Lie group analysis of the new extended KP and three-dimensional KP-BBM equations

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Declaration

I YANGA GAXELA, student number 22301178, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mahikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR. YANGA GAXELA

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:...

PROF C.M. KHALIQUE

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Dedication

To my beautiful children, GOFAONE LUMINATHI GOITIRWANG, NHLANHLA SIPHOSETHU NGWENYA, my tutees from Tiger Kloof Combined School: MOTHEO SESUMA, LOPANG SERAPELWANE, THAMSANQA SIWA, THAMSANQA VELELA, RETHABILE SHUPING, MOFENYI LEKHOE, THUTO SEJAKE, KARABO SETHIBE, GAELEBALE SEBOLA, KABO SEGANG, OMPABALETSI SETHA, NOXOLO TSHAYINCA. "The world is your oyster, it's up to you to find the pearls".

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Abstract

In this dissertation, we examine three partial differential equations that are nonlinear. Namely, nonlinear filtration equation, the new extended Kadomtsev-Petviashvili equation (eKP) and the Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation (KP-BBM). Nonlinear filtration equation is used as an illustrative example. The main equations of this dissertation are then examined. These equation's exact solutions are generated by Lie group analysis. The underlying equation's conservation laws are also determined using the multiplier method, Ibragimov's method and Noether theorem.

Keywords: The new extended Kadomtsev-Petviashvili equation; Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation; Lie group analysis; Conservation laws; Multiplier method; Ibragimov's method; Noether theorem.

Introduction

Nonlinear partial differential equations (NLPDEs) have been used for many years to represent a variety of nonlinear multidimensional systems that are seen in a wide range of natural events. Many scholars have been exploring the topic of NLPDEs in recent years since it is critical in understanding the complicated behaviours of these systems. NLPDEs have been shown to be quite important in today's world.

As a result of the aforementioned, numerous important approaches for generating exact solutions to NLPDEs have been created by various scientists. These include the ansatz method [1], the homogeneous balance method [2], the Bäcklund transformation [3], the inverse scattering transform method [4], the Darboux transformation [5], the simplest equation method [6], the Hirota bilinear method [7], the $(G'/G)-$expansion method [8], the Kudryashov's method [9], the Jacobi elliptic function expansion method [10] and also the Lie symmetry method [11–17], to mention but a few.

Sophus Lie, a great Norwegian mathematician in the late 1800s, pioneered a groundbreaking symmetry-based method for obtaining differential equation solutions, which is now known as Lie group analysis. As a result, exact solutions to differential equations can now be found in a more systematic manner. Several researchers have published a substantial amount of research that is based on Lie's work [11–20].

Conservation laws have been observed to be established and entrenched natural laws that have been seen by many researchers in a variety of scientific domains. Conservation of electric charge, conservation of linear momentum in an isolated system, conservation of mechanical energy in the absence of dissipative forces, conservation of energy, and many more conservation laws are frequent in this regard. Furthermore, in the field of applied mathematics, conservation laws are crucial in evaluating the extent to which differential equations are integrable, reducing and solving partial differential equations (PDEs), and developing numerical methods, among other things. For example, see [21–28] and the references contained therein.

This dissertation's structure can be summarized as follows:

In Chapter one, we cover some basic concepts such as Lie group analysis, the simplest method, Kudryashov's method, Noether's theorem, the multiplier approach, and Ibragimov's conservation theorem.

In Chapter two, as an illustrative example, we look at the nonlinear filtration equation. In order to build the commutator table, we first compute Lie point symmetries of the equation. We also provide the corresponding one-parameter Lie group of transformations. The group-invariant solutions of nonlinear filtration equation are then computed. Finally, using the multiplier method, we obtain the conservation laws.

In Chapter three, we obtain exact solutions of the new extended Kadomtsev-Petviashvili equation. The first step in accomplishing this will be to compute Lie symmetries. Conservation laws are to be derived by the application of Ibragimov's conservation theorem.

In Chapter four, we study the Kadomtsev-Petviashvili Benjamin-Bona-Mahony (KP-BBM) equation. We start by computing Lie point symmetries and then use them to construct invariant solutions. Thereafter, symmetry reductions are obtained. Additionally, using the multiplier method and Ibragimov's conservation theorem, we obtain conservation laws.

In Chapter five, a summary of the findings presented in the dissertation is given and suggestions for future research work are made.

Bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter, we briefly discuss several fundamental Lie theory ideas that we use in our dissertation. Additionally, we provide several techniques for obtaining exact solutions and determining conservation laws for the NLPDEs that will be investigated in the dissertation.

1.1 Introduction

In the latter half of the nineteenth century, Marius Sophus Lie (1842-1899), a known and esteemed Norwegian mathematician, created a ground-breaking symmetry-based approach for solving differential equations. This new technique, which is now often referred to as Lie group analysis, provides a more systematic and reliable manner of producing exact solutions to differential equations. Several books based on Lie group analysis have recently been published, see for instance [11–16]. As a result, the aforementioned books are where many of the definitions and findings provided in this Chapter are found.

1.2 Continuous one-parameter groups

Suppose $x = (x^1, \ldots, x^n)$ is the independent variable with coordinates x^i and $u = (u^1, \ldots, u^m)$ is the dependent variable with coordinates u^α (n and m finite). We consider the following change of the variables in x and u:

$$T_a : \bar{x}^i = f^i(x, u, a), \ \bar{u}^\alpha = \phi^\alpha(x, u, a), \tag{1.1}$$

where a is a real parameter which continuously takes values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A continuous one-parameter (local) Lie group of transformations in the space of variables x and u is a set G of transformations (1.1) which satisfies the following properties:

(i) If T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) There exists $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that from (i) the associativity property is satisfied. The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if the group composition law is additive, i.e. $\phi(a, b) = a + b$.

1.3 Prolongation of point transformations and group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_iD_j(u^\alpha), \ldots, \tag{1.3}$$

where the operator of total differentiation is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n. \tag{1.4}$$

The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$.

In group analysis, all variables $x, u, u_{(1)}, \ldots$ are considered functionally independent and are connected only by the differential relations (1.3). Therefore the u_s^α are called differential variables.

Considering a pth-order PDE, namely

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

1.3.1 Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to Lie's theory, finding the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x}^i \approx x^i + a\,\xi^i(x,u), \quad \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x,u), \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Consequently, we have

$$\xi^i(x,u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x,u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

We now introduce the symbol of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^\alpha \approx (1 + a\,X)u,$$

where the differential operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{1.9}$$

is known as the infinitesimal operator or generator of the group G. We say that X is an admitted operator of (1.5) or X is an infinitesimal symmetry of equation (1.5), if the group G is admitted by (1.5).

We now show how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiation in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \ldots.$$

Let us now apply (1.10) and (1.6)

$$D_i(\phi^\alpha) \;=\; D_i(f^j)\bar{D}_j(\bar{u}^\alpha)$$

$$= \quad D_i(f^j)\bar{u}^\alpha_j. \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u^\beta_i \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}^\alpha_j = \frac{\partial \phi^\alpha}{\partial x^i} + u^\beta_i \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}^\alpha_j$ can be represented as functions of $x, u, u_{(i)}$, for small a, i.e., (1.12) is locally invertible:

$$\bar{u}^\alpha_i = \psi^\alpha_i(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u^\alpha_i. \tag{1.13}$$

The transformations in $(x, u, u_{(1)})$ space given by (1.6) and (1.13) form a one-parameter group called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We now let

$$\bar{u}^\alpha_i \approx u^\alpha_i + a\zeta^\alpha_i \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14). Higher-order prolongations of G, viz., $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

1.3.2 Prolonged generators

Using (1.11) together with (1.7) and (1.14) we obtain

$$\begin{aligned}
D_i(f^j)(\bar{u}^\alpha_j) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u^\alpha_j + a\zeta^\alpha_j) &= D_i(u^\alpha + a\eta^\alpha) \\
u^\alpha_i + a\zeta^\alpha_i + au^\alpha_j D_i\xi^j &= u^\alpha_i + aD_i\eta^\alpha \\
\zeta^\alpha_i &= D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j), \quad \text{(sum on } j).
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Similarly, one can obtain the second prolongation

$$\zeta^\alpha_{ij} \quad = \quad D_j(\eta^\alpha_i) - u^\alpha_{ik} D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} \;=\; D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}\,j} D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \ldots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_i} \quad \text{(sum on } i, \alpha), \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p} \frac{\partial}{\partial u^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1, \end{aligned} \tag{1.18}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}. \tag{1.19}$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \tag{1.20}$$

is a Lie point symmetry of the pth-order PDE (1.5), if

$$X^{[p]}\, E\big|_{E=0} = 0, \tag{1.21}$$

where the symbol $\big|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 Equation (1.21) that determines all the infinitesimal symmetries of (1.5) is called the determining equation.

Definition 1.4 A one-parameter group G of continuous transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(p)}) = 0, \tag{1.22}$$

where E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x,u)$ is called an invariant of the group of transformations (1.1) if

$$F(\bar{x},\bar{u}) \equiv F(f^i(x,u,a),\, \phi^\alpha(x,u,a)) = F(x,u), \tag{1.23}$$

identically in x, u and a.

Theorem 1.1 A necessary and sufficient condition for a function $F(x,u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.24}$$

Thus every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants and consequently we take, as basic invariants, the left-hand side $n-1$ first integrals

$$J_1(x,u) = c_1, \ldots, J_{n-1}(x,) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^n(x,u)}.$$

Theorem 1.2 If the infinitesimal transformation (1.7) or its operator X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x},\bar{u}) \tag{1.25}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.5.1 Kudryashov's method

This technique is employed to obtain exact solutions of NLPDEs, and is detailed in [9]. See also [29,30].

Consider the NLPDE

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0. \tag{1.26}$$

We recall the algorithm involved in Kudryashov's method.

Step 1. The substitution $u(x,t) = U(\phi)$, $\phi = kx + \omega t$, with constants k and ω, transforms equation (1.26) to the ordinary differential equation

$$E_2(U, \omega U', kU', \omega^2 U'', k^2 U'', \dots) = 0. \tag{1.27}$$

Step 2. Suppose that the exact solutions of equation (1.27) is presented as

$$U(\phi) = \sum_{n=0}^{N} a_n Q^n(\phi), \tag{1.28}$$

where a_n $(n = 0, 1, 2, \dots, N)$ are constants that need to be determined such that $a_N \neq 0$, and $Q(\phi)$ becomes the solution of the first-order nonlinear ODE

$$Q_\phi(\phi) = Q^2(\phi) - Q(\phi). \tag{1.29}$$

Equation (1.29) has the solution

$$Q(\phi) = \frac{1}{1 + e^\phi}. \tag{1.30}$$

Step 3. Next we substitute the value of $U(\phi)$ into equation (1.27) and then use equation (1.29) to generate an equation which involves the powers of Q.

Step 4.

Equating various powers of Q to zero yields the system of algebraic equations

$$P_n(a_N, a_{N-1}, \dots, a_0, k, \omega, \dots) = 0, \quad (n = 0, \dots, N). \tag{1.31}$$

Step 5. Finally, the solution of the system of algebraic equations produces the values of coefficients $a_0, a_1, \dots, a_{N-1}, a_N$ and relations for parameters of equation (1.27). As a result, we obtain exact solutions of equation (1.27) in the form expressed in equation (1.28).

1.5.2 The simplest equation method

In this subsection we recall the simplest equation method developed by Kudryashov [31, 32] for finding exact solutions of NLPDEs. Several researchers have recently applied this method to various NLPDEs and it has been shown that this method provides a very effective and powerful mathematical tool for solving many of these equations in various fields of applied sciences. See, for example, the papers [33–37]. The basic steps of the method are as follows:

Consider the NLPDE of the form

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{yy} \dots) = 0. \tag{1.32}$$

The transformation

$$u(t, x, y) = F(z), \quad z = k_1 t + k_2 x + k_3 y + k_4, \tag{1.33}$$

reduces equation (1.32) to an ordinary differential equation

$$E_2(F(z), k_1 F'(z), k_2 F'(z), k_3 F'(z), k_1^2 F''(z), k_2^2 F''(z), k_3^2 F''(z), \dots) = 0. \tag{1.34}$$

The simplest equations that we use here are the Bernoulli equation namely,

$$H'(z) = aH(z) + bH^2(z), \tag{1.35}$$

the Riccati equation viz.,

$$G'(z) = aG^2(z) + bG(z) + c, \tag{1.36}$$

where a, b and c are constants [31, 35, 37]. We look for solutions of the nonlinear ordinary differential equation (1.34) that are of the form

$$F(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{1.37}$$

where $G(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure and $A_0, \dots, A_M$ are parameters to be determined.

The solution of Bernoulli equation (1.35) that we use here is given by

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}, \tag{1.38}$$

where C is a constant of integration. For the Riccati equation (1.36), the solutions to be used are

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right] \tag{1.39}$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \tag{1.40}$$

with $\theta = \sqrt{b^2 - 4ac}$ and C, a constant of integration.

1.6 Conservation laws

Conservation laws are a product of the physics discipline [12, 13]. It is of immense importance because conservation laws give physical and conserved quantities for all solutions $u(t,x)$. It should be noted that they are helpful in determining the accuracy for solving PDEs and stability of numerical methods. A local conservation law for a given PDE is a continuity equation

$$D_t T^t + D_x T^x = 0,$$

where T^t and T^x are respectively the conserved density and the spatial flux functions of t, x, u and the derivatives of u. Here D_t and D_x represent the total derivatives operators with respect to independent variables t and x respectively. Suppose that there exists a function $\Phi(t, x, u, u_t, u_x, ...)$ such that the conserved vector $(T^t, T^x) = (D_x\Phi, -D_t\Phi)$ holds for every solution $u(t,x)$, then, this conservation law is said to be locally equivalent. A nontrivial conservation law can be expressed in a general form as

$$\frac{d}{dt}\int_{\Psi} T^t dx = -T^x|_{\partial\Psi}$$

with $\Psi \subseteq \mathbb{R}$ a fixed spatial domain.

Conservation laws are important in various fields of applications [38, 39]. Precisely, they are highly essential in the study of solutions, integrability as well as in developing numerical solutions for PDEs. In view of this, several systematic approaches have been developed for calculating conservation laws such as the direct construction method popularly referred to as multiplier or non-variational derivatives approach, the symmetry or adjoint symmetry pair method.

Consider an rth-order system of PDEs which contain n independent variables $x = (x^1, x^2, \ldots, x^n)$ as well as m dependent variables $u = (u^1, u^2, \ldots, u^m)$, which are given by

$$E_\alpha(x, u, u_{(1)}, u_{(2)} \ldots, u_{(r)}) = 0, \quad \alpha = 1, \ldots, m \tag{1.41}$$

with $u_{(i)}$ denoting the collection of all i-th order partial derivatives of u. Moreover, n-tuple vector which is defined as $T = (T^1, T^2, \ldots, T^n)$, $T^j \epsilon\, \mathcal{A}$, $j = 1, \ldots, n$, ($\mathcal{A}$ is the space of differential functions) is a conserved vector of (1.41) if T^i satisfies equation

$$D_i T^i|_{(1.41)} = 0. \tag{1.42}$$

1.6.1 The multiplier approach

The multiplier approach has been employed by several researchers. See for example [13, 17, 38, 40–45]. It is noteworthy that for a given differential system, a local conservation law arises from a linear combination formed by local multipliers or characteristics with each of the differential equations in the system, such that the multipliers Λ_α are functions of the independent and dependent variables and are of a finite number of derivatives with respect to the dependent variables of the said system of differential equations.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.43}$$

holds identically. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.44}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{j_1}\ldots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}, \alpha = 1,\ldots,m. \tag{1.45}$$

Once the multipliers have been obtained, one can determine conserved vectors by invoking equation (1.43) as illustrated in [41].

1.6.2 Noether's theorem

The well-known theorem of an eminent, outstanding researcher and mathematician, Amalia 'Emmy' Noether [46] presented in 1918, plays an important fundamental role in various branches of theoretical physics due to the fact that it provides a highly straight-forward connection between the conservation laws of a physical theory and the invariances of the variational integral whose Euler-Lagrange equations are the governing equations of that theory. In recent times, many researchers have applied Noether's theorem in various fields such as mechanics, in finding conservation laws of PDEs. See, for example [47–49]. It may be said that the theorem has placed the Lagrangian formulation in a position of primacy. Furthermore, the theorem brought into existence a situation whereby the search for conservation laws and selection rules have been reduced to a robust systematic study of the symmetries of a theory as well as the corresponding invariances of its Lagrangian. We begin with Euler-Langrange equations given as

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, ..., m, \tag{1.46}$$

where we have $\mathcal{L}(x, u, u_{(1)})$ as a first-order Lagrangian, that is, it involves the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ only, along with the independent variables $x = (x^1, ..., x^n)$ and the dependent variables $u = (u^1, ..., u^m)$.

Noether's theorem states that suppose that the variational integral with the Lagrangian $\mathcal{L}(x, u, u_{(1)})$ is invariant under a group G with a generator defined as

$$X = \xi^i(x, u, u_{(1)}, ...)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, ...)\frac{\partial}{\partial u^\alpha}, \tag{1.47}$$

then the vector field $C = (C^1, \ldots, C^n)$ defined by [47]

$$C^k = \mathcal{L}\tau^k + (\xi^\alpha - \psi^\alpha_{x^j}\tau^j)\left[\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\Big(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^l x^k}}\Big)\right] + \sum_{l=k}^{n}(\eta^\alpha_l - \psi^\alpha_{x^l x^j}\tau^j)\frac{\partial \mathcal{L}}{\partial \psi^\alpha_{x^k x^l}} - B^k \tag{1.48}$$

gives a conservation law for the Euler-Langrange equations (1.46), that is, obeys the equation $\text{div}C \equiv D_k(C^k) = 0$ for all solutions of system (1.41) that is

$$D_k(C^k)|_{(1.41)} = 0. \tag{1.49}$$

Any vector field C^k satisfying equation (1.49) is referred to as a *conserved vector* for equation (1.41).

1.6.3 The new conservation theorem due to Ibragimov

This relatively new method for generating the conservation laws states a general formula on conservation laws [50] for arbitrary partial differential equations by combining Lie symmetry operator and adjoint together with formal Lagrangians. Recently, this method has been put to use by several researchers, for instance [39, 51–53]. The substance of the new conservation theorem due to Ibragimov is the fact that we can obtain a conservation law from every Lie generator, Lie-Bäcklund and non-local symmetry of a system of differential equations. We consider the system of NLPDEs (1.41) and its adjoint equations given by

$$E^*_\alpha(x, u, v, \ldots, v_{(s)}, u_{(s)}) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta E_\beta) = 0, \tag{1.50}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator (1.45) and m new dependent variables $v = (v^1, \ldots, v^m)$.

Theorem 1.3 Consider a system of m equations (1.41). The adjoint system given by (1.50), inherits the symmetries of the system (1.41). Namely, if the system (1.41) admits a point transformation group with an operator (1.20), then the adjoint system (1.50) admits the generator (1.20) extended to the variable v^α by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha} \tag{1.51}$$

with appropriately chosen $\eta_*^\alpha = \eta_*^\alpha(x, u, v)$.

In [50], the coefficients η_*^α in (1.51) are given by

$$\eta_*^\alpha = -\left[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)\right], \tag{1.52}$$

where λ_β^α can be computed by utilising the equation

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \tag{1.53}$$

We can obtain a conserved vector, for instance, for a third-order Lagrangian by applying the formula

$$\begin{aligned} C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\ + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots \right] + D_j D_k(W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}} + \cdots , \end{aligned} \tag{1.54}$$

where $\mathcal{L}$ is the Lagrangian of the system E and E^* that is given by

$$\mathcal{L} = v^\alpha E_\alpha \tag{1.55}$$

and W^α is the Lie characteristic function defined as

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \; \alpha = 1, \ldots, m. \tag{1.56}$$

1.7 Concluding remarks

In this chapter, we gave a brief introduction to Lie group analysis. We presented solution methods for obtaining the exact solutions of PDEs. Additionally, a summary of approaches to determining the conservation laws was included. The various techniques deliberated upon in this section will be utilized throughout this dissertation.

Chapter 2

Symmetry solutions and conservation laws of the filtration equation: an illustrative example

In this Chapter, we study the nonlinear filtration equation. We compute the Lie point symmetries and then generate the commutator table for the symmetries. We further use Lie equations to obtain a one-parameter group of transformations for each of the point symmetries obtained. Thereafter, we utilize the symmetries to obtain group-invariant solutions of the nonlinear filtration equation. Finally, we derive the conservation laws of the equation using the multiplier approach.

2.1 Introduction

Diffusion processes appear in many procedures in physics. These include plasma physics, kinetic theory of gases, solid state and transport in a porous medium. One of the mathematical models for diffusion processes is the nonlinear filtration equation [16] given as

$$u_t + u_t u_x^2 - u_{xx} = 0, \tag{2.1}$$

where $u = u(t, x)$.

