Group actions and ergodic theory on Banach function spaces

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It all starts here ™

Acknowledgments

There are several people without whom this work would never have been done.

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This thesis belongs to all of you too.

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Summary

This thesis is an account of our study of two branches of dynamical systems theory, namely the mean and pointwise ergodic theory.

In our work on mean ergodic theorems, we investigate the spectral theory of integrable actions of a locally compact abelian group on a locally convex vector space. We start with an analysis of various spectral subspaces induced by the action of the group. This is applied to analyse the spectral theory of operators on the space generated by measures on the group. We apply these results to derive general Tauberian theorems that apply to arbitrary locally compact abelian groups acting on a large class of locally convex vector spaces which includes Fréchet spaces. We show how these theorems simplify the derivation of Mean Ergodic theorems.

Next we turn to the topic of pointwise ergodic theorems. We analyse the Transfer Principle, which is used to generate weak type maximal inequalities for ergodic operators, and extend it to the general case of σ-compact locally compact Hausdorff groups acting measure-preservingly on σ-finite measure spaces. We show how the techniques developed here generate various weak type maximal inequalities on different Banach function spaces, and how the properties of these function spaces influence the weak type inequalities that can be obtained. Finally, we demonstrate how the techniques developed imply almost sure pointwise convergence of a wide class of ergodic averages.

Our investigations of these two parts of ergodic theory are unified by the techniques used - locally convex vector spaces, harmonic analysis, measure theory - and by the strong interaction of the final results, which are obtained in greater generality than hitherto achieved.

Keywords

Tauberian theorems, harmonic analysis, group action, spectral theory, mean ergodic theory, Transfer Principle, maximal inequalities, Banach function spaces, pointwise ergodic theory

Groep aksies en ergodiese teorie op Banach funksieruimtes

Samevatting

Hierdie tesis is 'n verslag van ons bestudering van twee takke van dinamiese stelselteorie, naamlik die middel en puntsgewyse teorie.

In ons werk in middel ergoiese teorie, ondersoek ons die spektraalteorie van integreerbare aksies van 'n lokaal kompakte abelse groep op 'n lokaal konvekse ruimte. Ons begin met 'n analise van verskeie spektrale deelruimtes wat deur die groep-aksie geïnduseer word. Dit word toegepas om die spektraalteorie van operatore op die ruimte voortgebring deur mate op die groep te analiseer. Ons pas hierdie resultate toe om algemene Tauberse stellings af te lei wat toepasbaar is op arbitrêre lokaal kompakte abelse groepe wat op 'n groot klas van lokaal kompakte vektorruimtes inwerk - 'n klas wat Fréchet ruimtes insluit. Ons toon aan hoe hierdie stellings die afleiding van middel ergodiese stellings vereenvoudig.

Daarna beskou ons die onderwerp van puntsgewyse ergodiese stellings. Ons analiseer die Oordragsbeginsel, wat gebruik word om swak-tipe maksimale ongelykhede vir ergodiese operatore voort te bring, en brei die tegniek uit na die algemene geval van σ-kompakte lokaal kompakte Hausdorff groepe wat maat-preserverend op 'n σ-eindige maatruimte inwerk. Ons toon aan hoe die tegnieke hier ontwikkel bring voort verskeie swak-tipe maksimale ongelykhede op verskillende Banach funksieruimtes, en hoe die eienskappe van hierdie funskieruimtes die swak-tipe ongelykhede wat verwerf word, beïnvloed. Laastens, wys ons hoe die tegnieke wat ontwikkel is byna-oral puntsgewyse konvergensie van 'n wye klas van ergodiese gemiddeldes impliseer.

Ons ondersoeking van hierdie twee dele van ergodiese teorie word geünieer deur die tegnieke wat gebruik word - lokaal konvekse ruimtes, harmoniese analise, maatteorie - en deur die sterk interaksie tussen die finale resultate, wat in groter algemeenheid as vantevore behaal word.

Sleutelwoorde

Tauber stellings, harmoniese analise, groep aksie, spektraalteorie, middel ergodiese teorie, Oordragsbeginsel, maksimale ongelykhede, Banach funksieruimtes, puntsgewyse ergodiese teorie

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Chapter 1

The scheme of this work

1.1 Overview of ergodic theory

Ergodic theory is a branch of dynamical systems theory where the long-term or asymptotic behaviour of the system is studied. The abstract theory of dynamical systems considered in this work is based on a distillation of experience in several fields where concrete dynamical systems are essential. Physics of course provides a great stimulus, as does number theory, where arguably ergodic theory has had even more success. The dynamical systems viewpoint is also prominent in the representation theory of topological groups [30] and even Ramsey theory [28]. One of the reasons for these developments is the power of the abstract formulation of dynamics and the ubiquity of questions of symmetry.

Perhaps surprisingly, it was in number theory, not physics, that ergodic questions concerning the asymptotic behaviour of systems first made an appearance. As recounted in [36], Nicole Oresme (c. 1320-1382) had already anticipated Weyl's equidistribution result described below, and in his book *Ad pauca respicientes* used it to prove that one cannot predict the position of planets long into the future, rendering astrology devoid of meaning. In more modern times, one of the first profound applications of the idea of examining the long term behaviour of a system was due to Gauss, in his work on continued fractions. He found a limiting distribution (i.e. a measure) that encodes the length of time the continued fraction algorithm will take to approximate a given $x \in (0, 1)$ by a rational number to a given degree of accuracy. (See for example [2]).

Of course the terms *ergodicity* and *entropy* were coined by the Austrian physicist Karl Boltzmann in his work on statistical mechanics in the 1870s. The *ergodic hypothesis* refers to the statement that if one takes the average of an observable over time or over the phase space at a particular instant, one gets the same answer. Boltzmann's goal was to develop statistical mechanics to the point where one could relate macroscopic phenomena to the behaviour of the myriad elementary particles that constitute the physical world. It is remarkable that he developed his theory in a time where the atomic hypothesis of matter was in doubt.

In his work on celestial mechanics in the last decade of the 19^{th} century, Henri Poincaré introduced new statistical techniques to the study of dynamical systems. He proved that if the system is closed (we would now say that the phase space is compact), then the system would return infinitely often and arbitrarily closely to any previous configuration. This notion is even more starkly expressed measure theoretically, in what is known as the Poincaré Recurrence Theorem.

Returning to number theory, Hermann Weyl studied distribution problems in number theory, greatly aiding the rise of statistical thinking in that deep and ancient branch of mathematics. Bohl (1909), Weyl (1910) and Sierpinski (1910) proved that if α is an irrational number, then the set $\{n\alpha \mod 1 : n \in \mathbb{N}\}$ is equidistributed according to Lebesgue measure. Later in 1916, Weyl proved the same thing for $\{n^2\alpha \mod 1\}$, as did Vinogradov for the set $\{p_n\alpha \mod 1\}$, where p_n denotes the n^{th} prime. All these questions can be posed as an ergodic problem of certain dynamical systems. Rosenblatt and Wierdl [40] is an excellent source for this material. At the forefront of this line of thinking, we have for instance Bourgain [5] and [6], and Buczolich and Mauldin [7]. These mathematicians consider far-reaching extensions of Birkhoff's original pointwise ergodic theorem. They show the pointwise convergence of averages sampled on certain subsets of the natural numbers, such as the perfect squares. The full ramifications of these results have yet to be fully explored.

In the 1930s, Erdős and Turán conjectured that any set of natural numbers with positive upper Banach density must contain arbitrarily long arithmetic progressions. This goes further than van der Waerden's theorem of 1927, which states that if the natural numbers are partitioned into a finite number of partitions, then at least one of those partitions has arbitrarily long arithmetic progressions. The conjecture by the two Hungarians was proved by another Hungarian, Szemerédi, in 1975. Hillel Furstenberg [18] proved this result using ergodic theory, in particular multiple recurrence [18]. In conjunction with Katnelson and Ornstein, the proof was simplified even further in [19]. The most spectacular success of this fusion of number theory and ergodic theory in recent years has been Green and Tao's proof that there are arbitrarily long arithmetic progressions in the primes [23].

Broadly speaking, a measure-theoretic dynamical system consists of three elements: a Set X, a group G, and an action α, continuous in some sense, that binds them together by mapping G into the group of automorphisms of X. In our work on mean ergodic theorems, we take X to be a locally convex topological vector space, G to be an abelian locally compact Hausdorff topological group and α to be a mapping of G into the bicontinuous linear automorphism group $\mathrm{Aut}(X)$. This is made precise in Definition 4.1.1. When dealing with pointwise ergodic theorems, X will be a σ-finite measure space, G a locally compact Hausdorff group and α a mapping of G into the group of measure preserving automorphisms of X. For the exact formulation, see Definition 5.1.1.

The research contained in this thesis appears in two articles, [10] and [11]. The first has already been published online on 25^{th} February 2013, and the second has been submitted and can be found at `arxiv:1309.0125 [math.DS]`. The material in Sections 2.1, part of 2.2, 2.4 and Chapter 4 appears in [10]. The material in Sections 2.3, part of 2.2 and Chapters 3 and 5 appears in [11]. See Sections 4.5 and 5.7 for further comments regarding the origin and attribution of the results.

1.2 Mean ergodic theorems

For the treatment of mean ergodic theorems, the aim of this thesis is to develop enough spectral theory of integrable group actions on locally convex vector spaces to prove Tauberian theorems, which are applicable to ergodic theory. A Tauberian theorem is one where, given the convergence of a sequence or series in one sense, the imposition of some condition on the sequence or series guarantees its convergence in a stronger sense. The condition added is called a Tauberian condition. An excellent overview of the subject, especially how it pertains to number theory, is given in Korevaar [29]. It is noteworthy that Wiener proved a general Tauberian theorem that has the Prime Number Theorem as a consequence. This proof, presented in [43], is still one of the easiest proofs of that theorem.

The Tauberian theorems proved in Section 4.4 apply to the situation where a general locally compact abelian group acts on certain types of barrelled spaces, and in particular all Fréchet spaces. This generalises the Tauberian theorem shown in [15], which applies only to the action of the integers on a Banach space. We use these theorems to simply derive mean ergodic theorems in a rather general context.

The bulk of our development of the mean ergodic theory consists of using spectral theory to transfer properties and constructions on a locally compact abelian group G to the topological vector space X on which it acts. Put another way, we use spectral theory to derive properties of a dynamical system from harmonic analytic properties of the group.

Firstly, we outline the correspondence between certain subsets of the Pontryagin dual $\widehat{G}$ and certain closed G-invariant subspaces of X, called *spectral subspaces*. This work was initiated by Beurling, closely followed by Godement in [21] and extended by many authors; we refer especially to [1],[51],[35] and [47].

Secondly, the action of G on X naturally generates many continuous linear mappings from X to itself, by associating to every finite Radon measure μ on G a continuous endomorphism on X, called α_μ. We discuss some characteristics of such maps, particularly how they relate to the spectral subspaces. Also important here is how properties of μ that can be determined by harmonic analysis are transferred to properties of α_μ.

The Tauberian theorems 4.3.1 and 4.3.2 are the focal point of our work on mean ergodic theorems, extending results of Dunford and Schwartz [15] to a very general setting. Mean ergodic theorems are proved for very general dynamical systems, involving arbitray locally compact Hausdorff groups acting on Fréchet spaces.

1.3 Pointwise ergodic theorems

Pointwise ergodic theorems have had an illustrious history spanning over 80 years since G.D. Birkhoff first proved the foundational result in 1931. The proof of his erogdic theorem has been so refined that one can give an elementary, leisurely demonstration in about two pages [26]. To work with more general ergodic averages however, it seems one must still rely on a different approach. Indeed, one of the main techniques that has been developed to prove pointwise ergodic theorems is supple enough to deal with pointwise convergence phenomena for a great variety of different ergodic averages, for different groups, function spaces and averages.

This is the technique of maximal operators. The idea is that once one estimates the behaviour of these maximal operators, proving the ergodic theorems becomes quite simple. (We explain the proof strategy in Section 5.6 in the form of a three-step programme). This is how the pointwise ergodic theorems are proved in [20] and [40]. It is also how results on entropy and information are obtained in [36, Chapter 6].

In fact, Wiener [49] developed a method, later greatly embellished by Calderón [8], for computing the requisite properties of the maximal operators, a method that is the central theme of this work: the Transfer Principle. It is our goal to extend the scope of this Principle and hence the scope of the maximal operator technique in proving pointwise ergodic theorems. The Transfer Principle refers to a body of techniques that allow one to transform certain types of operators acting on function spaces over G to corresponding transferred operators acting on function spaces over Ω, in such a way that many essential properties of the operator are preserved.

With regards to pointwise theorems and the technique of maximal operators, we have four aims: firstly, to extend the Transfer Principle to a larger class of groups, measure spaces and operators, secondly to broaden the reach of the techniques used to determine the weak type of the transferred operator, thirdly to demonstrate how properties of the function spaces on which the operators are defined influence the weak type inequalities of the transferred operator, and finally to outline how these results can be used to derive a number of new pointwise ergodic theorems.

We define the Transfer Principle in quite a general setting (Definition 5.1.2). If the operator to be transferred - call it T - is linear, then G and Ω need only be σ-finite (Definition 5.1.7) and if it is sublinear, then it must have separable and metrisable range (Definiton 5.1.6). The determination of the weak type of the transferred operator - call it $T^{\#}$- rests on results requiring Ω to be countably generated and resonant as defined just before Proposition 3.4.1. This fulfills the first aim.

Computing the weak type of the transferred operator $T^{\#}$ is achieved with Corollary 5.5.2, especially in combination with Lemma **??**. A noteworthy feature of these results is that they show how the most important factors determining the weak type of $T^{\#}$ are the fundamental functions associated with the function spaces defining the weak type of T. In this way, the problem of computing the weak type of the transferred operator is reduced to computations involving certain well-behaved real-valued functions. In particular, these results allow us to estimate the weak type of the maximal operator associated with ergodic averages over a wide class of rearrangement invariant Banach function spaces. This completes the second aim.

For the third aim, we use such properties as the Boyd and fundamental indices of a rearrangement invariant Banach function space X and show how straightforward weak type (p,p) inequalities of T can be transferred to weak type inequalites for $T^{\#}$ acting on X.

Finally, we address the fourth aim by proving pointwise ergodic theorems, transferring information obtained using Fourier analysis on the group to properties of the ergodic averages, and information on the function space on which they act, that is encoded in the fundamental function. The main results are Theorem 5.6.6 and Corollaries 5.6.7 and 5.6.8.

The importance of the Transfer Principle in ergodic theory has long been appreciated - see the excellent overview given in [3]. Apart from Calderón's seminal paper [8], this principle is treated in some detail in the monograph [9] and employed extensively in [40]. In [34] the author makes use of Orlicz spaces to prove results about the pointwise convergence of ergodic averages along certain subsets of the natural numbers. In [16] the Transfer Principle of Coifman and Weiss is extended to Orlicz spaces with weight for group actions that are uniformly bounded in a sense determined by the weighted Orlicz space.

1.4 Plan of the work on mean ergodic theorems

The plan of our work on mean ergodic theorems is as follows. Sections 2.4 and 2.1 contain some basic material on harmonic analysis and topological vector spaces. First, there is a brief discussion on the harmonic analysis required and includes extensions of known results, most notably Theorem 2.4.1. There follows some work on locally convex topological vector spaces and vector-valued measures. These results form the core of the techniques used to transfer information from the group to the topological vector space upon which it acts.

In Section 4.1, we discuss integrable actions of G on E. We shall do so using general topological considerations and employing a little measure theory of vector-valued measures, in the hope that it will bring some clarity to the idea (Definition 4.1.1). This definition elaborates on an idea introduced in [1] and is discussed elsewhere, such as in [35], [47] and [51]. In this work, we stress the continuity properties that a group action may have, and how such continuity properties can be analysed using vector-valued measure theory. Next we introduce *spectral subspaces* by providing the definitions that appear in [1], [35] and [21], namely Definitions 4.1.6 and 4.1.8. We demonstrate that they are in fact the same. There is a third kind of spectral subspace given in Definition 4.1.10. It is important because it is directly related to a given finite Radon measure and provides a link to the associated operator. We show how this type of spectral subspace is related to the first two mentioned. Finally we show how to employ the tool of spectral synthesis in harmonic analysis to analyse spectral subspaces. Here the highlight is Theorem 4.1.15.

We discuss properties of operators on E induced by finite Radon measures on G in Section 4.2. A major theme is how properties of the Fourier transform of a measure determine how the associated operator will act on spectral subspaces. This underlines the intuition that the Fourier transform on the group side of an action corresponds to spectral spaces on the vector space side. For the development of the Tauberian theorems, we need to know how to transfer convergence properties of sequences of measures to convergence properties of sequences of operators. This is done in Proposition 4.2.3. We prove these results by applying our knowledge of the relationship between a convergent sequence of measures and its sequence of Fourier transforms as set out in Section 2.4, as well as the link between spectral synthesis and spectral subspaces.

Having developed enough spectral theory, we come to the highlight of this work: the Tauberian theorems 4.3.1 and 4.3.2. Apart from being generally applicable to situations where a locally compact abelian group G acts on a Fréchet space X, it also handles general topologies of the action − where the action is continuous in the weak or strong operator topologies as well as intermediate topologies. We also discuss some general cases in Remark 4.3.3 where the hypotheses of the Tauberian theorems are automatically satisfied.

In Section 4.4 we show how, from the Tauberian results, we can quickly deduce mean ergodic theorems for general locally compact abelian groups acting on Fréchet spaces.

1.5 Plan of the work on pointwise ergodic theorems

Let us briefly describe the organisation of the our work on pointwise ergodic theorems. In Section 5.1, we define the Transfer Principle and analyse it in some detail. This involves quite intricate measure-theoretic considerations, including the development of a theory of locally Bochner integrable functions in parallel with the classical theory of Bochner integrable functions.

In Section 3.1 we bring to mind some basic constructions and definitions in the theory of rearrangement invariant Banach function spaces. We emphasise how in the general theory a central role is played by the fundamental function of such spaces, and how a great deal of their structure and behaviour is reflected in this function. Propositions 3.4.2 and 3.4.3 extend work of O'Neil [33] on tensor and integral products by showing how under certain conditions the hypotheses of Theorems 8.15 and 8.18 in [33] can be weakened. We also estimate other integrals that arise naturally for functions on product spaces (Proposition 3.4.1).

Section 5.5 contains the main results for estimating the weak type of the transferred operator, namely Corollary 5.5.2, and is based on the work of the previous two sections including an extension of an inequality of Kolmogorov (Theorem 5.4.1).

In the final part of the work, Section 5.6, we explain a general method for deriving pointwise ergodic theorems from maximal inequalities. This reduces the tack of proving almost everywhere convergence to checking certain group theoretic properties of the desired average using harmonic analysis, in particular the Fourier transform. We show again how properties of the averages, the space acted upon and the nature of the action interact to yield the almost everywhere convergence of ergodic averages.

Chapter 2

Harmonic analysis and locally convex vector spaces

The mean and pointwise ergodic theorems that are our ultimate goal depend on a thorough understanding of two very subtle subjects, namely that of locally convex topological vector spaces and harmonic analysis. Both have been central to the development of 20^{th} century mathematics and are associated with the great names in analysis and number theory. We bring to mind some of the most important objects and constructions in both.

For the material on locally convex spaces we draw principally from [38], but also [24] and [45] for more advanced information. For harmonic analysis we follow Folland [17], Rudin [41] and Katznelson [25]. Folland also deals with nonabelian groups, taking the representation theoretic viewpoint, whereas Rudin covers abelian group theory in detail, going more deeply into the structure theory of such groups.

Because we are dealing with topological group actions, quite a bit needs to be said about the way measure spaces (such as the group) interact with vector spaces. This leads us to vector-valued measure theory, which is masterfully treated by Diestel and Uhl [14] and Ryan [44].

In Section 2.1 we discuss the algebraic concept of a dual pair of Banach spaces and how one can use it to define three of the most important topologies on a vector space: the weak, Mackey and strong topologies. These will play an essential role in the mean ergodic theory.

In Section 2.2 we develop extensions of the standard vector-valued measure theory.

In the mean ergodic theory, this extension is crucial to understanding the different ways in which a group can act on a vector space. Section 4.1 relies heavily upon it. In the pointwise ergodic theory, it will be crucial to working with the transfer principle, in particular proving properties of the transferred operator in Section 5.1.

In Section 2.3 we shall use this theory of vector-valued measures combined with results on injective and projective tensor products of locally convex vector spaces to work with product measure spaces. This will be needed in the pointwise theory where we must frequently work with the Cartesian product of the group with another measure space. The vector-valued measure approach allows us to effectively deal with these questions.

Section 2.4 covers harmonic analysis, including extensions of known results and applications of the convex space theory to spaces that naturally arise out of the harmonic analysis itself.

We shall denote the Haar measure on the locally compact group G by the symbol h.

One final notational convention: if A is a measurable subset of a measure space (Ω, μ), for brevity we shall write $|A| := \mu(A)$. Likewise, if K is a measurable subset of the locally compact group G, we shall denote by $|K|$ the Haar measure of K.

2.1 Locally convex vector spaces

We now mention some aspects of the theory of locally convex topological vector spaces. A pair of complex vector spaces (E, E') is said to be a *dual pair* if E' can be viewed as a separating set of functionals on E and vice versa. For example, a Banach space X and its dual X^* are in duality.

The spaces E and E' induce upon each other certain topologies via their duality. We denote the smallest such topology, the weak topology, by $\sigma(E, E')$, and the largest, the strong topology, by $\beta(E, E')$. The latter is generated by sets of the form $A^\circ = \{x \in E \ : \ |\langle x, a\rangle| \leq 1 \text{ for all } a \in A\}$, as A ranges over all $\sigma(E', E)$-bounded subsets of E'. The set A° is called the *polar* of A. There is also the Mackey topology, called $\tau(E, E')$, which is the finest locally convex topology on E such that under this topology, E' is exactly the set all continuous linear functionals on E.

Suppose (E, E') and (F, F') are dual pairs. The set of all $\sigma(E, E') - \sigma(F, F')$-continuous linear mappings between topological vector spaces E and F is denoted by $\mathcal{L}_\omega(E,F)$. The set of all $\beta(E,E')-\beta(F,F')$-continuous linear mappings between topological vector spaces E and F is denoted by $\mathcal{L}_\sigma(E,F)$.

Now any linear map $T:E\to F$ is $\sigma(E,E')-\sigma(F,F')$-continuous if and only if it is $\tau(E,E')-\tau(F,F')$-continuous. Also, if $T:E\to F$ is $\sigma(E,E')-\sigma(F,F')$-continuous, then it is $\beta(E,E')-\beta(F,F')$-continuous. Hence $\mathcal{L}_\omega(E,F)\subset\mathcal{L}_\sigma(E,F)$.

A linear map from a Fréchet space X to a locally convex topological vector space is continuous if and only if it is bounded. Hence the set $B(X)$ of all bounded linear mappings from X to itself is precisely the set of all $\tau-\tau$ and hence $\sigma-\sigma$-continuous linear mappings. On a Fréchet space, the τ and β topologies are the same, so in this case we have $\mathcal{L}_\omega(X)=\mathcal{L}_\sigma(X)=B(X)$. Because of this, we automatically view a Fréchet space as a pair in duality, given as (X,X^*).

A set $A\subset E$ is said to be *bounded* if for every neighbourhood $V\subset E$ of the identity, there is a $\lambda>0$ such that $A\subseteq\lambda V$.

The weak operator topology (WOT) and the strong operator topology (SOT) can be described in terms of dual pairs. The pair $(\mathcal{L}_\omega(E),E\otimes E')$ is in duality via the bilinear form

$$\langle T,x\otimes y\rangle:=y(T(x)).$$

(Here $\otimes$ denotes the algebraic tensor product between the two vector spaces.) Then the WOT on $\mathcal{L}_\omega(E)$ is generated by the polars of all finite subsets of $E\otimes E'$. The SOT is generated by polars of the form $A\otimes B$ where A is a finite subset of E and B a ξ-equicontinuous subset of E'. (This condition on B means that for every $\epsilon>0$, there is a ξ-neighbourhood V of 0 in E such that $b(V)\subseteq[-\epsilon,\epsilon]$ for all $b\in B$.)

The theory of locally convex topological vector spaces is at the same time notoriously tricky and vastly general. Let us include two short results demonstrating some of the techniques required to prove facts about different locally convex toplogies, and how these topologies interact.

Lemma 2.1.1. *Let (E,E') and (F,F') be two pairs of vector spaces in duality. Suppose that $t:E\to F$ and its transpose $t':F'\to E'$ are both bijective linear maps. Then t is a homeomorphism between E and F when both are given their weak topologies, their Mackey topologies or their strong topologies.*

Proof. By [38, Ch. II, Prop 12, p38], both t and its inverse t^{-1} are continuous when E and F have their weak topologies.

Hence by [38, Ch. III, Prop 14, p62], both t and its inverse are continuous when E and F have their Mackey topologies.

Finally, suppose that both E and F have their strong topologies. We show that t is $\beta(E,E') - \beta(F,F')$-continuous. If $U \subset F$ is a neighbourhood in F, then under the $\beta(F,F')$-topology there is a $B \subset F'$ such that $B^\circ \subseteq U$. Now as shown above, $t' : F' \to E'$ is $\sigma(F',F) - \sigma(E',E)$-continuous, so $t'(B)$ is $\sigma(E',E)$-bounded in E'.

By [38, Ch. II, Lemma 6, p39], $(t^{-1}(B^\circ))^\circ = t'(B^{\circ\circ})$. By [38, Ch. II, Theorem 4, p35] and [38, Ch. IV, Lemma 1, p44], $B^{\circ\circ} \subseteq F'$ is also bounded. Hence by the weak continuity of t^{-1}, so is $(t^{-1}(B^\circ))^\circ$.

Therefore $t^{-1}(B^\circ)$ is a neighbourhood in the $\beta(E,E')$-topology, proving the continuity of t^{-1}. In exactly the same way we can prove the strong continuity of t. □

Lemma 2.1.2. *Let (E,E') be a dual pair with E barreled and let $(t_\gamma)_{\gamma\in\Gamma}$ be a net of continuous functionals in the $\beta(E,E')$-topology. Suppose that $t_\gamma \to t$ pointwise on E. Then t is also $\beta(E,E')$-continuous.*

Proof. This is essentially the Banach-Steinhaus theorem. By [38, Ch. IV, Theorem 3, p69], the set $\{t_\gamma\}$ is equicontinuous on E. This means that for any $\epsilon > 0$, there is a neighbourhood $U \subset E$ such that $|t_\gamma(U)| \le \epsilon$ for all $\gamma \in \Gamma$. Then if $x \in U$,

$$t(x) = \lim_{\gamma\to\infty} t_\gamma(x) \in [-\epsilon, \epsilon].$$

Thus t is continuous too. □

Finally, we frame a well-known mode of convergence in terms of a particular topology constructed by the techniques developed in [38]. In the sequel, we shall require several such reformulations, so it might be of use to see the process in action in a simple case. Recall that $M(X)$ is the space of all Radon measures on X.

Lemma 2.1.3. *If X is a locally compact Hausdorff space and $(f_n) \subset C_b(X)$ is a bounded sequence of continuous functions converging uniformly on compact subsets of X, then (f_n) is convergent in the $\sigma(C_b(X), M(X))$-topology.*

Proof. For any $Y \subset X$ and $g \in C_b(X)$, let $\|g\|_Y = \sup\{|g(y)| \ : \ y \in Y\}$. Let $A = \sup\{\|f_n\|_X\}$. Obviously, (f_n) converges to a continuous function f, and $\|f\|_X \le A$.

For any $\mu \in M(X)$, we must show that

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \to 0$$

as $n \to \infty$.

Let $\epsilon > 0$ be given. As μ is a finite Radon measure on X, there is a compact subset $K \subset X$ such that $\mu(X \backslash K) < \epsilon/4A$. Furthermore, (f_n) converges uniformly to f on K, and so there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n - f\|_K < \epsilon/2\mu(K)$. Hence for all $n \geq N$,

$$\begin{aligned} \left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \leq \int_X |f_n - f| \, d\mu &= \int_{X \backslash K} |f_n - f| \, d\mu + \int_K |f_n - f| \, d\mu \\ &\leq \|f_n - f\|_{X \backslash K} \mu(X \backslash K) + \|f_n - f\|_K \mu(K) \\ &< 2A.\epsilon/4A + (\epsilon/2\mu(K))\mu(K) \\ &= \epsilon. \end{aligned}$$

□

2.2 Basics of vector-valued measure theory

We make some remarks on measurable vector-valued functions on a measure space (Ω, μ). We require an extension of the theories of Bochner and Pettis integrable functions to functions taking values not in a Banach space, but more general locally convex vector spaces. We will also need a theory of *locally* Bochner integrable functions. Let (Ω, μ) be a σ-finite measure space and E a complete locally convex vector space whose topology is defined by the family $\{p_\alpha\}_{\alpha \in \Lambda}$ of seminorms. Here the theory and proofs closely follow the standard treatments for Banach space-valued functions, such as [44] or [14]. A *μ-simple measurable function* $f : \Omega \to E$ is a function $f = \sum_{i=1}^{N} \chi_{E_i} x_i$, where $E_1, \ldots, E_N$ are μ-measurable subsets of Ω and $x_1, \ldots, x_N \in E$. A function $f : \Omega \to E$ is said to be *μ-measurable* if there is a sequence of μ-simple measurable functions (f_n) that converges μ-almost everywhere to f.

A function $f : \Omega \to E$ is said to be *μ-weakly measurable* if the scalar-valued function $e'f$ is μ-measurable for every $e' \in E'$ and *Borel measurable* if for every open subset $\mathcal{O}$ of E, $f^{-1}(\mathcal{O})$ is a measurable subset of Ω. Finally, f is *μ-essentially $\langle$separably/ metrisably$\rangle$ valued* if there is a μ-measurable subset A of Ω whose complement has measure 0, such that $f(A)$ is contained in a $\langle$separable/metrisable$\rangle$ subspace of E.

It is worth noting that there are locally convex separable vector spaces that are not metrisable. The strict inductive limit topology discussed in [38, Section VII.1] can be used to construct such topologies.

Theorem 2.2.1. ***(Pettis Measurability Theorem)*** *For a σ-finite measure space (Ω, μ) and dual pair (E, E'), the following are equivalent for a μ-essentially metrisably valued function $f : \Omega \to E$:*

1. *f is μ-measurable*
2. *f is μ-weakly measurable and essentially separably valued*
3. *f is Borel μ-measurable and essentially separably valued.*

The proof of this theorem is a straightforward adaptation of the proof of the Banach space-valued proof presented in [44]. In particular, if E is separable and metrisable, the measurability and weak measurablility of a function are equivalent.

In the proof of this Theorem in [44], the following fact is quoted without proof. It remains crucial when handling metrisably valued measurable functions, so we record it here for completeness.

Lemma 2.2.2. *In a separable metric space X, any open set U is the union of countably many closed balls.*

Proof. Let $\{x_i\}$ be a countable dense subset of U. For each x_i, define

$$\begin{aligned} d_i &= d(x_i, U^c) = \inf\{d(x_i, u') : u' \in U^c\} \\ r_i &= d_i(1 - 1/2^i) \\ B_i &= \{x \in X : d(x, x_i) \leq r_i\}. \end{aligned}$$

Then the sets B_i are closed and $B_i \subset U_i$. The proof will be complete when we show that $U = \cup_i B_i$.

Take any $u \in U$. If u is an isolated point, then $u \in \{x_i\}$ and so $u \in B_i$ for some i. Otherwise, let $d = d(u, U^c)$ and $S = \{i \ : \ d(u, x_i) < d/2\}$. By the triangle inequality, for any $u' \in U^c$ and $i \in S$, we have

$$d(u, u') \leq d(u, x_i) + d(x_i, u').$$

Taking infimums over the u' first on the left and then on the right of the inequality and using the fact that $i \in S$, we get $d < d/2 + d_i$ and so

$$d/2 < d_i.$$

As S is infinite, we can find an i such that $d/2 < r_i$. Hence $d(x_i, u) < r_i$ and $u \in B_i$. □

We define the integral of a simple measurable function $s = \sum_{i=1}^{N} \chi_{E_i} x_i$ over a set $A \subset \Omega$ of finite measure by defining

$$\int_A s \; d\mu = \sum_{i=1}^{n} \mu(A \cap E_i) x_i.$$

Turning to the question of the integrability of vector-valued functions, we shall require two different integrals. We define the integral of a μ-simple measurable function $f = \sum_{i=1}^{N} \chi_{E_i} x_i$ to be

$$\int_\Omega f \; d\mu = \int_\Omega \sum_{i=1}^{N} x_i \chi_i \; d\mu = \sum_{i=1}^{N} \mu(E_i) x_i \tag{2.1}$$

Definition 2.2.3. *Consider a σ-finite measure space (Ω, μ) and dual pair (E, E') under a topology ξ and let $f : \Omega \to E$ be a vector valued function.*

1. *If f is μ-weakly measurable, we say that it is μ-**Pettis integrable** if for every μ-measurable subset $A \subset \Omega$ there is an element $\int_A f \; d\mu \in E$ such that for all $e' \in E'$,*
$$\Big\langle \int_A f \; d\mu, e' \Big\rangle = \int_A \langle f(\omega), e' \rangle \; d\mu(\omega). \tag{2.2}$$
2. *If f is μ-measurable and Pettis integrable, we say that it is μ-**Bochner integrable** if there exists a sequence (f_n) of μ-simple measurable functions converging a.e. to f, such that for every equicontinuous subset $\mathcal{A} \subset E'$*
$$\int_\Omega \sup_{e' \in \mathcal{A}} |\langle f(x) - f_n(x), e' \rangle| \; d|\mu|(x) \longrightarrow 0 \tag{2.3}$$
as $n \to \infty$.

In the definition of the Bochner integral, the hypothesis that the function is Pettis integrable ensures that the sequence $(\int_\Omega f_n d\mu)$ is not only Cauchy, but convergent. We define the Bochner integral of a μ-Bochner integrable function by the limit

$$\int_\Omega f(x) \; d\mu(x) = \lim_{n \to \infty} \int_\Omega f_n(x) \; d\mu(x).$$

The proof of the existence of this limit and its independence from the particular sequence of μ-simple measurable functions chosen, works exactly as in the Banach-valued case.

