Let $1 < r < r_0$. We start by proving (3.2), except for the first inclusion.

The second inclusion is trivial. If $f \in \mathcal{D}_r$, then f has an analytic extension to $r\mathbb{D}$, so that, in particular, f is analytic on $\overline{\mathbb{D}}$.

The argument to show that $H_m(\overline{\mathbb{D}}) \subset \text{Dom}(T_\Omega)$ is similar to that in the scalar case [10, Theorem 6.2]. Note that each $f \in H_m(\overline{\mathbb{D}})$ is also in $\mathcal{D}_{r'}$ for some $1 < r'$ sufficiently close to 1. Hence it suffices to show that $\mathcal{D}_r \subset \text{Dom}(T_\Omega)$. Let $f \in \mathcal{D}_r$. Since Ω is rational, Ωf is meromorphic on $r\mathbb{D}$ with finitely many poles, each of finite multiplicity. Computing the residues of the poles of each of the entries in the vector function Ωf it is easy to write Ωf in the form $g + \rho$ with $g \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$, showing that $f \in \text{Dom}(T_\Omega)$. Hence, we have proved the second inclusion.

Next we show that T_Ω maps $\mathcal{D}_r$ into $\mathcal{D}_r$ and that (3.3) holds. Let $f \in \mathcal{D}_r$. Hence f has an analytic extension to $r\mathbb{D}$ and $|f(z)|^p$ is integrable on $r\mathbb{T}$, that is, $f \in H_m^p(r\mathbb{T})$. By (3.2), $f \in \text{Dom}(T_\Omega)$ and hence $\Omega f = g + \rho$ for some $g \in L_m^p$ and $\rho \in \text{Rat}_0^m(\mathbb{T})$. Write $g = g_+ + g_-$ with $g_+ \in H_m^p$ and $g_- \in K_m^p$, so that $T_\Omega f = g_+$. Since f is analytic on $r\mathbb{D}$ and Ω and ρ are rational with no poles in $\mathfrak{A}_r \setminus \mathbb{T}$, it follows that $g = \Omega f - \rho$ must be analytic in $\mathfrak{A}_r \setminus \mathbb{T}$ with the poles on $\mathbb{T}$ all having finite multiplicity. However, $g \in L_m^p$, and hence cannot have poles of finite multiplicity on $\mathbb{T}$. Thus g is analytic in $\mathfrak{A}_r$. Hence, g_+ is analytic on $r\mathbb{D}$ and g_- on $\mathbb{C} \setminus \overline{r^{-1}\mathbb{D}}$. Since $r < r_0$, by an argument similar to that in the first part of the proof, it follows that $g_+ \in H_m^p(r\mathbb{T})$ and $g_- \in K_m^p(r^{-1}\mathbb{T})$. This implies that $g_{+,r}(z) = g_+(rz)$ and $g_{-,r}(z) = g_-(rz)$ define functions in H_m^p and K_m^p , respectively. In particular, $g_+ \in \mathcal{D}_r$ and it follows that T_Ω maps $\mathcal{D}_r$ into $\mathcal{D}_r$. Moreover, ρ is a rational matrix function with poles only in $\mathbb{T}$, so that ρ_r only has poles inside $\mathbb{D}$ and we obtain that $\rho_r \in K_m^p$. Note further that on $\mathbb{T}$

$$\begin{aligned}\Omega_r(z)f_r(z) &= \Omega(rz)f(rz) = g(rz) + \rho(rz) = g_r(z) + \rho_r(z) \\ &= g_{+,r}(z) + g_{-,r}(z) + \rho_r(z).\end{aligned}$$

Therefore, we have

$$T_{\Omega_r}f_r = \mathbb{P}(\Omega_rf_r) = \mathbb{P}(g_{+,r} + g_{-,r} + \rho_r) = g_{+,r} = (T_{\Omega}f)_r.$$

