Performance evaluation using Data Envelopment Analysis (DEA)

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Executive Summary

The study proposes the use of Data Envelopment Analysis (DEA) to analyse the performance of Decision Making Units (DMUs). DEA models are optimisation models, where you have an objective function to minimise or maximise based on data. Efficiency in DEA heavily relies on this concept. However, the optimal value is a single value which changes as the sample data changes. In other words, DMUs may be deemed efficient under one data set and inefficient under another. Due to the sample data nature used in statistics and DEA is a data driven method, the question is: How does one conclude whether a particular DMU is or is not efficient? This study attempts to answer this question by creating confidence intervals for such DMUs using the non-parametric bootstrap resampling method.

The core concept of bootstrapping involves using resampled data from an original sample to mimic the process of making inferences about a larger population. Since the actual population remains unknown, determining the accuracy of a sample statistic compared to its true population value is challenging. However, bootstrapping resolves this issue by treating the sample as a proxy for the population, allowing for measurable assessments of inference quality when applying the resampled data to the original sample. More formally, the bootstrap works by treating inference of the true probability distribution, given the original data, as being analogous to an inference of the empirical distribution, given the resampled data. The accuracy of inferences regarding the true probability distribution using the resampled data can be assessed because we know its estimate. If the estimate is a reasonable approximation to the true probability distribution, then the quality of inference on the true probability distribution can in turn be inferred.

This study explores the concept of efficiency and its evolution within the framework of Data Envelopment Analysis (DEA), motivated by the need for objective performance evaluation across various domains. A comprehensive theoretical foundation is laid out, detailing key DEA models and mathematical extensions, including those that accommodate ratio data which are critical for real-world applications. The study also incorporates the non-parametric bootstrap resampling method to enhance the robustness of the efficiency analysis. The models considered include the Banker, Charnes, and Cooper (BCC) model, the Slacks-Based Measure (SBM) model, and the Additive (ADD) model.

The DEA methodology is applied to two primary domains: professional football and finance. In the context of football, the study evaluates the performance of players during the 2020/2021 season across 18 top leagues and competitions worldwide. It critiques the current FIFA awards selection process, which heavily relies on subjective expert voting and often favours players from European teams or those in attacking positions. DEA is proposed as a more objective alternative, focusing on individual player performance based on 20 selected variables such as minutes played, assists, penalty goals, and clean sheets. The DEA results aligned with the FIFA rankings for 4 of the 11 nominees: Robert Lewandowski, Lionel Messi, Mohammed Salah, and Karim Benzema, whilst offering contrasting evaluations for others like Cristiano Ronaldo and Neymar. Ultimately, the models supported Lewandowski's win but highlighted inconsistencies in the current selection system.

In the financial domain, DEA models are employed to evaluate the performance of companies listed on the Johannesburg Stock Exchange (JSE) Top 40. The goal is to identify efficient companies based on fundamental performance metrics, such as dividends per share, current ratio, and quick ratio, rather than stock price behaviour. This approach is especially beneficial for investors prioritising dividend income. The results showed that both classical and bootstrap versions of the SBM and BCC models were consistent in selecting efficient companies, while the bootstrap version of the Additive model was overly conservative, labeling all companies as inefficient.

Building on the financial analysis, the study applies portfolio theory to the DEA identified efficient companies. Optimal portfolios are constructed and tested using out-of-sample data, with their performance compared to that of the JSE Top 40 index. While the index outperformed most portfolios, the medium-risk portfolio derived from the classical Additive model exceeded the index in terms of profitability. This portfolio included stocks such as Amplats, Anglogold, Bats, Capitec, Richemont, Firstrand, and Pepkorh.

Overall, the research demonstrates the value of DEA as a rigorous, data-driven tool for evaluating performance in both sports and financial markets, offering a more objective foundation for decision-making and recognition.

Keywords: Data envelopment analysis, decision making units, efficiency, non-parametric bootstrapping method, optimisation.

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Chapter 1

Introduction

1.1 Background

In 1957, Michael J. Farrell published an article in the prestigious journal of the Royal Statistical Society, titled: "The measurement of productive efficiency"(Farrell, 1957), which has become a seminal contribution to the field of efficiency measures and productivity. (Farrell, 1957) postulated that any production system could be described as a way of transforming inputs (resources) into outputs (products), and the productive efficiency of the system could be measured by the ratio of the weighted sum of the outputs to the weighted sum of the inputs. The weighting factors would be considered acceptable only if this ratio were:

(i) Non-negative.

(ii) Less than or equal to 1.

Systems with a productivity score of one would be considered efficient and serve as benchmarks for the other systems. The 20 years that followed the publication of Farrell's paper has been devoted to the discussion of the problem of the "fair" choice of these weights. This is a problem because given say n systems or firms that consume m inputs to produce s outputs. Suppose that the analyst is interested in knowing which firms have been relatively more efficient than others using the weighted ratio explained above. It should be noted that these weights reflect the relative importance associated with each input and output in evaluating a firm's productive efficiency. Suppose that the systems or firms of interest are automotive companies, such as Volkswagen, Ford, Mercedes Benz, etc. These companies have their own unique appeal to customers in terms of the types of cars that they offer. For example BMW is known for its high performance vehicles, whereas Mercedes Benz is known for its comfortable and a solid feel type of vehicles. It is not unreasonable then to assume that these companies would assign relatively larger weights to these attributes or outputs when evaluating their efficiency, this comes from the fact that these are the outputs that make them stand out against each other or other competitors in the industry such as Ford, Mazda, Hyundai, etc. The problem becomes clear since, for example BMW can choose an arbitrarily large value for the weight associated with a certain output such as fuel consumption rate without bounds or justification, apart from the fact that fuel consumption rate is one of the attributes that make their cars stand out from the competition. Putting it in another way, BMW may choose an arbitrary small weight associated with an input such as the cost of the steel used to manufacture the engine, because they spend less compared to the competition or to to maximise the aforementioned ratio without limits. This example illustrates the fact that a firm may select these weights in a manner that is biased and unfair to the other firms, so as to make itself more efficient.

The question then is the following: to an impartial outside observer or analyst with no bias, what would be a "fair" choice of these weights for each input and output variable in order to be able to compute the efficiencies and thus be able to compare the companies? Charnes et al. (1978) attempted to answer this question. What they proposed, as often happens in articles that become classics, was the break of a paradigm. For the first time, a model was conceived to use Farrell's formulation in such a way that different entities were allowed to assign different weights to inputs and outputs in a "fair" manner.

Thus different entities would be allowed to assign weights that best represented their own values and goals, and to avoid unfair manipulation of the weights to make any entity efficient, Charnes et al. (1978) had the clever idea of imposing a constraint to their model. The ideal choice of weights to a given entity would be the one that would maximise its own efficiency under the constraint, such that if the same weights were assigned to each one of all other entities present in the analysis (i.e., all entities whose efficiencies were to be measured and compared), none of them would have Farrell's ratio greater than one. A set of weights that would lead to an efficiency value greater than one would be called infeasible. This constraint is logical given that a set of weights that would allow for an entity to have an efficiency value greater than one would contradict Farrell's definition.

Charnes et al. (1978) proposed the problem of choosing the set of weights for each entity as an optimisation problem. Each entity of the list of entities under analysis would have their ideal set of weights determined according to the above logic. The set of weights assigned to each entity would be the one which maximised its efficiency, bringing it as close to one as possible. Charnes et al. (1978) also provided a geometrical interpretation of the model's solution, showing that if the entities had the values of their inputs and outputs represented on a graph, then efficient entities would be located on the border (frontier) of the plotted points. If these border points were connected by line segments or planes, there would be a frontier that would envelop all other entities under analysis. The region located within this frontier was called the Production Possibility Set (PPS) and it designated the space in which that kind of production was possible.

The search for optimal weights corresponded to the pursuit of that border, frontier or envelope, which is the reason why such models would later be called Data Envelopment

Analysis models (Milion and Alves, 2012). Data Envelopment Analysis (DEA) is a mathematical programming technique that has found a number of practical applications for measuring the performance of similar units such as a set of hospitals, schools, banks, etc. According to Paradi et al. (2018), DEA represents a type of frontier method, which focuses on identifying and evaluating the highest-performing or most efficient entities within a given sample those that lie on the "frontier" of performance. In DEA, this frontier reflects the best observed performance among the decision-making units being assessed, but it does not claim to represent the absolute or theoretical optimum. Frontier methods differ from regression techniques, which aim to model the average behaviour within a dataset.

These frontier approaches fall into two categories: parametric and non-parametric. Parametric methods involve defining a specific functional form for the frontier, which may or may not incorporate random noise, while non-parametric methods make no assumptions about the shape of the frontier beforehand. DEA is an example of a non-parametric approach that makes a single simple assumption of piecewise linear connections of units on the frontier. It should be noted that Farrell's definition of efficiency defined above was limited to production systems such as manufacturing companies where a distinction between inputs (resources) and outputs (products) is clear. However DEA has been used outside of this class of systems. With other types of systems or entities, it is not always clear which variables are inputs and which variables are outputs. In these cases the convention is that inputs are the variables whose increase would result in a system being less efficient and outputs are variables whose increase would result in a system being more efficient. Stated another way, inputs are variables whose increase would make the weighted ratio smaller and whose decrease would make the ratio larger. Outputs are variables whose increase would make the ratio larger and whose decrease would make the ratio smaller. It should be noted that efficiency is a measure of performance of a system, there maybe other measures of performance but the focus of this study is on the efficiency as defined in the DEA literature.

As an efficiency measurement and evaluation methodology, DEA is particularly useful in cases where sample units termed Decision-Making Units (DMUs) use multiple inputs and outputs and are operating under comparable conditions (Paradi et al., 2018). DEA primarily measures technical efficiency, i.e. focusing on levels of inputs relative to outputs, as opposed to economic efficiency which would also consider market prices. The use of levels of inputs and outputs is another powerful characteristic of DEA in that it can incorporate inputs and outputs in the natural units in which they are measured and does not require them to be converted to the same units of measure. DEA permits the evaluated DMUs to appear to be as good as possible, a feature that can be deemed as providing a fair evaluation of the DMUs. This characteristic stems from the optimisation underpinning of DEA where DEA assigns the highest efficiency rating to each DMU compared with the set of DMUs being analysed. It essentially gives the "benefit of the doubt" to each unit. Another advantage of DEA is that it suggests explicit improvement targets for inefficient DMUs namely the benchmark or point on the frontier to which it is being compared in order to measure its efficiency (Paradi et al., 2018).

Furthermore this frontier point will be defined as the linear combination of one or more actual DMUs that are efficient (i.e. on the efficiency frontier). The inefficient DMU is presented with a relevant set of efficient DMUs, called its reference set (sometimes referred to as the efficient reference set). The reference set represents the specific efficient DMUs against which the inefficient DMU is judged to be inefficient, and changes to improve the inefficient DMU can be most directly determined by analysing differences between the inefficient DMU and its reference set. Identifying the amount of excess inputs consumed or potential increase in outputs possible in inefficient units compared to the DMUs in the efficient reference set may be the most powerful and useful feature of DEA. This perspective offered by DEA is unique, in that it is not provided by any other method (Paradi et al., 2018).

It is clear that DEA is an extremely powerful analytic and management tool. It is believed that it has been underutilised and hopefully this study will open the path to greater utilisation. Since DEA was first introduced in 1978 in its present form, researchers in a number of fields have quickly recognized that it is an excellent and easily used methodology for modelling operational processes for performance evaluations. Such evaluations take a variety of forms in inputs and outputs. Examples found in include: number of yellow cards obtained per match, goals scored per match, number of assists per match, etc. In cricket you have batting strike rate, bowling average, bowling strike rate, etc. In minimising you have the discharge rates, recovery rates of patients, and so on. These measures are stated in the form of a ratio $\frac{output}{input}$. This is a widely used way to assess efficiency. Productivity is typically measured as a ratio, especially when evaluating the performance of workers or employees. Common examples include "output per worker hour" or "output per employee", where output such as sales or profit is placed in the numerator. These types of metrics are often called partial productivity measures because they focus on only one input at a time. In contrast, total factor productivity measures aim to evaluate performance by considering all inputs and outputs together in a single ratio. Shifting from partial to total productivity helps prevent mistakenly crediting improvements to one input when they may actually result from changes in another (Cooper et al., 2006).

1.2 Problem statement

The traditional DEA models employ inputs and outputs as the data to be used when estimating the efficiency score. It is clear then that the efficiency score is dependent on the given data set. In other words, a DMU may be deemed to be efficient under one set of data and inefficient under another set for the same input and output variables. This implies that the efficiency score is a random variable and the study is interested in estimating the sampling distribution of this statistic. Landete et al. (2017) proposed a methodology where the inputs and outputs were treated as random variables. However their method is based on the argument that, in any DEA study, the inputs and outputs to be selected for the study are uncertain and should be modelled by a Bernoulli random variable, with each variable taking on the value 1 with probability p and the value 0 with probability $1 - p$.

Olesen (2002) incorporates randomness in DEA differently, by focusing on the restriction of the optimal weights or multipliers. Traditional DEA models tend to assign a weight of zero to certain inputs and outputs which creates a problem since this suggests that the associated variables are not significant when assessing the performance of a particular DMU. To solve this issue, Olesen (2002) suggested the use of confidence intervals for the output multipliers. These confidence intervals are also known as the probabilistic assurance regions in the output space. This study differs from the work done by Landete et al. (2017) in the sense that the focus is not on the uncertainty of inputs and outputs, these are chosen based on experience and are deemed necessary for the performance evaluation of DMUs. It also differs from the work of Olesen (2002) in the sense that the confidence intervals suggested by this study refer to the objective function of the DEA model. The randomness aspect comes from the values that the input and output variables take on, since the objective function depends on these variables, it is a random variable as well. This is the main focus of this study.

The motivation behind this study is two fold, firstly it is to investigate the discriminatory power between the traditional DEA model and the non-parametric bootstrap DEA model that is proposed. Additionally, the efficiency score is unknown beforehand unlike the input and output data sets, it needs to be estimated by solving the optimisation problem. The problem faced by traditional DEA models is that they produce a single value as a measure of efficiency, this study proposes that it is much better to construct a confidence interval for the efficiency score. This suggests estimating the sampling distribution of the computed efficiency scores and constructing the confidence interval of the efficiency scores. The confidence interval approach is more reliable and easy to interpret than a point estimate. The approach proposed will be illustrated with applications in finance and football.

1.3 Aim and objectives of the study

1.3.1 Aim

The aim of the study is to apply the non-parametric bootstrap technique to DEA models and evaluate the performance of DMUs, specifically these models will be applied to finance and football data.

1.3.2 Objectives

The objectives of this research study are:

- To investigate the bootstrap resampling technique approach to DEA modelling, and compare the obtained results with those of a traditional DEA model.
- To apply the bootstrap based DEA and traditional DEA approaches for assessing the performance of football players with the purpose of using the approach as an alternative way of selecting best performing players for the FIFA awards, and as a team selection method.
- To apply the bootstrap based DEA and traditional DEA approaches to measure the overall performance of JSE listed companies, using key indicators of performance with the purpose of helping the potential investor in making sound investment decisions.

1.4 Research contributions

1.4.1 Simulation based DEA modelling

The objective is to perform random sampling from the empirical distribution, and using the generated data to compute the sample estimates to be used as new inputs and outputs. This study will employ the statistical technique that uses resampling with replacement from the original data to create bootstrap samples, allowing for the estimation of standard errors, confidence intervals, and hypothesis testing without making assumptions about the underlying data distribution.

Traditional DEA models face the limitation of providing only a single efficiency score. This study aims to address that limitation by constructing a confidence interval for the efficiency score. Using confidence intervals offers a more robust, reliable, and interpretable alternative to relying solely on a point estimate.

1.4.2 Evaluating the performance of soccer players competing in major football leagues.

FIFA is responsible for a variety of activities in football including the annual Best FIFA Football awards, which recognise both individual and team achievements in international association football. In particular the awards of public interest are:

- The Best FIFA Men's Player
- The Best FIFA Women's Player
- The Best FIFA Men's Coach
- The Best FIFA Women's Coach
- The Best FIFA Men's Goalkeeper
- The Best FIFA Women's Goalkeeper
- FIFA Fifpro Men's World 11

The awards are bestowed on the basis of on pitch performance and general conduct on and off the pitch for players, and on the basis of the on pitch performance and general behaviour of their teams on and off the pitch for coaches. Nominations for the awards are compiled by FIFA in collaboration with football stakeholders. The nominations as well as the submitted information supporting such nominations is reviewed by a panel of FIFA football experts. A panel of experts in men's football and a panel of experts in women's football compile a shortlist of nominees for the respective categories, which are published on FIFA.com shortly thereafter in order to enable a public vote among fans registered on FIFA.com. The winners of The Best FIFA Men/Women's Player, The Best FIFA Men/Women's Coach and The Best FIFA Men/Women's Goalkeeper awards are selected by an international jury comprising of the current coaches of all men/women's national teams (one per team), the current captains of all men/women's national teams (one per team), one specialist journalist from each territory represented by a national team and fans from all over the world registered on FIFA.com (FIFA Website, 2021).

These selection criteria from the nominees to the eventual winners is decided upon by the so called football "experts" mentioned above. There is a lot of subjectivity and bias involved in this process since the deciding jury tends to vote for their favourite players or fellow countrymen, irrespective of whether they performed well or not in that particular season. These awards consider both on and off field performance, the former is easier to determine based on historic data but even in this case, it is not always a straightforward decision. For example the 2010-2011 season where the top 3 finalists for the Best FIFA Men's player came from the same Barcelona team, namely Lionel Messi, Andres Iniesta and Xavi Hernandez. Messi emerged victorious not without controversy as many felt that Iniesta should have won due to his contribution in Spain's successful world cup campaign and Barcelona's domestic success.

It is also evident that this award in particular tends to favour forwards as opposed to defenders and midfielders, which is supported by the fact that only 3 defenders have won it in the history of the award. It is true that off field performance of players should be factored in the selection criteria since these players are role models whose bad behaviour off the field sends a wrong message to those who look up to them. It is highly likely that these criteria is the reason why Luis Suarez never made the 23 men shortlist in the 2010-2011 season. It is difficult to quantify off field performance and decide which act or deed deserves more weight, for example how does a panellist or a fan decide which one weighs more between Messi's involvement in a tax evasion scandal and his charity work? And to what extent does bad behaviour off the field lead to a loss of a nomination? It is for these reasons that this study will only be limited to on-field analysis of player performance, which is arguably more important.

However, this study offers an alternative method for ranking player performance which is based on available data. Another issue with these awards is that they are awarded to individuals in a team sport and the success or failure of a club often determines whether a player receives the award or not which should not be the case. The argument used to support this tendency is that a successful team is a result of the contribution from the players. Even though this is true the question is how do you measure a players contribution to the team? By the number of goals scored? Assists made? Or number of tackles made? But these are just individual player measures. To make the point clear, if for example Lionel Messi and Eden Hazard won 3 big titles with their respective teams (club or country) but Hazard does better than Messi in goals scored and assists made. Does this mean Hazard's contribution to the team is better than Messi's ? It is tempting to say yes but if one considers the potency and talent Messi has compared to his peers and how opposition teams tend to use 2 to 3 defenders to mark him, which results in fewer scoring and assist opportunities and high likelihood of injury.

It is not a fair statement to say Hazard has contributed more to the team as there are other factors that need to be taken into account. Hence a quantitative analysis that takes into account all these factors and gives an objective or fair assessment of performance on and off the field is required. Data Envelopment Analysis is a suitable tool for this as it will allow midfielders and defenders who have been historically disadvantaged by the voting system a fair chance. This is true not just for the Best FIFA Men's award but for all of the above mentioned awards. Since the selection criteria for these awards forms part of FIFA's business model, this study proposes a methodology to challenge this particular aspect of the business model.

1.4.3 Evaluating the performance of JSE listed companies.

The JSE Limited is the largest stock exchange in Africa. It is located in Sandton, Johannesburg, South Africa, after it moved from downtown Johannesburg in 2000. In 2003 the JSE had an estimated 473 listed companies and a market capitalisation of US \$182.6 billion, as well as an average monthly traded value of US \$6.399 billion. As of March 2022, the market capitalisation of the JSE was at US \$1.36 trillion.

The JSE provides a market where securities can be traded freely under a regulated procedure. It does not only channels funds into the economy, but also provides investors with returns on investments in the form of dividends. The exchange successfully fulfils its main function; the raising of primary capital by rechannelling cash resources into productive economic activity, thus building the economy while enhancing job opportunities and wealth creation.

Potential investors are looking for companies that are performing well in general. A popular measure of company performance is the share price as determined by the market. However, the share price alone does not give a whole picture of how good or poorly a company is performing. This is largely due to the fact the share price is primarily determined by the supply and demand of market participants. In other words, the share's "intrinsic" value is usually different from that determined by the market. This can be misleading to investors as to how well or badly the company is doing. This is not surprising since the stock market is a speculative game that is suited for speculators who do not care about the company's fundamentals, but only the short term results based on the forces that drive the market share price. How then does an investor decide whether a particular company is a good choice for investing or not without relying on the share price? Well, the investor can hire a financial expert such as an Accountant or Auditor to analyse the financial statements of the company in question. But the investor is typically interested in multiple companies in various industries and so he might have to pay the Accountant a fortune for analysing all such companies. The number of JSE listed companies alone is over 400.

Another approach which is more holistic is hiring a DEA expert, to analyse a group of companies simultaneously, comparing their productivity and gaining insight into how operations can be improved for less productive companies. The DEA approach is useful for a long term investor who is interested in how the company is performing (relative to others in the same sector) from an operations point of view, and not merely what the market indicates. The results of DEA analysis are even more important for dividend paying companies since dividends are a true indicator of company performance compared to the share price. The DEA approach attractively ranks companies by assigning each an efficiency score. This is useful when wanting to construct an investment portfolio as it will give the investor an idea of where to put the most money relative to how well a company performs as indicated by the efficiency score.

It should be noted that DEA analysis for measuring company performance should be used as a tool in conjunction with other methods of portfolio selection and risk management in finance and not in isolation as it does not account for risk associated with stock market volatility. However it does have the advantage of using multiple input and output variables which are important in evaluating company performance, and offers insight as to how a company's performance can be improved relative to the "best" performing companies. It is evident that such information is valuable to both potential investors and the company itself. This insight is unique to the DEA methodology. These characteristics are in contrast to the traditional mean-risk models of portfolio selection, which only consider two variables, i.e. return and risk (JSE Website, 2022).

1.5 Significance of the study

The goal of performance evaluation is to improve the operations of DMUs with the objective of increasing efficiency. Performance evaluation is a complicated process due to the existence of multiple attributes of DMUs (performers). This study proposes that DEA is a suitable tool for handling such problems. Additionally randomness is inherent in DEA data and this study captures this by using statistical models within the DEA framework. The criteria used in awarding the best performing players in football is often biased and unfair since the voting panel consists of coaches, former players, referees, and football journalist who tend to select or vote for their favourite players regardless of deserving performance. These awards tend to favour forwards and midfielders since they bring more entertainment to the sport. Defenders and goalkeepers tend to be overlooked. DEA has the ability to rank the players based on actual on-field performance which will give a fair assessment for all players. This rationale extends to any field where the performers are judged against each other, hence the analysis will be extended to JSE listed companies to construct portfolios for potential investors.

1.6 Ethical considerations

The study will be submitted to the North-West University Ethics Committee. The study will not make use of surveys which usually involve personal information that may be subjected to the Protection of Personal Information Act (POPIA).

- Instead, secondary data that is accessible will be used.
- The data will consist of soccer player statistics which are available at: https://footystats.org/
- The JSE listed companies' data such as earnings per share, yearly turnover, total sales, etc, which is information that is available on the IRESS website: https://login.iress.co.za/. Hence, the researchers are of the view that the entities of interest will not be exposed to any harm or violation of confidentiality, and the study is deemed a no risk study.

1.7 Study outline

- Chapter 1: This Chapter provides an introduction, background and motivation behind the formulation of the Data Envelopment Analysis methodology. The problem that the study attempts to solve within DEA is stated and the intended areas of application of DEA are also stated.
- Chapter 2: This Chapter explores the literature of traditional non-parametric DEA models and the non-parametric bootstrap method.
- Chapters 3: This Chapter provides analysis and results of the models discussed in Chapter 2, incorporating the bootstrap technique within the DEA framework. This analysis will be based on data from football.
- Chapter 4: This Chapter provides analysis and results of the models discussed in Chapter 2, incorporating the bootstrap technique within the DEA framework. This analysis will be based on data from finance.
- Chapter 5: This Chapter is an extension of Chapter 4. The objective is to select portfolios (from DEA efficient companies) based on the mean-risk criterion with the objective of investing.
- Chapter 6: This Chapter provides conclusions and suggestions for future research.
- Appendix: This Chapter gives an overview of mathematical optimisation which is the foundation for the DEA methodology.

Chapter 2

Data Envelopment Analysis

This Chapter discusses DEA theoretical concepts and mathematical models that are relevant to the aims and objectives outlined in Chapter 1. This Chapter begins by exploring the basic concepts of DEA. It goes on to explore basic DEA models along with the extensions of these models that accommodate ratio data. This is important since data in real life comes in ratio form. The Chapter ends by exploring the non-parametric bootstrap resampling method which is important to the study as explained in Chapter 1.

2.1 Basics of DEA concepts

2.1.1 Mathematical formulation of DEA

Data Envelopment Analysis (DEA) utilises mathematical programming techniques which can handle a large number of variables and constraints. This relaxes the requirements that are often encountered when one is limited to choosing only a few inputs and outputs because the techniques employed will otherwise encounter difficulties (Ramanathan, 2006).

Let x_i and y_r represent particular inputs and outputs, respectively. Thus x_i represent the i^{th} input and y_r represents the r^{th} output of Decision Making Units (DMUs). Let the total number of inputs and outputs be represented by m and s respectively where $m, s > 0$. In DEA, multiple inputs and outputs are linearly aggregated using weights. Thus, the virtual input of a DMU is obtained as the linear weighted sum of all its inputs:

$$\text{virtual input} = \sum_{i=1}^{m} v_i x_i \ , \tag{2.1}$$

where v_i is the weight assigned to input x_i during the aggregation. Similarly, the virtual output of a DMU is obtained as the linear weighted sum of all its outputs:

$$\text{virtual output} = \sum_{r=1}^{s} u_r y_r \ , \tag{2.2}$$

where u_r is the weight assigned to output y_r during the aggregation. Given these virtual inputs and outputs, the efficiency of the DMU in converting the inputs to outputs can be defined as the ratio of outputs to inputs:

$$\text{technical efficiency} = \frac{\sum_{r=1}^{s} u_r y_r}{\sum_{i=1}^{m} v_i x_i} \tag{2.3}$$

The efficiency score (2.3) is sought to be maximised with an upper limit of 1. Those DMUs which attain an efficiency score of 1 would be deemed efficient and would lie on the boundary of the efficiency frontier and act as benchmarks for other DMUs (Ramanathan, 2006).

2.1.2 Efficiency frontier

It should be noted that Farrel's concept of efficiency initially dealt with the productivity of labour in agriculture. However this study deals primarily with technical efficiency. Technical efficiency means the ability of a firm or unit to obtain maximum output from a given set of inputs. Allocative efficiency refers to the ability of a firm or unit to use inputs in optimal proportions, given their respective prices (Farrell, 1957). The efficiency frontier represents a standard of performance that the firms not on the frontier should try to achieve. This efficiency frontier forms the basis of efficiency measurement. The efficiency frontier envelops the available data. Hence, the term Data Envelopment Analysis. Figure 2.1 shows an example of a DEA efficiency frontier for 2 inputs and 1 output with 6 DMUs, A to F. Points A through F represent combinations of inputs 1 and 2 that result in 1 unit of output being produced. These points constitute what in DEA is known as the production possibility set (PPS), a set of all possible combinations of inputs and outputs. This concept is explained further later in the Chapter.

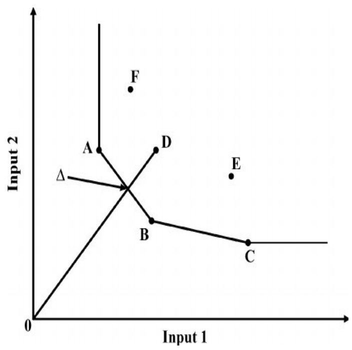


Figure 2.1: DEA efficiency frontier.

In the DEA literature, DMUs A, B, and C are called efficient while DMUs E, F, and D are inefficient, DMUs A, B, and C lie on the efficiency frontier, and hence are the most efficient. It should be noted that this does not mean that their performance cannot be improved. It may or may not be possible. The given data does not give any idea regarding the extent to which their performance can be improved. These are the best DMUs with regards to the given data. As no other DMUs shows better performance, it should be assumed that their performance is the best achievable. The performance of all other DMUs is rated in relation to this best achievable performance. Thus, relative efficiencies are considered, not absolute efficiencies. It should be noted that Δ represents an efficient input combination, a concept that will be explained in detail in the following sections (Ramanathan, 2006)

2.1.3 Inputs and outputs

Suppose there are n DMU_s: DMU_1, DMU_2,..., DMU_n. According to Cooper et al. (2006), DMUs are selected as follows:

(i) Numerical data are available for each input and output for all DMU_s.

(ii) The items (inputs, outputs and choice of DMUs) should reflect an analyst's or a manager's interest in the components that will enter into the relative efficiency evaluations of the DMUs.

(iii) In principle, smaller input amounts are preferable and larger output amounts are preferable so the efficiency scores should reflect these principles.

(iv) The measurement units of the different inputs and outputs need not be congruent. Some may involve number of persons, or areas of floor space, money expended, etc.

Now m input items and s output items are selected with the properties noted in (i) and (ii). Let the input and output data for DMU_j be $(x_{1j}, x_{2j}, ..., x_{mj})$ and $(y_{1j}, y_{2j}, ..., y_{sj})$ respectively. The input data matrix X and the output data matrix Y can be arranged as follows:

$$\text{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \tag{2.4}$$

$$\text{Y} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{s1} & y_{s2} & \cdots & y_{sn} \end{pmatrix}, \tag{2.5}$$

where X is an $(m\text{x}n)$ input matrix and Y is an $(s\text{x}n)$ output matrix. The choice of which inputs and outputs to include and the form of these measures should be an easy one at first glance. But upon examination, the problem is more complex. In a practical operation, what is used up is an input while what is produced is an output. However, there are instances where the variable is undesirable but forms part of the efficiency or productivity evaluation process, for example pollution or bad loans. Some inputs can also be outputs, such as medical students providing hospital patient care while simultaneously receiving a medical education through their work in the hospital. Paradi et al. (2018) and Cook and Zhu (2006) give a non exhaustive list of how to decide if certain variables should be treated as inputs or outputs:

(i) When selecting variables, rather than thinking about what is an input or output, examine each variable and decide if minimising or maximising it would result in a DMU appearing to be more efficient, productive or whatever the model is designed to show. The goal is to be able to give good advice to the DMU manager on how to improve operations.

(ii) The measure that has characteristics as an input and an output can be split into two and create a positive and negative variable where the DMUs can record their bad loans as something to minimise while the good loans are maximised.

(iii) A variable that has characteristics as an input and an output can be split into two and create a positive and negative variable where the DMUs can record their bad loans as something to minimise while the good loans are maximised.

(iv) Another approach is to net the two and use that as an output, assuming of course the amount or value of good loans is larger than the bad ones. But this is less acceptable as the size of the bad loans can be buried and be "invisible" to the managers. On the other hand, netting these two variables may be advantageous if the analyst is attempting to show the growth of the variable in spite of losses that are also there.

(v) In many cases two or more variables are highly correlated such as salaries and hours worked. A manager might insist on including both highly correlated variables, even if one is redundant, to ensure the model accurately reflects the production process from their perspective

(vi) Variables in a ratio form or percentages violate a strict interpretation of convexity in the production possibility set (that is. that it should be possible to produce all linear combinations of actual DMUs consistent with the return to scale (RTS) assumption). Mixing ratios and absolute input or output amounts can generate results that mathematically satisfy the model but are not meaningful in practice.

The next sections focus on describing the basic CCR, BCC, ADD, and SBM models. Input and output orientation for the CCR and BCC models are discussed along with the primal and dual versions of these. The study also extends these models to accommodate ratio data. The bootstrap resampling technique is also discussed.

2.2 CCR model

2.2.1 CCR input oriented model

Charnes et al. (1978) introduced the basic DEA model known as the CCR (Charnes, Cooper, and Rhodes) model. In this model the ratio of outputs to inputs is used to measure the relative efficiency of $\text{DMU}_j = \text{DMU}_o$ to be evaluated relative to the ratios of all of the other DMUs. The CCR construction can be interpreted as the reduction of the multiple-output/multiple-input situation (for each DMU) to that of a single "virtual" output and "virtual" input. For a particular DMU the ratio of this single virtual output to single virtual input provides a measure of efficiency that is a function of the multipliers. In the mathematical programming terminology this ratio which is to be maximised, forms the objective function for the particular DMU being evaluated. A set of normalising constraints (one for each DMU) reflects the condition that the virtual output to virtual input ratio of every DMU, including that of $\text{DMU}_j = \text{DMU}_o$ must be less than or equal to unity. Given the following variables:

- n = number of DMUs
- s = number of outputs
- m = number of inputs
- u_r = output multiplier variable
- v_i = input multiplier variable

The mathematical programming problem may thus be stated as (Cooper et al. (2011)) :

$$
\begin{aligned}
\max\ \text{w} = \quad & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \quad \text{over } u_r, v_i \in \mathbb{R} \\
\text{subject to} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \quad \text{for } j = 1, 2, ..., n. \\
& u_r, v_i \geq 0\ \forall\, i,\ r.
\end{aligned}
\tag{2.6}
$$

This ratio form generalises the engineering science definition of efficiency from a single output to a single input and does so without requiring the use of a priori chosen set of weights. The above ratio form yields an infinite number of solutions. If (u^*, v^*) is an optimal solution, then $(\alpha u^*, \alpha v^*)$ is also an optimal solution for any scalar $\alpha > 0$.

However, the transformation developed by Charnes and Cooper (1962) for linear fractional programming selects a solution (u, v) where $\sum_{i=1}^{m} v_i x_{io} = 1$ and yields the equivalent primal (2.7) linear programming problem and its dual (2.8) in which the change of variables from (u, v) to (μ, v) is a result of the "Charnes-Cooper" transformation:

$$\begin{aligned} \max\ \ z &= \sum_{r=1}^{s} \mu_r y_{ro} \\ \text{subject to } & \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0,\ j = 1, 2, ..., n \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \mu_r, v_i \geq 0\ \forall\, i,\ r. \end{aligned} \tag{2.7}$$

$$\begin{aligned} \min z &= \theta \\ \text{subject to } & \theta x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0, \quad \forall\, i \\ & \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro},\ \forall\, r \\ & \lambda_j \geq 0,\ \forall\, j. \\ & \theta \quad \text{unrestricted.} \end{aligned} \tag{2.8}$$

Due to the non-zero (semi-positive) assumption for the data, the second constraint of (2.8) forces $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$ to be non-zero because $\boldsymbol{y_o} = (y_{1o}, ..., y_{so}) \geq \mathbf{0}$ and $\boldsymbol{y_o} \neq \mathbf{0}$. Hence, the first constraint of (2.8) θ must be greater than zero. By virtue of the dual theorem of linear programming it is true that $z^* = \theta^*$. Hence either problem may be used. A feasible solution to (2.8) always exists. For example setting $\theta = 1$ and $\lambda_o^* = 1$ (where λ_o^*) corresponds to DMU_o), and $\lambda_j^* = 0\, \forall j \neq o$ satisfies all constraints. Moreover this solution implies $0 < \theta^* \leq 1$. The optimal value θ^* represents an efficiency score for the DMU under evaluation. The process is repeated for each DMU_j, that is., solve (2.8) with $(X_o, Y_o) = (X_j, Y_j)$ where (X_j, Y_j) with components x_{jo}, y_{jo}. DMUs for which $\theta^* < 1$ are inefficient, while DMUs for which $\theta^* = 1$ are boundary points. Some boundary points may be "weakly" efficient due to the presence of non-zero slack variables. This may appear to be worrisome because alternate optima may have non-zero slacks in some solutions, but not in others. However, one can avoid being concerned even in such cases by invoking the following linear program in which the slacks are taken to their maximal values and the value of θ^* is fixed from the optimal solution of model (2.8).

$$
\begin{aligned}
\max \ \rho = \ & \sum_{i=1}^{m} s_i^- \sum_{r=1}^{s} s_r^+, \ \text{ over } \ \lambda_j, \ s_i^-, \ s_r^+ \ \in \mathbb{R} \\
\text{subject to } \ & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta^* x_{io} \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro} \\
& \lambda_j, \ s_i^-, \ s_r^+ \ \geq 0 \, \forall \, i, j, r
\end{aligned} \tag{2.9}
$$

It should be noted that the constraint space of (2.8) defines the production possibility set P, that is :

$$
P = \left\{ (X, Y) | X \geq \sum_j \lambda_j X_j, \ Y \leq \sum_j \lambda_j Y_j, \ \lambda_j \geq 0 \right\},
$$

where (X, Y) represent the input and output vectors for the DMU under evaluation such that X can produce Y, λ_j is the dual variable for DMU$_j$, and X_j, Y_j represent the inputs and outputs of DMU$_j$. The constraints of (2.8) require that the activity $(\theta \boldsymbol{x_o}, \boldsymbol{y_o})$ to belong to P, while the objective seeks the minimum θ that reduces the input vector $\boldsymbol{x_o}$ radially to $\theta \boldsymbol{x_o}$ while remaining in P. In (2.8), the goal is to look for an activity in P that guarantees at least the output level $\boldsymbol{y_o}$ of DMU$_o$ in all components while reducing the input vector $\boldsymbol{x_o}$ proportionally (radially) to a value as small as possible. Note that (2.8) and (2.9) are also referred to as phase I and II in the DEA literature and an optimal solution $(\boldsymbol{\lambda^*}, \boldsymbol{s^{-*}}, \boldsymbol{s^{+*}})$ of phase II is referred to as the maximum slack solution. These developments lead to the following definitions of efficiency.

Definition 2.2.1.1 (CCR efficiency). The performance of DMU$_o$ is fully efficient if and only if $\theta^* = 1$ and $s_i^{-*} = s_r^{+*} = 0 \ \forall i, r$.
It should be noted that there exists an equivalent definition of CCR efficiency based on model (2.7).
Definition 2.2.1.2 (CCR Efficiency) DMU$_o$ is CCR efficient if and only if $z^* = 1$ and there exists at least one optimal $(\boldsymbol{v^*}, \boldsymbol{u^*})$, with $\boldsymbol{v^*} > \boldsymbol{0}$ and $\boldsymbol{u^*} > \boldsymbol{0}$.
Definition 2.2.1.3 (Weak Efficiency). The performance of DMU$_o$ is weakly efficient if and only if both $\theta^* = 1$ and $s_i^{-*} \neq 0$ and or $s_r^{+*} \neq 0 \ \forall i, r$.
Definition 2.2.1.4 (Reference Set). For an inefficient DMU$_o$, its reference set E_o is defined based on the maximum slack solution obtained from phase II by:

$$
E_o = \left\{ j | \lambda_j^* > 0 \right\} \left(j \epsilon \left\{ 1, ..., n \right\} \right) \tag{2.10}
$$

According to Mostafaeipour et al. (2020), the efficiency of $(\boldsymbol{x_o}, \boldsymbol{y_o})$ for DMU$_o$ can be improved if the input values are reduced radially by the ratio θ^* and the input excesses recorded in $\boldsymbol{s^{-*}}$ are eliminated. Efficiency can be attained if the output values are augmented by the amount represented by the output shortfalls in $\boldsymbol{s^+}$. Thus a method for improving an inefficient DMU is obtained. The gross input improvement $\Delta \boldsymbol{x_o}$ and output improvement $\Delta \boldsymbol{y_o}$ can be calculated from:

$$\Delta \boldsymbol{x_o} = \boldsymbol{x_o} - \left(\theta^* \boldsymbol{x_o} - \boldsymbol{s}^{-*}\right) = \left(1 - \theta^*\right)\boldsymbol{x_o} + \boldsymbol{s}^{-*} \tag{2.11}$$

$$\Delta \boldsymbol{y_o} = \boldsymbol{s}^{+*} \tag{2.12}$$

Hence there is a formula for improvement which is called the CCR projection:

$$\hat{\boldsymbol{x}}_o = \boldsymbol{x_o} - \Delta \boldsymbol{x_o} = \theta^* \boldsymbol{x_o} - \boldsymbol{s}^{-*} = \sum_{j \in E_o} \boldsymbol{\lambda}_j^* \boldsymbol{x_j} \leq \boldsymbol{x_o} \tag{2.13}$$

$$\hat{\boldsymbol{y}}_o = \boldsymbol{y_o} + \Delta \boldsymbol{y_o} = \boldsymbol{y_o} + \boldsymbol{s}^{+*} = \sum_{j \in E_o} \boldsymbol{\lambda}_j^* \boldsymbol{y_j} \geq \boldsymbol{y_o} \tag{2.14}$$

The CCR projections identify the point either as a positive combination of other DMUs with $\boldsymbol{x_o} \geq \hat{\boldsymbol{x}}_o$ and $\hat{\boldsymbol{y}}_o \geq \boldsymbol{y_o}$ unless $\theta^* = 1$ and all slacks are 0 in which case $\boldsymbol{x_o} = \hat{\boldsymbol{x}}_o$ and $\hat{\boldsymbol{y}}_o = \boldsymbol{y_o}$ so (2.13) and (2.14) identify a new DMU positioned on the efficiency frontier. Conversely, the point associated with the new DMU evaluates the performance of DMU$_o$ as exhibiting input excesses $\boldsymbol{x_o} - \hat{\boldsymbol{x}}_o$ and output shortfalls $\hat{\boldsymbol{y}}_o - \boldsymbol{y_o}$.The relationship between the primal (2.7) and the dual (2.8) constraints and variables are displayed in vector form in Table 2.1 below:

Table 2.1: Primal and Dual Correspondences (CCR-I)

Multiplier Form	**Envelopment Form**	**Envelopment Form**	**Multiplier Form**
Constraint (2.7)	**Variable (2.8)**	**Constraint (2.8)**	**Variable (2.7)**
$\boldsymbol{v}\boldsymbol{x_o} = 1$	θ	$\theta \boldsymbol{x_o} - X\boldsymbol{\lambda} \geq \boldsymbol{0}$	$\boldsymbol{v} \geq \boldsymbol{0}$
$-\boldsymbol{v}X + \boldsymbol{u}Y \leq \boldsymbol{0}$	$\boldsymbol{\lambda} \geq \boldsymbol{0}$	$Y\boldsymbol{\lambda} \geq \boldsymbol{y_o}$	$\boldsymbol{u} \geq \boldsymbol{0}$

The Multiplier form represents the dual problem of maximising efficiency with respect to the weights, while the Envelopment form corresponds to minimising inputs or maximising outputs with respect to the DMU data.

2.2.2 CCR output oriented model

Alternatively, one can adopt an output orientation and consider the ratio of virtual input to virtual output. This would reorient the objective from max to min to obtain:

$$\min\ \theta = \frac{\sum_{i=1}^{m} v_i x_{io}}{\sum_{r=1}^{s} u_r y_{ro}}, \quad \text{over } u_r, v_i \in \mathbb{R}$$

$$\text{subject to} \quad \frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{r=1}^{s} u_r y_{rj}} \geq 1 \quad \text{for } j = 1, 2, ..., n. \tag{2.15}$$

$$u_r, v_i \geq 0\ \forall\ i,\ r.$$

Again the Charnes and Cooper (1962) transformation for linear fractional programs yields the primal [multiplier model (2.16)] below with the associated dual [envelopment model (2.17)]:

$$\begin{aligned} \min\ q &= \sum_{i=1}^{m} \nu_i x_{io} \\ \text{subject to } & \sum_{i=1}^{m} \nu_i x_{ij} - \sum_{r=1}^{s} \mu_r y_{rj} \geq 0 \\ & \sum_{r=1}^{s} \mu_r y_{ro} = 1 \\ & \mu_r, \nu_i \geq 0\ \forall\ i,\ r. \end{aligned} \tag{2.16}$$

$$\begin{aligned} \max\ \ y &= \ \phi \\ \text{subject to }\ & x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0 \\ & \sum_{j=1}^{n} y_{rj}\lambda_j \geq \phi y_{ro} \\ & \lambda_j \geq 0\ \forall\ j. \end{aligned} \tag{2.17}$$

Similarly to models (2.8) and (2.9) , the following linear program is invoked in which the slacks are taken to their maximal values and the value of ϕ^* is fixed from the optimal solution of model (2.17):

$$\begin{aligned} \max \quad \rho =\ & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+, \ \text{ over } \lambda_j,\ s_i^-,\ s_r^+ \ \in\ \mathbb{R} \\ \text{subject to }\ & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \\ & \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = \phi^* y_{ro} \\ & \lambda_j,\ s_i^-,\ s_r^+,\ \geq 0\ \forall i, j, r \end{aligned} \tag{2.18}$$

The previous input-oriented definition of DEA efficiency is modified to the following output-oriented version:

Definition 2.2.2.1: DMU$_o$ is efficient if and only if $\phi = 1$ and $s_i^- * = s_r^+ * = 0$ for all i and r. DMU_0 is weakly efficient if $\phi = 1$ and $s_i^- * \neq 0$ and (or) $s_r^+ * \neq 0$ for some i and r in some alternate optima.

The relationship between the primal (2.16) and the dual (2.17) constraints and variables are displayed in vector form in Table 2.2.

Table 2.2: Primal and Dual Correspondences (CCR-O)

Multiplier form Constraint (2.16)	**Envelopment form Variable (2.17)**	**Envelopment form Constraint (2.17)**	**Multiplier form Variable (2.16)**
$\mu \boldsymbol{y_o} = 1$	ϕ	$\boldsymbol{x_o} - X\lambda \geq \mathbf{0}$	$\nu \geq \mathbf{0}$
$-\mu Y + \nu X \geq \mathbf{0}$	$\lambda \geq \mathbf{0}$	$Y\lambda \geq \phi \boldsymbol{y_o}$	$\mu \geq \mathbf{0}$

The Multiplier form represents the dual problem of maximising efficiency with respect to the weights, while the Envelopment form corresponds to minimising inputs or maximising outputs with respect to the DMU data.

2.3 BCC model

2.3.1 BCC input oriented model

The BCC model, introduced by Banker et al. (1984), extends the CCR model by accommodating variable returns to scale (VRS), allowing for more flexible efficiency assessments. According to Cook and Seiford (2009) the BCC ratio model differs from (2.6) by way of an additional variable, that is:

$$
\begin{aligned}
\max\ \beta \ &= \frac{\sum_{r=1}^{s} u_r y_{ro} - u_o}{\sum_{i=1}^{m} v_i x_{io}}, \quad \text{over } u_r, v_i, u_o \in \mathbb{R} \\
\text{subject to } &\sum_{r=1}^{s} u_r y_{rj} - u_o - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \ \forall j \qquad (2.19) \\
&u_r, v_i \geq 0 \ \forall\, i, r. \\
&u_o \text{ unrestricted in sign}
\end{aligned}
$$

where u_o is an intercept term that allows the efficiency frontier to account for variable returns to scale, distinguishing the BCC model from the CCR model. The primal (2.20) linear programming equivalent of (2.19) and its dual (2.21) are given as:

$$
\begin{aligned}
\max \rho = & \sum_{r=1}^{s} \mu_r y_{r0} - \mu_o \\
\text{subject to } & \sum_{r=1}^{s} \mu_r y_{rj} - \mu_o - \sum_{i=1}^{m} \nu_i x_{ij} \leq 0 \;\; \forall j \\
& \sum_{i=1}^{m} \nu_i x_{i0} = 1 \\
& \mu_r, \nu_i \geq 0 \; \forall \; i, r \\
& \mu_o \text{ unrestricted.}
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\min y \; = & \; \theta \\
\text{subject to } & \theta x_{io} - \sum_{j=1}^{n} x_{ij} \lambda_j \geq 0 \;, \forall \; i \\
& \sum_{j=1}^{n} y_{rj} \lambda_j \geq y_{ro} \;, \forall \; r \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \;, \forall \; j. \\
& \theta \text{ unrestricted.}
\end{aligned}
\tag{2.21}
$$

The unitary constraint of (2.21) ensures that the reference set is a convex combination of DMUs, which is essential for modelling VRS. Similarly to models (2.17) and (2.18) , the following linear program is invoked in which the slacks are taken to their maximal values and the value of θ^* is fixed from the optimal solution of model (2.21):

$$
\begin{aligned}
\max \quad \rho = & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+, \;\; \text{over} \;\; \lambda_j, \; s_i^-, \; s_r^+, u_0 \; \in \; \mathbb{R} \\
\text{subject to } & \sum_{j=1}^{n} x_{ij} \lambda_j + s_i^- \; = \theta^* x_{i0} \\
& \sum_{j=1}^{n} y_{rj} \lambda_j - s_r^+ = y_{ro} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j, \; s_i^-, \; s_r^+, \geq 0 \; \forall i, j, r
\end{aligned}
\tag{2.22}
$$

The evaluations from the CCR and BCC models are also related to each other as follows. An optimal solution for (2.21) is represented by: $(\theta_B^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$, where $\boldsymbol{s}^{-*}$ and $\boldsymbol{s}^{+*}$ represent the maximal input excesses and output shortfalls, respectively. It is noted that θ_B^* is not less than the optimal objective value θ_C^* of the CCR model (2.8), since (2.21) imposes one additional constraint $\mathbf{1'}\boldsymbol{\lambda} = 1$, so its feasible region is a subset of the feasible region for the CCR (2.8) model.

Definition 2.3.1.1 (BCC Efficiency). If an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ obtained in the two-phase process for (2.18) satisfies $\theta^* = 1$ and has no slack $(\boldsymbol{s}^{-*} = \mathbf{0}, \boldsymbol{s}^{+*} = \mathbf{0})$, then the DMU_0 is called BCC-efficient, otherwise it is BCC-inefficient.

Definition 2.3.1.2 (Reference Set) For a BCC-inefficient DMU_o, its reference set is defined as E_o based on an optimal solution $\boldsymbol{\lambda}^*$ by:

$$E_o = \{j|\lambda_j^* > 0\}(j \in \{1, ..., n\}) \tag{2.23}$$

While the reference set for the BCC model is defined similarly to the CCR model, the inclusion of the VRS constraint generally results in different optimal weights and reference DMUs. Similarly to the CCR model, there exists a formula for improving an inefficient DMU_o, this improvement is known as the BCC-projection:

$$\hat{\boldsymbol{x}}_o = \theta_B^* \boldsymbol{x_o} - \boldsymbol{s}^{-*} \tag{2.24}$$

$$\hat{\boldsymbol{y}}_o = \boldsymbol{y_o} + \boldsymbol{s}^{+*} \tag{2.25}$$

The BCC projection, denoted as $(\hat{\boldsymbol{x}}_o, \hat{\boldsymbol{y}}_o)$, represents the target input-output levels that would make the DMU efficient under VRS assumptions." The relationship between the primal (2.20) and the dual (2.21) constraints and variables are displayed in vector form in Table 2.3 below:

Table 2.3: Primal and Dual Correspondences (BCC-I)

Multiplier form Constraint (2.20)	**Envelopment form Variable (2.21)**	**Envelopment form Constraint (2.21)**	**Multiplier form Variable (2.20)**
$v\boldsymbol{x_o} = 1$	θ	$\theta_B \boldsymbol{x_o} - X\lambda \geq \mathbf{0}$	$v \geq \mathbf{0}$
$-vX + \mu Y - \mathbf{1'}\mu_o \leq \mathbf{0}$	$\lambda \geq \mathbf{0}$	$Y\lambda \geq \boldsymbol{y_o}$	$\mu \geq \mathbf{0}$
		$\mathbf{1'}\lambda = 1$	μ_o

The multiplier form represents the dual problem of maximising efficiency with respect to the weights, while the Envelopment form corresponds to minimising inputs or maximising outputs with respect to the DMU data.

2.3.2 BCC output oriented model

In contrast to the input-oriented model, the output-oriented BCC model seeks to maximise outputs while maintaining current input levels. Similar to model (2.15) one can consider the ratio of inputs to outputs which leads to the BCC output oriented model as follows:

$$\begin{aligned}
\min\ \theta \ &= \frac{\sum_{i=1}^{m} v_i x_{i0} - v_0}{\sum_{r=1}^{s} u_r y_{ro}}, \quad \text{over } u_r, v_i, v_o \in \mathbb{R} \\
\text{subject to} \ &\frac{\sum_{i=1}^{m} v_i x_{ij} - v_o}{\sum_{r=1}^{s} u_r y_{rj}} \geq 1 \ \forall \ j \\
&u_r, v_i \geq 0 \ \forall \ i, \ r. \\
&v_o \text{ unrestricted.}
\end{aligned} \tag{2.26}$$

v_o is an intercept term that allows the efficiency frontier to account for variable returns to scale, distinguishing the BCC model from the CCR model The primal (2.27) linear programming equivalent of (2.26) and its dual (2.28) are thus given by:

$$\begin{aligned}
\min \gamma = \ &\sum_{i=1}^{m} v_i x_{i0} - v_o \\
\text{subject to} \ &\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} \mu_r y_{rj} - v_o \geq 0 \ \forall j \\
&\sum_{r=1}^{s} \mu_r y_{r0} = 1 \\
&\mu_r, v_i \geq 0 \ \forall \ i, r \\
&v_o \text{ unrestricted.}
\end{aligned} \tag{2.27}$$

$$
\begin{aligned}
& \max \ \psi \\
& \text{subject to } \ x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0 \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq \psi y_{ro} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \ \forall \ j.
\end{aligned} \tag{2.28}
$$

ψ is the output multiplier variable for the DMU under evaluation. Similarly to models (2.20) and (2.21) , the following linear program is invoked in which the slacks are taken to their maximal values and the value of ψ^* is fixed from the optimal solution of model (2.28):

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = \psi^* y_{ro} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j, \ s_i^-, \ s_r^+, \geq 0 \ \forall \ i, j, r
\end{aligned} \tag{2.29}
$$

Definition 2.3.2.1 (BCC efficiency). If an optimal solution $(\psi^*, \boldsymbol{\lambda^*}, \boldsymbol{s^{-*}}, \boldsymbol{s^{+*}})$ obtained in the two-phase process for (2.29) satisfies $\psi^* = 1$ and has no slacks $(\boldsymbol{s^{-*} = 0}, \boldsymbol{s^{+*} = 0})$, then DMU_o is called BCC-efficient, otherwise it is BCC-inefficient.

The relationship between the primal (2.27) and the dual (2.28) constraints and variables are displayed in vector form in Table 2.4.

Table 2.4: Primal and Dual Correspondences (BCC-O)

Multiplier form (2.27)	**Envelopment form (2.28)**	**Envelopment form (2.28)**	**Multiplier form (2.27)**
$\boldsymbol{\mu y_o} = 1$	ψ	$\boldsymbol{x_o} - X\boldsymbol{\lambda} \geq \boldsymbol{0}$	$\boldsymbol{\mu} \geq \boldsymbol{0}$
$\boldsymbol{\nu}X - \boldsymbol{\mu}Y - \boldsymbol{1}'\nu_o \geq \boldsymbol{0}$	$\boldsymbol{\lambda} \geq \boldsymbol{0}$	$Y\boldsymbol{\lambda} \geq \psi\boldsymbol{y_o}$	$\boldsymbol{\nu} \geq \boldsymbol{0}$
		$\boldsymbol{1}'\boldsymbol{\lambda} = 1$	ν_o

The Multiplier form represents the dual problem of maximising efficiency with respect to the weights, while the Envelopment form corresponds to minimising inputs or maximising outputs with respect to the DMU data.

2.4 Additive model

The Additive model offers an alternative efficiency evaluation by directly minimising the sum of input surpluses and output shortfalls. The preceding models required a distinction to be made between input-oriented and output-oriented models. Now however, both orientations are combined in a single model, called the Additive model. There are several types of the Additive model, one of them being given by:

$$\begin{aligned}
\max z = & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{i0} \ , \ i = 1, 2, ..., m \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro} \ , \ r = 1, 2, ..., s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j, \ s_i^-, \ s_r^+, \geq 0 \ \forall i, j, r
\end{aligned} \tag{2.30}$$

The objective function z quantifies the sum of the input slack and output shortfall variables, ideally these should all be zero since they represent unused inputs and over produced outputs. This model considers the input excess and the output shortfalls simultaneously in arriving at a point on the efficiency frontier. Taking these considerations into account one can obtain a definition of efficiency as follows for the Additive model. Let the optimal solution be $(\lambda^*, \boldsymbol{s^{-*}}, \boldsymbol{s^{+*}})$. The definition of efficiency for an efficient DMU in the Additive model is then given by:

Definition 2.4.1 (ADD efficiency). DMU$_0$ is ADD efficient if and only if $\boldsymbol{s^{-*} = 0}$ and $\boldsymbol{s^{+*} = 0}$.

It is noted that the Additive model (2.30) does not measure the efficiency score θ^* explicitly but it is implicitly present in the slacks $\boldsymbol{s^{-*}}$ and $\boldsymbol{s^{+*}}$. Moreover whereas θ^* reflects only weak efficiency, the objective function in (2.27) reflects all inefficiencies that the model can identify in both inputs and outputs.