Lemma 2.2.4. *If f is μ-Bochner integrable from the measure space (Ω, μ) into the locally convex vector space E, then for any equicontinuous $\mathcal{A} \subset E'$ we have*

$$\sup_{e' \in \mathcal{A}} \left|\langle \int_\Omega f \; d\mu, e' \rangle\right| \leq \int_\Omega \sup_{e' \in \mathcal{A}} |\langle f(x), e' \rangle| \; d|\mu|(x). \tag{2.4}$$

Equivalently, one can define local Bochner integrability in terms of the seminorms definining the topology on E. In this language, f is locally Bochner integrable if there exists a sequence of measurable simple functions (f_n) converging almost everywhere to f and satisfying

$$\lim_{n \to \infty} \int_A p_\alpha(f - f_n) = 0$$

for every finitely measurable subset $A \subset \Omega$ and every seminorm p_α.

These integrability concepts will be crucial to understanding continuity properties of the action of a group on a locally convex vector space and will be used in Definition 4.1.1.

2.3 Product spaces via vector-valued measure theory

On the space of measurable functions over a measure space (Ω, μ) we can define a family of seminorms $p_A(f) := \int_A |f| \; d\mu$ as A ranges over all subsets of Ω of finite measure. Those measurable functions for which $p_A(f)$ is finite for all A of finite measure form the locally convex vector space of *locally integrable functions*. This space is called $L^{\mathrm{loc}}(\Omega)$ and is topologised by the family $\{p_A\}$.

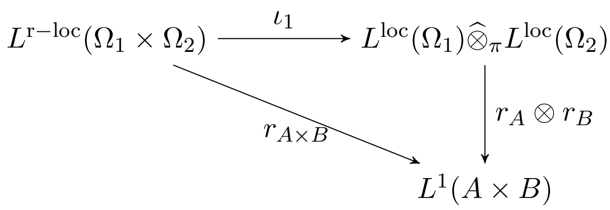
If we have two measure spaces (Ω_1, μ_1) and (Ω_2, μ_2) then we can consider the family of seminorms $p_{A \times B}$ defined on the set of measurable functions on $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ by setting $p_{A \times B}(f) = \int_{A \times B} |f| \; d\mu_1 \times \mu_2$ and define $L^{\mathrm{r-loc}}(\Omega_1 \times \Omega_2)$, the space of all *rectangular locally integrable functions*, to consist of those measurable functions for which all such seminorms are finite. As with the locally integrable functions, we use the family $\{p_{A \times B}\}$ to define the topology on $L^{\mathrm{r-loc}}(\Omega_1 \times \Omega_2)$.

In the category of locally convex spaces, it is possible to topologise the tensor product of two spaces in many ways. Two such tensor topologies can be distinguished: the projective tensor product used above and the injective tensor product. Here is one of two results on projective tensor products that we shall need.

Proposition 2.3.1. *If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two σ-finite measure spaces, then there is a continuous linear injection $\iota_1 : L^{\mathrm{r-loc}}(\Omega_1 \times \Omega_2) \to L^{\mathrm{loc}}(\Omega_1) \widehat{\otimes}_\pi L^{\mathrm{loc}}(\Omega_2)$.*

Proof. The space $L^{\text{loc}}(\Omega_1)$ may be recognised as the projective limit of the family of spaces $\{L^1(A)\}$, as A ranges over all finitely measurable subsets in Σ_1 and mappings $\{r_{BA} : L^1(B) \to L^1(A)\}$ where $B \supseteq A$ and r_{BA} is the restriction map. Similarly for $L^{\text{loc}}(\Omega_2)$. Hence by [24, Theorem 15.2], $L^{\text{loc}}(\Omega_1)\widehat{\otimes}_\pi L^{\text{loc}}(\Omega_2)$ is the projective limit of the family $L^1(A)\widehat{\otimes}_\pi L^1(B)$, as A and B range over all finitely measurable subsets of Ω_1 and Ω_2 respectively.

Now $L^1(A)\widehat{\otimes}_\pi L^1(B) \simeq L^1(A \times B)$ as shown for example in [44]. As the restriction maps $r_{A\times B} : L^{\text{r}-\text{loc}}(\Omega_1 \times \Omega_2) \to L^1(A \times B)$ are well-defined and continuous, from the universal property of the projective limit (as described in [31, Chapter 3.4] or [48, Appendix L]) there is a unique continuous linear mapping $\iota_1 : L^{\text{r}-\text{loc}}(\Omega_1 \times \Omega_2) \to L^{\text{loc}}(\Omega_1)\widehat{\otimes}_\pi L^{\text{loc}}(\Omega_2)$ such that the following diagrams commute for all finitely measurable $A \in \Sigma_1$ and $B \in \Sigma_2$.



It is clear from this diagram that ι_1 is injective: if f and g are distinct elements of $L^{\text{r}-\text{loc}}(\Omega_1 \times \Omega_2)$, then there is some rectangle $A \times B$ on which they are not equal a.e.; this means that $r_{A\times B}(f - g) \neq 0$ and so $\iota_1(f - g) \neq 0$. □

We now note the result about the injective product that we shall employ, namely that if X is locally compact Hausdorff and E is a complete locally convex vector space then

$$C(X, E) \simeq C(X)\widehat{\otimes}_\epsilon E$$

where $C(X, E)$ and $C(X)$ denote respectively the continuous E-valued and $\mathbb{C}$-valued functions on X equipped with the compact-open topology and $\widehat{\otimes}_\epsilon$ denotes the completion of the injective tensor product of the two spaces. This result is proved in [24, Corollary 3, Section 16.6].

The locally convex space of all locally Bochner integrable E-valued functions is called $L^{\text{loc}}(\Omega, E)$ and its topology is given by the family of seminorms $\pi_{A,\alpha}$ defined by $\pi_{A,\alpha}(f) = \int_A p_\alpha(f)\, d\mu$, where A is a subset of Ω of finite measure and p_α is one of the seminorms that defines the topology on E.

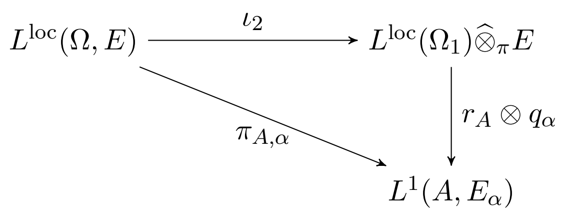
In the case where E is a Banach space, it is well-known that $L^1(\Omega, E) \simeq L^1(\Omega)\widehat{\otimes}_\pi E$, which is proved in detail in [44] and [14]. From this fact, we can construct a similar, though weaker, relation between $L^{\text{loc}}(\Omega, E)$ and $L^{\text{loc}}(\Omega)\widehat{\otimes}_\pi E$ where E is a complete locally convex space, which is given in the following lemma.

Lemma 2.3.2. *There is a naturally defined continuous injective mapping* $\iota_2 : L^{\text{loc}}(\Omega, E) \to L^{\text{loc}}(\Omega)\widehat{\otimes}_\pi E$.

Proof. We shall work once more with restriction maps as we did in Proposition 2.3.1. Any locally convex space E is the projective limit of a family of Banach spaces E_α. For every α, let $q_\alpha : E \mapsto E_\alpha$ be the continuous linear mapping induced by the projective limit. Then we can define $q'_\alpha : L^{\text{loc}}(\Omega, E) \to L^{\text{loc}}(\Omega, E_\alpha)$ to be the mapping $f \mapsto q_\alpha \circ f$.

For any subset $A \subseteq \Omega$ of finite measure, define $r_A : L^{\text{loc}}(\Omega, E) \to L^{\text{loc}}(A, E)$ to be the restriction map $f \mapsto f\big|_A$.

Now we can define maps $\pi_{A,\alpha} : L^{\text{loc}}(\Omega, E) \to L^1(A, E_\alpha)$ by setting $\pi_{A,\alpha} := r_A \circ q'_\alpha = q'_\alpha \circ r_A$. On the other hand, there are the mappings $r_A \otimes q_\alpha : L^{\text{loc}}(\Omega)\widehat{\otimes}_\pi E \to L^1(A)\widehat{\otimes}_\pi E_\alpha$. Identifying $L^1(A)\widehat{\otimes}_\pi E_\alpha$ and $L^1(A, E_\alpha)$, we see that $L^{\text{loc}}(\Omega)\widehat{\otimes}_\pi E$ is the projective limit of the family $\{L^1(A, E_\alpha)\}$. By the universal property of the projective limit, there must be a unique mapping $\iota_2 : L^{\text{loc}}(\Omega, E) \to L^{\text{loc}}(G)\widehat{\otimes}_\pi E$ such that the diagrams



commute for all α and finitely measurable A. From these diagrams the injectivity of ι_2 is clear. □

There is a strong link between the spaces $L^{\text{r}-\text{loc}}(G \times \Omega)$, $L^{\text{loc}}(\Omega, L^{\text{loc}}(G))$ and $L^{\text{loc}}(G)\widehat{\otimes}_\pi L^{\text{loc}}(\Omega)$ that builds on Proposition 2.3.1 and Lemma 2.3.2. (Here G is a locally compact Hausdorff group acting on the measure space Ω.) To establish this link, we need the following elementary lemma.

Lemma 2.3.3. *If* (Ω_1, μ_1) *and* (Ω_2, μ_2) *are* σ*-finite measure spaces then the collection of rectangular simple functions with support of finite measure is dense in* $L^{\mathrm{r-loc}}(\Omega_1 \times \Omega_2)$.

Proof. Let $f \in L^{\mathrm{r-loc}}(\Omega_1 \times \Omega_2)$ be a $[0, \infty]$-valued function. To prove the Lemma, it suffices to show that there is an increasing sequence (f_n) of non-negative rectangular simple functions with support of finite measure such that $\lim_{n\to\infty} f_n(\omega_1, \omega_2) = f(\omega_1, \omega_2)$ $\mu_1 \times \mu_2$-a.e., because then by the Monotone Convergence Theorem

$$\lim_{n\to\infty} \int_{A\times B} |f - f_n| \, d\mu_1 \times \mu_2 = 0$$

for any $\mu_1 \times \mu_2$-finite rectangle $A \times B$. By [42, Theorem 1.17] there is an increasing sequence of simple functions (g_n) converging pointwise a.e. to f. We may assume that each g_n has $\mu_1 \times \mu_2$-finite support: as $\Omega_1 \times \Omega_2$ is σ-finite, there is an increasing sequence of finitely measurable sets (A_n) whose union is all of $\Omega_1 \times \Omega_2$. Then the sequence $(g_n \chi_{A_n})$ has all the desired properties.

Let us observe how a simple function with finite-measured support can be approximated by a rectangular simple function of finite support. Fix an $\epsilon > 0$. If $g = \sum_{i=1}^m \chi_{E_i} c_i$ is a simple function with finite-measured support, the measurability of each E_i in $\Omega_1 \times \Omega_2$ implies that there is a set $E_i' \subseteq E_i$ such that E_i' is the union of finitely many rectangles and $\mu_1 \times \mu_2(E_i \backslash E_i') < \epsilon/2^i$. Then $g' = \sum_{i=1}^m \chi_{E_i'} c_i$ is a rectangular simple function and g' differs from g on a set of measure at most ϵ, and $g' \leq g$.

Starting from the sequence (g_n), define f_n to be the approximation of g_n as described in the previous paragraph such that $\mathrm{supp}(f_{n+1}) \supseteq \mathrm{supp}(f_n)$ and

$$\mathrm{supp}(g_n) \backslash \mathrm{supp}(f_n) < 1/n$$

for all $n \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} f_n(\omega_1, \omega_2) = \lim_{n\to\infty} g_n(\omega_1, \omega_2) = f(\omega_1, \omega_2)$$

for a.e. $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, proving the Lemma. □

Define $\iota_3 : L^{\mathrm{r-loc}}(G \times \Omega) \to L^{\mathrm{loc}}(\Omega, L^{\mathrm{loc}}(G))$ by setting $f \mapsto F$ where $F(\omega) = f_\omega$ and where $f_\omega(t) := f(t, \omega)$. If f is a rectangular simple function then f_ω is a simple function on G for every $\omega \in \Omega$ and so for any subsets $A \subset \Omega$ and $B \subset G$ of finite measure,

$$\int_A \int_B |f_\omega(t)| \, dt \, d\mu(\omega) < \infty,$$

which shows that $\iota_3(f)$ is a simple function in $L^{\mathrm{loc}}(\Omega, L^{\mathrm{loc}}(G))$. Furthermore, as by Lemma 2.3.3 $f \in L^{\mathrm{loc}}(G \times \Omega)$ is the limit of rectangular simple functions, so is $F := \iota_3(f)$, implying the measurability of F. Denoting by p_B the seminorm $g \mapsto \int_B |g|\, dh$ on $L^{\mathrm{loc}}(G)$,

$$\int_A p_B(F(\omega))\, d\mu(\omega) = \int_A \int_B |f|\, dt d\mu(\omega) < \infty,$$

proving the well-definedness and continuity of ι_3. Clearly ι_3 is injective. The mappings ι_1, ι_2 and ι_3 are related by the following diagram.

$$\begin{array}{ccc} L^{\mathrm{r-loc}}(G \times \Omega) & \xrightarrow{\iota_1} & L^{\mathrm{loc}}(\Omega) \widehat{\otimes}_\pi L^{\mathrm{loc}}(G) \\ & \searrow^{\iota_3} & \uparrow \iota_2 \\ & & L^{\mathrm{loc}}(\Omega, L^{\mathrm{loc}}(G)) \end{array}$$

The commutativity of this diagram can be checked for simple functions and tensors. As these collections are dense in their respective spaces, the continuity of the arrows yields the desired commutativity.

There is also a relation between the spaces $L^{\mathrm{loc}}(\Omega, C(G))$ and $L^{\mathrm{loc}}(\Omega) \widehat{\otimes}_\epsilon C(G) \simeq C(G, L^{\mathrm{loc}}(\Omega))$ that is of interest.

Lemma 2.3.4. *The space $L^{\mathrm{loc}}(\Omega, C(G))$ may be continuously embedded in $C(G, L^{\mathrm{loc}}(\Omega))$.*

Proof. Consider the mapping $\iota_2 : L^{\mathrm{loc}}(\Omega, C(G)) \to L^{\mathrm{loc}}(\Omega) \widehat{\otimes}_\pi C(G)$ given in Lemma 2.3.2, the canonical injection $i : L^{\mathrm{loc}}(\Omega) \widehat{\otimes}_\pi C(G) \to L^{\mathrm{loc}}(\Omega) \widehat{\otimes}_\epsilon C(G)$ and the isomorphism $L^{\mathrm{loc}}(\Omega) \widehat{\otimes}_\epsilon C(G) \simeq C(G, L^{\mathrm{loc}}(\Omega))$. The composition of these maps gives us the required continuous linear injection of $L^{\mathrm{loc}}(\Omega, C(G))$ into $C(G, L^{\mathrm{loc}}(\Omega))$. □

Lemma 2.3.5. *Let f be an element of $L^{\mathrm{loc}}(\Omega, C(G))$. Then the function $f'(t, \omega) := f(\omega)(t)$ defined on $(G \times \Omega, h \times \mu)$ is measurable.*

Furthermore, if f has metrisable range then f' satisfies the following restricted Fubini theorem: for any $A \subseteq \Omega$ of finite measure and compact $K \subset G$,

$$\int_A \int_K f'(t, \omega)\, dt d\mu = \int_K \int_A f'(t, \omega)\, d\mu dt. \tag{2.5}$$

Proof. The function f is measurable, which means by definition that there is a sequence (f_n) of $C(G)$-valued simple functions converging μ-a.e. to f. For a $C(G)$-valued simple function g, the function $g'(t,\omega) := g(\omega)(t)$ is easily seen to be measurable.

Now for μ-a.e. $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$ in $C(G)$. In particular, for any $t \in G$, $f_n(\omega)(t) \to f(\omega)(t)$. This is synonymous with the expression $f'_n(t,\omega) \to f'(t,\omega)$, which implies the measurability of f'.

We turn now to the Fubini-type result. Let $\mathcal{M}$ be the subspace (in the relative topology) of all elements in $L^{\mathrm{loc}}(\Omega, C(G))$ with metrisable range. Consider the following diagram.

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\int_A} & C(G) \\
{\scriptstyle \int_K}\downarrow & & \downarrow{\scriptstyle \int_K} \\
L^{\mathrm{loc}}(\Omega) & \xrightarrow[\int_A]{} & \mathbb{C}
\end{array}
$$

Here A is a subset of Ω of finite measure and K is a compact subset of G. The maps are naturally defined: $\int_A : L^{\mathrm{loc}}(\Omega, C(G)) \to C(G)$ sends an F to $\int_A F(\omega)\ d\mu$ which is well-defined because of the local integrability of F; $\int_K : L^{\mathrm{loc}}(\Omega, C(G)) \to L^{\mathrm{loc}}(\Omega)$ sends an F to $\widetilde{F}(\omega) := \int_K F(\omega)\ dh$. The well-definedness of this map depends on the Pettis Measurability Theorem 2.2.1, because the functional $f \mapsto \int_K f\ dt$ is a continuous linear functional on $C(G)$. The maps $\int_A : L^{\mathrm{loc}}(\Omega) \to \mathbb{C}$ and $\int_K : C(G) \to \mathbb{C}$ are defined in the obvious way.

So the arrows of the diagram are all continuous linear mappings. It is easy to see that the diagram commutes for all simple functions in $L^{\mathrm{loc}}(\Omega, C(G))$. As the simple functions form a dense subset, the commutativity of the diagram and the validity of (2.5) is proved. □

Alternatively, in the case that G is second countable, one can use the following lemma to prove the measurability of f'. It is an extension of [42, Ch. 7, exercise 8a)]. One of the reasons for including this result is to contrast the direct approach in measure theory with our approach of using vector valued measure theory. Indeed the previous lemma is stronger than the next one.

Lemma 2.3.6. *Let X be a second countable metric space with Borel σ-algebra Σ_X and Y a measure space with σ-algebra Σ_Y. Let f be a complex-valued function defined on $X \times Y$. Suppose that for almost every $x \in X$, the cross sections $f_x(y) := f(x, y)$ are measurable and that for almost every $y \in Y$ the cross sections $f^y(x) := f(x, y)$ are continuous. Then f is measurable with respect to the σ-algebra generated by $\Sigma_X \times \Sigma_Y$.*

Proof. We shall suppose, without loss of generality, that f is real-valued. For any $a \in \mathbb{R}$, we must show that $f^{-1}[a, \infty)$ is a measurable subset of $X \times Y$. We do this by constructing measurable subsets $E_{n,m} \subset X \times Y$ for all $m, n \in \mathbb{N}$ such that

$$\bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} E_{m,n} = f^{-1}[a, \infty).$$

For the remainder of the proof, let Q be a countable dense subset of X. For each $n \in \mathbb{N}$, let V_i be the open ball of radius $1/2n$ centred on $q_i \in Q$. Define $\mathcal{A}_n$ to be the countable collection of pairwise disjoint measurable subsets of X with nonempty interior of diameter less than $1/n$ where we set $A_1 = V_1$ and for $i > 1$, set $A_i = V_i \backslash (A_1 \bigcup \ldots \bigcup A_{n-1})$. We remove empty sets and renumber if necessary.

Now for any $A \in \mathcal{A}_n$, let f_A be a function on Y defined by

$$f_A(y) = \inf_{q\in Q\bigcap A} \{f_q(y)\} = \inf_{q\in Q\bigcap A} \{f(q, y)\}.$$

As f_A the the infimum of a countable collection of measurable functions, it is itself measurable.

For any $m, n \in \mathbb{N}$ we define

$$E_{m,n} = \bigcup_{A\in\mathcal{A}_n} A \times f_A^{-1}\big[a - \frac{1}{m}, \infty\big),$$

which is clearly a measurable subset of $X \times Y$. We shall show that $\bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} E_{n,m} \subseteq f^{-1}[a, \infty)$.

If $(x, y) \in \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} E_{n,m}$, then for some n and all m, $(x, y) \in E_{n,m}$. Now $x \in A$ for some $A \in \mathcal{A}_n$. For any $q \in A \bigcap Q$, $f_q \geq f_A$, so

$$f_q^{-1}\big[a - \frac{1}{m}, \infty\big) \supseteq f_A^{-1}\big[a - \frac{1}{m}, \infty\big)$$

for all $m \in \mathbb{N}$. Therefore $f_A(y) \geq a - 1/m$ and so $f_q(y) \geq a - 1/m$ for all $q \in A \bigcap Q$ and $m \in \mathbb{N}$. As some subsequence in $Q \bigcap A$ converges to x and as f^y is continuous on X, this means that $f(x,y) \geq a - 1/m$ for all $m \in \mathbb{N}$. Hence $f(x,y) \geq a$ and $(x,y) \in f^{-1}[a,\infty)$.

Now we complete the proof by demonstrating the reverse inclusion $f^{-1}[a,\infty) \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} E_{n,m}$. If $(x,y) \in f^{-1}[a,\infty)$, then for any $m \in \mathbb{N}$, $f(x,y) > a - 1/m$. Hence if we set $U = \{\omega : f(\omega, y) > a - 1/m\}$, then U is an open subset of X, owing to the continuity of f^y. Clearly $x \in U$. Let $n \in \mathbb{N}$ be a number such that $d(x, U^c) > 1/n$. Note that $x \in A$ for some $A \in \mathcal{A}_{2n}$.

In fact $A \subset U$, because for any $a \in A$ and $u' \in U^c$,

$$d(a,u') + \frac{1}{2n} \geq d(a,u') + d(a,x) \geq d(x,u') > \frac{1}{n}.$$

Hence $d(a,u') > 1/2n$ for all $u' \in U^c$, so $a \notin U^c$: that is, $a \in U$. Consequently, $f_A(y) = \inf_{q\in A\bigcap Q}\{f_q(y)\} \geq a - 1/m$, so $(x,y) \in A \times f_A^{-1}[a - 1/m, \infty)$. Therefore $(x,y) \in E_{2n,m}$ for all $m \in \mathbb{N}$ and so $(x,y) \in \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} E_{n,m}$, proving the lemma. □

All the different integrability conditions used in this section can be unified by the following concept. Let (Ω, Σ, μ) be a measure space and let $\mathcal{A} \subset \Sigma$ be an algebra of measurable sets. A measurable function f on Ω is called *$\mathcal{A}$-integrable* if

$$\int_A |f| \, d\mu$$

is finite for every $A \in \mathcal{A}$. On the product space $(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)$, when $\mathcal{A}$ is the algebra of rectangles $A \times B$ where A and B have finite measure, the $\mathcal{A}$-integrable functions on $\Omega_1 \times \Omega_2$ are the rectangular locally integrable functions. On $G \times \Omega$, where $\mathcal{A}$ consists of all rectangles $K \times A$ where $K \subset G$ is compact and $A \subset \Omega$ has finite measure, the functions considered in Lemma 2.3.5 are $\mathcal{A}$-measurable.

2.4 Harmonic Analysis

We develop here the harmonic analysis of abelian locally compact Hausdorff groups that we shall require for the mean ergodic theory. Thereafter, we discuss some locally convex topologies on vector spaces.

By $M(G)$ we shall mean the Banach *-algebra of all finite Radon measures on G, where the multiplication of measures is given by their convolution. The closed ideal of all those measures absolutely continuous with respect to the Haar measure is the

Banach algebra $L^1(G)$. By $\widehat{G}$ we shall mean the Pontryagin dual of G consisting of all continuous characters of G; we will call continuous characters simply 'characters' in what follows. We denote by $\widehat{\mu}$ the Fourier transform of a measure $\mu \in M(G)$ and by $\nu(\mu) = \{\xi \in \widehat{G} : \widehat{\mu}(\xi) = 0\}$ the *null-set* of μ.

We will need some results concerning the convergence of measures given the convergence of their Fourier series. We recall some elements of the representation theory of groups, as presented in Chapter 3 of [17]. We denote by $\mathcal{P} \subset C_b(\widehat{G})$ the set of continuous functions of positive type. Such functions are also known as positive-definite functions. (See [17] for the theory of functions of positive type, and both [17] and [41] for material on positive-definite functions.)

Set $\mathcal{P}_0 = \{\phi \in \mathcal{P} : \|\phi\|_\infty \leq 1\}$. This set, viewed as a subset of the unit ball of $L^1(G)^*$, is weak*-compact.

We have the following extension of [41, Theorem 1.9.2]:

Theorem 2.4.1. *Let (μ_n) be a bounded sequence of Radon measures on G and let K be a closed subset of $\widehat{G}$, such that K is the closure of its interior. If $(\widehat{\mu}_n)$ converges uniformly on compact subsets of K to a function ϕ, then there is a bounded Radon measure μ such that $\widehat{\mu} = \phi$ on K.*

Proof. Without loss of generality, we may assume that $(\mu_n) \subset M_1^+(G)$, the set of positive Radon measures of norm no greater than 1. Also, for those closed $K \subset \widehat{G}$ as in the hypotheses, the space $C_b(K)$ may be identified with a norm-closed subspace of $L^\infty(K, m)$, where m is the Haar measure on $\widehat{G}$. In the sequel, all L^∞-spaces will be taken with respect to the Haar measure and so we shall simply write $L^\infty(K)$ for $L^\infty(K, m)$.

First we prove the result for compact K. Note that $\mathcal{P}_0 \subset C_b(\widehat{G})$, which can be identified with a closed subset of $L^\infty(\widehat{G})$. Furthermore, $\mathcal{P}_0$ is absolutely convex and closed in the weak*-topology on $L^\infty(\widehat{G})$ and hence closed in the finer norm topology. Now consider the restriction map $R : L^\infty(G) \to L^\infty(K)$. This map is not only norm-continuous, but weak*-continuous. Hence $R(\mathcal{P}_0)$ is weak*-compact and absolutely convex in $L^\infty(K)$, which implies that it is also norm-closed (This follows from [39, Prop. 8, e34] and the fact that the weak* and norm topologies on $L^\infty(G)$ are respectively the topologies $\sigma(L^\infty(G), L^1(G))$ and $\beta(L^\infty(G), L^1(G)))$. Of course, $R(\mathcal{P}_0) \subset C(K)$, which can be identified with a closed subset of $L^\infty(K)$.

By Bochner's Theorem (cf [41] or [17]), the Fourier transform gives a bijection between $M_1^+(G)$ and $\mathcal{P}_0(\widehat{G})$. This means that $(\widehat{\mu}_n|_K) \subset R(\mathcal{P}_0)$. By hypothesis, this sequence converges uniformly to some $\phi \in R(\mathcal{P}_0)$ and so there is a $\tilde{\phi} \in \mathcal{P}_0$ such that $R(\tilde{\phi}) = \phi$. Consequently there is a $\mu \in M_1^+(G)$ such that $\widehat{\mu} = \tilde{\phi}$ and so $\widehat{\mu}|_K = \phi$.

This proves the result when K is compact. To prove it in the general case when K is closed, we use the above result as well as the fact that Bochner's Theorem states that the Fourier transform is in fact a homeomorphism when $M_1^+(G)$ and $\mathcal{P}_0(\widehat{G})$ are each given their weak*-topologies.

As $(\widehat{\mu}_n|_K)$ is bounded and converges uniformly on compact subsets of K, its limit ϕ is continuous and bounded on K. Let $\mathcal{C}$ be the collection of all compact subsets of K which are the closures of their interiors. For any $C \in \mathcal{C}$, define

$$S(C, \phi) = \{\mu \in M_1^+(G) : \widehat{\mu}|_C = \phi|_C\} \subset M_1^+(G).$$

As proved above, $S(C, \phi)$ is nonempty. Furthermore, $S(C, \phi)$ is a weak*-compact subset of $M_1^+(G)$. To see this, set

$$B_C(\phi) = \{f \in L^\infty(G) : f|_C = \phi|_C \ a.e.\}$$

and note that $B_C(\phi)$ is weak*-closed. Hence $\mathcal{P}_0 \cap B_C(\phi)$ is weak*-compact. Finally $\mathcal{P}_0 \cap B_C(\phi)$ is the image of $S(C, \phi)$ under the Fourier transform, so $S(C, \phi)$ must be weak*-compact as well, by Bochner's Theorem.

Now the collection $\{S(C, \phi) : C \in \mathcal{C}\}$ is a collection of nonempty weak*-compact subsets of $M_1^+(G)$. This collection has the finite intersection property, because if $C_1, C_2, \ldots, C_n \in \mathcal{C}$, then

$$S(C_1, \phi) \cap S(C_2, \phi) \cap \ldots \cap S(C_n, \phi) = S(C_1 \cup C_2 \cup \ldots \cup C_n, \phi),$$

which is of course also nonempty. Hence the intersection of all the sets $S(C, \phi)$ is nonempty. Let μ be in this intersection.

Then $\widehat{\mu}|_C = \phi|_C$ for every $C \in \mathcal{C}$; hence $\widehat{\mu}|_K = \phi$. □

We take the opportunity to demonstrate the power of the techniques developed for constructing various topologies on vector spaces. Indeed, one can build such a topology to encode the convergence of Fourier transforms handled in the last result.

Recall that $G_\flat$, the Bohr compactification of G, is a compact group in which G can be embedded continuously. It is constructed by first considering $\widehat{G}$ under the discrete topology. (This is denoted $\widehat{G}_d$.) The Pontryagin dual of this discrete group is a compact group, namely $G_\flat$. Recall also that $AP(G)$, the set of almost periodic functions on G, is simply the norm-closure in $C_b(G)$ of all the trigonometric polynomials. It so happens that $AP(G)$ is precisely the set of continuous bounded functions on G that can be extended to continuous functions on $G_\flat$. So $M(G_\flat)$ is the dual of $AP(G) = C(G_\flat)$.

By [41, Theorem 1.9.1], $M(G)$ may be identified with a subspace of $M(G_\flat)$.

Theorem 2.4.2. *Let (μ_n) be a bounded sequence of Radon measures on G such that $(\widehat{\mu}_n)$ converges pointwise. Then (μ_n) is Cauchy in the $\sigma(M(G), AP(G))$-topology. The sequence has a limit point in $M(G_\flat)$.*

In particular, if $(\widehat{\mu}_n)$ converges uniformly on compact subsets of $\widehat{G}$, then (μ_n) is convergent in $M(G)$.

Proof. Let $\Phi(\xi) = \lim_{n\to\infty} \widehat{\mu}_n(\xi)$ for all $\xi \in \widehat{G}$. By the Fourier Uniqueness Theorem, there is a measure on $G_\flat$, say μ, such that $\widehat{\mu} = \Phi$. Note that the $\sigma(M(G), AP(G))$-topology is exactly the restriction of the weak*-topology on $M(G_\flat)$ to $M(G)$. Define $D(\widehat{G})$ to be the collection of all finite linear combinations of all Dirac measures on $\widehat{G}$. Then the pair $(C(\widehat{G}_d), D(\widehat{G}))$ is a dual pair, where $\widehat{G}_d$ has the discrete topology. The Fourier transform $\mathcal{F} : M(G_\flat) \to C(\widehat{G}_d)$ has transpose $\mathcal{F}' : D(\widehat{G}) \to AP(G)$. So by [38], $\mathcal{F}$ is weakly convergent in this topology.

Now (μ_n) is relatively weak*-compact in $M(G_\flat)$ because this sequence is norm-bounded. It has a convergent subsequence, (μ_{n_k}). But then $(\widehat{\mu}_{n_k})$ is convergent in $M(\widehat{G}_d)$ in the $\sigma(M(\widehat{G}_d), D(\widehat{G}))$-topology, which means that $(\widehat{\mu}_{n_k})$ converges pointwise on $\widehat{G}$ to Φ. Hence $\mu_{n_k} \to \mu$. As all convergent subsequences of (μ_n) have the same limit point, (μ_n) is convergent to a point in $M(G_\flat)$. It follows at once that (μ_n) is Cauchy in the $\sigma(M(G), AP(G))$-topology.

For the second part of the theorem, note that by 2.4.1, the limit μ constructed above belongs in fact to $M(G)$. □

There is an ideal of $L^1(G)$ that will be important for our purposes: $K(G)$, the set of all functions in $L^1(G)$ whose Fourier transforms have compact support. In [17] and [41], it is shown that the ideal $K(G)$ is a norm-dense subset of $L^1(G)$.

Turning to closed ideals $\mathcal{I}$ of $L^1(G)$, we can define the null-set $\nu(\mathcal{I})$ as we did for individual functions and measures:

$$\nu(\mathcal{I}) = \{\xi \in \widehat{G} : \widehat{f}(\xi) = 0, \text{ for all } f \in \mathcal{I}\}.$$

Thus to each closed ideal in $L^1(G)$ we can assign a unique closed subset of $\widehat{G}$. However the converse is not in general true: for a closed subset K of $\widehat{G}$, there is usually more than one ideal whose null-set is K. Among all such ideals, two can be singled out: the largest, $\iota_+(K)$ consisting of all $f \in L^1(G)$ such that $\widehat{f}$ is 0 on K, and the smallest, $\iota_-(K)$, consisting of all $f \in L^1(G)$ such that $\widehat{f}$ is 0 on some *open neighbourhood* of K. From the definitions, it is clear that $\iota_-(K) \subseteq \iota_+(K)$. In [17] and [41] it is proved that if $\mathcal{I}$ is a closed ideal in $L^1(G)$ with $\nu(\mathcal{I}) = K$, then

$$\iota_-(K) \subseteq \mathcal{I} \subseteq \iota_+(K).$$

There are some sets K for which there is only one associated ideal. Such closed sets are called *sets of synthesis*, or *S-sets* for short. In this case, we shall call $\iota(K)$ the unique ideal associated with the S-set K. The fact that such sets have only one closed ideal in $L^1(G)$ associated with them will be used often in what follows.

Spectral synthesis will play a large part in the sequel. References for this material are [41, §7.8] and [17, §4.6]. To fix notation, we make a few remarks here. Any weak*-closed translation-invariant subset T of $L^\infty(G)$ has a *spectrum*, denoted $\sigma(T)$, consisting of all characters contained in T. The spectrum is always closed in $\widehat{G}$.

The following theorem is crucial in the use of spectral synthesis. It is a slight restatement of [21, Théorème F, p132]. The second part is proved in [41, Theorem 7.8.2e)] (in [17, Proposition 4.75], a special case is shown.)

Theorem 2.4.3. ***(Spectral Approximation Theorem)*** *Let V be a weak*-closed translation-invariant subspace of $L^\infty(G)$ with spectrum $\sigma(V) = \Lambda$. Then for any open set U containing Λ, any $f \in V$ can be weak*-approximated by trigonometric polynomials formed from elements of U.*

Furthermore, if Λ is an S-set, then any $f \in V$ can be weak-approximated by trigonometric polynomials formed from elements of Λ.*

We need the following modification of [21, Théorème A, p124].

