

A study of certain multi-dimensional partial differential equations using Lie symmetry analysis

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Thesis accepted in fulfilment of the requirements for the degree
Doctor of Philosophy in *Applied Mathematics* at the North-West
University

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**A STUDY OF CERTAIN
MULTI-DIMENSIONAL PARTIAL
DIFFERENTIAL EQUATIONS USING
LIE SYMMETRY ANALYSIS**

by

LETLHOGONOLO DADDY MOLELEKI (18045510)

Thesis submitted for the degree of Doctor of Philosophy in Applied
Mathematics at the Mafikeng Campus of the North-West University

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Supervisor: Professor C M Khalique

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Declaration

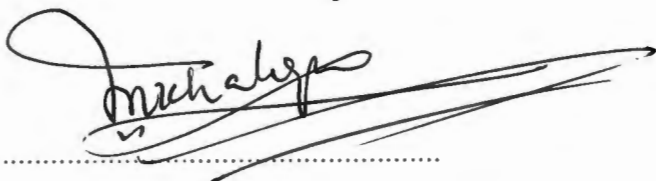
I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: 

MR LETLHOGONOLO DADDY MOLELEKI

Date: 19/03/2019

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed: 

PROF C.M. KHALIQUE

Date: 19 MAR 2019

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

L.D. Moleleki, C.M. Khalique, Solutions and conservation laws of a $(2+1)$ -dimensional Boussinesq equation, Abstract and Applied Analysis, Volume 2013, article ID 548975.

Chapter 3

L.D. Moleleki, C.M. Khalique, Solutions and conservation laws of a $(3+1)$ -dimensional Boussinesq equation, Advances in Mathematics Physics, Volume 2014, article ID 672679.

Chapter 4

L.D. Moleleki, C.M. Khalique, Solutions and conservation laws of a generalized $(3+1)$ -dimensional Kawahara equation, submitted to Open Physics

Chapter 5

L.D. Moleleki, I. Simbanefayi, C.M. Khalique, Exact solutions and conservation laws of a $(3+1)$ -dimensional KP-Boussinesq equation, submitted to Communications in Nonlinear Science and Numerical Simulation

Chapter 6

L.D. Moleleki, C.M. Khalique, Travelling wave solutions and conservation laws of a $(3+1)$ -dimensional BKP-Boussinesq equation, submitted to Mathematical Methods in the Applied Sciences

Chapter 7

L.D. Moleleki, C.M. Khalique, Exact solutions and conservation laws of the first generalized extended $(3+1)$ -dimensional Jimbo-Miwa equation, submitted to Physica A

Chapter 8

L.D. Moleleki, T. Motsepa, C.M. Khalique, Solutions and conservation laws of the generalized second extended (3+1)-dimensional Jimbo-Miwa equation, submitted to Results in Physics

Chapter 9

L.D. Moleleki, C.M. Khalique, Travelling wave solutions and conservation laws of the combined KdV-negative-order KdV equation, submitted to Optik

Chapter 10

L.D. Moleleki, C.M. Khalique, Lagrangian formulation of the Calogero-Bogoyavlenskii-Schiff equation, Proceedings of the International Conference on Numerical Analysis and Applied Mathematics 2018 (ICNAAM-2018), accepted and to appear in AIP Conference Proceedings of ICNAAM 2018

Dedication

I dedicate this work to my mother, Miss Lefi M Moleleki, and everyone who contributed to my studies. To my lovely wife, Mrs Selinah M Moleleki, and to my special baby boy, Refentse Letlhogonolo Junior Moleleki, and family.

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Abstract

In this thesis we study certain nonlinear multi-dimensional partial differential equations which are mathematical models of various physical phenomena of the real world. Closed-form solutions and conservation laws are obtained for such equations using various methods.

The multi-dimensional partial differential equations that are investigated in this thesis are $(2+1)$ and $(3+1)$ -dimensional Boussinesq equations, a generalized $(3+1)$ -dimensional Kawahara equation, a $(3+1)$ -dimensional KP-Boussinesq equation, a $(3+1)$ -dimensional BKP-Boussinesq equation, two extended $(3+1)$ -dimensional Jimbo-Miwa equations, the combined KdV-negative-order KdV equation and the Calogero-Bogoyavlenskii-Schiff equation.

Exact solutions of the $(2+1)$ -dimensional and $(3+1)$ -dimensional Boussinesq equations are obtained using the Lie symmetry method along with the simplest equation method. The solutions obtained are solitary waves and non-topological soliton. Conservation laws for both equations are constructed using the new conservation theorem due to Ibragimov.

Lie symmetry analysis together with Kudryashov's method is used to obtain travelling wave solutions for the generalized $(3+1)$ -dimensional Kawahara equation. Conservation laws are derived using the multiplier approach.

Lie symmetry method is employed to perform symmetry reductions on the $(3+1)$ -dimensional generalized KP-Boussinesq equation and thereafter Kudryashov's method is used to obtain exact solutions. Conservation laws are constructed using Ibragimov's theorem.

Exact solutions of the $(3+1)$ -dimensional BKP-Boussinesq equation are constructed using symmetry reductions and (G'/G) -expansion method. The new

conservation theorem is employed to obtain conservation laws.

Lie symmetry method together with the (G'/G) -expansion method and the simplest equation method are used to derive exact solutions of two generalized extended $(3 + 1)$ -dimensional Jimbo-Miwa equations. Conservation laws are constructed using Ibragimov's method.

The (G'/G) -expansion method is used to obtain travelling wave solutions of a combined KdV-negative-order KdV equation. Multiplier approach is employed to derive the conservation laws.

Noether's theorem is employed to construct conservation laws for the Calogero-Bogoyavlenskii-Schiff equation.

Introduction

Most natural phenomena of the real world are modelled by nonlinear partial differential equations (NLPDEs). Such equations can seldom be solved by an analytic method. In contrast the linear differential equations have a particularly good algebraic structure to their solutions, which makes them solvable. Unfortunately, for NLPDEs there is no general theory which can be applied to obtain exact closed-form solutions. However, scientists have developed geometric methods and dynamical systems theory which play prominent roles in the study of differential equations. Such theories deal with the long-term qualitative behaviour of dynamical systems and do not focus on finding precise solutions to the equations. Nevertheless, various methods have also been established by the researchers which provide exact solutions to NLPDEs.

Some of these methods are Hirota's bilinear transformation method [1], the inverse scattering method [2], the simplest equation method [3–5], the sine-cosine method [6], the tanh-coth method [7], Kudryashov's method [8, 9], the tanh-function method [10], the Darboux transformation [11], the (G'/G) -expansion method [12, 13], the Bäcklund transformation [14], and Lie symmetry methods [15–23].

Lie symmetry theory, originally developed by Marius Sophus Lie (1842-1899), a Norwegian mathematician, around the middle of the nineteenth century, is based

upon the study of the invariance under one parameter Lie group of point transformations [15–23]. The theory is highly algorithmic and is one of the most powerful methods to find exact solutions of differential equations be it linear or nonlinear. It has been applied to many scientific fields such as classical mechanics, relativity, control theory, quantum mechanics, numerical analysis, to name but a few.

Conservation laws can be described as fundamental laws of nature, which have extensive applications in various fields of scientific study such as physics, chemistry, biology, engineering, and so on. They have many uses in the study of differential equations [24–34]. Conservation laws have been used to prove global existence theorems and shock wave solutions to hyperbolic systems. They have been applied to problems of stability and have been used in scattering theory and elasticity [18]. Comparison of several different methods for computing conservation laws can be found in [32].

This thesis is structured as follows:

In Chapter one we present preliminaries on Lie symmetry analysis and conservation laws of partial differential equations. Also some methods for finding exact solutions of differential equations are given that will be needed in our study.

In Chapter two Lie symmetries as well as the simplest equation method is used to obtain exact solutions of the $(2+1)$ -dimensional Boussinesq equation. Moreover, conservation laws are derived by using the new conservation theorem due to Ibragimov.

Chapter three presents exact solutions of the $(3+1)$ -dimensional Boussinesq equation with the aid of Lie point symmetries as well as the simplest equation method. Furthermore, the conservation laws for the equation are constructed by utilizing the new conservation theorem due to Ibragimov.

In Chapter four exact solutions of a nonlinear evolution partial differential equa-

tion, namely the generalized (3+1)-dimensional Kawahara equation are obtained with the aid of Lie symmetries in conjunction with the Kudryashov's method. Moreover, the conservation laws for this equation are derived by using the multiplier method.

Chapter five studies the exact solutions of the (3+1)-dimensional generalized KP-Boussinesq using symmetry reductions and Kudryashov's method. Furthermore, conservation laws for the equation are derived using Ibragimov's conservation theorem.

In Chapter six exact solutions for the (3+1)-dimensional BKP-Boussinesq equation are obtained with the aid of Lie symmetry reductions, direct integration as well as the (G'/G) -expansion method. Thereafter we construct conservation laws by employing Ibragimov's conservation theorem.

Chapter seven and eight study the exact solutions of two generalized extended (3+1)-dimensional Jimbo-Miwa equation using symmetry reductions of the equations along with direct integration, the (G'/G) -expansion and simplest equation methods. Also conservation laws were computed for both equations by invoking the conservation theorem due to Ibragimov.

In Chapter nine we use the (G'/G) -expansion method to find exact solutions of a combined KdV-negative-order KdV equation and derive conservation laws using the multiplier method.

Chapter ten deals with obtaining the conservation laws for the Calogero-Bogoyavlenskii-Schiff equation using Noether's theorem. Noether point symmetries are first computed and then Noether's theorem is used to derive the associated conserved vectors.

Finally, in Chapter eleven a summary of the results of the thesis are presented and future work is deliberated.

Bibliography is given at the end.

Chapter 1

Preliminaries

In this chapter we give some basic methods of Lie symmetry analysis and conservation laws of partial differential equations (PDEs). We also present some methods for obtaining exact solutions of differential equations, which will be used in this thesis.

1.1 Introduction

Sophus Lie (1842-1899) was one of the most important mathematicians of the nineteenth century. He realised that many of the methods for solving differential equations could be unified using group theory and further developed a symmetry-based approach to obtaining exact solutions of differential equations. Symmetry methods have great power and generality. In fact, nearly all well-known techniques for solving differential equations are special cases of Lie's methods. Recently, many good books have appeared in the literature in this field. We mention a few here, Ovsiannikov [15], Stephani [16], Bluman and Kumei [17], Olver [18], Ibragimov [19–21], Cantwell [22] and Mahomed [23]. Definitions and results given in this

chapter are taken from these books.

