On the Double Frobenius Groups and their Characters

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Abstract

The Double Frobenius group is the result of the action of a Frobenius group $\overline{H} = NH$, with kernel N and complement H, on a finite group G. If the action of $\overline{H}$ on G is such that, N acts fixed point free on G and GN is also a Frobenius group with kernel G and complement N, then $\overline{G} = GNH = G{:}(N{:}H) = (G{:}N){:}H$ is a double Frobenius group. In this study we briefly describe the structure of the double Frobenius group and then construct in general two double Frobenius groups which have the form $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$, where n is a prime such that $2^n - 1$ is a Mersenne prime and $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$, where $2 \leq r \in \mathbb{N}$ respectively. We then proceed to analyse the two double Frobenius groups mentioned above, calculating the conjugacy classes, Fischer matrices and character table of the groups. The study is concluded by demonstrating these calculations of the conjugacy classes, Fischer matrices and character tables of two examples of each type of double Frobenius group, namely, $2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$ and $2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$ for the type $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ with $n = 3$ and $n = 5$ respectively, and $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ and $2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$ for the type $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ with $r = 2$ and $r = 3$ respectively.

Preface

The work covered in this dissertation was done by the author under the supervision of Prof. Jamshid Moori, School of Mathematical Sciences, University of North West, Mahikeng (2013-2017). The use of the work of others however has been duly acknowledged throughout the dissertation.

.... 21/JUNE/2017

Signature(Student) Date

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Signature(Supervisor) Date

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Dedication

DEDICATED TO MY PARENTS SIGA AND SIVAGAMI PERUMAL.
YOU ARE ALWAYS IN MY THOUGHTS.

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List of Notation

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integer numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{F}$	a field
$\mathbb{F}^*$	multiplicative group of $\mathbb{F}$
$\mathbb{F}_q$	Galois field of q elements
$U(\mathbb{Z}_n)$	the group of units of $\mathbb{Z}_n$
V	vector space
$\dim$	dimension of a vector space
det	determinant of a matrix
tr	trace of a matrix
G	a finite group
$e, 1_G$	identity of G
$\lvert G\rvert$	order of G
$o(g)$	order of $g \in G$
$\cong$	isomorphism of groups
$H \leq G$	H is a subgroup of G
$[G : H]$	index of H in G
$N \trianglelefteq G$	N is a normal subgroup of G
$N \times H$, $\bigotimes$	direct product of groups
$N{:}H$	split extension of N by H
G/N	quotient group
$[g]$, C_g	conjugacy class of g in G
$C_G(g)$	centralizer of $g \in G$
G_x, $\mathrm{Stab}_G(x)$	stabilizer of $x \in X$ when G acts on X
x^G	orbit of $x \in X$
$\lvert \mathrm{Fix}(g)\rvert$	number of elements in a set X fixed by $g \in G$ under the group action
$\mathrm{Aut}(G)$	automorphism group of G

$\mathrm{Holo}(G)$	holomorph of G
$[x, y]$	commutator of x and y in G
G'	derived or commutator subgroup of G
$Z(G)$	center of G
D_{2n}	dihedral group consisting of $2n$ elements
$\mathrm{Syl}_p(G)$	set of Sylow p−subgroups of G
$\mathbb{Z}_n$	group $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ under addition modulo n
S_n	symmetric group of n objects
A_n	alternating group of n objects
$GL(n, \mathbb{F})$	general linear group over a field $\mathbb{F}$
$GL(n, q)$	finite general linear group over $\mathbb{F}_q$
$SL(n, \mathbb{F})$	special linear group
$\mathrm{Aff}(n, \mathbb{F})$	affine group
$PSL(n, \mathbb{F})$	projective special linear group
$U_n(\mathbb{F})$	unitary group
χ	character of finite group
χ_ρ	character afforded by a representation ρ of G
1	trivial character
$\deg$	degree of a representation or character
$\mathrm{Irr}(G)$	set of ordinary irreducible characters of G
$\chi\uparrow_H^G$	character induced from subgroup H to G
$\chi\downarrow_H^G$	character restricted from a group G to its subgroup H
$\rho \uparrow G$	induced representation from subgroup H to group G
$\rho \downarrow G$	restriction of representation ρ of group G to subgroup H
$\chi(1_G)$	degree of character χ
$\overline{\chi(g)}$	conjugate of character value $\chi(g)$
$\widetilde{\chi}$	lift of character χ
$\widehat{\chi}$	character of factor group G/N
$\mathcal{C}(G)$	algebra of class functions of a group G

$\mathbb{F}[G]$	group algebra of a finite group G over a field $\mathbb{F}$
ϕ^g	conjugate class function/character
$I_G(\phi)$	inertia group of a character ϕ
$cf(H)$	set of class functions of group H
$\widehat{S}$	matrix of a representation S
$\langle\, ,\, \rangle$	inner product of class functions or group generated by two elements
$\otimes$	tensor product of representations
$\oplus, \bigoplus$	direct sum
x^g	action of g on x (gx or xg) when group G acts on set X
$x \sim y$	x is equivalent to y or x is conjugate to y
H^g	conjugate of H
$c(G)$	number of conjugacy classes of group G
$\phi(G)$	Frattini subgroup of a group G
$F(G)$	the Fitting subgroup of a finite group G
$\mathbb{F}_{p,q}$	group of order pq generated by p and q
$\langle x \rangle$	cyclic group generated by x
$Ker\phi$	kernel of a homomorphism ϕ
$Im\phi$	image of a function ϕ
(a, b)	greatest common divisor of a and b
$GK(G)$	prime graph of a finite group G
$\Gamma(G)$	commuting graph of a finite group G
$\omega(G)$	spectrum of a finite group G
$\pi(G) = \pi(\|G\|)$	the set of prime divisors of the order of a finite group G
$OC(G)$	the set of order components of the prime graph of a finite group G
$s(G)$	the number of connected components of the prime graph of a finite group

1

Introduction

All groups and the sets on which they act in this study are finite.

The automorphism group of a group G, denoted by $\mathrm{Aut}(G)$, is the set of all automorphisms of G under the group operation of composition. An automorphism ϕ of G is called *inner* if it is conjugation by some element of G, otherwise, it is *outer*. Finding the automorphism group of an arbitrary finite group G in general is not an easy task. However, if G is an elementary abelian group of order $q = p^n$, p a prime, then $\mathrm{Aut}(G) \cong GL(n, q)$ and if G is a cyclic group of order n, then $\mathrm{Aut}(G) \cong U(\mathbb{Z}_n)$, $U(\mathbb{Z}_n)$ being the group of units of $\mathbb{Z}_n$.

In this study we consider the case of a Frobenius group $\overline{H} = NH$ acting as a group of automorphisms on a group G. Here it is not a requirement that $\overline{H}$ be the automorphism group of G. In fact, in this study $\overline{H}$ is a subgroup of the automorphism group of G. In the Frobenius group $\overline{H} = NH$, N is the kernel and H is the complement. Chapter 3 of the thesis contains a detailed description of Frobenius groups. Much of the content in this chapter is from the Masters thesis of the author [32]. We give details of the structure, properties and characteristics of the Frobenius group and its characters. A good supply of examples is also included. The case where a Frobenius group $\overline{H} = NH$ acts by automorphisms on a group G has received some study in recent years. In this situation various properties (parameters) of G are found to be close to the corresponding properties of $C_G(H)$ and H. These properties include the *order*, *rank*, *exponent* and *nilpotency class* of G. These are described in [16]. Some results concerning the Fitting height and Fitting series of G are obtained by Khukhro in [17]. In [18], Khukhro and Makarenko generalize some results regarding the nilpotency class of G obtained by the authors and Shumyatsky in [16]. They also mention Lie rings with Frobenius group of automorphisms. Many of the studies and results in this regard were prompted by Mazurov's problem in the Kourovka Notebook [23]. Problem 17.72 in the Kourovka Notebook : Let AB be a Frobenius group with kernel A and complement B. Suppose that AB acts on a finite group G so that GA is also a Frobenius group with kernel G and complement A.

1. Is the nilpotency class of G bounded in terms of $|B|$ and the class of $C_G(B)$?

2. Is the exponent of G bounded in terms of $|B|$ and the exponent of $C_G(B)$?

We mention these results and state some theorems in Chapter 4 without going into details. We are interested in the case where a Frobenius group $\overline{H} = NH$ with kernel N and complement H acts as a group of automorphisms on a group G such that GN is also a Frobenius group. So suppose a Frobenius group $\overline{H} = NH$ with kernel N and complement H acts as a group of automorphisms on a group G such that the kernel N acts fixed point freely in the action of $\overline{H}$ on G, i.e. $C_G(N) = \{1_G\}$. If GN is also a Frobenius group with kernel G and complement N, then we say that $\overline{G} = GNH$ is a double Frobenius group. Sometimes we say 2-Frobenius group as opposed to double Frobenius group.

The group $\overline{G} = GNH$ is an example of a product of the groups G, N and H which are all subgroups of $\overline{G}$. Also, $\overline{G}$ can be represented as $\overline{G} = G{:}NH$ or $\overline{G} = GN{:}H$. Therefore, $\overline{G}$ is also an example of a split extension. In Chapter 4 of the thesis we define the double Frobenius group and describe some of its properties and structure. Also in Chapter 4 we describe the motivation for this study of the double Frobenius group. In recent years graphs associated with finite groups have received much attention. Some of these graphs are the *generating graph*, the *vanishing prime graph*, the *commuting graph*, the *Cayley graph*, the *character degree graph*, and the *prime graph* of a finite group G. Of these graphs, the *prime graph* or Gruenberg-Kegel graph has been the subject of most attention in interest and research. First mention of the prime graph of a finite group appears in unpublished manuscripts of Gruenberg and Kegel. All of the above mentioned graphs, in particular the prime graph of a group G and more recently the character degree graph are now being used to better understand the structure of the group G. Definitions and some terms and concepts are described in Chapter 4. If G is a finite group then we construct the prime graph of G as follows : the vertices of the graph are the primes dividing the order of G. Two vertices p and q are joined by an edge pq if and only if G contains an element of order pq. We denote the prime graph of a group G by $GK(G)$ or $\Gamma(G)$. The prime graph of a group may be connected or disconnected. Hence, the prime graph of the group may have one or more components. Both Frobenius and double Frobenius groups appear in the study of the prime graphs of finite groups. In Chapter 4 we state one of the key classification theorems (The Gruenberg-Kegel Theorem) of the prime graphs of finite groups with more than one component. The Gruenberg-Kegel Theorem, Theorem 4.4.1, proved in the paper by Williams [39], states that if G is a finite group with a prime graph having more than one component, then G must be one of the following three types: (i) a Frobenius group (ii) a double Frobenius group (iii) an extension of a nilpotent $\pi_1(G)$-group by a group A, where $L \trianglelefteq A \trianglelefteq Aut(L)$, L is a simple non-abelian group with $s(G) \leq s(L)$ ($s(G)$ and $s(L)$ here are the number of connected components (as described in Section 4.4) of G and L respectively), and A/L is a $\pi_1(G)$-group. The Gruenberg-Kegel Theorem implies the complete description of *solvable* groups with disconnected prime graphs. These are exactly Frobenius or double Frobenius groups, see Corollary 4.4.2. At the end of Chapter 4, we describe a method of Aleeva in [2] in which he uses the prime graph of a finite group (finite simple group) to construct two double Frobenius groups. We describe the constructions as Example 11 and Example 12 in Chapter 5.

Whenever we discover or define a "new" group, our first task is to search for examples of the group in the vast library of known groups. i.e. Symmetric groups, Alternating groups, Finite simple groups, p-groups, etc. We find that the symmetric group S_4 is an example of a double Frobenius group and it is also of the smallest order of a double Frobenius group that we can have. So S_3 and S_4 are the smallest Frobenius (see Example 3, Section 3.4 of Frobenius groups) and double Frobenius groups respectively. For the double Frobenius group S_4, we have that $S_4 = V_4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2) = (V_4{:}\mathbb{Z}_3){:}\mathbb{Z}_2$. Here we have the natural action of the Frobenius group $\overline{H} = NH = S_3 \cong SL(2,2) \cong GL(2,2)$, where $N \cong \mathbb{Z}_3$ and $H \cong \mathbb{Z}_2$, on the elementary abelian group V_4. Since $N \cong \mathbb{Z}_3$ acts fixed point freely on V_4 in the action of $\overline{H} \cong S_3$ on V_4, and $V_4{:}\mathbb{Z}_3 \cong A_4$ is a Frobenius group, $V_4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ is a double Frobenius group.

Further investigation leads us to believe that double Frobenius groups may be scarce amongst the library of known finite groups. This conclusion leads us to the subject of Chapter 5 - the construction of double Frobenius groups.

Not only is group construction a rewarding and interesting exercise, but it also adds to our stockpile of groups. In Chapter 5 we describe three methods of constructing double Frobenius groups. To construct a double Frobenius group we start by going back to our definition of the double Frobenius group, Definition 4.2.1. Our definition requires that we start from a Frobenius group. We therefore look for Frobenius groups amongst the finite groups. Frobenius groups appear frequently as maximal subgroups of the finite simple groups. However, we can actually do better. With certain conditions satisfied, Lemma 3.4.1 guarantees Frobenius subgroups in $PSL(n, q)$.

In Method 1 with $q = 2$ and $n \in \Omega$ where Ω is the set of primes such that $2^n - 1$ is a Mersenne prime, we construct a double Frobenius group which has the form $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$. In Method 2, we again turn to the finite simple groups for a Frobenius subgroup. This time we look inside $PSL(2, q)$ for the Frobenius group. In Corollary 2.2 of King's article [19], King shows that when q is even, two of the maximal subgroups of $PSL(2, q)$ are Dihedral groups of order $2(q-1)$ and $2(q+1)$. This gives us the Frobenius group we seek. The Dihedral group D_{2m} is Frobenius when m is odd, see example 1 in section 3.4. In Method 2 we construct double Frobenius groups of the form $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ for some $r \in \mathbb{N}, r \geq 2$.

The automorphism group of an elementary abelian group is the general linear group. Method 3 requires us finding a Frobenius subgroup inside the general linear group $GL(n, q)$. With the natural action of $GL(n, q)$ on the n dimensional vector space over a field of q elements, namely q^n, q a power of a prime p, we can construct a double Frobenius group of the form $q^n{:}\overline{H} = q^n{:}(N{:}H)$, where $\overline{H} = NH$ is a Frobenius subgroup of $GL(n, q)$ with kernel N and complement H. It should be noted that the constructions described in Method 1 and Method 2 are general constructions. That is, the construction described in Method 1 will generate a double Frobenius group for a particular value of $n \in \Omega$ and the construction described in Method 2 will generate a double Frobenius group for a chosen value of $2 \leq r \in \mathbb{N}$. The construction in Method 3 depends on the existence of or finding a Frobenius subgroup inside $GL(n, q)$. Chapter 5 concludes with brief descriptions of examples of the double Frobenius groups constructed. Detailed analysis of the general constructions follow in Chapter 6 and Chapter 7.

In Chapter 6 we analyse the double Frobenius group $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ constructed by Method 1. We describe the conjugacy classes of the group, the general form of the Fischer matrices of the group and finally the character table in general. Chapter 7 follows the same pattern as Chapter 6, this time we analyse the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ constructed in Method 2. The construction we describe in Method 3 depends on the existence of a Frobenius subgroup inside $GL(n, q)$. These constructions are described in Examples (7-10) of Chapter 5.

Chapter 8 is the examples chapter. In this chapter we analyse the examples we gave brief descriptions of in Chapter 5. Detailed analysis is done of the examples generated by Methods 1 and 2 (Examples 1-4). The analysis consists of the computation of the conjugacy classes, Fischer matrices and character tables of the groups. For the other examples, we give brief descriptions (some more detailed) of the double Frobenius groups constructed.

Throughout this study we refer to group theoretic concepts and results. Much of this content is elementary finite group theory. We include this in the Preliminaries which is Chapter 2 of the thesis. Also included in this chapter is the essentials of character theory and the theory of the Fischer matrices. We make reference to the content here in the construction of the Fischer matrices in Chapter 6 and 7.

We conclude this introduction by mentioning that the material in Chapter 5 regarding the construction of the different types of double Frobenius groups is original work. Also original is most of the material in Chapters 6 and 7 which deal with the construction of the Fischer matrices and character tables of the double Frobenius groups of the type $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ and $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$. The Preliminaries now follows.

2

Preliminaries

2.1 Introduction

The primary focus of this study is the **Double Frobenius** group. These groups are solvable. In this introduction we will present some definitions and results and where necessary proofs of some results on solvable groups and related ideas. Different authors use different descriptions in their definition of the double Frobenius group. We will define the double Frobenius group resulting from the action of one group on another. In particular the group acting is a Frobenius group. Therefore we will include a section on Frobenius groups and present in a fair amount of detail known results of Frobenius groups.

We begin however with some group theoretic terms and concepts and some results of finite group theory.

Definition 2.1.1. ***(Exponent)****([35]). The exponent of a group is the lowest common multiple of all the orders of the elements in the group.*

Definition 2.1.2. ***(Automorphism Group)****([35]). Let G be a group. Then the set of all isomorphisms of G onto G forms a group with respect to composition of maps. It is called the Automorphism group of G and is denoted by Aut G.*

Definition 2.1.3. ***(Inn G, Out G)****([35]). Let G be a group. To each $g \in G$ there is associated an automorphism τ_g of G, defined by conjugation*

$$\tau_g : x \mapsto gxg^{-1} \; \forall x \in G.$$

The automorphism τ_g of G is called the inner automorphism of G induced by g or conjugation of G by g. An automorphism which is not inner is called an outer automorphism of G. The set of all inner automorphisms of G is a group denoted by $\mathrm{Inn}\ G$. Clearly $\mathrm{Inn}\ G \trianglelefteq \mathrm{Aut}\ G$. An automorphism of G that is not inner is called outer; the quotient group $\mathrm{Out}\ G = \mathrm{Aut}\ G/\mathrm{Inn}\ G$ is called the outer automorphism group of G although its elements are not automorphisms.

Definition 2.1.4. ***(Invariant, Characteristic and Normal Subgroup)*** *([35]). Let* G *be a group and* $H \leq G$*. Let* A *be a non-empty set of automorphisms of* G*. We say that* H *is an* A−invariant *subgroup of* G *if*

$$h^{\alpha} \in H \quad \forall\, h \in H \text{ and } \forall\, \alpha \in A.$$

The subgroup H *is said to be characteristic in* G *if*

$$h^{\alpha} \in H \quad \forall\, h \in H \text{ and } \forall\, \alpha \in \text{Aut } G.$$

and H *is normal in* G *if*

$$h^{\alpha} \in H \quad \forall\, h \in H \text{ and } \forall\, \alpha \in \text{Inn } G.$$

Definition 2.1.5. ***(Centralizer, Normalizer of a Subgroup)*** *Let* G *be a group and* $H \leq G$*. Then*

$$C_G(H) = \{g \in G : gh = hg \ \ \forall\, h \in H\}$$

is the centralizer of H *in* G*. Also*

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

is the normalizer of H *in* G*.*

Theorem 2.1.1. *Let* H *be a subgroup of a group* G*. Then*

$$C_G(H) \trianglelefteq N_G(H) \text{ and } N_G(H)/C_G(H) \cong\leq \text{Aut } H.$$

PROOF:*If* $g \in N_G(H)$*, let* g^{τ} *denote the function* $h \mapsto ghg^{-1}$*. Then* g^{τ} *is an automorphism of* H*. Also,* $\tau : N_G(H) \to \text{Aut } H$ *is a homomorphism whose kernel is* $C_G(H)$*. The result now follows from the First Isomorphism Theorem.* ■

Definition 2.1.6. ***(Fixed point free Automorphism)*** *([22]). An automorphism* α *of a group is said to have a* ***fixed point*** g *in* G *if* $\alpha(g) = g$*. If* 1_G *is the only fixed point of* α*, then* α *is called* ***fixed point free*** *on* G*. A subgroup* Φ *of* Aut(G) *is said to be* ***fixed point free*** *on* G *if every element* φ *in* $\Phi \backslash \{1_G\}$ *is fixed point free.*

Definition 2.1.7. ***(Elementary Abelian Group)*** *An abelian group* A *is said to be elementary if there is a prime* p *such that* $a^p = 1$ *for every* $a \in A$*.*

Theorem 2.1.2. *Let* A *be an abelian group. The the following are equivalent:*

1. A *is elementary.*

2. *There is a prime* p *and a vector space* V *over* Z_p *such that* $A \cong (V, +)$*.*

PROOF: See [36, Theorem 7.40]. ■

Lemma 2.1.3. *[3]. Let* E *be a finite abelian group of exponent* p *a prime. For* $n \in \mathbb{N}$, $|E| = p^n$ *and* $\mathrm{Aut}\, E \cong GL(n, p)$.

PROOF: See Alperin [3]. ■

Proposition 2.1.4. *([22]). Let* Φ *be a fixed point free automorphism group on a group* G*, such that no non-identity element of* G *is fixed by* Φ*. Then* $|\Phi|$ *divides* $|G| - 1$.

PROOF: Since all automorphisms of Φ are fixed point free, the orbit of each $1_G \neq x \in G$ under Φ is of size $|\Phi|$. Thus $G \backslash \{1_G\}$ is partitioned into sets of size $|\Phi|$ and $|\Phi|$ is a divisor of $|G| - 1$. ■

Lemma 2.1.5. *[22]. Let* $(G, +)$ *be an elementary abelian group of order* p^n *for some prime* p*. There is a cyclic fixed point free automorphism group of order* k *on* G *if and only if* $k \mid p^n - 1$.

PROOF: See [22, Corollary 5.4]. ■

Proposition 2.1.6. *[22]. Let* Φ *be a fixed point free automorphism group on the additive group* $(N, +)$*. Then the semi-direct product* $G = \Phi : N$ *is a Frobenius group with complement* Φ *and kernel* N.

PROOF: See [22, Proposition 7.3]. ■

Lemma 2.1.7. ***(Dedekind Law)*** *Let* H*,* K *and* L *be subgroups of a group* G *with* $H \leq L$*. Then* $HK \cap L = H(K \cap L)$ *where we don't assume that* HK *or* $H(K \cap L)$ *is a subgroup.*

PROOF: See Rotman, page 37 [36]. ■.

Theorem 2.1.8. *Let* G *be a cyclic group. Then the automorphism group of* G *is abelian.*

PROOF: See Gorenstein, page 12 [8]. ■

Theorem 2.1.9. *An abelian group* G *is cyclic if and only if all its Sylow subgroups are cyclic.*

PROOF: See Gorenstein, page 9 [8]. ■

2.1.1 Solvable Groups

Double Frobenius groups are always solvable (proved later on). So in this first section we present some results on solvable groups and related ideas.

Definition 2.1.8. *(**Series**)([26]). A series of a group* G *is a finite sequence* $G_0, G_1, ..., G_n$ *of subgroups of* G *such that*

$$\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G. \quad (*)$$

The factor groups G_{i+1}/G_i *are called factors of (∗).The number of factors of order greater than 1 is called the length of (∗).*

Definition 2.1.9. *(**Normal Series**)([26]). A normal series of a group* G *is a finite sequence* $G_0, G_1, ..., G_n$ *of subgroups of* G *such that*

$$\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G. \quad (*)$$

and each $G_i \trianglelefteq G$.

Definition 2.1.10. *(**Composition Series**)([26]). A composition series is a series*

$$\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G. \quad (*)$$

such that G_i *is a maximal normal subgroup of* G_{i+1}.

Proposition 2.1.10. *If* $H \trianglelefteq G$, *then* H *is a maximal normal subgroup if and only if* G/H *is simple.*

PROOF: Easy and omitted. ■

Proposition 2.1.11. *A series is a composition series if and only if its factors are simple groups.*

PROOF: Follows from the Proposition 2.1.10. ■

Proposition 2.1.12. *(i)A composition series is a series of maximal length.*
(ii)Every finite group has a composition series.

PROOF: See Moori [26]. ■

Theorem 2.1.13. *(**Schreier, 1926**). Let* G *be a group. Then any two series of* G *have refinements which are equivalent.*

PROOF: See Moori [26]. ■

Theorem 2.1.14. *(**Jordan-Holder Theorem,1868 and 1889**). Any two composition series of a group* G *are equivalent.*

PROOF: See Moori [26]. ■

Definition 2.1.11. *(**Solvable Groups**)([26]). A group* G *is said to be solvable (or soluble) if it has a series whose factors are all abelian. Such a series is called a solvable series or abelian series. Groups that are not solvable are said to be insolvable*

Galois introduced the notion of solvability of groups in connection with solving polynomial equations by radicals.

Remark 2.1.1.

- Non-abelian simple groups are insolvable since $\{1_G\} \trianglelefteq G$ is the only series and $G/\{1_G\} \cong G$ is non-abelian.
- Every abelian group is solvable.
- S_n is solvable for $n \leq 4$ - S_3 and S_4 are two examples of non-abelian solvable groups.
- The Dihedral group D_{2n} is solvable for all n.
- S_n for $n \geq 5$ is not solvable.
  Consider the normal series $S_n \trianglerighteq A_n \trianglerighteq \{1_G\}$ $(*)$. Since the factors of $(*)$ are $\mathbb{Z}_2$ and A_n which are simple, $(*)$ is a composition series for S_n. Since by the Jordan Holder Theorem any other composition series for S_n is equivalent to $(*)$, $\mathbb{Z}_2$ and A_n are the only composition factors. The result follows now from Theorem 2.1.18 below.

Definition 2.1.12. ***(Derived length)*** *([33]). If G is a solvable group, the length of a shortest abelian series in G is called the derived length of G. Thus G has derived length 0 if and only if it has order 1. Also the groups with derived length at most 1 are just the abelian groups. A solvable group with derived length at most 2 is said to be metabelian.*

Definition 2.1.13. ***(Metacyclic)*** *([35],[37]). A group G is called metacyclic if it has a cyclic normal subgroup L such that G/L is cyclic. Equivalently we may say that a group G is metacyclic if and only if G/G' and G' are cyclic. Every Dihedral group for example is metacyclic.*

Theorem 2.1.15. *If G is a solvable group and $H \leq G$, then H is also solvable.*

PROOF: See Moori [26]. ■

Theorem 2.1.16. *If G is a solvable group and $H \trianglelefteq G$, then G/H is also solvable.*

PROOF: See Moori [26]. ■

Theorem 2.1.17. *Let G be a group and $H \trianglelefteq G$. If H and G/H are solvable, then G is solvable.*

PROOF: See Moori [26]. ■

Theorem 2.1.18. *A non-trivial finite group G is solvable if and only if it has a composition series whose factors are cyclic groups of prime order.*

PROOF: See Moori [26]. ■

Theorem 2.1.19. *Let $G = H \times K$ such that H and K are solvable. Then G is also solvable.*

PROOF: Since $G/H \cong K$, G/H is solvable and the result now follows from Proposition 2.1.17. ■

Theorem 2.1.20. *If* G *is a finite p-group, then* G *is solvable.*

PROOF: If $|G| = 1$, then clearly G is solvable. So assume $|G| > 1$. Since G is a p-group, $Z(G) \neq \{1_G\}$ and hence $|G/Z(G)| < |G|$. Now since $G/Z(G)$ is a finite p-group, by induction hypothesis $G/Z(G)$ is solvable. Since $Z(G)$ is abelian group, it is solvable. Now the solvability of $G/Z(G)$ and $Z(G)$ implies G solvable. ■

Proposition 2.1.21. *If* N *and* M *are solvable subgroups of* G *with* $N \trianglelefteq G$*, then* MN *is a solvable subgroup of* G.

PROOF: $N \trianglelefteq G$ implies that $MN \leq G$ and $N \trianglelefteq MN$. By the Isomorphism Theorems, $MN/N \cong M/M \cap N$. Since $M \cap N \trianglelefteq M$ and M is solvable, by Proposition 2.1.16, $M/M \cap N$ is solvable. Therefore, MN/N solvable and N solvable implies that MN solvable by Proposition 2.1.17. ■

We also have the following:

Proposition 2.1.22. *The product of two normal solvable subgroups of a group is a normal solvable subgroup.*

PROOF: Proof follows from Proposition 2.1.21. Also if N and M are normal subgroups of a group G, then NM is a normal subgroup of G. ■

It follows that every finite group G has a unique maximal normal solvable subgroup, namely the product of all normal solvable subgroups, the *solvable radical* of G.

Definition 2.1.14. *A group having a series*

$$\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_n = G. \quad (*)$$

with each factor group G_{i+1}/G_i *cyclic and each* G_i *normal in* G *is called a* ***Super-Solvable*** *group.*

Solvable groups need not be Super solvable. The Alternating group A_4 is a solvable group having series

$$\{1_G\} \trianglelefteq \{1_G, (12)(34)\} \trianglelefteq V_4 \trianglelefteq A_4,$$

but the subgroup $\{1_G, (12)(34)\}$ is not normal in A_4.

The Feit-Thompson Theorem is used to prove both statements in the following lemma [See *Pacific Journal of Mathematics,13 (1963), 775-1029*].

Lemma 2.1.23. *(i)Every finite group of odd order is solvable.*
(ii)Every finite non-abelian simple group has even order.

The derived subgroup of a group, also called the commutator subgroup is the subgroup generated by all the commutators in G. By repeatedly forming derived subgroups a descending sequence of fully invariant subgroups is generated. A subgroup H of a group G is said to be *fully-invariant* in G if $H^\alpha \leq H \; \forall \alpha \in \mathrm{End}\, G$. Fully invariant subgroups are *characteristic* subgroups and hence normal subgroups.

Definition 2.1.15. ***(Derived Series)****([35]). Let* G *be a group. We define subgroups* $G^{(n)}$ *of* G*, one for each non-negative integer* n*, recursively as follows:*

$$G^{(0)} = G$$

and for each $n > 0$*,*

$$G^{(n)} = [G^{(n-1)}\,,\ G^{(n-1)}] = (G^{(n-1)})'.$$

Thus $G^{(1)} = G'$, $G^{(2)} = G''$, $G^{(3)} = G'''$*, etc.*
By definition,

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq ...$$

This descending sequence of characteristic subgroups is called the derived series of G.

All the factors $G^{(n)}/G^{(n+1)}$ of the derived series are abelian and the first of these factors G/G' is the largest abelian quotient of G. If, for some n, $G^{(n)} = G^{(n+1)}$ then $G^{(n)} = G^{(r)}$ for every $r \geq n$. In this case we say that the derived series terminates. The derived series of a finite group must terminate. This is not necessarily true for infinite groups. However, the next result shows that if G is solvable then the derived series of G terminates in $\{1_G\}$.

Theorem 2.1.24. *([35]). Let* G *be a group. Then* G *is solvable if and only if* $G^{(n)} = \{1_G\}$.

PROOF: See [35, Theorem 7.52]. ■

Definition 2.1.16. ***(Derived Length)****. Let* G *be a solvable group with derived series,*

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq ...$$

Let n *be the least integer such that* $G^{(n)} = \{1_G\}$*. Then* n *is called the derived length of* G.

2.1.2 Nilpotent Groups

A Double Frobenius group contains two Frobenius groups as subgroups. Every Frobenius group contains a non-trivial normal subgroup, namely the kernel. Frobenius kernels are nilpotent, See Proposition 3.2.16. In this section we present some results on nilpotent groups.

Definition 2.1.17. *(Nilpotent Group)([33]). A group is called nilpotent if it has a central series, that is, a normal series* $\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G$ *such that* G_{i+1}/G_i *is contained in the center of* G/G_i $\forall i$.

Definition 2.1.18. *(Nilpotent class)([33]). Let* G *be a nilpotent group. Then the length of the shortest central series of* G *is called the nilpotent class of* G.

A nilpotent group of class 0 has order 1, while nilpotent groups of class at most 1 are abelian. Nilpotent groups are solvable and if $G \neq \{1_G\}$ is nilpotent , then $Z(G) \neq \{1_G\}$. See [38, Theorem 5.3.4].

Theorem 2.1.25. *A finite p-group is nilpotent.*

PROOF: See [35, Theorem 5.1.3]. ■

Theorem 2.1.26. *A group* G *is nilpotent if and only if it is the direct product of its Sylow subgroups.*

PROOF: See [10, Theorem 3.5]. ■

Lemma 2.1.27. *If* H *and* K *are normal nilpotent subgroups of a group* G*, then so is* HK.

PROOF: See [11, Lemma 1.1]. ■

Lemma 2.1.28. *If* G *is a nilpotent group and* $\{1_G\} \neq N \lhd G$*, then* $N \cap Z(G) \neq \{1_G\}$.

PROOF: See [35, Theorem 5.2.1]. ■

Theorem 2.1.29. *([35]). If* G *is a nilpotent group, then all subgroups and all quotient groups of* G *are nilpotent.*

PROOF: See [36, Theorem 7.46]. ■

Theorem 2.1.30. *([35]). Let* G *be a group such that* $G = H \times K$*. If* H *and* K *are nilpotent then* G *is nilpotent.*

PROOF: See [36, Theorem 7.49]. ■

2.1.3 The Frattini and Fitting Subgroups of a Finite Group

Every finite group contains two important characteristic and hence normal subgroups, namely the Frattini subgroup and the Fitting subgroup. We will denote the Frattini subgroup by $\phi(G)$ and the Fitting subgroup by $F(G)$ where G is a finite group. The Fitting subgroup in particular features prominently in definitions and descriptions of the Double Frobenius group. We include here some definitions and results concerning these subgroups.

Definition 2.1.19. *(**Frattini Subgroup**). Let G be a group. The Frattini subgroup of G is defined to be the intersection of all the maximal subgroups of G. If G has no maximal subgroups then we will adopt the convention that $\phi(G) = G$. Therefore,*

$$\phi(G) = \bigcap_{M \in \mathcal{M}} M,$$

*where M is a **maximal subgroup** of G and $\mathcal{M}$ is the collection of all maximal subgroups of G. If $G \neq \{1_G\}$ and G is finite, then G certainly has at least one maximal subgroup. Every proper subgroup of G lies in a maximal subgroup. Since any automorphism of G sends a maximal subgroup into a maximal subgroup, the set $\mathcal{M}$ is invariant by any automorphism, and so is $\phi(G)$. This shows that $\phi(G)$ is a characteristic subgroup and since characteristic subgroups are normal, we have that $\phi(G) \trianglelefteq G$.*

Definition 2.1.20. *(**Fitting Subgroup**). Let G be a group. The subgroup of G generated by all its nilpotent normal subgroups is a nilpotent normal subgroup of G. This subgroup is thus the unique maximal nilpotent normal subgroup of G. It is called the Fitting subgroup of G. For a given group G, F(G) may be trivial. The finite groups with this property are the semisimple groups (groups with no non-trivial normal abelian subgroups). However, if G is a solvable group, then F(G) is non-trivial, since the non-trivial minimal normal subgroup in an abelian series of G is abelian, nilpotent, and is therefore contained in F(G).*

Theorem 2.1.31. *Let G be a group. If G is solvable, then $C_G(F(G)) \leq F(G)$.*

PROOF: See [10, Theorem 1.3]. ■

2.1.4 The Fitting Series of a Solvable Group

Let G be a solvable group and $G \neq \{1_G\}$, then we know that $F(G) \neq \{1_G\}$. Since F(G) is a non-trivial normal subgroup of G, we can construct an ascending sequence of normal subgroups of G beginning with the Fitting subgroup of G as the first non-trivial member of the sequence. The resulting sequence is called the Fitting series of G.

Let $F_0(G) = \{1_G\}$ and $F_1(G) = F(G)$.
Now $F(G) \trianglelefteq G$, so let $\gamma_1 : G \to G/F_1(G)$ be the natural homomorphism. If $F(G) = G$, then γ_1 is the trivial homomorphism and we get the series $\{1_G\} \leq G$. So assume $F(G) \neq G$. Then $G/F_1(G)$ is non-trivial and the $F(G/F_1(G)) \trianglelefteq G/F_1(G)$.

Since $F(G/F_1(G)) \trianglelefteq G/F_1(G)$, $F(G/F_1(G)) = F_2(G)/F_1(G)$ where $F_1(G) \trianglelefteq F_2(G)$ and $F_2(G) \trianglelefteq G$.

If $F_2(G) = G$, then we have the series $\{1_G\} \trianglelefteq F_1(G) \trianglelefteq G$. If $F_2(G) \neq G$, let $\gamma_2 : G \to G/F_2(G)$ be the natural homomorphism. Then $F(G/F_2(G)) \trianglelefteq G/F_2(G)$.

Since $F(G/F_2(G)) \trianglelefteq G/F_2(G)$, $F(G/F_2(G)) = F_3(G)/F_2(G)$ where $F_2(G) \trianglelefteq F_3(G)$ and $F_3(G) \trianglelefteq G$.

If $F_3(G) = G$, then we have the series $\{1_G\} \trianglelefteq F_1(G) \trianglelefteq F_2(G) \trianglelefteq G$. If $F_3(G) \neq G$, then continuing in this fashion we can construct the series

$$\{1_G\} \trianglelefteq F_1(G) \trianglelefteq F_2(G) \trianglelefteq F_3(G) \trianglelefteq \ldots$$

This ascending series of subgroups of G is called the Fitting series of G.

If G is solvable then $F_n(G) = G$ for some n.

Note 2.1.1. The Fitting series described above can be generated by a recursive formula as follows. For a group G let $F_0(G) = \{1_G\}$ and for each positive integer n, let $F_n(G)/F_{n-1}(G) = F(G/F_{n-1}(G))$. Then

$$\{1_G\} = F_0(G) \leq F_1(G) \leq F_2(G) \leq \ldots,$$

is the Fitting series of G.

Definition 2.1.21. ***(Fitting height)****([35]). Let G be a solvable group with Fitting series*

$$\{1_G\} \trianglelefteq F_1(G) \trianglelefteq F_2(G) \trianglelefteq F_3(G) \trianglelefteq \ldots$$

Then the least integer n for which $F_n(G) = G$ is called the Fitting height (or nilpotent length) of G.

Example 2.1.1.

1. A group G has Fitting height 1 if and only if G is nilpotent.
   If G has Fitting height 1 then G has the Fitting series

   $$\{1_G\} \trianglelefteq F_1(G) = G$$

   and hence G is nilpotent.
   If G is nilpotent then $F_1(G) \subseteq G \subseteq F_1(G) \Longrightarrow F_1(G) = G$. Therefore, G has Fitting series

   $$\{1_G\} \trianglelefteq F_1(G) = G,$$

   and the Fitting height is thus 1.

2. The Fitting height of $G = S_3$ is 2. The Fitting series is

   $$\{1_G\} \trianglelefteq Z_3 \trianglelefteq S_3.$$

   Here, $Z_3 = F_1(G) \trianglelefteq F_2(G) = S_3$. The successive factor groups are Z_3 and $Z_2 \cong S_3/Z_3$ both of which are nilpotent groups.

3. The Fitting height of S_4 is 3. The Fitting series is

   $$\{1_G\} \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4.$$

   Here, $V_4 = F_1(G)$, $F_2(G) = A_4$, $F_3(G) = S_4$. The factor groups are V_4 , $Z_3 \cong A_4/V_4$ and $Z_2 \cong S_4/A_4$ all of which are nilpotent groups.

2.1.5 Group Extensions

The Frobenius group covered in the next section is an example of a split extension. The construction of the double Frobenius group also involves split extensions. So in this section we briefly discuss group extensions.

We start with a definition.

Definition 2.1.22. *Let $\overline{G}$ be a group, let N and G be subgroups of $\overline{G}$ such that*

1. *N is normal in $\overline{G}$*

2. *$\overline{G} = NG$*

3. *$N \cap H = \{1_{\overline{G}}\}$.*

Then $\overline{G}$ is called a semidirect product of N by G.

Note that the terms split extension and semidirect product are used interchangeably to mean one and the same thing.

For $\overline{G}$ a semidirect product of N by G, every element in $\overline{G}$ can be expressed uniquely in the form ng, where $n \in N$ and $g \in G$ and multiplication of elements in $\overline{G}$ is given by

$$(n_1 g_1)(n_2 g_2) = n_1 n_2^{g_1} g_1 g_2,$$

where $n^g = gng^{-1}$. Also there is a homomorphism $\theta : G \longrightarrow \mathrm{Aut}(N)$ given by $\theta(g) = \theta_g$, where $g \in G$, $\theta_g : N \longrightarrow N$ is defined by $\theta_g(n) = gng^{-1}$ and θ_g is an automorphism of N. Hence, G acts on N.

Definition 2.1.23. *Let $\overline{G}$, N and G be as defined above and $\theta : G \longrightarrow \mathrm{Aut}(N)$ a homomorphism. Then the semidirect product $\overline{G}$ of N by G is said to realize θ if $\theta_g(n) = n^g \; \forall n \in N, \; g \in G$.*

Remark 2.1.2. For $\overline{G}$ a semidirect of N by G, $\overline{G}$ is isomorphic to a semidirect product of N by G that realizes θ for some $\theta : G \longrightarrow \mathrm{Aut}(N)$.

We discuss now the method of coset analysis which we use later in Chapters 5 and 6. The technique works for both split and non-split extensions and was developed and first used by Moori in [27] and [28]. We use the method described in Mpono [29].

First we define a **lifting**.

Definition 2.1.24. (Lifting) *If $\overline{G}$ is a split extension of N by G, then $\overline{G} = \cup_{g \in G} Ng$, so G may be regarded as a right transversal for N in $\overline{G}$ (that is, a complete set of right coset representatives of N in $\overline{G}$). Now suppose $\overline{G}$ is any extension of N by G, not necessarily split, then, since $\overline{G}/N \cong G$, there is an onto homomorphism $\lambda : \overline{G} \to G$ with kernel N. For $g \in G$ define a* **lifting** *of g to be an element $\overline{g} \in \overline{G}$ such that $\lambda(\overline{g}) = g$.*

Let $\overline{G} = N.G$ where N is an abelian normal subgroup of $\overline{G}$.