Theorem 2.4.4. *Let K be a compact subset of $\widehat{G}$ and let μ be a measure in $M(G)$ whose Fourier transform $\widehat{\mu}$ does not vanish on K. Then there exists a $g \in L^1(G)$ satisfying $\widehat{g}(\xi) = \dfrac{1}{\widehat{\mu}(\xi)}$ for all $\xi \in K$.*

Proof. By [17, Lemma 4.50] or [41, Theorem 2.6.2], there exists a summable h on G such that $\widehat{h} = 1$ on K. As $\mu * h \in L^1(G)$, we can apply [21, Théorème A, p124] to obtain a $g \in L^1(G)$ such that

$$\widehat{g}(\xi) = \frac{1}{\widehat{\mu * h}(\xi)}$$

for all $\xi \in K$. From the above equation and the fact that $\widehat{h} = 1$ on K, we see that this g is the one required. □

As with Theorem 2.4.1, we will often be in a position where we must infer properties of a summable function μ from knowledge of its Fourier transform on a compact subset $K \subset \widehat{G}$. Of course, there will be in general many functions whose Fourier transforms agree on K.

To clarify the situation, we make use of a quotient space construction. Using $\iota_+(K)$, the largest ideal with nullset K, we can form the quotient $L^1(G)/\iota_+(K)$. Let $[\mu]$ denote an element of this quotient; it is an equivalence class consisting of all functions ν such that $\widehat{\mu}|_K = \widehat{\nu}|_K$.

The following lemma exemplifies some of the techniques employed when working with quotient spaces of $L^1(G)$, and will come in handy when proving the main result, Theorem 4.3.1.

Lemma 2.4.5. *Let (φ_n) be a sequence in $L^1(G)$ and let K be a compact subset of $\widehat{G}$. If $([\varphi_n]) \subset L^1(G)/\iota_+(K)$ is a relatively weakly compact sequence and $\lim_{n\to\infty} \widehat{\varphi}_n(\xi)$ exists for each $\xi \in K$, then $([\varphi_n])$ is weakly convergent.*

Proof. Suppose $([\varphi_n])$ was not weakly convergent. Being relatively weakly compact, it must then contain two weakly convergent subsequences $([\varphi_{n_k}])$ and $([\varphi_{n_\ell}])$ with different limits $[\mu]$ and $[\nu]$ respectively.

The Gelfand transform $\mathcal{F}_K : L^1(G)/\iota_+(K) \to C(K)$, given by $\varphi \mapsto \widehat{\varphi}|_K$, is norm-continuous and hence weakly continuous. Therefore $(\widehat{\varphi}_{n_k})$ and $(\widehat{\varphi}_{n_\ell})$ are weakly convergent in $C(K)$. By [13, Ch VII, Theorem 2], this means that $\lim_{k\to\infty} \widehat{\varphi}_{n_k}(\xi) = \widehat{\mu}(\xi)$ and $\lim_{\ell\to\infty} \widehat{\varphi}_{n_\ell}(\xi) = \widehat{\nu}(\xi)$. By hypothesis, then, $\widehat{\mu} = \widehat{\nu}$ on K and so $[\mu] = [\nu]$ in $L^1(G)/\iota_+(K)$, a contradiction. Hence $([\varphi_n])$ is weakly convergent. □

We are also able to prove an interesting result that sheds light on the topological vector space structure of the space $L^1(G)/\iota_+(K)$, a space which lies at the heart of our Tauberian theorems. This result, Theorem 2.4.7, will not have a direct influence on our work, but serves to illustrate the richness of the vector spaces that emerge from harmonic analytic considerations. The next Lemma appears in [43, Theorem 4.9]. We state it here for completeness.

Lemma 2.4.6. *Let E be a Banach space with dual E' and let $A \subset E$ be a closed subspace. Then the dual of A may be identified with E'/A°.*

It is well known that $A' = E'/A^\perp$. Since A is a linear subspace, it is moreover easy to verify that $A^\perp = A^\circ$.

The above result inspires the following proof.

Theorem 2.4.7. *If K is a compact subset of $\widehat{G}$, then $L^1(G)/\iota_+(K)$ is the dual of*

$$\overline{sp}\{\widehat{f} \ : \ f \in L^1(\widehat{G}), \ supp\ f \subset K\}.$$

Proof. Note that $C_0(G)^* = M(G)$ and set

$$\begin{aligned} A &= \overline{\mathrm{sp}}\{\widehat{f} \ : \ f \in L^1(\widehat{G}), \ \mathrm{supp}\ f \subset K\} \subset C_0(G) \\ M_K(G) &= \{\mu \in M(G) \ : \ \widehat{\mu} \equiv 0 \text{ on } K\}. \end{aligned}$$

We start by showing that $A^\circ = M_K(G)$. For any $f \in L^1(\widehat{G})$ and $\mu \in M(G)$, we compute:

$$\begin{aligned} \langle \widehat{f}, \mu \rangle &= \int_G \widehat{f}\, d\mu \\ &= \int_G \int_{\widehat{G}} f(\xi) \overline{\langle \xi, x \rangle}\, d\xi d\mu \\ &= \int_{\widehat{G}} f(\xi) \int_G \overline{\langle \xi, x \rangle}\, d\mu d\xi \\ &= \int_{\widehat{G}} f(\xi) \overline{\widehat{\mu}}\, d\xi. \end{aligned}$$

So if $\langle \widehat{f}, \mu \rangle = 0$ for all $\widehat{f} \in A$, then $\widehat{\mu} \equiv 0$ on K. So $M_K(G) \supset A^\circ$. On the other hand, if $\mu \in M_K(G)$, then by the above calculation $\langle \widehat{f}, \mu \rangle = 0$ for all $\widehat{f} \in A$ and so $\mu \in A^\circ$.

Consequently, by Lemma 2.4.6, $A^\circ \simeq M(G)/M_K(G)$. All that remains is to show that $M(G)/M_K(G) \simeq L^1(G)/\iota_+(K)$.

Consider the inclusion map $i : L^1(G) \to M(G)$ and the quotient map $q : M(G) \to M(G)/M_K(G)$. Let $T = q \circ i$. We shall prove two things: T is surjective and ker $T = \iota_+(K)$. Together, this means that

$$L^1(G)/\iota_+(K) \simeq M(G)/M_K(G).$$

First T is surjective: let $h \in L^1(G)$ such that $\widehat{h} \equiv 1$ on K. For any $\mu \in M(G)$, $\mu - \mu * h \in M_K(G)$, so $[\mu] = [\mu * h] \in M(G)/M_K(G)$. But now $\mu * h \in L^1(G)$, and $T(\mu * h) = [\mu]$.

Next, we compute the kernel of T. If $f \in \iota_+(K)$, then $\widehat{f} \equiv 0$ on K, so $f \in M_K(G)$. Hence $\iota_+(K) \subset \ker T$. On the other hand, if $f \in \ker T$, then $f \in M_K(G)$ by definition of T. So $f \in \iota_+(K)$. □

Chapter 3

Rearrangement invariant Banach function spaces

This Chapter is about function spaces, in particular rearrangement invariant Banach function spaces. The main reference works that set out the theory in detail include especially Bennett and Sharpley [4], but also O'Neil [33] and Sharpley [46]. Rao and Ren focus on the important subclass of Orlicz spaces in their book [37]. Section 3.1 deals with some basic definitions.

We focus on the fundamental function associated with a Banach function space in Sections 3.2 and 3.3, as well as certain canonically constructed function spaces derived from them. Using these spaces we can define the so-called *weak type* of an operator, which is crucial for understanding the concept of a maximal inequality in Chapter 5.

When working out the Transfer Principle and pointwise ergodic theorems, we will need the integral estimates of Section 3.4. These results, requiring the measure theoretic understanding of products of measure spaces gained in the previous Chapter, plus the knowledge of functions spaces from Section 3.1, form the technical heart of the work on the Transfer Principle and pointwise theorems.

3.1 Basic definitions and constructions

We start by recalling the definition of a rearrangement invariant Banach function space (hereafter referred to as a r.i. BFS) over a resonant measure space (Ω, μ). Our source for this material is mainly [4], and also [33].

Recall first that given a μ-a.e. finite measurable function f on (Ω, μ), the *distribution function* $s \mapsto m(f, s)$ is defined by

$$m(f, s) = \mu(\{\omega \in \Omega : |f(\omega)| > s\})$$

for all $s \geq 0$. The decreasing rearrangement f^* is defined as

$$f^*(s) = \inf\{t : m(f, t) \leq s\}.$$

Two measurable functions f and g are *equimeasurable* if we have $m(f, s) = m(g, s)$ for all $s \geq 0$. Note that the equimeasurability of f and g is the same as stating that $f^*(t) = g^*(t)$ for all $t \geq 0$. By [4, Definition 2.2.3], the space (Ω, μ) is said to be *resonant* if for each measurable finite a.e. functions f and g, the identity

$$\int_0^\infty f^*(t) g^*(t)\ dt = \sup \int_\Omega |f \widetilde{g}|\ d\mu$$

holds as $\widetilde{g}$ ranges over all functions equimeasurable with g. As a special case of [4, Theorem 2.2.6], let us mention that σ-finite nonatomic measure spaces and completely atomic measure spaces, with all atoms having the same measure, are resonant.

One also defines a primitive maximal operator $f \mapsto f^{**}$ as

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(t)\ dt.$$

We call f^{**} the *double decreasing rearrangement* of f.

When the function norm ρ that defines a Banach function space has the property that $\rho(f) = \rho(g)$ for all equimeasurable functions f and g, the Banach space is called *rearrangement invariant* - see [4, Definitions 1.1, 4.1].

For any r.i. BFS X we define another r.i. BFS X', called the associate space, to be the subset of the a.e.-finite measurable functions f on (Ω, μ) for which $\|f\|_{X'}$ is finite, where

$$\|f\|_{X'} = \sup \big\{ \big| \int_\Omega f(\omega) g(\omega)\ d\mu(\omega) \big| : g \in X,\ \|g\|_X \leq 1 \big\}.$$

We shall also have need of another Banach space $X_b \subseteq X$, which is the closure in X of the set of all simple functions in X. This is not in general a r.i. BFS itself, but is useful for the role it plays in the duality theory.

Associated with any rearrangement invariant BFS X, there is a *fundamental function* $\varphi_X : [0,\infty) \to [0,\infty)$ defined by

$$\varphi_X(t) = \|\chi_E\|_X,$$

where E is any subset of Ω such that $\mu(E) = t$. By the rearrangement invariance of X, this function is well-defined. It is a *quasi-concave* function, as explained in [4, Definition 2.5.6]. Such functions are automatically continuous on $(0,\infty)$, as proved in [4, Corollary 2.5.3]. They also have the following useful property.

Lemma 3.1.1. *Quasiconcave functions are subadditive.*

Proof. Indeed, if $s \leq t$ are two nonnegative real numbers, the fact that the mapping $t \mapsto \dfrac{\varphi(t)}{t}$ is nonincreasing implies that

$$\begin{aligned} \frac{\varphi(s+t)}{s+t} &\leq \frac{\varphi(t)}{t} \leq \frac{\varphi(s)}{s} \\ \varphi(s+t) &\leq \left(1 + \frac{s}{t}\right)\varphi(t) \\ &= \varphi(t) + \frac{\varphi(t)}{t}s \\ &\leq \varphi(t) + \varphi(s). \end{aligned}$$

□

We denote by φ_X^* the associate fundamental function of φ_X, where $\varphi_X^* = \varphi_{X'}$. We shall often make use of the identity

$$\varphi_X(t)\varphi_X^*(t) = t$$

for all $t \geq 0$, as proved in [4, Theorem 2.5.2].

Recall that a Young's function is a convex, nondecreasing function $\Phi : [0,\infty] \to [0,\infty]$ for which $\Phi(0) = 0$, $\lim_{x\to\infty} \Phi(x) = \infty$ and which is neither identically zero nor infinite valued on all of $(0,\infty)$.

One class of spaces that we shall study is that of Orlicz spaces. The theory of these important spaces, which include the standard L^p-spaces, is developed in [37]. In [4] and [33], they are also studied in some depth. We bring to mind the most salient features of their construction. The Luxemburg norm $\|\cdot\|_{L(\Phi)}$ is defined by a Minkowski functional on the set of all finite a.e. measurable functions on (Ω,μ) by the formula

$$\|f\|_{L(\Phi)} = \inf\left\{k^{-1} : \int_\Omega \Phi(k|f|)\, d\mu \leq 1\right\}.$$

The set of all f for which $\|f\|_{L(\Phi)} < \infty$ is the Orlicz space $L(\Phi)$.

By [4, Theorem 2.5.13], for any rearrangement invariant Banach function space X we have the embeddings

$$\Lambda(X) \hookrightarrow X \hookrightarrow M(X)$$

where the embeddings have norm 1.

There is also another Young's function, called the complementary Young's function. This is the function Ψ defined by

$$\Psi(x) = \sup_{y>0}\{xy - \Phi(y)\}.$$

Using this complementary Young's function, it is possible to define another, equivalent norm on the Orlicz space $L(\Phi)$. To this end, define the *Orlicz norm* $\|\cdot\|^{L(\Phi)}$ on the space of measurable functions f on (Ω, μ) by setting

$$\|f\|^{L(\Phi)} = \sup\Big\{ \int_0^\infty f^*(s) g^*(s)\, ds : \|g\|_{L(\Psi)} \leq 1 \Big\}. \tag{3.1}$$

Now the Orlicz and Luxemburg norms on the Orlicz space $L(\Phi)$ are equivalent. In fact it is proved in [4, Theorem 4.8.14] that

$$\|f\|_{L(\Phi)} \leq \|f\|^{L(\Phi)} \leq 2\|f\|_{L(\Phi)}. \tag{3.2}$$

We shall often have need of the inverse of Young's functions and (less frequently) fundamental functions. The following definition makes the notion precise.

Definition 3.1.2. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone left-continuous function. for any $t \in \mathbb{R}^+$, we form the sets $\mathcal{S}_t = \{x : \varphi(x) \leq t\}$ and $\mathcal{I}_t = \{x : \varphi(x) > t\}$. These sets form a Dedekind cut of $\mathbb{R}^+$ and we define $\varphi^{-1}(t)$ to number determined by this cut.*

Hence we may write $\varphi^{-1}(t) = \sup \mathcal{S}_t = \inf \mathcal{I}_t$ if φ is nonincreasing.

Now for an Orlicz space $L(\Phi)$ equipped with the Luxemburg norm, its fundamental function φ is related to Φ by the equation

$$\varphi(t) = 1/\Phi^{-1}(1/t) \tag{3.3}$$

for all $0 < t \leq |\Omega|$ as shown in [4, Lemma 4.8.17]. In the sequel, given a Young's function Φ we define the *fundamental function associated to* Φ to be the quasconcave function defined by (3.3).

Given a r.i. BFS X over (Ω, μ), we canonically associate two other r.i. BFSs with X, apart from the Orlicz space. If φ_X is the fundamental function associated with X, define the first space $M(X)$ to consist of all the measurable functions f over Ω such that

$$\|f\|_{M(X)} = \sup_{s>0} f^{**}(s)\varphi_X(s) < \infty.$$

The space $M(X)$ is a Banach space with the norm $\|\cdot\|_{M(X)}$. This is the largest r.i. BFS with fundamental function φ_X. In other words, if Y is any other r.i. BFS with fundamental function φ_X, then Y is contractively embedded in $M(X)$. Note that as the quasiconcave function φ_X is the only property of X required to construct $M(X)$, we may just as well denote this space by $M(\varphi_X)$.

The second space is the associate of $M(X')$. We define $\Lambda(X)$ to consist of all measurable functions f over Ω such that

$$\|f\|_{\Lambda(X)} = \sup\Big\{ \int_0^\infty f^*(s)g^*(s)\; ds : \; \|g\|_{M(\varphi_X^*)} \leq 1\Big\} < \infty.$$

The set $\Lambda(X)$ is a Banach space with norm $\|\cdot\|_{\Lambda(X)}$.

As in the case of $M(X)$, we can just as well write $\Lambda(\varphi_X)$, as φ_X is the only property of X employed in the construction of $\Lambda(X)$. This is the smallest r.i. BFS with fundamental function φ_X; if Y is any other r.i. BFS with this fundamental function then there is a continuous injection of $\Lambda(X)$ into Y.

There is another function space, denoted $M^*(X)$, that can be constructed from a given r.i. BFS X. Although complete, this space is in general not normable and consists of all those finite a.e. measurable functions f on (Ω, μ) for which the quasi-norm $\|\cdot\|_{M^*(X)}$ defined by

$$\|f\|_{M^*(X)} = \sup_{s>0} f^*(s)\varphi_X(s)$$

is finite. Again, note that the only property of X required for this construction is its fundamental function φ_X. While $M^*(X)$ is not necessarily a Banach space, it is a quasi-Banach space, in that $\|f\|_{M^*(X)} = 0$ if and only if $f = 0$ a.e., $\|\lambda f\|_{M^*(X)} = |\lambda|\|f\|_{M^*(X)}$ for all complex λ and

$$\|f + g\|_{M^*(X)} \leq 2(\|f\|_{M^*(X)} + \|g\|_{M^*(X)})$$

for all $f, g \in M^*(X)$. This space was introduced in [46] - see also [4, Ch. 4, exercise 21].

We provide a useful equivalent definition of the $M^*(X)$-norm.

Lemma 3.1.3. *If φ is a fundamental function, then* $\sup_{t>0} f^*(t)\varphi(t) = \sup_{s>0} s\varphi(m(f,s))$.

Proof. We follow [22, Proposition 1.4.5.16, p46]. Given $s > 0$, pick $\epsilon \in (0, s)$. As $f^*(m(f,s) - \epsilon) > s$,

$$\sup_{t>0} f^*(t)\varphi(t) \geq f^*(m(f,s) - \epsilon)\varphi(m(f,s) - \epsilon) > s\varphi(m(f,s) - \epsilon).$$

Because φ is continuous on $(0, \infty)$, as $\epsilon \to 0$, we obtain

$$\sup_{t>0} f^*(t)\varphi(t) \geq s\varphi(m(f,s))$$

for all $s > 0$, which proves that $\sup_{t>0} f^*(t)\varphi(t) \geq \sup_{s>0} s\varphi(m(f,s))$.

Conversely, given $t > 0$, if $f^*(t) > 0$ pick $\epsilon \in (0, f^*(t))$. Then $m(f, f^*(t) - \epsilon)) > t$, meaning that

$$\sup_{s>0} s\varphi(m(f,s)) \geq (f^*(t) - \epsilon)\varphi(m(f, f^*(t) - \epsilon)).$$

As φ is nondecreasing, we have $(f^*(t) - \epsilon)\varphi(m(f, f^*(t) - \epsilon)) \geq (f^*(t) - \epsilon)\varphi(t)$. Letting $\epsilon \to 0$, we obtain

$$\sup_{s>0} s\varphi(m(f,s)) \geq f^*(t)\varphi(t)$$

for all $t > 0$. If $f^*(t) = 0$, this inequality is trivially satisfied. Hence $\sup_{t>0} f^*(t)\varphi(t) \leq \sup_{s>0} s\varphi(m(f,s))$. □

Let us make some remarks on operators between function spaces, in particular on the weak type of an operator. There are two standard definitions of this concept. Let X and Y be rearrangement invariant BFSs. We say that a sublinear operator T has *Marcinkiewicz weak type* (X, Y) if T maps X into $M^*(Y)$ and that T has *Lorentz weak type* (X, Y) if it maps $\Lambda(X)$ into $M^*(Y)$. Clearly if T is of Marcinkiewicz weak type (X, Y) then it is of Lorentz weak type (X, Y). In the sequel, we shall write 'weak type' for 'Marcinkiewicz weak type.'

For an operator T of weak type (X, Y) there exists a $c > 0$ such that $\|Tf\|_{M^*(Y)} \leq c\|f\|_X$. The smallest value of c for which this equation holds is called the norm of T.

As we shall mostly be working with Λ-, M- and Orlicz-spaces, let us fix some terminology for dealing with the weak types associated with these kinds of spaces.

Definition 3.1.4. ***(Weak type)*** *Let Φ_A and Φ_B be Young's functions with associated fundamental functions φ_A and φ_B respectively. We say that a sublinear operator T has Λ-, M- or L-weak type (φ_A, φ_B) if it respectively maps $\Lambda(\Phi_A)$, $M(\Phi_A)$ or $L(\Phi_A)$ into $M^*(\varphi_B)$.*

Bear in mind that if an operator is of M-weak type (φ_A, φ_B), then it is automatically of L- and Λ-weak types (φ_A, φ_B) too.

3.2 Manipulations of fundamental functions

It is known (cf. [4, Propositions 2.5.10, 2.5.11]) that a fundamental function may be replaced with a concave fundamental function for most purposes. On the other hand, a Young's function is always convex, and if one considers (3.3), one sees that the associated fundamental function is automatically concave. (In other words, if we consider an Orlicz space with the Luxemburg norm, its fundamental function is concave). This is a consequence of the next Lemma, which will also show how in certain cases one can start with a fundamental function and associate with it a Young's function.

Lemma 3.2.1. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing convex (respectively concave) function. Then φ^{-1} is nondecreasing and concave (respectively convex).*

Furthermore, $t \mapsto 1/\varphi(1/t)$ is nondecreasing and convex (respectively concave).

Proof. Suppose that φ is convex, let $x, y \in \mathbb{R}^+$ and $0 \leq \lambda \leq 1$. We define three subsets of $\mathbb{R}^+$ as follows.

$$\begin{aligned} S_\Sigma &= \{s : \varphi(s) \leq \lambda x + (1-\lambda)y\} \\ S_x &= \{s : \varphi(s) \leq x\} \\ S_y &= \{s : \varphi(s) \leq y\}. \end{aligned}$$

Fix an $\epsilon > 0$ and let $t_x \in S_x$ and $t_y \in S_y$, with $t_x \geq \sup S_x - \epsilon$ and $t_y \geq \sup S_y - \epsilon$. Then from the convexity of φ, we can estimate

$$\begin{aligned} \varphi(\lambda t_x + (1-\lambda)t_y) &\leq \lambda\varphi(t_x) + (1-\lambda)\varphi(t_y) \\ &\leq \lambda x + (1-\lambda)y. \end{aligned}$$

Hence $\lambda t_x + (1-\lambda)t_y \in S_\Sigma$ and $\sup S_\Sigma \geq \lambda \sup S_x + (1-\lambda)\sup S_y - \epsilon$. As ϵ is arbitrary, Definition 3.1.2 shows us that

$$\varphi^{-1}(\lambda x + (1-\lambda)y) \geq \lambda\varphi^{-1}(x) + (1-\lambda)\varphi^{-1}(y).$$

For the second part, suppose that φ is concave, and again that $x, y \in \mathbb{R}^+$ and $0 \le \lambda \le 1$. We define three subsets of $\mathbb{R}^+$ as follows.

$$\begin{aligned} I_\Sigma &= \{s : \varphi(s) > \lambda x + (1-\lambda)y\} \\ I_x &= \{s : \varphi(s) > x\} \\ I_y &= \{s : \varphi(s) > y\}. \end{aligned}$$

Fix an $\epsilon > 0$ and let $t_x \in S_x$ and $t_y \in S_y$, with $t_x \le \inf I_x + \epsilon$ and $t_y \le \inf I_y + \epsilon$. Then from the concavity of φ, we get that

$$\begin{aligned} \varphi(\lambda t_x + (1-\lambda)t_y) &\ge \lambda\varphi(t_x) + (1-\lambda)\varphi(t_y) \\ &> \lambda x + (1-\lambda)y. \end{aligned}$$

Hence $\lambda t_x + (1-\lambda)t_y \in I_\Sigma$ and $\inf I_\Sigma \le \lambda \inf I_x + (1-\lambda)\inf S_y + \epsilon$. As ϵ is arbitrary, the result follows as before.

To prove the second part of the lemma, let $I(t) = 1/t$ for $t > 0$. set $\phi(t) = I \circ \varphi \circ I(t)$. As concave and convex functions have derivatives in $L^1(\mathbb{R}^+)$, we may compute:

$$\begin{aligned} \phi'(t) &= I'(\varphi \circ I)(t).\varphi'(I(t)).I'(t) \\ &= -(\varphi \circ I)^{-2}(t)\varphi'(1/t)(-t^{-2}). \end{aligned}$$

If φ is concave, then φ' is nonincreasing, so $t \mapsto \varphi'(1/t)$ is nondecreasing. Clearly $t \mapsto (-t^{-2})\varphi^{-2}(1/t)$ is nondecreasing, because $t \mapsto \varphi(t)/t$ is nonincreasing. Hence ϕ' is nondecreasing and so ϕ is concave.

If φ is convex, the same reasoning shows that ϕ is convex. □

From this lemma, we immediately deduce that if φ is concave, then $t \mapsto 1/\varphi^{-1}(1/t)$ is convex. Moreover if $\lim_{t\to 0}\varphi(t) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$, then $1/\varphi^{-1}(1/t)$ is a Young's function.

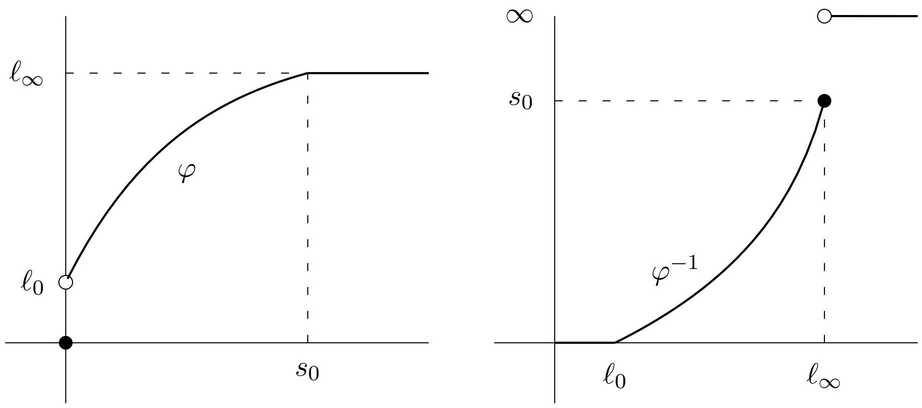
Finally, to round off the results pertaining to the taking of inverses, we have the following result on double inverses.

Lemma 3.2.2. *Let φ be a nondecreasing function, either concave or convex, on $(0, \infty)$. Then for all $t > 0$, $(\varphi^{-1})^{-1}(t) = \varphi(t)$.*

Proof. We shall prove the result for φ concave on $(0, \infty)$; the proof in the convex case is virtually identical. Let us define $\ell_0 = \lim_{t\to 0}\varphi(t)$ and $\ell_\infty = \lim_{t\to\infty}\varphi(t)$. If

φ is constant on an interval $[a,b)$ then φ is constant on $[a,\infty)$, for if $z=\varphi(a)$ and $s=\sup\{\xi:\varphi(\xi)=z\}<\infty$, then for $\epsilon>0$ small enough, $\varphi(a/2+(s+\epsilon)/2)<\varphi(a)/2+\varphi(s+\epsilon)/2$, contradicting the concavity of φ. With this in mind, define $s_0=\infty$ if φ is strictly increasing and $s_0=\inf\{\xi:\varphi(\xi)=\ell_\infty\}$.

The proof follows from studying the following two graphs.



Indeed, for all $s>\ell_\infty$, $\varphi^{-1}(s)=\sup\{t:\varphi(t)\le s\}=\infty$. For all $s\le\ell_0$, $\varphi^{-1}(s)=\inf\{t:\varphi(t)>s\}=0$. Also, φ is injective on $(0,s_0]$ as it is strictly increasing on this interval.

Hence $(\varphi^{-1})^{-1}(t)=\varphi(t)$ for all $t\in(0,s_0]$. For $t>s_0$, $(\varphi^{-1})^{-1}(t)=\sup\{s:\varphi^{-1}(s)\le t\}=\ell_\infty=\varphi(t)$. □

3.3 Comparison of fundamental functions

It should be quite clear that a lot rests upon the analysis of the various fundamental functions associated with r.i. BFSs. Indeed, based on techniques for comparing quasiconcave functions developed in the sequel, we will derive our results on the weak type of the transferred operator.

The growth properties of a Young's function have great bearing on the properties of the associated Orlicz space. The same holds more generally for the growth properties of a fundamental function and its associated BFSs. This part of the work will be concerned with translating some standard growth conditions on Young's functions into conditions on fundamental functions. Then we shall analyse these conditions in terms of inequalities prominent in O'Neil's work [33].

Recall from [37] that a Young's function Φ is said to satisfy the Δ_2 condition (globally), denoted $\Phi \in \Delta_2$ ($\Phi \in \Delta_2$ (globally)) if

$$\Phi(2x) \leq K\Phi(x), \qquad x \geq x_0 \geq 0 \ (x_0 = 0)$$

for some absolute constant $K > 0$. The Young's function Φ satisfies the ∇_2 condition (globally), denoted $\Phi \in \nabla_2$ ($\Phi \in \nabla_2$ (globally)) if

$$\Phi(x) \leq \frac{1}{2\ell}\Phi(\ell x), \qquad x \geq x_0 \geq 0 \ (x_0 = 0)$$

for some $\ell > 1$.

We now state these definitions in terms of fundamental functions.

Definition 3.3.1. *A fundamental function φ is said to satisfy the Δ_2 condition (globally), denoted $\varphi \in \Delta_2$ ($\varphi \in \Delta_2$ (globally)) if*

$$\varphi(Kx) \geq 2\varphi(x), \qquad x \geq x_0 \geq 0 \ (x_0 = 0) \tag{3.4}$$

for some absolute constant $K > 0$.

A fundamental function φ is said to satisfy the ∇_2 condition (globally), denoted $\varphi \in \nabla_2$ ($\varphi \in \nabla_2$ (globally)) if

$$\varphi(x) \geq \frac{2}{\ell}\varphi(\ell x), \qquad x \geq x_0 \geq 0 \ (x_0 = 0) \tag{3.5}$$

for some absolute constant $\ell > 1$.

Suppose that φ satisfies the Δ_2 condition. Note that as φ is nondecreasing, if $K < 1$, then $\varphi(Kx) \leq \varphi(x)$, so $2\varphi(x) \leq \varphi(x)$ for every $x \geq x_0$ - a contradiction as $\varphi(x) = 0$ if and only if $x = 0$. Hence $K \geq 1$. But then $\varphi(Kx) \leq K\varphi(x)$, which follows from the nonincreasing behaviour of $x \mapsto \varphi(x)/x$. Hence $2\varphi(x) \leq \varphi(Kx) \leq K\varphi(x)$. This shows that $K \geq 2$.

Likewise, we can provide a crude bound for ℓ in the case that φ satisfies the ∇_2 condition. As $\ell > 1$, $\varphi(\ell x) \geq \varphi(x) \geq \frac{2}{\ell}\varphi(\ell x)$ for every $x \geq x_0$, so $1 \geq 2/\ell$, which implies that $\ell \geq 2$.

Note that the same reasoning applies to the selection of K and ℓ for Young's functions respectively satisfying the conditions Δ_2 and ∇_2. Indeed, let $\Phi \in \Delta_2$. By [37, Corollary I.3.2], we may write

$$\Phi(x) = \int_0^x \phi(t)\, dt$$

for some nondecreasing left continuous function $\phi : [0,\infty) \to [0,\infty]$ for which $\phi(0) = 0$. Hence

$$\begin{aligned}\Phi(2x) &= \int_0^x \phi(t)\ dt + \int_x^{2x} \phi(t)\ dt \\ &\geq \int_0^x \phi(t)\ dt + \int_0^x \phi(t)\ dt \\ &= 2\Phi(x)\end{aligned}$$

on account of the fact that ϕ is nondecreasing. Therefore $2\Phi(x) \leq \Phi(2x) \leq K\Phi(x)$, implying that $K \geq 2$.

If $\Phi \in \nabla_2$ and $\Phi(x) \leq (1/2\ell)\Phi(\ell x)$ for some $\ell < 2$, note that by the convexity of Φ and the identity $\Phi(0) = 0$,

$$\Phi(\ell x) = \Phi((1-\ell/2)0 + (\ell/2)2x) \leq \frac{\ell}{2}\Phi(2x).$$

Consequently, $\Phi(x) \leq (1/2\ell)\Phi(\ell x) \leq (1/2\ell)(\ell/2)\Phi(2x) = (1/4)\Phi(2x)$. So Φ satisfies the ∇_2 condition with $\ell = 2$.

Proposition 3.3.2. *Suppose that the quasiconcave function φ satisfies the Δ_2 condition globally. Then there exist constants $1 \geq \epsilon > 0$ and $A > 0$ such that for all $y \geq 1$, $0 < p < \epsilon$ and $x \in \mathbb{R}^+$,*

$$\varphi(yx) \geq Ay^p\varphi(x).$$

Furthermore, if φ satisfies the ∇_2 condition globally, then there exist constants $1 > \epsilon \geq 0$ and $B > 0$ such that for all $y \geq 1$, $\epsilon \leq p < 1$ and $x \in \mathbb{R}^+$, it holds that

$$\varphi(yx) \leq By^p\varphi(x).$$

Proof. Suppose that φ satisfies the Δ_2 condition. As noted in the remarks preceding this Proposition, we can choose a $K \geq 2$ such that (3.4) holds. Set $\epsilon = \ln 2/\ln K$ and let $y \geq 1$. By iterating the definition of the Δ_2 condition, we see that for any $n \in \mathbb{N}$ and $x \geq 0$,

$$2^n\varphi(x) \leq \varphi(K^n x).$$

There is a natural number n such that $K^{n-1} \leq y < K^n$. Define λ such that $y/K^n = \lambda$. Then $1/K \leq \lambda < 1$, which allows the following computation:

$$\begin{aligned}\varphi(yx) = \varphi(\lambda K^n x) &\geq \lambda\varphi(K^n x) \\ &\geq \lambda 2^n\varphi(x). \qquad (3.6)\end{aligned}$$

Now $y/\lambda = K^n$ implies that $n = \dfrac{\ln y - \ln\lambda}{\ln K}$ and

$$\begin{aligned} 2^n = e^{n\ln 2} &= e^{\frac{\ln y}{\ln K}\ln 2 - \frac{\ln\lambda}{\ln K}\ln 2} \\ 2^n &= y^{\frac{\ln 2}{\ln K}}\lambda^{-\frac{\ln 2}{\ln K}} \\ \lambda 2^n &= y^{\frac{\ln 2}{\ln K}}\lambda^{1-\frac{\ln 2}{\ln K}}. \end{aligned}$$

Setting $A = (1/K)^{1-\ln 2/\ln K}$, we see that

$$\lambda 2^n \geq A y^{\epsilon}. \tag{3.7}$$

Combining (3.6) and (3.7), for any $0 < p < \epsilon$ and $y \geq 1$, we have

$$\varphi(yx) \geq A y^p \varphi(x).$$

Now suppose that φ satisfies the ∇_2 condition, that is, inequality (3.5) with $\ell \geq 2$. Define $\epsilon = 1 - \ln 2/\ln\ell$ and let $y \geq 1$. There is a natural number n such that $\ell^{n-1} \leq y < \ell^n$. Set $\lambda = y/\ell^n$. Then $1/\ell \leq \lambda < 1$ and

$$\begin{aligned} \varphi(yx) = \varphi(\lambda\ell^n x) &\leq \varphi(\ell^n x) \\ &\leq \frac{\ell^n}{2^n}\varphi(x). \end{aligned} \tag{3.8}$$

Now $n = \dfrac{\ln y - \ln\lambda}{\ln\ell}$, so

$$\begin{aligned} \frac{\ell^n}{2^n} = e^{n\ln\ell - n\ln 2} &= e^{\ln y - \ln\lambda - (\ln y - \ln\lambda)\frac{\ln 2}{\ln\ell}} \\ &= e^{\left(1-\frac{\ln 2}{\ln\ell}\right)\ln y/\lambda} \\ &= y^{1-\frac{\ln 2}{\ln\ell}}\lambda^{\frac{\ln 2}{\ln\ell}-1}. \end{aligned} \tag{3.9}$$

Because $0 < \ln 2/\ln\ell \leq 1$ and $1/\ell \leq \lambda < 1$, we have that $\lambda^{\ln 2/\ln\ell - 1} \leq \ell^{1-\ln 2/\ln\ell}$. Hence for any $1 > p \geq 1 - \ln 2/\ln\ell = \epsilon$, from (3.9) we have

$$\frac{\ell^n}{2^n} \leq B y^p,$$

where $B = \ell^{1-\ln 2/\ln\ell}$. Substituting this into (3.8) yields the result. □

The inequalities $\varphi(yx) \geq A y^p \varphi(x)$ and $\varphi(yx) \leq B y^p \varphi(x)$ obtained in the Proposition above are instances of *tail growth* conditions: they are valid for y large enough. A larger class of fundamental functions satisfying such conditions is provided in [46], which we now recall.