2.2 Solutions of the nonlinear filtration equation

In this section, we obtain group-invariant solutions of the nonlinear filtration equation (2.1) by using its Lie point symmetries. Thus, we start by first finding the Lie point symmetries of the nonlinear filtration equation and thereafter using the derived symmetries to compute invariant solutions of the equation.

2.2.1 Lie point symmetries

Equation (2.1) admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.2}$$

if and only if

$$X^{[2]}\left(u_t + u_t u_x^2 - u_{xx}\right)\Big|_{(2.1)} = 0. \tag{2.3}$$

Here $X^{[2]}$ is the second prolongation of (2.2) defined by

$$X^{[2]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}. \tag{2.4}$$

Thus, from equation (2.3) we obtain

$$\left\{\zeta_t(1+u_x^2) + \zeta_x(2u_t u_x) + \zeta_{xx}(-1)\right\}\Big|_{(2.1)} = 0, \tag{2.5}$$

where ζ_t, ζ_x and ζ_{xx} are given by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi)
\end{aligned}$$

and

$$\begin{aligned}
D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_x} + \cdots,\\
D_x &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \cdots
\end{aligned} \tag{2.6}$$

are the total derivatives with respect to t and x respectively. Substituting the values of ζ_t, ζ_x and ζ_{xx} in equation (2.5), we obtain

$$\begin{aligned}&\xi_{uu}u_x^3 - \xi_t u_x^3 - 3u_t\xi_u u_x^3 + \eta_t u_x^2 + 3u_t\eta_u u_x^2 - 2u_t\xi_x u_x^2 - u_t\tau_t u_x^2 - 3u_t^2\tau_u u_x^2 - \eta_{uu}u_x^2\\ &+ 2\xi_{xu}u_x^2 + u_t\tau_{uu}u_x^2 + 2u_t\eta_x u_x - \xi_t u_x - u_t\xi_u u_x - 2u_t^2\tau_x u_x + 2\tau_u u_{tx}u_x + 3\xi_u u_{xx}u_x\\ &- 2\eta_{xu}u_x + \xi_{xx}u_x + 2u_t\tau_{xu}u_x + \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u + 2\tau_x u_{tx} - \eta_u u_{xx} + 2\xi_x u_{xx}\\ &+ u_t\tau_u u_{xx} - \eta_{xx} + u_t\tau_{xx}|_{(2.1)} = 0. \end{aligned} \tag{2.7}$$

Replacing u_{xx} by $u_t + u_t u_x^2$ in equation (2.7) gives

$$\begin{aligned}&u_t u_x^2\tau_{uu} + 2\tau_u u_x u_{tx} + 2u_t u_x\tau_{xu} + 2\tau_x u_{tx} + u_t\tau_{xx} - u_x^2\eta_{uu} - 2u_x\eta_{xu} + u_x^3\xi_{uu}\\ &+ 2u_x^2\xi_{xu} + u_x\xi_{xx} - \eta_{xx} + \eta_t - \tau_t u_t + \eta_t u_x^2 + 2u_t\eta_u u_x^2 + 2u_t u_x\eta_x - \xi_t u_x^3\\ &- \xi_t u_x + 2u_t\xi_u u_x + 2u_t\xi_x - \tau_t u_t u_x^2 - 2u_t^2\tau_u u_x^2 - 2u_t^2 u_x\tau_x = 0. \end{aligned} \tag{2.8}$$

Splitting equation (2.8) on the derivatives of u, we obtain an overdetermined system of linear homogeneous PDEs:

$$\tau_u = 0, \tag{2.9}$$

$$\tau_x = 0, \tag{2.10}$$

$$\xi_t = 0, \tag{2.11}$$

$$\eta_t = 0, \tag{2.12}$$

$$\eta_{xu} = 0, \tag{2.13}$$

$$\eta_{xx} = 0, \tag{2.14}$$

$$\xi_u + \eta_x = 0, \tag{2.15}$$

$$2\xi_x - \tau_t = 0, \tag{2.16}$$

$$2\eta_u - \tau_t = 0. \tag{2.17}$$

From equations (2.9) and (2.10), we obtain

$$\tau = a(t), \tag{2.18}$$

where $a(t)$ is an arbitrary function of t. From equation (2.11) we have

$$\xi = A(x, u), \tag{2.19}$$

where $A(x,u)$ is an arbitrary function of x and u. From equation (2.12) we have

$$\eta = B(x,u), \tag{2.20}$$

where $B(x,u)$ is an arbitrary function depending on x and u. In equation (2.16), we substitute the values of ξ and τ to have

$$2A_x(x,u) - a'(t) = 0. \tag{2.21}$$

Subsequently, equation (2.21) yields

$$A(x,u) = \frac{1}{2}a'(t) + C(u), \tag{2.22}$$

where $C(u)$ is an arbitrary function of u. Therefore, equation (2.22) becomes

$$\xi = \frac{1}{2}a'(t)x + C(u). \tag{2.23}$$

Proceeding in the same way, equation (2.17) leads to

$$\eta = \frac{1}{2}a'(t)u + D(x), \tag{2.24}$$

where $D(x)$ is an arbitrary function of x. We substitute the value of η into equation (2.14) and obtain

$$D''(x) = 0. \tag{2.25}$$

Integrating equation (2.25) with respect to x twice yields

$$D(x) = C_1 x + C_2, \tag{2.26}$$

where C_1 and C_2 are arbitrary constants of integration. Therefore, (2.24) becomes

$$\eta = \frac{1}{2}a'(t)u + C_1 x + C_2. \tag{2.27}$$

By substituting the values of ξ and η into (2.15) we obtain

$$C'(u) + C_1 = 0. \tag{2.28}$$

Integrating equation (2.28) with respect to u yields

$$C(u) = -C_1 u + C_3, \tag{2.29}$$

where C_3 is an arbitrary constant. Therefore from (2.23), we have

$$\xi = \frac{1}{2}a'(t)x - C_1 u + C_3. \tag{2.30}$$

In addition, substituting the value of η into equation (2.12), we get

$$a''(t) = 0. \tag{2.31}$$

Integrating (2.31) with respect to t, we obtain

$$a(t) = C_4 t + C_5, \tag{2.32}$$

where C_4 and C_5 are arbitrary constants. Thus, the values of τ, ξ and η are

$$\begin{aligned}
\tau &= C_4 t + C_5,\\
\xi &= \frac{1}{2}C_4 x - C_1 u + C_3,\\
\eta &= \frac{1}{2}C_4 u + C_1 x + C_2.
\end{aligned}$$

Hence, the Lie point symmetries of the nonlinear filtration equation (2.1) are

$$X_1 = \frac{\partial}{\partial t}, \tag{2.33}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{2.34}$$

$$X_3 = \frac{\partial}{\partial u}, \tag{2.35}$$

$$X_4 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \tag{2.36}$$

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{2.37}$$

2.2.2 Commutator table for the symmetries

We now calculate the commutation relations for all the symmetry generators. We first compute $[X_5, X_1]$. By the definition of the Lie bracket, we have

$$\begin{aligned}
[X_5, X_1] &= X_5 X_1 - X_1 X_5\\
&= \left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\right)
\end{aligned}$$

$$\begin{aligned} &= -2\frac{\partial}{\partial t} \\ &= -2X_1. \end{aligned}$$

Proceeding in a similar manner we compute other commutation relations. In a tabular form, these commutation relations are given below.

Table 2.1: Commutator table of Lie algebra of the nonlinear filtration equation (2.1)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	$2X_1$
X_2	0	0	0	X_3	X_2
X_3	0	0	0	$-X_2$	X_3
X_4	0	$-X_3$	X_2	0	0
X_5	$-2X_1$	$-X_2$	$-X_3$	0	0

2.2.3 One-parameter groups of transformations

The corresponding one-parameter groups of transformations can be obtained using the Lie equations

$$\begin{aligned} \frac{d\bar{t}}{da} &= \tau(t,x,u), \quad \bar{t}|_{a=0} = t, \\ \frac{d\bar{x}}{da} &= \xi(t,x,u), \quad \bar{x}|_{a=0} = x, \\ \frac{d\bar{u}}{da} &= \eta(t,x,u), \quad \bar{u}|_{a=0} = u. \end{aligned}$$

We now compute the one-parameter group of transformations for each Lie point symmetry of the nonlinear filtration equation. For each X_i, let T_{a_i} be the corresponding group. We first calculate the one-parameter group corresponding to infinitesimal generator X_1, namely

$$X_1 = \frac{\partial}{\partial t}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 1,\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = 0,\ \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = 0,\ \bar{u}|_{a=0} = u.$$

Solving the above equations we get

$$\bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{u} = u.$$

Thus the one-parameter group T_{a_1} corresponding to the operator X_1 is given by

$$T_{a_1} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u).$$

For the infinitesimal generator $X_2 = \partial/\partial x$, the Lie equations are

$$\frac{d\bar{t}}{da} = 0,\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = 1,\ \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = 0,\ \bar{u}|_{a=0} = u. \tag{2.38}$$

Solving equations (2.38), we obtain

$$\bar{t} = t, \quad \bar{x} = x + a_2, \quad \bar{u} = u.$$

Thus the one-parameter group T_{a_2} corresponding to the operator X_2 is given by

$$T_{a_2} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u).$$

Likewise the one-parameter group T_{a_3} corresponding to $X_3 = \partial/\partial u$ is given by

$$T_{a_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, u + a_3).$$

The Lie equations for the infinitesimal generator X_4, namely

$$X_4 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$$

are

$$\frac{d\bar{t}}{da} = 0,\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = -\bar{u},\ \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = \bar{x},\ \bar{u}|_{a=0} = u.$$

Solving the above equations, we get

$$\bar{t} = t, \quad \bar{x} = x\cos a_4 - u\sin a_4, \quad \bar{u} = x\sin a_4 + u\cos a_4.$$

Thus the one-parameter group T_{a_4} corresponding to the operator X_4 is given by

$$T_{a_4} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x\cos a_4 - u\cos a_4, x\sin a_4 + u\cos a_4).$$

Finally, we compute one-parameter group corresponding to infinitesimal generator X_5, namely

$$X_5 \quad = \quad 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

The corresponding Lie equations are

$$\frac{d\bar{t}}{da} = 2\bar{t},\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = \bar{x},\ \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = \bar{u},\ \bar{u}|_{a=0} = u. \tag{2.39}$$

Solving equations (2.39), one obtains

$$\bar{t} = te^{2a_5}, \quad \bar{x} = xe^{a_5}, \quad \bar{u} = ue^{a_5}.$$

Therefore the one-parameter group T_{a_5} corresponding to the operator X_5 is given by

$$T_{a_5} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{2a_5}, xe^{a_5}, ue^{a_5}).$$

Thus, the one-parameter groups of transformations related to the five point symmetries of (2.1) are:

$$\begin{aligned}
T_{a_1} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u),\\
T_{a_2} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u),\\
T_{a_3} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x, u + a_3),\\
T_{a_4} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x\cos a_4 - u\sin a_4, x\sin a_4 + u\cos a_4),\\
T_{a_5} &: (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{2a_5}, xe^{a_5}, ue^{a_5}).
\end{aligned}$$

2.2.4 Construction of group-invariant solutions of (2.1)

Consider a Lie point symmetry

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.40}$$

of the nonlinear filtration equation (2.1). The group-invariant solutions under the one-parameter group generated by X are obtained as explained below. We calculate two linearly independent invariants

$$J_1 = \phi(t,x) \text{ and } J_2 = \psi(t,x,u)$$

by solving the first-order quasi-linear PDE

$$X(J) \equiv \tau(t,x,u)\frac{\partial J}{\partial t} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0$$

with characteristic equations otherwise known as the associated Lagrange system

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}.$$

Then we write

$$J_2 = f(J_1), \tag{2.41}$$

where f is a function of J_1. Thereafter, we solve (2.41) for u. Finally, the expression of u is substituted in equation (2.1) and an ordinary differential equation (ODE) is obtained for the unknown function f.

Let us now illustrate the above method by considering the five Lie point symmetries of (2.1), namely X_1, X_2, X_3, X_4 and X_5 and construct group-invariant solutions under these symmetry operators.

Case 1. We first calculate the group-invariant solution under the symmetry operator X_1. The operator X_1 is given by

$$X_1 = \frac{\partial}{\partial t}.$$

The characteristics equations associated with X_1 are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which provide the two invariants $J_1 = x$ and $J_2 = u$. Thus, the group-invariant solution is given by $J_2 = \phi(J_1)$, i.e.,

$$u = \phi(x).$$

Substituting this value of u in (2.1), we obtain

$$\phi''(x) = 0. \tag{2.42}$$

Thus the nonlinear filtration equation (2.1) reduces to second-order ODE (2.42). Integrating the above equation twice, we obtain

$$\phi(x) = c_1 x + c_2,$$

where c_1 and c_2 are an arbitrary constants of integration. Hence the group-invariant solution of (2.1) under X_1 is given by

$$u(t, x) = c_1 x + c_2.$$

Case 2. We now obtain the group-invariant solution under the symmetry operator

$$X_2 = \frac{\partial}{\partial x}.$$

The Lagrangian system associated with X_2 is

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0},$$

which provide the invariants $J_1 = t$ and $J_2 = u$. Thus the group-invariant solution is given by $J_2 = \phi(J_1)$, i.e.,

$$u = \phi(t).$$

Substituting this value of u in equation (2.1), we obtain

$$\phi'(x) = 0.$$

The solution to the obtained ODE is

$$\phi(t) = c_3,$$

where c_3 is an arbitrary constant of integration. Hence the group-invariant solution of equation (2.1) under X_2 is given by

$$u(t, x) = c_3.$$

Case 3. The Lie point symmetry X_3 defined by

$$X_3 = \frac{\partial}{\partial u}$$

does not provide us with a group-invariant solution.

Case 4. Let us now construct the group-invariant solution under the symmetry generator

$$X_4 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}.$$

The characteristic equations associated with X_4 are

$$\frac{dt}{0} = \frac{dx}{-u} = \frac{du}{x}.$$

Thus, one invariant is $J_1 = t$. The other is obtained from the equation

$$\frac{dx}{-u} = \frac{du}{x}$$

and calculated as

$$J_2 = x^2 + u^2.$$

Consequently, the group-invariant solution is $J_2 = \phi(J_1)$, i.e.,

$$u = \sqrt{\phi(t) - x^2}. \tag{2.43}$$

Then differentiating u with respect to t and x we obtain,

$$u_t = -\frac{\phi'(t)}{2\sqrt{\phi(t) - x^2}}, \; u_x = -\frac{x}{\sqrt{\phi(t) - x^2}}, \; u_{xx} = -\frac{\phi(t)}{\sqrt{(\phi(t) - x^2)^3}}.$$

Substitution of the above values of u_t, u_x and u_{xx} in (2.1) gives us the ODE

$$-\frac{\phi'(t)}{2\sqrt{\phi(t) - x^2}}\left\{1 + \left(-\frac{x}{\sqrt{\phi(t) - x^2}}\right)^2\right\} = -\frac{\phi(t)}{\sqrt{(\phi(t) - x^2)^3}}.$$

On simplifying the above ODE, we obtain $\phi'(t) = -2$, which on integration leads to

$$\phi(t) = -2t + C,$$

where C is a constant of integration. Hence the group-invariant solution under the symmetry X_4 is given by

$$u(t,x) = \sqrt{C - 2t - x^2}. \tag{2.44}$$

Case 5. We now calculate the group-invariant solution under the symmetry generator X_5, viz.,

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

The associated Lagrangian system is

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{u}.$$

Therefore one of the invariants is $J_1 = x/\sqrt{t}$. The other is obtained from the equation

$$\frac{dt}{2t} = \frac{du}{u},$$

and is given by $J_2 = u/\sqrt{t}$.

Consequently, the group-invariant solution of (2.1) under X_5 is $J_2 = \phi(J_1)$, i.e.,

$$u = \sqrt{t}\,\phi\left(\frac{x}{\sqrt{t}}\right). \tag{2.45}$$

Differentiating u with respect to t and x we obtain

$$u_t = \frac{1}{2}t^{-\frac{1}{2}}\phi\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2}xt^{-1}\phi'\left(\frac{x}{\sqrt{t}}\right),\ \ u_x = \phi'\left(\frac{x}{\sqrt{t}}\right),\ \ u_{xx} = \frac{1}{\sqrt{t}}\phi''\left(\frac{x}{\sqrt{t}}\right).$$

Substituting u_t, u_x and u_{xx} into nonlinear filtration equation (2.1), we obtain

$$\frac{1}{\sqrt{t}}\phi''\left(\frac{x}{\sqrt{t}}\right) = \left\{\frac{1}{2}t^{-\frac{1}{2}}\phi\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2}xt^{-1}\phi'\left(\frac{x}{\sqrt{t}}\right)\right\}\left\{1 + \phi'^2\left(\frac{x}{\sqrt{t}}\right)\right\},$$

which simplifies to an ODE of the form

$$2\phi''(\lambda) = \left\{1 + \phi'^2(\lambda)\right\}\left\{\phi(\lambda) - \lambda\phi'(\lambda)\right\},\ \ \lambda = x/\sqrt{t}. \tag{2.46}$$

Thus the nonlinear filtration equation (2.1) reduces to the ODE (2.46) under the symmetry X_5.

2.3 Conservation laws of the nonlinear filtration equation

In this section we derive conservation laws for the nonlinear filtration equation (2.1) by employing the multiplier method. This is achieved by using the first-order multiplier $\Lambda(t, x, u, u_x)$. Firstly we recall the Euler-Lagrange operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{tx}} - \cdots , \tag{2.47}$$

where the total derivatives D_t and D_x are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ttx} \frac{\partial}{\partial u_{tx}} + \cdots , \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{txx} \frac{\partial}{\partial u_{tx}} + \cdots . \end{aligned} \tag{2.48}$$

Following the procedure described in Section 1.6.1, the determining equation for the multiplier $\Lambda(t, x, u, u_x)$ is

$$\frac{\delta}{\delta u}[\Lambda(t, x, u, u_x)(u_t + u_t u_x^2 - u_{xx})] = 0. \tag{2.49}$$

Expanding the equation (2.49) we get

$$\begin{aligned} & u_t \Lambda_u - D_t \Lambda - D_x u_t \Lambda_{u_x} + u_t u_x^2 \Lambda_u + D_t u_x^2 \Lambda - 2 D_x u_t u_x \Lambda \\ & - D_x u_t u_x^2 \Lambda_{u_x} - u_{xx} \Lambda_u + D_x u_{xx} \Lambda_{u_x} - D_x^2 \Lambda = 0. \end{aligned} \tag{2.50}$$

Substituting the values of D_t, D_x from equation (2.48) into (2.50) gives

$$\begin{aligned} & u_t \Lambda_u - \left(\frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots \right) \Lambda \\ & - \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots \right) u_t \Lambda_{u_x} \\ & + u_t u_x^2 \Lambda_u + \left(\frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots \right) u_x^2 \Lambda \\ & - 2 \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots \right) u_t u_x \Lambda \\ & - \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots \right) u_t u_x^2 \Lambda_{u_x} \end{aligned}$$

$$
\begin{aligned}
&- u_{xx}\Lambda_u + \left(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots\right)u_{xx}\Lambda_{u_x} \\
&- \left(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots\right)\left(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x}\right. \\
&\left. + u_{tx}\frac{\partial}{\partial u_t} + \cdots\right)\Lambda = 0. \qquad (2.51)
\end{aligned}
$$

On expanding (2.51), we then have

$$
\begin{aligned}
&u_t\Lambda_u - \Lambda_t - u_t\Lambda_u - u_{tx}\Lambda_{u_x} - u_t\Lambda_{xu_x} - u_tu_x\Lambda_{uu_x} - u_{tx}\Lambda_{u_x} - u_tu_{xx}\Lambda_{u_xu_x} \\
&+ u_tu_x^2\Lambda_u - u_x^2\Lambda_t - u_tu_x^2\Lambda_u - 2u_{tx}u_x\Lambda - u_{tx}u_x^2\Lambda_{u_x} - 2u_tu_x\Lambda_x - 2u_tu_x^2\Lambda_u \\
&- 2u_xu_{tx}\Lambda - u_tu_{xx}\Lambda - 2u_tu_xu_{xx}\Lambda_{u_x} - u_tu_x^2\Lambda_{xu_x} - u_tu_x^3\Lambda_{uu_x} - u_{tx}u_x^2\Lambda_{u_x} \\
&- 2u_tu_xu_{xx}\Lambda_{u_x} - u_tu_x^2u_{xx}\Lambda_{u_xu_x} - u_x\Lambda_u + u_{xx}\Lambda_{xu_x} + u_xu_{xx}\Lambda_{uu_x} + u_{xx}^2\Lambda_{u_xu_x} \\
&+ u_{xxx}\Lambda_{u_x} - \Lambda_{xx} - 2u_x\Lambda_{xu} - u_{xx}\Lambda_{xu_x} - u_{xx}\Lambda_u - 2u_xu_{xx}\Lambda_{uu_x} - u_{xx}\Lambda_{xu_x} \\
&- u_{xx}^2\Lambda_{u_xu_x} - u_{xxx}\Lambda_{u_x} = 0. \qquad (2.52)
\end{aligned}
$$