Conservation laws for PDEs are constructed using three different approaches; the multiplier method [24], the new conservation theorem due to Ibragimov [35] and Noether's theorem [36].

1.2 Continuous one-parameter groups

Let $x = (x^1, \dots, x^n)$ be the independent variables with coordinates x^i and $u = (u^1, \dots, u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a :

$$T_a : \bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \phi^\alpha(x, u, a), \quad (1.1)$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

- (i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$
(Closure)
- (ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)
- (iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that
 $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\bar{\bar{x}}^i \equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)),$$

$$\bar{u}^\alpha \equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \quad (1.2)$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

1.3 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \dots, \quad (1.3)$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n \quad (1.4)$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, \dots, m, \quad i, j = 1, \dots, n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\dots$.

In group analysis all variables $x, u, u_{(1)} \dots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [19].

We now consider a p th-order PDE(s), namely

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(p)}) = 0. \quad (1.5)$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\begin{aligned} \bar{x}^i &= f^i(x, u, a), \quad f^i|_{a=0} = x^i, \\ \bar{u}^\alpha &= \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \end{aligned} \quad (1.6)$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x}^i \approx x^i + a \xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a \eta^\alpha(x, u) \quad (1.7)$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left. \frac{\partial f^i}{\partial a} \right|_{a=0}, \quad \eta^\alpha(x, u) = \left. \frac{\partial \phi^\alpha}{\partial a} \right|_{a=0}. \quad (1.8)$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a X)x, \quad \bar{u}^\alpha \approx (1 + a X)u,$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (1.9)$$

This differential operator X is known as the infinitesimal operator or generator of the group G . If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j) \bar{D}_j, \quad (1.10)$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \dots$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j) \bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j) \bar{u}_j^\alpha. \end{aligned} \quad (1.11)$$

This

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta} \right) \bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \quad (1.12)$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, a for small a , ie., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi_i^\alpha|_{a=0} = u_i^\alpha. \quad (1.13)$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \quad (1.14)$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G , viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\ (\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\ u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (\text{sum on } j). \end{aligned} \quad (1.15)$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad (\text{sum on } k). \quad (1.16)$$

By induction (recursively)

$$\zeta_{i_1, i_2, \dots, i_p}^\alpha = D_{i_p}(\zeta_{i_1, i_2, \dots, i_{p-1}}^\alpha) - u_{i_1, i_2, \dots, i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad (\text{sum on } j). \quad (1.17)$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \dots, G^{[p]}$. The corresponding prolonged generators are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad (\text{sum on } i, \alpha),$$

$$X^{[p]} = X^{[p-1]} + \zeta_{i_1, \dots, i_p}^\alpha \frac{\partial}{\partial u_{i_1, \dots, i_p}^\alpha} \quad p \geq 1,$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (1.18)$$

is a *point symmetry* of the p th-order PDE (1.5), if

$$X^{[p]}(E_\alpha) = 0 \quad (1.19)$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]} E_\alpha|_{E_\alpha=0} = 0, \quad (1.20)$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant

(has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, u_{(1)}^-, \dots, u_{(p)}^-) = 0, \quad (1.21)$$

where the function E_α is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \quad (1.22)$$

identically in x, u and a .

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X F \equiv \xi^i(x, u) \frac{\partial F}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial F}{\partial u^\alpha} = 0. \quad (1.23)$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n - 1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \dots, J_{n-1}(x, u) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \dots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \dots = \frac{du^n}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \quad (1.24)$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x, \quad \bar{u}^\alpha|_{a=0} = u.$$

1.6 Conservation laws

Conservation laws can be described as fundamental laws of nature, which have extensive applications in various fields of scientific study such as physics, chemistry, biology, engineering. They have many uses in the study of differential equations [24–34]. Conservation laws have been used to prove global existence theorems and shock wave solutions to hyperbolic systems. They have been applied to problems of stability and have been used in scattering theory and elasticity [18]. In [32] a comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics was presented.

1.6.1 Fundamental operators and their relationship

Consider a k th-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, namely,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m. \quad (1.25)$$

Here $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, ..., k th-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$, ..., respectively, with the total derivative operator with respect to x^i defined by [19]

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n.$$

The Euler-Lagrange operator, for each α , is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \quad (1.26)$$

and the Lie-Bäcklund operator operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \quad (1.27)$$

where $\mathcal{A}$ is the space of differential functions. The operator (1.27) can be written in terms of Lie characteristic function as

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad (1.28)$$

where

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{ji_1 \dots i_s}^\alpha, \quad s > 1 \end{aligned}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^i u_i^\alpha.$$

The Lie-Bäcklund operator (1.28) in characteristic form can be written as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}$$

and the Noether operators associated with the Lie-Bäcklund symmetry operator X are defined as

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n,$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.26) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \dots D_{j_s} \frac{\partial}{\partial u_{i j_1 j_2 \dots j_s}^\alpha}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m,$$

and the Euler-Lagrange, Euler-Lagrange and Noether operators are connected by the operator identity [19]

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i.$$

The n -tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a conserved vector of (1.25) if T^i satisfies

$$D_i T^i|_{(1.25)} = 0,$$

which defines a local conservation law of system (1.25).

1.6.2 The new conservation theorem due to Ibragimov

Consider the k th-order system of PDEs (1.25). The system of adjoint equations to (1.25) is defined by [35]

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (1.29)$$

where

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \dots, m, \quad v = v(x) \quad (1.30)$$

and $v = (v^1, v^2, \dots, v^m)$ are new dependent variables.

The system of equations (1.25) is known as self-adjoint if the substitution of $v = u$ into the system of adjoint equations (1.29) yields the same system (1.25).

Let us now assume the system of equations (1.25) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \quad (1.31)$$

Then the system of adjoint equations (1.29) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}, \quad \eta_*^\alpha = -[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)], \quad (1.32)$$

where the operator (1.32) is an extension of (1.31) to the variable v^α and the λ_β^α are obtainable from

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \quad (1.33)$$

We now state the following theorem:

Theorem 3.1. [35] Every Lie point, Euler-Lagrange and non local symmetry (1.31) admitted by the system of equations (1.25) gives rise to a conservation law for the system consisting of the equation (1.25) and the adjoint equation (1.29), where the components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s} (W^\alpha) \frac{\delta L}{\delta u_{i_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (1.34)$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x, u, \dots, u_{(k)}). \quad (1.35)$$

1.6.3 Multiplier method

The multiplier approach is an effective algorithmic for finding the conservation laws for partial differential equations with any number of independent and dependent variables. Authors in [24] gave this algorithm by using the multipliers presented in [18]. A local conservation law of a given differential system arises from a linear combination formed by local multipliers (characteristics) with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the given differential equation system.

The advantage of this approach is that it does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that [24]

$$\Lambda_\alpha E_\alpha = D_i T^i \quad (1.36)$$

holds identically. The right hand side of (1.36) is a divergence expression. The

determining equations for the multiplier Λ_α is [24]

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \quad (1.37)$$

1.6.4 Noether's theorem

It is known that for any PDE the conservation laws admitted by the PDE can be derived by a direct computational method [24]. This method is similar to Lie's method for determining the symmetries admitted by the PDE. However, when a PDE has a Lagrangian formulation, the celebrated Noether's theorem [36–38] provides a sophisticated and useful way of determining conservation laws. Certainly it gives a clear formula for finding a conservation law once a Noether symmetry associated with a Lagrangian is known for an Euler-Lagrange equation.

Definition 1.6 (Noether symmetry) A Lie-Bäcklund operator X of the form (1.27) is called a Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$, if there exists a vector $B^i = (B^1, \dots, B^n)$, $B^i \in \mathcal{A}$ such that

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i) \quad (1.38)$$

if $B^i = 0$ ($i = 1, \dots, n$), then X is called a Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$.

Theorem 1.4 (Noether Theorem) For any Noether symmetry generator X associated with a given Lagrangian $\mathcal{L} \in \mathcal{A}$, there corresponds a vector

$T = (T^1, \dots, T^n)$, $T^i \in \mathcal{A}$, given by

$$T^i = N^i(\mathcal{L}) - B^i, \quad i = 1, \dots, n, \quad (1.39)$$

which is a conserved vector of the Euler-Lagrange differential equations $\delta\mathcal{L}/\delta u^\alpha = 0$, $\alpha = 1, \dots, m$.

In the Noether approach, we find the Lagrangian $\mathcal{L}$ and then equation (1.38) is used to determine the Noether symmetries. Then, equation (1.39) will yield the corresponding Noether conserved vectors.

1.7 Exact solutions

In this section we present some solution methods which will be used in this thesis to determine exact/closed-form solutions of differential equations.

1.7.1 The simplest equation method

We first present the simplest equation method for finding exact solutions of nonlinear partial differential equations [3–5]. This method has been used successfully by many researchers to find exact solutions of PDEs in various fields of applied sciences.

We now describe this method briefly.

Consider the nonlinear partial differential equation

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{yy} \dots) = 0. \quad (1.40)$$

Using the transformation

$$u(t, x, y) = F(z), \quad z = k_1 t + k_2 x + k_3 y + k_4, \quad (1.41)$$

where $k_1, \dots, k_4$ are arbitrary constants, we reduce equation (1.40) to an ordinary differential equation

$$E_2[F(z), k_1 F'(z), k_2 F'(z), k_3 F'(z), k_1^2 F''(z), k_2^2 F''(z), k_3^2 F''(z), \dots] = 0. \quad (1.42)$$

The simplest equations that we use in this method are the Bernoulli equation

$$G'(z) = aG(z) + bG^2(z) \quad (1.43)$$

and the Riccati equation

$$G'(z) = aG^2(z) + bG(z) + c, \quad (1.44)$$

where a, b and c are constants. We look for solutions of equation (1.42) that are of the form

$$F(z) = \sum_{i=0}^M A_i (G(z))^i, \quad (1.45)$$

where $G(z)$ satisfies the Bernoulli or Riccati equation. Here M is a positive integer that is determined by the balancing procedure and $A_0, \dots, A_M$ are parameters to be determined.

The solution of Bernoulli equation (1.43) is

$$G(z) = a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\},$$

where C is a constant of integration. For the Riccati equation (1.44), we use the solutions

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \quad (1.46)$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \quad (1.47)$$

with $\theta = \sqrt{b^2 - 4ac}$ and C is a constant of integration.