- For each conjugacy class $[g]$ in G with representative $g \in G$, we analyze the coset $N\overline{g}$, where $\overline{g}$ is a lifting of g in $\overline{G}$ and $\overline{G} = \cup_{g \in G} N\overline{g}$.
- To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$, we define

$$C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\}.$$

  Then $C_{\overline{g}}$ is the stabilizer of $N\overline{g}$ in $\overline{G}$ under the action by conjugation of $\overline{G}$ on the set of cosets $N\overline{g}$, and hence $C_{\overline{g}}$ is a subgroup of $\overline{G}$.
- If $\overline{G} = N{:}G$ then we can identify $C_{\overline{g}}$ with $C_g = \{x \in \overline{G} \mid x(Ng) = (Ng)x\}$, where the lifting of $g \in \overline{G}$ is g itself since $G \leq \overline{G}$.
- The conjugacy classes of $\overline{G}$ will be determined by the action by conjugation of $\overline{G}$, for each conjugacy class $[g]$ of G, on the elements of $N\overline{g}$.
- To act $\overline{G}$ on the elements of $N\overline{g}$, we first act N and then act $\{\overline{h} \mid h \in C_G(g)\}$ where $\overline{h}$ is a lifting of h in G.
- We describe the action in two steps:

  1. The action of N on $N\overline{g}$:
     Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$\begin{aligned} x \in C_N(n\overline{g}) &\iff x(n\overline{g})x^{-1} = n\overline{g}\,, \\ &\iff xnx^{-1}x\overline{g}x^{-1} = n\overline{g}\,, \\ &\iff x\overline{g}x^{-1} = \overline{g}\,, \text{ (since } N \text{ is abelian)} \\ &\iff x \in C_N(\overline{g})\,. \end{aligned}$$

     Thus $C_N(\overline{g})$ fixes every element of $N\overline{g}$. Now let $|C_N(\overline{g})| = k$. Then under the action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, \ldots, Q_k$ where $|Q_i| = [N : C_N(\overline{g})] = \frac{|N|}{k}$ for $i \in \{1, 2, \ldots, k\}$.

  2. The action of $\{\overline{h} \mid h \in C_G(g)\}$ on $N\overline{g}$:
     Since the elements of $N\overline{g}$ are now in orbits $Q_1, Q_2, \ldots, Q_k$ from step (1) above, we only

act $\{\overline{h} \mid h \in C_G(g)\}$ on these k orbits. Suppose that under this action f_j of these orbits $Q_1, Q_2, \ldots, Q_k$ fuse together to form one orbit $\triangle_j$, then the f_j's obtained this way satisfy

$$\sum_j f_j = k \text{ and } |\triangle_j| = f_j \times \frac{|N|}{k} .$$

Thus for $x \in \triangle_j$, we obtain that

$$\begin{aligned} |[x]_{\overline{G}}| &= |\triangle_j| \times |[g]_G| \qquad (2.1)\\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\overline{G}|}{k|C_G(g)|} . \end{aligned}$$

Thus,

$$\begin{aligned} |C_{\overline{G}}(x)| = \frac{|\overline{G}|}{|[x]_{\overline{G}}|} &= |\overline{G}| \times k\frac{|C_G(g)|}{f_j\,|\overline{G}|} \\ &= \frac{k\,|C_G(g)|}{f_j} . \qquad (2.2) \end{aligned}$$

Therefore, to calculate the conjugacy classes of $\overline{G} = N.G$, we find the values of k and the f_j's for each class representative $g \in G$.

Remark 2.1.3. In the case of $\overline{G} = N{:}G$ a split extension, we analyse the coset Ng instead of $N\overline{g}$ since in this case $G \leq \overline{G}$. Under the action of N on Ng, we always assume that $g \in Q_1$. Also instead of acting $\{\overline{h} \mid h \in C_G(g)\}$ on the k orbits $Q_1, Q_2, \ldots, Q_k$, we just act $C_G(g)$ on these orbits. Since $g \in Q_1$, then $C_G(g)$ always fixes Q_1 and we always have $f_1 = 1$. Hence,

$$k = \sum_j f_j = 1 + \sum_m f_m,$$

where the sum is taken over all m such that $g \notin Q_m$.

In the following theorems and remark we discuss techniques useful in determining the orders of elements of $\overline{G} = N : G$.

Theorem 2.1.32. *([29]). Let* $\overline{G} = G{:}\overline{H}$ *and* $g\overline{h} \in \overline{G}$ *where* $g \in G$ *and* $\overline{h} \in \overline{H}$ *such that* $o(\overline{h}) = m$ *and* $o(g\overline{h}) = k$. *Then* m *divides* k.

PROOF: We have that

$$1_{\overline{G}} = (g\overline{h})^k) = gg^{\overline{h}}g^{\overline{h}^2}g^{\overline{h}^3} \cdots\cdots g^{\overline{h}^{k-1}}\overline{h}^k.$$

Since $\overline{H}$ acts on G and $g \in G$, we have g, $g^{\overline{h}}$, $g^{\overline{h}^2}$, $\cdots\cdots$, $g^{\overline{h}^{k-1}} \in G$. Hence, $gg^{\overline{h}}g^{\overline{h}^2}g^{\overline{h}^3} \cdots\cdots g^{\overline{h}^{k-1}} \in G$. Thus we must have that $gg^{\overline{h}}g^{\overline{h}^2}g^{\overline{h}^3} \cdots\cdots g^{\overline{h}^{k-1}} = 1_G$. Therefore m divides k. ■

Theorem 2.1.33. *([29]). Let* $\overline{G} = G{:}\overline{H}$ *be such that* G *is an elementary abelian p-group, where* p *is a prime. Let* $g\overline{h} \in \overline{G}$ *where* $g \in G$ *and* $\overline{h} \in \overline{H}$ *such that* $o(\overline{h}) = m$ *and* $o(g\overline{h}) = k$*. Then either* $k = m$ *or* $k = pm$.

PROOF: Since G is an elementary p-group and $g \in G$, then we have that $o(g) = 1$ or $o(g) = p$. Suppose that $g \neq 1_G$, then $o(g) = p$. Now we have that

$$(g\overline{h})^m = gg^{\overline{h}}g^{\overline{h}^2}g^{\overline{h}^3} \cdots\cdots g^{\overline{h}^{m-1}}\overline{h}^m.$$

Since $\overline{h}^m = 1_{\overline{H}}$, we deduce that $(g\overline{h})^m \in G$. If $(g\overline{h})^m = 1_G$, then k must divide m and Theorem 2.1.32 implies that $k = m$. If $(g\overline{h})^m \neq 1_G$, then $o((g\overline{h})) = p$ and hence $(g\overline{h})^{pm} = 1_G$. Thus we obtain that $k|pm$ and $pm = ks$ for some positive integer s. However, from Theorem 2.1.32 we have that $k = mt$ for some positive integer t. Since $o(g\overline{h}) = k$ and $(g\overline{h}) \neq 1_G$, we have $m \neq k$ and hence $t \neq 1$. Now $pm = ks$ and $k = mt$ implies that $pm = mst$ and hence that $p = st$. Since p is prime and $t \neq 1$, we must have $p = t$ and $s = 1$. The result now follows since $k = pm$. ■

Remark 2.1.4. ([29]). Let $\overline{G} = G{:}\overline{H}$, where G is an elementary abelian p-group. Let $g\overline{h} \in \overline{G}$ with $g \in G$ and $\overline{h} \in \overline{H}$ such that $o(\overline{h}) = m$ and $o(g\overline{h}) = k$, then we have that

$$(g\overline{h})^m = gg^{\overline{h}}g^{\overline{h}^2}g^{\overline{h}^3} \cdots\cdots g^{\overline{h}^{m-1}}\overline{h}^m.$$

Since $\overline{h}^m = 1_{\overline{H}}$, we have that $(g\overline{h})^m = \omega$, where $\omega \in G$ and

$$\omega = gg^{\overline{h}}.\cdots\cdots.g^{\overline{h}^{m-1}}.$$

By Theorem 2.1.33, we have that if $\omega = 1_G$ then $k = m$ and if $\omega \neq 1_G$ then $k = pm$.

2.1.6 Representation Theory and Characters of Finite Groups

There are two kinds of representations; *permutation* and *matrix*. Cayley's Theorem, which asserts that any group G can be embedded into the Symmetric group S_G, is an example of a permutation representation. We are interested here in matrix representations.

Definition 2.1.25. *Let* G *be a group. Any homomorphism* $\rho : G \longrightarrow GL(n, \mathbb{F})$*, where* $GL(n, \mathbb{F})$ *is the general linear group consisting of all* $n \times n$ *non-singular matrices is called a* **matrix representation** *or simply a representation of* G*. If* $\mathbb{F} = \mathbb{C}$*, then* ρ *is called an ordinary representation. The integer* n *is called the* **degree** *of* ρ*. Two representations* ρ *and* σ *are said to be* **equivalent** *if there exists* $P \in GL(n, \mathbb{F})$ *such that* $\sigma(g) = P\rho(g)P^{-1}$, $\forall g \in G$.

We will restrict our work to ordinary representations.

Definition 2.1.26. **(Character)** *Let* $\rho : G \longrightarrow GL(n, \mathbb{C})$ *be a representation of a group* G*. Then* ρ *affords a complex valued function* $\chi_\rho : G \longrightarrow \mathbb{C}$ *defined by* $\chi_\rho(g) = trace(\rho(g))$, $\forall g \in G$*. The function* χ_ρ *is called a* **character** *afforded by the representation* ρ *of* G *or simply a character of* G*. The integer* n *is called the* **degree** *of* χ_ρ*. If* $n = 1$*, then* χ_ρ *is said to be* **linear**.

Note 2.1.2. For any group G, consider the function $\rho : G \to GL(1,\mathbb{C})$ given by $\rho(g) = 1$, $\forall g \in G$. It is clear that ρ is a representation of G and $\chi_\rho(g) = 1$, $\forall g \in G$. The character χ_ρ is called the *trivial* character and it may also be denoted by 1.

Definition 2.1.27. (**Class Function**) *If* $\phi : G \to \mathbb{C}$ *is a function that is constant on conjugacy classes of a group* G, *that is* $\phi(g) = \phi(xgx^{-1})$, $\forall x \in G$, *then we say that* ϕ *is a class function.*

Proposition 2.1.34. *A character is a class function.*

PROOF: Immediate since similar matrices have the same trace. ■

Definition 2.1.28. *Let* $f : G \to GL(n,\mathbb{F})$ *be a representation of* G *over* $\mathbb{F}$. *Let* $S = \{f(g) | g \in G\}$. *Then* $S \subseteq GL(n,\mathbb{F})$. *We say that* f *is* **reducible, fully reducible** *or* **completely reducible** *if* S *is reducible, fully reducible or completely reducible.*

We state below two important results in representation theory, namely Maschke's Theorem and Schur's Lemma. The proof of both these results can be found in Moori [26].

Theorem 2.1.35. (**Maschke's Theorem**) *Let* $\rho : G \to GL(n,\mathbb{F})$ *be a representation of a group* G. *If the characteristic of* $\mathbb{F}$ *is zero or does not divide* $|G|$, *then* $\rho = \bigoplus_{i=1}^{r} \rho_i$, *where* ρ_i *are irreducible representations of* G.

PROOF: See Moori [26]. ■

Theorem 2.1.36. (**Schur's Lemma**) *Let* ρ *and* ϕ *be two irreducible representations of degree* n *and* m *respectively, of a group* G *over a field* $\mathbb{F}$. *Assume that there exists an* $m \times n$ *matrix* P *such that* $P\rho(g) = \phi(g)P$ *for all* $g \in G$. *Then either* $P = 0_{m\times n}$ *or* P *is non-singular so that* $\rho(g) = P^{-1}\phi(g)P$ *(that is* ρ *and* ϕ *are equivalent representations of* G*).*

PROOF: See Moori [26]. ■

Definition 2.1.29. (**Inner Product**) *Let* G *be a group. Over* $\mathcal{C}(G)$, *the set of all class functions on* G, *we define an inner product*

$$\langle\ ,\ \rangle : \mathcal{C}(G) \times \mathcal{C}(G) \to \mathbb{C} \text{ by } \langle \psi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\phi(g)},$$

where $\overline{\phi(g)}$ *is the complex conjugate of* $\phi(g)$.

In the following Proposition we list some properties of characters of a group.

Proposition 2.1.37. *([26]).*

1. *Let* χ_ρ *be the character afforded by an irreducible representation* ρ *of a group* G. *Then* $\langle \chi_\rho, \chi_\rho \rangle = 1$.

2. *If* χ_ρ *and* $\chi_{\rho'}$ *are irreducible characters of two non equivalent representations of* G, *then* $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$.

3. *If* $\rho \cong \bigoplus_{i=1}^{k} d_i \rho_i$, *then* $\chi_\rho = \sum_{i=1}^{k} d_i \chi_{\rho_i}$.

4. *If* $\rho \cong \bigoplus_{i=1}^{k} d_i \rho_i$, *then* $d_i = \langle \chi_\rho, \chi_{\rho_i} \rangle$.

PROOF: See Moori [26] or G. James [13]. ■

Proposition 2.1.38. *([26]). Let* χ_ρ *be the character afforded by a representation* ρ *of a group* G. *Then* ρ *is irreducible if and only if* $\langle \chi_\rho, \chi_\rho \rangle = 1$.

PROOF: See G. James [13]. ■

The following counting result counts the number of irreducible characters of a group.

Theorem 2.1.39. *The number of irreducible characters of a group* G *is equal to the number of conjugacy classes of* G.

PROOF: See G. James [13] or Moori [26]. ■

Proposition 2.1.40. *The number of linear characters of a group* G *is given by* $|G|/|G'|$, *where* G' *is the derived subgroup of* G.

PROOF: See Moori [25]. ■

The Character Table and Orthogonality Relations

The irreducible characters of a finite group are class functions, and the number of them by Theorem 2.1.39 is equal to the number of conjugacy classes of the group. A table recording the values of all the irreducible characters of the group is called a **character table** of the group.

Definition 2.1.30. (**Character Table**) *The character table of a group* G *is a square matrix whose columns correspond to the conjugacy classes of* G *and whose rows correspond to the irreducible characters of* G.

The character table is a useful tool which can be used to make inferences about the group. The simplicity, normality and solvability as well as the center and commutator of the group can also be determined from the character table.

The following Propositions contains some useful results about the values of the irreducible characters in the character table of a group G.

Proposition 2.1.41. *([26]).*

1. $\chi(1_G) \mid |G|$, $\forall \chi \in \mathrm{Irr}(G)$.

2. $\sum_{i=1}^{|\mathrm{Irr}(G)|} (\chi_i(1_G))^2 = |G|$.

3. *If* $\chi \in \mathrm{Irr}(G)$, *then* $\overline{\chi} \in \mathrm{Irr}(G)$, *where* $\overline{\chi}(g) = \overline{\chi(g)}$, $\forall g \in G$.

4. $\chi(g^{-1}) = \overline{\chi(g)}$, $\forall g \in G$. *In particular if* $g^{-1} \in [g]$, *then* $\chi(g) \in \mathbb{R}$, $\forall \chi$.

PROOF: See Moori [26]. ■

The rows and columns of the character table also satisfy orthogonality relations which we state in the next theorem.

Theorem 2.1.42. *([26]). Let* $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_k\}$ *and* $\{g_1, g_2, \dots, g_k\}$ *be a collection of representatives for the conjugacy classes of a group* G. *For each* $1 \leq i \leq k$ *let* $C_G(g_i)$ *be the centralizer of* g_i. *Then we have the following:*

1. *The row orthogonality relation:*
For each $1 \leq i, j \leq k$,
$$\sum_{r=1}^{k} \frac{\chi_i(g_r)\overline{\chi_j(g_r)}}{|C_G(g_r)|} = \langle \chi_i, \chi_j \rangle = \delta_{ij}.$$

2. *The column orthogonality relation:*
For each $1 \leq i, j \leq k$,
$$\sum_{r=1}^{k} \frac{\chi_r(g_i)\overline{\chi_r(g_j)}}{|C_G(g_i)|} = \delta_{ij}.$$

PROOF: (1) Using Proposition 2.1.37(2) we have
$$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \frac{1}{|G|} \sum_{r=1}^{k} \frac{|G|}{|C_G(g_r)|} \chi_i(g_r)\overline{\chi_j(g_r)} = \sum_{r=1}^{k} \frac{\chi_i(g_r)\overline{\chi_j(g_r)}}{|C_G(g_r)|}.$$

(2) For fixed $1 \leq s \leq k$, define $\psi_s : G \longrightarrow \mathbb{C}$ by $\psi_s(g) = \begin{cases} 1 & \text{if } g \in [g_s], \\ 0 & \text{otherwise.} \end{cases}$
It is clear that ψ_s is a class function on G. Since $\mathrm{Irr}(G)$ form an orthonormal basis for $\mathcal{C}(G)$, there exists $\lambda_t's \in \mathbb{C}$ such that $\psi_s = \sum_{t=1}^{k} \lambda_t \chi_t$. Now for $1 \leq j \leq k$ we have
$$\lambda_j = \langle \psi_s, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_s(g)\overline{\chi_j(g)} = \sum_{t=1}^{k} \frac{\psi_s(g_t)\overline{\chi_j(g_t)}}{|C_G(g_t)|} = \frac{\overline{\chi_j(g_s)}}{|C_G(g_s)|}.$$

Hence $\psi_s = \sum_{j=1}^{k} \frac{\overline{\chi_j(g_s)}}{|C_G(g_s)|}\chi_j$. Thus we have the required formula:

$$\delta_{st} = \psi_s(g_t) = \sum_{j=1}^{k} \frac{\chi_j(g_t)\overline{\chi_j(g_s)}}{|C_G(g_t)|}.$$

■

Definition 2.1.31. **(Transversal)** *Let* G *be a group. Let* $H \leq G$*. By a* **right transversal** *of* H *in* G *we mean a set of representatives for the right cosets of* H *in* G.

Lifting of Characters

We present here a method for constructing characters of a group G when G has a normal subgroup N. Assuming that the irreducible characters of the factor group G/N are known, the idea here is to construct characters of G by a process known as *lifting of characters.*

Definition 2.1.32. **(Kernel)** *Let* χ *be a character of a group* G *afforded by a representation* ρ *of* G*. Then*

$$\mathrm{Ker}(\rho) = \mathrm{Ker}(\chi) = \{g \in G \mid \chi(g) = \chi(1_G)\} \trianglelefteq G.$$

Also if $N \leq G$ such that N is an intersection of the kernel of irreducible characters of G, then $N \trianglelefteq G$.

Proposition 2.1.43. *([26]). Let* G *be a group. Let* $N \trianglelefteq G$ *and* $\widetilde{\chi}$ *be a character of* G/N*. The function* $\chi : G \to \mathbb{C}$ *defined by* $\chi(g) = \widetilde{\chi}(gN), \forall g \in G$ *is a character of* G *with* $\deg(\chi) = \deg(\widetilde{\chi})$*. Moreover, if* $\widetilde{\chi} \in \mathrm{Irr}(G/N)$*, then* $\chi \in \mathrm{Irr}(G)$.

PROOF: Suppose that $\tilde{\rho} : G/N \to GL(n, \mathbb{C})$ is a representation which affords the character $\widetilde{\chi}$. Define the function $\rho : G \to GL(n, \mathbb{C})$ by $\rho(g) = \tilde{\rho}(gN), \forall g \in G$. Then ρ defines a representation on G since

$$\rho(gh) = \tilde{\rho}(ghN) = \tilde{\rho}(gNhN) = \tilde{\rho}(gN)\tilde{\rho}(hN) = \rho(g)\rho(h), \forall g, h \in G.$$

Hence the character χ, which is afforded by ρ, satisfies

$$\chi(g) = \mathrm{trace}(\rho(g)) = \mathrm{trace}(\tilde{\rho}(gN)) = \widetilde{\chi}(gN)\ \forall g \in G.$$

So χ is a character of G. The degree of χ is

$$\deg(\chi) = \chi(1_G) = \widetilde{\chi}(1_G N) = \widetilde{\chi}(N) = \deg(\widetilde{\chi}).$$

Let T be a transversal of N in G. Then

$$\begin{aligned}
1 = \langle \tilde{\chi}, \tilde{\chi} \rangle &= \frac{1}{|G/N|} \sum_{gN \in G/N} \tilde{\chi}(gN)\tilde{\chi}(gN)^{-1} \\
&= \frac{1}{|G|} \sum_{gN \in G/N} |N|\, \tilde{\chi}(gN)\tilde{\chi}(gN)^{-1} \\
&= \frac{1}{|G|} \sum_{g \in T} |N|\, \tilde{\chi}(gN)\tilde{\chi}(g^{-1}N) \\
&= \frac{1}{|G|} \sum_{g \in T} |N|\, \chi(g)\chi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\
&= \langle \chi, \chi \rangle.
\end{aligned}$$

■

Induction and Restriction of Characters

Let G be a group, $H \leq G$. If $\rho : G \to GL(n, \mathbb{C})$ is a representation of G, then $\rho \downarrow H : H \to GL(n, \mathbb{C})$ given by $(\rho \downarrow H)(h) = \rho(h), \forall h \in H$, is a representation of H. We say that $\rho \downarrow H$ is the **restriction** of ρ to H. If χ_ρ is the character of ρ, then $\chi_\rho \downarrow H$ is the character of $\rho \downarrow H$. We refer to $\chi_\rho \downarrow H$ as the **restriction** of χ_ρ to H.

Remark 2.1.5. It is clear that $\deg(\rho) = \deg(\rho \downarrow H)$. However, ρ irreducible does not imply (in general) that $\rho \downarrow H$ is irreducible.

Theorem 2.1.44. *([26]). Let* G *be a group,* $H \leq G$. *Let* ψ *be a character of* H. *Then there is an irreducible character* χ *of* G *such that* $\langle \chi \downarrow H, \psi \rangle_H \neq 0$.

PROOF: See Moori [26]. ■

Theorem 2.1.45. *([26]). Let* G *be a group,* $H \leq G$. *Let* $\chi \in \mathrm{Irr}(G)$ *and let* $\mathrm{Irr}(H) = \{\psi_1, \psi_2, \ldots, \psi_r\}$. *Then* $\chi \downarrow H = \sum_{i=1}^{r} d_i \psi_i$, *where* $d_i \in \mathbb{N} \cup \{0\}$ *and* $\sum_{i=1}^{r} d_i^2 \leq [G : H]$. $(*)$
Moreover, we have equality in $(*)$ *if and only if* $\chi(g) = 0\ \forall g \in G \backslash H$.

PROOF: We have

$$\sum_{i=1}^{r} d_i^2 = \langle \chi \downarrow H, \chi \downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h).\overline{\chi(h)}\ .$$

Since χ is irreducible,

$$\begin{aligned} 1 = \langle \chi, \chi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \chi(g).\overline{\chi(g)} \\ &= \frac{1}{|G|} \sum_{g \in H} \chi(g).\overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in G-H} \chi(g).\overline{\chi(g)} \\ &= \frac{|H|}{|G|} \sum_{i=1}^{r} d_i^2 + K, \end{aligned}$$

where $K = \frac{1}{|G|} \sum_{g \in G-H} \chi(g)\overline{\chi(g)}$. Since $K = \frac{1}{|G|} \sum_{g \in G-H} |\chi(g)|^2, \ K \geq 0$.

Thus

$$\frac{|H|}{|G|} \sum_{i=1}^{r} d_i^2 = 1 - K \leq 1,$$

so

$$\sum_{i=1}^{r} d_i^2 \leq |G|/|H| = [G : H].$$

Also

$$K = 0 \text{ if and only if } |\chi(g)|^2 = 0 \ \forall g \in G - H.$$

Hence $K = 0$ if and only if $\chi(g) = 0, \ \forall g \in G - H$. ■

Induced Representations

Theorem 2.1.46. *([26]). Let* G *be a group. Let* $H \leq G$ *and* T *be a representation of* H *of degree* n. *Extend* T *to* G *by* $T^0(g) = T(g)$ *if* $g \in H$ *and* $T^0(g) = 0_{n \times n}$ *if* $g \notin H$. *Let* $\{x_1, x_2, \ldots, x_r\}$ *be a right transversal of* H *in* G. *Define* $T \uparrow G$ *by*

$$(T \uparrow G)(g) := \begin{pmatrix} T^0(x_1 g x_1^{-1}) & T^0(x_1 g x_2^{-1}) & \cdots & T^0(x_1 g x_r^{-1}) \\ T^0(x_2 g x_1^{-1}) & T^0(x_2 g x_2^{-1}) & \cdots & T^0(x_2 g x_r^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ T^0(x_r g x_1^{-1}) & T^0(x_r g x_2^{-1}) & \cdots & T^0(x_r g x_r^{-1}) \end{pmatrix} = \left(T^0(x_i g x_j^{-1})\right)_{i,j=1}, \ \forall g \in G.$$

Then $T \uparrow G$ *is a representation of* G *of degree* nr.

PROOF : See Moori [26]. ■

Definition 2.1.33. (Induced Representation/Character) *The representation* $T \uparrow G$ *defined above is said to be* **induced** *from the representation* T *of* H. *Let* ϕ *be the character afforded by* T.

Then the character afforded by $T \uparrow G$ *is called the* **induced character** *from* ϕ *and is denoted by* ϕ^G. *If we extend* ϕ *to* G *by* $\phi^0(g) = \phi(g)$ *if* $g \in H$ *and* $\phi^0(g) = 0$ *if* $g \notin H$, *then*

$$\phi^G(g) = \mathrm{trace}\big((T \uparrow G)(g)\big) = \sum_{i=1}^{r} \mathrm{trace}\big(T^0(x_i g x_i^{-1})\big) = \sum_{i=1}^{r} \phi^0(x_i g x_i^{-1}).$$

Note also that $\phi^G(1_G) = nr = \frac{|G|}{|H|}.\phi(1)$.

Proposition 2.1.47. *([26]). The values of the induced character* ϕ^G *are given by*

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^0(xgx^{-1}) \ \ ,\forall g \in G.$$

PROOF: See Moori [26]. ■

Proposition 2.1.48. *([26]). Let* G *be a group. Let* $H \leq G$. *Assume that* ϕ *is a character of* H *and* $g \in G$. *Let* $[g]$ *denote the conjugacy class of* G *containing* g.

1. *If* $H \cap [g] = \emptyset$, then $\phi^G(g) = 0$,
2. *if* $H \cap [g] \neq \emptyset$, then $\phi^G(g) = \big|C_G(g)\big| \sum_{i=1}^{m} \frac{\phi(x_i)}{\big|C_H(x_i)\big|}$,

where $x_1, x_2, \ldots, x_m$ *are representatives of classes of* H *that fuse to* $[g]$. *That is* $H \cap [g]$ *breaks up into* m *conjugacy classes of* H *with representatives* $x_1, x_2, \ldots, x_m$.

PROOF: By Proposition 2.1.47 we have

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^0(xgx^{-1}).$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ for all $x \in G$, so $\phi^0(xgx^{-1}) = 0 \ \forall x \in G$ and $\phi^G(g) = 0$. Now suppose that $H \cap [g] \neq \emptyset$. As x runs over G, xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so

$$\begin{aligned}
\phi^G(g) &= \frac{1}{|H|} \times |C_G(g)| \sum_{y \in [g]} \phi^0(y) \\
&= \frac{|C_G(g)|}{|H|} \sum_{y \in [g] \cap H} \phi(y) \\
&= \frac{|C_G(g)|}{|H|} \sum_{i=1}^{m} \big[H : C_H(x_i)\big] \phi(x_i) \\
&= |C_G(g)| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_H(x_i)|}.
\end{aligned}$$

■

The Frobenius Reciprocity Law

Definition 2.1.34. (Induced Class Function) *Let* G *be a group. Let* $H \leq G$ *and* ϕ *be a class function on* H*. Then the* ***induced class function*** ϕ^G *on* G *is defined by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}),$$

where ϕ^0 *coincides with* ϕ *on* H *and is zero otherwise.*

Note also that

$$\begin{aligned}\phi^G(ygy^{-1}) &= \frac{1}{|H|}\sum_{x\in G}\phi^0(xygy^{-1}x^{-1}) = \frac{1}{|H|}\sum_{x\in G}\phi^0((xy)g(xy)^{-1}) \\ &= \frac{1}{|H|}\sum_{z\in G}\phi^0(zgz^{-1}) = \phi^G(g).\end{aligned}$$

Thus ϕ^G is also a class function on G.

Note 2.1.3. Let G be group. If $H \leq G$ and ϕ is a class function on G, then $\phi \downarrow H$ is a class function on H.

Induction and Restriction of characters are related by the following result.

Theorem 2.1.49. (Frobenius Reciprocity) *Let* G *be a group. Let* $H \leq G$*,* ϕ *be a class function on* H *and* ψ *a class function on* G*. Then*

$$\langle \phi, \psi \downarrow H\rangle_H = \langle \phi^G, \psi\rangle_G.$$

PROOF:

$$\begin{aligned}\langle \phi^G, \psi\rangle_G &= \frac{1}{|G|}\sum_{g\in G}\phi^G(g)\,.\,\overline{\psi(g)} \\ &= \frac{1}{|G|}\sum_{g\in G}\Big(\frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1})\Big)\,.\,\overline{\psi(g)} \\ &= \frac{1}{|G||H|}\sum_{g\in G}\sum_{x\in G}\phi^0(xgx^{-1})\,.\,\overline{\psi(g)}\;. \qquad (2.3)\end{aligned}$$

Let $y = xgx^{-1}$. Then as g runs over G, xgx^{-1} runs through G. Also since ψ is a class function on G, $\psi(y) = \psi(xgx^{-1}) = \psi(g)$. Thus by 2.3 above we have

$$\begin{aligned}\langle \phi^G, \psi\rangle_G &= \frac{1}{|G||H|}\sum_{y\in G}\sum_{x\in G}\phi^0(y)\overline{\psi(y)} \\ &= \frac{1}{|G||H|}\sum_{x\in G}\Big(\sum_{y\in G}\phi^0(y)\overline{\psi(y)}\Big) \\ &= \frac{1}{|G||H|}\,.\,|G|\sum_{y\in G}\phi^0(y)\overline{\psi(y)} \\ &= \frac{1}{|H|}\sum_{y\in H}\phi(y)\overline{\psi(y)} = \langle \phi, \psi \downarrow H\rangle_H\;.\end{aligned}$$

■

Normal Subgroups

Definition 2.1.35. (**Conjugate Class Function/Representation**) *Let* G *be a group. Let* $N \trianglelefteq G$. *If* ϕ *is a class function on* N, *for each* $g \in G$ *define* $\phi^g(n) = \phi(gng^{-1})$, $n \in N$. *The function* ϕ^g *is said to be* ***conjugate*** *to* ϕ *in* G. *Also if* P *is a representation of* $N \trianglelefteq G$, *the* ***conjugate representation*** *is* P^g *given by* $P^g(n) = P(gng^{-1})$.

Proposition 2.1.50. *([26]). Let* G *be a group. Let* $N \trianglelefteq G$ *and* ϕ, ψ *class functions on* N. *Let* $x, y \in G$. *Then*

1. ϕ^x *is a class function on* N ;
2. $(\phi^x)^y = \phi^{xy}$;
3. $\langle \phi^x, \psi^y \rangle = \langle \phi, \psi \rangle$;
4. $\langle \chi \downarrow N, \phi^x \rangle = \langle \chi \downarrow N, \phi \rangle$ where χ is a class function on G ;
5. *If* ϕ *is a character, then so is* ϕ^x.

PROOF: See Moori [26]. ■

Proposition 2.1.51. *([26]). Let* $g, h \in G$. *Then* $g \sim h$ *if and only if* $\chi(g) = \chi(h)$ *for all characters* χ *of* G.

PROOF: See Moori [26]. ■

Corollary 2.1.52. *([26]). If* $\mathrm{Irr}(G) = \{\chi_i \mid i = 1, 2, \ldots, r\}$, then $\cap_{i=1}^{r} \mathrm{Ker}(\chi_i) = \{1_G\}$.

PROOF: If $g \in \cap_{i=1}^{r}\mathrm{Ker}(\chi_i)$, then $\chi_i(g) = \chi_i(1_G)$ $\forall i = 1, 2, \ldots, r$. Hence $\chi(g) = \chi(1_G)$ for all characters χ of G. So $g \sim 1_G$ by Proposition 2.1.51. Thus $g = 1_G$. ■

Theorem 2.1.53. *([26]). Let* G *be a group. Let* $N \trianglelefteq G$. *Then there exist some irreducible characters* $\chi_1, \chi_2, \ldots, \chi_s$ *of* G *such that* $N = \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$.

PROOF: Let $\mathrm{Irr}(G/N) = \{\widehat{\chi_1}, \widehat{\chi_2}, \ldots, \widehat{\chi_s}\}$. Then by Corollary 2.1.52, we have

$$\cap_{i=1}^{s}\mathrm{Ker}(\widehat{\chi_i}) = \{1_{G/N}\} = \{N\}.$$

Let χ_i be the lift to G of $\widehat{\chi_i}$ (that is $\chi_i(g) = \widehat{\chi_i}(gN)$, for all $g \in G$). We claim $N = \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$: Since $\chi_i(n) = \widehat{\chi_i}(nN) = \widehat{\chi_i}(N) = \chi_i(1_G)$, we have $n \in \mathrm{Ker}(\chi_i)$ so $N \subseteq \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$. Now let $g \in \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$. Then

$$\widehat{\chi_i}(N) = \chi_i(1_G) = \chi_i(g) = \widehat{\chi_i}(gN) \quad , i = 1, 2, \ldots, s$$

imply that $gN \in \cap_{i=1}^{s}\mathrm{Ker}(\widehat{\chi_i}) = \{N\}$. So $g \in N$ and hence $\cap_{i=1}^{s}\mathrm{Ker}(\chi_i) \subseteq N$. Thus $N = \cap_{i=1}^{s}\mathrm{Ker}(\chi_i)$.
■

Definition 2.1.36. *Suppose that ψ is a character of a group G, and that χ is an irreducible character of G. We say that χ is a* **constituent** *of ψ if $\langle\psi,\chi\rangle \neq 0$. Thus, the constituents of ψ are the irreducible characters χ_i of G for which the integer d_i in the expression $\psi = d_1\chi_1 + \ldots + d_k\chi_k$ is non-zero.*

Theorem 2.1.54. **(Clifford Theorem)** *Let G be a group. Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Let ϕ be an irreducible constituent of $\chi \downarrow N$ and let $\phi_1, \phi_2, \ldots, \phi_k$ (where $\phi = \phi_1$) be the distinct conjugates of ϕ in G. Then*

$$\chi \downarrow N = e \sum_{i=1}^{k} \phi_i \ , \quad \text{where } e = \langle \chi \downarrow N, \phi\rangle_N.$$

PROOF: Let $n \in N$. Then

$$\phi^G(n) = \frac{1}{|N|}\sum_{x\in G} \phi^0(xnx^{-1}) = \frac{1}{|N|}\sum_{x\in G} \phi(xnx^{-1}) = \frac{1}{|N|}\sum_{x\in G} \phi^x(n) \ ,$$

where we have used the fact that $xnx^{-1} \in N$, $\forall x \in G$. Now if $\psi \in \mathrm{Irr}(N)$ and $\psi \notin \{\phi_1, \phi_2, \ldots, \phi_k\}$, then $\langle\sum_{x\in G} \phi^x, \psi\rangle_N = 0$ whence $\langle(\phi^G) \downarrow N, \psi\rangle_N = 0$. Using the Frobenius Reciprocity theorem we get

$$0 = \langle(\phi^G) \downarrow N, \psi\rangle_N = \langle\phi^G, \psi^G\rangle_G$$

and

$$0 \neq \langle\phi, \chi \downarrow N\rangle_N = \langle\phi^G, \chi\rangle_G.$$

Thus

$$\langle\chi, \psi^G\rangle_G = 0 \ ; \ \text{so } \langle\chi \downarrow N, \psi\rangle_N = 0.$$

Hence

$$\chi \downarrow N = \sum_{i=1}^{k} \langle\chi \downarrow N, \phi_i\rangle_N \phi_i.$$

Now by Proposition 2.1.50(4) we have

$$\langle\chi \downarrow N, \phi_i\rangle_N = \langle\chi \downarrow N, \phi\rangle_N = e \text{ for all } i = 1, 2, \ldots, k.$$

Thus

$$\chi \downarrow N = \sum_{i=1}^{k} e\, \phi_i = e \sum_{i=1}^{k} \phi_i.$$

■

Definition 2.1.37. **(Inertia Group)** *Let G be a group. Let $N \trianglelefteq G$ and let $\phi \in \mathrm{Irr}(N)$. Then the* ***inertia group*** *of ϕ is defined by*

$$I_G(\phi) := \{g \in G \mid \phi^g = \phi\}.$$

Proposition 2.1.55. *([26]). Let* G *be a group. Let* $N \trianglelefteq G$, $\phi \in \mathrm{Irr}(N)$. *Then* $\phi^G \in \mathrm{Irr}(G)$ *if and only if* $I_G(\phi) = N$.

PROOF: Let $g, k \in G$. Then $\phi^g = \phi^k$ if and only if $\phi^{gk^{-1}} = \phi$ if and only if $gk^{-1} \in I_G(\phi)$ if and only if $I_G(\phi).g = I_G(\phi).k$. So if $\{t_1, t_2, \ldots, t_m\}$ is a right transversal for $I_G(\phi)$ in G then $\phi^{t_1}, \phi^{t_2}, \ldots, \phi^{t_m}$ is a complete set of distinct conjugates of ϕ in G. Now for any $g \in G$ we have

$$\phi^G(g) = \frac{1}{|N|}\sum_{x \in G} \phi^0(xgx^{-1}) = \frac{1}{|N|}\sum_{y \in I}\sum_{j=1}^{m} \phi^0(yt_j g t_j^{-1} y^{-1}) \ , \ \text{where } I = I_G(\phi).$$

Thus $\forall\, n \in N$

$$\begin{aligned}(\phi^G \downarrow N)(n) &= \frac{1}{|N|}\sum_{y \in I}\sum_{j=1}^{m} \phi(yt_j n t_j^{-1} y^{-1}) \\ &= \frac{1}{|N|}|\,I\,|\sum_{j=1}^{m} \phi(t_j n t_j^{-1}) \\ &= [I:N]\sum_{j=1}^{m} \phi^{t_j}(n).\end{aligned}$$

(Note: We have used the fact that $yt_jnt_j^{-1}y^{-1} \in N$, $\forall y \in I$, $\forall t_j$). Hence

$$\langle \phi^G \downarrow N, \phi\rangle_N = [I:N]\sum_{j=1}^{m} \langle \phi^{t_j}, \phi\rangle_N = [I:N] \ ,$$

since $\phi^{t_j} \neq \phi$ if $j \neq 1$ and ϕ^{t_j} are irreducible (because ϕ is and by Proposition 2.1.50(3)). Now by the Frobenius Reciprocity theorem we have

$$\langle \phi^G, \phi^G\rangle_G = \langle \phi^G \downarrow N, \phi\rangle_N = [I:N].$$

So ϕ^G is irreducible if and only if $[I:N] = 1$. Hence ϕ^G is irreducible if and only if $N = I_G(\phi)$. ■

Proposition 2.1.56. *([26]). Let* G *be group. Assume that* $G = N : H$. *That is* G *is a split extension of* N *by* H. *Let* $\phi \in \mathrm{Irr}(N)$. *Then* $I_G(\phi) = N : I_H(\phi)$. *Hence* $\phi^G \in \mathrm{Irr}(G)$ *if and only if* $I_H(\phi) = \{1_H\}$.

PROOF: Since $N \leq I_G(\phi)$ and $N \trianglelefteq G$, $N \trianglelefteq I_G(\phi)$. Let $g \in I_G(\phi)$. Then $g \in G$ and $g = nh$ where $n \in N$ and $h \in H$. So

$$\phi = \phi^g = \phi^{nh} = (\phi^n)^h = \phi^h.$$

Hence $h \in I_H(\phi)$; so $I_G(\phi) \subseteq NI_H(\phi)$. Similarly we can show that $NI_H(\phi) \subseteq I_G(\phi)$. Thus $NI_H(\phi) = I_G(\phi)$. Since $I_H(\phi) \subseteq H$ and $H \cap N = \{1_G\}$, $N \cap I_H(\phi) = \{1_G\}$. Thus $I_G(\phi) = N : I_H(\phi)$. Now $I_G(\phi) = N : I_H(\phi) = N$ if and only if $I_H(\phi) = \{1_G\}$. The result now follows by Proposition 2.1.55. ■

2.1.7 Fischer Matrices

Much of the theory covered in Chapter 6 and Chapter 7 regarding the Fischer matrices and characters tables of the double Frobenius groups $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ and $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ is based on the theory covered here. The main purpose of this section is the theory behind construction of the Fischer matrices and thereafter the character tables of the two double Frobenius groups mentioned above. We start first with some theorems and results which will be important for the theory of the Fischer matrices.

Let $\overline{G} = N{:}G$ be an extension of a group N by a group G. Let $\theta \in \mathrm{Irr}(N)$, where $N \trianglelefteq \overline{G}$. Then $\overline{G}$ permutes the irreducible characters of N by $g : \theta \to \theta^g$. Since N acts trivially on $\mathrm{Irr}(N)$, $\mathrm{Irr}(N)$ is permuted by $\overline{G}/N$, by $gN : \theta \to \theta^g$.

As a consequence of Clifford's Theorem, we have the following theorem whose proof we refer to [38].

Theorem 2.1.57. *([38]) Let $N \trianglelefteq \overline{G}$, $\theta \in \mathrm{Irr}(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$, the inertia group of θ in $\overline{G}$. Then induction to $\overline{G}$ maps the irreducible characters of $\overline{H}$ that contain θ in their restriction to N faithfully onto the irreducible characters of $\overline{G}$ which contain θ in their restriction to N.*

PROOF: See [38]. ∎

Theorem 2.1.57 shows that to find the irreducible characters of $\overline{G}$ that contain θ in their restriction to N, it suffices to find the irreducible characters of $\overline{H} = I_{\overline{G}}(\theta)$ that contain θ in their restriction. If θ can be extended to an irreducible character ψ of $\overline{H}$, then the relevant characters of $\overline{H}$ can be obtained by using Gallagher's Theorem which we state here.

Theorem 2.1.58. *(Gallagher)([12]). Let $\theta \in \mathrm{Irr}(N)$, where $N \trianglelefteq \overline{G}$. If θ extends to an irreducible character $\psi \in \mathrm{Irr}(\overline{H}$ then as β ranges over all irreducible characters of $\overline{H}$ that contain N in their kernel, $\beta\psi$ ranges over all irreducible characters of $\overline{H}$ that contain θ in their restriction.*

PROOF: See [38]. ∎

So if $\overline{G}$ is an extension of N by G and every irreducible character of N can be extended to its inertia group in $\overline{G}$, then by using Theorem 2.1.57 and Theorem 2.1.58, the characters of $\overline{G}$ can be obtained in the following way.

Let $\theta_1, \theta_2, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$. For each i, let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and let $\psi_i \in \mathrm{Irr}(\overline{H}_i)$ with $\psi_i|N = \theta_i$. Now each irreducible character of $\overline{G}$ contains some θ_i in its restriction to N by Clifford's Theorem, so by Theorem 2.1.57 and Theorem 2.1.58, we have that

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\psi_i)^{\overline{G}} : \beta \in \mathrm{Irr}(\overline{H}_i), N \subset \ker(\beta)\}.$$

Hence, the characters of $\overline{G}$ fall into blocks, with each block corresponding to an inertia group.

