EXACT SOLUTIONS AND CONSERVATION LAWS OF A GENERALIZED ITO-TYPE COUPLED KdV SYSTEM, BOUSSINESQ SYSTEM OF KdV-KdV TYPE AND COUPLED KdV SYSTEM

by

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Declaration

I, Tshepo E. Mogorosi, student number 21408114, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

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This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

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This dissertation has been submitted with my approval as a University co-supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

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Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 2

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Chapter 3

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Dedication

I would like to say special thanks to my family and friends for their moral support, advice and encouragement while compiling my dissertation.

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Abstract

In this dissertation, three systems of nonlinear partial differential equations, namely, the generalized Ito-type coupled KdV system, Boussinesq system of KdV-KdV type equation and coupled KdV system are studied.

The exact solutions of the Boussinesq system of KdV-KdV type equations and the coupled KdV system are obtained. The Lie symmetry approach along with the simplest equation method is used to obtain these solutions.
Moreover, conservation laws for all the three systems are derived using the Noether's theorem. The three systems are third-order systems of PDEs and do not have Lagrangian. The transformation $u \to U_x$, $v \to V_x$ are used to convert the above mentioned systems to fourth-order systems in U, V variables, which have Lagrangians. Noether's approach is then used to derive the conservation laws. Finally, the conservation laws are expressed in the variables u, v and they constitute the conservation laws for the third-order systems. Infinitely many nonlocal conserved quantities are obtained for these three systems.

List of Acronyms

PDE:	Partial differential equation
ODE:	Ordinary differential equation
KdV:	Korteweg-de Vries
NLPDE:	Nonlinear partial differential equation

Introduction

The study of nonlinear partial differential equations (NLPDEs) has made a substantial progress during the last few decades (see for example [1–16]). In particular, its integrability aspects have been studied. A variety of techniques, such as solitary wave ansatz method [7, 8], the inverse scattering transform method [9], the Hirota's bilinear method [10], the homogeneous balance method [11] and Lie symmetry methods [12–16], have been used in the literature to find exact solutions of NLPDEs.

Among the above mentioned methods, the Lie symmetry approach developed by Sophus Lie (1842-1899) is one of the most effective methods to determine solutions of NLPDEs. It also plays a vital role in the reduction of differential equations. The method centres around the algebra of one parameter Lie groups of transformations that are admitted by the PDE. When the Lie point symmetries have been determined, the reduction of the PDE is standard and may lead to exact (symmetry invariant) solutions [13, 17].

The notion of conservation laws plays an important role in the study of NLPDEs. The constructive way of finding conservation laws is by means of Noether's theorem [18]. This theorem states that for Euler-Lagrange differential equations, each Noether symmetry associated with the Lagrangian there corresponds a conservation law. The application of Noether's theorem depends upon the knowledge of a suitable Lagrangian. Conservation laws have been extensively used in studying the existence, uniqueness and stability of solutions of NLPDEs (see for example [19, 20]).

Many methods such as the Laplace direct method [21] and characteristics method [22] have been developed to obtain conservation laws for differential equations which do not have a Lagrangian. The direct method [21] has been successfully applied to obtain conservation laws for such differential equations. The characteristics method introduced by Steudel [22], involves writing a conservation law in characteristic form, where the characteristics are the multipliers of the differential equations.

In this research project we will obtain exact solutions and conservation laws for the following three nonlinear systems of PDEs:

The generalized Ito-type coupled KdV system [23] is given by

$$\begin{aligned} u_t + \alpha u u_x + \beta v v_x + \gamma u_{xxx} &= 0, \\ v_t + \beta u v_x + \beta v u_x &= 0, \end{aligned}$$

where α, β and γ are arbitrary constants. Here the space and time variables x and t represent the position and the elapsed time, respectively along the channel, where $v(x,t)$ is the horizontal velocity and $u(x,t)$ is the deviation of the free surface from its rest position. The generalized Ito-type coupled KdV system describes the interaction process of two long waves that have infinitely many conserved quantities.

The Boussinesq system of KdV-KdV type equation [24], is given by

$$\begin{aligned} u_t + v_x + v u_x + u v_x + a v_{xxx} &= 0, \\ v_t + u_x + v v_x + c u_{xxx} &= 0, \end{aligned}$$

where a and c are arbitrary constants. The independent variable x represents position along the channel, t is proportional to elapsed time, $u(x,t)$ is the deviation of the free surface from its rest position at the coordinate x at time t while $v(x,t)$ is the horizontal velocity. The Boussinesq system of KdV-KdV type equation describes the two-way propagation of small-amplitude, long wavelength gravity waves on the water in a canal.

The coupled KdV system [25]

$$
\begin{aligned}
v_t + u_x + \beta u_x v + \beta u v_x + \gamma u_{xxx} &= 0, \\
u_t + v_x + \alpha v v_x + \beta u u_x + \lambda v_{xxx} &= 0,
\end{aligned}
$$

where α, β, γ and λ are arbitrary constants, describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. The independent variable x represents position along the channel, t is proportional to elapsed time, $u(x,t)$ is the deviation of the free surface from its rest position at the coordinate x at time t while $v(x,t)$ is the horizontal velocity.

The outline of the research project is as follows:

In Chapter one, the basic definitions and theorems concerning the Lie group methods are recalled.

Chapter two deals with the symmetry reductions of a generalized Ito-type coupled KdV system using Lie group analysis. Furthermore, conservation laws of this system are constructed using Noether's theorem.

In Chapter three, Lie group analysis is applied along with the simplest equation method [26–29] to obtain exact solutions for the Boussinesq system of KdV-KdV type equation and its conservation laws are derived using Noether's approach [18].

In Chapter four, we obtain exact solutions of the coupled KdV system using Lie symmetry method together with the simplest equation method. Furthermore, conservation laws for the system are constructed using Noether's approach [18].

Finally, in Chapter five, a summary of the results of the dissertation are presented and future work is discussed.

A bibliography is presented at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter, we present basic definitions and theorems that will be used throughout this dissertation. This includes the algorithm to determine the Lie point symmetries and Noether point symmetries of partial differential equations.

1.1 Introduction of Lie group theory

In this section we give a brief introduction to Lie group theory of partial differential equations. Lie group analysis originated in the late nineteenth century by an outstanding mathematician Sophus Lie (1842-1899). He discovered that majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory which is based on the invariance of the differential equations under a continuous group of symmetries. The mathematical ideas of Lie's theory are presented in several books, see for example, Bluman and Kumei [12], Olver [13], Ovsiannikov [14], Ibragimov [1, 15] and Stephani [30].

1.1.1 Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variable with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variable with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a), \;\; \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law.* A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.1.2 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u^\alpha), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots , \quad i = 1, ..., n, \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$,$u_{(5)}$ $\cdots$.

We note that in group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables and a pth-order PDE(s) is given as

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x,u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x,u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots.$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha. \end{aligned} \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta\frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta\frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, for small a, ie., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= q_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j).
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\dots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\dots,i_{p-1}}^\alpha) - u_{i_1,i_2,\dots,i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$
\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha), \\
&\cdot \\
&\cdot \\
&\cdot \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1,
\end{aligned}
\tag{1.18}
$$

where

$$
X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.19}
$$

1.1.3 Group admitted by system of PDEs

The operator

$$
X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v} \tag{1.20}
$$

is said to be a (generator of) point symmetry of the third-order system of PDEs

$$
E(t,x,u,v,u_t,v_t,u_x,v_x,u_{tt},v_{tt},u_{xx},v_{xx},u_{xxx},v_{xxx}) = 0, \tag{1.21}
$$

if

$$
X^{[3]}(E) = 0, \tag{1.22}
$$

whenever $E = 0$. This can also be written as (symmetry condition)

$$
X^{[3]} E\big|_{E=0} = 0, \tag{1.23}
$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.2 Equation (1.23) is called the determining equation of (1.21), because it determines all the infinitesimal symmetries of equation (1.21).

The theorem below enables us to construct some solutions of (1.21) from known one.

Theorem 1.2 A symmetry of equation (1.21) transforms any solution of (1.21) into another solution of the same equation.

Proof: It follows from the fact that a symmetry of an equation leaves invariant that equation.

1.1.4 Lie algebras

Let X_1 and X_2 be any two operators defined by

$$X_1 = \xi_1^1(t,x,u,v)\frac{\partial}{\partial t} + \xi_1^2(t,x,u,v)\frac{\partial}{\partial x} + \eta_1^1(t,x,u,v)\frac{\partial}{\partial u} + \eta_1^2(t,x,u,v)\frac{\partial}{\partial v}$$

and

$$X_2 = \xi_2^1(t,x,u,v)\frac{\partial}{\partial t} + \xi_2^2(t,x,u,v)\frac{\partial}{\partial x} + \eta_2^1(t,x,u,v)\frac{\partial}{\partial u} + \eta_2^2(t,x,u,v)\frac{\partial}{\partial v}.$$

Definition 1.3 (Commutator) The commutator of X_1 and X_2, written as $[X_1, X_2]$, is defined by the formula $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.4 (Lie algebra) A Lie algebra is a vector space L of operators with the following property : For all $X_1, X_2 \in L$, the commutator $[X_1, X_2] \in L$.

The dimension of a Lie algebra is the dimension of the vector space L.

Theorem 1.3 The set of all solutions of any determining equation forms a Lie algebra.

1.2 Fundamental operators and their relationships

In this section we give some basic definitions and theorems concerning Noether point symmetries and conservation laws which we utilize later in the dissertation.

Consider a kth-order system of PDEs of n independent variables $x = (x^1, ..., x^n)$ and m dependent variables $u = (u^1, ..., u^m)$ given by

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.24}$$

where $u_{(1)}, ..., u_{(k)}$ denote the collection of all first, second and kth order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$, ..., where $\alpha = 1 \cdots, m$ and $i, j, \cdots = 1, \cdots, n$, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + \cdots. \tag{1.25}$$

Definition 1.5 The Lie-Bäcklund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta_{i_1 i_2 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \tag{1.26}$$

where $\zeta_{i_1 i_2 \ldots i_s}^\alpha$ are defined by

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha \\ \zeta_{i_1 \ldots i_s}^\alpha &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha. \end{aligned} \tag{1.27}$$

The Lie-Bäcklund operator (1.26) in characteristic form is

$$X = \xi^i \frac{\partial}{\partial x^i} + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \tag{1.28}$$

in which W^α is the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{1.29}$$

Definition 1.6 If there exist a function $L(x, u, u_{(1)}, u_{(2)}, ...u_{(k-1)}) \in \mathcal{A}$ (space of differential functions), such that the system of PDEs (1.24) are equivalent to

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, ..., m, \tag{1.30}$$

where the Euler-Lagrange operator $\delta/\delta u_i^\alpha$ is

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha}, \quad \alpha = 1, 2, ..., m, \tag{1.31}$$

then L is called a Lagrangian of (1.24) and equations (1.30) are the corresponding Euler-Lagrange differential equations.