2.5 SBM model

In order to estimate the efficiency of a DMU, the following fractional program in $\boldsymbol{\lambda}$, $\boldsymbol{s^-}$ and $\boldsymbol{s^+}$ is formulated:

$$\min\ \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{i0}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{r0}}}$$

$$\text{subject to}\quad \boldsymbol{x_o} = X\boldsymbol{\lambda} + \boldsymbol{s^-} \tag{2.31}$$

$$\boldsymbol{y_o} = Y\boldsymbol{\lambda} - \boldsymbol{s^+}$$

$$\boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{s^-} \geq \mathbf{0},\ \boldsymbol{s^+} \geq \mathbf{0}.$$

In this model, it is assumed that $X \geq O$. If $x_{io} = 0$, then the term $\frac{s_i^-}{x_{io}}$ is deleted in the objective function. If $y_{ro} \leq 0$, then it is replaced by a very small positive number so that the term $\frac{s_r^+}{y_{ro}}$ plays the role of a penalty. It is true that:

$$0 \leq \rho \leq 1 \tag{2.32}$$

The above holds since $s_i^- \leq x_{io}$ for very i so that $0 \leq \frac{s_i^-}{x_{io}} \leq 1 (i = 1, ..., m)$ with $\frac{s_i^-}{x_{io}} = 1$ only if the evidence shows that only a zero amount of this input was required. It then follows that

$$0 \leq \frac{\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{m} \leq 1 \tag{2.33}$$

This relation does not hold for outputs since an output shortfall represented by a nonzero slack can exceed the corresponding amount of output produced. In any case it is true that:

$$\frac{\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}}{s} \geq 0 \tag{2.34}$$

These represent ratios of average input and output mix inefficiencies with the upper limit $\rho = 1$ reached in (2.29) only if slacks are zero in all inputs and outputs. Model (2.29) can be transformed into the program below by introducing a positive scalar variable t:

$$\min\ \tau\ =\ t - \frac{1}{m}\sum_{i=1}^{m} \frac{ts_i^-}{x_{io}}$$

$$\text{subject to}\ t + \frac{1}{s}\sum_{r=1}^{s} \frac{ts_r^+}{y_{ro}} = 1$$

$$\boldsymbol{x_o} = X\boldsymbol{\lambda} + \boldsymbol{s^-} \tag{2.35}$$

$$\boldsymbol{y_o} = Y\boldsymbol{\lambda} - \boldsymbol{s^+}$$

$$\boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{s^-} \geq \mathbf{0},\ \boldsymbol{s^+} \geq \mathbf{0},\ t > 0$$

Defining $\boldsymbol{S^-} = t\boldsymbol{s^-}, \boldsymbol{S^+} = t\boldsymbol{s^+}$, and $\boldsymbol{\Lambda} = t\boldsymbol{\lambda}$. The SBM model becomes the following linear program in $t, \boldsymbol{S^+}, \boldsymbol{S^-}, \boldsymbol{\Lambda}$:

$$\begin{aligned} \min \ \tau &= t - \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^-}{x_{io}} \\ \text{subject to } \ t &+ \frac{1}{s}\sum_{r=1}^{s} \frac{S_r^+}{y_{ro}} = 1 \\ t\boldsymbol{x_o} &= X\boldsymbol{\Lambda} + \boldsymbol{s^-} \\ t\boldsymbol{y_o} &= Y\boldsymbol{\Lambda} - \boldsymbol{s^+} \\ \boldsymbol{\Lambda} &\geq \boldsymbol{0}, \ \boldsymbol{s^-} \geq \boldsymbol{0}, \ \boldsymbol{s^+} \geq \boldsymbol{0}, \ t > 0 \end{aligned} \tag{2.36}$$

It is noted that $t > 0$ by virtue of the first constraint. This means that the transformation is reversible. Thus let an optimal solution of (2.32) be given by: $(\tau^*, t^*, \boldsymbol{\Lambda}^*, \boldsymbol{S^{-*}}, \boldsymbol{S^{+*}})$. An optimal solution of (2.27) is obtained as:

$$\rho^* = \tau^*, \boldsymbol{\lambda^*} = \frac{\boldsymbol{\Lambda^*}}{t^*}, \boldsymbol{s^{-*}} = \frac{\boldsymbol{S^{-*}}}{t^*}, \boldsymbol{s^{+*}} = \frac{\boldsymbol{S^{+*}}}{t^*}. \tag{2.37}$$

From this optimal solution, whether a DMU is SBM efficient is decided as follows:

Definition 2.6.1 (SBM Efficiency) A DMU $(\boldsymbol{x_o}, \boldsymbol{y_o})$ is SBM efficient if and only if $\rho^* = 1$.

This condition is equivalent to requiring that $\boldsymbol{s^{-*}} = \boldsymbol{s^{+*}} = \boldsymbol{0}$, that is no input excesses and no output shortfalls in an optimal solution. For an SMB inefficient DMU $(\boldsymbol{x_o}, \boldsymbol{y_o})$, there is the following expressions:

$$\boldsymbol{x_o} = X\boldsymbol{\lambda^*} + \boldsymbol{s^{-*}} \tag{2.38}$$

$$\boldsymbol{y_o} = Y\boldsymbol{\lambda^*} - \boldsymbol{s^{+*}} \tag{2.39}$$

The DMU $(\boldsymbol{x_o}, \boldsymbol{y_o})$ can be improved and becomes efficient by deleting the input excesses and augmenting the output shortfalls. This is accomplished by the following formulae (SBM projection):

$$\hat{\boldsymbol{x}}_o = \boldsymbol{x_o} - \boldsymbol{s^{-*}} = \boldsymbol{X\lambda^*} \tag{2.40}$$

$$\hat{\boldsymbol{y}}_o = \boldsymbol{y_o} + \boldsymbol{s^{+*}} = \boldsymbol{Y\lambda^*} \tag{2.41}$$

which are the same for the Additive model. The reference set for $(\boldsymbol{x_o}, \boldsymbol{y_o})$ is defined as:

Definition 2.6.2 (Reference set): The set of indices corresponding to positive λ_js is called the reference set for $(\boldsymbol{x_o}, \boldsymbol{y_o})$. For multiple optimal solutions, the reference set is not unique but any solution can be selected. Let $\boldsymbol{R_o}$ be the reference set designated by

$$R_o = \{j | \lambda_j^* > 0\} (j \in \{1, ..., n\}) \tag{2.42}$$

Then using R_o, the improved activity $(\hat{\boldsymbol{x}}_{\boldsymbol{o}}, \hat{\boldsymbol{y}}_{\boldsymbol{o}})$ is expressed by:

$$\hat{\boldsymbol{x}}_o = \sum_{j \epsilon R_o} \lambda_j^* \boldsymbol{x_j} \tag{2.43}$$

$$\hat{\boldsymbol{y}}_o = \sum_{j \epsilon R_o} \lambda_j^* \boldsymbol{y_j} \tag{2.44}$$

This means that $(\hat{\boldsymbol{x}}_{\boldsymbol{o}}, \hat{\boldsymbol{y}}_{\boldsymbol{o}})$, a point on the efficiency frontier is expressed as a positive combination of the members of the reference set R_o, each member which is also efficient. The dual program of (2.34) can be expressed as follows with the dual variables $\zeta \, \epsilon R$, $\boldsymbol{v} \in R^m$, and $\boldsymbol{u} \in R^s$:

$$\begin{aligned}
&\max \ \zeta \\
&\text{subject to } \ \zeta + \boldsymbol{v x_o} - \boldsymbol{u y_o} = 1 \\
&-\boldsymbol{v}X + \boldsymbol{u y_o} \leq \boldsymbol{0} \\
&\boldsymbol{v} \geq \frac{1}{m}\left(\frac{1}{\boldsymbol{x_o}}\right) \\
&\boldsymbol{u} \geq \frac{\zeta}{s}\left(\frac{1}{\boldsymbol{y_o}}\right) \\
&\boldsymbol{u} \geq \boldsymbol{0}, \ \boldsymbol{v} \geq \boldsymbol{0}, \ \zeta \in \mathbb{R}.
\end{aligned} \tag{2.45}$$

In Table 2.5, some important topics to consider in choosing between basic DEA models are summarised. In this table, "semi-p" (semi-positive) means non-negative with at least one positive element in the data for each DMU, and "free" permits negative, zero or positive data. Although some DEA models assume positive data, this assumption can be relaxed as exhibited in the table. In the BCC-I(O) model, outputs(inputs) are free due to the translation invariance theorem. In the case of the SBM, non-positive outputs can be replaced by a very small positive number and non-positive input terms can be neglected for consideration in the objective function. "Tech or Mix" indicates whether the model measures "technical efficiency" or "mix efficiency". "CRS" and "VRS" mean constant and variable returns to scale, respectively. The returns to scale of ADD and SBM depends on the added convexity constraint $\boldsymbol{1'\lambda} = 1$. Model selection is one of the problems to be considered in DEA up to, and including, choices of multiple models to test whether or not a result is dependent on the models (or methods) used.

Table 2.5: Summary of Model Characteristics

Model		**CCR-I**	**CCR-O**	**BCC-I**	**BCC-O**	**ADD**	**SBM**
Data	X	Semi-p	Semi-p	Semi-p	Free	Free	Semi-p
	Y	Free	Free	Free	Semi-p	Free	Free
Trans.	X	No	No	No	Yes	Yes[a]	No
Invariance	Y	No	No	Yes	No	Yes[a]	No
Units invariance		Yes	Yes	Yes	Yes	No	Yes
θ^*		$[0,1]$	$[0,1]$	$(0,1]$	$(0,1]$	No	$[0,1]$
Tech. or Mix		Tech	Tech	Tech	Tech	Mix	Mix
Returns to Scale		CRS	CRS	VRS	VRS	C(V)RS[b]	C(V)RS

(a): The Additive model is translation invariant only when the convexity constraint is added.

(b): C(V)RS means Constant or Variable returns to scale according to whether or not the convexity constraint is included.

2.6 Dimensionality in DEA

Although DEA is regarded as non-parametric, the sample size can be an issue of great importance in determining the efficiency scores for the evaluated units, empirically, when the use of too many inputs and outputs may result in a significant number of DMUs being rated as efficient. In the DEA literature, empirical rules have been established to avoid too many DMUs being rated as efficient. These empirical thresholds relate the number of variables to the number of observations. When the number of DMUs is below the empirical threshold levels, the discriminatory power among the DMUs may weaken, which leads to the data set not being suitable to apply traditional DEA models. In the literature, the lack of discrimination is often referred to as the "curse of dimensionality" (Charles et al., 2019).

The lack of discriminating power has important implications, as in practice it can limit the managerial insights that can be drawn (Ghasemi et al., 2019). Nevertheless, the literature indicates some empirical rules regarding the number of DMUs versus the number of inputs and outputs. For example, Golany and Roll (1989) suggest that the number of DMUs should be at least twice the number of inputs and outputs. Raab and Litchty (2002) suggest that the number of DMUs should be at least three times the number of inputs and outputs; and Dyson et al. (2001) suggest that the number of DMUs should be at least twice the product of the number of inputs and the number of outputs. Yet, another empirical rule of thumb which can provide guidance is, in line with Cooper et al. (2007), $n \geq \max(m \times s, 3(m+s))$, where n is the number of DMUs, m is the number of inputs, and s is the number of outputs.

Figure 2.2 shows the number of DMUs that would be required in the case of each empirical rule of thumb mentioned above. The observation to be made is that, even for the case of 12 inputs and 12 outputs, the number of DMUs required becomes very high, ranging between 48 and 288, depending on the rule of thumb used. This may turn out to be a problem in real-life applications, where a high number of DMUs may simply just not be available.

Cook et al. (2014) emphasised that in statistical regression analysis, the sample size is crucial since it aims to estimate the average behaviour of a group of DMUs. In contrast, when employing DEA as a benchmarking tool, the focus shifts to the individual performance of each DMU, making the sample size or number of evaluated DMUs potentially irrelevant. However, it is important to note that if the number of DMUs falls below certain threshold levels, the ability to differentiate between them may diminish.

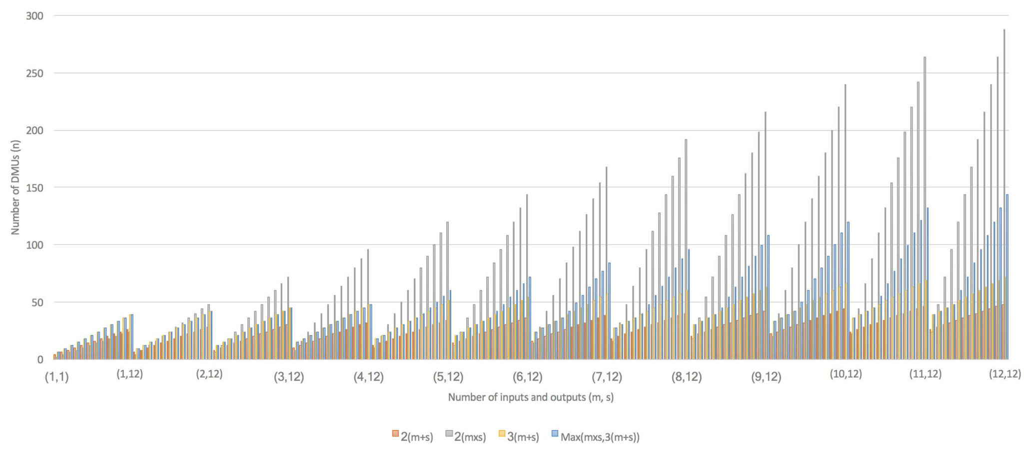


Figure 2.2: Comparison of the empirical rules of thumb.

2.7 DEA models for ratio data

Most performance measuring variables in sports come in a form of ratios, such as goals scored per game, number of wickets per match, number of runs scored per game, etc. The following example illustrates the importance of using ratios in football: Suppose Messi and Neymar score 30 and 38 goals across all competitions in a season for PSG. Messi played 20 matches and Neymar played 7 more than Messi. Even though Neymar has scored more goals than Messi but Messi's goals to match ratio is 1.5 and Neymar's is 1.41 which means that Messi is considered to be more efficient as far as goal scoring is concerned. The production possibility set (PPS) as defined earlier in this Chapter is unknown and is estimated from the observed data. Convexity is one of the underlying assumptions for estimating the PPS. However Emrouznejad and Amin (2009) showed that the standard DEA models can not be used directly if at least one of the input or output data is in the form of a ratio. Consequently the standard DEA models as introduced earlier in this Chapter need to be modified to accommodate the ratio data. This is formalised by the following theorem:

Theorem 2.7.1: The standard DEA models can not be used directly if at least one of the input or output data is in the form of a ratio.

Proof: Suppose you have n DMUs consuming a single input-ratio to produce an absolute single output, say $y_j = 1$ for each $j = 1, ..., n$. Further assume the input-ratio of DMU$_j$ is denoted by: $x_j = \frac{n_j}{d_j}$ for each $j = 1, ..., n$. The standard input oriented PPS can be written as follows:

$$PPS = \left\{(x, y) : x \geq \sum_{j=1}^{n} \lambda_j x_j, \ y \leq 1, \ \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \ j = 1, ..., n\right\}$$

, Considering the actual convex combination of DMUs defined by:

$$x^0 = \frac{\sum_{j=1}^{n} \lambda_j n_j}{\sum_{j=1}^{n} \lambda_j d_j}, \; \sum_{j=1}^{n} \lambda_j = 1, \; \lambda_j \geq 0, \; j = 1, ..., n$$

Let $\lambda_j = \frac{1}{n} \, \forall j$ and note that $(x^0, 1)$ may not be in the PPS because

$$\frac{n_1 + ... + n_n}{d_1 + ... + d_n} \geq \frac{1}{n}\left(\frac{n_1}{d_1} + ... + \frac{n_n}{d_n}\right)$$

is not valid for any arbitrary given set of numbers n_j and d_j $\big(j = 1, ..., n\big)$.

2.8 DEA models with outputs and inputs as ratios

Consider a set of n DMUs, where DMU$_j$ is associated with m inputs and s outputs. The standard input oriented DEA model to measure the efficiency of DMU$_o$ is given by:
Model 1: Standard input oriented BCC model:

$$\begin{aligned} \min \;\; & z = \;\; \theta \\ \text{subject to} \;\; & \sum_{j=1}^{n} x_{ij}\lambda_j - \theta x_{io} \leq 0 \;, \forall \; i \\ & \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro} \;, \forall \; r \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \;, \forall \; j. \end{aligned} \tag{2.46}$$

Assume that the k^{th} output is in the form of a ratio, where $1 \leq k \leq s$. Suppose that y_{kj} for unit j is calculated from the numerator and denominator of n_{kj} and d_{kj}, that is: $y_{kj} = \frac{n_{kj}}{d_{kj}}$, $\big(j = 1, ..., n\big)$. According to Theorem 2.7.1, **Model 1** cannot be used since one of the outputs is a ratio. Taking into account the correct convexity for the ratio variables which is the *ratio of convex combination of numerator to the convex combination of denominator* rather than a simple convex combination of the ratio variable, the correct convex combination is:

$$\sum_{j=1}^{n} y_{kj}\lambda_j = \frac{\sum_{j=1}^{n} n_{kj}\lambda_j}{\sum_{j=1}^{n} d_{kj}\lambda_j} \tag{2.47}$$

However, in the standard DEA model the convex combination is:

$$\sum_{j=1}^{n} y_{kj}\lambda_j = \sum_{j=1}^{n} \frac{n_{kj}}{d_{kj}}\lambda_j \tag{2.48}$$

Therefore the convexity assumption when assessing DMU$_o$ should be taken into the model as follows:

$$\frac{\sum_{j=1}^{n} n_{kj}\lambda_j}{\sum_{j=1}^{n} d_{kj}\lambda_j} \geq \frac{n_{ko}}{d_{ko}} = y_{ko} \tag{2.49}$$

Hence **Model 1** for DMU$_o$ should be written in the following form:

Model 2: Input oriented BCC model with output ratio variable included:

$$\begin{aligned} & \min z = \theta \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij}\lambda_j - \theta x_{io} \leq 0 \,, \forall\, i \\ & \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro} \,, \forall\, r \neq k \\ & \sum_{j=1}^{n} n_{kj}\lambda_j - y_{ko} \sum_{j=1}^{n} d_{kj}\lambda_j \geq 0 \,, r = k \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \,, \forall\, j. \end{aligned} \tag{2.50}$$

A similar argument can be made for output oriented DEA models with output as a ratio as follows:

Model 3: Output oriented BCC model with output ratio variable included.

$$\begin{aligned}
& \max\ \psi \\
\text{subject to } & x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0 \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq \psi y_{ro},\ \forall r \neq k \\
& \sum_{j=1}^{n} n_{kj}\lambda_j - y_{ko}\psi \sum_{j=1}^{n} d_{kj}\lambda_j,\ r = k \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0\ \forall\ j.
\end{aligned} \tag{2.51}$$

Now, one considers DEA models with at least one input variable as a ratio. Assume that the p^{th} input is a ratio variable such that x_{pj} for unit j is calculated from the numerator and denominator of n_{pj} and d_{pj}, that is, $x_{pj} = \frac{n_{pj}}{d_{pj}}$, then using a similar argument to the output ratio case leads to the following model:

Model 4: Input oriented BCC model with input ratio variable included.

$$\begin{aligned}
& \min\ z = \quad \theta \\
\text{subject to } & \theta x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0\ ,\forall\ i \neq p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro}\ ,\forall\ r \\
& \sum_{j=1}^{n} n_{pj}\lambda_j - x_{po}\theta \sum_{j=1}^{n} d_{pj}\lambda_j \leq 0,\ i = p \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0\ ,\forall\ j.
\end{aligned} \tag{2.52}$$

The output oriented model is also modified to accommodate the input ratio variable.

Model 5: Output oriented BCC model with input ratio variable included:

$$\begin{aligned}
\max \ z = \ & \psi \\
\text{subject to } \ & x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0, \ \forall\, i \neq p \\
& \sum_{j=1}^{n} n_{pj}\lambda_j - x_{po} \sum_{j=1}^{n} d_{pj}\lambda_j \leq 0, \ i = p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq \psi y_{ro} \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \ \forall\, j.
\end{aligned} \tag{2.53}$$

Now, we consider the case where the DEA model has both inputs and outputs as ratio variables. Suppose that the k^{th} output $y_{kj} = \frac{\overline{y}_{kj}}{\underline{y}_{kj}}$ and the p^{th} input $x_{pj} = \frac{\overline{x}_{pj}}{\underline{x}_{pj}}$.

The modified input oriented model is given by:

Model 6. Input oriented BCC model with input and output ratio variables included:

$$\begin{aligned}
\min \ z = \ & \theta \\
\text{subject to } \ & \sum_{j=1}^{n} x_{ij}\lambda_j - \theta x_{io} \leq 0 \ , \forall\, i \neq p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro} \ , \forall\, r \neq k \\
& \sum_{j=1}^{n} \overline{y}_{kj}\lambda_j - y_{ko} \sum_{j=1}^{n} \underline{y}_{kj}\lambda_j \geq 0 \ , r = k \\
& \sum_{j=1}^{n} \overline{x}_{pj}\lambda_j - \underline{x}_{po}\theta \sum_{j=1}^{n} \underline{x}_{pj}\lambda_j \leq 0, \ i = p \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \ , \forall\, j.
\end{aligned} \tag{2.54}$$

It should be noted that:

$$\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \frac{\overline{y}_{ko}}{\underline{y}_{ko}} = y_{ko}$$

$$\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \theta\frac{\overline{x}_{po}}{\underline{x}_{po}} = \theta x_{po}$$

An output oriented DEA model is formulated in a similar manner as follows:

Model 7: Output oriented BCC model with input and output ratio variables included:

$$\begin{aligned}
\max\ & z = \ \psi \\
\text{subject to}\ & x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j \geq 0,\ \forall\, i \neq p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq \psi y_{ro},\ \forall\, r \neq k \\
& \sum_{j=1}^{n} \overline{x}_{pj}\lambda_j - \underline{x}_{po} \sum_{j=1}^{n} \underline{x}_{pj}\lambda_j \leq 0,\ i = p \qquad (2.55) \\
& \sum_{j=1}^{n} n_{kj}\lambda_j - y_{ko}\psi \sum_{j=1}^{n} d_{kj}\lambda_j,\ r = k \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0\ \forall\, j.
\end{aligned}$$

It should be noted that:

$$\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \psi\frac{\overline{y}_{ko}}{\underline{y}_{ko}} = \psi y_{ko}$$

$$\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \frac{\overline{x}_{po}}{\underline{x}_{po}} = x_{po}$$

Although these modifications are expressed using the BCC model, they also apply to the

Additive and SBM models since they are all based on the same production possibility set.

The next section explores the possibility of including randomness in the determination of efficient units based on the DEA methodology. As can be seen from (2.55) the optimal solution is a single value which will be different under a different set of inputs and outputs. We would like to be able to estimate the distribution of the objective function and create confidence intervals, hence the necessity of the bootstrap method.

2.9 Bootstrap resampling technique

The term "bootstrapping" refers to the concept of "pulling oneself up by using one's bootstraps ", a phrase apparently first used in *The Singular Travel, Campaigns and Adventures of Baron Munchausen* by Rudolph Erich Raspe in 1786. The derivative of the same term is used in a similar manner to describe the process of "booting" a computer by a sequence of software increments loaded into memory at power-up (Horowitz, 2019).

In statistics, "bootstrapping" refers to making inferences about a sampling distribution of a statistic by "resampling" the sample itself with replacement, as if it were a finite population."Resampling" as a concept was first used by Fisher (1935) in his famous randomization test. The "bootstrap" as sampling with replacement and its Monte Carlo approximate form was first presented in a Stanford University technical report by Brad Efron in 1977. This report led to his famous paper in the *Annals of Statistics* in 1979.

The "bootstrap" is one of a number of techniques that is now part of the broad umbrella of non-parametric statistics that are commonly called resampling methods. Bootstrapping was made practical through the use of the Monte Carlo approximation, but it too goes back to the beginning of computers in the early 1940s. However, 1979 is a critical year for the bootstrap because that is when Brad Efron's paper in the *Annals of Statistics* was published (Efron, 1992). Efron had defined a resampling procedure that he coined as bootstrap. He constructed it as a simple approximation to the Jackknife (an early resampling method that was developed by John Tukey) (Chernick and LaBudde, 2014).

Two of the most important problems in applied statistics are the determination of an estimator for a particular parameter θ of interest and the evaluation of the accuracy of that estimator through estimates of the standard error of the estimator and the determination of confidence intervals for the parameter. When Efron introduced his version of the "bootstrap", he was particularly motivated by these two problems. Most important was the estimation of the standard error of the parameter estimator, particularly when the estimator was complex and standard approximations were either not appropriate or too inaccurate (Chernick, 2011).

2.9.1 Confidence interval for θ using the bootstrap (*basic percentile method*)

Suppose $X_1, X_2, ..., X_n$ are identically and independently distributed random variables from an unknown distribution function F. Let $\hat{\theta}(X_1, X_2, ..., X_n)$ be an estimator for the population parameter θ. We need to construct a $100(1-\alpha)\%$ confidence interval for θ. This is achieved by sampling with *replacement* B samples (normally B is chosen reasonably large, e.g, $B = 1000$), each of size n, from $X_1, X_2, ..., X_n$. For each of the B samples calculate the statistic $\hat{\theta}^*$.

$$\begin{array}{lll} (1) & X_1^*, X_2^*, ..., X_n^* & \text{calculate } \hat{\theta}_1^* \\ (2) & X_1^*, X_2^*, ..., X_n^* & \text{calculate } \hat{\theta}_2^* \\ & \vdots & \\ (B) & X_1^*, X_2^*, ..., X_n^* & \text{calculate } \hat{\theta}_B^* \end{array} \tag{2.56}$$

The next step is to obtain the order statistics:

$$\hat{\theta}^*_{(1)} \leq \hat{\theta}^*_{(2)} \leq ... \leq \hat{\theta}^*_{(B)} \tag{2.57}$$

A $100(1-\alpha)\%$ confidence interval for θ is then given by:

$$\left[\hat{\theta}^*_{(r)}; \hat{\theta}^*_{(s)}\right] \tag{2.58}$$

where

$$r = \left\lfloor (B+1)\left(\frac{\alpha}{2}\right)\right\rfloor \tag{2.59}$$

and

$$s = \left\lceil (B+1)\left(1-\frac{\alpha}{2}\right)\right\rceil \tag{2.60}$$

2.9.2 Estimating the variability of estimators by the bootstrap method

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution F. We would like to investigate the variability of our estimator $\hat{\theta}$, where:

$$\hat{\theta} = \hat{\theta}(X_1, X_2, ..., X_n) \tag{2.61}$$

For example, $\hat{\theta} = \tilde{X}$ (sample median) or $\hat{\theta} = \bar{X}$ (sample mean).

The study is faced with two problems:

1. We might not know F, and
2. Even if we know F, $\hat{\theta}$ may be a very complicated function of $X_1, X_2, ..., X_n$

The case where F is known

In this case $\text{Var}(\hat{\theta})$ can be estimated by Monte Carlo simulation as follows:

1. Generate a sample $X_1, X_2, ..., X_n$ from F and calculate $\hat{\theta} = \hat{\theta}(X_1, X_2, ..., X_n)$. Denote this by $\hat{\theta}_1$
2. Repeat step (1) a large number of times, say MC times, to obtain $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_{MC}$
3. If MC is large enough, the variance of $\hat{\theta}$ can be approximated by

$$\text{Var}(\hat{\theta}) \approx \frac{1}{MC} \sum_{i=1}^{MC} (\hat{\theta}_i - \hat{\theta}(\cdot))^2, \tag{2.62}$$

where:

$$\hat{\theta}(\cdot) = \frac{1}{MC} \sum_{i=1}^{MC} \hat{\theta}_i \tag{2.63}$$

The case where F is unknown

If F is unknown, the bootstrap provides an alternative method to estimate the variance of $\hat{\theta}$. As already explained, a bootstrap sample is obtained from the original sample by placing equal probability on each observation. Sampling is done with replacement , that is, each time an observation is drawn for the bootstrap sample its value is recorded and the element is returned to the original sample. Sampling with replacement is considered to be an independent type of sampling since the probability of drawing an element in not dependent on the elements already drawn.

Drawing a sample of size n, with replacement, from the original sample $X_1, X_2, ..., X_n$ is equivalent to drawing a sample from F_n (The empirical distribution function). Applying the bootstrap method is therefore similar to the Monte Carlo method simulation, except that we draw a sample from F_n and not from F. To summarise, we draw with replacement B samples (each of size n) from $X_1, X_2, ..., X_n$. For each of the B samples, we calculate the statistics of interest:

$$\begin{array}{lll} (1) & X_1^*, X_2^*, ..., X_n^* & \text{calculate } \hat{\theta}_1^* \\ (2) & X_1^*, X_2^*, ..., X_n^* & \text{calculate } \hat{\theta}_2^* \\ & \vdots & \\ (B) & X_1^*, X_2^*, ..., X_n^* & \text{calculate } \hat{\theta}_B^* \end{array} \tag{2.64}$$

The variance of $\hat{\theta}$ is then estimated by:

$$S_{\hat{\theta}}^2 = \frac{1}{B} \sum_{i=1}^{B} \left(\hat{\theta}_i^* - \hat{\theta}(\,\cdot\,) \right)^2 \tag{2.65}$$

where

$$\hat{\theta}(\cdot) = \frac{1}{B} \sum_{i=1}^{B} \hat{\theta}_i \tag{2.66}$$

Estimating the sampling distribution F of $\hat{\theta}(\cdot)$ is illustrated with an example from Rice (2009) on platinum data by using the bootstrap to approximate the sampling distribution of the 20% trimmed mean and its standard error. To this end, 1000 samples of size $n = 26$ were drawn randomly with replacement from the collection of 26 values. A histogram of the 1000 trimmed means is displayed in Figure 2.3. The standard deviation of the 1000 values was 0.64, which is the estimated standard error of the 20% trimmed mean.

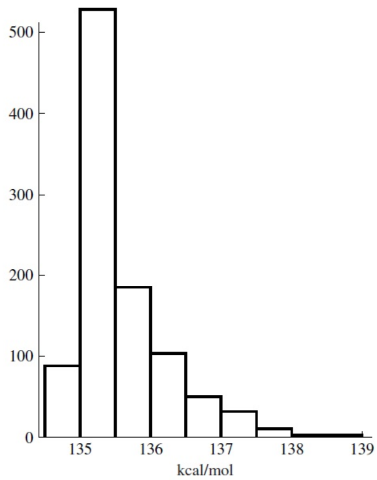


Figure 2.3: Histogram of 1000 bootstrap 20% trimmed means.

In practice the underlying distribution F of the sample data $X_1, X_2, ..., X_n$ is unknown which makes the bootstrap resampling technique very useful. Additionally the empirical distribution function F_n is an unbiased estimator of the true distribution function F, meaning on average F_n produces estimates to F that are correct, this is what makes the bootstrap method so powerful in the sense that in cases where the underlying distribution F of an estimator $\hat{\theta}$ is known the bootstrap method is a very good approximation. This gives confidence in using it for cases where F or $\hat{\theta}$ is a complicated function (Rice, 2009). The concept of bootstrapping is very handy in the next Chapters. It will be explained how it relates to solving the problem encountered by DEA models.

Chapter 3

DEA application in Football

3.1 Introduction

In this Chapter, the study investigates the results of the DEA method applied in football. The data consists of football players from 18 competitive leagues across Europe, South America, and Africa. The objective is to evaluate or measure player performance using DEA models. As explained in Chapter 1, the concept of efficiency is useful in ranking DMUs especially when it comes to player selection or the awarding of FIFA individual awards. This study focuses on the latter. In particular, the players are grouped according to their playing positions such that goalkeepers are grouped together, defenders are grouped together, etc. This is sensible since the corresponding inputs (resources) and outputs (expectations) differ for goalkeepers, defenders, midfielders, and forwards. Additionally, the roles that goalkeepers, defenders, midfielders, and forwards play are different, hence it is sensible to group them separately and measure efficiency for each group with the objective of picking the best and deserving players for the FIFA awards without discrimination. This study makes no distinction between say an attacking midfielder and a defensive midfielder or between a winger and an out and out striker,etc. This is due to the lack of sufficient data, since for example the number of crosses made by a winger would be considered an output but not necessarily for a striker. There is no available data for the number of completed crosses for each winger. To make the study feasible the variables chosen are common for both roles, this is true for all types of similar playing positions. The results are presented as follows: firstly the traditional DEA models' results are shown and secondly, the bootstrap DEA models' results are shown. Since the data consists mostly of ratios, the suitable DEA models are those described in Chapter 2.

3.2 Problem description

The traditional DEA models employ inputs and outputs as the data to be used when estimating the efficiency score. It is clear that the efficiency score is dependent on the set of inputs and outputs. In other words, a DMU may be deemed to be efficient under one set of data and inefficient under another set (for the same input and output variables). Landete et al. (2017) proposed a methodology where the inputs and outputs were treated as random variables. However their method is based on the argument that, in any DEA study, the inputs and outputs to be selected for the study are uncertain and should be modelled by a Bernoulli random variable, with each variable taking on the value 1 with probability p and the value 0 with probability $1 - p$.

Olesen (2002) incorporates randomness in DEA differently, by focusing on the restriction of the optimal weights or multipliers. Traditional DEA models tend to assign a weight of zero to certain inputs and outputs which creates a problem since this suggests that the associated variables are not significant when assessing the performance of a particular DMU. To solve this issue, Olesen (2002) suggested the use of confidence intervals for the output multipliers. These confidence intervals are also known as the probabilistic assurance regions in the output space. This study differs from the work done by Landete et al. (2017) in the sense that there is no uncertainty about the inclusion of the inputs and outputs since these are chosen based on experience and are deemed necessary for the performance evaluation of DMUs. It also differs from the work of Olesen (2002) in the sense that the confidence intervals suggested by this study refer to the objective function of the DEA model. The randomness aspect comes from the values that the input and output variables take on, since the objective function depends on these variables it is a random variable as well. This is the main focus of this study.

The motivation behind this study is two fold, firstly it is to investigate the discriminatory power between the traditional DEA model and the non-parametric bootstrap DEA model that is proposed. The efficiency score is unknown beforehand unlike the input and output data sets, it needs to be estimated by solving the optimisation problem. The problem faced by traditional DEA models is that they produce a single value as a measure of efficiency. This study proposes that it is much better to construct a confidence interval for the efficiency score. This suggests estimating the sampling distribution of the computed efficiency scores and estimating the confidence interval of the efficiency score. The confidence interval approach is more reliable and easy to interpret than a point estimate. The approach proposed will be illustrated with an application in football.

3.3 Methods

In this section the author presents the methods used to investigate the results of the traditional SBM, BCCI, and Additive models, along with the bootstrap based SBM, BCCI, and Additive DEA model's results. The data was collected for the 2020/2021 season. This data consists of 16340 players who participated across 18 top football leagues and cup competitions in the world, these are: Bundesliga, Caf champions league, Carabao Cup, Copa America, Copa del rey, Copa Italy, Copa de Espana, DFB Pokal, DFL Super Cup, Europa league, FA Cup, Coupe de France, Italy supercoppa, Laliga, Ligue 1, Premier league, Serie A, UEFA champions league. 20 variables of interest are selected, these include minutes played overall, assists overall, penalty goals, clean sheets overall, etc.

The following tables show the variables (inputs and outputs) that were selected to analyse the performance of goalkeepers, defenders, midfielders and forwards. The choice of inputs and outputs is based solely on the availability of the data. DEA models suffer from the curse of dimensionality where an appropriate combination of inputs and outputs should not exceed the number of DMUs under consideration. Additional variables would not be an issue here since the number of players are in the thousands.

Table 3.1: Inputs and Outputs for Goalkeepers

Inputs	**Outputs**
minutes played overall	clean sheets overall
appearances overall	minutes per conceded overall
conceded overall	minutes per card overall
yellow cards overall	-
red cards overall	-
minutes per match	-

Table 3.2: Inputs and Outputs for Defenders

Inputs	**Outputs**
minutes played overall	minutes per conceded overall
appearances overall	minutes per card overall
yellow cards overall	-
red cards overall	-
minutes per match	-
cards per 90 overall	-

Table 3.3: Inputs and Outputs for Midfielders

Inputs	**Outputs**
minutes played overall	goals overall
appearances overall	assists overall
yellow cards overall	penalty goals
red cards overall	assists per 90 overall
minutes per match	goals per 90 overall
cards per 90 overall	minutes per card overall
penalty misses	-
minutes per goal overall	-
minutes per assist overall	-

Table 3.4: Inputs and Outputs for Forwards

Inputs	**Outputs**
minutes played overall	goals overall
appearances overall	assists overall
yellow cards overall	penalty goals
red cards overall	assists per 90 overall
minutes per match	goals per 90 overall
cards per 90 overall	minutes per card overall
penalty misses	-
minutes per goal overall	-
minutes per assist overall	-

It should be noted that the choice of inputs and outputs is not limited to the ones selected here, these were chosen based on the availability of sufficient data. Since, the study is dealing with a mixture of nominal and ratio variables in terms of inputs and outputs, the SBM, BCCI and Additive models used in this section are the following:

SBM Model:

$$\begin{aligned}
\min \;\; & \tau = t - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{x_{io}} \\
\text{subject to} \;\; & t + \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^+}{y_{ro}} = 1 \\
& \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = tx_{i0} \;,\; \forall\, i \neq p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = ty_{ro} \;,\; \forall\, r \neq k \\
& \sum_{j=1}^{n} \overline{x}_{pj}\Lambda_j + s_p^- \sum_{j=1}^{n} \underline{x}_{pj}\Lambda_j = tx_{po}\sum_{j=1}^{n} \underline{x}_{pj}\Lambda_j,\; \forall\, i = p \\
& \sum_{j=1}^{n} \overline{y}_{kj}\Lambda_j - s_k^+ \sum_{j=1}^{n} \underline{y}_{kj}\Lambda_j = ty_{ko}\sum_{j=1}^{n} \underline{y}_{kj}\Lambda_j,\; \forall\, r = k \\
& \sum_{j=1}^{n} \Lambda_j = 1 \\
& t \;, \Lambda_j,\; s_i^-,\; s_r^+,\; \geq 0 \;\forall i, j, r, p, k
\end{aligned} \tag{3.1}$$

It should be noted that:

$$\frac{\sum_{j=1}^{n} \overline{y}_{kj}\Lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\Lambda_j} \geq t\frac{\overline{y}_{ko}}{\underline{y}_{ko}} = ty_{ko}$$

$$\frac{\sum_{j=1}^{n} \overline{x}_{pj}\Lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\Lambda_j} \leq t\frac{\overline{x}_{po}}{\underline{x}_{po}} = tx_{po}$$

BCCI Model:

$$\begin{aligned} \min\ z = \ & \theta \\ \text{subject to}\ & \sum_{j=1}^{n} x_{ij}\lambda_j - \theta x_{io} \leq 0\ , \forall\ i \neq p \\ & \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro}\ , \forall\ r \neq k \\ & \sum_{j=1}^{n} \overline{y}_{kj}\lambda_j - y_{ko}\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j \geq 0\ , r = k \\ & \sum_{j=1}^{n} \overline{x}_{pj}\lambda_j - \underline{x}_{po}\theta\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j \leq 0,\ i = p \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0\ , \forall\ j. \end{aligned} \tag{3.2}$$

It should be noted that:

$$\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \frac{\overline{y}_{ko}}{\underline{y}_{ko}} = y_{ko}$$

$$\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \theta\frac{\overline{x}_{po}}{\underline{x}_{po}} = \theta x_{po}$$

Additive Model:

$$
\begin{aligned}
\max \phi = & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \ , \forall\ i \neq p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro} \ , \forall\ r \neq k \\
& \sum_{j=1}^{n} \overline{y}_{kj}\lambda_j - y_{ko} \sum_{j=1}^{n} \underline{y}_{kj}\lambda_j - s_r^+ = 0 \ , r = k \\
& \sum_{j=1}^{n} \overline{x}_{pj}\lambda_j - \underline{x}_{po} \sum_{j=1}^{n} \underline{x}_{pj}\lambda_j + s_i^- = 0, \ i = p \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \ , \forall \ j.
\end{aligned}
\tag{3.3}
$$

It should be noted that:

$$
\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \frac{\overline{y}_{ko}}{\underline{y}_{ko}} = y_{ko}
$$

$$
\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \frac{\overline{x}_{po}}{\underline{x}_{po}} = x_{po}
$$

3.4 DEA Results for the Classical SBM model

This section presents the results for the classical SBM model applied to goalkeepers, defenders, midfielders and forwards. The data analysis was performed using Python 3.12.

3.4.1 DEA results for the classical SBM model for goalkeepers.

Table 3.5 shows the SBM efficiency score (column 3) where a value of one indicates efficiency. It should be noted that the goalkeeper data consists of 1021 players but the results are presented for the first 50 players since we are dealing with a large data set, and the results are for illustrative purposes. The same is done for defenders, midfielders, and forwards. It should be noted that a goalkeeper's role is well defined. When it comes to defenders, the study makes no distinction whether a player is a centre back (CB), left back (LB), right back (RB) etc, they are all regarded as defenders. The same applies for midfielders. When it comes to forwards, the study makes no distinction between the wingers and the strikers, they are all considered forwards.

Table 3.5: SBM efficiency score for the first 50 goalkeepers

	Full Name	**Nationality**	τ^*	**DMU Status**
1	Aaron McCarey	Republic of Ireland	0.5333	SBM inefficient
2	Aaron Ramsdale	England	0.4454	SBM inefficient
3	Aarón Escandell Banacloche	Spain	0.5760	SBM inefficient
4	Adrián	Spain	1.0000	SBM efficient
5	Adrián San Miguel del Castillo	Spain	0.2323	SBM inefficient
6	Agustín Federico Marchesín	Argentina	0.5831	SBM inefficient
7	Aiden Stone	England	1.0000	SBM efficient
8	Aitor Fernández Abarisketa	Spain	0.2403	SBM inefficient
9	Alban Lafont	France	0.2597	SBM inefficient
10	Alberto Paleari	Italy	0.3438	SBM inefficient
11	Aldo Junior Simoncini	San Marino	0.5196	SBM inefficient
12	Alejandro Remiro Gargallo	Spain	0.7520	SBM inefficient
13	Alessandro Berardi	Italy	0.6667	SBM inefficient
14	Alessio Cragno	Italy	0.4717	SBM inefficient
15	Alex Bass	England	0.4002	SBM inefficient
16	Alex Cairns	England	0.3581	SBM inefficient
17	Alex McCarthy	England	0.2124	SBM inefficient
18	Alex Meret	Italy	0.3785	SBM inefficient
19	Alex Palmer	England	0.1988	SBM inefficient
20	Alex Smithies	England	0.5333	SBM inefficient
21	Alexander Meyer	Germany	0.6528	SBM inefficient
22	Alexander Nübel	Germany	0.5449	SBM inefficient
23	Alexandre Oukidja	France	0.1973	SBM inefficient
24	Alexandros Paschalakis	Greece	0.6926	SBM inefficient
25	Aleš Hruška	Czech Republic	0.5500	SBM inefficient
26	Alfonso Pastor Vacas	Spain	0.4375	SBM inefficient
27	Alfred Gomis	Senegal	0.4685	SBM inefficient
28	Alisson Becker	Brazil	0.8630	SBM inefficient
29	Alphonse Areola	France	0.3457	SBM inefficient
30	Amjhad Nazih	France	0.5333	SBM inefficient
31	Anatolii Trubin	Ukraine	0.4871	SBM inefficient
32	Andoni Zubiaurre Dorronsoro	Spain	1.0000	SBM efficient
33	Andrea Consigli	Italy	0.5012	SBM inefficient
34	Andreas Linde	Sweden	0.5046	SBM inefficient
35	Andreas Luthe	Germany	0.7889	SBM inefficient
36	Andrew Lonergan	England	0.5333	SBM inefficient
37	Andriy Lunin	Ukraine	1.0000	SBM efficient
38	Andriy Pyatov	Ukraine	0.4375	SBM inefficient
39	Andrés Fernández	Spain	0.3093	SBM inefficient
40	Andrés Tomás Prieto Albert	Spain	0.4375	SBM inefficient
41	Andy Fisher	England	0.6875	SBM inefficient
42	Anthony Lopes	Portugal	0.3925	SBM inefficient
43	Anthony Racioppi	Switzerland	0.2464	SBM inefficient
44	Antonio Mirante	Italy	0.2404	SBM inefficient
45	Antonio Sivera Salvá	Spain	0.3737	SBM inefficient
46	Artur Boruc	Poland	0.5333	SBM inefficient
47	Ashley Maynard-Brewer	Australia	1.0000	SBM efficient
48	Asmir Begović	Bosnia and Herzegovina	0.2560	SBM inefficient
49	Aynsley Pears	England	1.0000	SBM efficient
50	Bailey Peacock-Farrell	England	0.1843	SBM inefficient

From Table 3.5, goalkeepers Adrian, Aiden Stone, Andoni Zubiaurre Dorronsoro, Ashley Maynard-Brewer, e.t.c, are the most efficient out of the 50 players according to the SBM model. The motivation of using DEA models is to select "best" performing players on merit regardless of the strength of the team they play for. These results can be used for purposes of selecting a team or individual player awards, such as best player of the season, best goalkeeper of the year, etc. The question then is how does one choose amongst these players. If one is selecting a team and has to choose between two goalkeepers that are both efficient, then they may both be selected with the other player being on the bench. If it is to award players then the award should be shared, as should be the case in the event of a tie. For SBM efficient defenders, midfielders, and forwards the same logic applied to SBM efficient goalkeepers applies to these group of players as well.

3.4.2 DEA results for the classical SBM model for defenders.

Table 3.6: SBM efficiency score for the first 50 defenders

	Full Name	Nationality	τ^*	DMU Status
1	Aaron Cresswell	England	0.6918	SBM inefficient
2	Aaron Hayden	England	0.6533	SBM inefficient
3	Aaron O'Driscoll	Republic of Ireland	1.0000	SBM efficient
4	Aaron Pierre	Grenada	0.4585	SBM inefficient
5	Aarón Martín	Spain	0.7016	SBM inefficient
6	Abbas Hüseynov	Azerbaijan	0.5917	SBM inefficient
7	Abdou Diallo	France	0.2971	SBM inefficient
8	Abdoulaye Bamba	Côte d'Ivoire	0.4177	SBM inefficient
9	Abdoulaye Ousame	France	1.0000	SBM efficient
10	Abdourahmane Barry	France	0.3614	SBM inefficient
11	Abzal Beysebekov	Kazakhstan	0.6114	SBM inefficient
12	Achraf Hakimi Mouh	Morocco	0.3296	SBM inefficient
13	Adam Bodzek	Germany	0.2925	SBM inefficient
14	Adam Crookes	England	0.7111	SBM inefficient
15	Adam Jackson	England	1.0000	SBM efficient
16	Adam Marušić	Montenegro	1.0000	SBM efficient
17	Adam Reach	England	0.6507	SBM inefficient
18	Adam Smith	England	0.5015	SBM inefficient
19	Adam Sušac	Croatia	0.5150	SBM inefficient
20	Adam Webster	England	0.4112	SBM inefficient
21	Adetayo Edun	England	0.5064	SBM inefficient
22	Adrián Hernández Hernández	Spain	0.8499	SBM inefficient
23	Adrián Marín Gómez	Spain	0.4478	SBM inefficient
24	Advan Kadušić	Bosnia and Herzegovina	0.7407	SBM inefficient
25	Ahmet Gürleyen	Germany	0.8861	SBM inefficient
26	Akinwale Joseph Odimayo	England	0.4829	SBM inefficient
27	Albert Vallci	Austria	0.6170	SBM inefficient
28	Alberto Moreno	Spain	0.7868	SBM inefficient
29	Alberto Rodríguez Baro	Spain	0.2115	SBM inefficient
30	Aleix Vidal	Spain	1.0000	SBM efficient
31	Alejandro Centelles Plaza	Spain	1.0000	SBM efficient
32	Alejandro Grimaldo García	Spain	0.4734	SBM inefficient
33	Aleksandar Dragović	Austria	0.4812	SBM inefficient
34	Aleksandar Ignjovski	Serbia	0.4221	SBM inefficient
35	Aleksandar Kolarov	Serbia	0.5899	SBM inefficient
36	Aleksandar Živanović	Serbia	0.6646	SBM inefficient
37	Aleksandr Martynovich	Belarus	0.5631	SBM inefficient
38	Aleksandr Pavlovets	Belarus	0.7500	SBM inefficient
39	Alessandro Bastoni	Italy	0.3651	SBM inefficient
40	Alessandro Buongiorno	Italy	0.3076	SBM inefficient
41	Alessandro Tripaldelli	Italy	0.6034	SBM inefficient
42	Alessio Romagnoli	Italy	0.2377	SBM inefficient
43	Alex Baptiste	England	0.5926	SBM inefficient
44	Alex Cini	Malta	0.9996	SBM inefficient
45	Alex Ferrari	Italy	0.3276	SBM inefficient
46	Alex Junior Christian	Haiti	1.0000	SBM efficient
47	Alex Nicolao Telles	Brazil	0.7501	SBM inefficient
48	Alex Pearce	Republic of Ireland	0.6995	SBM inefficient
49	Alex Sandro	Brazil	0.3316	SBM inefficient
50	Alexander Hahn	Germany	0.4822	SBM inefficient

The table consists of the first 50 defenders in alphabetical order. The majority of players are SBM-inefficient.

3.4.3 DEA results for the classical SBM model for midfielders.

Table 3.7: SBM efficiency score for the first 50 midfielders

	Full Name	Nationality	τ^*	DMU Status
1	Aaron Greene	England	0.7223	SBM inefficient
2	Aaron Hickey	Scotland	0.4375	SBM inefficient
3	Aaron Hunt	Germany	0.4197	SBM inefficient
4	Aaron Morley	England	0.7202	SBM inefficient
5	Aaron Opoku	Germany	0.2503	SBM inefficient
6	Aaron Ramsey	Wales	0.3179	SBM inefficient
7	Aaron Wan-Bissaka	England	0.1639	SBM inefficient
8	Aaron Wildig	England	1.0000	SBM efficient
9	Abdellah Zoubir	France	1.0000	SBM efficient
10	Abdoulaye Doucouré	France	0.1855	SBM inefficient
11	Abdoulaye Touré	France	0.2998	SBM inefficient
12	Achraf Drif	France	0.8333	SBM inefficient
13	Adam Clayton	England	0.8283	SBM inefficient
14	Adam David Lallana	England	0.1426	SBM inefficient
15	Adam May	England	0.7272	SBM inefficient
16	Adam Ounas	Algeria	1.0000	SBM efficient
17	Adam Phillips	England	1.0000	SBM efficient
18	Adam Randell	England	0.7444	SBM inefficient
19	Adama Traoré Diarra	Spain	0.1494	SBM inefficient
20	Adel Taarabt	Morocco	1.0000	SBM efficient
21	Ademipo Odubeko	England	0.8161	SBM inefficient
22	Adil Aouchiche	France	0.4098	SBM inefficient
23	Adil Elmoueden	Germany	0.6458	SBM inefficient
24	Adnan Januzaj	Belgium	1.0000	SBM efficient
25	Adrian Bernabe Garcia	Spain	0.8395	SBM inefficient
26	Adrian Fein	Germany	1.0000	SBM efficient
27	Adrian Grbic	Austria	1.0000	SBM efficient
28	Adrian Stanilewicz	Germany	1.0000	SBM efficient
29	Adrian Zöfelt	Germany	0.7447	SBM inefficient
30	Adrien Hunou	France	1.0000	SBM efficient
31	Adrien Rabiot	France	0.1351	SBM inefficient
32	Adrien Sebastian Perruchet Silva	Portugal	0.0610	SBM inefficient
33	Adrien Tameze	France	0.1458	SBM inefficient
34	Adrián Andrés Cubas	Argentina	0.1104	SBM inefficient
35	Aeron Edwards	Wales	0.7421	SBM inefficient
36	Afeez Aremu	Nigeria	0.5797	SBM inefficient
37	Ahmed Eissa El Mohamady	Egypt	0.7761	SBM inefficient
38	Ahmet Arslan	Germany	1.0000	SBM efficient
39	Aihen Muñoz Capellán	Spain	0.1543	SBM inefficient
40	Aimar Oroz	Spain	0.7817	SBM inefficient
41	Aimen Moueffek	France	0.6374	SBM inefficient
42	Ainsley Maitland-Niles	England	1.0000	SBM efficient
43	Ajdin Hrustic	Australia	0.8056	SBM inefficient
44	Akaki Gogia	Germany	0.8056	SBM inefficient
45	Alan Browne	Republic of Ireland	1.0000	SBM efficient
46	Alan Judge	Republic of Ireland	1.0000	SBM efficient
47	Alan Patrick Lourenço	Brazil	0.3119	SBM inefficient
48	Albert Gudmundsson	Iceland	1.0000	SBM efficient
49	Alberto Grassi	Italy	0.6717	SBM inefficient
50	Alberto Soro Álvarez	Spain	1.0000	SBM efficient

The table consists of the first 50 midfielders in alphabetical order. The majority of players are SBM-inefficient.

3.4.4 DEA results for the classical SBM model for forwards.

Table 3.8: SBM efficiency score for the first 50 forwards

	Full Name	Nationality	τ^*	DMU Status
1	Aaron Anthony Connolly	Republic of Ireland	0.0683	SBM inefficient
2	Aaron Collins	Wales	0.7379	SBM inefficient
3	Aaron Jarvis	England	1.0000	SBM efficient
4	Aaron Martin	England	0.1920	SBM inefficient
5	Aaron Rowe	England	0.2199	SBM inefficient
6	Aaron Seydel	Germany	0.1384	SBM inefficient
7	Abdoulay Diaby	Mali	1.0000	SBM efficient
8	Abdoulaye Diallo	Senegal	1.0000	SBM efficient
9	Abdul Majeed Waris	Ghana	0.1169	SBM inefficient
10	Aboubakar Kamara	France	0.0597	SBM inefficient
11	Adalberto Peñaranda Maestre	Venezuela	1.0000	SBM efficient
12	Adam Armstrong	England	1.0000	SBM efficient
13	Adam Zreľák	Slovakia	0.7530	SBM inefficient
14	Adan George	England	0.7901	SBM inefficient
15	Ademola Lookman	England	0.0723	SBM inefficient
16	Admir Mehmedi	Switzerland	1.0000	SBM efficient
17	Adolfo Julián Gaich	Argentina	1.0000	SBM efficient
18	Adrian Tabarcea Petre	Romania	1.0000	SBM efficient
19	Adriel D'Avila Ba Loua	Côte d'Ivoire	0.2419	SBM inefficient
20	Adrien Truffert	France	0.0481	SBM inefficient
21	Adrián Gallardo Valdés	Spain	0.2523	SBM inefficient
22	Affamefuna-Michael Ifeadigo	Nigeria	1.0000	SBM efficient
23	Agustín Gonzalo Torassa	Argentina	0.1126	SBM inefficient
24	Ahmed Hassan Mahgoub	Egypt	1.0000	SBM efficient
25	Aiden O'Brien	Republic of Ireland	0.7405	SBM inefficient
26	Aitor Ruibal García	Spain	0.0773	SBM inefficient
27	Alassane Pléa	France	1.0000	SBM efficient
28	Alberto Cerri	Italy	1.0000	SBM efficient
29	Alberto Perea Correoso	Spain	0.0643	SBM inefficient
30	Albian Ajeti	Switzerland	1.0000	SBM efficient
31	Albion Vrenezi	Germany	0.0703	SBM inefficient
32	Alejandro Blanco Sánchez	Spain	1.0000	SBM efficient
33	Alejandro Millán Iranzo	Spain	1.0000	SBM efficient
34	Aleksa Vukanović	Serbia	0.7030	SBM inefficient
35	Aleksandar Mitrović	Serbia	0.1057	SBM inefficient
36	Aleksandar Vujačić	Montenegro	1.0000	SBM efficient
37	Aleksey Shchetkin	Kazakhstan	0.7556	SBM inefficient
38	Alex Addai	England	0.7418	SBM inefficient
39	Alex Gilliead	England	0.7315	SBM inefficient
40	Alex Iwobi	Nigeria	0.0570	SBM inefficient
41	Alex Mighten	England	0.7491	SBM inefficient
42	Alex Samuel	Wales	0.3878	SBM inefficient
43	Alexander Isak	Sweden	1.0000	SBM efficient
44	Alexander MacDonald	Scotland	0.7063	SBM inefficient
45	Alexander Mesa Travieso	Spain	0.7980	SBM inefficient
46	Alexander Sørloth	Norway	0.1261	SBM inefficient
47	Alexandre Lacazette	France	1.0000	SBM efficient
48	Alexandre Mendy	France	0.8161	SBM inefficient
49	Alexis Claude Maurice	France	1.0000	SBM efficient
50	Alexis Sanchez	Chile	1.0000	SBM efficient

The table consists of the first 50 forwards in alphabetical order. There is an almost even split between SBM-inefficient and SBM-efficient players.

3.5 DEA results for the classical BCCI model

This section presents the results for the classical BCCI model applied to goalkeepers, defenders, midfielders and forwards.

3.5.1 DEA results for the classical BCCI model for goalkeepers.

The following table show results of this model for goalkeepers. It can be mentioned again that an efficiency score of 1 implies that the particular player is BCC-efficient otherwise it is deemed BCC-inefficient.