Let $\overline{G}$ be an extension of N by G, with the property that every irreducible character of N can be extended to its inertia group. Then as we have seen above

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\psi_i)^{\overline{G}} : \beta \in Irr(\overline{H}_i), N \subset ker(\beta)\}.$$

Using this result, we will construct the character table of $\overline{G}$ by constructing a matrix for each conjugacy class of G. These are the Fischer matrices and together with the character tables of factor groups of the inertia groups, we can construct the character table of $\overline{G}$.

Remark 2.1.6. Let $\theta_1, \theta_2, \dots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$, and let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and $H_i = \overline{H}_i/N$. Let ψ_i be an extension of θ_i to $\overline{H}_i$. We take $\theta_1 = 1_N$, so $\overline{H}_1 = \overline{G}$ and $H_1 = G$. Consider a conjugacy class $[g]$ of G with representative g.

Let $X(g) = \{x_1, x_2, \dots, x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy classes of elements of the coset $N\overline{g}$. Take $x_1 = \overline{g}$.

Let $R(g)$ be a set of pairs (i, y) where $i \in \{1, 2, \dots, t\}$ such that H_i contains an element of $[g]$ and y ranges over representatives of the conjugacy classes of H_i that fuse to $[g]$. Corresponding to this $y \in H_i$, let $\{y_{l_m}\}$ be representatives of conjugacy classes of $\overline{H}_i$ that contain liftings of y.

If $\beta \in Irr(\overline{H}_i)$ with $N \subset Ker(\beta)$, then β has been lifted from some $\hat{\beta} \in Irr(H_i)$, with $\hat{\beta}(y) = \beta(y_{l_m})$ for any lifting y_{l_m} of y. For convenience we will write $\beta(y)$ for $\hat{\beta}(y)$. Using the formula for induced characters, Proposition 2.1.48, we have

$$\begin{aligned}(\psi_i\beta)^{\overline{G}}(x_j) &= \sum_{y:(i,y)\in R(g)} \sum_m{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_m})|}(\psi_i\beta)(y_{l_m}) \\ &= \sum_{y:(i,y)\in R(g)} \sum_m{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_m})|}\psi_i(y_{l_m})\hat{\beta}(y) \\ &= \sum_{y:(i,y)\in R(g)} \Big(\sum_m{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_m})|}\psi_i(y_{l_m})\Big)\beta(y)\end{aligned}$$

where by $\sum_m'$ we mean the sum over those m for which y_{l_m} is conjugate to x_j in $\overline{G}$. We now define the Fischer matrix $M(g) = \left(a^j_{(i,y)}\right)$ where

$$a^j_{(i,y)} = \sum_m{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_m})|}\psi_i(y_{l_m}). \qquad (2.1)$$

The columns of this matrix are indexed by $X(g)$ and the rows of this matrix are indexed by $R(g)$.

$M(g)$ is the Fisher-Clifford matrix of $\overline{G}$ corresponding to the coset $N\overline{g}$. Fischer showed that $M(g)$ is necessarily a square non-singular matrix. The size of $M(g)$ is $p \times c(g)$ where p is the number of conjugacy classes of elements of the inertia factors H_i's for $1 \leq i \leq t$ which fuse to $[g]$ in G and $c(g)$ is the number of conjugacy classes of elements of $\overline{G}$ corresponding to the coset $N\overline{g}$.

We then have that

$$(\psi_i\beta)^{\overline{G}}(x_j) = \sum_{y:(i,y)\in R(g)} a^j_{(i,y)}\beta(y). \tag{2.2}$$

The rows of $M(g)$ can be divided into blocks. Each block corresponds to an inertia group. Denote the submatrix corresponding to H_i by $M_i(g)$, and let $C_i(g)$ be the portion of the character table of H_i consisting of columns corresponding to the classes that fuse to [g]. Then by equation (2.2), the characters of $\overline{G}$ at the classes represented by $X(g)$ obtained from inducing characters of $\overline{H_i}$ are given by the matrix product $C_i(g).M_i(g)$.

Note 2.1.4. 1. In matrix terms $M(g)$ is given by

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix}$$

where $M_i(g)$ is the submatrix corresponding to the inertia group $\overline{H_i}$ and its inertia factor H_i. If $H_i \cap [g] = \emptyset$, then $M_i(g)$ will not exist and $M(g)$ does not contain $M_i(g)$.

2. The partial character table of $\overline{G}$ on the classes $\{x_1, x_2, \ldots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where C_i and M_i are as described above. We note also that the number of irreducible characters of $\overline{G}$ is the sum of the number of irreducible characters of the inertia factors H_i's.

2.1.8 Properties of Fischer Matrices

We describe here properties of the Fischer matrices which we will construct in Chapters 6 and 7. We follow the thesis of Whitney [38].

Lemma 2.1.59. ***(Brauer's Lemma).*** *Let* K *be a group of automorphisms of a group* G. *Then* K *also acts on* $\mathrm{Irr}(G)$ *and the number of orbits of* K *on* $\mathrm{Irr}(G)$ *is the same as that on the conjugacy classes of* G.

PROOF: See [38]. ∎

Lemma 2.1.60. *Let* K *be a group of automorphisms of a group* G*, so* K *acts on* $\mathrm{Irr}(G)$ *and on the conjugacy classes of* G *with the same number of orbits on each by Brauer's Lemma above. Consider the matrix below describing these actions.*

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array} \begin{array}{c} \begin{array}{ccccccc} l_1 & l_2 & \cdots & l_j & \cdots & l_t \end{array} \\ \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2t} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{it} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tj} & \cdots & a_{tt} \end{pmatrix} \end{array}$$

where $a_{1j} = 1$ *for* $j = 1, 2, ..., t$,
l_j*'s are lengths of orbits of* K *on the conjugacy classes of* G,
s_i*'s are lengths of orbits of* K *on* $\mathrm{Irr}(G)$,
a_{ij} *is the sum of* s_i *irreducible characters of* G *on the element* x_j*, where* x_j *is an element of the orbit of length* l_j.
Then the following relations hold for $i, i' \in \{1, 2, \ldots, t\}$*:*

$$\sum_{j=1}^{t} a_{ij}\overline{a_{i'j}}l_j = |K|s_i\delta_{ii'}.$$

PROOF: Let $\widetilde{s_i}$ denote the sum of s_i irreducible characters of G, so $\widetilde{s_i}(x_j) = a_{ij}$. Then $\langle \widetilde{s_i}, \widetilde{s_{i'}} \rangle = |G|^{-1} \sum_{j=1}^{t} l_j \widetilde{s_i}(x_j)\overline{\widetilde{s_{i'}}(x_j)} = |G|^{-1} \sum_{j=1}^{t} l_j a_{ij}\overline{a_{i'j}}$. But by orthogonality of irreducible characters, $\langle s_i, s_{i'} \rangle = \delta_{ii'} s_i$, so $\sum_{j=1}^{t} l_j a_{ij}\overline{a_{i'j}} = |G|s_i\delta_{ii'}$. ■

Now let $M(g) = \left(a^j_{(1,y)}\right)$ be the Fischer matrix for $\overline{G} = N.G$ at $g \in G$. We present $M(g)$ with corresponding "weights" for columns and rows as follows:

$$\begin{array}{c} \\ |C_{H_1}(g)| \\ \\ |C_{H_2}(y)| \\ |C_{H_2}(y')| \\ \vdots \\ \\ |C_{H_i}(y)| \\ \vdots \\ \\ |C_{H_t}(y)| \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \cdots & |C_{\overline{G}}(x_{c(g)})| \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ --- & --- & --- & --- \\ a^1_{(2,y)} & a^2_{(2,y)} & \cdots & a^{c(g)}_{(2,y)} \\ a^1_{(2,y')} & a^2_{(2,y')} & \cdots & a^{c(g)}_{(2,y')} \\ \vdots & & & \\ --- & --- & --- & --- \\ a^1_{(i,y)} & a^2_{(i,y)} & \cdots & a^{c(g)}_{(i,y)} \\ \vdots & & & \\ --- & --- & --- & --- \\ a^1_{(t,y)} & a^2_{(t,y)} & \cdots & a^{c(g)}_{(t,y)} \\ \vdots & & & \end{array} \right) \end{array}$$

The matrix $M(g)$ is divided into blocks (separated by the dashed lines), each corresponding to an inertia group. Note that $a^j_{(1,g)} = 1$ for all $j \in \{1, 2, \ldots, c(g)\}$.

Proposition 2.1.61. ***(column orthogonality)***

$$\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|.$$

PROOF: The partial character table of $\overline{G}$ at classes $x_1, x_2, \ldots, x_{c(g)}$ is

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where $C_i(g)$,$M_i(g)$ are as defined in Note 2.1.5.

By column orthogonality of the character table of $\overline{G}$, we have

$$
\begin{aligned}
|C_{\overline{G}}(x_j)|\delta_{jj'} &= \sum_{i=1}^{t}\sum_{\beta_i\in \mathrm{Irr}(H_i)}\Big(\sum_{y:(i,y)\in R(g)} a^j_{(i,y)}\beta_i(y)\overline{\sum_{y':(i,y')\in R(g)} a^{j'}_{(i,y')}\beta_i(y')}\Big)\\
&= \sum_{i=1}^{t}\sum_{\beta_i\in \mathrm{Irr}(H_i)}\Big(\sum_{y} a^j_{(i,y)}\overline{a^{j'}_{(i,y)}}\beta_i(y)\overline{\beta_i(y)}+\sum_{y}\sum_{y'\neq y} a^j_{(i,y)}\overline{a^{j'}_{(i,y')}}\beta_i(y)\overline{\beta_i(y')}\Big)\\
&= \sum_{i=1}^{t}\Big(\sum_{y} a^j_{(i,y)}\overline{a^{j'}_{(i,y)}}\sum_{\beta_i\in \mathrm{Irr}(H_i)}\beta_i(y)\overline{\beta_i(y)}\sum_{y}\sum_{y'\neq y} a^j_{(i,y)}\overline{a^{j'}_{(i,y')}}\sum_{\beta_i\in \mathrm{Irr}(H_i)}\beta_i(y)\overline{\beta_i(y')}\Big)\\
&= \sum_{i=1}^{t}\Big(\sum_{y} a^j_{(i,y)}\overline{a^{j'}_{(i,y)}}|C_{H_i}(y)|+0\Big)\\
&= \sum_{(i,y)\in R(g)} a^j_{(i,y)}\overline{a^{j'}_{(i,y)}}|C_{H_i}(y)|.
\end{aligned}
$$

■

Proposition 2.1.62. *([38]) For the identity 1_G, of G, the matrix $M(1_G)$ is the matrix with rows equal to orbit sums of the action of $\overline{G}$ on $\mathrm{Irr}(N)$ with duplicate columns discarded.*
For this matrix we have $a^j_{(i,1)} = [G : H]$, and an orthogonality relation for rows:

$$\sum_{j=1}^{t} a^j_{(i,1)}a^j_{(i',1)}|C_{\overline{G}}(x_j)|^{-1} = \delta_{ii'}|C_{H_i}(1)|^{-1} = \delta_{ii'}|H_i|^{-1}.$$

PROOF: The $(i,1)$, j^{th} entry of $M(1_G)$ is

$$a^j_{(i,1)} = \sum_m \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_m})|}\psi_i(y_{l_m})$$

where we sum over representatives of conjugacy classes of $\overline{H_i}$ that fuse to $[x_j]$ in $\overline{G}$. Therefore $a^j_{(i,1)} = \psi_i^{\overline{G}}(x_j)$. By Theorem 2.1.57 $\psi_i^{\overline{G}}$ is an irreducible character of $\overline{G}$, and $\langle \psi_i^{\overline{G}}|_N, \theta_i\rangle = \langle \psi_i|_N, \theta_i\rangle = 1$. Therefore, by Cliffords Theorem, $\psi_i^{\overline{G}}|_N = \sum_\alpha \chi_\alpha$, where we sum over all $\chi_\alpha \in \mathrm{Irr}(N)$ in the orbit containing θ_i. Now $x_j \in N$, and $a^j_{(i,1)} = \sum_\alpha \chi_\alpha(x_j)$. The orthogonality relation will follow from Lemma 2.1.60. ■

Note 2.1.5. If $\overline{G}$ is a split extension of N by G where N is elementary abelian, then $M(g)$ is the matrix of orbit sums of C_g (C_g, as defined in Section 2.1.5) acting on the rows of the character table of a certain factor group of N with duplicate columns discarded. For these matrices, with N elementary abelian and any extension, we have $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, and we have an orthogonality relation for rows which follows from Lemma 2.1.60:

$$
\begin{aligned}
\sum_{j=1}^{c(g)} m_j a^j_{(i,y)}\overline{a^j_{(i',y')}} &= \delta_{(i,y)(i',y')}|C_G(g)||C_{H_i}(y)|^{-1}|N|\\
&= \delta_{(i,y)(i',y')}a^1_{(i,y)}|N|,
\end{aligned}
$$

where $m_j = [C_g : C_{\overline{G}}(x_j)]$. From Section 2.1.5, m_j is the length of the orbit Ω of C_g, hence, $m_j = \frac{f \times |N|}{k}$.

The results contained in the Note 2.1.5 and Proposition 2.1.61 and Proposition 2.1.62 will be used in later Chapters. We list them in the following Theorem.

Theorem 2.1.63. *For a Fischer matrix* $M(g) = \left(a^j_{(i,y)}\right)$ *of* $\overline{G} = N.G$, *we have the following relations.*

1. $a^j_{(1,g)} = 1$ *for all* $j \in \{1, 2, \ldots, c(g)\}$

2. $\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$.

3. $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, *for* N *elementary abelian*

4. $\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} = \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|$, *for* N *elementary abelian.*

Note 2.1.6. Let $\overline{G} = N{:}G$ be a split extension and N an elementary abelian $2-$group. Then for $g \in G$, a lifting of g is g itself. Then C_g acts on N/M where $M = Im(\phi_g)$ (See Remark 5.2.8 in Mpono [29]). The Fischer-Clifford matrix $M(g)$ is then given by

$$M(g) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2j} & \cdots & d_{2c(g)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{i1} & d_{i2} & d_{i3} & \cdots & d_{ij} & \cdots & d_{ic(g)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{t1} & d_{t2} & d_{t3} & \cdots & d_{tj} & \cdots & d_{tc(g)} \end{pmatrix}$$

where the d_{ij}'s are the orbit sums of C_g acting on the rows of the character table of N/M.

The following three Propositions are from Mpono [29]. We state them without proof.

Proposition 2.1.64. $d_{i1} \in \mathbb{N}$ *for all* $i \in \{2, 3, \ldots, t\}$ *where* d_{i1} *are from the Fischer matrix* $M(g)$ *in Note 2.1.6.*

PROOF: See [29]. ■

For $j \geq 2$, we obtain that

$$d_{ij} = \sum_{\chi \in \Delta_i} \chi(\overline{x_j}),$$

where Δ_i's are the orbits of C_g acting on $Irr(N/M)$ and where $\overline{x_j} \in N/M$ is a representative of the j-th orbit under the action of C_g on the elements of N/M. Since $\chi(\overline{x_j}) \in \{-1, 1\}$, we have $d_{ij} \in \mathbb{Z}$.

Proposition 2.1.65. $d_{ij} \equiv d_{i1}$ *(mod 2) for all* $j \geq 2$.

PROOF: See [29]. ■

Note 2.1.7. Since $d_{ij} \in \mathbb{Z}$, we deduce that the Fischer matrix $M(g)$ will have integer entries d_{ij} such that $d_{i1} \geq |d_{ij}|$ and $d_{ij} \equiv d_{i1} (\text{mod } 2)$. If $d_{i1} = n$ for some $n \in \mathbb{N}$, then for $j \geq 2$ we have $d_{ij} \in \{\pm 1, \pm 3, ..., \pm n\}$ if n is odd and $d_{ij} \in \{0, \pm 2, \pm 4, ..., \pm n\}$ if n is even. For a fixed n there are $n + 1$ possible values for each d_{ij} with $j \geq 2$. Note also that $\sum_i d_{i1} = |N/M| = k$.

Proposition 2.1.66. *For any* j*-th column of* $M(g)$ *for which* $j \geq 2$*, we obtain that* $\sum_i d_{ij} = 0$.

PROOF: See [29]. ■

3

Frobenius Groups

3.1 Introduction

This chapter contains most of the main results on Frobenius groups mentioned in the Masters thesis of the author. Frobenius groups play an important role in finite group theory as point stabilizers of Zassenhaus groups (doubly transitive permutation groups in which some non-identity element fixes two points but none fixes three). If G is a Zassenhaus group on a set Ω and $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$, then G_α is a Frobenius group with complement $G_{\alpha\beta}$. If K is the Frobenius kernel of G_α, then the non-identity elements of K are exactly those elements of G which fix α and no other point. It follows easily that $G_\alpha = N_G(K)$, see Proposition 3.2.8 below.

Definition 3.1.1. *([20]). Let* G *be a* transitive permutation group *on a set* Ω *with* $|\Omega| > 1$. *Then* G *is said to be a Frobenius Group on* Ω *if:*

1. $G_\alpha \neq \{1_G\}$ *for any* $\alpha \in \Omega$.
2. $G_\alpha \cap G_\beta = \{1_G\}$ *for all* $\alpha, \beta \in \Omega$ *and* $\alpha \neq \beta$.

Note 3.1.1. G_α here is the *stabilizer* of $\alpha \in \Omega$.

Remark 3.1.1. Although we have defined them as permutation groups, Frobenius groups have numerous equivalent descriptions. The following proposition is one of several characterizations of Frobenius groups.

Proposition 3.1.1. *([10]). A group* G *is a Frobenius group if and only if it has a proper subgroup* $H \neq \{1_G\}$ *such that* $H \cap H^x = \{1_G\}$ *for all* $x \in G - H$.

PROOF: Assume G acts on Ω. Take $\alpha \in \Omega$ and let $G_\alpha = H \neq \{1_G\}$.
Now for any $x \in G - H$, $\alpha^x \neq \alpha$ (since if $\alpha^x = \alpha$, then $x \in G_\alpha = H$ which is a contradiction).
Let $1_G \neq y \in H$. We will show that $y \notin H^x$, $\forall x \in G - H$. Since $y \in H$, we have $\alpha^y = \alpha$.
Let $\alpha^x = \beta$ for any $x \in G - H$ and $\beta \in \Omega$. Then $\beta = \alpha^x \neq \alpha$. Now if $(\alpha^x)^y = \alpha^x$ then

$y \in G_\beta$ contradicting part (2) of definition 3.1.1. Therefore

$$(\alpha^x)^y \neq \alpha^x \Rightarrow \alpha^{xyx^{-1}} \neq \alpha^{1_G} = \alpha \Rightarrow xyx^{-1} \notin G_\alpha$$
$$\Rightarrow y \notin x^{-1}G_\alpha x = x^{-1}Hx = H^{x^{-1}} = H^{x'} \text{ for } x' \in G.$$

So $H \cap H^x = \{1_G\}\ \forall x \in G - H$.
Conversely, set $\Omega = \{xH :\ x \in G\}$ where $\{1_G\} < H < G$. Then G acts *transitively* on Ω by the *Generalised Cayley Theorem.* First we show that for any $\alpha \in \Omega, G_\alpha \neq 1_G$. Let $\alpha = g_0H$ for some $g_0 \in G$. Then

$$\begin{aligned} G_\alpha &= \{g \in G\ : gg_0H = g_0H\} = \{g \in G\ : g_0^{-1}gg_0H = H\} \\ &= \{g \in G\ : g_0^{-1}gg_0 \in H\} = \{g \in G\ : g \in\ g_0Hg_0^{-1}\} \\ &= H^{g_0} \neq \{1_G\}\ \text{ (since } H \neq \{1_G\}). \end{aligned}$$

So $G_\alpha \neq \{1_G\}$ for any $\alpha \in \Omega$. Now let $\alpha = xH$ and $\beta = yH$ for $\alpha, \beta \in \Omega$ and $x, y \in G$ such that $\alpha \neq \beta$. Then by the above argument $G_\alpha = H^x$ and $G_\beta = H^y$. We know that $H \cap H^t = \{1_G\}\ \forall t \in G \backslash H$. We just need to show that $H^x \cap H^y = \{1_G\}$. So suppose that $g \in H^x$ and $g \in H^y$ for $g \in G$. Then $g = xh'x^{-1} = yh''y^{-1}$ for $h', h'' \in H$, which implies that $h' = x^{-1}yh''y^{-1}x = x^{-1}yh''(x^{-1}y)^{-1}$. Let $w = x^{-1}y$. Then $w \in G\backslash H$ since if $w \in H$, then $x^{-1}y \in H$ which implies that $x^{-1}yH = H$ and hence that $yH = xH$, that is $\alpha = \beta$, which is a contradiction. Thus $h' \in H$ and $h' \in H^w$ implies $h' \in H \cap H^w = \{1_G\}$. Therefore, $g = xh'x^{-1} = x\{1_G\}x^{-1} = 1_G$. Hence, $H^x \cap H^y = \{1_G\}$. ■

Corollary 3.1.2. *([10]). If* G *is a Frobenius group and* $H \leq G$ *is the stabilizer of a point then* $N_G(H) = H$.

PROOF: Let $\{1_G\} < H < G$. We know that $H \subseteq N_G(H)$ since $gHg^{-1} = H\ \forall g \in H$. Let $g \in N_G(H)$ and suppose that $g \notin H$. Since $g \in N_G(H), gHg^{-1} = H\ \forall g \in G\backslash H$. But G is a Frobenius group. Therefore $H^g \cap H = \{1_G\}\ \forall g \in G\backslash H$. This implies that $H^g = H = \{1_G\}\ \forall g \in G\backslash H$ contradicting the fact that $H \neq \{1_G\}$. Therefore $g \in H$ and hence $N_G(H) \subseteq H$ and the result follows. ■

Note 3.1.2. ([10]). If G is a Frobenius group on Ω and $H = G_\alpha$ for some $\alpha \in \Omega$ then H is called the **Frobenius Complement** in G. Denote by N^* the set of all $x \in G$ having **no fixed points** in Ω and set $N = N^* \cup \{1_G\}$. Then $N = \big(G\backslash \cup \{H^x : x \in G\}\big) \cup \{1_G\}$ and we call N the **Frobenius Kernel** of G.

In 1901, Frobenius proved that Frobenius **kernels** of Frobenius groups are normal subgroups. We state without proof the result below.

Proposition 3.1.3. *([10]). Suppose* G *is a Frobenius group with complement* H *and kernel* N*. Then*

1. $|N| = [G : H] > 1$.

2. If $K \trianglelefteq G$ *with* $K \cap H = \{1_G\}$ *then* $K \subseteq N$.

PROOF: 1. Since $N_G(H) = H$ by Corollary 3.1.2, there are $[G:H]$ distinct conjugates of H in G. So

$$\begin{aligned} \left|\cup\{H^x : x \in G\}\right| &= [G:H] \times \left(|H|-1\right)+1 \\ &= \frac{|G|}{|H|} \times \left(|H|-1\right)+1 \\ &= |G| - \frac{|G|}{|H|}+1 \\ &= |G| - [G:H]+1. \end{aligned}$$

Now since $N = \left(G \backslash \cup \{H^x : x \in G\}\right) \cup \{1_G\}$, we have

$$\begin{aligned} |N| &= |G| - \left(|G| - [G:H]+1\right)+1 \\ &= |G| - |G| + [G:H] - 1 + 1 \\ &= [G:H]. \end{aligned}$$

Also $\{1_G\} < H < G$ implies $[G:H] > 1$.
2. Let $1_G \neq k \in K$. Suppose $k \notin N$. Then $k \in H^z$ for some $z \in G \backslash H$. So $k = zhz^{-1}$ for some $h \in H$. So $z^{-1}kz = h \in H$ and since $K \trianglelefteq G$ we have $z^{-1}kz \in K$. Therefore $z^{-1}kz \in H \cap K = \{1_G\}$ and hence $k = 1_G$ which is a contradiction. Therefore $k \in N$ and $K \subseteq N$. ∎

Theorem 3.1.4. *([10]). If* G *is a Frobenius group with complement* H *and kernel* N *then* N *is a normal subgroup of* G.

PROOF: See Perumal [32]. ∎

Corollary 3.1.5. *If* G *is a Frobenius group with kernel* N *and complement* H *then* G *is a semi-direct product of* N *by* H.

PROOF: We know that $N = \left(G \backslash \cup \{H^x : x \in G\}\right) \cup \{1_G\}$ and $N \cap H = \{1_G\}$. Also $N \trianglelefteq G$ and $H \leq G$. So

$$|NH| = \frac{|N| \times |H|}{|N \cap H|} = \frac{(G:H) \times |H|}{1} = |G|.$$

Therefore $G = NH$. Thus G is a semi-direct product of N by H. ∎

3.2 Structure of Frobenius Groups

Proposition 3.2.1. *([10]). Suppose that* G *is a Frobenius group with complement* H *and kernel* N. *If* $1_G \neq x \in N$ *then* $C_G(x) \leq N$.

PROOF: See Perumal [32]. ∎

Proposition 3.2.2. *([10]). Suppose that* G *is a Frobenius group with complement* H *and kernel* N*. Then* $o(\mathsf{H})\big|o(\mathsf{N})-1$.

PROOF: Now G acts by conjugation on N. Restricting this action to H, the complement H acts by conjugation on N. Let $1_G \neq x \in N$. Then $H_x = \{h \in H : x^h = x\} = \{h \in H : hxh^{-1} = x\} = C_H(x)$. Now by Proposition 3.2.1, $C_G(x) \leq N$. Since $C_H(x) \subseteq C_G(x) \leq N$, $C_H(x) \leq N$. But G is a Frobenius group and $H \cap N = \{1_G\}$. Therefore $C_H(x) = \{1_G\}$. Now by the Orbit Stabilizer Theorem, we have that $|x^H| = [H : H_x]$. So $|x^H| = \frac{|H|}{|H_x|} = \frac{|H|}{1} = |H|$. Since the H - orbits partition N, $N \backslash \{1_G\}$ is a union of H - orbits each of size $|H|$. Therefore $|N| - 1 = \alpha|H|$ where α is the number of orbits. This implies that $|H|\,\big||N| - 1$. ■

Note 3.2.1. A subgroup H of a group G is called a **Hall** subgroup if $|H|$ and $[G : H]$ are relatively prime. Thus, by Proposition 3.1.3 and the following Corollary, in a Frobenius group the **order of the complement** H **and the kernel** N **are always relatively prime.**

Corollary 3.2.3. *The complement* H *of a Frobenius group* G *is a Hall subgroup of* G *and the kernel* N *is a normal Hall subgroup of* G.

PROOF: By Proposition 3.2.2, we have $|H|\,\big||N| - 1$. So $|H| \times \alpha = |N| - 1$ for some $\alpha \in \mathbb{N}$. So $|N| - \alpha\ |H| = 1$ implies that $(|N|\ ,\ |H|) = 1$. But by Proposition 3.1.3, we have $|N| = [G : H]$. So $\big([G : H]\ ,\ |H|\ \big) = 1$ and hence that H is a Hall subgroup of G. Since $G = NH$ and $|G| = |N| \times |H|$, $\frac{|G|}{|N|} = |H|$. So $[G : N] = |H|$. Hence by above, $\big(|N|\ ,\ [G : N]\ \big) = 1$. This implies that N is a normal Hall subgroup of G. ■

Theorem 3.2.4. *([10]). A finite group* G *is a Frobenius group if and only if it has a non-trivial proper normal subgroup* N *such that if* $1_G \neq x \in N$ *then* $C_G(x) \leq N$.

PROOF: If G is A Frobenius group, then by Proposition 3.2.1, we have $C_G(x) \leq N$.
Conversely suppose now that a finite group G has a non-trivial proper normal subgroup N such that if $1_G \neq x \in N$ then $C_G(x) \leq N$. First we show that N is a normal Hall subgroup of G. Suppose that N is not a normal Hall subgroup of G. There exists a prime p such that $p\big||N|$ and $p\big|[G : N]$. Let $|G| = p^{\alpha}q$ and $|N| = p^{\beta}q'$ with $\alpha > \beta$ and $(p, q) = 1 = (p, q')$. Let P be a **Sylow** p - subgroup of N and let Q be a **Sylow** p - subgroup of G with $\{1_G\} \leq P \leq Q$ and $Q \neq P$. Then $|P| = p^{\beta}$ and $|Q| = p^{\alpha}$. Since Q is a non-trivial p - group, the centre of Q is non-trivial.
Clearly $P \leq Q \cap N$. Now $Q \cap N \leq N$ and $Q \cap N \leq Q$. Therefore $Q \cap N$ is a p - subgroup of N. Now since P is a maximal p - subgroup of N we must have that $Q \cap N \leq P$ and hence $P = Q \cap N$. Let $x \in Z(Q)$ with $o(x) = p$. Then $xg = gx\ \forall g \in Q$. So $Q \subseteq C_G(x)$. Now if $x \in P$ then $x \in N$ and $C_G(x) \leq N$ implies that $Q \subseteq C_G(x) \subseteq N$ which is a contradiction, since $Q \cap N = P$. Suppose now $x \notin P$. Then for any $1_G \neq y \in P$ we have $y \in Q$ ($\because P \leq Q$) and hence $xy = yx$. Therefore $x \in C_G(y)$. Now $1_G \neq y \in N$ implies that $C_G(y) \leq N$ by Proposition 3.2.1. Hence, $x \in N$, so that $x \in Q \cap N = P$, which is a contradiction. Hence, N must be a normal Hall subgroup of G.

By the Schur-Zassenhaus Theorem there is a complement H to N in G such that $G = NH$ and $N \cap H = \{1_G\}$. Let $x \in G\backslash H$ and suppose that $H \cap H^x \neq \{1_G\}$. Since $G = NH$, we can write $x = nh$ with $x \in N$ and $h \in H$.
Then

$$\begin{aligned} H^x &= H^{nh} = nh(H)(nh)^{-1} \\ &= nh(H)h^{-1}n^{-1} = n(hHh^{-1})n^{-1} = nHn^{-1} = H^n. \end{aligned}$$

So $H \cap H^n \neq \{1_G\}$ and there exists $1_G \neq y \in H \cap H^n$ such that $y \in H$ and $y = nh'n^{-1}$ for some $1_G \neq h' \in H$. So

$$nh'n^{-1} \in H \Rightarrow (nh'n^{-1})h'^{-1} \in H \Rightarrow n(h'n^{-1}h'^{-1}) \in H.$$

But $n(h'n^{-1}h'^{-1}) \in N$ since $h'n^{-1}h'^{-1} \in N$ ($\because N \trianglelefteq G$).
Therefore $nh'n^{-1}h'^{-1} \in N \cap H = \{1_G\}$ and hence $nh' = h'n$. This implies that $h' \in C_G(x) \leq N$, which is a contradiction since $h' \neq 1_G$. Therefore $H \cap H^x = \{1_G\}$ $\forall x \in G\backslash H$ and by Proposition 3.1.1, G is a Frobenius group. ∎

Theorem 3.2.5. *([10]).*

1. *Suppose that $|G| = mn$ with $(m, n) = 1$, that either $x^n = 1_G$ or $x^m = 1_G$ $\forall x \in G$ and that $N = \{x \in G : x^n = 1_G\} \trianglelefteq G$. Then G is a Frobenius group with kernel N.*

2. *Conversely, if G is a Frobenius group with kernel N and complement H, and if $|N| = n$, $|H| = m$, then either $x^n = 1_G$ or $x^m = 1_G$ $\forall x \in G$ and $N = \{x \in G : x^n = 1_G\}$.*

PROOF: See Perumal [32]. ∎

Proposition 3.2.6. *([10]). Suppose G is a Frobenius group with kernel N and complement H and that $\{1_G\} \neq N_1 \leq N, \{1_G\} \neq H_1 \leq H$, with $H_1 \leq N_G(N_1)$. Then $G_1 = N_1H_1$ is a Frobenius group with kernel N_1 and complement H_1.*

PROOF: See Perumal [32]. ∎

Proposition 3.2.7. *([14]). Let G be a Frobenius group with kernel N and let K be a subgroup of G. Then one of the following must occur.*

1. $K \subseteq N$.

2. $K \cap N = \{1_G\}$.

3. *K is a Frobenius group with kernel $N \cap K$.*

PROOF: (1) Let $M = N \cap K$ and assume that neither (1) nor (2) holds. Then $M \neq \{1_G\}$ and $M \neq K$. We have that $M \trianglelefteq K$. Now let $1_G \neq x \in M$, then $x \in N$, so by Proposition 3.2.1, $C_G(x) \subseteq N$. Also $C_K(x) \subseteq C_G(x) \subseteq N$ and $C_K(x) \subseteq K$. So $C_K(x) \subseteq N \cap K = M$. Hence, by Theorem 3.2.4, K is Frobenius with kernel $N \cap K$. ■

Proposition 3.2.8. *([14]). Let $K \neq \{1_G\}$ be a subgroup of G such that $K \neq N_G(K)$ and $C_G(x) \subseteq K \; \forall \; 1_G \neq x \in K$. Then $N_G(K)$ is a Frobenius group with Frobenius kernel K.*

PROOF: It is clear that $K \trianglelefteq N_G(K)$. If $1_G \neq x \in K$ then by hypothesis $C_{N_G(K)}(x) \subseteq C_G(x) \subseteq K$. So by Theorem 3.2.4, $N_G(K)$ is a Frobenius group with kernel K. ■

Proposition 3.2.9. *([10]). Suppose G is a Frobenius group with kernel N and complement H, and that $K \leq N; K \neq N$ and $K \trianglelefteq G$. Then G/K is Frobenius with kernel N/K.*

PROOF: By the Correspondence Theorem since $K \leq N$ and $K \trianglelefteq G, N/K \trianglelefteq G/K$. The index of N/K in G/K is $[G/K : N/K]$ and

$$\begin{aligned}
[G/K : N/K] &= |G/K|/|N/K| \\
&= (|G|/|K|) \times (|K|/|N|) \\
&= |G|/|N| \\
&= [G:N] = |H| \text{ by Proposition 3.1.3.}
\end{aligned}$$

The order of N/K is $|N/K| = |N|/|K| = \frac{[G:H]}{|K|}$ by Proposition 3.1.3. Now $[G : H]$ and $|H|$ are relatively prime since H is a Hall subgroup of G. Let $\frac{[G:H]}{|K|} = n$ and $|H| = m$.
Now if p is a prime and $p|m$ and $p|n$ then $p||H|$ and $p|[G:H]$. But this is a contradiction because H is a Hall subgroup. Therefore $(m, n) = 1$ which implies that N/K is a normal Hall subgroup of G/K. We show now that $N/K = \{xK \in G/K : (xK)^n = 1_{G/K}\}$. If $1_{G/K} \neq xK \in G/K$ then since $|N/K| = n, xK \in N/K$ implies that $(xK)^n = 1_{G/K}$. Let $1_{G/K} \neq xK \in G/K$ and $(xK)^n = 1_{G/K}$. Suppose now that $(xK) \notin N/K$. Then $x \notin N$ and since G is Frobenius either $x \in H$ or $x = n'h$ for some $1_G \neq n' \in N$ and $1_G \neq h \in H$. If $x \in H$ then since $|H| = m, x^m = 1_G$. So $x^mK = K$ implies that $(xK)^m = 1_{G/K}$ and hence that $o(xK)|m$. But this a contradiction since $o(xK)|n$ and $(m, n) = 1$. If $x = n'h$ then $x^n = n''h^n$ for some $n'' \in N$. (See the proof of Theorem 3.2.5, part(2)). Now $(xK)^n = 1_{G/K}$ implies that $x^nK = K$ and hence that $x^n \in K \leq N$. Since $x^n \in N$, $n''h^n \in N$ which implies that $h^n \in N$. Since $h \neq 1_G$, $h^n = 1_G$ implies that $o(h)|n$ which is a contradiction since $o(h)|m$ and $(m, n) = 1$. Thus we must have that $N/K = \{xK \in G/K : (xK)^n = 1_{G/K}\}$ and by part(1) of Theorem 3.2.5 , G/K is Frobenius with kernel N/K. ■

Lemma 3.2.10. *([12]). Let $N \trianglelefteq G$, $H \leq G$ with $NH = G$ and $N \cap H = \{1_G\}$. Then the following are equivalent.*

1. $C_G(n) \leq N \;\; \forall 1_G \neq n \in N$.

2. $C_H(n) = \{1_G\}\ \forall 1_G \neq n \in N$.

3. $C_G(h) \leq H\ \forall 1_G \neq h \in H$.

4. *Every* $x \in G\backslash N$ *is conjugate to an element of* H.

5. *If* $1_G \neq h \in H$, *then* h *is conjugate to every element of* Nh.

6. H *is a Frobenius complement in* G.

PROOF: See Perumal [32]. ■

Lemma 3.2.11. *([37]). If* G *is a Frobenius group with kernel* N *and* $K \trianglelefteq G$ *,then either* $K \subseteq N$ *or* $N \subseteq K$.

PROOF: Assume that $K \nsubseteq N$. Let H be a complement of N and let $x \in K\backslash N$.
First we show that $C_G(x) \cap N = \{1_G\}$. Suppose that $C_G(x) \cap N \neq \{1_G\}$ and let $y \in C_G(x) \cap N$. Since $y \in N$, by Proposition 3.2.1 $C_G(x) \leq N$. Also $y \in C_G(x)$ implies that $xy = yx$. But this implies that $x \in C_G(y) \leq N$ which is a contradiction. Thus $C_G(x) \cap N = \{1_G\}$.
Now $NC_G(x) \leq G$ and $|NC_G(x)|\,\big|\,|G|$. So

$$|NC_G(x)|\,k = |G| \ \Rightarrow\ \frac{|C_G(x)|\,|N|}{|C_G(x) \cap N|}k = |G| \ \Rightarrow\ |N|\,|C_G(x)|\,k = |N|\,|H|,$$

for some $k \in \mathbb{N}$. This implies that $|C_G(x)|\,\big|\,|H|$.
Since $|C_G(x)|\,\big|\,|H|$ implies that $|C_G(x)|\,q = |H|$ for some $q \in \mathbb{N}$, and $|G| = |C_G(x)|\,|[x]|$, we have that $|G|\,q = |C_G(x)|\,[x]|\,q = |H|\,|[x]|$. This implies that $|N|\,|H|\,q = |[x]|\,|H|$ and hence that $|N|\,q = |[x]|$. Thus $|N|\,\big|\,|[x]|$. Since $[x] \subseteq K$, $|N|\,\big|\,|K|$.
Now let $|N| = p^{\alpha}z$, then $|K| = p^{\alpha}kz$ for some $k, z \in \mathbb{N}$. Also since N is a normal Hall subgroup of G, $|G| = p^{\alpha}z'$ for $z' \in \mathbb{N}$. Let $P \in Syl_p(K)$ then $P \in Syl_p(G)$. If $Q \in Syl_p(N)$ then Q is conjugate to P in G which implies that $Q = gPg^{-1}$ for some $g \in G$. So $P = g^{-1}Qg \leq N$ (since $N \trianglelefteq G$). So every Sylow p - subgroup of K such that $p\,\big|\,|N|$ is contained in N. Hence, $N \subseteq K$. ■

Theorem 3.2.12. *([10]). If* G *is a Frobenius group with kernel* N *and complement* H, *then no subgroup of* H *is Frobenius.*

PROOF: See Perumal [32]. ■

Theorem 3.2.13. *([37]). The Frobenius kernel of a Frobenius group is unique.*

PROOF: Let G be a Frobenius group with Frobenius kernels N and N_1 with H a complement of N. By Lemma 3.2.11, without any loss of generality, we may assume that $N \subseteq N_1$. Let $K = H \cap N_1 \trianglelefteq H$. Since $N \subseteq N_1$ and $N \trianglelefteq G, N \trianglelefteq N_1$. Therefore $NK \leq N_1$. Let $n_1 \in N_1$. Then $n_1 = nh$ for $1_G \neq n \in N$ and $1_G \neq h \in H$, (since $n_1 \in G$). So $h = n^{-1}n_1 \in N_1$ (since $n^{-1} \in N \subseteq N_1$ and

$n_1 \in N_1$). So $h \in N_1 \cap H = K$. Since $h \in K$, $n_1 = nh \in NK$ which implies that $N_1 \subseteq NK$. Therefore $N_1 = NK$. If $1_G \neq x \in K$, then since N_1 is a Frobenius kernel, by Proposition 3.2.1, $C_G(x) \leq N_1$. Also $x \in K$ implies that $x \in H$ and by Lemma 3.2.10, for $1_G \neq x \in H, C_G(x) \subseteq H$. So we have for $1_G \neq x \in K, C_G(x) \subseteq H \cap N_1 = K$. Since $C_H(x) = C_G(x) \subseteq K$ for $1_G \neq x \in K$, we have that H is a Frobenius group with kernel K by Theorem 3.2.4. But this contradicts Theorem 3.2.12. Hence we have $N = N_1$. ■

Proposition 3.2.14. *([10]). Suppose that* G *is Frobenius with kernel* N *and complement* H*, and that* p, q *are primes in* $\mathbb{N}$*, not necessarily distinct. If* $K \leq H$ *and* $|K| = pq$ *then* K *is cyclic.*

PROOF: See Perumal [32]. ■

Proposition 3.2.15. *([10]). Suppose* G *is a Frobenius group with complement* H*. Let* $P \in Syl_p(H)$ *then*

1. *If* $p = 2$*, then* P *is cyclic or generalized quaternion.*
2. *If* $p \neq 2$*, then* P *is cyclic.*

PROOF: See Perumal [32]. ■

The following result was proved by J.Thompson in 1959.

Proposition 3.2.16. *Frobenius kernels are nilpotent.*

PROOF: See Passman [31, Theorem 17.4]. ■

The result in Proposition 3.2.16 implies that Frobenius kernels are solvable since every finite nilpotent group is solvable.

3.3 The Center, Commutator, Frattini and Fitting Subgroups of a Frobenius Group

We describe here briefly the Center, Commutator, Frattini and Fitting subgroups of a Frobenius group.

3.3.1 The Center

Lemma 3.3.1. *The center of a Frobenius group is trivial.*

PROOF: Let G be a Frobenius group. Now $Z(G) \leq C_G(x)$ for $x \in G$. Since $C_G(x) \leq N\ \forall x \in N$, $Z(G) \leq N$. Suppose now that $1_G \neq x \in Z(G)$, then since $Z(G) \leq N, x \notin H$. (Since $H \cap N = \{1_G\}$). Since G is Frobenius, $H^x \cap H = \{1_G\}\ \forall x \in G \backslash H$. But $x \in Z(G)$ implies that $H^x = H$ which is a contradiction. ■

3.3.2 The Commutator Subgroup

Let G be a Frobenius group with kernel N and complement H.