Definition 3.3.3. *We define two classes of fundamental functions as follows.*

1. *$\varphi \in \mathcal{U}$ if for some $0 < \alpha < 1$, there are positive constants A and δ such that*

$$\varphi(ts) \quad \leq \quad At^{\alpha}\varphi(s) \quad if \ t \geq \delta.$$

The $\mathcal{U}$-index of φ, denoted $\rho_{\mathcal{U}}^{\varphi}$, is the infimum of all α for which the above inequality obtains.

2. *$\varphi \in \mathcal{L}$ if for some $\alpha > 0$, there are positive constants A and δ such that*

$$\varphi(ts) \quad \geq \quad At^{\alpha}\varphi(s) \quad if \ t \geq \delta.$$

The $\mathcal{L}$-index of φ, denoted $\rho_{\mathcal{L}}^{\varphi}$, is the supremum of all α for which the above inequality obtains.

We shall often require the equivalent forms of this definition as provided in the next result.

Lemma 3.3.4. *Let φ be a fundamental function.*

1. *$\varphi \in \mathcal{U}$ if and only if $A\varphi(uv) \geq v^{\alpha}\varphi(u)$ for some constants $A, \delta > 0$ and $u > 0, v \leq \delta$.*

2. *$\varphi \in \mathcal{L}$ if and only if $A\varphi(uv) \leq v^{\alpha}\varphi(u)$ for some constants $A, \delta > 0$ and $u > 0, v \leq \delta$.*

Moreover, by adjusting A if necessary, we may always take $\delta = 1$.

Proof. Suppose $\varphi \in \mathcal{U}$ so that $\varphi(ts) \leq At^{\alpha}\varphi(s)$ if $t \geq \delta$. By the change of variables $u = ts$ and $v = 1/t$, we get $A\varphi(uv) \geq v^{\alpha}\varphi(u)$ for $u > 0, v \leq 1/\delta$. Part (2) is treated with the same change of variables.

For the last part, suppose $\varphi \in \mathcal{U}$ so that $\varphi(ts) \leq At^{\alpha}\varphi(s)$ if $t \geq \delta$. If $\delta \leq 1$, then this inequality is certainly true for all $t \geq 1$. If $\delta > 1$, then for all $t \geq 1, s > 0$,

$$\varphi(ts) \leq \varphi(\delta ts) \leq A(\delta t)^{\alpha}\varphi(s)$$

because φ is nondecreasing. Setting $A' = A\delta^{\alpha}$, we have proven that $\varphi(ts) \leq A't^{\alpha}\varphi(s)$ if $t \geq 1$. The same reasoning can be used for the case $\varphi \in \mathcal{L}$. □

We have thus shown that $\mathcal{U}$ and $\mathcal{L}$ defined above are identical to the classes as defined in [46]. Furthermore, the $\mathcal{U}$- and $\mathcal{L}$- indices of φ are equal to the fundamental indices introduced by Zippin and defined in [46]. We will prove this later in the chapter.

Proposition 3.3.2 shows that if φ satisfies the Δ_2 condition, then $\varphi \in \mathcal{L}$ and if φ satisfies the ∇_2 condition then $\varphi \in \mathcal{U}$.

In a far-reaching extension of the $\mathcal{U}$ and $\mathcal{L}$ classes, we shall use relations between three quasiconcave functions φ_A, φ_B and φ_C that satisfy one of the inequalities

$$\varphi_C(st) \leq \theta\varphi_A(t)\varphi_B(s) \text{ for all } t, s > 0 \tag{3.10}$$

$$\varphi_C(st) \leq \theta\varphi_A(t)\varphi_B(s), \;\; s > 0, t \geq \delta \tag{3.11}$$

$$\varphi_C(st) \leq \theta\varphi_A(t)\varphi_B(s), \;\; s > 0, t \leq \delta \tag{3.12}$$

for some $\delta > 0$. Inequalities such as (3.10) are common and have been extensively studied by O'Neil in [33] and [32].

The inequalities (3.11) and (3.12), which are more general that (3.10), shall be the basis for our analysis of the maximal inequalities for a transfer operator. Note that the classes $\mathcal{U}$ and $\mathcal{L}$ satisfy (3.11) and (3.12) respectively. Inequalities of these types are common for fundamental functions. For example, note that for any quasiconcave function φ we have

$$\varphi(st) \leq \varphi(s)t$$

for all $s > 0$ and $t \geq 1$, which is true because $t \mapsto \varphi(t)/t$ is nonincreasing. Another general O'Neil-type inequality appears in Lemma 3.3.5.

We recall the construction of the Boyd indices of a r.i. BFS X which is defined by a function norm ρ. First, by the Luxemburg Representation Theorem [4, Theorem 2.4.10] there is a (not necessarily unique) r.i. BFS $\overline{X}$ over the positive reals with Lebesgue measure, defined by a function norm $\overline{\rho}$ which is related to ρ by the formula

$$\overline{\rho}(f^*) = \rho(f)$$

for every $f \in X$. Now for each $t \in \mathbb{R}^+$ we define the *dilation operator* E_t by

$$(E_t g)(s) = g(st)$$

for all $s \in \mathbb{R}^+$ and g a measurable and finite a.e. function on $[0, \infty)$. Let $h_X(t)$ denote the operator norm of $E_{1/t}$: that is, $h_X(t) = \|E_{1/t}\|_{\mathcal{B}(\overline{X})}$ for $t > 0$. Define $h_X(0) = 0$. In [4, Section 3.5], the authors thoroughly develop the basics of the theory, including the fact that h_X is submultiplicative. Note that it is also quasiconcave, for by [4, Proposition 3.5.11], h_X is nondecreasing and $h_X(t)/t = h_{X'}(1/t)$, which is nonincreasing. Furthermore, $h_X(t) > 0$ for $t > 0$, which is a consequence of the next Lemma. The Lemma also provides an interesting general example of the O'Neil-type inequality (3.10).

Lemma 3.3.5. *If X is a r.i. BFS then for all $s, t > 0$,*

$$\begin{aligned} \varphi_X(st) &\leq h_X(t)\varphi_X(s) \\ \varphi_X(st) &\geq h^*_{X'}(t)\varphi_X(s). \end{aligned}$$

Proof. Consider the function space $\overline{X}$ over the positive reals given by the Luxemburg Representation Theorem mentioned above and fix $s, t \in \mathbb{R}^+$. Note that

$$\begin{aligned} \varphi_X(st) &= \|\chi_{(0,st)}\|_{\overline{X}} \\ &= \|E_{1/t}\chi_{(0,s)}\|_{\overline{X}} \\ &\leq \|E_{1/t}\|_{\mathcal{B}(\overline{X})}\|\chi_{(0,s)}\|_{\overline{X}} \\ &= h_X(t)\varphi_X(s). \end{aligned}$$

From this inequality, we immediately deduce that $\varphi^*_X(st) \geq h^*_X(t)\varphi^*_X(s)$. As $\varphi^*_X = \varphi'_X$, this can be written as $\varphi_{X'}(st) \geq h^*_X(t)\varphi_{X'}(s)$. Equivalently, $\varphi_X(st) \geq h^*_{X'}(t)\varphi_X(s)$. □

It is worth mentioning that by [4, Proposition 3.5.11], $h^*_{X'}(t) = 1/h_X(1/t)$. Note also that as h_X is submultiplicative, $h^*_{X'}$ is *supermultiplicative*, in that $h^*_{X'}(st) \geq h^*_{X'}(s)h^*_{X'}(t)$. We have shown that any fundamental function φ_X can be bounded above by a submultiplicative and below by a supermultiplicative function (up to a constant factor), in that

$$\varphi(1)h^*_{X'}(t) \leq \varphi(t) \leq \varphi(1)h_X(t).$$

The *lower and and Boyd indices* of X, denoted respectively by $\underline{\alpha}_X$ and $\overline{\alpha}_X$ are given by

$$\underline{\alpha}_X = \lim_{t\to 0^+} \frac{\ln h_X(t)}{\ln t}, \qquad \overline{\alpha}_X = \lim_{t\to\infty} \frac{\ln h_X(t)}{\ln t}.$$

For a r.i. BFS X with fundamental function φ, if $M(t, X) = \sup_{s>0} \varphi(st)/\varphi(s)$, recall than Zippin [50] defines the *fundamental indices* as

$$\underline{\beta}_X = \lim_{t\to 0^+} \frac{\ln M(t, X)}{\ln t}, \qquad \overline{\beta}_X = \lim_{t\to\infty} \frac{\ln M(t, X)}{\ln t}.$$

Recall from Definition 3.3.3 the definitions of the $\mathcal{L}$- and $\mathcal{U}$-indices of a fundamental function. Note that the fundamental indices of a fundamental function are always defined, even if the $\mathcal{L}$- or $\mathcal{U}$- indices are not. The relation between the Boyd and fundamental indices given in the next Lemma is well known - see for example [4, Chapter 3, exercise 14].

Lemma 3.3.6. *Let X be a r.i. BFS with fundamental function φ. Then*

$$0 \leq \underline{\alpha}_X \leq \underline{\beta}_X \leq \overline{\beta}_X \leq \overline{\alpha}_X \leq 1.$$

Moreover, $\varphi \in \mathcal{U}$ if and only if $\overline{\beta}_X < 1$ and in this case $\overline{\beta}_X = \rho_{\mathcal{U}}^{\varphi}$, and $\varphi \in \mathcal{L}$ if and only if $\underline{\beta}_X > 0$ and in this case $\underline{\beta}_X = \rho_{\mathcal{L}}^{\varphi}$.

Proof. From Lemma 3.3.5 it is easy to see that $M(t, X) \leq h_X(t)$. Hence if $t > 1$, $\ln M(t, X)/\ln t \leq \ln h_X(t)/\ln t$, so $\overline{\beta}_X \leq \overline{\alpha}_X$. On the other hand, if $t < 1$, $\ln M(t, X)/\ln t \geq \ln h_X(t)/\ln t$, so $\underline{\beta}_X \geq \underline{\alpha}_X$. The fact that $M(t, X)$ is nondecreasing shows that $\underline{\beta}_X \leq \overline{\beta}_X$.

The rest of the Lemma follows directly from the definition of the $\mathcal{U}$- and $\mathcal{L}$-indices. □

3.4 Estimates of integrals and function norms

When working with maximal inequalities, there are certain integrals that we will need to estimate. The following Proposition covers the cases that we will need.

First, some terminology, following [27]: consider a measure space (Ω, Σ, μ) and a countable collection $\mathcal{D} \subset \Sigma$ of measurable subsets of μ-finite measure. The σ-algebra $\sigma(\mathcal{D})$ generated by $\mathcal{D}$ is contained in Σ. If for any $F \in \Sigma$ there is a $D \in \sigma(\mathcal{D})$ such that $F \Delta D$ has null measure, where $F \Delta D$ denotes the symmetric difference between D and F, we say that (Ω, Σ, μ) is *countably generated modulo null sets*, or just *countably generated*. We call $\mathcal{D}$ the *generators* of Σ. Moreover, we may assume that $\mathcal{D}$ is an algebra, for if $\mathcal{D}$ is countable, so is the algebra it generates. If $\mathcal{D}$ is an algebra of sets that generates Σ in the above sense, it is easy to see that if $F \subset \Omega$ is any μ-finite subset and $\epsilon > 0$, then there is a $D \in \mathcal{D}$ such that $\mu(D \Delta F) < \epsilon$ and $|\mu(D) - \mu(F)| < \epsilon$.

Proposition 3.4.1. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be resonant spaces with Ω_2 countably generated. Let Φ_A, Φ_B and Φ_C be Young's functions and φ_A, φ_B and φ_C be their respective associated fundamental functions satisfying*

$$\theta\varphi_A(st) \geq \varphi_B(s)\varphi_C(t) \tag{3.13}$$

for all $s, t > 0$ and some $\theta > 0$. Let f be a measurable function on $\Omega_1 \times \Omega_2$ and $E \subset \Omega_1$ a subset of finite measure.

1) *If $f \in M(\Phi_A)$, then*

$$\frac{\varphi_C(|E|)}{|E|} \int_E \|f_{\omega_1}\|_{M(\Phi_B)} \, d\mu_1(\omega_1) \quad \leq \quad 4e^3\theta\|f\|_{M(\Phi_A)}.$$

2) *If $f \in \Lambda(\Phi_A)$ and $\lim_{t\to 0} \varphi_B^*(t) = 0$, then*

$$\frac{\varphi_C(|E|)}{|E|} \int_E \|f_{\omega_1}\|_{\Lambda(\Phi_B)} \, d\mu_1(\omega_1) \quad \leq \quad 6\theta\|f\|_{\Lambda(\Phi_A)}.$$

3) *If $f \in L(\Phi_A)$ and $\lim_{t\to 0} \varphi_B^*(t) = 0$, then*

$$\frac{\varphi_C(|E|)}{|E|} \int_E \|f_{\omega_1}\|_{L(\Phi_B)} \, d\mu_1(\omega_1) \quad \leq \quad \theta\|f\|_{L(\Phi_A)}.$$

This theorem is to some extent an adaptation of [33, Theorem 8.18] to the needs of the present program. In particular here g is replaced by χ_E and $h(\omega) = \int f(\omega_1, \omega_2)g(\omega_2)d\mu(\omega_2)$ by $\int \|f_{\omega_1}\|_{X(\Phi_B)}d\mu(\omega_2)$.

As the proof of this Proposition relies heavily on [33, Theorem 8.18], it is worth mentioning that the condition on the fundamental functions given there, namely $\Phi_A^{-1}(st)\Phi_B^{-1}(t) \leq \theta t\Phi_C^{-1}(s)$, can with the help of (3.3) and the identity $\varphi_B(t)\varphi_B^*(t) = t$ be written in the equivalent form

$$\theta\varphi_A(st) \geq \varphi_B^*(t)\varphi_C(s).$$

Proof. Let $\mathcal{D}$ be a countable algebra that generates (Ω_2, μ_2).

Suppose $f \in M(\Phi_A)$. For any subset $\Delta \subset \Omega_2$ of finite measure, define h_Δ by

$$h_\Delta(\omega_1) = \frac{1}{\varphi_B^*(|\Delta|)} \int_\Delta |f(\omega_1, \omega_2)| \, d\mu_2(\omega_2).$$

Thus $h_\Delta(\omega_1) = \displaystyle\int_{\Omega_2} |f|(\omega_1, \omega_2)(\chi_\Delta(\omega_2)/\varphi_B^*(|\Delta|)) \, d\mu_2(\omega_2)$. Note that (3.13) can be written in the form

$$\theta\varphi_A(st) \geq (\varphi_B^*)^*(s)\varphi_C(t)$$

because for any fundamental function φ, $(\varphi^*)^* = \varphi$. We apply [33, Theorem 8.18, part 1°] to conclude that $h_\Delta \in M(\Phi_C)$, with $\|h_\Delta\|_{M(\Phi_C)} \le 4e^3\theta\|f\|_{M(\Phi_A)}$. We also used the obvious fact that $\|\chi_\Delta/(\varphi_B^*(|\Delta|))\|_{\Lambda(\varphi_B^*)} = 1$.

Now define

$$\tilde{h} = \sup_{\Delta\in\mathcal{D}} h_\Delta.$$

As $\tilde{h}$ is the supremum of a countable number of functions, it is itself a measurable function.

For any $\Delta \in \mathcal{D}$ and μ_1-almost every $\omega_1 \in \Omega_1$,

$$\begin{aligned}\widetilde{h}(\omega_1) &= \frac{1}{\varphi_B^*(|\Delta|)}\int_\Delta |f_{\omega_1}|\,d\mu_2 = \frac{1}{|\Delta|}\int_\Delta |f_{\omega_1}|\,d\mu_2.\varphi_B(|\Delta|)\\ &\le f_{\omega_1}^{**}(|\Delta|)\varphi_B(|\Delta|) \le \|f_{\omega_1}\|_{M(\Phi_B)},\end{aligned}$$

by definition of the norm $\|\cdot\|_{M(\Phi_B)}$. Hence $\widetilde{h}(\omega_1) \le \|f_{\omega_1}\|_{M(\Phi_B)}$ a.e.

On the other hand for any fixed $\epsilon > 0$, there is a $t > 0$ such that $f_{\omega_1}^{**}(t)\varphi_B(t) > \|f_{\omega_1}\|_{M(\Phi_B)} - \epsilon$. As (Ω_2, μ_2) is a resonant space, by [4, Proposition 2.3.3], there is a subset F such that $|F| = t$ and

$$\frac{1}{|F|}\int_F |f_{\omega_1}|\,d\mu_2 > f_{\omega_1}^{**}(t) - \epsilon/\varphi_B(t).$$

Hence

$$\frac{1}{\varphi_B^*(|F|)}\int_F |f_{\omega_1}|\,d\mu_2 > f_{\omega_1}^{**}(t)\varphi_B(t) - \epsilon.$$

Because $\mathcal{D}$ is dense in the Borel σ-algebra, there is a $\Delta \in \mathcal{D}$ such that

$$\left|\frac{1}{\varphi_B^*(|\Delta|)}\int_\Delta |f_{\omega_1}|\,d\mu_2 - \frac{1}{\varphi_B^*(|F|)}\int_F |f_{\omega_1}|\,d\mu_2\right| < \epsilon.$$

Therefore

$$\begin{aligned}h_\Delta(\omega_1) &= \frac{1}{\varphi_B^*(|\Delta|)}\int_\Delta |f_{\omega_1}|\,d\mu_2\\ &> \frac{1}{\varphi_B^*(|F|)}\int_F |f_{\omega_1}|\,d\mu_2 - \epsilon\\ &> f_{\omega_1}^{**}(t)\varphi_B(t) - 2\epsilon > \|f_{\omega_1}\|_{M(\Phi_B)} - 3\epsilon,\end{aligned}$$

whence

$$\widetilde{h}(\omega_1) > \|f_{\omega_1}\|_{M(\Phi_B)} - 3\epsilon.$$

As $\epsilon > 0$ was arbitrary, it is clear that $\widetilde{h}(\omega_1) \geq \|f_{\omega_1}\|_{M(\Phi_B)}$. So we have proved that $\tilde{h}(\omega_1) = \|f_{\omega_1}\|_{M(\Phi_B)}$ for almost all $\omega_1 \in \Omega_1$. Also, $\|\tilde{h}\|_{M(\Phi_C)} \leq 4e^3\theta\|f\|_{M(\Phi_A)}$ because as we have already shown, $\|h_\Delta\|_{M(\Phi_C)} \leq 4e^3\theta\|f\|_{M(\Phi_A)}$ for all $\Delta \in \mathcal{D}$. Combining these two facts yields part 1) of the Proposition.

For the second part, we shall follow a similar strategy to that of the first part. Consider the space $M(\varphi_B^*)_b$ over Ω_2, which is the closure of the space of all simple functions in $M(\varphi_B^*)$ whose support has finite measure. The condition $\lim_{t\to 0}\varphi_B^*(t) = 0$ means that by [4, Theorem 2.5.5], $M(\varphi_B^*)_b$ is separable and that $(M(\varphi_B^*)_b)^* = \Lambda(\Phi_B)$. Let $\mathcal{D}$ be a countable dense subset of the unit ball of $M(\varphi_B^*)_b$. By the above remarks, this is a norming set for $\Lambda(\Phi_B)$, in that for any $g \in \Lambda(\Phi_B)$, we have

$$\|g\|_{\Lambda(\Phi_B)} = \sup_{\delta\in\mathcal{D}} \int_{\Omega_2} |g(\omega_2)\delta(\omega_2)| \; d\mu_1(\omega_2).$$

Now for each $\delta \in \mathcal{D}$, define the functions

$$\begin{aligned} h_\delta(\omega_1) &= \int_{\Omega_2} |f(\omega_1,\omega_2)\delta(\omega_2)| \; d\mu_2(\omega_2) \\ \tilde{h}(\omega_1) &= \sup_{\delta\in\mathcal{D}} h_\delta(\omega_1). \end{aligned}$$

Note that as $\tilde{h}$ is the supremum of a countable number of measurable functions, it is itself measurable.

By [33, Theorem 8.18, part 3°], $\|h_\delta\|_{L(\Phi_C)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}\|\delta\|_{M(\varphi_B^*)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}$. Hence $\|\tilde{h}\|_{L(\Phi_C)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}$.

On the other hand, for each $\omega_1 \in \Omega_1$,

$$\begin{aligned} \tilde{h}(\omega_1) &= \sup_{\delta\in\mathcal{D}} \int_{\Omega_2} |f(\omega_1,\omega_2)\delta(\omega_2)| \; d\mu_2(\omega_2) \\ &= \|f_{\omega_1}\|_{\Lambda(\Phi_B)} \end{aligned}$$

where the last equality is true on account of $\mathcal{D}$ being a norming subset of $M(\varphi_B^*)_b$ for $\Lambda(\Phi_B)$.

Hence if $E \subset \Omega_1$ is any set of finite measure, then by Hölder's inequality

$$\begin{aligned} \frac{\varphi_C(|E|)}{|E|} \int_E \|f_{\omega_1}\|_{\Lambda(\Phi_B)} \; d\mu_1(\omega_1) &\leq \frac{\varphi_C(|E|)}{|E|} \|\tilde{h}\|_{L(\Phi_C)} \|\chi_E\|_{L(\Phi_C^*)} \\ &= \|\tilde{h}\|_{L(\Phi_C)} \leq 6\theta\|f\|_{\Lambda(\Phi_A)}, \end{aligned}$$

proving part 2).

For the third part, let Ψ_B denote the Young's function complementary to Φ_B and note that because $L(\Phi_B)$ is an Orlicz space with the Luxemburg norm, its associate space is the Orlicz space $L(\Psi_B)$ under the Orlicz norm and with fundamental function φ_B^*.

The proof now proceeds as for the second part. Because $\lim_{t\to\infty} \varphi_B^*(t) = 0$, by [4, Theorem 2.5.5], $L(\Psi_B)_b$ is separable and that $(L(\Psi_B)_b)^* = L(\Phi_B)$. Let $\mathcal{D}$ be a countable dense subset of the unit ball of $L(\Psi_B)_b$ and define as before the functions h_δ and $\widetilde{h}$. By [33, Theorem 8.18, part 2°],

$$\|h_\delta\|_{L(\Phi_C)} \leq \theta\|f\|_{L(\Phi_A)}\|\delta\|_{L(\Phi_B)} \leq \theta\|f\|_{L(\Phi_A)}\|\delta\|^{L(\Phi_B)} \leq \theta\|f\|_{L(\Phi_A)},$$

where $\|\cdot\|_{L(\Phi_B)}$ and $\|\cdot\|^{L(\Phi_B)}$ denote the Luxemburg and Orlicz norms respectively and we used the fact that by (3.2), $\|\delta\|_{L(\Phi_B)} \leq \|\delta\|^{L(\Phi_B)} \leq 1$. Hence $\|\widetilde{h}\|_{L(\Phi_C)} \leq \theta\|f\|_{L(\Phi_A)}$.

For each $\omega_1 \in \Omega_1$,

$$\begin{aligned} \tilde{h}(\omega_1) &= \sup_{\delta\in\mathcal{D}} \int_{\Omega_2} |f(\omega_1,\omega_2)\delta(\omega_2)|\ d\mu_2(\omega_2) \\ &= \|f_{\omega_1}\|_{L(\Phi_B)} \end{aligned}$$

where the last equality is true on account of $\mathcal{D}$ being a norming subset of $L(\Psi_B)_b$ for $L(\Phi_B)$.

For a subset $E \subset \Omega_1$ of finite measure, Hölder's inequality reveals that

$$\frac{\varphi_C(|E|)}{|E|}\int_E \|f_{\omega_1}\|_{L(\Phi_B)}\ d\mu_1(\omega_1) \quad\leq\quad \frac{\varphi_C(|E|)}{|E|}\|\tilde{h}\|_{L(\Phi_C)}\|\chi_E\|_{L(\Phi_C^*)} \leq \theta\|f\|_{L(\Phi_A)},$$

proving part 3). □

In the above Proposition, we used the condition $\theta\varphi_A(st) \geq \varphi_B(s)\varphi_C(t)$ for all $s,t > 0$. In [33] the author devotes quite a bit of effort to proving that such a condition is the best possible for his Theorems 8.15 and 8.18. However, we shall need versions of these results where the condition is not satisified for all $s,t > 0$ but only for $t \geq \delta$ for some δ. We now state and prove the results corresponding to the aforementioned Theorems 8.15 and 8.18 of O'Neil. The extension of these results here are that the inequalities governing the fundamental functions, namely (3.14) and (3.16), do not have to apply for all $s,t > 0$, but only for $s > 0, t \geq \delta$ for some $\delta > 0$. This naturally widens the possibilities for funcamental functions comparable by such inequalities. On the other hand, now we cannot state results for arbitrary simple tensors of functions, but only tensors between an arbitrary function and a characteristic function.

Proposition 3.4.2. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be measure spaces. Let Φ_A, Φ_B and Φ_C be Young's functions and φ_A, φ_B and φ_C be their respective fundamental functions. Suppose that f is a measurable function on (Ω_1, μ_1) and K is a subset of Ω_2 of finite measure. If*

$$h(\omega_1, \omega_2) = f(\omega_1)\chi_K(\omega_2)$$

then h is measurable. Suppose that there is a $\delta > 0$ such that

$$\varphi_C(st) \leq \theta\varphi_A(s)\varphi_B(t) \text{ for all } s > 0, t \geq \delta. \tag{3.14}$$

Then for K large enough that $\varphi_B(|K|) \geq \delta$, we have

1) if $f \in L(\Phi_A)$ then $h \in L(\Phi_C)$ and $\|h\|_{L(\Phi_C)} \leq \theta\|f\|_{L(\Phi_A)}\varphi_B(|K|)$

2) if $f \in M(\Phi_A)$ then $h \in M(\Phi_C)$ and $\|h\|_{M(\Phi_C)} \leq 3\theta\|f\|_{M(\Phi_A)}\varphi_B(|K|)$

3) if $f \in \Lambda(\Phi_A)$ then $h \in \Lambda(\Phi_C)$ and $\|h\|_{\Lambda(\Phi_C)} \leq 2\theta\|f\|_{\Lambda(\Phi_A)}\varphi_B(|K|)$.

Proof. The inequality (3.14) is equivalent to $\theta\Phi_C^{-1}(st) \geq \Phi_A^{-1}(s)\Phi_B^{-1}(t)$ for all $s > 0, t \leq 1/\delta$. Using the fact that for a Young's function Φ and $x \geq 0$, we have $x \leq \Phi^{-1}(\Phi(x))$ and $\Phi(\Phi^{-1}(x)) \leq x$, we compute for all $s > 0, t \leq 1/\delta$:

$$\begin{aligned} st &\leq \Phi_A^{-1}(\Phi_A(s))\Phi_B^{-1}(\Phi_B(t)) \leq \theta\Phi_C^{-1}(\Phi_A(s)\Phi_B(t)) \\ \Phi_C(st/\theta) &\leq \Phi_C(\Phi_C^{-1}(\Phi_A(s)\Phi_B(t))) \leq \Phi_A(s)\Phi_B(t). \end{aligned}$$

Proof of 1) Suppose without loss of generality that $\|f\|_{L(\Phi_A)} = 1$. For a fixed $\omega_1 \in \Omega_1$, set $s = |f(\omega_1)|$ and $t = 1/\varphi_B(|K|)$. As $t \leq 1/\delta$, we have

$$\Phi_C(|f(\omega_1)|/\theta\varphi_B(|K|)) = \Phi_C(st/\theta) \leq \Phi_A(|f(\omega_1)|)\Phi_B(1/\varphi_B(|K|)).$$

Hence

$$\begin{aligned} &\int_{\Omega_1\times\Omega_2} \Phi_C(|h(\omega_1,\omega_2)|/\theta\varphi_B(|K|))\, d\mu_1 \times \mu_2 \\ =\ &\int_{\Omega_1\times\Omega_2} \Phi_C(|f(\omega_1)\chi_K(\omega_2)|/(\theta\varphi_B(|K|)))\, d\mu_1 \times \mu_2 \\ =\ &|K|\int_{\Omega_1} \Phi_C(|f(\omega_1)|/\theta\varphi_B(|K|))\, d\mu_1(\omega_1) \\ \leq\ &|K|\Phi_B(1/\varphi_B(|K|))\int_{\Omega_1} \Phi_A(|f(\omega_1)|)\, d\mu_1(\omega_1) \\ \leq\ &1. \end{aligned}$$

The last inequality follows from $\Phi_B(1/\varphi_B(|K|)) = \Phi_B(\Phi_B^{-1}(1/|K|)) \leq 1/|K|$ and $\|f\|_{L(\Phi_A)} = 1$. By definition of the Luxemburg norm, $\|h\|_{L(\Phi_C)} \leq \theta\varphi_B(|K|)$, giving us the desired estimate.

Proof of 2) As in the proof of 1), set $g = \chi_K/\varphi_B(|K|)$ and suppose that $\|f\|_{M(\Phi_A)} = 1$. Then $\|g\|_{M(\Phi_B)} = 1$ and by [33, Lemma 7.1 part 1°],

$$m(h/\theta, z) = \int_K m(f, z\theta/|g(\omega_2)|)\ d\mu_2(\omega_2) = |K| m(f, z\theta\varphi_B(|K|)).$$

Then for any $a > 0$, $\int_a^\infty m(h/\theta, z)\ dz = |K| \int_a^\infty m(f, z\theta\varphi_B(|K|))\ dz$. With the change of variables $u = z\theta\varphi_B(|K|)$, we get

$$\begin{aligned}\int_a^\infty m(h/\theta, z)\ dz &= \frac{|K|}{\theta\varphi_B(|K|)} \int_{z\theta\varphi_B(|K|)}^\infty m(f, u)\ du \\ &\leq a|K|/\Phi_A(a\theta\varphi_B(|K|)), \end{aligned} \tag{3.15}$$

where the last inequality follows from [33, Lemma 8.14].

Set $s = a\theta\varphi_B(|K|)$ and $t = 1/\varphi_B(|K|)$. As in the proof of part 1), we turn to the inequality $\Phi_C(rs/\theta) \leq \Phi_A(s)\Phi_B(t)$. Hence

$$\begin{aligned}\Phi_C(a) &\leq \Phi_A(a\theta\varphi_B(|K|))\Phi_B(1/\varphi_B(|K|)) \\ &\leq \Phi_A(a\theta\varphi_B(|K|))\frac{1}{|K|}\end{aligned}$$

where we have used that fact that $\Phi_B(1/\varphi_B(|K|)) \leq 1/|K|$. Combining this with (3.15), we obtain $\int_a^\infty m(h/\theta, z)\ dz \leq a/\Phi_C(a)$. Hence by [33, Lemma 8.14], $h \in M(\Phi_C)$ and $\|h\|_{M(\Phi_C)} \leq 3\theta$.

Proof of 3) We compute:

$$\begin{aligned}\|h\|_{\Lambda(\varphi_C)} &\leq 2\int_0^\infty \varphi_C(m(h,t))\ dt && \text{by [33, Theorem 8.5 part 1°]} \\ &= 2\int_0^\infty \varphi_C(|K| m(f, t\varphi_B(|K|)))\ dt && \text{by (3.15)} \\ &\leq 2\theta\varphi_B(|K|)\int_0^\infty \varphi_A(m(f, t\varphi_B(|K|)))\ dt && \text{by (3.14)} \\ &= 2\theta \int_0^\infty \varphi_A(m(f,s))\ ds && \text{by a change of variables} \\ &\leq 2\theta\|f\|_{\Lambda(\varphi_A)} && \text{by [33, Theorem 8.5 part 2°].} \quad \square\end{aligned}$$

Proposition 3.4.3. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be measure spaces. Let Φ_A, Φ_B and Φ_C be Young's functions and φ_A, φ_B and φ_C be their respective fundamental functions.*

Suppose that f is a measurable function on $\Omega_1 \times \Omega_2$ and K is a subset of Ω_2 of finite measure. If

$$h(\omega_1) = \int_{\Omega_2} f(\omega_1, \omega_2)\chi_K(\omega_2)\ d\mu_2(\omega_2)$$

then h is measurable. Suppose that there is a $\delta > 0$ such that

$$\varphi_A(st) \geq \theta\varphi_C(s)\varphi_B^*(t) \text{ for all } s > 0, t \geq \delta. \tag{3.16}$$

Then for K large enough that $\varphi_B(|K|) \geq \delta$, we have

1) *if $f \in L(\Phi_A)$ then $h \in L(\Phi_C)$ and $\|h\|_{L(\Phi_C)} \leq (4/\theta)\|f\|_{L(\Phi_A)}\varphi_B(|K|)$*

2) *if $f \in \Lambda(\Phi_A)$ then $h \in \Lambda(\Phi_C)$ and $\|h\|_{\Lambda(\Phi_C)} \leq 3\theta\|f\|_{\Lambda(\Phi_A)}\varphi_B(|K|)$*

3) *if $f \in M(\Phi_A)$ then $h \in M(\Phi_C)$ and $\|h\|_{M(\Phi_C)} \leq 2\theta\|f\|_{M(\Phi_A)}\varphi_B(|K|)$.*

Proof. The proofs of all three parts are similar, relying on associate spaces for the norm estimates required. We shall use Proposition 3.4.2 together with Hölder-type estimates involving associate spaces. Using the identity $\varphi(t)\varphi^*(t) = t$, we reformulate (3.16):

$$\begin{aligned} \frac{st}{\varphi_A^*(st)} &\geq \theta\frac{s}{\varphi_C^*(s)}\frac{t}{\varphi_B(t)} \\ \varphi_A^*(st) &\leq (1/\theta)\varphi_C^*(s)\varphi_B(t). \end{aligned} \tag{3.17}$$

Let Ψ_A, Ψ_B and Ψ_C be the Young's functions complementary to Φ_A, Φ_B and Φ_C respectively. The Orlicz spaces $L(\Psi_A), L(\Psi_B)$ and $L(\Psi_C)$ with Orlicz norms are respectively associate spaces of $L(\Phi_A), L(\Phi_B)$ and $L(\Phi_C)$ with Luxemburg norms and hence their fundamental functions are φ_A^*, φ_B^* and φ_C^* respectively.