By simplifying equation (2.52) we obtain

$$
\begin{aligned}
&\Lambda_t + 2u_{tx}\Lambda_{u_x} + u_t\Lambda_{xu_x} + u_tu_x\Lambda_{uu_x} + u_tu_{xx}\Lambda_{u_xu_x} + u_x^2\Lambda_t + 4u_{tx}u_x\Lambda \\
&+ 2u_{tx}u_x^2\Lambda_{u_x} + 2u_tu_x\Lambda_x + 2u_tu_x^2\Lambda_u + 2u_tu_{xx}\Lambda + 4u_tu_xu_{xx}\Lambda_{u_x} \\
&+ u_tu_x^2\Lambda_{xu_x} + u_tu_x^3\Lambda_{uu_x} + u_tu_x^2u_{xx}\Lambda_{u_xu_x} + 2u_{xx}\Lambda_u + u_xu_{xx}\Lambda_{uu_x} + \Lambda_{xx} \\
&+ 2u_x\Lambda_{xu} + u_{xx}\Lambda_{xu_x} = 0. \qquad (2.53)
\end{aligned}
$$

Since Λ depends only on t, x, u and u_x and is independent of other derivatives of u, then the coefficients of those derivatives can be equated to zero. Thus we obtain the following set of equations:

$$u_{tx}: \quad (1 + u_x^2)\Lambda_{u_x} + 2u_x\Lambda = 0, \qquad (2.54)$$

$$u_tu_{xx}: \quad (1 + u_x^2)\Lambda_{u_xu_x} + 4u_x\Lambda_{u_x} + 2\Lambda = 0, \qquad (2.55)$$

$$u_t: \quad (u_x^2 - 1)u_x\Lambda_{uu_x} + (1 + u_x^2)\Lambda_{xu_x} - 2u_x\Lambda_x - 2u_x^2\Lambda_u = 0, \qquad (2.56)$$

$$u_{xx}: \quad 2\Lambda_u + u_x\Lambda_{uu_x} + \Lambda_{xu_x} = 0, \qquad (2.57)$$

$$\text{Rest}: \quad \Lambda_t + \Lambda_{xx} + u_x^2\Lambda_t + 2u_x\Lambda_{xu} = 0. \qquad (2.58)$$

The integration of equation (2.54) gives

$$\Lambda(t, x, u, u_x) = \frac{A(t, x, u)}{1 + u_x^2}, \qquad (2.59)$$

where $A(t,x,u)$ is an arbitrary function of t, x and u. Differentiating (2.59) with respect to u_x yields

$$\Lambda_{u_x} = -\frac{2u_x A}{(1+u_x^2)^2}. \tag{2.60}$$

Further differentiation of (2.60) with respect to x and u respectively gives

$$\Lambda_{xu_x} = \frac{-2u_x A_x}{(1+u_x^2)^2}, \quad \Lambda_{uu_x} = \frac{-2u_x A_u}{(1+u_x^2)^2}.$$

Substituting the values of Λ_{xu_x} and Λ_{uu_x} into (2.56), we obtain

$$\left(\frac{{u_x}^2-1}{{u_x}^2+1}\right) u_x^2 A_u + 2u_x A_x + {u_x}^2 A_x = 0, \tag{2.61}$$

which simplifies to

$$A_u u_x^3 + (1+u_x^2)\, A_x = 0. \tag{2.62}$$

Since A is an arbitrary function of t, x and u, splitting (2.62) with respect to the powers of u_x we obtain

$$u_x^3 : A_u = 0, \tag{2.63}$$

$$u_x^2 : A_x = 0. \tag{2.64}$$

Therefore from the above equations we get

$$A(t,x,u) = B(t), \tag{2.65}$$

where B is an arbitrary function of t. Consequently

$$\Lambda(t,x,u,u_x) = B(t). \tag{2.66}$$

Substituting (2.66) into equation (2.58) gives

$$\Lambda_t + u_x^2 \Lambda_t = 0. \tag{2.67}$$

Thus, from (2.67), we have

$$(1+u_x^2)B'(t) = 0. \tag{2.68}$$

Integrating (2.68) with respect to t we get

$$B(t) = \frac{C}{1+u_x^2}.$$

Hence, substituting the value of $B(t)$ in (2.66) gives the result

$$\Lambda(t,x,u,u_x) = \frac{C}{1+u_x^2}, \tag{2.69}$$

where C is an arbitrary constant of integration. Assuming that $C = 1$, the multiplier for the nonlinear filtration equation (2.1) is obtained as

$$\Lambda(t,x,u,u_x) = \frac{1}{1+u_x^2}. \tag{2.70}$$

The multiplier Λ has the property

$$\Lambda\left[u_t(1+u_x^2) - u_{xx}\right] = D_tT^t + D_xT^x, \tag{2.71}$$

where $T^x = T^x(t,x,u,u_x)$ and $T^t = T^t(t,x,u,u_x)$. Expanding equation (2.71) we have

$$\frac{1}{1+u_x^2}\left[u_t(1+u_x^2) - u_{xx}\right] = T^t_t + T^t_u u_x + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}$$

or

$$u_t - \frac{u_{xx}}{1+u_x^2} = T^t_t + T^t_u u_x + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}.$$

Since u_t, u_{tx} and u_{xx} are independent variables, splitting the above equation yields

$$u_t: \; T^t_u = 1, \tag{2.72}$$

$$u_{tx}: \; T^t_{u_x} = 0, \tag{2.73}$$

$$u_{xx}: \; T^x_{u_x} = -\frac{1}{1+u_x^2}, \tag{2.74}$$

$$\text{rest}: \; T^t_t + T^x_x + T^x_u u_x = 0. \tag{2.75}$$

Integrating equation (2.73) with respect to u_x gives

$$T^t = C(t,x,u), \tag{2.76}$$

where $C(t,x,u)$ is an arbitrary function of integration. From equation (2.72), after integrating with respect to u, we get

$$C(t,x,u) = u + D(t,x), \tag{2.77}$$

where $D(t,x)$ is an arbitrary function of integration, therefore

$$T^t = u + D(t,x). \tag{2.78}$$

Integrating equation (2.74) with respect to u_x, one obtains

$$T^x = -\tan^{-1}(u_x) + E(t,x,u), \tag{2.79}$$

where $E(t,x,u)$ is an arbitrary function depending on its arguments. By substituting the value of T^t and T^x into (2.75) gives

$$D_t(t,x) + E_x(t,x,u) + E_u(t,x,u)u_x = 0. \tag{2.80}$$

Splitting the above equation with respect to derivatives of u yields

$$\begin{aligned} u_x: &\;\; E_u(t,x,u) = 0, \\ rest: &\;\; D_t(t,x) + E_x(t,x,u) = 0. \end{aligned}$$

From the above equations, we take D and E to be zero because they contribute to the trivial part of the conservation law. Thus, the conservation law of the nonlinear filtration equation (2.1) associated with the multiplier $\Lambda = 1/(1+u_x^2)$ is given by

$$\begin{aligned} T^t &= u, \\ T^x &= -\tan^{-1}(u_x). \end{aligned}$$

2.4 Concluding remarks

In this chapter we computed Lie point symmetries of the nonlinear filtration equation (2.1). We constructed commutator table for all Lie point symmetries obtained for the equation. Moreover, we presented groups of transformations associated with all the symmetries. Thereafter we obtained the group-invariant solutions of the nonlinear filtration equation under all its symmetries. Finally, we proceeded to derive the conservation laws for the nonlinear filtration equation using the multiplier approach.

Chapter 3

Solutions and conservation laws of the new extended KP equation

In this Chapter we study the new extended Kadomtsev-Petviashvili (eKP) equation [54]. We first find the Lie symmetries of the eKP equation. Thereafter we present the group-invariant solutions. Furthermore we derive the conservation laws of the equation by engaging Ibragimov's theorem.

3.1 Introduction

The two-dimensional Kadomtsev-Petviashvili (KP) equation was developed by Kadomtsev and Petviashvili in 1970 [55]. The KP equation has soliton solutions through the inverse scattering transform, just like the Korteweg-de Vries (KdV) equation. Other nonlinear models with the same properties are the sine-Gordon equation and the nonlinear Schrödinger equation. It is still one of the most extensively researched integrable equations in (2+1)-dimensions [56]. The KP equation was proposed to deal with slowly varying perturbation wave in dispersion media. Since then, it has been thoroughly studied in the literature mathematically and physically in a variety of science fields. These fields include plasma physics, solid state physics, fiber optics, propagation of waves, oceanography and many other areas [57–60]. Recently, there has been a lot of interest in researching both the integrable and non-integrable extended variants of the KP equation. These expanded models stimulated studies that produced a number of encouraging findings and provided insight into some of the physical characteristics of scientific and engineering applications. The standard KP equation reads [61]

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0.$$

Recently, the authors in [56] investigated an extended KP equation of the form

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + \lambda u_{tt} + \mu u_{ty} = 0, \tag{3.1}$$

where additional terms namely λu_{tt} and μu_{ty} were introduced into the standard KP equation. More recently, an additional term νu_{tx} was introduced in [54] and equation (3.1) now becomes

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + \lambda u_{tt} + \mu u_{ty} + \nu u_{tx} = 0. \tag{3.2}$$

In order to highlight the newly constructed KP equation's integrability, the Painlevé analysis and WTC-Kruskal approach were used [62,63]. Consequently, the developed eKP equation (3.2) takes the form [54]

$$6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6{u_x}^2 + u_{xxxx} - u_{yy} = 0, \tag{3.3}$$

where α and β are non-zero real numbers, which we study in this chapter.

3.2 Exact solutions of (3.3)

Firstly we compute Lie symmetries of the eKP equation (3.3). Thereafter, we perform symmetry reduction and obtain exact solutions of the equation.

3.2.1 Lie point symmetries of (3.3)

The vector field

$$\mathbf{X} = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \psi(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of equation (3.3) if

$$\mathbf{X}^{[4]}\Delta|_{\Delta=0} = 0, \tag{3.4}$$

where

$$\Delta \equiv 6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6u_x^2 + u_{xxxx} - u_{yy}$$

and $\mathbf{X}^{[\mathbf{4}]}$ is the fourth prolongation of $\mathbf{X}$ defined as

$$\begin{aligned}\mathbf{X}^{[\mathbf{4}]} = & \mathbf{X} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} \\ & + \zeta_{xxxx} \frac{\partial}{\partial u_{xxxx}}.\end{aligned} \tag{3.5}$$

Here ζ_x, ζ_{tx}, ζ_{ty}, ζ_{tt}, ζ_{xx}, ζ_{yy} and ζ_{xxxx} are determined by

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) - u_y D_t(\psi), \\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi), \\
\zeta_{tx} &= D_x(\zeta_t) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi), \\
\zeta_{ty} &= D_y(\zeta_t) - u_{ty} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\psi), \\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi), \\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi) - u_{ty} D_t(\psi), \\
\zeta_{yy} &= D_y(\zeta_y) - u_{ty} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\psi), \\
\zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{txxx} D_x(\tau) - u_{xxxx} D_x(\xi) - u_{xxxy} D_x(\psi),
\end{aligned}$$

where $D_t,\ D_x$ and D_y are the total derivatives described as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots, \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \cdots.
\end{aligned}$$

On expanding the above ζs, we obtain their respective values as

$$\begin{aligned}
\zeta_t &= \eta_t + \eta_u u_t - \tau_t u_t - \tau_u u_t^2 - \xi_t u_x - \xi_u u_t u_x - \psi_u u_t u_y - \psi_t u_y, \\
\zeta_x &= \eta_x + \eta_u u_x - \tau_x u_t - \tau_u u_t u_x - \xi_u u_x^2 - \xi_x u_x - \psi_x u_y - \psi_u u_x u_y,
\end{aligned}$$

$$
\begin{aligned}
\zeta_y =&\, \eta_y + \eta_u u_y - \tau_y u_t - \tau_u u_t u_y - \xi_u u_x u_y - \xi_y u_x - \psi_y u_y - \psi_u u_y^2, \\
\zeta_{tt} =&\, \eta_{tt} - u_t^2 u_x \xi_{uu} - 2u_t u_{ty} \psi_u + u_{tt}\eta_u - u_t^3 \tau_{uu} - 2u_t^2 \tau_{tu} - 3u_{tt} u_t \tau_u - \tau_{tt} u_t \\
&+ u_t^2 \eta_{uu} - 2u_t u_{tx} \xi_u - 2u_t u_x \xi_{tu} - u_{tt} u_x \xi_u - 2u_{tx}\xi_t - u_x \xi_{tt} - u_t^2 u_y \psi_{uu} \\
&+ 2u_t \eta_{tu} - 2u_t u_y \psi_{tu} - u_{tt} u_y \psi_u - 2u_{ty}\psi_t - u_y \psi_{tt} - 2u_{tt}\tau_t, \\
\zeta_{tx} =&\, u_t u_x \eta_{uu} - u_t \xi_{uu} {u_x}^2 - \xi_{tu} u_x^2 - u_{xx}\xi_u u_x - u_{tx}\xi_u u_x - u_{tx}\tau_u u_x - u_{xy}\psi_u u_x \\
&+ \eta_{tu} u_x - {u_t}^2 \tau_{uu} u_x - u_y u_t \psi_{uu} u_x - u_t \xi_{xu} u_x - u_t \tau_{tu} u_x - u_y \psi_{tu} u_x - \xi_{tx} u_x \\
&+ u_{tx}\eta_u - u_{xx} u_t \xi_{,\,u} - 2u_t u_{tx}\tau_u - u_{xy} u_t \psi_u - u_y u_{tx}\psi_u - u_{xx}\xi_x - u_{tx}\tau_x \\
&+ \eta_{tx} - u_{xy}\psi_x - {u_t}^2 \tau_{xu} - u_y u_t \psi_{xu} - u_{xx}\xi_t - u_{tx}\tau_t - u_{xy}\psi_t - u_t \tau_{tx} \\
&+ u_t \eta_{xu} - u_y \psi_{tx}, \\
\zeta_{xx} =&\, \eta_{xx} + \eta_u u_{xx} + 2\eta_{xu} u_x + \eta_{uu} u_x^2 - 2\tau_x u_{tx} - \tau_{xx} u_t - \tau_u u_t u_{xx} - \tau_{uu} u_t u_x^2 \\
&- 2\tau_u u_x u_{tx} - 2\tau_{xu} u_t u_x - \xi_{xx} u_x - 2\xi_x u_{xx} - \xi_{uu} u_x^3 - 2\xi_{xu} u_x^2 - 3\xi_u u_x u_{xx} \\
&- 2\psi_x u_{xy} - \psi_{xx} u_y - \psi_u u_y u_{xx} - 2\psi_u u_x u_{xy} - 2\psi_{xu} u_x u_y - \psi_{uu} u_x^2 u_y, \\
\zeta_{yy} =&\, \eta_{yy} + \eta_u u_{yy} + 2\eta_{yu} u_y + \eta_{uu} u_y^2 - 2\tau_y u_{ty} - \tau_u u_t u_{yy} - 2\tau_u u_y u_{ty} - \tau_{yy} u_t \\
&- 2\tau_{yu} u_t u_y - \tau_{uu} u_t u_y^2 - 2\xi_y u_{xy} - \xi_u u_x u_{yy} - 2\xi_u u_y u_{xy} - \xi_{yy} u_x \\
&- 2\xi_{yu} u_x u_y - \xi_{uu} u_x u_y^2 - 2\psi_y u_{yy} - 3\psi_u u_y u_{yy} - \psi_{yy} u_y - 2\psi_{yu} u_y^2 - \psi_{uu} u_y^3, \\
\zeta_{xxxx} =&\, \eta_{xxxx} + 4\eta_{xu} u_{xxx} + \eta_u u_{xxxx} + 3\eta_{uu} u_{xx}^2 + 4u_x u_{xxx}\eta_{uu} + 4u_x \eta_{xxxu} + 6\eta_{uuu} u_x^2 u_{xx} \\
&+ 6\eta_{xxu} u_{xx} + 6\eta_{xxuu} u_x^2 + 12\eta_{xuu} u_x u_{xx} + 4\eta_{xuuu} u_x^3 + \eta_{uuuu} u_x^4 - 12\tau_{xu} u_x u_{txx} \\
&- 4\tau_{xxx} u_{tx} - 4\tau_{xu} u_t u_{xxx} - \tau_{xxxx} u_t - 4\tau_u u_x u_{txxx} - 12u_x \tau_{uu} u_{tx} u_{xx} \\
&- 4\tau_{uu} u_t u_x u_{xxx} - 12\tau_{xuu} u_t u_x u_{xx} - 6\tau_{uuu} u_t u_x^2 u_{xx} - 12\tau_{xu} u_{tx} u_{xx} - 6\tau_u u_{xx} u_{txx} \\
&- \tau_u u_t u_{xxxx} - 6\tau_{xxu} u_t u_{xx} - 6\tau_{xx} u_{txx} - 12\tau_{xxu} u_x u_{tx} - 4\tau_x u_{txxx} - 4\tau_{xxxu} u_t u_x \\
&- 4\tau_u u_{tx} u_{xxx} - 6\tau_{uu} u_x^2 u_{txx} - 4\tau_{uuu} u_x^3 u_{tx} - 12\tau_{xuu} u_x^2 u_{tx} - 3\tau_{uu} u_t u_{xx}^2 \\
&- 4\tau_{xuuu} u_t u_x^3 - \tau_{uuuu} u_t u_x^4 - 6\tau_{xxuu} u_t u_x^2 - 4\xi_x u_{xxxx} - 10\xi_{uu} u^2 u_x^2 u_{xxx} \\
&- 6\xi_{xx} u_{xxx} - 16\xi_{xu} u_x u_{xxx} - 18\xi_{xxu} u_x u_{xx} - 5\xi_u u_x u_{xxxx} - 12\xi_{xu} u_{xx}^2 \\
&- 10\xi_u u_{xx} u_{xxx} - 4\xi_{xxx} u_{xx} - 10\xi_{uuu} u_x^3 u_{xx} - \xi_{xxxx} u_x - 4\xi_{xxxu} u_x^2 - 24\xi_{xuu} u_x^2 u_{xx} \\
&- 4\xi_{xuuu} u_x^4 - 6\xi_{xxuu} u_x^3 - 15\xi_{uu} u_x u_{xx}^2 - \xi_{uuuu} u_x^5 - \psi_u u_y y_{xxxx} - 4\psi_u u_x u_{xxxy} \\
&- 4\psi_x u_{xxxy} - 4\psi_u u_{xy} u_{xxx} - 12\psi_{xu} u_x u_{xxy} - 6\psi_{xx} u_{xxy} - 12\psi_{uu} u_x u_{xx} u_{xy} \\
&- 4\psi_{uu} u_x u_y u_{xxx} - 6u_{xx}\psi_u u_{xxy} - 3\psi_{uu} u_y u_{xx}^2 - 12\psi_{xu} u_{xx} u_{xy} - 4\psi_{xu} u_y u_{xxx}
\end{aligned}
$$

$$
\begin{aligned}
&- 6\psi_{uu}u^2u_x^2u_{xxy} - 12\psi_{xuu}u_xu_yu_{xx} - 6\psi_{uuu}u_yu_x^2u_{xx} - 4\psi_{xxxu}u_xu_y \\
&- 12\psi_{xxu}u_xu_{xy} - 6\psi_{xxu}u_yu_{xx} - \psi_{xxxx}u_y - 4\psi_{xxx}u_{xy} - 12\psi_{xuu}u_x^2u_{xy} \\
&- 4\psi_{xuuu}u^3u_x^3u_y - 6\psi_{xxuu}u_x^2u_y - \psi_{uuuu}u_x^4u_y - 4\psi_{uuu}u_x^3u_{xy}.
\end{aligned}
$$

Expanding (3.4) we obtain

$$
\zeta_{tx} + 12u_x\zeta_x + 6u_{xx}\eta + 6u\zeta_{xx} + \zeta_{xxxx} - \zeta_{yy} - \frac{\alpha^2}{4}\zeta_{tt} + \alpha\zeta_{ty} + \beta\zeta_{tx}\Big|_{\Delta=0} = 0.
$$

Substituting the values of ζs in the above equation, replacing u_{xxxx} by $6uu_{xx} - (\alpha^2u_{tt}/4) + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6u_x^2 - u_{yy}$ and then splitting on various derivatives of u, we obtain the following system of linear homogeneous PDEs:

$$
\begin{aligned}
&\tau_x = 0,\ \tau_u = 0,\ \xi_u = 0,\ \psi_x = 0,\ \psi_u = 0,\ \eta_{uu} = 0,\ 2\eta_{xu} - 3\xi_{xx} = 0, \\
&2\tau_y + 2\alpha\xi_x - \alpha\tau_t = 0, 2\xi_y - \alpha\xi_t - (\beta+1)\psi_t = 0, \\
&\alpha\eta_u - 4\tau_y + 2\alpha(\tau_t - \xi_x) = 0,\ \alpha(-2\psi_y + 2\tau_t + \alpha\psi_t) - 4\tau_y = 0, \\
&8\eta_{yu} - 4\psi_{yy} - \alpha(4\eta_{tu} - 4\psi_{ty} + \alpha\psi_{tt}) = 0, \\
&\alpha\left(\xi_t\alpha^2 - 2\xi_y\alpha - 2(\beta+1)\xi_x + 2\beta\tau_t + 2\tau_t\right) - 8(\beta+1)\tau_y = 0, \\
&6\alpha\eta - 24u\tau_y - \alpha(12u\xi_x - 5\xi_{xxx} - 12u\tau_t + \beta\xi_t + \xi_t) = 0, \\
&4\eta_{ty}\alpha - \eta_{tt}\alpha^2 - 4\eta_{yy} + 24u\eta_{xx} + 4\eta_{xxxx} + 4\beta\eta_{tx} + 4\eta_{tx} = 0, \\
&\tau_{tt}\alpha^2 - 2\eta_{tu}\alpha^2 + 4\eta_{yu}\alpha - 4\tau_{ty}\alpha + 4\tau_{yy} + 6\beta\xi_{xx} + 6\xi_{xx} = 0, \\
&\xi_{tt}\alpha^2 - 4\xi_{ty}\alpha + 4\xi_{yy} + 48\eta_x + 48u\xi_{xx} + 20\xi_{xxxx} + 4\beta\eta_{tu} + 4\eta_{tu} - 4\beta\xi_{tx} - 4\xi_{tx} = 0.
\end{aligned} \tag{3.6}
$$

After solving the above system of PDEs (3.6) with the aid of Mathematica, we get the following Lie point symmetries for eKP equation (3.3):

$$
\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{X}_2 &= \frac{\partial}{\partial x}, \\
\mathbf{X}_3 &= \frac{\partial}{\partial y}, \\
\mathbf{X}_4 &= 6\alpha^2 t\frac{\partial}{\partial t} + \left(3\alpha^2x - 6\beta t - 6t\right)\frac{\partial}{\partial x} + \left(6\alpha t + 9\alpha^2 y\right)\frac{\partial}{\partial y} \\
&\quad - \left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right)\frac{\partial}{\partial u},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{X}_5 = & \left(24\alpha t^2 + 12\alpha^2 ty\right)\frac{\partial}{\partial t} + \left(12\alpha tx - 12\beta ty - 12ty + 6\alpha^2 xy\right)\frac{\partial}{\partial x} \\
& + \left(12\alpha ty - 12t^2 + 9\alpha^2 y^2\right)\frac{\partial}{\partial y} + \left(2\alpha\beta x - 24\alpha tu - 12\alpha^2 uy\right. \\
& \left. + 2\alpha x - 2\beta^2 y - 4\beta y - 2y\right)\frac{\partial}{\partial u}.
\end{aligned}
$$

Note that the Lie point symmetries $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ represent translation symmetries.