Definition 1.7 A Lie-Bäcklund operator X is a Noether symmetry generator associated with a Lagrangian L of (1.30) if there exists a gauge function $B = (B^1, B^2, ..., B^n)$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i) \quad i = 1, 2, ..., n. \tag{1.32}$$

Definition 1.8 A conserved vector of (1.24) is an n-tuple $T^i = (T^1, T^2, ..., T^n)$, $T^i \in \mathcal{A}$ defined by

$$T^i = B^i - \xi^i L - W^\alpha \frac{\delta L}{\delta u_i^\alpha} - \sum_{s\geq 1} D_{i_1...i_s}(W^\alpha)\frac{\delta L}{\delta u^\alpha_{i i_1...i_s}}, \quad i = 1, 2, ..., n, \tag{1.33}$$

such that

$$D_i T^i = 0, \tag{1.34}$$

holds for all solutions of (1.24).

1.3 Conclusion

In this chapter we presented some basic definitions and concepts concerning Lie group analysis, Noether symmetries and conservation laws of PDEs. These included the algorithm to determine the Lie point symmetries, Noether point symmetries and conservation laws of PDEs.

Chapter 2

Symmetry reduction and conservation laws of generalized Ito-type coupled KdV system

In this chapter we use Lie symmetry method to determine the Lie point symmetries of a generalized Ito-type coupled KdV system [23] given by

$$\begin{aligned} u_t + \alpha u u_x + \beta v v_x + \gamma u_{xxx} &= 0, \\ v_t + \beta u v_x + \beta v u_x &= 0. \end{aligned} \tag{2.1}$$

We also derive the conservation laws for system (2.1) using Noether theorem [18]. Here α, β and γ are arbitrary constants. It is well known that coupled nonlinear systems in which a KdV structure is embedded occur naturally in shallow water wave problems [31, 32]. When $\alpha = -6$, $\beta = -2$ and $\gamma = -1$, the system (2.1) is called Ito's system and it describes the interaction process of two internal long waves [33]. It should be noted that in the absence of the effect of v, system (2.1) reduces to the ordinary KdV equation. In [34] it has been shown that this Ito's system can be a member of a bi-Hamiltonian integrable hierarchy. The numerical methods for this system are very limited [23]. Xu and Shu [32] however developed local discontinuous Galerkin methods for the Ito's system and proved the stability of these methods and as a result showed some good numerical results. Recently, in [23], the generalized Ito-type coupled KdV system (2.1) was constructed as a multi-symplectic Hamiltonian partial differential equation by introducing some new variables and multi-symplectic numerical methods were applied to investigate this system. Part of the work of this chapter has been published [35].

2.1 Symmetry reduction of generalized Ito-type coupled KdV system

2.1.1 Lie point symmetries of system (2.1)

The symmetry group of a generalized Ito-type coupled KdV system (2.1) will be generated by the vector field of the form

$$\Psi = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}. \qquad (2.2)$$

Applying the third prolongation $\mathrm{pr}^{(3)}\Psi$ [13] to (2.1) we obtain the overdetermined system of linear partial differential equations (PDEs)

$$\begin{aligned}
\xi^1_x &= 0,\\
\xi^1_u &= 0,\\
\xi^1_v &= 0,\\
\xi^2_u &= 0,\\
\xi^2_v &= 0,\\
\eta^1_v &= 0,\\
\eta^1_{uu} &= 0,\\
\eta^2_u &= 0,\\
\eta^1_{xu} - \xi^2_{xx} &= 0,\\
\xi^1_t - 3\xi^2_x &= 0,
\end{aligned}$$

$$\beta v\eta^1_x + \eta^2_t + \beta u\eta^2_x = 0,$$

$$\beta\eta^1 - \beta u\xi^2_x + \beta u\xi^1_t - \xi^2_t = 0,$$

$$\alpha u\eta^1_x + \beta v\eta^2_x + \eta^1_t + \gamma\eta^1_{xxx} = 0,$$

$$\eta^2 - v\xi^2_x + v\xi^1_t - v\eta^1_u + v\eta^2_v = 0,$$

$$\eta^2 - v\xi^2_x + v\xi^1_t + v\eta^1_u - v\eta^2_v = 0,$$

$$\alpha\eta^1 + \alpha u\xi^1_t - \alpha u\xi^2_x - \xi^2_t + 3\gamma\eta^1_{xxu} - \gamma\xi^2_{xxx} = 0.$$

Solving the resultant overdetermined system of linear partial differential equations we obtain the following three Lie point symmetries:

$$\begin{aligned}
\Psi_1 &= \frac{\partial}{\partial t}, \\
\Psi_2 &= \frac{\partial}{\partial x}, \\
\Psi_3 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}.
\end{aligned}$$

We now compute the Lie brackets of the above point symmetries for all symmetry generators. By definition of the Lie bracket [1], we have

$$[\Psi_i, \Psi_j] = \Psi_i, \Psi_j - \Psi_j, \Psi_i \tag{2.3}$$

and

$$[\Psi_i, \Psi_j] = -[\Psi_j, \Psi_i]. \tag{2.4}$$

We now compute the commutation relations for all the symmetry generators. First we compute $[\Psi_1, \Psi_2]$. Using (2.3) we have

$$\begin{aligned}
[\Psi_1, \Psi_2] &= \Psi_1\Psi_2 - \Psi_2\Psi_1 \\
&= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\right) - \left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
[\Psi_1, \Psi_3] &= \Psi_1\Psi_3 - \Psi_3\Psi_1 \\
&= \frac{\partial}{\partial t}\left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}\right) - \left(3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}\right)\frac{\partial}{\partial t} \\
&= 3\Psi_1.
\end{aligned}$$

Likewise, one can obtain the commutation relations between other vector fields. In **Table 1** below we present the commutator table of the Lie algebra of system (2.1). Here the entry in row i and column j is represented by $[\Psi_i, \Psi_j]$:

Table 1.

$[\Psi_i, \Psi_j]$	Ψ_1	Ψ_2	Ψ_3
Ψ_1	0	0	$3\Psi_1$
Ψ_2	0	0	Ψ_2
Ψ_3	$-3\Psi_1$	$-\Psi_2$	0

2.1.2 Symmetry reduction of system (2.1)

In this subsection we will use symmetries obtained in the above section to reduce system (2.1) into ODEs. In order to obtain symmetry reduction and exact solutions, one has to solve the associated characteristics equations

$$\frac{dt}{\xi^1(t,x,u,v)} = \frac{dx}{\xi^2(t,x,u,v)} = \frac{du}{\eta^1(t,x,u,v)} = \frac{dv}{\eta^2(t,x,u,v)}.$$

We consider the following cases:

Case 1.

Let us calculate the invariant solution for the operator Ψ_3, viz.,

$$\Psi_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}. \tag{2.5}$$

Let

$$\Psi_3 J = 3t\frac{\partial J}{\partial t} + x\frac{\partial J}{\partial x} - 2u\frac{\partial J}{\partial u} - 2v\frac{\partial J}{\partial v}.$$

The characteristic system is

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{du}{-2u} = \frac{dv}{-2v}.$$

Solving

$$\frac{dt}{3t} = \frac{dx}{x},$$

we obtain $z = xt^{-1/3}$. Similarly, solving for

$$\frac{dt}{3t} = \frac{du}{-2u} \quad \text{and} \quad \frac{dt}{3t} = \frac{dv}{-2v},$$

one gets $J_1 = ut^{2/3}$ and $J_2 = vt^{2/3}$. Therefore we obtain the group-invariant solution

$$u(t,x) = t^{-2/3}E(z) \quad \text{and} \quad v(t,x) = t^{-2/3}F(z),$$

where $z = xt^{-1/3}$ is an invariant of the symmetry Ψ_3 and the functions $E(z)$ and $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} 3\gamma E'''(z) + 3\alpha E(z)E'(z) + 3\beta F(z)F'(z) - zE'(z) - 2E(z) &= 0, \\ 3\beta E(z)F'(z) + 3\beta F(z)E'(z) - zF'(z) &= 0. \end{aligned} \tag{2.6}$$

Case 2.

Let us consider the linear combination of the translations symmetries $\Psi_1 = \partial/\partial t$ and $\Psi_2 = \partial/\partial x$, namely the symmetry

$$\Psi = \frac{\partial}{\partial t} + k\frac{\partial}{\partial x},$$

where k is a constant.

Let

$$\Psi J = \frac{\partial J}{\partial t} + k\frac{\partial J}{\partial x} + 0\frac{\partial J}{\partial u} + 0\frac{\partial J}{\partial v}.$$

The characteristic system is

$$\frac{dt}{1} = \frac{dx}{k} = \frac{du}{0} = \frac{dv}{0}.$$

Therefore, solving

$$\frac{dt}{1} = \frac{dx}{k},$$

we get $z = x - kt$. In a similar procedure, solving for

$$\frac{dt}{1} = \frac{du}{0} \quad \text{and} \quad \frac{dt}{1} = \frac{dv}{0},$$

one obtains $J_1 = u$ and $J_2 = v$. Therefore we obtain the group-invariant solution

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z),$$

where $z = x - kt$ is an invariant of the symmetry $\Psi_1 + k\Psi_2$ and the functions $E(z)$, $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} \gamma E'''(z) + \alpha E(z)E'(z) + \beta F(z)F'(z) - kE'(z) &= 0, \\ \beta E(z)F'(z) + \beta F(z)E'(z) - kF'(z) &= 0. \end{aligned} \tag{2.7}$$

Case 3.

For invariance under symmetry Ψ_2, one obtains an invariant $z = x$ and the group-invariant solution is of the form

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z),$$

where the functions $E(z)$ and $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} \gamma E'''(z) + \alpha E(z)E'(z) + \beta F(z)F'(z) &= 0, \\ \beta E(z)F'(z) + \beta F(z)E'(z) &= 0. \end{aligned} \tag{2.8}$$

2.2 Conservation laws of a generalized Ito-type coupled KdV system

In this subsection we will construct the conservation laws of the generalized Ito-type coupled KdV system [23]

$$\begin{aligned} u_t + \alpha u u_x + \beta v v_x + \gamma u_{xxx} &= 0, \\ v_t + \beta u v_x + \beta v u_x &= 0, \end{aligned} \tag{2.9}$$

where α, β and γ are arbitrary constants. It should be noted that the generalized Ito-type coupled KdV system does not have a Lagrangian. In order to apply the

Noether's approach we use the transformations $u \to U_x$, $v \to V_x$, then system (2.9) becomes

$$\begin{aligned} U_{tx} + \alpha U_x U_{xx} + \beta V_x V_{xx} + \gamma U_{xxxx} = 0,\\ V_{xt} + \beta V_x U_{xx} + \beta U_x V_{xx} = 0. \end{aligned} \tag{2.10}$$

It can be readily verified that L given by

$$L = \frac{1}{2}\left\{\gamma U_{xx}^2 - \frac{\alpha}{3}U_x^3 - \beta U_x V_x^2 - U_t U_x - V_t V_x\right\}, \tag{2.11}$$

is a Lagrangian for (2.10). This is because L satisfies

$$\frac{\delta L}{\delta U} = 0 \quad \text{and} \quad \frac{\delta L}{\delta V} = 0,$$

where $\delta/\delta U$ and $\delta/\delta V$ are Euler-Lagrange operators defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_t^2\frac{\partial}{\partial U_{tt}} + D_x^2\frac{\partial}{\partial U_{xx}} + D_x D_t\frac{\partial}{\partial U_{tx}} - \cdots \tag{2.12}$$

and

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} + D_t^2\frac{\partial}{\partial V_{tt}} + D_x^2\frac{\partial}{\partial V_{xx}} + D_x D_t\frac{\partial}{\partial V_{tx}} - \cdots. \tag{2.13}$$