1. By Lemma 3.2.10, part (5), for all $n \in N$ and $1_G \neq h \in H$, there exists $g \in G$ such that $h^g = nh$. Hence

$$h^g = ghg^{-1} = nh \Rightarrow ghg^{-1}h^{-1} = n$$
$$\Rightarrow [g,h] = n \Rightarrow N \subseteq G'.$$

2. Also if the complement H has prime order (and hence abelian), then $H \cong G/N$ is abelian and $N \trianglelefteq G$ implies that $G' \subseteq N$. So by (1) above we have that $N = G'$.

3.3.3 The Frattini Subgroup

Let G be a Frobenius group with kernel N and complement H. Now if $N \trianglelefteq G$ with G finite, then $N \leq \phi(G)$ if and only if there is no proper subgroup H of G such that $G = NH$ (see Rodrigues [34]). Since in a Frobenius group the complement H is a proper subgroup of G, the above result and the result of Lemma 3.2.11 implies that $\phi(G) \leq N$.

3.3.4 Fitting Subgroup

Lemma 3.3.2. *Let G be a Frobenius group with kernel N and complement H. Let $F = F(G)$, then* N=F.

PROOF: Since N is a normal nilpotent subgroup of G, $N \subseteq F$. Since $\{1_G\} \neq N \trianglelefteq G$ and F is nilpotent, $N \cap Z(F) \neq \{1_G\}$ by Theorem 2.1.28. Let $\{1_G\} \neq g \in N \cap Z(F)$. Then $xg = gx$ $\forall x \in F$ since $g \in Z(F)$. But $xg = gx$ $\forall x \in F$ implies that $F \subseteq C_G(g)$. Since $g \in N$ and N is the kernel, $C_G(g) \leq N$ by Theorem 3.2.4. Therefore, $F \subseteq N$ and $F = F(G) = N$. ■

Note 3.3.1. 1. Frobenius groups are not nilpotent, since if $G = NH$ is a nilpotent Frobenius group with kernel N, then $N \cap Z(G) \neq \{1_G\}$ by Theorem 2.1.28, is a contradiction since Frobenius groups have a trivial center.

2. If G is a Frobenius group and the order of G is odd, then by the Feit-Thompson Theorem, G is solvable.

3. If the complement H of a Frobenius group G is solvable, then G is solvable. (Since, H solvable implies that G/N is solvable. By Proposition 3.2.16, N the Frobenius kernel is solvable. Since G/N is solvable, and $N \trianglelefteq G$ is solvable, G is solvable by Proposition 2.1.17.

4. If the complement H of a Frobenius group G has odd order, then G is solvable. This follows from the Feit-Thompson Theorem and (3) above.

5. Also, the last line in 3.3.3 above implies that $\phi(G) \leq F(G)$. Of course this is a known result for finite groups, namely if G is a finite group, then $\phi(G) \subset F(G)$. See Scott [33, Theorem 7.4.4.].

3.4 Examples of Frobenius Groups

1. The Dihedral Group D_{2q} where q is odd is a Frobenius group. Let $G = D_{2q}$ with q odd. Then

$$\begin{aligned} G &= \langle a, b : a^q = b^2 = 1_G, bab = a^{-1} \rangle \\ &= \{1_G, a, a^2, \ldots\ldots, a^{q-1}, b, ab, \ldots\ldots, a^{q-1}b\}. \end{aligned}$$

Now $o(a) = q$ and $o(b) = 2$. Let $\langle a \rangle = N$ and $\langle b \rangle = H = \{1_G, b\}$. Since N has index 2 in G, it is normal in G. Now $N \trianglelefteq G, H \leq G$, so $NH \leq G$ and $|NH| = \frac{|N||H|}{|N \cap H|} = 2q$. Therefore $G = N : H$. This implies that H is the complement of N in G. To show that D_{2q} is a Frobenius group, we must show that H is a Frobenius complement in G. We just need to show that $H \cap H^x = \{1_G\}\ \forall x \in G \backslash H$.

Now $x \in G$ implies that $x = a^k$ or $x = a^k b$ for $0 < k \leq q-1$. If $x = a^k$, then $H^x = \{1_G, a^k b a^{-k}\}$. But since $ba^k = a^{-k}b\ \forall k \in \mathbb{N}$, $H^x = \{1_G, a^k(a^k b)\} = \{1_G, a^{2k}b\}$. Suppose now that $H \cap H^x \neq \{1_G\}$ for some $x \in G \backslash H$. Then $H \cap H^x = \{1_G, b\}$. Therefore $a^{2k}b = b$ implies that $a^{2k} = 1_G$. So $q|2k$ and $q|k$. Therefore $qk' = k$ for some $k' \in \mathbb{N}$. Hence, $x = a^k = a^{qk'} = 1_G$ which is a contradiction. If $x = a^k b$ then $H^x = \{1_G, (a^k b)b(a^k b)^{-1}\} = \{1_G, a^k b a^{-k}\}$. So by the argument used above we get the contradiction $x = 1_G$. Therefore $H \cap H^x = \{1_G\}\ \forall x \in G \backslash H$. This implies that H is a Frobenius complement in D_{2q}.

2. If p and q are primes and G is a non-abelian group of order pq, then G is Frobenius.
We can assume that p and q are distinct primes, since if $p = q$ then $|G| = p^2$ and G is abelian contradicting the hypothesis. So let $|G| = pq$ with $p > q$. If $q = 2$ then $|G| = 2p$ with p odd. In this case G is either cyclic or $G = D_{2p}$. If G is cyclic then G is abelian contrary to hypothesis. If $G = D_{2p}$ then by Example 1, G is Frobenius.
So assume $|G| = pq, p > q, p \neq 2$. By Cauchy's Theorem G has an element of order p and an element of order q. Let $a \in G$ such that $o(a) = p$, then $\langle a \rangle = P$ is a Sylow p - subgroup of G. If $b \in G$ such that $o(b) = q$, then $\langle b \rangle = Q$ is a Sylow q - subgroup of G. If n_p is the number of Sylow p - subgroups of G, then $n_p \equiv 1 (\text{mod } p)$ and $n_p | q$. So $n_p = 1$ or $n_p = q$. If $n_p = q$ then $q \equiv 1 (\text{mod } p)$ and hence $q - 1 = kp$ for $k \in \mathbb{Z}$. This is not possible since $p > q$. Therefore $n_p = 1$ which implies that $P \trianglelefteq G$. If n_q is the number of Sylow q - subgroups of G, then $n_q \equiv 1 (\text{mod } q)$ and $n_q | p$. So $n_q = 1$ or $n_q = p$. If

$n_q = 1$, then $Q \trianglelefteq G$. Since $P \cap Q = \{1_G\}$ and $PQ \leq G$, $|PQ| = pq$ and hence $G = P : Q$. So $G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. Therefore G is abelian contrary to hypothesis. Therefore $n_q = p$. So there are p Sylow q - subgroups each of order q in G. Now $Q \in Syl_q(G)$ implies that $xQx^{-1} \in Syl_q(G)\ \forall x \in G$. Let $Syl_q(G) = \{Q_i : 1 \leq i \leq p\}$. Since $Q_i \cap Q_j \leq Q_i$ for $i \neq j$, we have that $Q_i \cap Q_j = \{1_G\}$. Therefore $Q \cap Q^x = \{1_G\}\ \forall x \in G \backslash Q$. This implies that Q is a Frobenius complement in G. So if p and q are primes and G is a non-abelian group of order pq, then G is Frobenius. The kernel is the Sylow p - subgroup generated by the element of order p and the complement is the Sylow q - subgroup generated by the element of order q.

3. If H is a non-trivial fixed point free group of automorphisms of a finite group N, then a semi-direct product of N by H is a Frobenius group. Let $H \leq Aut(N)$. Since H is a fixed point free non-trivial group of automorphisms of N, $\forall 1_G \neq h \in H$ and $\forall 1_G \neq n \in N, n^h \neq n$. Therefore $C_H(n) = \{1_G\}$ and so by Lemma 3.2.10, G = N:H is Frobenius (since H is a Frobenius complement in G).

4. The semi-direct product $G = \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ for p a prime is Frobenius, $p \neq 2$.

   Firstly the $Aut(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. Let $\mathbb{Z}_p = \langle a \rangle$. Each $\alpha \in Aut(\mathbb{Z}_p)$ is determined by $\alpha(a)$. Therefore $Aut(\mathbb{Z}_p) = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{p-1}\}$ where we define $\alpha_i(a) = a^i$ for $i = 1, 2, \ldots, p-1$. Let $\mathbb{Z}_p^*$ be the multiplicative group of non-zero elements of $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. Define

$$\psi : Aut(\mathbb{Z}_p) \mapsto \mathbb{Z}_p^* \text{ by } \psi(\alpha_i) = \bar{i}$$

   Then ψ is an automorphism so that $Aut(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$. Since the non-zero elements of a finite field are a cyclic group, $Aut(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. We just need to show now that $Aut(\mathbb{Z}_p)$ is fixed point free and the result will then follow from Example 3 above. We have defined $\alpha_i : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ by $\alpha_i(a) = a^i$ with $1 \leq i \leq p-1$. If $i = 1$ then $\alpha_1(a) = a$ which implies that $\alpha_1(a^i) = a^i\ \forall\ 1 \leq i \leq p-1$. This implies that $\alpha_1 = 1_{Aut(\mathbb{Z}_p)}$. So if $\alpha_i(a) = a^i$ with $2 \leq i \leq p-1$ then each $\alpha_i \in Aut(\mathbb{Z}_p)$ maps onto a different non-identity element of $\mathbb{Z}_p$. This implies that $Aut(\mathbb{Z}_p)$ is fixed point free. Therefore Z_{p-1} is a non-trivial fixed point free group of automorphisms of $\mathbb{Z}_p$ and by Example 3, the split extension $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ is Frobenius.

5. If p is a prime, q not necessarily prime and $q|p-1$, then we write $\mathbb{F}_{p,q}$ for the group of order pq with presentation: $\mathbb{F}_{p,q} = \langle a, b | a^p = b^q = 1, b^{-1}ab = a^u \rangle$ where u is an element of order q in $\mathbb{Z}_p^*$. Then $\mathbb{F}_{p,q}$ is Frobenius. Let $N = \langle a \rangle$ and $H = \langle b \rangle$. Then $|N| = p$ and N is cyclic. Also $|H| = q$ and since $q|p-1$, $(p, q) = 1$. Let $N \in Syl_p(\mathbb{F}_{p,q})$. By Sylow's theorem $n_p \equiv 1(mod p)$ and $n_p|q$. So $n_p = 1 + kp$ and $n_p k' = q$ for $k, k' \in \mathbb{Z}$. So $q = k'(1 + kp)$. But $q < p$. Therefore we must have that $k = 0$. So $n_p = 1$ and $N \trianglelefteq \mathbb{F}_{p,q}$. Therefore $|NH| = \frac{|N||H|}{|N \cap H|} = |N|\,|H| = pq$. So $\mathbb{F}_{p,q} = NH$ and we have that $\mathbb{F}_{p,q}$ is a split extension of N by

H. We now show that $H \leq Aut(N)$ and that H is a fixed point free group of automorphisms of N and thus by Example 3 the result will follow.
Now the action of H on N is given by the relation $b^{-1}ab = a^u$ where u is an element of order q in $\mathbb{Z}_{p-1}$. Since $u \in \mathbb{Z}_{p-1}$, $u \in Aut(\mathbb{Z}_p)$. If u fixes $a \in N$ then $b^{-1}ab = a$ or $bab^{-1} = a$. The definition of the multiplication in $\mathbb{F}_{p,q}$ now gives:

$$(ab)(a'b') \;=\; aba'b^{-1}bb' = a(ba'b^{-1})bb' = aa'^{b}bb' = (aa')(bb') \quad (\text{since } a'^{b} = a').$$

Hence, $N : H = N \times H$ which implies that $\mathbb{F}_{p,q}$ is abelian and hence not Frobenius. Therefore each $u \in \mathbb{Z}_{p-1}$ sends each $a \in N$ to a different , non-identity element of N. Hence, each $b \in H$ induces a fixed point free automorphism of N. Thus by Example 3 above, the semi direct product $\mathbb{F}_{p,q}$ is a Frobenius group.

6. S_3 is a Frobenius group. S_n is not Frobenius for $n > 4$. Since the only proper normal subgroup of S_n is A_n , if S_n wants to be Frobenius then it must equal a **split extension** of A_n by $\mathbb{Z}_2$ where A_n is the kernel and $\mathbb{Z}_2$ is the complement. But since the order of the kernel and the complement are relatively prime, S_n can be Frobenius if and only if $|A_n|$ is odd. This can only happen if $n = 2$ or $n = 3$. Hence, S_n is not Frobenius for $n > 4$. Now $S_1 = \{1_G\}$ and $S_2 \cong \mathbb{Z}_2$. Also $S_3 \cong D_6$ and this is the smallest Frobenius group since the group of order six is the smallest non-abelian group (Frobenius groups are non-abelian).
If $n = 4$ then the normal subgroups of S_4 are S_4, V_4, A_4 and $\{e\}$. Since the kernel is a non-trivial proper subgroup, only V_4 and A_4 can be kernels. However, in a Frobenius group the order of the complement divides the order of the kernel less one (see Proposition 3.2.2). This implies that neither A_4 $(2 \nmid 11)$ nor V_4 $(6 \nmid 3)$ can be a kernel in S_4. Hence S_4 is not Frobenius.

7. The Alternating Group A_4 is Frobenius. We will show that $A_4 = N : H$, where $N \cong V_4$ and $H \cong \mathbb{Z}_3$. Let $N = V_4 = \{1_G, \alpha, \beta, \gamma\}$ where $\alpha = (12)(34), \beta = (13)(24)$ and $\gamma = (14)(23)$, and $H = \langle a \rangle = \langle (123) \rangle$.
Now $V_4 \trianglelefteq A_4$ since V_4 is a union of conjugacy classes of A_4. Therefore $V_4\langle(123)\rangle \leq A_4$. Also it is clear that $V_4 \cap \langle a \rangle = \{1_G\}$ so,

$$|V_4\langle(123)\rangle| = \frac{|V_4| \times |\langle(123)\rangle|}{|V_4 \cap \langle(123)\rangle|} = |V_4| \times |\langle(123)\rangle| = |A_4|$$

So $A_4 = V_4$:H. The subgroup H generated by a 3-cycle in A_4 is a Sylow 3 - subgroup. By Sylow's Theorem there are four conjugates of H in A_4. That is, $[A_4 : N_{A_4}(H)] = 4$ which implies that $N_{A_4}(H) = H$. Therefore $H^x = H$ if and only if $x \in H$ which implies that $H^x \neq H \; \forall x \in G \backslash H$. Therefore $H^x \cap H = \{1_G\} \; \forall x \in G \backslash H$. Hence H is a Frobenius complement in A_4.

Frobenius groups appear prominently as maximal subgroups of the finite simple groups. For example we have the following:

Lemma 3.4.1. *([9]). Let* L *be a finite simple group* $PSL(n,q) = L_n(q)$ *and* $d = \gcd(q-1,n)$. *Then*

1. *if there exists a primitive prime divisor* r *of* $q^n - 1$, *then* L *contains a Frobenius subgroup with kernel of order* r *and complement of order* n,

2. L *contains a Frobenius subgroup with kernel of order* q^{n-1} *and cyclic complement of order* $\frac{q^{n-1}-1}{d}$.

PROOF: See Lucido [21, Lemma 7]. ■

3.5 Characters of Frobenius Groups

We describe here the characters of a Frobenius group. The main results are given by Theorem 3.5.4, Note 3.5.1, Theorem 3.5.5 and Corollary 3.5.6.

Let $S = \{1, 2, \ldots, n\}$ and $X = S \times S$. Let σ be a permutation of X and $A = [a_{1,j}]$ an $n \times n$ matrix over a field $\mathbb{F}$. Define $A^\sigma = [b_{i,j}]$ where $b_{i,j} = a_{kl}$ with $(k,l) = (i,j)^\sigma$. Since $A^{\sigma\tau} = (A^\sigma)^\tau$ for any other permutation τ of X, any permutation action on X determines a permutation action on the set of $n \times n$ matrices.

Proposition 3.5.1. *([10]). Suppose that* G *is a permutation group on* $X = S \times S$ *as above,* $\mathbb{F} \subseteq \mathbb{C}$ *and A is an invertible* $n \times n$ *matrix over* $\mathbb{F}$. *Suppose further that for each* $\sigma \in G$ *the matrix* A^σ *can be obtained from A either by permuting the rows of A or by permuting the columns of A, so* G *can be viewed either as a permutation group* G_r *on the set of rows of A or* G_c *on the set of columns of A. Then the permutation characters* θ_r *and* θ_c *of* G_r *and* G_c *respectively, are equal.*

PROOF: If $\sigma \in G$ then there are permutation matrices $\sigma(R)$ and $\sigma(C)$ for which $\sigma(R)A = \sigma(A) = A\sigma(C)$. In fact, $\sigma \mapsto \sigma(R)^t$ and $\sigma \mapsto \sigma(C)$ are permutation representations of G_r and G_c respectively. Thus $A^{-1}\sigma(R)A = \sigma(C)$. So trace $\sigma(R)^t =$ trace $\sigma(C)$ $\forall \sigma \in G$ and $\theta_r = \theta_c$. ■

Corollary 3.5.2. *In the setting of Brauer's lemma the number of orbits of* G_r *and* G_c *are equal.*

PROOF: Since $\theta_r = \theta_c$ we have that $\langle \theta_r, 1_G \rangle = \langle \theta_c, 1_G \rangle$.
But
$$\theta_r(g) = \theta_c(g) \; \forall g \in G \Rightarrow \frac{1}{|G|}\sum_{g \in G} \theta_r(g) = \frac{1}{|G|}\sum_{g \in G} \theta_c(g).$$
So the number of orbits of G_r equals the number of orbits of G_c. ■

Proposition 3.5.3. *([10]). If* G *is a Frobenius group with kernel* N *and if* $\chi_1 \neq \chi \in \mathrm{Irr}(N)$ *then* χ *has inertia group* $I_G(\chi) = N$.

PROOF: If A is the character table of N then G acts on the rows of A by **conjugating characters** and on the columns of A by **conjugating the conjugacy classes** of N. If $x \in G\backslash N, \chi \in Irr(N)$ and L is a conjugacy class of N, then $\chi^x(l) = \chi(l^x)\ \forall l \in L$. Choose $x \in G\backslash N$ and suppose $L^x = L$ for a conjugacy class L of N. Let $L = [y]$ where $1_G \neq y \in N$. Thus $y^x \in L$, so $y^x = y^n$ for some $n \in N$. So $y^{n^{-1}x} = y$ which implies that $n^{-1}x \in C_G(y)$ and hence that $x \in nC_G(y)$. But then $x \in N$ since by Proposition 3.2.1, $C_G(y) \leq N$. This is a contradiction. Therefore $y = 1_G$ and $L=\{1_G\}$. Thus $\theta_c(x) = 1\ \forall x \in G\backslash N$, and so by Brauers Lemma $\theta_r(x) = 1$. But this implies that $\chi^x \neq \chi$ if $\chi_1 \neq \chi \in Irr(N)$. Hence, if $\chi_1 \neq \chi \in Irr(N)$, then $Stab_G(\chi)$ contains elements from N only. This implies that $I_G(\chi) = N$. ■

Note 3.5.1. If G is Frobenius with kernel N and complement H then $G/N \cong H$, so any character of H can be viewed as a character of G/N, hence also as a character of G by **lifting**. In particular, $Irr(H)$ can be viewed as a subset of $Irr(G)$ in a natural way.

Theorem 3.5.4. *Suppose that G is Frobenius with complement H and kernel N.*

1. *If $\phi_1 \neq \phi \in Irr(N)$, then $\phi^G \in Irr(G)$.*
2. *If $\psi \in Irr(G)$, then either $N \subset ker\psi$ or $\psi = \phi^G$ for some irreducible character $\phi_1 \neq \phi$ of N.*
3. *If $\psi \in Irr(G)$, such that $ker\psi \not\supset N$ and ρ is the **regular representation** of H, then $\psi|_H = n\rho$ where $n \in \mathbb{N}$.*

PROOF: (1) Let $\phi_1 \neq \phi \in Irr(N)$. Then by Proposition 3.5.3, $I_G(\phi) = N$. But this implies that $\phi^G \in Irr(G)$ (see Moori, Proposition 5.7. [26]).
(2) Let $\psi|_N = \sum a_i\phi_i$ with $\phi_i \in Irr(N)$. If some $a_i \neq 0$ for $i \neq 1$, then by the Frobenius Reciprocity Theorem we have that $\langle \phi_i^G, \psi\rangle = \langle \phi_i, \psi|_N\rangle = a_i \neq 0$ and since by (1) $\phi_i^G \in Irr(G)$, we have $a_i = 1$ and $\phi_i^G = \psi$. If all $a_i = 0$ for $i \neq 1$ then $\psi|_N = a_1\phi_1$, $\psi(x) = a_1\ \forall x \in N$. Hence, $N \subset ker\psi$. So $Irr(G) = Irr(H) \cup \{\phi^G : \phi_1 \neq \phi \in Irr(N)\}$.
(3) By part (2), there is an irreducible character ϕ of N such that $\psi = \phi^G$. Now $\phi^G(y) = 0\ \forall y \in H\backslash\{1_G\}$ and also $\phi^G(1_G) = [G:N]\,\phi(1_G) = \rho(1_G)\phi(1_G)$. Thus $\psi|_H(y) = \phi^G(y) = \rho(y)\phi(y)\ \forall y \in H$, so that $\psi|_H = n\rho$ where $n = \phi(1_G)$ is a positive integer. ■

The following notes are consequences of Theorem 3.5.4

Note 3.5.2. If G is Frobenius with kernel N and complement H, then,

1. from Theorem 3.5.4(2) the irreducible characters of G are of 2 types; those with kernel containing N and those induced from non-trivial irreducible characters of N,
2. also from Theorem 3.5.4(3) the order of H divides the degree of the induced character ϕ^G for $\phi_1 \neq \phi \in Irr(N)$. That is $|H|\,|\phi^G(1_G)$. Furthermore, if $\phi_1 \neq \phi \in Irr(N)$ is a linear character then $|H| = \phi^G(1_G)$.

Theorem 3.5.5. *([10]). Suppose that* G *is a Frobenius group with kernel* N *and complement* H*, and that* ϕ, θ *are non-trivial irreducible characters of* N*. Then* $\phi^G = \theta^G$ *if and only if* $\theta \in \mathrm{Orb}_H(\phi) = \Delta_\phi$*. Furthermore,* $|\Delta_\phi| = |H|$*, so* G *has* $\frac{c(N)-1}{|H|}$ *distinct irreducible characters of the form* $\phi^G, \phi_1 \neq \phi \in \mathrm{Irr}(N)$*. (Here* $c(N)$ *is the number of conjugacy classes of* N*).*

PROOF: By Theorem 3.5.4, $\theta, \phi \in \mathrm{Irr}(N)$ imply that $\theta^G, \phi^G \in \mathrm{Irr}(G)$. Suppose that $\theta^G = \phi^G$. By Frobenius reciprocity we have that: $\langle \phi^G|_N, \theta\rangle_H = \langle \phi^G, \theta^G\rangle = 1$. So θ is an irreducible constituent of $\phi^G|_N$. So by Clifford Theorem

$$\phi^G|_N = \sum_{\theta_i \in \Delta_\phi} \theta_i. \tag{3.1}$$

Also $\langle \phi^G|_N, \phi\rangle_N = \langle \phi^G, \phi^G\rangle = 1$. So ϕ is an irreducible constituent of $\phi^G|_N$ and by Clifford Theorem

$$\phi^G|_N = \sum_{\phi_i \in \Delta_\phi} \phi_i. \tag{3.2}$$

Now (3.1) and (3.2) imply that $\sum_{\theta_i \in \Delta_\phi} \theta_i = \sum_{\phi_i \in \Delta_\phi} \phi_i$. Therefore some $\theta_i = \phi_j$, which implies that there exists $y \in G$ such that $\theta = \phi^y$. That is $\theta \in \Delta_\phi$. Note that $\theta \in \Delta_\phi$, implies that any conjugate of θ will also be in Δ_ϕ.
So

$$\begin{aligned}
\phi^G(g) &= \frac{1}{|N|}\sum_{x\in G} \phi^0(xgx^{-1}) \\
&= \frac{1}{|N|}\sum_{x\in G} \phi^x(g) \\
&= \frac{1}{|N|}\sum_{x\in G} \theta^{y^{-1}x}(g) \\
&= \frac{1}{|N|}\sum_{z\in G} \theta^z(g) \\
&= \frac{1}{|N|}\sum_{z\in G} \theta^0(zgz^{-1}) = \theta^G(g).
\end{aligned}$$

Thus $\theta^G = \phi^G$. By the Orbit Stabilizer Theorem we have that $|\mathrm{Orb}_H(\phi)||I_H(\phi)| = |H|$. Since $\phi^G \in \mathrm{Irr}(G)$, $I_H(\phi) = 1_G$ which implies that $|\Delta_\phi| = |H|$. ■

Corollary 3.5.6. *If* G *is Frobenius with complement* H *and kernel* N *then* $c(G) = c(H) + \frac{c(N)-1}{|H|}$.

PROOF: This follows from Theorem 3.5.4, part(2) and Theorem 3.5.5. Since $\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \{\phi^G : \phi_1 \neq \phi \in \mathrm{Irr}(N)\}$ and since the number of irreducible characters equals the number of conjugacy classes, the equation above gives $c(G) = c(H) + \frac{c(N)-1}{|H|}$. ■

3.5.1 Coset Analysis Applied to the Frobenius Group

We begin by making the following note which applies in this section and the next.

Note 3.5.3. In Chapter 2 in our description of coset analysis, we used the conventional notation for the split extension $\overline{G} = N{:}G$. In our Frobenius group we have used the notation $G = N{:}H$. In this section, we will use the conventional notation.

So let $\overline{G} = N : G$ be a Frobenius group with kernel N and complement G. Since the extension is split, a **lifting** of $g \in \overline{G}$ is g itself since $G \leq \overline{G}$. So $\overline{G} = \cup_{g \in G} Ng$. So in a Frobenius group, for step(1) of coset analysis we have that $C_N(\overline{g}) = C_N(g) = \{1_G\}$ by Lemma 3.2.10. So $k = 1$ here and under the action of N, Ng remains intact. Since $k = 1$, in step(2) we now have that $f_j = 1$ so that $|\Delta_j| = |N|$ and equation (2.1) now implies that: $\left|[x]_{\overline{G}}\right| = |N|.\left|[g]_G\right|$. This is the same result we obtained in Lemma 3.2.10, $(3 \Rightarrow 4)$. Also since $k = f_j = 1$, equation (2.2) now implies that: $\left|C_{\overline{G}}(x)\right| = |C_G(g)|$, which is the same result we obtained in the proof of Lemma 3.2.10. Note that in the equation: $\left|C_{\overline{G}}(x)\right| = |C_G(g)|$, the element x on the left hand side is in Δ_j and the g on the right hand side is in the coset Ng. Since the coset remains intact, $\Delta_j = Ng$ and $x \in Ng$. So we may choose this x to be g. This will give: $\left|C_{\overline{G}}(g)\right| = |C_G(g)|$.

3.5.2 Fischer Matrices of the Frobenius group

Having defined and described the Fischer matrices in Chapter 2, we now describe the Fischer matrices for a Frobenius group.

Let $\overline{G} = N{:}G$ be a Frobenius group with kernel N and complement G. By Proposition 3.5.3 we know that $\forall \chi_1 \neq \chi \in Irr(N)$, the inertia group of χ is N. Since $I_{\overline{G}}(\chi_1) = \overline{G}$, in a Frobenius group, the inertia factors are G and $\{1_G\}$ corresponding to the inertia groups $\overline{G}$ and N respectively. Let $X(g)$ and $R(g)$ be as defined in Chapter 2. We have defined the Fischer matrix $M(g)$ for $g \in G$ as a matrix whose rows are indexed by $R(g)$ and columns by $X(g)$. We now find $X(g)$ and $R(g)$ for a Fischer matrix in a Frobenius group.

If $\underline{g = 1_G}$, then $X(g)$ is made up of the class representatives of the conjugacy classes of $\overline{G}$ that come from N (since $g = 1_G$ implies that $Ng = N$). These are representatives of the $(m+1)$ orbits (m non-trivial orbits and the trivial orbit) of G on N, where $m = \frac{|N|-1}{|G|}$.

For $\underline{g = 1_G}$, the inertia factors H_i for $i = \{1, 2, \ldots, t\}$ contain $[g]$, where $t = m + 1$ orbits of G on N which is the same as the number of orbits of $\overline{G}$ on $Irr(N)$. The conjugacy classes of the H_i that fuse to $[g]$ is the singleton conjugacy classes containing the identity 1_G. So $y = 1_G$. Thus $R(g)$ contains the $t = m + 1$ ordered pairs $(i, 1)$ where $i \in \{1, 2, \ldots, t\}$. Therefore the Fischer matrix $M(1_G)$ in a Frobenius group is an $(m+1) \times (m+1)$ matrix where m is the number of non-trivial orbits of G on N. Now the entries in this matrix are given by:
$M(1_G) = [a^j_{(i,1)}] = \psi_i^{\overline{G}}(x_j)$, where $x_j \in X(g)$ for $j = 1, 2, \ldots, t = m + 1$ and $\psi_i \in Irr(N)$ is a representative of the $t = m + 1$ orbits of $\overline{G}$ on Irr(N).
Note that the $\psi_i^{\overline{G}}$ here is the induction of a character ψ_i which is the extension of $\theta_i \in Irr(N)$ to

$I_{\overline{G}}(\theta_i)$. But since $I_{\overline{G}}(\theta_i) = N \; \forall \theta_1 \neq \theta_i \in Irr(N)$, ψ_i is the same as θ_i for each i. So $\psi_i^{\overline{G}}$ is just the induction of $\psi_i \in Irr(N)$ which we know from Theorem 3.5.4 is an irreducible character of $\overline{G}$.

If $g \neq 1_G$, then $X(g)$ is made up of the representatives of $\overline{G}$ conjugacy classes of elements of Ng. But by Lemma 3.2.10, every element of Ng is conjugate to g. So the conjugacy class of $\overline{G}$ that contains g will contain Ng. So this entire coset is contained in the conjugacy class of $\overline{G}$ which has g as a representative. The coset therefore contributes to only this conjugacy class of $\overline{G}$ with representative g. Therefore $X(g) = \{g\}$.

For $g \neq 1_G$, only $H_1 = G$ contains an element of $[g]$. So $i = 1$ and $y = g$ only. So $R(g) = \{(1, g)\} \; \forall g \neq 1_G$. Therefore $\forall g \neq 1_G$, the Fischer matrix $M(g)$ is a 1×1 matrix. The entry of this matrix is given by: $M(g) = \left[a^j_{(1,g)}\right] = \psi_1^{\overline{G}}(g) = 1$

In summary then :

In a Frobenius group $\overline{G} = N{:}G$, the Fischer matrix $M(1_G)$ is an $(m+1) \times (m+1)$ matrix where m is the number of non-trivial orbits of G on N, and $M(g) \; \forall g \neq 1_G$ is just 1.

4

The Double Frobenius Group

4.1 Introduction

In the previous chapter we discussed the Frobenius group in some detail. We consider now the case when a Frobenius group acts on a finite group G. In this case a Frobenius group NH (kernel N and complement H) acts as a group of automorphisms on a finite group G. Numerous results have been obtained here regarding certain properties or parameters of G. Some of these properties include order, rank (here, a group has rank at most r if each of its finitely generated subgroups can be generated by r elements), fitting height, nilpotency class and exponent. We briefly mention some of these results omitting details. We are interested here in the case when a Frobenius group NH (kernel N and complement H) acts as a group of automorphisms on a finite group G, with $C_G(N) = \{1_G\}$ such that GN is also a Frobenius group.

4.2 Finite Groups having a Frobenius Group of Automorphisms

Suppose a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that the kernel F acts fixed point freely, that is $C_G(F) = \{1_G\}$. Then numerous results have been obtained with regard to properties or parameters of G. These properties include the order, the rank, the Fitting height, the nilpotency class and the exponent. These properties of G are related to the $|H|$, $C_G(H)$ and the exponent of $C_G(H)$. Detailed descriptions are given in [16], [17] and [18]. We state these results without proof. Proofs are given in [16] , [17] and [18].

Theorem 4.2.1. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = \{1_G\}$. Then*

1. $|G| = |C_G(H)|^{|H|}$
2. *the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$*
3. *If $C_G(H)$ is nilpotent, then G is nilpotent.*

PROOF: See [16, Theorem 2.7]. ∎

Theorem 4.2.2. *Suppose that a finite group* G *admits a Frobenius group of automorphisms* FH *with kernel* F *and complement* H *such that* $C_G(F) = \{1_G\}$ *and* $C_G(H)$ *has exponent e. Then the exponent of* G *is bounded solely in terms of e and* $|FH|$.

PROOF: See [16, Theorem 3.4]. ∎

Theorem 4.2.3. *Suppose that a finite group* G *admits a Frobenius group of automorphisms* FH *with cyclic kernel* F *of order* n *and complement* H *of order* q *such that* $C_G(F) = \{1_G\}$ *and* $C_G(H)$ *is nilpotent of class* c. *Then* G *is nilpotent of* (c, q)*-bounded class. i.e.* G *is nilpotent and the class of* G *is bounded solely in terms of* $c=$ *nilpotency class of* $C_G(H)$ *and* $q = |H|$.

PROOF: See [16, Theorem 5.8]. ∎

Theorem 4.2.4. *Suppose that a finite group* G *admits a Frobenius group of automorphisms* FH *with kernel* F *and complement* H *such that* $C_G(F) = \{1_G\}$. *Then the Fitting height of* G *is equal to the Fitting height of* $C_G(H)$.

PROOF: See [17, Theorem 2.1]. ∎

We also have the following lemmas.

Lemma 4.2.5. *Suppose that a finite group* G *admits a Frobenius group of automorphisms* FH *with kernel* F *and complement* H *such that* $C_G(F) = \{1_G\}$. *Then for each prime* p *dividing* $|G|$ *there is a unique* FH*-invariant Sylow* p *subgroup of* G.

PROOF: See [17, Lemma 1.2]. ∎

Lemma 4.2.6. *Suppose that* V *is a vector space over any field* k *admitting a finite Frobenius group of linear transformations* FH *with kernel* F *and complement* H *such that* $C_V(F) = 0$. *Then* V *is a free* kH *module.*

PROOF: See [18, Lemma 2.2]. ∎

Lemma 4.2.7. *Let* FH *be a Frobenius group with kernel* F *and complement* H. *In any action of* FH *with non-trivial action of* F *the complement acts faithfully.*

PROOF: See [17, Lemma 2.2]. ∎

In this study we are interested in the case where a Frobenius group NH with kernel N and complement H acts on a finite group G such that $C_G(N) = \{1_G\}$ and GN is also a Frobenius group with kernel G and complement N. In this case the group GNH which we will denote by $\overline{G}$ is called a double Frobenius group.

Definition 4.2.1. *([16]).(**Double Frobenius Group**). Let* G *be a finite group. Let* NH *be a Frobenius group with kernel* N *and complement* H. *If* G *admits* NH *as a Frobenius group of automorphisms with* $C_G(N) = \{1_G\}$ *such that* GN *is also a Frobenius group with kernel* G *and complement* N, *then* $\overline{G} = GNH$ *is called a double Frobenius group.*

Note: Some authors use 2-Frobenius as opposed to double Frobenius.

Note 4.2.1.

1. In the group $\overline{G} = GNH$, GN and NH are Frobenius subgroups of $\overline{G}$, GN has kernel G and complement N, NH has kernel N and complement H. Therefore, N acts both as a kernel and a complement in $\overline{G}$. The subgroup N is sometimes referred to as the upper kernel(or lower complement) and the subgroup G as the lower kernel.

2. G, N, H, NH and GN are all subgroups of $\overline{G}$, $G \trianglelefteq \overline{G}$ and $GN \trianglelefteq \overline{G}$. The subgroups N and H are Frobenius complements of GN and NH respectively and since Frobenius complements are self normalizing, that is, $N_{GN}(N) = N$ and $N_{NH}(H) = H$ (See Corollary 3.1.2), they are not normal in $\overline{G}$.

3. If $\overline{G} = GNH$ is a double Frobenius group, then $\overline{G} = G{:}NH = GN{:}H$, $\overline{G}/G \cong NH$ and $\overline{G}/GN \cong H$. Since, $GN \cap NH = N(G \cap NH)$ (applying the Dedekind Law, Lemma 2.1.7 to $N, G, NH \leq \overline{G}$), we have that $N \subseteq GN \cap NH$. Then there is $1_{\overline{G}} \neq n \in N \subseteq GN \cap NH$, $1_G \neq n' \in N$, $1_{\overline{G}} \neq x \in (G \cap NH)$, such that $n = n'x$. This implies that $x \in G \cap N = 1_{\overline{G}}$. Hence, $G \cap NH = \{1_{\overline{G}}\}$ and by the Second Isomorphism Theorem since $G \trianglelefteq \overline{G}$ and $NH \leq \overline{G}$, we have that $NH = NH/(G \cap NH) \cong GNH/G = \overline{G}/G$.

4. Similarly since, $NH \cap GN = N(GN \cap H)$ (applying the Dedekind Law, Lemma 2.1.7 to $N, H, GN \leq \overline{G}$), we have $N \subseteq NH \cap GN$. Then there is $1_{\overline{G}} \neq n \in NH \cap GN$, $1_G \neq n' \in N$, $1_{\overline{G}} \neq x \in GN \cap H$, such that $n = n'x$. This implies that $x \in N \cap H = 1_{\overline{G}}$. Hence, $GN \cap H = \{1_{\overline{G}}\}$ and by the Second Isomorphism Theorem since $GN \trianglelefteq \overline{G}$ and $H \leq \overline{G}$, we have that $H = H/(GN \cap H) \cong GNH/GN = \overline{G}/GN$.

Lemma 4.2.8. *([11]. Let* $\overline{G} = GNH$ *be a double Frobenius group with* G *and* N *Frobenius kernels to complements* N *and* H *in Frobenius groups* GN *and* NH *respectively. Then* N *and* H *are both cyclic subgroups of* $\overline{G}$. *Furthermore,* N *has odd order and* G *is not cyclic.*

PROOF: Since N is a kernel to NH, it is nilpotent. Also N is a complement to GN. Therefore all its Sylow subgroups are cyclic or generalized quaternion (see Proposition 3.2.15). If $|N|$ is even, then the Sylow 2-subgroup is generalized quaternion. Since Q_n has a unique involution, this involution must be in the center of GN. This is a contradiction since Frobenius groups have a trivial center. Therefore $|N|$ is odd and since N is nilpotent, N is cyclic.

Now $N \trianglelefteq NH$ and since N is a kernel in NH, $C_{NH}(N) \leq N$ by Theorem 3.2.4. Also

$$H \cong NH/N \cong \leq Aut(N)$$

by Theorem 2.1.1. Since N is cyclic, Aut(N) is abelian by Theorem 2.1.8. Therefore, H is abelian and by Proposition 3.2.15 all its Sylow subgroups are cyclic. Hence, by Theorem 2.1.9 H is cyclic.

Since G is the kernel of GN and $C_{GN}(G) \leq G$ we have that

$$\overline{G}/G \cong \leq Aut(G)$$

by Theorem 2.1.1. If G is cyclic, its automorphism group is abelian by Theorem 2.1.8. But $\overline{G}/G \cong NH$ which is Frobenius group. Therefore G cannot be cyclic. ■

Lemma 4.2.9. *([6]). If $\overline{G}$ is a double Frobenius group, then $\overline{G}$ is solvable.*

PROOF: Let $\overline{G} = GNH$ be a double Frobenius group with GN and NH Frobenius groups. Since $N \cong GN/G$ is nilpotent, GN/G is solvable. Now G is solvable since G is nilpotent. So $G \trianglelefteq \overline{G}$ and both G and GN/G solvable implies by Theorem 2.1.17 that GN is solvable. Also $H \cong \overline{G}/GN$ is isomorphic to a subgroup of the automorphism group of N which is cyclic by Lemma 4.2.8. Therefore, $\overline{G}/GN$ is abelian and hence solvable. Applying Theorem 2.1.17 again, since $GN \trianglelefteq \overline{G}$ and GN and $\overline{G}/GN$ are both solvable , $\overline{G}$ is solvable. ■

Lemma 4.2.10. *The center of a double Frobenius group is trivial*

PROOF: Let $\overline{G} = GNH$ be a double Frobenius group and suppose that $1_G \neq x \in Z(\overline{G})$. Then $xy = yx\ \forall y \in \overline{G}$. But, $xy = yx\ \forall y \in \overline{G}$ implies that $xy = yx\ \forall y \in GN$. This implies that $x \in Z(GN)$. This is a contradiction since, GN is a Frobenius group and $Z(GN) = \{1_G\}$ by Lemma 3.3.1. Hence, the center of $\overline{G}$ is trivial. ■.

Lemma 4.2.11. *Let $\overline{G} = GNH$ be a double Frobenius group. Then G is the Fitting subgroup of $\overline{G}$. i.e. $G = F(G)$.*

PROOF: Since $G \trianglelefteq \overline{G}$ and G is nilpotent, $G \subseteq F(G)$. Now $G \trianglelefteq \overline{G}$ implies that $G \trianglelefteq F(G)$ and since $F(G)$ is nilpotent, by Lemma 2.1.28, $G \cap Z(F(G)) \neq \{1_{\overline{G}}\}$. So let $1_{\overline{G}} \neq g \in G \cap Z(F(G))$. Then $xg = gx\ \forall x \in F(G)$ since $g \in Z(F(G))$. But $xg = gx\ \forall x \in F(G)$ and $g \in G$ implies that $F(G) \subseteq C_{\overline{G}}(g) = C_G(g) \subseteq G$ by Theorem 3.2.4. Therefore $G = F(G)$. ■

4.3 The Fitting Series in a Double Frobenius Group

Let $\overline{G} = GNH$ be a double Frobenius group. Since $\overline{G}$ is solvable, we can construct the Fitting series as described in Section 1.1.4.