We shall have need of $L(\Psi_A), L(\Psi_B)$ and $L(\Psi_C)$ equipped with Luxemburg norms. Let $\overline{\varphi}_A, \overline{\varphi}_B$ and $\overline{\varphi}_C$ be their respective fundamental functions. By [4, Lemma 4.8.16]

$$1/w \leq \Phi_A^{-1}(1/w)\Psi_A^{-1}(1/w) \leq 2/w$$

for all $w > 0$. As $\varphi_A(w) = 1/\Phi_A^{-1}(1/w)$ and $\overline{\varphi}_A(w) = 1/\Psi_A^{-1}(1/w)$,

$$w \geq \varphi_A(w)\overline{\varphi}_A(w) \geq w/2.$$

Therefore

$$\varphi_A^*(w) \geq \overline{\varphi}_A(w) \geq \varphi_A^*(w)/2.$$

The same inequalities hold of course for $\overline{\varphi}_C$. From (3.17), we derive

$$\overline{\varphi}_A(st) \leq (2/\theta)\overline{\varphi}_C(s)\varphi_B(t). \tag{3.18}$$

Proof of 1) By Definition of the Orlicz norm (3.1),

$$\begin{aligned}\|h\|^{L(\Phi_C)} &= \sup\Big\{\int_{\Omega_1} |h(\omega_1)\nu(\omega_1)|\, d\mu_1(\omega_1) : \|\nu\|_{L(\Psi_C)} \leq 1\Big\} \\ &= \sup\Big\{\int_{\Omega_1\times\Omega_2} |f(\omega_1,\omega_2)\chi_K(\omega_2)\nu(\omega_1)|\, d\mu_1 \times \mu_2 : \|\nu\|_{L(\Psi_C)} \leq 1\Big\}.\end{aligned}$$

By part 1) of Proposition 3.4.2 and (3.18), $\|\chi_K \otimes \nu\|_{L(\Psi_A)} \leq (2/\theta)\|\nu\|_{L(\Psi_C)}\varphi_B(|K|)$. Hence for some ν with $\|\nu\|_{L(\Psi_C)} \leq 1$,

$$\begin{aligned}\int_{\Omega_1} |h(\omega_1)\nu(\omega_1)|\, d\mu_1(\omega_1) &\leq \|f\|^{L(\Phi_A)}.(2/\theta)\varphi_B(|K|) \\ &\leq 2\|f\|_{L(\Phi_A)}(2/\theta)\varphi_B(|K|)\end{aligned}$$

where we used the estimate given in (3.2) in the last line. Using (3.2) again, we finally obtain

$$\|h\|_{L(\Phi_C)} \leq (4/\theta)\|f\|_{L(\Phi_A)}\varphi_B(|K|).$$

Proof of 2) By [33, Theorem 8.5 part 3°], because the associate space of $\Lambda(\Phi_C)$ is $M(\Psi_C)$,

$$\begin{aligned}\|h\|_{\Lambda(\Phi_C)} &= \sup\Big\{\int_{\Omega_1} |h(\omega_1)\nu(\omega_1)|\, d\mu_1(\omega_1) : \|\nu\|_{M(\Psi_C)} \leq 1\Big\} \\ &= \sup\Big\{\int_{\Omega_1\times\Omega_2} |f(\omega_1,\omega_2)\chi_K(\omega_2)\nu(\omega_1)|\, d\mu_1 \times \mu_2 : \|\nu\|_{M(\Psi_C)} \leq 1\Big\}.\end{aligned}$$

By Proposition 3.4.2 and (3.17), $\|\chi_K \otimes \nu\|_{M(\Psi_A)} \leq 3\theta\|\nu\|_{M(\Psi_C)}\varphi_B(|K|)$. Hence for some ν with $\|\nu\|_{\Lambda(\Psi_C)} \leq 1$,

$$\int_{\Omega_1} |h(\omega_1)\nu(\omega_1)|\, d\mu_1(\omega_1) \leq 3\theta\|f\|_{\Lambda(\Phi_A)}\varphi_B(|K|),$$

which proves that $\|h\|_{\Lambda(\Phi_C)} \leq 3\theta\|f\|_{\Lambda(\Phi_A)}\varphi_B(|K|)$.

Proof of 3) By [33, Theorem 8.7], because the associate space of $M(\Phi_C)$ is $\Lambda(\Psi_C)$,

$$\begin{aligned}\|h\|_{M(\Phi_C)} &= \sup\Big\{\int_{\Omega_1} |h(\omega_1)\nu(\omega_1)|\, d\mu_1(\omega_1) : \|\nu\|_{\Lambda(\Psi_C)} \leq 1\Big\} \\ &= \sup\Big\{\int_{\Omega_1\times\Omega_2} |f(\omega_1,\omega_2)\chi_K(\omega_2)\nu(\omega_1)|\, d\mu_1 \times \mu_2 : \|\nu\|_{\Lambda(\Psi_C)} \leq 1\Big\}.\end{aligned}$$

By Proposition 3.4.2 and (3.17), $\|\chi_K \otimes \nu\|_{\Lambda(\Psi_A)} \leq 2\theta\|\nu\|_{\Lambda(\Psi_C)}\varphi_B(|K|)$. Hence for some ν with $\|\nu\|_{M(\Psi_C)} \leq 1$,

$$\int_{\Omega_1} |h(\omega_1)\nu(\omega_1)|\, d\mu_1(\omega_1) \quad\leq\quad 2\theta\|f\|_{M(\Phi_A)}\varphi_B(|K|),$$

which proves that $\|h\|_{M(\Phi_C)} \leq 2\theta\|f\|_{M(\Phi_A)}\varphi_B(|K|)$. □

We end off this Chapter with a result that is to Λ-spaces what [4, Lemma 8.14] is for M-spaces. One difference is that the latter result is an equivalence, whereas our result is merely an implication. It would be interesting to know if the converse holds as well.

Lemma 3.4.4. *Suppose that f is a measurable function on some measure space (Ω, μ), that φ is a fundamental function and that $f \in \Lambda(\varphi)$. Then*

$$\varphi\Big(\int_z^\infty m(f,y)\frac{dy}{y}\Big) \leq \frac{e}{z(e-1)}\|f\|_{\Lambda(\varphi)}.$$

Proof. We compute:

$$\begin{aligned}
\varphi\Big(\int_z^\infty m(f,y)\frac{dy}{y}\Big) &= \varphi\Big(\sum_{n=0}^\infty \int_{ze^n}^{ze^{n+1}} m(f,y)\frac{dy}{y}\Big) \\
&\leq \sum_{n=0}^\infty \varphi\Big(\int_{ze^n}^{ze^{n+1}} m(f,y)\frac{dy}{y}\Big) \text{ (as } \varphi \text{ is subadditive)} \\
&\leq \sum_{n=0}^\infty \varphi\big(m(f,ze^n)\big) \text{ as } \int_{ze^n}^{ze^{n+1}} \frac{dy}{y} = 1 \\
&\leq \sum_{n=-1}^\infty \frac{1}{ze^n(e-1)} \int_{ze^n}^{ze^{n+1}} \varphi(m(f,y))\, dy \\
&\leq \frac{e}{z(e-1)} \sum_{n=-1}^\infty \int_{ze^n}^{ze^{n+1}} \varphi(m(f,y))\, dy \\
&= \frac{e}{z(e-1)} \int_{z/e}^\infty \varphi(m(f,y))\, dy \\
&\leq \frac{e}{z(e-1)}\|f\|_{\Lambda(\varphi)}
\end{aligned}$$

where the last inequality follows from [33, Theorem 8.5, 2°]. □

Chapter 4

Mean ergodic theorems

Having laid the foundations in the previous two Chapters, in this Chapter and the next we produce the main results of this thesis. The mean ergodic theorems produced here in Section 4.4 are derived from yet more general results, namely Theorems 4.3.1 and 4.3.2 of Section 4.3. As explained in Section 1.2, these are Tauberian theorems: starting from the convergence of a sequence of averages on the group, we end up with the convergence of averages on the space acted upon by the group.

In this way, these results echo the Transfer Principle of the next Chapter in spirit. They arise of course through very different techniques. The essence of the procedure for transferring the convergence from group to vector space is presented in Section 4.2, where we show how one can define operators on the vector space starting from Radon measures on the group.

In doing so, we build upon the results of Section 4.1, which contain a careful analysis of different types of group actions, based upon continuity considerations.

Finally, in Section 4.4, we can easily derive general mean ergodic theorems.

4.1 Integrable Actions and Spectral Subspaces

We first describe the general type of Group Actions that shall concern us. In [35] for example, the author uses the central concept of an integrable action. Earlier in [21], Godement considered bounded group actions on Banach spaces in order to study Tauberian theorems. However, we shall work more generally, considering actions on locally convex vector spaces. Many of these ideas are important in Operator Theory and so expositions of various aspects of this material can be found in [35] and [47]. We differentiate between two types of integrability − weak and strong − and express our definition in the language of vector-valued integration theory. We use [38] as our reference for the theory of locally convex topological vector spaces.

We take as our starting point the concept of an integrable action, given in Definition 4.1.1. We pay special attention to the different topologies on $\mathcal{L}_\omega(E)$ and how this affects the continuity properties of α. From there, as in Arveson's work [1], we define various kinds of spectral subspaces in Definitions 4.1.6 and 4.1.8, stressing their equivalence. Other kinds of spectral subspaces are considered in Definition 4.1.10. In this section, we will stress the importance of S-sets, a notion from harmonic analysis that fruitfully links all these different kinds of spectral subspaces. One advantage of this is that, depending on the situation, it will be easier to recognise invariant subspaces as being spectral subspaces of one of these types; the general theory presented below will show how to view each of these subspaces in the light of the other, complementary definitions.

Definition 4.1.1. *An* ***action*** α *of a locally compact group* G *on a dual pair of topological vector spaces* (E, E') *is a homomorphism* $t \mapsto \alpha_t$ *from* G *into* $\mathcal{L}_\omega(E)$.

The action α *is a* ***weak action*** *if it is bounded and continuous when* $\mathcal{L}_\omega(E)$ *has the* WOT. *The action* α *is a* ***strong action*** *if it is bounded and continuous when* $\mathcal{L}_\omega(E)$ *has the* SOT.

We call a weak action α *a* ***weak integrable action*** *if for each* $x \in E$, *the function* $t \mapsto \alpha_t(x)$ *is* μ*-Pettis integrable for every finite Radon measure* μ *on* G.

We call a strong action α *a* ***strong integrable action*** *if for each* $x \in E$, *the function* $t \mapsto \alpha_t(x)$ *is* μ*-Bochner integrable for every finite Radon measure* μ *on* G.

From the definitions it is immediate that the transposed map $t \mapsto \alpha'_t$ of an action on E is an action on E' and that α' is weak or strong integrable if and only if α has that property. Indeed, we work with the space $\mathcal{L}_\omega(E)$ because it contains an operator T if and only if the transpose T' lies in $\mathcal{L}_\omega(E')$. With a view to our applications in Section 4.4, recall from Section 2.1 that if E is a Fréchet space, then $\mathcal{L}_\omega(E) = \mathcal{L}_\sigma(E)$.

It is also clear that an integrable action yields a map, also called α, from $M(G)$ to $\mathcal{L}_\omega(E)$, sending μ to α_μ, where α_μ is the Pettis integral of equation (2.2) in Definition 2.2.3:

$$\langle \alpha_\mu(x), y \rangle = \int_G \langle \alpha_t(x), y \rangle \, d\mu(t).$$

The validity of this equation for the action α is the definition of an integrable action in [1] and [35].

In the sequel, all actions will be weak actions unless otherwise stated. Note that all other cases covered in Definition 4.1.1 are strengthenings of this.

Definition 4.1.2. *For each $x \in E$ and $y \in E'$, we define the function $\eta_{x,y} : G \to \mathbb{C}$ by*

$$\eta_{x,y} : t \mapsto \langle \alpha_t(x), y \rangle. \tag{4.1}$$

Note that each $\eta_{x,y}$ is in $C_b(G) \subset L^\infty(G)$.

For each $x \in E$, we also define a weak-closed subspace E_x of $L^\infty(G)$ by*

$$E_x = \{\eta_{x,y} : y \in E'\}^{-wk*}. \tag{4.2}$$

Note that E_x is translation-invariant. Indeed, for any $x \in G$,

$$\eta_{x,y}(t+s) = \langle \alpha_{t+s}(x), y \rangle = \langle \alpha_t(x), \alpha'_s(y) \rangle$$

and so the function $t \mapsto \eta_{x,y}(t+s) \in E_x$.

Part of the importance of the above definition stems from the fact that the well-defined map $\eta : E \otimes E' : x \otimes y \mapsto \eta_{x,y}$ is the transpose of $\alpha : M(G) \to \mathcal{L}_\omega(E) : \mu \mapsto \alpha_\mu$.

Lemma 4.1.3. *Let E be a convex vector space with topology ξ and dual E'. Let α be an action of G on E.*

1. *If α is weak integrable then the map $M(G) \to \mathcal{L}_\omega(E)$ defined by $\mu \mapsto \alpha_\mu$ is weak-WOT and norm-SOT continuous.*

2. *If α is strong integrable then the map $M(G) \to \mathcal{L}_\omega(E)$ defined by $\mu \mapsto \alpha_\mu$ is weak-SOT and norm-SOT continuous.*

Proof. For the first part, define as above $\eta : E \otimes E' \to C_b(G)$ by $\eta(x \otimes y) = \eta_{x,y}$. By definition of the Pettis integral, η is the transpose of $\alpha : M(G) \to \mathcal{L}_\omega(E)$. As $C_b(G)$ may be identified with a subspace of $M(G)^*$, by [38, Ch II, Prop. 12 p38], α is $\sigma(M(G), M(G)^*) - \sigma(\mathcal{L}_\omega(E), E \otimes E')$-continuous, that is, weak-WOT continuous. As noted in Section 2.1, this means that α is also $\beta(M(G), M(G)^*) - \beta(\mathcal{L}_\omega(E), E \otimes E')$-continuous, that is, norm-SOT continuous.

For the second part, recall that a neighbourhood base of the SOT topology on $\mathcal{L}_\omega(E)$ is given by sets of the form $W(A,V)$, where A is a finite subset of E, V is an absolutely convex ξ-neighbourhood in E and $W(A,V) = \{T \in \mathcal{L}_\omega(E) : T(A) \subseteq V\}$. To prove the result, we must show that for every such $W(A,V)$ there is a weak neighbourhood U of $M(G)$ such that U is mapped into $W(A,V)$.

As α is a strong bounded action, for each $x \in E$ and V as above, $f_{x,V} : t \mapsto \sup_{e' \in V^\circ} |\langle \alpha_t(x), e' \rangle|$ is bounded and continuous. In fact, the strong boundedness of α implies that for each $x \in E$, there is an $M \in \mathbb{R}^+$ such that $\{\alpha_t(x) : t \in G\} \subset M.W(A,V)$. The polar of the finite subset $\{f_{x,V} : x \in A\} \subset C_b(G) \subset M(G)^*$ is a weak-neighbourhood in $M(G)$. Call this set U. Then by (2.4) of Lemma 2.2.4, $(1/M)U \subset W(A,V)$.

From the above, the norm-SOT continuity is trivial. □

We now show that, in the case for Fréchet spaces, mild hypotheses on the action actually ensure far more - that it is in fact integrable. Recall that a Fréchet space X with dual X^* has the Mackey topology $\tau(X,X^*)$ and is metrisable and complete.

Proposition 4.1.4. *Let G be a locally compact σ-compact abelian group and X a Fréchet space with dual X^*. If $(\alpha_t)_{t\in G}$ is a family of continuous isomorphisms from X to itself such that the mapping $t \mapsto \alpha_t$ from G into $B(X)$ is continuous and bounded when $B(X)$ has the* WOT*, then α is a weak integrable action of G on the dual pair (X,X^*).*

Proof. In the sequel, fix an $x \in X$. From the hypotheses, the map $t \mapsto \alpha_t(x)$ is continuous when X has its weak topology. Hence, if K is a compact subset of G, the set $\alpha_K(x) = \{\alpha_t(x) : t \in K\}$ is weakly compact. As a Fréchet space is barrelled, by [38, Ch IV, Corollary 3, p66], the closed convex hull of $\alpha_K(x)$, denoted by $\overline{co}\big(\alpha_K(x)\big)$, is also weakly compact. Suppose that μ is a Radon probability measure on K. By [13, Theorem 1 p148], there is a unique $x_{K,\mu} \in \overline{co}\big(\alpha_K(x)\big)$ such that

$$\langle x_{K,\mu}, y \rangle = \int_K \langle \alpha_t(x), y \rangle \; d\mu(t)$$

for all $y \in X^*$. By the same token, if $\mu \in M(G)$ is not a probability measure, there exists a unique $x_{K,\mu} \in \|\mu\| \overline{co}\big(\alpha_K(x)\big)$.

Now fix a $\mu \in M(G)$ and a sequence of compact sets $K_n \subset G$ whose union is all of G. Write $x_n = x_{K_n,\mu}$ for each $n \in \mathbb{N}$. We will show that the sequence (x_n) is Cauchy in X under the $\tau(X,X^*)$-topology and hence convergent.

A neighbourhood base of 0 in the Mackey topology is by definition given by the polar sets Y°, where $Y \subset X^*$ is $\sigma(X^*, X)$-compact and absolutely convex.

Now as the map $t \mapsto \alpha_t(x)$ is bounded, the orbit set $\{\alpha_t(x) \ : \ t \in G\}$ is bounded in all topologies of the dual pair (X, X^*), including the Mackey topology. Hence there is an $M \in \mathbb{R}$ such that

$$|\langle \alpha_t(x), y\rangle| \leq M$$

for all $t \in G$ and $y \in Y$.

Take $N \in \mathbb{N}$ such that $|\mu|(G \backslash K_N) \leq 1/M$. Then for $n > m > N$,

$$\begin{aligned} |\langle x_n - x_m, y\rangle| &= \Big|\int_{K_n} \langle \alpha_t(x), y\rangle \ d\mu(t) - \int_{K_m} \langle \alpha_t(x), y\rangle \ d\mu(t)\Big| \\ &= \Big|\int_{K_n \backslash K_m} \langle \alpha_t(x), y\rangle \ d\mu(t)\Big| \\ &\leq \int_{K_n \backslash K_m} |\langle \alpha_t(x), y\rangle| \ d|\mu|(t) \\ &\leq \int_{G \backslash K_m} |\langle \alpha_t(x), y\rangle| \ d|\mu|(t) \leq 1. \end{aligned}$$

Hence $x_n - x_m \in Y^\circ$ and (x_n) is Cauchy in X under the Mackey topology. As a Fréchet space is complete in this topology, the sequence has a limit. Call its limit $\alpha_\mu(x)$. We have shown that

$$\langle \alpha_\mu(x), y\rangle = \lim_{n\to\infty} \int_{K_n} \langle \alpha_t(x), y\rangle \ d\mu(t) = \int_G \langle \alpha_t(x), y\rangle \ d\mu(t).$$

Therefore the action is weak integrable. □

Proposition 4.1.5. *Let α be a strong action of a locally compact σ-compact abelian group G on a Fréchet space X. Then α is a strong integrable action for any finite Radon measure.*

Proof. Let μ be a finite Radon measure and $x \in X$. We are going to show that $f(t) = \alpha_t(x)$ is μ-measurable by constructing a sequence of μ-simple measurable functions converging a.e. to it. Fix an $\epsilon > 0$ in all the constructions that follow. As G is σ-compact, there is a compact $K \subset G$ such that $\mu(G \backslash K) < \epsilon$. Because α is strongly continuous, $\alpha_K(x) = \{\alpha_t(x) \ : t \in K\}$ is compact and so for any open neighbourhood U of 0, there is a finite set $t_1, \ldots, t_n \in G$ such that the sets $\alpha_{t_1}(x) + U, \ldots, \alpha_{t_n}(x) + U$

cover $\alpha_K(x)$. Let $E_1 = \alpha_K(x) \cap (\alpha_{t_1}(x) + U)$ and $E_i = \big(\alpha_K(x) \cap (\alpha_{t_i}(x) + U)\big) \backslash E_{i-1}$ for $i = 2, \ldots, n$. Define the μ-simple function

$$f_{U,K}(t) = \sum_{i=1}^{n} \alpha_{t_i}(x)\chi_{E_i}(t).$$

Then $\mu(\{t \in G : f(t) - f_{U,K}(t) \notin U\}) < \epsilon$. As X is metrisable we may choose a decreasing sequence of open neighbourhoods of 0, say (U_i), that generate the topology. Owing to the σ-compactness of G, we can choose an increasing sequence of compact subsets of G, say (K_i), whose union is all of G and such that $\mu(G\backslash K_i) < 1/i$. Define

$$f_i := f_{U_i,K_i}.$$

This sequence of μ-measurable functions converges a.e. to f. (Note that the functions f_i do not depend on ϵ for their construction).

Next we show that f is μ-Bochner integrable. Take any equicontinuous set $\mathcal{A} \subset X^*$. Its polar $\mathcal{A}^\circ$ is a neighbourhood of 0 in X and so there is an $N_1 \in \mathbb{N}$ such that $U_n \subset (\epsilon/2|\mu|(G))\mathcal{A}^\circ$ for all $n \geq N_1$. Let $M = \sup_{t \in G} \sup_{e' \in \mathcal{A}} |\langle f(t), e' \rangle|$. This value is finite because the boundedness of the action ensures that f is bounded too. There is an $N_2 \in \mathbb{N}$ such that $M/n < \epsilon/2$ for any $n \geq N_2$.

For any $n \geq N = \max\{N_1, N_2\}$, we compute:

$$\begin{aligned}
& \int_\Omega \sup_{e' \in \mathcal{A}} |\langle f(t) - f_n(t), e' \rangle| \; d|\mu|(t) \\
\leq \; & \int_{K_i} \sup_{e' \in \mathcal{A}} |\langle f(t) - f_n(t), e' \rangle| \; d|\mu|(t) + \int_{G\backslash K_i} \sup_{e' \in \mathcal{A}} |\langle f(t) - f_n(t), e' \rangle| \; d|\mu|(t) \\
\leq \; & \int_{K_i} \frac{\epsilon}{2|\mu|(G)} \; d|\mu|(t) + \int_{G\backslash K_i} \sup_{e' \in \mathcal{A}} |\langle f(t), e' \rangle| \; d|\mu|(t) + \int_{G\backslash K_i} \sup_{e' \in \mathcal{A}} |\langle f_n(t), e' \rangle| \; d|\mu|(t) \\
\leq \; & \frac{\epsilon}{2} + \frac{M}{n} + 0 \\
\leq \; & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

As ϵ is arbitrary throughout the above constructions, we see that (2.3) of Definition 2.2.3 is satisfied. Hence α is a strong integrable action. □

Now we turn to the definition and elementary characteristics of *spectral subspaces*. In the sequel, α will denote a weak integrable action, unless otherwise specified. We define certain closed subspaces of E and E', which are α-invariant.

Definition 4.1.6. ***(Arveson) [1]*** *For each open subset $\Omega \in \widehat{G}$ define the* ***spectral R-subspace*** *$R^{\alpha}(\Omega)$ as the $\sigma(E, E')$-closure in E of the linear span of the elements $\alpha_f(x)$, where $x \in E$, $f \in K(G)$ and supp $\widehat{f} \subset \Omega$. Similarly we define $R^{\alpha'}(\Omega)$ in E'.*

For each closed subset Λ of $\widehat{G}$ we define the ***spectral M-subspace*** *$M^{\alpha}(\Lambda)$ as the polar (or equivalently the annihilator) of $R^{\alpha'}(\widehat{G}\backslash\Lambda)$. Similarly we define $M^{\alpha'}(\Lambda)$ in E'.*

If we need to emphasise the space on which G acts, we will write $M^{\alpha}_{E}(\Lambda)$ and $R^{\alpha}_{E}(\Omega)$.

Looking at the definition of $M^{\alpha}(\Lambda)$ given above, $x \in M^{\alpha}(\Lambda)$ if and only if $\langle x, \alpha'_f(y)\rangle = 0$ for all $y \in E'$ and all f in $K(G)$ with supp $\widehat{f} \subset \widehat{G}\backslash\Lambda$; i.e. if $\alpha_f(x) = 0$.

In [35, Theorem 8.1.4], the author lists several elementary properties of these subspaces, several of which also appear in [1] and [51].

Definition 4.1.7. *Let V be a $\sigma(E, E')$-closed subspace of E. We define $\gamma(V)$ to be the weak*-closure in $L^{\infty}(G)$ of the subspace*

$$\{\eta_{x,y} : x \in V, y \in E'\}.$$

Furthermore, we define the ***spectrum of*** $\mathbf{V}$ *to be $\sigma(V)$, where $\sigma(V) = \sigma(\gamma(V))$.*

Although the conclusion of the following Proposition is a fairly natural one to make, the author is unaware of a proof or statement of this fact in the literature.

Definition 4.1.8. ***(Godement) [21]*** *For each closed subset $\Lambda \subset \widehat{G}$, we define $\Gamma(\Lambda)$ to be the set of all $x \in E$ such that the spectrum of the set E_x defined in Definition 4.1.2, is contained in Λ.*

The subspace $\Gamma(\Lambda)$ is invariant under the action α. This is because if $x \in \Gamma(\Lambda)$ then the spectrum of E_x is contained in Λ and $E_{\alpha_t(x)} = E_x$.

Again it is clear that $\gamma(V)$ is a translation-invariant subspace of $L^{\infty}(G)$.

The definitions of $\Gamma(\Lambda)$ and $M^{\alpha}(\Lambda)$ for a given closed subset Λ of $\widehat{G}$ are in fact equal. We have produced both here because their constructions provide a slightly different emphasis, which will be useful when proving the main theorems and discussing examples.

Proposition 4.1.9. *For a given closed subset Λ of $\widehat{G}$, $\Gamma(\Lambda) = M^{\alpha}(\Lambda)$.*

Proof. Let $x \in \Gamma(\Lambda)$. Take any $g \in K(G)$ such that supp $\widehat{g}$ is in $\widehat{G}\backslash\Lambda$, and any $y \in E'$. We show that

$$\langle x, \alpha'_g(y)\rangle = \int_G \langle \alpha_t(x), y\rangle g(t)\ dt = 0. \tag{4.3}$$

From this equation, we see at once that $x \in \big(R^{\alpha'}(\widehat{G}\backslash\Lambda)\big)^\circ = M^\alpha(\Lambda)$ and so $\Gamma(\Lambda) \subseteq M^\alpha(\Lambda)$. Now let U be an open subset containing Λ such that $\widehat{g}$ vanishes on U. As the space of functions in $L^\infty(G)$ vanishing on Λ is weak*-closed, we can apply the Spectral Approximation Theorem (Theorem 2.4.3). For any $\epsilon > 0$ we can find a trigonometric polynomial $\sum_{n=0}^N a_n\langle t, \xi_n\rangle$ where $\xi_1, \ldots, \xi_n \in U$ such that

$$\Big|\int_G \langle \alpha_t(x), y\rangle g(t)\ dt - \int_G \sum_{n=0}^N a_n\langle t, \xi_n\rangle g(t)\ dt\Big| < \epsilon$$

and as $\displaystyle\int_G \sum_{n=0}^N a_n\langle t, \xi_n\rangle g(t)\ dt = \sum_{n=0}^N a_n\widehat{g}(\xi_n) = 0$, we can conclude that

$$\Big|\int_G \langle \alpha_t(x), y\rangle g(t)\ dt\Big| < \epsilon.$$

As ϵ is arbitrary, (4.3) is proved.

For the reverse inclusion, take any $x \in M^\alpha(\Lambda)$ and $y \in E'$. If $x \notin \Gamma(\Lambda)$, then $\sigma(E_x) \not\subseteq \Lambda$. This means that there is a character $\xi \in \sigma(E_x)\backslash\Lambda$ which can be weak*-approximated by a finite combination of functions $\eta_{x,y_1}, \ldots, \eta_{x,y_n}$. Any such combination is again of the form $\eta_{x,y}$, where y is a linear combination of $y_1, \ldots, y_n$. We can in fact find a net y'_i in E' such that η_{x,y'_i} converges in the weak* topology to ξ . Hence $\langle \alpha_t(x), y'_i\rangle \to \xi(t)$ as $i \to \infty$ and for any $f \in L^1(G)$,

$$\langle x, \alpha'_f(y'_i)\rangle = \int_G \langle \alpha_t(x), y'_i\rangle f(t)\ dt \to \overline{\widehat{\overline{f}}}(\xi)$$

as $i \to \infty$. But we can find an $f \in K(G)$ such that $\overline{\widehat{\overline{f}}}(\xi) \neq 0$ and $\overline{\widehat{\overline{f}}}$ is 0 on an open neighbourhood of Λ not containing ξ. Thus $\langle x, \alpha'_f(y_i)\rangle \neq 0$, contradicting the fact that $x \in M^\alpha(\Lambda) = \big(R^{\alpha'}(\widehat{G}\backslash\Lambda)\big)^\circ$.

This contradiction shows that $M^\alpha(\Lambda) \subseteq \Gamma(\Lambda)$, and the proof is complete. □

Apart from the invariant subspaces described above, there are other invariant subspaces that will be useful to us.

Definition 4.1.10. *Let $\mu \in M(G)$. The nullspace of μ is given by*

$$N(\mu) = \{x \in E : \alpha_\mu(x) = 0\}.$$

Similarly,

$$N'(\mu) = \{x \in E' : \alpha'_\mu(x) = 0\}.$$

The range space of μ is given by

$$R(\mu) = \{\alpha_\mu(x) : x \in E\}^{-\sigma}$$

where $-\sigma$ denotes the $\sigma(E, E')$-closure of the space. One can likewise define

$$R'(\mu) = \{\alpha'_\mu(x) : x \in E'\}^{-\sigma}.$$

Again, it is easy to see that these spaces are invariant under the action α of G. We will have need of the following straightforward relations between these four sets.

Lemma 4.1.11. *The following equalities hold between the spaces $N(\mu), R(\mu), N'(\mu)$ and $R'(\mu)$ defined above:*

$$\begin{aligned} N(\mu) &= \left(R'(\mu)\right)^\circ \\ R(\mu) &= \left(N'(\mu)\right)^\circ. \end{aligned}$$

Proof. If $x \in \left(R'(\mu)\right)^\circ$, then by definition, for any $y \in E'$,

$$0 = \langle x, \alpha'_\mu(y)\rangle = \langle \alpha_\mu(x), y\rangle.$$

Hence $\alpha_\mu(x) = 0$ and $x \in N(\mu)$. Thus $\left(R'(\mu)\right)^\circ \subseteq N(\mu)$.

On the other hand, if $x \in N(\mu)$, then for any $y \in E'$, $\langle x, \alpha'_\mu(y)\rangle = 0$. As the set $\{\alpha'_\mu(y) : y \in E'\}$ is by definition $\sigma(E', E)$-dense in $R'(\mu)$, we see that for any $z \in R'(\mu)$, $\langle x, z\rangle = 0$. Hence $N(\mu) \subseteq \left(R'(\mu)\right)^\circ$.

Putting the two inclusions together, $N(\mu) = \left(R'(\mu)\right)^\circ$.

The second equation is proved in the same manner as the first. □

Although the subspaces given in Definition 4.1.10 are similar to the spectral subspaces specified in Definition 4.1.6, they are not in general the same. Whether or not they are equal depends on the structure of the nullset $\nu(\mu)$ of the measure. To prove the results linking the two types of subspaces, we first define certain types of ideals.

Definition 4.1.12. *Let* $x \in E$ *and set* $\mathcal{I}_x = \{f \in L^1(G) : \alpha_f(x) = 0\}$*. Similarly for* $x \in E'$ *we define* $\mathcal{I}^x$*.*

These closed ideals are called the *isotropy ideals*, to borrow a term from the study of group actions on sets. Takesaki uses them in [47] as the basis for his analysis of spectral subspaces.

Lemma 4.1.13. *Let* μ *be a measure in* $M(G)$*. The inclusion* $N(\mu) \subseteq M^\alpha(\nu(\mu))$ *always holds. If, furthermore, the null set* $\nu(\mu)$ *is an* S*-set, then* $N(\mu) = M^\alpha(\nu(\mu))$*.*

Proof. Let $x \in N(\mu)$: this means $\alpha_\mu(x) = 0$. Now take any $\alpha'_f(y) \in R^{\alpha'}(\widehat{G}\backslash\nu(\mu))$ and set $K = \operatorname{supp} \widehat{f} \subset \{\xi : \widehat{\mu}(\xi) \neq 0\}$. By Theorem 2.4.4, there is an $h \in L^1(G)$ such that $\widehat{\mu}\widehat{h} = 1$ on K. Hence $(h * \mu * f)\widehat{\ } = \widehat{f}$ and so by the Fourier Uniqueness Theorem, $h * \mu * f = f$. We conclude that

$$\langle x, \alpha'_f(y)\rangle = \langle \alpha_f(x), y\rangle = \langle \alpha_h \alpha_\mu \alpha_f(x), y\rangle = 0,$$

which shows that $x \in M^\alpha(\nu(\mu))$.