Consider the vector field

$$\begin{aligned} X = \quad & \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U}\\ & + \eta^2(t,x,U,V)\frac{\partial}{\partial V}, \end{aligned} \tag{2.14}$$

which has the second extension

$$\begin{aligned} X^{[2]} = \quad & \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U}\\ & + \eta^2(t,x,U,V)\frac{\partial}{\partial V} + \zeta_t^1\frac{\partial}{\partial U_t} + \zeta_t^2\frac{\partial}{\partial V_t} + \zeta_x^1\frac{\partial}{\partial U_x} + \zeta_x^2\frac{\partial}{\partial V_x} + \ldots, \end{aligned} \tag{2.15}$$

where

$$\zeta_t^1 = D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), \quad \zeta_x^1 = D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2),$$

$$\zeta_t^2 = D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), \quad \zeta_x^2 = D_x(\eta^2) - V_t D_x(\xi^1) - V_x D_x(\xi^2)$$

and

$$D_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + V_t\frac{\partial}{\partial V} + U_{tt}\frac{\partial}{\partial U_t} + V_{tt}\frac{\partial}{\partial V_t} + U_{tx}\frac{\partial}{\partial U_x} + V_{tx}\frac{\partial}{\partial V_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + V_x\frac{\partial}{\partial V} + U_{xx}\frac{\partial}{\partial U_x} + V_{xx}\frac{\partial}{\partial V_x} + U_{tx}\frac{\partial}{\partial U_t} + V_{tx}\frac{\partial}{\partial V_t} + \cdots$$

The Lie-Bäcklund operator X defined in equation (2.14) is a Noether operator corresponding to the Lagrangian L if it satisfies

$$X(L) + L[D_t(\xi^1) + D_x(\xi^2)] = D_t(B^1) + D_x(B^2), \tag{2.16}$$

where $B^1(t,x,U,V)$, $B^2(t,x,U,V)$ are the gauge terms.

The expansion of equation (2.16) with the Lagrangian (2.11) yields

$$\begin{aligned}
&-\frac{1}{2}U_x[\eta_t^1 + U_t\eta_U^1 + V_t\eta_V^1 - U_t\xi_t^1 - U_t^2\xi_U^1 - U_tV_t\xi_V^1 - U_x\xi_t^2 \\
&-U_tU_t\xi_U^2 - U_xV_t\xi_V^2] - \frac{1}{2}V_x[\eta_t^2 + U_t\eta_U^2 + V_t\eta_V^2 - V_t\xi_t^1 - U_tV_t\xi_U^1 \\
&-V_t^2\xi_V^1 - V_x\xi_t^2 - U_tV_x\xi_U^2 - V_tV_x\xi_V^2] - \\
&(\frac{\alpha}{2}U_x^2 + \frac{\beta}{2}V_x^2 + \frac{1}{2}U_t)[\eta_x^1 + U_x\eta_U^1 + V_x\eta_V^1 \\
&-U_t\xi_x^1 - U_tU_x\xi_U^1 - U_tV_x\xi_V^1 - U_x\xi_x^2 - U_x^2\xi_U^2 - U_xV_x\xi_V^2] \\
&-(\frac{1}{2}V_t + \beta U_xV_x)[\eta_x^2 + U_x\eta_U^2 + V_x\eta_V^2 - V_t\xi_x^1 - U_xV_t\xi_U^1 \\
&-V_tV_x\xi_V^1 - V_x\xi_x^2 - U_xV_x\xi_U^2 - V_x^2\xi_V^2] + \gamma U_{xx}[D_x^2\eta^1 - U_tD_x^2\xi^1 \\
&-U_xD_x^2\xi^2 - 2U_{tx}(\xi_x^1 + U_x\xi_U^1 + V_x\xi_V^1) - 2U_{xx}(\xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)] \\
&+\frac{1}{2}[\gamma U_{xx}^2 - \frac{\alpha}{3}U_x^3 - \beta U_xV_x^2 - U_tU_x - V_tV_x][\xi_t^1 + U_t\xi_U^1 \\
&+V_t\xi_V^1 + \xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2] \\
&= B_t^1 + U_tB_U^1 + V_tB_V^1 + B_x^2 + U_xB_U^2 + V_xB_V^2.
\end{aligned} \tag{2.17}$$

Splitting the above system of partial differential equations with respect to different combinations of derivatives of U and V results in an overdetermined systems of linear partial differential equations for ξ^1, ξ^2, η^1, η^2, B^1 and B^2:

$$\xi^1_t = 0, \tag{2.18}$$
$$\xi^1_x = 0, \tag{2.19}$$
$$\xi^1_U = 0, \tag{2.20}$$
$$\xi^1_V = 0, \tag{2.21}$$
$$\xi^2_U = 0, \tag{2.22}$$
$$\xi^2_V = 0, \tag{2.23}$$
$$\xi^2_x = 0, \tag{2.24}$$
$$\eta^1_{xx} = 0, \tag{2.25}$$
$$\eta^1_U = 0, \tag{2.26}$$
$$\eta^1_V = 0, \tag{2.27}$$
$$\eta^2_x = 0, \tag{2.28}$$
$$\eta^2_U = 0, \tag{2.29}$$
$$\eta^2_V = 0, \tag{2.30}$$
$$\xi^2_t - \alpha\eta^1_x = 0, \tag{2.31}$$
$$(\alpha-\beta)\eta^1_x = 0, \tag{2.32}$$
$$B^1_U = -\frac{1}{2}\eta^1_x, \tag{2.33}$$
$$B^1_V = -\frac{1}{2}\eta^2_x, \tag{2.34}$$
$$B^2_U = -\frac{1}{2}\eta^1_t, \tag{2.35}$$
$$B^2_V = -\frac{1}{2}\eta^2_t, \tag{2.36}$$
$$B^1_t + B^2_x = 0. \tag{2.37}$$

Solving the above systems of PDEs prompt the following two cases:

Case 1. $\alpha \neq \beta$, where α, β are non-zero constants.

In this case, after solving equations (2.18), (2.19), (2.20) and (2.21) we deduce that

$$\xi^1 = C_1, \tag{2.38}$$

where C_1 is an arbitrary constant. Solving equations (2.22), (2.23) and (2.24) we get

$$\xi^2 = A(t), \tag{2.39}$$

where $A(t)$ is an arbitrary function of t. Solving equations (2.26) and (2.27) we obtain

$$\eta^1 = Q(t,x), \tag{2.40}$$

where $Q(t,x)$ is an arbitrary function of t and x. Substituting equation (2.40) into equation (2.25) and integrating twice with respect to x, we obtain

$$\eta^1 = D(t)x + E(t). \tag{2.41}$$

Solving equations (2.28), (2.29) and (2.30) yields

$$\eta^2 = F(t), \tag{2.42}$$

where $F(t)$ is an arbitrary function of t. Substituting (2.41) into (2.32) we get

$$D(t) = 0. \tag{2.43}$$

Hence

$$\eta^1 = E(t). \tag{2.44}$$

Substituting the values of ξ^2 and η^1 from equations (2.39) and (2.44)
into (2.33) gives

$$A(t) = C_2, \tag{2.45}$$

thus

$$\xi^2 = C_2, \tag{2.46}$$

where C_2 is an arbitrary constant. Substituting the value of η^1 from equation (2.44) into equation (2.33) and integrating once with respect to U we obtain

$$B^1 = G(t,x,V). \tag{2.47}$$

Substituting the values of η^2 and B^1 from (2.42) and (2.47) into (2.34) and integrating once with respect to V we get

$$G(t,x,V) = a(t,x). \tag{2.48}$$

Now replacing the values of $G(t,x,V)$ back into equation (2.47) we obtain

$$B^1 = a(t,x), \tag{2.49}$$

where $a(t,x)$ is an arbitrary functions of t and x. Substituting the value of η^1 from equation (2.44) into equation (2.35) and integrate with respect U we obtain

$$B^2 = -\frac{1}{2}UE'(t) + H(t,x,V). \tag{2.50}$$

Inserting the values of η^2 and B^2 from equations (2.47) and (2.50) into equation (2.36) and integrate with respect to V we obtain

$$H(t,x,V) = -\frac{1}{2}VF'(t) + b(t,x). \tag{2.51}$$

Substituting the value of $H(t,x,V)$ back into equation (2.50) yields

$$B^2 = -\frac{1}{2}UE'(t) - \frac{1}{2}VF'(t) + b(t,x), \tag{2.52}$$

where $b(t,x)$ is an arbitrary functions of t and x. Substituting the values of B^1 and B^2 into equation (2.37) we get

$$a_t + b_x = 0. \tag{2.53}$$

Thus, we obtain the solution for the systems of PDEs as:

$$\begin{aligned}
\xi^1 &= C_1,\\
\xi^2 &= C_2,\\
\eta^1 &= E(t),\\
\eta^2 &= F(t),\\
B^1 &= a(t,x),\\
B^2 &= -\frac{1}{2}UE'(t) - \frac{1}{2}VF'(t) + b(t,x),\\
a_t + b_x &= 0.
\end{aligned} \tag{2.54}$$

Therefore we get the following Noether symmetries and gauge terms:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0, \\
X_2 &= \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0, \\
X_{E(t)} &= E(t)\frac{\partial}{\partial U}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}UE'(t), \\
X_{F(t)} &= F(t)\frac{\partial}{\partial V}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}VF'(t).
\end{aligned}
$$

Here we have set $a = 0$ and $b = 0$ as they contribute to the trivial part of the conserved vector.

We now recall the formula for conserved vectors for the second-order Lagrangian (2.11) [18, 36] viz.,

$$
\begin{aligned}
T^1 \;=\; & -B^1 + \xi^1 L + W^1\left[\frac{\partial L}{\partial U_t} - D_t\frac{\partial L}{\partial U_{tt}} - D_x\frac{\partial L}{\partial U_{tx}}\dots,\right] \\
& + W^2\left[\frac{\partial L}{\partial V_t} - D_t\frac{\partial L}{\partial V_{xt}} - D_x\frac{\partial L}{\partial V_{tt}}\dots,\right] \\
& + D_t(W^1)\frac{\partial L}{\partial U_{tt}} + D_t(W^2)\frac{\partial L}{\partial V_{tt}},
\end{aligned}
\tag{2.55}
$$

$$
\begin{aligned}
T^2 \;=\; & -B^2 + \xi^2 L + W^1\left[\frac{\partial L}{\partial U_x} - D_t\frac{\partial L}{\partial U_{xt}} - D_x\frac{\partial L}{\partial U_{xx}}\dots,\right] \\
& + W^2\left[\frac{\partial L}{\partial V_x} - D_t\frac{\partial L}{\partial V_{xt}} - D_x\frac{\partial L}{\partial V_{xx}}\dots,\right] \\
& + D_x(W^1)\frac{\partial L}{\partial U_{xx}} + D_x(W^2)\frac{\partial L}{\partial V_{xx}},
\end{aligned}
\tag{2.56}
$$

where $W^1 = \eta^1 - U_t\xi^1 - U_x\xi^2$ and $W^2 = \eta^2 - V_t\xi^1 - V_x\xi^2$ are the characteristic functions.