Let $F_0(\overline{G}) = \{1_G\}$ and $F_1(\overline{G}) = G$ by Lemma 4.2.11. Then

$$F_2(\overline{G})/F_1(\overline{G}) = F(\overline{G}/F_1(\overline{G})) \implies F_2(\overline{G})/G = F(\overline{G}/G) \implies F_2(\overline{G})/G = N.$$

So $F_2(\overline{G}) = GN$. Now

$$F_3(\overline{G})/F_2(\overline{G}) = F(\overline{G}/F_2(\overline{G})) \implies F_3(\overline{G})/GN = F(\overline{G}/GN) \implies F_3(\overline{G})/GN = H.$$

So $F_3(\overline{G}) = GNH$.

Therefore the Fitting series is

$$\{1_G\} \trianglelefteq G \trianglelefteq GN \trianglelefteq GNH = \overline{G}$$

.

This construction of a normal series in a double Frobenius group gives two alternate descriptions of a double Frobenius group.

Definition 4.3.1. *([15]). A finite group $\overline{G}$ is called a double Frobenius group if it has a normal series $\{1_G\} \trianglelefteq G \trianglelefteq GN \trianglelefteq \overline{G}$, where GN and $\overline{G}/G$ are Frobenius groups with kernels G and GN/G, respectively.*

We also have the following description of the group which follows from Lemma 4.2.11 and section 3.2.1 above.

Definition 4.3.2. *([11]). A group $\overline{G}$ is a double Frobenius group if GN and $\overline{G}/G$ are Frobenius groups, where $G = F(\overline{G})$ and $GN/G = F(\overline{G}/G)$.*

4.4 Prime Graphs of Finite Groups and the Double Frobenius Group

4.4.1 Introduction

The first references of the *prime graph* of a finite group correspond to unpublished manuscripts of K.W.Gruenberg and O.Kegel that introduced this graph (also called the Gruenberg-Kegel graph) in the mid seventies. Gruenberg and Kegel gave a characterization of finite groups with a disconnected prime graph. This result and the classification of finite simple groups led to other results. For example, the problem of the *recognizability* of a finite group by *spectrum* or by *prime graph*. The prime graphs of finite groups has been a widely researched topic in the last few decades. One of our aims in the study of the double Frobenius group stems from the appearance and occurrence of these groups in the study of prime graphs of finite groups. Double Frobenius groups and Frobenius groups appear prominently in the study of the prime graphs of finite groups. In this section we give some definitions and briefly describe some of the results.

Definition 4.4.1. *([1]). Let* G *be a finite group. The prime graph* GK(G) *of* G *is defined as follows: the vertex set of* GK(G) *is* $\pi(G)$, *the set of all prime divisors of the order of* G, *and two distinct vertices* p *and* q *in* $\pi(G)$, *are adjacent by an edge* pq *if and only if* G *possesses an element of order* pq.

The *spectrum* $\omega(G)$ of a group G is the set of all element orders of G. The set $\omega(G)$ determines the prime graph GK(G) of G. Two groups are *isospectral* if their spectra coincide. A finite group G is said to be recognizable by spectrum (respectively by prime graph), if for any finite group H with $\omega(H) = \omega(G)$ (respectively $GK(H) = GK(G)$), we have $H \cong G$. Groups with the same spectra have coincident prime graphs.

Denote by $\pi(G) = \pi(|G|)$ the prime divisors of the order of G. Denote by $s(G)$ the number of connected components of GK(G) and denote by $\pi_i = \pi_i(G)$, where $i = 1, 2, \ldots, s(G)$, the ith connected component of GK(G). Then, the order of G can be expressed as the product of the positive integers $m_1, m_2, \ldots, m_{s(G)}$, where $\pi(m_i) = \pi_i$, $1 \leq i \leq s(G)$. That is, $|G| = \prod_{i=1}^{s(G)} m_i$. The positive integers $m_1, m_2, ..., m_{s(G)}$ are called the *order components* of G and the set $OC(G) = \{m_1, m_2, \ldots, m_{s(G)}\}$ is called the set of order components of G. We also remark that if the order of G is even, then we denote the component containing 2 by π_1.

The following classification theorem is a result of Gruenberg and Kegel in an unpublished manuscript of 1975. The theorem appears in Williams [39]. We omit the proof which is given in Williams [39].

Theorem 4.4.1. *([21]).(Gruenberg-Kegel Theorem)*

If G *is a group with a disconnected prime graph, then one of the following statements holds:*

1. G *is a Frobenius group*

2. G *is a double Frobenius group*

3. G *is an extension of a nilpotent* $\pi_1(G)$*-group by a group* A, *where* $S \trianglelefteq A \trianglelefteq Aut(S)$, S *is a simple non-abelian group with* $s(G) \leq s(S)$, *and* A/S *is a* $\pi_1(G)$*-group.*

PROOF: See Williams [38, Theorem A]. ■

We also have the following Corollary.

Corollary 4.4.2. *([39], [15]). If* G *is solvable with more than two graph components, then* G *is either Frobenius or double Frobenius. If* G *is double Frobenius then* G *has exactly two components.*

PROOF: See Williams [39] or Khosravi [15]. ■

The following two lemmas show that the *commuting graphs* of Frobenius and double Frobenius groups are disconnected. The *commuting graph*, denoted by $\Gamma(G)$, of a group G is the graph which has vertices the non-central elements of G and two distinct vertices of $\Gamma(G)$ are adjacent if and only if they commute.

Lemma 4.4.3. *([30]). If G is a Frobenius group, then the commuting graph $\Gamma(G)$ is disconnected.*

PROOF: Suppose K is a Frobenius complement of G. Then $C_G(k) \leq K$ for all $1_G \neq k \in K$ by Lemma 3.2.10. Hence the vertices of $\Gamma(G)$ in $K - \{1_G\}$ are only connected to vertices in $K - \{1_G\}$ and this implies that $\Gamma(G)$ is disconnected with one of the connected components having vertices in $K - \{1_G\}$. ■

Lemma 4.4.4. *([30]). If G is a double Frobenius group, then $\Gamma(G)$ is disconnected.*

PROOF: Let K and L be normal subgroups of G such that L is a Frobenius group with kernel K and G/K is a Frobenius group with kernel L/K. Let J be a complement to K in L. Then $L = KJ$, $L/K \cong J$ and $N_G(J) = M$. Then by the Frattini Argument, $G = MK$ and $M \bigcap K = \{1_G\}$. Hence, M is a complement to K in G. Thus, $M \cong G/K$ is a Frobenius group. We consider the subgraph of $\Gamma(G)$ spanned by the elements of $J - \{1_G\}$ and claim that this is disconnected from $G - J$. So let $1_G \neq j \in J$ and consider $C_G(j)$. Since L and M are Frobenius groups, by Theorem 3.2.4, $C_M(j) \leq J$ and $C_L(j) \leq J$. Let $x \in C_G(j)$. Now $x = mk$ for some $m \in M$ and $1_G \neq k \in K$. Then

$$j^x = j^{mk} = mkj(mk)^{-1} = m(kjk^{-1})m^{-1} = j$$

and

$$j^k = j^{m^{-1}}.$$

Now $j^{m^{-1}} \in M$ since $M = N_X(J)$ and $j^k \in L$. Therefore,

$$j^{m^{-1}} = j^k \in L \cap M = JK \cap M = J(K \cap M) = J,$$

where we have applied the Dedekind Law to subgroups $J \leq M$, K and M of G.

Since M is a Frobenius group with complement J, $m^{-1} \in J$ and since L is a Frobenius group with complement J, $k \in J$. This follows from Corollary 3.1.2. Thus, we have that $x \in J$ and $C_G(j) \leq J$ proving our claim. ■

Remark 4.4.1. 1. Aleeva in [2] shows that the list of simple groups isospectral to Frobenius groups is exhausted by the groups $L_3(3)$ and $U_3(3)$. For a simple group S isomorphic to $L_3(3)$ or $U_3(3)$, Aleeva constructs Frobenius groups G such that $\omega(S) = \omega(G)$ (examples 1 and 2 in [2]). Also in [2], Aleeva shows that the only simple groups isospectral to double Frobenius groups are groups isomorphic to $U_3(3)$ or $S_4(3)$. However, the existence problem for double Frobenius groups G with $\omega(S) = \omega(G)$, where S is isomorphic to $U_3(3)$ or $S_4(3)$ remains open.

2. Moghaddamfar shows in [24] that if S is a non-abelian finite simple group with disconnected prime graph GK(S), with an exception of $U_4(2)$ and $U_5(2)$, and G is a finite group with OC(G) = OC(S), then G is neither Frobenius nor double Frobenius. For a group S isomorphic to $U_4(2)$ or $U_5(2)$, Moghaddamfar constructs examples of double Frobenius groups G such that OC(G) = OC(S). The construction will be described in chapter 5, Example 11 and Example 12.

5

Constructing Double Frobenius Groups

5.1 Introduction

In this section we shall describe a few methods of constructing a double Frobenius group. The constructions in method 1 and 2 are general and they give us double Frobenius groups of a certain type. In each construction we will list a few examples and in the following two chapters we will do a detailed analysis of the examples, viz, finding its conjugacy classes, Fischer matrices and character table.

Before describing the methods we make the following Remark.

Remark 5.1.1. The number $2^p - 1$ where p is a prime is a Mersenne prime denoted by M_p. As of January 2016, forty nine Mersenne primes have been discovered. The number M_p is prime if $p \in \Omega$ where
$\Omega = \{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4453, 4223, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281\}$.

We describe now our methods of construction.

5.2 Method 1

Consider $\mathsf{PSL}(n, q)$ with $q = 2$. If $n \in \Omega$ in Remark 5.1.1 above, then $2^n - 1$ is prime. Now by Lemma 3.4.1, $\mathsf{PSL}(n, 2)$ contains a Frobenius subgroup with kernel of order $2^n - 1$ and cyclic complement of order n. So, $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n \leq \mathsf{PSL}(n, 2)$ for $n \in \Omega$ as in Remark 5.1.1.

Let $(G, +)$ be an elementary abelian group of order $|G| = 2^n$ where $n \in \Omega$. By the natural action of $\mathsf{GL}(n, 2) \cong \mathsf{PSL}(n, 2)$ on $G = 2^n$, we have that $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n) \leq 2^n{:}\mathsf{GL}(n, 2)$. Since $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$

is Frobenius, if the split extension $2^n{:}\mathbb{Z}_{2^n-1}$ is a Frobenius group, then $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ is a double Frobenius group. This follows from Proposition 2.1.6 and Lemma 2.1.5.

The following examples give some double Frobenius groups using Method 1.

Example 1: $\overline{G} = V_4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.
$\overline{G} = V_4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ is a double Frobenius group. Let $V_4 \cong G$, $\mathbb{Z}_3 \cong N$ and $\mathbb{Z}_2 \cong H$. Then $\overline{G} = GNH = G{:}NH$, $NH = \mathbb{Z}_3{:}\mathbb{Z}_2 \cong S_3$ and $GN = V_4{:}\mathbb{Z}_3 \cong A_4$ are both Frobenius groups by Section 3.4. The $\mathrm{Aut}\ V_4 \cong GL(2,2) \cong PSL(2,2) \cong S_3$ and there is a natural action of S_3 on the elementary abelian group V_4. Since S_3 has only one conjugacy class of elements of order three, the subgroup $N \cong \mathbb{Z}_3$, by Lemma 2.1.5 acts fixed point free in this action of S_3 on the vector space $G = V_4$. Hence, by Proposition 2.1.6, $\overline{G} = V_4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ is a double Frobenius group.

Example 2: $\overline{G} = 2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$.
Let $n = 3 \in \Omega$. Then we claim that $\overline{G} = 2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$ is a double Frobenius group. Let $G \cong 2^3$, $N \cong \mathbb{Z}_7$, $H \cong \mathbb{Z}_3$, then $\overline{G} = GNH = G{:}NH$. Now $NH = \mathbb{Z}_7{:}\mathbb{Z}_3$ is a maximal Frobenius subgroup of $L_3(2) \approx L_2(7) \cong GL(3,2)$. Since $L_3(2)$ has a single conjugacy class of elements of order seven, the subgroup $N \cong \mathbb{Z}_7$, by Lemma 2.1.5 acts fixed point free in the action of NH on the vector space $G \cong 2^3$. Hence, by Proposition 2.1.6, $\overline{G} = 2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$ is a double Frobenius group. This double Frobenius group is a maximal subgroup of the Sporadic Janko Group J_1.

Example 3: $\overline{G} = 2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$.
Let $n = 5 \in \Omega$. Then we claim that $\overline{G} = 2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$ is a double Frobenius group. Let $G \cong 2^5$, $N \cong \mathbb{Z}_{31}$, $H \cong \mathbb{Z}_5$, then $\overline{G} = GNH = G{:}NH$. Now $NH = \mathbb{Z}_{31}{:}\mathbb{Z}_5$ is a maximal Frobenius subgroup in $L_5(2) \cong GL(5,2)$ and since $L_5(2)$ has a single conjugacy class of elements of order thirty one, the subgroup $N \cong \mathbb{Z}_{31}$, by Lemma 2.1.5 acts fixed point free in the action of NH on the vector space $G \cong 2^5$. Hence, by Proposition 2.1.6, $\overline{G} = 2^5{:}(\mathbb{Z}_{32}{:}\mathbb{Z}_5)$ is a double Frobenius group.

5.3 Method 2

We aim to construct double Frobenius groups using $PSL(n, q)$ with $n = 2$ and q even. For q even, $PSL(2, q)$ has maximal subgroups which are Dihedral groups of order $2(q-1)$ or $2(q+1)$, (See King [19]).

Since q is even, $q-1$ is odd so the maximal subgroup of order $2(q-1)$ is a Frobenius group since the

Dihedral group D_{2m} is Frobenius if m is odd, see Section 3.4. Let $q = 2^r$, $r \in \mathbb{N}$, then the Frobenius group $D_{2(q-1)}$ has the form $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$. Now for q even, $PSL(2,q) \cong SL((2,q) \leq GL(2,q)$. The natural action of $GL(2,q)$ on the elementary abelian group of order q^2 implies that $q^2{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2) \leq q^2{:}GL(2,q)$

Therefore now, $(2^r)^2{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2) \leq (2^r)^2{:}GL(2,2^r)$. Since 2^r-1 divides $2^{2r}-1$ (because $2^{2r}-1 = (2^r-1)\times(2^r+1)$), if $2^{2r}{:}\mathbb{Z}_{2^r-1}$ is a Frobenius group, then by Proposition 2.1.6 and Lemma 2.1.5, $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ is a double Frobenius group.

The following examples give double Frobenius groups using Method 2.

Example 4: $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.
Let $r = 2$, then $\overline{G} = GNH = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ is a double Frobenius group. We have $G \cong 2^4$, $N \cong \mathbb{Z}_3$ and $H \cong \mathbb{Z}_2$. We have that $NH = \mathbb{Z}_3 : \mathbb{Z}_2 \cong S_3$ is a maximal Frobenius subgroup of $PSL(2,4)$ and since $PSL(2,4)$ has a single conjugacy class of elements of order three, the subgroup $N \cong \mathbb{Z}_3$, by Lemma 2.1.5 acts fixed point free in the action of NH on $G \cong 2^4$. Hence, by Proposition 2.1.6, $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ is a double Frobenius group.

Example 5: $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$.
Let $r = 3$, then we claim that $\overline{G} = GNH = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$ is a double Frobenius group. We have $G \cong 2^6$, $N \cong \mathbb{Z}_7$ and $H \cong \mathbb{Z}_2$. Now $NH = \mathbb{Z}_7{:}\mathbb{Z}_2$ is a maximal Frobenius subgroup of $PSL(2,8)$. The $|PSL(2,8)| = 2^3 \times 3^2 \times 7$ and in $PSL(2,8)$ a Sylow 7-subgroup is isomorphic to $\mathbb{Z}_7$. By Lemma 2.1.5, $N \cong \mathbb{Z}_7$ acts fixed point free in the action of NH on the vector space $G \cong 2^6$. Hence, by Proposition 2.1.6, $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$ is a double Frobenius group.

Example 6: $\overline{G} = 2^{10}{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_2)$.
Let $r = 5$, then $\overline{G} = GNH = 2^{10}{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_2)$ is a double Frobenius group. We have $G \cong 2^{10}$, $N \cong \mathbb{Z}_{31}$ and $H \cong \mathbb{Z}_2$. Note that $NH = \mathbb{Z}_{31}{:}\mathbb{Z}_2$ is a maximal Frobenius subgroup of $PSL(2,32)$. The $|PSL(2,32)| = 2^5 \times 3 \times 11 \times 31$ and in $PSL(2,32)$ a Sylow 31-subgroup is isomorphic to $\mathbb{Z}_{31}$. By Lemma 2.1.5, $N \cong \mathbb{Z}_{31}$ acts fixed point free in the action of NH on the vector space $G \cong 2^{10}$. Hence, by Proposition 2.1.6, $\overline{G} = 2^{10}{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_2)$ is a double Frobenius group.

For the maximal subgroup of order $2(q+1)$, a similar argument holds and this time we construct a double Frobenius group which has the form $2^{2r}{:}(\mathbb{Z}_{2^r+1}{:}\mathbb{Z}_2)$.

Before describing Method 3 we make the following two Remarks.

Remark 5.3.1. 1. If $NH = N{:}H$ is a Frobenius subgroup of $GL(n,q)$ with kernel N and complement H and N acts fixed point free on $V = q^n$, such that no non-identity element of $V = q^n$ is fixed by N, then $\overline{G} = q^n{:}(N{:}H)$ is a double Frobenius group.

2. If the kernel N acts fixed point free on $V = q^n$, then N acts fixed point free on $V \times V \times \ldots \times V$, m times say, (since, if there exists $1_N \neq n \in N$ such that $(a_1, a_2, \ldots, a_m)^n = (a_1, a_2, \ldots, a_m)$ then $(a_1^n, a_2^n, \ldots, a_m^n) = (a_1, a_2, \ldots, a_m)$, which implies that $a_i^n = a_i$ for all $i = 1, 2, \ldots, m$ which is a contradiction). Thus, $\overline{G} = (q^n \times q^n \times \ldots \times q^n){:}(N{:}H)$ is a double Frobenius group, where there are m factors in the product $(q^n \times q^n \times \ldots \times q^n)$.

Remark 5.3.2. The Affine group $A(n,q)$ where q is a power of a prime p is the point stabilizer of the action of the general linear group $GL(n,q)$ on the non-zero vectors of the vector space $V \cong q^n$. So $A(n,q) = \big(GL(n,q)\big)_\alpha \leq GL(n,q)$ where $\alpha \in V^* = V - \{0\}$. Therefore $A(n,q) = q^{n-1} : GL(n-1,q)$ and $|A(n,q)| = q^{n-1} \times |GL(n-1,q)|$. Let $g \in GL(n,q)$ and suppose $o(g) = t$. If $t \nmid |A(n,q)|$, then any $\mathbb{Z}_t \cong \langle t \rangle$ acts fixed point free on the vector space $V \cong q^n$.

Theorem 5.3.1. *If A is a linear group acting on $V = q^n$, such that there exists a Frobenius subgroup $NH = N{:}H \leq A$ with kernel N and complement H, and N acting fixed point free on $V = q^n$, that is, no non-identity element of $V = q^n$ is fixed by N, then*

1. *$\overline{G} = q^n{:}(N : H)$ is a double Frobenius group.*

2. *$\overline{G} = (q^n \times q^n \times \ldots \times q^n){:}(N{:}H)$ is a double Frobenius group where there are m factors in the product $(q^n \times q^n \times \ldots \times q^n)$.*

PROOF:

1. Since N acts fixed point free on $V = q^n$, by Proposition 2.1.4, $|N|$ divides $q^n - 1$. Then by Proposition 2.1.6, $GN = G{:}N$ is a Frobenius group and hence, $\overline{G} = GNH$ is a double Frobenius group.

2. We know that $|N|$ divides $q^n - 1$ and by Remark 5.3.1(2), N acts fixed point free on $q^n \times q^n \times \ldots \times q^n$, where there are m factors in the product $q^n \times q^n \times \ldots \times q^n$. Now $q^{nm} - 1 =$

$(q^n)^m - 1 = (q^n - 1)((q^n)^{m-1} + (q^n)^{m-2} + ... + 1)$. Since $|N|$ divides $q^n - 1$, the $|N|$ divides $q^{nm} - 1$. Then by Proposition 2.1.6, $GN = G{:}N$ is a Frobenius group where $G \cong V = q^{nm}$ and hence, $\overline{G} = GNH$ is a double Frobenius group.

■

5.4 Method 3

The construction of the double Frobenius group here rests on the existence of or finding a Frobenius subgroup inside the general linear group. Theorem 5.3.1 is then applied to obtain a double Frobenius group.

The following examples are constructed from Method 3.

Example 7: In the general linear group $GL(4,3)$, there exists a Frobenius subgroup of the form $NH = \mathbb{Z}_5{:}\mathbb{Z}_4$. The natural action of $GL(4,3)$ on the four dimensional vector space $V \cong 3^4$ over a field of three elements, implies that $NH = \mathbb{Z}_5{:}\mathbb{Z}_4$ acts on the vector space $V \cong 3^4$ and $3^4{:}(\mathbb{Z}_5{:}\mathbb{Z}_4) \leq 3^4{:}GL(4,3)$. Now $|A(4,3)| = 3^3 \times |GL(3,3)|$ where $A(4,3)$ is the Affine subgroup of $GL(4,3)$. Therefore, $|A(4,3)| = 3^6 \times 2^5 \times 13$ and thus $5 \nmid |A(4,3)|$. By Remark 5.3.2 this implies that if $o(g) = 5$ for $g \in GL(4,3)$, then g has no fixed points and any $\mathbb{Z}_5 \cong \langle g \rangle$ acts fixed point free on the vector space $V \cong 3^4$. By Lemma 2.1.5 and Proposition 2.1.6, $3^4{:}\mathbb{Z}_5$ is a Frobenius group and hence, $3^4{:}(\mathbb{Z}_5{:}\mathbb{Z}_4)$ is a double Frobenius group.

Example 8 : By applying part(2) of Theorem 5.3.1, we can construct a double Frobenius group of the form $(3^4 \times 3^4){:}(\mathbb{Z}_5{:}\mathbb{Z}_4)$.

Example 9: In the general linear group $GL(5,3)$, there exists a Frobenius subgroup of the form $NH = \mathbb{Z}_{11}{:}\mathbb{Z}_5$. The natural action of $GL(5,3)$ on the five dimensional vector space $V \cong 3^5$, over a field of three elements, implies that $NH = \mathbb{Z}_{11}{:}\mathbb{Z}_5$ acts on the vector space $V \cong 3^5$ and $3^5{:}(\mathbb{Z}_{11}{:}\mathbb{Z}_5) \leq 3^5{:}GL(5,3)$. Now $|A(5,3)| = 3^4 \times |GL(4,3)|$ where $A(5,3)$ is the Affine subgroup of $GL(5,3)$. Therefore, $|A(5,3)| = 3^{10} \times 2^9 \times 5 \times 13$ and thus $11 \nmid |A(5,3)|$. By Remark 5.3.2 this implies that if $o(g) = 11$ for $g \in GL(5,3)$, then g has no fixed points and any $\mathbb{Z}_{11} \cong \langle g \rangle$ acts fixed point free on the vector space $V \cong 3^5$. By Lemma 2.1.5 and Proposition 2.1.6, $3^5{:}\mathbb{Z}_{11}$ is a Frobenius group and hence, $3^5{:}(\mathbb{Z}_{11}{:}\mathbb{Z}_5)$ is a double Frobenius group.

Example 10: By applying part(2) of Theorem 5.3.1, we can construct a double Frobenius group of the form $(3^5 \times 3^5 \times 3^5){:}(\mathbb{Z}_{11}{:}\mathbb{Z}_5)$.

For the final two examples of double Frobenius groups we refer to the construction alluded to in Remark 4.4.1

Example 11: $\overline{G} = (2^4 \times 3^4){:}(\mathbb{Z}_5{:}\mathbb{Z}_4)$.

There exists a double Frobenius group $\overline{G}$ with $OC(\overline{G}) = OC(U_4(2))$. In the general linear groups $GL(4,2)$ and $GL(4,3)$, there exists a Frobenius group $NH = \mathbb{Z}_5{:}\mathbb{Z}_4$. For $GL(4,2)$, there is the natural action on the four dimensional vector space $V_1 \cong 2^4$ over a field of two elements and for $GL(4,3)$, there is the natural action on the four dimensional vector space $V_2 \cong 3^4$ over the field of three elements. Now $|A(4,2)| = 2^3 \times |GL(3,2)|$ where $A(4,2)$ is the Affine subgroup of $GL(4,2)$. Therefore, $|A(4,2)| = 2^6 \times 3 \times 7$ and thus $5 \nmid |A(4,2)|$. By Remark 5.3.2 this implies that if $o(g) = 5$ for $g \in GL(4,2)$, then g has no fixed points and any $\mathbb{Z}_5 \cong \langle g \rangle$ acts fixed point free on the vector space $V \cong 2^4$. Similarly, $|A(4,3)| = 3^3 \times |GL(3,3)|$ where $A(4,3)$ is the Affine subgroup of $GL(4,3)$. Therefore, $|A(4,3)| = 3^6 \times 2^5 \times 13$ and thus $5 \nmid |A(4,3)|$. This implies that if $o(g) = 5$ for $g \in GL(4,3)$, then g has no fixed points and any $\mathbb{Z}_5 \cong \langle g \rangle$ acts fixed point free on the vector space $V \cong 3^4$. Since $\mathbb{Z}_5$ acts fixed point freely on the vector spaces $V_1 \cong 2^4$ and $V_2 \cong 3^4$, $\mathbb{Z}_5$ acts fixed point free on $(V_1 \times V_2)$. Since if the kernel N acts fixed point free on vector spaces $V_1 \cong q_1^n$ and $V_2 \cong q_2^n$, then N acts fixed point free on $V_1 \times V_2$ because if there exists $1_N \neq n \in N$ such that $(a,b)^n = (a,b)$ for $a \in V_1$, $b \in V_2$, then $(a^n, b^n) = (a,b)$ which implies that $a^n = a$ and $b^n = b$ which is a contradiction. By Lemma 2.1.5 and Proposition 2.1.6 we have that $(2^4 \times 3^4){:}\mathbb{Z}_5{:}\mathbb{Z}_4$ is a double Frobenius group.

Example 12: $\overline{G} = (2^{10} \times 3^5){:}(\mathbb{Z}_{11}{:}\mathbb{Z}_5)$.

There exists a double Frobenius group $\overline{G}$ with $OC(\overline{G}) = OC(U_5(2))$. In the general linear groups $GL(10,2)$ and $GL(5,3)$, there exists a Frobenius group $NH = \mathbb{Z}_{11}{:}\mathbb{Z}_5$. For $GL(10,2)$, there is the natural action on the ten dimensional vector space $V_1 \cong 2^{10}$ over a field of two elements and for $GL(5,3)$, there is the natural action on the five dimensional vector space $V_2 \cong 3^5$ over a field of three elements. Now $|A(10,2)| = 2^9 \times |GL(9,2)|$ where $A(10,2)$ is the Affine subgroup of $GL(10,2)$. Therefore, $|A(10,2)| = 2^{45} \times 3^5 \times 5^2 \times 7^3 \times 17 \times 31 \times 127$ and thus $11 \nmid |A(10,2)|$. Remark 5.3.2 now implies that if $o(g) = 11$ for $g \in GL(10,2)$, then g has no fixed points and any $\mathbb{Z}_{11} \cong \langle g \rangle$ acts fixed point free on the vector space $V \cong 2^{10}$. Similarly, $|A(5,3)| = 3^4 \times |GL(4,3)|$ where $A(5,3)$ is the Affine subgroup of $GL(5,3)$. Therefore, $|A(5,3)| = 2^9 \times 3^{10} \times 5 \times 13$ and thus $11 \nmid |A(5,3)|$. This implies that if $o(g) = 11$ for $g \in GL(5,3)$, then g has no fixed points and any $\mathbb{Z}_{11} \cong \langle g \rangle$ acts fixed point free on the vector space $V \cong 3^5$. Since $\mathbb{Z}_{11}$ acts fixed point freely on the vector spaces $V_1 \cong 2^{10}$ and $V_2 \cong 3^5$, $\mathbb{Z}_{11}$ acts fixed point free on $(V_1 \times V_2)$ as in Example 11 above. By Lemma 2.1.5 and Proposition 2.1.6 we have that $(2^{10} \times 3^5){:}\mathbb{Z}_{11}{:}\mathbb{Z}_5$ is a double Frobenius group.

6

Fischer Matrices and Character Table of $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$

In this section we shall determine the conjugacy classes, Fischer matrices and character table of the double Frobenius group $2^n{:}(\mathbb{Z}_{2^n-1}{:}\ \mathbb{Z}_n)$ constructed in the previous chapter. We give a general description of the conjugacy classes, Fischer matrices and character table of the group.

Let $\overline{G} = GNH$ be a double Frobenius group with GN and NH Frobenius groups. Consider now the double Frobenius group $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ constructed in the previous chapter. Let $\overline{G} = GNH = 2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$. Then, $G \cong 2^n$, $N \cong \mathbb{Z}_{2^n-1}$ and $H \cong \mathbb{Z}_n$. Let $\overline{H} = NH$. Two examples of this type, namely $2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$ and $2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$ are given in Chapter eight.

6.1 The Group $\overline{H} = \mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$

We will first determine the conjugacy classes of the Frobenius group $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$.

6.1.1 Conjugacy Classes of $\overline{H}$

Now $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$ is a Frobenius group with kernel $N = \mathbb{Z}_{2^n-1}$ and complement $H = \mathbb{Z}_n$. Since H acts on N with fixed point free action, from the proof of Proposition 3.2.2, we have that the number of non-trivial orbits of H on N is given by $\alpha = \frac{|N|-1}{|H|}$ and the length of each orbit is given by $|H|$. Therefore here the orbits of $\mathbb{Z}_n$ on $\mathbb{Z}_{2^n-1}$ have lengths 1 and n. Using the method of coset analysis, see Chapter two, Section 2.1.5, we analyse the coset Nh for each $h \in H$ and find the values of k where k is the size of the stabilizer in N of h. The values of k can be determined from the action of H on N. Since this action is fixed point free, $k = 2^n - 1$ for $h = 1_H$ and $k = 1$ for $h \neq 1_H$.

For $h = 1_H$, $k = 2^n - 1$ and $f_1 = 1$ and $f_i = n$ $\forall i \in \{2, 3, \ldots, \alpha + 1\}$.

For $h = 1_H$, $k = 2^n - 1$, $f_i = 1$:

$$|C_{\overline{H}}(x)| = \frac{(2^n - 1) \times n}{1} = |\overline{H}|.$$

So for $f = 1$, we have the identity class of $\overline{H}$.

For $h = 1_H$, $k = 2^n - 1$, $f_i = n$:

$$|C_{\overline{H}}(x)| = \frac{(2^n-1)\times n}{n} = |N|.$$

So for $f_i = n$, we have a class of $\overline{H}$ containing x with $o(x) = 2^n - 1$. The size of the conjugacy class is

$$|[x]_{\overline{H}}| = \frac{|\overline{H}|}{|C_{\overline{H}}(x)|} = \frac{(2^n-1)\times n}{2^n-1} = n.$$

For $h \neq 1_H$ we have $k = 1$, $f = 1$:

$$|C_{\overline{H}}(x)| = \frac{1\times n}{1} = n.$$

Therefore,

$$|[x]_{\overline{H}}| = \frac{|\overline{H}|}{|C_{\overline{H}}(x)|} = \frac{(2^n-1)\times n}{n} = 2^n - 1.$$

So, for each coset Nh where $h \neq 1_H$, there is a unique class of $\overline{H}$ containing h.

Note 6.1.1. The number of conjugacy classes in a Frobenius group is given by Corollary 3.5.6.

So in the Frobenius group $\overline{H} = \mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$,

$$c(\overline{H}) = c(H) + \alpha = c(H) + \frac{c(N)-1}{|H|} = n + \frac{2^n-2}{n} = \frac{n^2+2^n-2}{n}.$$

In the previous section we determined that in the Frobenius group $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$, there are conjugacy classes of size n, size (2^n-1) and of course the identity class. Clearly the number of classes of size (2^n-1) must be $(n-1)$, the number of non-identity elements in H. Therefore we must have that:

$$|\overline{H}| = (2^n-1)\times n = 1 + (n-1)(2^n-1) + \beta \times n,$$

where β is the number of conjugacy classes of size n. Solving for β in the equation above gives $\beta = \frac{2^n-2}{n}$.

Table 6.1 summarizes the results discussed above.

The full list of conjugacy classes based on coset analysis is given in Table 6.2.

Table 6.1: Summary of results

Class$[\overline{g}]$	$[1]$	$[x_h]$	$[x_n]$
$\|[\overline{g}]\|$	1	2^n-1	n
$o(\overline{g})$	1	n	2^n-1
$\|C_{\overline{G}}(\overline{g})\|$	$(2^n-1)n$	n	2^n-1
No of classes	1	$n-1$	$\frac{2^n-2}{n}$

where $x_h \in H$ and $x_n \in a^iH - H$, for $i = 1, 2, \ldots, 2^n-1$ and $N = \langle a \rangle$.

Table 6.2: Conjugacy Classes of $\overline{H}$

H	$\overline{H} = NH$	$o(\overline{h})$	$\|C_{\overline{H}}(\overline{h})\|$
1_H	1	1	$(2^n-1)n$
	$(2^n-1)A_1$	2^n-1	2^n-1
	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$
	$(2^n-1)A_\alpha$	2^n-1	2^n-1
b	nA	n	n
b^2	$(nA)^2$	n	n
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
b^{n-1}	$(nA)^{n-1}$	n	n

where $(2^n-1)A_i$ is the conjugacy class containing elements of order 2^n-1 and (nA) is the conjugacy class containing elements of order n using the notation of the ATLAS [5]. Also here $H = \langle b \rangle$.

6.1.2 Character Table of $\overline{H}$

To construct the character table of $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$ we use the following results from the characters of Frobenius groups. We list the main results below.

1. The number of conjugacy classes and hence irreducible characters of $\overline{H}$ by Corollary 3.5.6, are given by: $c(\overline{H}) = \frac{c(N)-1}{|H|} + c(H) = \alpha + n = \frac{2^n-2}{n} + n$.

2. If $\phi_1 \neq \phi \in Irr(N)$ then by Proposition 3.5.3, ϕ has inertia group $I_{\overline{H}}(\phi) = N$.

3. If $\phi_1 \neq \phi \in Irr(N)$, then by Theorem 3.5.4, $\phi^{\overline{H}} \in Irr(\overline{H})$.

4. If $\psi \in \mathrm{Irr}(\overline{H})$, then by Theorem 3.5.4, either $N \subset \ker\psi$ or $\psi = \phi^{\overline{H}}$ for some irreducible character $\phi_1 \neq \phi$ of N.

5. By Note 3.5.2, the irreducible characters of $\overline{H}$ are of 2 types; those with kernel containing N and those induced from non-trivial irreducible characters of N.

6. By Note 3.5.2, the order of H divides the degree of the induced character $\phi^{\overline{H}}$ for $\phi_1 \neq \phi \in \mathrm{Irr}(N)$. That is, $|H|\,|\phi^{\overline{H}}(1_{\overline{H}})$. Furthermore, if $\phi_1 \neq \phi \in \mathrm{Irr}(N)$ is a linear character then $|H| = \phi^{\overline{H}}(1_{\overline{H}})$.

7. By Theorem 3.5.5, $\overline{H}$ has $\alpha = \frac{c(N)-1}{|H|}$ distinct irreducible characters of the form $\phi^{NH}, \phi_1 \neq \phi \in \mathrm{Irr}(N)$.

8. From Section 3.5.2, we know that in the Frobenius group $\overline{H} = NH$, the Fischer matrix $M(1_H)$ is a $(\alpha+1)\times(\alpha+1)$ matrix where α is the number of non-trivial orbits of H on N and $M(h)\ \forall\, h \neq 1_H$ is just 1. The entries in the matrix $M(1_H)$ is given by: $M(1_H) = \left[c^j(i,1)\right] = \psi_i^{\overline{H}}(a_j)$ where $a_j \in X(g)$ for $j = 1, 2, \ldots, \alpha$.

9. From the discussion in Section 3.5.2,

$$\psi^{\overline{H}}(a_j) = \left|C_{\overline{H}}(a_j)\right| \sum_{i=1}^{m} \frac{\phi(x_i)}{\left|C_N(x_i)\right|},$$

where $\phi \in \mathrm{Irr}(N)$, $[a_j]$ is the conjugacy class of $\overline{H}$ containing a_j and $x_1, x_2, \ldots, x_m$ are class representatives for the classes of N that fuse to $[a_j]$.
Since, $\left|C_{\overline{H}}(a_j)\right| = 2^n - 1$ and $\left|C_N(x_i)\right| = |N| = 2^n - 1$, $\phi^{\overline{H}}(a_j) = \sum_{i=1}^{m} \phi(x_i)$. Therefore the entries of the matrix $M(1_H)$ are the orbit sums of the action of H on N.

10. We know that $\mathbb{Z}_n$ acts fixed point free on $\mathbb{Z}_{2^n-1}$, the number of non-trivial orbits is given by $\alpha = \frac{|N|-1}{|H|} = \frac{2^n-2}{n}$ and the length of each orbit is given by $|H| = n$. Also by Euler's Theorem, if $n \neq 2$, then $2^{n-1} \equiv 1 \pmod{n}$. Since $\alpha = \frac{2^n-2}{n} = \frac{2\times(2^{n-1}-1)}{n}$, α must be even. Also by Fermat's Theorem we have $a^{(2^n-1)-1} = 1 \pmod{2^n-1}$ and this implies that $a^{2^n-2} = 1 \pmod{2^n-1}$. Therefore, $a^{2(2^{n-1}-1)} = 1 \pmod{2^n-1}$ and thus $(a^{2^{n-1}-1})^2 = 1 \pmod{2^n-1}$. Since $o(a^i) \neq 2\ \ \forall\, a^i \in \mathbb{Z}_{2^n-1}$, $a^{2^{n-1}-1} = 1$ and we have that $a^{2^{n-1}} \equiv a \pmod{2^n-1}$. The left hand side of this congruence indicates that for any $a^i \in \mathbb{Z}_{2^n-1}$, even powers of a^i generate the orbits of $\mathbb{Z}_n$ on $\mathbb{Z}_{2^n-1}$. That is, the action of $\mathbb{Z}_n$ on $\mathbb{Z}_{2^n-1}$ is given by $ba^ib^{-1} = (a^i)^2$.

In fact we can describe the orbits Θ_j for $j = 1, 2, \ldots, \alpha$ as follows:

$$\Theta_1 = \{a, a^2, a^4, \ldots, a^{2^{n-1}}\},$$

$$\Theta_2 = \{a^3, a^6, a^{12}, \ldots, a^{3(2^{n-1})}\},$$

$$\Theta_3 = \{a^5, a^{10}, a^{20}, \ldots, a^{5(2^{n-1})}\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Theta_\alpha = \{a^{2^{n-1}-1}, \ldots\ldots, a^{(2^{n-1}-1)(2^{n-1})}\}.$$

We notice also that the α orbits can be paired as Θ_j and $\Theta_{j'}$ say, such that the inverse of each $a^i \in \mathbb{Z}_{2^n-1}$ in the orbit Θ_j is found in the orbit $\Theta_{j'}$.

11. Now let $p_j = \sum_i \phi(x_i)$ for $j = 1, 2, \ldots, \alpha$, then $p_{j'} = \sum_i \phi(x_i^{-1}) = \sum_i \overline{\phi(x_i)}$.

The following table shows the conjugacy classes, characters and respective character values for a portion of the character table of $\overline{H}$

	$(2^n-1)A_1$	$(2^n-1)A_2$	$(2^n-1)A_3$	$\ldots\ldots\ldots$	$(2^n-1)A_\alpha$
χ_{n+1}	p_1	p_2	p_3	$\ldots\ldots\ldots$	p_α
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{n+\alpha}$	p_α	$p_{\alpha-1}$	$p_{\alpha-2}$	$\ldots\ldots\ldots$	p_1

We produce now the character tables of $\mathbb{Z}_{2^n-1}$ and $\mathbb{Z}_n$ and then the character table of the Frobenius group $\overline{H}$.

Table 6.3: Character Table of $\mathbb{Z}_n$

Classes	e	b	b^2	$\ldots$	$\ldots$ b^{n-1}
χ_1	1	1	1	1	1 1
χ_2	1	q	q^2	$\ldots$	$\ldots$ q^{n-1}
$\vdots$					$\vdots$
$\vdots$					$\vdots$
χ_n	1	q^{n-1}	q^{n-2}	$\ldots$	$\ldots$ q

where $q = e^{\frac{2\pi i}{n}}$

Table 6.4: Character Table of $\mathbb{Z}_{2^n-1}$

Classes	e	a	a^2	$\dots$	$\dots$	a^{2^n-2}
ϕ_1	1	1	1	1	1	1
ϕ_2	1	p	p^2	$\dots$	$\dots$	p^{2^n-1}
$\vdots$						$\vdots$
$\vdots$						$\vdots$
ϕ_{2^n-1}	1	p^{2^n-1}	p^{2^n-2}	$\dots$	$\dots$	p

where $p = e^{\frac{2\pi i}{2^n-1}}$

Table 6.5: Character Table of $\overline{H} = NH$

$[g]$	(1)	$(2^n-1)A_1 \dots\dots\dots (2^n-1)A_\alpha$	$(nA) \dots\dots\dots (nA)^{n-1}$
$\lvert C_{\overline{H}}(g)\rvert$	$(2^n-1)n$	$2^n-1 \dots\dots\dots 2^n-1$	$n \dots\dots\dots n$
χ_1	1	$1 \dots\dots\dots 1$	$1 \dots\dots\dots 1$
χ_2	1	$1 \dots\dots\dots 1$	$q \dots\dots\dots q^{n-1}$
$\vdots$	$\vdots$	$\vdots \dots\dots\dots \vdots$	$\vdots \dots\dots\dots \vdots$
$\vdots$	$\vdots$	$\vdots \dots\dots\dots \vdots$	$\vdots \dots\dots\dots \vdots$
$\vdots$	$\vdots$	$\vdots \dots\dots\dots \vdots$	$\vdots \dots\dots\dots \vdots$
χ_n	1	$1 \dots\dots\dots 1$	$q^{n-1} \dots\dots\dots q$
χ_{n+1}	n	$p_1 \dots\dots\dots p_\alpha$	$0 \dots\dots\dots 0$
$\vdots$	$\vdots$	$\vdots \dots\dots\dots \vdots$	$\vdots \dots\dots\dots \vdots$
$\chi_{n+\alpha}$	n	$p_\alpha \dots\dots\dots p_1$	$0 \dots\dots\dots 0$

where $q = e^{\frac{2\pi i}{n}}$ and p_j for $j = 1$, ..., α are as given in section 6.1.2 (11).