1.7.2 Kudryashov's method

In this section we present Kudryashov's method for finding exact solutions of nonlinear partial differential equations, which has been described in [8].

We now recall this method and give its description. Suppose we have a nonlinear partial differential equation for $u(t, x)$, in the form

$$E_1[u, u_t, u_x, u_{tt}, u_{xx}, \dots] = 0, \quad (1.48)$$

where E_1 is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The transformation

$$u(t, x) = F(p), \quad p = k_1 x + k_2 t \quad (1.49)$$

reduces equation (1.48) to the nonlinear ordinary differential equation

$$E_2[F(p), k_1 F'(p), k_2 F'(p), k_1 F''(p), k_2 F''(p), \dots] = 0. \quad (1.50)$$

We assume that the solution of equation (1.50) can be expressed as

$$F(p) = \sum_{i=0}^M A_i (H(p))^i, \quad (1.51)$$

where

$$H(p) = \frac{1}{1 + \cosh(p) + \sinh(p)} = \frac{1}{1 + \exp(p)} \quad (1.52)$$

satisfies the equation

$$H'(p) = H^2(p) - H(p) \quad (1.53)$$

and M is the positive integer found by the balancing procedure and $A_0, \dots, A_M$ are parameters to be determined.

We then substitute the function $F(p)$ into the ODE (1.50) and use equation (1.53). Equating coefficients of different powers of H to zero we obtain a system of algebraic equations in A_i . Solving these algebraic equations yields the values of the parameters A_i .

1.7.3 The (G'/G) -expansion method

The (G'/G) -expansion method for finding exact solutions of nonlinear partial differential equations was introduced in [12]. The description of this method is as follows:

Consider a nonlinear partial differential equation, say, in two independent variables t and x , given by

$$P(u, u_x, u_t, u_{tt}, u_{xt}, u_{xx} \cdots) = 0, \quad (1.54)$$

where $u(t, x)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. As a first step we use the transformation $u(t, x, y) = F(z)$, $z = k_1 t + k_2 x + k_3 y + k_4$ to reduce equation (1.54) to the ordinary differential equation, say

$$E_2[F(z), k_1 F'(z), k_2 F'(z), k_3 F'(z), k_1^2 F''(z), k_2^2 F''(z), k_3^2 F''(z), \cdots] = 0. \quad (1.55)$$

We assume that the solution of (1.55) can be expressed by a polynomial in (G'/G) as follows:

$$U(z) = \sum_{i=0}^m \alpha_i \left(\frac{G'}{G} \right)^i, \quad (1.56)$$

where $G = G(z)$ satisfies the second-order linear ordinary differential equation

$$G'' + \lambda G' + \mu G = 0, \quad (1.57)$$

with α_i , $i = 0, 1, 2, \cdots, m$, λ and μ are constants to be determined. The positive integer m is determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ordinary differential equation (1.55).

By substituting (1.56) into (1.55) and using the second-order ordinary differential equation (1.57), collecting all terms with same order of (G'/G) together, the left-hand side of (1.55) is converted into another polynomial in (G'/G) . Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_0, \cdots, \alpha_m, \nu, \lambda, \mu$.

Finally, assuming that the constants can be obtained by solving the above algebraic equations, since the general solution of (1.57) is known, then substituting the constants and the general solutions of (1.57) into (1.56) we obtain travelling wave solutions of the nonlinear partial differential equation (1.54).

1.8 Conclusion

In this chapter we presented a brief introduction to the Lie symmetry analysis and conservation laws of PDEs and presented some results which will be used throughout this thesis. We also presented the algorithm to determine the Lie point symmetries and conservation laws of PDEs. We also recalled certain methods that were used to determine the exact solutions which will be studied in this work.

Chapter 2

Solutions and conservation laws of a (2+1)-dimensional Boussinesq equation

2.1 Introduction

In this chapter we consider the (2+1)-dimensional Boussinesq equation given by

$$u_{tt} - u_{xx} - u_{yy} - \alpha(u^2)_{xx} - u_{xxxx} = 0, \quad (2.1)$$

which describes the propagation of gravity waves on the surface of water, in particular it describes the head-on collision of an oblique wave. In [39] the authors used a generalized transformation in homogeneous balance method and found some explicit solitary wave solutions of the (2+1)-dimensional Boussinesq equation. Applied homotopy perturbation method was used in [40] to construct numerical solutions of (2.1). Extended ansatz method was employed in [41] to derive exact periodic solitary wave solutions. Recently, the Hirota bilinear method was used in [42] to obtain two soliton solutions.

Here Lie group analysis in conjunction with the simplest equation method [3, 5] is employed to obtain some exact solutions of (2.1). In addition to this, conservation laws will be derived for (2.1) using the new conservation theorem due to Ibragimov [35].

This work has been published. See [43].

2.2 Solutions of (2.1)

In this section we obtain exact solutions of (2.1) using Lie group analysis along with the simplest equation method.

2.2.1 Exact solutions using Lie point symmetries

We first calculate the Lie point symmetries of (2.1) and latter use the translation symmetries to construct the exact solutions.

Lie point symmetries

The symmetry group of the (2+1)-dimensional Boussinesq equation (2.1) will be generated by the vector field of the form

$$R = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u},$$

where ξ^i , $i = 1, 2, 3$ and η depend on x , y , t and u . Applying the fourth prolongation $R^{[4]}$ to (2.1) we obtain an overdetermined system of linear partial differential equations. Solving this resultant system one obtains the following five Lie point symmetries:

$$R_1 = \frac{\partial}{\partial x}$$

$$\begin{aligned}
R_2 &= \frac{\partial}{\partial t} \\
R_3 &= \frac{\partial}{\partial y} \\
R_4 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \\
\text{and } R_5 &= -2\alpha t \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} + (1 + 2\alpha u) \frac{\partial}{\partial u}.
\end{aligned}$$

We now utilize the symmetry $R = R_1 + R_2 + cR_3$, where c is a constant, and reduces the Boussinesq equation (2.1) to a PDE in two independent variables. Solving the associated Lagrange system for R , we obtain the following three invariants:

$$f = y - ct, \quad g = t - x, \quad \theta = u. \quad (2.2)$$

Now treating θ as the new dependent variable and f and g as new independent variables, the Boussinesq equation (2.1) transforms to

$$(1 - c^2)\theta_{ff} + 2c\theta_{fg} + 2\alpha\theta_g^2 + 2\alpha\theta\theta_{gg} + \theta_{gggg} = 0, \quad (2.3)$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (2.3) and transform it to an ordinary differential equation (ODE).

The equation (2.3) has the following three symmetries:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial g}, \\
\Gamma_2 &= \frac{\partial}{\partial f}, \\
\Gamma_3 &= (2\alpha f - 2\alpha f c^2) \frac{\partial}{\partial f} + (\alpha c f - c^2 \alpha g + \alpha g) \frac{\partial}{\partial g} + (c^2 + 2c^2 \alpha \theta - 2\alpha \theta) \frac{\partial}{\partial \theta}.
\end{aligned}$$

The combination of the first two translational symmetries, $\Gamma = \Gamma_1 + \nu\Gamma_2$, where ν is a constant, yields the two invariants

$$z = f - \nu g, \quad \psi = \theta,$$

which give rise to a group invariant solution $\psi = \psi(z)$ and consequently using these invariants, (2.3) is transformed into the fourth-order nonlinear ODE

$$2\alpha\nu^2\psi'^2 + (1 - c^2 - 2c\nu)\psi'' + 2\alpha\nu^2\psi\psi'' + \nu^4\psi'''' = 0. \quad (2.4)$$

Integrating the above equation four times and taking the constants of integration to be zero (because we are looking for soliton solutions) and reverting back to the original variables, we obtain the following group-invariant solutions of the Boussinesq equation (2.1):

$$u(x, y, t) = \frac{A_1}{A_2} \text{sech}^2 \left[\frac{\sqrt{A_1}}{2} (B \pm z) \right], \quad (2.5)$$

where B is a constant of integration and

$$\begin{aligned} A_1 &= \frac{c^2 + 2\nu c - 1}{\nu^4}, \\ A_2 &= \frac{2\alpha}{3\nu^2}, \\ z &= \nu x + y - (c + \nu)t. \end{aligned}$$

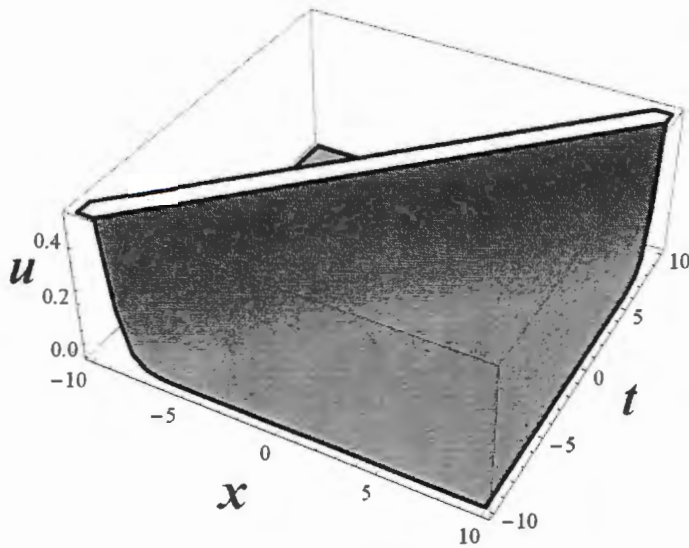


Figure 2.1: Profile of solution (2.5)

2.2.2 Exact solutions of (2.1) using simplest equation method

In this section we employ the simplest equation method [3,5] to solve the nonlinear ODE (2.4). This will then give us the exact solutions for our Boussinesq equation (2.1). The simplest equations that we will use in our work are the Bernoulli and Riccati equations.