Table 3.9: BCCI efficiency score for the first 50 goalkeepers

	Full Name	Nationality	θ^*	DMU Status
1	Aaron McCarey	Republic of Ireland	1.0000	BCC efficient
2	Aaron Ramsdale	England	0.0233	BCC inefficient
3	Aarón Escandell Banacloche	Spain	0.0833	BCC inefficient
4	Adrián	Spain	0.0526	BCC inefficient
5	Adrián San Miguel del Castillo	Spain	0.2500	BCC inefficient
6	Agustín Federico Marchesín	Argentina	0.1111	BCC inefficient
7	Aiden Stone	England	1.0000	BCC efficient
8	Aitor Fernández Abarisketa	Spain	0.0333	BCC inefficient
9	Alban Lafont	France	0.0250	BCC inefficient
10	Alberto Paleari	Italy	0.2000	BCC inefficient
11	Aldo Junior Simoncini	San Marino	0.5000	BCC inefficient
12	Alejandro Remiro Gargallo	Spain	0.0208	BCC inefficient
13	Alessandro Berardi	Italy	1.0000	BCC efficient
14	Alessio Cragno	Italy	0.0286	BCC inefficient
15	Alex Bass	England	0.5000	BCC inefficient
16	Alex Cairns	England	0.3333	BCC inefficient
17	Alex McCarthy	England	0.0294	BCC inefficient
18	Alex Meret	Italy	0.0357	BCC inefficient
19	Alex Palmer	England	0.2000	BCC inefficient
20	Alex Smithies	England	1.0000	BCC efficient
21	Alexander Meyer	Germany	0.0263	BCC inefficient
22	Alexander Nübel	Germany	0.3333	BCC inefficient
23	Alexandre Oukidja	France	0.0256	BCC inefficient
24	Alexandros Paschalakis	Greece	1.0000	BCC efficient
25	Aleš Hruška	Czech Republic	0.2000	BCC inefficient
26	Alfonso Pastor Vacas	Spain	1.0000	BCC efficient
27	Alfred Gomis	Senegal	0.0303	BCC inefficient
28	Alisson Becker	Brazil	0.0227	BCC inefficient
29	Alphonse Areola	France	0.0256	BCC inefficient
30	Amjhad Nazih	France	1.0000	BCC efficient
31	Anatolii Trubin	Ukraine	0.1000	BCC inefficient
32	Andoni Zubiaurre Dorronsoro	Spain	1.0000	BCC efficient
33	Andrea Consigli	Italy	0.0263	BCC inefficient
34	Andreas Linde	Sweden	0.0625	BCC inefficient
35	Andreas Luthe	Germany	0.5000	BCC inefficient
36	Andrew Lonergan	England	1.0000	BCC efficient
37	Andriy Lunin	Ukraine	0.5000	BCC inefficient
38	Andriy Pyatov	Ukraine	1.0000	BCC efficient
39	Andrés Fernández	Spain	0.0556	BCC inefficient
40	Andrés Tomás Prieto Albert	Spain	0.5000	BCC inefficient
41	Andy Fisher	England	1.0000	BCC efficient
42	Anthony Lopes	Portugal	0.0250	BCC inefficient
43	Anthony Racioppi	Switzerland	0.0476	BCC inefficient
44	Antonio Mirante	Italy	0.0588	BCC inefficient
45	Antonio Sivera Salvá	Spain	0.2500	BCC inefficient
46	Artur Boruc	Poland	0.2500	BCC inefficient
47	Ashley Maynard-Brewer	Australia	1.0000	BCC efficient
48	Asmir Begović	Bosnia and Herzegovina	0.2000	BCC inefficient
49	Aynsley Pears	England	1.0000	BCC efficient
50	Bailey Peacock-Farrell	England	0.1250	BCC inefficient

The table consists of the first 50 goalkeepers in alphabetical order. There is an almost even split between BBC-inefficient and BCC-efficient players.

3.5.2 DEA results for the classical BCCI model for defenders

Table 3.10: BCCI efficiency score for the first 50 defenders

	Full Name	**Nationality**	θ^*	**DMU Status**
1	Aaron Cresswell	England	0.0238	BCC inefficient
2	Aaron Hayden	England	0.2500	BCC inefficient
3	Aaron O'Driscoll	Republic of Ireland	0.5000	BCC inefficient
4	Aaron Pierre	Grenada	0.2500	BCC inefficient
5	Aarón Martín	Spain	0.0476	BCC inefficient
6	Abbas Hüseynov	Azerbaijan	0.1250	BCC inefficient
7	Abdou Diallo	France	0.0256	BCC inefficient
8	Abdoulaye Bamba	Côte d'Ivoire	0.0476	BCC inefficient
9	Abdoulaye Ousame	France	1.0000	BCC efficient
10	Abdourahmane Barry	France	0.0769	BCC inefficient
11	Abzal Beysebekov	Kazakhstan	0.5000	BCC inefficient
12	Achraf Hakimi Mouh	Morocco	0.0185	BCC inefficient
13	Adam Bodzek	Germany	0.0270	BCC inefficient
14	Adam Crookes	England	0.5000	BCC inefficient
15	Adam Jackson	England	0.3333	BCC inefficient
16	Adam Marušić	Montenegro	0.0196	BCC inefficient
17	Adam Reach	England	0.1667	BCC inefficient
18	Adam Smith	England	0.2000	BCC inefficient
19	Adam Sušac	Croatia	0.1667	BCC inefficient
20	Adam Webster	England	0.0286	BCC inefficient
21	Adetayo Edun	England	0.1667	BCC inefficient
22	Adrián Hernández Hernández	Spain	1.0000	BCC efficient
23	Adrián Marín Gómez	Spain	0.0526	BCC inefficient
24	Advan Kadušić	Bosnia and Herzegovina	0.5000	BCC inefficient
25	Ahmet Gürleyen	Germany	0.5000	BCC inefficient
26	Akinwale Joseph Odimayo	England	0.2000	BCC inefficient
27	Albert Vallci	Austria	0.2000	BCC inefficient
28	Alberto Moreno	Spain	0.1111	BCC inefficient
29	Alberto Rodríguez Baro	Spain	0.0526	BCC inefficient
30	Aleix Vidal	Spain	0.0400	BCC inefficient
31	Alejandro Centelles Plaza	Spain	1.0000	BCC efficient
32	Alejandro Grimaldo García	Spain	0.1429	BCC inefficient
33	Aleksandar Dragović	Austria	0.0909	BCC inefficient
34	Aleksandar Ignjovski	Serbia	0.0714	BCC inefficient
35	Aleksandar Kolarov	Serbia	0.0909	BCC inefficient
36	Aleksandar Živanović	Serbia	0.3333	BCC inefficient
37	Aleksandr Martynovich	Belarus	0.1111	BCC inefficient
38	Aleksandr Pavlovets	Belarus	1.0000	BCC efficient
39	Alessandro Bastoni	Italy	0.0204	BCC inefficient
40	Alessandro Buongiorno	Italy	0.0556	BCC inefficient
41	Alessandro Tripaldelli	Italy	0.0714	BCC inefficient
42	Alessio Romagnoli	Italy	0.0256	BCC inefficient
43	Alex Baptiste	England	0.5000	BCC inefficient
44	Alex Cini	Malta	1.0000	BCC efficient
45	Alex Ferrari	Italy	0.0625	BCC inefficient
46	Alex Junior Christian	Haiti	0.3333	BCC inefficient
47	Alex Nicolao Telles	Brazil	0.0400	BCC inefficient
48	Alex Pearce	Republic of Ireland	0.5000	BCC inefficient
49	Alex Sandro	Brazil	0.0233	BCC inefficient
50	Alexander Hahn	Germany	0.1250	BCC inefficient

The table consists of the first 50 defenders in alphabetical order. The majority of players are BCC-inefficient.

3.5.3 DEA results for the classical BCCI model for midfielders

Table 3.11: BCCI efficiency score for the first 50 midfielders

	Full Name	Nationality	θ^*	DMU Status
1	Aaron Greene	England	0.3333	BCC inefficient
2	Aaron Hickey	Scotland	0.0833	BCC inefficient
3	Aaron Hunt	Germany	0.0357	BCC inefficient
4	Aaron Morley	England	0.3333	BCC inefficient
5	Aaron Opoku	Germany	0.0435	BCC inefficient
6	Aaron Ramsey	Wales	0.0333	BCC inefficient
7	Aaron Wan-Bissaka	England	0.0185	BCC inefficient
8	Aaron Wildig	England	0.2000	BCC inefficient
9	Abdellah Zoubir	France	0.1111	BCC inefficient
10	Abdoulaye Doucouré	France	0.0294	BCC inefficient
11	Abdoulaye Touré	France	0.0333	BCC inefficient
12	Achraf Drif	France	0.5000	BCC inefficient
13	Adam Clayton	England	1.0000	BCC efficient
14	Adam David Lallana	England	0.0323	BCC inefficient
15	Adam May	England	0.3333	BCC inefficient
16	Adam Ounas	Algeria	0.0400	BCC inefficient
17	Adam Phillips	England	0.1667	BCC inefficient
18	Adam Randell	England	0.5000	BCC inefficient
19	Adama Traoré Diarra	Spain	0.0244	BCC inefficient
20	Adel Taarabt	Morocco	0.1429	BCC inefficient
21	Ademipo Odubeko	England	1.0000	BCC efficient
22	Adil Aouchiche	France	0.0286	BCC inefficient
23	Adil Elmoueden	Germany	0.5000	BCC inefficient
24	Adnan Januzaj	Belgium	0.0278	BCC inefficient
25	Adrian Bernabe Garcia	Spain	1.0000	BCC efficient
26	Adrian Fein	Germany	0.3333	BCC inefficient
27	Adrian Grbic	Austria	0.0278	BCC inefficient
28	Adrian Stanilewicz	Germany	0.1250	BCC inefficient
29	Adrian Zöfelt	Germany	0.5000	BCC inefficient
30	Adrien Hunou	France	0.0417	BCC inefficient
31	Adrien Rabiot	France	0.0213	BCC inefficient
32	Adrien Sebastian Perruchet Silva	Portugal	0.0385	BCC inefficient
33	Adrien Tameze	France	0.0294	BCC inefficient
34	Adrián Andrés Cubas	Argentina	0.0333	BCC inefficient
35	Aeron Edwards	Wales	0.5000	BCC inefficient
36	Afeez Aremu	Nigeria	0.0588	BCC inefficient
37	Ahmed Eissa El Mohamady	Egypt	0.0588	BCC inefficient
38	Ahmet Arslan	Germany	0.0556	BCC inefficient
39	Aihen Muñoz Capellán	Spain	0.0526	BCC inefficient
40	Aimar Oroz	Spain	0.5000	BCC inefficient
41	Aimen Moueffek	France	0.0500	BCC inefficient
42	Ainsley Maitland-Niles	England	0.0286	BCC inefficient
43	Ajdin Hrustic	Australia	0.5000	BCC inefficient
44	Akaki Gogia	Germany	0.5000	BCC inefficient
45	Alan Browne	Republic of Ireland	0.3333	BCC inefficient
46	Alan Judge	Republic of Ireland	0.3333	BCC inefficient
47	Alan Patrick Lourenço	Brazil	0.1111	BCC inefficient
48	Albert Gudmundsson	Iceland	0.1250	BCC inefficient
49	Alberto Grassi	Italy	0.0435	BCC inefficient
50	Alberto Soro Álvarez	Spain	0.0357	BCC inefficient

The table consists of the first 50 midfielders in alphabetical order. The majority of players are BCC-inefficient.

3.5.4 DEA results for the classical BCCI model for forwards

Table 3.12: BCCI efficiency score for the first 50 forwards

	Full Name	Nationality	θ^*	DMU Status
1	Aaron Anthony Connolly	Republic of Ireland	0.0588	BCC inefficient
2	Aaron Collins	Wales	0.5000	BCC inefficient
3	Aaron Jarvis	England	0.2500	BCC inefficient
4	Aaron Martin	England	0.3333	BCC inefficient
5	Aaron Rowe	England	0.5000	BCC inefficient
6	Aaron Seydel	Germany	0.0714	BCC inefficient
7	Abdoulay Diaby	Mali	0.3333	BCC inefficient
8	Abdoulaye Diallo	Senegal	0.2500	BCC inefficient
9	Abdul Majeed Waris	Ghana	0.0625	BCC inefficient
10	Aboubakar Kamara	France	0.0400	BCC inefficient
11	Adalberto Peñaranda Maestre	Venezuela	0.3333	BCC inefficient
12	Adam Armstrong	England	0.3333	BCC inefficient
13	Adam Zreľák	Slovakia	0.2000	BCC inefficient
14	Adan George	England	1.0000	BCC efficient
15	Ademola Lookman	England	0.0286	BCC inefficient
16	Admir Mehmedi	Switzerland	0.2000	BCC inefficient
17	Adolfo Julián Gaich	Argentina	0.0500	BCC inefficient
18	Adrian Tabarcea Petre	Romania	0.2500	BCC inefficient
19	Adriel D'Avila Ba Loua	Côte d'Ivoire	0.3333	BCC inefficient
20	Adrien Truffert	France	0.0286	BCC inefficient
21	Adrián Gallardo Valdés	Spain	0.5000	BCC inefficient
22	Affamefuna-Michael Ifeadigo	Nigeria	0.5000	BCC inefficient
23	Agustín Gonzalo Torassa	Argentina	0.3333	BCC inefficient
24	Ahmed Hassan Mahgoub	Egypt	0.1429	BCC inefficient
25	Aiden O'Brien	Republic of Ireland	0.5000	BCC inefficient
26	Aitor Ruibal García	Spain	0.0333	BCC inefficient
27	Alassane Pléa	France	0.0833	BCC inefficient
28	Alberto Cerri	Italy	0.0385	BCC inefficient
29	Alberto Perea Correoso	Spain	0.0345	BCC inefficient
30	Albian Ajeti	Switzerland	0.1429	BCC inefficient
31	Albion Vrenezi	Germany	0.0323	BCC inefficient
32	Alejandro Blanco Sánchez	Spain	0.0625	BCC inefficient
33	Alejandro Millán Iranzo	Spain	0.5000	BCC inefficient
34	Aleksa Vukanović	Serbia	0.2000	BCC inefficient
35	Aleksandar Mitrović	Serbia	0.0323	BCC inefficient
36	Aleksandar Vujačić	Montenegro	0.3333	BCC inefficient
37	Aleksey Shchetkin	Kazakhstan	0.5000	BCC inefficient
38	Alex Addai	England	0.2000	BCC inefficient
39	Alex Gilliead	England	0.5000	BCC inefficient
40	Alex Iwobi	Nigeria	0.0278	BCC inefficient
41	Alex Mighten	England	0.5000	BCC inefficient
42	Alex Samuel	Wales	0.5000	BCC inefficient
43	Alexander Isak	Sweden	0.0227	BCC inefficient
44	Alexander MacDonald	Scotland	0.2500	BCC inefficient
45	Alexander Mesa Travieso	Spain	0.5000	BCC inefficient
46	Alexander Sørloth	Norway	0.1250	BCC inefficient
47	Alexandre Lacazette	France	0.0233	BCC inefficient
48	Alexandre Mendy	France	1.0000	BCC efficient
49	Alexis Claude Maurice	France	0.0270	BCC inefficient
50	Alexis Sanchez	Chile	0.0256	BCC inefficient

The table consists of the first 50 forwards in alphabetical order. The majority of players are BCC-inefficient.

3.6 DEA results for the classical ADD model

This section presents the results for the classical Additive model for goalkeepers, defenders, midfielders and forwards.

3.6.1 DEA results for the classical ADD model for goalkeepers

It should be mentioned again that an efficiency score 0 implies that the particular player is ADD-efficient otherwise it is deemed ADD-inefficient. This applies to all groups of players.

Table 3.13: ADD efficiency score for the first 50 goalkeepers

	Full Name	**Nationality**	ϕ^*	**DMU Status**
1	Aaron McCarey	Ireland	122.6667	ADD inefficient
2	Aaron Ramsdale	England	117.7297	ADD inefficient
3	Aarón Escandell Banacloche	Spain	156.3864	ADD inefficient
4	Adrián	Spain	0.0000	ADD efficient
5	Adrián San Miguel del Castillo	Spain	329.0000	ADD inefficient
6	Agustín Federico Marchesín	Argentina	350.3292	ADD inefficient
7	Aiden Stone	England	0.0000	ADD efficient
8	Aitor Fernández Abarisketa	Spain	2595.9298	ADD inefficient
9	Alban Lafont	France	1923.4394	ADD inefficient
10	Alberto Paleari	Italy	365.8833	ADD inefficient
11	Aldo Junior Simoncini	San Marino	198.6667	ADD inefficient
12	Alejandro Remiro Gargallo	Spain	770.7982	ADD inefficient
13	Alessandro Berardi	Italy	54.0000	ADD inefficient
14	Alessio Cragno	Italy	126.7472	ADD inefficient
15	Alex Bass	England	166.3111	ADD inefficient
16	Alex Cairns	England	236.4311	ADD inefficient
17	Alex McCarthy	England	2087.7463	ADD inefficient
18	Alex Meret	Italy	1974.5537	ADD inefficient
19	Alex Palmer	England	492.5909	ADD inefficient
20	Alex Smithies	England	122.6667	ADD inefficient
21	Alexander Meyer	Germany	95.1723	ADD inefficient
22	Alexander Nübel	Germany	190.2500	ADD inefficient
23	Alexandre Oukidja	France	2675.4330	ADD inefficient
24	Alexandros Paschalakis	Greece	166.0000	ADD inefficient
25	Aleš Hruška	Czech Republic	137.0000	ADD inefficient
26	Alfonso Pastor Vacas	Spain	138.7500	ADD inefficient
27	Alfred Gomis	Senegal	119.4006	ADD inefficient
28	Alisson Becker	Brazil	36.7010	ADD inefficient
29	Alphonse Areola	France	1831.9234	ADD inefficient
30	Amjhad Nazih	France	122.6667	ADD inefficient
31	Anatolii Trubin	Ukraine	167.9640	ADD inefficient
32	Andoni Zubiaurre Dorronsoro	Spain	0.0000	ADD efficient
33	Andrea Consigli	Italy	117.6036	ADD inefficient
34	Andreas Linde	Sweden	166.5269	ADD inefficient
35	Andreas Luthe	Germany	171.0000	ADD inefficient
36	Andrew Lonergan	England	122.6667	ADD inefficient
37	Andriy Lunin	Ukraine	61.3333	ADD inefficient
38	Andriy Pyatov	Ukraine	138.7500	ADD inefficient
39	Andrés Fernández	Spain	1376.1321	ADD inefficient
40	Andrés Tomás Prieto Albert	Spain	218.5000	ADD inefficient
41	Andy Fisher	England	91.5000	ADD inefficient
42	Anthony Lopes	Portugal	1850.9992	ADD inefficient
43	Anthony Racioppi	Switzerland	1768.7835	ADD inefficient
44	Antonio Mirante	Italy	857.6810	ADD inefficient
45	Antonio Sivera Salvá	Spain	273.5474	ADD inefficient
46	Artur Boruc	Poland	243.0000	ADD inefficient
47	Ashley Maynard-Brewer	Australia	0.0000	ADD efficient
48	Asmir Begović	Bosnia	410.0000	ADD inefficient
49	Aynsley Pears	England	0.0000	ADD efficient
50	Bailey Peacock-Farrell	England	702.0000	ADD inefficient

The table consists of the first 50 goalkeepers in alphabetical order. The majority of players are ADD-inefficient.

3.6.2 DEA results for the classical ADD model for defenders

Table 3.14: ADD efficiency score for the first 50 defenders

	Full Name	Nationality	ϕ^*	DMU Status
1	Aaron Cresswell	England	0.0000	ADD efficient
2	Aaron Hayden	England	0.0000	ADD efficient
3	Aaron O'Driscoll	Ireland	0.0000	ADD efficient
4	Aaron Pierre	Grenada	0.0000	ADD efficient
5	Aarón Martín	Spain	0.0000	ADD efficient
6	Abbas Hüseynov	Azerbaijan	0.0000	ADD efficient
7	Abdou Diallo	France	0.0000	ADD efficient
8	Abdoulaye Bamba	Côte d'Ivoire	0.0000	ADD efficient
9	Abdoulaye Ousame	France	0.0000	ADD efficient
10	Abdourahmane Barry	France	381.8206	ADD inefficient
11	Abzal Beysebekov	Kazakhstan	0.0000	ADD efficient
12	Achraf Hakimi Mouh	Morocco	0.0000	ADD efficient
13	Adam Bodzek	Germany	0.0000	ADD efficient
14	Adam Crookes	England	45.1041	ADD inefficient
15	Adam Jackson	England	0.0000	ADD efficient
16	Adam Marušić	Montenegro	0.0000	ADD efficient
17	Adam Reach	England	0.0000	ADD efficient
18	Adam Smith	England	24.4117	ADD inefficient
19	Adam Sušac	Croatia	0.0000	ADD efficient
20	Adam Webster	England	908.8630	ADD inefficient
21	Adetayo Edun	England	0.0000	ADD efficient
22	Adrián Hernández Hernández	Spain	0.0000	ADD efficient
23	Adrián Marín Gómez	Spain	0.0000	ADD efficient
24	Advan Kadušić	Bosnia	0.0000	ADD efficient
25	Ahmet Gürleyen	Germany	0.0000	ADD efficient
26	Akinwale Joseph Odimayo	England	175.6968	ADD inefficient
27	Albert Vallci	Austria	0.0000	ADD efficient
28	Alberto Moreno	Spain	0.0000	ADD efficient
29	Alberto Rodríguez Baro	Spain	845.0944	ADD inefficient
30	Aleix Vidal	Spain	0.0000	ADD efficient
31	Alejandro Centelles Plaza	Spain	0.0000	ADD efficient
32	Alejandro Grimaldo García	Spain	0.0000	ADD efficient
33	Aleksandar Dragović	Austria	0.0000	ADD efficient
34	Aleksandar Ignjovski	Serbia	117.6877	ADD inefficient
35	Aleksandar Kolarov	Serbia	0.0000	ADD efficient
36	Aleksandar Živanović	Serbia	0.0000	ADD efficient
37	Aleksandr Martynovich	Belarus	0.0000	ADD efficient
38	Aleksandr Pavlovets	Belarus	0.0000	ADD efficient
39	Alessandro Bastoni	Italy	0.0000	ADD efficient
40	Alessandro Buongiorno	Italy	0.0000	ADD efficient
41	Alessandro Tripaldelli	Italy	0.0000	ADD efficient
42	Alessio Romagnoli	Italy	1314.1879	ADD inefficient
43	Alex Baptiste	England	0.0000	ADD efficient
44	Alex Cini	Malta	0.0000	ADD efficient
45	Alex Ferrari	Italy	684.2493	ADD inefficient
46	Alex Junior Christian	Haiti	0.0000	ADD efficient
47	Alex Nicolao Telles	Brazil	0.0000	ADD efficient
48	Alex Pearce	Ireland	0.0000	ADD efficient
49	Alex Sandro	Brazil	1077.1835	ADD inefficient
50	Alexander Hahn	Germany	0.0000	ADD efficient

The table consists of the first 50 defenders in alphabetical order. The majority of players are ADD-efficient.

3.6.3 DEA results for the classical ADD model for midfielders

Table 3.15: ADD efficiency score for the first 50 midfielders

	Full Name	Nationality	ϕ^*	DMU Status
1	Aaron Greene	England	322.6667	ADD inefficient
2	Aaron Hickey	Scotland	678.6630	ADD inefficient
3	Aaron Hunt	Germany	1762.2090	ADD inefficient
4	Aaron Morley	England	364.0000	ADD inefficient
5	Aaron Opoku	Germany	1778.5127	ADD inefficient
6	Aaron Ramsey	Wales	2103.5006	ADD inefficient
7	Aaron Wan-Bissaka	England	6974.5767	ADD inefficient
8	Aaron Wildig	England	453.6736	ADD inefficient
9	Abdellah Zoubir	France	1089.3624	ADD inefficient
10	Abdoulaye Doucouré	France	4148.5571	ADD inefficient
11	Abdoulaye Touré	France	2605.2743	ADD inefficient
12	Achraf Drif	France	48.0000	ADD inefficient
13	Adam Clayton	England	90.0000	ADD inefficient
14	Adam David Lallana	England	5048.8784	ADD inefficient
15	Adam May	England	256.0000	ADD inefficient
16	Adam Ounas	Algeria	721.9181	ADD inefficient
17	Adam Phillips	England	639.2969	ADD inefficient
18	Adam Randell	England	228.0000	ADD inefficient
19	Adama Traoré Diarra	Spain	4657.3751	ADD inefficient
20	Adel Taarabt	Morocco	466.2635	ADD inefficient
21	Ademipo Odubeko	England	118.0000	ADD inefficient
22	Adil Aouchiche	France	1928.7346	ADD inefficient
23	Adil Elmoueden	Germany	158.1650	ADD inefficient
24	Adnan Januzaj	Belgium	0.0000	ADD efficient
25	Adrian Bernabe Garcia	Spain	74.0000	ADD inefficient
26	Adrian Fein	Germany	0.0000	ADD efficient
27	Adrian Grbic	Austria	0.0000	ADD efficient
28	Adrian Stanilewicz	Germany	31.5116	ADD inefficient
29	Adrian Zöfelt	Germany	225.0000	ADD inefficient
30	Adrien Hunou	France	0.0000	ADD efficient
31	Adrien Rabiot	France	4561.6033	ADD inefficient
32	Adrien Sebastian	Portugal	5147.5564	ADD inefficient
33	Adrien Tameze	France	4356.0184	ADD inefficient
34	Adrián Andrés Cubas	Argentina	4436.3553	ADD inefficient
35	Aeron Edwards	Wales	255.0000	ADD inefficient
36	Afeez Aremu	Nigeria	493.8630	ADD inefficient
37	Ahmed Eissa	Egypt	608.3423	ADD inefficient
38	Ahmet Arslan	Germany	125.6757	ADD inefficient
39	Aihen Muñoz Capellán	Spain	2681.8453	ADD inefficient
40	Aimar Oroz	Spain	87.0000	ADD inefficient
41	Aimen Moueffek	France	600.8148	ADD inefficient
42	Ainsley Maitland-Niles	England	3458.8330	ADD inefficient
43	Ajdin Hrustic	Australia	63.0000	ADD inefficient
44	Akaki Gogia	Germany	63.0000	ADD inefficient
45	Alan Browne	Ireland	223.7742	ADD inefficient
46	Alan Judge	Ireland	221.6083	ADD inefficient
47	Alan Patrick Lourenço	Brazil	1213.2337	ADD inefficient
48	Albert Gudmundsson	Iceland	294.3748	ADD inefficient
49	Alberto Grassi	Italy	844.3922	ADD inefficient
50	Alberto Soro Álvarez	Spain	0.0000	ADD efficient

The table consists of the first 50 midfielders in alphabetical order. The majority of players are ADD-inefficient.

3.6.4 DEA results for the classical ADD model for forwards

Table 3.16: ADD efficiency score for the first 50 forwards

	Full Name	**Nationality**	ϕ^*	**DMU Status**
1	Aaron Anthony	Ireland	1959.2879	ADD inefficient
2	Aaron Collins	Wales	160.0000	ADD inefficient
3	Aaron Jarvis	England	31.8078	ADD inefficient
4	Aaron Martin	England	450.8022	ADD inefficient
5	Aaron Rowe	England	430.7500	ADD inefficient
6	Aaron Seydel	Germany	579.1463	ADD inefficient
7	Abdoulay Diaby	Mali	32.1321	ADD inefficient
8	Abdoulaye Diallo	Senegal	331.6647	ADD inefficient
9	Abdul Majeed Waris	Ghana	1287.2940	ADD inefficient
10	Aboubakar Kamara	France	1624.6926	ADD inefficient
11	Adalberto Peñaranda	Venezuela	32.6612	ADD inefficient
12	Adam Armstrong	England	321.3002	ADD inefficient
13	Adam Zreľák	Slovakia	66.4000	ADD inefficient
14	Adan George	England	180.0000	ADD inefficient
15	Ademola Lookman	England	3561.9656	ADD inefficient
16	Admir Mehmedi	Switzerland	0.0000	ADD efficient
17	Adolfo Julián Gaich	Argentina	777.6145	ADD inefficient
18	Adrian Tabarcea Petre	Romania	284.4145	ADD inefficient
19	Adriel D'Avila	Côte d'Ivoire	329.3126	ADD inefficient
20	Adrien Truffert	France	4373.6689	ADD inefficient
21	Adrián Gallardo Valdés	Spain	355.6500	ADD inefficient
22	Affamefuna-Michael	Nigeria	170.6768	ADD inefficient
23	Agustín Gonzalo	Argentina	489.8362	ADD inefficient
24	Ahmed Hassan	Egypt	51.5595	ADD inefficient
25	Aiden O'Brien	Ireland	137.5000	ADD inefficient
26	Aitor Ruibal García	Spain	2798.0867	ADD inefficient
27	Alassane Pléa	France	0.0000	ADD efficient
28	Alberto Cerri	Italy	219.2256	ADD inefficient
29	Alberto Perea Correoso	Spain	2945.7612	ADD inefficient
30	Albian Ajeti	Switzerland	166.6434	ADD inefficient
31	Albion Vrenezi	Germany	2962.6561	ADD inefficient
32	Alejandro Blanco Sánchez	Spain	1421.9267	ADD inefficient
33	Alejandro Millán Iranzo	Spain	0.0000	ADD efficient
34	Aleksa Vukanović	Serbia	288.4000	ADD inefficient
35	Aleksandar Mitrović	Serbia	1975.4495	ADD inefficient
36	Aleksandar Vujačić	Montenegro	13.0363	ADD inefficient
37	Aleksey Shchetkin	Kazakhstan	76.0000	ADD inefficient
38	Alex Addai	England	79.6000	ADD inefficient
39	Alex Gilliead	England	271.0000	ADD inefficient
40	Alex Iwobi	Nigeria	3386.5718	ADD inefficient
41	Alex Mighten	England	94.0000	ADD inefficient
42	Alex Samuel	Wales	180.1250	ADD inefficient
43	Alexander Isak	Sweden	3289.2334	ADD inefficient
44	Alexander MacDonald	Scotland	296.7500	ADD inefficient
45	Alexander Mesa Travieso	Spain	34.0000	ADD inefficient
46	Alexander Sørloth	Norway	866.5103	ADD inefficient
47	Alexandre Lacazette	France	1922.5287	ADD inefficient
48	Alexandre Mendy	France	58.0000	ADD inefficient
49	Alexis Claude Maurice	France	1961.3059	ADD inefficient
50	Alexis Sanchez	Chile	1107.8706	ADD inefficient

The table consists of the first 50 forwards in alphabetical order. The majority of players are ADD-inefficient.

3.7 DEA results for the bootstrap based DEA models

This section presents the results for the DEA models based on the bootstrap method. Before getting to the results, an explanation of how the bootstrap resampling method can be adapted within the optimisation framework or DEA to be more specific needs to be given. The contribution of this study is to use the non parametric bootstrap resampling technique to simulate DEA inputs and outputs for each DMU. The motivation behind this is because computation of DEA efficiency scores for various models is based on some sample of inputs and outputs. This score will differ for different samples which makes the score an estimate of the "true" efficiency score which is unknown. The functional form of this estimate is the estimator which is a random variable. When using the BCC multiplier model, the estimator of the efficiency score for the reference DMU is given by:

$$\hat{\theta} = \frac{\sum_{r=1}^{s} u_r y_{ro} - u_o}{\sum_{i=1}^{m} v_i x_{io}} \tag{3.4}$$

Equation (3.4) shows that a different sample of inputs and outputs yields a different efficiency score. Instead of using a point estimate for the efficiency score, this study proposes using a confidence interval. Let $X_{1j}, X_{2j}, ..., X_{mj}$ and $Y_{1j}, Y_{2j}, ..., Y_{sj}$ be independently and identically distributed (iid) random variables from unknown distributions F and G. Where X_{ij} represents the i^{th} input of DMU$_j$ and Y_{rj} represents the r^{th} output of DMU$_j$, with $j = 1, 2, ..., n$, $i = 1, 2, ..., m$, and $r = 1, 2, ..., s$. Let $\hat{\theta} = \hat{\theta}(X_{10}, X_{20}, ..., X_{m0}, Y_{10}, Y_{20}, ..., Y_{s0})$ be an estimator for the "true" efficiency score θ. The objective is to construct a $100(1 - \alpha)\%$ confidence interval for θ using the non parametric bootstrap approach. One would like to estimate the variance or standard error of the estimator $\hat{\theta}(X_{10}, X_{20}, ..., X_{m0}, Y_{10}, Y_{20}, ..., Y_{s0})$ from sample to sample as this will give us an idea of how reliable the estimator is. We can use the bootstrap resampling technique to do this, since in practice the underlying distribution of the sample data is unknown.

It should be noted that the sets of inputs and outputs for goalkeepers, defenders, midfielders, and forwards remains unchanged from the previous section. This section and the succeeding sections incorporate the non-parametric bootstrap method within DEA as explained in Chapter 2. We are now in a position to approximate the standard error of $\hat{\tau}$ and use the bootstrap distribution to approximate the confidence interval for the unknown parameter τ. As stated earlier, we do this by sampling with replacement from the given inputs $X_{1j}, X_{2j}, ..., X_{mj}$ and outputs $Y_{1j}, Y_{2j}, ..., Y_{sj}$ data sets for DMU$_j$. For each of the $B = 1000$ bootstrap samples, we calculate $\hat{\tau}^*$ which is the value of the objective function under the SBM model, that is for each DMU we solve the optimisation problem B times to obtain $\hat{\tau}_1^* \leq \hat{\tau}_2^* \leq ... \leq \hat{\tau}_B^*$.

3.7.1 DEA results for the bootstrap based SBM model for goalkeepers

The SBM model used is the one stated at the beginning of the method section, the same one used for the classical results. Traditionally one would solve this optimisation problem for each DMU and obtain an optimal value or efficiency score $0 \leq \hat{\tau} \leq 1$. However we are not interested in a point estimate or single value of the efficiency score since this will vary depending on the sample data used, in other words the efficiency score from the traditional SBM model is an estimator $\hat{\tau}$ of the true score τ. For each DMU we need to approximate a confidence interval for τ. Each DMU_j (in this case goalkeepers) has a set of inputs and outputs denoted by $X_{1j}, X_{2j}, ..., X_{mj}$ and $Y_{1j}, Y_{2j}, ..., Y_{sj}, \quad j = 1, 2, ..., n$. For each DMU, we then:

1. Sample B samples with replacement from $X_{1j}, X_{2j}, ..., X_{mj}$
2. Sample B samples with replacement from $Y_{1j}, Y_{2j}, ..., Y_{sj}$
3. For each of the B samples in step 1 and 2, solve the above optimisation problem to obtain $\hat{\tau}_1^*, \hat{\tau}_2^*, ..., \hat{\tau}_B^*$
4. Sort the estimators in ascending order: $\hat{\tau}_1^* \leq \hat{\tau}_2^* ... \leq \hat{\tau}_B^*$
5. A $100(1-\alpha)\%$ confidence interval for τ is then given by $\left[\hat{\tau}_{(r)}^*; \hat{\tau}_{(s)}^*\right]$
6. The variance of $\hat{\tau}$ is then estimated by $S_{\hat{\tau}}^2 = \frac{1}{B} \sum_{i=1}^{B} \left(\hat{\tau}_i^* - \hat{\tau}(\,\cdot\,)\right)^2$

The following table shows the results of the SBM bootstrap based DEA methodology for goalkeepers. The standard error (stde) is the standard deviation of all the bootstrap estimates $\hat{\tau}_1^*, \hat{\tau}_2^*, ..., \hat{\tau}_B^*$ calculated from the B bootstrap samples. This is done for each DMU. The motivation for using a confidence interval was described in Chapter 2. We make the assertion that a DMU must be deemed SBM efficient if and only if the bootstrap confidence interval contains 1. This can be interpreted as follows: given a confidence level α, the $(1-\alpha)\%$ bootstrap confidence interval will contain 1 $(1-\alpha)\%$ percent of the time, i.e, the DMU in question is efficient $(1-\alpha)\%$ percent of the time. This definition is much better than the traditional one of deciding efficiency based on one sample $X_{1j}, X_{2j}, ..., X_{mj}, Y_{1j}, Y_{2j}, ..., Y_{sj}$ and one score $\boldsymbol{\tau^*}$. It should be noted that the procedure described for the SBM model applies for the BCCI model as well, for the ADD model the confidence interval must contain 0 for the DMU to be deemed ADD-efficient.

Table 3.17: Bootstrap SBM model results for the first 50 goalkeepers

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron McCarey	Ireland	0.1174	[0.004, 1.0]	SBM efficient
2	Aaron Ramsdale	England	0.1595	[0.0, 1.0]	SBM efficient
3	Aarón Escandell	Spain	0.1356	[0.0, 0.795]	SBM inefficient
4	Adrián	Spain	0.1249	[0.003, 1.0]	SBM efficient
5	Adrián San Miguel	Spain	0.1474	[0.001, 1.0]	SBM efficient
6	Agustín Federico	Argentina	0.1594	[0.004, 1.0]	SBM efficient
7	Aiden Stone	England	0.0000	[1.0, 1.0]	SBM efficient
8	Aitor Fernández	Spain	0.1555	[0.005, 1.0]	SBM efficient
9	Alban Lafont	France	0.1142	[0.0, 0.101]	SBM inefficient
10	Alberto Paleari	Italy	0.1310	[0.0, 0.671]	SBM inefficient
11	Aldo Junior	San Marino	0.1161	[0.003, 1.0]	SBM efficient
12	Alejandro Remiro	Spain	0.1096	[0.002, 1.0]	SBM efficient
13	Alessandro Berardi	Italy	0.0000	[1.0, 1.0]	SBM efficient
14	Alessio Cragno	Italy	0.1313	[0.0, 0.505]	SBM inefficient
15	Alex Bass	England	0.1217	[0.005, 1.0]	SBM efficient
16	Alex Cairns	England	0.1137	[0.016, 1.0]	SBM efficient
17	Alex McCarthy	England	0.1499	[0.0, 1.0]	SBM efficient
18	Alex Meret	Italy	0.1522	[0.012, 1.0]	SBM efficient
19	Alex Palmer	England	0.1308	[0.0, 0.669]	SBM inefficient
20	Alex Smithies	England	0.1187	[0.006, 1.0]	SBM efficient
21	Alexander Meyer	Germany	0.1080	[0.0, 0.169]	SBM inefficient
22	Alexander Nübel	Germany	0.1686	[0.0, 1.0]	SBM efficient
23	Alexandre Oukidja	France	0.1013	[0.001, 0.205]	SBM inefficient
24	Alexandros Paschalakis	Greece	0.0831	[0.169, 1.0]	SBM efficient
25	Aleš Hruška	Czech	0.1348	[0.0, 0.669]	SBM inefficient
26	Alfonso Pastor	Spain	0.1158	[0.009, 1.0]	SBM efficient
27	Alfred Gomis	Senegal	0.1368	[0.0, 0.674]	SBM inefficient
28	Alisson Becker	Brazil	0.1164	[0.0, 0.284]	SBM inefficient
29	Alphonse Areola	France	0.1022	[0.0, 0.162]	SBM inefficient
30	Amjhad Nazih	France	0.1176	[0.007, 1.0]	SBM efficient
31	Anatolii Trubin	Ukraine	0.1360	[0.0, 0.771]	SBM inefficient
32	Andoni Zubiaurre	Spain	0.0000	[1.0, 1.0]	SBM efficient
33	Andrea Consigli	Italy	0.1194	[0.0, 0.171]	SBM inefficient
34	Andreas Linde	Sweden	0.1469	[0.0, 1.0]	SBM efficient
35	Andreas Luthe	Germany	0.0845	[0.169, 1.0]	SBM efficient
36	Andrew Lonergan	England	0.1214	[0.006, 1.0]	SBM efficient
37	Andriy Lunin	Ukraine	0.0901	[0.002, 0.833]	SBM inefficient
38	Andriy Pyatov	Ukraine	0.1172	[0.007, 1.0]	SBM efficient
39	Andrés Fernández	Spain	0.1549	[0.005, 1.0]	SBM efficient
40	Andrés Tomás	Spain	0.1206	[0.006, 1.0]	SBM efficient
41	Andy Fisher	England	0.1106	[0.006, 1.0]	SBM efficient
42	Anthony Lopes	Portugal	0.1610	[0.0, 1.0]	SBM efficient
43	Anthony Racioppi	Switzerland	0.1619	[0.006, 1.0]	SBM efficient
44	Antonio Mirante	Italy	0.1221	[0.0, 0.356]	SBM inefficient
45	Antonio Sivera	Spain	0.1679	[0.001, 1.0]	SBM efficient
46	Artur Boruc	Poland	0.1605	[0.001, 1.0]	SBM efficient
47	Ashley Maynard	Australia	0.1075	[0.002, 1.0]	SBM efficient
48	Asmir Begović	Bosnia	0.1545	[0.001, 1.0]	SBM efficient
49	Aynsley Pears	England	0.1135	[0.004, 1.0]	SBM efficient
50	Bailey Peacock	England	0.1574	[0.004, 1.0]	SBM efficient

The table consists of the first 50 goalkeepers in alphabetical order. The majority of players are SBM-efficient.

3.7.2 DEA Results for the bootstrap based SBM model for defenders

Table 3.18: Bootstrap SBM model results for the first 50 defenders

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Cresswell	England	0.2327	[0.001, 1.0]	SBM efficient
2	Aaron Hayden	England	0.1822	[0.009, 1.0]	SBM efficient
3	Aaron O'Driscoll	Ireland	0.1476	[0.065, 1.0]	SBM efficient
4	Aaron Pierre	Grenada	0.2145	[0.007, 0.037]	SBM inefficient
5	Aarón Martín	Spain	0.2372	[0.0, 1.0]	SBM efficient
6	Abbas Hüseynov	Azerbaijan	0.2339	[0.003, 1.0]	SBM efficient
7	Abdou Diallo	France	0	[0.014, 1.0]	SBM efficient
8	Abdoulaye Bamba	Côte d'Ivoire	0.2340	[0.001, 0.138]	SBM inefficient
9	Abdoulaye Ousame	France	0.1348	[0.078, 1.0]	SBM efficient
10	Abdourahmane Barry	France	0.2379	[0.0, 0.05]	SBM inefficient
11	Abzal Beysebekov	Kazakhstan	0.1203	[0.097, 1.0]	SBM efficient
12	Achraf Hakimi Mouh	Morocco	0.2054	[0.013, 1.0]	SBM efficient
13	Adam Bodzek	Germany	0	[0.004, 0.148]	SBM inefficient
14	Adam Crookes	England	0.1728	[0.033, 1.0]	SBM efficient
15	Adam Jackson	England	0.2194	[0.001, 0.011]	SBM inefficient
16	Adam Marušić	Montenegro	0.2277	[0.002, 0.273]	SBM inefficient
17	Adam Reach	England	0.2374	[0.002, 1.0]	SBM efficient
18	Adam Smith	England	0.2041	[0.01, 1.0]	SBM efficient
19	Adam Sušac	Croatia	0.0923	[0.164, 1.0]	SBM efficient
20	Adam Webster	England	0.2293	[0.002, 1.0]	SBM efficient
21	Adetayo Edun	England	0.1773	[0.022, 1.0]	SBM efficient
22	Adrián Hernández	Spain	0.1793	[0.026, 1.0]	SBM efficient
23	Adrián Marín Gómez	Spain	0.2253	[0.005, 1.0]	SBM efficient
24	Advan Kadušić	Bosnia	0.2243	[0.006, 1.0]	SBM efficient
25	Ahmet Gürleyen	Germany	0.0490	[0.376, 0.878]	SBM inefficient
26	Akinwale Joseph Odimayo	England	0.1780	[0.016, 1.0]	SBM efficient
27	Albert Vallci	Austria	0.1720	[0.013, 1.0]	SBM efficient
28	Alberto Moreno	Spain	0.2006	[0.02, 1.0]	SBM efficient
29	Alberto Rodríguez	Spain	0.2129	[0.016, 1.0]	SBM efficient
30	Aleix Vidal	Spain	0.2364	[0.001, 1.0]	SBM efficient
31	Alejandro Centelles	Spain	0.1906	[0.008, 0.75]	SBM inefficient
32	Alejandro Grimaldo	Spain	0.2119	[0.009, 1.0]	SBM efficient
33	Aleksandar Dragović	Austria	0.2009	[0.016, 1.0]	SBM efficient
34	Aleksandar Ignjovski	Serbia	0.2285	[0.004, 0.116]	SBM inefficient
35	Aleksandar Kolarov	Serbia	0.1755	[0.024, 1.0]	SBM efficient
36	Aleksandar Živanović	Serbia	0.1466	[0.047, 1.0]	SBM efficient
37	Aleksandr Martynovich	Belarus	0.2413	[0.0, 0.031]	SBM inefficient
38	Aleksandr Pavlovets	Belarus	0.0000	[1.0, 1.0]	SBM efficient
39	Alessandro Bastoni	Italy	0.2217	[0.004, 1.0]	SBM efficient
40	Alessandro Buongiorno	Italy	0.2254	[0.004, 0.454]	SBM inefficient
41	Alessandro Tripaldelli	Italy	0.2407	[0.001, 0.169]	SBM inefficient
42	Alessio Romagnoli	Italy	0.2201	[0.004, 0.285]	SBM inefficient
43	Alex Baptiste	England	0.1504	[0.064, 1.0]	SBM efficient
44	Alex Cini	Malta	0.2003	[0.007, 0.598]	SBM inefficient
45	Alex Ferrari	Italy	0.2298	[0.001, 1.0]	SBM efficient
46	Alex Junior Christian	Haiti	0.1999	[0.018, 0.578]	SBM inefficient
47	Alex Nicolao Telles	Brazil	0.2381	[0.001, 1.0]	SBM efficient
48	Alex Pearce	Ireland	0.2189	[0.008, 1.0]	SBM efficient
49	Alex Sandro	Brazil	0.2234	[0.004, 1.0]	SBM efficient
50	Alexander Hahn	Germany	0.2063	[0.01, 1.0]	SBM efficient

Table 3.18 consists of the first 50 defenders. The majority of players are SBM-efficient.

3.7.3 DEA results for the bootstrap based SBM model for midfielders

Table 3.19: Bootstrap SBM model results for the first 50 midfielders

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Greene	England	0.0000	[1.0, 1.0]	SBM efficient
2	Aaron Hickey	Scotland	0.0000	[1.0, 1.0]	SBM efficient
3	Aaron Hunt	Germany	0.1824	[0.008, 1.0]	SBM efficient
4	Aaron Morley	England	0.0000	[1.0, 1.0]	SBM efficient
5	Aaron Opoku	Germany	0.1348	[0.048, 1.0]	SBM efficient
6	Aaron Ramsey	Wales	0.1103	[0.029, 1.0]	SBM efficient
7	Aaron Wan-Bissaka	England	0.1275	[0.008, 1.0]	SBM efficient
8	Aaron Wildig	England	0.0849	[0.119, 1.0]	SBM efficient
9	Abdellah Zoubir	France	0.1035	[0.034, 1.0]	SBM efficient
10	Abdoulaye Doucouré	France	0.1363	[0.013, 1.0]	SBM efficient
11	Abdoulaye Touré	France	0.1940	[0.013, 1.0]	SBM efficient
12	Achraf Drif	France	0.0000	[1.0, 1.0]	SBM efficient
13	Adam Clayton	England	0.0000	[1.0, 1.0]	SBM efficient
14	Adam David Lallana	England	0.1835	[0.03, 0.669]	SBM inefficient
15	Adam May	England	0.0000	[1.0, 1.0]	SBM efficient
16	Adam Ounas	Algeria	0.1050	[0.023, 1.0]	SBM efficient
17	Adam Phillips	England	0.0721	[0.248, 1.0]	SBM efficient
18	Adam Randell	England	0.0000	[1.0, 1.0]	SBM efficient
19	Adama Traoré Diarra	Spain	0.1250	[0.012, 1.0]	SBM efficient
20	Adel Taarabt	Morocco	0.0987	[0.08, 1.0]	SBM efficient
21	Ademipo Odubeko	England	0.0000	[1.0, 1.0]	SBM efficient
22	Adil Aouchiche	France	0.1123	[0.029, 1.0]	SBM efficient
23	Adil Elmoueden	Germany	0.0000	[1.0, 1.0]	SBM efficient
24	Adnan Januzaj	Belgium	0.1904	[0.016, 1.0]	SBM efficient
25	Adrian Bernabe Garcia	Spain	0.0000	[1.0, 1.0]	SBM efficient
26	Adrian Fein	Germany	0.0679	[0.21, 1.0]	SBM efficient
27	Adrian Grbic	Austria	0.0778	[0.139, 1.0]	SBM efficient
28	Adrian Stanilewicz	Germany	0.0000	[1.0, 1.0]	SBM efficient
29	Adrian Zöfelt	Germany	0.0000	[1.0, 1.0]	SBM efficient
30	Adrien Hunou	France	0.0789	[0.144, 1.0]	SBM efficient
31	Adrien Rabiot	France	0.1244	[0.007, 1.0]	SBM efficient
32	Adrien Sebastian Silva	Portugal	0.1400	[0.022, 1.0]	SBM efficient
33	Adrien Tameze	France	0.0806	[0.116, 1.0]	SBM efficient
34	Adrián Andrés Cubas	Argentina	0.0687	[0.13, 1.0]	SBM efficient
35	Aeron Edwards	Wales	0.0000	[1.0, 1.0]	SBM efficient
36	Afeez Aremu	Nigeria	0.0000	[1.0, 1.0]	SBM efficient
37	Ahmed Eissa Fattah	Egypt	0.0000	[1.0, 1.0]	SBM efficient
38	Ahmet Arslan	Germany	0.0792	[0.16, 1.0]	SBM efficient
39	Aihen Muñoz Capellán	Spain	0.1457	[0.029, 1.0]	SBM efficient
40	Aimar Oroz	Spain	0.0000	[1.0, 1.0]	SBM efficient
41	Aimen Moueffek	France	0.0000	[1.0, 1.0]	SBM efficient
42	Ainsley Maitland-Niles	England	0.1451	[0.019, 1.0]	SBM efficient
43	Ajdin Hrustic	Australia	0.0000	[1.0, 1.0]	SBM efficient
44	Akaki Gogia	Germany	0.0000	[1.0, 1.0]	SBM efficient
45	Alan Browne	Ireland	0.0797	[0.153, 1.0]	SBM efficient
46	Alan Judge	Ireland	0.0597	[0.329, 1.0]	SBM efficient
47	Alan Patrick Lourenço	Brazil	0.1267	[0.069, 1.0]	SBM efficient
48	Albert Gudmundsson	Iceland	0.0890	[0.142, 1.0]	SBM efficient
49	Alberto Grassi	Italy	0.0000	[1.0, 1.0]	SBM efficient
50	Alberto Soro Álvarez	Spain	0.1155	[0.029, 1.0]	SBM efficient

The table consists of the first 50 midfielders in alphabetical order. The majority of players are SBM-efficient.

3.7.4 DEA results for the bootstrap based SBM model for forwards

Table 3.20: Bootstrap SBM model results for the first 50 forwards

	Full Name	**Nat**	**Stde**	**CI(95%)**	**DMU Status**
1	Aaron Anthony Connolly	Ireland	0.0739	[0.091, 1.0]	SBM efficient
2	Aaron Collins	Wales	0.0000	[1.0, 1.0]	SBM efficient
3	Aaron Jarvis	England	0.0630	[0.225, 1.0]	SBM efficient
4	Aaron Martin	England	0.0492	[0.333, 1.0]	SBM efficient
5	Aaron Rowe	England	0.0483	[0.334, 1.0]	SBM efficient
6	Aaron Seydel	Germany	0.0571	[0.298, 1.0]	SBM efficient
7	Abdoulay Diaby	Mali	0.0615	[0.23, 1.0]	SBM efficient
8	Abdoulaye Diallo	Senegal	0.0614	[0.255, 1.0]	SBM efficient
9	Abdul Majeed Waris	Ghana	0.0672	[0.171, 1.0]	SBM efficient
10	Aboubakar Kamara	France	0.1183	[0.002, 1.0]	SBM efficient
11	Adalberto Peñaranda Maestre	Venezuela	0.0647	[0.227, 1.0]	SBM efficient
12	Adam Armstrong	England	0.0340	[0.37, 1.0]	SBM efficient
13	Adam Zreľák	Slovakia	0.0000	[1.0, 1.0]	SBM efficient
14	Adan George	England	0.0000	[1.0, 1.0]	SBM efficient
15	Ademola Lookman	England	0.1216	[0.0, 1.0]	SBM efficient
16	Admir Mehmedi	Switzerland	0.1027	[0.002, 1.0]	SBM efficient
17	Adolfo Julián Gaich	Argentina	0.1317	[0.001, 1.0]	SBM efficient
18	Adrian Tabarcea Petre	Romania	0.1133	[0.001, 1.0]	SBM efficient
19	Adriel D'Avila Ba Loua	Côte d'Ivoire	0.0499	[0.304, 1.0]	SBM efficient
20	Adrien Truffert	France	0.1057	[0.0, 1.0]	SBM efficient
21	Adrián Gallardo Valdés	Spain	0.0464	[0.323, 1.0]	SBM efficient
22	Affamefuna-Michael Ifeadigo	Nigeria	0.0596	[0.227, 1.0]	SBM efficient
23	Agustín Gonzalo Torassa	Argentina	0.1098	[0.008, 1.0]	SBM efficient
24	Ahmed Hassan Mahgoub	Egypt	0.0998	[0.016, 1.0]	SBM efficient
25	Aiden O'Brien	Ireland	0.0000	[1.0, 1.0]	SBM efficient
26	Aitor Ruibal García	Spain	0.1406	[0.0, 1.0]	SBM efficient
27	Alassane Pléa	France	0.1105	[0.0, 1.0]	SBM efficient
28	Alberto Cerri	Italy	nan	[0.002, 1.0]	SBM efficient
29	Alberto Perea Correoso	Spain	0.0619	[0.063, 1.0]	SBM efficient
30	Albian Ajeti	Switzerland	0.1202	[0.003, 1.0]	SBM efficient
31	Albion Vrenezi	Germany	0.1044	[0.001, 1.0]	SBM efficient
32	Alejandro Blanco Sánchez	Spain	nan	[0.0, 1.0]	SBM efficient
33	Alejandro Millán Iranzo	Spain	0.0805	[0.194, 1.0]	SBM efficient
34	Aleksa Vukanović	Serbia	0.0000	[1.0, 1.0]	SBM efficient
35	Aleksandar Mitrović	Serbia	0.2217	[0.0, 1.0]	SBM efficient
36	Aleksandar Vujačić	Montenegro	0.0483	[0.272, 1.0]	SBM efficient
37	Aleksey Shchetkin	Kazakhstan	0.0000	[1.0, 1.0]	SBM efficient
38	Alex Addai	England	0.0000	[1.0, 1.0]	SBM efficient
39	Alex Gilliead	England	0.0000	[1.0, 1.0]	SBM efficient
40	Alex Iwobi	Nigeria	0.0914	[0.043, 1.0]	SBM efficient
41	Alex Mighten	England	0.0000	[1.0, 1.0]	SBM efficient
42	Alex Samuel	Wales	0.0414	[0.381, 1.0]	SBM efficient
43	Alexander Isak	Sweden	0.1074	[0.0, 1.0]	SBM efficient
44	Alexander MacDonald	Scotland	0.0000	[1.0, 1.0]	SBM efficient
45	Alexander Mesa Travieso	Spain	0.0000	[1.0, 1.0]	SBM efficient
46	Alexander Sørloth	Norway	0.0503	[0.248, 1.0]	SBM efficient
47	Alexandre Lacazette	France	0.2170	[0.0, 1.0]	SBM efficient
48	Alexandre Mendy	France	0.0000	[1.0, 1.0]	SBM efficient
49	Alexis Claude Maurice	France	0.1149	[0.0, 1.0]	SBM efficient
50	Alexis Sanchez	Chile	0.1268	[0.0, 1.0]	SBM efficient

The table consists of the first 50 forwards in alphabetical order. All 50 players are SBM-efficient.

3.7.5 DEA results for the bootstrap based BCCI model for goalkeepers

The BCCI model used in this section is the one stated at the beginning of the method section, the same one used for the classical results. As stated earlier a DMU whose confidence interval includes 1 is deemed BCC-efficient contains otherwise it is deemed BCC-inefficient.