6.2 Conjugacy Classes of 2^n:($\mathbb{Z}_{2^n-1}$:$\mathbb{Z}_n$)

We determine now the conjugacy classes of the double Frobenius group $\overline{G} = GNH = G{:}NH$ where $N = \langle a \rangle \cong \mathbb{Z}_{2^n-1}$ and $H = \langle b \rangle \cong \mathbb{Z}_n$.

The action of NH on G is determined by the actions of a and b where $o(a) = 2^n - 1$ and $o(b) = n$. Now N acts on G fixed point free such that GN is a Frobenius group. The number of non-trivial orbits of N on G is 1, since

$$m = \frac{|G|-1}{|N|} = \frac{2^n-1}{2^n-1} = 1,$$

and the length of this orbit equals $|N| = 2^n - 1$. Therefore N has two orbits on G of lengths 1 and $2^n - 1$ respectively. Hence, NH also has only these two orbits. Since N acts fixed point free on G, for any non-identity element $y \in N$, we have $k = 1$ where k is the size of the stabilizer in G of y. For $y = 1_N$, the identity of NH, we have $k = 2^n$ and $f_1 = 1$ and $f_2 = 2^n - 1$.

For $y = 1_N$, $k = 2^n$, $f_1 = 1$:

$$|C_{\overline{G}}(x)| = \frac{2^n \times |C_{\overline{H}}(y)|}{1} = 2^n(2^n - 1)n = |\overline{G}|.$$

So this gives us the identity class of $\overline{G}$.

For $y = 1_N$, $k = 2^n$, $f_2 = 2^n - 1$:

$$|C_{\overline{G}}(x)| = \frac{2^n \times |C_{\overline{H}}(y)|}{2^n - 1} = 2^n n$$

and we have that

$$|[x]_{\overline{G}}| = 2^n - 1.$$

For $y \neq 1_N$, $k = 1$, $f = 1$:

$$|C_{\overline{G}}(x)| = \frac{|C_{\overline{H}}(y)|}{1} = 2^n - 1$$

and we have that $|[x]_{\overline{G}}| = 2^n n$.

Now since NH is Frobenius, $|H| \mid (|N| - 1)$ and $n \mid 2^n - 1$. Since GN is Frobenius, $|N| \mid (|G| - 1)$ and therefore we must have that $n \mid 2^n - 2$. Thus in the action of H on G, b fixes two points and permutes the remaining $2^n - 2$ elements of G in $\frac{2^n-2}{n}$ orbits of length n. So b commutes with an involution $g \in G$ and each $b^j \in H$ produces two conjugacy classes of $\overline{G}$. One class contains elements of order n and the other class contain elements of order 2n. In both cases,

$$|C_{\overline{G}}(x)| = \frac{2 \times |C_{\overline{H}}(h)|}{1} = 2n.$$

Remark 6.2.1. From the above discussion we note that if n is odd then $\overline{G}$ has only one class of involutions.

Note 6.2.1. We derive here a general formula which gives the number of conjugacy classes for the double Frobenius group $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$.

The conjugacy classes of the group are listed in Table 6.6 below. From the table we note that the identity class of $\overline{H} = NH$ produces two conjugacy classes in $\overline{G} = G{:}(NH)$. Each class of the type $(2^n - 1)A_i$ for $i = \{1, 2, \ldots, \alpha\}$ produces a unique class in $\overline{G}$. So the number of such classes will be $\alpha = \frac{2^n-2}{n}$. Each class of the type $(nA)^j$ for $j = \{1, 2, \ldots, n-1\}$ produces two classes of $\overline{G}$. So the number of conjugacy classes of $\overline{G}$ is given by:

$$c(\overline{G}) = 2 + \alpha + 2(n - 1) = 2n + \alpha = \frac{2n^2 + 2^n - 2}{n}.$$

Therefore,

$$c(\overline{G}) = 2n + \alpha = n + (n + \alpha) = c(H) + c(NH).$$

Table 6.6: Conjugacy Classes of $\overline{G} = G{:}(NH)$

$\overline{H} = NH$	$\overline{G} = G{:}(NH)$	$o(\overline{g})$	$\lvert C_{\overline{G}}(\overline{g})\rvert$
1	1	1	$2^n(2^n-1)n$
	2A	2	$2^n n$
$(2^n-1)A_1$	$(2^n-1)A_1$	2^n-1	2^n-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2^n-1)A_\alpha$	$(2^n-1)A_\alpha$	2^n-1	2^n-1
b	nA	n	$2n$
	$(2n)A$	$2n$	$2n$
b^2	$(nA)^2$	n	$2n$
	$(2n)A_2$	$2n$	$2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
b^{n-1}	$(nA)^{n-1}$	n	$2n$
	$(2n)A_{n-1}$	$2n$	$2n$

6.3 Fischer Matrices of $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$

Let $\overline{G} = G{:}(NH) = 2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$. To construct the Fischer matrices of $\overline{G}$ we consider the action of $\overline{G}$ on the $\mathrm{Irr}(G)$. Let $\theta \in \mathrm{Irr}(G)$, then $\overline{G}$ permutes $\mathrm{Irr}(G)$ by $g : \theta \mapsto \theta^g$ for $g \in \overline{G}$. Since G acts trivially on $\mathrm{Irr}(G)$, $\mathrm{Irr}(G)$ is permuted by $\overline{G}/G \cong NH$, by $gG : \theta \mapsto \theta^g$. Now $\forall\theta \in \mathrm{Irr}(G)$, define

$$\overline{P} = \{x \in \overline{G} : \theta^x = \theta\} = I_{\overline{G}}(\theta)$$
$$P = \{y \in NH : \theta^y = \theta\} = I_{NH}(\theta).$$

Then it can be shown that $\overline{P} = G{:}P$. From the previous section we know that the action of NH on G produces two orbits of length 1 and $2^n - 1$. Therefore by Brauer's Lemma, Lemma 2.1.59, NH has two orbits on $\mathrm{Irr}(G)$ of length 1 and $2^n - 1$. So the inertia groups are $\overline{P}_1 = \overline{G}$ with $P_1 = NH$ and $\overline{P}_2 = G{:}P_2$ with $P_2 \leq NH$ and $[NH{:}P_2] = 2^n - 1$. Therefore, $P_2 \cong H$ and $\overline{P}_2 = G{:}H$.

We construct Fischer matrices of $\overline{G} = 2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ for each conjugacy class of $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$. From the previous section the number of conjugacy classes of $\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n$ is given by $c(NH) = \frac{n^2+2^n-2}{n}$. Therefore the number of Fischer matrices we can get for the double Frobenius group $2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$ is $\frac{n^2+2^n-2}{n}$.

For each conjugacy class of NH, there is a corresponding Fischer matrix. The Fischer matrices are constructed using the results of Section 2.1.7

$g = 1_{NH}$:

This class produces two conjugacy classes in $\overline{G}$ and the corresponding Fischer matrix is a (2×2) matrix. The Fischer matrix corresponding to the identity element of NH is

$$M(1_{NH}) = \begin{bmatrix} 1 & 1 \\ 2^n - 1 & -1 \end{bmatrix}.$$

$g \in (2^n - 1)A_i$:

Each of these conjugacy classes produces a unique conjugacy class of $\overline{G}$. The corresponding Fischer matrix is a (1×1) matrix. There are $\alpha = \frac{2^n-2}{n}$ of these (1×1) matrices.

$g \in (nA)^j$:

Each of these conjugacy classes produces two conjugacy classes in $\overline{G}$ and the corresponding Fischer matrix is a (2×2) matrix. There are $(n-1)$ of these (2×2) matrices. So,

$$M(g) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

6.4 Character Table of $\overline{G} = 2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$

We describe here the character table of $\overline{G} = 2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$. To construct the character table of this group we will use the character table of the Frobenius group $\overline{H}$, the Fischer matrices constructed in the previous section and the character tables of the inertia factors $P_1 = \overline{H} = NH$ and $P_2 = H$. Each factor corresponds to a block of the character table of $\overline{G}$. We construct these blocks by multiplying columns of the character tables of P_1 and P_2 by relevant rows of the Fischer matrices following the results described in Section 2.1.7.

For $g = 1_{NH}$, the corresponding Fischer matrix is a (2×2) matrix given by:

$$\begin{bmatrix} 1 & 1 \\ 2^n - 1 & -1 \end{bmatrix}.$$

The corresponding column from NH is the first column which is the $(n+\alpha) \times 1$ matrix given by:

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ n \\ \vdots \\ \vdots \\ n \end{bmatrix}$$

This column has n entries of 1 and α entries of n.
To obtain the P_1 block of the character table of $\overline{G}$, we multiply the above matrix by the first row of the Fischer matrix corresponding to $g = 1_{NH}$.

We get:

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ n \\ \vdots \\ \vdots \\ n \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & 1 \\ n & n \\ \vdots & \vdots \\ \vdots & \vdots \\ n & n \end{bmatrix}$$

To get the P_2 block of the character table of $\overline{G}$, we multiply the first column of the character table of $H = \mathbb{Z}_n$ with the second row of the Fischer matrix corresponding to $g = 1_{NH}$. We get:

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 2^n - 1 & -1 \end{bmatrix} = \begin{bmatrix} 2^n - 1 & -1 \\ 2^n - 1 & -1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 2^n - 1 & -1 \end{bmatrix}$$

The P_1 and P_2 blocks gives us the first two columns of the character table of $\overline{G}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & 1 \\ n & n \\ \vdots & \vdots \\ \vdots & \vdots \\ n & n \\ --- & ---- \\ 2^n-1 & -1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 2^n-1 & -1 \end{bmatrix}$$

Next from the character table of $\overline{H} = NH$ we have α columns each of which has size $(n+\alpha)$. We have α Fischer matrices of size 1×1 corresponding to these classes. So multiplying the α columns by 1, we get the α columns of size $(n+\alpha)$ in the P_1 block of the character table of $\overline{G}$. In the P_2 block we have n zeros under these α columns.

The P_1 and P_2 blocks gives the following columns of the character table of $\overline{G}$:

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ p_1 \\ \vdots \\ \vdots \\ p_\alpha \\ --- \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ p_2 \\ \vdots \\ \vdots \\ p_{\alpha-1} \\ --- \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ p_\alpha \\ \vdots \\ \vdots \\ p_1 \\ --- \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Finally from the character table of $\overline{H} = NH$ we have the $(n-1)$ columns each of size $(n+\alpha)$ given by:

$$\begin{bmatrix} 1 \\ q \\ \vdots \\ \vdots \\ q^{n-1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ q^2 \\ \vdots \\ \vdots \\ q^{n-2} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ q^{n-1} \\ \vdots \\ \vdots \\ q \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

The Fischer matrices corresponding to these classes are given by:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So multiplying each of these $(n-1)$ columns by the first row of the Fischer matrix gives the P_1 block in the character table of $\overline{G}$ given by:

$$\begin{bmatrix} 1 & 1 \\ q & q \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-1} & q^{n-1} \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ q^2 & q^2 \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-2} & q^{n-2} \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 1 \\ q^{n-1} & q^{n-1} \\ \vdots & \vdots \\ \vdots & \vdots \\ q & q \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

For the P_2 block of the character table of $\overline{G}$, we multiply the n columns from the character table of $H = \mathbb{Z}_n$ by the second row of the Fischer matrix. We get:

$$\begin{bmatrix} 1 & -1 \\ q & -q \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-1} & -q^{n-1} \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ q^2 & -q^2 \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-2} & -q^{n-2} \end{bmatrix}, \ldots, \begin{bmatrix} 1 & -1 \\ q^{n-1} & -q^{n-1} \\ \vdots & \vdots \\ \vdots & \vdots \\ q & -q \end{bmatrix}.$$

The P_1 and P_2 blocks gives the following columns of the character table of $\overline{G}$:

$$\begin{bmatrix} 1 & 1 \\ q & q \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-1} & q^{n-1} \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ --- & --- \\ 1 & -1 \\ q & -q \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-1} & -q^{n-1} \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ q^2 & q^2 \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-2} & q^{n-2} \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ --- & --- \\ 1 & -1 \\ q^2 & -q^2 \\ \vdots & \vdots \\ \vdots & \vdots \\ q^{n-2} & -q^{n-2} \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 1 \\ q^{n-1} & q^{n-1} \\ \vdots & \vdots \\ \vdots & \vdots \\ q & q \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ --- & --- \\ 1 & -1 \\ q^{n-1} & -q^{n-1} \\ \vdots & \vdots \\ \vdots & \vdots \\ q & -q \end{bmatrix}$$

Table 6.7: Character Table of $\overline{G} = GNH = 2^n{:}(\mathbb{Z}_{2^n-1}{:}\mathbb{Z}_n)$

	(1)	(2A)	$(2^n-1)A_1$	$(2^n-1)A_\alpha$	(nA)	$(2n)A_1$	……	$(nA)^{n-1}$	$(2n)A_{n-1}$
χ_1	1	1	1	1	1	1		1	1
χ_2	1	1	1	1	q	q		q^{n-1}	q^{n-1}
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
χ_n	1	1	1	1	q^{n-1}	q^{n-1}		q	q
χ_{n+1}	n	n	p_1	p_α	0	0		0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
$\chi_{n+\alpha}$	n	n	p_α	p_1	0	0		0	0
ϕ_1	2^n-1	-1	0	0	1	-1		1	-1
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮
ϕ_n	2^n-1	-1	0	0	q^{n-1}	$-q^{n-1}$		q	$-q$

where $q = e^{\frac{2\pi i}{n}}$, p_j for $j = 1, \ldots, \alpha$ are as given in section 6.1.2 (11) and $[(2n)A_i]^2 = (nA)^i$.

7

Fischer Matrices and Character Table of $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

In this section we shall determine the conjugacy classes, Fischer matrices and character table of the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ constructed in the Chapter 5. We give a general description of the conjugacy classes, Fischer matrices and character table of the group.

Let $\overline{G} = GNH$ be a double Frobenius group with GN and NH Frobenius groups. Consider now the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ constructed in the Chapter 4. Let $\overline{G} = GNH = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$. Then, $G \cong 2^{2r}$, $N \cong \mathbb{Z}_{2^r-1}$ and $H \cong \mathbb{Z}_2$. Let $\overline{H} = NH$.

7.1 The Group $\overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$

We will first determine the conjugacy classes of the Frobenius group $\overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$.

7.1.1 Conjugacy Classes of $\overline{H}$

We know that $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$ is a Frobenius group with kernel $N = \langle a \rangle \cong \mathbb{Z}_{2^r-1}$ and complement $H = \langle b \rangle \cong \mathbb{Z}_2$. Since H acts on N with fixed point free action, from the proof of Proposition 3.2.2, we have that the number of non-trivial orbits of H on N is given by $\alpha = \frac{|N|-1}{|H|}$ and the length of each orbit is given by $|H|$. Therefore, here the orbits of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^r-1}$ have lengths 1 and 2. Using the method of coset analysis, see [38], we analyse the coset Nh for each $h \in H$ and find the values of k where k is the order of the stabilizer in N of h. The values of k can be determined from the action of H on N. Since this action is fixed point free, $k = 2^r - 1$ for $h = 1_H$ and $k = 1$ for $h = b$.

For $h = 1_H$, $k = 2^r - 1$, $f_1 = 1$ and $f_i = 2$ $\forall i \in \{2, 3, \ldots, \alpha + 1\}$.

For $h = 1_H$, $k = 2^r - 1$, $f_1 = 1$:

$$|C_{\overline{H}}(x)| = \frac{(2^r - 1) \times 2}{1} = |\overline{H}|.$$

So for $f_1 = 1$, we have the identity class of $\overline{H}$.

For $h = 1_H$, $k = 2^r - 1$, $f_i = 2$:

$$|C_{\overline{H}}(x)| = \frac{(2^r - 1) \times 2}{2} = |N|.$$

So for $f_i = 2$, we have a class of $\overline{H}$ containing x with $o(x) = 2^r - 1$. The size of the conjugacy class is

$$|[x]_{\overline{H}}| = \frac{|\overline{H}|}{|C_{\overline{H}}(x)|} = \frac{(2^r - 1) \times 2}{2^r - 1} = 2.$$

Note that there are $\frac{2^r-2}{2} = 2^{r-1} - 1$ such classes in $\overline{H}$.

For $h = b$ we have $k = 1$, $f = 1$:

$$|C_{\overline{H}}(x)| = \frac{1 \times 2}{1} = 2.$$

Therefore,

$$|[x]_{\overline{H}}| = \frac{|\overline{H}|}{|C_{\overline{H}}(x)|} = \frac{(2^r - 1) \times 2}{2} = 2^r - 1.$$

So, for the coset Nb there is a unique involutary class of $\overline{H}$ containing h.

From the above we deduce that there are $1 + (2^{r-1} - 1) + 1 = 2^{r-1} + 1$ classes in $\overline{H}$. This number can be confirmed by Corollary 3.5.6, since $\overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$ and we have that

$$c(\overline{H}) = c(H) + \alpha = c(H) + \frac{c(N) - 1}{|H|} = 2 + \frac{2^r - 2}{2} = 2^{r-1} + 1.$$

The results discussed above are summarized in Table 7.1.

Table 7.1:

Class$[\overline{g}]$	$[1]$	$[x_h]$	$[x_{n_i}]$
$\|[\overline{g}]\|$	1	$2^r - 1$	2
$o(\overline{g})$	1	2	$2^r - 1$
$\|C_{\overline{G}}(\overline{g})\|$	$(2^r - 1)2$	2	$2^r - 1$
No of classes	1	1	$2^{r-1} - 1$

where $x_h \in H$ and $x_{n_i} \in a^iH - H$, for $i = 1, 2, \ldots, |N| - 1$

The full list of conjugacy classes based on coset analysis is given in the Table 7.2.

Table 7.2: Conjugacy Classes of $\overline{H}$

H	$\overline{H} = NH$	$o(\overline{h})$	$\lvert C_{NH}(\overline{h})\rvert$
1_H	1	1	$(2^r-1)2$
	$(2^r-1)A_1$	c_1	2^r-1
	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$
	$(2^r-1)A_\alpha$	c_α	2^r-1
b	$2A$	2	2

where $(2^r-1)A_i$ is the conjugacy class containing elements of order 2^r-1 and $(2A)$ is the conjugacy class containing elements of order 2 using the notation of the ATLAS. Also here $H = \langle b \rangle$ and c_i for $1 = 1, 2, ..., \alpha$ divides the order of $N = 2^r - 1$.

7.1.2 Character Table of $\overline{H}$

To construct the character table of $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$ we use the following results from the characters of Frobenius groups. We list the main results below.

1. The number of conjugacy classes and hence irreducible characters of $\overline{H}$ by Corollary 3.5.6, are given by: $c(\overline{H}) = \frac{c(N)-1}{|H|} + c(H) = \alpha + 2 = 2^{r-1} + 1$.
2. If $\phi_1 \neq \phi \in Irr(N)$ then by Proposition 3.5.3, ϕ has inertia group $I_{\overline{H}}(\phi) = N$.
3. If $\phi_1 \neq \phi \in Irr(N)$, then by Theorem 3.5.4, $\phi^{\overline{H}} \in Irr(\overline{H})$.
4. If $\psi \in Irr(G)$, then by Theorem 3.5.4, either $N \subset ker\psi$ or $\psi = \phi^{\overline{H}}$ for some irreducible character $\phi_1 \neq \phi$ of N.
5. By Note 3.5.2, the irreducible characters of $\overline{H}$ are of 2 types; those with kernel containing N, namely χ_1 and χ_2 of degree 1 and those induced from non-trivial irreducible characters of N, namely $\chi_3, \chi_4, \ldots, \chi_{2+\alpha}$ of degree 2.
6. By Note 3.5.2, the order of H divides the degree of the induced character $\phi^{\overline{H}}$ for $\phi_1 \neq \phi \in Irr(N)$. That is $|H|\,|\phi^{\overline{H}}(1_{\overline{H}})$. Furthermore, if $\phi_1 \neq \phi \in Irr(N)$ is a linear character then $|H| = \phi^{\overline{H}}(1_{\overline{H}})$.
7. By Theorem 3.5.5, $\overline{H}$ has $\alpha = \frac{c(N)-1}{|H|}$ distinct irreducible characters of the form $\phi^{\overline{H}}$, where $\phi_1 \neq \phi \in Irr(N)$.
8. From Section 3.5.2, we know that in the Frobenius group $\overline{H} = NH$, the Fischer matrix $M(1_H)$ is a $(\alpha+1) \times (\alpha+1)$ matrix where α is the number of non-trivial orbits of H on N and

$M(h)\ \forall\ h \neq 1_H$ is just 1. The entries in the matrix $M(1_H)$ is given by: $M(1_H) = [c^j(i,1)] = \psi_i^{\overline{H}}(a_j)$ where $a_j \in X(g)$ for $j = 1, 2, \ldots, \alpha$

9. From the discussion in Section 3.5.2, we have

$$\psi^{\overline{H}}(a_j) = |C_{\overline{H}}(a_j)| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_N(x_i)|},$$

where $\phi \in Irr(N)$, $[a_j]$ is the conjugacy class of $\overline{H}$ containing a_j and $x_1, x_2, \ldots, x_m$ are class representatives for the classes of N that fuse to $[a_j]$.

Since $|C_{\overline{H}}(a_j)| = 2^r - 1$ and $|C_N(x_i)| = |N| = 2^r - 1$, $\phi^{\overline{H}}(a_j) = \sum_{i=1}^{m} \phi(x_i)$. Therefore the entries of the matrix $M(1_H)$ are the orbit sums of the action of H on N.

10. We know that $\mathbb{Z}_2$ acts fixed point free on $\mathbb{Z}_{2^r-1}$, the number of non-trivial orbits is given by $\alpha = \frac{|N|-1}{|H|} = 2^{r-1} - 1$ and the length of each orbit is given by $|H| = 2$. Also, since $\overline{H}$ is dihedral, the action of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^r-1}$ is given by $ba^ib^{-1} = a^{-i}$.

The orbits are: $\Theta_j = \{a^i,\ a^{-i}\}$ for $j = 1, 2, \ldots, \alpha$ and $i = 1, 2, \ldots, \alpha$.

11. Now let $p_j = \sum_{i=1}^{2} \phi(x_i) = [c^j(i,1)]$, for $j = 2, 3, \ldots, \alpha$, then $p_j = \phi(x_i) + \phi(x_i^{-1})$. So $p_j = \phi(x_i) + \overline{\phi(x_i)} = 2t$, where t is the real part of $\exp(\frac{2\pi i}{2^r-1})$.

The following table gives the partial character table of $\overline{H}$, namely the values of the induced characters of degree 2, $\chi_3, \chi_4, \ldots, \chi_{2+\alpha}$, on classes $((2^r-1)A_i\ ,\ i = 1, 2, \ldots, \alpha)$.

	$(2^r-1)A_1$	$(2^r-1)A_2$	$(2^r-1)A_3$	....	$(2^r-1)A_\alpha$
χ_3	p_1	p_2	p_3	....	p_α
χ_4	p_2	p_3	p_4	....	$p_{\alpha-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{2+\alpha}$	p_α	$p_{\alpha-1}$	$p_{\alpha-2}$	....	p_1

$$\text{where } p_j = \sum_{j=1}^{2} t_j = 2t_j \text{ and } t_j = \exp(\tfrac{2\pi i}{2^r-1})$$

We now produce the character tables of $\mathbb{Z}_2$ and $\mathbb{Z}_{2^r-1}$ and then the character table of the Frobenius group $\overline{H}$.

Table 7.3: Character Table of $\mathbb{Z}_2$

Classes	e	b
χ_1	1	1
χ_2	1	-1

Table 7.4: Character Table of $\mathbb{Z}_{2^r-1}$

Classes	e	a	a^2	$\dots$	$\dots\ a^{2^r-2}$
ϕ_1	1	1	1	1	1 1
ϕ_2	1	p	p^2	$\dots$	$\dots\ p^{2^r-2}$
$\vdots$					$\vdots$
$\vdots$					$\vdots$
ϕ_{2^r-1}	1	p^{2^r-2}	p^{2^r-3}	$\dots$	$\dots\ p$

where $p = e^{\frac{2\pi i}{2^r-1}}$

Table 7.5: Character Table of $\overline{H} = NH$

(g)	(1)	$(2^r-1)A_1 \dots\dots\dots (2^r-1)A_\alpha$	$(2A)$
$\lvert C_{\overline{H}}(g)\rvert$	$(2^r-1)2$	$2^r-1 \dots\dots\dots 2^r-1$	2
χ_1	1	$1 \dots\dots\dots 1$	1
χ_2	1	$1 \dots\dots\dots 1$	-1
χ_3	2	$p_1 \dots\dots\dots p_\alpha$	0
χ_4	2	$p_2 \dots\dots\dots p_{\alpha-1}$	0
$\vdots$	$\vdots$	$\vdots \dots\dots\dots \vdots$	$\vdots$
$\chi_{2+\alpha}$	2	$p_\alpha \dots\dots\dots p_1$	0

where $p_j = \sum_{j=1}^{2} t_j = 2t_j$ and $t_j = \exp(\frac{2\pi i}{2^r-1})$

Remark 7.1.1. For $n \in PSL(2,q) \cong SL(2,q)$, q even $(q = 2^r)$, let $n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{F}_q^*$

and $o(\lambda) = q - 1$. Then there exists an involution $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq b = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ for $x,\ y,\ z,\ t \in \mathbb{F}_q$ such that $bnb^{-1} = n^{-1}$ and $\langle n,\ b\rangle \cong \mathbb{Z}^{2^r-1}{:}\mathbb{Z}_2$.

For $b = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the following conditions apply:

- $xt - yz = 1$ .... (1)
- $b^2 = \begin{pmatrix} x^2+yz & xy+yt \\ xz+zt & yz+t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .... (2)
- $bnb^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ .... (3)

From equation (2) above , we have that:

- $x^2 + yz = 1$ .... (4)
- $xy + yt = 0$ .... (5)
- $xz + zt = 0$ .... (6)
- $zy + t^2 = 1$ .... (7)

From equation (5) above, we have that $y(x+t) = 0$ and that $y = 0$ or $x + t = 0$. Similarly from equation (6), we have $z(x+t) = 0$ and $z = 0$ or $x + t = 0$. From these two equations we have the following six cases to consider.

- Case 1: $y = 0$ and $x + t = 0$.
- Case 2: $z = 0$ and $x + t = 0$.
- Case 3: $y = z = 0$.
- Case 4: $y = 0$.
- Case 5: $z = 0$.
- Case 6: $x + t = 0$.

We now consider each case:

1. Case 1: $y = 0$ and $x + t = 0$. If $x + t = 0$ then $x = -t = t$ since in prime field $\mathbb{F}_2, -1 \equiv 1$. Now $y = 0$ implies by equation (4) that $x^2 = 1$ and $x = 1$. This gives us $b = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.

   Now $bnb^{-1} = n^{-1}$ implies that $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$.

   So, $\begin{pmatrix} \lambda & 0 \\ z\lambda + z\lambda^{-1} & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$. Therefore, we must have $z\lambda + z\lambda^{-1} = 0$ and $z(\lambda + \lambda^{-1}) = 0$ implies $z = 0$ since $\lambda + \lambda^{-1} = 0$ implies that $o(\lambda) = 2$ which we can not have.

   Hence, this gives $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

2. Case 2: $z = 0$ and $x + t = 0$. Just as in Case 1 above we arrive at the conclusion that $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

3. Case 3: If $y = z = 0$, then as in Cases 1 and 2 above, we get $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

4. Case 4: If $y = 0$ then equation (4) implies that $x = 1$. So b has the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. This case is similar to Case 1 above and we get: $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

5. Case 5: $z = 0$ gives us a similar conclusion as in Case 4 above where we get $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

6. Case 6: If $x + t = 0$, then $x = t$. There are two cases to consider here: (i)$x = t \neq 0$ and (ii)$x = t = 0$.

   If $x = t \neq 0$, then b has the form $\begin{pmatrix} x & y \\ z & x \end{pmatrix}$. Then for $n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\lambda \neq 0$, $bnb^{-1} = n^{-1}$ gives:

   $$\begin{pmatrix} x & y \\ z & x \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} x & y \\ z & x \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

   So,

   $$\begin{pmatrix} x^2\lambda + zy\lambda^{-1} & xy\lambda + xy\lambda^{-1} \\ xz\lambda + xz\lambda^{-1} & yz\lambda + x^2\lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

   So this gives us the following: $xy\lambda + xy\lambda^{-1} = 0$ .... (8). Then $xy(\lambda + \lambda^{-1}) = 0$ and $\lambda + \lambda^{-1} = 0$ or $xy = 0$. If $\lambda + \lambda^{-1} = 0$, then $\lambda = \lambda^{-1}$ and $o(\lambda) = 2$ which is not possible since λ is an element in a cyclic group of odd order. Therefore, $xy = 0$ and since $x \neq 0$,

$y = 0$. Similarly using the equation $xz\lambda + xz\lambda^{-1} = 0$, we arrive at the conclusion $z = 0$. So for this first case $x = t \neq 0$ implies that $y = z = 0$ and equation (1) gives $x = 1$. Hence, $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction. For the second case, if $x + t = 0$ and $x = t = 0$, then equations (1) and (4) implies that $yz = 1$. Thus, b has the form $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$.

Then $z = y^{-1}$ and $b = \begin{pmatrix} 0 & \lambda^i \\ \lambda^{-i} & 0 \end{pmatrix}$ for $\lambda^i \in \mathbb{F}_q^*$.

Before describing the conjugacy classes of the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$,we make the following Note.

Note 7.1.1. 1. $|PSL(2, q)| = q^3 - q$ if q is even.

2. If q is even, then $PSL(2, q) \cong SL(2, q)$.

3. The group $SL(2, q)$, $q = 2^t$, $t \geq 1$ has $q + 1$ distinct conjugacy classes, see Basheer [4]. These classes are described in the Table 7.6 below.

Table 7.6: Conjugacy Classes of $SL(2, q)$, q even.

Class	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
Rep. of Class	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & r + r^q \end{pmatrix}$
No. of Classes	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
$\lvert C_{SL(2,2^t)}(g)\rvert$	$q^3 - q$	q	$q - 1$	$q + 1$
$\lvert C_g\rvert$	1	$q^2 - 1$	$q(q + 1)$	$q(q - 1)$

where $\alpha \in \mathbb{F}_q^*$, $\alpha = \epsilon^k$, $k \neq 0$ and if $\mathbb{F}_{q^2}^* = \langle\theta\rangle$, then $r = \theta^{(q-1)j}$ for $j = 1, 2, \ldots, \frac{q}{2}$.

7.2 Conjugacy Classes of $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

We determine now the conjugacy classes of the double Frobenius group $\overline{G} = GNH = G{:}NH$ where $N = \langle a\rangle \cong \mathbb{Z}_{2^r-1}$ and $H = \langle b\rangle \cong \mathbb{Z}_2$.

To determine the conjugacy classes of $\overline{G}$, we consider the cosets $\overline{h}G$ where $\overline{h} \in NH = \overline{H}$.

The coset 1G :

Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^{2r}$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}, C_{\overline{H}}(\overline{h}) = \overline{H}$ on G.

Now let $nh \in \overline{H} = NH$ for $1_{\overline{H}} \neq n \in N$, $1_{\overline{H}} \neq h \in H$. Then for $g \in G$, $g^{nh} = nhgh^{-1}n^{-1} = n(hgh^{-1})n^{-1} = n(g^h)n^{-1} = (g^h)^n$.

Therefore to act $\overline{H}$ on G, we act $h \in H$ first and then act $n \in N$.

Now $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2 \leq PSL(2,2^r) \cong SL(2,2^r) \leq GL(2,2^r)$ and $H = \langle b\rangle$ where b is an involution in $PSL(2,2^r)$. The group $SL(2,q)$, q even, $(q = 2^r, r \geq 1)$ has $q+1$ distinct conjugacy classes. Of these $q+1$ classes there is only one class of involutions. The size of this class is q^2-1 and for any involution $b \in SL(2,q)$, b has the form $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$, where $x, z \in \mathbb{F}_q^*$. See the Note 7.1.1.

Now $G \cong V_2(\mathbb{F}_q)$, the vector space of dimension two over the field of $q = 2^r$ elements. So $G = \{0, \lambda^i e_1, \lambda^j e_2, (\lambda^i e_1 + \lambda^j e_2)\}$ for $i, j = \{0,1,2,\dots,q-2\}$, where $\lambda^i, \lambda^j \in \mathbb{F}_q^*$, and $\{e_1, e_2\}$ is a basis of G, with $e_1^2 = 1$, $e_2^2 = 1$. So in the action of $\overline{H}$ on G, we look at the action of $b = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$ on the elements of $G \cong 2^{2r} = \{0, \lambda^i e_1, \lambda^j e_2, (\lambda^i e_1 + \lambda^j e_2)\}$ followed by the action of N on these orbits (the orbits of b on G).

Now H acts on the 2^{2r} elements of G fixing (including the identity) 2^r elements and permuting the remaining $2^{2r} - 2^r$ elements in orbits of length two. So the elements of G are now in $\frac{2^{2r}-2^r}{2}$ orbits of length two plus the 2^r fixed points. Acting N on these 2^r fixed points and the $\frac{2^{2r}-2^r}{2}$ orbits gives the following:

1. Each of the (2^r-1) (identity excluded) fixed points fuses with $\frac{2^r-2}{2} = 2^{r-1}-1$ of the two cycles to form an orbit of size (2^r-1). There are (2^r-1) of these orbits.
2. The remaining $2^{2r} - \{(2^r-1)(2^r-1)+1\} = 2(2^r) - 2$ elements fuse to form an orbit of size $2(2^r-1)$.

So the action of $C_{\overline{H}}(\overline{h}) = \overline{H}$ on G, gives the following: one orbit of length one, one orbit of length $2(2^r-1)$ and (2^r-1) orbits of length (2^r-1).

Therefore we have: $k = 2^{2r}$ and $f_1 = 1$, $f_i = 2^r - 1$ for $i = 2,3,\dots,2^r-1$ and $f_{2^r+1} = 2(2^r-1)$.

$k = 2^{2r},\ f_1 = 1$:

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^{2r} \times (2^r - 1) \times 2}{1} = |\overline{G}|,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^{2r} \times (2^r - 1) \times 2} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

$k = 2^{2r},\ f_i = 2^r - 1$ for $i = 2, 3, \ldots, 2^r - 1$.

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^r - 1} = 2 \times 2^{2r},$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times 2^{2r}} = 2^r - 1.$$

$k = 2^{2r},\ f_i = 2(2^r - 1)$:

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times (2^r - 1)} = 2^{2r},$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^{2r}} = 2 \times (2^r - 1).$$

Therefore the identity coset $1G$ produces the following conjugacy classes of $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$: the identity conjugacy class, one class of size $2(2^r - 1)$ and $(2^r - 1)$ classes of size $(2^r - 1)$. The order of the non -identity elements in all the classes from the identity coset is two.

The coset bG :

When G acts on the coset bG, it partitions the coset into 2^r orbits of size 2^r. The 2^r orbits of the action of G on the coset bG consists of the orbit containing b, and $(2^r - 1)$ remaining orbits each containing $b\lambda^i e_1$ for $i = 0, 1, \ldots, q-2$ where $q = 2^r$. The orbit containing b also contains $b\lambda^i(e_1 + e_2)$ for $i = 0, 1, \ldots, q-2$. Each of the orbits containing $b\lambda^i e_1$ also contain $b\lambda^i e_2$ for $i = 0, 1, \ldots, q-2$ and a two cycle $\{(\lambda^i e_1 + \lambda^j e_2)\ ,\ (\lambda^j e_1 + \lambda^i e_2)\}$ for $i \neq j$ and $i, j = 0, 1, \ldots, q-2$. We now act the centralizer of $b \in H$, $C_{\overline{H}}(b)$ on these 2^r orbits. Since $C_{\overline{H}}(b) = \langle b \rangle$, the action of the centralizer is just the action of $\langle b \rangle$.
When b acts on G, it fixes zero and each $\lambda^i(e_1 + e_2)$ for $i = 0, 1, \ldots, q-2$ and permutes the remaining $2^{2r} - 2^r$ elements of G into $\frac{2^{2r}-2^r}{2}$ orbits of length two. These $\frac{2^{2r}-2^r}{2}$ orbits are of the form $(\lambda^i e_1\ ,\ \lambda^i e_2)$ for $i = 0, 1, \ldots, q-2$ and $\{(\lambda^i e_1 + \lambda^j e_2)\ ,\ (\lambda^j e_1 + \lambda^i e_2)\}$ for $i \neq j$ and $i, j = 0, 1, \ldots, q-2$.

This implies that when $\langle b\rangle = C_{\overline{H}}(b)$ acts on the 2^r orbits, it permutes the elements in each of the 2^r orbits.

$k = 2^r,\ f_i = 1$ for $i = 1, 2, \ldots, 2^r$:

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{2^r \times 2}{1} = 2 \times 2^r,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times 2^r} = 2^r \times (2^r - 1).$$

Therefore, the coset bG produces 2^r conjugacy classes of $\overline{G}$ each of size $2^r(2^r - 1)$. We determine now the orders of the elements in these 2^r conjugacy classes. Using Remark 2.1.4, we have that $\omega = gg^b$ for $g \in G,\ b \in H$. Now if $\omega = 1$, then $g^{-1} = g = g^b$ since $o(g) = 2$. Therefore, if b fixes $g \in G$, then $o(\overline{g}) = 2$ for $gb = \overline{g} \in \overline{G}$. If $\omega \neq 1$, then $o(\overline{g}) = 4$. Since the conjugacy class containing b is the class that has the fixed points, from the discussion above the order of the elements in this class is two (as would be expected since the class has the element b in it). The order of the elements in the remaining $(2^r - 1)$ conjugacy classes is four.

The coset a_iG : for $i = 1, 2, \ldots, m$ where m is the number of non-trivial orbits of the action of H on N. Now the action of G on the coset a_iG for $i = 1, 2, \ldots, m$ produces a single orbit of length 2^{2r}. The centralizer of $a_i \in N$, $C_{\overline{H}}(a_i) = \langle a_i\rangle$ acts fixed point free on the orbit permuting the elements in the orbit. Therefore, for each $a_i \in N$ for $i = 1, 2, \ldots, m$, there is a single conjugacy class of $\overline{G}$. There are m such classes.

$k = 1,\ f = 1$:

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times (2^r - 1)}{1} = 2^r - 1,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^r - 1} = 2 \times 2^{2r}.$$

The coset a_iG, produces a single conjugacy class of $\overline{G}$ of size $2(2^{2r})$. The order of the elements in the class is $2^r - 1$.

Note 7.2.1. From the discussion above we can determine the number of conjugacy classes of $\overline{G}$. The classes produced by the identity coset are the identity class, $(2^r - 1)$ classes of size $(2^r - 1)$ and one class of size $2(2^r - 1)$. The coset bG produces 2^r classes of size $2^r(2^r - 1)$. Each of the cosets a_iG for $i = 1, 2, \ldots, m$ where $m = \frac{|N|-1}{H} = \frac{2^r-2}{2} = 2^{r-1} - 1$ is the number of non-trivial orbits of H

on N produces a single conjugacy class of size $2(2^{2r})$. There are m such classes.

$$c(\overline{G}) = (1 + 2^r - 1 + 1) + (2^r) + (2^{r-1} - 1) = 2 \times 2^r + 2^{r-1} = 2^{r+1} + 2^{r-1}.$$

The full list of conjugacy classes based on coset analysis is given in Table 7.7.

Table 7.7: Conjugacy Classes of $\overline{G} = G{:}(NH)$

$\overline{H} = NH$	$\overline{G} = G{:}(NH)$	$o(\overline{g})$	$\|[g]\|$	$\|C_{\overline{G}}(\overline{g})\|$
1	1	1	1	$2^{2r}(2^r-1)2$
	$(2A)_1$	2	$2^r - 1$	$2(2^{2r})$
	$(2A)_2$	2	$2^r - 1$	$2(2^{2r})$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(2A)_{2^r-1}$	2	$2^r - 1$	$2(2^{2r})$
	$(2A)_{2^r+1}$	2	$2(2^r-1)$	2^{2r}
b	$(2B)$	2	$2^r(2^r-1)$	$2(2^r)$
	$(4A)_1$	4	$2^r(2^r-1)$	$2(2^r)$
	$(4A)_2$	4	$2^r(2^r-1)$	$2(2^r)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(4A)_{2^r-1}$	4	$2^r(2^r-1)$	$2(2^r)$
a_1	$(2^r-1)A_1$	c_1	$2(2^{2r})$	$2^r - 1$
a_2	$(2^r-1)A_2$	c_2	$2(2^{2r})$	$2^r - 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	$(2^r-1)A_m$	c_α	$2(2^{2r})$	$2^r - 1$

The following Proposition and Remark will be used to construct the Fischer matrices of the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$.

Proposition 7.2.1. *([29]). If* G *is elementary abelian and* $M = \mathrm{Im}(\phi_g)$ *where* ϕ_g *is an endomorphism of* G *defined by* $\phi_g : x \mapsto xgx^{-1}g^{-1}$ *for* $x \in G$*, then* $[G : M] = k$ *where* k *is the number of elements of* G *fixed by a class representative* $\overline{h} \in \overline{H}$ *where* $\overline{G} = G{:}\overline{H}$.

PROOF: The orbits $Q_1, Q_2, \ldots, Q_k$ of G acting on $\overline{h}G$ are the same as the orbits $D_1, D_2, \ldots, D_k$ of M acting on $\overline{h}G$ by left multiplication (See Remark (5.2.6) in Mpono, [29]). Also the orbits $D_1, D_2, \ldots, D_k$ can be identified with the elements of G/M. Then it follows that $G/M = [G : M] = k$. ■

Remark 7.2.1. If G is an elementary abelian p-group, then from coset analysis for the group $\overline{G} = G{:}\overline{H}$, we obtain $k = p^m$ for $0 \leq m \leq n$, where $|G| = p^n$ and k is the number of elements of G fixed by a class representative $\overline{h}$ of $\overline{H}$. Suppose for some class representative $\overline{h} \in \overline{H}$, we have the orbits $Q_1, Q_2, \ldots, Q_k$ of the action of G on $\overline{h}G$. Then for $h \in C_{\overline{H}}(\overline{h})$, suppose that acting h on the orbits $Q_1, Q_2, \ldots, Q_k$, we get $f_1 = f_2 = \ldots = f_k = 1$ and that the entries of the first column of $M(\overline{h})$ are 1. Then in this case, the Fischer matrix $M(\overline{h})$ coincides with the character table of the abelian group G/M of order $k = p^m$.