Solutions of (2.1) using the Bernoulli equation as the simplest equation

The balancing procedure gives $M = 2$ so the solutions of (2.4) are of the form

$$F(z) = A_0 + A_1G + A_2G^2. \quad (2.6)$$

Inserting (2.6) into (2.4) and using the Bernoulli equation (1.43) and thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of

six equations in terms of A_0, A_1, A_2 , namely

$$\begin{aligned}
& -120\nu^4 A_2 b^4 - 20\alpha\nu^2 A_2^2 b^2 = 0, \\
& -336\nu^4 A_2 a b^3 - 36\alpha\nu^2 A_2^2 a b - 24\nu^4 A_1 b^4 - 24\alpha\nu^2 A_1 A_2 b^2 = 0, \\
& -A_1 a^2 + 2\nu A_1 a^2 c - \nu A_1 a^4 + A_1 a^2 c^2 - 2\alpha\nu^2 A_1 A_2 a^2 = 0, \\
& -16\alpha\nu^2 A_2^2 a^2 + 12\nu A_2 b^2 c - 6\alpha\nu^2 A_1^2 b^2 - 60\nu^4 A_1 a b^3 - 12\alpha\nu^2 A_0 A_1 b^2 \\
& \quad -330\nu^4 A_2 a^2 b^2 - 42\alpha\nu^2 A_1 a b A_2 + 6A_2 b^2 c^2 - 6A_2 b^2 = 0, \\
& -15\nu^4 A_1 a^3 b + 8\nu A_2 a^2 c - 3A_1 a b - 4A_2 a^2 + 4A_2 a^2 c^2 - 4\alpha\nu^2 A_1^2 a^2 - 6\alpha\nu^2 A_0 A_1 a b \\
& \quad -16\nu^4 A_2 a^4 + 6A_1 a b c \nu - 8\alpha\nu^2 A_0 A_2 a^2 + 3A_1 a b c^2 = 0, \\
& -18\alpha\nu^2 A_1 A_2 a^2 - 10\alpha\nu^2 A_1^2 a b - 4\alpha\nu^2 A_0 A_1 b^2 + 10A_2 a b c^2 + 4\nu A_1 b^2 c + 20\nu A_2 a b c \\
& \quad -2A_1 b^2 + 2A_1 b^2 c^2 - 20\alpha\nu^2 A_0 A_2 a b - 10A_2 a b - 130\nu^4 A_2 a^3 b - 50\nu^4 A_1 a^2 b^2 = 0.
\end{aligned}$$

With the aid of Mathematica, solving the above system of algebraic equations, one possible solution for A_0, A_1 and A_2 is

$$\begin{aligned}
A_0 &= \frac{-(1 - c^2 - 2c\nu + a^2\nu^4)}{2\alpha\nu^2}, \\
A_1 &= \frac{-6ab\nu^2}{\alpha}, \\
A_2 &= \frac{-6b^2\nu^2}{\alpha}.
\end{aligned}$$

Thus, reverting back to the original variables, a solution of (2.1) is

$$\begin{aligned}
u(t, x, y) &= A_0 + A_1 a \left\{ \frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b \cosh[a(z + C)] - b \sinh[a(z + C)]} \right\} + \\
& A_2 a^2 \left\{ \frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b \cosh[a(z + C)] - b \sinh[a(z + C)]} \right\}^2, \tag{2.7}
\end{aligned}$$

where $z = \nu x + y - (c + \nu)t$ and C is an arbitrary constant of integration.

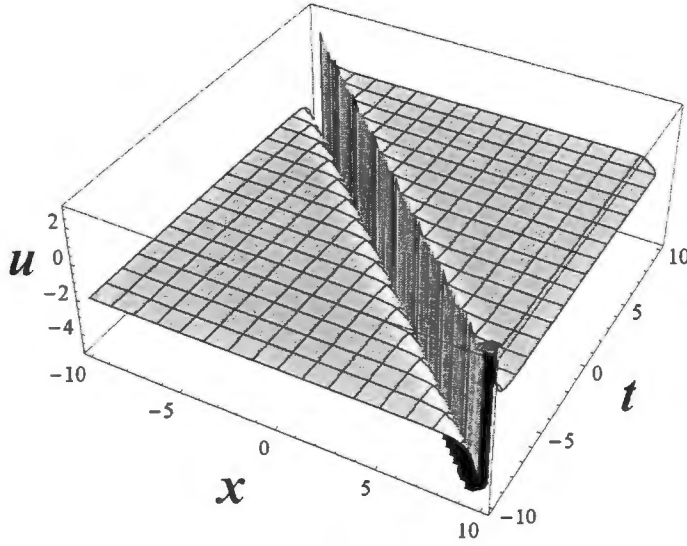


Figure 2.2: Profile of solution (2.7)

Solutions of (2.1) using the Riccati equation as the simplest equation

The balancing procedure yields $M = 2$ so the solutions of (2.4) takes the form

$$F(z) = A_0 + A_1 G + A_2 G^2. \quad (2.8)$$

Inserting (2.8) into (2.4) and making use of the Riccati equation (1.44), we obtain algebraic system of equations in terms of A_0 , A_1 and A_2 by equating the coefficients of powers of G^i to zero. The resulting algebraic equations are

$$\begin{aligned} -120\nu^4 A_2 b^4 - 20\alpha\nu^2 A_2^2 b^2 &= 0, \\ -36\alpha\nu^2 A_2^2 ab - 336\nu^4 A_2 ab^3 - 24\nu^4 A_1 b^4 - 24\alpha\nu^2 A_1 A_2 b^2 &= 0, \\ -32\alpha\nu^2 A_2^2 bd - 6\alpha\nu^2 A_1^2 b^2 - 240\nu^4 A_2 b^3 d + 6A_2 b^2 c^2 + 12A_2 b^2 c\nu - 6A_2 b^2 \\ -12\alpha\nu^2 A_0 A_1 b^2 - 42\alpha\nu^2 A_1 A_2 ab - 16\alpha\nu^2 A_2^2 a^2 - 60\nu^4 A_1 ab^3 - 330A_2 a^2 b^2 &= 0, \\ -16\nu^4 A_2 bd^3 - 14\nu^4 A_2 a^2 d^2 + 2A_1 acd\nu - A_1 ad + 4A_2 cd^2\nu + A_1 ac^2 d - 8\nu^4 A_1 abd^2 \\ + 2A_2 c^2 d^2 - \nu^4 A_1 a^3 d - 2\alpha\nu^2 A_0 A_1 ad - 4\alpha\nu^2 A_0 A_2 d^2 - 2\alpha\nu^2 A_1^2 d^2 - 2A_2 d^2 &= 0, \end{aligned}$$

$$\begin{aligned}
& 2A_1b^2c^2 - 28\alpha\nu^2A_2^2ad - 20\alpha\nu^2A_0A_2ab - 36\alpha\nu^2A_1A_2bd - 18\alpha\nu^2A_1A_2a^2 \\
& -10\alpha\nu^2A_1^2ab - 10A_2ab + 10A_2abc^2 + 4A_1b^2c\nu - 40\nu^4A_1b^3d - 4\alpha\nu^2A_0A_1b^2 \\
& -50\nu^4A_1a^2b^2 + 20A_2abc\nu - 2A_1b^2 - 130\nu^4A_2a^3b - 440\nu^4A_2ab^2d = 0, \\
& 2A_1a^2c\nu - 6A_2ad - \nu^4A_1a^4 + 12A_2acd\nu - 6\alpha\nu^2A_1^2ad + 6A_2ac^2d + A_1a^2c^2 \\
& -12\alpha\nu^2A_1A_2d^2 - 4\alpha\nu^2A_0A_1bd - 120\nu^4A_2abd^2 + 4A_1bcd\nu - 2\alpha\nu^2A_0A_1a^2 \\
& -12\alpha\nu^2A_0A_2ad - 16\nu^4A_1b^2d^2 - A_1a^2 - 30\nu^4A_2a^3d - 2A_1bd + 2A_1bc^2d \\
& -22\nu^4A_1a^2bd = 0, \\
& -8\alpha\nu^2A_0A_2a^2 + 3A_1abc^2 - 8A_2bd + 6A_1abc\nu - 3A_1ab - 6\alpha\nu^2A_0A_1ab \\
& -136\nu^4A_2b^2d^2 - 4A_2a^2 - 12\alpha\nu^2A_2^2d^2 + 8A_2a^2c\nu - 16\alpha\nu^2A_0A_2bd - 232\nu^4A_2a^2bd \\
& -8\alpha\nu^2A_1^2bd + 16A_2bcd\nu - 15\nu^4A_1a^3b - 16\nu^4A_2a^4 - 60\nu^4A_1ab^2d + 8A_2bc^2d \\
& +4A_2a^2c^2 - 30\alpha\nu^2A_1A_2ad - 4\alpha\nu^2A_1^2a^2 = 0.
\end{aligned}$$

Solving the above equations, we get

$$\begin{aligned}
A_0 &= \frac{-8bd\nu^4 - a^2\nu^4 + c^2 + 2c\nu - 1}{2\nu^2\alpha}, \\
A_1 &= \frac{-6ab\nu^2}{\alpha}, \\
A_2 &= \frac{-6b^2\nu^2}{\alpha}
\end{aligned}$$

and consequently, the solutions of (2.1) are

$$\begin{aligned}
u(t, x, y) &= A_0 + A_1 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left[\frac{1}{2}\theta(z + C) \right] \right\} + \\
& A_2 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left[\frac{1}{2}\theta(z + C) \right] \right\}^2
\end{aligned} \tag{2.9}$$

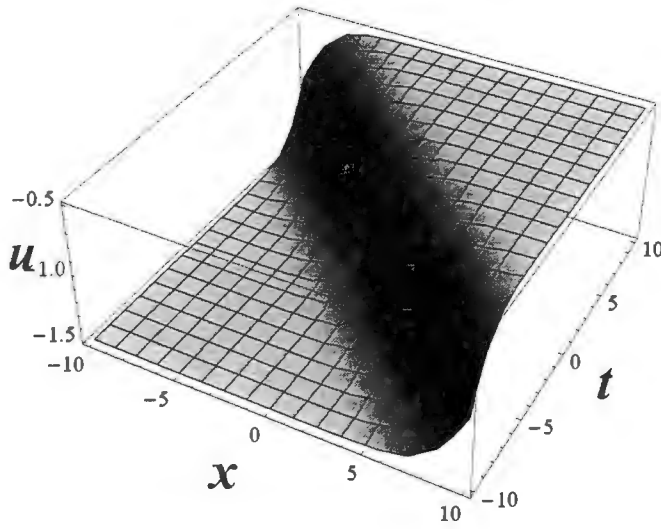


Figure 2.3: Profile of solution (2.9)

and

$$\begin{aligned}
 u(t, x, y) = & A_0 + A_1 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2b}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\} \\
 & + A_2 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left(\frac{1}{2} \theta z \right) + \right. \\
 & \left. \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2b}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\}^2, \tag{2.10}
 \end{aligned}$$

where $z = \nu x + y - (c + \nu)t$ and C is an arbitrary constant of integration.