Table 3.21: Bootstrap BCCI model results for the first 50 goalkeepers

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron McCarey	Ireland	0.1399	[0.0, 1.0]	BCCI efficient
2	Aaron Ramsdale	England	0.1353	[0.0, 1.0]	BCCI efficient
3	Aarón Escandell Banacloche	Spain	0.1272	[0.0, 1.0]	BCCI efficient
4	Adrián	Spain	0.0025	[0.0, 0.1429]	BCCI inefficient
5	Adrián San Miguel del Castillo	Spain	0.0086	[0.0, 0.25]	BCCI inefficient
6	Agustín Federico Marchesín	Argentina	0.0029	[0.0, 0.125]	BCCI inefficient
7	Aiden Stone	England	0.1391	[0.0, 1.0]	BCCI efficient
8	Aitor Fernández Abarisketa	Spain	0.0002	[0.0, 0.0333]	BCCI inefficient
9	Alban Lafont	France	0.0318	[0.0, 0.5]	BCCI inefficient
10	Alberto Paleari	Italy	0.0087	[0.0, 0.2]	BCCI inefficient
11	Aldo Junior Simoncini	San Marino	0.0405	[0.0, 0.5]	BCCI inefficient
12	Alejandro Remiro Gargallo	Spain	0.0001	[0.0, 0.0217]	BCCI inefficient
13	Alessandro Berardi	Italy	0.1406	[0.0, 1.0]	BCCI efficient
14	Alessio Cragno	Italy	0.1146	[0.0, 1.0]	BCCI efficient
15	Alex Bass	England	0.0368	[0.0, 0.5]	BCCI inefficient
16	Alex Cairns	England	0.0161	[0.0, 0.3333]	BCCI inefficient
17	Alex McCarthy	England	0.0151	[0.0, 0.3333]	BCCI inefficient
18	Alex Meret	Italy	0.0002	[0.0, 0.0357]	BCCI inefficient
19	Alex Palmer	England	0.0052	[0.0, 0.2]	BCCI inefficient
20	Alex Smithies	England	0.1297	[0.0, 1.0]	BCCI efficient
21	Alexander Meyer	Germany	0.1365	[0.0, 1.0]	BCCI efficient
22	Alexander Nübel	Germany	0.0403	[0.0, 0.5]	BCCI inefficient
23	Alexandre Oukidja	France	0.1122	[0.0003, 1.0]	BCCI efficient
24	Alexandros Paschalakis	Greece	0.1352	[0.0, 1.0]	BCCI efficient
25	Aleš Hruška	Czech	0.1214	[0.0, 1.0]	BCCI efficient
26	Alfonso Pastor Vacas	Spain	0.1297	[0.0, 1.0]	BCCI efficient
27	Alfred Gomis	Senegal	0.1431	[0.0, 1.0]	BCCI efficient
28	Alisson Becker	Brazil	0.1231	[0.0, 1.0]	BCCI efficient
29	Alphonse Areola	France	0.0293	[0.0, 0.5]	BCCI inefficient
30	Amjhad Nazih	France	0.1369	[0.0, 1.0]	BCCI efficient
31	Anatolii Trubin	Ukraine	0.1301	[0.0, 1.0]	BCCI efficient
32	Andoni Zubiaurre Dorronsoro	Spain	0.1431	[0.0, 1.0]	BCCI efficient
33	Andrea Consigli	Italy	0.1406	[0.0, 1.0]	BCCI efficient
34	Andreas Linde	Sweden	0.1317	[0.0, 1.0]	BCCI efficient
35	Andreas Luthe	Germany	0.0347	[0.0, 0.5]	BCCI inefficient
36	Andrew Lonergan	England	0.1324	[0.0, 1.0]	BCCI efficient
37	Andriy Lunin	Ukraine	0.2460	[0.0, 1.0]	BCCI efficient
38	Andriy Pyatov	Ukraine	0.1293	[0.0, 1.0]	BCCI efficient
39	Andrés Fernández	Spain	0.0005	[0.0, 0.0556]	BCCI inefficient
40	Andrés Tomás Prieto Albert	Spain	0.0345	[0.0, 0.5]	BCCI inefficient
41	Andy Fisher	England	0.1440	[0.0, 1.0]	BCCI efficient
42	Anthony Lopes	Portugal	0.0314	[0.0, 0.5]	BCCI inefficient
43	Anthony Racioppi	Switzerland	0.0004	[0.0, 0.0476]	BCCI inefficient
44	Antonio Mirante	Italy	0.0311	[0.0, 0.5]	BCCI inefficient
45	Antonio Sivera Salvá	Spain	0.0111	[0.0, 0.25]	BCCI inefficient
46	Artur Boruc	Poland	0.0183	[0.0, 0.3333]	BCCI inefficient
47	Ashley Maynard-Brewer	Australia	0.2165	[0.0, 1.0]	BCCI efficient
48	Asmir Begović	Bosnia	0.0056	[0.0, 0.2]	BCCI inefficient
49	Aynsley Pears	England	0.2079	[0.0, 1.0]	BCCI efficient
50	Bailey Peacock-Farrell	England	0.0022	[0.0, 0.125]	BCCI inefficient

Table 3.21 consists of the first 50 goalkeepers. The majority of players are BCCI-efficient.

3.7.6 DEA results for the bootstrap based BCCI model for defenders

Table 3.22: Bootstrap BCCI model results for the first 50 defenders

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Cresswell	England	0.0753	[0.0, 0.8508]	BCCI inefficient
2	Aaron Hayden	England	0.0369	[0.0, 0.9655]	BCCI inefficient
3	Aaron O'Driscoll	Ireland	0.0038	[0.0, 0.0]	BCCI inefficient
4	Aaron Pierre	Grenada	0.1073	[0.1001, 1.0]	BCCI efficient
5	Aarón Martín	Spain	0.0983	[0.0, 1.0]	BCCI efficient
6	Abbas Hüseynov	Azerbaijan	0.0621	[0.0, 0.9937]	BCCI inefficient
7	Abdou Diallo	France	0.0653	[0.0526, 0.9857]	BCCI inefficient
8	Abdoulaye Bamba	Côte d'Ivoire	0.0498	[0.0, 0.7507]	BCCI inefficient
9	Abdoulaye Ousame	France	0.0064	[0.0, 0.0]	BCCI inefficient
10	Abdourahmane Barry	France	0.0679	[0.0, 0.8244]	BCCI inefficient
11	Abzal Beysebekov	Kazakhstan	0.0097	[0.0, 0.4308]	BCCI inefficient
12	Achraf Hakimi Mouh	Morocco	0.0844	[0.0, 0.8889]	BCCI inefficient
13	Adam Bodzek	Germany	0.0843	[0.0, 0.8896]	BCCI inefficient
14	Adam Crookes	England	0.0066	[0.0, 0.3111]	BCCI inefficient
15	Adam Jackson	England	0.0427	[0.0, 1.0]	BCCI efficient
16	Adam Marušić	Montenegro	0.0582	[0.0, 0.6805]	BCCI inefficient
17	Adam Reach	England	0.0452	[0.0, 0.9244]	BCCI inefficient
18	Adam Smith	England	0.0125	[0.0, 0.4416]	BCCI inefficient
19	Adam Sušac	Croatia	0.0124	[0.0, 0.0]	BCCI inefficient
20	Adam Webster	England	0.1133	[0.0, 0.9069]	BCCI inefficient
21	Adetayo Edun	England	0.0377	[0.0, 0.8333]	BCCI inefficient
22	Adrián Hernández	Spain	0.0109	[0.0, 0.4179]	BCCI inefficient
23	Adrián Marín Gómez	Spain	0.0748	[0.0, 0.8274]	BCCI inefficient
24	Advan Kadušić	Bosnia	0.0114	[0.0, 0.3111]	BCCI inefficient
25	Ahmet Gürleyen	Germany	0.0029	[0.0, 0.0]	BCCI inefficient
26	Akinwale Joseph	England	0.0481	[0.0, 0.96]	BCCI inefficient
27	Albert Vallci	Austria	0.0427	[0.0, 0.9333]	BCCI inefficient
28	Alberto Moreno	Spain	0.0274	[0.0, 0.7556]	BCCI inefficient
29	Alberto Rodríguez Baro	Spain	0.0800	[0.0714, 1.0]	BCCI efficient
30	Aleix Vidal	Spain	0.0379	[0.0, 0.6652]	BCCI inefficient
31	Alejandro Centelles	Spain	0.0276	[0.0, 1.0]	BCCI efficient
32	Alejandro Grimaldo	Spain	0.0182	[0.0, 0.6178]	BCCI inefficient
33	Aleksandar Dragović	Austria	0.0193	[0.0, 0.7273]	BCCI inefficient
34	Aleksandar Ignjovski	Serbia	0.0473	[0.0, 0.8406]	BCCI inefficient
35	Aleksandar Kolarov	Serbia	0.0262	[0.0, 0.7214]	BCCI inefficient
36	Aleksandar Živanović	Serbia	0.0122	[0.0, 0.3333]	BCCI inefficient
37	Aleksandr Martynovich	Belarus	0.0853	[0.0, 1.0]	BCCI efficient
38	Aleksandr Pavlovets	Belarus	0.0184	[0.0, 0.0]	BCCI inefficient
39	Alessandro Bastoni	Italy	0.0817	[0.0, 0.9639]	BCCI inefficient
40	Alessandro Buongiorno	Italy	0.0801	[0.0, 0.9297]	BCCI inefficient
41	Alessandro Tripaldelli	Italy	0.0861	[0.0, 1.0]	BCCI efficient
42	Alessio Romagnoli	Italy	0.1149	[0.0, 0.9333]	BCCI inefficient
43	Alex Baptiste	England	0.0086	[0.0, 0.3111]	BCCI inefficient
44	Alex Cini	Malta	0.0323	[0.0, 1.0]	BCCI efficient
45	Alex Ferrari	Italy	0.0674	[0.0, 0.7714]	BCCI inefficient
46	Alex Junior Christian	Haiti	0.0571	[0.3792, 1.0]	BCCI efficient
47	Alex Nicolao Telles	Brazil	0.0784	[0.0, 1.0]	BCCI efficient
48	Alex Pearce	Ireland	0.0095	[0.0, 0.3333]	BCCI inefficient
49	Alex Sandro	Brazil	0.0606	[0.0, 0.7568]	BCCI inefficient
50	Alexander Hahn	Germany	0.0160	[0.0, 0.3111]	BCCI inefficient

The table consists of the first 50 defenders in alphabetical order. The majority of players are BCCI-inefficient.

3.7.7 DEA results for the bootstrap based BCCI model for midfielders

Table 3.23: Bootstrap BCCI model results for the first 50 midfielders

	Full Name	**Nat**	**Stde**	**CI(95%)**	**DMU Status**
1	Aaron Greene	England	0.0012	[0.0, 0.0]	BCCI inefficient
2	Aaron Hickey	Scotland	0.0088	[0.0, 0.2381]	BCCI inefficient
3	Aaron Hunt	Germany	0.0359	[0.0, 0.5378]	BCCI inefficient
4	Aaron Morley	England	0.0017	[0.0, 0.0]	BCCI inefficient
5	Aaron Opoku	Germany	0.0722	[0.0, 0.976]	BCCI inefficient
6	Aaron Ramsey	Wales	0.0758	[0.0, 1.0]	BCCI efficient
7	Aaron Wan-Bissaka	England	0.0232	[0.0, 0.5556]	BCCI inefficient
8	Aaron Wildig	England	0.0149	[0.0, 0.3513]	BCCI inefficient
9	Abdellah Zoubir	France	0.0408	[0.0, 1.0]	BCCI efficient
10	Abdoulaye Doucouré	France	0.0441	[0.0, 0.8824]	BCCI inefficient
11	Abdoulaye Touré	France	0.0794	[0.0, 1.0]	BCCI efficient
12	Achraf Drif	France	0.0143	[0.0, 0.0]	BCCI inefficient
13	Adam Clayton	England	0.0084	[0.0, 0.0]	BCCI inefficient
14	Adam David Lallana	England	0.0416	[0.0, 0.9677]	BCCI inefficient
15	Adam May	England	0.0029	[0.0, 0.0]	BCCI inefficient
16	Adam Ounas	Algeria	0.0455	[0.0, 1.0]	BCCI efficient
17	Adam Phillips	England	0.0037	[0.0, 0.2088]	BCCI inefficient
18	Adam Randell	England	0.0040	[0.0, 0.0]	BCCI inefficient
19	Adama Traoré Diarra	Spain	0.0399	[0.0, 0.7317]	BCCI inefficient
20	Adel Taarabt	Morocco	0.0194	[0.0, 0.4487]	BCCI inefficient
21	Ademipo Odubeko	England	0.0097	[0.0, 0.0]	BCCI inefficient
22	Adil Aouchiche	France	0.0688	[0.0, 0.8571]	BCCI inefficient
23	Adil Elmoueden	Germany	0.0092	[0.0, 0.3846]	BCCI inefficient
24	Adnan Januzaj	Belgium	0.0664	[0.0, 0.8333]	BCCI inefficient
25	Adrian Bernabe Garcia	Spain	0.0109	[0.0, 0.0]	BCCI inefficient
26	Adrian Fein	Germany	0.0324	[0.0, 0.7258]	BCCI inefficient
27	Adrian Grbic	Austria	0.0324	[0.0, 0.8333]	BCCI inefficient
28	Adrian Stanilewicz	Germany	0.0090	[0.0, 0.0]	BCCI inefficient
29	Adrian Zöfelt	Germany	0.0034	[0.0, 0.0]	BCCI inefficient
30	Adrien Hunou	France	0.0349	[0.0, 0.9877]	BCCI inefficient
31	Adrien Rabiot	France	0.0870	[0.0, 1.0]	BCCI efficient
32	Adrien Silva	Portugal	0.0737	[0.0, 1.0]	BCCI efficient
33	Adrien Tameze	France	0.0318	[0.0, 0.8824]	BCCI inefficient
34	Adrián Andrés Cubas	Argentina	0.0731	[0.0, 1.0]	BCCI efficient
35	Aeron Edwards	Wales	0.0019	[0.0, 0.0]	BCCI inefficient
36	Afeez Aremu	Nigeria	0.0260	[0.0, 0.8615]	BCCI inefficient
37	Ahmed Eissa Fattah	Egypt	0.0102	[0.0, 0.0588]	BCCI inefficient
38	Ahmet Arslan	Germany	0.0272	[0.0, 0.8207]	BCCI inefficient
39	Aihen Muñoz Capellán	Spain	0.0163	[0.0, 0.4066]	BCCI inefficient
40	Aimar Oroz	Spain	0.0052	[0.0, 0.0]	BCCI inefficient
41	Aimen Moueffek	France	0.0113	[0.0, 0.3333]	BCCI inefficient
42	Ainsley Maitland-Niles	England	0.0380	[0.0, 0.8571]	BCCI inefficient
43	Ajdin Hrustic	Australia	0.0018	[0.0, 0.0]	BCCI inefficient
44	Akaki Gogia	Germany	0.0036	[0.0, 0.0]	BCCI inefficient
45	Alan Browne	Ireland	0.0221	[0.0, 0.4018]	BCCI inefficient
46	Alan Judge	Ireland	0.0181	[0.0, 0.4592]	BCCI inefficient
47	Alan Patrick Lourenço	Brazil	0.0187	[0.0, 0.5684]	BCCI inefficient
48	Albert Gudmundsson	Iceland	0.0129	[0.0, 0.5322]	BCCI inefficient
49	Alberto Grassi	Italy	0.0118	[0.0, 0.5852]	BCCI inefficient
50	Alberto Soro Álvarez	Spain	0.0623	[0.0, 1.0]	BCCI efficient

The table consists of the first 50 midfielders in alphabetical order. The majority of players are BCCI-inefficient.

3.7.8 DEA results for the bootstrap based BCCI model for forwards

Table 3.24: Bootstrap BCCI model results for the first 50 forwards

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Anthony Connolly	Ireland	0.0213	[0.0, 0.6381]	BCCI inefficient
2	Aaron Collins	Wales	0.0018	[0.0, 0.0]	BCCI inefficient
3	Aaron Jarvis	England	0.0055	[0.0, 0.0]	BCCI inefficient
4	Aaron Martin	England	0.0030	[0.0, 0.194]	BCCI inefficient
5	Aaron Rowe	England	0.0052	[0.0, 0.2444]	BCCI inefficient
6	Aaron Seydel	Germany	0.0183	[0.0, 0.6431]	BCCI inefficient
7	Abdoulay Diaby	Mali	0.0083	[0.0, 0.0]	BCCI inefficient
8	Abdoulaye Diallo	Senegal	0.0021	[0.0, 0.2035]	BCCI inefficient
9	Abdul Majeed Waris	Ghana	0.0133	[0.0, 0.4377]	BCCI inefficient
10	Aboubakar Kamara	France	0.0388	[0.0, 0.767]	BCCI inefficient
11	Adalberto Peñaranda	Venezuela	0.0065	[0.0, 0.0]	BCCI inefficient
12	Adam Armstrong	England	0.0055	[0.0, 0.3333]	BCCI inefficient
13	Adam Zreľák	Slovakia	0.0012	[0.0, 0.0]	BCCI inefficient
14	Adan George	England	0.0066	[0.0, 0.0]	BCCI inefficient
15	Ademola Lookman	England	0.0738	[0.0, 1.0]	BCCI efficient
16	Admir Mehmedi	Switzerland	0.0194	[0.0, 0.2626]	BCCI inefficient
17	Adolfo Julián Gaich	Argentina	0.0197	[0.0, 0.5732]	BCCI inefficient
18	Adrian Tabarcea Petre	Romania	0.0182	[0.0, 0.3116]	BCCI inefficient
19	Adriel D'Avila Ba Loua	Côte d'Ivoire	0.0035	[0.0, 0.1477]	BCCI inefficient
20	Adrien Truffert	France	0.0214	[0.0, 0.5422]	BCCI inefficient
21	Adrián Gallardo Valdés	Spain	0.0024	[0.0, 0.1453]	BCCI inefficient
22	Affamefuna-Michael	Nigeria	0.0091	[0.0, 0.1444]	BCCI inefficient
23	Agustín Gonzalo	Argentina	0.0216	[0.0, 0.6667]	BCCI inefficient
24	Ahmed Hassan	Egypt	0.0230	[0.0, 0.5409]	BCCI inefficient
25	Aiden O'Brien	Ireland	0.0019	[0.0, 0.0]	BCCI inefficient
26	Aitor Ruibal	Spain	0.0284	[0.0, 0.6667]	BCCI inefficient
27	Alassane Pléa	France	0.0391	[0.0, 1.0]	BCCI efficient
28	Alberto Cerri	Italy	0.0356	[0.0, 0.7692]	BCCI inefficient
29	Alberto Perea	Spain	0.0180	[0.0, 0.4483]	BCCI inefficient
30	Albian Ajeti	Switzerland	0.0336	[0.0, 0.9286]	BCCI inefficient
31	Albion Vrenezi	Germany	0.0444	[0.0, 0.8125]	BCCI inefficient
32	Alejandro Blanco	Spain	0.0526	[0.0, 1.0]	BCCI efficient
33	Alejandro Millán	Spain	0.0107	[0.0, 0.0]	BCCI inefficient
34	Aleksa Vukanović	Serbia	0.0010	[0.0, 0.0]	BCCI inefficient
35	Aleksandar Mitrović	Serbia	0.0805	[0.0, 1.0]	BCCI efficient
36	Aleksandar Vujačić	Montenegro	0.0005	[0.0, 0.0]	BCCI inefficient
37	Aleksey Shchetkin	Kazakhstan	0.0085	[0.0, 0.0]	BCCI inefficient
38	Alex Addai	England	0.0023	[0.0, 0.0]	BCCI inefficient
39	Alex Gilliead	England	0.0013	[0.0, 0.0]	BCCI inefficient
40	Alex Iwobi	Nigeria	0.0099	[0.0, 0.3822]	BCCI inefficient
41	Alex Mighten	England	0.0055	[0.0, 0.0]	BCCI inefficient
42	Alex Samuel	Wales	0.0075	[0.0, 0.275]	BCCI inefficient
43	Alexander Isak	Sweden	0.0466	[0.0, 1.0]	BCCI efficient
44	Alexander MacDonald	Scotland	0.0013	[0.0, 0.0]	BCCI inefficient
45	Alexander Mesa Travieso	Spain	0.0086	[0.0, 0.0]	BCCI inefficient
46	Alexander Sørloth	Norway	0.0198	[0.0, 0.609]	BCCI inefficient
47	Alexandre Lacazette	France	0.0170	[0.0, 0.4651]	BCCI inefficient
48	Alexandre Mendy	France	0.0099	[0.0, 0.0]	BCCI inefficient
49	Alexis Claude Maurice	France	0.0236	[0.0, 0.5922]	BCCI inefficient
50	Alexis Sanchez	Chile	0.0779	[0.0, 1.0]	BCCI efficient

The table consists of the first 50 forwards in alphabetical order. The majority of players are BCCI-inefficient.

3.7.9 DEA results for the bootstrap based Additive model for goalkeepers

The additive model used in this section is the one described earlier in the method section. As stated earlier a DMU is deemed ADD-efficient if and only if the confidence interval contains 0 otherwise it is deemed ADD-inefficient.

Table 3.25: Bootstrap ADD model results for the first 50 goalkeepers

	Full Name	**Nat**	**Stde**	**CI(95%)**	**DMU Status**
1	Aaron McCarey	Ireland	4821.5011	[9.5572, 228.0887]	ADD inefficient
2	Aaron Ramsdale	England	870.2707	[30.3732, 127.2671]	ADD inefficient
3	Aarón Escandell	Spain	4423.8369	[27.3743, 156.2856]	ADD inefficient
4	Adrián	Spain	1462.6826	[9.8339, 125.6861]	ADD inefficient
5	Adrián San Miguel	Spain	2435.6234	[10.8794, 86.6381]	ADD inefficient
6	Agustín Federico	Argentina	2320.1567	[12.5714, 120.2462]	ADD inefficient
7	Aiden Stone	England	325.8805	[0.0, 60.9671]	ADD efficient
8	Aitor Fernández	Spain	880.7008	[12.9365, 113.4926]	ADD inefficient
9	Alban Lafont	France	637.1809	[29.0941, 118.478]	ADD inefficient
10	Alberto Paleari	Italy	11299.5524	[9.8093, 172.6944]	ADD inefficient
11	Aldo Junior	San Marino	6529.9582	[9.7042, 282.5199]	ADD inefficient
12	Alejandro Remiro	Spain	1360.7417	[15.1987, 114.7046]	ADD inefficient
13	Alessandro Berardi	Italy	299.5158	[5.099, 57.0112]	ADD inefficient
14	Alessio Cragno	Italy	2450.6478	[34.8442, 128.2767]	ADD inefficient
15	Alex Bass	England	2434.5707	[8.775, 93.9839]	ADD inefficient
16	Alex Cairns	England	3501.4014	[6.3068, 98.0479]	ADD inefficient
17	Alex McCarthy	England	426.9768	[33.2507, 114.6476]	ADD inefficient
18	Alex Meret	Italy	1216.9611	[14.116, 109.8349]	ADD inefficient
19	Alex Palmer	England	3791.2452	[10.6522, 89.3667]	ADD inefficient
20	Alex Smithies	England	1761.6578	[9.2508, 112.3434]	ADD inefficient
21	Alexander Meyer	Germany	7030.0554	[35.1608, 132.8549]	ADD inefficient
22	Alexander Nübel	Germany	6239.1152	[10.1706, 150.5357]	ADD inefficient
23	Alexandre Oukidja	France	315.9245	[36.3835, 114.0377]	ADD inefficient
24	Alexandros Paschalakis	Greece	263.5571	[9.4862, 75.5134]	ADD inefficient
25	Aleš Hruška	Czech	10040.1521	[15.3474, 252.9493]	ADD inefficient
26	Alfonso Pastor Vacas	Spain	1702.4092	[9.0772, 110.4183]	ADD inefficient
27	Alfred Gomis	Senegal	1870.1662	[32.0394, 131.5481]	ADD inefficient
28	Alisson Becker	Brazil	406.8583	[37.1223, 122.4275]	ADD inefficient
29	Alphonse Areola	France	2542.7850	[34.1498, 122.5313]	ADD inefficient
30	Amjhad Nazih	France	2405.4023	[8.7246, 122.9573]	ADD inefficient
31	Anatolii Trubin	Ukraine	9160.7099	[30.2172, 125.2287]	ADD inefficient
32	Andoni Zubiaurre	Spain	352.7769	[0.0, 60.9671]	ADD efficient
33	Andrea Consigli	Italy	692.7651	[34.6058, 125.2475]	ADD inefficient
34	Andreas Linde	Sweden	4129.4114	[30.0501, 129.3066]	ADD inefficient
35	Andreas Luthe	Germany	309.2673	[9.7468, 81.5828]	ADD inefficient
36	Andrew Lonergan	England	2415.6877	[8.7325, 125.3972]	ADD inefficient
37	Andriy Lunin	Ukraine	11622.8519	[11.4254, 510.1002]	ADD inefficient
38	Andriy Pyatov	Ukraine	1887.2210	[9.0917, 111.8434]	ADD inefficient
39	Andrés Fernández	Spain	3338.3301	[12.1655, 116.1511]	ADD inefficient
40	Andrés Tomás	Spain	10458.4884	[8.7646, 272.8908]	ADD inefficient
41	Andy Fisher	England	9733.2126	[4.4241, 379.0589]	ADD inefficient
42	Anthony Lopes	Portugal	1533.2256	[33.6075, 119.9249]	ADD inefficient
43	Anthony Racioppi	Switzerland	1454.4512	[12.5904, 109.5153]	ADD inefficient
44	Antonio Mirante	Italy	1853.3087	[28.6798, 117.981]	ADD inefficient
45	Antonio Sivera	Spain	6677.0000	[10.3212, 139.6444]	ADD inefficient
46	Artur Boruc	Poland	5612.8817	[12.5762, 134.8272]	ADD inefficient
47	Ashley Maynard	Australia	2599.3840	[9.3274, 169.0658]	ADD inefficient
48	Asmir Begović	Bosnia	4734.3038	[11.7818, 100.3929]	ADD inefficient
49	Aynsley Pears	England	4383.8966	[9.4868, 192.8613]	ADD inefficient
50	Bailey Peacock-Farrell	England	4162.2702	[11.3941, 90.174]	ADD inefficient

The table consists of the first 50 goalkeepers. The majority of players are ADD-inefficient.

3.7.10 DEA results for the bootstrap based Additive model for defenders

Table 3.26: Bootstrap ADD model results for the first 50 defenders

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Cresswell	England	55.4017	[52.0, 10258.0]	ADD inefficient
2	Aaron Hayden	England	15.7130	[8.0, 907.0]	ADD inefficient
3	Aaron O'Driscoll	Ireland	3.2177	[7.0, 47.0]	ADD inefficient
4	Aaron Pierre	Grenada	15.4014	[7.0, 872.0]	ADD inefficient
5	Aarón Martín	Spain	38.5496	[45.0, 5016.0]	ADD inefficient
6	Abbas Hüseynov	Azerbaijan	16.0046	[10.0, 907.0]	ADD inefficient
7	Abdou Diallo	France	47.3986	[42.0, 7431.0]	ADD inefficient
8	Abdoulaye Bamba	Côte d'Ivoire	37.4193	[34.0, 4761.0]	ADD inefficient
9	Abdoulaye Ousame	France	3.3693	[5.0, 49.0]	ADD inefficient
10	Abdourahmane Barry	France	25.0838	[24.0, 2180.0]	ADD inefficient
11	Abzal Beysebekov	Kazakhstan	10.9179	[9.0, 458.0]	ADD inefficient
12	Achraf Hakimi Mouh	Morocco	53.8335	[74.0, 9668.0]	ADD inefficient
13	Adam Bodzek	Germany	39.5128	[48.0, 5244.0]	ADD inefficient
14	Adam Crookes	England	13.0715	[6.0, 633.0]	ADD inefficient
15	Adam Jackson	England	13.3200	[4.0, 688.0]	ADD inefficient
16	Adam Marušić	Montenegro	59.6598	[84.0, 11566.0]	ADD inefficient
17	Adam Reach	England	18.8765	[9.0, 1264.0]	ADD inefficient
18	Adam Smith	England	17.2783	[12.0, 1080.0]	ADD inefficient
19	Adam Sušac	Croatia	3.7195	[9.0, 62.0]	ADD inefficient
20	Adam Webster	England	49.8362	[44.0, 8418.0]	ADD inefficient
21	Adetayo Edun	England	17.1162	[16.0, 1089.0]	ADD inefficient
22	Adrián Hernández	Spain	8.6916	[6.0, 273.0]	ADD inefficient
23	Adrián Marín Gómez	Spain	22.7519	[23.0, 1742.0]	ADD inefficient
24	Advan Kadušić	Bosnia	12.9466	[5.0, 633.0]	ADD inefficient
25	Ahmet Gürleyen	Germany	13.3095	[0.0, 634.0]	ADD efficient
26	Akinwale Joseph	England	15.5540	[9.0, 906.0]	ADD inefficient
27	Albert Vallci	Austria	16.5747	[11.0, 1030.0]	ADD inefficient
28	Alberto Moreno	Spain	13.2303	[12.0, 638.0]	ADD inefficient
29	Alberto Rodríguez Baro	Spain	30.7783	[29.0, 3160.0]	ADD inefficient
30	Aleix Vidal	Spain	32.0485	[19.0, 3416.0]	ADD inefficient
31	Alejandro Centelles Plaza	Spain	10.2377	[3.0, 368.0]	ADD inefficient
32	Alejandro Grimaldo	Spain	22.8572	[16.0, 1767.0]	ADD inefficient
33	Aleksandar Dragović	Austria	26.1454	[22.0, 2296.0]	ADD inefficient
34	Aleksandar Ignjovski	Serbia	21.6058	[12.0, 1569.0]	ADD inefficient
35	Aleksandar Kolarov	Serbia	24.5180	[25.0, 2124.0]	ADD inefficient
36	Aleksandar Živanović	Serbia	15.6194	[10.0, 902.0]	ADD inefficient
37	Aleksandr Martynovich	Belarus	24.9962	[18.0, 2254.0]	ADD inefficient
38	Aleksandr Pavlovets	Belarus	5.7308	[1.0, 114.0]	ADD inefficient
39	Alessandro Bastoni	Italy	57.2916	[66.0, 10635.0]	ADD inefficient
40	Alessandro Buongiorno	Italy	30.3405	[33.0, 3181.0]	ADD inefficient
41	Alessandro Tripaldelli	Italy	20.1179	[27.0, 1444.0]	ADD inefficient
42	Alessio Romagnoli	Italy	48.7691	[75.0, 7941.0]	ADD inefficient
43	Alex Baptiste	England	12.9923	[8.0, 635.0]	ADD inefficient
44	Alex Cini	Malta	8.9002	[3.0, 279.0]	ADD inefficient
45	Alex Ferrari	Italy	29.6825	[23.0, 3018.0]	ADD inefficient
46	Alex Junior	Haiti	6.8738	[9.0, 178.0]	ADD inefficient
47	Alex Nicolao Telles	Brazil	40.0102	[45.0, 5304.0]	ADD inefficient
48	Alex Pearce	Ireland	12.8640	[7.0, 632.0]	ADD inefficient
49	Alex Sandro	Brazil	50.1564	[69.0, 8186.0]	ADD inefficient
50	Alexander Hahn	Germany	25.2695	[17.0, 2254.0]	ADD inefficient

The table consists of the first 50 defenders. The majority of players are ADD-inefficient.

3.7.11 DEA results for the bootstrap based Additive model for midfielders

Table 3.27: Bootstrap ADD model results for the first 50 midfielders

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Greene	England	2.7038	[2.0, 28.0]	ADD inefficient
2	Aaron Hickey	Scotland	3.6753	[4.0, 49.0]	ADD inefficient
3	Aaron Hunt	Germany	3.9636	[14.0, 77.0]	ADD inefficient
4	Aaron Morley	England	2.7977	[2.0, 30.0]	ADD inefficient
5	Aaron Opoku	Germany	3.3853	[10.0, 64.0]	ADD inefficient
6	Aaron Ramsey	Wales	3.9425	[13.0, 77.0]	ADD inefficient
7	Aaron Wan-Bissaka	England	5.2739	[17.0, 135.0]	ADD inefficient
8	Aaron Wildig	England	3.0877	[4.0, 44.0]	ADD inefficient
9	Abdellah Zoubir	France	3.2693	[18.0, 61.0]	ADD inefficient
10	Abdoulaye Doucouré	France	4.5597	[16.0, 101.0]	ADD inefficient
11	Abdoulaye Touré	France	4.1192	[13.0, 85.0]	ADD inefficient
12	Achraf Drif	France	1.5413	[1.0, 10.0]	ADD inefficient
13	Adam Clayton	England	1.8281	[1.0, 13.0]	ADD inefficient
14	Adam David Lallana	England	4.2934	[40.0, 100.0]	ADD inefficient
15	Adam May	England	2.5545	[0.0, 26.0]	ADD efficient
16	Adam Ounas	Algeria	3.9630	[20.0, 77.0]	ADD inefficient
17	Adam Phillips	England	2.9727	[4.0, 41.0]	ADD inefficient
18	Adam Randell	England	2.3899	[0.0, 23.0]	ADD efficient
19	Adama Traoré Diarra	Spain	4.4999	[32.0, 107.0]	ADD inefficient
20	Adel Taarabt	Morocco	3.1634	[4.0, 48.0]	ADD inefficient
21	Ademipo Odubeko	England	1.9934	[0.0, 17.0]	ADD efficient
22	Adil Aouchiche	France	4.0197	[20.0, 84.0]	ADD inefficient
23	Adil Elmoueden	Germany	2.4179	[2.0, 24.0]	ADD inefficient
24	Adnan Januzaj	Belgium	3.8649	[14.0, 70.0]	ADD inefficient
25	Adrian Bernabe Garcia	Spain	1.7387	[1.0, 13.0]	ADD inefficient
26	Adrian Fein	Germany	2.0726	[6.0, 22.0]	ADD inefficient
27	Adrian Grbic	Austria	3.8027	[8.0, 59.0]	ADD inefficient
28	Adrian Stanilewicz	Germany	2.8330	[2.0, 30.0]	ADD inefficient
29	Adrian Zöfelt	Germany	2.3699	[1.0, 24.0]	ADD inefficient
30	Adrien Hunou	France	3.4184	[6.0, 49.0]	ADD inefficient
31	Adrien Rabiot	France	4.7100	[26.0, 108.0]	ADD inefficient
32	Adrien Silva	Portugal	4.2536	[42.0, 102.0]	ADD inefficient
33	Adrien Tameze	France	4.9535	[9.0, 108.0]	ADD inefficient
34	Adrián Andrés Cubas	Argentina	5.0272	[9.0, 108.0]	ADD inefficient
35	Aeron Edwards	Wales	2.4834	[0.0, 24.0]	ADD efficient
36	Afeez Aremu	Nigeria	3.4337	[4.0, 43.0]	ADD inefficient
37	Ahmed Eissa Fattah	Egypt	4.1223	[4.0, 58.0]	ADD inefficient
38	Ahmet Arslan	Germany	2.7457	[6.0, 34.0]	ADD inefficient
39	Aihen Muñoz Capellán	Spain	4.3050	[9.0, 83.0]	ADD inefficient
40	Aimar Oroz	Spain	1.8619	[1.0, 14.0]	ADD inefficient
41	Aimen Moueffek	France	3.8243	[3.0, 51.0]	ADD inefficient
42	Ainsley Maitland-Niles	England	4.9562	[9.0, 105.0]	ADD inefficient
43	Ajdin Hrustic	Australia	1.6740	[1.0, 12.0]	ADD inefficient
44	Akaki Gogia	Germany	1.7136	[0.0, 12.0]	ADD efficient
45	Alan Browne	Ireland	2.5592	[9.0, 33.0]	ADD inefficient
46	Alan Judge	Ireland	2.0485	[6.0, 22.0]	ADD inefficient
47	Alan Patrick Lourenço	Brazil	2.8769	[22.0, 51.0]	ADD inefficient
48	Albert Gudmundsson	Iceland	2.9481	[5.0, 38.0]	ADD inefficient
49	Alberto Grassi	Italy	4.2503	[2.0, 60.0]	ADD inefficient
50	Alberto Soro Álvarez	Spain	3.5991	[14.0, 61.0]	ADD inefficient

The table consists of the first 50 midfielders. The majority of players are ADD-inefficient.

3.7.12 DEA results for the bootstrap based Additive model for forwards

Table 3.28: Bootstrap ADD model results for the first 50 forwards

	Full Name	Nat	Stde	CI(95%)	DMU Status
1	Aaron Anthony Connolly	Ireland	3.3740	[13.0, 63.0]	ADD inefficient
2	Aaron Collins	Wales	2.1859	[1.0, 19.0]	ADD inefficient
3	Aaron Jarvis	England	2.4889	[2.0, 24.0]	ADD inefficient
4	Aaron Martin	England	2.5238	[3.0, 32.0]	ADD inefficient
5	Aaron Rowe	England	2.4024	[9.0, 30.0]	ADD inefficient
6	Aaron Seydel	Germany	2.8248	[5.0, 38.0]	ADD inefficient
7	Abdoulay Diaby	Mali	2.1945	[2.0, 20.0]	ADD inefficient
8	Abdoulaye Diallo	Senegal	2.6940	[3.0, 34.0]	ADD inefficient
9	Abdul Majeed Waris	Ghana	3.4864	[6.0, 56.0]	ADD inefficient
10	Aboubakar Kamara	France	3.8740	[8.0, 64.0]	ADD inefficient
11	Adalberto Peñaranda	Venezuela	2.2809	[2.0, 21.0]	ADD inefficient
12	Adam Armstrong	England	2.3260	[3.0, 27.0]	ADD inefficient
13	Adam Zreľák	Slovakia	1.7835	[2.0, 13.0]	ADD inefficient
14	Adan George	England	2.3114	[0.0, 19.0]	ADD inefficient
15	Ademola Lookman	England	4.6358	[13.0, 102.0]	ADD inefficient
16	Admir Mehmedi	Switzerland	2.8741	[8.0, 40.0]	ADD inefficient
17	Adolfo Julián Gaich	Argentina	3.6683	[7.0, 56.0]	ADD inefficient
18	Adrian Tabarcea Petre	Romania	2.7701	[9.0, 39.0]	ADD inefficient
19	Adriel D'Avila Ba Loua	Côte d'Ivoire	2.2945	[7.0, 26.0]	ADD inefficient
20	Adrien Truffert	France	4.4464	[28.0, 105.0]	ADD inefficient
21	Adrián Gallardo Valdés	Spain	2.3024	[9.0, 27.0]	ADD inefficient
22	Affamefuna-Michael Ifeadigo	Nigeria	2.5169	[2.0, 26.0]	ADD inefficient
23	Agustín Gonzalo Torassa	Argentina	2.5840	[4.0, 35.0]	ADD inefficient
24	Ahmed Hassan Mahgoub	Egypt	2.1792	[4.0, 23.0]	ADD inefficient
25	Aiden O'Brien	Ireland	2.0930	[1.0, 19.0]	ADD inefficient
26	Aitor Ruibal García	Spain	4.0047	[13.0, 82.0]	ADD inefficient
27	Alassane Pléa	France	3.3981	[12.0, 55.0]	ADD inefficient
28	Alberto Cerri	Italy	2.9908	[6.0, 40.0]	ADD inefficient
29	Alberto Perea Correoso	Spain	3.9770	[13.0, 80.0]	ADD inefficient
30	Albian Ajeti	Switzerland	2.6159	[4.0, 31.0]	ADD inefficient
31	Albion Vrenezi	Germany	4.1141	[21.0, 86.0]	ADD inefficient
32	Alejandro Blanco Sánchez	Spain	3.3830	[27.0, 66.0]	ADD inefficient
33	Alejandro Millán Iranzo	Spain	1.2118	[2.0, 8.0]	ADD inefficient
34	Aleksa Vukanović	Serbia	2.6979	[2.0, 28.0]	ADD inefficient
35	Aleksandar Mitrović	Serbia	3.9268	[17.0, 77.0]	ADD inefficient
36	Aleksandar Vujačić	Montenegro	1.2365	[3.0, 8.0]	ADD inefficient
37	Aleksey Shchetkin	Kazakhstan	1.8009	[1.0, 14.0]	ADD inefficient
38	Alex Addai	England	1.9122	[0.0, 14.0]	ADD inefficient
39	Alex Gilliead	England	2.5169	[0.0, 25.0]	ADD inefficient
40	Alex Iwobi	Nigeria	4.0820	[25.0, 87.0]	ADD inefficient
41	Alex Mighten	England	1.9207	[0.0, 16.0]	ADD inefficient
42	Alex Samuel	Wales	1.9528	[2.0, 20.0]	ADD inefficient
43	Alexander Isak	Sweden	4.8027	[14.0, 103.0]	ADD inefficient
44	Alexander MacDonald	Scotland	2.7015	[0.0, 28.0]	ADD inefficient
45	Alexander Mesa Travieso	Spain	1.4218	[0.0, 9.0]	ADD inefficient
46	Alexander Sørloth	Norway	2.6682	[11.0, 42.0]	ADD inefficient
47	Alexandre Lacazette	France	4.4438	[16.0, 90.0]	ADD inefficient
48	Alexandre Mendy	France	1.6742	[0.0, 11.0]	ADD inefficient
49	Alexis Claude Maurice	France	4.4707	[22.0, 90.0]	ADD inefficient
50	Alexis Sanchez	Chile	4.0564	[16.0, 73.0]	ADD inefficient

The table consists of the first 50 forwards. All 50 players are ADD-inefficient.

3.8 DEA results for the "best" 15 players by FIFA for the 2020/2021 season

As stated earlier the data used for this study was taken from the 2020/2021 season in football. It is interesting to note that the players in Table 3.29 were selected by FIFA for the 2020/2021 "best" team. Meaning each player selected was the "best" in that position.

Table 3.29: FIFA "best" team for the 2020/2021 season

	Player	**Club(s)**	**Nationality**	**Position**
1	Robert Lewandowski	Bayern Munich	Poland	forward
2	Lionel Messi	Barcelona & Paris-Saint-Germain	Argentina	forward
3	Cristiano Ronaldo	Juventus & Manchester United	Portugal	forward
4	Gianluigi Donnarumma	AC Milan & Paris Saint Germain	Italy	goalkeeper
5	David Alaba	Bayern Munich & Real Madrid	Austria	defender
6	Leonardo Bonucci	Juventus	Italy	defender
7	Kevin De Bruyne	Manchester City	Belgium	midfielder
8	Ruben Dias	Manchester City	Portugal	defender
9	N'Golo Kante	Chelsea	France	midfielder
10	Jorginho	Chelsea	Italy	midfielder
11	Erling Haaland	Borrusia Dortmund	Norway	forward

Table 3.30 shows the 11 players who were shortlisted or nominated for the "best" men's player of the year for the 2020/2021 season. The players are ranked based on points with the eventual winner (Robert Lewandowski) in first position. It would be interesting to see what our classical and bootstrap based DEA models conclude regarding the competence of these players in Table 3.29 and 3.30.

Table 3.30: Winner and nominees of the FIFA "best" male player 2020/2021 season

	Player	**Club(s)**	**Nationality**	**Position**	**Points**
1	Robert Lewandowski	Bayern Munich	Poland	forward	48
2	Lionel Messi	Barcelona & PSG	Argentina	forward	44
3	Mohamed Salah	Liverpool	Egypt	forward	39
4	Karim Benzema	Real Madrid	France	forward	30
5	N'Golo Kante	Chelsea	France	midfielder	24
6	Jorginho	Chelsea	Italy	midfielder	24
7	Cristiano Ronaldo	Juventus & Manchester United	Portugal	forward	23
8	Kylian Mbappe	Paris-Saint-Germain	France	forward	16
9	Kevin De Bruyne	Manchester City	Belgium	midfielder	11
10	Neymar	Paris-Saint-Germain	Brasil	forward	10
11	Erling Haaland	Borrusia Dortmund	Norway	forward	7

Table 3.31 and 3.32 show the results of the classical and bootstrap based SBM, BCCI, and ADD models for the players listed above. It should be noted that some players occur in both lists above but will feature only once in the following tables. It is interesting to note that amongst the forwards: Messi, Mbappe, Salah, and Haaland attain efficiency on 2 (SBM and Additive models) out of the 3 efficiency measures used (in Table 3.31) which makes a compelling case for these players to be considered the best forwards during the 2020/2021 season. Amongst the midfielders in Table 3.31, it is only Kevin De Bruyne

who attains efficiency on 2 (SBM and Additive models) out of 3 measures which makes a compelling case for the best midfielder. Amongst the defenders only Ruben Dias attains efficiency on 2 (SBM and Additive) out of the 3 measures which makes a compelling case for best defender.There is only one goalkeeper and he does not attain efficiency on any of the measures. As stated earlier the forwards mentioned above would have to share the award or number 1 spot. It does not make sense to group forwards, midfielders, defenders, and goalkeepers to decide on who wins or loses since the roles and expectations from players are different, for example the goalkeeper's main role is to prevent goals and the forwards's main role is to score goals and provide assists which are contrasting expectations. From Table 3.32, it is interesting to note that only the goalkeeper (Donnarumma) attains ADD efficiency using the definition stated above. This is in contrast with the results of Table 3.31 where attaining efficiency was more common amongst the players.

The study is now in a good position to make a comparison between the DEA results (classical and bootstrap) and the FIFA voting system results as depicted in Table 3.30. We will not make a comparison with Table 3.29 since there is no ranking there (from best to worst). Table 3.33 shows the ranking of players in Table 3.30 based on the DEA models (SBM, BCC,ADD) and FIFA voting system.

Table 3.31: Ranking of players using DEA model's results and FIFA results

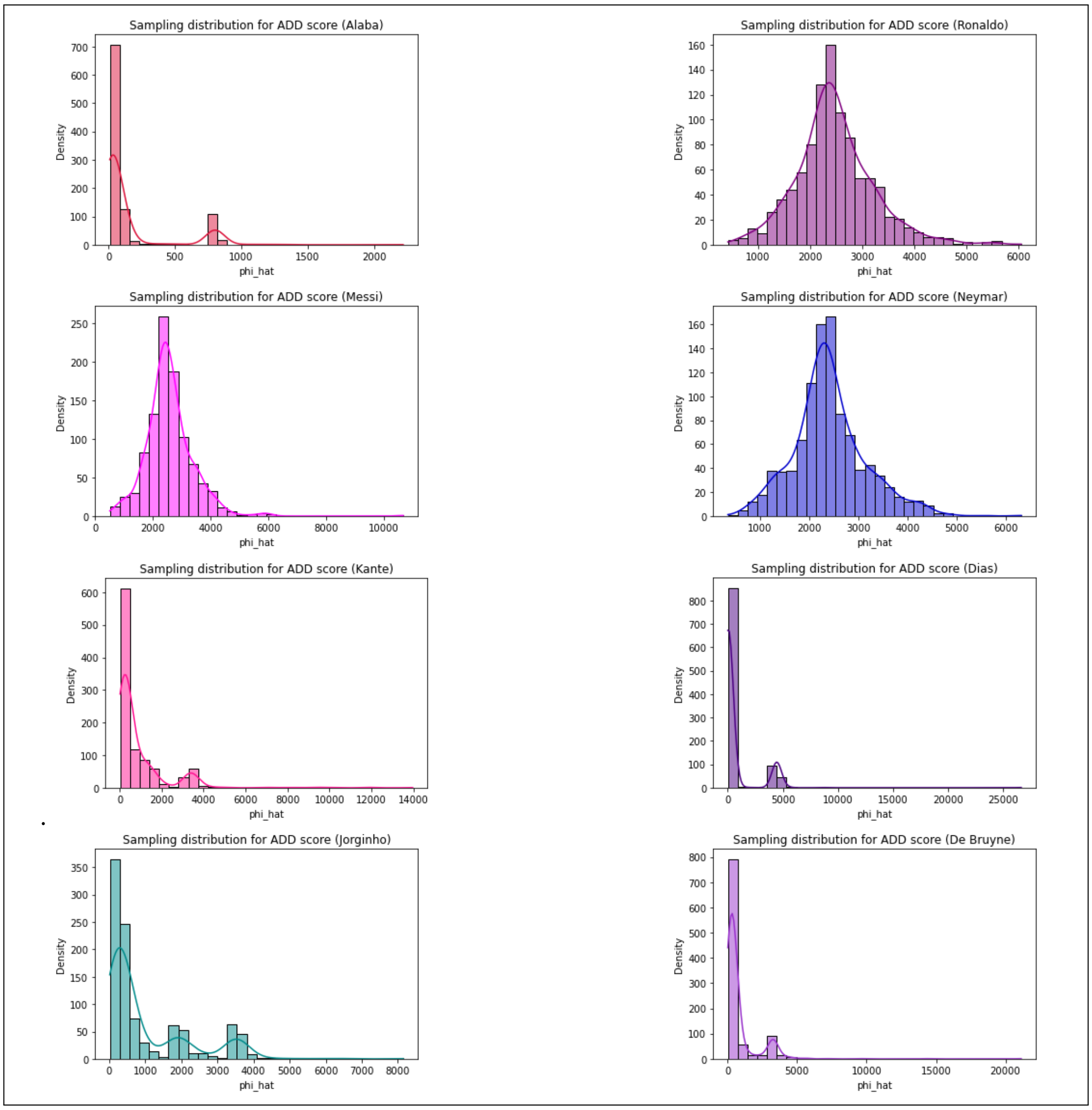
Player	**FIFA**	**C-SBM**	**C-BCC**	**C-ADD**	**B-SBM**	**B-BCC**	**B-ADD**	**Points**
Robert	1	2	1	2	1	11	11	29
Messi	2	1	7	1	1	11	11	34
Salah	3	1	7	1	1	11	11	35
Benzema	4	3	4	3	1	11	11	37
Kante	5	6	8	7	1	11	11	50
Jorginho	6	6	3	6	1	11	11	46
Ronaldo	7	4	5	4	1	11	11	43
Mbappe	8	1	6	1	1	11	11	39
Kevin	9	1	8	1	1	11	11	42
Neymar	10	5	5	5	1	11	11	48
Haaland	11	1	2	1	1	11	11	38

For the classical SBM and BCC models, the higher the score the higher the ranking. For the classical ADD model, the lower the score the higher the ranking. Since the bootstrap models consist of intervals, the score will be 1 or 11 depending on whether the player is efficient or not as shown in Table 3.32. C-SBM is the classical SBM model score, etc. B-SBM is the bootstrap SBM score, etc. Points is the total score across the columns, where the player with the lowest score is declared the winner.

According to the DEA models' results Robert Lewandowski deserved the prestigious FIFA best player award for the 2020/2021 season as shown in the last column of Table 3.33. The results agree with the FIFA voting system for the first 4 players in Table 3.33 but differ for the rest. These results do not take away the value of using a player's metrics to evaluate a player's performance as opposed to FIFA's voting criteria.

3.9 Estimated Sampling distribution of the Additive Model's $\hat{\phi}$ for the 15 players.

It should be noted that one of the objectives is to estimate the sampling distribution of the estimator $\hat{\phi}$ for each of the 15 players. We do this by plotting the histogram (kernel density) of the bootstrap estimates $\hat{\phi}_1^*, \hat{\phi}_2^*, ...\hat{\phi}_B^*$ $(B = 1000)$ which we used to construct the confidence intervals above. The sampling distribution is only estimated for the Additive model due to the nature of the results for the other 2 models which consists of very small values that are not easy to visualise, as can be seen from the confidence intervals of the BCCI and SBM models. Figure 3.1 shows the results.



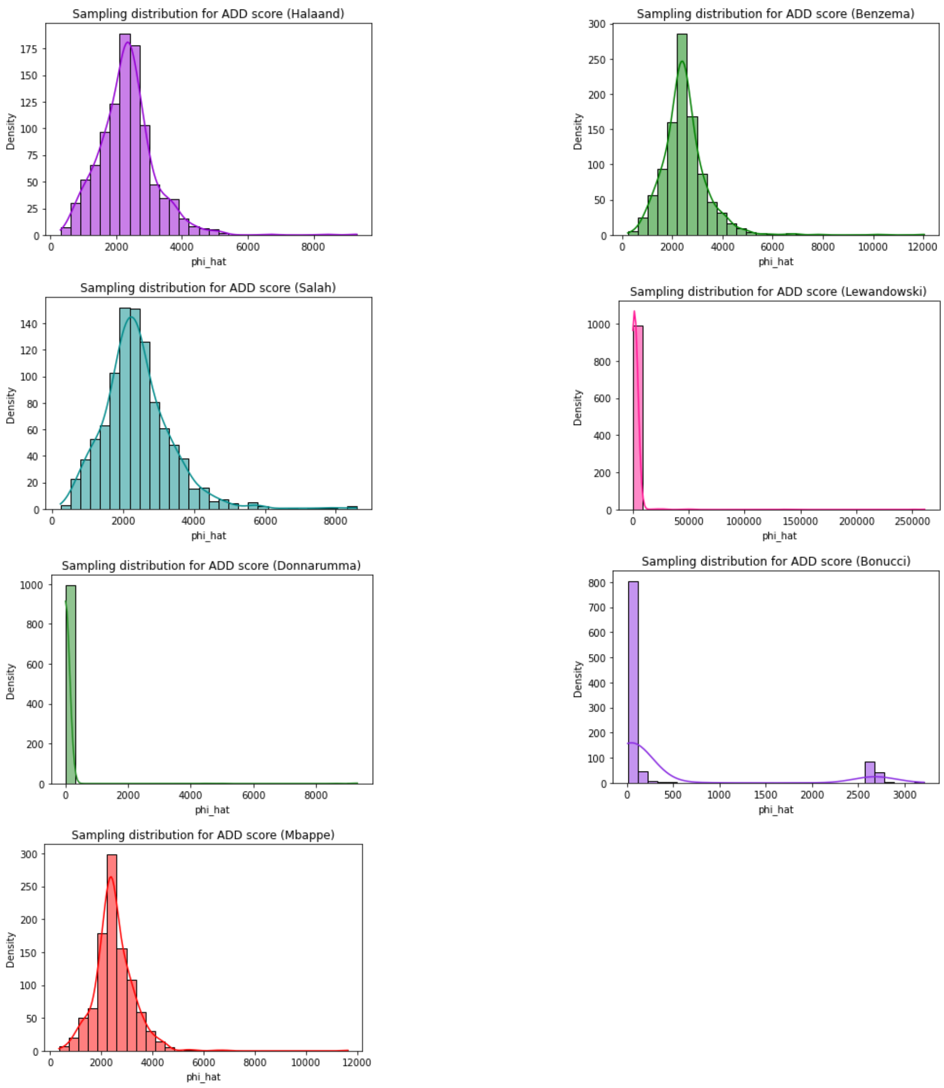


Figure 3.1: Estimated Sampling distribution of the Additive Model's $\hat{\boldsymbol{\phi}}$ for the 15 players.

From Figure 3.1 it is worth noting that the estimated sampling distribution for $\hat{\boldsymbol{\phi}}$ resembles the normal distribution for all forwards (Messi, Neymar, Ronaldo, Haaland, Lewandowski, Salah, Mbappe, Benzema), for the midfielders (De Bruyne, Kante and Jorginho) the estimated sampling distribution for $\hat{\boldsymbol{\phi}}$ resembles that of an exponential distribution. One can take things a step further with these results by calculating probabilities, expected values, etc, but the objective of this study was to show that bootstrap based efficiency scores give a much better picture than point estimates.

3.10 Summary and conclusion

In this Chapter we explored the application of classical and bootstrap based DEA models to football data. The data was collected for the 2020/2021 season. This data consisted of 16340 players who participated across 18 top football leagues and cup competitions in the world, these are: Bundesliga. Caf champions league. Carabao Cup. Copa America. Copa del rey. Copa italy. Copa de Espana. DFB Pokal. DFL Super Cup. Europa league. FA Cup. Coupe de France. Italy supercoppa. Laliga. Ligue 1. Premier league. Serie A. UEFA champions league. 20 variables of interest were selected, these include minutes played overall, assists overall, penalty goals, clean sheets overall, etc. The main interest was on the nominated players for the FIFA awards which are voted by industry experts. The results showed that the ranking criteria provided by the DEA models agree with the FIFA ranking system for 4 out of the 11 players nominated for the best player award. Robert Lewandowski was the overall winner. The value of DEA models' results lies in their capability to allow a user to make decisions based on measurable variables which quantify individual player performance, with no bias or favouritism.

Chapter 4

DEA application in Finance

4.1 Introduction

This Chapter focuses on the application of the DEA methodology in finance. In particular, the aim is to use DEA model to evaluate the performance of companies on the Johannesburg Stock Exchange (JSE) Top 40. The objective is to create a portfolio by using DEA as a selection mechanism. This can assist investors to make informed decisions when selecting stocks based on the actual performance of a company and not on the behaviour of the stock price in a market. This study argues that this strategy is particularly useful for investors looking for stocks that offer dividends because payment of dividends does not rely on the stock's behaviour on the market but on how the company is performing fundamentally. If a company is performing well, it is more likely to reward its investors or stakeholders through dividends. This study argues that the DEA model are able to select these well performing companies by using efficiency as a selection criteria.