3.2.2 Symmetry reductions and solutions of (3.3)

Case 1. We consider a linear combination $\mathbf{X} = \mathbf{X_1} + a\mathbf{X_2} + b\mathbf{X_3}$, where a and b are constants. It can be seen that symmetry $\mathbf{X}$ yields the following three invariants:

$$
p = x - at, \;\; q = y - bt, \;\; u = \Phi(p,q). \tag{3.7}
$$

These invariants reduce equation (3.3) into a NLPDE of the form

$$
\begin{aligned}
& \Phi_{pppp} - \left(a + \frac{1}{4}\alpha^2 a^2 + a\beta\right)\Phi_{pp} + 6\Phi\Phi_{pp} - \left(b + a\alpha + b\beta + \frac{1}{2}ab\alpha^2\right)\Phi_{pq} + 6\Phi_p^2 \\
& - \left(1 + \frac{1}{4}b^2\alpha^2 + b\alpha\right)\Phi_{qq} = 0,
\end{aligned} \tag{3.8}
$$

where Φ is a function of p and q. Equation (3.8) have the following point symmetries:

$$
\begin{aligned}
& \Gamma_1 = \frac{\partial}{\partial p}, \quad \Gamma_2 = \frac{\partial}{\partial q}, \\
& \Gamma_3 = \left\{3(b\alpha + 2)^2 p + (3ab\alpha^2 + 6a\alpha + 6b\beta + 6b)q\right\}\frac{\partial}{\partial p} + 6(b\alpha + 2)^2 q\frac{\partial}{\partial q} \\
& \qquad + \left(2ab\alpha\beta - 6b^2\alpha^2 u - b^2\beta^2 - 2b^2\beta - b^2 - 24b\alpha u + 2ab\alpha + 4a\beta - 24u\right)\frac{\partial}{\partial \Phi}.
\end{aligned}
$$

Likewise we make use of a linear combination of translation symmetries Γ_1 and Γ_2, written as

$$
\Gamma = \Gamma_1 + c\Gamma_2 \tag{3.9}
$$

with c a constant. Using the characteristic equations associated with Γ we obtain two invariants

$$
z = q - cp, \;\; \Phi = \Psi(z), \tag{3.10}
$$

which reduce equation (3.8) to the fourth order NLODE

$$c^4\Psi''''(z) + A\Psi''(z) + 6c^2\left\{\Psi'^2(z) + \Psi(z)\Psi''(z)\right\} = 0 \tag{3.11}$$

with $z = (ac - b)t - cx + y$ and the constant

$$A = \frac{abc\alpha^2}{2} - \frac{\alpha^2 a^2 c^2}{4} - \frac{b^2\alpha^2}{4} + bc - ac^2 + ac\alpha - b\alpha - ac^2\beta + bc\beta - 1.$$

Integrating the NLODE (3.11), we have

$$c^4\Psi''' + A\Psi' + 6c^2\Psi\Psi' + \mathbf{k_0} = 0, \tag{3.12}$$

where $\mathbf{k_0}$ is an integration constant. Assuming $\mathbf{k_0}$ to be zero and integrating (3.12) gives

$$c^4\Psi'' + A\Psi + 3c^2\Psi^2 + \mathbf{k_1} = 0, \tag{3.13}$$

where $\mathbf{k_1}$ is the integration constant. Multiplying (3.13) by Ψ' and integrating the resultant NLODE, we have

$$\frac{1}{2}c^4\Psi'^2 + \frac{1}{2}A\Psi^2 + c^2\Psi^3 + \mathbf{k_1}\Psi + \mathbf{k_2} = 0, \tag{3.14}$$

where $\mathbf{k_2}$ is a constant.

Solution in terms of the elliptic integral function

We rewrite (3.14) in the form

$$\Psi'^2 + \frac{2}{c^2}\Psi^3 + \frac{A}{c^4}\Psi^2 + \frac{2\mathbf{k_1}}{c^4}\Psi + \frac{2\mathbf{k_2}}{c^4} = 0. \tag{3.15}$$

Suppose that v_1, v_2 and v_3 are real roots ($v_1 > v_2 > v_3$) of the cubic equation

$$\Psi^3 + \frac{A}{2c^2}\Psi^2 + \frac{\mathbf{k_1}}{c^2}\Psi + \frac{\mathbf{k_2}}{c^2} = 0 \tag{3.16}$$

that satisfy the conditions

$$v_1v_2v_3 = -\frac{\mathbf{k_2}}{c^2},\ v_1v_2 + v_1v_3 + v_2v_3 = \frac{\mathbf{k_1}}{c^2},\ v_1 + v_2 + v_3 = -\frac{A}{2c^2},$$

then equation (3.15) is written as

$$\Psi'^2 = -\frac{2}{c^2}(\Psi - v_1)(\Psi - v_2)(\Psi - v_3). \tag{3.17}$$

Hence, (3.17) has the solution [64]

$$\Psi(z) = v_2 + (v_1 - v_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{v_1 - v_3}{2c^2}}\,(z - z_0)\,\middle|\,K^2\right\}, K^2 = \frac{v_1 - v_2}{v_1 - v_3}, \tag{3.18}$$

where z_0 is a constant and cn is the Jacobi cosine function. Thus by reverting to the original variables, we obtain the solution of (3.3) as

$$u(t, x, y) = v_2 + (v_1 - v_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{v_1 - v_3}{2c^2}}\,(z - z_0)\,\middle|\,K^2\right\}, K^2 = \frac{v_1 - v_2}{v_1 - v_3}, \tag{3.19}$$

where $z = (ac - b)t - cx + y$.

Figure 3.1 demonstrates the solution (3.19) for the values $a = -4, b = -0.2, c = 1.6, t = -14, v_1 = 60, v_2 = 20.05, v_3 = -60, z_0 = 0$.

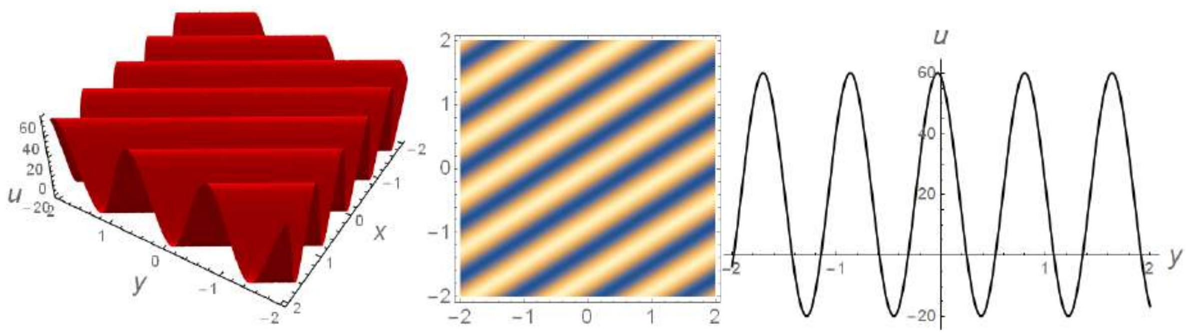

Figure 3.1: The 3D and 2D solution profiles of (3.19).

Solution of (3.3) using Kudryashov's method

We now solve the fourth order NLODE (3.11) using Kudryashov's technique which has been fully described in Chapter one. To utilize the method, we begin by assuming the solution of (3.11) to be of the form

$$\Psi(z) = \sum_{i=0}^{M} a_i Y^i(z), \tag{3.20}$$

where $Y(z)$ satisfies the Riccati equation

$$Y'(z) = Y^2(z) - Y(z) \tag{3.21}$$

whose solution is

$$Y(z) = \frac{1}{1+e^{z}}. \tag{3.22}$$

Applying the balancing procedure to (3.11), we get $M = 2$. Thus, the solution (3.20) can be written as

$$\Psi(z) = a_0 + a_1 Y(z) + a_2 Y^2(z). \tag{3.23}$$

Substituting the value of $\Psi(z)$ into equation (3.11) and making use of (3.21), we get

$$\begin{aligned}
&c^4 a_1 Y(z) + A a_1 Y(z) + 72\, c^2 a_1 Y^5(z) a_2 - 126\, c^2 a_1 Y^4(z) a_2 \\
&+ 54\, c^2 a_1 Y^3(z) a_2 + 12\, c^2 a_1 Y^3(z) a_0 - 18\, c^2 a_1 Y^2(z) a_0 \\
&+ 6\, c^2 a_1 Y(z) a_0 + 36\, c^2 a_2 Y^4(z) a_0 - 60\, c^2 a_2 Y^3(z) a_0 - 3\, A a_1 Y^2(z) \\
&+ 24\, c^2 a_2 Y^2(z) a_0 - 60\, c^4 a_1 Y^4(z) + 24\, c^4 a_1 Y^5(z) - 15\, c^4 a_1 Y^2(z) \\
&+ 50\, c^4 a_1 Y^3(z) + 120\, c^4 a_2 Y^6(z) - 336\, c^4 a_2 Y^5(z) - 30\, c^2 {a_1}^2 Y^3(z) \\
&+ 330\, c^4 a_2 Y^4(z) - 130\, c^4 a_2 Y^3(z) + 16\, c^4 a_2 Y^2(z) + 2\, A a_1 Y^3(z) \\
&+ 6\, A a_2 Y^4(z) - 10\, A a_2 Y^3(z) + 4\, A a_2 Y^2(z) + 18\, c^2 {a_1}^2 Y^4(z) \\
&+ 12\, c^2 {a_1}^2 Y^2(z) + 60\, c^2 {a_2}^2 Y^6(z) - 108\, c^2 {a_2}^2 Y^5(z) + 48\, c^2 {a_2}^2 Y^4(z) = 0.
\end{aligned} \tag{3.24}$$

After splitting (3.24) with respect to the like powers of $Y(z)$, we have

$$\begin{aligned}
Y^6(z) &: \; 2\, c^4 a_2 + c^2 {a_2}^2 = 0, \\
Y^5(z) &: \; 2c^4 a_1 - 24\, c^4 a_2 + 6\, c^2 a_1 a_2 - 9\, c^2 {a_2}^2 = 0, \\
Y^4(z) &: \; 55\, c^4 a_2 - 10\, c^4 a_1 + 6\, c^2 a_0 a_2 + 3\, c^2 {a_1}^2 - 21\, c^2 a_1 a_2 + 8\, c^2 {a_2}^2 + A a_2 = 0, \\
Y^3(z) &: \; 25\, c^4 a_1 - 65\, c^4 a_2 + 6\, c^2 a_0 a_1 - 30\, c^2 a_0 a_2 - 15\, c^2 {a_1}^2 + 27\, c^2 a_1 a_2 \\
&\quad + A a_1 - 5\, A a_2 = 0, \\
Y^2(z) &: \; 16\, c^4 a_2 - 15\, c^4 a_1 - 18\, c^2 a_0 a_1 + 24\, c^2 a_0 a_2 + 12\, c^2 {a_1}^2 - 3\, A a_1 + 4\, A a_2 = 0, \\
Y(z) &: \; c^4 a_1 + 6\, c^2 a_0 a_1 + A a_1 = 0.
\end{aligned} \tag{3.25}$$

With the aid of Maple, the solution of the system (3.25) is

$$a_0 = -\frac{c^4 + A}{c^2}, \;\; a_1 = 2\, c^2, \;\; a_2 = -2\, c^2. \tag{3.26}$$

Thus the solution (3.23) is written as

$$\Psi(z) = -\frac{A}{c^2} - \frac{c^2\,(1+e^{2z})}{(1+e^{z})^2}.$$

Therefore, reverting to the original variables, we get

$$u(t,x,y) = -\frac{A}{c^2} - \frac{c^2\left\{1+e^{2((ac-b)t-cx+y)}\right\}}{\left\{1+e^{((ac-b)t-cx+y)}\right\}^2}, \tag{3.27}$$

where $A = (abc\alpha^2)/2 - (\alpha^2 a^2 c^2)/4 - (b^2\alpha^2)/4 + bc - ac^2 + ac\alpha - b\alpha - ac^2\beta + bc\beta - 1$.

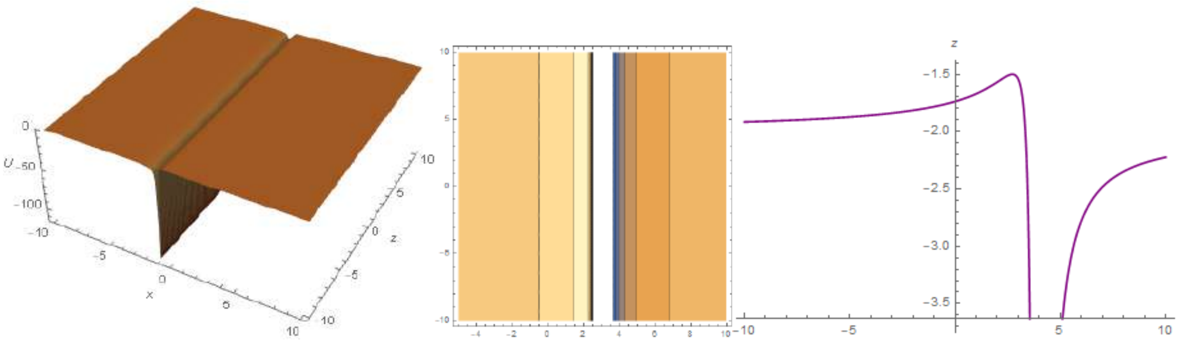

Figure 3.2: The 3D and 2D solution profiles of (3.27).

Figure 3.2 demonstrates the solution (3.27) for the values $a = 0.9, b = 1.5, c = -1, A = 1, t = 1, y = 3$.

Case 2. We consider Lie point symmetry X_4, namely

$$X_4 = 6\alpha^2 t\frac{\partial}{\partial t} + \left(3\alpha^2 x - 6\beta t - 6t\right)\frac{\partial}{\partial x} + \left(6\alpha t + 9\alpha^2 y\right)\frac{\partial}{\partial y} - \left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right)\frac{\partial}{\partial u}.$$

The Lagrangian system associated with X_4 is

$$\frac{dt}{6\alpha^2 t} = \frac{dx}{3\alpha^2 x - 6\beta t - 6t} = \frac{dy}{6\alpha t + 9\alpha^2 y} = \frac{du}{-\beta^2 - 2\beta - 6\alpha^2 u - 1}. \tag{3.28}$$

From (3.28) we consider

$$\frac{dt}{6\alpha^2 t} = \frac{dx}{3\alpha^2 x - 6\beta t - 6t}, \tag{3.29}$$

which leads to the linear first-order differential equation

$$\frac{dx}{dt} - \frac{x}{2t} = -\frac{1+\beta}{\alpha^2}. \tag{3.30}$$

Solving the differential equation, we obtain an invariant solution

$$J_1 = \frac{\alpha^2 x + 2t(\beta+1)}{\sqrt{t}\alpha^2}.$$

Moreover, from system (3.28), we use

$$\frac{dt}{6\alpha^2 t} = \frac{dy}{6\alpha t + 9\alpha^2 y} \tag{3.31}$$

and get the linear first-order differential equation

$$\frac{dy}{dt} - \frac{3y}{2t} = \frac{1}{\alpha}. \tag{3.32}$$

Solving (3.32) we obtain invariant J_2 as

$$J_2 = \frac{\alpha\, y + 2\, t}{t^{3/2}\alpha}.$$

Finally, solving

$$\frac{dt}{6\alpha^2 t} = \frac{du}{-\beta^2 - 2\beta - 6\alpha^2 u - 1},$$

we obtain the third invariant

$$J_3 = \frac{(6\, u\alpha^2 + \beta^2 + 2\,\beta + 1)\, t}{6\alpha^2}. \tag{3.33}$$

Invariant J_3, is then written as a function of the other two invariants J_1 and J_2, that is $J_3 = G(J_1, J_2)$. Thus

$$\frac{(6\, u\alpha^2 + \beta^2 + 2\,\beta + 1)\, t}{6\alpha^2} = G\left(\frac{\alpha^2 x + 2t(\beta+1)}{\sqrt{t}\alpha^2}, \frac{\alpha\, y + 2\, t}{t^{3/2}\alpha}\right), \tag{3.34}$$

which gives

$$u(t,x,y) = -\frac{\beta^2 + 2\beta + 1}{6\alpha^2} + \frac{1}{t} G\left(\frac{\alpha^2 x + 2t(\beta+1)}{\sqrt{t}\alpha^2}, \frac{\alpha\, y + 2\, t}{t^{3/2}\alpha}\right). \tag{3.35}$$

Substituting the value of $u(t,x,y)$ from (3.35) into equation (3.3) yields

$$\begin{aligned}
&\alpha^2 q^2 G_{qq} - 7\alpha^2 q G_q - 9\alpha^2 p^2 G_{pp} - 2\left(3\alpha^2 pq - \frac{8\beta}{\alpha} - \frac{8}{\alpha}\right) G_{pq} - 27\alpha^2 p G_p \\
&- 8\alpha^2 G(p,q) + 96 G_{qq} G(p,q) + 96 G_p^2 + 16 G_{qqqq} = 0,
\end{aligned} \tag{3.36}$$

where

$$p = \frac{\alpha^2 x + 2t(\beta+1)}{\sqrt{t}\alpha^2} \text{ and } q = \frac{\alpha\, y + 2\, t}{t^{3/2}\alpha}.$$

Thus we have performed a symmetry reduction on the eKP (3.3) using X_4. We note that equation (3.36) does not possess any symmetry and hence we cannot perform symmetry reduction on (3.36).

Case 3. We now consider symmetry X_5, viz.,

$$\begin{aligned}\mathbf{X}_5 = {} & \left(24\alpha t^2 + 12\alpha^2 ty\right)\frac{\partial}{\partial t} + \left(12\alpha tx - 12\beta ty - 12ty + 6\alpha^2 xy\right)\frac{\partial}{\partial x} \\ & + \left(12\alpha ty - 12t^2 + 9\alpha^2 y^2\right)\frac{\partial}{\partial y} + \big(2\alpha\beta x - 24\alpha tu - 12\alpha^2 uy \\ & +2\alpha x - 2\beta^2 y - 4\beta y - 2y\big)\frac{\partial}{\partial u}.\end{aligned}$$

We were unable to find invariants for this symmetry.