Finally, we show that $T_{\Omega}^{-1}(\mathcal{D}_r) \subset \mathcal{D}_r$. Since $0 \in \mathcal{D}_r$, this proves in particular that $\text{Ker } T_{\Omega} \subset \mathcal{D}_r$ and hence the first inclusion of (3.2). To prove the inclusion, we require the Wiener-Hopf type factorization from [11, Theorems 1.1 and 1.2], namely Ω can be factored as

$$\Omega(z) = \Omega_{-}(z)\Xi(z)\Omega_{+}(z), \quad \text{with } \Xi(z) = z^{-k}\Omega_{\circ}(z)P_0(z) \quad (3.4)$$

for some integer $k \geq 0$, $\Omega_{+}, \Omega_{\circ}, \Omega_{-} \in \text{Rat}^{m \times m}$ and $P_0 \in \mathcal{P}^{m \times m}$ such that Ω_{-} and Ω_{-}^{-1} are both minus functions (i.e., no poles outside $\mathbb{D}$), Ω_{+} and Ω_{+}^{-1} are both plus functions (i.e., no poles in $\overline{\mathbb{D}}$), $\Omega_{\circ} = \text{Diag}_{j=1}^m(\phi_j)$ with $\phi_j \in \text{Rat}(\mathbb{T})$ having no zeroes and having poles only on $\mathbb{T}$ (in [11, Theorem 1.1], $\phi_j \in \text{Rat}$ can have zeroes on $\mathbb{T}$, but this cannot occur since Ω has no zeroes on $\mathbb{T}$), and P_0 a lower triangular polynomial with $\det(P_0(z)) = z^N$ for some integer $N \geq 0$. It then follows from Theorem 1.3 in [11] that

$$T_{\Omega} = T_{\Omega_{-}}T_{\Xi}T_{\Omega_{+}} \quad \text{and} \quad T_{\Omega_{-}}^{-1} = T_{\Omega_{-}^{-1}}, \quad T_{\Omega_{+}}^{-1} = T_{\Omega_{+}^{-1}}.$$

To show that $T_{\Omega}^{-1}(\mathcal{D}_r) \subset \mathcal{D}_r$, it suffices to show that $T_{\Xi}^{-1}(\mathcal{D}_r) \subset \mathcal{D}_r$, $T_{\Omega_{+}}^{-1}(\mathcal{D}_r) \subset \mathcal{D}_r$ and $T_{\Omega_{-}}^{-1}(\mathcal{D}_r) \subset \mathcal{D}_r$. The latter two inclusions follow from the fact that $T_{\Omega_{-}}$ and $T_{\Omega_{+}}$ are invertible with inverses $T_{\Omega_{-}}^{-1} = T_{\Omega_{-}^{-1}}$ and $T_{\Omega_{+}}^{-1} = T_{\Omega_{+}^{-1}}$ along with the argument from the previous paragraph applied to $T_{\Omega_{-}^{-1}}$ and $T_{\Omega_{+}^{-1}}$ showing that $\mathcal{D}_r$ is an invariant subspace for these two operators; for the latter, note that Ω_{-} and Ω_{+} do not have zeroes and poles on the annulus $\mathfrak{A}_r$. Hence it remains to show that $T_{\Xi}^{-1}(\mathcal{D}_r) \subset \mathcal{D}_r$.

Let $f \in \text{Dom}(T_{\Xi})$ such that $T_{\Xi}f \in \mathcal{D}_r$. Since $f \in \text{Dom}(T_{\Xi})$, we can apply Lemma 2.1 from [12] to conclude that $\Xi(z)f(z) = z^{-k}h(z) + \eta(z)$ with $h \in H_m^p$ and $\eta = (\eta_1, \dots, \eta_m)^T \in \text{Rat}_0^m(\mathbb{T})$ of the form $\eta_j = r_j/q_j \in \text{Rat}_0(\mathbb{T})$ with q_j the denominator of the j -th diagonal element of Ω_0 . Write $z^{-k}h(z) = h_{-}(z) + h_{+}(z)$ with $h_{+} \in H_m^p$ and $h_{-} \in K_m^p$. It is clear that h_{-} is analytic on $\mathbb{C} \setminus \{0\}$. Moreover, $h_{+} = T_{\Xi}f$, so that $h_{+} \in \mathcal{D}_r$, by assumption. Since $\det \Xi \neq 0$, we have $f = \Xi^{-1}h_{+} + \Xi^{-1}h_{-} + \Xi^{-1}\eta$. Note that Ξ^{-1} has no poles on the annulus $\mathfrak{A}_r$ and no zeroes on $\mathfrak{A}_r \setminus \mathbb{T}$. Since Ξ^{-1} , f , h_{+} and h_{-} don't have poles on $\mathbb{T}$, neither can $\Xi^{-1}\eta$. It follows that $\Xi^{-1}\eta$, $\Xi^{-1}h_{+}$ and $\Xi^{-1}h_{-}$ are all analytic on $\mathfrak{A}_r$. Therefore, f is analytic on $\mathfrak{A}_r$. However, $f \in H_m^p$, so that f in fact is analytic on $r\mathbb{T}$. Using that Ξ^{-1} is rational, $g_{+} \in \mathcal{D}_r$, g_{-} and ρ are continuous on $r\mathbb{T}$ it follows that f is p -integrable on $r\mathbb{T}$, and hence $f \in \mathcal{D}_r$. $\square$