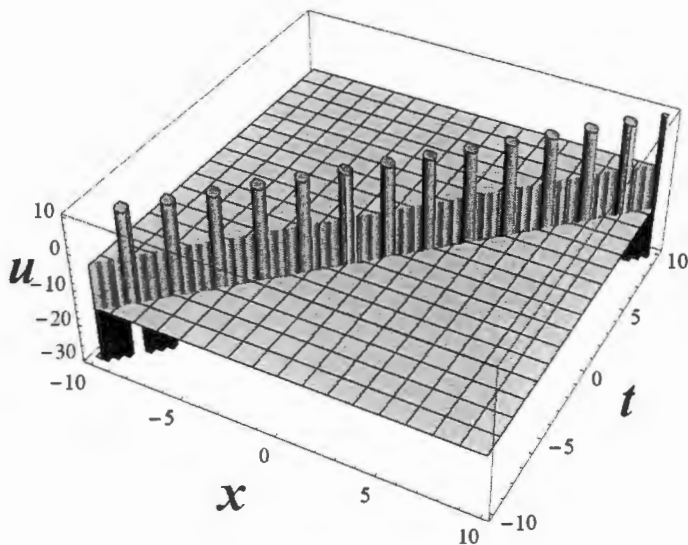


Figure 2.4: Profile of solution (2.10)

2.3 Conservation laws

In this subsection, we obtain conservation laws of (2+1)-dimensional Boussinesq equation

$$u_{tt} - u_{xx} - u_{yy} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx} = 0. \quad (2.11)$$

Recall that the equation (2.11) admits the following five Lie point symmetry generators:

$$\begin{aligned} R_1 &= \frac{\partial}{\partial x} \\ R_2 &= \frac{\partial}{\partial t} \\ R_3 &= \frac{\partial}{\partial y} \\ R_4 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \end{aligned}$$

$$\text{and } R_5 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} + (1 + 2\alpha u) \frac{\partial}{\partial u}.$$

We now find five conserved vectors corresponding to each of these five Lie point symmetries.

The adjoint equation of (2.11), by invoking (1.30), is

$$E^*(t, x, u, v, \dots, u_{xxxx}, v_{xxxx}) = \frac{\delta}{\delta u} \left[v(u_{tt} - u_{xx} - u_{yy} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx}) \right] = 0, \quad (2.12)$$

where $v = v(t, x, y)$ is a new dependent variable and (3.13) gives

$$v_{tt} - v_{xx} - v_{yy} - 2\alpha u v_{xx} - v_{xxxx} = 0. \quad (2.13)$$

It is obvious from the adjoint equation (2.13) that equation (2.11) is not self-adjoint. By recalling (1.35), we get the following Lagrangian for the system of equations (2.11) and (2.13):

$$L = v(u_{tt} - u_{xx} - u_{yy} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx}). \quad (2.14)$$

(i) We first consider the Lie point symmetry generator $R_1 = \partial/\partial x$. It can be verified from (1.32) that the operator Y_1 is the same as R_1 and the Lie characteristic function is $W = -u_x$. Thus, by using (1.34), the components T^i , $i = 1, 2, 3$, of the conserved vector $T = (T^1, T^2, T^3)$ are given by

$$\begin{aligned} T^1 &= u_x v_t - v u_{tx}, \\ T^2 &= v u_{tt} - v u_{yy} - u_x v_x - 2\alpha u u_x v_x - u_x v_{xxx} + v_{xx} u_{xx} - v_x u_{xxx}, \\ T^3 &= -u_x v_y + v u_{xy}. \end{aligned}$$

Remark. The conserved vector T contains the arbitrary solution v of the adjoint equation (2.13) and hence gives an infinite number of conservation laws.

The same remark applies to all the following four cases.

(ii) Now for the second symmetry generator $R_2 = \partial/\partial t$, we have $W = -u_t$. Hence, by invoking (1.34), the symmetry generator R_2 gives rise to the following components of the conserved vector:

$$\begin{aligned} T^1 &= -vu_{xx} - vu_{yy} - 2\alpha vu_x^2 - 2\alpha uvu_{xx} - vu_{xxxx} + u_tv_t, \\ T^2 &= -u_tv_x + 2\alpha vu_tu_x - 2\alpha uu_tv_x - u_tv_{xxx} + vu_{tx} + 2\alpha uvu_{tx} + v_{xx}u_{tx} - v_xu_{txx} \\ &\quad + vu_{txxx}, \\ T^3 &= -v_yu_t + vu_{ty} \end{aligned}$$

(iii) The third symmetry generator $R_3 = \partial/\partial y$, gives $W = -u_y$ and the corresponding components of the conserved vector are

$$\begin{aligned} T^1 &= v_tu_y - vu_{ty}, \\ T^2 &= -u_yv_x + 2\alpha vu_yu_x - 2\alpha uu_yv_x - u_yv_{xxx} + vu_{xy} + 2\alpha uvu_{xy} + v_{xx}u_{xy} - v_xu_{xxy} \\ &\quad + vu_{xxyy}, \\ T^3 &= vu_{tt} - vu_{xx} - 2\alpha vu_x^2 - 2\alpha uvu_{xx} - vu_{xxxx} - u_yv_y. \end{aligned}$$

(iv) For the symmetry generator $R_4 = y\partial/\partial t + t\partial/\partial y$ the components of the conserved vector, as before, are given by

$$\begin{aligned} T^1 &= -yvu_{xx} - yvu_{yy} - 2\alpha yvu_x^2 - 2\alpha yuvu_{xx} - yvu_{xxxx} + yu_tv_t + tu_yv_t \\ &\quad - vu_y - tvu_{ty}, \\ T^2 &= -yu_tv_x + 2\alpha yvu_tu_x - 2\alpha yuu_tv_x - yu_tv_{xxx} - tv_xu_y + 2\alpha tvu_yu_x - 2\alpha tvv_xu_y \\ &\quad - tu_yv_{xxx} + yvu_{tx} + 2\alpha yuvu_{tx} + yu_{tx}v_{xx} + tvu_{xy} + 2\alpha tvuvu_{xy} + tu_{xy}v_{xx} \\ &\quad - yv_xu_{txx} - tv_xu_{xxy} + yvu_{txxx} + tvu_{xxyy}, \\ T^3 &= tvu_{tt} - tvu_{xx} - 2\alpha tvu_x^2 - 2\alpha tvuvu_{xx} - tvu_{xxxx} - yv_yu_t - tv_yu_y + vu_t + yvu_{ty}. \end{aligned}$$

(v) Finally, for the symmetry generator

$$R_5 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} + (1 + 2\alpha u) \frac{\partial}{\partial u}$$

the value of Y_5 is different than R_5 and is given by

$$Y_5 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} + (1 + 2\alpha u) \frac{\partial}{\partial u} - v\alpha \frac{\partial}{\partial v}.$$

In this case the Lie characteristic function is $W = 1 + 2\alpha u + 2\alpha t u_t + x\alpha u_x + 2\alpha y u_y$.

So using (1.34), one can obtain the conserved vector T whose components are given by

$$\begin{aligned} T^1 &= 2\alpha t v u_{xx} + 2\alpha t v u_{yy} + 4\alpha^2 t v u_x^2 + 4\alpha^2 t v u u_{xx} + 2\alpha t v u_{xxx} - v_t - 2\alpha u v_t \\ &\quad - 2\alpha t u_t v_t - x\alpha u_x v_t - 2\alpha y u_y v_t + 4\alpha v u_t + x\alpha v u_{tx} + 2\alpha y v u_{ty}, \\ T^2 &= -x\alpha v u_{tt} + x\alpha v u_{yy} + 2\alpha x u v u_{xx} + v_x + 4\alpha u v_x + v_{xxx} + 4\alpha^2 u^2 v_x + 2\alpha u v_{xxx} \\ &\quad - 8\alpha t v u_x u_t + 2\alpha t u_t v_x + 4\alpha^2 t v u_t u_x + 4\alpha^2 t u u_t v_x + 2\alpha t u_t v_{xxx} + x\alpha u_x v_x \\ &\quad + 2\alpha^2 x u u_x v_x + x\alpha u_x v_{xxx} - 8\alpha^2 y v u_x u_y + 2\alpha y u_y v_x + 4\alpha^2 y v u_x u_y + 4\alpha^2 y u u_y v_x \\ &\quad + 2\alpha y u_y v_{xxx} - 2\alpha t v u_{tx} - 5\alpha v u_x - 2\alpha y v u_{xy} - 4\alpha^2 t v u_{tx} - 10\alpha^2 u v u_x \\ &\quad - 2\alpha^2 x u v u_{xx} - 4\alpha^2 y u v u_{xy} - 3\alpha u_x v_{xx} - 2\alpha t u_{tx} v_{xx} - x\alpha u_{xx} v_{xx} - 2\alpha y u_{xy} v_{xx} \\ &\quad + 2\alpha t v_x u_{txx} + 4\alpha v_x u_{xx} + x\alpha v_x u_{xxx} + 2\alpha y v_x u_{xxy} - 2\alpha t v u_{txx} - 5\alpha v u_{xxx} \\ &\quad - 2\alpha y v u_{xxy}, \\ T^3 &= -2\alpha y v u_{tt} + 2\alpha y v u_{xx} + 4\alpha^2 y v u_x^2 + 4\alpha^2 y v u u_{xx} + 2\alpha y v u_{xxx} + v_y + 2\alpha u v_y \\ &\quad + 2\alpha t u_t v_y + x\alpha u_x v_y + 2\alpha y u_y v_y - 4\alpha v u_y - x\alpha v u_{xy} - 2\alpha t v u_{ty}. \end{aligned}$$

2.4 Conclusion

In this chapter Lie symmetries as well as the simplest equation method were used to obtain exact solutions of the (2+1)-dimensional Boussinesq equation (2.1). The solutions obtained were solitary waves and non-topological soliton. Moreover, the conservation laws for the (2+1)-dimensional Boussinesq equation were also derived by using the new conservation theorem due to Ibragimov.