3.3 Conservation laws of the eKP equation

We now construct conservation laws for the eKP equation (3.3) by employing Ibragimov's theorem. Firstly we write the adjoint equation in the form

$$F^* \equiv \frac{\delta}{\delta u}\left\{v\left(6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6u_x^2 + u_{xxxx} - u_{yy}\right)\right\} = 0, \tag{3.37}$$

where $v = v(t, x, y)$ and the Euler-Lagrange operator $\delta/\delta u$ is defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_xD_t\frac{\partial}{\partial u_{tx}} - \cdots, \tag{3.38}$$

with the total derivatives $D_t,\ D_x$ and D_y being given by

$$\begin{aligned}D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t} + \cdots, \\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + u_{ty}\frac{\partial}{\partial u_t} + v_{ty}\frac{\partial}{\partial v_t} + \cdots.\end{aligned}$$

Expanding (3.37), we obtain the adjoint equation as

$$6v_{xx}u - \frac{1}{4}\alpha^2 v_{tt} + (\beta + 1)v_{tx} + \alpha v_{ty} + v_{xxxx} - v_{yy} = 0. \tag{3.39}$$

Equation (3.3) and its adjoint (3.39) have a second-order Lagrangian

$$\mathcal{L} = v\big(6uu_{xx} - \frac{1}{4}\alpha^2 u_{tt} + \beta u_{tx} + u_{tx} + \alpha u_{ty} + 6{u_x}^2 - u_{yy}\big) + v_{xx}u_{xx}. \tag{3.40}$$

Applying Ibragimov's theorem [14], we deduce that conserved vectors associated with symmetries of equation (3.3) are given by

$$
\begin{aligned}
C^i =& \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\
&+ D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \cdots \right] + D_j D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}} + \cdots , \qquad (3.41)
\end{aligned}
$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, \ldots, m$ is the Lie characteristic function and $\mathcal{L}$ is the Lagrangian. We have five cases.

Case 1. Let us consider Lie point symmetry $\mathbf{X_1} = \partial/\partial t$. The Lie characteristic functions are $W^1 = -u_t$ and $W^2 = -v_t$. Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_1}$ is

$$
\begin{aligned}
C_1^t =& \frac{1}{2}\alpha u_{ty} v + \frac{1}{2}\beta u_{tx} v + 6{u_x}^2 v - u_{yy} v + 6 u_{xx} u v + \frac{1}{2} u_{tx} v - \frac{1}{4}\alpha^2 u_t v_t + \frac{1}{2}\beta u_t v_x \\
&+ \frac{1}{2} u_t v_x + \frac{1}{2}\alpha u_t v_y + u_{xx} v_{xx}, \\
C_1^x =& 6 u_t v_x u - \frac{1}{2}\beta u_{tt} v - 6 u_t u_x v - 6 u_{tx} u v - \frac{1}{2} u_{tt} v + \frac{1}{2}\beta u_t v_t + u_t v_{xxx} + v_t u_{xxx} \\
&- v_{xx} u_{tx} - u_{xx} v_{tx} + \frac{1}{2} u_t v_t, \\
C_1^y =& \frac{1}{2}\alpha u_t v_t - \frac{1}{2}\alpha u_{tt} v + u_{ty} v - u_t v_y.
\end{aligned}
$$

Case 2. For symmetry $\mathbf{X_2} = \partial/\partial x$, the Lie characteristic functions are $W^1 = -u_x$ and $W^2 = -v_x$. Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_2}$ is

$$
\begin{aligned}
C_2^t =& \frac{1}{4}\alpha^2 u_{tx} v - \frac{1}{2}\alpha u_{xy} v - \frac{1}{2}\beta u_{xx} v - \frac{1}{2} u_{xx} v - \frac{1}{4}\alpha^2 v_t u_x + \frac{1}{2}\beta u_x v_x \\
&+ \frac{1}{2}\alpha u_x v_y + \frac{1}{2} u_x v_x, \\
C_2^x =& \alpha u_{ty} v + \frac{1}{2}\beta u_{tx} v - u_{yy} v + 6 u_x v_x u + \frac{1}{2} u_{tx} v + \frac{1}{2} v_t u_x - \frac{1}{4}\alpha^2 u_{tt} v \\
&+ \frac{1}{2}\beta v_t u_x - u_{xx} v_{xx} + u_{xxx} v_x + u_x v_{xxx}, \\
C_2^y =& u_{xy} v - \frac{1}{2}\alpha u_{tx} v + \frac{1}{2}\alpha v_t u_x - u_x v_y.
\end{aligned}
$$

Case 3. For symmetry $\mathbf{X_3} = \partial/\partial y$, the Lie characteristic functions are $W^1 = -u_y$ and $W^2 = -v_y$ and so the conserved vector corresponding to the Lie point symmetry

$\mathbf{X_3}$ is

$$
\begin{aligned}
C_3^t =& \frac{1}{4}\alpha^2 u_{ty}v - \frac{1}{2}\alpha u_{yy}v - \frac{1}{2}\beta u_{xy}v - \frac{1}{2}u_{xy}v - \frac{1}{4}\alpha^2 v_t u_y + \frac{1}{2}\beta u_y v_x + \frac{1}{2}u_y v_x + \frac{1}{2}\alpha u_y v_y,\\
C_3^x =& 6u_y v_x u - \frac{1}{2}\beta u_{ty}v - 6u_x u_y v - 6u_{xy}uv - \frac{1}{2}u_{ty}v + \frac{1}{2}\beta v_t u_y + \frac{1}{2}v_t u_y - u_{xx}v_{xy}\\
&+ u_{xxx}v_y - v_{xx}u_{xy} + u_y v_{xxx},\\
C_3^y =& \frac{1}{2}\alpha u_{ty}v - \frac{1}{4}\alpha^2 u_{tt}v + \beta u_{tx}v + 6{u_x}^2 v + 6u_{xx}uv + u_{tx}v + \frac{1}{2}\alpha v_t u_y\\
&+ u_{xx}v_{xx} - u_y v_y.
\end{aligned}
$$

Case 4. For the symmetry

$$
\mathbf{X_4} = 6\alpha^2 t\frac{\partial}{\partial t} + \left(3\alpha^2 x - 6\beta t - 6t\right)\frac{\partial}{\partial x} + \left(6\alpha t + 9\alpha^2 y\right)\frac{\partial}{\partial y} - \left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right)\frac{\partial}{\partial u},
$$

the Lie characteristic functions are

$$
W^1 = -\left(\beta^2 + 2\beta + 6\alpha^2 u + 1\right) - 6\alpha^2 t u_t - \left(3\alpha^2 x - 6\beta t - 6t\right)u_x - \left(6\alpha t + 9\alpha^2 y\right)u_y,
$$

and

$$
W^2 = -\left(\beta^2 + 2\beta + 6\alpha^2 v + 1\right) - 6\alpha^2 t v_t - \left(3\alpha^2 x - 6\beta t - 6t\right)v_x - \left(6\alpha t + 9\alpha^2 y\right)v_y.
$$

Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_4}$ is

$$
\begin{aligned}
C_4^t =& 3\alpha^4 v u_t - \frac{3}{2}\alpha^4 u v_t - \frac{9}{4}\alpha^4 y u_y v_t - \frac{3}{4}\alpha^4 x u_x v_t - \frac{3}{2}\alpha^4 t u_t v_t + \frac{9}{4}\alpha^4 y v u_{ty}\\
&+ \frac{3}{4}\alpha^4 x v u_{tx} - 6\alpha^3 v u_y + 3\alpha^3 u v_y + \frac{9}{2}\alpha^3 y u_y v_y - \frac{9}{2}\alpha^3 y v u_{yy} + \frac{3}{2}\alpha^3 x v_y u_x\\
&- \frac{3}{2}\alpha^3 x v u_{xy} + 3\alpha^3 t v_y u_t - \frac{3}{2}\alpha^3 t u_y v_t + \frac{9}{2}\alpha^3 t v u_{ty} + 36\alpha^2 t v {u_x}^2 + 3\alpha^2 t u_y v_y\\
&- 9\alpha^2 t v u_{yy} - 6\beta\alpha^2 v u_x - 6\alpha^2 v u_x + 3\beta\alpha^2 u v_x + 3\alpha^2 u v_x + \frac{9}{2}\alpha^2 y u_y v_x\\
&+ \frac{9}{2}\alpha^2\beta y u_y v_x + \frac{3}{2}\alpha^2 x u_x v_x + \frac{3}{2}\alpha^2\beta x u_x v_x - \frac{9}{2}\alpha^2 y v u_{xy} - \frac{3}{2}\alpha^2 x v u_{xx}\\
&- \frac{3}{2}\beta\alpha^2 x v u_{xx} + 36\alpha^2 t u v u_{xx} + 6\alpha^2 t u_{xx} v_{xx} + 3\alpha^2 t v_x u_t + 3\beta\alpha^2 t v_x u_t\\
&- \frac{1}{4}\beta^2\alpha^2 v_t - \frac{1}{2}\beta\alpha^2 v_t + \frac{3}{2}\alpha^2 t u_x v_t + \frac{3}{2}\beta\alpha^2 t u_x v_t - \frac{1}{4}\alpha^2 v_t + \frac{3}{2}\alpha^2 t v u_{tx}\\
&+ \frac{3}{2}t\beta\alpha^2 v u_{tx} + \frac{1}{2}\beta^2\alpha v_y + \beta\alpha v_y + \frac{1}{2}v_y\alpha - 3\alpha t v_y u_x - 3t\beta\alpha v_y u_x\\
&+ 3\alpha t u_y v_x + 3\beta\alpha t u_y v_x + \frac{1}{2}\beta^3 v_x + \frac{3}{2}\beta^2 v_x + \frac{3}{2}\beta v_x - 3\beta^2 t u_x v_x - 3t u_x v_x\\
&- 6\beta t u_x v_x + \frac{1}{2}v_x + 3\beta^2 t v u_{xx} + 3t v u_{xx} + 6\beta t v u_{xx},
\end{aligned}
$$

$$\begin{aligned}
C_4^x =& 3\alpha^3 xvu_{ty} - \frac{3}{4}\alpha^4 xvu_{tt} - 3\alpha^2 xvu_{yy} - 90\alpha^2 uvu_x - 54\alpha^2 yvu_y u_x \\
&+ 36\alpha^2 u^2 v_x + 18\alpha^2 xuu_x v_x - 54\alpha^2 yuvu_{xy} - 3\alpha^2 v_x u_{xx} - 9\alpha^2 yv_{xy} u_{xx} \\
&- 9\alpha^2 u_x v_{xx} - 9\alpha^2 yu_{xy} v_{xx} - 3\alpha^2 xu_{xx} v_{xx} + 9\alpha^2 yv_y u_{xxx} + 3\alpha^2 xv_x u_{xxx} \\
&+ 6\alpha^2 uv_{xxx} + 9\alpha^2 yu_y v_{xxx} + 3\alpha^2 xu_x v_{xxx} - 6\beta\alpha^2 vu_t - 6\alpha^2 vu_t \\
&- 36\alpha^2 tvu_x u_t + 36\alpha^2 tuv_x u_t + 6\alpha^2 tv_{xxx} u_t + 3\beta\alpha^2 uv_t + 3\alpha^2 uv_t + \frac{9}{2}\alpha^2 yu_y v_t \\
&+ \frac{9}{2}\beta\alpha^2 yu_y v_t + \frac{3}{2}\alpha^2 xu_x v_t + \frac{3}{2}\beta\alpha^2 xu_x v_t + 6tu_{xxx} v_t\alpha^2 + 3tu_t v_t\alpha^2 \\
&- \frac{9}{2}\alpha^2 yvu_{ty} - \frac{9}{2}\beta\alpha^2 yvu_{ty} + \frac{3}{2}\alpha^2 xvu_{tx} + \frac{3}{2}\beta\alpha^2 xvu_{tx} - 36\alpha^2 tuvu_{tx} \\
&- \frac{3}{2}\alpha^2 tvu_{tt} - \frac{3}{2}\beta\alpha^2 tvu_{tt} - 3\beta\alpha vu_y - 3\alpha vu_y - 36\alpha tvu_y u_x + 36\alpha tuu_y v_x \\
&- 6\alpha tv_{xy} u_{xx} - 6\alpha tu_{xy} v_{xx} + 6\alpha tv_y u_{xxx} + 6\alpha tu_y v_{xxx} + 3\alpha tu_y v_t + 3\beta\alpha tu_y v_t \\
&+ 6tvu_{yy} + 6\beta tvu_{yy} - 3\beta^2 vu_x - 6\beta vu_x - 3vu_x + 6\beta^2 uv_x + 12\beta uv_x + 6uv_x \\
&- 36tuu_x v_x - 36\beta tuu_x v_x + 6tu_{xx} v_{xx} + 6\beta tu_{xx} v_{xx} - 6tv_x u_{xxx} - 6\beta tv_x u_{xxx} \\
&+ 2\beta v_{xxx} - 6tu_x v_{xxx} - 6\beta tu_x v_{xxx} + v_{xxx} + \frac{1}{2}\beta^3 v_t + \frac{3}{2}\beta^2 v_t + \frac{3}{2}\beta v_t \\
&- 3tu_x v_t - 6\beta tu_x v_t + \frac{1}{2}v_t - 3\beta^2 tvu_{tx} - 3tvu_{tx} - 6\beta tvu_{tx} - 9\beta\alpha tvu_{ty} \\
&- 6\alpha^2 tu_{xx} v_{tx} + 3\beta\alpha^2 tu_t v_t + 54\alpha^2 yuu_y v_x + \beta^2 v_{xxx} - 9\alpha tvu_{ty} \\
&- 36\alpha tuvu_{xy} - 6\alpha^2 tv_{xx} u_{tx} - 3\beta^2 tu_x v_t,
\end{aligned}$$

$$\begin{aligned}
C_4^y =& 3\alpha^3 uv_t - \frac{9}{4}\alpha^4 yvu_{tt} - 6\alpha^3 vu_t + \frac{9}{2}\alpha^3 yu_y v_t + \frac{3}{2}\alpha^3 xu_x v_t + 3\alpha^3 tu_t v_t \\
&- \frac{3}{2}\alpha^3 xvu_{tx} - \frac{9}{2}\alpha^3 tvu_{tt} + 54\alpha^2 yvu_x^2 + 12\alpha^2 vu_y - 6\alpha^2 uv_y - 9\alpha^2 yu_y v_y \\
&+ 3\alpha^2 xvu_{xy} + 54\alpha^2 yuvu_{xx} + 9\alpha^2 yu_{xx} v_{xx} - 6\alpha^2 tv_y u_t + 3\alpha^2 tu_y v_t - 6t\beta vu_{xy} \\
&+ 9\alpha^2 tvu_{ty} + 9\alpha^2 yvu_{tx} + 9\beta\alpha^2 yvu_{tx} + 36\alpha tvu_x^2 - 6\alpha tu_y v_y + 3\beta\alpha vu_x \\
&+ 36\alpha tuvu_{xx} + 6\alpha tu_{xx} v_{xx} + \frac{1}{2}\beta^2\alpha v_t + \beta\alpha v_t - 3\alpha tu_x v_t - 3t\beta\alpha u_x v_t + \frac{1}{2}\alpha v_t \\
&+ 9\alpha tvu_{tx} + 9\beta\alpha tvu_{tx} - \beta^2 v_y - 2\beta v_y - v_y + 6tv_y u_x + 6\beta tv_y u_x - 6tvu_{xy} \\
&- 3\alpha^2 xv_y u_x + \frac{9}{2}\alpha^3 yvu_{ty} + 3\alpha vu_x.
\end{aligned}$$

Case 5. Finally for the symmetry

$$\mathbf{X_5} = \left(24\alpha t^2 + 12\alpha^2 ty\right)\frac{\partial}{\partial t} + \left(12\alpha tx - 12\beta ty - 12ty + 6\alpha^2 xy\right)\frac{\partial}{\partial x}$$

$$+\left(12\alpha ty-12t^2+9\alpha^2y^2\right)\frac{\partial}{\partial y}+\left(2\alpha\beta x-24\alpha tu-12\alpha^2uy\right.$$
$$\left.+2\alpha x-2\beta^2y-4\beta y-2y\right)\frac{\partial}{\partial u},$$

the Lie characteristic functions are

$$\begin{aligned}W^1=&\left(2\alpha\beta x-24\alpha tu-12\alpha^2uy+2\alpha x-2\beta^2y-4\beta y-2y\right)-\left(24\alpha t^2+12\alpha^2ty\right)u_t\\&-\left(12\alpha tx-12\beta ty-12ty+6\alpha^2xy\right)u_x-\left(12\alpha ty-12t^2+9\alpha^2y^2\right)u_y,\end{aligned}$$

and

$$\begin{aligned}W^2=&\left(2\alpha\beta x-24\alpha tv-12\alpha^2vy+2\alpha x-2\beta^2y-4\beta y-2y\right)-\left(24\alpha t^2+12\alpha^2ty\right)v_t\\&-\left(12\alpha tx-12\beta ty-12ty+6\alpha^2xy\right)v_x-\left(12\alpha ty-12t^2+9\alpha^2y^2\right)v_y.\end{aligned}$$

Hence the conserved vector corresponding to the Lie point symmetry $\mathbf{X_5}$ is

$$\begin{aligned}C_5^t=&6\alpha^4yvu_t-3\alpha^4yuv_t-\frac{9}{4}\alpha^4y^2u_yv_t-\frac{3}{2}\alpha^4xyu_xv_t-3\alpha^4tyu_tv_t\\&-12\alpha^3yvu_y+6\alpha^3yuv_y+\frac{9}{2}\alpha^3y^2u_yv_y-\frac{9}{2}\alpha^3y^2vu_{yy}+3\alpha^3xyv_yu_x\\&+6\alpha^3tyv_yu_t+\frac{1}{2}\alpha^3xv_t+\frac{1}{2}\alpha^3\beta xv_t-6\alpha^3tuv_t-3\alpha^3tyu_yv_t-3\alpha^3txu_xv_t\\&+9\alpha^3tyvu_{ty}+3\alpha^3txvu_{tx}+72\alpha^2tyvu_x^2-24\alpha^2tvu_y-\alpha^2xv_y-\alpha^2\beta xv_y\\&+6\alpha^2tyu_yv_y-18\alpha^2tyvu_{yy}-12\alpha^2yvu_x-12\alpha^2\beta yvu_x+6\alpha^2txv_yu_x\\&+6\alpha^2\beta yuv_x+\frac{9}{2}\alpha^2y^2u_yv_x+\frac{9}{2}\alpha^2\beta y^2u_yv_x+3\alpha^2xyu_xv_x+3\alpha^2\beta xyu_xv_x\\&-6\alpha^2txvu_{xy}-\frac{9}{2}\alpha^2\beta y^2vu_{xy}-3\alpha^2xyvu_{xx}-3\alpha^2\beta xyvu_{xx}+72\alpha^2tyuvu_{xx}\\&+12\alpha^2tyu_{xx}v_{xx}+12\alpha^2t^2v_yu_t+6\alpha^2tyv_xu_t+6\alpha^2\beta tyv_xu_t-\frac{1}{2}\alpha^2\beta^2yv_t\\&-\alpha^2\beta yv_t+3\alpha^2t^2u_yv_t+3\alpha^2tyu_xv_t+3\alpha^2\beta tyu_xv_t+9\alpha^2t^2vu_{ty}+3\alpha^2tyvu_{tx}\\&+144\alpha t^2vu_x^2+\alpha\beta^2yv_y+\alpha yv_y+2\alpha\beta yv_y-6\alpha t^2u_yv_y-18\alpha t^2vu_{yy}\\&-6\alpha tyv_yu_x-6\alpha\beta tyv_yu_x-\beta^2\alpha xv_x-\alpha xv_x-2\alpha\beta xv_x+12\alpha tuv_x\\&+6\alpha\beta tyu_yv_x+6\alpha txu_xv_x+6\alpha\beta txu_xv_x-6\alpha txvu_{xx}-6\alpha\beta txvu_{xx}\\&+24\alpha t^2u_{xx}v_{xx}+12\alpha t^2v_xu_t+12\alpha\beta t^2v_xu_t+12\alpha t^2vu_{tx}+12\alpha\beta t^2vu_{tx}\\&+yv_x+3\beta yv_x-6t^2u_yv_x-6\beta t^2u_yv_x-6\beta^2tyu_xv_x-6tyu_xv_x-12\beta tyu_xv_x\\&+6t^2vu_{xy}+6\beta t^2vu_{xy}+6\beta^2tyvu_{xx}+6tyvu_{xx}+12\beta tyvu_{xx}-12\alpha\beta tvu_x\end{aligned}$$