This study will consider the traditional BCCI, SBM, and Additive model' results along with the bootstrap based BCCI, SBM, and Additive model' results. Before we get to the results, an explanation of how the bootstrap resampling method can be adapted within the optimisation framework or DEA to be more specific needs to be given. The contribution of this study is to use the non parametric bootstrap resampling technique to simulate DEA inputs and outputs for each DMU. The motivation behind this is based on the fact that computation of DEA efficiency scores for various model is based on some sample of inputs and outputs. This score will differ for different samples which makes the score an estimate of the "true" efficiency score which is unknown. The functional form of this estimate is the estimator which is a random variable. For example when using the BCC multiplier model, the estimator of the efficiency score for the reference DMU is given by

$$\hat{\theta} = \frac{\sum_{r=1}^{s} u_r y_{ro} - u_o}{\sum_{i=1}^{m} v_i x_{io}} \tag{4.1}$$

It is clear from this formula that a different sample of inputs and outputs yields a different efficiency score. Instead of using a point estimate for the efficiency score, this study proposes using a confidence interval. Let $X_{1j}, X_{2j}, ..., X_{mj}$ and $Y_{1j}, Y_{2j}, ..., Y_{sj}$ be i.i.d random variables from unknown distributions F and G. Where X_{ij} represents the i^{th} input of DMU$_j$ and Y_{rj} represents the r^{th} output of DMU$_j$, with $j = 1, 2, ..., n$, $i = 1, 2, ..., m$, and $r = 1, 2, ..., s$. Let $\hat{\theta} = \hat{\theta}(X_{10}, X_{20}, ..., X_{m0}, Y_{10}, Y_{20}, ..., Y_{s0})$ be an estimator for the "true" efficiency score θ. The objective is to construct a $100(1-\alpha)\%$ confidence interval for θ using the non parametric bootstrap approach. Additionally one would like to estimate the variance or standard error of the estimator $\hat{\theta}(X_{10}, X_{20}, ..., X_{m0}, Y_{10}, Y_{20}, ..., Y_{s0})$ from sample to sample as this will give us an idea of how reliable the estimator is. We can use the bootstrap resampling technique to do this, since in practice the underlying distribution of the sample data is unknown.

4.2 Problem description

The traditional DEA model employ inputs and outputs as the data to be used when estimating the efficiency score. It is clear then that the efficiency score is dependent on the given data set. In other words, a DMU may be deemed to be efficient under one set of data and inefficient under another set (for the same input and output variables). Since the efficiency score is a function of these variables, it is a random variable as well. Landete et al. (2017) proposed a methodology where the inputs and outputs were treated as random variables. However their method is based on the argument that, in any DEA study, the inputs and outputs to be selected for the study are uncertain and should be modelled by a Bernoulli random variable, with each variable taking on the value 1 with probability p and the value 0 with probability $1 - p$.

Olesen (2002) incorporates randomness in DEA differently, by focusing on the restriction of the optimal weights or multipliers. Traditional DEA model tend to assign a weight of zero to certain inputs and outputs which creates a problem since this suggests that the associated variables are not significant when assessing the performance of a particular DMU. To solve this issue, Olesen (2002) suggested the use of confidence intervals for the output multipliers. These confidence intervals are also known as the probabilistic assurance regions in the output space. This study differs from the work done by Landete et al. (2017) in the sense that there is no uncertainty about the inclusion of the inputs and outputs since these are chosen based on experience and are deemed necessary for the performance evaluation of DMUs. It also differs from the work of Olesen (2002) in the sense that the confidence intervals suggested by this study refer to the objective function of the DEA model. The randomness aspect comes from the values that the input and output variables take on.

The motivation behind this study is two fold, firstly it is to investigate the discriminatory power between the traditional DEA model and the non-parametric bootstrap DEA model that is proposed. Additionally, the efficiency score is unknown beforehand unlike the input and output data sets, it needs to be estimated by solving the optimisation problem. The problem faced by traditional DEA model is that they produce a single value as a measure of efficiency, this study proposes that it is much better to construct a confidence interval for the efficiency score. This suggests estimating the sampling distribution of the computed efficiency scores and estimating the confidence interval of the efficiency score. The confidence interval approach is more reliable and easy to interpret than a point estimate. The approach proposed will be illustrated with an application in finance.

4.3 Methods

In this section the classical and bootstrap based DEA model are explained and they are applied to finance data, the rest of the section deals with the generated results. As mentioned earlier, the study looks at the JSE Top 40 for the period 2015-2023. Table 4.1 shows the listed companies of interest:

Table 4.1: JSE Top 40 listed companies

	Code	Short Name	Full Name
1	ABG	ABSA	Absa Group Ltd.
2	ARI	ARM	African Rainbow Minerals Ltd.
3	AMS	AMPLATS	Anglo American Platinum Ltd.
4	AGL	ANGLO	Anglo American plc
5	ANG	ANGGOLD	AngloGold Ashanti Ltd.
6	ANH	AB INBEV	Anheuser-Busch InBev SA/NV
7	APN	ASPEN	Aspen Pharmacare Holdings Ltd.
8	BHG	BHP	BHP Group Ltd.
9	BID	BIDCORP	Bid Corporation Ltd.
10	BTI	BATS	British American Tobacco plc
11	CPI	CAPITEC	Capitec Bank Holdings Ltd.
12	CLS	CLICKS	Clicks Group Ltd.
13	CFR	RICHEMONT	Compagnie Financière Richemont SA
14	DSY	DISCOVERY	Discovery Ltd.
15	EXX	EXXARO	Exxaro Resources Ltd.
16	FSR	FIRSTRAND	FirstRand Ltd.
17	GLN	GLENCORE	Glencore plc
18	GFI	GFIELDS	Gold Fields Ltd.
19	IMP	IMPLATS	Impala Platinum Holdings Ltd.
20	INP	INVPLC	Investec plc
21	KIO	KUMBA	Kumba Iron Ore Ltd.
22	MEI	MEDICLINIC	Mediclinic International plc
23	MNP	MONDIPLC	Mondi plc
24	MTN	MTN GROUP	MTN Group Ltd.
25	NPN	NASPERS-N	Naspers Ltd.
26	NED	NEDBANK	Nedbank Group Ltd.
27	NRP	NEPIROCK	NEPI Rockcastle N.V.
28	NPH	NORTHAM	Northam Platinum Holdings Ltd.
29	PPH	PEPKORH	Pepkor Holdings Ltd.
30	PRX	PROSUS	Prosus N.V.
31	REM	REMGRO	Remgro Ltd.
32	SLM	SANLAM	Sanlam Ltd.
33	SOL	SASOL	Sasol Ltd.
34	SHP	SHOPRIT	Shoprite Holdings Ltd.
35	SSW	SIBANYE-S	Sibanye Stillwater Ltd.
36	S32	SOUTH32	South32 Ltd.
37	SBK	STANBANK	Standard Bank Group Ltd.
38	BVT	BIDVEST	The Bidvest Group Ltd.
39	VOD	VODACOM	Vodacom Group Ltd.
40	WHL	WOOLIES	Woolworths Holdings Ltd.

The following inputs and outputs in Table 4.2 were identified as variables of interest and used to analyse the performance of the JSE Top 40. It should be noted that all the variables used are in a form of a ratio or percentage. The choice of inputs and outputs is based solely on the availability of the data. DEA model suffer from the curse of dimensionality where an appropriate combination of inputs and outputs should not exceed the number of DMUs under evaluation. Since this study only looks at 40 companies, this should be taken into consideration.

Table 4.2: Inputs and outputs for the JSE Top 40 companies

Inputs	Outputs
Debt / Assets	Current Ratio
Debt / Equity	Dividend / Share (c)
Long-Term Loans (%)	Dividend Cover
Price / Book Value	Dividend Yield (%)
Price / Earnings	Earnings / Share (C)
Price / Share (C)	Earnings Yield (%)
Total Debt / Cash Flow	Quick Ratio
Price / Cash	Retention Rate
-	Return On Equity (%)
-	Return on Capital Employed

It should be noted that the sample inputs and outputs for each company have been aggregated for convenience, meaning for each company over the 2015-2023 period the data has been transformed into simple averages to make computation easier. For example the price/share for ABSA is R15381.00, this is an average over the 9 year period. Since we are dealing with ratio variables in terms of inputs and outputs, the SBM, BCCI and Additive model used in this section are the following:

SBM model:

$$
\begin{aligned}
\min \ \tau = t - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{x_{io}} & \\
\text{subject to } \ t + \frac{1}{s}\sum_{r=1}^{s}\frac{S_r^+}{y_{ro}} = 1 & \\
\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = tx_{i0} \ , \ \forall\, i \neq p & \\
\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = ty_{ro} \ , \ \forall\, r \neq k & \qquad (4.2)\\
\sum_{j=1}^{n} \overline{x}_{pj}\Lambda_j + s_p^- \sum_{j=1}^{n} \underline{x}_{pj}\Lambda_j = tx_{po} \sum_{j=1}^{n} \underline{x}_{pj}\Lambda_j, \ \forall\, i = p & \\
\sum_{j=1}^{n} \overline{y}_{kj}\Lambda_j - s_k^+ \sum_{j=1}^{n} \underline{y}_{kj}\Lambda_j = ty_{ko} \sum_{j=1}^{n} \underline{y}_{kj}\Lambda_j, \ \forall\, r = k & \\
\sum_{j=1}^{n} \Lambda_j = 1 & \\
t\ , \Lambda_j,\ s_i^-,\ s_r^+, \geq 0 \ \forall i, j, r, p, k &
\end{aligned}
$$

It should be noted that:

$$\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \frac{\overline{y}_{ko}}{\underline{y}_{ko}} = y_{ko}$$

$$\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \theta\frac{\overline{x}_{po}}{\underline{x}_{po}} = x_{po}$$

BCCI model:

$$\begin{aligned}
&\min \ z = \ \theta \\
\text{subject to } \ &\sum_{j=1}^{n} x_{ij}\lambda_j - \theta x_{io} \leq 0 \,, \forall\, i \neq p \\
&\sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro} \,, \forall\, r \neq k \\
&\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j - y_{ko}\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j \geq 0 \,, r = k \\
&\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j - \underline{x}_{po}\theta\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j \leq 0, \ i = p \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0 \,, \forall\, j.
\end{aligned} \tag{4.3}$$

It should be noted that:

$$\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \frac{\overline{y}_{ko}}{\underline{y}_{ko}} = y_{ko}$$

$$\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \theta\frac{\overline{x}_{po}}{\underline{x}_{po}} = \theta x_{po}$$

Additive model:

$$
\begin{aligned}
\max \phi = & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{subject to } & \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \text{ ,} \forall\, i \neq p \\
& \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro} \text{ ,} \forall\, r \neq k \\
& \sum_{j=1}^{n} \overline{y}_{kj}\lambda_j - y_{ko} \sum_{j=1}^{n} \underline{y}_{kj}\lambda_j - s_r^+ = 0 \text{ ,} r = k \\
& \sum_{j=1}^{n} \overline{x}_{pj}\lambda_j - \underline{x}_{po} \sum_{j=1}^{n} \underline{x}_{pj}\lambda_j + s_i^- = 0, \ i = p \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 \text{ ,} \forall \ j.
\end{aligned}
\tag{4.4}
$$

It should be noted that:

$$
\frac{\sum_{j=1}^{n} \overline{y}_{kj}\lambda_j}{\sum_{j=1}^{n} \underline{y}_{kj}\lambda_j} \geq \frac{\overline{y}_{ko}}{\underline{y}_{ko}} = y_{ko}
$$

$$
\frac{\sum_{j=1}^{n} \overline{x}_{pj}\lambda_j}{\sum_{j=1}^{n} \underline{x}_{pj}\lambda_j} \leq \frac{\overline{x}_{po}}{\underline{x}_{po}} = x_{po}
$$

4.4 DEA results for the classical model

This section presents results for the classical SBM, BCCI and Additive DEA model for the JSE Top 40 companies. The data analysis was done using Python 3.12. A DMU is deemed SBM-efficient if and only if $\tau^* = 1$ otherwise it is deemed SBM-inefficient. A DMU is deemed BCC-efficient if and only if $\theta^* = 1$ otherwise it is deemed BCC-inefficient. A DMU is deemed ADD-efficient if and only if $\phi^* = 0$ otherwise it is deemed ADD-inefficient.

4.4.1 DEA results for the classical SBM model (for the JSE Top 40 companies)

Table 4.3 shows the results of the classical SBM model for the JSE Top 40 companies.

Table 4.3: Classical SBM model results for the JSE Top 40 companies

	Code	Short Name	Company Name	τ^*	DMU Status
1	ABG	ABSA	Absa Group Ltd.	0.1176	SBM inefficient
2	ARI	ARM	African Rainbow Minerals Ltd.	1.0000	SBM efficient
3	AMS	AMPLATS	Anglo American Platinum Ltd.	1.0000	SBM efficient
4	AGL	ANGLO	Anglo American plc	0.3832	SBM inefficient
5	ANG	ANGGOLD	AngloGold Ashanti Ltd.	1.0000	SBM efficient
6	ANH	AB INBEV	Anheuser-Busch InBev SA/NV	0.1387	SBM inefficient
7	APN	ASPEN	Aspen Pharmacare Holdings Ltd.	0.0401	SBM inefficient
8	BHG	BHP	BHP Group Ltd.	1.0000	SBM efficient
9	BID	BIDCORP	Bid Corporation Ltd.	0.0001	SBM inefficient
10	BTI	BATS	British American Tobacco plc	1.0000	SBM efficient
11	CPI	CAPITEC	Capitec Bank Holdings Ltd.	1.0000	SBM efficient
12	CLS	CLICKS	Clicks Group Ltd.	0.0001	SBM inefficient
13	CFR	RICHEMONT	Compagnie Financière Richemont SA	1.0000	SBM efficient
14	DSY	DISCOVERY	Discovery Ltd.	0.0000	SBM inefficient
15	EXX	EXXARO	Exxaro Resources Ltd.	1.0000	SBM efficient
16	FSR	FIRSTRAND	FirstRand Ltd.	1.0000	SBM efficient
17	GLN	GLENCORE	Glencore plc	1.0000	SBM efficient
18	GFI	GFIELDS	Gold Fields Ltd.	0.0873	SBM inefficient
19	IMP	IMPLATS	Impala Platinum Holdings Ltd.	1.0000	SBM efficient
20	INP	INVPLC	Investec plc	1.0000	SBM efficient
21	KIO	KUMBA	Kumba Iron Ore Ltd.	1.0000	SBM efficient
22	MEI	MEDICLINIC	Mediclinic International plc	0.0057	SBM inefficient
23	MNP	MONDIPLC	Mondi plc	0.0377	SBM inefficient
24	MTN	MTN GROUP	MTN Group Ltd.	1.0000	SBM efficient
25	NPN	NASPERS-N	Naspers Ltd.	0.0018	SBM inefficient
26	NED	NEDBANK	Nedbank Group Ltd.	1.0000	SBM efficient
27	NRP	NEPIROCK	NEPI Rockcastle N.V.	0.0030	SBM inefficient
28	NPH	NORTHAM	Northam Platinum Holdings Ltd.	0.0187	SBM inefficient
29	PPH	PEPKORH	Pepkor Holdings Ltd.	1.0000	SBM efficient
30	PRX	PROSUS	Prosus N.V.	1.0000	SBM efficient
31	REM	REMGRO	Remgro Ltd.	0.0031	SBM inefficient
32	SLM	SANLAM	Sanlam Ltd.	0.0031	SBM inefficient
33	SOL	SASOL	Sasol Ltd.	0.0208	SBM inefficient
34	SHP	SHOPRIT	Shoprite Holdings Ltd.	0.4979	SBM inefficient
35	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	1.0000	SBM efficient
36	S32	SOUTH32	South32 Ltd.	1.0000	SBM efficient
37	SBK	STANBANK	Standard Bank Group Ltd.	1.0000	SBM efficient
38	BVT	BIDVEST	The Bidvest Group Ltd.	0.0276	SBM inefficient
39	VOD	VODACOM	Vodacom Group Ltd.	0.0148	SBM inefficient
40	WHL	WOOLIES	Woolworths Holdings Ltd.	1.0000	SBM efficient

From Table 4.3, it is worth noting that almost half (19) of the companies attain SBM efficiency across different sectors such as mining, banking, fast-moving consumer goods, e.t.c, which is encouraging from a diversification perspective for potential investors.

4.4.2 DEA results for the classical BCCI model (for the JSE Top 40 companies)

Table 4.4 shows the results of the classical BCCI model for the JSE Top 40 companies.

Table 4.4: Classical BCCI model results for the JSE Top 40 companies

	Code	Short Name	Company Name	θ^*	DMU Status
1	ABG	ABSA	Absa Group Ltd.	0.6799	BCC inefficient
2	ARI	ARM	African Rainbow Minerals Ltd.	1.0000	BCC efficient
3	AMS	AMPLATS	Anglo American Platinum Ltd.	1.0000	BCC efficient
4	AGL	ANGLO	Anglo American plc	0.5387	BCC inefficient
5	ANG	ANGGOLD	AngloGold Ashanti Ltd.	0.4671	BCC inefficient
6	ANH	AB INBEV	Anheuser-Busch InBev SA/NV	0.2923	BCC inefficient
7	APN	ASPEN	Aspen Pharmacare Holdings Ltd.	0.4580	BCC inefficient
8	BHG	BHP	BHP Group Ltd.	0.5532	BCC inefficient
9	BID	BIDCORP	Bid Corporation Ltd.	0.0078	BCC inefficient
10	BTI	BATS	British American Tobacco plc	0.0156	BCC inefficient
11	CPI	CAPITEC	Capitec Bank Holdings Ltd.	0.0111	BCC inefficient
12	CLS	CLICKS	Clicks Group Ltd.	0.0065	BCC inefficient
13	CFR	RICHEMONT	Compagnie Financière Richemont SA	0.0086	BCC inefficient
14	DSY	DISCOVERY	Discovery Ltd.	0.0170	BCC inefficient
15	EXX	EXXARO	Exxaro Resources Ltd.	0.0332	BCC inefficient
16	FSR	FIRSTRAND	FirstRand Ltd.	1.0000	BCC efficient
17	GLN	GLENCORE	Glencore plc	0.6865	BCC inefficient
18	GFI	GFIELDS	Gold Fields Ltd.	0.8552	BCC inefficient
19	IMP	IMPLATS	Impala Platinum Holdings Ltd.	0.3213	BCC inefficient
20	INP	INVPLC	Investec plc	0.8920	BCC inefficient
21	KIO	KUMBA	Kumba Iron Ore Ltd.	0.3412	BCC inefficient
22	MEI	MEDICLINIC	Mediclinic International plc	0.4955	BCC inefficient
23	MNP	MONDIPLC	Mondi plc	0.5433	BCC inefficient
24	MTN	MTN GROUP	MTN Group Ltd.	1.0000	BCC efficient
25	NPN	NASPERS-N	Naspers Ltd.	0.1297	BCC inefficient
26	NED	NEDBANK	Nedbank Group Ltd.	0.6420	BCC inefficient
27	NRP	NEPIROCK	NEPI Rockcastle N.V.	0.3676	BCC inefficient
28	NPH	NORTHAM	Northam Platinum Holdings Ltd.	0.3819	BCC inefficient
29	PPH	PEPKORH	Pepkor Holdings Ltd.	1.0000	BCC efficient
30	PRX	PROSUS	Prosus N.V.	0.1240	BCC inefficient
31	REM	REMGRO	Remgro Ltd.	0.4221	BCC inefficient
32	SLM	SANLAM	Sanlam Ltd.	0.6653	BCC inefficient
33	SOL	SASOL	Sasol Ltd.	0.5927	BCC inefficient
34	SHP	SHOPRIT	Shoprite Holdings Ltd.	0.5345	BCC inefficient
35	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	1.0000	BCC efficient
36	S32	SOUTH32	South32 Ltd.	0.8095	BCC inefficient
37	SBK	STANBANK	Standard Bank Group Ltd.	1.0000	BCC efficient
38	BVT	BIDVEST	The Bidvest Group Ltd.	0.5003	BCC inefficient
39	VOD	VODACOM	Vodacom Group Ltd.	0.5464	BCC inefficient
40	WHL	WOOLIES	Woolworths Holdings Ltd.	0.7633	BCC inefficient

From Table 4.4, it is worth noting that there are less than 10 BCC-efficient companies which indicates the discriminatory power of the BCCI model is more compared to the SBM model.

4.4.3 DEA results for the classical Additive model (for the JSE Top 40 companies)

Table 4.5 shows the results of the classical ADD model for the JSE Top 40 companies.

Table 4.5: Classical ADD model results for the JSE Top 40 companies

	Code	Short Name	Company Name	ϕ^*	DMU Status
1	ABG	ABSA	Absa Group Ltd.	4549.7213	ADD inefficient
2	ARI	ARM	African Rainbow Minerals Ltd.	0.0000	ADD efficient
3	AMS	AMPLATS	Anglo American Platinum Ltd.	0.0000	ADD efficient
4	AGL	ANGLO	Anglo American plc	2310.5418	ADD inefficient
5	ANG	ANGGOLD	AngloGold Ashanti Ltd.	0.0000	ADD efficient
6	ANH	AB INBEV	Anheuser-Busch InBev SA/NV	2485.5115	ADD inefficient
7	APN	ASPEN	Aspen Pharmacare Holdings Ltd.	19153.9476	ADD inefficient
8	BHG	BHP	BHP Group Ltd.	0.0000	ADD efficient
9	BID	BIDCORP	Bid Corporation Ltd.	33371.8833	ADD inefficient
10	BTI	BATS	British American Tobacco plc	0.0000	ADD efficient
11	CPI	CAPITEC	Capitec Bank Holdings Ltd.	0.0000	ADD efficient
12	CLS	CLICKS	Clicks Group Ltd.	23886.1065	ADD inefficient
13	CFR	RICHEMONT	Compagnie Financière Richemont SA	0.0000	ADD efficient
14	DSY	DISCOVERY	Discovery Ltd.	17923.0525	ADD inefficient
15	EXX	EXXARO	Exxaro Resources Ltd.	0.0000	ADD efficient
16	FSR	FIRSTRAND	FirstRand Ltd.	0.0000	ADD efficient
17	GLN	GLENCORE	Glencore plc	0.0000	ADD efficient
18	GFI	GFIELDS	Gold Fields Ltd.	3191.0361	ADD inefficient
19	IMP	IMPLATS	Impala Platinum Holdings Ltd.	0.0000	ADD efficient
20	INP	INVPLC	Investec plc	0.0000	ADD efficient
21	KIO	KUMBA	Kumba Iron Ore Ltd.	0.0000	ADD efficient
22	MEI	MEDICLINIC	Mediclinic International plc	7134.3303	ADD inefficient
23	MNP	MONDIPLC	Mondi plc	22393.1470	ADD inefficient
24	MTN	MTN GROUP	MTN Group Ltd.	0.0000	ADD efficient
25	NPN	NASPERS-N	Naspers Ltd.	236361.1841	ADD inefficient
26	NED	NEDBANK	Nedbank Group Ltd.	0.0000	ADD efficient
27	NRP	NEPIROCK	NEPI Rockcastle N.V.	3089.5544	ADD inefficient
28	NPH	NORTHAM	Northam Platinum Holdings Ltd.	4011.3790	ADD inefficient
29	PPH	PEPKORH	Pepkor Holdings Ltd.	0.0000	ADD efficient
30	PRX	PROSUS	Prosus N.V.	0.0000	ADD efficient
31	REM	REMGRO	Remgro Ltd.	13208.8396	ADD inefficient
32	SLM	SANLAM	Sanlam Ltd.	2644.2620	ADD inefficient
33	SOL	SASOL	Sasol Ltd.	23903.6709	ADD inefficient
34	SHP	SHOPRIT	Shoprite Holdings Ltd.	10399.7319	ADD inefficient
35	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	0.0000	ADD efficient
36	S32	SOUTH32	South32 Ltd.	0.0000	ADD efficient
37	SBK	STANBANK	Standard Bank Group Ltd.	0.0000	ADD efficient
38	BVT	BIDVEST	The Bidvest Group Ltd.	14039.4352	ADD inefficient
39	VOD	VODACOM	Vodacom Group Ltd.	6823.6727	ADD inefficient
40	WHL	WOOLIES	Woolworths Holdings Ltd.	0.0000	ADD efficient

From Table 4.5, it is worth noting that almost more than half (21) of the DMUs are ADD efficient which indicates that the discriminatory power of the ADD model is similar to the SBM model. We are now in a position to present the bootstrap based results for the three model (SBM, BCCI, ADD) using the JSE Top 40 companies.

4.5 DEA results for the bootstrap based model

It should be noted that the sets of inputs and outputs are the same as those used in the classical model. This section and the succeeding sections incorporate the non-parametric bootstrap method within DEA as explained in Chapter 2. We are now in a position to create confidence intervals for the estimator $\hat{\tau} = \hat{\tau}(X_1, X_2, ..., X_n)$ under the different DEA model (BCC,SBM, ADD). As stated earlier, we do this by sampling with replacement from the given inputs $X_{1j}, X_{2j}, ..., X_{mj}$ and outputs $Y_{1j}, Y_{2j}, ..., Y_{sj}$ data sets for DMU_j. For each of the $B = 1000$ bootstrap samples, we calculate $\hat{\tau}^*$ which is the value of the objective function under the different model, that is for each DMU we solve the optimisation problem B times to obtain $\hat{\tau}_1^* \leq \hat{\tau}_2^* \leq ... \leq \hat{\tau}_B^*$.

Traditionally one would solve the DEA model for each DMU and obtain an optimal value or efficiency score $0 \leq \tau^* \leq 1$. However we are not interested in a point estimate or single value of the efficiency score since this will vary depending on the data set used, in other words the efficiency score from the traditional BCC model is an estimator $\hat{\tau}$ of the true score τ. For each DMU we need to construct a confidence interval for τ. Each DMU_j (in this case goalkeepers) has a set of inputs and outputs denoted by $X_{1j}, X_{2j}, ..., X_{mj}$ and $Y_{1j}, Y_{2j}, ..., Y_{sj}, \quad j = 1, 2, ..., n$. For each DMU, we then:

1. Sample B samples with replacement from $X_{1j}, X_{2j}, ..., X_{mj}$
2. Sample B samples with replacement from $Y_{1j}, Y_{2j}, ..., Y_{sj}$
3. For each of the B samples in step 1 and 2, solve the above optimisation problem to obtain $\hat{\tau}_1^*, \hat{\tau}_2^*, ..., \hat{\tau}_B^*$
4. Sort the estimators in ascending order: $\hat{\tau}_1^* \leq \hat{\tau}_2^* ... \leq \hat{\tau}_B^*$
5. A $100(1 - \alpha)\%$ confidence interval for τ is then given by $\left[\hat{\tau}_{(r)}^*; \hat{\tau}_{(s)}^*\right]$
6. The variance of $\hat{\tau}$ is then estimated by $S_{\hat{\tau}}^2 = \frac{1}{B}\sum_{i=1}^{B}\left(\hat{\tau}_i^* - \hat{\tau}(\cdot)\right)^2$

The standard error (stde) is the standard deviation of all the bootstrap estimates $\hat{\tau}_1^*, \hat{\tau}_2^*, ..., \hat{\tau}_B^*$ calculated from the B bootstrap samples. This is done for each DMU. Again the 95% confidence interval is calculated using the algorithm described above for each DMU. The motivation for using confidence intervals was described in Chapter 2. We make the assertion that a DMU must be deemed SBM efficient if and only if the confidence interval contains 1, similarly a DMU must be BCCI efficient if and only if the confidence interval contains 1, and ADD efficient if and only if the confidence interval contains 0. This can be interpreted as follows: given a confidence level α, the $(1 - \alpha)\%$ efficiency confidence interval will contain 1 or 0 $(1 - \alpha)\%$ percent of the time, in other words the DMU in question is efficient $(1 - \alpha)\%$ percent of the time. This definition is much better than the traditional one of deciding efficiency based on one sample $X_{1j}, X_{2j}, ..., X_{mj}, Y_{1j}, Y_{2j}, ..., Y_{sj}$ and one score $\boldsymbol{\tau^*}$, $\boldsymbol{\theta^*}$, $\boldsymbol{\phi^*}$.

4.5.1 DEA results for the bootstrap based SBM model (for the JSE Top 40 companies)

Table 4.6 presents the results of the bootstrap SBM model for the JSE Top 40 companies. The results indicate an almost even split between SBM-efficient and inefficient companies.

Table 4.6: Bootstrap SBM model results for the JSE Top 40.

	Code	Short Name	Company Name	Stde	CI(95%)	Status
1	ABG	ABSA	Absa Group Ltd.	0.0010	[0.0, 0.011]	SBM inefficient
2	ARI	ARM	African Rainbow Minerals Ltd.	0.0479	[0.0, 1.0]	SBM efficient
3	AMS	AMPLATS	Anglo American Platinum Ltd.	0.0165	[0.0, 0.005]	SBM inefficient
4	AGL	ANGLO	Anglo American plc	0.0247	[0.0, 0.695]	SBM inefficient
5	ANG	ANGGOLD	AngloGold Ashanti Ltd.	0.0016	[0.0, 0.024]	SBM inefficient
6	ANH	AB INBEV	Anheuser-Busch InBev SA/NV	0.0467	[0.0, 1.0]	SBM efficient
7	APN	ASPEN	Aspen Pharmacare Holdings Ltd.	0.0020	[0.0, 0.013]	SBM inefficient
8	BHG	BHP	BHP Group Ltd.	0.0319	[0.0, 1.0]	SBM efficient
9	BID	BIDCORP	Bid Corporation Ltd.	0.0440	[0.0, 1.0]	SBM efficient
10	BTI	BATS	British American Tobacco plc	0.0430	[0.0, 1.0]	SBM efficient
11	CPI	CAPITEC	Capitec Bank Holdings Ltd.	0.0014	[0.0, 0.003]	SBM inefficient
12	CLS	CLICKS	Clicks Group Ltd.	0.0405	[0.0, 1.0]	SBM efficient
13	CFR	RICHEMONT	Compagnie Financière SA	0.0022	[0.0, 0.002]	SBM inefficient
14	DSY	DISCOVERY	Discovery Ltd.	0.0021	[0.0, 0.02]	SBM inefficient
15	EXX	EXXARO	Exxaro Resources Ltd.	0.0326	[0.0, 1.0]	SBM efficient
16	FSR	FIRSTRAND	FirstRand Ltd.	0.0333	[0.0, 1.0]	SBM efficient
17	GLN	GLENCORE	Glencore plc	0.0277	[0.0, 0.923]	SBM inefficient
18	GFI	GFIELDS	Gold Fields Ltd.	0.0337	[0.0, 1.0]	SBM efficient
19	IMP	IMPLATS	Impala Platinum Holdings Ltd.	0.0219	[0.0, 0.344]	SBM inefficient
20	INP	INVPLC	Investec plc	0.0178	[0.0, 0.241]	SBM inefficient
21	KIO	KUMBA	Kumba Iron Ore Ltd.	0.0024	[0.0, 0.001]	SBM inefficient
22	MEI	MEDICLINIC	Mediclinic International plc	0.0252	[0.0, 0.692]	SBM inefficient
23	MNP	MONDIPLC	Mondi plc	0.0147	[0.0, 0.302]	SBM inefficient
24	MTN	MTN GROUP	MTN Group Ltd.	0.0251	[0.0, 0.757]	SBM inefficient
25	NPN	NASPERS-N	Naspers Ltd.	0.0489	[0.0, 1.0]	SBM efficient
26	NED	NEDBANK	Nedbank Group Ltd.	0.0291	[0.0, 1.0]	SBM efficient
27	NRP	NEPIROCK	NEPI Rockcastle N.V.	0.0371	[0.0, 1.0]	SBM efficient
28	NPH	NORTHAM	Northam Platinum Holdings Ltd.	0.0511	[0.0, 1.0]	SBM efficient
29	PPH	PEPKORH	Pepkor Holdings Ltd.	0.0376	[0.0, 1.0]	SBM efficient
30	PRX	PROSUS	Prosus N.V.	0.0178	[0.0, 0.16]	SBM inefficient
31	REM	REMGRO	Remgro Ltd.	0.0205	[0.0, 0.411]	SBM inefficient
32	SLM	SANLAM	Sanlam Ltd.	0.0182	[0.0, 0.39]	SBM inefficient
33	SOL	SASOL	Sasol Ltd.	0.0022	[0.0, 0.003]	SBM inefficient
34	SHP	SHOPRIT	Shoprite Holdings Ltd.	0.0389	[0.0, 1.0]	SBM efficient
35	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	0.0174	[0.0, 0.036]	SBM inefficient
36	S32	SOUTH32	South32 Ltd.	0.0094	[0.0, 0.046]	SBM inefficient
37	SBK	STANBANK	Standard Bank Group Ltd.	0.0148	[0.0, 0.037]	SBM inefficient
38	BVT	BIDVEST	The Bidvest Group Ltd.	0.0435	[0.0, 1.0]	SBM efficient
39	VOD	VODACOM	Vodacom Group Ltd.	0.0455	[0.0, 1.0]	SBM efficient
40	WHL	WOOLIES	Woolworths Holdings Ltd.	0.0014	[0.0, 0.022]	SBM inefficient

4.5.2 DEA results for the bootstrap based BCCI model (for the JSE Top 40 companies)

Tables 4.7 presents the results of the bootstrap BCCI model for the JSE Top 40 companies. The results indicate that the majority of companies are BCCI-inefficient.

Table 4.7: Bootstrap BCCI model results for the JSE Top 40.

	Code	Short Name	Company Name	Stde	CI(95%)	Status
1	ABG	ABSA	Absa Group Ltd.	0.0000	[0.6761, 0.6862]	BCCI inefficient
2	ARI	ARM	African Rainbow Minerals Ltd.	0.0398	[0.4645, 1.0]	BCCI efficient
3	AMS	AMPLATS	Anglo American Platinum Ltd.	0.0000	[0.9909, 1.0]	BCCI efficient
4	AGL	ANGLO	Anglo American plc	0.0129	[0.2985, 0.6154]	BCCI inefficient
5	ANG	ANGGOLD	AngloGold Ashanti Ltd.	0.0086	[0.254, 0.5295]	BCCI inefficient
6	ANH	AB INBEV	Anheuser-Busch InBev SA/NV	0.0004	[0.2461, 0.2923]	BCCI inefficient
7	APN	ASPEN	Aspen Pharmacare Holdings Ltd.	0.1451	[0.0008, 0.9636]	BCCI inefficient
8	BHG	BHP	BHP Group Ltd.	0.0667	[0.0003, 0.9676]	BCCI inefficient
9	BID	BIDCORP	Bid Corporation Ltd.	0.0933	[0.0006, 0.7315]	BCCI inefficient
10	BTI	BATS	British American Tobacco plc	0.0048	[0.0003, 0.1834]	BCCI inefficient
11	CPI	CAPITEC	Capitec Bank Holdings Ltd.	0.1003	[0.0002, 0.7431]	BCCI inefficient
12	CLS	CLICKS	Clicks Group Ltd.	0.1179	[0.0009, 0.8467]	BCCI inefficient
13	CFR	RICHEMONT	Compagnie Financière SA	0.0640	[0.0013, 0.6159]	BCCI inefficient
14	DSY	DISCOVERY	Discovery Ltd.	0.0308	[0.0013, 0.4689]	BCCI inefficient
15	EXX	EXXARO	Exxaro Resources Ltd.	0.0471	[0.0013, 0.6621]	BCCI inefficient
16	FSR	FIRSTRAND	FirstRand Ltd.	0.1614	[0.0032, 1.0]	BCCI efficient
17	GLN	GLENCORE	Glencore plc	0.0888	[0.0, 0.6865]	BCCI inefficient
18	GFI	GFIELDS	Gold Fields Ltd.	0.0578	[0.0019, 0.8552]	BCCI inefficient
19	IMP	IMPLATS	Impala Platinum Holdings Ltd.	0.0166	[0.0017, 0.3469]	BCCI inefficient
20	INP	INVPLC	Investec plc	0.0606	[0.002, 0.892]	BCCI inefficient
21	KIO	KUMBA	Kumba Iron Ore Ltd.	0.0472	[0.0005, 0.7136]	BCCI inefficient
22	MEI	MEDICLINIC	Mediclinic International plc	0.1218	[0.0019, 0.8573]	BCCI inefficient
23	MNP	MONDIPLC	Mondi plc	0.0546	[0.0006, 0.9339]	BCCI inefficient
24	MTN	MTN GROUP	MTN Group Ltd.	0.0817	[0.0015, 1.0]	BCCI efficient
25	NPN	NASPERS-N	Naspers Ltd.	0.0214	[0.0001, 0.587]	BCCI inefficient
26	NED	NEDBANK	Nedbank Group Ltd.	0.0339	[0.0009, 0.642]	BCCI inefficient
27	NRP	NEPIROCK	NEPI Rockcastle N.V.	0.0993	[0.0014, 0.7899]	BCCI inefficient
28	NPH	NORTHAM	Northam Platinum Holdings Ltd.	0.0379	[0.0018, 0.5116]	BCCI inefficient
29	PPH	PEPKORH	Pepkor Holdings Ltd.	0.1374	[0.0099, 1.0]	BCCI efficient
30	PRX	PROSUS	Prosus N.V.	0.0213	[0.0001, 0.5152]	BCCI inefficient
31	REM	REMGRO	Remgro Ltd.	0.0626	[0.001, 0.9805]	BCCI inefficient
32	SLM	SANLAM	Sanlam Ltd.	0.0565	[0.0027, 0.7307]	BCCI inefficient
33	SOL	SASOL	Sasol Ltd.	0.0258	[0.0005, 0.5927]	BCCI inefficient
34	SHP	SHOPRIT	Shoprite Holdings Ltd.	0.1060	[0.001, 0.9179]	BCCI inefficient
35	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	0.0944	[0.0035, 1.0]	BCCI efficient
36	S32	SOUTH32	South32 Ltd.	0.0838	[0.0057, 0.8095]	BCCI inefficient
37	SBK	STANBANK	Standard Bank Group Ltd.	0.1014	[0.0012, 1.0]	BCCI efficient
38	BVT	BIDVEST	The Bidvest Group Ltd.	0.0501	[0.0009, 0.8207]	BCCI inefficient
39	VOD	VODACOM	Vodacom Group Ltd.	0.0557	[0.0013, 0.8338]	BCCI inefficient
40	WHL	WOOLIES	Woolworths Holdings Ltd.	0.0950	[0.0029, 0.9828]	BCCI inefficient

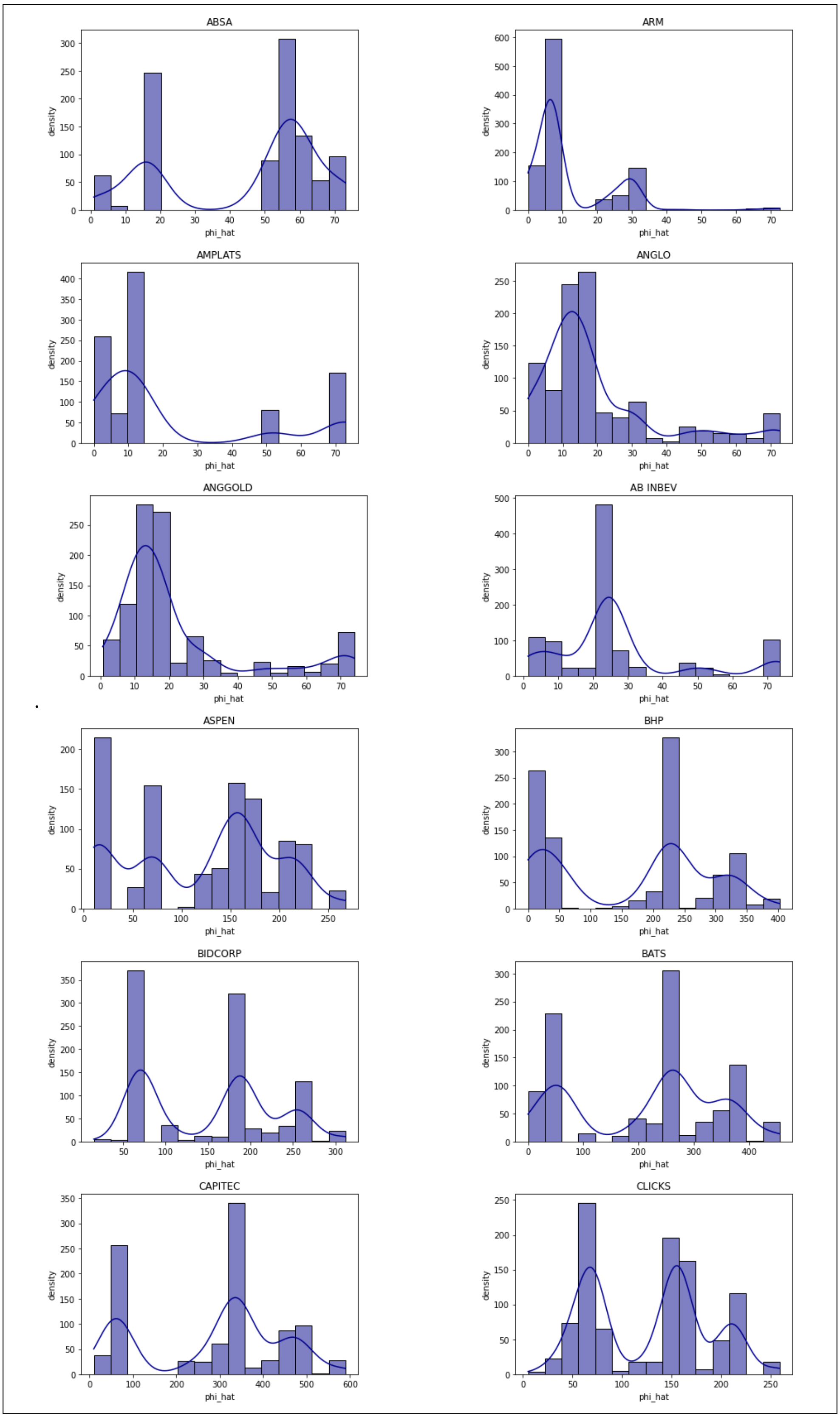
4.5.3 DEA results for the bootstrap based Additive model (for the JSE Top 40 companies)

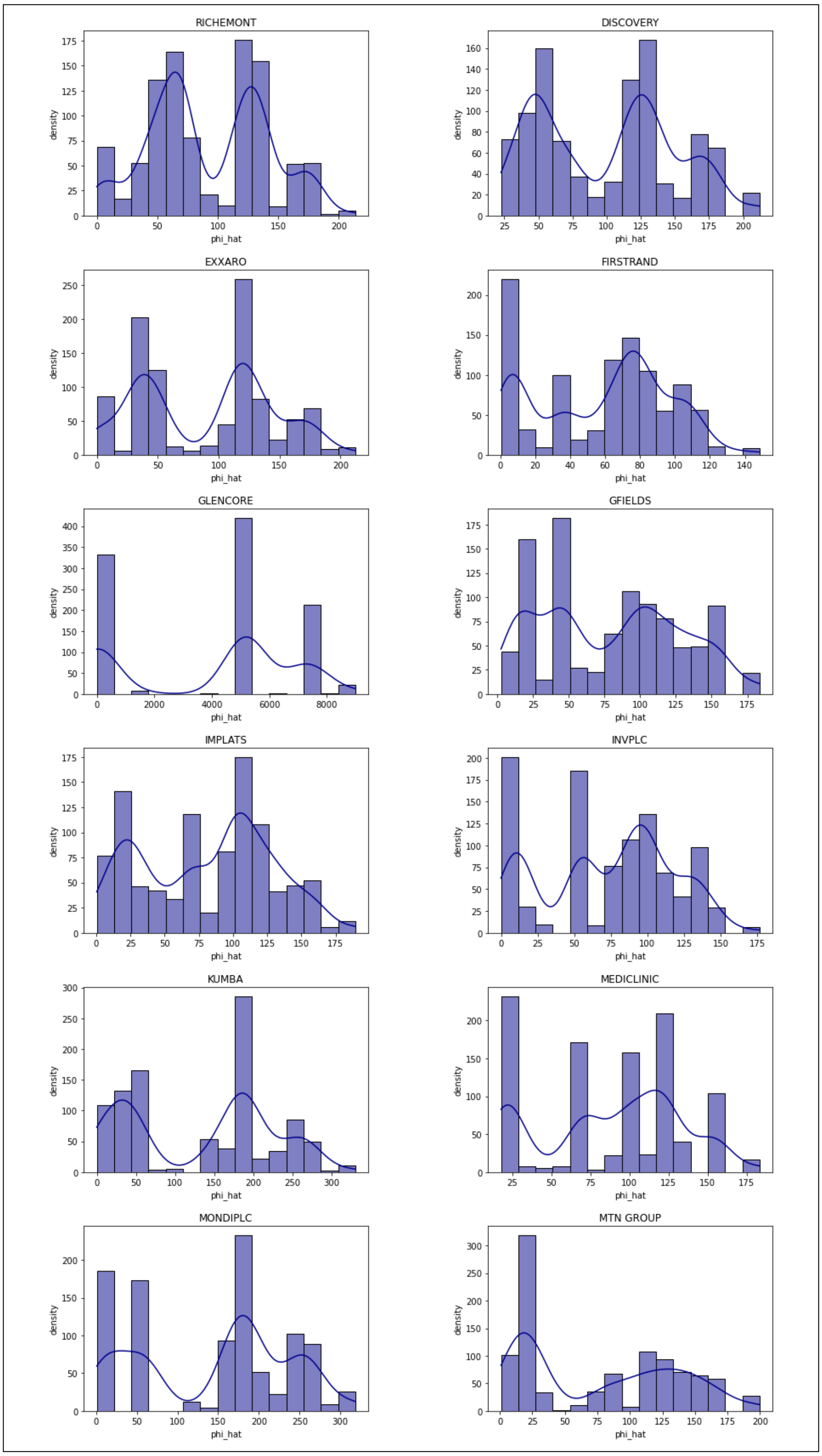
Tables 4.8 presents the results of the bootstrap ADD model for the JSE Top 40 companies.

Table 4.8: Bootstrap ADD model results for the JSE Top 40.

	Code	**Company Name**	**Stde**	**CI(95%)**	**Status**
1	ABG	ABSA	22.4838	[3.3027, 72.6972]	ADD inefficient
2	ARI	ARM	12.0373	[0.3517, 30.4188]	ADD inefficient
3	AMS	AMPLATS	25.7771	[0.5666, 72.7674]	ADD inefficient
4	AGL	ANGLO	17.5706	[0.5875, 72.4692]	ADD inefficient
5	ANG	ANGGOLD	19.2026	[0.6294, 72.5019]	ADD inefficient
6	ANH	AB INBEV	18.9369	[1.3554, 72.5197]	ADD inefficient
7	APN	ASPEN	73.9165	[10.6405, 222.441]	ADD inefficient
8	BHG	BHP	123.2662	[1.2349, 354.9698]	ADD inefficient
9	BID	BIDCORP	75.2193	[61.0766, 277.7607]	ADD inefficient
10	BTI	BATS	130.4813	[4.6239, 433.8525]	ADD inefficient
11	CPI	CAPITEC	159.5736	[44.6003, 568.843]	ADD inefficient
12	CLS	CLICKS	59.0837	[32.7189, 215.9294]	ADD inefficient
13	CFR	RICHEMONT	48.1285	[4.4024, 179.4816]	ADD inefficient
14	DSY	DISCOVERY	49.7273	[26.4193, 178.1831]	ADD inefficient
15	EXX	EXXARO	55.0186	[2.0009, 178.2885]	ADD inefficient
16	FSR	FIRSTRAND	37.1170	[3.4674, 118.6757]	ADD inefficient
17	GLN	GLENCORE	2964.1052	[24.523, 7342.4009]	ADD inefficient
18	GFI	GFIELDS	48.6548	[8.6369, 155.6361]	ADD inefficient
19	IMP	IMPLATS	47.6108	[3.8701, 160.5091]	ADD inefficient
20	INP	INVPLC	43.7223	[2.5589, 149.0031]	ADD inefficient
21	KIO	KUMBA	92.9146	[1.7884, 273.002]	ADD inefficient
22	MEI	MEDICLINIC	45.3125	[19.0983, 155.492]	ADD inefficient
23	MNP	MONDIPLC	91.4689	[3.5735, 307.788]	ADD inefficient
24	MTN	MTN GROUP	59.0749	[0.9527, 191.6696]	ADD inefficient
25	NPN	NASPERS-N	278.5388	[20.1694, 806.8759]	ADD inefficient
26	NED	NEDBANK	71.0171	[2.9745, 215.504]	ADD inefficient
27	NRP	NEPIROCK	50.1698	[16.7867, 173.668]	ADD inefficient
28	NPH	NORTHAM	45.8795	[6.0441, 156.8893]	ADD inefficient
29	PPH	PEPKORH	28.6782	[9.4699, 94.187]	ADD inefficient
30	PRX	PROSUS	181.7918	[25.885, 526.7361]	ADD inefficient
31	REM	REMGRO	61.6270	[16.1885, 202.2801]	ADD inefficient
32	SLM	SANLAM	38.1973	[13.8565, 135.1187]	ADD inefficient
33	SOL	SASOL	95.6435	[2.6905, 293.3323]	ADD inefficient
34	SHP	SHOPRIT	64.4047	[4.3876, 202.665]	ADD inefficient
35	SSW	SIBANYE-S	34.2957	[11.9714, 124.3987]	ADD inefficient
36	S32	SOUTH32	28.4986	[18.7858, 107.2909]	ADD inefficient
37	SBK	STANBANK	61.2310	[3.1361, 190.6683]	ADD inefficient
38	BVT	BIDVEST	66.6676	[4.1899, 213.7907]	ADD inefficient
39	VOD	VODACOM	54.8795	[2.2289, 180.7113]	ADD inefficient
40	WHL	WOOLIES	39.0959	[4.2019, 132.9357]	ADD inefficient

It is interesting to note that none of the companies attain boot-ADD efficiency . It should also be noted that one of the objectives is to estimate the sampling distribution of the estimator $\hat{\boldsymbol{\phi}}$, the optimal solution or objective function value for each of the 40 companies. This is done by plotting the histogram (kernel density) of the bootstrap estimates $\hat{\phi}_1^*, \hat{\phi}_2^*, ..., \hat{\phi}_B^*$ $(B = 1000)$ which we used to construct the confidence intervals above. The sampling distribution is only estimated for the Additive model due to the nature of the results for the other 2 model which consists of very small values that are not easy to visualise, as can be seen from the confidence intervals of the BCC and SBM model. Figure 4.1 shows the results.

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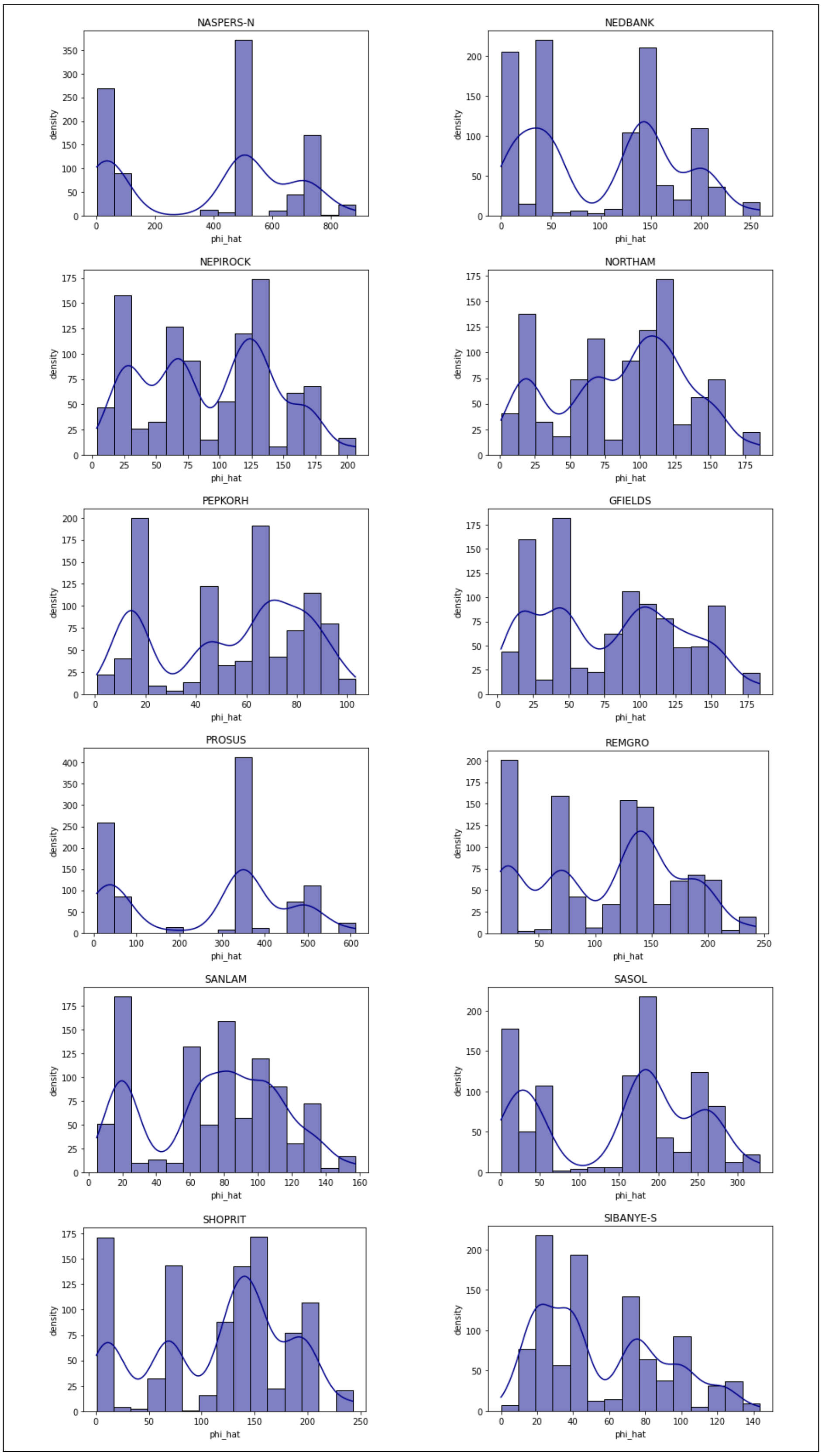
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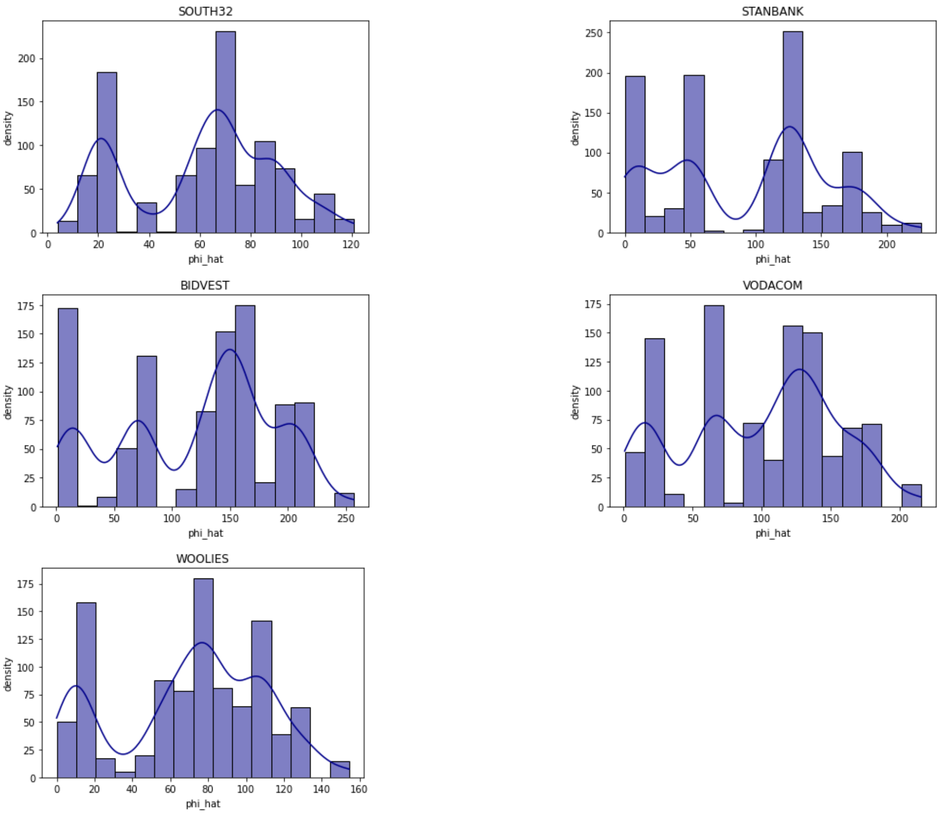

Figure 4.1: Estimated sampling distribution of the ADD model's $\hat{\boldsymbol{\phi}}$ for the JSE Top 40.

4.6 Summary and conclusion

This Chapter explored the application of classical and bootstrap based DEA model to finance data. The data consists of selected fundamental metrics for the JSE Top 40 listed companies for the period 2015-2023. These include: dividends per share, current ratio, quick ratio, e.t.c. The classical and bootstrap model's (SBM and BCC) results were similar in their selection of efficient companies. The classical ADD model's results were more discriminatory than the bootstrap ADD model's results as the latter evaluated all companies to be inefficient.

Chapter 5

DEA application to portfolio optimisation

5.1 Introduction

This Chapter focuses on the application of portfolio theory to DEA-efficient companies. It builds on the results from Chapter 4. The objective is to eventually invest in securities of different companies listed in the JSE Top 40. What the DEA models have given us are companies that perform “well” when looking at the fundamental metrics.

5.2 Problem description

When investing one has to look at the potential gains(return) and losses (risk) of the investment being considered, and make informed decisions accordingly. One must know which assets to select and the amount to risk per investment. From the DEA models, we were able to narrow down efficient companies from inefficient ones by looking at each model. We are now in a position to use the tools of portfolio theory to select portfolios or a combination of stocks that will “guarantee” profitability and protection from the adverse market movements.