Chapter 3

Symmetries, travelling wave solutions and conservation laws or a (3+1)-dimensional Boussinesq equation

3.1 Introduction

In this chapter we study the (3+1)-dimensional Boussinesq equation

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} - \alpha(u^2)_{xx} - u_{xxxx} = 0. \quad (3.1)$$

Several authors have studied this equation, in [44], the author obtained one-periodic wave solution, two-periodic wave solutions and soliton solution for (3.1) by means of Hirota's bilinear method and the Riemann theta function. Wazwaz [45] employed a combination of Hirota's method and Hereman's method to formally study (3.1) and derived two soliton solution of (3.1). Some other work concerning symmetries and exact solutions of some Boussinesq equations can be seen in [46–49].

For this chapter we use Lie group method along with the simplest equation method [3, 5] to construct some exact solutions of (3.1). Furthermore, we employ the new conservation theorem due to Ibragimov [35] to derive conservation laws for (3.1).

This work has been published in [50].

3.2 Travelling wave solutions of (3.1)

We obtain exact solutions of (3.1) using Lie group method along with the simplest equation method.

3.2.1 Non-topological soliton solutions using Lie point symmetries

The vector field

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u},$$

where ξ^i , $i = 1, 2, 3, 4$ and η depend on t, x, y, z and u , is a generator of Lie point symmetries of the (3+1)-dimensional Boussinesq equation (3.1) if and only if

$$X^{[4]}(u_{tt} - u_{xx} - u_{yy} - u_{zz} - \alpha(u^2)_{xx} - u_{xxxx})|_{(3.1)} = 0. \quad (3.2)$$

Here $X^{[4]}$ is the fourth prolongation of the vector field X . The invariance condition (3.2) yields the determining equations, which are a system of linear partial differential equations. Solving this system we obtain the following eight Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \\ X_2 &= \frac{\partial}{\partial t} \end{aligned}$$

$$\begin{aligned}
X_3 &= \frac{\partial}{\partial y} \\
X_4 &= \frac{\partial}{\partial z} \\
X_5 &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \\
X_6 &= z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z} \\
X_7 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \\
\text{and } X_8 &= -2\alpha t \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} + (1 + 2\alpha u) \frac{\partial}{\partial u}.
\end{aligned}$$

To obtain the Non-topological soliton solution of (3.1), we use the combination of the four translation symmetries, namely, $X = X_1 + X_2 + X_3 + \mu X_4$, where μ is a constant. Solving the associated Lagrange system for X , we obtain the four invariants

$$g = t - x, \quad f = t - y, \quad h = \mu t - z, \quad \theta = u. \quad (3.3)$$

Now considering θ as the new dependent variable and g, f and h as new independent variables, (3.1) transforms to a nonlinear PDE in three independent variables, viz.,

$$2\mu\theta_{fh} + 2\mu\theta_{gh} + 2\theta_{fg} + (\mu^2 - 1)\theta_{hh} - 2\alpha\theta_g^2 - 2\alpha\theta\theta_{gg} - \theta_{gggg} = 0. \quad (3.4)$$

The Lie point symmetries of (3.4) are

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial g}, \\
\Gamma_2 &= \frac{\partial}{\partial f}, \\
\Gamma_3 &= \frac{\partial}{\partial h}, \\
\Gamma_4 &= (2\alpha f\mu^3 - 4\alpha h\mu^2 - 2\alpha f\mu) \frac{\partial}{\partial h} + (2\mu^2\alpha f - \alpha\mu^2g - \alpha f - 3h\alpha\mu) \frac{\partial}{\partial g} \\
&\quad + (2\alpha\mu^2\theta + \mu^2 + 1) \frac{\partial}{\partial \theta}, \\
\Gamma_5 &= (\alpha f\mu^3 - \alpha f\mu - 3\alpha h\mu^2) \frac{\partial}{\partial h} + (\mu^2\alpha f - \alpha\mu^2g - \alpha f - 2h\alpha\mu) \frac{\partial}{\partial g}
\end{aligned}$$

$$-(\alpha f \mu^2) \frac{\partial}{\partial f} + (2\alpha \mu^2 \theta + \mu^2 + 1) \frac{\partial}{\partial \theta}.$$

The use of the combination $\Gamma = \Gamma_1 + \Gamma_2 + \beta \Gamma_3$, (β is a constant) of the three translation symmetries, gives us the three invariants

$$r = f - g, \quad w = \beta f - h, \quad \theta = \phi. \quad (3.5)$$

Treating ϕ as the new dependent variable and r and w as new independent variables, (3.4) transforms to

$$(\mu^2 - 2\mu\beta - 1)\phi_{ww} - 2\beta\phi_{rw} - 2\phi_{rr} - 2\alpha\phi_r^2 - 2\alpha\phi\phi_{rr} - \phi_{rrrr} = 0, \quad (3.6)$$

which is a nonlinear PDE in two independent variables. Equation (3.6) has three Lie point symmetries, namely

$$\begin{aligned} \Sigma_1 &= \frac{\partial}{\partial w}, \\ \Sigma_2 &= \frac{\partial}{\partial r}, \\ \Sigma_3 &= (4w\mu\alpha\beta + 2w\alpha - 2w\mu^2\alpha) \frac{\partial}{\partial w} + (w\alpha\beta + 2\mu r\alpha\beta + r\alpha - \mu^2 r\alpha) \frac{\partial}{\partial r} \\ &\quad + (\beta^2 - 4\mu\beta - 2 + 2\mu^2 - 4\alpha\phi\mu\beta - 2\alpha\phi + 2\alpha\mu^2\phi) \frac{\partial}{\partial \phi}, \end{aligned}$$

and the symmetry $\Sigma = \Sigma_1 + \delta \Sigma_2$ (δ is a constant) provides the two invariants

$$\xi = \delta w - r, \quad \phi = \psi,$$

which gives rise to a group invariant solution $\psi = \psi(\xi)$. Using these invariants, the PDE (3.6) transforms to

$$(\mu^2 \delta^2 - 2\mu\beta \delta^2 - \delta^2 + 2\beta\delta - 2)\psi'' - 2\alpha\psi'^2 - 2\alpha\psi\psi'' - \psi'''' = 0, \quad (3.7)$$

which is a fourth-order nonlinear ODE. This ODE can be integrated easily. Integrating it four times while choosing the constants of integration to be zero (because

we are looking for soliton solutions) and then reverting back to our original variables t, x, y, z, u , we obtain the following group-invariant (nontopological soliton) solutions of the Boussinesq equation (3.1):

$$u(x, y, t, z) = \frac{A_1}{A_2} \text{sech}^2 \left[\frac{\sqrt{A_1}}{2} (B \pm \xi) \right],$$

where B is a constant of integration and

$$\begin{aligned} A_1 &= \mu^2 \delta^2 - 2\mu\beta\delta^2 - \delta^2 + 2\beta\delta - 2, \\ A_2 &= \frac{2\alpha}{3}, \\ \xi &= \delta z + (1 - \beta\delta)y - x + (\delta\beta - \delta\mu)t. \end{aligned}$$

3.2.2 Exact solutions of (3.1) using simplest equation method

We now use the simplest equation method to obtain more solutions of the nonlinear ODE (3.7), which will then give us more exact solutions for our Boussinesq equation (3.1). The simplest equations that we will use in our work are the Bernoulli and Riccati equations.

Solutions of (3.1) using the Bernoulli equation as the simplest equation

In this case the balancing procedure yields $M = 2$ so the solutions of (3.7) have the form

$$F(k) = A_0 + A_1 G + A_2 G^2. \quad (3.8)$$

Inserting (2.6) into (3.7) and using the Bernoulli equation and then equating the coefficients of powers of G^i to zero gives us the following algebraic system of six equations:

$$-20\alpha A_2^2 b^2 - 120A_2 b^4 = 0,$$

$$\begin{aligned}
& -336A_2ab^3 - 24A_1b^4 - 36\alpha A_2^2ab - 24\alpha A_1b^2A_2 = 0, \\
& -2A_1a^2 - 2A_1a^2\delta^2\beta\mu - A_1a^4 + 2A_1a^2\delta\beta + A_1a^2\delta^2\mu^2 - \delta^2A_1a^2 - 2\alpha A_0A_1a^2 = 0, \\
& -6\alpha A_1^2b^2 + 6A_2b^2\delta^2\mu^2 - 12\alpha A_0A_2b^2 - 6\delta^2A_2b^2 + 12A_2b^2\delta\beta - 12A_2b^2 \\
& -16\alpha A_2^2a^2 - 12A_2b^2\delta^2\beta\mu - 330A_2a^2b^2 - 42\alpha A_1aA_2b - 60A_1ab^3 = 0, \\
& -6A_1ab\delta^2\beta\mu - 16A_2a^4 - 6\alpha A_0A_1ab + 3A_1ab\delta^2\mu^2 - 8\alpha A_0A_2a^2 - 8A_2a^2\delta^2\beta\mu \\
& -4\alpha A_1^2a^2 - 3\delta^2A_1ab + 6A_1ab\delta\beta - 15A_1a^3b - 8A_2a^2 - 6A_1ab + 4A_2a^2\delta^2\mu^2 \\
& -4\delta^2A_2a^2 + 8A_2a^2\delta\beta = 0, \\
& 10A_2ab\delta^2\mu^2 - 130A_2a^3b - 20A_2ab - 4A_1b^2\delta^2\beta\mu + 4A_1b^2\delta\beta - 4A_1b^2 \\
& -4\alpha A_0A_1b^2 + 20A_2ab\delta\beta - 50A_1a^2b^2 - 10\delta^2A_2ab - 10\alpha A_1^2ab - 18\alpha A_1a^2A_2 \\
& -2\delta^2A_1b^2 + 2A_1b^2\delta^2\mu^2 - 20\alpha A_0A_2ab - 20A_2ab\delta^2\beta\mu = 0.
\end{aligned}$$

These equations can be solved with the aid of Mathematica and one possible solution for A_0 , A_1 and A_2 is

$$\begin{aligned}
A_0 &= \frac{-2 - 2\delta^2\beta\mu - a^2 + 2\delta\beta + \delta^2\mu^2 - \delta^2}{2\alpha}, \\
A_1 &= \frac{-6ab}{\alpha}, \\
A_2 &= \frac{-6b^2}{\alpha}.
\end{aligned}$$

Consequently, returning to the original variables, a solution of (3.1) is

$$\begin{aligned}
u(t, x, y, z) &= A_0 + A_1a \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\} + \\
&A_2a^2 \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\}^2, \quad (3.9)
\end{aligned}$$

where $\xi = \delta z + (1 - \alpha\delta)y - x + (\delta\beta - \delta\mu)t$ and C is an arbitrary constant of integration.