Lemma 3.2. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$. Then T_{Ω} is Fredholm if and only if Ω has no zeroes on $\mathbb{T}$. Assume T_{Ω} is Fredholm and let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$. Then T_{Ω_r} is bounded and Fredholm for each $1 < r < r_0$.*

Proof. For $1 < r < r_0$, by definition of r_0 it is clear that Ω_r has no poles and no zeroes on $\mathbb{T}$, so that T_{Ω_r} is bounded and Fredholm. For T_{Ω} the result is not included in [11] but follows from the results proved there. Indeed, consider a Wiener-Hopf type decomposition of Ω as in the proof of Proposition 3.1, e.g., as in (3.4). Since Ω_{+}^{-1} is a plus function and Ω_{-}^{-1} is a minus function, they do not have poles on $\mathbb{T}$. Also, since $\det P_0(z) = z^N$ for some integer $N \geq 0$, P_0^{-1} as a function in $\text{Rat}^{m \times m}$ can only have a pole at 0, so that P_0 also does not have zeroes on $\mathbb{T}$. This shows that the zeroes of Ω on $\mathbb{T}$ correspond to the zeroes of $\Omega_{\circ}$. Since $\Omega_{\circ}$ is a diagonal matrix function, its zeroes correspond to the zeroes of its diagonal elements, which are all on $\mathbb{T}$, by construction. Since Ω has no zeroes on $\mathbb{T}$, this implies that the numerators of the diagonal

elements of Ω_o are constant, assuming the numerators and denominators are co-prime. The statement for T_Ω now follows from the fact that T_Ω is Fredholm if and only if the numerators (assuming co-primeness) in Ω_o are constant, according to Theorem 1.4 in [11]. $\square$

As a consequence of Proposition 3.1, it is easy to show that the dimensions of the kernels of T_Ω and T_{Ω_r} are the same.

Corollary 3.3. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and assume that Ω has no zeroes on $\mathbb{T}$. Let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$. For each $1 < r < r_0$ we have*

$$\text{Ker } T_\Omega = \{f_{1/r} : f \in \text{Ker } T_{\Omega_r}\} \quad \text{and} \quad \text{Ker } T_{\Omega_r} = \{f_r : f \in \text{Ker } T_\Omega\}.$$

In particular, we have $\dim \text{Ker } T_\Omega = \dim \text{Ker } T_{\Omega_r}$.

Proof. Note that the formula for $\text{Ker } T_{\Omega_r}$ makes sense, since $\text{Ker } T_\Omega \subset \mathcal{D}_r$. Since the map $f \mapsto f_r$ defines a bijection from $\mathcal{D}_r$ onto H_m^p , with inverse map $h \mapsto h_{1/r}$, it suffices to prove one of the two formulas. From Proposition 3.1 it follows that

$$f \in \text{Ker } T_\Omega \subset \mathcal{D}_r \iff 0 = T_\Omega f \iff 0 = (T_\Omega f)_r = T_{\Omega_r} f_r.$$

This proves the formula for $\text{Ker } T_{\Omega_r}$. $\square$

With a bit more work we can prove a similar result for the codimensions of the ranges of T_Ω and T_{Ω_r} .