5.3 Literature review on portfolio optimisation

5.3.1 Financial portfolio risk

Investing is a financial activity that involves risk. It is the commitment of funds for a return expected to be realised in the future. In this study investments may be made to the JSE Top 40 companies. When investing there is the possibility that the actual return may vary from the expected return or result into a loss (negative return), that possibility is the risk involved in the investment (Suresh, 2013). Financial risk has two components: uncertainty and exposure, uncertainty refers to the probability of facing the risk, exposure is the amount of the financial loss if the risk is realised. For a portfolio of securities, the total risk comprises of the systematic(undiversifiable) risk and unsystematic(diversifiable) risk. Systematic risk refers to the movement of the "whole" financial market, this implies that even with a perfectly diversified portfolio, there is some risk that cannot be avoided.

Unsystematic risk is the risk associated with the individual asset and it differs from asset to asset. Unlike systematic risk, it can be diversified away by including a large number of assets from different financial sectors in the portfolio. The difference between systematic and unsystematic risk can be illustrated by the following example: Suppose there are two investors A and B, investor A buys ten different securities subject to a certain volume size that amounts to USD 100,000, and investor B buys one security that amounts to the same USD 100,000. If the market goes against investor A (systematic risk), then one, two or more currencies may lose value but it is highly unlikely that all ten will be affected at the same time. Hence investor A will suffer a loss but still make some profit on the unaffected securities. On the other hand, if the market goes against investor B on the single security, then the investor suffers a loss. The reason is that investor B's portfolio has more unsystematic risk that needs to be diversified away (Karadag, 2008).

There are various measures or models of portfolio risk, the most popular being variance, mean absolute deviation(MAD) semi mean absolute deviation (SMAD), value-at-risk (VaR) and conditional-value-at-risk (CVaR). A brief overview of each of these measures is given and their advantages and disadvantages are highlighted.

(i) Variance: It is a global measure of risk that penalises returns below and above the mean. Hiroshi and Yamazaki (1991) argues that despite the mathematical plausibility of variance as a risk measure, it is not popular amongst some investors since it fails to capture an investor's true perception of risk, the normality assumption of security returns is often violated, and many non-zero weights occur in the efficient portfolio making the portfolio financially challenging to manage. The variance of a portfolio is often expressed by the following equation:

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \tag{5.1}$$

where w_i , w_j are the invested proportions in security i and j and σ_{ij} is the covariance between security i and j

(ii) Mean-absolute-deviation: Due to the symmetric nature of variance as a risk measure, Hiroshi and Yamazaki (1991) attempted to remove this undesirable property by introducing the mean absolute deviation (MAD) risk measure :

$$\delta(\boldsymbol{w}) = E\Big[\Big|R_{\boldsymbol{w}} - \mu_{\boldsymbol{w}}\Big|\Big] = E\Big[\Big|\sum_{j=1}^{n} w_j R_j - E\big[\sum_{j=1}^{n} w_j R_j\big]\Big|\Big] \tag{5.2}$$

$R_{\boldsymbol{w}}$ and $\mu_{\boldsymbol{w}}$ are the return and mean return of the portfolio over the investment period. The MAD measures the average of the absolute value of the difference between the random variable and its expected value. With respect to the variance, the MAD considers absolute values instead of squared values.

(iii) Semi-mean-absolute-deviation: The MAD accounts for all deviations of the return of the portfolio from its expected value. However, one may sensibly think that any rational investor would consider real risk only the deviations below the expected value. In other words, the variability of the portfolio return above the mean should not be penalized since investors are concerned with under-performance rather than over-performance of a portfolio. Charnes and Cooper (1962) modified the MAD in order to consider only the deviations below the expected value. The SMAD is thus expressed as:

$$\xi(\boldsymbol{w}) = E\Big[max\Big\{0, \mu_{\boldsymbol{w}} - R_{\boldsymbol{w}}\Big\}\Big] = E\Big[max\Big\{0, E\Big\{\sum_{j=1}^{n} w_j R_j\Big\} - \sum_{j=1}^{n} w_j R_j\Big\}\Big] \tag{5.3}$$

(iv) Value-at-risk: The value at risk VaR at $\alpha \in (0,1)$ confidence level of the return distribution is one of the most popular downside risk measures in mathematical finance, it was introduced in the late 1930's by financial firms indicating the amount of minimum loss at a given confidence level α. By definition with respect to a specified probability level α, the VaR_α of a portfolio is the lowest amount β such that with probability α the loss will not exceed β.

Rockafellar and Uryasev (2000) define a loss function $f(\boldsymbol{x}, \boldsymbol{y})$ to be the loss (negative portfolio return) associated with the decision vector $\boldsymbol{x}$, to be chosen from a certain subset X of $\mathbb{R}^n$ and the random vector $\boldsymbol{y} \epsilon \mathbb{R}^m$.The vector $\boldsymbol{x}$ is interpreted as representing a portfolio, with X as the set of permissible portfolios but other interpretations are possible. The vector $\boldsymbol{y}$ represents uncertainties that affect the loss. The probability of $f(\boldsymbol{x}, \boldsymbol{y})$ not exceeding a threshold γ is given by:

$$\psi(\boldsymbol{x}, \gamma) = \int_{f(\boldsymbol{x},\boldsymbol{y}) \leq \gamma} p(\boldsymbol{y}) d\boldsymbol{y} \tag{5.4}$$

As a function of γ for a fixed $\boldsymbol{x}$, $\psi(\boldsymbol{x}, \gamma)$ is the cumulative distribution function for the loss associated with $\boldsymbol{x}$. It completely determines the behaviour of this random variable and is fundamental in defining VaR. The VaR_α values for the loss random variable associated with decision vector $\boldsymbol{x}$ and any specified probability level α in $(0,1)$ is denoted by $\gamma_\alpha(\boldsymbol{x})$. Mathematically it is expressed as:

$$\gamma_\alpha(\boldsymbol{x}) = \min\Big\{\gamma \,\epsilon\, \mathbb{R} : \psi(\boldsymbol{x}, \gamma) \geq \alpha\Big\} \tag{5.5}$$

(v) Conditional-value-at-risk: Since VaR_α of a portfolio is the lowest amount β such that with probability α the loss will not exceed β, CVaR_α is the conditional expectation of losses above that amount β. Three values of α that are commonly used in the literature are: 0.9, 0.95, and 0.99. Mathematically CVaR$_\alpha$ denoted by $\phi_\alpha(\boldsymbol{x})$ is expressed as:

$$\phi_\alpha(\boldsymbol{x}) = (1-\alpha)^{-1}\int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma_\alpha(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y} \tag{5.6}$$

Figure 5.1 shows the VaR and CVaR associated with the loss distribution. These two quantities occur on the right tail of the distribution where the extreme losses are.

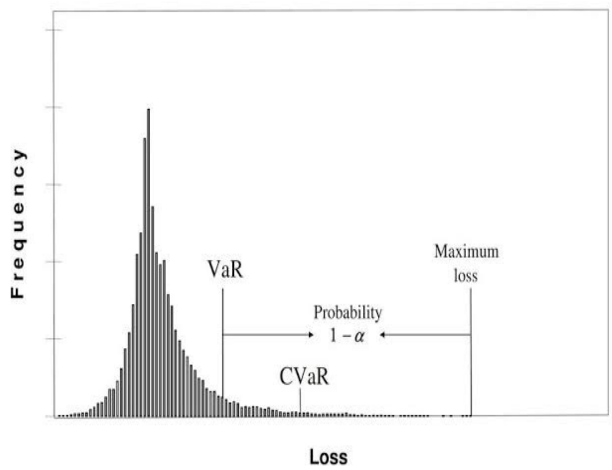

Figure 5.1: VaR$_\alpha$ and CVaR$_\alpha$ of the loss distribution

In this study we will make use of the CVaR$_\alpha$ as a risk measure of choice due to the desirable properties it has. Pflug (2000) gives the following non-exhaustive list:

(i) CVaR$_\alpha$ is coherent: satisfies properties of monotonicity, sub-additivity, homogeneity, and translational invariance.

(ii) CVaR$_\alpha$ is positively homogenous: Let $f : X \to \mathbb{R} = \text{CVaR}_\alpha$, then $f(tx) = tkf(x)$ for every $x \in X$ and every $t \in \mathbb{R}^{++}$

(iii) CVaR$_\alpha$ is convex: The second derivative of CVaR$_\alpha$ is non-negative.

(iv) CVaR$_\alpha$ is monotonic with respect to stochastic dominance of order 1 and 2.

According to Pflug (2000) , $\gamma_\alpha(\boldsymbol{x})$ and $\phi_\alpha(\boldsymbol{x})$ can be characterised in-terms of the function F_α on $X \times \mathbb{R}$ given by :

$$F_\alpha(\boldsymbol{x},\gamma) = \gamma + (1-\alpha)^{-1}\int_{\boldsymbol{y}\in\mathbb{R}^m} \left[f(\boldsymbol{x},\boldsymbol{y})-\gamma\right]^+ p(\boldsymbol{y})d\boldsymbol{y} \tag{5.7}$$

where $[t]^+ = t$ when $t > 0$ and $[t]^+ = 0$ when $t \leq 0$. The important feature of F_α in optimisation is that it is convex which is a key property that guarantees global optimality of solutions. Unlike variance, MAD, SMAD, it is not straightforward to obtain a $\gamma_\alpha(\boldsymbol{x})$ or $\phi_\alpha(\boldsymbol{x})$ optimal portfolio by using the definition directly. Rockafellar and Uryasev (2000) proved an important result which makes it easier to optimise a CVaR$_\alpha$ portfolio. The result is stated in the following theorem:

Theorem 5.1.1: Minimising the CVaR$_\alpha$ of the loss associated with $\boldsymbol{x} \in$ X is equivalent to minimising $F_\alpha(\boldsymbol{x},\gamma)$ over all $(\boldsymbol{x},\gamma) \in X \times \mathbb{R}$, in the sense that :

$$\min_{\boldsymbol{x}\in X} \phi_\alpha(\boldsymbol{x}) = \min_{(\boldsymbol{x},\gamma)\ \in\ X\times\mathbb{R}} F_\alpha(\boldsymbol{x},\gamma) \tag{5.8}$$

Furthermore, $F_\alpha(\boldsymbol{x},\gamma)$ is convex with respect to $(\boldsymbol{x},\gamma)$ and $\phi_\alpha(\boldsymbol{x})$ is convex with respect to $\boldsymbol{x}$ when $f(\boldsymbol{x},\boldsymbol{y})$ is convex with respect to $\boldsymbol{x}$, in which case if the constraints are such that X is a convex set, the joint minimisation is a case of convex programming. According to the above theorem, it is not necessary, for the purpose of determining a portfolio $\boldsymbol{x}$ that yields minimum CVaR$_\alpha$, to work directly with the function $\phi_\alpha(\boldsymbol{x})$, which may be hard to do because of the nature of its definition in terms of the VaR$_\alpha$ value $\gamma_\alpha(\boldsymbol{x})$ and the often troublesome mathematical properties of this value. Instead one can operate on the far simpler expression $F_\alpha(\boldsymbol{x},\gamma)$ which is convex in the variable γ and in most cases it is also convex in $(\boldsymbol{x},\gamma)$.

Proof of Theorem 5.1.1

The proof relies on the fact that the minimisation of $F_\alpha(\boldsymbol{w},\gamma)$ with respect to $(\boldsymbol{w},\gamma) \in W\times \mathbb{R}$ can be done by first minimising over $\gamma \in \mathbb{R}$ for a fixed $\boldsymbol{w}$ and then minimising the result over $\boldsymbol{w} \in W$. Justification of the convexity claim starts with the observation that $F_\alpha(\boldsymbol{w},\gamma)$ is convex with respect to $(\boldsymbol{w},\gamma)$ whenever the integrand $[f(\boldsymbol{w},\boldsymbol{r})-\gamma]^-$ in the formula for $F_\alpha(\boldsymbol{w},\gamma)$ is itself convex with respect to $(\boldsymbol{w},\gamma)$. For each $\boldsymbol{r}$, this integrand is the composition of the function $(\boldsymbol{w},\gamma) \longrightarrow f(\boldsymbol{w},\boldsymbol{r})-\gamma$ with the non-decreasing function function $t \longrightarrow [t]^-$ and by the rules of Cooper et al. (2006), it is convex as long as the function $(\boldsymbol{w},\gamma) \longrightarrow f(\boldsymbol{w},\boldsymbol{r})-\gamma$ is convex. $f(\boldsymbol{w},\boldsymbol{r})-\gamma$ is convex when $f(\boldsymbol{w},\boldsymbol{r})$ is convex with respect to $\boldsymbol{w}$. Since in this setting the function $f(\boldsymbol{w},\boldsymbol{r})$ represents the portfolio return that is convex in the variable $\boldsymbol{w}$, the result of Theorem 5.1.1 follows.

5.3.2 Conditional value-at-risk model

As stated previously, the performance function in connection with VaR$_\alpha$ and CVaR$_\alpha$ is :

$$F_\alpha(\boldsymbol{x},\gamma) = \gamma + (\alpha)^{-1}\int_{\boldsymbol{x}\in\mathbb{R}^m} \left[\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{y}-\gamma\right]^+ p(\boldsymbol{y})d\boldsymbol{y} \tag{5.9}$$

It is important to observe that in this setting, $F_\alpha(\boldsymbol{x},\gamma)$ is convex as a function of $(\boldsymbol{x},\gamma)$

not just γ.

If one considers the feasible set of portfolios:

$$X = \left\{ \boldsymbol{x} : \boldsymbol{x}'\boldsymbol{\mu} = \rho,\ \boldsymbol{x}'\boldsymbol{e} = 1, \boldsymbol{x} \geq \boldsymbol{0} \right\} \tag{5.10}$$

This set X is convex (in fact "polyhedral" due to linearity of all the constraints). As a result the problem of minimising $F_\alpha(\boldsymbol{x}, \gamma)$ over $X \times \mathbb{R}$ subject to (5.10) is one of convex programming which guarantees globality of optimal solutions. Considering the kind of approximation of F_α that is obtained by sampling from the probability distribution of $\boldsymbol{y}$. A sample set $\boldsymbol{y_1}, \boldsymbol{y_2}, ..., \boldsymbol{y_n}$ yields the approximate function :

$$\tilde{F}_\alpha(\boldsymbol{x}, \gamma) = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} [\boldsymbol{x}^T \boldsymbol{y_t} - \gamma]^+ \tag{5.11}$$

In terms of auxiliary variables u_t for $t = 1, 2, ..., T$, minimising F_α is equivalent to minimising the linear expression:

$$z = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} u_t \tag{5.12}$$

subject to the constraints : $u_t \geq 0$ and $\boldsymbol{x}^T \boldsymbol{y_t} + \gamma + u_t \geq 0$.

Hence the CVaR$_\alpha$ minimisation problem is formulated as :

$$\begin{aligned}
\text{minimise } \ & z = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} u_t, \ \text{ over } x_i, u_t \in \mathbb{R}^+ \\
\text{subject to } \ & \sum_{i=1}^{n} x_i \mu_i = \rho \\
& \sum_{i=1}^{n} x_i = 1 \\
& \boldsymbol{x}^T \boldsymbol{r_t} + \gamma + u_t \geq 0 \\
& u_t \geq 0, \ \ t = 1, 2, ..., T \\
& x_i \geq 0 \ \ i = 1, 2, ..., n
\end{aligned} \tag{5.13}$$

Figure 5.2 shows the set of efficient portfolios generated by the CVaR model, where the efficient frontier is indicated by the solid line:

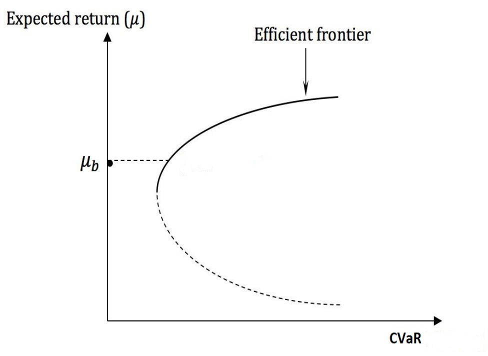

Figure 5.2: CVaR Efficient Frontier.

5.4 Data and results

This section deals with the selection of portfolios to invest in subject to the trade-off between risk and return. The analysis was done using Python 3.12. This study uses the expected return as a measure of reward and CVaR_α as a measure of risk.

5.4.1 Data

The data consists of fundamental metrics for the JSE Top 40 companies as stated in Chapter 4. These are listed in Table 5.1.

Table 5.1: Inputs and Outputs for the JSE Top 40

Inputs	**Outputs**
Debt / Assets	Current Ratio
Debt / Equity	Dividend / Share (c)
Long-Term Loans (%)	Dividend Cover
Price / Book Value	Dividend Yield (%)
Price / Earnings	Earnings / Share (C)
Price / Share (C)	Earnings Yield (%)
Total Debt / Cash Flow	Quick Ratio
Price / Cash	Retention Rate
-	Return On Equity (%)
-	Return on Capital Employed

5.4.2 Portfolio selection of the DEA efficient (classical) companies

This study will start with the results of the classical SBM, BCCI, and Additive models by using the DMUs that were deemed efficient as a starting point, these results were generated in Chapter 4.

5.4.3 Portfolio selection of the SBM efficient (classical) companies

Table 5.2 shows the SBM-efficient companies whose daily returns are used for the analysis. The returns are generated from closing prices for the period 15-Feb-2013 to 17-Feb-2023.

Table 5.2: SBM efficient JSE Top 40 companies

	Code	Short Name	Company Name	τ^*	DMU Status
1	ARI	ARM	African Rainbow Minerals Ltd.	1.0000	SBM efficient
2	AMS	AMPLATS	Anglo American Platinum Ltd.	1.0000	SBM efficient
3	ANG	ANGGOLD	AngloGold Ashanti Ltd.	1.0000	SBM efficient
4	BTI	BATS	British American Tobacco plc	1.0000	SBM efficient
5	CPI	CAPITEC	Capitec Bank Holdings Ltd.	1.0000	SBM efficient
6	EXX	EXXARO	Exxaro Resources Ltd.	1.0000	SBM efficient
7	FSR	FIRSTRAND	FirstRand Ltd.	1.0000	SBM efficient
8	GLN	GLENCORE	Glencore plc	1.0000	SBM efficient
9	IMP	IMPLATS	Impala Platinum Holdings Ltd.	1.0000	SBM efficient
10	INP	INVPLC	Investec plc	1.0000	SBM efficient
11	KIO	KUMBA	Kumba Iron Ore Ltd.	1.0000	SBM efficient
12	MTN	MTN GROUP	MTN Group Ltd.	1.0000	SBM efficient
13	NED	NEDBANK	Nedbank Group Ltd.	1.0000	SBM efficient
14	PPH	PEPKORH	Pepkor Holdings Ltd.	1.0000	SBM efficient
15	PRX	PROSUS	Prosus N.V.	1.0000	SBM efficient
16	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	1.0000	SBM efficient
17	S32	SOUTH32	South32 Ltd.	1.0000	SBM efficient
18	SBK	STANBANK	Standard Bank Group Ltd.	1.0000	SBM efficient
19	WHL	WOOLIES	Woolworths Holdings Ltd.	1.0000	SBM efficient

It should be noted that due to the lack of sufficient data for the given period, some companies in Table 5.2 were omitted namely: GLENCOR, PROSUS, SIBANYE-S, and SOUTH32. So the listed companies are not part of the portfolio selection process. It should be noted that portfolio selection is performed on 60 % of the data. The remaining 40 % will be used for backtesting analysis.

Figure 5.3 shows the efficient frontier curve which comprises of 1000 portfolios of the companies in Table 5.2.

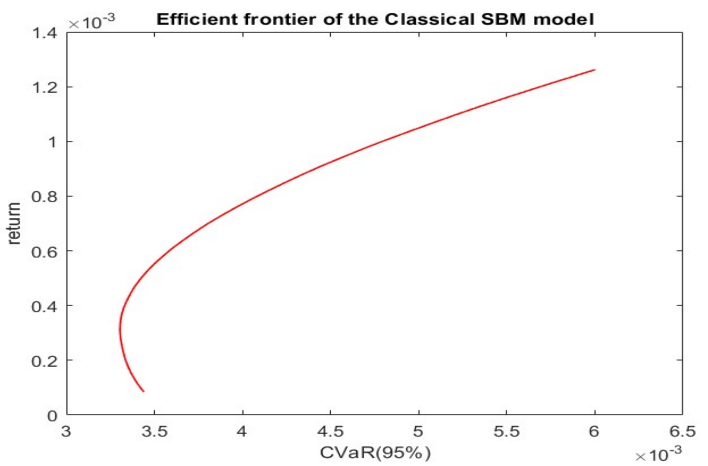

Figure 5.3: Efficient frontier of the classical SBM-Efficient companies

This curve shows the trade-off between risk and return and an investor has the opportunity to select a portfolio that suits his/her investment strategy. This curve demonstrates the well known fact in investment theory: "The higher the risk the higher the reward".

5.4.4 Portfolio selection of the BCCI efficient (classical) companies

This study performs a similar analysis for the BCCI efficient (Classical) companies listed in Table 5.3. Due to lack of sufficient data SIBANYE-S is omitted from the analysis.

Table 5.3: BCCI efficient JSE Top 40 companies

	code	**short name**	**company name**	θ^*	**DMU status**
1	ARI	ARM	African Rainbow Minerals Ltd.	1.0000	BCC efficient
2	AMS	AMPLATS	Anglo American Platinum Ltd.	1.0000	BCC efficient
3	FSR	FIRSTRAND	FirstRand Ltd.	1.0000	BCC efficient
4	MTN	MTN GROUP	MTN Group Ltd.	1.0000	BCC efficient
5	PPH	PEPKORH	Pepkor Holdings Ltd.	1.0000	BCC efficient
6	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	1.0000	BCC efficient
7	SBK	STANBANK	Standard Bank Group Ltd.	1.0000	BCC efficient

Figure 5.4 shows the efficient frontier curve which comprises of 1000 portfolios of the companies in Table 5.3.

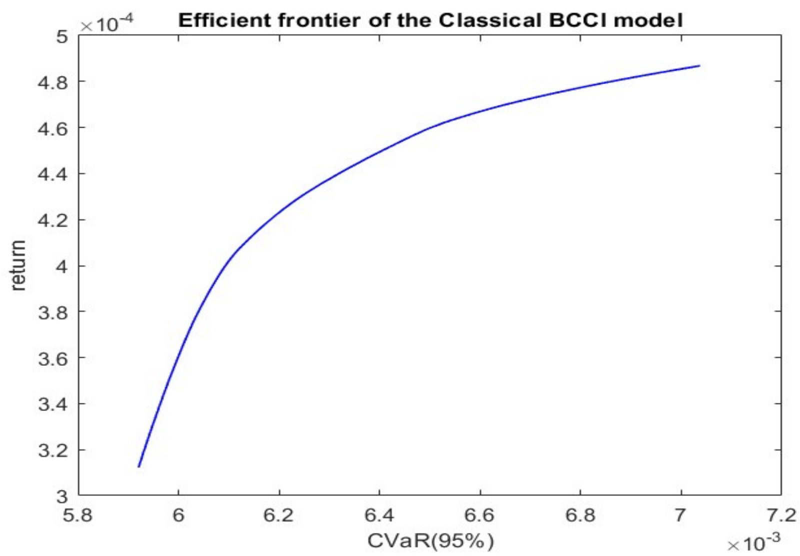

Figure 5.4: Efficient frontier of the Classical BCCI Efficient companies

This study performs a similar analysis for the ADD-efficient (classical) companies listed in Table 5.4. Due to lack of sufficient data: SIBANYE-S, GLENCORE, PROSUS, SOUTH32, BHP are omitted from the analysis.

Table 5.4: ADD efficient JSE Top 40 companies

	Code	**Short Name**	**Company Name**	θ^*	**DMU Status**
1	ARI	ARM	African Rainbow Minerals Ltd.	0.0000	ADD efficient
2	AMS	AMPLATS	Anglo American Platinum Ltd.	0.0000	ADD efficient
3	ANG	ANGGOLD	AngloGold Ashanti Ltd.	0.0000	ADD efficient
4	BHG	BHP	BHP Group Ltd.	0.0000	ADD efficient
5	BTI	BATS	British American Tobacco plc	0.0000	ADD efficient
6	CPI	CAPITEC	Capitec Bank Holdings Ltd.	0.0000	ADD efficient
7	CFR	RICHEMONT	Compagnie Financière Richemont SA	0.0000	ADD efficient
8	EXX	EXXARO	Exxaro Resources Ltd.	0.0000	ADD efficient
9	FSR	FIRSTRAND	FirstRand Ltd.	0.0000	ADD efficient
10	GLN	GLENCORE	Glencore plc	0.0000	ADD efficient
11	IMP	IMPLATS	Impala Platinum Holdings Ltd.	0.0000	ADD efficient
12	INP	INVPLC	Investec plc	0.0000	ADD efficient
13	KIO	KUMBA	Kumba Iron Ore Ltd.	0.0000	ADD efficient
14	MTN	MTN GROUP	MTN Group Ltd.	0.0000	ADD efficient
15	NED	NEDBANK	Nedbank Group Ltd.	0.0000	ADD efficient
16	PPH	PEPKORH	Pepkor Holdings Ltd.	0.0000	ADD efficient
17	PRX	PROSUS	Prosus N.V.	0.0000	ADD efficient
18	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	0.0000	ADD efficient
19	S32	SOUTH32	South32 Ltd.	0.0000	ADD efficient
20	SBK	STANBANK	Standard Bank Group Ltd.	0.0000	ADD efficient
21	WHL	WOOLIES	Woolworths Holdings Ltd.	0.0000	ADD efficient

Figure 5.5 shows the efficient frontier curve which comprises of 1000 portfolios of the companies in Table 5.4.

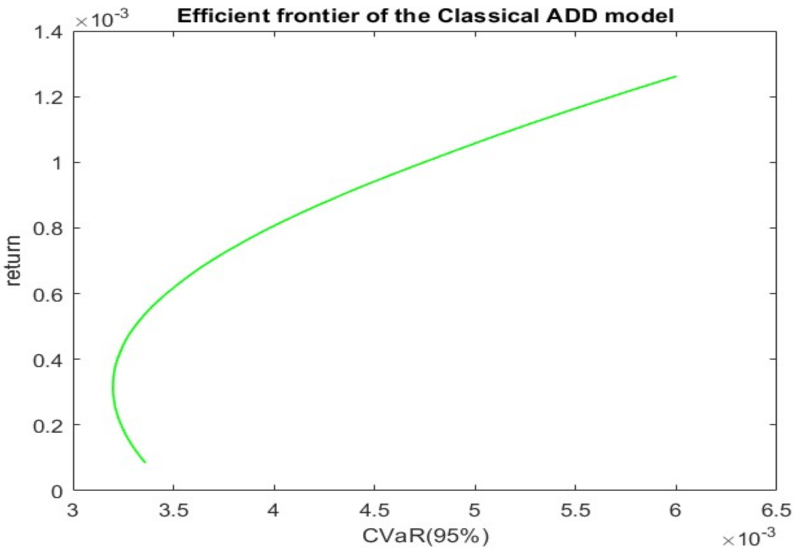

Figure 5.5: Efficient frontier of the Classical ADD Efficient companies

5.4.5 Portfolio selection of the DEA bootstrap efficient companies

Now, the results for bootstrap based models are presented. It should be noted that the ADD model will not feature in this analysis since none of the companies were deemed efficient. Table 5.5 shows the bootstrap SBM-efficient companies.

Table 5.5: Bootstrap SBM efficient companies

	Code	Short Name	Company Name	Stde	CI(95%)	Status
1	ARI	ARM	African Rainbow Minerals Ltd.	0.0479	[0.0, 1.0]	SBM efficient
2	ANH	AB INBEV	Anheuser-Busch InBev SA/NV	0.0467	[0.0, 1.0]	SBM efficient
3	BHG	BHP	BHP Group Ltd.	0.0319	[0.0, 1.0]	SBM efficient
4	BID	BIDCORP	Bid Corporation Ltd.	0.0440	[0.0, 1.0]	SBM efficient
5	BTI	BATS	British American Tobacco plc	0.0430	[0.0, 1.0]	SBM efficient
6	CLS	CLICKS	Clicks Group Ltd.	0.0405	[0.0, 1.0]	SBM efficient
7	EXX	EXXARO	Exxaro Resources Ltd.	0.0326	[0.0, 1.0]	SBM efficient
8	FSR	FIRSTRAND	FirstRand Ltd.	0.0333	[0.0, 1.0]	SBM efficient
9	GFI	GFIELDS	Gold Fields Ltd.	0.0337	[0.0, 1.0]	SBM efficient
10	NPN	NASPERS-N	Naspers Ltd.	0.0489	[0.0, 1.0]	SBM efficient
11	NED	NEDBANK	Nedbank Group Ltd.	0.0291	[0.0, 1.0]	SBM efficient
12	NRP	NEPIROCK	NEPI Rockcastle N.V.	0.0371	[0.0, 1.0]	SBM efficient
13	NPH	NORTHAM	Northam Platinum Holdings Ltd.	0.0511	[0.0, 1.0]	SBM efficient
14	PPH	PEPKORH	Pepkor Holdings Ltd.	0.0376	[0.0, 1.0]	SBM efficient
15	SHP	SHOPRIT	Shoprite Holdings Ltd.	0.0389	[0.0, 1.0]	SBM efficient
16	BVT	BIDVEST	The Bidvest Group Ltd.	0.0435	[0.0, 1.0]	SBM efficient
17	VOD	VODACOM	Vodacom Group Ltd.	0.0455	[0.0, 1.0]	SBM efficient

Figure 5.6 shows the efficient frontier of the boot-SBM efficient companies with the exception of BHG due to lack of sufficient data.

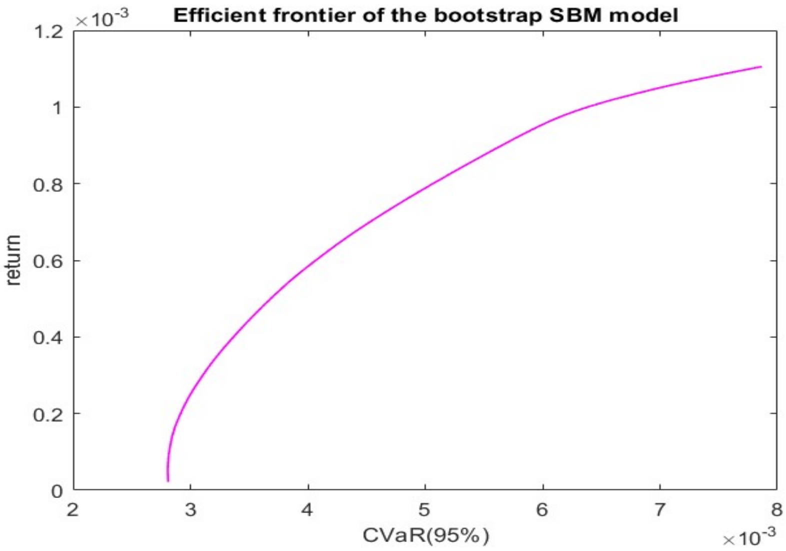

Figure 5.6: Efficient frontier of the Bootstrap-SBM-Efficient companies

Similar analysis for the bootstrap BCCI-efficient companies in Table 5.6 is performed next. Due to lack of sufficient data SIBANYE-S is omitted from the analysis.

Table 5.6: Bootstrap BCCI efficient companies.

	Code	Short Name	Company Name	Stde	CI(95%)	Status
1	ARI	ARM	African Rainbow Minerals Ltd.	0.0398	[0.4645, 1.0]	BCCI efficient
2	AMS	AMPLATS	Anglo American Platinum Ltd.	0.0000	[0.9909, 1.0]	BCCI efficient
3	FSR	FIRSTRAND	FirstRand Ltd.	0.1614	[0.0032, 1.0]	BCCI efficient
4	MTN	MTN GROUP	MTN Group Ltd.	0.0817	[0.0015, 1.0]	BCCI efficient
5	PPH	PEPKORH	Pepkor Holdings Ltd.	0.1374	[0.0099, 1.0]	BCCI efficient
6	SSW	SIBANYE-S	Sibanye Stillwater Ltd.	0.0944	[0.0035, 1.0]	BCCI efficient
7	SBK	STANBANK	Standard Bank Group Ltd.	0.1014	[0.0012, 1.0]	BCCI efficient

Figure 5.7 shows the efficient frontier of the boot-BCCI efficient companies with the exception of SIBANYE-S due to lack of sufficient data.

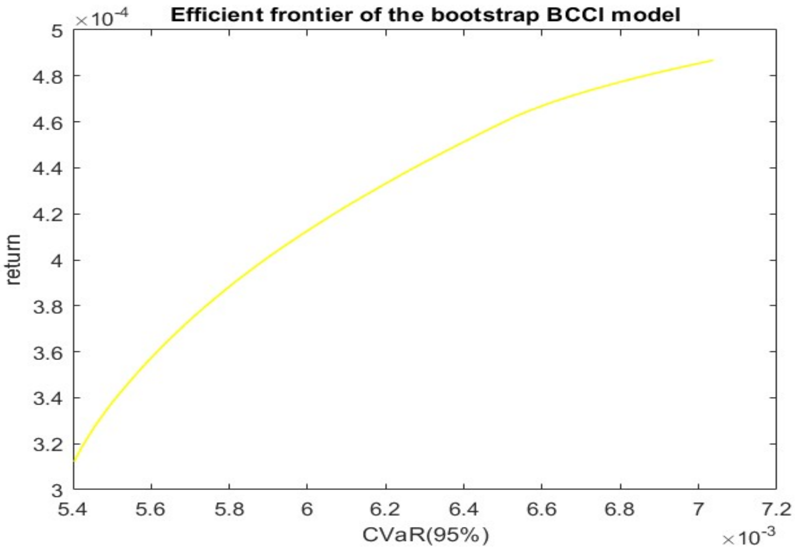

Figure 5.7: Efficient frontier of the Bootstrap BCCI-Efficient companies

5.5 Backtesting analysis

As stated earlier portfolio optimisation is performed on 60% of the data (daily returns), the remaining 40% will be used to perform backtesting analysis.

The reasoning behind using the 60-40 % split is due to the fact that in general the training data set should be bigger than the testing set as suggested by (Siegfried, 1995). Out of the 1000 portfolios, 3 are selected for each model, namely the portfolios corresponding with the: 1^{st} , 500^{th}, and 1000^{th} return, the so called "extreme" portfolios.

The efficient portfolios only inform the investor about which stocks to select and how much to invest. In principle the optimal weights "ensure" that the risk exposed to is as small as possible. The main objective of investing is to make money, the optimisation models in this study make use of the mean as the return, but in a real investment setting profits and losses are observed daily. How is it known if these portfolios are profitable in a "real" investment period?

An attempt to answer this question is provided by performing backtesting analysis on some portion of the data which is assumed to be what the market looks like if one were to invest during that period. The objective is to compare the profitability of the three portfolios to see which is more profitable during this period. In particular it will be interesting to see how these portfolios compare with the JSE Top 40 index which is used for comparison. The returns data consists of 2500 daily returns. The portfolio selection was done with 1500 points and the backtesting analysis was done on 1000 points. In this study, it is assumed that the investor employs the buy and hold strategy, and that there is no rebalancing of the portfolios for the whole "investment" period.

5.5.1 Backtesting analysis of the DEA efficient (classical) companies

Figure 5.8 shows the classical SBM "extreme" portfolios for the mean-CVaR model upon which the backtesting analysis is performed.

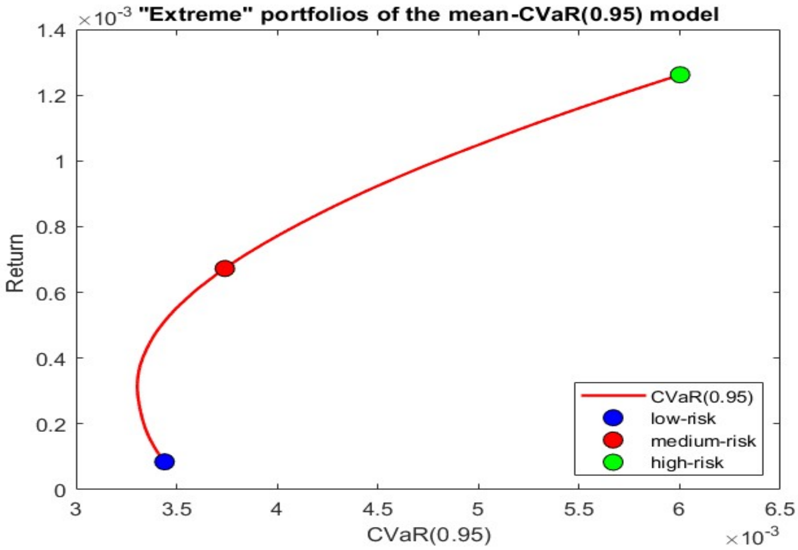

Figure 5.8: Mean-CVaR(0.95) "extreme" portfolios.

Table 5.6 shows the optimal weights corresponding with these portfolios.

Table 5.7: "Extreme" portfolios of the Classical SBM Efficient companies

Code	Low risk portfolio	Medium risk portfolio	High risk portfolio
ARI	0.01371	0	0
AMS	0	0.03013	0
ANG	0.07631	0.0521	0
BTI	0.3886	0.283	0
CPI	0.06277	0.488	1.0
EXX	0.02452	0	0
FSR	0	0.04706	0
IMP	0	0	0
INP	0.08191	0.03444	0
KIO	0.004813	0	0
MTN	0.05209	0	0
NED	0.04684	0	0
PPH	0.1047	0.06527	0
SBK	0.04111	0	0
WHL	0.1026	0	0

It is assumed that the investor has an initial balance of C = R100000 in the account and buys and holds equities according to the "extreme" portfolios. Using the formulae:

$$\mathrm{R}_p = \mathrm{C}\sum_{j=1}^{n} w_j R_{tj} \ , \qquad t = 1, 2, ..., T. \tag{5.14}$$

and

$$\mathrm{V}_p = \mathrm{C}\sum_{j=1}^{n} w_j \left(1 + R_{tj}\right) \quad , \qquad t = 1, 2, ..., T. \tag{5.15}$$

to denote the daily portfolio return and value, the results are shown in Figures 5.9 and 5.10.

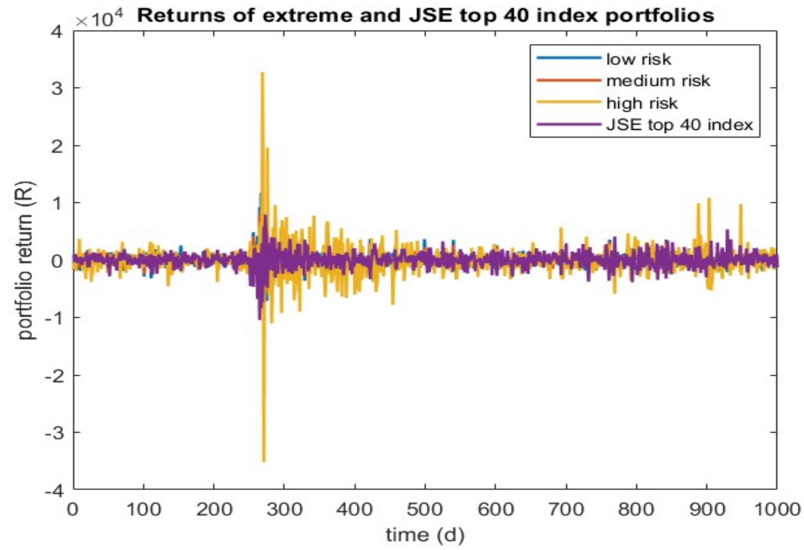

Figure 5.9: Returns(in monetary values) of SBM extreme and JSE Top 40 index portfolios.

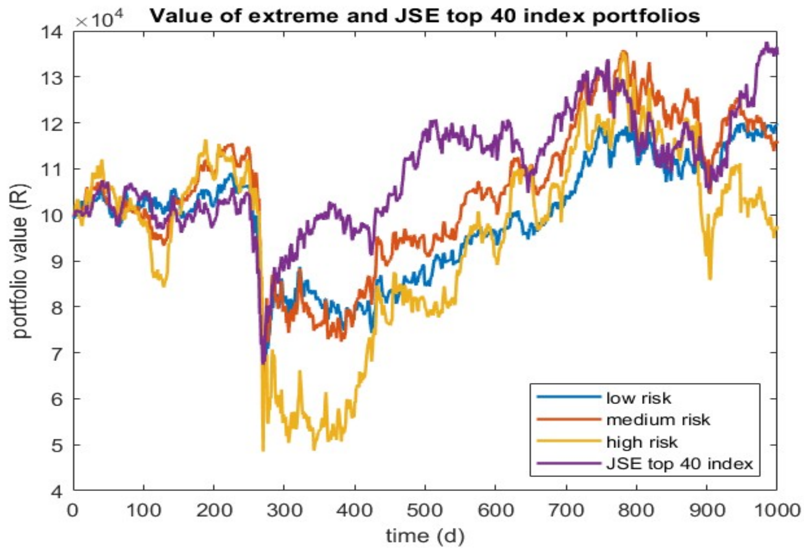

Figure 5.10: Value of SBM extreme and JSE Top 40 index portfolios.

It is difficult to tell from Figure 5.10 which portfolio performs better than the others. Table 5.8 shows the minimum, maximum, and average return of the 4 portfolios. It is clear from Table 5.8 that the JSE Top 40 index outperforms the "extreme" portfolios.

Table 5.8: Classical SBM extreme and JSE Top 40 portfolios

Portfolio	**Minimun Return**	**Maximum Return**	**Mean Return**
Low risk portfolio	63660.0	1.202e+5	99530.0
Medium risk portfolio	63440.0	1.358e+5	1.051e+5
High risk portfolio	48500.0	1.355e+5	95740.0
JSE Top 40 index	67400.0	1.376e+5	1.095e+5

A similar analysis for the classical BCCI model is performed next. Figure 5.11 shows the classical BCCI "extreme" portfolios upon which the backtesting analysis is performed.

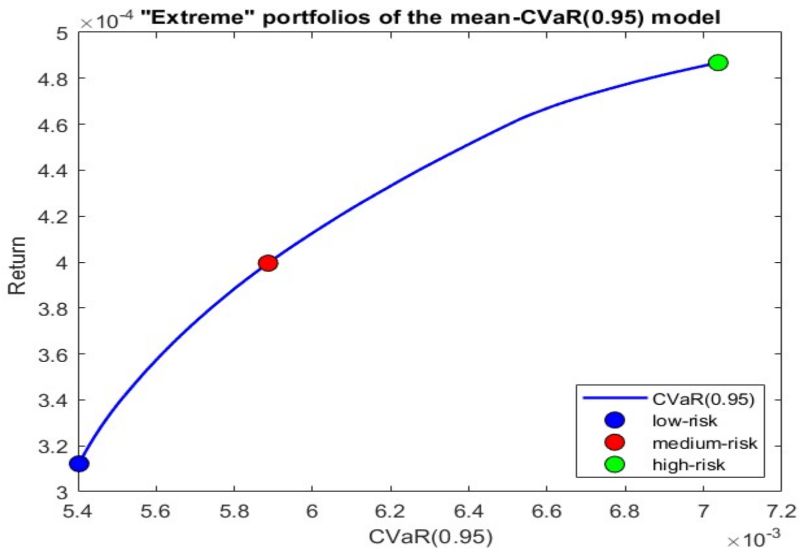

Figure 5.11: Mean-CVaR(0.95) "extreme" portfolios.

Table 5.9 shows the optimal weights corresponding with these portfolios.

Table 5.9: "Extreme" portfolios of the Classical BCCI Efficient companies

Code	**Low Risk Portfolio**	**Medium Risk Portfolio**	**High Risk Portfolio**
ARI	0	0	0
AMS	0.1641	0.1607	0
FSR	0.3992	0.6805	1.0
MTN	0	0	0
PPH	0.2319	0.06659	0
SBK	0.2048	0.09222	0

Figures 5.12 and 5.13 show the return and value of the portfolios in Table 5.9

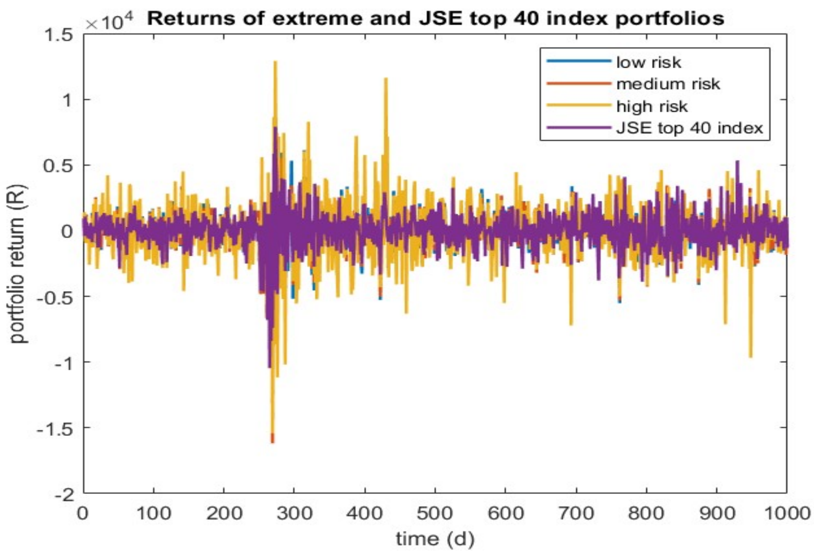

Figure 5.12: Returns (in monetary values) of BCCI extreme and JSE Top 40 index portfolios

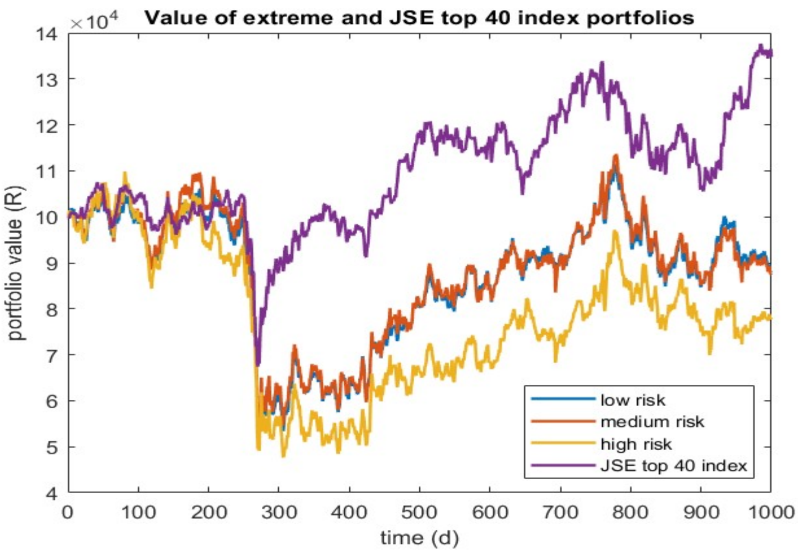

Figure 5.13: Value of BCCI extreme and JSE Top 40 index portfolios

Table 5.10 shows the minimum, maximum, and average return of the 4 portfolios. Again based on Table 5.10 the JSE Top 40 index outperforms the BCCI model's extreme portfolios.

Table 5.10: Classical BCCI extreme and JSE Top 40 portfolios.

Portfolio	**Minimun Return**	**Maximum Return**	**Mean Return**
Low risk portfolio	51290.0	1.114e+5	87560.0
Medium risk portfolio	51280.0	1.137e+5	88200.0
High risk portfolio	47610.0	1.099e+5	77670.0
JSE Top 40 index	67400.0	1.376e+5	1.095e+5

This study performs a similar analysis for the classical ADD model. Figure 5.14 shows the classical ADD "extreme" portfolios upon which the back-testing analysis is performed.

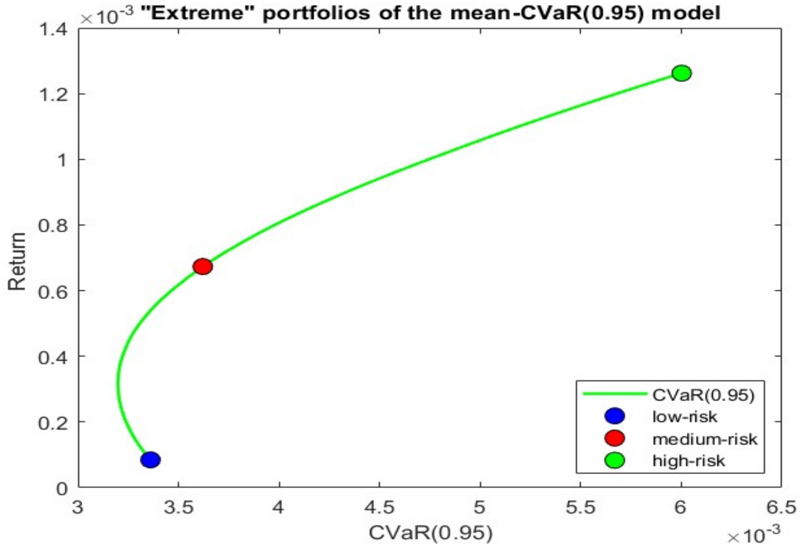

Figure 5.14: Mean-CVaR(0.95) "extreme" portfolios.

Table 5.11 shows the optimal weights corresponding with these portfolios.

Table 5.11: "Extreme" portfolios of the Classical ADD-Efficient companies.

Code	**Low Risk Portfolio**	**Medium Risk Portfolio**	**High Risk Portfolio**
ARI	0.006945	0	0
AMS	0	0.02813	0
ANG	0.07391	0.04643	0
BTI	0.3263	0.2159	0
CPI	0.126	0.1615	0
CFR	0.04701	0.4776	1.0
EXX	0.02183	0	0
FSR	0	0.02022	0
IMP	0	0	0
INP	0.04552	0	0
KIO	0.006056	0	0
MTN	0.05808	0	0
NED	0.04982	0	0
PPH	0.09769	0.05024	0
SBK	0.04451	0	0
WHL	0.0964	0	0

Figures 5.15 and 5.16 show the return and value of the portfolios in Table 5.11

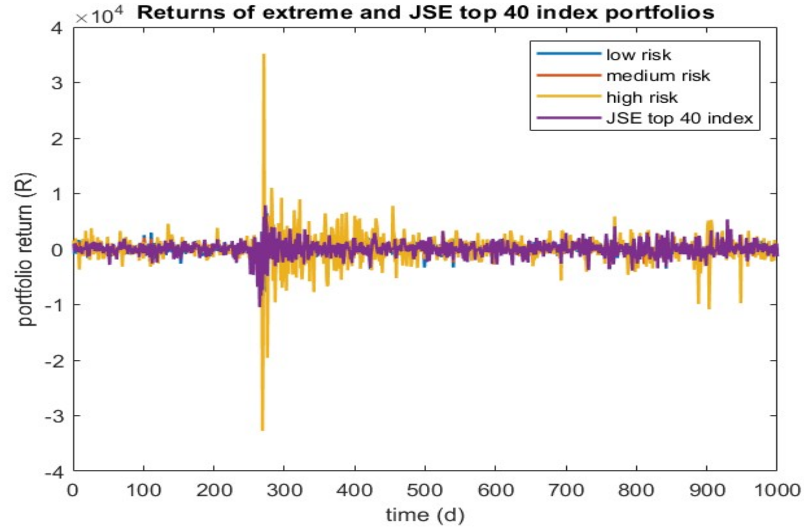

Figure 5.15: Returns (in monetary values) of ADD extreme and JSE Top 40 index portfolios

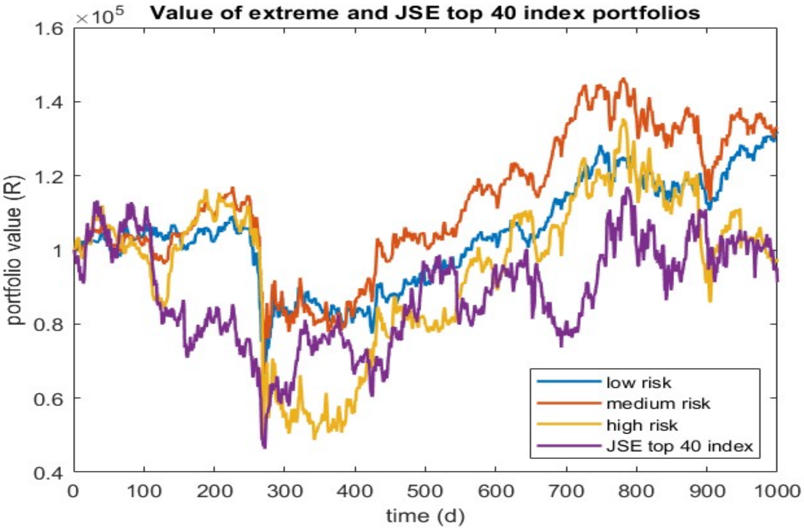

Figure 5.16: Value of ADD extreme and JSE Top 40 index portfolios

Table 5.12 shows the minimum, maximum, and average return of the 4 portfolios. Based on the results in Table 5.12, the medium risk portfolio outperforms the other portfolios.

Table 5.12: Classical ADD extreme and JSE Top 40 portfolios.

Portfolio	**Minimun Return**	**Maximum Return**	**Mean Return**
Low risk portfolio	66410.0	1.318e+5	1.042e+5
Medium risk portfolio	67130.0	1.465e+5	1.125e+5
High risk portfolio	48500.0	1.355e+5	95740.0
JSE Top 40 index	67400.0	1.376e+5	1.095e+5

5.5.2 Backtesting analysis of the DEA bootstrap efficient companies

This study performs a similar analysis for the bootstrap SBM model. Figure 5.17 shows the boot-SBM "extreme" portfolios upon which the backtesting analysis is performed.

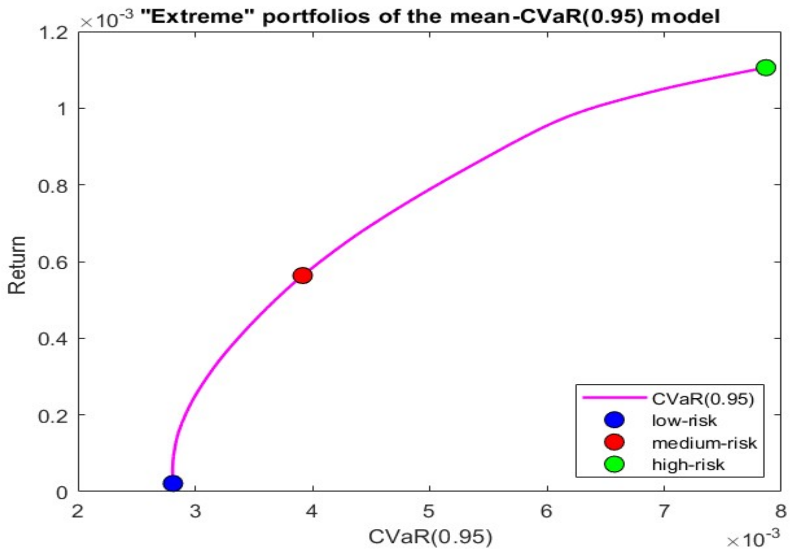

Figure 5.17: Mean-CVaR(0.95) "extreme" portfolios.

Table 5.13 shows the optimal weights corresponding with these portfolios.

Table 5.13: "Extreme" portfolios of the bootstrap SBM-Efficient companies.

Code	Low Risk Portfolio	Medium Risk Portfolio	High Risk Portfolio
ARI	0.006421	0	0
EXX	0.01828	0	0
FSR	0	0.02914	0
NED	0.04722	0	0
PPH	0.05178	0.02937	0
ANH	0.1175	0	0
BID	0.09175	0.09387	0
CLS	0.07377	0.2847	0
GFI	0.0319	0.02794	0
NPN	0.02814	0.2923	1.0
NRP	0.1048	0.00901	0
NPH	0.02581	0.03847	0
SHP	0.03442	0	0
BVT	0.00261	0	0
VOD	0.1245	0	0
BTI	0.241	0.1952	0

Figures 5.18 and 5.19 show the return and value of the portfolios in Table 5.13

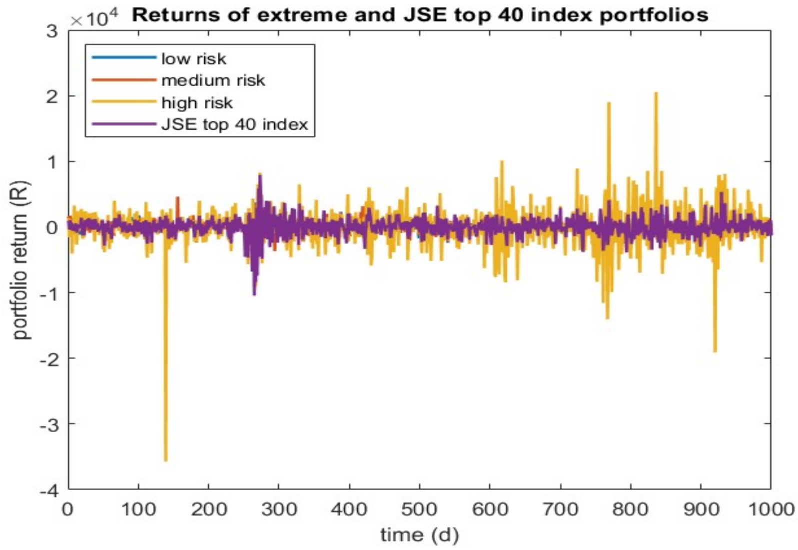

Figure 5.18: Returns (in monetary values) of boot-SBM extreme and JSE Top 40 index portfolios

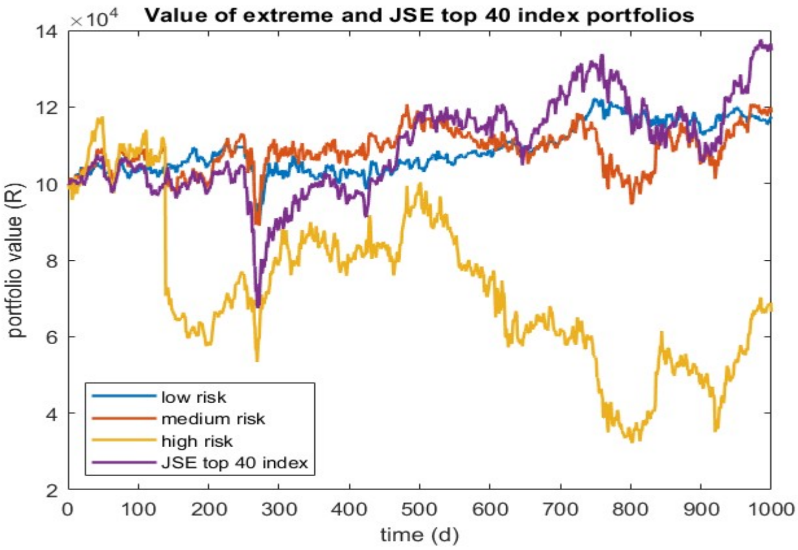

Figure 5.19: Value of boot-SBM extreme and JSE Top 40 index portfolios

Table 5.14 shows the minimum, maximum, and average return of the 4 portfolios. Based on the results in Table 5.14, the medium risk portfolio outperforms the other portfolios.