Solutions of (3.1) using the Riccati equation as the simplest equation

Here the balancing procedure gives $M = 2$ so the solutions of (3.7) are of the form

$$F(z) = A_0 + A_1G + A_2G^2. \quad (3.10)$$

Substituting (3.10) into (3.7) and using the Riccati equation, and as before, we obtain the following algebraic system of equations in terms of A_0 , A_1 and A_2 :

$$\begin{aligned}
& -20\alpha A_2^2 b^2 - 120A_2 b^4 = 0, \\
& -336A_2 ab^3 - 24A_1 b^4 - 24\alpha A_1 b^2 A_2 - 36\alpha A_2^2 ab = 0, \\
& -42\alpha A_1 a A_2 b - 6\delta^2 A_2 b^2 - 12A_2 b^2 - 330A_2 a^2 b^2 - 6\alpha A_1^2 b^2 - 16\alpha A_2^2 a^2 \\
& + 12A_2 b^2 \delta \beta - 32\alpha A_2^2 db + 6A_2 b^2 \delta^2 \mu^2 - 12A_2 b^2 \delta^2 \beta \mu - 240A_2 b^3 d \\
& - 60A_1 ab^3 - 12\alpha A_0 A_2 b^2 = 0, \\
& -2A_1 ad \delta^2 \beta \mu - 2\delta^2 A_2 d^2 + 2A_1 ad \delta \beta - 4A_2 d^2 - A_1 a^3 d - 4\alpha A_0 A_2 d^2 \\
& - 2\alpha A_1^2 d^2 - 2\alpha A_0 A_1 ad - \delta^2 A_1 ad - 16A_2 bd^3 + 2A_2 d^2 \delta^2 \mu^2 + 4A_2 d^2 \delta \beta \\
& - 8A_1 abd^2 - 4A_2 d^2 \delta^2 \beta \mu - 2A_1 ad + A_1 ad \delta^2 \mu^2 - 14A_2 a^2 d^2 = 0, \\
& -20A_2 ab \delta^2 \beta \mu - 18\alpha A_1 a^2 A_2 - 36\alpha A_1 d A_2 b - 10\delta^2 A_2 ab - 20A_2 ab \\
& - 10\alpha A_1^2 ab - 2\delta^2 A_1 b^2 + 4A_1 b^2 \delta \beta - 4A_1 b^2 \delta^2 \beta \mu - 4A_1 b^2 - 130A_2 a^3 b \\
& - 440A_1 adb^2 + 10A_2 ab \delta^2 \mu^2 - 28\alpha A_2^2 dc - 4\alpha A_0 A_1 b^2 - 50A_1 a^2 b^2 \\
& + 2A_1 b^2 \delta^2 \mu^2 - 20\alpha A_0 A_2 ab + 20A_2 ab \delta \beta - 40A_1 db^3 = 0, \\
& -12A_1 da \delta^2 \beta \mu - 4A_1 bd \delta^2 \beta \mu + 2A_1 bd \delta^2 \mu^2 + 6A_2 da \delta^2 \mu^2 \\
& + 4A_1 bd \delta \beta + 12A_2 da \delta \beta - 4\alpha A_0 A_1 bd - 12\alpha A_0 A_2 da - 2A_1 a^2 \delta^2 \beta \mu \\
& + A_1 a^2 \delta^2 \mu^2 + 2A_1 a^2 \delta \beta - 2\delta^2 A_1 bd - 6\delta^2 A_1 da - 6\alpha A_1^2 da - 12\alpha A_1 d^2 A_2 \\
& - 2\alpha A_0 A_1 a^2 - 120A_2 ad^2 b - 22A_1 a^2 bd - A_1 a^4 - 2A_1 a^2 - 4A_1 bd - \delta^2 A_1 a^2 \\
& - 12A_2 da - 16A_1 b^2 d^2 - 30A_2 a^3 d = 0, \\
& -16A_2 db \delta^2 \beta \mu - 6A_1 ab \delta^2 \beta \mu + 8A_2 db \delta^2 \mu^2 - 8A_2 a^2 \delta^2 \beta \mu \\
& + 6A_1 ab \delta \beta + 16A_2 db \delta \beta - 30\alpha A_1 d A_2 a - 6\alpha A_0 A_1 ab - 16\alpha A_0 A_2 db \\
& + 3A_1 ab \delta^2 \mu^2 + 4A_2 a^2 \delta^2 \mu^2 + 8A_2 a^2 \delta \beta - 3\delta^2 A_1 ab - 8\delta^2 A_2 db \\
& - 8\alpha A_1^2 db - 8\alpha A_0 A_2 a^2 - 60A_1 adb^2 - 232A_2 a^2 db - 8A_2 a^2 - 16A_2 a^4 - 4\alpha A_2^2 a^2 \\
& - 15A_1 a^3 b - 16A_2 db - 136A_2 b^2 d^2 - 12\alpha A_2^2 d^2 - 4\delta^2 A_2 a^2 - 6A_2 ab = 0.
\end{aligned}$$

Solving the above equations, yields

$$\begin{aligned} A_0 &= \frac{-a^2 - 8bd - \delta^2 - 2 + 2\delta\beta + \delta^2\mu^2 - 2\delta^2\beta\mu}{2\alpha}, \\ A_1 &= \frac{-6ab}{\alpha}, \\ A_2 &= \frac{-6b^2}{\alpha} \end{aligned}$$

and consequently, the solutions of (3.1) are

$$\begin{aligned} u(t, x, y, z) &= A_0 + A_1 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left[\frac{1}{2} \theta (\xi + C) \right] \right\} + \\ &A_2 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left[\frac{1}{2} \theta (\xi + C) \right] \right\}^2 \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} u(t, x, y, z) &= A_0 + A_1 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left(\frac{1}{2} \theta \xi \right) + \frac{\operatorname{sech} \left(\frac{\theta \xi}{2} \right)}{C \cosh \left(\frac{\theta \xi}{2} \right) - \frac{2b}{\theta} \sinh \left(\frac{\theta \xi}{2} \right)} \right\} + \\ &A_2 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left(\frac{1}{2} \theta \xi \right) + \frac{\operatorname{sech} \left(\frac{\theta \xi}{2} \right)}{C \cosh \left(\frac{\theta \xi}{2} \right) - \frac{2b}{\theta} \sinh \left(\frac{\theta \xi}{2} \right)} \right\}^2, \end{aligned}$$

where $\xi = \delta z + (1 - \alpha\delta)y - x + (\delta\beta - \delta\mu)t$ and C is an arbitrary constant of integration.

3.3 Conservation laws for (3.1)

We utilize the new conservation theorem due to Ibragimov [35] to obtain conservation laws for the (3+1)-dimensional Boussinesq equation (3.1) written as

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx} = 0. \quad (3.12)$$

For details of notations, definitions and theorems the reader is referred to [35].

In Section 2.1 we derived the following eight Lie point symmetries of equation (3.12):

$$X_1 = \frac{\partial}{\partial x}$$

$$\begin{aligned}
X_2 &= \frac{\partial}{\partial t} \\
X_3 &= \frac{\partial}{\partial y} \\
X_4 &= \frac{\partial}{\partial z} \\
X_5 &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \\
X_6 &= z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z} \\
X_7 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \\
\text{and } X_8 &= -2\alpha t \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} + (1 + 2\alpha u) \frac{\partial}{\partial u}.
\end{aligned}$$

Corresponding to each of these eight Lie point symmetries we shall construct eight conserved vectors. By definition [35] the adjoint equation of (3.12), is given by

$$\begin{aligned}
&E^*(t, x, u, v, \dots, u_{xxxx}, v_{xxxx}) \\
&= \frac{\delta}{\delta u} \left[v(u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx}) \right] = 0,
\end{aligned}$$

which gives

$$v_{tt} - v_{xx} - v_{yy} - v_{zz} - 2\alpha u v_{xx} - v_{xxxx} = 0. \quad (3.13)$$

Here $v = v(t, x, y, z)$ is a new dependent variable. Clearly, equation (3.12) is not self-adjoint. The Lagrangian for the system of equations (3.12) and (3.13) is given by

$$\mathcal{L} = v(u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx}). \quad (3.14)$$

(i) Consider first the translation symmetry $X_1 = \partial/\partial x$. In this case the operator Y_1 [35] is the same as X_1 and the Lie characteristic function $W = -u_x$. Thus the components [35] T^i , $i = 1, 2, 3, 4$, of the conserved vector $T = (T^1, T^2, T^3, T^4)$ are given by

$$\begin{aligned}
T^1 &= u_x v_t - v u_{tx}, \\
T^2 &= v u_{tt} - v u_{yy} - v u_{zz} - u_x v_x - 2\alpha u u_x v_x - u_x v_{xxx} + v_{xx} u_{xx} - v_x u_{xxx},
\end{aligned}$$

$$T^3 = -u_x v_y + v u_{xy},$$

$$T^4 = -u_x v_z + v u_{xz}.$$

(ii) The second translation symmetry $X_2 = \partial/\partial t$, gives $W = -u_t$. Hence the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$T^1 = -v u_{xx} - v u_{yy} - v u_{zz} - 2\alpha v u_x^2 - 2\alpha u v u_{xx} - v u_{xxx} + u_t v_t,$$

$$T^2 = 2\alpha v u_t u_x - u_t v_x - 2\alpha u u_t v_x - u_t v_{xxx} + v u_{tx} + 2\alpha u v u_{tx} + v_{xx} u_{tx} - v_x u_{txx} + v u_{txxx},$$

$$T^3 = -v_y u_t + v u_{ty},$$

$$T^4 = -v_z u_t + v u_{tz}.$$

(iii) For the third symmetry $X_3 = \partial/\partial y$, we have $W = -u_y$ and the corresponding components of the conserved vector are

$$T^1 = v_t u_y - v u_{ty},$$

$$T^2 = -u_y v_x + 2\alpha v u_y u_x - 2\alpha u u_y v_x - u_y v_{xxx} + v u_{xy} + 2\alpha u v u_{xy} + v_{xx} u_{xy} - v_x u_{xxy} + v u_{xxxy},$$

$$T^3 = v u_{tt} - v u_{xx} - v u_{zz} - 2\alpha v u_x^2 - 2\alpha u v u_{xx} - v u_{xxx} - u_y v_y,$$

$$T^4 = -v_z u_y + v u_{yz}.$$

(iv) The fourth symmetry $X_4 = \partial/\partial z$, gives $W = -u_z$ and the corresponding components of the conserved vector are

$$T^1 = v_t u_z - v u_{tz},$$

$$T^2 = -u_z v_x + 2\alpha v u_z u_x - 2\alpha u u_z v_x - u_z v_{xxx} + v u_{xz} + 2\alpha u v u_{xz} + v_{xx} u_{xz} - v_x u_{xxz} + v u_{xxxz},$$

$$T^3 = -v_y u_z + v u_{yz},$$

$$T^4 = vu_{tt} - vu_{xx} - vu_{yy} - 2\alpha vu_x^2 - 2\alpha uvu_{xx} - vu_{xxx} - u_z v_z.$$