Corollary 3.4. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and assume that Ω has no zeroes on $\mathbb{T}$. Let $r_0 > 1$ be such that Ω has no zeroes in the annulus $\mathfrak{A}_{r_0}$ and no poles in $\mathfrak{A}_{r_0} \setminus \mathbb{T}$. Let $1 < r < r_0$ and let $\mathcal{X}$ be a complement of $\text{Ran } T_{\Omega_r}$ in H_m^p . Then*

$$\mathcal{X}_{1/r} := \{h_{1/r} : h \in \mathcal{X}\}$$

is a complement of $\text{Ran } T_\Omega$. In particular, we have

$$\text{codim } \text{Ran } T_\Omega = \text{codim } \text{Ran } T_{\Omega_r}.$$

Proof. Note that by assumption T_Ω and T_{Ω_r} are both Fredholm. Hence they have closed ranges and $\mathcal{X}$ is finite dimensional. By Proposition 3.1 we have that

$$\{g_r : g \in T_\Omega(\mathcal{D}_r)\} = \text{Ran } T_{\Omega_r},$$

and thus

$$T_\Omega(\mathcal{D}_r) = (\text{Ran } T_{\Omega_r})_{1/r} := \{g_{1/r} : g \in \text{Ran } T_{\Omega_r}\}.$$

Since $\text{Ran } T_{\Omega_r} + \mathcal{X}$ is a direct sum, the same is true for $(\text{Ran } T_{\Omega_r})_{1/r} + \mathcal{X}_{1/r} = T_\Omega(\mathcal{D}_r) + \mathcal{X}_{1/r}$, since $h \in (\text{Ran } T_{\Omega_r})_{1/r} \cap \mathcal{X}_{1/r}$ implies that $h \in \mathcal{D}_r$ and $h_r \in \text{Ran } T_{\Omega_r} \cap \mathcal{X} = \{0\}$, so that $h = 0$. Also, since $\text{Ran } T_{\Omega_r} + \mathcal{X} = H_m^p$, we have that $T_\Omega(\mathcal{D}_r) + \mathcal{X}_{1/r} = \mathcal{D}_r$. We claim that $\text{Ran } T_\Omega + \mathcal{X}_{1/r}$ is also a direct sum. Indeed, let $h \in \text{Ran } T_\Omega \cap \mathcal{X}_{1/r}$. Since $h \in \text{Ran } T_\Omega$, we have $h = T_\Omega f$ for some $f \in \text{Dom}(T_\Omega)$. Moreover, we have $h \in \mathcal{D}_r$, because $\mathcal{X}_{1/r} \subset \mathcal{D}_r$. But then Proposition 3.1 implies that also $f \in \mathcal{D}_r$. Hence $h \in T_\Omega(\mathcal{D}_r) \cap \mathcal{X}_{1/r} = \{0\}$. Finally, note that

$$\mathcal{D}_r = T_\Omega(\mathcal{D}_r) + \mathcal{X}_{1/r} \subset \text{Ran } T_\Omega + \mathcal{X}_{1/r} \subset H_m^p,$$

and that $\text{Ran } T_\Omega + \mathcal{X}_{1/r}$ is closed, since $\text{Ran } T_\Omega$ and $\mathcal{X}_{1/r}$ are both closed and $\mathcal{X}_{1/r}$ finite dimensional, using [5, Proposition III.4.3]. The fact that $\mathcal{D}_r$ is dense now implies that $\text{Ran } T_\Omega + \mathcal{X}_{1/r} = H_m^p$ is a direct sum decomposition of H_m^p . $\square$

Proof of Theorem 1.2. The claims follow directly by combining the results of Lemma 3.2 and Corollaries 3.3 and 3.4. $\square$

4. Invertibility of T_Ω and the formula for T_Ω^{-1}

In this section we prove Theorem 1.3 and Proposition 1.4. That invertibility of T_Ω corresponds to the existence of a solution to the Riccati equation (1.3) so that $A_\circ$ and $\alpha_\circ$ in (1.4) are stable, and hence the pseudo-canonical factorization of Ω , follows easily from the results of the previous sections. To obtain the formula for T_Ω and its block matrix representation in terms of the realization (1.2) requires more work. We shall first present the operator factorization of T_Ω corresponding to the pseudo-canonical factorization.