Table 5.14: boot-SBM extreme and JSE Top 40 portfolios

Portfolio	**Minimun Return**	**Maximum Return**	**Mean Return**
Low risk portfolio	89000.0	1.222e+5	1.087e+5
Medium risk portfolio	94500.0	1.408e+5	1.098e+5
High risk portfolio	32190.0	1.175e+5	72500.0
JSE Top 40 index	67400.0	1.376e+5	1.095e+5

A similar analysis for the bootstrap BCCI model is performed next. Figure 5.20 shows the boot-BCCI "extreme" portfolios upon which the backtesting analysis is performed.

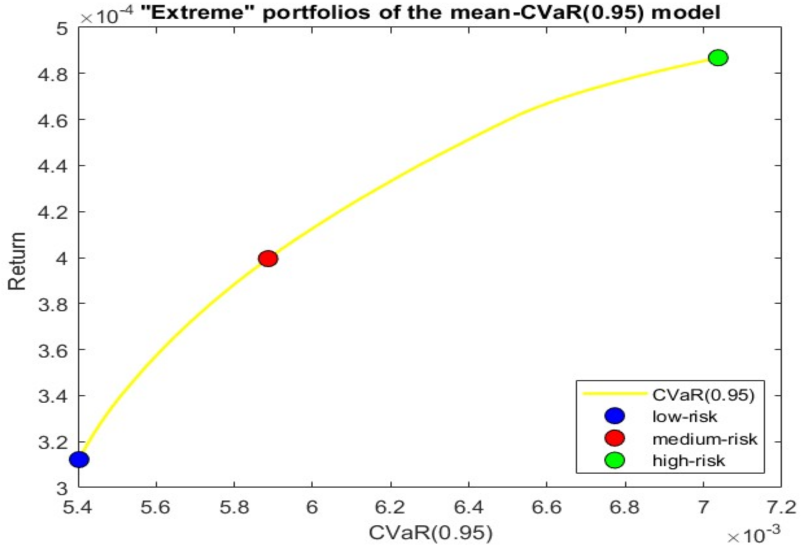

Figure 5.20: Mean-CVaR(0.95) "extreme" portfolios.

Table 5.14 shows the optimal weights corresponding with these portfolios.

Table 5.15: "Extreme" portfolios of the bootstrap- BCCI-Efficient companies.

Code	Low Risk Portfolio	Medium Risk Portfolio	High Risk Portfolio
ARI	0	0	0
AMS	0.1641	0.1607	0
FSR	0.3992	0.6805	1.0
MTN	0	0	0
PPH	0.2048	0.09222	0
SBK	0.2319	0.06659	0

Figures 5.21 and 5.22 show the return and value of the portfolios in Table 5.15

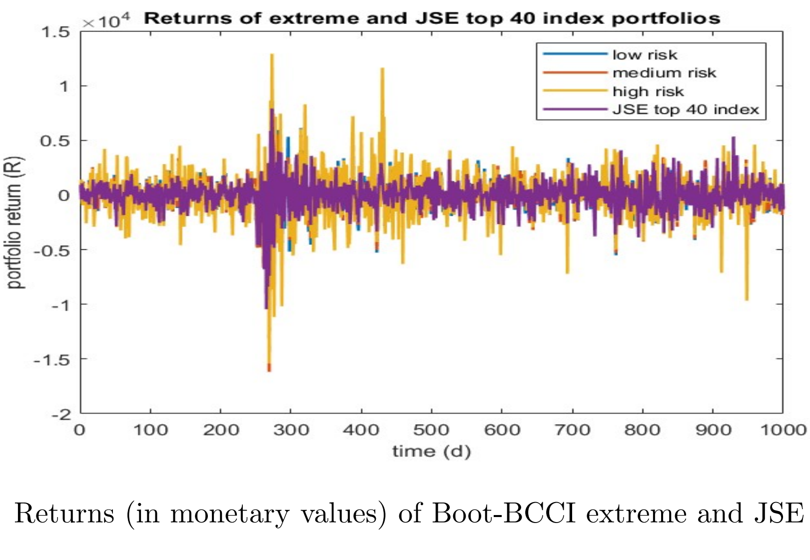

Figure 5.21: Returns (in monetary values) of Boot-BCCI extreme and JSE Top 40 index portfolios

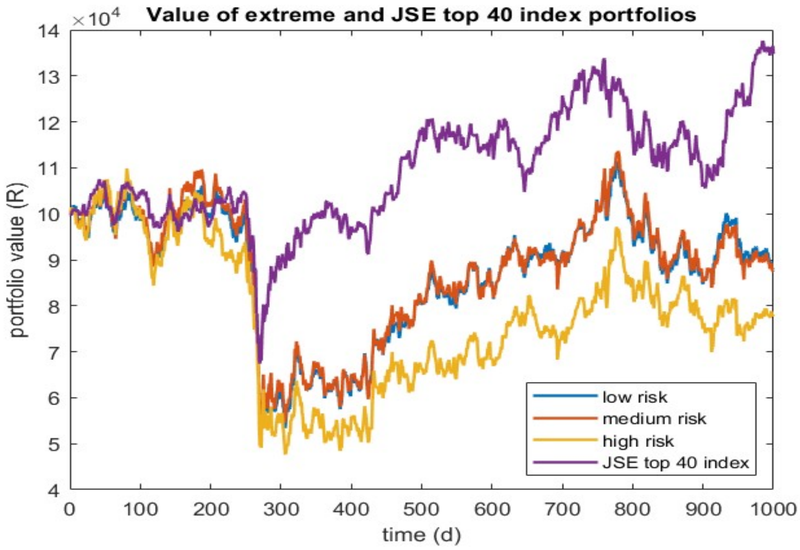

Figure 5.22: Value of Boot-BCCI extreme and JSE Top 40 index portfolios

Table 5.16 shows the minimum, maximum, and average return of the 4 portfolios. Based on the results in Table 5.16 the JSE Top 40 index outperforms the boot-BCCI extreme portfolios.

Table 5.16: Boot-BCCI extreme and JSE Top 40 portfolios.

Portfolio	**Minimun Return**	**Maximum Return**	**Mean Return**
Low risk portfolio	51290.0	1.114e+5	87560.0
Medium risk portfolio	51280.0	1.137e+5	88200.0
High risk portfolio	47610.0	1.099e+5	77670.0
JSE Top 40 index	67400.0	1.376e+5	1.095e+5

From the back-testing analysis we reach the conclusion that the medium risk portfolio associated with the classical ADD model and the medium risk portfolio associated with the bootstrap SBM model outperform the JSE Top 40 index based on profitability. In all the othe instances the JSE Top 40 was superior. Whilst it may seem obvious that the JSE Top 40 is a superior portfolio one must take into account the risk exposure to the market that the JSE index fails to protect an investor from as opposed to the mean-CVaR portfolios. In this analysis we only compared profitability and not risk exposure since the JSE Top 40 index has no method or way to mitigate this. The companies in the index are selected solely based on the market capitalisation.

5.6 Summary and conclusion

This Chapter was an application of DEA to portfolio optimisation. It is an extension of Chapter 4 where DEA-efficient companies are used to construct mean-CVaR$_{0.95}$ portfolios with the objective of investing in the stock market. The data used is the same as that used in Chapter 4. Only 3 portfolios were selected for each DEA model: the low-risk, medium-risk, and high-risk portfolios. The performance of these portfolios was compared to that of the JSE Top 40 index. When performing backtesting analysis on out of sample data to see how the DEA efficient companies fared against the JSE Top 40 index, the medium risk portfolio of the classical ADD model outperformed the index on profitability. In all other cases the index was superior. This medium risk portfolio consisted of the following stocks or companies: Amplats, Anggold, Bats, Capitec, Richemont, Firstrand, and Pepkorh. Due to the curse of dimensionality in DEA, certain variables were ommitted to accomodate the 40 selected companies It would be interesting to see how these results would differ if more than 40 companies were used which would imply more inputs and outputs. This is an idea that can be explored for future research.

Chapter 6

Conclusion

DEA has proven to be a successful tool for measuring performance of DMUs. DEA also has the ability to show how inefficient units can achieve efficiency. Selecting well performing DMUs can be a daunting task as there are often too many variables and other factors to consider. However DEA makes this task easier which is one of the motivations for using it in this study. The study set out to apply the bootstrap resampling technique to DEA. The objective was to extend the standard DEA models to include randomness in the objective functions of the DEA models. This then led to the estimation of the standard errors and confidence intervals for various efficiency scores under selected DEA models. Three DEA models were considered: The Banks, Charnes, and Cooper (BCC) model, the Slacks Based model (SBM), and the Additive (ADD) model. The study used these models to evaluate the performance of football players competing in the biggest leagues in the world, and the Johannesburg stock Exchange (JSE) top 40 listed companies. The study then proceeded to compare the standard DEA models' results with the bootstrap based DEA models' results.

Chapter 3 explored the application of classical and bootstrap based DEA models to football data. The objective was to evaluate individual player performance, in particular to challenge the FIFA ranking system used to nominate and award players for the Ballon d'or. The data was collected for the 2020/2021 season. This data consisted of 16340 players who participated across 18 top football leagues and cup competitions in the world, these are: Bundesliga. Caf champions league. Carabao Cup. Copa America. Copa del rey. Copa italy. Copa de Espana. DFB Pokal. DFL Super Cup. Europa league. FA Cup. Coupe de France. Italy supercoppa. Laliga. Ligue 1. Premier league. Serie A. UEFA champions league. 20 variables of interest were selected, these include minutes played overall, assists overall, penalty goals, clean sheets overall,etc. The main interest was on the nominated players for the FIFA awards which are voted by industry experts. The results showed that the ranking criteria provided by the DEA models agree with the FIFA ranking system for 4 out of the 11 players nominated for the best player award. With Robert Lewandowski being the overall winner.

Chapter 4 explored the application of classical and bootstrap based DEA models to finance data. The data consisted of selected fundamental metrics for the JSE Top 40 listed companies for the period 2015-2023. These included: dividends per share, current ratio, quick ratio,etc. The classical and bootstrap models's (SBM and BCC) results were similar in their selection of efficient companies. The classical ADD model's results were more dicriminatory than the bootstrap ADD model's results as the latter evaluated all companies to be inefficient. The bootstrap resampling technique made it possible to construct confidence intervals for efficiency scores which are more robust than point estimates given by the classical DEA models

Chapter 5 was an application of DEA to portfolio optimisation. It was an extension of Chapter 4 where DEA-efficient companies are used to construct mean-$\text{CVaR}_{0.95}$ portfolios with the objective of investing in the stork market. The data used is the same as that used in Chapter 4. Only 3 portfolios were selected for each DEA model: the low-risk, medium-risk, and high-risk portfolios. The performance of these portfolios was compared to that of the JSE Top 40 index. When performing back-testing analysis on out of sample data to see how the DEA efficient companies fared against the JSE Top 40 index, the medium risk portfolio of the classical ADD model outperformed the index on profitability. In all other cases the index was superior. This medium risk portfolio consisted of the following stocks or companies: Amplats, Anggold, Bats, Capitec, Richemont, Firstrand, and Pepkorh. Due to the curse of dimensionality in DEA, certain variables were ommitted to accommodate the 40 selected companies It would be interesting to see how these results would differ if more than 40 companies were used which would imply more inputs and outputs. This is an idea that can be explored for future research.

This study has made a strong case for the application of the non-parametric bootstrap method to constructing confidence intervals for assessing the efficiency of DMUs. Confidence intervals are more robust than point estimates. DEA results are data dependent, meaning they are only as reliable as the original sample dictates. The results generated in this study are not universal as a different sample would probably yield similar or contrasting results. What this study was able to accomplish was to extend the idea of a single efficiency score to a more robust confidence interval and treat the efficiency score as a random variable whose sampling distribution can be estimated using the non-parametric bootstrapping method.

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Appendix A

Mathematical Optimisation Methods

Mathematical Optimisation Overview

Data Envelopment Analysis (DEA) relies heavily on mathematical optimisation, and this chapter delves into the fundamentals of this crucial discipline. With roots tracing back centuries, optimisation has experienced a significant revival with the advent of computing technology, now permeating various fields such as economics, management science, logistics, robotics, engineering design, and signal processing. Optimisation encompasses a broad spectrum of disciplines, including calculus of variations, operations research (focusing on optimising decision-making processes), and optimal control theory. Notably, optimisation techniques diverge into two primary categories based on whether the variables involved are continuous or discrete..

Typically, if you minimise a function $f(\boldsymbol{x})$ with $\boldsymbol{x} \in \mathbb{R}^n$ you get a continuous optimisation problem. While if $\boldsymbol{x} \in \mathbb{Z}^n$, you get a discrete or combinatorial optimisation problem. Inspite of appearances, continuous optimisation problems are often "easier" than the discrete counterparts, one can use the idea of a derivative which is very useful from the theoretical as well as algorithmic point of view. Combinatorial optimisation is natural and essential in many problems in operations research. This is a domain where, besides rigorous theoretical results, there are numerous "heuristic" methods essential to obtaining good algorithmic performance.

Mathematical formulation of the optimisation problem

One may consider the optimisation problem:

$$\text{minimise } f(\boldsymbol{x}) \tag{A.1}$$

$$\text{subject to } \boldsymbol{x} \in \boldsymbol{\Omega} \tag{A.2}$$

The function $f : \Omega \to \mathbb{R}$ to be minimised is a real-valued function called the objective function or cost function. The vector $\boldsymbol{x}$ is an n-vector of independent variables: $\boldsymbol{x} = (x_1, x_2, ..., x_n)' \in \boldsymbol{\Omega}$. The variables $x_1, x_2, ..., x_n$ are often referred to as decision variables, the set $\boldsymbol{\Omega}$ is a subset of $\mathbb{R}^n$ called the constraint set or feasible set. The optimisation problem in $(A.1)$ can be viewed as a decision problem that involves finding the "best" vector $\boldsymbol{x}$ of the decision variables over all possible vectors in $\boldsymbol{\Omega}$. The "best" vector meaning the one that results in the smallest value of the objective function. This vector is called the minimiser of f over Ω. It is possible that there may be many minimisers. In this case, finding any of the minimisers will suffice. There are also optimisation problems that require maximisation of the objective function, in which case a maximiser is sought. minimisers and maximisers are also called extremisers (Allaire, 2007).

According to Allaire (2007): "Maximisation problems, however can be represented equivalently in the same minimisation form as in $(A.1)$ because maximising f is equivalent to minimising $-f$. Therefore, the attention can be confined to minimisation problems without loss of generality. The problem above is a general form of a constrained optimisation problem, because the decision variables are constrained to be in the constraint set $\boldsymbol{\Omega} \in \mathbb{R}^n$. If $\boldsymbol{\Omega} = \mathbb{R}^n$, then the problem is referred to as an unconstrained optimisation problem. In what follows, basic properties of the general optimisation problem in $(A.1)$ are discussed, which include the unconstrained case. In considering the general optimisation problem in $(A.1)$, a distinction is made between two kinds of minimisers, as specified by the following definition."

Definition A.1.1 [Kinds of Minimisers] Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function defined on some set $\Omega \subset \mathbb{R}^n$. A point $\boldsymbol{x}^* \in \Omega$ is a local minimiser of f over Ω if there exists $\varepsilon > 0$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)\ \forall \boldsymbol{x} \in \Omega \setminus \{\boldsymbol{x}^*\}$ and $||\boldsymbol{x} - \boldsymbol{x}^*|| < \varepsilon$. A point $\boldsymbol{x}^* \in \Omega$ is a global minimiser of f over Ω if $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) \forall \boldsymbol{x} \in \Omega \setminus \{\boldsymbol{x}^*\}$. If in the definitions above $\geq$ is replaced with $>$ then there is a strict local minimiser and a strict global minimiser respectively. Figure A.1 illustrates the definition for the case of $n = 1$, x_1 is a strict global minimiser, x_2 is a strict local minimiser, x_3 is a local (not strict) minimiser.

If $\boldsymbol{x}^*$ is a global minimiser of f over Ω, then $f(\boldsymbol{x}^*) = \min_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x})$ and $\boldsymbol{x}^* = \arg\min_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x})$. If the minimization is unconstrained, then $\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x})$ or $\boldsymbol{x}^* = \arg\min f(\boldsymbol{x})$. In other words, given a real-valued function f, the notation $\arg\min f(\boldsymbol{x})$ denotes the argument that minimises the function f (a point in the domain of f), assuming that such a point is unique (if there is more than one such point, it is picked arbitrarily). Strictly speaking, an optimisation problem is solved only when a global minimiser is found. However, global minimisers are, in general, difficult to find. Therefore, in practice, one often has to be

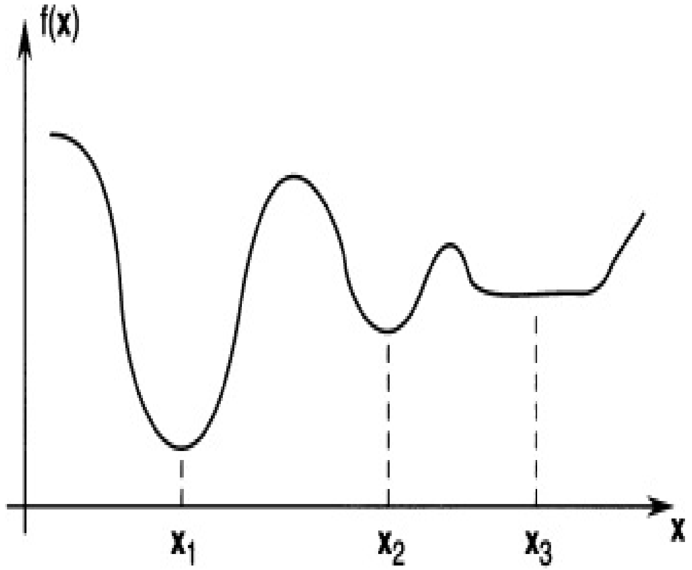

Figure A.1: Illustration of the kind of minimisers for $n = 1$

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satisfied with finding local minimisers.

Necessary conditions for local optima

In this section conditions for a point $\boldsymbol{x}^*$ to be a local minimiser are determined. Derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ are used.

Definition A.2.1 [First Derivative] The first-order derivative of f is: $\boldsymbol{D}f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]$. It is worth noting that the gradient ∇f is just the transpose of $\boldsymbol{D}f$, that is: $\nabla f = (\boldsymbol{D}f)'$.

Definition A.2.2 [Second Derivative] The second derivative of f also called the Hessian of f is:

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{D}^2 f(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\boldsymbol{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{x}) & \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\boldsymbol{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\boldsymbol{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\boldsymbol{x}) \end{pmatrix}$$

Given an optimisation problem with constraint set Ω, a minimiser may lie either in the interior or on the boundary of Ω. To study the case where it lies on the boundary, the notion of feasible directions is needed.

Definition A.2.3 [Feasible Direction]: A vector $\boldsymbol{d} \in \mathbb{R}^n, \boldsymbol{d} \neq \boldsymbol{0}$, is a feasible direction at $\boldsymbol{x} \in \Omega$ if there exists $\alpha_0 > 0$ such that $\boldsymbol{x} + \alpha\boldsymbol{d} \in \Omega\ \forall\alpha \in [0, \alpha_0]$. Figure A.2 illustrates this notion of feasible directions in $\mathbb{R}^2$, d_1 is a feasible direction, d_2 is not a feasible direction.

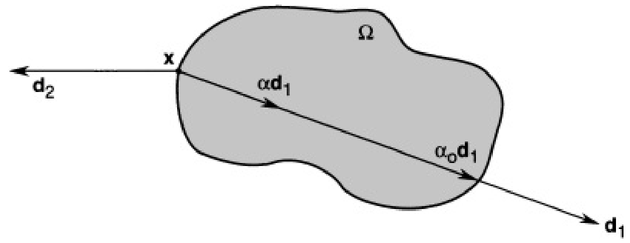

Figure A.2: Illustration of a feasible direction in $\mathbb{R}^2$.

One dimensional search methods

In this section, one is interested in the problem of minimising an objective function $f : \mathbb{R} \to \mathbb{R}$ (i.e., a one-dimensional problem). The approach is to use an iterative search algorithm, also called a line search method. One-dimensional search methods are of interest for the following reasons. First, they are special cases of search methods used in multivariable problems. Second, they are used as part of general multivariable algorithms. In an iterative algorithm, one starts with an initial candidate solution $x^{(0)}$ and generate a sequence of iterates $x^{(1)}, x^{(2)}, \ldots$. For each iteration $k = 0, 1, 2, \ldots$, the next point $x^{(k+1)}$ depends on $x^{(k)}$ and the objective function f. The algorithm may use only the value of f at specific points, or perhaps its first derivative f', or even its second derivative f''. In this study the following algorithms are covered:

1. Golden section method (uses only f)
2. Fibonacci method (uses only f)
3. Bisection method (uses only f')
4. Secant method (uses only f')
5. Newton's method (uses f' and f'')

Golden section search method

This search method makes it possible to determine the minimiser of an objective function $f : \mathbb{R} \to \mathbb{R}$ over a closed interval, say $[a_0, b_0]$. The only property that is assumed of the objective function f is that it is unimodal, which means that f has only one local minimiser. An example of such a function is depicted in the Figure A.3.

This method is based on evaluating the objective function at different points in the interval $[a_0, b_0]$. These points are chosen in such a way that an approximation to the minimiser of f may be achieved in as few evaluations as possible. The goal of the Golden section search method is to narrow the range progressively until the minimiser is "boxed in" with sufficient accuracy. Consider a unimodal function f of one variable and the interval $[a_0, b_0]$. If one evaluates f at only one intermediate point of the interval, one cannot

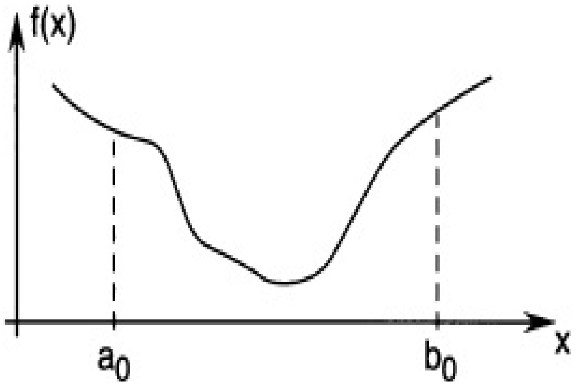

Figure A.3: Unimodal function

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narrow the range within which the minimiser is located. f has to be evaluated at two intermediate points, as illustrated in Figure A.4. The intermediate points are chosen in such a way that the reduction in the range is symmetric, in the sense that $a_1 - a_0 = b_0 - b_1 = \rho(b_0 - a_0)$, where $\rho < \frac{1}{2}$

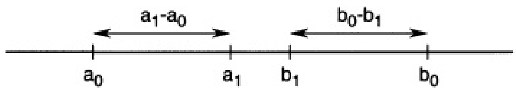

Figure A.4: Illustration of the Golden search section method with two intermediate points

f is then evaluated at the intermediate points. If $f(a_1) < f(b_1)$ then the minimiser must lie in the range $[a_0, b_1]$. If on the other hand $f(a_1) \geq f(b_1)$ then the minimiser is located in the range $[a_1, b_0]$. Figure A.5 illustrates the former case.

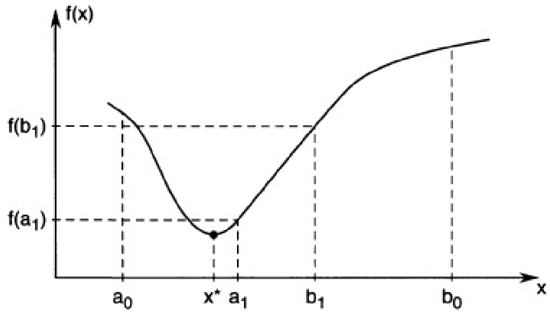

Figure A.5: Golden section search method: The case where. $f(a_1) < f(b_1)$: the minimiser $x^* \in [a_0, b_1]$

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Starting with the reduced range of uncertainty, one can repeat the process and similarly find two new points, say a_2 and b_2, using the same value of $\rho < \frac{1}{2}$ as before. However, one would like to minimise the number of objective function evaluations while reducing the width of the uncertainty interval. Suppose for example that $f(a_1) < f(b_1)$ as in the figure above then it is known that $x^* \in [a_0, b_1]$. Since a_1 is already in the uncertainty interval and $f(a_1)$ is already known, one can make a_1 coincide with b_2. Thus, only one new evaluation of f at a_2 would be necessary. The goal is to find a value of ρ that results in only one new evaluation of f, (see Figure A.6).

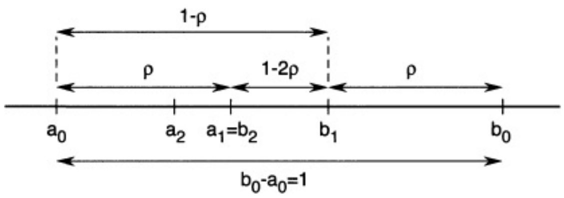

Figure A.6: Finding a value of ρ resulting in only one new evaluation of f

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In a more general sense, one can imagine that the original range $[a_0, b_0]$ is of unit length. To minimise the number of new evaluations of of f we can select ρ so that:

$$\rho(b_1 - a_0) = b_1 - b_2. \tag{A.3}$$

Because $b_1 - a_0 = 1 - \rho$ and $b_1 - b_2 = 1 - 2\rho$, it is true that $\rho(1 - \rho) = 1 - 2\rho$. The quadratic equation can be written as:

$$\rho^2 - 3\rho + 1 = 0. \tag{A.4}$$

The solutions are:

$$\rho_1 = \frac{3 + \sqrt{5}}{2} \rho_2 = \frac{3 - \sqrt{5}}{2} \tag{A.5}$$

Beacuse $\rho < \frac{1}{2}$, this implies that:

$$\rho = \frac{3 - \sqrt{5}}{2} \approx 0.382 \tag{A.6}$$

Now $1 - \rho = \frac{\sqrt{5}-1}{2}$ and $\frac{\rho}{1-\rho} = \frac{3-\sqrt{5}}{\sqrt{5}-1} = \frac{\sqrt{5}-1}{2} = \frac{1-\rho}{1}$. That is, $\frac{\rho}{1-\rho} = \frac{1-\rho}{1}$.

Thus, dividing a range in the ratio of ρ to $1 - \rho$ has the effect that the ratio of the shorter segment to the longer equals the ratio of the longer sum of the two. Using the Golden section rule means that at every stage of the uncertainty range reduction (except the first), the objective function f need only be evaluated at one new point. The uncertainty range is reduced by the ratio $1 - \rho \approx 0.61803$ at every stage. Hence, N steps of reduction using the golden section method reduces the range by the factor $(1 - \rho)^N \approx (0.61803)^N$ (Chong and Zak, 2013)

Fibonacci method

The Golden section method uses the same value of ρ throughout. Suppose one wants to vary the value of ρ from stage to stage, so that at the k^{th} stage in the reduction process a value of ρ_k is used, at the next stage a value of ρ_{k+1} is used, and so on. As in the Golden section search, the goal of the Fibonacci method is to select successive values of ρ_k, $0 \le \rho_k \le \frac{1}{2}$, such that only one new function evaluation is required at each stage. To derive the strategy for selecting evaluation points, one may consider the Figure A.7. From this figure, it is seen that it is sufficient to choose ρ_k such that $\rho_{k+1}(1 - \rho_k) = 1 - 2\rho_k$ (Chong and Zak, 2013).

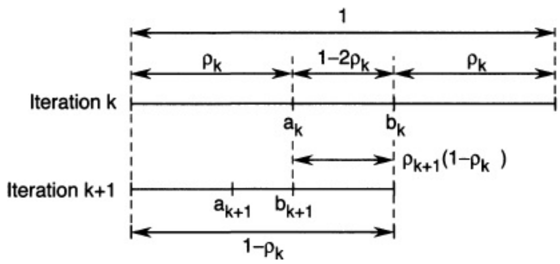

Figure A.7: Illustration for selecting evaluation points for the Fibonacci method.

After some manipulations, one obtains $\rho_{k+1} = 1 - \frac{\rho_k}{1-\rho_k}$. There are many sequences $\rho_1, \rho_2, ...$ that satisfy the law of formation above and the condition that $0 \le \rho_k \le \frac{1}{2}$. For example, the sequence $\rho_1 = \rho_2 = \rho_3 = ... = \frac{(3-\sqrt{5})}{2}$ satisfies the conditions above and gives rise to the Golden section method. Suppose that one is given a sequence $\rho_1, \rho_2, ...$ satisfying the conditions above and this sequence is used in the search algorithm. Then, after N iterations of the algorithm, the uncertainty range is reduced by a factor of

$$(1 - \rho_1)(1 - \rho_2)...(1 - \rho_N) \tag{A.7}$$

Depending on the sequence $\rho_1, \rho_2, ...$ one gets a different reduction factor. The natural question is as follows: What sequence $\rho_1, \rho_2, ...$ minimises the reduction factor above? This problem is a constrained optimisation problem that can be formulated as:

$$\text{minimise} \quad (1 - \rho_1)(1 - \rho_2)...(1 - \rho_N) \tag{A.8}$$

$$\text{subject to} \quad \rho_{k+1} = 1 - \frac{\rho_k}{1 - \rho_k},\ k = 1, 2, ..., N-1 \tag{A.9}$$

$$0 \le \rho_k \le \frac{1}{2},\ k = 1, 2, ..., N \tag{A.10}$$

Before the solution to problem (A.2) is given, it is necessary to introduce the Fibonacci sequence $F_1, F_2, ...$This sequence is defined as follows. First, let $F_{-1} = 0$ and $F_0 = 1$ by convention. Then for $k \ge 0$, $F_{k+1} = F_k + F_{k-1}$. Some values of elements in the Fibonacci sequence are:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
1	2	3	5	8	13	21	34

It turns out that the solution to the optimisation problem (A.7) is:

$$\rho_1 = 1 - \frac{F_N}{F_{N+1}}, \rho_2 = 1 - \frac{F_{N-1}}{F_N}, ...\rho_k = 1 - \frac{F_{N-k+1}}{F_{N-k+2}} \rho_N = 1 - \frac{F_1}{F_2} \tag{A.11}$$

where the F_k are the elements of the Fibonacci sequence. The resulting algorithm is called the Fibonacci search method. In the Fibonacci search method, the uncertainty range is reduced by the factor:

$$(1 - \rho_1)(1 - \rho_2) \dots (1 - \rho_N) = \frac{F_N}{F_{N+1}} \frac{F_{N-1}}{F_N} \dots \frac{F_1}{F_2} = \frac{F_1}{F_{N+1}} = \frac{1}{F_{N+1}} \tag{A.12}$$

Since the Fibonacci method uses the optimal values of $\rho_1, \rho_2, ...$, the reduction factor above is less than that of the Golden section method. In other words, the Fibonacci method is better than the Golden section method in that it gives a smaller final uncertainty range.

It is worth pointing out that there is an anomaly in the final iteration of the Fibonacci search method, because $\rho_N = 1 - \frac{F_1}{F_2} = \frac{1}{2}$. Recall that two intermediate points are needed at each stage, one that comes from a previous iteration and another that is a new evaluation point. However, with $\rho_N = \frac{1}{2}$, the two intermediate points coincide in the middle of the uncertainty interval, and therefore the uncertainty range cannnot be reduced any further.

To get around this problem, the new evaluation for the last iteration is performed using $\rho_N = \frac{1}{2} - \varepsilon$, where ε is a small number. In other words, the new evaluation point is just to the left or right of the midpoint of the uncertainty interval. This modification to the Fibonacci method is of no significant practical consequence. As a result of the modification above, the reduction in the uncertainty range at the last iteration may be either $1 - \rho_N = \frac{1}{2}$ or $1 - (\rho_N - \varepsilon) = \frac{1}{2} + \varepsilon = \frac{1+2\varepsilon}{2}$, depending on which of the two points has the smaller objective function value. Therefore, in the worst case, the reduction factor in the uncertainty range for the Fibonacci method is $\frac{1+2\varepsilon}{F_{N+1}}$ (Chong and Zak, 2013).

Bisection method

The goal of the Bisection method is to find the minimiser of an objective function $f : \mathbb{R} \to \mathbb{R}$ over an interval $[a_0, b_0]$. As before , it is assumed that the objective function f is unimodal. Further, suppose that f is continuously differentiable and that values of the derivative f' can be used as a basis for reducing the uncertainty interval. The Bisection method is a simple algorithm for successively reducing the uncertainty interval based on evaluations of the derivative.

To begin, let $x^{(0)} = (a_0 + b_0)/2$ be the midpoint of the initial uncertainty interval. Next, evaluate $f'(x^{(0)})$. If $f'(x^{(0)}) > 0$, then it is deduced that the minimiser lies to the left of $x^{(0)}$. In other words, the uncertainty interval is reduced to $[a_0, x^{(0)}]$. On the other hand , if $f'(x^{(0)}) < 0$, then it is deduced that the minimiser lies to the to the right of $x^{(0)}$. In this case, the uncertainty interval is reduced to $[x^{(0)}, b_0]$. Finally, if $f'(x^{(0)}) = 0$, then $x^{(0)}$ is declared to be the minimiser of f and the search is terminated. With the new uncertainty interval computed, the process is repeated iteratively. At each iteration k, the midpoint of the uncertainty interval is Golden. Call this point $x^{(k)}$. Depending on the sign of $f'(x^{(k)})$ (assuming that it is nonzero), the uncertainty interval is reduced to the left or right of $x^{(k)}$. If at any iteration k, $f'(x^{(k)}) = 0$, then $x^{(k)}$ is declared as the minimiser and the search is terminated.

Two salient features distinguish the Bisection method from the Golden section and Fibonacci methods. First, instead of using values of f, the Bisection methods uses values of f'. Second, at each iteration, the length of the uncertainty interval is reduced by a factor of $\frac{1}{2}$. Hence, after N steps, the range is reduced by a factor of $\left(\frac{1}{2}\right)^N$. This factor is smaller than in the Golden section and Fibonacci methods (Chong and Zak, 2013).

Newton's method

Suppose one is confronted with the problem of minimising a function f of a single real variable x. It is assumed that at each point $x^{(k)}$, one can determine $f(x^{(k)})$, $f'(x^{(k)})$, and $f''(x^{(k)})$. A quadratic function can be fitted through $x^{(k)}$ that matches its first and second derivatives with that of the function f. This quadratic function has the form:

$$q(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) + \frac{1}{2}f''(x^{(k)})(x - x^{(k)})^2 \tag{A.13}$$

It is noted that $q(x^{(k)}) = f(x^{(k)})$, $q'(x^{(k)}) = f'(x^{(k)})$, and $q''(x^{(k)}) = f''(x^{(k)})$. Then instead of minimising f, its approximation q is minimised instead. The first-order necessary condition for a minimiser of q yields

$$q'(x) = f'(x^{(k)}) + f''(x^{(k)})(x - x^{(k)} \tag{A.14}$$

Setting $x = x^{(k+1)}$, one obtains:

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})} \tag{A.15}$$

It is worth noting that Newton's method works well if $f'(x) > 0$ everywhere. However if $f'(x) < 0$ for some x, Newton's method may fail to converge to the minimiser. Newton's method can also be viewed as a way to drive the first derivative of of f to zero. Indeed if one sets $g(x) = f'(x)$, a formula for the iterative solution of the equation $g(x) = 0$ is obtained as: $x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})}$. In other words, Newton's method can be used for root finding. Newton's method for solving equations of the form $g(x) = 0$ is also referred to as Newton's method of tangents. This name is easily justified if one looks at a geometric interpretation of the method when applied to the solution of the equation $g(x) = 0$ as shown in Figure A.8.

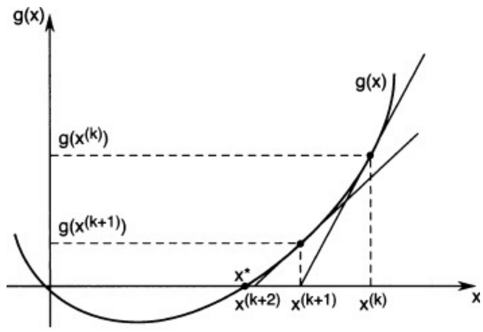

Figure A.8: Newton's method of tangents.

If one draws a tangent to $g(x)$ at the given point $x^{(k)}$, then the tangent line intersects the x-axis at the point $x^{(k+1)}$, which is expected to be closer to the root x^* of $g(x) = 0$. Note that the slope of $g(x)$ at $x^{(k)}$ is $g'(x^{(k)}) = \frac{g(x^{(k)})}{x^{(k)} - x^{(k+1)}}$. Hence $x^{(k+1)} = x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})}$. Newton's method of tangents may fail if the first approximation to the root is such that the ratio $\frac{g(x^{(0)})}{g'(x^{(0)})}$ is not small enough, as demonstrated in Figure A.9. Thus, an initial approximation to the root is very important (Chong and Zak, 2013) .

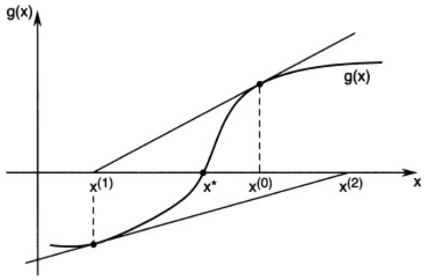

Figure A.9: Example of Newton's method of tangents failing to converge to the root x^* of $g(x) = 0$.

Secant method

Newton's method for minimising f uses second derivatives of f:

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^k)} \tag{A.16}$$

If the second derivative is not available, one may attempt to approximate it using the first derivative information. In particular one may approximate $f''(x^{(k)})$ above with

$$\frac{f'(x^{(k)}) - f'(x^{(k-1)})}{x^{(k)} - x^{(k-1)}} \tag{A.17}$$

Using the foregoing approximation of the second derivative, one obtains the algorithm

$$x^{(k+1)} = x^{(k)} - \frac{x^{(k)} - x^{(k-1)}}{f'(x^{(k)}) - f'(x^{(k-1)})} f'(x^{(k)}) \tag{A.18}$$

called the secant method. Note that the algorithm requires two initial points to start it, which are denoted by $x^{(-1)}$ and $x^{(0)}$. The secant algorithm can be represented in the following equivalent form

$$x^{(k+1)} = \frac{f'(x^{(k)})x^{(k-1)} - f'(x^{(k-1)})x^{(k)}}{f'(x^{(k)}) - f'(x^{(k-1)})} \tag{A.19}$$

It is worth noting that, like Newton's method, the secant method does not directly involve values of $f(x^{(k)})$. Instead, it tries to drive the derivative f' to zero. In fact, as was done for the Newton's method, one can interpret the secant method as an algorithm for solving equations of the form $g(x) = 0$. Specifically, the secant algorithm for finding a root of the equation $g(x) = 0$ takes the form

$$x^{(k+1)} = x^{(k)} - \frac{x^{(k)} - x^{(k-1)}}{g(x^{(k)}) - g(x^{(k-1)})} g(x^{(k)}) \tag{A.20}$$

or equivalently

$$x^{(k+1)} = \frac{g(x^{(k)})x^{(k-1)} - g(x^{(k-1)})x^{(k)}}{g(x^{(k)}) - g(x^{(k-1)})} \tag{A.21}$$

The secant method for root finding is illustrated in Figure A.10. Unlike Newton's method, which uses the slope of g to determine the next point, the secant method uses the "secant" between the $(k-1)^{th}$ and k^{th} points to determine the $(k+1)^{th}$ point (Chong and Zak, 2013).

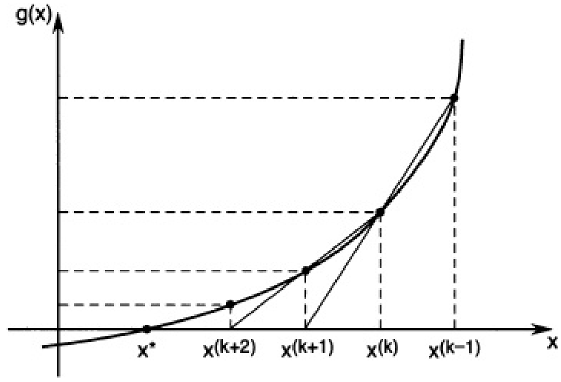

Figure A.10: Secant method for root finding

Non-linear constrained optimisation

The attention is now turned to methods that are used to solve a class of non-linear constrained optimisation problems that can be formulated as

$$\text{minimise } f(\boldsymbol{x}) \tag{A.22}$$

$$\text{subject to} \quad h_i(\boldsymbol{x}) = 0,\ i = 1, \ldots, m \tag{A.23}$$

$$g_j(\boldsymbol{x}) \leq 0,\ j = 1, \ldots, p \tag{A.24}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, and $m \geq n$. In vector notation, the problem above can be represented in the following standard form:

$$\text{minimise } f(\boldsymbol{x}) \tag{A.25}$$

$$\text{subject to} \quad \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.26}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \tag{A.27}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$. The following terminology is adopted.

Definition A.4.1 Any point satisfying the above constraints is called a feasible point. The set of all feasible points $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\}$ is called a feasible set. The study first considers constrained optimisation problems with only equality constraints (Chong and Zak, 2013).

Problems with equality constraints

These problems can be formulated as:

$$\text{minimise } f(\boldsymbol{x}) \tag{A.28}$$

$$\text{subject to} \quad \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.29}$$

where $\boldsymbol{x} \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}, \boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{h} = [h_1, ..., h_m]'$ and $m \leq n$. It is assumed that the function $\boldsymbol{h}$ is continously differentiable, that is $\boldsymbol{h} \in \subset^1$.

Definition A.4.1.1 A point $\boldsymbol{x}^*$ satisfying the constraints $h_1(\boldsymbol{x}^*) = 0, ..., h_m(\boldsymbol{x}^* = 0)$ is said to be a regular point of the constraints if the gradient vectors $\nabla h_1(\boldsymbol{x}^*), ..., \nabla h_m(\boldsymbol{x}^*)$ are linearly independent. Let $\boldsymbol{D}_h(\boldsymbol{x}^*)$ be the Jacobian matrix of $\boldsymbol{h}$ at $\boldsymbol{x}^*$ given by:

$$\boldsymbol{Dh}(\boldsymbol{x}^*) = \begin{pmatrix} \boldsymbol{D}h_1(\boldsymbol{x}^*) \\ \vdots \\ \boldsymbol{D}h_m(\boldsymbol{x}^*) \end{pmatrix} = \begin{pmatrix} \nabla h_1(\boldsymbol{x}^*)' \\ \vdots \\ \nabla h_m(\boldsymbol{x}^*)' \end{pmatrix} \tag{A.30}$$

Then, $\boldsymbol{x}^*$ is regular if and only if rank $\boldsymbol{Dh}(\boldsymbol{x}^*) = m$ (i.e., the Jacobian matrix is of full rank). The set of equality constraints $h_1(\boldsymbol{x}) = 0, ..., h_m(\boldsymbol{x}) = 0, h_i : \mathbb{R}^n \to \mathbb{R}$, describes a surface $S = \{\boldsymbol{x} \in \mathbb{R}^n : h_1(\boldsymbol{x}) = 0, ..., h_m(\boldsymbol{x}) = 0\}$. Assuming that the points in S are regular, the dimension of the surface S is $n - m$.

Tangent and normal spaces

At this point, it is important to introduce the notion of Tangent and Normal spaces.

Definition A.4.2.1 A curve C on a surface S is a set of points $\{\boldsymbol{x}(t) \in S : t \in (a, b)\}$, continuously parametrised by $t \in (a, b)$, that is, $\boldsymbol{x} : (a, b) \to S$ is a continuous function.

A graphical illustration of the definition of a curve is given in Figure A.11. The definition of a curve implies that all the points on the curve satisfy the equation describing the surface. The curve C passes through a point $\boldsymbol{x}^*$ if there exists $t^* \in (a, b)$ such that:

$$\boldsymbol{x}(t^*) = \boldsymbol{x}^* \tag{A.31}$$

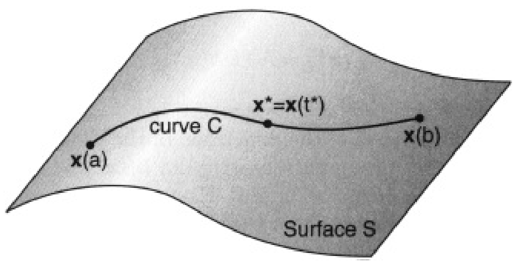

Figure A.11: Curve on a surface.

Intuitively, one can think of a curve $C = \{\boldsymbol{x}(t) \in S : t \in (a, b)\}$ as the path traversed by a point $\boldsymbol{x}$ traveling on the surface S. The position of the point at time t is given by $\boldsymbol{x}(t)$.

Definition A.4.2.2 The curve $C = \{\boldsymbol{x}(t) \in S : t \in (a, b)\}$ is differentiable if:

$$\dot{\boldsymbol{x}}(t) = \frac{d\boldsymbol{x}}{dt}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} \tag{A.32}$$

exists for all $t \in (a, b)$.

The curve $\boldsymbol{C} = \{\boldsymbol{x}(t) \in S : t \in (a, b)\}$ is twice differentiable if:

$$\ddot{\boldsymbol{x}}(t) = \frac{d^2\boldsymbol{x}}{dt^2}(t) = \begin{pmatrix} \ddot{x}_1(t) \\ \vdots \\ \ddot{x}_n(t) \end{pmatrix} \tag{A.33}$$

exists for all $t \in (a, b)$.

It should be noted that both $\dot{\boldsymbol{x}}(t)$ and $\ddot{\boldsymbol{x}}(t)$ are n-dimensional vectors. One can think of $\dot{\boldsymbol{x}}(t)$ and $\ddot{\boldsymbol{x}}(t)$ as the velocity and acceleration of a point traversing the curve $\boldsymbol{C}$ with position $\boldsymbol{x}(t)$ at time t. The vector $\dot{\boldsymbol{x}}(t)$ points in the direction of the instantaneous motion of $\boldsymbol{x}(t)$. Therefore, the vector $\dot{\boldsymbol{x}}(t^*)$ is tangent to the curve $\boldsymbol{C}$ at $\boldsymbol{x}^*$ as shown in Figure A.12:

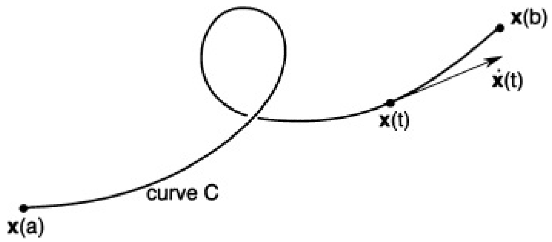

Figure A.12: Geometric interpretation of the differentiability of a curve.

The notion of a tangent space can thus be introduced by the following definition.

Definition A.4.2.3 The tangent space at a point $\boldsymbol{x}^*$ on the surface $S = \{x \in \mathbb{R}^n : \boldsymbol{h}(x) = \boldsymbol{0}\}$ is the set $T(\boldsymbol{x}^*) = \{\boldsymbol{y} : \boldsymbol{Dh}(\boldsymbol{x}^*)\boldsymbol{y} = \boldsymbol{0}\}$.

Assuming that $\boldsymbol{x}^*$ is regular, the dimension of the tangent space is $n - m$, where m is the number of equality constraints $h_i(\boldsymbol{x}^*) = 0$. It should be noted that the tangent space passes through the origin. However, it is often convenient to picture the tangent space as a plane that passes through the point $\boldsymbol{x}^*$. For this, the tangent plane at $\boldsymbol{x}^*$ is defined to be the set:

$$TP(\boldsymbol{x}^*) = T(\boldsymbol{x}^*) + \boldsymbol{x}^* = \{\boldsymbol{x} + \boldsymbol{x}^* : \boldsymbol{x} \in T(\boldsymbol{x}^*)\}.$$

Intuitively, one would expect the definition of the tangent space at a point on a surface to be the collection of all "tangent vectors" to the surface at that point. It has been shown that the derivative of a curve on a surface at a point is a tangent vector to the curve, and hence to the surface. The intuition above agrees with the definition whenever $\boldsymbol{x}^*$ is regular, as stated in the next theorem.

Theorem A.4.2.1 Suppose that $\boldsymbol{x}^* \in S$ is a regular point and $T(\boldsymbol{x}^*)$ is the tangent space at $\boldsymbol{x}^*$. Then, $\boldsymbol{y} \in T(\boldsymbol{x}^*))$ if and only if there exists a differentiable curve in S passing through $\boldsymbol{x}^*$ with derivative $\boldsymbol{y}$ at $\boldsymbol{x}^*$.

Figure A.13 illustrates the notion of a tangent plane, and Figure A.14, the relationship between the tangent plane and the tangent space.

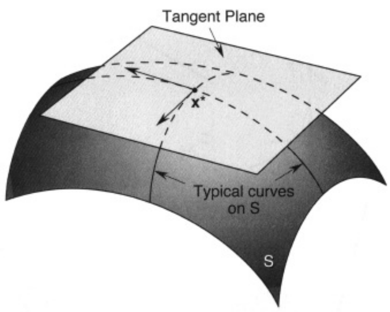

Figure A.13: Tangent plane to the surface S at the point $\boldsymbol{x}^*$.

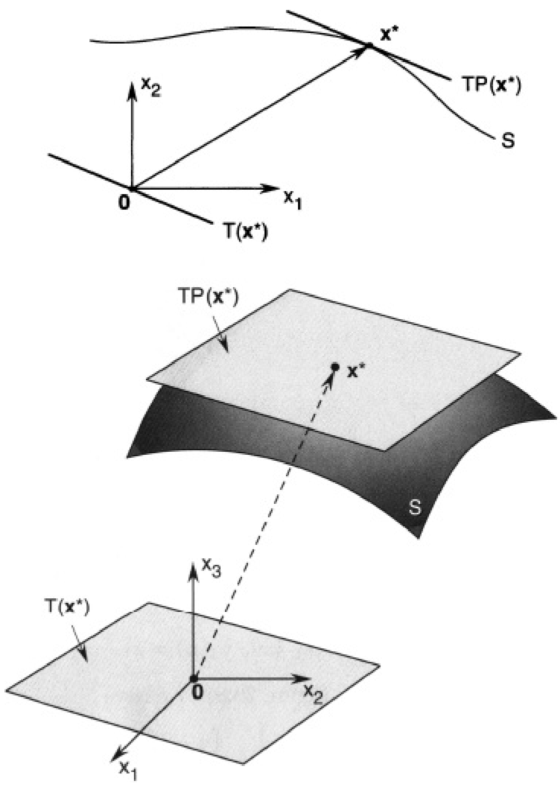

Figure A.14: Tangent spaces and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The notion of a normal space is now introduced.

Definition A.4.2.4 The normal space $N(\boldsymbol{x}^*)$ at a point $\boldsymbol{x}^*$ on the surface $S = x \in \mathbb{R}^n : \boldsymbol{h}(\boldsymbol{x}) = 0$ is the set $N(\boldsymbol{x}^*) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = D\boldsymbol{h}(\boldsymbol{x}^*)' z, z \in \mathbb{R}^n\right\}$.

One can express the normal space $N(\boldsymbol{x}^*)$ as $N(\boldsymbol{x}^*) = R\big(D\boldsymbol{h}(\boldsymbol{x}^*)'\big)$ that is, the range of the matrix $D\boldsymbol{h}(\boldsymbol{x}^*)'$. It should be noted that the normal space $N(\boldsymbol{x}^*)$ is the subspace of $\mathbb{R}^n$

spanned by the vectors $\nabla h_1(\boldsymbol{x}^*), ..., \nabla h_m(\boldsymbol{x}^*)$, that is, $N(\boldsymbol{x}^*) = \text{span}[\nabla h_1(\boldsymbol{x}^*), ..., \nabla h_m(\boldsymbol{x}^*)] = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = z_1 \nabla h_1(\boldsymbol{x}^*) + ... + z_m \nabla h_m(\boldsymbol{x}^*), z_1, ..., z_m \in \mathbb{R}^n\right\}$

According to Chong and Zak (2013): "The normal space contains the zero vector. Assuming that $\boldsymbol{x}^*$ is regular, the dimension of the normal space $N(\boldsymbol{x}^*)$ is m. As in the case of the tangent space, it is often convenient to picture the normal space $N(\boldsymbol{x}^*)$ as passing through the point $\boldsymbol{x}^*$ (rather than through the origin of $\mathbb{R}^n$). For this, the normal plane at $\boldsymbol{x}^*$ is defined as the set $NP(\boldsymbol{x}^*) = N(\boldsymbol{x}^*) + \boldsymbol{x}^* = \left\{\boldsymbol{x} + \boldsymbol{x}^* \in \mathbb{R}^n : \boldsymbol{x} \in N(\boldsymbol{x}^*)\right\}$. Figure 2.20 shows the normal space and plane in $\mathbb{R}^3$ (i.e., $n = 3, m = 1$)."

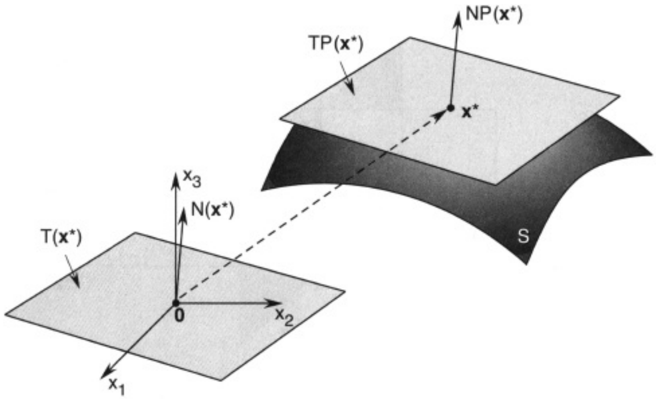

Figure A.15: Normal space in $\mathbb{R}^3$.

Lagrange condition

In this section, the first-order necessary condition (FONC) for extremum problems with constraints is presented. The result is the well known Lagrange's theorem. To better understand the idea underlying this theorem, functions of two variables and only one equality constraint are considered. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the constraint function. Recall that at each point $\boldsymbol{x}$ of the domain, the gradient vector $\nabla h(\boldsymbol{x})$ is orthogonal to the level set that passes through that point. Indeed, if one chooses a point $\boldsymbol{x}^* = \left(x_1^*, x_2^*\right)'$ such that $h(\boldsymbol{x}^*) = 0$, and assume that $\nabla h(\boldsymbol{x}^*) \neq 0$. The level set through the point $\boldsymbol{x}^*$ is the set $\{\boldsymbol{x} : h(\boldsymbol{x}) = 0\}$. This this level set is then parameterized in a neighborhood of $\boldsymbol{x}^*$ by a curve $\{\boldsymbol{x}(t)\}$, that is, a continuously differentiable function $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^2$ such that:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{A.34}$$

, $t \in (a, b), \boldsymbol{x}^* = \boldsymbol{x}(t^*), \dot{\boldsymbol{x}}(t^*) \neq \boldsymbol{0}, t^* \in (a, b)$ (Chong and Zak, 2013).