To prove the second part of the lemma, we must prove the reverse inclusion under the additional hypothesis that $\nu(\mu)$ is an S-set. Let $x \in M^\alpha(\nu(\mu))$. Consider $f \in K(G)$ such that f has compact support in $\widehat{G}\backslash\nu(\mu)$. By definition, $M^\alpha(\nu(\mu)) = \big(R^{\alpha'}(\widehat{G}\backslash\nu(\mu))\big)^\circ$, so

$$0 = \langle x, \alpha'_f(y)\rangle = \langle \alpha_f(x), y\rangle$$

for all $y \in E'$. Hence it must be that $\alpha_f(x) = 0$; so $f \in \mathcal{I}_x$. Now because $\nu(\mu)$ is an S-set, the set of all $f \in K(G)$ with $\operatorname{supp} \widehat{f} \subset \widehat{G}\backslash\nu(\mu)$ generates $\iota(\nu(\mu))$, the unique ideal with nullset $\nu(\mu)$. Therefore $\mathcal{I}_x \supseteq \iota(\nu(\mu))$.

Let $(g_i)_{i\in\Gamma}$ be an approximate identity for $L^1(G)$. We have that $\mu * g_i \in \mathcal{I}_x$ for each $i \in \Gamma$. As $\mu * g_i \to \mu$ in norm, $\alpha_\mu = \lim_{i\to\infty} \alpha_{\mu * g_i}$ in the SOT and in fact

$$\alpha_\mu(x) = \lim_{i\to\infty} \alpha_{\mu * g_i}(x) = 0.$$

Therefore $x \in N(\mu)$ and $M^\alpha(\nu(\mu)) \subseteq N(\mu)$. □

In discussing the properties of operators induced by measures in the next Section 4.2, we will need to know how to approximate the functions $\eta_{x,y}$ by trigonometric polynomials. This is presented in Theorem 4.1.15. To prove this theorem, we proceed via the following calculation of the spectra of certain invariant subspaces.

Lemma 4.1.14. *Let μ be a finite Radon measure on G and Λ be a closed subset of $\widehat{G}$. The spectra of the subspaces $M^\alpha(\Lambda)$ and $N(\nu(\mu))$ are given by*

$$\sigma(M^\alpha(\Lambda)) = \Lambda \tag{4.4}$$

$$\sigma(N(\mu)) \subseteq \nu(\mu). \tag{4.5}$$

Proof. Equation (4.4) is derived directly from Definitions 4.1.8, 4.1.7 and Proposition 4.1.9.

For (4.5), note that by Lemma 4.1.13 $N(\mu) \subseteq M^\alpha(\nu(\mu))$ and so by (4.4), $\sigma(N(\mu)) \subseteq \nu(\mu)$. □

Theorem 4.1.15. *Let μ be a finite Radon measure on G, $x \in N(\mu)$ and y any element in E'.*

Then for any open neighbourhood U containing $\nu(\mu)$, $\eta_{x,y}$ can be weak- approximated by a finite linear combination of characters in U.*

Furthermore, if $\nu(\mu)$ is an S-set, each $\eta_{x,y}$ can be weak-approximated by a finite linear combination of characters in $\nu(\mu)$.*

Proof. If $x \in N(\mu)$ then $E_x \subset \gamma(N(\mu))$ and $\sigma(E_x) \subset \sigma(N(\mu))$, by Definitions 4.1.7 and 4.1.8. So by Lemma 4.1.14, $\sigma(E_x) \subset \nu(\mu)$. Hence, by the Spectral Approximation Theorem 2.4.3, for any $y \in E'$, $\eta_{x,y}$ can be approximated by finite linear combinations of characters from U.

For the second part, if $\nu(\mu)$ is an S-set, then reasoning as above but appealing to the second part of Theorem 2.4.3, the result follows at once. □

The space $L^1(G)/\iota_+(K)$ can be used to underscore the inherent dual characteristics that arise between the group and the space upon which it acts, via the group action α. Indeed, take a weak integrable action α and define

$$\eta : M^\alpha(K) \otimes E' \to \iota_+(K)^\circ \;:\; \eta(x \otimes y) = \langle \alpha_t(x), y \rangle = \eta_{x,y}(t).$$

By Definitions 4.1.8, 4.1.8 and Proposition 4.1.9, this is well-defined. (Note that $\iota_+(K)^\circ$ is the weak*-closure of the characters in K in $L^\infty(G)$; see [17]).

We have that $\alpha : L^1(G)/\iota_+(K) \to \mathcal{L}_\omega(M^\alpha(K))$ and η are each other's transpose. Pictorially:

$$\begin{array}{ccc} \iota_+(K)^\circ & \xleftarrow{\ \eta\ } & M^\alpha(K)\otimes E' \\ L^1(G)/\iota_+(K) & \xrightarrow[\alpha]{} & \mathcal{L}_\omega(M^\alpha(K)). \end{array}$$

This result is stated using the WOT. The analogous result for the SOT is also true and can be proved in the same way.

The fact that $M^\alpha(\{\xi\})$ is complemented in E is already known; it may be found for example in [51]. It can easily be seen that the Mean Ergodic Theorem for the fixed point space is just a special consequence of this result. Indeed, the fixed point subspace is just the eigenspace corresponding to the eigenvalue 1. These matters are fully dealt with in Proposition 4.4.1.

4.2 Operators on Spectral Subspaces

A large class of operators on a vector space can be induced via the integrable action by finite Radon measures on the group. In this section we discuss how properties of the measures relate to properties of the corresponding operators. In particular, we are interested in what can be gleaned from the Fourier transform of the measures and how to handle sequences of measures and their associated operators.

Let us take a sequence of bounded $L^1(G)$-functions (φ_n). We are looking for conditions on the functions φ_n, $n \in \mathbb{N}$, which cause the corresponding operators α_{φ_n} to converge.

Lemma 4.2.1. *Let U be an open subset of $\widehat{G}$ and μ, ν be finite Radon measures such that $\widehat{\mu} = \widehat{\nu}$ on U. Then the operators α_μ and α_ν are equal on $M^\alpha(K)$ for any compact subset K of U.*

In particular, let $\mu \in M(G)$ such that $\widehat{\mu} \equiv 1$ on U. If $x \in M^\alpha(K)$ then $\alpha_\mu(x) = x$.

Proof. First of all, there is an open subset V of U with compact closure such that $K \subset V \subset \overline{V} \subset U$ and an $h \in L^1(G)$ such that $\widehat{h} = 1$ on $\overline{V}$. Then $h * \mu$ and $h * \nu$ are in $L^1(G)$ and the Fourier transform of $h * \mu - h * \nu$ is zero on $V \supset K$. It follows that

$$\operatorname{supp}(h * \mu - h * \nu)\widehat{\ } \subset \widehat{G} \backslash V \subset \widehat{G} \backslash K.$$

Thus for $x \in M^\alpha(K)$ we have by definition,

$$\alpha_{h*\mu}(x) = \alpha_{h*\nu}(x).$$

Now this equality holds for *any* $h \in L^1(G)$ because $\widehat{\mu}\big|V = \widehat{\nu}\big|V$, so we get

$$\alpha_h(\alpha_\mu(x)) = \alpha_{h*\mu}(x) = \alpha_{h*\nu}(x) = \alpha_h(\alpha_\nu(x)),$$

and the arbitrariness of h yields the claim. □

Corollary 4.2.2. *Let K be a compact set in $\widehat{G}$ and let U be an open set containing K. Furthermore, let $\mu \in M(G)$ such that $\widehat{\mu}$ is never 0 on U. Then α_μ is invertible on $M^\alpha(K)$ and its inverse is continuous.*

Proof. There is an open set V with compact closure such that $K \subset V \subset \overline{V} \subset U$ and $\widehat{\mu}$ is never 0 on $\overline{V}$. By Theorem 2.4.4, there is a function $g \in L^1(G)$ such that $\widehat{g}\widehat{\mu} = 1$ on $\overline{V}$. By Lemma 4.2.1, $\alpha_{g*\mu}(x) = x$ for all $x \in M^\alpha(K)$. Because $\alpha_{g*\mu} = \alpha_g\alpha_\mu$, the inverse of α_μ on $M^\alpha(K)$ is α_g. □

Proposition 4.2.3. *Let α be an action of a locally compact abelian Hausdorff group G on the dual pair (E, E'). Let $K \subset \widehat{G}$ be a compact set and let (μ_n) be a sequence of functions in $L^1(G)$ such that the sequence $([\mu_n]) \subset L^1(G)/\iota_-(K)$ is weakly convergent.*

If α is weak integrable, then there is a function Φ in $L^1(G)$ such that the sequence (α_{μ_n}) converges to α_Φ on $M^\alpha(K)$ in the WOT.

If α is strong integrable, then (α_{μ_n}) converges to α_Φ on $M^\alpha(K)$ in the SOT.

Proof. Suppose that α is weak integrable. We shall show that the map

$$\alpha_K : L^1(G)/\iota_-(K) \to \mathcal{L}_\omega(M^\alpha(K)) : [\mu] \mapsto \alpha_\mu\big|_{M^\alpha(K)}$$

is well-defined and weak-WOT continuous. If μ and ν are in $L^1(G)$ such that $[\mu] = [\nu] \in L^1(G)/\iota_-(K)$, then $\widehat{\mu} = \widehat{\nu}$ on some open set containing K and so by Lemma 4.2.1, $\alpha_\mu = \alpha_\nu$ on $M^\alpha(K)$. Hence α_K is well-defined.

Now by Definition 4.1.6, the dual of $M^\alpha(K)$ is the quotient space $E'/R^{\alpha'}(\widehat{G}\backslash K)$. For any $y \in E'$, let $[y]$ denote the equivalence class of y in $E'/R^{\alpha'}(\widehat{G}\backslash K)$. The spaces $L_\omega(M^\alpha(K))$ and $M^\alpha(K) \otimes E'/R^{\alpha'}(\widehat{G}\backslash K)$ are in duality via the bilinear form $\langle T, x \otimes [y]\rangle = \langle Tx, [y]\rangle$ and the map

$$\eta_K : M^\alpha(K) \otimes E'/R^{\alpha'}(\widehat{G}\backslash K) \to C_b(G) : x \otimes [y] \mapsto \eta_{x,y},$$

is well-defined. By Proposition 4.1.9, $\eta_{x,y} \in \iota_+(K)^\circ \subseteq \iota_-(K)^\circ$, the dual of $L^1(G)/\iota_-(K)$. From this we see that η_K is the transpose of α_K. Hence by [38, Ch II, Prop. 12 p38] and the fact that η_K is the transpose of α_K, the map α_K is weak-WOT continuous.

As $([\mu_n])$ is weakly convergent to $[\Phi]$ say, the sequence $(\alpha_{\mu_n}|_{M^\alpha(K)})$ is convergent to $\alpha_\Phi|_{M^\alpha(K)}$ in the WOT on $L_\omega(M^\alpha(K))$.

If α is strong integrable and $\mathcal{A} \subset E'/R^{\alpha'}(\widehat{G}\backslash K)$ is ξ-equicontinuous, for any $x \in M^\alpha(K)$, the maps $t \mapsto \eta_{x,y}(t)$ where $[y] \in \mathcal{A}$ are uniformly bounded and so is $t \mapsto \sup_{[y]\in\mathcal{A}} |\langle \alpha_t(x), [y]\rangle|$. As in the proof of Lemma 4.1.3, this implies that α_K is weak-SOT continuous, and hence that (α_{μ_n}) converges to α_Φ in the SOT. □

Lemma 4.2.4. *Let $K \subset \widehat{G}$ be a compact S-set and let μ, ν be finite Radon measures such that $\widehat{\mu} = \widehat{\nu}$ on K. Then the operators α_μ and α_ν are equal on $M^\alpha(K)$.*

Proof. First of all, note that there is an $h \in L^1(G)$ such that $\widehat{h} = 1$ on a neighbourhood of K. Thus $\widehat{\mu * h} = \widehat{\nu * h}$ on K. Note that we convolve both μ and ν by $h \in L^1(G)$. Now $L^1(G)$ is a closed ideal of $M(G)$ (cf [41, Theorem 1.3.4 p16]), so both $h * \mu$ and $h * \nu$ are in $L^1(G)$. Hence the lemma is proved for all finite Radon measures μ, ν if it is proved whenever μ and ν are functions in $L^1(G)$.

Now fix an $x \in M^\alpha(K)$ and a $y \in E'$. As K is an S-set, by Theorem 4.1.15, for any $\epsilon > 0$ and any $f \in L^1(G)$, there is a trigonometric polynomial $\sum_{i=0}^{n} c_i \overline{\langle t, \xi_i\rangle}$ with $\xi_i \in K$ such that

$$|\int_G \eta_{x,y}(t) f(t)\ dt - \int_G \sum_{i=0}^{n} c_i \overline{\langle t, \xi_i\rangle} f(t)\ dt| < \epsilon.$$

Taking $f = \mu - \nu$, we have

$$\begin{aligned} \big|\langle \alpha_\mu(x), y\rangle - \langle \alpha_\nu(x), y\rangle\big| &= \big|\int_G \langle \alpha_t(x), y\rangle (\mu - \nu)(t)\ dt\big| \\ &< \epsilon + \big|\sum_{i=0}^{n} c_i \big(\widehat{\mu} - \widehat{\nu}\big)(\xi_i)\big| = \epsilon \end{aligned}$$

due to the equality of $\widehat{\mu}$ and $\widehat{\nu}$ on K. As ϵ and y are arbitrary, we have shown that for any $x \in M^\alpha(K)$, $\alpha_\mu(x) = \alpha_\nu(x)$. □

Corollary 4.2.5. *Let $K \subset \widehat{G}$ be a compact S-set and let μ be a finite Radon measure such that $\widehat{\mu}$ never vanishes on K. Then the restriction of α_μ to $M^\alpha(K)$ is invertible.*

Proof. By Theorem 2.4.4, there is a $g \in L^1(G)$ such that $\widehat{g}\widehat{\mu} = 1$ on K, and so by Lemma 4.2.4, $\alpha_{g*\mu}(x) = x$ for $x \in M^\alpha(K)$. Thus α_g is the inverse of α_μ on $M^\alpha(K)$. □

Remark 4.2.6. *One virtue of proving these results for the abstract dual pair (E, E') is that all these results remain true if E and E' are swapped around. In particular, Proposition 4.2.3 now states that $\alpha'_{\varphi_n} \to \alpha'_\Phi$ on $M^{\alpha'}(K)$ in the* WOT *or* SOT, *according as the action α is weak or strong integrable.*

4.3 Tauberian Theorems for Ergodic Theory

The Tauberian theorems of this section are the culmination of our development of the spectral theory of integrable actions given in the previous sections.

Theorem 4.3.1. *Let α be a weak integrable action of a locally compact abelian Hausdorff group G on a barrelled space E with dual E'. Let $\mu \in M(G)$ be such that $\nu(\mu)$ is a compact S-set and let (φ_n) be a sequence in $M(G)$ such that*

1. *$\{\alpha_{\varphi_n}(x)\}$ is relatively weakly compact*
2. *$([\varphi_n]) \subset L^1(G)/\iota(\nu(\mu))$ is relatively weakly compact*
3. *for each $\xi \in \nu(\mu)$, $\lim_{n\to\infty} \widehat{\varphi}_n(\xi)$ exists, and for some $A > 0$, $\lim_{n\to\infty} \widehat{\varphi}_n(\xi) > A$ for all $\xi \in \nu(\mu)$*
4. *$\alpha_{\mu*\varphi_n} \to 0$ in the* WOT.

Then we have that:

(1′) (α_{φ_n}) converges in the WOT *to an invertible operator on $N(\mu)$, and 0 on $R(\mu)$*

(2′) $E = R(\mu) \oplus N(\mu)$.

Note that because $\nu(\mu)$ is an S-set, $\iota_+(\nu(\mu)) = \iota_-(\nu(\mu))$, so we write $\iota(\nu(\mu))$ for this ideal, as explained on our introduction to S-sets before Theorem 2.4.3.

Proof. Because $\nu(\mu)$ is compact, without loss of generality, we may assume that $(\varphi_n) \subset L^1(G)$, by replacing it if necessary by the sequence $(\varphi_n * h)$, where $h \in L^1(G)$ such that $\widehat{h}$ is identically 1 on $\nu(\mu)$. We prove the result in three steps:

1. (α_{φ_n}) converges weakly to an invertible operator on $N(\mu)$
2. $R(\mu) \cap N(\mu) = \{0\}$ and (α_{φ_n}) converges weakly to 0 on $R(\mu)$
3. (α_{φ_n}) converges weakly to an operator on E and $R(\mu) \oplus N(\mu) = E$.

Step 1. By Lemma 2.4.5, hypotheses (2) and (3) imply that $([\varphi_n])$ is weakly convergent. So by Proposition 4.2.3, the sequence (α_{φ_n}) converges on $M^\alpha(\nu(\mu))$ in the WOT to an operator α_Φ, where $\Phi \in L^1(G)$. By hypothesis (3), $\widehat{\Phi}$ doesn't vanish on $\nu(\mu)$, so by Corollary 4.2.5, α_Φ is invertible on $M^\alpha(\nu(\mu))$. Finally, note that $N(\mu) = M^\alpha(\nu(\mu))$ by Lemma 4.1.13.

Step 2. As both $R(\mu)$ and $N(\mu)$ are α-invariant subspaces of E, so is $R(\mu) \cap N(\mu)$. As noted above, α_Φ restricted to $N(\mu)$ is invertible. Hence α_Φ restricted to $R(\mu) \cap N(\mu)$ is also invertible. Pick any $y \in R(\mu) \cap N(\mu)$.

The sequence (α_{φ_n}) is pointwise bounded on $N(\mu)$ and each one is continuous in the $\tau(E, E')$-topology on E. As E is barrelled, this is exactly the strong topology on E and we may use the Banach−Steinhaus theorem [38, Ch IV, Theorem 3 p69] to conclude that (α_{φ_n}) is equicontinuous on $N(\mu)$. In other words, for any weak neighbourhood V of 0 in $N(\mu)$, there is a $\tau(E, E')$-neighbourhood U such that

$$\alpha_{\varphi_n}(U) \subset V/3$$

for all $n \in \mathbb{N}$. Furthermore, as $R(\mu)$ is the closure of the space of elements of the form $\alpha_\mu(e)$ for $e \in E$, we can find a $y' \in E$ such that $\alpha_\mu(y') - y \in U$.

Hence $\alpha_{\varphi_n}\alpha_\mu(y') - \alpha_{\varphi_n}(y) \in V/3$ for all $n \in \mathbb{N}$. By hypothesis (4), there exists an N_1 such that $\alpha_{\varphi_n}\alpha_\mu(y') \in V/3$ for all $n \geq N_1$. Because $\alpha_{\varphi_n} \to \alpha_\Phi$ in the WOT on $N(\mu)$, there exists an N_2 such that $\alpha_{\varphi_n}(y) - \alpha_\Phi(y) \in V/3$ for all $n \geq N_2$. Hence

$$\alpha_\Phi(y) = \alpha_{\varphi_n}\alpha_\mu(y) - \big(\alpha_{\varphi_n}\alpha_\mu(y) - \alpha_{\varphi_n}(y)\big) - \big(\alpha_{\varphi_n}(y) - \alpha_\Phi(y)\big) \in V/3 + V/3 + V/3 = V$$

for all $n \geq \max\{N_1, N_2\}$. As V is arbitrary, $\alpha_\Phi(y) = 0$.

But since y was an arbitrary element of $R(\mu) \cap N(\mu)$ with α_Φ invertible on $N(\mu)$, this ensures that $R(\mu) \cap N(\mu) = \{0\}$.

The same technique shows that $\alpha_{\varphi_n} \to 0$ in the WOT on $R(\mu)$. For any weak neighbourhood V, there is a $\tau(E, E')$-neighbourhood U such that $\alpha_{\varphi_n}(U) \subset V/2$, as we have seen above. Furthermore, there is a $y' \in E$ such that $\alpha_\mu(y') - y \in U$. As $\alpha_{\varphi_n * \mu} \to 0$ weakly, there is an $N \in \mathbb{N}$ such that $\alpha_{\varphi_n}\alpha_\mu(y) \in V/2$ for all $n \geq N$. Hence

$$\begin{aligned} \alpha_{\varphi_n}(y) &= \alpha_{\varphi_n}(y) - \alpha_{\varphi_n}(\alpha_\mu(y)) + \alpha_{\varphi_n}(\alpha_\mu(y)) \\ &\in V/2 + V/2 = V \end{aligned}$$

for all $n \geq N$; hence $\alpha_{\varphi_n}(y) \to 0$ as $n \to \infty$.

Step 3. First we show that $(\alpha_{\varphi_n}(x))$ converges weakly for every $x \in E$. As this sequence is relatively weakly compact, if it is not convergent, we can find two subsequences with different limits:

$$\alpha_{\varphi_{n_i}}(x) \to x_0 \text{ and } \alpha_{\varphi_{n_j}}(x) \to x_1$$

with $x_0 \neq x_1$. As $\lim_{i\to\infty} \alpha_\mu \alpha_{\varphi_{n_i}}(x) = 0 = \lim_{j\to\infty} \alpha_\mu \alpha_{\varphi_{n_j}}(x)$ by hypothesis, x_0 and x_1 are in $N(\mu)$.

So $x_0 - x_1 \notin R(\mu)$, because $R(\mu) \cap N(\mu) = \{0\}$. This means that there is a y in $R(\mu)^\circ$ such that $\langle x_0 - x_1, y\rangle \neq 0$. By Lemma 4.1.11, $R(\mu)^\circ = N'(\mu)$ and

$$\begin{aligned} \langle x_0, y\rangle &= \lim_{n\to\infty} \langle \alpha_{\varphi_{n_i}}(x), y\rangle \\ &= \lim_{n\to\infty} \langle x, \alpha'_{\varphi_{n_i}}(y)\rangle \\ &= \langle x, \alpha'_\Phi(y)\rangle \end{aligned}$$

where in the last equality we invoked the Remark 4.2.6 of the previous section. Similarly,

$$\langle x_1, y\rangle = \langle x, \alpha'_\Phi(y)\rangle$$

and so $\langle x_0, y\rangle = \langle x_1, y\rangle$, which is a contradiction. Hence $(\alpha_{\varphi_n}(x))$ is weakly convergent for all $x \in E$.

We define $T(x) = \lim_{n\to\infty} \alpha_{\varphi_n}(x)$, so that T is continuous by the Banach-Steinhaus Theorem and the range of T is $N(\mu)$ by hypothesis (4). Furthermore, $\ker(T) = R(\mu)$, for if $x \in \ker(T)$, then for all $y \in E'$,

$$0 = \langle Tx, y\rangle = \langle x, T'y\rangle.$$

As $T'y \in N'(\mu)$, $x \in \big(N(\mu)\big)^\circ = R(\mu)$, so $\ker(T) \subseteq R(\mu)$. By the hypotheses of the theorem and the definition of T, $R(\mu) \subseteq \ker(T)$.

Now if $\rho \in L^1(G)$ such that $\widehat{\rho}\widehat{\Phi} = 1$ on $\nu(\mu)$, then α_ρ and α_Φ are inverses on $N(\mu)$ and so the operator $P = \alpha_\rho T$ is a projection whose range is $N(\mu)$ and whose kernel is $\ker(T) = R(\mu)$. This proves that $E = R(\mu) \oplus N(\mu)$. □

We can prove Theorem 4.3.1 for other topologies on $L_\omega(E)$.

Theorem 4.3.2. *Let α be a strong integrable action of a locally compact abelian Hausdorff group G on a barrelled space E with dual E'. Let $\mu \in M(G)$ such that $\nu(\mu)$ is an S-set and let (φ_n) be a sequence in $M(G)$ such that*

1. $\{\alpha_{\varphi_n}(x)\}$ *is relatively weakly compact*
2. $([\varphi_n]) \subset L^1(G)/\iota(\nu(\mu))$ *is relatively weakly compact*
3. $\lim_{n\to\infty} \widehat{\varphi}_n(\xi) > A$ *for some* $A > 0$ *and all* $\xi \in \nu(\mu)$
4. $\alpha_{\mu*\varphi_n} \to 0$ *in the* SOT.

Then we have that:

(1′) (α_{φ_n}) *converges in the* SOT *to an invertible operator on* $N(\mu)$*, and* 0 *on* $R(\mu)$

(2′) $E = R(\mu) \oplus N(\mu)$.

Proof. All parts of Theorem 4.3.2 but the strong convergence of (α_{φ_n}) to T follow immediately from Theorem 4.3.1. But $E = R(\mu) \oplus N(\mu)$ and by Proposition 4.2.3 the convergence is strong on $N(\mu)$ and by hypothesis it is also strong on $R(\mu)$. □

Remark 4.3.3. *We now make some remarks on further generalisations as well as specific situations where the hypotheses of the Tauberians theorems can always be shown to hold.*

Different operator topologies: *The above theorem remains true when the* SOT *on* $L_\omega(E)$ *is replaced by any weaker topology in the following sense. If* $\mathcal{A}$ *is a collection of* $\sigma(E', E)$*-bounded subsets of* E'*, we can form the topology of* $\mathcal{A}$*-convergence on* $L_\omega(E)$ *given by the neighbourhood base*

$$W_{A,V} = \{L \in L_\omega(E) : L(A^\circ) \subseteq V\},$$

where $A \in \mathcal{A}$ *and* V *is a bounded set in* E*. Then* (α_{φ_n}) *will converge in the topology of* $\mathcal{A}$*-convergence to an operator invertible on* $N(\mu)$ *and* 0 *on* $R(\mu)$*.*

Reflexive spaces: *The condition that* $\{\alpha_{\varphi_n}(x)\}$ *be relatively weakly compact is routinely satisfied in a number of general cases: for instance, if* $(\varphi_n) \in M(G)$ *is bounded, then* $\{\alpha_{\varphi_n}(x)\}$ *is weakly bounded for all* $x \in E$ *and if* E *is in addition reflexive, then* $\{\alpha_{\varphi_n}(x)\}$ *is automatically relatively weakly compact.*

Relatively weakly compact sequences: *If in the above theorems the sequence* $(\varphi_n) \subset L^1(G)$ *is relatively weakly compact, by Lemma 4.1.3 the set* $\{\alpha_{\varphi_n}\}$ *is relatively weakly compact in the* WOT *and so for any* $x \in E$*,* $\{\alpha_{\varphi_n}(x)\}$ *is relatively weakly compact in* E*. Also, as the quotient map from* $L^1(G)$ *to* $L^1(G)/\iota_+(K)$ *is weakly continuous, the sequence* $([\varphi_n])$ *is also relatively weakly compact in* $L^1(G)/\iota_+(K)$*.*

Thus the relative weak compactness of (φ_n) *ensures that the first two hypotheses of the Tauberian theorems are satisfied.*

4.4 Applications to Ergodic Theorems

In this section, we show how to use the Tauberian theorems 4.3.1 and 4.3.2 to prove results in ergodic theory. By a judicious choice of the measures μ and φ_n, we can quickly prove several Mean Ergodic theorems.

By $F(E)$ we mean the (closed) subspace of all α-invariant elements in E. By Lemma 4.2.4, the elements of $M^\alpha(\{0\})$ are fixed because $\widehat{\delta_t} = \widehat{\delta_0}$ on $\{0\}$, so $\alpha_t = \alpha_0 = id$ on $M^\alpha(\{0\})$ for all $t \in G$. Hence $M^\alpha(\{0\}) \subseteq F(E)$. On the other hand, if $x \in F(E)$, then for any $t \in G$ and $y \in E'$, $\langle \alpha_t(x), y\rangle = \langle x, y\rangle$, so by Definition 4.1.8, $x \in \Gamma(\{0\})$, which equals $M^\alpha(\{0\})$ by Proposition 4.1.9. Hence $F(E) = M^\alpha(\{0\})$.

Let us now discuss a generalisation of the classical Mean Ergodic Theorem in the context of Fréchet spaces. Suppose that X is a Fréchet space and that T is a power-bounded automorphism of X − that is, for any bounded subset C of X, there is a bounded subset B such that $T^n(C) \subseteq B$ for all n. By Proposition 4.1.5, T induces a strong integrable action α of the group $\mathbb{Z}$ on X. Suppose that the convex hull of $\{T^n(x) : n \in \mathbb{Z}\}$ is weakly relatively compact for each $x \in X$. (This is always true if X is reflexive, for example, because then every weakly bounded set is weakly relatively compact, as shown in [38]).

Then the Mean Ergodic Theorem states that there is a projection P_F of X onto $F(X)$ and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i \longrightarrow P_F \tag{4.6}$$

in the SOT. To prove this, on $\mathbb{Z}$ define the measures $\mu = \delta_0 - \delta_1$ and $\varphi_n = \dfrac{1}{n}\chi_{[0,n-1]}$ for each $n \in \mathbb{N}$, where $\chi_{[0,n-1]}$ is the characteristic function of the set $\{0, 1, \ldots, n-1\}$. As $\mu * \varphi_n$ converges to 0 in norm, by Lemma 4.1.3 we conclude that $\alpha_{\mu*\varphi_n} \to 0$ in the SOT.

Now $\nu(\mu) = \{1\}$, where 1 is the identity element of $\mathbb{T}$. Being a singleton, $\{1\}$ is an S-set. (The fact a singleton is an S-set is a direct consequence of [17, Corollary 4.67]). Furthermore, on $\{1\}$, we see that obviously $\lim_{n\to\infty} \widehat{\varphi}_n(1) = 1$, and $[\varphi_n] = [\varphi_m]$ in $L^1(\mathbb{Z})/\iota(\{0\})$, which is one-dimensional, for all $n, m \in \mathbb{N}$.

As the convex hulls of the orbits $\{T^n(x) : n \in \mathbb{Z}\}$ are weakly relatively compact, so are the sets $\{\alpha_{\varphi_n}(x) : n \in \mathbb{N}\}$ for all $x \in X$. Indeed, by the theory of vector-valued integration outlined in [43], because $\|\varphi_n\| \leq 1$ for all $n \in \mathbb{N}$, $\alpha_{\varphi_n}(x)$ lies in the closure of the convex hull of $\{T^n(x) : n \in \mathbb{Z}\}$. Hence all the hypotheses of Theorem 4.3.2 are satisfied; this theorem hence establishes the validity of (4.6).

Similarly for actions of $\mathbb{R}$ on X, we obtain the formula

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} \alpha_t(x)\ dt \longrightarrow P_F(x). \tag{4.7}$$

Here we set $\mu(x) = xe^{-x^2}$ and $\varphi_n = \dfrac{1}{2n}\chi_{[-n,n]}$ and follow the same steps as in the proof of (4.6).

Firstly condition 1) of Theorem 4.3.2 is satisfied using precisely the same reasoning as above.

We start by computing the Fourier transform of $f(x) = e^{-x^2}$. Note that the function

$$s \mapsto \int_{-\infty}^{\infty} e^{-(x+is)^2} dx$$

is constant. Its derivative is

$$\int_{-\infty}^{\infty} 2i(x+is)e^{-(x+is)^2} dx = \int_{-\infty}^{\infty} i\frac{d}{dx} e^{-(x+is)^2} dx = 0.$$

Using completion of the square we compute:

$$\begin{aligned} \widehat{f}(y) &= \int_{-\infty}^{\infty} e^{-x^2} e^{-ixy} dx \\ &= e^{y^2/4} \int_{-\infty}^{\infty} e^{-(x+iy/2)^2}\ dx \\ &= e^{y^2/4} \int_{-\infty}^{\infty} e^{-x^2}\ dx \\ &= \sqrt{\pi} e^{y^2/4}. \end{aligned}$$

As $\mu(x) = -f'(x)/2$,

$$\widehat{\mu}(y) = \pi\sqrt{\pi} i y e^{y^2/4}$$

and hence $\nu(\mu) = \{0\}$.

Let us verify condition 3). Clearly $\widehat{\varphi}_n(0) = \int_{-\infty}^{\infty} \varphi_n(x) dx = 1$.

Let us verify condition 2). As $L^1(\mathbb{R})/\iota(\nu(\mu))$ is one dimensional, $[\varphi_n] = [\varphi_m]$ in $L^1(\mathbb{R})/\iota(\nu(\mu))$ for all $n, m \in \mathbb{N}$.

Let us verify condition 4). We compute:

$$\begin{aligned}
\mu * \varphi_n(x) &= \int_{-\infty}^{\infty} \varphi_n(t)\mu(x-t)\ dx \\
&= = \frac{1}{2n}\int_{-n}^{n}(x-t)e^{-(x-t)^2}\ dt \\
&= -\frac{1}{2n}\int_{-n}^{n} ye^{-y^2}\ dy \text{ (with the substitution } y = x-t) \\
&= -\frac{1}{2n}\Big[-\frac{1}{2}\Big]e^{-y^2}\Big|_{-n}^{n} \\
&= \frac{1}{4n}\big[e^{-n^2} - e^{-n^2}\big] \\
&= 0.
\end{aligned}$$

Hence $\varphi_n * \mu = 0$ for all n and so $\alpha_{\varphi_n * \mu} = 0$.

We can now apply Theorem 4.3.2 to prove (4.7).

It is possible to extend this technique to all projections onto eigenspaces of the group action. Recall that $x \in X$ is an *eigenvector* corresponding to the eigenvalue $\xi \in \widehat{G}$ if $\alpha_t(x) = \langle t, \xi\rangle x$ for all $t \in G$. Using the same arguments as where we showed that $M^\alpha(\{0\})$ is the fixed point space of the action, it is possible to show that $M^\alpha(\{\xi\})$ is the eigenspace with eigenvalue ξ.

We shall prove that it is a consequence of our Tauberian theorem that there is a projection P_ξ of X onto $M^\alpha(\{\xi\})$, and that it can be computed by an ergodic limit in the SOT. In the case of an action of $\mathbb{Z}$ given by a power bounded automorphism as above, the formula can be determined explicitly:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\langle i, \xi\rangle T^i(x) \longrightarrow P_\xi(x).$$

To prove it, we take $\mu = \delta_0 - \langle 1, \xi\rangle\delta_1$ and $\varphi_n(i) = \frac{1}{n}\langle i, \xi\rangle\chi_{[0,n-1]}(i)$ for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Using an approximation result in harmonic analysis, we can prove these ideas in full generality.