Lemma 4.1. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ and no zeroes on $\mathbb{T}$ be given by the minimal realization (1.2) with A stable and α semi-stable. Assume that Ω admits a pseudo-canonical factorization $\Omega = \Psi\Theta$ as in item (ii) of Theorem 1.1, with Θ and Ψ as in (1.6). Then T_Ω is given by*

$$T_\Omega = T_\Psi T_\Theta.$$

In particular, T_Θ is bounded and has a bounded inverse $T_\Theta^{-1} = T_{\Theta^{-1}}$. Moreover, T_Ψ admits an upper triangular block Toeplitz matrix representation with respect to the standard basis of H_m^p given by

$$[T_\Psi]_{i,j} = 0 \text{ if } j < i, \quad [T_\Psi]_{i,j} = \delta \text{ if } j = i, \quad [T_\Psi]_{i,j} = \gamma \alpha^{j-i-1} \beta_\circ \text{ if } j > i. \quad (4.1)$$

Proof. Since A and $A_\circ$ are both stable, it follows that Θ and Θ^{-1} have no poles inside the closed unit disc $\overline{\mathbb{D}}$, that is, both are plus functions. It then follows from Lemma 6.1 in [11] (with $\Omega = \Psi$ and $V = \Theta$) that $T_\Omega = T_\Psi T_\Theta$. It is then also clear that T_Θ is bounded with bounded inverse $T_\Theta^{-1} = T_{\Theta^{-1}}$. Hence, it remains to determine the block matrix representation of T_Ψ . For this purpose, we compute $T_\Psi z^n$. That should produce a polynomial, and the coefficients of that polynomial, augmented with zeroes, give the n -th block column of the block matrix representation of T_Ψ . By successive applications of the formula

$$(zI - \alpha)^{-1} = z^{-1}I + z^{-1}\alpha(zI - \alpha)^{-1}$$

we see that

$$\begin{aligned} \Psi(z)z^n &= \left(\delta + \sum_{j=0}^{n-1} z^{-j-1} \gamma \alpha^j \beta_\circ + z^{-n} \gamma \alpha^n (zI - \alpha)^{-1} \beta_\circ \right) z^n \\ &= \delta z^n + \sum_{j=0}^{n-1} z^{n-j-1} \gamma \alpha^j \beta_\circ + \gamma \alpha^n (zI - \alpha)^{-1} \beta_\circ. \end{aligned}$$

Now, because α has all its eigenvalues in the closed unit disc $\overline{\mathbb{D}}$, the function $\gamma \alpha^n (zI - \alpha)^{-1} \beta_\circ$ can be written as the sum of a function in $K_{m \times m}^p$ and a function in $\text{Rat}_0^{m \times m}(\mathbb{T})$. Thus $T_\Psi z^n = \delta z^n + \sum_{j=0}^{n-1} z^{n-j-1} \gamma \alpha^j \beta_\circ$. This shows that the block matrix representation of T_Ψ is indeed given by (4.1). $\square$

Proof of Theorem 1.3. First assume T_Ω is invertible. By Theorem 1.2, it follows that T_{Ω_r} is invertible for all $1 < r < r_0$, with r_0 as in Theorem 1.2. It then follows from Proposition 2.1 that a solution Q to the Riccati equation (1.3) exists, and reasoning as in the proof of Theorem 1.1 it follows that for this solution Q the matrices A_o and α_o in (1.4) are stable.

Conversely, if the Riccati equation (1.3) has a solution Q such that A_o and α_o are stable, then Theorem 1.1 provides a pseudo-canonical factorization of Ω , and, due to the stability of A_o and α_o , this factorization extends to a canonical factorization of Ω_r for $r > 1$ small enough. It then follows from Proposition 2.1 that T_{Ω_r} is invertible, and, consequently, that T_Ω is invertible, by Theorem 1.2.