Using equations (2.55) and (2.56) together with X_1, we obtain the conserved vector

$$
\begin{aligned}
T_1^1 &= L - U_t\Big(-\frac{1}{2}U_x\Big) - V_t\Big(-\frac{1}{2}V_x\Big)\\
&= L + \frac{1}{2}U_tU_x + \frac{1}{2}V_tV_x\\
&= \frac{\gamma}{2}U_{xx}^2 - \frac{\alpha}{6}U_x^3 - \frac{\beta}{2}U_xV_x^2 - \frac{1}{2}U_tU_x - \frac{1}{2}V_tV_x + \frac{1}{2}U_tU_x + \frac{1}{2}V_tV_x\\
&= \frac{\gamma}{2}U_{xx}^2 - \frac{\alpha}{6}U_x^3 - \frac{\beta}{2}U_xV_x^2,\\
T_1^2 &= -U_t\Big(-\frac{\alpha}{2}U_x^2 - \frac{\beta}{2}V_x^2 - \frac{1}{2}U_t - \gamma U_{xxx}\Big) - V_t\Big(-\beta U_xV_x - \frac{1}{2}V_t\Big) + D_x(-U_t).\gamma U_{xx}\\
&= \frac{\alpha}{2}U_tU_x^2 + \frac{\beta}{2}U_tV_x^2 + \gamma U_tU_{xxx} + \beta U_xV_tV_x + \frac{1}{2}U_t^2 + \frac{1}{2}V_t^2 - U_{tx}U_{xx}.
\end{aligned} \tag{2.57}
$$

Invoking the inverse transformation $U \to \int udx$, $V \to \int vdx$ to the equation (2.57) yields

$$
\begin{aligned}
T_1^1 &= \frac{\gamma u_x^2}{2} - \frac{\alpha u^3}{6} - \frac{\beta uv^2}{2},\\
T_1^2 &= \frac{\alpha u^2}{2}\int u_tdx + \frac{\beta v^2}{2}\int u_tdx + \gamma u_{xx}\int u_tdx + \beta uv\int v_tdx - \gamma u_tu_x\\
&\quad + \frac{1}{2}\Big[\int u_tdx\Big]^2 + \frac{1}{2}\Big[\int v_tdx\Big]^2.
\end{aligned} \tag{2.58}
$$

Following the above procedure, we obtain the following conservation laws associated with X_2 and $X_{E(t),F(t)}$ respectively as

$$
\begin{aligned}
T_2^1 &= \frac{u^2}{2} + \frac{v^2}{2},\\
T_2^2 &= -\frac{\gamma u_x^2}{2} + \frac{\alpha u^3}{3} + \beta uv^2 + \gamma uu_{xx},
\end{aligned} \tag{2.59}
$$

$$
\begin{aligned}
T_{(E,F)}^1 &= -\frac{u}{2}E(t) - \frac{v}{2}F(t),\\
T_{(E,F)}^2 &= -\beta uvF(t) - E(t)\Big[\frac{\alpha u^2}{2} + \frac{\beta v^2}{2} + \gamma u_{xx}\Big] - \frac{1}{2}E(t)\int u_tdx\\
&\quad - \frac{1}{2}F(t)\int v_tdx + \frac{1}{2}E'(t)\int udx + \frac{1}{2}F'(t)\int vdx.
\end{aligned} \tag{2.60}
$$

Here we extract two special cases from conserved vector (2.60).

Choosing $E(t) = 1$ and $F(t) = 0$ gives a nonlocal conserved vector

$$
\begin{aligned}
T_3^1 &= -\frac{u}{2},\\
T_3^2 &= -\frac{\alpha u^2}{2} - \frac{\beta v^2}{2} - \gamma u_{xx} - \frac{1}{2}\int u_tdx
\end{aligned} \tag{2.61}
$$

and for $E(t)=0$ and $F(t)=1$, we obtain a nonlocal conserved vector

$$T_4^1 = -\frac{v}{2},$$
$$T_4^2 = -\beta uv - \frac{1}{2}\int v_t dx. \tag{2.62}$$

Therefore we note that for different values of arbitrary functions $E(t)$ and $F(t)$, one obtains infinitely many nonlocal conserved vectors.

Case 2. $\alpha = \beta$, where α, β are non-zero constants.

In this case we obtain the following Noether symmetries and gauge terms

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0,\\
X_2 &= \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0,\\
X_3 &= \alpha t\frac{\partial}{\partial x} + x\frac{\partial}{\partial U}, \quad B^1 = -\frac{1}{2}U, \quad B^2 = 0,\\
X_{E(t)} &= E(t)\frac{\partial}{\partial U}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}UE'(t),\\
X_{F(t)} &= F(t)\frac{\partial}{\partial V}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}VF'(t).
\end{aligned}$$

In the similar fashion we have set $a=0$ and $b=0$ as they contribute to the trivial part of the conserved vector, also setting $u = U_x$, $v = V_x$. The independent conserved vectors for the system (2.9) corresponding to X_1, X_2, X_3 and $X_{E(t),F(t)}$ respectively are

$$\begin{aligned}
T_1^1 &= \frac{\gamma u_x^2}{2} - \frac{\alpha u^3}{6} - \frac{\beta uv^2}{2},\\
T_1^2 &= \frac{\alpha u^2}{2}\int u_t dx + \frac{\beta v^2}{2}\int u_t dx + \gamma u_{xx}\int u_t dx + \beta uv\int v_t dx - \gamma u_t u_x\\
&\quad +\frac{1}{2}\left[\int u_t dx\right]^2 + \frac{1}{2}\left[\int v_t dx\right]^2,
\end{aligned} \tag{2.63}$$

$$\begin{aligned}
T_2^1 &= \frac{u^2}{2} + \frac{v^2}{2},\\
T_2^2 &= -\frac{\gamma u_x^2}{2} + \frac{\alpha u^3}{3} + \beta uv^2 + \gamma u u_{xx},
\end{aligned} \tag{2.64}$$

$$\begin{aligned}
T_3^1 &= \frac{\alpha t u^2}{2} + \frac{\alpha t v^2}{2} + \frac{1}{2}\int u dx,\\
T_3^2 &= -\frac{\alpha\gamma t u_x^2}{2} + \frac{\alpha^2 t u^3}{3} + \alpha^2 t uv^2 + t\alpha\gamma u u_{xx},
\end{aligned} \tag{2.65}$$

$$
\begin{aligned}
T^1_{(E,F)} &= -\frac{u}{2}E(t) - \frac{v}{2}F(t),\\
T^2_{(E,F)} &= -\beta uvF(t) - E(t)\left[\frac{\alpha u^2}{2} + \frac{\beta v^2}{2} + \gamma u_{xx}\right] - \frac{1}{2}E(t)\int u_t dx\\
&\quad -\frac{1}{2}F(t)\int v_t dx + \frac{1}{2}E'(t)\int u dx + \frac{1}{2}F'(t)\int v dx.
\end{aligned}
\tag{2.66}
$$

We now extract two special cases from the conserved vector (2.66).

Letting $E(t) = 1$ and $F(t) = 0$ gives a nonlocal conserved vector

$$
\begin{aligned}
T^1_4 &= -\frac{u}{2},\\
T^2_4 &= -\frac{\alpha u^2}{2} - \frac{\beta v^2}{2} - \gamma u_{xx} - \frac{1}{2}\int u_t dx
\end{aligned}
\tag{2.67}
$$

and for $E(t) = 0$ and $F(t) = 1$ gives a nonlocal conserved vector

$$
\begin{aligned}
T^1_5 &= -\frac{v}{2},\\
T^2_5 &= -\beta uv - \frac{1}{2}\int v_t dx.
\end{aligned}
\tag{2.68}
$$

Remark: For the arbitrary values of $E(t)$ and $F(t)$ infinitely many nonlocal conservation laws exist for system (2.9). It should be noted that for the special values α, β and γ, namely $\alpha = -6$, $\beta = -2$ and $\gamma = -1$, we retrieve the three constants of motion F_0, F_1 and F_2 obtained in [37].

2.3 Conclusion

In this chapter we performed some symmetry reduction of a generalized Ito-type coupled KdV system using Lie group analysis. We also obtained conservation laws for a generalized Ito-type coupled KdV system. This system does not have a Lagrangian. In order to apply Noether's theorem, we transformed the system into the fourth-order system (2.10), which admitted a Lagrangian (2.11). Noether's approach was then used to derive the conservation laws in U and V variables. Finally, by reverting back to our original variables u and v using the inverse transformation $U \to \int udx$, $V \to \int vdx$, we obtained the conservation laws for the third-order generalized Ito-type coupled KdV system (2.9). The conservation laws for the generalized Ito-type coupled KdV system consist of some local and infinite number of nonlocal conserved vectors.

Chapter 3

Exact solutions and conservation laws of the Boussinesq system of KdV-KdV type equation

In this chapter, we will use the Lie symmetry method together with simplest equation method [26–29] to find the solutions of the Boussinesq system of KdV-KdV type equation [24], namely

$$\begin{aligned} u_t + v_x + vu_x + uv_x + av_{xxx} &= 0, \\ v_t + u_x + vv_x + cu_{xxx} &= 0, \end{aligned} \tag{3.1}$$

where a and c are arbitrary constants. We also derive the conservation laws for (3.1) using Noether theorem [18]. The classical Boussinesq systems was first derived by Boussinesq to describe the two-way propagation of small amplitude, long wavelength gravity waves on the surface of water in a canal. We will refer to it as the KdV-KdV system because of the special form of the dispersive terms (the terms with third-order derivatives). The system is an approximation to the full two-dimensional Euler equations for surface wave propagation along a channel with a flat bottom filled with an ideal fluid. In [38], fully discrete Galerkin-finite element method was used to find numerical solutions of classical Boussinesq systems.