Proposition 4.4.1. *Let G be a σ-compact locally compact abelian group and α a weak integrable action of G on the dual pair (E, E') where E is a Fréchet space, such that the convex hulls of the orbits $\{\alpha_t(x) : t \in G\}$ are weakly relatively compact for each $x \in X$.*

If $\xi \in \widehat{G}$, then there is a projection P_ξ of E onto $M^\alpha(\{\xi\})$ and a bounded sequence φ_n of functions in $L^1(G)$ such that

$$\alpha_{\varphi_n} \to P_\xi$$

in the WOT*. In particular, each $M^\alpha(\{\xi\})$ is a complemented subspace of E.*

Proof. Let $\mu \in M(G)$ such that $\nu(\mu) = \{\xi\}$ and W_n, $n \in \mathbb{N}$ a sequence of open neighbourhoods of $\{\xi\}$ with compact closure such that $\cap W_n = \{\xi\}$. By [41, Theorem 2.6.3 p49], we can choose a bounded sequence $(\varphi_n) \subset L^1(G)$ such that $\|\varphi_n * \mu\|_1 < 1/n$, $\widehat{\varphi}_n(\xi) = 1$ and supp $\widehat{\varphi_n} \subset W_n$ for all $n \in \mathbb{N}$.

We see that $(\varphi_n * \mu)$ converges to 0 in norm and hence that $\alpha_{\varphi_n * \mu} \to 0$ in the SOT (and hence certainly in the WOT). Clearly $\widehat{\varphi}_n$ is convergent on $\nu(\mu) = \{\xi\}$, as $\widehat{\varphi}_n(\xi) = 1$. Furthermore, $[\varphi_n] = [\varphi_m] \in L^1(G)/\iota(\{\xi\})$, which is one-dimensional, for all $n, m \in \mathbb{N}$.

Because $(\varphi_n) \subset L^1(G)$ is bounded and the convex hull of $\{\alpha_t(x) : t \in G\}$ is relatively weakly compact for each $x \in X$, $\alpha_{\varphi_n}(x)$ lies in the compact closure of the convex hull of $\|\varphi_n\|_1\{\alpha_t(x) : t \in G\}$. Hence $\{\alpha_{\varphi_n}(x) : n \in \mathbb{N}\}$ is also relatively weakly compact for each $x \in E$.

Applying Theorem 4.3.1, the result follows. □

4.5 Notes and Remarks

The author's thinking on the proof of mean ergodic theorems in a general context - that is, for more general groups and vector spaces - evolved from studying the Tauberian theorems of [15] on the one hand, and harmonic analysis from [17] on the other. The first link was made in Lemma 4.1.14. Indeed, this was the first novel result in all this work obtained by the author. In attempting to copy Dunford and Schwarz's proof of their Tauberian theorem, the author was led to the study of the correct kinds of locally convex vector spaces.

Prof. L. Labuschagne immediately noticed an egregious error in an early draft of [10], which is worth repeating, if only as an illustration of the subtleties involved in working with convex vector spaces. Suppose that (E, E') is a dual pair and that $X \subset E$ is a dense subspace. Then the $\beta(X, E')$-topology is in general strictly stronger than the relative $\beta(E, E')$ topology induced on X. At first, the author believed that they were identical topologies.

This serves to illustrate how strong the β-topology is. It is essential for applications of the Banach-Steinhaus theorem, which is why it is so important for the proof of the Tauberian Theorems 4.3.1 and 4.3.2, as a look at the proofs of those results will show.

On the group theoretic side, it has long been known that certain invariant subspaces of the space acted upon serve as a kind of analogue of the Fourier transform on the group. There are several definitions discussed in the text, due to Arveson [1], Godement [21] and Takesaki [47]. These are compared and shown to be essentially the same in Section 4.1. However, the slightly different viewpoints combined with the innovation of using sets of synthesis, allowed the Tauberian theorems to have greater scope than they may otherwise have had.

Chapter 5

Maximal inequalities and pointwise ergodic theorems

The pointwise ergodic theory of this thesis is more intricate than the mean ergodic theory. Anyone familiar with von Neumann's mean ergodic theorem (1929) and Birkhoff's pointwise theorem (1931) will find in this an echo of a long standing discrepancy in difficulty.

We start by formalising the notion of the Transfer Principle and establishing some of its basic properties in Sections 5.1, 5.2 and 5.3. This work requires the full force of the vector-valued measure theory and our understanding of product measure spaces developed in Sections 2.2 and 2.3.

Thereafter our goal is the very general maximal inequalities of Corollary 5.5.2. The proof of the Corollaries is based on a generalisation of an inequality of Kolmogorov, which in a sense provides a local interpretation of what a maximal inequality is: this is Theorem 5.4.1. This plus Lemma 5.5.1 and Proposition 3.4.1 constitute the proof of these Corollaries.

Finally, in Section 5.6 we first demonstrate how one can use these Corollaries to get maximal inequalities from various properties of the function spaces combined with simpler maximal inequalities on the group. This requires some work on Definition **??** to link the seminorms provided there to more standard BFS norms. It is for this end that Propositions 3.4.2 and 3.4.3 were developed. Thereafter we can derive pointwise ergodic theorems in a wide variety of situations, over very general abelian groups and ergodic averages. The key here is a standard three step strategy for using maximal inequalities to derive the pointwise theorems.

5.1 The Transfer Principle

In the sequel, all groups shall be considered to be multiplicative, with identity 1.

Definition 5.1.1. *A dynamical system consists of a locally compact group G, a measure space (Ω, μ) and an action of G on Ω denoted by α, which is a group homomorphism from G into the group of all invertible measurable mappings from Ω onto itself. This action is measure-preserving in the sense that for any measurable subset $A \subseteq \Omega$ and $g \in G$, $\mu(\alpha_g^{-1}(A)) = \mu(A)$. Furthermore, the map $\tilde{\alpha} : G \times \Omega \to \Omega : t \times \omega \mapsto \alpha_t(\omega)$ is also measurable. The data is summarised by these four objects:*

$$(\Omega, \mu, G, \alpha).$$

The condition that $\tilde{\alpha} : G \times \Omega \to \Omega : t \times \omega \mapsto \alpha_t(\omega)$ be measurable is equivalent to stating that if f is a measurable function of Ω, then the function $F(t, \omega) := f(\alpha_t(\omega))$ is measurable on $G \times \Omega$ because $F = f \circ \widetilde{\alpha}$. With a slight abuse of notation, then, for any measurable function f on Ω and $t \in G$ we can define $\alpha_t(f)$ by setting $\alpha_t(f)(\omega) := f(\alpha_t(\omega))$ for a μ-almost all $\omega \in \Omega$.

The first order of business is to specify what the Transfer Principle is and to which operators the procedure applies. A word on terminology. If T is a mapping between locally convex vector spaces, we say that T has *metrisable range* if the relative topology on the range is metrisable. An operator T whose domain is some linear subspace of the measurable functions on (Ω, μ) and mapping into the measurable functions on a measure space (Ω_1, μ_1) is said to be *sublinear* if for any f and g in the domain of T and complex λ, we have $|T(f+g)| \leq |T(f)| + |T(g)|$ and $|T(\lambda f)| = |\lambda||T(f)|$.

Definition 5.1.2. ***(Transferable operators)*** *Let T be an operator having the class of locally integrable functions over G as domain and having $C(G)$, the space of continuous functions on G given the compact-open topology, as range. It is called **transferable** if it satisfies the following conditions:*

1. *T is either a sublinear mapping with metrisable range or a continuous linear mapping*

2. *T is **semilocal**: there exists an open neighbourhood U of $1 \in G$ with compact closure such that if* $\mathrm{supp} f$ *is contained in a set V, then* $\mathrm{supp} T(f)$ *is contained in UV.*

3. *T is **translation invariant**: for all $t \in G$ and $f \in L^{\mathrm{loc}}(G)$,*

$$\tau_t \circ T(f) = T \circ \tau_t(f),$$

where $\tau_t(f)$ is the function defined by $s \mapsto f(ts)$ for all locally integrable f and $s \in G$.

Note that if $C(G)$ is metrisable under the compact-open topology, then T automatically has metrisable range. For instance, this occurs when G is second countable and metrisable. Moreover, if $\{g_n\}$ is a countable dense subset of G we can form the countable collection of closed balls $\{\overline{B}(g_n, q) : n \in \mathbb{N}, q \in \mathbb{Q}\}$, and as any compact set can be covered by a finite number of these balls, we can see that $C(G)$ under the compact-open topology is a Fréchet space.

For a given transferable operator T, the remainder of this section is devoted to the construction of the transfer of T, denoted by $T^{\#}$, and some of its basic properties. It shall be defined as the composition of maps that arise naturally in the study of vector-valued measure theory, which we shall define and analyse. These constructions will allow us to handle the delicate measure theory on product spaces that shall appear.

We shall pay special attention to separability and countability properties of the space (Ω, μ) and group G and how they affect the transfer operator. As we will see in the sequel, if Ω and G are σ-finite, the transfer operator will be well-defined. If T is also metrisably valued (whether sublinear or linear) we will be able to write down the construction of the Transfer Principle in an even more intuitively direct way, as given in Remark 5.1.9. Indeed, on a first reading, one may skip directly to this Remark as it is this construction that shall be used in the rest of the paper.

We now turn to some tensor constructions of functions that will be necessary when defining and working with the transfer operator.

Definition 5.1.3. *Given a dynamical system (G, α, Ω, μ) as in Definition 5.1.1, let f and g be measurable functions on G and Ω respectively. The **α-skew tensor** $f \otimes_\alpha g$ is a measurable function on $G \times \Omega$ defined by*

$$f \otimes_\alpha g(t, \omega) = f(t) g(\alpha_t(\omega)).$$

There is a strong link between the skew tensor product and the standard tensor product of two functions that will come in handy.

Lemma 5.1.4. *Given f and g as above, the functions $f \otimes_\alpha g$ and $f \otimes g$ on $G \times \Omega$ are equimeasurable.*

Proof. Let $\lambda \in \mathbb{R}^+$ be fixed and define the following sets:

$$\begin{aligned} E &= \{(t,\omega) \in G \times \Omega : |f \otimes_\alpha g(t,\omega)| > \lambda\} \\ E' &= \{(t,\omega) \in G \times \Omega : |f \otimes g(t,\omega)| > \lambda\}. \end{aligned}$$

Moreover, for a fixed $t \in G$, we define

$$\begin{aligned} E_t &= \{\omega \in \Omega : |g(\alpha_t(\omega))| > \lambda/|f(t)|\} \\ E'_t &= \{\omega \in \Omega : |g(\omega)| > \lambda/|f(t)|\}. \end{aligned}$$

Now because $\alpha_t(g)$ and g are equimeasurable,

$$\mu(E_t) = m(\alpha_t(g), \lambda/|f(t)|) = m(g, \lambda/|f(t)|) = \mu(E'_t).$$

Furthermore, $h \times \mu(E) = \int_G \mu(E_t)dt = \int_G \mu(E'_t)dt = h \times \mu(E')$. Hence

$$m(f \otimes_\alpha g, \lambda) = m(f \otimes g, \lambda).$$

□

If f is a measurable function on Ω, we define

$$F := \otimes_{\alpha,G}(f) := \chi_G \otimes_\alpha f. \tag{5.1}$$

In other words, $F(t,\omega) = f(\alpha_t(\omega))$. This function is measurable on $G \times \Omega$. To see this, recall from Definition 5.1.1 that $\tilde{\alpha} : G \times \Omega \to \Omega : t \times \omega \mapsto \alpha_t(\omega)$ is measurable, which implies that $F = f \circ \tilde{\alpha}$ is measurable too.

We define the Banach space $L^{1+\infty}(\Omega)$ to be the set of a.e.-finite measurable functions on (Ω, μ) that can be written in the form $f+g$, where $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$. The norm on $L^{1+\infty}(\Omega)$ is given by

$$\|h\|_{L^{1+\infty}(\Omega)} = \inf\{\|f\|_{L^1} + \|g\|_{L^\infty} : f \in L^1(\Omega),\ g \in L^\infty(\Omega),\ h = f + g\}.$$

Such a space is a rearrangement invariant Banach function space, as discussed in Section 3.1.

Lemma 5.1.5. *If $f \in L^{1+\infty}(\Omega)$, then F is rectangular-locally integrable on $G \times \Omega$. Furthermore $\otimes_{\alpha,G}$ is a continuous mapping from $L^{1+\infty}(\Omega)$ to $L^{\mathrm{r-loc}}(G \times \Omega)$.*

Proof. Let us write $f = g_1 + g_2$, where $g_1 \in L^1(\Omega)$ and $g_2 \in L^\infty(\Omega)$. Now for any subsets $K \subset G$ and $A \subset \Omega$ of finite measure, we must show that $\int_{K\times A} |F| \; dh \times \mu$ is finite, where as per our convention, h denotes the Haar measure on G.

Note that for any $t \in G$, the measure-invariance of α ensures that

$$\begin{aligned}
\int_A |f|(\alpha_t(\omega)) \; d\mu(\omega) &= \int_{\alpha_{t^{-1}}(A)} |f|(\omega) \; d\mu(\omega) \\
&\le \int_{\alpha_{t^{-1}}(A)} |g_1|(\omega) \; d\mu(\omega) + \int_{\alpha_{t^{-1}}(A)} |g_2|(\omega) \; d\mu(\omega) \\
&\le \|g_1\|_1 + |A|\|g_2\|_\infty .
\end{aligned}$$

By Fubini's theorem and the measurability of F,

$$\begin{aligned}
\int_{K\times A} |F| \; dh \times \mu &= \int_K \int_A |f(\alpha_t(\omega))| \; d\mu(\omega) dh(t) \\
&\le \int_K \|g_1\|_1 + |A|\|g_2\|_\infty \; dh \\
&= |K|(\|g_1\|_1 + |A|\|g_2\|_\infty) < \infty .
\end{aligned}$$

Hence F is rectangular locally integrable. Furthermore, as

$$|K|(\|g_1\|_1 + |A|\|g_2\|_\infty) < |K|(1 + |A|)(\|g_1\|_1 + \|g_2\|_\infty),$$

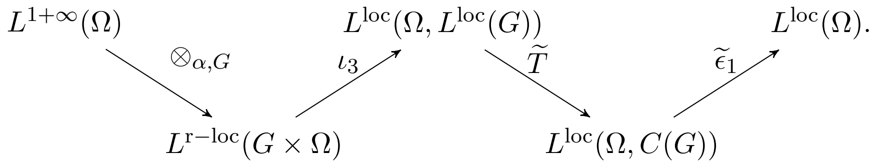
we have

$$\int_{K\times A} |F| \; dh \times \mu \le |K|(1 + |A|)\|f\|_{1+\infty},$$

which implies the continuity of $\otimes_{\alpha,G}$. □

Now we are in a position to define the transfer operator. We do so first for the case of transferable operators that are sublinear with metrisable range, then for the linear case. Thereafter we show that the two definitions agree for linear transferable operators with metrisable range.

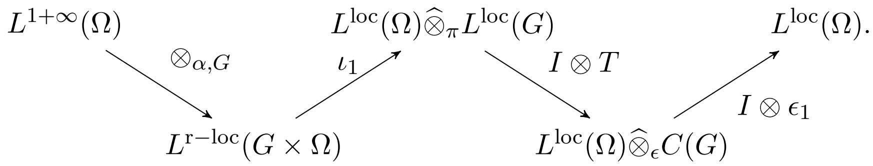
Definition 5.1.6. ***(Sublinear transfer operators with metrisable range)*** *Let T be a sublinear transferable operator on $L^{\mathrm{loc}}(G)$ with metrisable range. We define the transfer operator $T^\#$ on $L^{1+\infty}(\Omega)$ as the composition*



Here $\widetilde{T}$ is defined as $\widetilde{T}(f) = T \circ f$ for all $f \in L^{\mathrm{loc}}(\Omega, L^{\mathrm{loc}}(G))$, $\epsilon_1 : C(G) \to \mathbb{C}$ is the evaluation map at $t = 1$ in G, and $\widetilde{\epsilon}_1(g) = \epsilon_1 \circ g$ for all $g \in L^{\mathrm{loc}}(\Omega, C(G))$.

Fix an $f \in L^{\mathrm{loc}}(\Omega, L^{\mathrm{loc}}(G))$. To ensure the well-definedness of $\widetilde{T}$, note that there is a sequence of simple functions (f_n) converging a.e. to f. Then $(\widetilde{T}(f_n))$ is a sequence of simple functions in $L^{\mathrm{loc}}(\Omega, C(G))$ converging a.e. to $\widetilde{T}(f)$. To ensure the well-definedness of $\widetilde{\epsilon}_1$ we invoke the fact that $\widetilde{T}(f)$ has metrisable range. The Pettis Measurability Theorem 2.2.1 then implies that as $\widetilde{T}(f)$ is weakly measurable and ϵ_1 is a continuous linear functional on $C(G)$, $\widetilde{\epsilon}_1 \circ \widetilde{T}(f)$ is indeed measurable. That it is locally integrable is now easy to confirm.

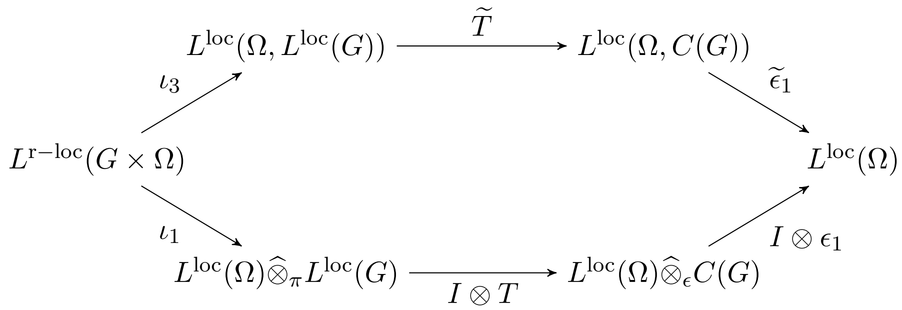
Definition 5.1.7. ***(Linear transfer operator)*** *Let T be a linear transferable operator on $L^{\mathrm{loc}}(G)$. We define the transfer operator $T^{\#}$ on $L^{1+\infty}(\Omega)$ as the composition*



As in Definition 5.1.6, $\epsilon_1 : C(G) \to \mathbb{C}$ is the evaluation map at $t = 1$ in G, so $I \otimes \epsilon_1$ maps $L^{\mathrm{loc}}(\Omega)\widehat{\otimes}_\epsilon C(G)$ to $L^{\mathrm{loc}}(\Omega) \otimes \mathbb{C}$ which is naturally isomorphic to $L^{\mathrm{loc}}(\Omega)$, which explains the slight abuse of notation in the diagram.

Lemma 5.1.8. *Let T be a linear and metrisably valued transferable operator. Then the two Definitions 5.1.6 and 5.1.7 agree.*

Proof. The proof is encapsulated in the following diagram, which combines the two diagrams of Definitions 5.1.6 and 5.1.7.



Its commutativity can be easily checked for simple functions in $L^{\text{r}-\text{loc}}(G \times \Omega)$, and so by continuity of all the arrows, the commutativity of the diagram follows. □

Now that the well-definedness of the Transfer Principle has been established and it is clear what countability assumptions are used, let us give a simpler, working definition. We emphasise that this working definition depends on the above analysis for its correctness.

Remark 5.1.9. *Let T be a sublinear transferable operator with metrisable range. Starting with a function $f \in L^{1+\infty}(\Omega)$, define as before $F(t,\omega) := f(\alpha_t(\omega))$ and $F'(t,\omega) := (T(F_\omega))(t)$, where F_ω is the cross section of F at $\omega \in \Omega$. Finally, we set $T^{\#}(f) := F'(1,\omega)$. Note that F' as defined above is in $L^{\text{loc}}(\Omega, C(G))$.*

Obviously, the steps followed in this Remark are exactly the steps used in Definition 5.1.6. As we will be working quite a bit with the functions F and F' given above, there is another point that must be made about them. Firstly F is measurable on $G \times \Omega$ and indeed rectangular locally integrable as shown in Lemma 5.1.5. For F', the situation is a little more tricky. By Lemma 2.3.5, F' is a well-defined measurable function. However, it is not necessarily locally integrable or even rectangular locally integrable. But when working out the maximal inequalities of Section 5.5, we shall need only the Fubini result of equation (2.5).

Note that of course in the case where G is second countable, one could use Lemma 2.3.6 instead of Lemma 2.3.5 to prove the measurability of F'.

5.2 The effect of semilocality and translation invariance

Let T be a transferable operator with metrisable range. We now describe some approximation properties of $T^{\#}$ that will be useful in the next section. We start by extending some constructions that we have used earlier. Let K be any measurable subset of G. As in equation (5.1), we can define the operator $\otimes_{\alpha,K}$ on the set of measurable functions on Ω by setting $F_K := \otimes_{\alpha,K}(f) := \chi_K \otimes_\alpha f$ on $G \times \Omega$ using Definition 5.1.3. Hence

$$F_K(t,\omega) = \begin{cases} f(\alpha_t(\omega)) & \text{if} \quad t \in K \\ 0 & \text{if} \quad t \notin K. \end{cases} \tag{5.2}$$

In this notation, $F_G = F$. We shall also use the notation $F'_K = \widetilde{T}(F_K)$ where $\widetilde{T}$ is as given in Definition 5.1.6. In particular, $F' = F'_G$. By Lemma 2.3.5, the functions F'_K are measurable on $G \times \Omega$.

Lemma 5.2.1. *Let T be a transferable operator with metrisable range and let U be the open neighbourhood satisfying the conditions stated in Definition 5.1.2(2). Let K and E be measurable subsets of G such that $EU^{-1} \subseteq K$. For any $f \in L^{1+\infty}(\Omega)$, and almost all $(t,\omega) \in E \times \Omega$, we have*

$$|F'(t,\omega)| \leq |F'_K(t,\omega)|. \tag{5.3}$$

Proof. First note that $F_{K^c} = (\chi_G - \chi_K) \otimes_\alpha f = \chi_G \otimes_\alpha f - \chi_K \otimes_\alpha f = F - F_K$. Consequently,

$$\begin{aligned} |F'| = |\widetilde{T}(F)| &= |\widetilde{T}(F - F_K + F_K)| \\ &\leq |\widetilde{T}(F_{K^c})| + |\widetilde{T}(F_K)| \\ &= |F'_{K^c}| + |F'_K|. \end{aligned}$$

By the semilocality of T, for almost every $\omega \in \Omega$, the measurable map $t \mapsto F'_{K^c}(t,\omega)$ has support in K^cU. Because $EU^{-1} \subseteq K$, it follows that $(K^cU) \cap E$ is empty since $a \in (K^cU) \cap E$ implies that there is a $b \in U$ so that $ab^{-1} \in K^c \cap (EU^{-1})$, which is impossible if $EU^{-1} \subseteq K$. Hence $|F'(t,\omega)| \leq |F'_K(t,\omega)|$ for almost all $\omega \in \Omega$ and all $t \in E$. □

Translation invariant operators, a class that includes all convolution operators, are automatically equimeasurability-preserving, in a sense made precise by the following Lemma.

Lemma 5.2.2. *Let T be a transferable operator with metrisable range. For any $f \in L^{1+\infty}(\Omega)$, all $s, t \in G$ and almost all $\omega \in \Omega$,*

$$F'(t, \alpha_s(\omega)) = F'(ts, \omega).$$

Moreover, for any $t_1, t_2 \in G$, the mappings $\omega \mapsto F'(t_1, \omega)$ and $\omega \mapsto F'(t_2, \omega)$ are equimeasurable.

Proof. By definition of F, for μ-a.e. ω and any $s, t \in G$, we have

$$F(t, \alpha_s(\omega)) = f \circ \alpha_t(\alpha_s(\omega)) = f \circ \alpha_{ts}(\omega) = F(ts, \omega).$$

Let $\tau_t : G \to G$ be defined by $\tau_t(s) = ts$ for all $s, t \in G$ as in Definition 5.1.2. By definition of F' and the translation-invariance of T, we have

$$\begin{aligned} F'(t, \alpha_s(\omega)) &= \widetilde{T} \circ F(t, \alpha_s(\omega)) \\ &= \widetilde{T} \circ F(ts, \omega) \\ &= \widetilde{T} \circ \tau_t \circ F(s, \omega) \\ &= \tau_t \circ \widetilde{T} \circ F(s, \omega) \\ &= F'(ts, \omega). \end{aligned}$$

Finally, let $s = t_1^{-1} t_2$ and $\lambda > 0$. Then as $F'(t_2, \omega) = F'(t_1, \alpha_s(\omega))$, we see that

$$\begin{aligned} \mu(\{\omega : |F'(t_2, \omega)| > \lambda\}) &= \mu(\{\omega : |F'(t_1, \alpha_s(\omega))| > \lambda\}) \\ &= \mu(\{\alpha_s^{-1}(\omega) : |F'(t_1, \omega)| > \lambda\}) \\ &= \mu(\{\omega : |F'(t_1, \omega)| > \lambda\}), \end{aligned}$$

proving the equimeasurability of the maps $\omega \mapsto |F'(t_1, \omega)|$ and $\omega \mapsto |F'(t_2, \omega)|$. □

The last two lemmas will both be needed in the proof of Lemma 5.5.1.

5.3 An example

One of the main sources of transferable operators in applications is convolution operators. The straightforward construction of ergodic averaging operators from convolution operators demonstrates the utility and ubiquity of the transfer operator construction.

Suppose that $\mathcal{O}$ is a measure-preserving automorphism on the measure space (Ω, μ). It induces an action α of $\mathbb{Z}$ on Ω via $\alpha(n) := \mathcal{O}^n$. If S is a finite subset of $\mathbb{Z}$, let T_S be the convolution operator defined on the space of all locally integrable functions f on $\mathbb{Z}$ by

$$T_S(f) := \frac{1}{|S|}\chi_{-S} * f,$$

where of course χ_{-S} is the characteristic function of $-S$. Bearing in mind that the set of locally integrable functions on $\mathbb{Z}$ is precisely the set of all complex-valued functions, we see that T_S is well-defined, linear and takes its values in $C(\mathbb{Z})$, which is metrisable. From the properties of convolution, it is clearly semi-local. Indeed, if $N = \max\{|s| : s \in S\}$ and f is a function with support in $[-M, M]$, then $T_S(f)$ will have support in $[-M - N, M + N]$.

Let us now determine the transfer operator $T_S^{\#}$. Let $f \in L^{1+\infty}(\Omega)$. With the help of Definition 5.1.6 and Lemma 2.3.5, we compute:

$$\begin{aligned} \widetilde{T_S}(F(t,\omega)) &= \widetilde{T_S}(f(\alpha_t(\omega))) \\ &= \frac{1}{|S|}\sum_{s\in S} f(\alpha_{t+s}(\omega)). \end{aligned}$$

Hence

$$T_S^{\#}(f)(\omega) = \widetilde{\epsilon}_0 \circ \widetilde{T_S}(F)(\omega) = \frac{1}{|S|}\sum_{s\in S} f(\alpha_s(\omega)),$$

which is a locally integrable function on Ω. (Note that we write $\widetilde{\epsilon}_0$ above because 0 is the identity element of $\mathbb{Z}$).

5.4 Kolmogorov's inequality for r.i. BFSs

The next two Sections are devoted to calculating the type of the transfer operator $T^{\#}$ from information on the type of T.

The following theorem will be useful in determining the weak type of an operator. It is an extension of Kolmogorov's criterion as found in [12].

Theorem 5.4.1. *Let (Ω, μ) be a measure space and let T be an operator on some class of measurable functions on Ω that maps into the set of measurable functions on Ω. Suppose that T is of weak type (X, Y) for r.i. BFSs X and Y and has norm c. Let φ be the fundamental function of the space Y. If $0 < \sigma < 1$ and A is any subset* *of Ω of finite measure, then for any $f \in X$ we have*

$$\int_A |Tf|^\sigma \, d\mu(x) \;\; \leq \;\; \frac{c^\sigma}{1-\sigma}\big[\varphi^*(|A|)\big]^\sigma |A|^{1-\sigma}\|f\|_X^\sigma. \tag{5.4}$$

Conversely, if T satisfies this inequality for some c and $0 < \sigma < 1$, and for each $f \in X$ and each $A \subset \Omega$ with finite measure, then T is of weak type (X, Y).

Proof. Suppose $0 < \sigma < 1$. As $t \mapsto \varphi(t)/t$ is nondecreasing, if $s \leq t$, we have the implications

$$\begin{aligned} \frac{\varphi(s)}{s} \geq \frac{\varphi(t)}{t} \quad &\Longrightarrow \quad \frac{\varphi(t)}{\varphi(s)} \leq \frac{t}{s} \Longrightarrow \frac{\varphi^\sigma(t)}{\varphi^\sigma(s)} \leq \big(\frac{t}{s}\big)^\sigma \\ &\Longrightarrow \quad \frac{s}{t}\varphi^\sigma(t) \leq \big(\frac{t}{s}\big)^{\sigma-1}\varphi^\sigma(s). \end{aligned}$$

Note that $\chi_{[1,\infty)}(t/s) = \chi_{(0,t]}(s)$ for all $s, t \in \mathbb{R}^+$. On the multiplicative group of the positive reals, we now compute, using the convolution of the functions $[(Tf)^\sigma]^*(x)\varphi^\sigma(x)$ and $x^{\sigma-1}\chi_{[1,\infty)}(x)$ at $t \in \mathbb{R}$:

$$\begin{aligned} \frac{\varphi^\sigma(t)}{t}\int_0^t [(Tf)^\sigma]^*(s)ds \;\; &\leq \;\; \int_0^t [(Tf)^\sigma]^*(s)\varphi^\sigma(s)\big(\frac{t}{s}\big)^{\sigma-1}\frac{ds}{s} \\ &= \;\; \int_0^\infty [(Tf)^\sigma]^*(s)\varphi^\sigma(s)\big(\frac{t}{s}\big)^{\sigma-1}\chi_{(0,t]}(s)\,\frac{ds}{s} \\ &= \;\; \int_0^\infty [(Tf)^\sigma]^*(s)\varphi^\sigma(s)\big(\frac{t}{s}\big)^{\sigma-1}\chi_{[1,\infty)}(t/s)\,\frac{ds}{s} \\ &= \;\; [(Tf)^\sigma]^*(x)\varphi^\sigma(x) * x^{\sigma-1}\chi_{[1,\infty)}(x)\big|_t. \end{aligned}$$

With this in hand, we exploit the inequality

$$\|[(Tf)^\sigma]^*(x)\varphi^\sigma(x) * x^{\sigma-1}\chi_{[1,\infty)}(x)\|_\infty \leq \|[(Tf)^\sigma]^*(x)\varphi^\sigma(x)\|_\infty \|x^{\sigma-1}\chi_{[1,\infty)}(x)\|_1.$$

As $\|x^{\sigma-1}\chi_{[1,\infty)}(x)\|_1 = \int_1^\infty s^{\sigma-2}ds = \frac{1}{\sigma-1}s^{\sigma-1}\big|_1^\infty = \frac{1}{1-\sigma}$, we have

$$\sup_{t>0}\frac{\varphi^\sigma(t)}{t}\int_0^t [(Tf)^\sigma]^*(s)ds \leq \frac{1}{1-\sigma}\sup_{t>0}[(Tf)^\sigma]^*(t)\varphi^\sigma(t) = \frac{1}{1-\sigma}\big[\sup_{t>0}(Tf)^*(t)\varphi(t)\big]^\sigma.$$

Here we used the fact that $(|f|^\sigma)^* = (|f|^*)^\sigma$ for all $0 < \sigma < \infty$ (see Prop 2.1.7 p41 of [4]). We thus have the following inequality.

$$\begin{aligned} \frac{\varphi^\sigma(t)}{t}\int_0^t [(Tf)^\sigma]^*(s)ds \;\; &\leq \;\; \frac{1}{1-\sigma}\|Tf\|_{M^*(Y)}^\sigma \\ &\leq \;\; \frac{c^\sigma}{1-\sigma}\|f\|_X^\sigma, \end{aligned}$$

by our hypothesis. It is obvious that $\int_A |Tf|^\sigma \, d\mu \le \int_0^{|A|} [(Tf)^\sigma]^*(s)ds$, and so we obtain

$$\begin{aligned}\int_A |Tf|^\sigma \, d\mu &\le \frac{c^\sigma}{1-\sigma}\frac{|A|}{\varphi^\sigma(|A|)}\|f\|_X^\sigma \\ &= \frac{c^\sigma}{1-\sigma}\big[\varphi^*(|A|)\big]^\sigma |A|^{1-\sigma}\|f\|_X^\sigma.\end{aligned}$$

To get the last equality, we used the identity $\varphi(|A|)\varphi^*(|A|) = |A|$.

To prove the converse, suppose that T satisfies (5.4) for some $0 < \sigma < 1$ and fix $\lambda > 0$. Consider a set $K \subset \{\omega : |Tf(\omega)| > \lambda\}$ of finite measure. By hypothesis,

$$|K| \le \int_K \frac{|Tf|^\sigma}{\lambda^\sigma} \, d\mu \le \frac{1}{\lambda^\sigma}\frac{c^\sigma}{1-\sigma}[\varphi^*(|K|)]^\sigma |K|^{1-\sigma}\|f\|_X^\sigma.$$

Consequently, the following computations are valid:

$$\begin{aligned}\frac{|K|^\sigma}{\varphi^*(|K|)^\sigma} &\le \frac{1}{\lambda^\sigma}\frac{c^\sigma}{1-\sigma}\|f\|_X^\sigma; \\ \varphi(|K|) &\le \frac{1}{\lambda}\frac{c}{(1-\sigma)^{1/\sigma}}\|f\|_X; \\ \lambda\varphi(m(|Tf|,\lambda)) &\le \frac{c}{(1-\sigma)^{1/\sigma}}\|f\|_X; \\ \|Tf\|_{M^*(Y)} &\le \frac{c}{(1-\sigma)^{1/\sigma}}\|f\|_X;\end{aligned}$$

where in the last line we used the identity $\|Tf\|_{M^*(Y)} = \sup_{\lambda>0} \lambda\varphi_Y(m(|Tf|,\lambda))$ of Lemma 3.1.3. This proves the converse. □

Note that the proof remains correct even if $\|\cdot\|_X$ is a seminorm.

Although we shall not need the concept here, this theorem, and especially the inequality (5.4), suggest a further refinement of the idea of weak-type operators.

Definition 5.4.2. *Let T be an operator on a BFS X on the measure space (Ω, μ) taking values in the space of measurable functions on (Ω, μ). Let $\mathcal{A}$ be a class of measurable μ-finite subsets of Ω. We say that T is of $\mathcal{A}$-weak type (X, Y) if*

$$\int_A |Tf|^\sigma \, d\mu(x) \le \frac{c^\sigma}{1-\sigma}\big[\phi^*|A|\big]^\sigma |A|^{1-\sigma}\|f\|_{\Lambda(\psi)}^\sigma$$

holds for all $A \in \mathcal{A}$.