From the factorization $T_\Omega = T_\Psi T_\Theta$ and the boundedness and boundedly invertibility of T_Θ , obtained in Lemma 4.1, it follows that invertibility of T_Ω corresponds to invertibility of T_Ψ and, moreover, $T_\Omega^{-1} = T_\Theta^{-1} T_\Psi^{-1} = T_{\Theta^{-1}} T_\Psi^{-1}$. Since Θ , Θ^{-1} and Ψ^{-1} have no poles on $\mathbb{T}$, the Toeplitz operators T_Θ , $T_{\Theta^{-1}}$ and $T_{\Psi^{-1}}$ are bounded and their block matrix representations are well understood. It remains to show that the block matrix representation of T_Ψ^{-1} and $T_{\Psi^{-1}}$ are the same, as this would prove that these bounded operators coincide on the subspace of polynomials $\mathcal{P}^m$ and equality would follow from their boundedness and the denseness of $\mathcal{P}^m$ in H_m^p . Indeed, once it is proved that $T_\Psi^{-1} = T_{\Psi^{-1}}$, then the block matrix representation of T_Ω^{-1} in (1.11) follows directly from $T_\Omega^{-1} = T_{\Theta^{-1}} T_{\Psi^{-1}}$ and the block matrix representations of $T_{\Theta^{-1}}$ and $T_{\Psi^{-1}}$. Hence, we need to show that the matrix entries with respect to the standard (block) basis of H_m^p of $T_{\Psi^{-1}} T_\Psi$ and $T_\Psi T_{\Psi^{-1}}$ are I_m on the diagonal and 0 elsewhere.

The block matrix representation of T_Ψ , in terms of the realization (1.2), is given by (4.1). The realization formula of Ψ^{-1} in item (ii) of Theorem 1.1, together with the stability of α_o shows that the block matrix representation of $T_{\Psi^{-1}}$ is an upper block triangular Toeplitz matrix that is determined by its first block row, which is given by

$$[\delta^{-1} \quad -\delta^{-1}\gamma\beta_o\delta^{-1} \quad -\delta^{-1}\gamma\alpha_o\beta_o\delta^{-1} \quad -\delta^{-1}\gamma\alpha_o^2\beta_o\delta^{-1} \quad \dots].$$

We first consider the block matrix representation of $T_{\Psi^{-1}} T_\Psi$. It is required to show that the (i, j) -th block entry $[T_{\Psi^{-1}} T_\Psi]_{ij}$ works out as

$$[T_{\Psi^{-1}} T_\Psi]_{ij} = 0 \text{ if } j < i, \quad [T_{\Psi^{-1}} T_\Psi]_{ij} = I_m \text{ if } j = i, \quad [T_{\Psi^{-1}} T_\Psi]_{ij} = 0 \text{ if } j > i.$$

The case where $j < i$ follows directly because the matrix representations of $T_{\Psi^{-1}}$ and T_Ψ are both block upper triangular, and the case $j = i$ follows because the block diagonal elements are each others inverses. Hence, it remains to consider the case where $j > i$. For this purpose, notice that

$$\alpha - \alpha_o = \beta_o \delta^{-1} \gamma \quad \text{and} \quad A - A_o = B D^{-1} C_o. \quad (4.2)$$

For $j > i$ we have

$$\begin{aligned} [T_{\Psi^{-1}} T_\Psi]_{ij} &= -\delta(\delta^{-1}\gamma\alpha_o^{j-i-1}\beta_o\delta^{-1}) - \sum_{k=0}^{j-i-2} \gamma\alpha^k\beta_o\delta^{-1}\gamma\alpha_o^{j-i-2-k}\beta_o\delta^{-1} + \\ &\quad + \gamma\alpha^{j-i-1}\beta_o\delta^{-1}. \end{aligned}$$

If $j = i + 1$, then the summation in the middle term of the right-hand side is empty, and it is easy to see that the right-hand side collapses to 0 by a direct application of the first identity in (4.2). For $j > i + 1$, using the first identity in (4.2), we see that

$$\sum_{k=0}^{j-i-2} \gamma\alpha^k\beta_o\delta^{-1}\gamma\alpha_o^{j-i-2-k}\beta_o\delta^{-1} = \sum_{k=0}^{j-i-2} \gamma\alpha^k(\alpha - \alpha_o)\alpha_o^{j-i-2-k}\beta_o\delta^{-1} =$$

$$\begin{aligned}
&= \sum_{k=0}^{j-i-2} \gamma \alpha^{k+1} \alpha_{\circ}^{j-i-2-k} \beta_{\circ} \delta^{-1} - \sum_{k=0}^{j-i-2} \gamma \alpha^k \alpha_{\circ}^{j-i-1-k} \beta_{\circ} \delta^{-1} \\
&= \sum_{k=1}^{j-i-1} \gamma \alpha^k \alpha_{\circ}^{j-i-1-k} \beta_{\circ} \delta^{-1} - \sum_{k=0}^{j-i-2} \gamma \alpha^k \alpha_{\circ}^{j-i-1-k} \beta_{\circ} \delta^{-1}.
\end{aligned} \tag{4.3}$$