3.1 Symmetries of the Boussinesq system of KdV-KdV type equation

In this section we determine the Lie point symmetry of Boussinesq system of KdV-KdV type system (3.1). The symmetry group of the Boussinesq system of KdV-KdV type equation will be generated by the vector field of the form

$$\Psi = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}. \quad (3.2)$$

Applying the third prolongation $\mathrm{pr}^{(3)}\Psi$ [13] to system (3.1) we obtain the overdetermined system of linear partial differential equations (PDEs):

$$\begin{aligned}
\xi^1_x &= 0,\\
\xi^1_u &= 0,\\
\xi^1_v &= 0,\\
\xi^2_u &= 0,\\
\xi^2_v &= 0,\\
\eta^1_{uu} &= 0,\\
\eta^1_{vv} &= 0,\\
\eta^2_{uu} &= 0,\\
\eta^2_{vv} &= 0,\\
\eta^1_{uv} &= 0,\\
\eta^2_{uv} &= 0,\\
\eta^1_{xv} &= 0,\\
\eta^2_{xu} &= 0,
\end{aligned}$$

$$c\eta^1_v - a\eta^2_u = 0,$$
$$\eta^1_{xu} - \xi^2_{xx} = 0,$$
$$\xi^2_{xx} - \eta^2_{xv} = 0,$$
$$2\xi^2_x + 3c\eta^1_{xxu} - c\xi^2_{xxx} = 0,$$
$$c\eta^1_{xxx} + \eta^1_x + \eta^2_t + v\eta^2_x = 0,$$
$$\xi^1_t - 3\xi^2_x - \eta^1_u - \eta^2_v = 0,$$
$$3\xi^2_x - \xi^1_t - \eta^1_u - \eta^2_v = 0,$$
$$a\eta^2_{xxx} + u\eta^2_x + \eta^2_x + v\eta^1_x + \eta^1_t = 0,$$
$$c\eta^1 - cv\xi^2_x + cv\xi^1_t - c\xi^2_t + c\eta^2_u + cu\eta^2_u - a\eta^2_u = 0,$$
$$\eta^2 + 2v\xi^2_x - \xi^2_t - v\eta^1_u - \eta^2_u - u\eta^2_u + \eta^1_v + v\eta^1_v = 0,$$
$$c\eta^1 + cu\xi^1_t + c\xi^1_t - cu\xi^2_x - c\xi^2_x - c\eta^1_u - cu\eta^1_u +$$
$$cv\eta^1_v - av\eta^2_u + uv\eta^2_v + c\eta^2_v + 3ac\eta^2_{xxv} - ac\xi^2_{xxx} = 0.$$

The general solution of the overdetermined system of linear PDEs with the aid of mathematical software, Maple, is given by

$$\begin{aligned}
\xi^1(t,x,u,v) &= C_1,\\
\xi^2(t,x,u,v) &= C_2 + tC_3,\\
\eta^1(t,x,u,v) &= 0,\\
\eta^2(t,x,u,v) &= C_3,
\end{aligned} \tag{3.3}$$

where C_1, C_2, C_3 are arbitrary constants. Thus the Lie point symmetries of the Boussinesq system of KdV-KdV type equation (3.1) are spanned by the following three vector fields

$$\begin{aligned}
\Psi_1 &= \frac{\partial}{\partial t},\\
\Psi_2 &= \frac{\partial}{\partial x},\\
\Psi_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}.
\end{aligned}$$

The commutation relations between these vector fields is given in **Table 2**, where the entry in row i and column j is represented by $[\Psi_i, \Psi_j]$:

Table 2. Commutator table of the Lie algebra of (3.1)

$[\Psi_i, \Psi_j]$	Ψ_1	Ψ_2	Ψ_2
Ψ_1	0	0	Ψ_3
Ψ_2	0	0	0
Ψ_3	$-\Psi_2$	0	0

3.1.1 Symmetry reduction and exact solution of the Boussinesq system of KdV-KdV type equation (3.1)

In this subsection we will use symmetries obtained in the above section to reduce system (3.1) into ODEs.

In order to obtain symmetry reduction and exact solutions, one has to solve the associated characteristics equations

$$\frac{dt}{\xi^1(t,x,u,v)} = \frac{dx}{\xi^2(t,x,u,v)} = \frac{du}{\eta^1(t,x,u,v)} = \frac{dv}{\eta^2(t,x,u,v)}.$$

We consider the following cases.

Case 1.

The symmetry $\Psi_1 + k\Psi_2$ gives rise to the group-invariant solution

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z), \tag{3.4}$$

where $z = x - kt$ is an invariant of the symmetry $\Psi_1 + k\Psi_2$. Substitution of (3.4) into system (3.1) results in the system of ODEs where $E(z)$ and $F(z)$ satisfy

$$\begin{aligned} aF'''(z) + E(z)F'(z) + F(z)E'(z) + F'(z) - kE'(z) &= 0, \\ cE'''(z) + F(z)F'(z) + E'(z) - kF'(z) &= 0. \end{aligned} \tag{3.5}$$

Case 2.

In this case, after solving the corresponding characteristic system for the symmetry $\Psi_3 + \lambda\Psi_1$, one obtains an invariant $z = x - t^2/(2\lambda)$ and the group-invariant solution of the form

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z) + \frac{t}{\lambda},$$

where the functions $E(z)$ and $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} aF'''(z) + E(z)F'(z) + F(z)E'(z) + F'(z) &= 0, \\ c\lambda E'''(z) + \lambda F(z)F'(z) + \lambda E'(z) + 1 &= 0. \end{aligned} \tag{3.6}$$

Case 3.

For the symmetry Ψ_2, one obtains an invariant $z = x$ and the group-invariant solution of the form

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z),$$

where the functions $E(z)$ and $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} aF'''(z) + E(z)F'(z) + F(z)E'(z) + F'(z) &= 0, \\ cE'''(z) + F(z)F'(z) + E'(z) &= 0. \end{aligned} \tag{3.7}$$

3.1.2 Exact solutions of system (3.1) using simplest equation method

The simplest equation method which was introduced by Kudryashov [26, 27] and modified by Vitanov [28] (see also [29]), will be used to find exact solutions for system (3.5). The simplest equations that will be used are the Bernoulli and Ricatti equations.

Let us consider the solutions of system (3.5) in the form

$$E(z) = \sum_{i=0}^{M} A_i (H(z))^i \quad \text{and} \quad F(z) = \sum_{i=0}^{M} B_i (H(z))^i, \tag{3.8}$$

where $H(z)$ satisfies the Bernoulli or Ricatti equation, M is a positive integer that can be determined by balancing procedure as in [28] and $A_0, \cdots, A_M$, $B_0, \cdots, B_M$, are parameters to be determined. We note that the Bernoulli and Ricatti equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

For the Bernoulli equation

$$H'(z) = qH(z) + pH^2(z), \tag{3.9}$$

where q and p are constants. As a result the solution of the Bernoulli equation is

$$H(z) = q\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right). \tag{3.10}$$

For the Ricatti equation

$$H'(z) = qH^2(z) + pH(z) + r, \tag{3.11}$$

we shall use the solutions

$$H(z) = -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right] \tag{3.12}$$

and

$$H(z) = -\frac{p}{2q} - \frac{\theta}{2q}\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \left(\frac{2p}{\theta}\right)\sinh\left(\frac{\theta z}{2}\right)} \tag{3.13}$$

with $\theta^2 = p^2 - 4qr > 0$ and C is a constant of integration.

Solutions of system (3.1) using the Bernoulli equation as the simplest equation

The balancing procedure [28] yields $M = 2$, so the solutions of system (3.5) are of the form

$$E(z) = A_0 + A_1H + A_2H^2 \quad \text{and} \quad F(z) = B_0 + B_1H + B_2H^2. \tag{3.14}$$

Substituting equation (3.14) into system (3.5), making use of the Bernoulli equation (3.9) and equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1, A_2 and B_0, B_1, B_2. These algebraic equations are

$$
\begin{aligned}
2pB_2^2 + 24cp^3A_2 &= 0,\\
4pA_2B_2 + 24ap^3B_2 &= 0,\\
2qB_2^2 + 6cp^3A_1 + 3pB_1B_2 + 54cqp^2A_2 &= 0,\\
cq^3A_1 + qB_0B_1 + qA_1 - kqB_1 &= 0,\\
aq^3B_1 + qA_1B_0 + qB_1 + qA_0B_1 - kqA_1 &= 0,\\
4qA_2B_2 + 6ap^3B_1 + 3pA_1B_2 + 3pA_2B_1 + 54aqp^2B_2 &= 0,\\
pB_1^2 + 38cpq^2A_2 + 3qB_1B_2 + 2pA_2 - 2kpB_2 + 2pB_0B_2 + 12cqp^2A_1 &= 0,\\
2pA_0B_2 - 2kpA_2 + 3qA_2B_1 + 3qA_1B_2 + 2pB_2 + 2pA_2B_0 + 38apq^2B_2 &\\
+2pA_1B_1 + 12aqp^2B_1 &= 0,\\
pB_0B_1 + pA_1 - 2kpB_2 - kpB_1 + 8cq^3A_2 + 7cpq^2A_1 + 2qB_0B_2 + qB_1^2 + 2qA_2 &= 0,\\
pA_0B_1 + pA_1B_0 + 8aq^3B_2 - 2kqA_2 - kpA_1 + 2qA_0B_2 + 2qA_1B_1 + pB_1 + 2qB_2 &\\
+7apq^2B_1 + 2qA_2B_0 &= 0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Maple, one obtains

$$
\begin{aligned}
A_1 &= -6apq,\\
A_2 &= -6ap^2,\\
B_1 &= \pm pq\sqrt{72ac},\\
B_2 &= \pm p^2\sqrt{72ac},\\
A_0 &= \frac{A_2B_2^2q^2 - 12B_2^2p^2 + 12A_2^2p^2}{12B_2^2p^2},\\
B_0 &= \frac{B_2^2q^2 - 12A_2p^2 + 12kp^2B_2}{12B_2p^2}.
\end{aligned}
$$

Hence the solution of system (3.1) is given by

$$u(t,x) = A_0 + A_1 q\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right)$$
$$+A_2 q^2\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right)^2, \tag{3.15}$$
$$v(t,x) = B_0 + B_1 q\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right)$$
$$+B_2 q^2\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right)^2, \tag{3.16}$$

where $z = x - kt$.

Solutions of (3.1) using the Ricatti equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of system (3.5) are of the form

$$E(z) = A_0 + A_1 H + A_2 H^2 \quad \text{and} \quad F(z) = B_0 + B_1 H + B_2 H^2. \tag{3.17}$$

Substituting equation (3.17) into system (3.5), making use of the Riccati equation (3.11) we obtain an algebraic system of equations in terms of A_0, A_1, A_2 and B_0, B_1, B_2, by equating all coefficients of the functions H^i to zero. The corresponding algebraic equations are

$$
\begin{aligned}
2B_2{}^2q+24cA_2q^3 &= 0,\\
4\,B_2A_2q+24\,aB_2q^3 &= 0,\\
54\,cA_2q^2p+6\,cA_1q^3+2\,B_2{}^2p+3\,B_1B_2q &= 0,\\
6\,aB_1q^3+54\,aB_2q^2p+3\,B_2A_1q+3\,B_1A_2q+4\,B_2A_2p &= 0,\\
2\,cA_1qr^2+cA_1p^2r+6\,cA_2pr^2+A_1r-kB_1r+B_0B_1r &= 0,\\
B_1r+2\,aB_1qr^2+aB_1p^2r+6\,aB_2pr^2+A_0B_1r-kA_1r+B_0A_1r &= 0,\\
2\,A_2q-2\,kB_2q+2\,B_0B_2q+3\,B_1B_2p+2\,B_2{}^2r+12\,cA_1q^2p+38\,cA_2qp^2 &\\
+B_1{}^2q+40\,cA_2q^2r &= 0,\\
cA_1p^3-kB_1p+2\,B_0B_2r+B_0B_1p+B_1{}^2r+A_1p+8\,cA_1qpr+2\,A_2r+14\,cA_2p^2r &\\
+16\,cA_2qr^2-2\,kB_2r &= 0,\\
2\,B_0A_2q+3\,B_2A_1p-2\,kA_2q+2\,B_1A_1q+40\,aB_2q^2r+4\,B_2A_2r+2\,A_0B_2q+3\,B_1A_2p &\\
+2\,B_2q+12\,aB_1q^2p+38\,aB_2qp^2 &= 0,\\
2\,A_2p+52\,cA_2qpr-2\,kB_2p+7\,cA_1qp^2+B_0B_1q+8\,cA_2p^3-kB_1q+2\,B_0B_2p+A_1q &\\
+B_1{}^2p+3\,B_1B_2r+8\,cA_1q^2r &= 0,\\
aB_1p^3+A_0B_1p+B_1p+B_0A_1p+2\,B_2r+8\,aB_1qpr-2\,kA_2r+2\,B_1A_1r+2\,B_0A_2r &\\
+16\,aB_2qr^2+2\,A_0B_2r-kA_1p+14\,aB_2p^2r &= 0,\\
B_0A_1q+A_0B_1q+8\,aB_2p^3+52\,aB_2qpr+7\,aB_1qp^2-2\,kA_2p+2\,B_1A_1p+2\,B_0A_2p+2\,B_2p &\\
+2\,A_0B_2p+3\,B_1A_2r-kA_1q+8\,aB_1q^2r+3\,B_2A_1r+B_1q &= 0.
\end{aligned}
$$