In particular if Ω is a product space and $\mathcal{A}$ is the class of all μ-finite measurable rectangles, we say that T is of rectangular weak-type (X, Y).

5.5 Computation of the weak type of the transferred operator

In the rest of the paper, we shall work with dynamical systems (G, Ω, μ, α) as given in Definition 5.1.1, where (Ω, μ) is a countably generated σ-finite and resonant measure space and G is a σ-finite locally compact group.

We can now state and prove results on the weak type of the transfer operator, which form one of the main themes of this work. The next lemma is the key technical ingredient. It is in this Lemma that the semilocality and equimeasurability-preserving properties of the transferable operator are used.

Lemma 5.5.1. *Let X, Y be r.i. BFSs over G with fundamental functions φ_X and φ_Y respectively and let T be a transferable operator of weak type (X, Y) with metrisable range. Let U be the open neighbourhood satisfying the conditions in Definition 5.1.2(2). Then for any subset $A \subset \Omega$ of finite measure, and $0 < \sigma < 1$, there is a compact neighbourhood $\widetilde{K}$ of the identity such that*

$$\frac{1}{|A|}\int_A |T^{\#}f|^{\sigma}(\omega)d\mu \quad \le \quad \frac{2c^{\sigma}}{1-\sigma}[\varphi_Y(|\widetilde{K}|)]^{-\sigma}\left(\frac{1}{|A|}\int_A \|(F_{\widetilde{K}U^{-1}})_{\omega}\|_X \; d\mu\right)^{\sigma},$$

where for each $\omega \in \Omega$ the cross section $(F_{\widetilde{K}U^{-1}})_{\omega}$ is the measurable function defined on G by $t \mapsto F_{\widetilde{K}U^{-1}}(t, \omega)$.

Proof. By the Fubini-type Lemma 2.3.5 and Lemma 5.2.1, which we employ by identifying E with $\widetilde{K}$ and K with $\widetilde{K}U^{-1}$, we have:

$$\begin{aligned}\int_{\widetilde{K}}\int_A |F'(t,\omega)|^{\sigma}\; d\mu dt &= \int_A\int_{\widetilde{K}} |F'(t,\omega)|^{\sigma}\; dt d\mu \\ &\le \int_A\int_{\widetilde{K}} |F'_{\widetilde{K}U^{-1}}(t,\omega)|^{\sigma}\; dt d\mu.\end{aligned}$$

As T is of weak type (X, Y), using Kolmogorov's criterion (5.4) we have that

$$\int_{\widetilde{K}} |F'_{\widetilde{K}U^{-1}}(t,\omega)|^{\sigma} dt \le \frac{c^{\sigma}}{1-\sigma}[\varphi^*_Y(|\widetilde{K}|)]^{\sigma}|\widetilde{K}|^{1-\sigma}\|(F_{\widetilde{K}U^{-1}})_{\omega}\|^{\sigma}_X.$$

From Jensen's inequality and the identity $\varphi^*_Y(t)\varphi_Y(t) = t$,

$$\begin{aligned}\int_{\widetilde{K}}\int_A |F'(t,\omega)|^{\sigma}\; d\mu dt &\le \frac{c^{\sigma}}{1-\sigma}[\varphi^*_Y(|\widetilde{K}|)]^{\sigma}|\widetilde{K}|^{1-\sigma}\int_A \|(F_{\widetilde{K}U^{-1}})_{\omega}\|^{\sigma}_X\; d\mu \\ &\le \frac{c^{\sigma}}{1-\sigma}[\varphi_Y(|\widetilde{K}|)]^{-\sigma}|\widetilde{K}||A|^{1-\sigma}\left(\int_A \|(F_{\widetilde{K}U^{-1}})_{\omega}\|_X\; d\mu\right)^{\sigma}.\end{aligned}$$

We rewrite this as

$$\frac{1}{|\widetilde{K}|}\int_{\widetilde{K}}\left[\frac{1}{|A|}\int_A |F'(t,\omega)|^\sigma\ d\mu\right]dt \leq \frac{c^\sigma}{1-\sigma}[\varphi_Y(|\widetilde{K}|)]^{-\sigma}\left(\frac{1}{|A|}\int_A \|(F_{\widetilde{K}U^{-1}})_\omega\|_X\ d\mu\right)^\sigma.$$

As

$$t \mapsto \frac{1}{|A|}\int_A |F'(t,\omega)|^\sigma d\mu$$

is continuous, and as $|F'(1,\omega)| = |T^\# f|(\omega)$ by definition, from the Lebesgue differentiation Theorem we obtain

$$\frac{1}{|A|}\int_A |T^\# f|^\sigma(\omega)d\mu \quad = \quad \lim_{\widetilde{K}\to\{1\}} \frac{1}{|\widetilde{K}|}\int_{\widetilde{K}}\left[\frac{1}{|A|}\int_A |F'(t,\omega)|^\sigma\ d\mu\right]dt.$$

Hence for some $\widetilde{K}$ small enough,

$$\begin{aligned}\frac{1}{|A|}\int_A |T^\# f|^\sigma(\omega)d\mu \quad &\leq \quad \frac{2}{|\widetilde{K}|}\int_{\widetilde{K}}\left[\frac{1}{|A|}\int_A |F'(t,\omega)|^\sigma\ d\mu\right]dt \\ &\leq \quad \frac{2c^\sigma}{1-\sigma}[\varphi_Y(|\widetilde{K}|)]^{-\sigma}\left(\frac{1}{|A|}\int_A \|(F_{\widetilde{K}U^{-1}})_\omega\|_X\ d\mu\right)^\sigma.\end{aligned}$$

□

Corollary 5.5.2. *Let (Ω,μ,G,α) be a dynamical system with (Ω,μ) countably generated and resonant and let T be a transferable operator of weak type (X,Y) and suppose that Φ_A and Φ_B are Young's functions with respective associated fundamental functions φ_A and φ_B satisfying*

$$\theta\varphi_A(st) \geq \varphi_X(s)\varphi_B(t)$$

for some $\theta > 0$ and all $s,t > 0$.

1) If X is an Orlicz space and $\lim_{t\to 0}\varphi^*_X(t) = 0$, then $T^\#$ is of weak type $(L(\Phi_A),\varphi_B)$.

2) If X is an M- space then $T^\#$ is of weak type $(M(\varphi_A),\varphi_B)$.

3) If X is a Λ-space and $\lim_{t\to 0}\varphi^*_X(t) = 0$, then $T^\#$ is of weak type $(\Lambda(\varphi_A),\varphi_B)$.

Proof. We prove part 3). Let $A \subset \Omega$ have finite measure and let U be the open neighbourhood guaranteed by Definition 5.1.2(2). Then for any $0 < \sigma < 1$, by Lemma 5.5.1, there is a compact neighbourhood of the identity $K \subset G$ such that

$$\frac{1}{|A|}\int_A |T^\# f|^\sigma(\omega)d\mu \quad \leq \quad \frac{2c^\sigma}{1-\sigma}[\varphi_Y(|K|)]^{-\sigma}\left(\frac{1}{|A|}\int_A \|(F_{KU^{-1}})_\omega\|_{\Lambda(X)}\ d\mu\right)^\sigma.$$

From Proposition 3.4.1, part 2),

$$\frac{\varphi_B(|A|)}{|A|}\int_A \|(F_{KU^{-1}})_\omega\|_{\Lambda(X)}\ d\mu(\omega) \le 6\theta\|F_{KU^{-1}}\|_{\Lambda(\Phi_A)}.$$

Combining these last two inequalities yields

$$\frac{1}{|A|}\int_A |T^{\#}f|^\sigma(\omega)d\mu \quad\le\quad \frac{2(6\theta c)^\sigma}{1-\sigma}[\varphi_Y(|K|)]^{-\sigma}[\varphi_B^*(|A|)]^\sigma|A|^{-\sigma}\|F_{KU^{-1}}\|^\sigma_{\Lambda(\Phi_A)} \tag{5.5}$$

A simple calculation shows that $\varphi_A(st) \le \varphi_A(s)\max(1,t)$. Now note that the condition in [33, Theorem 8.15] on the Young's functions may be rephrased as $\varphi_A(st) \le \theta\varphi_{A_0}(s)\varphi_{B_0}(t)$ for all $s,t>0$. Hence, that theorem is applicable here. Together with Lemma 5.1.4 we may conclude that $\|F_{KU^{-1}}\|_{\Lambda(\Phi_A)} \le \max(1,|KU^{-1}|)\|f\|_{\Lambda(\Phi_A)}$ and so

$$\begin{aligned}\frac{1}{|A|}\int_A |T^{\#}f|^\sigma(\omega)d\mu \quad&\le\quad \frac{2(6\theta c)^\sigma}{1-\sigma}[\varphi_Y(|K|)]^{-\sigma}[\varphi_B^*(|A|)]^\sigma|A|^{-\sigma}\max(1,|KU^{-1}|^\sigma)\|f\|^\sigma_{\Lambda(\Phi_A)}\\ &=\quad \frac{c_0^\sigma}{1-\sigma}[\varphi_B^*(|A|)]^\sigma|A|^{-\sigma}\|f\|^\sigma_{\Lambda(\Phi_A)}\end{aligned}$$

where $c_0 = 2^{1/\sigma}6\theta c\varphi_Y(|K|)^{-1}\max(1,|KU^{-1}|)$.

Therefore

$$\int_A |T^{\#}f|^\sigma(\omega)\ d\mu \quad\le\quad \frac{c_0^\sigma}{1-\sigma}[\varphi_B^*(|A|)]^\sigma|A|^{1-\sigma}\|f\|^\sigma_{\Lambda(\Phi_A)},$$

and so by Theorem 5.4.1, $T^{\#}$ is of weak type $(\Lambda(\Phi_A),\varphi_B)$. □

5.6 Pointwise ergodic theorems via the transfer principle

In the last part of this work, we turn to the derivation of pointwise ergodic theorems. Henceforth, in the dynamical system (Ω,μ,G,α), not only will (Ω,μ) be countably generated and resonant, but G will be an *abelian, additive, second countable* locally compact Hausdorff group with identity element 0, and X will be a r.i. BFS over the countably generated and resonant measure space (Ω,μ). We shall work with transfer operators generated by sequences of convolution operators. That is, we consider a sequence (T_n) of operators on $L^{\text{loc}}(G)$ given by

$$T_n(f) = k_n * f, \tag{5.6}$$

where $k_n \in L^1(G)$ is bounded and has bounded support, and f is locally integrable. It is easy to see that the operators T_n are semilocal. As $k_n * f$ is a continuous function and $C(G)$ is metrisable, we see that these operators satisfy the definition of linear transferable operators with metrisable range given in Definition 5.1.2. We also define $Tf := \sup_n |T_n(f)|$. Using the Transfer Principle and given information about the functions k_n and the space X, we show that the transferred operators $T_n^{\#}$ satisfy a pointwise convergence theorem: that is, $T_n^{\#} f(\omega)$ converges a.e. for all $f \in X$ as n tends to infinity.

To achieve this goal, our strategy is the following three step programme.

1. Given the weak type of the operator T, find the weak type of $T^{\#}$.

2. In the domain of $T^{\#}$ computed in step (1), identify a dense subset D for which the pointwise convergence of $(T_n^{\#} f)$ can be verified for all $f \in D$.

3. Use an appropriate version of Banach's Principle to extend the a.e. convergence of step (2) to the whole domain of $T^{\#}$.

To do step (1), we shall use results such as the following:

Proposition 5.6.1. *Let (T_n) be a sequence of operators given by (5.6). Suppose there are Young's functions Φ_A, Φ_B, Φ_C, Φ_D and Φ_E with associated fundamental functions $\varphi_A, \ldots, \varphi_E$ satisfying*

$$\begin{aligned} \varphi_C(t)\varphi_B^*(s) &\leq \theta_1 \varphi_A(st) \\ \varphi_A(st) &\leq \theta_2 \varphi_D(s)\varphi_E(t) \end{aligned}$$

for all $s, t > 0$.

Suppose further that there are measurable functions ℓ_0 and ℓ_1 on G such that $\sup |k_n(s^{-1})| = \ell_0(s)\ell_1(s)$, $\ell_0 \in L(\Phi_B)$ and $\ell_1 \in L(\Phi_E)$. Then the operator $T^{\#}$ defined by $T^{\#} f(\omega) = \sup_{n\in\mathbb{N}} |T_n^{\#} f|(\omega)$ is a sublinear operator mapping $L(\Phi_D)$ into $L(\Phi_C)$.

Proof. For each $N \in \mathbb{N}$, let $S_N f(\omega) := \max_{1\leq n\leq N} |T_n^{\#} f|(\omega)$. By [33, Theorem 8.18],

$$\begin{aligned} \|S_N f\|_{L(\Phi_C)} &\leq \| \max_{1\leq n\leq N} | \int_G k_n(s^{-1}) f(\alpha_s(\omega))\, ds| \|_{L(\Phi_C)} \\ &\leq \| \int_G \max_{1\leq n\leq N} |k_n(s^{-1})||f(\alpha_s(\omega))| ds\|_{L(\Phi_C)} \\ &\leq \theta \|\ell_1\|_{L(\Phi_B)} \|\ell_0 \otimes_\alpha f\|_{L(\Phi_A)} \\ &= \theta_1 \|\ell_1\|_{L(\Phi_B)} \|\ell_0 \otimes f\|_{L(\Phi_A)} (\text{by Lemma 5.1.4}) \\ &\leq \theta_1\theta_2 \ell_1\|_{L(\Phi_B)} \|\ell_0\|_{L(\Phi_E)} \|f\|_{L(\Phi_D)} \end{aligned}$$

where the final inequality follows from [33, Theorem 8.15]. □

For step (3), we prove the following variation on the theme of [4, Corollary 4.5.8] and [20, Theorem 1.1.1], which provides the final link between maximal inequalities and pointwise ergodic theorems.

The following result in essence amounts to an extension of that part of the Banach principle cycle of ideas that we need to complete our program.

Proposition 5.6.2. *Let X and Y be r.i. BFSs over a measure space (Ω, μ). Let (T_n) be a sequence of linear operators on X and define the maximal operator T by $T(f) = \sup_n |T_n(f)|$. If*

1. *there is a dense subset $D \subseteq X$ such that for all $f \in D$, $(T_n(f)(\omega))$ converges for μ-a.e. $\omega \in \Omega$,*
2. *T is of weak-type (X, Y),*

then $(T_n(f)(\omega))$ converges for μ-a.e. $\omega \in \Omega$ and all $f \in X$.

Proof. Define the oscillation $\mathcal{O}_f$ of $f \in X$ as follows. For any $\omega \in \Omega$ set

$$\mathcal{O}_f(\omega) = \limsup_{n,m\to\infty} |T_n(f)(\omega) - T_m(f)(\omega)|.$$

Clearly the linearity of the operators T_n implies that $\mathcal{O}_f(\omega) \le \mathcal{O}_g(\omega) + \mathcal{O}_{f-g}(\omega)$.

For any $g \in D$, $(T_n g)$ converges μ-a.e. and thus $\mathcal{O}_g = 0$ μ-a.e.

Pick an $f \in X$. Now for any $\eta > 0$, there is a $g \in D$ such that $\|f - g\|_X < \eta$ and

$$\mu(\{\omega : \mathcal{O}_f(\omega) > \delta\}) \le \mu(\{\omega : \mathcal{O}_{f-g}(\omega) > \delta\}).$$

Furthermore, by the definition of the oscillation, $\mathcal{O}_f(\omega) \le 2T(f)(\omega)$ a.e. Similarly for $\mathcal{O}_{f-g}$. Hence

$$\begin{aligned}\mu(\{\omega : \mathcal{O}_f(\omega) > \delta\}) &\le \mu(\{\omega : 2T(f-g)(\omega) > \delta\}) \\ &= m(2T(f-g), \delta).\end{aligned}$$

As T is of weak-type (X, Y), $\|2T(f-g)\|_{M^*(Y)} \le 2\beta\|f-g\|_X < 2\beta\eta$ where β depends only on T. Rewriting this using Lemma 3.1.3,

$$\sup_{s>0} s\varphi_Y(m(2T(f-g), s)) \le 2\beta\eta,$$

In particular, $\delta\varphi_Y(m(2T(f-g),\delta)) \le 2\beta\eta$. Therefore

$$\varphi_Y(\mu(\{\omega : \mathcal{O}_f(\omega) > \delta\})) \le \frac{2\beta\eta}{\delta}.$$

As η is arbitrary, $\varphi_Y(\mu(\{\omega : \mathcal{O}_f(\omega) > \delta\})) = 0$. Because a fundamental function is 0 only at the origin, $\mu(\{\omega : \mathcal{O}_f(\omega) > \delta\}) = 0$. Because δ is arbitrary, $\mathcal{O}_f = 0$ μ-a.e. which implies that $(T_n f)$ does indeed converge μ-a.e. □

The fundamental result towards completing step (2) of the three-step programme is given in Proposition 5.6.4. From this, many interesting pointwise ergodic theorems can be deduced, given further information on the nature of X. We start by constructing subsets D of X for whose elements a.e. convergence is easy to check. To this end, for any $f \in L^1(G)$ and $x \in X$, we define

$$\alpha_f(x) = \int_G \alpha_t(x) f(t)\, dt, \tag{5.7}$$

where the integral is a Bochner integral. Because the action of G on (Ω, μ) is measure-preserving, on any r.i. BFS the automorphism α_t is an isometry and so $\alpha_f(x) \in X$ too.

Note that the above equation actually gives a bounded bilinear mapping from $L^1(G) \times X$ into X, given by $(f, x) \mapsto \alpha_f(x)$.

Definition 5.6.3. *Let Y be a set of measurable functions on (Ω, μ) and $\mathcal{L} \subseteq L^1(G)$. Define*

$$D_X(Y, \mathcal{L}) = \{\alpha_f(x) : f \in \mathcal{L},\ x \in X \cap Y\},$$

which is a subset of X. In particular, if $\mathcal{F}_0$ consists of those integrable functions on G with vanishing integral, we shall simply write D_X for $D_X(L^\infty(\Omega), \mathcal{F}_0)$.

Proposition 5.6.4. *Let (Ω, μ, G, α) be a dynamical system and (T_n) a sequence of convolution operators given by (5.6). Suppose that $(k_n * \phi)$ converges weakly in L^1 for all $\phi \in L^1(G)$ with vanishing integral.*

Then given a r.i. BFS X, the sequence $(T_n^{\#} f)$ converges a.e. for every $f \in D_X$.

Proof. We begin by describing $T_n^{\#}$ explicitly, using the construction of Remark 5.1.9. Let $f \in X$. For almost every $\omega \in \Omega$, the function $t \mapsto f(\alpha_t\omega)$ is locally integrable by Lemma 5.1.5 and so by definition of T_n we have

$$T_n f(\alpha_t\omega) = \int_G k_n(-s) f(\alpha_{s-t}\omega)\, ds.$$

Because we have assumed that k_n is bounded and has bounded support, the integral converges for any locally integrable f, and in particular for any $f \in X$. Setting $t = 0$, we obtain

$$T_n^{\#} f(\omega) = \int_G k_n(-s) f(\alpha_s \omega)\, ds. \tag{5.8}$$

We prove that for any $f \in D_X$, the sequence $(T_n^{\#}(f))$ converges a.e. By definition of D_X, there exists a $g \in L^\infty(\Omega) \cap X$ and $\psi \in L^1(G)$ with vanishing integral, such that $f = \alpha_\psi(g)$. We compute:

$$\begin{aligned} T_n^{\#} f(\omega) &= \int_G k_n(-t) f(\alpha_t \omega)\, dt \\ &= \int_G k_n(-t) \int_G g(\alpha_{t+s}\omega)\psi(s)\, ds\, dt \\ &= \int_G g(\alpha_t \omega) \int_G k_n(s-t)\psi(s)\, ds\, dt. \end{aligned}$$

As the inner integrals converge weakly in $L^1(G)$, and bearing in mind that $g \in L^\infty(\Omega)$, we have proved that $(T_n^{\#}(f))$ converges a.e. □

In the light of Propositions 5.6.2 and 5.6.4, to complete the three-step programme and prove pointwise ergodic theorems, we indicate situations where we can use the space D_X to construct dense subsets of X.

First, we must establish a lemma that will allow us to construct invariant subsets for a given dynamical system. Recall from [4, Definition 1.3.1] that a function norm on a r.i. BFS X over a measure space (Ω, μ) is *absolutely continuous* if for every nested decreasing sequence (A_n) of measurable subsets of Ω such that $\chi_{A_n} \to 0$ μ-a.e. as $n \to \infty$, and any $f \in X$, we have $\|f\chi_{A_n}\|_X \to 0$ as $n \to \infty$.

Lemma 5.6.5. *Let X be a r.i. BFS over a σ-finite measure space (Ω, μ), f be a function in X and $\lambda > 0$. If X has absolutely continuous norm then $m(f, \lambda) < \infty$.*

In particular, the conclusion holds if X is reflexive.

Proof. Suppose to the contrary that there is a $\lambda > 0$ and $f \in X$ such that $m(f, \lambda) = \infty$. Set $\Lambda = \{\omega : |f(\omega)| > \lambda\}$. As Ω is σ-finite there is an increasing sequence of finite-measure sets K_n such that $\cup K_n = \Lambda$. Set $C_n = \Lambda \backslash K_n$. Clearly $\mu(C_n) = \infty$.

Because $\lambda\chi_\Lambda \le |f|$, $\lambda\chi_\Lambda$, and hence χ_Λ itself, are in X. So are the characteristic functions χ_{C_n} for all n.

But as $C_n \downarrow \emptyset$, the hypothesis that X has absolutely continuous norm implies that

$$\lim_{n\to\infty} \|\chi_{C_n}\|_X = \lim_{n\to\infty} \|\chi_\Lambda \chi_{C_n}\|_X = 0. \tag{5.9}$$

However, it is easy to compute that $\chi^{**}_{C_n}(t) = 1$ for all $t > 0$ and so

$$\varphi_X(1) \le \sup_{t>0} \chi^{**}_{C_n}(t)\varphi_X(t) = \|\chi_{C_n}\|_{M(X)} \le \|\chi_{C_n}\|_X.$$

This contradicts (5.9) and so we conclude that our hypothesis $m(f,\lambda) = \infty$ is false.

In the case that X is reflexive, note that by [4, Corollary 1.4.4], its norm is absolutely continuous. □

Theorem 5.6.6. *In the setup of Proposition 5.6.4, suppose that X is reflexive and that $T^\#$ has weak type (X, Y). Then $(T^\#_n f)$ converges a.e. for every $f \in X$.*

Proof. By hypothesis, $T^\#$ is of weak type (X, Y). The plan of the proof is to execute steps (2) and (3) of the three step programme: identify a dense subset D of X and use Proposition 5.6.4 to show the a.e. convergence of the sequence $(T^\#_n f)$ for $f \in D$, prove that D is dense in X, and then finally invoke Proposition 5.6.2 to show that $(T^\#_n f)$ converges a.e. for all $f \in X$.

We consider the set $D = D_X + F$, where F is the subspace of all fixed points of the action α in X. We have already seen in Proposition 5.6.4 that $(T^\#_n(f))$ converges a.e. for all $f \in D_X$. Similarly, if $g \in F$, it is easy to see from (5.8) that as $t \mapsto g(\alpha_t\omega) = g(\omega)$ for almost every $\omega \in \Omega$, $T^\#_n g(\omega) = g(\omega) \displaystyle\int_G k_n(-s)\, ds$ a.e. From the assumption that the sequence $\big(\int_G k_n(s)\, ds\big)$ converges, it follows that $\big(T^\#_n g(\omega)\big)$ converges a.e. for all $g \in F$. Therefore $(T^\#_n(f))$ converges a.e. for all $f \in D$.

The next task is to demonstrate that D is dense in X. By [4, Corollary 1.4.3], the associate space X' is also the dual of X. Let $\ell \in X'$ be orthogonal to D. We must show that $\ell = 0$ μ-a.e. We may assume that ℓ is real-valued, because if ℓ is orthogonal to D, so is $\overline{\ell}$.

For any $f = \alpha_\psi(g) \in D_X \subseteq D$ we have

$$\begin{aligned} 0 = \int_\Omega \ell(\omega) f(\omega)\ d\mu(\omega) &= \int_\Omega \ell(\omega) \int_G g(\alpha_t\omega)\psi(t)\ dtd\mu(\omega) \\ &= \int_\Omega \int_G \ell(\omega) g(\alpha_t\omega)\psi(t)\ dtd\mu(\omega) \\ &= \int_\Omega \int_G \ell(\alpha_{-t}\omega) g(\omega)\psi(t)\ dtd\mu(\omega) \\ &= \int_\Omega g(\omega) \int_G \psi(t)\ell(\alpha_{-t}\omega)\ dtd\mu(\omega). \end{aligned}$$

The absolute continuity of the norm of X and [4, Theorem 1.3.11] imply that $X_b = X$. Hence $L^\infty(\Omega) \cap X$ is dense in X. By definition of D_X, g is an arbitrary element of $L^\infty(\Omega) \cap X$ and therefore

$$\int_G \psi(t)\ell(\alpha_{-t}\omega)\ dt = 0.$$

Now we use some basic ideas from spectral synthesis, as presented in [17, Section 4.6] and [41, Section 7.8]. We write

$$\mathcal{F}_0^\perp = \big\{\xi \in L^\infty(G) : \int_G \psi(t)\xi(t)\ dt = 0 \text{ for all } \psi \in \mathcal{F}_0\big\}.$$

Clearly $\mathcal{F}_0^\perp$ is a translation-invariant subspace of $L^\infty(G)$ and $\mathcal{F}_0$ is a closed ideal in $L^1(G)$. Furthermore,

$$\nu(\mathcal{F}_0) = \{\xi \in \widehat{G}\ : \widehat{f}(\xi) = 0 \text{ for all } f \in \mathcal{F}_0\} = \{0\}.$$

To see this, first note that for any $f \in L^1(G)$, $\int_G f(t)\ dt = \widehat{f}(0)$, so the fact that each $f \in \mathcal{F}_0$ has vanishing integral means that $0 \in \nu(\mathcal{F}_0)$.

Now suppose there was a $\xi \in \widehat{G}$, $\xi \neq 0$, such that $\xi \in \nu(\mathcal{F}_0)$. Then $f \in \mathcal{F}_0$ implies that $\widehat{f}(\xi) = 0$. In other words, ker $\overline{\xi} \supseteq \mathcal{F}_0 =$ker $\overline{0}$.

But as $\overline{0}$ and $\overline{\xi}$ are linear functionals, this means that ker $\overline{0} =$ker $\overline{\xi}$, and that there is a non zero $\lambda \in \mathbb{C}$ such that $\overline{0} = \lambda\overline{\xi}$.

Now take any two $f, g \in L^1(G)$ such that $\overline{0}(f), \overline{0}(g) \neq 0$. Then

$$\lambda\overline{\xi}(f * g) = \overline{0}(f * g) = \overline{0}(f)\overline{0}(g) = \lambda^2\overline{\xi}(f * g),$$

so $\lambda = 1$. Hence $\overline{0} = \overline{\xi}$ and $\xi = 0$, a contradiction.

Recall that for any translation-invariant subspace $\mathcal{M} \subseteq L^\infty(G)$, the *spectrum* $\sigma(\mathcal{M})$ is the set of all continuous characters in $\mathcal{M}$:

$$\sigma(\mathcal{M}) = \mathcal{M} \cap \widehat{G}.$$

By [17, Proposition 4.73], $\sigma(\mathcal{F}_0^\perp) = \nu(\mathcal{F}_0) = \{0\}$. By [17, Proposition 4.75 b)], $\mathcal{F}_0^\perp$ is the linear span of the constant character and so consists of the constant functions.

From this analysis we conclude that $t \mapsto \ell(\alpha_{-t}\omega) \in \mathcal{F}_0^\perp$ is a constant for a.e. $\omega \in \Omega$. We can write this fact as $\alpha_t(\ell) = \ell$ for all $t \in G$.

To complete the proof that D is dense in X, we must show that $\ell = 0$ a.e. Fix a $\lambda > 0$ and define $\Lambda = \{\omega : \ell(\omega) > \lambda\}$. By the reflexivity of $X' = X^*$ and Lemma 5.6.5, $|\Lambda| \leq m(f, \lambda) < \infty$. This set is invariant under α (because $\alpha_t(\ell) = \ell$ for all $t \in G$) and of course $\chi_\Lambda \in X$. So *a fortiori* $\chi_\Lambda \in F$. As ℓ is orthogonal to F as well, we have

$$0 = \int_\Omega \chi_\Lambda \ell \, d\mu \geq \lambda |\Lambda|,$$

which implies that $|\Lambda| = 0$. As this holds for all $\lambda > 0$, we have proved that $\ell \leq 0$ μ-a.e. But now the same argument applied to the sets $\Lambda' = \{\omega : \ell(\omega) < -\lambda\}$, where λ is an arbitrary positive number, shows that $|\Lambda'| = 0$ for all $\lambda > 0$, and so $\ell = 0$ μ-a.e.

Thus we have shown that any function in X' orthogonal to D must be the zero function, proving the density of D in X. We have already proved that $(T_n^\#(f))$ converges a.e. for all $f \in D$. Applying Proposition 5.6.2, the Theorem is proved. □

If we have more information about the action α, we can relax the conditions on the space X. In the next corollary, given additional assumptions about the group action, we shall not need the reflexivity of X.

Corollary 5.6.7. *In the setup of Theorem 5.6.6, if X has absolutely continuous norm and the fixed point space F is finite dimensional, then $(T_n^\# f)$ converges a.e. for every $f \in X$.*

In particular, the result holds if the dynamical system is ergodic.

Proof. As F is finite dimensional, we can find a finite number of mutually disjoint subsets $\{A_n\}_{n=1}^N$ such that F is spanned by χ_{A_n} for $1 \leq n \leq N$. Indeed if $f \in F$, then for every $\lambda > 0$, if we set $E_\lambda = \{\omega : |f(\omega)| > \lambda\}$, then $\chi_{E_\lambda} \in F$. As F is finite dimensional, there can only be a finite number of different sets E_λ, and so f is a linear combination of a finite number of characteristic functions, making f a simple function. Hence F is spanned by a collection of simple functions. As F is finite dimensional, such a collection must be finite.

By [4, Corollary 1.4.3], $X' = X^*$. Reasoning as in Theorem 5.6.6, we consider the set $D = D_X + F$, where F is the subspace of all fixed points of the action α in

X and let $\ell \in X'$ orthogonal to D. To prove the density of D in X, we must again show that $\ell = 0$ μ-a.e.

Firstly, as shown in the proof of Theorem 5.6.6, ℓ is invariant under the action α. Hence for any $\lambda > 0$, $\{\omega : |\ell(\omega)| > \lambda\}$ is an invariant subset of Ω. This implies that $\{\omega : |\ell(\omega)| > \lambda\} \cap A_n$ is invariant and is either empty or equal to A_n. Hence $\{\omega : |\ell(\omega)| > \lambda\}$ either equals the empty set, A_n, or A_n^c. From this we conclude that ℓ is a (possibly trivial) linear combination of the characteristic functions χ_{A_n} and $\chi_{A_n^c}$ for $1 \leq n \leq N$. Note that as $\chi_{A_n^c} \in F$, it is a linear combination of the functions $\{\chi_{A_n}\}_{i=1}^N$, so ℓ is really just a linear combination of the functions χ_{A_n}.

As ℓ is orthogonal to F, this implies that $\ell = 0$ a.e. Hence D is dense in X. Applying Proposition 5.6.2, the Corollary is proved.

It is easy to prove that the action of G on (Ω, μ) is ergodic if and only if the only invariant function is the characteristic function χ_Ω. In this case, F has dimension one if Ω has finite measure, for then $\chi_\Omega \in F$, and $F = \{0\}$ if Ω has infinite measure. This is because if there was a non-zero $f \in F$, then as we have seen above, $\{\omega : |f(\omega)| > \lambda\}$ is an invariant subset of Ω for each $\lambda > 0$, contradicting the ergodicity of the action. □

Finally, let us mention another way to obtain a dense subset of a r.i. BFSs on which the pointwise convergence of the ergodic averages can readily be checked.

Corollary 5.6.8. *In the setup of Theorem 5.6.6, if $T^\#$ is also of weak type (E, Z) for r.i. BFSs E and Z where E is reflexive and $X \cap E$ is dense in X, then $(T_n^\# f)$ converges a.e. for every $f \in X$.*

Proof. By Theorem 5.6.6, $(T_n^\# f)$ converges a.e. for all $f \in Y$. Hence we have a dense subset of X, namely $X \cap Y$, on which the ergodic averages converge pointwise. Applying Proposition 5.6.2 finishes the proof. □

If for example $X = L^1(\Omega)$ and $Y = Z = L^p(\Omega)$ for some $1 < p < \infty$, this Corollary is applicable. A special case of this setup in given in [8, Theorem 3].

5.7 Notes and Remarks

Proposition 5.6.1 is due to Prof L. Labuschagne. He also corrected an earlier statement in Proposition 3.4.1 and is also responsible for the current proof of part (3) of Theorem 3.4.2, which is a lot shorter than the one originally offered. He and an external examiner spotted an error in an earlier draft of Section 5.5 and was instrumental in finding a fix to the problem. These are just a few instances of his influence which can be felt on every page of this work.

With three exceptions, all the results proved in this thesis are original (to the author's knowledge). One exception is the duly attributed first part of Lemma 3.3.6. Nevertheless, it has a novel proof. The others are Lemmas 2.2.2 and 3.1.1, which we included for completeness: many quote the results, though without proof.

Let us make a few remarks on the development of the ideas used in the pointwise ergodic theory.

The author's original intention was to follow Calderón's lead in [8] and prove his Theorem 2 for more general r.i. BFSs and locally compact abelian groups. Following Calderón's proof strategy closely, the author could only prove Corollary 5.5.2, in a restricted sense: the measure space (Ω, μ) had to have finite measure and the fundamental function had to be submultiplicative (i.e. essentially satisfy the Δ' condition. The breakthrough occurred in two stages: first was the generalisation of Kolmogorov's inequality 5.4.1, from its classical statement in [12]. A different proof strategy also had to be found.

The power of Kolmogorov's inequality is that it allows the determination of the weak type of a operator to be related to an integral over a finitely measurable subset - a localisation if you will. However, one must then estimate the integral of a function whose value is the norm of a cross section. This is why Proposition 3.4.1 was developed. Note that it is to apply this result that the most stringent condition on (Ω, μ) has to be imposed: that it be a countably generated measure space.

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