Inserting this formula back into the formula for $[T_{\Psi^{-1}} T_{\Psi}]_{ij}$, it follows that in the first summation in (4.3) the term $k = 0$ is added, while in the second summation the term $k = j - i - 1$ is added, so that $[T_{\Psi^{-1}} T_{\Psi}]_{ij} = 0$, as claimed. A similar computation, using the second identity of (4.2), shows that the block matrix representation of $T_{\Psi} T_{\Psi^{-1}}$ also corresponds to the block matrix representation of $I_{H_m^p}$. $\square$

Finally, we turn to the proof of the last result in the introduction. Proposition 1.4 is stated for $p = 2$, but the lemma which we require for the proof also works for $p \neq 2$. For $r > 1$, define the invertible linear map

$$\Upsilon : \mathcal{D}_r \rightarrow H_m^p, \quad \Upsilon : f \mapsto f_r, \quad \text{with inverse} \quad \Upsilon^{-1} : H_m^p \rightarrow H_m^p, \quad \Upsilon^{-1} : f \mapsto f_{r^{-1}}.$$

Lemma 4.2. *Let $\Omega \in \text{Rat}^{m \times m}$ with $\det \Omega \neq 0$ be given by the minimal realization (1.2) with A stable and α semi-stable. Define r_0 as in Theorem 1.2, $\mathcal{O}_{C,A}$ as in (1.12) and $\mathcal{C}_{\alpha,\beta}$ as in (1.13), and define $\mathcal{O}_{C_r,A_r}$ and $\mathcal{C}_{\alpha_r,\beta_r}$ analogously, where $C_r, A_r, \alpha_r, \beta_r$ are defined as in (2.3) and $1 < r < r_0$. Then $\mathcal{O}_{C,A}$, $\mathcal{O}_{C_r,A_r}$ and $\mathcal{C}_{\alpha_r,\beta_r}$ are bounded, the range of $\mathcal{O}_{C,A}$ is contained in $\mathcal{D}_r$ and $\mathcal{D}_r$ is contained in $\text{Dom}(\mathcal{C}_{\alpha,\beta})$. Moreover, we have*

$$r\mathcal{C}_{\alpha_r,\beta_r} = \mathcal{C}_{\alpha,\beta}\Upsilon^{-1}, \quad \mathcal{O}_{C_r,A_r} = r\Upsilon\mathcal{O}_{C,A} \quad \text{and} \quad T_{\Omega_r} = \Upsilon T_{\Omega}\Upsilon^{-1}.$$

Furthermore, in case T_{Ω} is invertible, then T_{Ω_r} is invertible as well and $T_{\Omega_r}^{-1} = \Upsilon T_{\Omega}^{-1}\Upsilon^{-1}$.

Proof. Since the realization (1.2) of Ω is minimal, it follows from the definition of r_0 that $A_r = rA$ is still stable. Since $r > 1$ and α is semi-stable, $\alpha_r = r^{-1}\alpha$ is stable. This implies the boundedness of $\mathcal{O}_{C,A}$, $\mathcal{O}_{C_r,A_r}$ and $\mathcal{C}_{\alpha_r,\beta_r}$, as well as the fact that $\mathcal{O}_{C,A}$ maps into $\mathcal{D}_r$. The identity $\mathcal{O}_{C_r,A_r} = r\Upsilon\mathcal{O}_{C,A}$ is straightforward from the definitions.