Solving the above system of algebraic equations with the aid of Maple, one obtains

$$
\begin{aligned}
A_1 &= -6apq,\\
A_2 &= -6aq^2,\\
B_1 &= \pm pq\sqrt{72ac},\\
B_2 &= \pm q^2\sqrt{72ac},\\
A_0 &= \frac{A_2B_2^2p^2-12B_2^2q^2+12A_2^2q^2+8rqA_2B_2^2}{12B_2^2q^2},\\
B_0 &= \frac{B_2^2p^2-12A_2q^2+12kq^2B_2+8rqB_2^2}{12B_2q^2}.
\end{aligned}
$$

Hence the solutions of system (3.1) are given by

$$
\begin{aligned}
u(t,x) &= A_0 + A_1\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\
&\quad + A_2\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
v(t,x) &= B_0 + B_1\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\
&\quad + B_2\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
u(t,x) &= A_0 + A_1\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\
&\quad + A_2\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
v(t,x) &= B_0 + B_1\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2c}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\
&\quad + B_2\left\{-\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (3.21)
\end{aligned}
$$

where $z = x - kt$.

3.2 Conservation laws for the Boussinesq system of KdV-KdV type equation

In this subsection we will construct the conservation laws of the Boussinesq KdV system

$$
\begin{aligned}
u_t + v_x + u_x v + u v_x + a v_{xxx} &= 0, \\
v_t + u_x + v v_x + c u_{xxx} &= 0, \qquad (3.22)
\end{aligned}
$$

where a and c are arbitrary constants. Here we will derive the conservation laws for system (3.22) using the Noether's theorem. In order to apply the Noether's approach we use the transformations $u \to U_x$, $v \to V_x$ then system (3.22) becomes

$$U_{tx} + V_{xx} + U_{xx}V_x + U_xV_{xx} + aV_{xxxx} = 0,$$
$$V_{xt} + U_{xx} + V_xV_{xx} + cU_{xxxx} = 0. \tag{3.23}$$

It can be readily verified that L given by

$$L = \frac{1}{2}\left\{cU_{xx}^2 + aV_{xx}^2 - U_x^2 - V_x^2 - U_xV_x^2 - U_tV_x - U_xV_t\right\}, \tag{3.24}$$

is a Lagrangian for system (3.23). Inserting L from (3.24) into equation (2.16) and expand yields

$$\begin{aligned}
&-\frac{1}{2}V_x[\eta_t^1 + U_t\eta_U^1 + V_t\eta_V^1 - U_t\xi_t^1 - U_t^2\xi_U^1 - U_tV_t\xi_V^1 - U_x\xi_t^2 \\
&-U_tU_t\xi_U^2 - U_xV_t\xi_V^2] - \frac{1}{2}U_x[\eta_t^2 + U_t\eta_U^2 + V_t\eta_V^2 - V_t\xi_t^1 - U_tV_t\xi_U^1 \\
&-V_t^2\xi_V^1 - V_x\xi_t^2 - U_tV_x\xi_U^2 - V_tV_x\xi_V^2] \\
&-(U_x + \frac{1}{2}V_x^2 + \frac{1}{2}V_t)[\eta_x^1 + U_x\eta_U^1 + V_x\eta_V^1 \\
&-U_t\xi_x^1 - U_tU_x\xi_U^1 - U_tV_x\xi_V^1 - U_x\xi_x^2 - U_x^2\xi_U^2 - U_xV_x\xi_V^2] \\
&-(V_x + \frac{1}{2}U_t + U_xV_x)[\eta_x^2 + U_x\eta_U^2 + V_x\eta_V^2 - V_t\xi_x^1 - U_xV_t\xi_U^1 \\
&-V_tV_x\xi_V^1 - V_x\xi_x^2 - U_xV_x\xi_U^2 - V_x^2\xi_V^2] + cU_{xx}[D_x^2\eta^1 - U_tD_x^2\xi^1 \\
&-U_xD_x^2\xi^2 - 2U_{tx}(\xi_x^1 + U_x\xi_U^1 + V_x\xi_V^1) - 2U_{xx}(\xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)] \\
&+aV_{xx}[D_x^2\eta^2 - V_tD_x^2\xi^1 - V_xD_x^2\xi^2 - 2V_{tx}(\xi_x^1 + U_x\xi_U^1 + V_x\xi_V^1) \\
&-2V_{xx}(\xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)] \\
&+\frac{1}{2}[cU_{xx}^2 + aV_{xx}^2 - U_x^2 - V_x^2 - U_xV_x^2 - U_tV_x - U_xV_t][\xi_t^1 + U_t\xi_U^1 \\
&+V_t\xi_V^1 + \xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2] \\
&= B_t^1 + U_tB_U^1 + V_tB_V^1 + B_x^2 + U_xB_U^2 + V_xB_V^2.
\end{aligned} \tag{3.25}$$

Splitting (3.25) with respect to different combinations of derivatives of U and V results in an overdetermined systems of equations for ξ^1, ξ^2, η^1, η^2, B^1 and B^2. The solution of the overdetermined system yields the following Noether symmetries and gauge terms

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0, \\
X_2 &= \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0, \\
X_{H(t)} &= H(t)\frac{\partial}{\partial U}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}VH'(t), \\
X_{J(t)} &= J(t)\frac{\partial}{\partial V}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}UJ'(t).
\end{aligned} \tag{3.26}
$$

The above results will now be used to find the components of the conserved vectors. Equations (2.55) and (2.56) together with (3.26) and $u = U_x$, $v = V_x$ yield the following independent conserved vectors for system (3.22).

$$
\begin{aligned}
T_1^1 &= \frac{cu_x^2}{2} + \frac{av_x^2}{2} - \frac{u^2}{2} - \frac{v^2}{2} - \frac{uv^2}{2}, \\
T_1^2 &= u\int u_t dx + v\int v_t dx + uv\int v_t dx + \frac{v^2}{2}\int u_t dx + \int u_t dx\int v_t dx \\
&\quad + cu_{xx}\int u_t dx + av_{xx}\int v_t dx - cu_t u_x - av_t v_x,
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
T_2^1 &= uv, \\
T_2^2 &= cuu_{xx} + avv_{xx} - \frac{cu_x^2}{2} - \frac{av_x^2}{2} + uv^2 + \frac{u^2}{2} + \frac{v^2}{2},
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
T_{(H,J)}^1 &= -\frac{v}{2}H(t) - \frac{u}{2}J(t), \\
T_{(H,J)}^2 &= -H(t)\left[u + \frac{v^2}{2} + cu_{xx}\right] - J(t)\left[v + uv + av_{xx}\right] - \frac{1}{2}J(t)\int u_t dx \\
&\quad - \frac{1}{2}H(t)\int v_t dx + \frac{1}{2}J'(t)\int u dx + \frac{1}{2}H'(t)\int v dx.
\end{aligned} \tag{3.29}
$$

Here we extract two special cases from conserved vector (3.29).

Choosing $H(t) = 1$ and $J(t) = 0$ gives a nonlocal conserved vector

$$
\begin{aligned}
T_3^1 &= -\frac{v}{2}, \\
T_3^2 &= -u - \frac{v^2}{2} - cu_{xx} - \frac{1}{2}\int v_t dx
\end{aligned} \tag{3.30}
$$

and for $H(t) = 0$ and $J(t) = 1$ gives a nonlocal conserved vector

$$
\begin{aligned}
T_4^1 &= -\frac{u}{2}, \\
T_4^2 &= -v - uv - av_{xx} - \frac{1}{2}\int u_t dx.
\end{aligned}
$$

Remark: For the arbitrary values of $H(t)$ and $J(t)$ infinitely many nonlocal conservation laws exist for system (3.22).

3.3 Conclusion

In this chapter we obtained the solutions of the Boussinesq system of KdV-KdV type equation by using its Lie point symmetries along with the simplest equation method. Solutions obtained were travelling wave solutions. Furthermore, conservation laws for the Boussinesq system of KdV-KdV type equation were obtained using Noether approach. The conservation laws consisted of some local and infinite number of nonlocal conserved vectors.

Chapter 4

Exact solutions and conservation laws of a coupled KdV system

In this chapter, we will use the Lie symmetry method together with simplest equation method [26–29] to find the solutions of the coupled KdV system of two nonlinear dispersive wave equations [25], namely

$$\begin{aligned} v_t + u_x + \beta u_x v + \beta u v_x + \gamma u_{xxx} &= 0, \\ u_t + v_x + \alpha v v_x + \beta u u_x + \lambda v_{xxx} &= 0, \end{aligned} \tag{4.1}$$

where α, β, γ and λ are arbitrary constants. Furthermore, we derive the conservation laws for the coupled KdV system using Noether theorem [18]. In [39], the coupled KdV system (4.1) has been indicated as models for long waves of small amplitude at the surface of an ideal fluid in a long rectangular channel with a flat bottom. This system is known to be a valid approximation of the full two-dimensional Euler equations for fluid motion under the influence of gravity in suitably small amplitude, long wavelength regimes.

4.1 Symmetries of a coupled KdV system

In this section we determine the Lie point symmetry of system (4.1). The symmetry group of the coupled KdV system will be generated by the vector field of the form

$$\Gamma = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}. \tag{4.2}$$

Applying the third prolongation $\mathrm{pr}^{(3)}\Gamma$ [13] to system (4.1) results in an overdetermined system of linear partial differential equations. Solving the overdetermined system of linear PDEs we obtain

$$\begin{aligned}
\xi^1(t,x,u,v) &= C_1,\\
\xi^2(t,x,u,v) &= C_2 + \beta t C_3,\\
\eta^1(t,x,u,v) &= 0,\\
\eta^2(t,x,u,v) &= C_3,
\end{aligned} \tag{4.3}$$

where C_1, C_2, C_3 are arbitrary constants. Thus the Lie point symmetries of the coupled KdV system (4.1) are spanned by the following three vector fields:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t},\\
\Gamma_2 &= \frac{\partial}{\partial x},\\
\Gamma_3 &= \beta t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}.
\end{aligned}$$

We now compute the commutation relations for all vector fields. By definition of the Lie bracket (2.3) we have

$$\begin{aligned}
[\Gamma_1, \Gamma_2] &= \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1\\
&= \left(\frac{\partial}{\partial t}\right)\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right)\\
&= 0.
\end{aligned}$$

Likewise, one can obtain the commutation relations between other vector fields. In **Table 3** below, we present the commutator table of the Lie algebra of system (4.1). Here the entry in row i and column j is represented by $[\Gamma_i, \Gamma_j]$:

Table 3.