$$+\frac{3}{2}\alpha^4xyvu_{tx}+12\alpha^3tvu_t-6\alpha^3t^2u_tv_t+12\alpha^2tuv_y-\frac{9}{2}\alpha^2y^2vu_{xy}$$
$$+3\alpha^2\beta tyvu_{tx}+3\beta^2yv_x+144\alpha t^2uvu_{xx}-\frac{1}{2}\alpha^2yv_t-12\alpha tvu_x$$
$$+6\alpha^2yuv_x+\beta^3yv_x-3\alpha^3xyvu_{xy}+6\alpha tyu_yv_x+\frac{9}{4}\alpha^4y^2vu_{ty}+12\beta\alpha tuv_x,$$

$$C_5^x=6\alpha^3xyvu_{ty}-\frac{3}{2}\alpha^4xyvu_{tt}-3\alpha^3txvu_{tt}-6\alpha^2xyvu_{yy}-180\alpha^2yuvu_x$$
$$+72\alpha^2yu^2v_x+54\alpha^2y^2uu_yv_x+36\alpha^2xyuu_xv_x-54\alpha^2y^2uvu_{xy}-6\alpha^2yv_xu_{xx}$$
$$-9\alpha^2y^2v_{xy}u_{xx}-18\alpha^2yu_xv_{xx}-9\alpha^2y^2u_{xy}v_{xx}-6\alpha^2xyu_{xx}v_{xx}$$
$$+12\alpha^2yuv_{xxx}+9\alpha^2y^2u_yv_{xxx}+6\alpha^2xyu_xv_{xxx}-12\alpha^2yvu_t-12\alpha^2\beta yvu_t$$
$$+72\alpha^2tyuv_xu_t+12\alpha^2tyv_{xxx}u_t+6\alpha^2yuv_t+6\alpha^2\beta yuv_t+\frac{9}{2}\alpha^2y^2u_yv_t$$
$$+3\alpha^2xyu_xv_t+3\alpha^2\beta xyu_xv_t+12\alpha^2tyu_{xxx}v_t+6\alpha^2tyu_tv_t+6\alpha^2\beta tyu_tv_t$$
$$+12\alpha^2txvu_{ty}-\frac{9}{2}\alpha^2\beta y^2vu_{ty}+3\alpha^2xyvu_{tx}+3\alpha^2\beta xyvu_{tx}-72\alpha^2tyuvu_{tx}$$
$$-12\alpha^2tyu_{xx}v_{tx}-3\alpha^2tyvu_{tt}-3\alpha^2\beta tyvu_{tt}-6\alpha yvu_y-6\alpha\beta yvu_y$$
$$+6\alpha xvu_x+6\alpha\beta xvu_x-360\alpha tuvu_x-72\alpha tyvu_yu_x+144tu^2\alpha v_x-12\alpha xuv_x$$
$$+72\alpha tyuu_yv_x+72\alpha txuu_xv_x-72\alpha tyuvu_{xy}-12\alpha tv_xu_{xx}-12\alpha tyv_{xy}u_{xx}$$
$$-36\alpha tu_xv_{xx}-12\alpha tyu_{xy}v_{xx}-12\alpha txu_{xx}v_{xx}+2\alpha v_{xx}+12\alpha tyv_yu_{xxx}$$
$$-2\alpha xv_{xxx}-2\alpha\beta xv_{xxx}+24\alpha tuv_{xxx}+12\alpha tyu_yv_{xxx}+12\alpha txu_xv_{xxx}$$
$$-144\alpha t^2vu_xu_t+144\alpha t^2uv_xu_t+24\alpha t^2v_{xxx}u_t-\alpha\beta^2xv_t-\alpha xv_t$$
$$+12\alpha\beta tuv_t+6\alpha tyu_yv_t+6\alpha\beta tyu_yv_t+6\alpha txu_xv_t+6\alpha\beta txu_xv_t$$
$$+12\alpha\beta t^2u_tv_t-18\alpha tyvu_{ty}-18\alpha\beta tyvu_{ty}+6\alpha txvu_{tx}+6\alpha\beta txvu_{tx}$$
$$-24\alpha t^2v_{xx}u_{tx}-24\alpha t^2u_{xx}v_{tx}-12\alpha t^2vu_{tt}-12\alpha\beta t^2vu_{tt}+12tvu_y$$
$$+12\beta tyvu_{yy}-6\beta^2yvu_x-6yvu_x-12\beta yvu_x+72t^2vu_yu_x+12\beta^2yuv_x$$
$$-72t^2uu_yv_x-72tyuu_xv_x-72\beta tyuu_xv_x+72t^2uvu_{xy}+12t^2v_{xy}u_{xx}$$
$$+12t^2u_{xy}v_{xx}+12tyu_{xx}v_{xx}+12\beta tyu_{xx}v_{xx}-12t^2v_yu_{xxx}-12tyv_xu_{xxx}$$
$$-12\beta tyv_xu_{xxx}+2\beta^2yv_{xxx}+2yv_{xxx}+4\beta yv_{xxx}-12t^2u_yv_{xxx}$$
$$+\beta^3yv_t+3\beta^2yv_t+yv_t+3\beta yv_t-6t^2u_yv_t-6\beta t^2u_yv_t-6\beta^2tyu_xv_t$$
$$+6t^2vu_{ty}+6\beta t^2vu_{ty}-6\beta^2tyvu_{tx}-6tyvu_{tx}-12\beta tyvu_{tx}$$

$$
\begin{aligned}
&- 54\alpha^2y^2vu_yu_x + 6\alpha^2xyv_xu_{xxx} - 72\alpha^2tyvu_xu_t + \frac{9}{2}\alpha^2\beta y^2u_yv_t - \frac{9}{2}\alpha^2y^2vu_{ty}\\
&- 12\alpha^2tyv_{xx}u_{tx} - 12\alpha\beta xuv_x + 12\alpha txv_xu_{xxx} - 12\beta tyu_xv_{xxx} + 12\alpha t^2u_tv_t\\
&- 12tyu_xv_{xxx} + 24\beta yuv_x - 36\alpha\beta tvu_t - 12\beta tyu_xv_t + 12tyvu_{yy}\\
&+ 2\beta\alpha v_{xx} + 12\alpha tuv_t - 144\alpha t^2uvu_{tx} - 36\alpha tvu_t - 12\alpha txvu_{yy} - 6tyu_xv_t\\
&+ 12yuv_x + 12\beta tvu_y - 2\alpha\beta xv_t + 24\alpha t^2u_{xxx}v_t + 9\alpha^2y^2v_yu_{xxx},
\end{aligned}
$$

$$
\begin{aligned}
C_5^y =\,& 6\alpha^3yuv_t - \frac{9}{4}\alpha^4y^2vu_{tt} - 12\alpha^3yvu_t + \frac{9}{2}\alpha^3y^2u_yv_t + 3\alpha^3xyu_xv_t\\
&+ \frac{9}{2}\alpha^3y^2vu_{ty} - 3\alpha^3xyvu_{tx} - 9\alpha^3tyvu_{tt} + 54\alpha^2y^2vu_x^2 + 24\alpha^2yvu_y\\
&- 9\alpha^2y^2u_yv_y - 6\alpha^2xyv_yu_x + 6\alpha^2xyvu_{xy} + 54\alpha^2y^2uvu_{xx} + 9\alpha^2y^2u_{xx}v_{xx}\\
&- 12\alpha^2tyv_yu_t - \alpha xv_t^2 - \alpha^2\beta xv_t + 12\alpha^2tuv_t + 6\alpha^2tyu_yv_t + 6\alpha^2txu_xv_t\\
&+ 12\alpha^2t^2u_tv_t + 18\alpha^2tyvu_{ty} + 9\alpha^2y^2vu_{tx} - 6\alpha^2txvu_{tx} + 9\alpha^2\beta y^2vu_{tx}\\
&+ 72\alpha tyvu_x^2 + 48\alpha tvu_y + 2\alpha xv_y + 2\alpha\beta xv_y - 24\alpha tuv_y - 12\alpha tyu_yv_y\\
&+ 6\alpha\beta yvu_x - 12\alpha txv_yu_x + 12\alpha txvu_{xy} + 72\alpha tyuvu_{xx} + 12\alpha tyu_{xx}v_{xx}\\
&+ \alpha\beta^2yv_t + \alpha yv_t + 2\alpha\beta yv_t - 6\alpha t^2u_yv_t - 6\alpha tyu_xv_t - 6\alpha\beta tyu_xv_t\\
&+ 18\alpha tyvu_{tx} + 18\alpha\beta tyvu_{tx} - 72t^2vu_x^2 + 2\beta^2v + 4\beta v + 2v - 2\beta^2yv_y\\
&- 4y\beta v_y + 12t^2u_yv_y - 12tvu_x - 12\beta tvu_x + 12tyv_yu_x + 12\beta tyv_yu_x\\
&- 12\beta tyvu_{xy} - 72t^2uvu_{xx} - 12t^2u_{xx}v_{xx} - 12t^2vu_{tx} - 12\beta t^2vu_{tx}\\
&+ 6\alpha^3tyu_tv_t - 12\alpha^2yuv_y - 24\alpha^2tvu_t - 9\alpha^2t^2vu_{tt} + 6\alpha yvu_x - 24\alpha t^2v_yu_t\\
&+ 18\alpha t^2vu_{ty} - 2yv_y - 12tyvu_{xy}.
\end{aligned}
$$

3.4 Concluding Remarks

In this Chapter we looked at the newly introduced extended Kadomtsev-Petviashvili (eKP) model from the literature [54]. This equation can be used in a variety of academic disciplines, including physics and mechanics. We computed the Lie point symmetries of the eKP equation using the theory of Lie groups. In addition, we applied the obtained symmetries to carry out symmetry reductions and obtained closed-form solutions of the underlying equation. The solutions found in this study were in terms of incomplete elliptic integral and exponential functions.

Additionally, for specific parametric values, the graphic display of some solutions was given. Finally we used Ibragimov's theorem to construct conservation laws for the model under study.

Chapter 4

Solutions and conservation laws of the generalized three-dimensional KP-BBM equation

In this Chapter we study a generalized (3+1)-dimensional Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation. We perform symmetry reductions and find exact solutions. Moreover, we derive conservation laws by employing two approaches; the multiplier method and Noether's theorem.

4.1 Introduction

It is argued that Benjamin-Bona-Mahony (BBM) equation is a model that is significantly preferable due to the fact that it precludes some knotty aspects of the KdV equation [65]. Wazwaz [66] suggested two variants of the BBM equation that were contrived in the Kadomtsev-Petviashvili (KP) sense. These equations are usually called KP-BBM equation and they read

$$\big(u_t + u_x - p(u^2)_x - qu_{txx}\big)_x + su_{yy} = 0 \tag{4.1}$$

and

$$(u_t + u_x - p(u^n)_x - q(u)^n_{txx})_x + su_{yy} = 0, \tag{4.2}$$

where we have p, q, s as real parameters as well as n denoting a positive integer [66]. Equations (4.1) and (4.2) have been investigated by various researchers using diverse techniques. In [66], Wazwaz obtained compactons, solitons, periodic solutions and solitary patterns by the tan-hyperbolic as well as sine-cosine approach. Crank-Nicholson finite difference technique was applied to construct approximate solutions [67]. In [68], Abdou utilized the extended mapping technique to find periodic solutions. Moreover, in [69, 70] the authors applied a bifurcation approach to gain various solitary waves of the equations. Lately, Kalim [71] contemplated a (3+1)-dimensional KP-BBM equation presented as

$$u_{tx} + au_{xx} + b\left(uu_x\right)_x - cu_{txxx} + du_{yy} + eu_{zz} = 0 \tag{4.3}$$

with parameters

$$a = L_x^2,\ b = \Delta/D,\ c = \left(DL_x/L\right)^2,\ d = L_y^2,\ e = L_z^2,\ \text{and}\ L = \sqrt{L_x^2 + L_y^2 + L_z^2}$$

with L_x, L_y and L_z, denoting the wavelength in the direction of x, y and z, separately. In addition, $u(t, x, y, z)$ depicts the wave amplitude function having temporal coordinate t alongside propagation distance x, y and z. Meanwhile, Δ connotes the wave amplitude along with D in the depth of water [72].

In this work, a generalized (3+1)-dimensional KP-BBM equation will be investigated. The equation reads [73]

$$u_{tx} + \alpha u_y u_{xx} + \beta u_x u_{xy} - bu_{txxx} + au_{xx} + cu_{yy} + du_{zz} = 0, \tag{4.4}$$

where α, β, a, b, c and d are non-zero constants. Xie and Li in [73], investigated a generalized (3+1)-dimensional KP-BBM equation (4.4), where they chose the nonlinear convection term as $u_x u_y$ that can be utilized to depict more dispersion effects. This in turn makes the equation more useful and meaningful than the two-dimensional case.

Using the Lie symmetry technique, we carry out symmetry reductions of equation (4.4). Thus, we acquire nonlinear differential equations (NODEs), which are to be solved by direct integration. In addition, the simplest equation method is utilized to construct some analytical solutions of (4.4). Thereafter, the two methods, namely the multiplier approach and Noether's theorem are used to derive conservation laws for the underlying equation.

4.2 Exact solutions of (4.4)

In this section we present exact solutions of the KP-BBM (4.4) by applying Lie symmetry method.

4.2.1 Lie point symmetries of (4.4)

We begin by determining the Lie point symmetries of (4.4). The symmetry group is generated by the vector field

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial z}, \tag{4.5}$$

where $(\tau, \xi, \phi, \psi, \eta)$ are functions of t, x, y, z and u. Then, the vector field X is said to be a point symmetry of equation (4.4), if

$$X^{[4]}\Delta\Big|_{\Delta=0} = 0, \tag{4.6}$$

with

$$\Delta \equiv u_{tx} + \alpha u_y u_{xx} + \beta u_x u_{xy} - b u_{txxx} + a u_{xx} + c u_{yy} + d u_{zz}.$$

Here $X^{[4]}$ is the fourth prolongation of X and can be derived from the general formula. Expanding the determining equation (4.6), we have

$$\zeta_{tx} + \alpha u_{xx}\zeta_y + \alpha u_y\zeta_{xx} + \beta u_{xy}\zeta_x + \beta u_x\zeta_{xy} - b\zeta_{txxx} + a\zeta_{xx} + c\zeta_{yy} + d\zeta_{zz}\Big|_{\Delta=0} = 0, \tag{4.7}$$

where the ζs are defined as

$$\zeta_x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi) - u_z D_x(\phi),$$
$$\zeta_y = D_y(\eta) - u_t D_y(\tau) - u_x D_y(\xi) - u_y D_y(\psi) - u_z D_y(\phi),$$

$$\zeta_{xy} = D_y(\zeta_x) - u_{tx}D_y(\tau) - u_{xx}D_y(\xi) - u_{xy}D_y(\psi) - u_{yz}D_y(\phi),$$

$$\zeta_{tx} = D_x(\zeta_t) - u_{tt}D_x(\tau) - u_{tx}D_x(\xi) - u_{ty}D_x(\psi) - u_{tz}D_t(\phi),$$

$$\zeta_{xx} = D_x(\zeta_x) - u_{tx}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\psi) - u_{xz}D_x(\phi),$$

$$\zeta_{yy} = D_y(\zeta_y) - u_{ty}D_y(\tau) - u_{xy}D_y(\xi) - u_{yy}D_y(\psi) - u_{yz}D_y(\phi),$$

$$\zeta_{zz} = D_z(\zeta_z) - u_{tz}D_z(\tau) - u_{xz}D_z(\xi) - u_{yz}D_z(\psi) - u_{zz}D_z(\phi),$$

$$\zeta_{txxx} = D_x(\zeta_{txx}) - u_{ttxx}D_x(\tau) - u_{txxx}D_x(\xi) - u_{txxy}D_x(\psi) - u_{txxz}D_x(\phi),$$

with the total derivatives D_t, D_x, D_y and D_z written as

$$\begin{aligned} D_t =& \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots , \\ D_x =& \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots , \\ D_y =& \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y} + u_{ty}\frac{\partial}{\partial u_t} + \cdots , \\ D_z =& \frac{\partial}{\partial z} + u_z\frac{\partial}{\partial u} + u_{tz}\frac{\partial}{\partial u_t} + u_{xz}\frac{\partial}{\partial u_x} + \cdots . \end{aligned} \tag{4.8}$$

Substituting the respective ζs in equation (4.7) and splitting the resulting equation on different derivatives of u, we obtain the determining PDEs

$$\tau_x = 0,\ \tau_y = 0,\ \tau_z = 0,\ \tau_u = 0, \tag{4.9}$$

$$\xi_t = 0,\ \xi_x = 0,\ \xi_y = 0,\ \xi_z = 0,\ \xi_u = 0, \tag{4.10}$$

$$\phi_t = 0,\ \phi_x = 0,\ \phi_y = 0,\ \phi_u = 0, \tag{4.11}$$

$$\psi_t = 0,\ \psi_x = 0,\ \psi_z = 0,\ \psi_u = 0, \tag{4.12}$$

$$\phi_{zz} = 0, \tag{4.13}$$

$$\psi_y - \phi_z = 0, \tag{4.14}$$

$$\tau_t - 2\phi_z = 0, \tag{4.15}$$

$$\eta_u + \phi_z = 0, \tag{4.16}$$

$$\eta_x = 0, \tag{4.17}$$

$$\eta_{zz} = 0, \tag{4.18}$$

$$\alpha\eta_y + 2a\,\phi_z = 0. \tag{4.19}$$

Equations (4.9) yield

$$\tau = A(t), \tag{4.20}$$

where $A(t)$ is an arbitrary function depending on t. Integrating equations (4.10) we obtain

$$\xi(t,x,y,z) = c_1, \tag{4.21}$$

where c_1 is an arbitrary constant. Solving equations (4.11) gives

$$\phi = B(z), \tag{4.22}$$

where B is an arbitrary function depending on z. On integrating equation (4.12), we get

$$\psi = C(y), \tag{4.23}$$

where C is an arbitrary function depending on y. Inserting (4.22) into (4.13) we obtain

$$B''(z) = 0 \tag{4.24}$$

and integrating the equation twice with respect to z yields

$$B(z) = c_2 z + c_3, \tag{4.25}$$

where c_2 and c_3 are arbitrary constants of integration. Thus

$$\phi = c_2 z + c_3. \tag{4.26}$$

Substituting the values of ϕ and ψ into equation (4.14) gives $C'(y) - c_2 = 0$, which has the solution

$$C(y) = c_2 y + c_4, \tag{4.27}$$

where c_4 is arbitrary constant of integration. Hence we have

$$\psi = c_2\, y + c_4. \tag{4.28}$$

Now, substituting the values of τ and ϕ into equation (4.15), we get $A'(t) - 2c_2 = 0$, whose solution is

$$A(t) = 2c_2 t + c_5,$$

where c_5 is the constant of integration. Therefore (4.20) becomes

$$\tau = 2c_2 t + c_5. \tag{4.29}$$

Differentiating(4.26) with respect to z and substituting the result into (4.16) gives

$$\eta_u = -\, c_2. \tag{4.30}$$

Integrating (4.30) with respect to u yields

$$\eta = -c_2\, u + E(t, x, y, z), \tag{4.31}$$

where E is a function of its arguments. Inserting (4.31) into equation (4.17) gives $E_x = 0$, thus $E = E(t, y, z)$. Therefore, equation (4.31) becomes

$$\eta = -c_2\, u + E(t, y, z), \tag{4.32}$$

where the arbitrary function E depends on (t, y, z). Substituting the value of η from (4.32) into (4.18), we obtain

$$E_{zz} = 0. \tag{4.33}$$

Integrating (4.33) twice with respect z, we get

$$E(t, y, z) = F(t, y)z + G(t, y), \tag{4.34}$$

where F and G are arbitrary functions depending on t and y. Thus, we have

$$\eta = -c_2\, u + F(t, y)z + G(t, y). \tag{4.35}$$

Substituting the derivatives of η and ϕ with respect to y and z respectively into equation (4.19) yields

$$\alpha z F_y + \alpha G_y + 2ac_2 = 0. \tag{4.36}$$

Now, since the functions F and G are independent of z, we split (4.36) on powers of z and this gives

$$z:\ F_y = 0, \tag{4.37}$$

$$\text{Rest}:\ \alpha G_y + 2ac_2 = 0. \tag{4.38}$$

Equation (4.37) implies that $F = F(t)$. Solving (4.38), we obtain

$$G(t, y) = -\frac{2}{\alpha} a\, c_2\, y + H(t), \tag{4.39}$$

where H is an arbitrary function of t. Hence, equation (4.35) becomes

$$\eta = -c_2\, u - \frac{2}{\alpha} a\, c_2\, y + F(t)z + H(t). \tag{4.40}$$

Thus, the infinitesimals of equation (4.4) are obtained as

$$\begin{aligned}
\tau =& 2c_2 t + c_5,\\
\xi =& c_1,\\
\phi =& c_2 z + c_3,\\
\psi =& c_2\, y + c_4,\\
\eta =& -\, c_2\, u + F(t)z - \frac{2}{\alpha} a\, c_2\, y + H(t).
\end{aligned} \tag{4.41}$$