7.3 Fischer Matrices of $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

In this section we will give a general description of the number of Fischer matrices and their form for the double Frobenius group $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$. For each conjugacy class of $\overline{H}$ there is a corresponding Fischer matrix. Therefore there are $2^{r-1}+1$ such matrices. The action of $\overline{H}$ on G produces 2^r+1 orbits. The Fischer matrix $M(1_{\overline{H}})$ corresponding to the identity coset is therefore a $\left((2^r+1)\times(2^r+1)\right)$ matrix. Since the action of $\overline{H}$ of G has 2^r+1 orbits, by Brauer's Lemma the action of $\overline{H}$ on $\mathrm{Irr}(G)$ has 2^r+1 orbits also. The lengths of the orbits are 1, (2^r-1) and $2(2^r-1)$. The number of orbits of each length is 1, (2^r-1) and 1 respectively. We can show that the orbits of the action of $\overline{H}$ on $\mathrm{Irr}(G)$ also have lengths 1, (2^r-1) and $2(2^r-1)$ and that the number of orbits is also 1, (2^r-1) and 1 respectively. Now when $\overline{H}$ acts on $\mathrm{Irr}(G)$, the possibilities for orbit lengths are: 1, 2, (2^r-1) and $2(2^r-1)$. Let the number of orbits of length one be a, the number of orbits of length two be b, the number of orbits of length 2^r-1 be c and the number of orbits of length $2(2^r-1)$ be d. Then

$$a+b+c+d = 2^r+1 \quad \cdots\cdots (1)$$

and

$$a+2b+(2^r-1)c+2(2^r-1)d = 2^{2r} \quad \cdots\cdots (2).$$

where $a, b, c, d \in \mathbb{N}$.

We find values for a, b, c, d. Note first that we can assume that $r \geq 2$ in equations 1 and 2 above since $r = 0$ and $r = 1$ give trivial cases for the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$.

We know that $a \geq 1$ since the action of $\overline{H}$ on $\mathrm{Irr}(G)$ fixes the identity character. We claim that $a = 1$. So suppose that $a > 1$. Then equation (2) implies that $(a-1)+2b+c(2^r-1)+2d(2^r-1) = 2^{2r}-1 \quad \cdots\cdots (3)$. So, $(a-1)+2b+c(2^r-1)+2d(2^r-1) = (2^r-1)(2^r+1) \quad \cdots\cdots (4)$. Since 2^r-1 divides the right hand side of equation (4), it must divide the left hand side also. Therefore, 2^r-1 divides $(a-1)+2b$. So $(2^r-1)\alpha = a-1+b$ for some $\alpha \in \mathbb{N}$. From this equation we get

$a+b=\alpha 2^r-\alpha+1$. Substituting this into equation (1) above gives $c+d=2^r(1-\alpha)+\alpha \cdots\cdots (5)$. Since $r \geq 2$, and $a,b,c,d \in \mathbb{N}$, equation (5) will only be true if $\alpha=0$ or $\alpha=1$.

$\underline{\alpha=0}$:

Then $a+b=1$ and since $a \neq 0$, $b=0$ and $a=1$. This is a contradiction since $a>1$.

$\underline{\alpha=1}$:

Then $a+b=2^r$ and $c+d=1$. Therefore there are two cases to consider:(i) $c=0$ and $d=1$ (ii) $c=1$ and $d=0$.

If $c=0$ and $d=1$, then equation (2) implies that $a+2b+2(2^r-1)=2^{2r}$ and hence $2^r+b+2(2^r-1)=2^{2r}$ after substituting for $a=2^r-b$. This gives $2^{2r}=b-2+2^r+2.2^r \cdots\cdots (6)$. Now 2^r divides the left hand side of equation (6) and must divide the right hand side also. Therefore, 2^r divides $b-2$ and $b=2^r\gamma+2$ for some $\gamma \in \mathbb{N}$. This is a contradiction since $b<2^r$.

If $d=0$ and $c=1$, then equation(2) implies that $a+2b+2^r-1=2^{2r}$ and hence $2^r+b+2^r-1=2^{2r}$ after substituting for $a=2^r-b$. This gives $2^{2r}=b-1+2.2^r \cdots\cdots (7)$. Now 2^r divides the left hand side of equation(7) and must divide the right hand side also. Therefore, 2^r divides $b-1$ and $b=2^r\delta+1$ for some $\delta \in \mathbb{N}$. This is a contradiction since $b<2^r$.

Therefore, $a=1$ as claimed. With $a=1$, from equation (1) we now have that $b+c+d=2^r$ and from equation (2) we have that $2b+(2^r-1)c+2(2^r-1)d=(2^r-1)(2^r+1)$. Since 2^r-1 divides the right hand side of this equation, it must divide the left hand side. This implies that 2^r-1 divides $2b$. Since 2^r-1 is odd, 2^r-1 must divide b. Thus, we have $(2^r-1)\epsilon=b$ for some $\epsilon \in \mathbb{N}$. From equation (1), $a+b+c+d=2^r+1$ which implies that $b+c+d=2^r$ and hence that $b \leq 2^r$. So, $(2^r-1)\epsilon \leq 2^r$ and since $r \geq 2$, this inequality is true only if $\epsilon=0$ or $\epsilon=1$.

If $\epsilon=1$, then $b=2^r-1$ and equation (1) now implies that $c+d=1$. There are two cases to consider.

$\underline{c=0\ ,d=1}$:

Equation (2) now implies that $2(2^r-1)+0+2(2^r-1)=(2^r+1)(2^r-1)$, which gives us $4=2^r+1$ which is false.

$\underline{c=1\ ,d=0}$:

Equation (2) now implies that $2(2^r-1)+2^r-1=(2^r+1)(2^r-1)$ and $3=2^r+1$ which is false for $r \geq 2$.

Thus we must have $\epsilon=0$ and hence $b=0$.

With $a=1$ and $b=0$, equation (1) and equation (2) now give $c+d=2^r$ and $c+2d=2^r+1$ respectively. Solving gives us $d=1$ and $c=2^r-1$.

Therefore $a=1=d$, $b=0$ and $c=2^r$. Therefore, when $\overline{H}$ acts on $\mathrm{Irr}(G)$, there is one orbit of length one, 2^r-1 orbits of length 2^r-1 and one orbit of length $2(2^r-1)$. This is the same number of orbits and orbit lengths as the action of $\overline{H}$ on G.

7.3.1 The Inertia Groups and Inertia Factor Groups of $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

Using the results of the section above, we can give a general description of the inertia groups and inertia factor groups for the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$.

Now denote the inertia groups by $\overline{P}$ and the inertia factor groups by P. Since the action of $\overline{H}$ on the Irr(G) has orbit lengths 1, (2^r-1), $2(2^r-1)$, the inertia groups are:

$$\overline{P}_1 = \overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1} : \mathbb{Z}_2)\ ,\ \overline{P}_2 = \overline{P}_3 = \cdots = \overline{P}_{2^r} = G{:}H = 2^{2r}{:}\mathbb{Z}_2\ ,\ \overline{P}_{2^r+1} = G = 2^{2r}.$$

The inertia factor groups are:

$$P_1 = \overline{H} = \mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2\ ,\ P_2 = P_3 = \cdots = P_{2^r} = H = \mathbb{Z}_2\ ,\ P_{2^r+1} = \{1_{\overline{H}}\}.$$

7.3.2 Fischer Matrices

There are (2^r+1) Fischer matrices, namely, $M(1_{\overline{H}})$, $M(b)$ and $M(a_i)$ for $i = 1, 2, \ldots, m$ where $m = 2^{r-1} - 1$ is the number of non-trivial orbits of the action of H on N.

$\underline{M(1_{\overline{H}}):}$

This is a $\big((2^r+1)\times(2^r+1)\big)$ matrix. The first row of the matrix is a row of 1's. The first column of the matrix consists of the (2^r+1) entries: $\big(1,\ 2^r-1,\ 2^r-1,\ \cdots\cdots\cdots, 2^r-1,\ 2(2^r-1)\big)^t$. There are (2^r-1) entries of (2^r-1). The last column of the matrix consists of the entries $\big(1,\ -1,\ -1, \cdots\cdots\cdots, 2^r-2\big)^t$. There are (2^r-1) entries of -1. The last row of the matrix consists of the entries $\big(2(2^r-1),\ -2,\ -2, \cdots\cdots\cdots, 2^r-2\big)$. The remainder of the matrix is a $\big(2^r-1\times 2^r-1\big)$ block whose rows are just a permutation of the entries $\big(2^r-1,\ -1,\ -1, \cdots\cdots\cdots, -1\big)$. This is the block denoted by X X X in the Fischer matrix shown below.

$$M(1_{\overline{H}}) = \left(\begin{array}{c|cccc|c} 1 & 1 & 1 & \ldots\ldots & 1 & 1 \\ \hline 2^r-1 & & & & & -1 \\ 2^r-1 & & & & & -1 \\ \cdots & & & & & \cdots \\ \cdots & X & X & X & X & \cdots \\ \cdots & & & & & \cdots \\ 2^r-1 & & & & & -1 \\ \hline 2(2^r-1) & -2 & -2 & \ldots\ldots & -2 & 2^r-2 \end{array}\right)$$

M(b) :

This is a $(2^r \times 2^r)$ matrix. By Remark 7.2.1, the Fischer matrix corresponding to $b \in \overline{H}$ coincides with the character table of the elementary abelian group of order $k = 2^m$ where k is number of fixed points of the action of b on G.

$$M(b) = \begin{pmatrix} \text{Character Table of} \\ \text{elementary abelian group} \\ \text{of order } 2^m \end{pmatrix}$$

$M(a_i)$:

These are just singleton matrices with entry 1 of which there are $m = 2^{r-1} - 1$ in number.

8

Examples

8.1 The Group $2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$

Let $\overline{G} = GNH$ be the double Frobenius group with $G \cong 2^3$, $N \cong \mathbb{Z}_7 = \langle a \rangle$ and $H \cong \mathbb{Z}_3 = \langle b \rangle$. In this section we apply the general theory developed in Chapter Six to this example. The conjugacy classes, Fischer matrices and character table of this group was done by Whitney [38] in her Masters thesis.

8.1.1 Conjugacy Classes of $\overline{H}$

Let $\overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_3$ be the Frobenius group with kernel $\mathbb{Z}_7$ and complement $\mathbb{Z}_3$. Now $\mathbb{Z}_3$ acts on $\mathbb{Z}_7$ fixed point free with orbits of lengths 1, 3 and 3 where the number of non-trivial orbits equals $\alpha = \frac{|\mathbb{Z}_7|-1}{|\mathbb{Z}_3|} = 2$.

The number of conjugacy classes of $\overline{H}$ is given by $\frac{3^2+2^3-2}{3} = 5$. These classes are made up of the identity class of $\overline{H}$, two classes of size 3 containing elements of order 7 and two classes of size 7 containing elements of order 3. These are displayed in the Table 8.1.

Table 8.1: Conjugacy Classes of $\overline{H}$

H	$\overline{H} = NH$	$o(\overline{h})$	$\lvert C_{NH}(\overline{h})\rvert$
1_H	1	1	21
	(7_1)	7	7
	(7_2)	7	7
b	(3_1)	3	3
b^2	(3_2)	3	3

where (7_i) for $i = 1, 2$ is the conjugacy class containing elements of order 7 and (3_i) for $i = 1, 2$ is the conjugacy class containing elements of order 3 using the notation of ATLAS. Also here $\mathbb{Z}_3 = \langle b \rangle$.

8.1.2 Character Table of $\overline{H}$

1. We know that $\overline{H}$ has 5 characters

2. The two non-trivial orbits of $\mathbb{Z}_3$ on $\mathbb{Z}_7$ are: $\Theta_1 = \{a, a^2, a^4\}$, $\Theta_2 = \{a^3, a^6, a^5\}$.

3. The character table of $\mathbb{Z}_3$ is given in Table 8.2 where $q = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$,
   $q^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = \overline{q}$

Table 8.2: Character table of $\mathbb{Z}_3$

	e	b	b^2
χ_1	1	1	1
χ_2	1	q	q^2
χ_3	1	q^2	q

4. The character table of $\mathbb{Z}_7$ is given in Table 8.3 where $p = e^{\frac{2\pi i}{7}}$

Table 8.3: Character table of $\mathbb{Z}_7$

	e	a	a^2	a^3	a^4	a^5	a^6
ϕ_1	1	1	1	1	1	1	1
ϕ_2	1	p	p^2	p^3	p^4	p^5	p^6
ϕ_3	1	p^2	p^3	p^4	p^5	p^6	p
ϕ_4	1	p^3	p^4	p^5	p^6	p	p^2
ϕ_5	1	p^4	p^5	p^6	p	p^2	p^3
ϕ_6	1	p^5	p^6	p	p^2	p^3	p^4
ϕ_7	1	p^6	p	p^2	p^3	p^4	p^5

5. The character table of $\overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_3$ is given in Table 8.4 where $q = e^{\frac{2\pi i}{3}}$, $p = e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{8\pi i}{7}}$

Table 8.4: Character table of $\mathbb{Z}_7$:$\mathbb{Z}_3$

$[g]$	(1)	(7_1)	(7_2)	(3_1)	(3_2)
$\|C_{\overline{H}}(g)\|$	21	7	7	3	3
χ_1	1	1	1	1	1
χ_2	1	1	1	q	$\overline{q}$
χ_3	1	1	1	$\overline{q}$	q
χ_4	3	p	$\overline{p}$	0	0
χ_5	3	$\overline{p}$	p	0	0

8.1.3 Conjugacy Classes of 2^3:($\mathbb{Z}_7$:$\mathbb{Z}_3$)

The identity of $\overline{H}$ produces two classes of $\overline{G}$, the identity class of $\overline{G}$ and a class of size 7 containing elements of order 2. Each of the classes (7_1) and (7_2) of $\overline{H}$ produces a unique class of $\overline{G}$. The size of each of these classes is 24 and they contain elements of order 7. Each of the classes (3_1) and (3_2) of $\overline{H}$ produces two classes of $\overline{G}$, both classes have size 28 with one class containing elements of order 3 and the other elements of order 6 respectively. The classes are listed in Table 8.5.

Table 8.5: Conjugacy classes of $\overline{G}$

H	$\overline{G}$	$o(\overline{g})$	$\|C_{\overline{G}}(\overline{g})\|$
1	1	1	168
	(2_1)	2	24
(7_1)	(7_1)	7	7
(7_2)	(7_2)	7	7
(3_1)	(3_1)	3	6
	(6_1)	6	6
(3_2)	(3_2)	3	6
	(6_2)	6	6

8.1.4 Fischer Matrices of $\overline{G}$

The number of Fischer matrices of $\overline{G}$ equals $\frac{3^2+2^3-2}{3} = 5$. For each conjugacy class of $\overline{H}$ there is a corresponding Fischer matrix. The identity class of $\overline{H}$ produces two classes of $\overline{G}$ and the Fischer matrix corresponding to this class is a (2×2) matrix given by: $M(1_{\overline{H}}) = \begin{pmatrix} 1 & 1 \\ 7 & -1 \end{pmatrix}$.

Each of the conjugacy classes (7_1) and (7_2) produces a unique class of $\overline{G}$, with corresponding Fischer matrix being a (1×1) matrix (1). There are 2 of these matrices.

Each of the conjugacy classes (3_1) and (3_2) produces two conjugacy classes of $\overline{G}$. The corresponding (2×2) Fischer matrices are given by: $M((3_1)) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $M((3_2)) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

8.1.5 The Character Table of $\overline{G} = 2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$

Table 8.6: Character table of $2^3{:}(\mathbb{Z}_7{:}\mathbb{Z}_3)$

	(1)	(2_1)	(7_1)	(7_2)	(3_1)	(6_1)	(3_2)	(6_2)
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	q	q	$\overline{q}$	$\overline{q}$
χ_3	1	1	1	1	$\overline{q}$	$\overline{q}$	q	q
χ_4	3	3	p	$\overline{p}$	0	0	0	0
χ_5	3	3	$\overline{p}$	p	0	0	0	0
χ_6	7	-1	0	0	1	-1	1	-1
χ_7	7	-1	0	0	q	$-q$	$\overline{q}$	$-\overline{q}$
χ_8	7	-1	0	0	$\overline{q}$	$-\overline{q}$	q	$-q$

$$q = e^{\frac{2\pi i}{3}}, \quad p = e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{8\pi i}{7}}$$

8.2 The Group $2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$

Let $\overline{G} = GNH$ be the double Frobenius group with $G \cong 2^5$, $N \cong \mathbb{Z}_{31} = \langle a \rangle$ and $H \cong \mathbb{Z}_5 = \langle b \rangle$. Let $\overline{H} = NH$.

8.2.1 Conjugacy Classes of $\overline{H}$

Let $\overline{H} = \mathbb{Z}_{31}{:}\mathbb{Z}_5$ be the Frobenius group with kernel $\mathbb{Z}_{31}$ and complement $\mathbb{Z}_5$. Now $\mathbb{Z}_5$ acts on $\mathbb{Z}_{31}$ fixed point free with orbits of length 1 and 5 and the number of non-trivial orbits equals $\alpha = \frac{|\mathbb{Z}_{31}|-1}{|\mathbb{Z}_5|} = 6$.

The number of conjugacy classes of $\overline{H}$ is given by $\frac{5^2+2^5-2}{3} = 11$. These classes are made up of the identity class of $\overline{H}$, six classes of size 5 containing elements of order 31 and four classes of size 31 containing elements of order 5. These are displayed in the Table 8.7.

Table 8.7: Conjugacy Classes of $\overline{H}$

H	$\overline{H} = NH$	$o(\overline{h})$	$\lvert C_{NH}(\overline{h})\rvert$
1_H	1	1	155
	(31_1)	31	31
	(31_2)	31	31
	(31_3)	31	31
	(31_4)	31	31
	(31_5)	31	31
	(31_6)	31	31
b	(5_1)	5	5
b^2	(5_2)	5	5
b^3	(5_3)	5	5
b^4	(5_4)	5	5

8.2.2 Character Table of $\overline{H}$

1. We know that $\overline{H}$ has 11 characters.

2. The six non-trivial orbits of $\mathbb{Z}_5$ on $\mathbb{Z}_{31}$ are:

$$\Theta_1 = \{a,\ a^2,\ a^4,\ a^8,\ a^{16}\},$$

$$\Theta_2 = \{a^3,\ a^6,\ a^{12},\ a^{24},\ a^{17}\},$$

$$\Theta_3 = \{a^5,\ a^{10},\ a^{20},\ a^9,\ a^{18}\},$$

$$\Theta_4 = \{a^{15},\ a^{30},\ a^{29},\ a^{27},\ a^{23}\},$$

$$\Theta_5 = \{a^7,\ a^{14},\ a^{28},\ a^{25},\ a^{19}\},$$

$$\Theta_6 = \{a^{11},\ a^{22},\ a^{13},\ a^{26},\ a^{21}\}.$$

3. Now let $p_1 = \phi(a) + \phi(a^2) + \phi(a^4) + \phi(a^8) + \phi(a^{16})$ (see section 6.1.2(9)). Then $p_1 = e^{\frac{2\pi i}{31}} + e^{\frac{4\pi i}{31}} + e^{\frac{8\pi i}{31}} + e^{\frac{16\pi i}{31}} + e^{\frac{32\pi i}{31}}$, where $\phi \in \mathrm{Irr}(N)$ and the a^i are the representatives of the conjugacy classes of $N = \mathbb{Z}_{31}$ that fuse to Θ_1 above.

   Similarly, $p_2 = \phi(a^3)+\phi(a^6)+\phi(a^{12})+\phi(a^{17})+\phi(a^{24}) = e^{\frac{6\pi i}{31}} + e^{\frac{12\pi i}{31}} + e^{\frac{24\pi i}{31}} + e^{\frac{34\pi i}{31}} + e^{\frac{48\pi i}{31}}$, where $\phi \in \mathrm{Irr}(N)$ and the a^i are the representatives of the conjugacy classes of $N = \mathbb{Z}_{31}$ that fuse to Θ_2 and

$p_3 = \phi(a^5) + \phi(a^9) + \phi(a^{10}) + \phi(a^{18}) + \phi(a^{20}) = e^{\frac{10\pi i}{31}} + e^{\frac{18\pi i}{31}} + e^{\frac{20\pi i}{31}} + e^{\frac{36\pi i}{31}} + e^{\frac{40\pi i}{31}}$, where $\phi \in \mathrm{Irr}(N)$ and the a^i are the representatives of the conjugacy classes of $N = \mathbb{Z}_{31}$ that fuse to Θ_3.

Also, $p_4 = \overline{p_1}$, $p_5 = \overline{p_2}$, $p_6 = \overline{p_3}$ (see section 6.1.2(9)).

Finally we note that in the character table of $\mathbb{Z}_{31}$:$\mathbb{Z}_5$, the conjugacy classes (31_i) for $i = 1, 2, 3, 4, 5, 6$ correspond to the orbits Θ_j for $j = 1, 2, 3, 4, 5, 6$.

4. The character table of $\mathbb{Z}_5$ is given in Table 8.8 where $q = e^{\frac{2\pi i}{5}}$.

Table 8.8: Character table of $\mathbb{Z}_5$

	e	b	b^2	b^3	b^4
χ_1	1	1	1	1	1
χ_2	1	q	q^2	q^3	q^4
χ_3	1	q^2	q^3	q^4	q
χ_4	1	q^3	q^4	q	q^2
χ_5	1	q^4	q	q^2	q^3

5. The character table of $\mathbb{Z}_{31}$ is given in Table 8.9 where $p = e^{\frac{2\pi i}{31}}$.

Table 8.9: Character table of $\mathbb{Z}_{31}$

	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	1	p	p^2	p^3	p^4	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}
ϕ_3	1	p^2	p^3	p^4	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}
ϕ_4	1	p^3	p^4	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}
ϕ_5	1	p^4	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}
ϕ_6	1	p^5	p^6	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}
ϕ_7	1	p^6	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}
ϕ_8	1	p^7	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}
ϕ_9	1	p^8	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}
ϕ_{10}	1	p^9	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}
ϕ_{11}	1	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}
ϕ_{12}	1	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}
ϕ_{13}	1	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}
ϕ_{14}	1	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}
ϕ_{15}	1	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}

	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}
Φ_{16}	1	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}
Φ_{17}	1	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}
Φ_{18}	1	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p
Φ_{19}	1	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}
Φ_{20}	1	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}
Φ_{21}	1	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}
Φ_{22}	1	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}
Φ_{23}	1	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}
Φ_{24}	1	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}
Φ_{25}	1	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}
Φ_{26}	1	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}
Φ_{27}	1	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}
Φ_{28}	1	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}
Φ_{29}	1	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}
Φ_{30}	1	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}
Φ_{31}	1	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}

	a^{16}	a^{17}	a^{18}	a^{19}	a^{20}	a^{21}	a^{22}	a^{23}	a^{24}	a^{25}	a^{26}	a^{27}	a^{28}	a^{29}	a^{30}
Φ_{1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Φ_{2}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}
Φ_{3}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p
Φ_{4}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}
Φ_{5}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}
Φ_{6}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}
Φ_{7}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}
Φ_{8}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}
Φ_{9}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}
Φ_{10}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}
Φ_{11}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}
Φ_{12}	p^{26}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}
Φ_{13}	p^{27}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}
Φ_{14}	p^{28}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}
Φ_{15}	p^{29}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}

	a^{16}	a^{17}	a^{18}	a^{19}	a^{20}	a^{21}	a^{22}	a^{23}	a^{24}	a^{25}	a^{26}	a^{27}	a^{28}	a^{29}	a^{30}
Φ_{16}	p^{30}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}
Φ_{17}	p	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}
Φ_{18}	p^{2}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}
Φ_{19}	p^{3}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}
Φ_{20}	p^{4}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}
Φ_{21}	p^{5}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}
Φ_{22}	p^{6}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}
Φ_{23}	p^{7}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}
Φ_{24}	p^{8}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}
Φ_{25}	p^{9}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}
Φ_{26}	p^{10}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}
Φ_{27}	p^{11}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}
Φ_{28}	p^{12}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}
Φ_{29}	p^{13}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}
Φ_{30}	p^{14}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}
Φ_{31}	p^{15}	p^{16}	p^{17}	p^{18}	p^{19}	p^{20}	p^{21}	p^{22}	p^{23}	p^{24}	p^{25}	p^{26}	p^{27}	p^{28}	p^{29}

6. The character table of $\mathbb{Z}_{31}{:}\mathbb{Z}_5$ is given in Table 8.10 where $q = e^{\frac{2\pi i}{5}}$ and p_i for $i = 1, 2, \ldots, 6$ are as given in Section 8.2.2, 3.

Table 8.10: Character table of $\mathbb{Z}_{31}$:$\mathbb{Z}_5$

$[g]$	(1)	(31_1)	(31_2)	(31_3)	(31_4)	(31_5)	(31_6)	(5_1)	(5_2)	(5_3)	(5_4)
$\lvert C_{\overline{H}}(g)\rvert$	155	31	31	31	31	31	31	5	5	5	5
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	q	q^2	q^3	q^4
χ_3	1	1	1	1	1	1	1	q^2	q^3	q^4	q
χ_4	1	1	1	1	1	1	1	q^3	q^4	q	q^2
χ_5	1	1	1	1	1	1	1	q^4	q	q^2	q^3
χ_6	5	p_1	p_2	p_3	p_4	p_5	p_6	0	0	0	0
χ_7	5	p_2	p_3	p_4	p_5	p_6	p	0	0	0	0
χ_8	5	p_3	p_4	p_5	p_6	p	p_2	0	0	0	0
χ_9	5	p_4	p_5	p_6	p	p_2	p_3	0	0	0	0
χ_{10}	5	p_5	p_6	p	p_2	p_3	p_4	0	0	0	0
χ_{11}	5	p_6	p	p_2	p_3	p_4	p_5	0	0	0	0

8.2.3 Conjugacy Classes of 2^5:($\mathbb{Z}_{31}$:$\mathbb{Z}_5$)

The identity of $\overline{H}$ produces two classes of $\overline{G}$, the identity class of $\overline{G}$ and a class of size 31 containing elements of order 2. Each of the classes (31_1), (31_2), (31_3), (31_4), (31_5) and (31_6) of $\overline{H}$ produces a unique class of $\overline{G}$. The size of the class is 160 and the class contains elements of order 31. Each of the classes (5_1), (5_2), (5_3) and (5_4) of $\overline{H}$ produces two classes of $\overline{G}$. Both classes have size 496 with one class containing elements of order 5 and the other elements of order 10. The classes are listed in Table 8.11.

Table 8.11: Conjugacy classes of $\overline{G}$

H	$\overline{G}$	$o(\overline{g})$	$\lvert C_{\overline{G}}(\overline{g})\rvert$
1	1	1	4960
	(2_1)	2	31
(31_1)	(31_1)	31	256
(31_2)	(31_2)	31	256
(31_3)	(31_3)	31	256
(31_4)	(31_4)	31	256
(31_5)	(31_5)	31	256
(31_6)	(31_6)	31	256
(5_1)	(5_1)	5	10
	(10_1)	10	10
(5_2)	(5_2)	5	10
	(10_2)	10	10
(5_3)	(5_3)	5	10
	(10_3)	10	10
(5_4)	(5_4)	5	10
	(10_4)	10	10

8.2.4 Fischer Matrices of $\overline{G}$

The number of Fischer matrices of $\overline{G}$ equals $\frac{5^2+2^5-2}{5} = 11$. For each conjugacy class of $\overline{H}$ there is a corresponding Fischer matrix. The identity class of $\overline{H}$ produces two classes of $\overline{G}$ and the Fischer matrix corresponding to this class is a (2×2) matrix given by: $M(1_{\overline{H}}) = \begin{pmatrix} 1 & 1 \\ 31 & -1 \end{pmatrix}$.

Each of the conjugacy classes (31_1), (31_2), (31_3), (31_4), (31_5) and (31_6) of $\overline{H}$ produces a unique class of $\overline{G}$. The corresponding Fischer matrix is a (1×1) matrix (1). There are 6 of these matrices.

Each of the conjugacy classes (5_1), (5_2), (5_3) and (5_4) of $\overline{H}$ produces two conjugacy classes of $\overline{G}$. The corresponding Fischer matrix is a (2×2) matrix given by: $M((5_i)) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $i = 1, 2, 3, 4$.

8.2.5 The Character Table of $\overline{G} = 2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$

Table 8.12: Character table of $2^5{:}(\mathbb{Z}_{31}{:}\mathbb{Z}_5)$

	(1)	(2_1)	(31_1)	(31_2)	(31_3)	(31_4)	(31_5)	(31_6)	(5_1)	(10_1)	(5_2)	(10_2)	(5_3)	(10_3)	(5_4)	(10_4)
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	q	q	q^2	q^2	q^3	q^3	q^4	q^4
χ_3	1	1	1	1	1	1	1	1	q^2	q^2	q^3	q^3	q^4	q^4	q	q
χ_4	1	1	1	1	1	1	1	1	q^3	q^3	q^4	q^4	q	q	q^2	q^2
χ_5	1	1	1	1	1	1	1	1	q^4	q^4	q	q	q^2	q^2	q^3	q^3
χ_6	5	5	p_1	p_2	p_3	p_4	p_5	p_6	0	0	0	0	0	0	0	0
χ_7	5	5	p_2	p_3	p_4	p_5	p_6	p	0	0	0	0	0	0	0	0
χ_8	5	5	p_3	p_4	p_5	p_6	p	p_2	0	0	0	0	0	0	0	0
χ_9	5	5	p_4	p_5	p_6	p	p_2	p_3	0	0	0	0	0	0	0	0
χ_{10}	5	5	p_5	p_6	p	p_2	p_3	p_4	0	0	0	0	0	0	0	0
χ_{11}	5	5	p_6	p	p_2	p_3	p_4	p_5	0	0	0	0	0	0	0	0
χ_{12}	31	−1	0	0	0	0	0	0	1	−1	1	−1	1	−1	1	−1
χ_{13}	31	−1	0	0	0	0	0	0	q	$-q$	q^2	$-q^2$	q^3	$-q^3$	q^4	$-q^4$
χ_{14}	31	−1	0	0	0	0	0	0	q^2	$-q^2$	q^3	$-q^3$	q^4	$-q^4$	q	$-q$
χ_{15}	31	−1	0	0	0	0	0	0	q^3	$-q^3$	q^4	$-q^4$	q	$-q$	q^2	$-q^2$
χ_{16}	31	−1	0	0	0	0	0	0	q^4	$-q^4$	q	$-q$	q^2	$-q^2$	q^3	$-q^3$

$q = e^{\frac{2\pi i}{5}}$ and p_i for $i = 1, 2, 3, 4, 5, 6$ are as given in Section 8.2.2(3).

8.3 The Group $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

Let $\overline{G} = GNH$ where $G \cong 2^{2r}$, $N \cong \mathbb{Z}_{2^r-1}$, $H \cong \mathbb{Z}_2$ and $NH \cong \overline{H}$. Let $r = 2$, then $G \cong 2^4$, $N \cong \mathbb{Z}_3 = \langle a \rangle$, $H \cong \mathbb{Z}_2 = \langle b \rangle$ and $NH \cong \overline{H} \cong \mathbb{Z}_3{:}\mathbb{Z}_2$. We know that $\mathbb{Z}_3{:}\mathbb{Z}_2 \leq PSL(2,4) \cong SL(2,4)$. We note also that $PSL(2,4)$ has a single class of involutions and a single class of elements of order three. Now $o(b) = 2$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where $\langle \lambda \rangle = \mathbb{F}_4^*$, $o(a) = 3$.

$G = \{0,\ e_1,\ e_2,\ (e_1+e_2),\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2,\ \lambda(e_1+e_2),\ \lambda^2(e_1+e_2),\ \lambda e_1 + e_2,\ \lambda^2 e_1 + e_2,\ e_1 + \lambda e_2,\ e_1 + \lambda^2 e_2,\ \lambda^2 e_1 + \lambda e_2,\ \lambda e_1 + \lambda^2 e_2\}$ and $GF(4) = \{0,\ 1,\ \lambda,\ \lambda^2\}$.

Now, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $be_1 = e_2$ and $be_2 = e_1$.

8.3.1 Conjugacy Classes of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.

To calculate the conjugacy classes of $\overline{G}$, we need the conjugacy classes of the group $\overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$. The conjugacy classes of $\overline{H}$ are represented in Table 8.13.

Conjugacy classes of : $\overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$.

Table 8.13: Conjugacy Classes of $\mathbb{Z}_3{:}\mathbb{Z}_2$

classes of $\mathbb{Z}_3 : \mathbb{Z}_2$	[1]	[b]	[a]
$\lvert C_G(g)\rvert$	6	2	3
$o(g)$	1	2	3
$\lvert[g]\rvert$	1	3	2

To calculate the conjugacy classes of $\overline{G}$ we use the method of coset analysis.

Conjugacy classes of : $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.

To determine the conjugacy classes of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$, we consider the cosets $\overline{h}G$ where $\overline{h} \in \overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$.

The coset 1G: Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^4$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}$, $C_{\overline{H}}(\overline{1_{\overline{H}}}) = \overline{H}$ on G.

To act $\overline{H}$ on G, we first act $b \in \mathbb{Z}_2$ and then act $a \in \mathbb{Z}_3$.

The action of b on G:

In this action, b fixes $\{0,\ (e_1+e_2),\ \lambda(e_1+e_2),\ \lambda^2(e_1+e_2)\}$ and permutes the remaining 12 elements in the following 2 cycles:

$\{e_1,\ e_2\}$, $\{\lambda e_1,\ \lambda e_2\}$, $\{\lambda^2 e_1,\ \lambda^2 e_2\}$, $\{(\lambda e_1 + e_2),\ (e_1 + \lambda e_2)\}$, $\{(\lambda^2 e_1 + e_2),\ (e_1 + \lambda^2 e_2\}$, $\{(\lambda^2 e_1 + \lambda e_2),\ (\lambda e_1 + \lambda^2 e_2\}$.

Now acting $a \in \mathbb{Z}_3$ on the 4 fixed points of b and the six 2 cycles we get:

a fixes the zero vector of G and when it acts on the orbits of b, each fixed point $\lambda^i(e_1 + e_2)$ for $i = 0,\ 1,\ 2$ fuses with a 2 cycle to form an orbit of size three as follows:

$$\Theta_1 = \{(e_1 + e_2),\ (\lambda^2 e_1 + \lambda e_2),\ (\lambda e_1 + \lambda^2 e_2\},$$
$$\Theta_2 = \{\lambda(e_1 + e_2),\ (\lambda^2 e_1 + e_2),\ (e_1 + \lambda^2 e_2)\},$$
$$\Theta_3 = \{\lambda^2(e_1 + e_2),\ (\lambda e_1 + e_2),\ (e_1 + \lambda e_2)\}.$$

The remaining orbits come together under the action of a to form the orbit Θ_4 of size six.

$$\Theta_4 = \{e_1,\ e_2,\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2\}.$$

Thus, the identity coset produces 5 orbits (conjugacy classes) of $\overline{G}$, viz, the singleton orbit containing the identity, three orbits of size three containing the remaining three fixed points, one in each orbit, and an orbit of size six.

Therefore we have:$k = 2^4$ and $f_1 = 1,\ f_2 = 3,\ f_3 = 3,\ f_4 = 3,\ f_5 = 6.$
<u>$k = 2^4,\ f_1 = 1:$</u>

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^4 \times 6}{1} = |\overline{G}|,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^4 \times 6}{2^4 \times 6} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

<u>$k = 2^4,\ f_i = 3:$</u> for $i = 2, 3, 4$

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^4 \times 6}{3} = 32,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{32} = 3.$$

This will give us three conjugacy classes of $\overline{G}$ of size three. The order of the elements in all three classes is two.

$k = 2^4,\ f_5 = 6:$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_5} = \frac{2^4 \times 6}{6} = 16,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{16} = 6.$$

This gives us a fifth conjugacy class from the identity coset of $\overline{G}$ of size six. The order of the elements in this class is two.

The coset bG :

First we act G on the coset bG. The action of G on the coset bG partitions the coset into four orbits of size four. The orbits are:

$$\Delta_1 = \{b,\ b(e_1 + e_2),\ b\lambda(e_1 + e_2),\ b\lambda^2(e_1 + e_2)\},$$

$$\Delta_2 = \{be_1,\ be_2,\ b(\lambda e_1 + \lambda^2 e_2),\ b(\lambda^2 e_1 + \lambda e_2)\},$$

$$\Delta_3 = \{b\lambda e_1,\ b\lambda e_2,\ b(\lambda^2 e_1 + e_2),\ b(e_1 + \lambda^2 e_2)\},$$

$$\Delta_4 = \{b\lambda^2 e_1,\ b\lambda^2 e_2,\ b(e_1 + \lambda e_2),\ b(\lambda e_1 + e_2)\}.$$

We also note that in the orbit Δ_1 the g entry of the element bg where $g \in G$ is the fixed point of the action of b on G, and in the orbits Δ_i for $i = 2, 3, 4$, the g entries of the element bg where $g \in G$ are the entries of the two cycles of the action of b on G. See the action of b on G above.

Next, we act the centralizer of b on the four orbits Δ_i for $i = 1, 2, 3, 4$. Now $C_{\overline{H}}(b) = \langle b \rangle$. Therefore the action of the centralizer is the same as the action of b. From the comment above, when b acts on the four orbits Δ_i for $i = 1, 2, 3, 4$, it permutes the elements in each orbit. Therefore for the coset bG we have $k = 4$ and $f_i = 1$ for $i = 1, 2, 3, 4$.

$k = 4,\ f_i = 1$ for $i = 1, 2, 3, 4:$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{4 \times 2}{1} = 8,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{8} = 12.$$

Therefore, the coset bG produces four conjugacy classes of $\overline{G}$ each of size twelve. The order of the elements in the first of these classes is two (the class containing b) and the order of the elements in the remaining three classes is four. This follows from Remark 2.1.4.

The coset aG :

Finally we act G on the coset aG. When G acts on the coset aG, it simply permutes the 16 elements in the coset producing an orbit of length sixteen. Next we act the centralizer of a on this orbit. But $C_{\overline{H}}(a) = \langle a \rangle$. Therefore the action of the centralizer is the same as the action of $\langle a \rangle$. When $\langle a \rangle$ acts on the orbit of length sixteen it permutes the elements in the orbit. Therefore, here $k = 1,\ f = 1$.

$k = 1,\ f = 1$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times 3}{1} = 3,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{3} = 32.$$

The coset aG, produces a single conjugacy class of $\overline{G}$ of size thirty two. The order of the elements in the class is three.

The full list of the conjugacy classes of $\overline{G}$ is described in Table 8.14.

Table 8.14: Conjugacy Classes of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

$\overline{H} = NH$	$\overline{G} = G{:}(NH)$	$o(\overline{g})$	$\lvert C_{\overline{G}}(\overline{g})\rvert$	power map(π^2)	power map(π^3)
1	1A	1	96	(1A)	(1A)
	(2A)	2	32	(1A)	(2A)
	(2B)	2	32	(1A)	(2B)
	(2C)	2	32	(1A)	(2C)
	(2D)	2	16	(1A)	(2D)
b	(2E)	2	8	(1A)	(2E)
	(4A)	4	8	(2A)	(4A)
	(4B)	4	8	(2B)	(4B)
	(4C)	4	8	(2C)	(4C)
a	(3A)	3	3	(3A)	(1A)

8.3.2 Fischer Matrices of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.

We construct the Fischer matrices of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$, for each conjugacy class of $\overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$. From the previous sections we know that there are three conjugacy classes of $\overline{H}$ and therefore three Fischer matrices of $\overline{G}$. For the Fischer matrix corresponding to the identity class of $\mathbb{Z}_3{:}\mathbb{Z}_2$ we look at the action of $\overline{H}$ on $G = 2^4$. There are five orbits of lengths 1, 3, 3, 3, 6. The Fischer matrix corresponding to the identity class is $M(1_{\overline{H}})$ which is a (5×5) matrix. Since the action of $\overline{H}$ on G has five orbits of lengths 1, 3, 3, 3 and 6, we know that the action of $\overline{H}$ on $\mathrm{Irr}(G)$ also produces five orbits of lengths 1, 3, 3, 3 and 6 as described in Section 7.3.

From Section 7.3.1 we have the following inertia and inertia factor groups. The inertia groups are:

$$\overline{P}_1 = \overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)\ ,\ \overline{P}_2 = \overline{P}_3 = \overline{P}_4 = G{:}H = 2^4{:}\mathbb{Z}_2\ ,\ \overline{P}_5 = G = 2^4.$$

The corresponding inertia factor groups are:

$$P_1 = NH = \mathbb{Z}_3{:}\mathbb{Z}_2\ ,\ P_2 = P_3 = P_4 = H = \mathbb{Z}_2\ ,\ P_5 = \{1_{\overline{H}}\}.$$

From the theory described in Section 7.3.2 we can construct the following Fischer matrices.

$\underline{M(1_{\overline{H}}):}$

$$M(1_{\overline{H}}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & -1 & -1 \\ 3 & -1 & 3 & -1 & -1 \\ 3 & -1 & -1 & 3 & -1 \\ 6 & -2 & -2 & -2 & 2 \end{pmatrix}.$$

The matrix corresponding to $b \in \mathbb{Z}_2$ is a 4×4 matrix $M(b)$ given by:

$$M(b) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Finally the third Fischer matrix is a (1×1) matrix with the singleton entry 1. This matrix $M(a)$ is given by :$M(a) = (1)$.

8.3.3 Character Table of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

We can now construct the character table of $\overline{G}$ using the Fischer matrices above and the character tables of the inertia factor groups $P_1 = \overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$ and $P_2 = P_3 = P_4 = H = \mathbb{Z}_2$.
We divide the character table of $\overline{G}$ into blocks as shown in the matrix below. Each block A_i, B_i, C_i for $i = 1, 2, 3, 4, 5$ corresponds to an inertia group $\overline{P}_i$. Also the A_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the identity coset $1G$, the B_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the coset bG and the C_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the coset aG.

A_1	B_1	C_1
A_2	B_2	C_2
A_3	B_3	C_3
A_4	B_4	C_4
A_5	B_5	C_5

First we need the character tables of $\overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$ and $H = \mathbb{Z}_2$

Character table $\overline{H}$:

$(\overline{h})$	(1)	(b)	(a)
$\lvert C_{\overline{H}}(\overline{h})\rvert$	6	2	3
$o(\overline{h})$	1	2	3
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1

Character table H :

(h)	(1)	(b)
$\lvert C_H(h)\rvert$	2	2
$o(h)$	1	2
χ_1	1	1
χ_2	1	−1

We now calculate the characters of $\overline{G}$, which fall into five blocks (A_i, for $i = 1, 2, 3, 4, 5$) with inertia groups $\overline{P}_1 = \overline{G}$, $\overline{P}_2 = G{:}H$, $\overline{P}_3 = G{:}H$, $\overline{P}_4 = G{:}H$, $\overline{P}_5 = G$ by using the Fischer matrices and inertia factor groups $P_1 = \overline{H}$, $P_2 = \mathbb{Z}_2$, $P_3 = \mathbb{Z}_2$, $P_4 = \mathbb{Z}_2$, $P_5 = \{1_{\overline{G}}\}$.