$[\Gamma_i, \Gamma_j]$	Γ_1	Γ_2	Γ_2
Γ_1	0	0	Γ_2
Γ_2	0	0	0
Γ_3	$-\beta\Gamma_2$	0	0

4.1.1 Symmetry reduction and exact solution of coupled KdV system (4.1)

In this section we will use symmetries obtained in the above section to reduce system (4.1) into system of ODEs. In order to obtain symmetry reduction and exact solutions, one needs to solve the associated characteristics equations

$$\frac{dt}{\xi^1(t,x,u,v)} = \frac{dx}{\xi^2(t,x,u,v)} = \frac{du}{\eta^1(t,x,u,v)} = \frac{dv}{\eta^2(t,x,u,v)}.$$

Here we will perform the symmetry reduction for the following three cases:

Case 1.

In this case, after solving the corresponding characteristic system for the symmetry $\Gamma_3 + a\Gamma_1$, one obtains an invariant $z = x - \beta t^2/(2a)$ and the group-invariant solution is of the form

$$u(t,x) = E(z) + \frac{t}{a} \quad \text{and} \quad v(t,x) = F(z),$$

where the functions $E(z)$ and $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} \gamma E'''(z) + \beta E(z)F'(z) + \beta F(z)E'(z) + E'(z) &= 0, \\ \lambda F'''(z) + \beta E(z)E'(z) + \alpha F(z)F'(z) + F'(z) + \frac{1}{a} &= 0. \end{aligned} \tag{4.4}$$

Case 2.

The symmetry $\Gamma_1 + k\Gamma_2$ gives rise to the group-invariant solution of the form

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z), \tag{4.5}$$

where $z = x - kt$ is an invariant of the symmetry $\Gamma_1 + k\Gamma_2$. Substitution of equation (4.5) into system (4.1) results in the system of ODEs where $E(z)$ and $F(z)$ satisfy

$$\begin{aligned} \gamma E'''(z) + \beta E(z)F'(z) + \beta F(z)E'(z) + E'(z) - kF'(z) &= 0, \\ \lambda F'''(z) + \beta E(z)E'(z) + \alpha F(z)F'(z) + F'(z) - kE'(z) &= 0. \end{aligned} \tag{4.6}$$

Case 3.

For the symmetry Γ_2, one obtains an invariant $z = x$ and the group-invariant solution is of the form

$$u(t,x) = E(z) \quad \text{and} \quad v(t,x) = F(z),$$

where the functions $E(z)$ and $F(z)$ satisfy the following system of ODEs:

$$\begin{aligned} \gamma E'''(z) + \beta E(z)F'(z) + \beta F(z)E'(z) + E'(z) &= 0, \\ \lambda F'''(z) + \beta E(z)E'(z) + \alpha F(z)F'(z) + F'(z) &= 0. \end{aligned} \tag{4.7}$$

4.1.2 Exact solutions of system (4.1) using simplest equation method

Let us consider the solutions of the system (4.6) in the form

$$E(z) = \sum_{i=0}^{M} A_i(H(z))^i \quad \text{and} \quad F(z) = \sum_{i=0}^{M} B_i(H(z))^i, \tag{4.8}$$

where $H(z)$ satisfies the Bernoulli or Ricatti equation.

Solutions of system (4.1) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of system (4.6) are of the form

$$E(z) = A_0 + A_1 H + A_2 H^2 \quad \text{and} \quad F(z) = B_0 + B_1 H + B_2 H^2. \tag{4.9}$$

Substituting equation (4.9) into system (4.6), making use of Bernoulli equation and equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1, A_2 and B_0, B_1, B_2. These algebraic equations are

$$\begin{aligned}
4\beta B_2A_2p + 24\gamma A_2p^3 &= 0,\\
24\lambda B_2p^3 + 2\beta {A_2}^2p + 2\alpha {B_2}^2p &= 0,\\
\gamma A_1q^3 + \beta B_0A_1q - kB_1q + \beta A_0B_1q + A_1q &= 0,\\
\lambda B_1q^3 - kA_1q + B_1q + \beta A_0A_1q + \alpha B_0B_1q &= 0,\\
54\gamma A_2p^2q + 3\beta B_1A_2p + 6\gamma A_1p^3 + 4\beta B_2A_2q + 3\beta B_2A_1p &= 0,\\
54\lambda B_2p^2q + 6\lambda B_1p^3 + 2\beta {A_2}^2q + 3\beta A_1A_2p + 2\alpha {B_2}^2q + 3\alpha B_1B_2p &= 0,\\
-2kB_2p + 2\beta B_0A_2p + 2\beta B_1A_1p + 38\gamma A_2pq^2 + 2A_2p &\\
+12\gamma A_1p^2q + 3\beta B_2A_1q + 3\beta B_1A_2q + 2\beta A_0B_2p &= 0,\\
2\beta A_0A_2p + \beta {A_1}^2p + 38\lambda B_2pq^2 + 2B_2p + \alpha {B_1}^2p &\\
+3\alpha B_1B_2q + 3\beta A_1A_2q + 12\lambda B_1p^2q - 2kA_2 + 2\alpha B_0B_2p &= 0,\\
-kB_1p + \beta A_0B_1p + 7\gamma A_1pq^2 - 2kB_2q + 2A_2q + 2\beta B_1A_1q &\\
+2\beta B_0A_2q + \beta B_0A_1p + 2\beta A_0B_2q + 8\gamma A_2q^3 + A_1p &= 0,\\
-kA_1p + \alpha B_0B_1p + \beta A_0A_1p + \alpha {B_1}^2q + B_1p + \beta {A_1}^2q &\\
+8\lambda B_2q^3 + 7\lambda B_1pq^2 - 2kA_2q + 2B_2q + 2\beta A_0A_2q + 2\alpha B_0B_2q &= 0.
\end{aligned}$$

Solving the above system of algebraic equations with the aid of maple, one obtains

$$\begin{aligned}
A_2 &= \frac{6\phi p^2}{\beta}, \\
A_1 &= \frac{6\phi pq}{\beta}, \\
A_0 &= \frac{\phi\gamma\alpha q^2 + \alpha\phi - \beta\phi - \beta\lambda\phi q^2 + 2k\gamma\alpha - 2k\lambda\beta}{2\beta(\gamma\alpha - \beta\lambda)}, \\
B_2 &= -\frac{6\gamma p^2}{\beta}, \\
B_1 &= -\frac{6\gamma pq}{\beta}, \\
B_0 &= -\frac{\gamma\beta - \gamma\beta\lambda q^2 - 2\beta\lambda + \gamma^2 q^2\alpha + \gamma\alpha}{2\beta(\gamma\alpha - \beta\lambda)}, \quad \text{where} \quad \phi = \pm\sqrt{\frac{2\gamma\lambda\beta - \alpha\gamma^2}{\beta}}.
\end{aligned}$$

As a result, we obtain the solution of system (4.1) as

$$\begin{aligned}
u(t,x) = A_0 + A_1 q&\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right) \\
&+ A_2 q^2\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right)^2, \qquad (4.10)\\
v(t,x) = B_0 + B_1 q&\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right) \\
&+ B_2 q^2\left(\frac{\cosh[q(z+C)] + \sinh[q(z+C)]}{1 - p\cosh[q(z+C)] - p\sinh[q(z+C)]}\right)^2, \qquad (4.11)
\end{aligned}$$

where $z = x - kt$.

Solutions of system (4.1) using the Ricatti equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of system (4.6) are of the form

$$E(z) = A_0 + A_1 H + A_2 H^2 \quad \text{and} \quad F(z) = B_0 + B_1 H + B_2 H^2. \qquad (4.12)$$

Substituting equation (4.12) into system (4.6) and making use of Ricatti equation, we obtain an algebraic system of equations in terms of A_0, A_1, A_2 and B_0, B_1, B_2, by equating all coefficients of the functions H^i to zero. The corresponding algebraic equations are

$$
\begin{aligned}
2B_2{}^2q+24cA_2q^3 &= 0,\\
4\,B_2A_2q+24\,aB_2q^3 &= 0,\\
54\,cA_2q^2p+6\,cA_1q^3+2\,B_2{}^2p+3\,B_1B_2q &= 0,\\
6\,aB_1q^3+54\,aB_2q^2p+3\,B_2A_1q+3\,B_1A_2q+4\,B_2A_2p &= 0,\\
2\,cA_1qr^2+cA_1p^2r+6\,cA_2pr^2+A_1r-kB_1r+B_0B_1r &= 0,\\
B_1r+2\,aB_1qr^2+aB_1p^2r+6\,aB_2pr^2+A_0B_1r-kA_1r+B_0A_1r &= 0,\\
2\,A_2q-2\,kB_2q+2\,B_0B_2q+3\,B_1B_2p+2\,B_2{}^2r+12\,cA_1q^2p+38\,cA_2qp^2 &\\
+B_1{}^2q+40\,cA_2q^2r &= 0,\\
cA_1p^3-kB_1p+2\,B_0B_2r+B_0B_1p+B_1{}^2r+A_1p+8\,cA_1qpr+2\,A_2r+14\,cA_2p^2r &\\
+16\,cA_2qr^2-2\,kB_2r &= 0,\\
2\,B_0A_2q+3\,B_2A_1p-2\,kA_2q+2\,B_1A_1q+40\,aB_2q^2r+4\,B_2A_2r+2\,A_0B_2q+3\,B_1A_2p &\\
+2\,B_2q+12\,aB_1q^2p+38\,aB_2qp^2 &= 0,\\
2\,A_2p+52\,cA_2qpr-2\,kB_2p+7\,cA_1qp^2+B_0B_1q+8\,cA_2p^3-kB_1q+2\,B_0B_2p+A_1q &\\
+B_1{}^2p+3\,B_1B_2r+8\,cA_1q^2r &= 0,\\
aB_1p^3+A_0B_1p+B_1p+B_0A_1p+2\,B_2r+8\,aB_1qpr-2\,kA_2r+2\,B_1A_1r+2\,B_0A_2r &\\
+16\,aB_2qr^2+2\,A_0B_2r-kA_1p+14\,aB_2p^2r &= 0,\\
B_0A_1q+A_0B_1q+8\,aB_2p^3+52\,aB_2qpr+7\,aB_1qp^2-2\,kA_2p+2\,B_1A_1p+2\,B_0A_2p+2\,B_2p &\\
+2\,A_0B_2p+3\,B_1A_2r-kA_1q+8\,aB_1q^2r+3\,B_2A_1r+B_1q &= 0.
\end{aligned}
$$