Consequently, the Lie point symmetries of equation (4.4) are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= \frac{\partial}{\partial y},\\
X_4 &= \frac{\partial}{\partial z},\\
X_5 &= 2\alpha t\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial y} + \alpha z\frac{\partial}{\partial z} - (2a\, y + \alpha u)\frac{\partial}{\partial u},\\
X_H &= H(t)\frac{\partial}{\partial u},\\
X_F &= zF(t)\frac{\partial}{\partial u}.
\end{aligned} \tag{4.42}$$

4.2.2 Symmetry reductions of (4.4)

Case 1. Consider the combination of translation symmetries X_1, X_2, X_3 and X_4 as

$$X = X_1 + X_2 + X_3 + \mu X_4,$$

where μ is a constant. The symmetry X gives the invariants

$$p = x - t,\ q = y - t,\ k = z - \mu t,$$

which lead to the group invariant solution $u(t,x,y,z) = G(p,q,k)$. This then transforms (4.4) into the NLPDE

$$
\begin{aligned}
&aG_{pp} - G_{pq} - \mu G_{kp} + \alpha G_q G_{pp} + \beta G_p G_{pq} + bG_{pppp} + bG_{pppq} + b\mu G_{pppk} \\
&- G_{pp} + cG_{qq} + dG_{kk} = 0. \qquad (4.43)
\end{aligned}
$$

Equation (4.43) have five symmetries namely,

$$
\Gamma_1 = \frac{\partial}{\partial p},\ \Gamma_2 = \frac{\partial}{\partial q},\ \Gamma_3 = \frac{\partial}{\partial k},\ \Gamma_4 = \frac{\partial}{\partial G},\ \Gamma_5 = k\frac{\partial}{\partial G}.
$$

Similarly, characteristic equations of the symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \gamma\Gamma_3$, where γ is a constant, provide the invariants $f = q - p$, $g = k - \gamma p$ and $G(p,q,k) = U(f,g)$. Hence, equation (4.43) transforms into

$$
\begin{aligned}
&2\alpha\gamma U_f U_{fg} + \alpha\gamma^2 U_f U_{gg} + \beta\gamma U_f U_{fg} + \beta\gamma U_g U_{ff} + \beta\gamma^2 U_g U_{fg} - 3b\mu\gamma U_{ffgg} \qquad (4.44) \\
&- 3b\mu\gamma^2 U_{fggg} - b\mu\gamma^3 U_{gggg} - \gamma^2 U_{gg} + \mu U_{fg} + aU_{ff} - \gamma U_{fg} + cU_{ff} + dU_{gg} \\
&+ b\gamma^4 U_{gggg} + a\gamma^2 U_{gg} + \beta U_f U_{ff} + 2a\gamma U_{fg} + \mu\gamma U_{gg} + \alpha U_f U_{ff} + 3b\gamma^2 U_{ffgg} \\
&+ b\gamma U_{fffg} + 3b\gamma^3 U_{fggg} - b\mu U_{fffg} = 0.
\end{aligned}
$$

Equation (4.44) gives three symmetries

$$
R_1 = \frac{\partial}{\partial f},\ R_2 = \frac{\partial}{\partial g},\ R_3 = \frac{\partial}{\partial U}.
$$

Furthermore, the combination $R_1 + \theta R_2$ with θ as a constant, gives invariant solution $U(f,g) = H(r)$ with $r = g - \theta f$. Finally, substituting $U(f,g) = H(r)$ into (4.44), transforms (4.44) into a NLODE

$$
\begin{aligned}
&2\alpha\gamma\theta^2 H'H'' - \alpha\gamma^2\theta H'H'' + 2\beta\gamma\theta^2 H'H'' - \beta\gamma^2\theta H'H'' - 3b\mu\gamma\theta^2 H'''' \qquad (4.45) \\
&+ 3b\mu\gamma^2\theta H'''' - b\mu\gamma^3 H'''' - \gamma^2 H'' - \mu\theta H'' + a\,\theta^2 H'' + \gamma\theta H'' + c\,\theta^2 H'' \\
&+ dH'' + b\gamma^4 H'''' + a\gamma^2 H'' - \beta\theta^3 H'H'' - 2a\gamma\theta H'' + \mu\gamma H'' - \alpha\theta^3 H'H'' \\
&+ 3b\gamma^2\theta^2 H'''' - b\gamma\theta^3 H'''' - 3b\gamma^3\theta H'''' + b\mu\theta^3 H'''' = 0.
\end{aligned}
$$

Equation (4.45) simplifies to

$$
(2\alpha\gamma\theta^2 - \alpha\gamma^2\theta + 2\beta\gamma\theta^2 - \beta\gamma^2\theta - \beta\theta^3 - \alpha\theta^3)H'H'' \qquad (4.46)
$$

$$+ (a\gamma^2 - 2a\gamma\theta + \mu\gamma + d - \gamma^2 - \mu\theta + a\,\theta^2 + \gamma\theta + c\,\theta^2)H''$$
$$+ (b\gamma^4 + 3b\mu\gamma^2\theta - b\mu\gamma^3 - 3b\mu\gamma\theta^2 + 3b\gamma^2\theta^2 - b\gamma\theta^3 - 3b\gamma^3\theta + b\mu\theta^3)H'''' = 0.$$

Thus, we have

$$\mathcal{A}H'H'' + \mathcal{B}H'' + \mathcal{C}H'''' = 0, \tag{4.47}$$

where

$$\mathcal{A} = 2\alpha\gamma\theta^2 - \alpha\gamma^2\theta + 2\beta\gamma\theta^2 - \beta\gamma^2\theta - \beta\theta^3 - \alpha\theta^3,$$
$$\mathcal{B} = a\gamma^2 - 2a\gamma\theta + \mu\gamma + d - \gamma^2 - \mu\theta + a\,\theta^2 + \gamma\theta + c\,\theta^2,$$
$$\mathcal{C} = b\gamma^4 + 3b\mu\gamma^2\theta - b\mu\gamma^3 - 3b\mu\gamma\theta^2 + 3b\gamma^2\theta^2 - b\gamma\theta^3 - 3b\gamma^3\theta + b\mu\theta^3.$$

4.2.3 Solution by direct integration

In this section, we seek to find the general solution of the KP-BBM equations (4.4) by directly integrating (4.47). Integrating (4.47) with respect to r yields

$$\frac{\mathcal{A}}{2}H'^2 + \mathcal{B}H' + \mathcal{C}H''' + K_0 = 0, \tag{4.48}$$

where K_0 is an arbitrary constant of integration. Let $H'(r) = \Phi(r)$, then equation (4.48) becomes

$$\frac{\mathcal{A}}{2}\Phi^2 + \mathcal{B}\Phi + \mathcal{C}\Phi'' + K_0 = 0. \tag{4.49}$$

To integrate (4.49), we first multiply by $\Phi'(r)$. Thereafter, we get

$$\frac{\mathcal{A}}{2}\Phi^2\Phi' + \mathcal{B}\Phi\Phi' + \mathcal{C}\Phi''\Phi' + K_0\Phi' = 0. \tag{4.50}$$

Integrating (4.50) with respect to r yields

$$\frac{\mathcal{C}}{2}\Phi'^2 + \frac{\mathcal{A}}{6}\Phi^3 + \frac{\mathcal{B}}{2}\Phi^2 + K_0\Phi + K_1 = 0, \tag{4.51}$$

where K_1 is an arbitrary constant. Then,

$$\Phi'^2 + \frac{\mathcal{A}}{3\mathcal{C}}\Phi^3 + \frac{\mathcal{B}}{\mathcal{C}}\Phi^2 + \frac{2}{\mathcal{C}}K_0\Phi + \frac{2}{\mathcal{C}}K_1 = 0. \tag{4.52}$$

Suppose that m_1, m_2 and m_3 are real roots ($m_1 > m_2 > m_3$) of a cubic equation

$$\Phi^3 + \frac{3\mathcal{B}}{\mathcal{A}}\Phi^2 + \frac{6}{\mathcal{A}}K_0\Phi + \frac{6}{\mathcal{A}}K_1 = 0, \tag{4.53}$$

that satisfies the conditions

$$m_1m_2m_3 = -\frac{6}{\mathcal{A}}K_1,\ m_1m_2 + m_1m_3 + m_2m_3 = \frac{6}{\mathcal{A}}K_0,\ m_1 + m_2 + m_3 = -\frac{3\mathcal{B}}{\mathcal{A}}.$$

Then equation (4.52) is written as

$$\Phi'^2 = -\frac{\mathcal{A}}{3\mathcal{C}}(\Phi - m_1)(\Phi - m_2)(\Phi - m_3)$$

and has the solution [64]

$$\Phi(r) = m_2 + (m_1 - m_2)\mathrm{cn}^2\left\{\sqrt{\frac{\mathcal{A}(m_1 - m_2)}{12\mathcal{C}}}(r - r_0)\,\middle|\,K^2\right\}, K^2 = \frac{m_1 - m_2}{m_1 - m_3}, \quad (4.54)$$

where r_0 is a constant and cn is the Jacobi cosine function.

Thus, by integrating (4.54) and returning to the original variables, we obtain the solution of (4.4) as

$$u(t, x, y, z) = \mathcal{B}_0\left[\mathrm{EllipticE}\left\{\mathrm{sn}(\mathcal{B}_1(r - r_0), K^2), K^2\right\}\right] + \left\{m_2 - (m_1 - m_2)\frac{1 - K^4}{K^4}\right\} \times (r - r_0) + k_1, \quad (4.55)$$

where

$$\mathcal{B}_0 = \sqrt{\frac{12\mathcal{C}(m_1 - m_2)^2}{(m_1 - m_3)\mathcal{A}K^8}},\quad \mathcal{B}_1 = \sqrt{\frac{\mathcal{A}(m_1 - m_2)}{12\mathcal{C}}},$$

$r = (\gamma - \mu)t + (\theta - \gamma)x - \theta y + z$ and k_1 is an integration constant. EllipticE[r,k] is the incomplete elliptic integral [74].

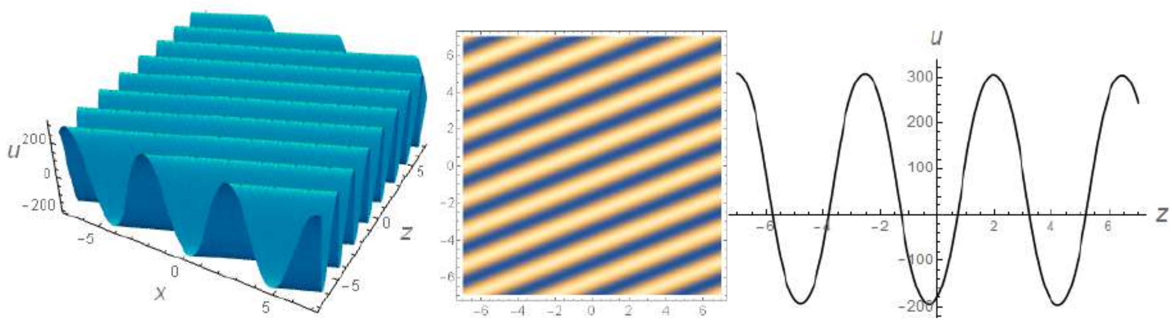

Figure 4.1: The 3D and 2D solution profiles of (4.55).

Figure 4.1 demonstrates the solution (4.55) for the values $\gamma = -4, \mu = 0.2, \theta = 0.6, t = -14, k_1 = 1, m_1 = 100, m_2 = 50.05, m_3 = -60, \mathcal{C} = 1, \mathcal{A} = 0.287$ and $r_0 = 0$.

4.2.4 Exact solutions of (4.4) using simplest equation method

In this subsection, we use the simplest equation method to solve the nonlinear fourth-order NLODE (4.47).

Solutions of (4.4) using the Riccati equation as the simplest equation

The balancing procedure yields $M = 1$. Thus, the solution of (4.47) from (1.37) is of the form

$$H(r) = a_0 + a_1 Y(r).$$

Inserting this value of $H(r)$ into (4.47) and making use of the Riccati equation (1.36) yields the following algebraic system of equations in terms of a_0 and a_1:

$$\mathcal{A}{a_1}^2 l^3 + 12\,\mathcal{C} a_1 l^4 = 0,$$

$$\mathcal{A}{a_1}^2 l^2 m + 12\,\mathcal{C} a_1 l^3 m = 0,$$

$$\mathcal{A}{a_1}^2 m n^2 + 8\,\mathcal{C} a_1 l m n^2 + \mathcal{C} a_1 m^3 n + \mathcal{B} a_1 m n = 0,$$

$$6\,\mathcal{A}{a_1}^2 l m n + \mathcal{A}{a_1}^2 m^3 + 60\,\mathcal{C} a_1 l^2 m n + 15\,\mathcal{C} a_1 l m^3 + 3\,\mathcal{B} a_1 l m = 0,$$

$$2\,\mathcal{A}{a_1}^2 l^2 n + 2\,\mathcal{A}{a_1}^2 l m^2 + 20\,\mathcal{C} a_1 l^3 n + 25\,\mathcal{C} a_1 l^2 m^2 + \mathcal{B} a_1 l^2 = 0,$$

$$2\,\mathcal{A}{a_1}^2 l n^2 + 2\,\mathcal{A}{a_1}^2 m^2 n + 16\,\mathcal{C} a_1 l^2 n^2 + 22\,\mathcal{C} a_1 l m^2 n + \mathcal{C} a_1 m^4 + 2\,\mathcal{B} a_1 l n + \mathcal{B} a_1 m^2 = 0.$$

Using Maple, we can solve the aforementioned system of algebraic equations and get

$$a_0 = a_0,\ a_1 = -12\,\frac{\mathcal{C} l}{\mathcal{A}},\ \mathcal{B} = 4\,\mathcal{C}(ln - m^2). \tag{4.56}$$

It follows that the solutions for the equation (4.4) using the Ricatti equation as the simplest equation are

$$u(t,x,y,z) = a_0 + a_1\left\{-\frac{m}{2l} - \frac{\omega}{2l}\tanh\left(-\frac{1}{2}\omega(r+D)\right)\right\} \tag{4.57}$$

and

$$u(t,x,y,z) = a_0 + a_1\left\{-\frac{m}{2l} - \frac{\omega}{2l}\tanh\left(\frac{\omega\, r}{2}\right) + \frac{\operatorname{sech}\left(\frac{\omega r}{2}\right)}{D\cosh\left(\frac{\omega r}{2}\right) - \frac{2\,l}{\omega}\sinh\left(\frac{\omega r}{2}\right)}\right\} \tag{4.58}$$

with $\omega = \sqrt{m^2 - 4ln}$, $r = (\gamma - \mu)t + (\theta - \gamma)x - \theta y + z$ and D a constant of integration. Figure 4.2 demonstrates the solution (4.57) for the values $\gamma = 0.2, \mu =$

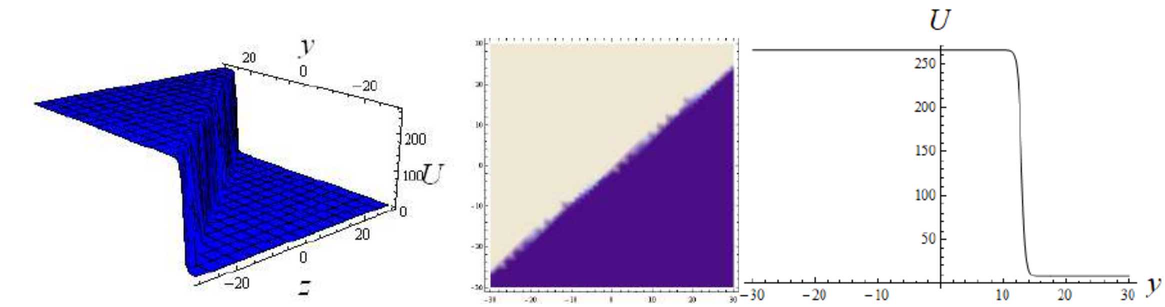

Figure 4.2: The 3D and 2D solution profiles of (4.57).

$0.3, \theta = 1.9, t = 6, \mathcal{C} = 1.9, D = 1.9, \mathcal{A} = 2, z = -1, m = 2, l = 0.09, a_0 = 10, a_1 = -11.4, \omega = 2.03$

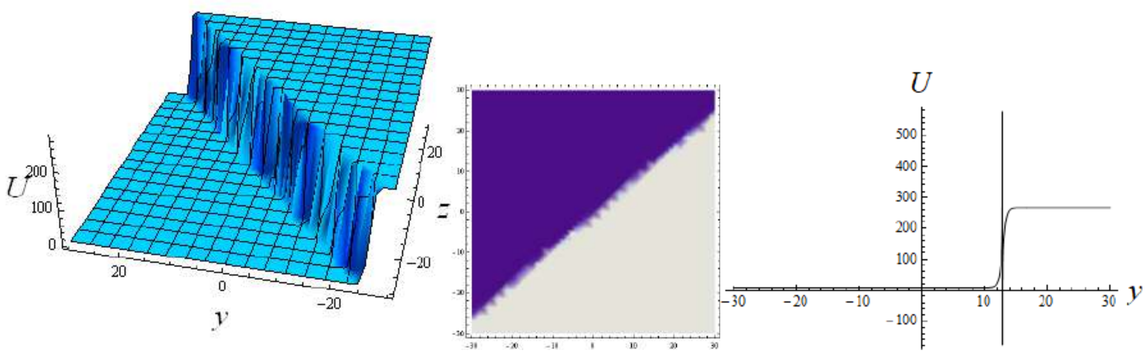

Figure 4.3: The 3D and 2D solution profiles of (4.58).

while Figure 4.3 demonstrates the solution (4.58) for the values $\gamma = 0.2, \mu = 0.3, \theta = 1.9, t = 6, \mathcal{C} = 1.9, D = 1.9, \mathcal{A} = 2, z = -1, m = 2, l = 0.09, a_0 = 10, a_1 = -11.4, \omega = 2.03$.

Solutions of (4.4) using the Bernoulli equation as the simplest equation.

In this case, the balancing procedure yields $M = 1$ and solutions of (4.47) are of the form

$$H(r) = a_0 + a_1 Y(r).$$

The insertion of the expression for $H(r)$ into (4.47) and making use of the Bernoulli equation (1.35) yields the following system of algebraic equations in terms of a_0, a_1:

$$\mathcal{A}{a_1}^2m^3 + 12\,\mathcal{C}a_1m^4 = 0,$$

$$\mathcal{C}a_1l^4 + \mathcal{B}a_1l^2 = 0,$$

$$\mathcal{A}{a_1}^2lm^2 + 12\,\mathcal{C}a_1lm^3 = 0,$$

$$2\,\mathcal{A}{a_1}^2l^2m + 25\,\mathcal{C}a_1l^2m^2 + \mathcal{B}a_1m^2 = 0,$$

$$\mathcal{A}{a_1}^2l^3 + 15\,\mathcal{C}a_1l^3m + 3\,\mathcal{B}a_1lm = 0.$$

The solution of the given system using Maple yields

$$a_0 = a_0,\; a_1 = -12\,\frac{\mathcal{C}m}{\mathcal{A}},\; \mathcal{B} = -Cl^2. \tag{4.59}$$

Therefore, the solution of the KP-BBM (4.4) is

$$u(t,x,y,z) = a_0 + \, a_1l\left\{\frac{\cosh\left[l(r+D)\right] + \sinh\left[l(r+D)\right]}{1 - m\cosh\left[l(r+D)\right] - m\sinh\left[l(r+D)\right]}\right\}, \tag{4.60}$$

where $r = (\gamma-\mu)t + (\theta-\gamma)x - \theta y + z$ and D is an arbitrary constant.

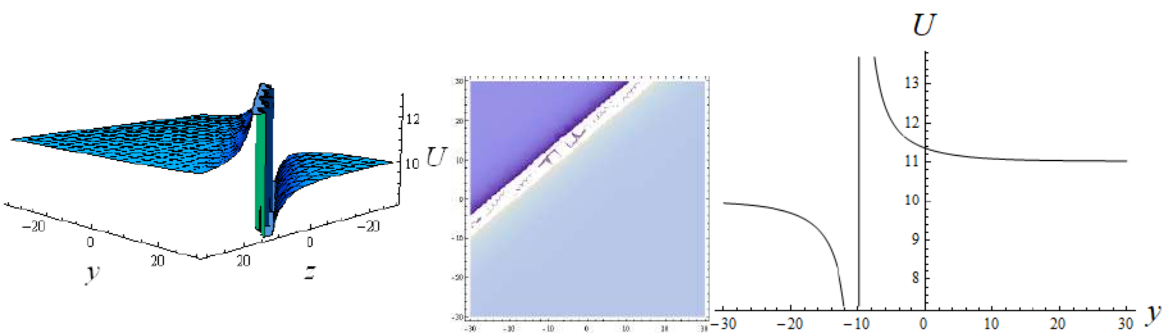

Figure 4.4: The 3D and 2D solution profiles of (4.60).

Figure 4.4 demonstrates the solution (4.60) for the values $\gamma = 0.2, \mu = 0.3, \theta = 1.9, t = 6, \mathcal{C} = 1.9, D = 0.9, \mathcal{A} = 2, z = -1, m = 2, l = 0.09, a_0 = 10, a_1 = -22.8, \omega = 2.03$.