We complete the character table of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ by multiplying rows of $M(g)$ for $g \in \{1_{\overline{H}}, b, a\}$ with sections of the character tables of the inertia factor groups corresponding to each $g \in \{1_{\overline{H}}, b, a\}$.

The first block of table above A_1 is the block corresponding to conjugacy classes from the identity $1_{\overline{H}}$. To obtain this block, we multiply the first column $C_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ = 1st column of $\overline{H}$ by $M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ = 1st row of $M(1_{\overline{H}})$.

We get:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

For the A_2 block, we multiply $C_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by $M_2 = \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \end{pmatrix}$ = 2nd row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_3 block, we multiply $C_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by $M_3 = \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \end{pmatrix}$ = 3rd row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \\ 3 & -1 & 3 & -1 & -1 \end{pmatrix}.$$

For the A_4 block, we multiply $C_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by $M_4 = \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \end{pmatrix}$ = 4th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \\ 3 & -1 & -1 & 3 & -1 \end{pmatrix}.$$

For the A_5 block, we multiply $C_5 = \begin{pmatrix} 1 \end{pmatrix}$ = 1st column of $\{1_{\overline{G}}\}$ by $M_5 = \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix}$ = 5th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix}.$$

For the next block, the B_i block for $i = 1, 2, 3, 4, 5$ of the character table of $\overline{G}$, we use the Fischer matrix $M(b)$. To complete the B_1 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$=2nd column of $\overline{H}$ by $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$= 1st row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the B_2 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=2nd column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$= 2nd row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

For the B_3 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=2nd column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$= 3rd row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

For the B_4 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=2nd column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$= 4th row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

For the B_5 block of the table, we will have a row of zeros since $P_5 \bigcap [b] = \emptyset$ and hence, $M_5(b)$ will not exist.

To complete the C_i block for $i = 1, 2, 3, 4, 5$ of the character table of $\overline{G}$, we use the Fischer matrix $M(a) = \{1_{\overline{G}}\}$. To complete the C_1 block of the table, we multiply $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ =3rd column of $\overline{H}$ by $\begin{pmatrix} 1 \end{pmatrix}$ = 1st row of $M(a)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

For the C_i blocks for $i = 2, 3, 4, 5$, we have zeros since $P_i \bigcap [a] = \emptyset$ for $i = 2, 3, 4, 5$ and therefore M_i for $i = 2, 3, 4, 5$ does not exist.

Character Table of 2^4:$(\mathbb{Z}_3$:$\mathbb{Z}_2)$:

$[\overline{g}]$	(1A)	(2A)	(2B)	(2C)	(2D)	(2E)	(4A)	(4B)	(4C)	(3A)
$\lvert[(\overline{g})]\rvert$	1	3	3	3	6	12	12	12	12	32
$\lvert C_{\overline{G}}(\overline{g})\rvert$	96	32	32	32	16	8	8	8	8	3
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	−1	−1	−1	−1	1
χ_3	2	2	2	2	2	0	0	0	0	−1
χ_4	3	3	−1	−1	−1	1	1	−1	−1	0
χ_5	3	3	−1	−1	−1	−1	−1	1	1	0
χ_6	3	−1	3	−1	−1	1	−1	1	−1	0
χ_7	3	−1	3	−1	−1	−1	1	−1	1	0
χ_8	3	−1	−1	3	−1	1	−1	−1	1	0
χ_9	3	−1	−1	3	−1	−1	1	1	−1	0
χ_{10}	6	−2	−2	−2	2	0	0	0	0	0

8.4 The Group $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$

Let $\overline{G} = GNH$ where $G \cong 2^{2r}$, $N \cong \mathbb{Z}_{2^r-1}$, $H \cong \mathbb{Z}_2$ and $NH \cong \overline{H}$. Let $r = 3$, then $G \cong 2^6$, $N \cong \mathbb{Z}_7 = \langle a \rangle$, $H \cong \mathbb{Z}_2 = \langle b \rangle$ and $NH \cong \overline{H} \cong \mathbb{Z}_7{:}\mathbb{Z}_2$. We know that $\mathbb{Z}_7{:}\mathbb{Z}_2 \leq PSL(2,8) \cong SL(2,8)$. Now in $PSL(2,8)$ a Sylow 7-subgroup is isomorphic to $\mathbb{Z}_7$. Now $o(b) = 2$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where $\langle \lambda \rangle = \mathbb{F}_8^*$, $o(a) = 7$.

$G = \{0,\ e_1,\ e_2,\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2,\ ,\ \lambda^3 e_1,\ \lambda^3 e_2,\ \lambda^4 e_1,\ \lambda^4 e_2,\ \lambda^5 e_1,\ \lambda^5 e_2,\ \lambda^6 e_1,\ \lambda^6 e_2,\ (e_1 + e_2),\ (e_1 + \lambda e_2),\ (e_1 + \lambda^2 e_2),\ (e_1 + \lambda^3 e_2),\ (e_1 + \lambda^4 e_2),\ (e_1 + \lambda^5 e_2),\ (e_1 + \lambda^6 e_2),\ (\lambda e_1 + e_2),\ (\lambda e_1 + \lambda e_2),\ (\lambda e_1 + \lambda^2 e_2),\ (\lambda e_1 + \lambda^3 e_2),\ (\lambda e_1 + \lambda^4 e_2),\ (\lambda e_1 + \lambda^5 e_2),\ (\lambda e_1 + \lambda^6 e_2),\ (\lambda^2 e_1 + e_2),\ (\lambda^2 e_1 + \lambda e_2),\ (\lambda^2 e_1 + \lambda^2 e_2),\ (\lambda^2 e_1 + \lambda^3 e_2),\ (\lambda^2 e_1 + \lambda^4 e_2),\ (\lambda^2 e_1 + \lambda^5 e_2),\ (\lambda^2 e_1 + \lambda^6 e_2),\ (\lambda^3 e_1 + e_2),\ (\lambda^3 e_1 + \lambda e_2),\ (\lambda^3 e_1 + \lambda^2 e_2),\ (\lambda^3 e_1 + \lambda^3 e_2),\ (\lambda^3 e_1 + \lambda^4 e_2),\ (\lambda^3 e_1 + \lambda^5 e_2),\ (\lambda^3 e_1 + \lambda^6 e_2),\ (\lambda^4 e_1 + e_2),\ (\lambda^4 e_1 + \lambda e_2),\ (\lambda^4 e_1 + \lambda^2 e_2),\ (\lambda^4 e_1 + \lambda^3 e_2),\ (\lambda^4 e_1 + \lambda^4 e_2),\ (\lambda^4 e_1 + \lambda^5 e_2),\ (\lambda^4 e_1 + \lambda^6 e_2),\ (\lambda^5 e_1 + e_2),\ (\lambda^5 e_1 + \lambda e_2),\ (\lambda^5 e_1 + \lambda^2 e_2),\ (\lambda^5 e_1 + \lambda^3 e_2),\ (\lambda^5 e_1 + \lambda^4 e_2),\ (\lambda^5 e_1 + \lambda^5 e_2),\ (\lambda^5 e_1 + \lambda^6 e_2),\ (\lambda^6 e_1 + e_2),\ (\lambda^6 e_1 + \lambda e_2),\ (\lambda^6 e_1 + \lambda^2 e_2),\ (\lambda^6 e_1 + \lambda^3 e_2),\ (\lambda^6 e_1 + \lambda^4 e_2).\ (\lambda^6 e_1 + \lambda^5 e_2),\ (\lambda^6 e_1 + \lambda^6 e_2).$

$GF(8) = \{0,\ 1,\ \lambda,\ \lambda^2,\ \lambda^3,\ \lambda^4,\ \lambda^5,\ \lambda^6\}$ where $\lambda^3 = \lambda + 1$, $\lambda^4 = \lambda^2 + \lambda$, $\lambda^5 = \lambda^3 + \lambda^2$ and $\lambda^6 = \lambda^2 + 1$.

Now, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $be_1 = e_2$ and $be_2 = e_1$.

8.4.1 Conjugacy Classes of the Group $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$.

To calculate the conjugacy classes of $\overline{G}$,we need the conjugacy classes of the group $\overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_2$. The conjugacy classes of $\overline{H}$ are represented in Table 8.15.

Table 8.15: Conjugacy Classes of $\mathbb{Z}_7{:}\mathbb{Z}_2$

classes of $\mathbb{Z}_7{:}\mathbb{Z}_2$	[1]	[b]	[a_1]	[a_2]	[a_3]
$\lvert C_G(g) \rvert$	6	2	3	3	3
$o(g)$	1	2	7	7	7
$\lvert [g] \rvert$	1	7	2	2	2

To calculate the conjugacy classes of $\overline{G}$ we use the method of coset analysis.

Conjugacy Classes of $2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$

To determine the conjugacy classes of $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$, we consider the cosets $\overline{h}G$ where $\overline{h} \in \overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_2$.

The coset $1G$:

Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^6$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}$, $C_{\overline{H}}(\overline{1_{\overline{H}}}) = \overline{H}$ on G.

To act $\overline{H}$ on G, we first act $b \in \mathbb{Z}_2$ and then act $a \in \mathbb{Z}_7$.

The action of b on G

In this action, b fixes $\{0, (e_1+e_2), \lambda(e_1+e_2), \lambda^2(e_1+e_2), \lambda^3(e_1+e_2), \lambda^4(e_1+e_2), \lambda^5(e_1+e_2), \lambda^6(e_1+e_2)\}$ and permutes the remaining 56 elements in the following 2 cycles:

$\{e_1, e_2\}$, $\{\lambda e_1, \lambda e_2\}$, $\{\lambda^2 e_1, \lambda^2 e_2\}$, $\{\lambda^3 e_1, \lambda^3 e_2\}$, $\{\lambda^4 e_1, \lambda^4 e_2\}$, $\{\lambda^5 e_1, \lambda^5 e_2\}$, $\{\lambda^6 e_1, \lambda^6 e_2\}$, $\{(e_1+\lambda e_2), (e_2+\lambda e_1)\}$, $\{(e_1+\lambda^2 e_2), (e_2+\lambda^2 e_1\}$, $\{(e_1+\lambda^3 e_2), (e_2+\lambda^3 e_1)\}$, $\{(e_1+\lambda^4 e_2), (e_2+\lambda^4 e_1)\}$, $\{(e_1+\lambda^5 e_2), (e_2+\lambda^5 e_1)\}$, $\{(e_1+\lambda^6 e_2), (e_2+\lambda^6 e_1)\}$, $\{(\lambda^2 e_1+\lambda e_2), (\lambda^2 e_2+\lambda e_1)\}$, $\{(\lambda^3 e_1+\lambda e_2), (\lambda^3 e_2+\lambda e_1)\}$, $\{(\lambda^4 e_1+\lambda e_2), (\lambda^4 e_2+\lambda e_1)\}$, $\{(\lambda^5 e_1+\lambda e_2), (\lambda^5 e_2+\lambda e_1)\}$, $\{(\lambda^6 e_1+\lambda e_2), (\lambda^6 e_2+\lambda e_1)\}$, $\{(\lambda^3 e_1+\lambda^2 e_2), (\lambda^3 e_2+\lambda^2 e_1)\}$, $\{(\lambda^4 e_1+\lambda^2 e_2), (\lambda^4 e_2+\lambda^2 e_1)\}$, $\{(\lambda^5 e_1+\lambda^2 e_2), (\lambda^5 e_2+\lambda^2 e_1)\}$, $\{(\lambda^6 e_1+\lambda^2 e_2), (\lambda^6 e_2+\lambda^2 e_1)\}$, $\{(\lambda^4 e_1+\lambda^3 e_2), (\lambda^4 e_2+\lambda^3 e_1)\}$, $\{(\lambda^5 e_1+\lambda^3 e_2), (\lambda^5 e_2+\lambda^3 e_1)\}$, $\{(\lambda^6 e_1+\lambda^3 e_2), (\lambda^6 e_2+\lambda^3 e_1)\}$, $\{(\lambda^5 e_1+\lambda^4 e_2), (\lambda^5 e_2+\lambda^4 e_1)\}$, $\{(\lambda^6 e_1+\lambda^4 e_2), (\lambda^6 e_2+\lambda^4 e_1)\}$, $\{(\lambda^6 e_1+\lambda^5 e_2), (\lambda^6 e_2+\lambda^5 e_1)\}$.

We now act a on the 8 fixed points of the action of b on G and the twenty eight 2 cycles. In this action, each fixed point of b (identity excluded), namely, $\lambda^i(e_1+e_2)$ for $i = 0, 1, \ldots, 6$ joins with three 2-cycles of b forming an orbit of size seven. These orbits are:

$\Theta_1 = \{(e_1+e_2), (\lambda^3 e_1+\lambda^4 e_2), (\lambda^3 e_2+\lambda^4 e_1), (\lambda^6 e_1+\lambda e_2), \lambda^6 e_2+\lambda e_1), (\lambda^5 e_1+\lambda^2 e_2), (\lambda^5 e_2+\lambda^2 e_1)\}$,

$\Theta_2 = \{(\lambda(e_1+e_2), (\lambda^4 e_1+\lambda^5 e_2), (\lambda^4 e_2+\lambda^5 e_1), (\lambda^2 e_1+e_2), (\lambda^2 e_2+e_1), (\lambda^3 e_1+\lambda^6 e_2), (\lambda^3 e_1+\lambda^6 e_2)\}$,

$\Theta_3 = \{\lambda^2(e_1+e_2), (\lambda^5 e_1+\lambda^6 e_2), (\lambda^5 e_2+\lambda^6 e_1), (\lambda e_1+\lambda^3 e_2), (\lambda e_2+\lambda^3 e_1), (\lambda^4 e_1+e_2), (\lambda^4 e_2+e_1)\}$,

$\Theta_4 = \{\lambda^3(e_1+e_2), (\lambda^6 e_1+e_2), (\lambda^6 e_2+e_1), (\lambda^2 e_1+\lambda^4 e_2), (\lambda^2 e_2+\lambda^4 e_1), (\lambda e_1+\lambda^5 e_2), (\lambda e_2+\lambda^5 e_1)\}$,

$\Theta_5 = \{\lambda^4(e_1+e_2), (\lambda e_1+e_2), (\lambda e_2+e_1), (\lambda^3 e_1+\lambda^5 e_2), (\lambda^3 e_2+\lambda^5 e_1), (\lambda^2 e_1+\lambda^6 e_2), (\lambda^2 e_2+\lambda^6 e_1)\}$,

$\Theta_6 = \{\lambda^5(e_1+e_2), (\lambda e_1+\lambda^2 e_2), (\lambda e_2+\lambda^2 e_1), (\lambda^4 e_1+\lambda^6 e_2), (\lambda^4 e_2+\lambda^6 e_1), (\lambda^3 e_1+e_2), (\lambda^3 e_2+e_1)\}$,

$\Theta_7 = \{\lambda^6(e_1+e_2), (\lambda^2 e_1+\lambda^3 e_2), (\lambda^2 e_2+\lambda^3 e_1), (\lambda^5 e_1+e_2), (\lambda^5 e_2+e_1), (\lambda e_1+\lambda^4 e_2), (\lambda e_2+\lambda^4 e_1)\}$.

The remaining $\lambda^i e_1$ and $\lambda^i e_2$ for $i = 0, 1, \ldots, 6$ come together under the action of a to form an orbit of size fourteen.

$$\Theta_8 = \{(e_1, \lambda^3 e_1, \lambda^6 e_1, \lambda^2 e_1, \lambda^5 e_1, \lambda e_1, \lambda^4 e_1), (e_2, \lambda^4 e_2, \lambda e_2, \lambda^5 e_2, \lambda^2 e_2, \lambda^6 e_2, \lambda^3 e_2)\}.$$

Thus, the identity coset produces 9 orbits (conjugacy classes) of $\overline{G}$, viz, the singleton orbit containing the identity, seven orbits of size seven containing the remaining seven fixed points, one in each orbit, and an orbit of size fourteen.

Therefore we have:$k = 2^6$ and $f_1 = 1,\ f_2 = 7,\ f_3 = 7,\ f_4 = 7,\ f_5 = 7,\ f_6 = 7,\ f_7 = 7,\ f_8 = 7,\ f_9 = 14$.

$\underline{k = 2^6,\ f_1 = 1:}$

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^6 \times 14}{1} = |\overline{G}|,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^6 \times 14}{2^6 \times 14} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

$\underline{k = 2^6,\ f_i = 7:}$ for $i = 2,3,4,5,6,7,8$

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^6 \times 14}{7} = 128,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^6 \times 14}{128} = 7.$$

This will give us seven conjugacy classes of $\overline{G}$ of size seven. The order of the elements in all seven classes is two.

$\underline{k = 2^6,\ f_9 = 14:}$

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_9} = \frac{2^6 \times 14}{14} = 64,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^6 \times 14}{64} = 14.$$

This gives us a ninth conjugacy class from the identity coset of $\overline{G}$ of size fourteen. The order of the elements in this class is two.

<u>The coset bG :</u>

First we act G on the coset bG. The action of G on the coset bG partitions the coset into eight orbits (since $k = 8$) of size eight. The orbits are:

$\Delta_1 = \{b,\ b(e_1+e_2),\ b\lambda(e_1+e_2),\ b\lambda^2(e_1+e_2),\ b\lambda^3(e_1+e_2),\ b\lambda^4(e_1+e_2),\ b\lambda^5(e_1+e_2),\ b\lambda^6(e_1+e_2)\},$

$\Delta_2 = \{be_1,\ be_2,\ b(\lambda e_1+\lambda^3 e_2),\ b(\lambda^3 e_1+\lambda e_2),\ b(\lambda^2 e_1+\lambda^6 e_2),\ b(\lambda^6 e_1+\lambda^2 e_2),\ b(\lambda^5 e_1+\lambda^4 e_2),\ b(\lambda^4 e_1+\lambda^5 e_2)\},$

$\Delta_3 = \{b(\lambda e_1),\ b(\lambda e_2),\ b(\lambda^4 e_1+\lambda^2 e_2),\ b(\lambda^2 e_1+\lambda^4 e_2),\ b(\lambda^6 e_1+\lambda^5 e_2),\ b(\lambda^5 e_1+\lambda^6 e_2),\ b(e_1+\lambda^3 e_2),\ b(\lambda^3 e_1+e_2)\}$

$\Delta_4 = \{b(\lambda^2 e_1),\ b(\lambda^2 e_2),\ b(\lambda^5 e_1+\lambda^3 e_2),\ b(\lambda^3 e_1+\lambda^5 e_2),\ b(e_1+\lambda^6 e_2),\ b(\lambda^6 e_1+e_2),\ b(\lambda^4 e_1+\lambda e_2),\ b(\lambda e_1+\lambda^4 e_2)\}$

$\Delta_5 = \{b(\lambda^3 e_1),\ b(\lambda^3 e_2),\ b(\lambda e_1+e_2),\ b(e_1+\lambda e_2),\ b(\lambda^5 e_1+\lambda^2 e_2),\ b(\lambda^2 e_1+\lambda^5 e_2),\ b(\lambda^6 e_1+\lambda^4 e_2),\ b(\lambda^4 e_1+\lambda^6 e_2)\}$

$\Delta_6 = \{b(\lambda^4 e_1),\ b(\lambda^4 e_2),\ b(\lambda e_1+\lambda^2 e_2),\ b(\lambda^2 e_1+\lambda e_2),\ b(\lambda^6 e_1+\lambda^3 e_2),\ b(\lambda^3 e_1+\lambda^6 e_2),\ b(\lambda^5 e_1+e_2),\ b(e_1+\lambda^5 e_2)\}$

$\Delta_7 = \{b(\lambda^5 e_1),\ b(\lambda^5 e_2),\ b(\lambda^4 e_1+e_2),\ b(e_1+\lambda^4 e_2),\ b(\lambda^6 e_1+\lambda e_2),\ b(\lambda e_1+\lambda^6 e_2),\ b(\lambda^3 e_1+\lambda^2 e_2),\ b(\lambda^2 e_1+\lambda^3 e_2)\}$

$\Delta_8 = \{b(\lambda^6 e_1),\ b(\lambda^6 e_2),\ b(e_1+\lambda^2 e_2),\ b(\lambda^2 e_1+e_2),\ b(\lambda^4 e_1+\lambda^3 e_2),\ b(\lambda^3 e_1+\lambda^4 e_2),\ b(\lambda^5 e_1+\lambda e_2),\ b(e_1+\lambda^5 e_2)\},$

We also note that in the orbit Δ_1 the g entry of the element bg where $g \in G$ is the fixed point of the action of b on G, and in the orbits Δ_i for $i = 2,3,4,5,6,7$, the g entries of the element bg where $g \in G$ are the entries of the two cycles of the action of b on G (when the elements in the orbit are paired as represented above). See the action of b on G above.

Next, we act the centralizer of b on the eight orbits Δ_i for $i = 1,2,3,4,5,6,7,8$. Now $C_{\overline{H}}(b) = \langle b \rangle$. Therefore the action of the centralizer is the same as the action of b. From the comment above, when b acts on the eight orbits Δ_i for $i = 1,2,3,4,5,6,7,8$, it permutes the elements in each orbit. Therefore for the coset bG we have $k = 8$ and $f_i = 1$ for $i = 1,2,3,4,5,6,7,8$.

$\underline{k = 8,\ f_i = 1}$ for $i = 1,2,3,4,5,6,7,8$.

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{8 \times 2}{1} = 16,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^6 \times 14}{16} = 56.$$

Therefore, the coset bG produces eight conjugacy classes of $\overline{G}$ each of size fifty six. The order of the elements in the first of these classes is two and the order of the elements in the remaining seven classes is four. See Remark 2.1.4

The coset $a_i G$: for $i = 1,2,3$

Finally we act G on the cosets $a_i G$. When G acts on the cosets $a_i G$, it simply permutes the 64 elements in the coset producing an orbit of length sixty four. Next we act the centralizer of a_i on this orbit. But $C_{\overline{H}}(a_i) = \langle a \rangle$. Therefore the action of the centralizer is the same as the action of $\langle a \rangle$. When $\langle a \rangle$ acts on the orbit of length sixty four it permutes the elements in the orbit. Therefore, here for the coset $a_1 G$, $k = 1,\ f = 1$.

$\underline{k = 1,\ f = 1:}$

$$|C_{\overline{G}}(x)| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times 7}{1} = 7,$$

$$|[x]_{\overline{G}}| = \frac{|\overline{G}|}{|C_{\overline{G}}(x)|} = \frac{2^6 \times 14}{7} = 128.$$

Similarly for the cosets a_2G and a_3G, we get two more conjugacy classes of $\overline{G}$ of length 128. The order of the elements in each of the three classes is seven.

The full list of conjugacy classes of $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$ is given in Table 8.16.

Table 8.16: Conjugacy Classes of $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$

$\overline{H} = \dot{N}H$	$\overline{G} = G : (NH)$	$o(\overline{g})$	$\lvert C_{\overline{G}}(\overline{g})\rvert$	power map(π^2)	power map(π^3)
1	1A	1	896	(1A)	(1A)
	(2A)	2	128	(1A)	(2A)
	(2B)	2	128	(1A)	(2B)
	(2C)	2	128	(1A)	(2C)
	(2D)	2	128	(1A)	(2D)
	(2E)	2	128	(1A)	(2E)
	(2F)	2	128	(1A)	(2F)
	(2G)	2	128	(1A)	(2G)
	(2H)	2	64	(1A)	(2H)
b	(2I)	2	16	(1A)	(2I)
	(4A)	4	16	(2A)	(4A)
	(4B)	4	16	(2B)	(4B)
	(4C)	4	16	(2C)	(4C)
	(4D)	4	16	(2D)	(4D)
	(4E)	4	16	(2E)	(4E)
	(4F)	4	16	(2F)	(4F)
	(4G)	4	16	(2G)	(4G)
a_1	(7A)	7	7	(7A)	(1A)
a_2	(7B)	7	7	(7A)	(1A)
a_3	(7C)	7	7	(7A)	(1A)

8.4.2 Fischer Matrices of $2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$.

We construct the Fischer matrices of $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$, for each conjugacy class of $\overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_2$. From the previous sections we know that there are five conjugacy classes of $\overline{H}$ and therefore five Fischer matrices of $\overline{G}$. For the Fischer matrix corresponding to the identity class of $\mathbb{Z}_7{:}\mathbb{Z}_2$ we look at the action of $\overline{H}$ on $G = 2^6$. There are nine orbits of lengths 1, 7, 7, 7, 7, 7, 7, 7, 14. The Fischer matrix corresponding to the identity class is $M(1_{\overline{H}})$ which is (9×9) matrix. Since the action of $\overline{H}$ on G has nine orbits of lengths 1, 7, 7, 7, 7, 7, 7, 7, 14, we know that the action of $\overline{H}$ on $\mathrm{Irr}(G)$ also produces nine orbits of lengths 1, 7, 7, 7, 7, 7, 7, 7, 14 as described in Section 7.3.

From Section 7.3.1 we have the following inertia and inertia factor groups. The inertia groups are:

$$\overline{P}_1 = \overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)\ ,\ \overline{P}_2 = \overline{P}_3 = \overline{P}_4 = \overline{P}_5 = \overline{P}_6 = \overline{P}_7 = \overline{P}_8 = G{:}H = 2^6{:}\mathbb{Z}_2\ ,\ \overline{P}_9 = G = 2^6.$$

The corresponding inertia factor groups are:

$$P_1 = \overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2\ ,\ P_2 = P_3 = P_4 = P_5 = P_6 = P_7 = P_8 = H = \mathbb{Z}_2\ ,\ P_9 = \{1_{\overline{H}}\}.$$

From the theory described in Section 7.3.2 we can construct the following Fischer matrices:

$\underline{M(1_{\overline{H}}):}$

$$M(1_{\overline{H}}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 7 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 7 & -1 & 7 & -1 & -1 & -1 & -1 & -1 & -1 \\ 7 & -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1 \\ 7 & -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \\ 7 & -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \\ 7 & -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 \\ 7 & -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \\ 14 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 6 \end{pmatrix}.$$

The matrix corresponding to $b \in \mathbb{Z}_2$ is a 8×8 matrix $M(b)$ given by:

$$M(b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

Finally the remaining three Fischer matrices are all (1×1) matrices with the singleton entry 1. These matrices are $M(a_1) = M(a_2) = M(a_3) = (1)$.

8.4.3 Character Table of $\overline{G} = 2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$

We can now construct the character table of $\overline{G}$ using these Fischer matrices and the character tables of the inertia factor groups $\overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_2$ and $H = \mathbb{Z}_2$.

We divide the character table of $\overline{G}$ into blocks as shown in the matrix below. Each block A_i, B_i, C_i, D_i, E_i for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ corresponds to an inertia group $\overline{P}_i$. Also the A_i blocks for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ come from the conjugacy classes produced by the identity coset $1G$, the B_i blocks for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ come from the conjugacy classes produced by the coset bG, the C_i blocks for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ come the conjugacy class produced by the coset a_1G, the D_i blocks for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ come from the conjugacy class produced by the coset a_2G and the E_i blocks for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ come from conjugacy class produced by the coset a_3G.

A_1	B_1	C_1	D_1	E_1
A_2	B_2	C_2	D_2	E_2
A_3	B_3	C_3	D_3	E_3
A_4	B_4	C_4	D_4	E_4
A_5	B_5	C_5	D_5	E_5
A_6	B_6	C_6	D_6	E_6
A_7	B_7	C_7	D_7	E_7
A_8	B_8	C_8	D_8	E_8
A_9	B_9	C_9	D_9	E_9

First we need the character tables of $\overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_2$ and $H = \mathbb{Z}_2$

Character table: $\overline{H}$

$(\overline{h})$	(1)	(b)	(a_1)	(a_2)	(a_3)
$\lvert C_{\overline{H}}(\overline{h})\rvert$	14	2	7	7	7
$o(\overline{h})$	1	2	7	7	7
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	1
χ_3	2	0	A	B	C
χ_4	2	0	B	C	A
χ_5	2	0	C	A	B

where $A=(e^{\frac{2\pi i}{7}}+e^{\frac{12\pi i}{7}})$, $B=(e^{\frac{4\pi i}{7}}+e^{\frac{10\pi i}{7}})$, $C=(e^{\frac{6\pi i}{7}}+e^{\frac{8\pi i}{7}})$.

Character table: H

(h)	(1)	(b)
$\lvert C_H(h)\rvert$	2	2
$o(h)$	1	2
χ_1	1	1
χ_2	1	−1

We now calculate the characters of $\overline{G}$, which fall into nine blocks (A_i, for $i = 1,2,\ldots,9$) with inertia groups $\overline{P}_1 = \overline{G}$, $\overline{P}_i = G{:}H$ for $1 = 2,3,\ldots,8$ and $\overline{P}_9 = G$ by using the Fischer matrices and inertia factor groups $P_1 = \overline{H} = \mathbb{Z}_7{:}\mathbb{Z}_2$, $P_i = \mathbb{Z}_2$ for $1 = 2,3,\ldots,8$ and $P_9 = \{1_{\overline{G}}\}$.

We complete the character table of $2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$ by multiplying rows of $M(g)$ for $g = \{1_{\overline{H}}, b, a_1, a_2, a_3\}$ with sections of the character tables of the inertia factor groups corresponding to each $g = \{1_{\overline{H}}, b, a_1, a_2, a_3\}$.

The first block of table above A_1 is the block corresponding to conjugacy classes from the identity $1_{\overline{H}}$. To obtain this block, we multiply the first column $C_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ = 1st column of $\overline{H}$ by

$M_1=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$= 1st row of $M(1_{\overline{H}})$.

We get:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

For the A_2 block, we multiply $C_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_2 = \begin{pmatrix} 7 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$ = 2nd row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 7 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_3 block, we multiply $C_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_3 = \begin{pmatrix} 7 & -1 & 7 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$ = 3rd row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & 7 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 & 7 & -1 & -1 & -1 & -1 & -1 & -1 \\ 7 & -1 & 7 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_4 block, we multiply $C_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_4 = \begin{pmatrix} 7 & -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$ = 4th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1 \\ 7 & -1 & -1 & 7 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_5 block, we multiply $C_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_5 = \begin{pmatrix} 7 & -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \end{pmatrix}$ = 5th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \\ 7 & -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_6 block, we multiply $C_6 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_6 = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \end{pmatrix}$ = 6th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \\ 7 & -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_7 block, we multiply $C_7 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_7 = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 \end{pmatrix}$ = 7th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 \\ 7 & -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 \end{pmatrix}.$$

For the A_8 block, we multiply $C_8 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = 1st column of $\mathbb{Z}_2$ by

$M_8 = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \end{pmatrix}$ = 8th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \\ 7 & -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \end{pmatrix}.$$

For the A_9 block, we multiply $C_9 = \begin{pmatrix} 1 \end{pmatrix}$ = 1st column of $\{1_{\overline{G}}\}$ by

$M_9 = \begin{pmatrix} 14 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 6 \end{pmatrix}$ = 9th row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 14 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 14 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 6 \end{pmatrix}.$$

For the next block, the B_i block for $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ of the character table of $\overline{G}$, we use the Fischer matrix $M(b)$. To complete the B_1 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ = 2nd column of $\overline{H}$ by

$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ = 1st row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To complete the B_2 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ = 2nd column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix}$ = 2nd row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

To complete the B_3 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ = 2nd column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \end{pmatrix}$ = 3rd row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

To complete the B_4 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ = 4th column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$ = 4th row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

To complete the B_5 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ = 5th column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}$ = 5th row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \end{pmatrix}.$$

To complete the B_6 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$= 6th column of $\mathbb{Z}_2$ by

$\begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}$= 6th row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

To complete the B_7 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$= 7th column of $\mathbb{Z}_2$ by

$\begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$= 7th row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

To complete the B_8 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$= 8th column of $\mathbb{Z}_2$ by

$\begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}$= 8th row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

For the B_9 block we will have a row of zeros here since, $P_9 \bigcap [b] = \emptyset$.

For the C_i blocks we use the Fischer matrices $M(a_1)$ which is a singleton (1×1) matrix.

For C_1 block, we multiply the third column of $\overline{H}$ by the first row of $M(a_1)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ A \\ B \\ C \end{pmatrix}\begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ A \\ B \\ C \end{pmatrix}.$$

For the blocks C_i for $i = 2, 3, \ldots, 9$, we will have a column of zeros since $P_i \bigcap [a_1] = \emptyset$ for

$i = 2, 3, \ldots, 9$.

For the D_i blocks we use the Fischer matrix $M(a_2)$ which is a singleton (1×1) matrix. For D_1 block, we multiply the fourth column of $\overline{H}$ by the first row of $M(a_2)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ B \\ C \\ A \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ B \\ C \\ A \end{pmatrix}.$$

For the blocks D_i for $i = 2, 3, \ldots, 9$, we will have a column of zeros since $P_i \bigcap [a_2] = \emptyset$ for $i = 2, 3, \ldots, 9$.

For the C_i blocks we use the Fischer matrices $M(a_1)$ which is a singleton (1×1) matrix.

For C_1 block, we multiply the third column of $\overline{H}$ by the first row of $M(a_1)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ A \\ B \\ C \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ A \\ B \\ C \end{pmatrix}.$$

For the blocks C_i for $i = 2, 3, \ldots, 9$, we will have a column of zeros since $P_i \bigcap [a_1] = \emptyset$ for $i = 2, 3, \ldots, 9$.

For the E_i blocks we use the Fischer matrix $M(a_3)$ which is a singleton (1×1) matrix. For E_1 block, we multiply the fifth column of $\overline{H}$ by the first row of $M(a_3)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ C \\ A \\ B \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ C \\ A \\ B \end{pmatrix}.$$

For the blocks E_i for $i = 2, 3, \ldots, 9$, we will have a column of zeros since $P_i \bigcap [a_3] = \emptyset$ for $i = 2, 3, \ldots, 9$.

$(\overline{g})$	(1A)	(2A)	(2B)	(2C)	(2D)	(2E)	(2F)	(2G)	(2H)	(2I)	(4A)	(4B)	(4C)	(4D)	(4E)	(4F)	(4G)	(7A)	(7B)	(7C)
$\|(\overline{g})\|$	1	7	7	7	7	7	7	7	14	16	16	16	16	16	16	16	16	7	7	7
$\|C_{\overline{G}}(\overline{g})\|$	896	128	128	128	128	128	128	128	64	16	16	16	16	16	16	16	16	7	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1
χ_3	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	1	1	1
χ_4	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	A	B	C
χ_5	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	B	C	A
χ_6	7	7	−1	−1	−1	−1	−1	−1	−1	1	−1	1	1	−1	−1	1	−1	C	B	A
χ_7	7	7	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	−1	1	1	−1	1	0	0	0
χ_8	7	−1	7	−1	−1	−1	−1	−1	−1	1	1	−1	1	−1	1	−1	−1	0	0	0
χ_9	7	−1	7	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1	1	1	0	0	0
χ_{10}	7	−1	−1	7	−1	−1	−1	−1	−1	1	−1	−1	1	1	−1	−1	1	0	0	0
χ_{11}	7	−1	−1	7	−1	−1	−1	−1	−1	−1	1	1	−1	−1	1	1	−1	0	0	0
χ_{12}	7	−1	−1	−1	7	−1	−1	−1	−1	1	1	1	−1	1	−1	−1	−1	0	0	0
χ_{13}	7	−1	−1	−1	7	−1	−1	−1	−1	−1	−1	−1	1	−1	1	1	1	0	0	0
χ_{14}	7	−1	−1	−1	−1	7	−1	−1	−1	1	−1	1	−1	−1	1	−1	1	0	0	0
χ_{15}	7	−1	−1	−1	−1	7	−1	−1	−1	−1	1	−1	1	1	−1	1	−1	0	0	0
χ_{16}	7	−1	−1	−1	−1	−1	7	−1	−1	1	1	−1	−1	−1	−1	1	1	0	0	0
χ_{17}	7	−1	−1	−1	−1	−1	7	−1	−1	−1	−1	1	1	1	1	−1	−1	0	0	0
χ_{18}	7	−1	−1	−1	−1	−1	−1	7	−1	1	−1	−1	−1	1	1	1	−1	0	0	0
χ_{19}	7	−1	−1	−1	−1	−1	−1	7	−1	−1	1	1	1	−1	−1	−1	1	0	0	0
χ_{20}	14	−2	−2	−2	−2	−2	−2	−2	6	0	0	0	0	0	0	0	0	0	0	0

Character table: $2^6{:}(\mathbb{Z}_7{:}\mathbb{Z}_2)$

Bibliography

[1] M. Akbari, M. Kheirabadi and A.R. Moghaddamfar, *Recognition by noncommuting graph of finite simple groups* $L_4(q)$, K.N.Toosi University of Technology, Tehran, Iran, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran, 2010.

[2] M.R. Aleeva, *On Finite Simple Groups with the Set of Element Orders as in a Frobenius Group or a Double Frobenius Group*, Mathematical Notes, **73**(3) (2003), 299-313.

[3] J.L. Alperin and R.B. Bell, *Groups and Representations*, Springer-Verlag, New York, Graduate Texts in Mathematics, 1995.

[4] A.B.M. Basheer, *Character Tables of the General Linear Group and some of it's Subgroups*, MSc Thesis, University of Kwa-Zulu Natal, 2008.

[5] J.H.Conway, R.T.Curtis, S.P.Norton, R.A.Parker and R.A.Wilson, *Altas of Finite Groups*, Clarendon Press, Oxford University Press, Eynsham, 1985.

[6] M.R. Darafsheh, Y. Farjami and A. Sadrudini, *A characterization property of the simple group* $PSL_4(5)$ *by the set of its element orders*, Archivum Mathematicum, **43**(1) (2007), 31-37.

[7] The GAP Group, *GAP - Groups, Algorithms and Programming*, Version 4.4.10; 2007.**http://www.gap-system.org**

[8] D. Gorenstein, *Finite Groups*, Harper and Row Publishers, New York, 1968.

[9] M.A. Grechkoseeva, M.S. Lucido, V.D. Mazurov, A.R. Moghaddamfar and A.V. Vasilev, *On Recognition of the Projective Special Linear groups over the binary field*, Siberian Electronic Mathematical Reports, 2004.

[10] L.C. Grove, *Groups and Characters*, John Wiley & Sons, New York, 1997.

[11] A. Gruber, T.M. Keller, M.L. Lewis, K. Naughton and B. Strasser, *A characterization of prime graphs of solvable groups*, Journal of Algebra **442** (2015), 397-422.

[12] I.M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, Sans Francisco, London, 1967.

[13] G. James and M. Liebeck, *Representations and Characters of Groups*, 2nd Edition, Cambridge Mathematical Textbooks, Cambridge University Press, 2001.

[14] G. Karpilovsky, *Group Representations Volume 1, Part A: Background Material*, Elsevier Science Publishers B.V., The Netherlands, 1992.

[15] B. Khosravi, *On the prime graphs of the automorphism groups of sporadic simple groups*, Archivum Mathematicum (BRNO), **45** (2009), 83-94.

[16] E.I. Khukhro, N.Yu.Makarenko and P.Shumyatsky, *Frobenius groups of automorphisms and their fixed points*, arXiv:1010.0343v1, 2010.

[17] E.I. Khukhro, *Fitting height of a finite group with a Frobenius group of automorphisms*, Journal of Algebra, **366** (2012), 1-11.

[18] E.I. Khukhro and N.Yu. Makarenko, *Finite groups and Lie rings with a metacyclic Frobenius group of automorphisms*, Journal of Algebra, **386** (2013),77-103.

[19] O. King, *The subgroup structure of finite classical groups in terms of geometric configurations,* British Combinatorial Conference, Surveys in Combinatorics, Cambridge University Press (2005), 29-56.

[20] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups*, Springer-Verlag, New York, 2004.

[21] N.V. Maslova, *On the Gruenberg-Kegel graphs of finite groups*, Krasovskii Institute of Mathematics and Mechanics, Proceedings of the 47th International Youth School-conference "Modern Problems in Mathematics and it's Applications", 2016.

[22] P. Mayr, *Finite Fixed Point Free Automorphism Groups*, Diploma Thesis, University of Linz, 1998.

[23] V.D. Mazurov and E.I. Khukhro, *Unsolved Problems in Group Theory, The Kourovka Notebook*, No.18, Russian Academy of Sciences, Siberian Division, Institute of Mathematics, Novosibirsk, 2014.

[24] A.R. Moghaddamfar, *A comparison of the order components in Frobenius and 2- Frobenius groups with finite simple groups*, Taiwanese Journal of Mathematics, **13**(1) (2009), 67-89.

[25] J. Moori, *Further Group Theory,* Lecture Notes, University of Kwazulu Natal, 2008.

[26] J. Moori, *Finite Groups and Representation Theory*, Lecture Notes, University of Kwa-Zulu Natal, Pietermaritzburg, 2008.

[27] J. Moori, *On the Groups* G^{+} *and* $\overline{G}$ *of the forms* $2^{10}:M_{22}$ *and* $2^{10}:\overline{M}_{22}$, PhD thesis, University of Birmingham, 1975.

[28] J. Moori, *On certain groups associated with the smallest Fischer group*, J. London Math. Soc.,**2** (1981), 61-67.

[29] Z.E. Mpono, *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal, Pietermaritzburg, 1998.

[30] C. Parker, *The commuting graph of a solvable group*, arXiv:1209.2279v1, 2012.

[31] D.E. Passman, *Permutation Groups*, Benjamin, New York, 1968.

[32] P. Perumal, *On the Theory of the Frobenius Groups*, MSc Thesis, University of Kwa-Zulu Natal, Pietermaritzburg, 2012.

[33] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag NewYork, Heidelberg, Berlin, Graduate Text in Mathematics, 1982.

[34] B. Rodrigues, *On the Theory and Examples of Group Extensions*, MSc Thesis, University of Kwa-Zulu Natal, Pietermaritzburg, 1999.

[35] J.S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[36] J.J. Rotman, *An Introduction to the Theory of Groups*, 4th Edition, Springer-Verlag New York, Graduate Text in Mathematics, 1995.

[37] W.R. Scott, *Group Theory*, Prentice Hall, New Jersey, 1964.

[38] N.S. Whitney, *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Kwazulu Natal, Pietermaritzburg, 1993.

[39] J.S. Williams, *Prime graph components of finite groups*, Journal of Algebra, **69** (1981),487-513.