Solving the above system of algebraic equations with the aid of Maple, one obtains

$$
\begin{aligned}
A_2 &= \frac{6\phi q^2}{\beta}, \\
A_1 &= \frac{6\phi pq}{\beta}, \\
A_0 &= \frac{\phi\gamma\alpha p^2 + \alpha\phi - \beta\phi - \beta\lambda\phi p^2 + 2k\gamma\alpha - 8\phi\lambda\beta rq + 8\gamma\phi\alpha rq - 2k\lambda\beta}{2\beta(\gamma\alpha - \beta\lambda)}, \\
B_2 &= -\frac{6\gamma q^2}{\beta}, \\
B_1 &= -\frac{6\gamma pq}{\beta}, \\
B_0 &= -\frac{\gamma\beta - \gamma\beta\lambda p^2 - 8\gamma r\lambda\beta q + 8\gamma^2 r\alpha q - 2\beta\lambda + \gamma^2 p^2\alpha + \gamma\alpha}{2\beta(\gamma\alpha - \beta\lambda)},
\end{aligned}
$$

where

$$
\phi = \pm\sqrt{\frac{2\gamma\lambda\beta - \alpha\gamma^2}{\beta}}.
$$

As a result, we obtain the solution of system (4.1) as

$$
\begin{aligned}
u(t,x) &= A_0 + A_1\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\
&\quad + A_2\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
v(t,x) &= B_0 + B_1\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\
&\quad + B_2\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2
\end{aligned} \tag{4.14}
$$

and

$$
\begin{aligned}
u(t,x) &= A_0 + A_1\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\
&\quad + A_2\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2,
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
v(t,x) &= B_0 + B_1\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\
&\quad + B_2\left\{ -\frac{p}{2q} - \frac{\theta}{2q}\tanh\left(\frac{z\theta}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2,
\end{aligned} \tag{4.16}
$$

where $z = x - kt$.

4.2 Conservation laws of the coupled KdV system

In this section we will construct the conservation laws of the coupled KdV system of two nonlinear dispersive wave equations [25], namely

$$\begin{aligned} v_t + u_x + \beta u_x v + \beta u v_x + \gamma u_{xxx} &= 0, \\ u_t + v_x + \alpha v v_x + \beta u u_x + \lambda v_{xxx} &= 0, \end{aligned} \tag{4.17}$$

where α, β, γ and λ are arbitrary constants. Here we will derive the conservation laws for system (4.17). In order to apply the Noether's approach we use the transformations $u \to U_x$, $v \to V_x$ then system (4.17) becomes

$$\begin{aligned} V_{tx} + U_{xx} + \beta U_x V_{xx} + \beta V_x U_{xx} + \gamma U_{xxxx} = 0, \\ U_{xt} + V_{xx} + \beta U_x U_{xx} + \alpha V_x V_{xx} + \lambda V_{xxxx} = 0. \end{aligned} \tag{4.18}$$

It can be readily verified that L given by

$$L = \frac{1}{2}\left\{ \gamma U_{xx}^2 + \lambda V_{xx}^2 - \frac{\alpha}{3} V_x^3 - U_x^2 - V_x^2 - \beta U_x^2 V_x - U_t V_x - U_x V_t \right\}, \tag{4.19}$$

is the Lagrangian for system (4.18).

The insertion of L (4.19) into (2.16) and expand gives

$$
\begin{aligned}
&-\frac{1}{2}V_x[\eta^1_t + U_t\eta^1_U + V_t\eta^1_V - U_t\xi^1_t - U_t^2\xi^1_U - U_tV_t\xi^1_V - U_x\xi^2_t \\
&-U_tU_t\xi^2_U - U_xV_t\xi^2_V] - \frac{1}{2}U_x[\eta^2_t + U_t\eta^2_U + V_t\eta^2_V - V_t\xi^1_t - U_tV_t\xi^1_U \\
&-V_t^2\xi^1_V - V_x\xi^2_t - U_tV_x\xi^2_U - V_tV_x\xi^2_V] - \\
&\left(\frac{1}{2}V_t + \beta U_xV_x + U_x\right)[\eta^1_x + U_x\eta^1_U + V_x\eta^1_V \\
&-U_t\xi^1_x - U_tU_x\xi^1_U - U_tV_x\xi^1_V - U_x\xi^2_x - U_x^2\xi^2_U - U_xV_x\xi^2_V] \\
&-\left(\frac{1}{2}U_t + V_x + \frac{\alpha}{2}V_x^2 + \frac{\beta}{2}U_x^2\right)[\eta^2_x + U_x\eta^2_U + V_x\eta^2_V - V_t\xi^1_x - U_xV_t\xi^1_U \\
&-V_tV_x\xi^1_V - V_x\xi^2_x - U_xV_x\xi^2_U - V_x^2\xi^2_V] + \gamma U_{xx}[D_x^2\eta^1 - U_tD_x^2\xi^1 \\
&-U_xD_x^2\xi^2 - 2U_{tx}(\xi^1_x + U_x\xi^1_U + V_x\xi^1_V) - 2U_{xx}(\xi^2_x + U_x\xi^2_U + V_x\xi^2_V)] \\
&+\lambda V_{xx}[D_x^2\eta^2 - V_tD_x^2\xi^1 - V_xD_x^2\xi^2 - 2V_{tx}(\xi^1_x + U_x\xi^1_U + V_x\xi^1_V) \\
&-2V_{xx}(\xi^2_x + U_x\xi^2_U + V_x\xi^2_V)] \\
&+\frac{1}{2}[\gamma U_{xx}^2 + \lambda V_{xx}^2 - \frac{\alpha}{3}V_x^3 - U_x^2 - V_x^2 - \beta U_x^2V_x - U_tV_x - U_xV_t][\xi^1_t + U_t\xi^1_U \\
&+V_t\xi^1_V + \xi^2_x + U_x\xi^2_U + V_x\xi^2_V] \\
&= B^1_t + U_tB^1_U + V_tB^1_V + B^2_x + U_xB^2_U + V_xB^2_V.
\end{aligned} \tag{4.20}
$$

Splitting (4.20) with respect to different combinations of derivatives of U and V results in an overdetermined systems of equations for ξ^1, ξ^2, η^1, η^2, B^1 and B^2. The solution of overdetermined system yields the following Noether symmetries and gauge terms

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0, \\
&X_2 = \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0, \\
&X_3 = \beta t\frac{\partial}{\partial x} + x\frac{\partial}{\partial U}, \quad B^1 = -\frac{1}{2}V, \quad B^2 = -U, \\
&X_{E(t)} = E(t)\frac{\partial}{\partial U}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}VE'(t), \\
&X_{F(t)} = F(t)\frac{\partial}{\partial V}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}UF'(t).
\end{aligned} \tag{4.21}
$$

Equations (2.55), (2.56) and (4.21) and $u = U_x$, $v = V_x$ yield the following independent conserved vectors for system (4.17):

$$
\begin{aligned}
T_1^1 &= \frac{\gamma u_x^2}{2} + \frac{\lambda v_x^2}{2} - \frac{\alpha v^3}{6} - \frac{u^2}{2} - \frac{v^2}{2} - \frac{\beta u^2 v}{2}, \\
T_1^2 &= \left[u + \beta uv + \gamma u_{xx}\right] \int u_t dx - \gamma u_t u_x - \lambda v_t v_x \\
&\quad + \left[\frac{\alpha v^2}{2} + \frac{\beta u^2}{2} + v + \lambda v_{xx}\right] \int v_t dx + \int u_t dx \int v_t dx; \qquad (4.22) \\
T_2^1 &= \beta tuv - \frac{xv}{2} + \frac{1}{2}\int v dx, \\
T_2^2 &= -\frac{t\gamma\beta u_x^2}{2} - \frac{t\lambda\beta v_x^2}{2} + t\gamma\beta u u_{xx} + t\lambda\beta v v_{xx} + \frac{t\alpha\beta v^3}{3} + \frac{t\beta u^2}{2} \\
&\quad + \frac{t\beta v^2}{2} + t\beta^2 u^2 v - x\beta uv - x\lambda u_{xx} + \lambda u_x - xu - \frac{x}{2}\int v_t dx \\
&\quad + \int u dx; \qquad (4.23) \\
T_3^1 &= uv, \\
T_3^2 &= \gamma u u_{xx} + \lambda v v_{xx} + \beta u^2 v - \frac{\gamma u_x^2}{2} - \frac{\lambda v_x^2}{2} + \frac{\alpha v^3}{3} + \frac{u^2}{2} + \frac{v^2}{2}; \qquad (4.24) \\
T_{(H,J)}^1 &= -\frac{u}{2}J(t) - \frac{v}{2}H(t), \\
T_{(H,J)}^2 &= \frac{1}{2}J'(t)\int u dx + \frac{1}{2}H'(t)\int v dx - H(t)\left[u + \beta uv + \gamma u_{xx} + \frac{1}{2}\int v_t dx\right] \\
&\quad - J(t)\left[v + \frac{\beta u^2}{2} + \frac{\alpha v^2}{2} + \lambda v_{xx} + \frac{1}{2}\int u_t dx\right]. \qquad (4.25)
\end{aligned}
$$

We now extract two particular cases from the conserved vector (4.25).

Letting $H(t) = 1$ and $J(t) = 0$ gives a nonlocal conserved vector

$$
\begin{aligned}
T_4^1 &= -\frac{v}{2}, \\
T_4^2 &= -u - \beta uv - \gamma u_{xx} - \frac{1}{2}\int v_t dx \qquad (4.26)
\end{aligned}
$$

and for $H(t) = 0$ and $J(t) = 1$ gives a nonlocal conserved vector

$$
\begin{aligned}
T_5^1 &= -\frac{u}{2}, \\
T_5^2 &= -v - \frac{\beta u^2}{2} - \frac{\alpha v^2}{2} - \lambda v_{xx} - \frac{1}{2}\int u_t dx. \qquad (4.27)
\end{aligned}
$$

Remark: For the arbitrary values of $H(t)$ and $J(t)$ infinitely many nonlocal conservation laws exist for system (4.17).

4.3 Conclusion

In this chapter the coupled KdV system was studied from the Lie symmetry analysis point of view. The Lie symmetries obtained were then used to transform coupled KdV system to nonlinear ordinary differential equations. Thereafter the simplest equation method was employed to construct exact solutions for coupled KdV system. We also employed the Noether's theorem to derive conservation laws for the coupled KdV system. The conservation laws for the coupled KdV system consist of some local and infinite number of nonlocal conserved vectors.

Chapter 5

Concluding remarks

In this research project we first recalled some important definitions and results from Lie group theory, which were later used in the dissertation. In Chapter two symmetry reduction of a generalized Ito-type coupled KdV system was performed. Furthermore, conservation laws of this system were derived using Noether's theorem. In Chapter three we obtained exact solutions of the Boussinesq system of KdV-KdV type equation by using the Lie symmetry method along with the simplest equation method. In addition, Noether's theorem was employed to obtain the conservation laws of the Boussinesq system of KdV-KdV type equation. Finally, in Chapter four exact solutions for the coupled KdV system were obtained using the Lie symmetry method along with the simplest equation method and the conservation laws of the same system were derived using Noether's theorem. In future, we will perform the Noether symmetry classification of certain nonlinear partial differential equations.

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