To show that $\mathcal{D}_r$ is in the domain of $\mathcal{C}_{\alpha,\beta}$ and that $r\mathcal{C}_{\alpha_r,\beta_r} = \mathcal{C}_{\alpha,\beta}\Upsilon^{-1}$ holds on H_m^p , let $f(z) = \sum_{k=0}^{\infty} f_k z^k \in H_m^p$, then f is integrable in $\mathbb{T}$ and so is the rational matrix function $(zI - \alpha_r)^{-1}\beta_r = \sum_{k=0}^{\infty} \alpha_r^k \beta_r z^k$. It follows from Lemma 1.5 on page 81 of [19] that

$$\begin{aligned}
\mathcal{C}_{\alpha_r,\beta_r} f &= \sum_{k=0}^{\infty} \alpha_r^k \beta_r f_k = \sum_{k=0}^{\infty} r^{-k-1} \alpha^k \beta f_k = r^{-1} \sum_{k=0}^{\infty} \alpha^k \beta r^{-k} f_k \\
&= r^{-1} \mathcal{C}_{\alpha,\beta} f_{1/r} = r^{-1} \mathcal{C}_{\alpha,\beta} \Upsilon^{-1} f.
\end{aligned}$$

The above computation shows in particular that $\Upsilon^{-1}f$ is in $\text{Dom}(\mathcal{C}_{\alpha,\beta})$ for each $f \in H_m^p$ so that $\mathcal{D}_r \subset \text{Dom}(\mathcal{C}_{\alpha,\beta})$.

The relation between T_{Ω} and T_{Ω_r} in (3.3) implies that $\Upsilon T_{\Omega}|_{\mathcal{D}_r} = T_{\Omega_r}\Upsilon$. Since the range of Υ^{-1} is equal to $\mathcal{D}_r$ and $\Upsilon\Upsilon^{-1} = I$, we have $T_{\Omega_r} = \Upsilon T_{\Omega}\Upsilon^{-1}$, as claimed.

Assume T_{Ω} is invertible. By Theorem 1.2, also T_{Ω_r} is invertible. By Proposition 3.1, T_{Ω} maps $\mathcal{D}_r$ into $\mathcal{D}_r$ and T_{Ω}^{-1} also maps $\mathcal{D}_r$ into $\mathcal{D}_r$. In particular, the operator $\Upsilon T_{\Omega}^{-1}\Upsilon^{-1}$ is a well-defined linear map on H_m^p , which is closed by [16, Problem III.5.7], and hence bounded by the closed graph theorem. To see that $T_{\Omega_r}^{-1} = \Upsilon T_{\Omega}^{-1}\Upsilon^{-1}$, note that if $f \in H_m^p$, then $\Upsilon^{-1}f$ and $T_{\Omega}\Upsilon^{-1}f$ are in $\mathcal{D}_r$ so that

$$\begin{aligned}
(\Upsilon T_{\Omega}^{-1}\Upsilon^{-1})(\Upsilon T_{\Omega}\Upsilon^{-1})f &= \Upsilon T_{\Omega}^{-1}\Upsilon^{-1}\Upsilon(T_{\Omega}\Upsilon^{-1}f) \\
&= \Upsilon T_{\Omega}^{-1}T_{\Omega}\Upsilon^{-1}f = \Upsilon\Upsilon^{-1}f = f.
\end{aligned}$$

Hence $(\Upsilon T_{\Omega}^{-1} \Upsilon^{-1})(\Upsilon T_{\Omega} \Upsilon^{-1}) = I$. Similarly one obtains $(\Upsilon T_{\Omega} \Upsilon^{-1})(\Upsilon T_{\Omega}^{-1} \Upsilon^{-1}) = I$. Hence, $T_{\Omega_r}^{-1} = \Upsilon T_{\Omega}^{-1} \Upsilon^{-1}$. $\square$

With the identities of the previous lemma, we can now prove Proposition 1.4.

Proof of Proposition 1.4. Set $p = 2$ and identify H_m^2 and ℓ_m^2 in the usual way. According to [6], the matrix Q in Proposition 2.1 is given by $Q = \mathcal{C}_{\alpha_r, \beta_r} T_{\Omega_r}^{-1} \mathcal{O}_{C_r, A_r}$. It was already noted in Proposition 2.1 that Q is independent of the value of $1 < r < r_0$. That also $Q = \mathcal{C}_{\alpha, \beta} T_{\Omega}^{-1} \mathcal{O}_{C, A}$ now follows directly from the identities derived in Lemma 4.2. $\square$

Declaration of competing interest

There are no competing interests.

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