Case 2. We consider the symmetry X_5 given as

$$X_5 = 2\alpha t\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial y} + \alpha z\frac{\partial}{\partial z} - (2a\,y + \alpha u)\frac{\partial}{\partial u}.$$

We compute invariants associated with X_5. The Lagrangian system associated with X_5 is

$$\frac{dt}{2\alpha t} = \frac{dy}{\alpha y} = \frac{dz}{\alpha z} = -\frac{du}{2\,a\,y + \alpha u}. \tag{4.61}$$

We first consider

$$\frac{dt}{2\alpha t} = \frac{dy}{\alpha y}. \tag{4.62}$$

Solving equation (4.62) by using variables separable method, we get invariant

$$J_1 = y/\sqrt{t}.$$

Secondly, from (4.61) we use

$$\frac{dt}{2\alpha t} = \frac{dz}{\alpha z} \tag{4.63}$$

and solving (4.63) yields

$$J_2 = z/\sqrt{t}.$$

Similarly from

$$\frac{dy}{\alpha y} = \frac{dz}{\alpha z}, \tag{4.64}$$

we use variables separable approach and obtain

$$J_3 = y/z.$$

Finally, to get invariant J_4 we use

$$\frac{dt}{2\alpha t} = -\frac{du}{2\,a\,y + \alpha u}. \tag{4.65}$$

Simplifying (4.65) we get a first-order linear differential equation of the form

$$\frac{du}{dt} + \frac{u}{2t} = -\frac{a\,y}{\alpha t}, \tag{4.66}$$

which solves to

$$J_4 = \frac{\sqrt{t}}{\alpha}\,(a\,y + \alpha u). \tag{4.67}$$

We now write J_4 as a function of other invariants namely, $J_1,\ J_2$ and J_3. Thus,

$$J_4 = F(J_1,\ J_2,\ J_3). \tag{4.68}$$

Substituting the value of J_4 from (4.68) into (4.67), we get

$$F(J_1,\ J_2,\ J_3) = \frac{\sqrt{t}}{\alpha}\,(a\,y + \alpha u). \tag{4.69}$$

Writing $u(t,x,y,z)$ in terms of the other variables in (4.69) gives

$$u(t,x,y,z) = \frac{1}{\sqrt{t}}F(J_1,\ J_2,\ J_3) - \frac{a\,y}{\alpha}. \tag{4.70}$$

Substituting $u(t,x,y,z)$ from equation (4.70) into (4.4) leads to

$$\begin{aligned} &F_{pq} - \alpha F_{pp}F - 2q\alpha F_{pp}F_q - k\alpha F_{pp}F_k - 2\beta F F_p - 2q\beta F_p F_{pq} - k\beta F_p F_{kq} \\ &- bF_{pppq} + 2cF + 10qcF_q + 4kcF_k + 4cq^2F_{qq} + 4cqkF_{kq} + ck^2F_{kk} + dF_{kk} = 0, \end{aligned} \tag{4.71}$$

where F in (4.71) is a function of p, q and k. Equation (4.71) does not have symmetries. Thus we have reduced the number of independent variable of the KP-BBM (4.4) by one.

Remark. We note that the symmetries X_H and X_F do not provide invariant solutions.

4.3 Conservation laws of (4.4)

In this section, we derive conservation laws of the KP-BBM (4.4). We use two different methods to find conservation laws. These are the multiplier method and Noether's theorem.

4.3.1 Conservation laws using multiplier method

We seek the zeroth-order multiplier $\mathcal{M} = \mathcal{M}(t,x,y,z,u)$, determined by using equation

$$\frac{\delta}{\delta u}\left\{\mathcal{M}\left(\alpha u_y u_{xx} + \beta u_y u_{xy} - bu_{txxx} + au_{xx} + cu_{yy} + du_{zz} + u_{tx}\right)\right\} = 0, \tag{4.72}$$

where $\delta/\delta u$ is the Euler operator, given in our case as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_y\frac{\partial}{\partial u_y} - D_x\frac{\partial}{\partial u_x} + D_tD_x\frac{\partial}{\partial u_{tx}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_z^2\frac{\partial}{\partial u_{zz}} + D_xD_y\frac{\partial}{\partial u_{xy}} + D_tD_x^3\frac{\partial}{\partial u_{txxx}},$$

where D_t, D_x, D_y, D_z are total derivative operators as defined in (4.8). Expanding (4.72) and splitting the result on appropriate derivatives of u yields the system of four multiplier determining equations:

$$\mathcal{M}_{zz} = 0,\ \mathcal{M}_x = 0,\ \mathcal{M}_y = 0,\ \mathcal{M}_u = 0.$$

The solution to the above obtained PDEs gives

$$\mathcal{M} = F(t)z + G(t),$$

where F and G are functions of t. The conserved quantities of equation (4.4) are derived by applying the divergence identity

$$\begin{aligned} &D_tT^t + D_xT^x + D_yT^y + D_zT^z \\ &= \{\mathcal{M}\left(\alpha u_y u_{xx} + \beta u_y u_{xy} - bu_{txxx} + au_{xx} + cu_{yy} + du_{zz} + u_{tx}\right)\} \end{aligned} \tag{4.73}$$

with T^t being conserved density and T^x, T^y, T^z spatial fluxes. Thus, using (4.73), we obtain the following conserved vectors corresponding to the two multipliers:

Case 1. For $\mathcal{M}_1 = F(t)z$, we have the corresponding conserved vector

$$\begin{aligned} T^t =&\frac{1}{2}zu_xF(t) - \frac{1}{4}bzu_{xxx}F(t), \\ T^x =&\frac{1}{2}\alpha zF(t)u_xu_y - \frac{1}{2}\alpha zF(t)uu_{xy} - \frac{1}{4}\beta zF(t)uu_{yy} + \frac{1}{4}\beta zF(t)u_y^2 \\ &+ \frac{1}{4}bzF'(t)u_{xx} - \frac{3}{4}bzF(t)u_{txx} + azF(t)u_x - \frac{1}{2}zF'(t)u + \frac{1}{2}zF(t)u_t, \\ T^y =&\frac{1}{2}\alpha zF(t)uu_{xx} + c\,zF(t)u_y + \frac{1}{4}\beta zF(t)u_xu_y + \frac{1}{4}\beta zF(t)uu_{xy}, \\ T^z =&dzF(t)u_z - dF(t)u. \end{aligned}$$

Case 2. For $\mathcal{M}_2 = G(t)$, we have the conserved vector given by

$$T^t = \frac{1}{2}u_xG(t) - \frac{1}{4}bu_{xxx}G(t),$$

$$
\begin{aligned}
T^x =& \frac{1}{2}\alpha G(t)u_x u_y - \frac{1}{2}\alpha G(t)uu_{xy} - \frac{1}{4}\beta G(t)uu_{yy} + \frac{1}{4}\beta G(t)u_y^2 + \frac{1}{4}bG'(t)u_{xx} \\
& - \frac{3}{4}bG(t)u_{txx} + aG(t)u_x - \frac{1}{2}G'(t)u + \frac{1}{2}G(t)u_t, \\
T^y =& \frac{1}{2}\alpha G(t)uu_{xx} + c\,G(t)u_y + \frac{1}{4}\beta G(t)u_x u_y + \frac{1}{4}\beta G(t)uu_{xy}, \\
T^z =& dG(t)u_z.
\end{aligned}
$$

4.3.2 Conservation laws using Noether's theorem

The KP-BBM equation (4.4) does not posses a Langrangian. However, if we let $\beta = 2\alpha$ the KP-BBM equation (4.4) then becomes

$$
u_{tx} + \alpha u_y u_{xx} + 2\alpha u_x u_{xy} - bu_{txxx} + au_{xx} + cu_{yy} + du_{zz} = 0. \tag{4.74}
$$

The standard Euler operator $\delta/\delta u$ is defined in (1.45). Now, $\delta\mathcal{L}/\delta u = 0$ for

$$
\mathcal{L} = -\frac{1}{2}a{u_x}^2 - \frac{1}{2}bu_{xx}u_{tx} - \frac{1}{2}c{u_y}^2 - \frac{1}{2}d{u_z}^2 - \frac{1}{2}u_t u_x - \frac{1}{2}\alpha {u_x}^2 u_y, \tag{4.75}
$$

therefore $\mathcal{L}$ is the Lagrangian for (4.74).

The Lie-Bäcklund operator

$$
X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u}, \tag{4.76}
$$

where the infinitesimals τ, ξ, ϕ and ψ are function of t, x, y, z and u is a Noether operator corresponding to the Lagrangian $\mathcal{L}$ if it satisfies

$$
\begin{aligned}
X^{[2]}(\mathcal{L}) + \mathcal{L}\left[D_t(\tau) + D_x(\xi) + D_y(\phi) + D_z(\psi)\right] =& D_t(B^1) + D_x(B^2) \\
& + D_y(B^3) + D_z(B^4),
\end{aligned} \tag{4.77}
$$

where B^1, B^2, B^3 and B^4 are gauge terms. We proceed to find all Noether symmetries of (4.74) and their gauge functions depending on t, x, y, z, and u. We expand (4.77) and split the result on the derivatives of u, thereby yielding a system of PDEs. Solving the obtained system of PDEs, we have the Noether point symmetries with their respective gauge functions as

$$
X_1 = \frac{\partial}{\partial t}, \;\; B^1 = 0, \;\; B^2 = 0, \;\; B^3 = 0, \;\; B^4 = 0,
$$

$$
\begin{aligned}
X_2 =& \frac{\partial}{\partial x}, \ \ B^1 = 0, \ \ B^2 = 0, \ \ B^3 = 0, \ \ B^4 = 0, \\
X_3 =& \frac{\partial}{\partial y}, \ \ B^1 = 0, \ \ B^2 = 0, \ \ B^3 = 0, \ \ B^4 = 0, \\
X_4 =& \frac{\partial}{\partial z}, \ \ B^1 = 0, \ \ B^2 = 0, \ \ B^3 = 0, \ \ B^4 = 0, \\
X_5 =& 2\alpha t\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial y} + \alpha z\frac{\partial}{\partial z} - (\alpha u + 2ay)\frac{\partial}{\partial u}, \ \ B^1 = 0, \ \ B^2 = 0, \\
& \quad B^3 = 2acu, \ \ B^4 = 0, \\
X_G =& G(t)\frac{\partial}{\partial u}, \ \ B^1 = 0, \ \ B^2 = -\frac{1}{2}vG'(t), \ \ B^3 = 0, \ \ B^4 = 0, \\
X_F =& zF(t)\frac{\partial}{\partial u}, \ \ B^1 = 0, \ \ B^2 = -\frac{1}{2}vzF'(t), \ \ B^3 = 0, \ \ B^4 = -duF(t).
\end{aligned}
$$

Corresponding to each of the Noether symmetries, we respectively obtain the following non-local conserved vectors:

$$
\begin{aligned}
T_1^t =& -\frac{1}{2}au_x^2 - \frac{1}{4}bu_tu_{xxx} - \frac{1}{4}bu_{xx}u_{tx} - \frac{1}{2}cu_y^2 - \frac{1}{2}du_z^2 - \frac{1}{2}\alpha u_x^2u_y, \\
T_1^x =& au_tu_x - \frac{3}{4}bu_tu_{,\,txx} + \frac{1}{2}bu_{tx}^2 + \frac{1}{4}bu_{tt}u_{xx} + \alpha u_tu_xu_y + \frac{u_t^2}{2}, \\
T_1^y =& cu_tu_y + \frac{1}{2}\alpha u_tu_x^2, \\
T_1^z =& du_tu_z;
\end{aligned}
$$

$$
\begin{aligned}
T_2^t =& \frac{1}{4}bu_{xx}^2 - \frac{1}{4}bu_{xxx}u_x + \frac{u_x^2}{2}, \\
T_2^x =& \frac{1}{2}au_x^2 + \frac{1}{4}bu_{xx}u_{tx} - \frac{3}{4}bu_xu_{txx} - \frac{1}{2}cu_y^2 - \frac{1}{2}du_z^2 + \frac{1}{2}\alpha u_x^2u_y, \\
T_2^y =& cu_xu_y + \frac{1}{2}\alpha u_x^3, \\
T_2^z =& du_xu_z;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t =& \frac{1}{4}bu_{xx}u_{xy} - \frac{1}{4}bu_{xxx}u_y + \frac{1}{2}u_xu_y, \\
T_3^x =& au_xu_y - \frac{3}{4}bu_yu_{txx} + \frac{1}{4}bu_{xx}u_{ty} + \frac{1}{2}bu_{tx}u_{xy} + \frac{1}{2}u_tu_y + \alpha u_xu_y^2, \\
T_3^y =& \frac{1}{2}cu_y^2 - \frac{1}{2}au_x^2 - \frac{1}{2}bu_{xx}u_{tx} - \frac{1}{2}du_z^2 - \frac{1}{2}u_tu_x, \\
T_3^z =& du_yu_z;
\end{aligned}
$$

$$
\begin{aligned}
T_4^t =& \frac{1}{4} b u_{xx} u_{xz} - \frac{1}{4} b u_{xxx} u_z + \frac{1}{2} u_x u_z, \\
T_4^x =& a u_x u_z + \frac{1}{4} b u_{xx} u_{tz} + \frac{1}{2} b u_{tx} u_{xz} - \frac{3}{4} b u_z u_{txx} + \frac{1}{2} u_t u_z + \alpha u_x u_y u_z, \\
T_4^y =& c u_y u_z + \frac{1}{2} \alpha u_x^2 u_z, \\
T_4^z =& \frac{1}{2} d u_z^2 - \frac{1}{2} a u_x^2 - \frac{1}{2} b u_{xx} u_{tx} - \frac{1}{2} c u_y^2 - \frac{1}{2} u_t u_x - \frac{1}{2} \alpha u_x^2 u_y;
\end{aligned}
$$

$$
\begin{aligned}
T_5^t =& \frac{1}{2} z \alpha u_x u_z - d t \alpha u_z^2 - \frac{1}{4} b z \alpha u_{xxx} u_z - c t \alpha u_y^2 - a t \alpha u_x^2 - \frac{1}{2} b t \alpha u_{xx} u_{tx} \\
& + a y u_x - t \alpha^2 u_y u_x^2 + \frac{1}{2} \alpha u u_x + \frac{1}{2} y \alpha u_y u_x + \frac{1}{4} b \alpha u_x u_{xx} + \frac{1}{4} b z \alpha u_{xz} u_{xx} \\
& + \frac{1}{4} b y \alpha u_{xy} u_{xx} - \frac{1}{2} a b y u_{xxx} - \frac{1}{4} b \alpha u u_{xxx} - \frac{1}{4} b y \alpha u_y u_{xxx} - \frac{1}{2} b t \alpha u_{xxx} u_t, \\
T_5^x =& 2 y u_x a^2 + \alpha u u_x a + z \alpha u_z u_x a + 3 y \alpha u_y u_x a + y u_t a + 2 t \alpha u_x u_t a \\
& + t \alpha u_t^2 - \frac{3}{2} b y u_{txx} a + b t \alpha u_{tx}^2 + y \alpha^2 u_y^2 u_x + \alpha^2 u u_y u_x + z \alpha^2 u_z u_y u_x \\
& + \frac{1}{2} \alpha u u_t + \frac{1}{2} z \alpha u_z u_t + \frac{1}{2} y \alpha u_y u_t + 2 t \alpha^2 u_y u_x u_t + \frac{3}{4} b \alpha u_{xx} u_t + \frac{1}{4} b z \alpha u_{xx} u_{tz} \\
& + \frac{1}{4} b y \alpha u_{xx} u_{ty} + \frac{1}{2} b \alpha u_x u_{tx} + \frac{1}{2} b z \alpha u_{xz} u_{tx} + \frac{1}{2} b y \alpha u_{xy} u_{tx} - \frac{3}{4} b \alpha u u_{txx} \\
& + \frac{1}{2} b t \alpha u_{xx} u_{tt} - \frac{3}{4} b z \alpha u_z u_{txx} - \frac{3}{4} b y \alpha u_y u_{txx} - \frac{3}{2} b t \alpha u_t u_{txx}, \\
T_5^y =& \frac{1}{2} z \alpha^2 u_x^2 u_z - \frac{1}{2} d y \alpha u_z^2 + c z \alpha u_y u_z + \frac{1}{2} c y \alpha u_y^2 + \frac{1}{2} a y \alpha u_x^2 + \frac{1}{2} \alpha^2 u u_x^2 \\
& + 2 a c y u_y + c \alpha u u_y + t \alpha^2 u_x^2 u_t + 2 c t \alpha u_y u_t - \frac{1}{2} y \alpha u_x u_t - \frac{1}{2} b y \alpha u_{xx} u_{tx}, \\
T_5^z =& \frac{1}{2} d z \alpha u_z^2 + 2 a d y u_z + d \alpha u u_z + d y \alpha u_y u_z - \frac{1}{2} c z \alpha u_y^2 \\
& + 2 d t \alpha u_t u_z - \frac{1}{2} a z \alpha u_x^2 - \frac{1}{2} z \alpha^2 u_y u_x^2 - \frac{1}{2} z \alpha u_x u_t - \frac{1}{2} b z \alpha u_{xx} u_{tx};
\end{aligned}
$$

$$
\begin{aligned}
T_G^t =& \frac{1}{4} b G(t) u_{xxx} - \frac{1}{2} G(t) u_x, \\
T_G^x =& \frac{3}{4} b G(t) u_{txx} - a G(t) u_x - \alpha G(t) u_x u_y - \frac{1}{2} G(t) u_t - \frac{1}{4} b G'(t) u_{xx}, \\
T_G^y =& - c G(t) u_y - \frac{1}{2} \alpha G(t) u_x^2, \\
T_G^z =& - d G(t) u_z;
\end{aligned}
$$

$$
T_F^t = \frac{1}{4} b z F(t) u_{xxx} - \frac{1}{2} z F(t) u_x,
$$

$$
\begin{aligned}
T_F^x =& \frac{3}{4} bzF(t)u_{txx} - azF(t)u_x - \alpha zF(t)u_x u_y - \frac{1}{2} zF(t)u_t - \frac{1}{4} bzF'(t)u_{xx}, \\
T_F^y =& - czF(t)u_y - \frac{1}{2}\alpha zF(t)u_x^2, \\
T_F^z =& - dzF(t)u_z.
\end{aligned}
$$

4.4 Concluding Remarks

In this Chapter we examined the three-dimensional KP-BBM equation, which was just recently published in the literature [73]. There are numerous disciplines in which this equation can be used. In order to execute symmetry reductions and create accurate solutions, we first reduced the equation using its Lie point symmetries. Besides, using the direct integration technique, we found an incomplete elliptic integral solution associated with the model (4.4). In addition, with the introduction of Riccati and Bernoulli equations, we utilized the simplest equation approach to secure more solutions to (4.4). Moreover, the solutions found were depicted with various diagrammatic representations by making an adequate choice of parameter values. Lastly, the multiplier method of conservation laws was used to generate conservation laws for this model. Additionally, we employed Noether's theorem to obtain conserved vectors for a particular instance of (4.4) where $\beta = 2\alpha$.

Chapter 5

Concluding remarks and future work

Nonlinear partial differential equations are used to simulate many of the physical systems found in fluid mechanics, materials science, elasticity, thermodynamics, biology, gas dynamics, and other fields. Therefore, it is crucial to research these systems in order to identify their precise solutions and conservation laws. The nonlinear filtration equation, the extended Kadomtsev-Petviashvili (eKP) and Kadomtsev-Petviashvili-Benjamin-Bona-Mahony equation are three nonlinear partial differential equations that have so been examined in this dissertation (KP-BBM).

The relevant literature that was consulted for this dissertation was presented in Chapter one. Different techniques for determining the precise solutions of nonlinear partial differential equations were presented, along with various techniques for determining conservation laws.

The nonlinear filtration equation was covered in Chapter two as an illustration. We derived the equation's Lie point symmetries, calculated their commutator tables, and produced one-parameter groups of transformations for them. Later, group-invariant solutions to the nonlinear filtration equation were constructed. Additionally, we used the multiplier method to derive the conservation laws for the equation.

We looked at the extended KP equation (3.3) in Chapter three by first determining its infinitesimal generators. We provided symmetry reductions and then presented group-invariant solutions to the extended KP problem. We also used the obtained symmetries to derive conservation laws for the equation.

In Chapter four we examined the three-dimensional KP-BBM equation, which was just recently published in the literature. There are many fields in which this equation can be utilized. In order to perform symmetry reductions and create accurate solutions, we first compute Lie symmetries. The simplest equation approach and incomplete elliptic integral were two techniques used to get solutions for (4.4). The multiplier method and Noether's theorem were used to generate conservation laws for this model. Additionally, we employed Noethers theorem to obtain conserved vectors for a particular instance of (4.4) where $\beta = 2\alpha$.

For future work we shall use the conservation laws to obtain exact solutions of these nonlinear partial differential equations.

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