Now it can be shown that $\nabla h(\boldsymbol{x}^*)$ is orthogonal to $\dot{\boldsymbol{x}}(t^*)$. h is constant on the curve $\{\boldsymbol{x}(t), t \in (a, b)\}$, i.e. for all $t \in (a, b)$, $h(\boldsymbol{x}(t)) = 0$. Hence for all $t \in (a, b)$, $\frac{d}{dt}h(\boldsymbol{x}(t)) = 0$. Applying the chain rule, one gets $\frac{d}{dt}h(\boldsymbol{x}(t)) = \nabla h(\boldsymbol{x}(t))'\dot{\boldsymbol{x}}(t) = 0$. Therefore, $\nabla h(\boldsymbol{x}^*)$ is orthogonal to $\dot{\boldsymbol{x}}(t^*)$. Now suppose that $\boldsymbol{x}^*$ is a minimiser of $f : \mathbb{R}^2 \to \mathbb{R}$ on the set $\{\boldsymbol{x} : h(\boldsymbol{x}) = 0\}$. It is claimed that $\nabla f(\boldsymbol{x}^*)$ is orthogonal to $\dot{x}(t^*)$. To see this, it is enough to observe that the composite function of t given by $\phi(t) = f(\boldsymbol{x}(t))$ achieves a minimum at t^*. Consequently, the FONC for the unconstrained extremum problem implies that

$\frac{d\phi}{dt}(t^*) = 0$ Applying the chain rule yields $\frac{d\phi}{dt}(t^*) = \nabla f(\boldsymbol{x}(t^*))'\dot{\boldsymbol{x}}(t^*) = \nabla f(\boldsymbol{x}^*)'\dot{\boldsymbol{x}}(t^*)$. Thus, $\nabla f(\boldsymbol{x}^*)$ is orthogonal to $\dot{\boldsymbol{x}}(t^*)$. The fact that $\dot{\boldsymbol{x}}(t^*)$ is tangent to the curve $\{\boldsymbol{x}(t)\}$ at $\boldsymbol{x}^*$ means that $\nabla f(\boldsymbol{x}^*)$ is orthogonal to the curve at $\boldsymbol{x}^*$ as shown in Figure 2.21.

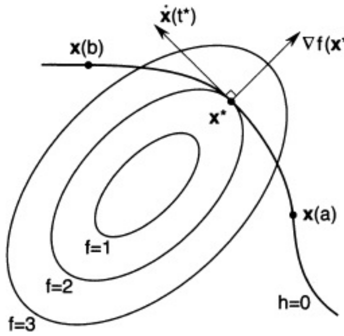

Figure A.16: Illustration of an orthogonal gradient $\nabla f(\boldsymbol{x}^*)$ to the curve $\{\boldsymbol{x}(t)\}$ at the point $\boldsymbol{x}^*$ that is a minimiser of f on the curve.

It is worth noting that $\nabla h(\boldsymbol{x}^*)$ is also orthogonal to $\dot{\boldsymbol{x}}(t^*)$. Therefore the vectors $\nabla h(\boldsymbol{x}^*)$ and $\nabla f(\boldsymbol{x}^*)$ are parallel, that is, $\nabla f(\boldsymbol{x}^*)$is a scalar multiple of $\nabla h(\boldsymbol{x}^*)$. The observations above allow the formulation of Lagrange's theorem for functions of two variables with one constraint.

Theorem A.4.3.1 Lagrange's Theorem for $n = 2$, $m = 1$. Let the point $\boldsymbol{x}^*$ be a minimiser of $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the constraint $h(\boldsymbol{x}) = 0$, $h : \mathbb{R}^2 \to \mathbb{R}$. Then, $\nabla f(\boldsymbol{x}^*)$ and $\nabla h(\boldsymbol{x}^*)$ are parallel. That is, if $\nabla h(\boldsymbol{x}^*) \neq 0$, then there exists a scalar λ^* such that $\nabla f(\boldsymbol{x}^*) + \lambda^* \nabla h(\boldsymbol{x}^*) = \boldsymbol{0}$. The λ^* in Theorem A.4.3.1 is referred to as the Lagrange multiplier. This theorem also holds for maximisers. Figure 2.22 below gives an illustration of Lagrange's theorem for the case where $\boldsymbol{x}^*$ is a maximiser of f over the set $\{\boldsymbol{x} : h(\boldsymbol{x}) = 0\}$

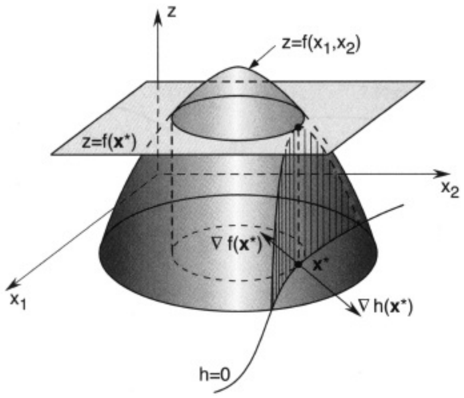

Figure A.17: Lagrange's theorem for $n = 2, m = 1$

Lagrange's theorem provides a FONC for a point to be a local minimiser. This condition, which is called the Lagrange condition, consists of two equations:

$$\nabla f(\boldsymbol{x}^*) + \lambda^* \nabla h(\boldsymbol{x}^*) = \boldsymbol{0} \tag{A.35}$$

$$h(\boldsymbol{x}^*) = 0 \tag{A.36}$$

It is worth noting that the Lagrange condition is necessary but not sufficient. Figure 2.23

illustrates a variety of points where the Lagrange condition is satisfied, including a case where the point is not an extremiser (neither a maximiser nor a minimiser).

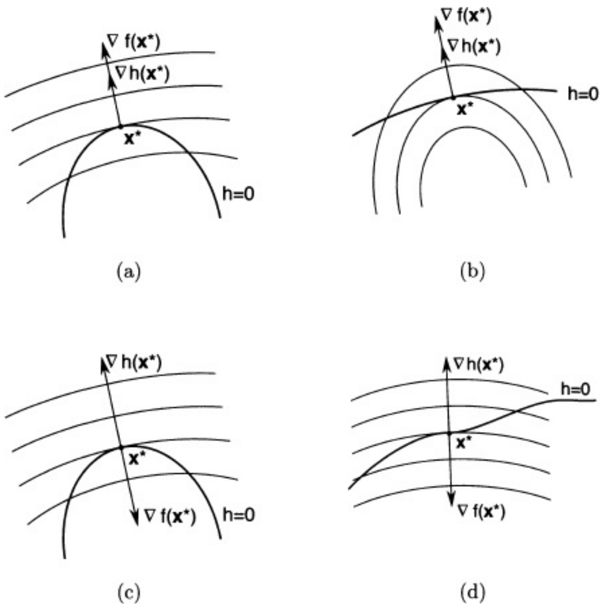

Figure A.18: Examples where the Lagrange condition is satisfied

Lagrange's theorem is then generalised for the case when $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{h(x)} : \mathbb{R}^n \to \mathbb{R}^m, m \leq n$.

Theorem A.4.3.2 Lagrange's Theorem Let $\boldsymbol{x}^*$ be a local minimiser (maximiser) of $f : \mathbb{R}^n \to \mathbb{R}$, subject to $\boldsymbol{h}(\boldsymbol{x}) = 0, \boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m, m \leq n$. Assuming that $\boldsymbol{x}^*$ is a regular point. Then, there exists $\lambda^* \in \mathbb{R}^m$ such that $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime} D\boldsymbol{h}(\boldsymbol{x}^*) = \boldsymbol{0}'$.

Lagrange's theorem states that if $\boldsymbol{x}^*$ is an extremiser, then the gradient of the objective function f can be expressed as a linear combination of the gradients of the constraints. The vector λ in Theorem 2.2.3.2 is referred to as the Lagrange multiplier vector, and its components as Lagrange multipliers. A a compact way to write the necessary condition is $\nabla f(\boldsymbol{x}^*) \in N(\boldsymbol{x}^*)$. If this condition fails, then $\boldsymbol{x}^*$ cannot be an extremiser. This situation is illustrated in Figure 2.24.

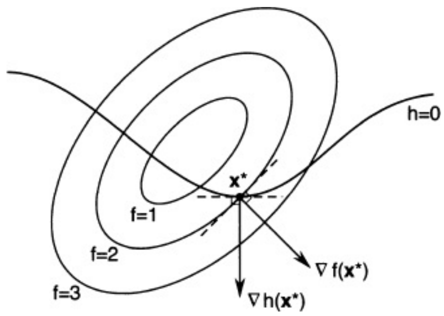

Figure A.19: Example where the Lagrange condition does not hold.

Lagrange's condition enables us to find points that are candidates for extremisers of the given objective function subject to equality constraints. These critical points are the only candidates because they are the only points that satisfy the Lagrange condition. To classify such critical points as minimisers, maximisers, or neither, a stronger condition is needed, possibly a necessary and sufficient condition. In the next section, a second-order necessary condition and a second-order sufficient condition for minimisers are discussed (Chong and Zak, 2013).

Second order conditions

It is assumed that $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable: $f, \boldsymbol{h} \in \subset^2$. Let $l(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}'\boldsymbol{h}(\boldsymbol{x}) = f(\boldsymbol{x}) + \lambda_1 h_1(\boldsymbol{x}) + \ldots + \lambda_m h_m(\boldsymbol{x})$ be the Lagrangian function. Let $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda})$ be the Hessian matrix of $l(\boldsymbol{x}, \boldsymbol{\lambda})$ with respect to $\boldsymbol{x}$: $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{F}(\boldsymbol{x}) + \lambda_1 \boldsymbol{H}_1(\boldsymbol{x}) + \ldots + \lambda_m \boldsymbol{H}_m(\boldsymbol{x})$, where $\boldsymbol{F}(\boldsymbol{x})$ is the Hessian matrix of f at $\boldsymbol{x}$ and $\boldsymbol{H}_k(\boldsymbol{x})$ is the Hessian matrix of h_k at $\boldsymbol{x}$, $k = 1, \ldots, m$, given by:

$$\boldsymbol{H}_k(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial^2 h_k}{\partial x_1^2}(\boldsymbol{x}) & \frac{\partial^2 h_k}{\partial x_2 \partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial^2 h_k}{\partial x_n \partial x_1}(\boldsymbol{x}) \\ \frac{\partial^2 h_k}{\partial x_1 \partial x_2}\boldsymbol{x} & \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{x}) & \cdots & \frac{\partial^2 h_k}{\partial x_n \partial x_2}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 h_k}{\partial x_1 \partial x_n}(\boldsymbol{x}) & \frac{\partial^2 h_k}{\partial x_2 \partial x_n}(\boldsymbol{x}) & \cdots & \frac{\partial^2 h_k}{\partial x_n^2}(\boldsymbol{x}) \end{pmatrix} \tag{A.37}$$

Theorem A.4.4.1 Second Order Necessary Conditions. Let $\boldsymbol{x}^*$ be a local minimiser of $f : \mathbb{R}^n \to \mathbb{R}$, subject to $\boldsymbol{h}(\boldsymbol{x}) = 0, \boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m, m \leq n$ and $f, \boldsymbol{h} \in \subset^2$. Suppose that $\boldsymbol{x}^*$ is regular . Then, there exists $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that:

1. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}\boldsymbol{D}h(\boldsymbol{x}^*) = \boldsymbol{0}'$.

2. For all $\boldsymbol{y} \in T(\boldsymbol{x}^*)$, it is true that $\boldsymbol{y}'\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)\boldsymbol{y} \geq 0$.

The conditions above are necessary, but not sufficient, for a point to be a local minimiser. Sufficient conditions for a point to be a strict local minimiser are now presented.

Theorem A.4.4.2 Second Order Sufficient Conditions Suppose that $f, \boldsymbol{h} \in \subset^2$ and there exists a point $\boldsymbol{x}^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that:

1. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}D\boldsymbol{h}(\boldsymbol{x}^*) = \boldsymbol{0}'$.

2. For all $\boldsymbol{y} \in T(\boldsymbol{x}^*)$, $\boldsymbol{y} \neq \boldsymbol{0}$, then $\boldsymbol{y}'\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)\boldsymbol{y} > 0$.

Then, $\boldsymbol{x}^*$ is a strict local minimiser of f subject to $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$

Theorem A.4.4.2 states that if a point $\boldsymbol{x}^*$ satisfies the Lagrange condition, and $\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is positive definite on $T(\boldsymbol{x}^*)$, then $\boldsymbol{x}^*$ is a strict local minimiser. A similar result to Theorem A.4.4.2 holds for a strict local maximiser, the only difference being that $\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ be negative definite on $T(\boldsymbol{x}^*)$(Chong and Zak, 2013).

Karush-Kuhn-Tucker condition

In the previous section, constrained optimisation problems involving only equality constraints were discussed. In this section extremum problems that also involve inequality constraints are discussed. The treatment in this section parallels that of the previous section. In particular, as it will be evident, problems with inequality constraints can also be treated using Lagrange multipliers. The following problem is considered:

$$\text{minimise} \;\; f(\boldsymbol{x}) \tag{A.38}$$

$$\text{subject to} \;\; \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.39}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}, \tag{A.40}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m, m \leq n$, and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$. For the general problem above, the following definitions are adopted:

Definition A.4.5.1 An inequality constraint $g_j(\boldsymbol{x}) \leq 0$ is said to be active at $\boldsymbol{x}^*$ if $g_j(\boldsymbol{x}^*) = 0$. It is inactive at $\boldsymbol{x}^*$ if $g_j(\boldsymbol{x}^*) < 0$. By convention, an equality constraint $h_i(\boldsymbol{x}) = 0$ is considered to be always active.

Definition A.4.5.2 Let $\boldsymbol{x}^*$ satisfy $\boldsymbol{h}(\boldsymbol{x}^*) = \boldsymbol{0}$, $\boldsymbol{g}(\boldsymbol{x}^*) \leq \boldsymbol{0}$, and let $j(\boldsymbol{x}^*)$ be the index set of active inequality constraints: $J(\boldsymbol{x}^*) = \{j : g_j(\boldsymbol{x}^*) = 0\}$.

Then $\boldsymbol{x}^*$ is said to be a regular point if the vectors $\nabla h_i(\boldsymbol{x}^*)$, $\nabla g_j(\boldsymbol{x}^*)$, $1 \leq i \leq m, j \in J(\boldsymbol{g}(\boldsymbol{x}^*))$ are linearly independent.

A FONC for a point to be a local minimiser is now provided. This condtion is called the Karush-Kuhn-Tucker (KKT) condition.

Theorem A.4.5.1 Karush-Kuhn-Tucker (KKT) Theorem. Let $f, \boldsymbol{h}, \boldsymbol{g} \in \subset^1$. Let $\boldsymbol{g}(\boldsymbol{x}^*)$ be a regular point and a local minimiser for the problem of minimising f subject to $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$. Then, there exists $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu} \in \mathbb{R}^p$ such that:

1. $\boldsymbol{\mu}^* \geq \boldsymbol{0}$

2. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}D\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\prime}D\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{0}'$

3. $\boldsymbol{\mu}^{*\prime}\boldsymbol{g}(\boldsymbol{x}^*) = 0$

In Theorem A.4.5.1, $\boldsymbol{\lambda}^*$ is referred to as the Lagrange multiplier vector and $\boldsymbol{\mu}^*$ as the Karush-Kuhn-Tucker (KKT) multiplier vector. Their components are referred to as the Lagrange multipliers and Karush-Kuhn-Tucker (KKT) multipliers, respectively. From the theorem it is important to note that $\mu_j^* \geq 0$ (by condition 1) and $g_j(\boldsymbol{x}^*) \leq 0$. Therefore, the condition

$$\boldsymbol{\mu}^{*\prime}\boldsymbol{g}(\boldsymbol{x}^*) = \mu_1^* g_1(\boldsymbol{x}^*) + \ldots + \mu_p^* g_p(\boldsymbol{x}^*) = 0 \tag{A.41}$$

implies that if $g_j(\boldsymbol{x}^*) < 0$, then $\mu_j^* = 0$, that is, for all $j \notin J(\boldsymbol{x}^*)$, it is true that $\mu_j^* = 0$. In other words, the KKT multipliers corresponding to inactive constraints are zero. The other KKT multipliers, $\mu_i^*, i \in J(\boldsymbol{x}^*)$, are nonnegative, they may or may not equal to zero.

Example A.4.5.1 A graphical illustration of the KKT theorem is given in the Figure below. In this two dimensional example, there are only inequality constraints $g_j(\boldsymbol{x}) \leq 0, j = 1, 2, 3$. It should be noted that the point $\boldsymbol{x}^*$ in Figure 2.25 is indeed a minimiser.

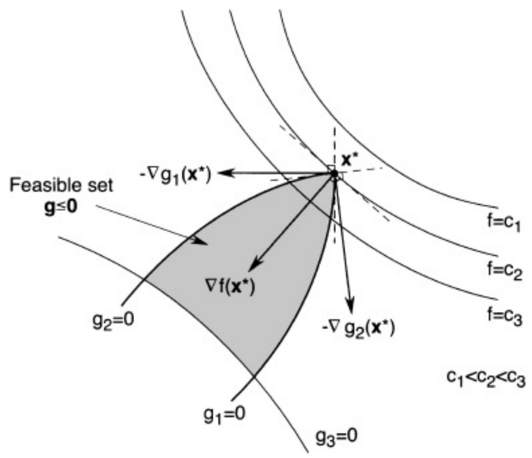

Figure A.20: Illustration of the Karush Kuhn Tucker (KKT) theorem.

The constraint $g_3(\boldsymbol{x}) \leq 0$ is inactive since $g_3 < 0$, hence $\mu_3^* = 0$. By the KKT theorem:

$$\nabla f(\boldsymbol{x}^*) + \mu_1^* \nabla g_1(\boldsymbol{x}^*) + \mu_2^* \nabla g_2(\boldsymbol{x}^*) = \boldsymbol{0} \tag{A.42}$$

, or equivalently

$$\nabla f(\boldsymbol{x}^*) = -\mu_1^* \nabla g_1(\boldsymbol{x}^*) - \mu_2^* \nabla g_2(\boldsymbol{x}^*) \tag{A.43}$$

, where $\mu_1^* > 0$ and $\mu_2^* > 0$. It is easy to interpret the KKT condition graphically for this example. Specifically, it can be seen from the figure above that $\nabla f(\boldsymbol{x}^*)$ must be a linear combination of the vectors $-\nabla g_1(\boldsymbol{x}^*)$ and $-\nabla g_2(\boldsymbol{x}^*)$ with positive coefficients. This is reflected exactly in the equation above, where the coefficients μ_1^* and μ_2^* are the KKT multipliers. The KKT condition is applied in the same way that any necessary condition is applied. Specifically, one searches for points satisfying the KKT condition and treats these points as candidate minimisers.

To summarise, the KKT condition consists of five parts (three equations and two inequalities):

1. $\boldsymbol{\mu}^* \geq \mathbf{0}$

2. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}^*) = \mathbf{0}'$.

3. $\boldsymbol{\mu}^{*\prime}\boldsymbol{g}(\boldsymbol{x}^*) = 0$

4. $\boldsymbol{h}(\boldsymbol{x}^*) = \mathbf{0}$

5. $\boldsymbol{g}(\boldsymbol{x}^*) \leq \mathbf{0}$

Second order conditions

As in the case of extremum problems with equality constraints, second-order necessary and sufficient conditions for extremum problems involving inequality constraints are given. For this, the following matrix is defined: $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{F}(\boldsymbol{x}) + [\boldsymbol{\lambda}\boldsymbol{H}(\boldsymbol{x})] + [\boldsymbol{\mu}\boldsymbol{G}(\boldsymbol{x})]$, where $\boldsymbol{F}(\boldsymbol{x})$ is the Hessian matrix of f at $\boldsymbol{x}$, and $[\boldsymbol{\lambda}\boldsymbol{H}(\boldsymbol{x})] = \lambda_1\boldsymbol{H}_1(\boldsymbol{x}) + \ldots + \lambda_m\boldsymbol{H}_m(\boldsymbol{x})$. Similarly $[\boldsymbol{\mu}\boldsymbol{G}(\boldsymbol{x})] = \mu_1\boldsymbol{G}_1(\boldsymbol{x}) + \ldots + \mu_p\boldsymbol{G}_p(\boldsymbol{x})$, where $\boldsymbol{G}_k(\boldsymbol{x})$ is the Hessian of g_k at $\boldsymbol{x}$, given by:

$$\boldsymbol{G}_k(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial^2 g_k}{\partial x_1^2}(\boldsymbol{x}) & \frac{\partial^2 g_k}{\partial x_2 \partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial^2 g_k}{\partial x_n \partial x_1}(\boldsymbol{x}) \\ \frac{\partial^2 g_k}{\partial x_1 \partial x_2}\boldsymbol{x} & \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{x}) & \cdots & \frac{\partial^2 g_k}{\partial x_n \partial x_2}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g_k}{\partial x_1 \partial x_n}(\boldsymbol{x}) & \frac{\partial^2 g_k}{\partial x_2 \partial x_n}(\boldsymbol{x}) & \cdots & \frac{\partial^2 g_k}{\partial x_n^2}(\boldsymbol{x}) \end{pmatrix} \tag{A.44}$$

The following theorem uses $T(\boldsymbol{x}^*) = \left\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}^*)\boldsymbol{y} = 0, j \in J(\boldsymbol{x}^*)\right\}$, that is, the tangent space to the surface defined by active constraints.

Theorem A.4.6.1 Second Order Necessary Conditions. Let $\boldsymbol{x}^*$ be a local minimiser of $f : \mathbb{R}^n \to \mathbb{R}$ subject to $\boldsymbol{h}(\boldsymbol{x}) = \mathbf{0}, \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}, \boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}, m \leq n, \boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$, and $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in \subset^2$. Suppose that $\boldsymbol{x}^*$ is regular. Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

1. $\boldsymbol{\mu}^* \geq \mathbf{0}$

2. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\prime}\boldsymbol{D}\boldsymbol{g}(\boldsymbol{x}^*) = \mathbf{0}'$

3. For all $\boldsymbol{y} \in T(\boldsymbol{x}^*), \boldsymbol{y}'\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{y} \geq 0$

The second order sufficient conditions for extremum problems involving inequality constraints are now discussed. In the formulation of the result , the following set is used: $\tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \left\{\boldsymbol{y} : \boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}^*)\boldsymbol{y} = \mathbf{0}, \boldsymbol{D}g_i(\boldsymbol{x}^*)\boldsymbol{y} = 0, i \in \tilde{J}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)\right\}$, where $\tilde{J}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \left\{i : g_i(\boldsymbol{x}^*) = 0, \boldsymbol{\mu}_i^* > \mathbf{0}\right\}$. It should be noted that $\tilde{J}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) \subset J(\boldsymbol{x}^*)$. This in turn implies that $T(\boldsymbol{x}^*) \subset \tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$.

Theorem A.4.6.2 Second Order Sufficient Conditions. Suppose that $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in \subset^2$ and there exist a feasible point $\boldsymbol{x}^* \in \mathbb{R}^n$ and vectors $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\boldsymbol{\mu} \in \mathbb{R}^p$ such that :

1. $\boldsymbol{\mu}^* \geq \mathbf{0}$

2. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\prime}\boldsymbol{D}\boldsymbol{g}(\boldsymbol{x}^*) = \mathbf{0}'$

3. $\boldsymbol{\mu}^{*\prime}\boldsymbol{g}(\boldsymbol{x}^*) = 0$

4. For all $\boldsymbol{y} \in \tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*), \boldsymbol{y} \neq \mathbf{0}$, then $\boldsymbol{y}'\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{y} > 0$.

Then, $\boldsymbol{x}^*$ is a strict local minimiser of f subject to $\boldsymbol{h}(\boldsymbol{x}) = \mathbf{0},\ \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}$

A similar result to Theorem 2.1.21.3 holds for a strict local maximiser, the only difference being that, one needs $\boldsymbol{\mu}^* \leq \mathbf{0}$ and that $\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ be negative definite on $\tilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$(Chong and Zak, 2013)

Optimisation problems

The optimisation problems examined so far are typically challenging to solve due to inherent complexities. These difficulties often stem from either the objective function or the constraints, or a combination of both. Even when the objective function is straightforward and exhibits desirable properties, the constraints can still pose significant obstacles to finding a solution. One approach to mitigating these challenges involves limiting consideration to problems with convex feasible regions. Although this framework doesn't encompass all significant real-world problems, it does enable the derivation of globally applicable results for this specific class of optimisation problems(Chong and Zak, 2013).

Convex optimisation problems

This section focuses on optimisation problems characterized by convex objective functions and convex constraint sets, collectively known as convex optimisation or convex programming problems. Notable examples of convex programming problems include linear programs and quadratic optimisation problems with linear constraints. Convex programming problems possess unique properties that make them particularly appealing. Most notably, any local minimum is also a global minimum, as demonstrated by the following theorem. Moreover, necessary first-order conditions become sufficient for minimization, simplifying the optimisation process(Chong and Zak, 2013).

Theorem A.4.7.1 Let $f : \Omega \to \mathbb{R}$ be a convex function defined on a convex set $\Omega \subset \mathbb{R}^n$. Then, a point is a global minimiser of f over Ω if and only if it is a local minimiser of f.

The following Theorems characterise sufficient conditions for a point $\boldsymbol{x}^*$ to be a global minimiser.

Lemma A.4.7.1 Let $f : \Omega \to \mathbb{R}$ be a convex function defined on a convex set $\boldsymbol{\Omega} \subset \mathbb{R}^n$ and $f \in \subset^1$ on an open convex set containing $\boldsymbol{\Omega}$. Suppose that the point $\boldsymbol{x}^* \in \Omega$ is such that for all $\boldsymbol{x} \in \Omega, \boldsymbol{x} \neq \boldsymbol{x}^*$, then $\boldsymbol{D}f(\boldsymbol{x}^*)(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0$. Then, $\boldsymbol{x}^*$ is a global minimiser of f over Ω.

Theorem A.4.7.2 Let $f : \Omega \to \mathbb{R}$ be a convex function defined on a convex set $\Omega \subset \mathbb{R}^n$ and $f \in \subset^1$ on an open convex set containing Ω. Suppose that the point $\boldsymbol{x}^* \in \Omega$ is such that for any feasible direction $\boldsymbol{d}$ at $\boldsymbol{x}^*$, $\boldsymbol{d}'\nabla f(\boldsymbol{x}^*) \geq 0$. Then, $\boldsymbol{x}^*$ is a global minimiser of f over Ω.

Corollary A.4.7.1 Let $f : \Omega \to \mathbb{R}$ be a convex function defined on a convex set $\Omega \subset \mathbb{R}^n$. Suppose that the point $\boldsymbol{x}^* \in \Omega$ is such that $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$. Then, $\boldsymbol{x}^*$ is a global minimiser of f over Ω.

Considering the following constrained optimisation problem:

$$\text{minimise } \ f(\boldsymbol{x}) \tag{A.45}$$

$$\text{subject to } \ \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.46}$$

It is assumed that the feasible set is convex. An example where this is the case is when

$$\boldsymbol{h}(\boldsymbol{x}) = A\boldsymbol{x} - \boldsymbol{b} \tag{A.47}$$

The following theorem states that provided the feasible set is convex, the Lagrange condition is sufficient for a point to be a minimiser.

Theorem A.4.7.8 Let $f : \Omega \to \mathbb{R}, f \in \subset^1$ be a convex function defined on a convex set

$$\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\right\} \tag{A.48}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{h} \in \subset^1$, and Ω is convex. Suppose that there exist $\boldsymbol{x}^* \in \Omega$ and $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that

$$\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*'}D\boldsymbol{h}(\boldsymbol{x}^*) = 0 \tag{A.49}$$

then, $\boldsymbol{x}^*$ is a global minimiser of f over Ω

Considering the general constrained optimisation problem:

$$\text{minimise } \quad f(\boldsymbol{x}) \tag{A.50}$$

$$\text{subject to } \ \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.51}$$

$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \tag{A.52}$$

As before, it is assumed that the feasible set is convex. This is the case if the two sets $\{\boldsymbol{x} : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\}$ and $\{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\}$ are convex, because the feasible set is the intersection of these two sets. The next theorem states that the KKT condition is sufficient for a point to be a minimiser to the problem above.

Theorem A.4.7.9 Let $f : \Omega \to \mathbb{R}, f \in \subset^1$ be a convex function defined on a convex set

$$\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}, \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\right\} \tag{A.53}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p, \boldsymbol{h}, \boldsymbol{g} \in \subset^1$ and Ω is convex. Suppose that there exists $\boldsymbol{x}^* \in \Omega, \boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$, such that

1. $\boldsymbol{\mu}^* \geq \mathbf{0}$

2. $\boldsymbol{D}f(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\prime}\boldsymbol{D}\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\prime}\boldsymbol{D}\boldsymbol{g}(\boldsymbol{x}^*) = \mathbf{0}'$

3. $\boldsymbol{\mu}^{*\prime}\boldsymbol{g}(\boldsymbol{x}^*) = \mathbf{0}$

Then, $\boldsymbol{x}^*$ is a global minimiser of f over $\boldsymbol{\Omega}$ (Chong and Zak, 2013).

Linear programming

According to Dongarra and Sullivan (2000): "Linear programming aims to optimise a linear objective function by identifying the optimal values of decision variables, subject to a set of linear constraints. This problem represents a specific instance of the broader class of constrained optimization problems. In this general framework, the objective is to locate a solution that simultaneously minimises the objective function and adheres to the given constraints. Solutions that satisfy all constraints are deemed feasible. Specifically, in linear programming, the objective function is linear, and the feasible region is defined by a system of linear equations and/or inequalities. This section explores techniques for resolving linear programming problems. These methods enable the identification of the optimal feasible solution amidst an infinite array of possibilities. Notably, despite the vast number of feasible points, the solution can be efficiently determined by examining a finite subset of feasible solutions, specifically the basic feasible solutions.

In theory, linear programming problems can be solved by exhaustively comparing all basic feasible solutions and identifying the one that optimizes the objective function, a method known as the brute-force approach. However, for most real-world problems, the sheer number of possible solutions renders this approach impractical, one can consider the following example:

Suppose that there is a small factory with 20 different machines producing 20 different parts. Assume that any of the machines can produce any part. It is also assumed that the time for producing each part on each machine is known. The problem then is to assign a part to each machine so that the overall production time is minimised. It can be seen that there are 20! possible assignments. The brute-force approach to solving this assignment problem would involve writing down all the possible assignments and then choosing the best one by comparing them. Suppose that a computer takes 1 μs (10^{-6} second) to determine each assignment.

Then, to find the best (optimal) assignment this computer would need 77,147 years (working 24 hours a day, 365 days a year) to find the best solution. An alternative approach to solving this problem is to use experienced planners to optimise this assignment problem Such an approach relies on heuristics. Heuristics come close, but give suboptimal answers. Heuristics that do reasonably well, with an error of, say, 10%, may still not be good enough. For example, in a business that operates on large volumes and a small profit margin, a 10% error could mean the difference between loss and profit. Efficient methods for solving linear programming problems became available in the late 1930's."

In 1939, Kantorovich presented a number of solutions to some problems related to production and transportation planning. During World War II, Koopmans contributed significantly to the solution of transportation problems. Kantorovich and Koopmans were awarded a Nobel prize in Economics in 1975 for their work on the theory of optimal allocation of resources. In 1947, Dantzig developed a new method for solving linear programs, known today as the simplex method. The simplex method is efficient and elegant and has been declared one of the 10 algorithms with the greatest influence on the development and practice of science and engineering in the twentieth century (Dongarra and Sullivan, 2000).

Standard form of linear programs

A linear program of the form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.54}$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{A.55}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.56}$$

is referred to as a linear program in standard form. Here $\boldsymbol{A}$ is an $m \times n$ matrix composed of real entries, $m < n$, $\text{rank}(\boldsymbol{A}) = m$. Without loss of generality, it is assumed that $\boldsymbol{b} \geq \boldsymbol{0}$. If a component of $\boldsymbol{b}$ is negative, say the i^{th} component, then the i^{th} constraint is multiplied by -1 to obtain a positive right-hand side. Theorems and solution techniques for linear programs are usually stated for problems in standard form. Other forms of linear programs can be converted to the standard form, as it is now shown. If a linear program is in the form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.57}$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \tag{A.58}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.59}$$

then by introducing surplus variables y_i, the original problem can be converted into the standard form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.60}$$

$$\text{subject to} \quad a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - y_i = b_i, \ i = 1, \ldots, m \tag{A.61}$$

$$x_1 \geq 0, \ldots, x_n \geq 0 \tag{A.62}$$

$$y_1 \geq 0, \ldots, y_m \geq 0 \tag{A.63}$$

In a more compact notation, the formulation above can be presented as:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.64}$$

$$\text{subject to} \quad [\boldsymbol{A}, -\boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{I}_m\boldsymbol{y} \tag{A.65}$$

$$\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0} \tag{A.66}$$

where I_m is the $m \times m$ identity matrix. If on the other hand, the constraints have the form:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \tag{A.67}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.68}$$

then by introducing slack variables y_i to convert the constraints into the form:

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}_m\boldsymbol{y} = \boldsymbol{b} \tag{A.69}$$

$$\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0} \tag{A.70}$$

where $\boldsymbol{y}$ is the vector of slack variables. It should be noted that neither surplus nor slack variables contribute to the objective function $\boldsymbol{c}'\boldsymbol{x}$ (Chong and Zak, 2013).

Basic solutions

Assuming that the linear program is in standard form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.71}$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{A.72}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.73}$$

Considering the system of equalities: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where rank(A) $=m$. In dealing with this system of equations, one frequently needs to consider a subset of the columns of the matrix $\boldsymbol{A}$. For convenience, one often reorders the columns of $\boldsymbol{A}$ so that the columns of interest appear first. Specifically, let $\boldsymbol{B}$ be a square matrix whose columns are m linearly independent columns of $\boldsymbol{A}$. If necessary, one reorders the columns of $\boldsymbol{A}$ so that the columns in $\boldsymbol{B}$ appear first: A has the form $\boldsymbol{A} = [\boldsymbol{B}, \boldsymbol{D}]$, where $\boldsymbol{D}$ is an $m \times (n - m)$ matrix whose columns are the remaining columns of $\boldsymbol{A}$. The matrix $\boldsymbol{B}$ is nonsingular, and thus one can solve the equation: $\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{b}$ for the m-vector $\boldsymbol{x}_B$. The solution is $\boldsymbol{x}_B = B^{-1}\boldsymbol{b}$.

Let $\boldsymbol{x}$ be the n-vector whose first m components are equal to $\boldsymbol{x}_B$ and the remaining components are equal to optimisero, that is, $\boldsymbol{x} = [\boldsymbol{x}_B, \boldsymbol{0}]$. Then, $\boldsymbol{x}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Definition A.5.2.1 The vector $\boldsymbol{x} = [\boldsymbol{x}_B, \boldsymbol{0}]$ is called a basic solution to $A\boldsymbol{x} = \boldsymbol{b}$ with respect to the basis $\boldsymbol{B}$. The components of the vector $\boldsymbol{x}_B$ are referred to as the basic variables and the columns of $\boldsymbol{B}$ as basic columns.

If some of the basic variables of a basic solution are optimal, then the basic solution is said to be a degenerate basic solution. A vector $\boldsymbol{x}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, is said to be a feasible solution. A feasible solution that is also basic is called a basic feasible solution. If the basic feasible solution is a degenerate basic solution, then it is called a degenerate basic feasible solution. It should be noted that in any basic feasible solution, $\boldsymbol{x}_B \geq \boldsymbol{0}$ (Chong and Zak, 2013).

Properties of basic solutions

It is necessary to discuss the importance of basic feasible solutions in solving linear programming (LP) problems. The fundamental theorem of LP, which states that when solving an LP problem, one needs only to consider the basic feasible solutions. This is because the optimal value (if it exists) is always achieved at a basic feasible solution. Before stating the theorem, the following definition is needed.

Definition A.5.3.1 Any vector $\boldsymbol{x}$ that yields the minimum value of the objective function $\boldsymbol{c}'\boldsymbol{x}$ over the set of vectors satisfying the constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is said to be an optimal feasible solution. An optimal feasible solution that is basic is said to be an optimal basic feasible solution.

Theorem A.5.3.1 The Fundamental Theorem of LP. Consider a linear program in standard form.

1 If there exists a feasible solution, then there exists a basic feasible solution.

2 If there exists an optimal feasible solution, then there exists an optimal basic feasible solution.

(Allaire, 2007)

The simplex method

Dongarra and Sullivan (2000) suggests considering a linear programming problem in standard form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.74}$$

$$\text{subject to } \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{A.75}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.76}$$

where the first m columns of $\boldsymbol{A}$ be the basic columns. The columns form a square $m \times m$ nonsingular matrix $\boldsymbol{B}$. The nonbasic columns of $\boldsymbol{A}$ form an $m \times (n-m)$ matrix $\boldsymbol{D}$. The cost vector is partitioned correspondingly as $\boldsymbol{c}' = [\boldsymbol{c}'_B, \boldsymbol{c}'_D]$. Then, the original linear program can be represented as follows:

$$\text{minimise } \boldsymbol{c}'_B\boldsymbol{x}_B + \boldsymbol{c}'_D\boldsymbol{x}_D \tag{A.77}$$

$$\text{subject to } \quad \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{D}\boldsymbol{x}_D = \boldsymbol{b} \tag{A.78}$$

$$\boldsymbol{x}_B \geq \boldsymbol{0},\ \boldsymbol{x}_D \geq \boldsymbol{0} \tag{A.79}$$

.

If $\boldsymbol{x}_D = \boldsymbol{0}$, then the solution $\boldsymbol{x} = [\boldsymbol{x}'_B, \boldsymbol{x}'_D]' = [\boldsymbol{x}'_B, \boldsymbol{0}']'$ is the basic feasible solution corresponding to the basis $\boldsymbol{B}$. It is clear that for this to be a solution, one needs $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$, that is the basic feasible solution is $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$,

The corresponding objective function value is $z_0 = \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{b}$. If on the other hand, $\boldsymbol{x}_D \neq \boldsymbol{0}$, then the solution $\boldsymbol{x} = [\boldsymbol{x}'_B, [\boldsymbol{x}'_D]'$ is not basic.

In this case $\boldsymbol{x}_B$ is given by $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D$, and the corresponding objective function value is $z = \boldsymbol{c}'_B(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D) + \boldsymbol{c}'_D\boldsymbol{x}_D = \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}'_D - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{D})\boldsymbol{x}_D$

Let $\boldsymbol{r}'_D = \boldsymbol{c}'_D - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{D}$, then $z = z_0 + \boldsymbol{r}'_D\boldsymbol{x}_D$. The elements of the vector $\boldsymbol{r}_D$ are the reduced cost coefficients corresponding to the nonbasic variables. If $\boldsymbol{r}_D \geq \boldsymbol{0}$, then the basic feasible solution corresponding to the basis $\boldsymbol{B}$ is optimal. If, on the other hand, a component of $\boldsymbol{r}_D$ is negative, then the value of the objective function can be reduced by increasing a corresponding component of $\boldsymbol{x}_D$, that is, by changing the basis. The development of the matrix form of the simplex method starts by adding the cost coefficient vector $\boldsymbol{c}'$ to the bottom of the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ as follows:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}' & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}'_B & \boldsymbol{c}'_B & 0 \end{bmatrix} \tag{A.80}$$

This matrix is refered to as the tableau of the given LP problem. The tableau contains all relevant information about the linear program. Suppose now that one wants to apply elementary row operations to the tableau such that the top part of the tableau corresponding to the augmented matrix $[A, \boldsymbol{b}]$ is transformed into canonical form. This corresponds to premultiplying the tableau by the matrix:

$$\begin{bmatrix} \boldsymbol{B}^{-1} & \boldsymbol{0} \\ \boldsymbol{0}' & 1 \end{bmatrix} \tag{A.81}$$

resulting in

$$\begin{bmatrix} \boldsymbol{B}^{-1} & \boldsymbol{0} \\ \boldsymbol{0}' & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}'_B & \boldsymbol{c}'_D & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{c}'_B & \boldsymbol{c}'_D & 0 \end{bmatrix} \tag{A.82}$$

According to Chong and Zak (2013):"One can then apply elementary row operations to the tableau above so that the entries of the last row corresponding to the basic columns become optimisero. Specifically, this corresponds to premultiplication of the tableau by the matrix $\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ -\boldsymbol{c}'_B & 1 \end{bmatrix}$

Resulting in

$$\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ -\boldsymbol{c}'_B & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{c}'_B & \boldsymbol{c}'_D & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}' & \boldsymbol{c}'_D - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix} \tag{A.83}$$

The resulting tableau is refered to as the canonical tableau corresponding to the basis $\boldsymbol{B}$. It should be noted that the first m entries of the last column of the canonical tableau, $\boldsymbol{B}^{-1}\boldsymbol{b}$, are the values of the basic variables corresponding to the basis $\boldsymbol{B}$. The entries of $\boldsymbol{c}'_D - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{D}$ in the last row are the reduced cost coefficients. The last element in the last row of the tableau,$-\boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{b}$, is the negative of the value of the objective function corresponding to the basic feasible solution.

Given an LP problem, one can in general construct many different canonical tableaus, depending on which columns are basic. Suppose that one has a canonical tableau corresponding to a particular basis. Consider the task of computing the tableau corresponding to another basis that differs from the previous basis by a single vector. This can be accomplished by applying elementary row operations to the tableau in a similar fashion as discussed above.

This operation is refered to as updating the canonical tableau. Working with the tableau is a convenient way of implementing the simplex algorithm, since updating the tableau immediately gives the values of both the basic variables and the reduced cost coefficients. In addition, the (negative of the) value of the objective function can be found in the lower right-hand corner of the tableau.

Two-phase simplex method

The simplex method requires starting with a tableau for the problem in canonical form, that is one needs an initial basic feasible solution. A brute-force approach to finding a starting basic feasible solution is to choose m basic columns arbitrarily and transform the tableau for the problem into canonical form. If the rightmost column is positive, then one has obtained a legitimate (initial) basic feasible solution. Otherwise, one would have to pick another candidate basis. Potentially, this brute-force procedure requires $\binom{n}{m}$ tries, and is therefore not practical. Certain LP problems have obvious initial basic feasible solutions. For example, if one has constraints of the form $\boldsymbol{Ax} \leq \boldsymbol{b}$ and adds m slack variables $z_1, ..., z_m$, then the constraints in standard form become

$$[A, I_m] \begin{bmatrix} \boldsymbol{x} \\ z \end{bmatrix} = \boldsymbol{b}, \begin{bmatrix} \boldsymbol{x} \\ z \end{bmatrix} \geq \boldsymbol{0} \tag{A.84}$$

where $\boldsymbol{z} = [z_1, ..., z_m]$. The obvious initial basic feasible solution is $\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix}$,

and the basic variables are the slack variables.

Suppose one is given a linear program in standard form:

$$\text{minimise } \boldsymbol{c'x} \tag{A.85}$$

$$\text{subject to } \quad \boldsymbol{Ax} = \boldsymbol{b} \tag{A.86}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.87}$$

In general, an initial basic feasible solution is not always apparent. One therefore needs a systematic method for finding an initial basic feasible solution for general LP problems so that the simplex method can be initialised. For this purpose, suppose that one is given an LP problem in standard form. Consider the following associated artificial problem:

$$\text{minimise } \sum_{i=1}^{m} y_i \tag{A.88}$$

$$\text{subject to } \quad [A, I_m] \begin{bmatrix} \boldsymbol{x} \\ z \end{bmatrix} = \boldsymbol{b} \tag{A.89}$$

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \geq \boldsymbol{0} \tag{A.90}$$

where $\boldsymbol{y} = [y_1, ..., y_m]'$. $\boldsymbol{y}$ is called the vector of artificial variables. It should be noted that the artificial problem has an obvious basic feasible solution $\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix}$,

One can therefore solve this problem using the simplex method.

Theorem A.5.5.1 The original LP problem has a basic feasible solution if and only if the associated artificial problem has an optimal feasible solution with objective function value of zero.

Suppose that the simplex method applied to the associated artificial problem has terminated with an objective function value of zero. Then this implies that all $y_i = 0, i = 1, \ldots, m$. Hence, assuming non degeneracy, the basic variables are in the first n components, that is none of the artificial variables are basic. Therefore, the first n components form a basic feasible solution to the original problem. One can then use this basic feasible solution (resulting from the artificial problem) as the initial basic feasible solution for the original LP problem (after deleting the components corresponding to artificial variables). Thus, using artificial variables one can attack a general linear programming problem by applying the two-phase simplex method. In phase I artificial variables and the artificial objective function are introduced and a basic feasible solution is sought. In phase II, the basic feasible solution resulting from phase I is used to initialised the simplex algorithm to solve the original LP problem.

Revised simplex method

Consider an LP problem in standard form with a matrix $\boldsymbol{A}$ of size $m \times n$. Suppose the simplex method is used to solve the problem. Experience suggests that if m is much smaller than n, then, in most instances, pivots will occur in only a small fraction of the columns of the matrix $\boldsymbol{A}$. The operation of pivoting involves updating all the columns of the tableau. However, if a particular column of $\boldsymbol{A}$ never enters any basis during the entire simplex procedure, then computations performed on this column are never used. Therefore, if m is much smaller than n, the effort expended on performing operations on many of the columns of $\boldsymbol{A}$ may be wasted. The revised simplex method reduces the amount of computation leading to an optimal solution by eliminating operations on columns of $\boldsymbol{A}$ that do not enter the bases.

To be specific, suppose that one is at a particular iteration in the simplex algorithm. Let $\boldsymbol{B}$ be the matrix composed of the columns of $\boldsymbol{A}$ forming the current basis, and let $\boldsymbol{D}$ be the matrix composed of the remaining columns of $\boldsymbol{A}$. The sequence of elementary row operations on the tableau leading to this iteration (represented by matrices $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$) corresponds to premultiplying $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{b}$ by $\boldsymbol{B}^{-1} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1$. In particular, the vector of current values of the basic variables is $\boldsymbol{B}^{-1}\boldsymbol{b}$. It should be noted that the computation of the current basic feasible solution does not require computation of $\boldsymbol{B}^{-1}\boldsymbol{D}$, all that is needed is the matrix $\boldsymbol{B}^{-1}$. In the revised simplex method, $\boldsymbol{B}^{-1}\boldsymbol{D}$ is not Golden. Instead, one only keeps track of the basic variables and the revised tableau, which is the tableau $[\boldsymbol{B}^{-1}, \boldsymbol{B}^{-1}\boldsymbol{b}]$.

It should be noted that this tableau is only of size $m \times (m+1)$ (compared to the tableau in the original simplex method, which is $m \times (n+1)$). To see how the revised tableau is updated, suppose one chooses the column $\boldsymbol{a}_q$ to enter the basis. Let $\boldsymbol{y}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q, \boldsymbol{y}_0 = [y_{01}, \ldots, y_{0m}] = \boldsymbol{B}^{-1}\boldsymbol{b}$, and $p = \left\{\arg\min_i \frac{y_{i0}}{y_{iq}} : y_{iq} > 0\right\}$. Then, to update the revised tableau, one forms the augmented revised tableau $[\boldsymbol{B}^{-1}, \boldsymbol{y}_0, \boldsymbol{y}_q]$ and pivots about the p^{th} element of the last column. It is claimed that the first $m + 1$ columns of the resulting matrix comprise the updated revised tableau. To see this, $\boldsymbol{B}^{-1}$ is written as $\boldsymbol{B}^{-1} = \boldsymbol{E}_k ... \boldsymbol{E}_1$ and let the matrix $\boldsymbol{E}_{k+1}$ represent the pivoting operation above, i.e. $\boldsymbol{E}_{k+1}\boldsymbol{y}_q = \boldsymbol{e}_p$.

Then, the updated augmented tableau resulting from the pivoting operation above is $[\boldsymbol{E}_{k+1}\boldsymbol{B}^{-1}, \boldsymbol{E}_{k+1}\boldsymbol{y}_0, \boldsymbol{e}_p]$. Let $\boldsymbol{B}_{new}$ be the new basis. Then, $\boldsymbol{B}_{new}^{-1} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1$. It should be noted that $B_{new}^{-1} = \boldsymbol{E}_{k+1}\boldsymbol{B}^{-1}$, and the values of the basic variables corresponding to $\boldsymbol{B}_{new}$ are given by $\boldsymbol{y}_{0new} = \boldsymbol{E}_{k+1}\boldsymbol{B}^{-1} = \text{Ek+1y0}$. Hence, the updated tableau is $[\boldsymbol{B}_{new}^{-1}, y_{0new}] = [\boldsymbol{E}_{k+1}\boldsymbol{B}^{-1}, \boldsymbol{E}_{k+1}\boldsymbol{y}_0]$.

The above discussion is summarised in the following algorithm.

Revised simplex method algorithm

1. Form a revised tableau corresponding to an initial basic feasible solution $[\boldsymbol{B}^{-1}, \boldsymbol{y}_0]$
2. Calculate the current reduced cost coefficients vector via $\boldsymbol{r}'_D = \boldsymbol{c}'_D - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{D}$.
3. If $r_j \geq 0$, $\forall j$, stop-the current basic feasible solution is optimal.
4. Else select a q such that $r_q < 0$ (the q corresponding to the most negative r_q) and compute $\boldsymbol{y}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q$.
5. If no $y_{iq} > 0$, stop-the problem is unbounded, else compute
   $p = \left\{\arg\min_i \frac{y_{i0}}{y_{iq}} : y_{iq} > 0\right\}$
6. Form the augmented revised tableau $[\boldsymbol{B}^{-1}, \boldsymbol{y}_0, \boldsymbol{y}_q]$ and pivot about the p^{th} element of the last column. Form the updated revised tableau by taking the first $m+1$ columns of the resulting augmented revised tableau (i.e., remove the last column).
7. Go to step 2.

As in the original simplex method, the two-phase method can be used to solve a given LP problem using the revised simplex method. In particular, one uses the revised tableau from the final step of phase I as the initial revised tableau in phase II."

Dual linear programs

According to Allaire (2007): "Associated with every linear programming problem is a corresponding dual linear programming problem. The dual problem is constructed from the cost and constraints of the original, or primal, problem. Being an LP problem, the dual can be solved using the simplex method. However, as shall be seen, the solution to the dual can also be obtained from the solution of the primal problem, and vice versa. Solving an LP problem via its dual may be simpler in certain cases, and also often provides further insight into the nature of the problem. In what follows, properties of duality and their significance are discussed. Suppose that one is given a linear programming problem of the form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.91}$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \tag{A.92}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.93}$$

The above is refered to as the primal problem. The corresponding dual problem is defined as:

$$\text{maximise } \boldsymbol{\lambda}'\boldsymbol{b} \tag{A.94}$$

$$\text{subject to} \quad \boldsymbol{\lambda}'\boldsymbol{A} \leq \boldsymbol{c}' \tag{A.95}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0} \tag{A.96}$$

The variable $\boldsymbol{\lambda} \in \mathbb{R}^m$ is known as the dual vector. It should be noted that the cost vector $\boldsymbol{c}$ in the primal has moved to the constraints in the dual. The vector $\boldsymbol{b}$ on the right hand side of $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ becomes part of the cost in the dual. Thus, the roles of $\boldsymbol{b}$ and $\boldsymbol{c}$ are reversed. The form of duality defined above is called the symmetric form of duality.

To define the dual of an arbitrary linear programming problem, the following procedure is used. First, the given linear programming problem is converted into an equivalent problem of the primal form shown above. Then, using the symmetric form of duality, the dual to the equivalent problem is constructed. The resulting problem is called the dual of the original problem. It should be noted that based on the definition of duality above, the dual of the dual problem is the primal problem. To see this, the dual problem is presented in the form.

$$\text{minimise } \boldsymbol{\lambda}'(-\boldsymbol{b}) \tag{A.97}$$

$$\text{subject to} \quad \boldsymbol{\lambda}'(-\boldsymbol{A}) \geq -\boldsymbol{c}' \tag{A.98}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0} \tag{A.99}$$

Therefore, by the symmetric form of duality, the dual to the above is

$$\text{maximise } (-\boldsymbol{c}')\boldsymbol{x} \tag{A.100}$$

$$\text{subject to} \quad (-\boldsymbol{A})\boldsymbol{x} \leq -\boldsymbol{b} \tag{A.101}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.102}$$

Upon rewriting, one gets the original primal problem. Now consider an LP problem in standard form. This form has equality constraints $\boldsymbol{A}\boldsymbol{x}$. To formulate the corresponding dual problem, one first converts the equality constraints into equivalent inequality constraints. Specifically, observe that $\boldsymbol{Ax} = \boldsymbol{b}$ is equivalent to $\boldsymbol{Ax} \geq \boldsymbol{b}$ and $-\boldsymbol{Ax} \geq -\boldsymbol{b}$. Thus, the original problem with the equality constraints can be written in the form:

$$\text{minimise } \boldsymbol{c}'\boldsymbol{x} \tag{A.103}$$

$$\text{subject to } \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \boldsymbol{x} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \tag{A.104}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{A.105}$$

The LP problem above is in the form of the primal problem in the symmetric form of duality. The corresponding dual is therefore

$$\text{maximise } [\boldsymbol{u}' \quad \boldsymbol{v}'] \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix} \tag{A.106}$$

$$\text{subject to } \quad [\boldsymbol{u}' \quad \boldsymbol{v}'] \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{bmatrix} \leq \boldsymbol{c}' \tag{A.107}$$

$$\boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0} \tag{A.108}$$

After simple manipulation the dual above can be represented as

$$\text{maximise } (\boldsymbol{u} - \boldsymbol{v})'\boldsymbol{b} \tag{A.109}$$

$$\text{subject to } \quad (\boldsymbol{u} - \boldsymbol{v})'\boldsymbol{A} \leq \boldsymbol{c}' \tag{A.110}$$

$$\boldsymbol{u}, \boldsymbol{v} \geq \boldsymbol{0} \tag{A.111}$$

Let $\boldsymbol{\lambda} = \boldsymbol{u} - \boldsymbol{v}$. Then, the dual problem becomes

$$\text{maximise } \boldsymbol{\lambda}'\boldsymbol{b} \tag{A.112}$$

$$\text{subject to } \quad \boldsymbol{\lambda}'\boldsymbol{A} \leq \boldsymbol{c}' \tag{A.113}$$

$$\boldsymbol{\lambda} \text{ unrestricted} \tag{A.114}$$

The dual for a primal problem in standard form has been developed. The form of duality above is known as the asymmetric form of duality. It should be noted that in the asymmetric form of duality, the dual of the dual is also the primal. This can be shown by reversing the arguments that were used to arrive at the asymmetric form of duality and using the symmetric form of duality. These forms of duality are summarised in Tables A.2 and A.3

Primal		Dual	
minimise	$\boldsymbol{c}'\boldsymbol{x}$	maximise	$\boldsymbol{\lambda}'\boldsymbol{b}$
subject to	$\boldsymbol{Ax} \geq \boldsymbol{b}$	subject to	$\boldsymbol{\lambda}'\boldsymbol{A} \leq \boldsymbol{c}'$
	$\boldsymbol{x} \geq \boldsymbol{0}$		$\boldsymbol{\lambda} \geq \boldsymbol{0}$

Table A.2: Symmetric form of duality.

Primal		Dual	
minimise	$\boldsymbol{c}'\boldsymbol{x}$	maximise	$\boldsymbol{\lambda}'\boldsymbol{b}$
subject to	$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$	subject to	$\boldsymbol{\lambda}'\boldsymbol{A} \leq \boldsymbol{c}'$
	$\boldsymbol{x} \geq \boldsymbol{0}$		$\boldsymbol{\lambda} \in \mathbb{R}^m$

Table A.3: Asymmetric form of duality.

Properties of dual problems

This section deals with some important results on dual linear programs. Thus the weak duality theorem is stated.

Theorem A.5.8.1 (Weak Duality Theorem). Suppose that $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible solutions to primal and dual LP problems, respectively (either in the symmetric or asymmetric form). Then, $\boldsymbol{c}'\boldsymbol{x} \geq \boldsymbol{\lambda}'\boldsymbol{b}$.

It follows from the weak duality theorem that if one is given the feasible primal and dual solutions with equal cost, then these solutions must be optimal in their respective problems. This is stated formally in the following theorem.

Theorem A.5.8.2 Suppose that $\boldsymbol{x}_0$ and $\boldsymbol{\lambda}_0$ are feasible solutions to the primal and dual, respectively (either in symmetric or asymmetric form). If $\boldsymbol{c}'\boldsymbol{x}_0 = \boldsymbol{\lambda}_0'\boldsymbol{b}$, then $\boldsymbol{x}_0$ and $\boldsymbol{\lambda}_0$ are optimal solutions to their respective problems.

Theorem A.5.8.2 can be interpreted as follows: The primal seeks to minimise its cost, and the dual seeks to maximise its cost. Because the weak duality lemma states that 'maximum $\leq$ minimum' each problem 'seeks to reach the other.' When their costs are equal for a pair of feasible solutions, both solutions are optimal. It turns out that the converse of Theorem A.5.8.2 is also true, and this is known as the strong duality theorem.

Theorem A.5.8.3 (Strong Duality Theorem). If the primal problem (either in symmetric or asymmetric form) has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.

To summarise the discussion relating the solutions of the primal and dual problems. If one has unbounded objective function values, then the other has no feasible solution. If one has an optimal feasible solution, then so does the other (and their objective function values are equal). One final case remains: What can be said if one (the primal, say) has no feasible solution? In this case the other (the dual, say) cannot have an optimal solution. However, is it necessarily the case that the dual is unbounded? The answer is no, If one of the problems has no feasible solution, then the other may or may not have a feasible solution.

The following theorem describes an alternative form of the relationship between the optimal solutions to the primal and dual problems. "

Theorem A.5.8.4 (Complementary Slackness Condition). The feasible solutions $\boldsymbol{x}$

and $\boldsymbol{\lambda}$ to a dual pair of problems (either in symmetric or asymmetric form) are optimal if and only if:

1. $(\boldsymbol{c}' - \boldsymbol{\lambda}'\boldsymbol{A})\boldsymbol{x} = 0.$
2. $\boldsymbol{\lambda}'(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0.$