(v) For the symmetry $X_5 = y\partial/\partial z - z\partial/\partial y$, we have $W = -yu_z + zu_y$ and the corresponding components of the conserved vector, as before, are given by

$$T^1 = yu_z v_t - zu_y v_t - yv u_{tz} + zv u_{ty},$$

$$T^2 = 2\alpha yvu_x u_z - yu_z v_x - 2\alpha yuu_z v_x - yu_z v_{xxx} - 4\alpha zv u_x u_y + zu_y v_x + 2\alpha zv u_y u_x \\ + 2\alpha zuu_y v_x + zu_y v_{xxx} + yv u_{xz} + 2\alpha yuvu_{xz} + yu_{xz} v_{xx} - zv u_{xy} - 2\alpha zv u_{xy} \\ - zu_{xy} v_{xx} - yv_x u_{xxx} - zv_x u_{xxy} - yv u_{xxx} - zv u_{xxx},$$

$$T^3 = -zv u_{tt} + zv u_{xx} + zv u_{zz} + 2\alpha zv u_x^2 + 2\alpha zv u_{xx} + zv u_{xxx} - yv_y u_z + zv_y u_y \\ + vu_t + yv u_{yz} + vu_z,$$

$$T^4 = yv u_{tt} - zv u_{xx} - zv u_{yy} - 2\alpha yv u_x^2 - 2\alpha yuvu_{xx} - yv u_{xxx} - yv_z u_z + zv_z u_y \\ - vu_y - zv u_{yz}.$$

(vi) Likewise, the symmetry $X_6 = z\partial/\partial t + t\partial/\partial z$, gives $W = -zu_t - tu_z$ and the corresponding components of the conserved vector are given by

$$T^1 = -zv u_{xx} - zv u_{yy} - zv u_{zz} - 2\alpha zv u_x^2 - 2\alpha zv u_{xx} - zv u_{xxx} + zv_t u_t + tv_t u_z \\ - vu_z - tv u_{tz},$$

$$T^2 = 2\alpha zv u_x u_t - zu_t v_x - 2\alpha zuu_t v_x - zu_t v_{xxx} + 2\alpha tv u_x u_z - tu_z v_x - 2\alpha tuu_z v_x \\ - tu_z v_{xxx} + 2vu_{tx} + 2\alpha zv u_{tx} + zu_{tx} v_{xx} + tv u_{xz} + 2\alpha tv u_{xz} \\ + tu_{xz} v_{xx} - zv_x u_{txx} - tv_x u_{xxx} + zv u_{txx} + tv u_{xxx},$$

$$T^3 = -zu_t v_y - tu_z v_y + zv u_{ty} + tv u_{yz},$$

$$T^4 = tv u_{tt} - tv u_{xx} - tv u_{yy} - 2\alpha tv u_x^2 - 2\alpha tv u_{xx} - tv u_{xxx} - zv_t u_z - tv_z u_z \\ + vu_t + zv u_{tz}.$$

(vii) As before, the symmetry $X_7 = y\partial/\partial t + t\partial/\partial y$, yields $W = -yu_t - tu_y$ and the corresponding components of the conserved vector are given by

$$T^1 = -yv u_{xx} - yv u_{yy} - yv u_{zz} - 2\alpha yv u_x^2 - 2\alpha yv u_{xx} - yv u_{xxx} + yv_t u_t + tv_t u_y$$

$$-vu_y - tvu_{ty},$$

$$\begin{aligned} T^2 = & 2\alpha yvu_xu_t - yu_tv_x - 2\alpha yuu_tv_x - yu_tv_{xxx} + 4\alpha tvu_xu_y - tu_yv_x - 2\alpha tuu_yv_x \\ & - 2\alpha tvu_xu_y - tu_yv_{xxx} + yvu_{tx} + 2\alpha yuvu_{tx} + yu_{tx}v_x + tvu_{xy} + 2\alpha tvuv_{xy} \\ & + tu_{xy}v_x - yv_xu_{tx} - tv_xu_{xy} + yvu_{txx} + tvu_{xxy}, \end{aligned}$$

$$\begin{aligned} T^3 = & tvu_{tt} - tvu_{xx} - tvu_{yy} - 2\alpha tvu_x^2 - 2\alpha tvuv_{xx} - tvu_{xxx} - yv_yu_t - tv_yu_y \\ & + vu_t + yvu_{ty}, \end{aligned}$$

$$T^4 = -yu_tv_z - tu_yv_z + yvu_{tz} + tvu_{yz}.$$

(viii) Finally, for the symmetry

$$X_8 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} + (1 + 2\alpha u) \frac{\partial}{\partial u}$$

the value of Y_8 is not the same as X_8 and in fact is given by

$$Y_8 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} + (1 + 2\alpha u) \frac{\partial}{\partial u} + \alpha v \frac{\partial}{\partial v}.$$

The Lie characteristic function $W = 1 + 2\alpha u + 2\alpha tu_t + x\alpha u_x + 2\alpha yu_y + 2\alpha zu_z$, and consequently, the conserved vector T has components given by

$$\begin{aligned} T^1 = & 2\alpha tvu_{xx} + 2\alpha tvu_{yy} + 2\alpha tvu_{zz} + 4\alpha^2 tvu_x^2 + 4\alpha^2 tvuu_{xx} + 2\alpha tvu_{xxx} - v_t \\ & - 2\alpha uv_t - 2\alpha tu_tv_t - \alpha xu_xv_t - 2\alpha yu_yv_t - 2\alpha zu_zv_t + 4\alpha vu_t + \alpha xvu_{tx} \\ & + 2\alpha yvu_{ty} + 2\alpha zvu_{tz}, \\ T^2 = & -\alpha xvu_{tt} + \alpha xvu_{yy} + \alpha xvu_{zz} - 3\alpha vu_x + v_x + 4\alpha uv_x + v_{xxx} - 8\alpha^2 uvu_x \\ & + 4\alpha^2 u^2v_x + 2\alpha uv_{xxx} + 2\alpha tu_tv_x - 4\alpha^2 tvu_tv_x + 4\alpha^2 tuu_tv_x + 2\alpha tu_tv_{xxx} \\ & + \alpha xu_xv_x + 2\alpha^2 xuu_xv_x + \alpha xu_xv_{xxx} + 2\alpha yu_yv_x - 4\alpha^2 yvu_yv_x + 4\alpha^2 yuu_yv_x \\ & + 2\alpha yu_yv_{xxx} + 2\alpha zu_zv_x - 4\alpha^2 zvu_zv_x + 4\alpha^2 zuu_zv_x + 2\alpha zu_zv_{xxx} - 2\alpha tvu_{tx} \\ & - 4\alpha^2 tvuv_{tx} - 2\alpha tu_{tx}v_{xx} - 2\alpha^2 uvu_x - 3\alpha u_xv_{xx} - \alpha xu_{xx}v_{xx} - 2\alpha yvu_{xy} \\ & - 4\alpha^2 yuvu_{xy} - 2\alpha yu_{xy}v_{xx} - 2\alpha zvu_{xz} - 4\alpha^2 zuvu_{xz} - 2\alpha zu_{xz}v_{xx} + 4\alpha v_xu_{xx} \\ & + 2\alpha tv_xu_{tx} + \alpha xv_xu_{xxx} + 2\alpha yv_xu_{xxy} + 2\alpha zv_xu_{xxz} - 2\alpha tvu_{txxx} - 5\alpha vu_{xxx} \end{aligned}$$

$$\begin{aligned}
& -2\alpha y v u_{xxxy} - 2\alpha z v u_{xxxz}, \\
T^3 = & -2\alpha y v u_{tt} + 2\alpha y v u_{xx} + 2\alpha y v u_{zz} + 4\alpha^2 y v u_x^2 + 4\alpha^2 y v u_{xx} + 2\alpha y v u_{xxx} + v_y \\
& + 2\alpha u v_y + 2\alpha t u_t v_y + \alpha x u_x v_y + 2\alpha y u_y v_y + 2\alpha z u_z v_y - 4\alpha v u_y - 2\alpha t v u_{ty} \\
& - \alpha x v u_{xy} - 2\alpha z v u_z, \\
T^4 = & -2\alpha z v u_{tt} + 2\alpha z v u_{xx} + 2\alpha z v u_{yy} + 4\alpha^2 z v u_x^2 + 4\alpha^2 z v u_{xx} + 2\alpha z v u_{xxx} + v_z \\
& + 2\alpha u v_z + 2\alpha t u_t v_z + \alpha x u_x v_z + 2\alpha y u_y v_z + 2\alpha z u_z v_z - 4\alpha v u_z - 2\alpha t v u_{tz} \\
& - \alpha x v u_{xz} - 2\alpha y v u_{yz}.
\end{aligned}$$

Remark. Each conserved vector T obtained above contains the arbitrary solution v of the adjoint equation (3.13) and hence gives an infinite number of conservation laws.

3.4 Conclusion

In this chapter exact solutions of the (3+1)-dimensional Boussinesq equation (3.1) were obtained with the aid of Lie point symmetries of (3.1) as well as the simplest equation method. The solutions obtained were solitary waves and non-topological soliton. Furthermore, the conservation laws for the (3+1)-dimensional Boussinesq equation (3.1) were also constructed by utilizing the new conservation theorem due to Ibragimov.

Chapter 4

Solutions and conservation laws of a generalized (3+1)-dimensional Kawahara equation

4.1 Introduction

The Kawahara type equations, which are nonlinear evolution equations, are of the form

$$u_t - u_{xxxxx} + H(u, u_x, u_{xx}, u_{xxx}) = 0. \quad (4.1)$$

These equations have been thoroughly studied in the last few decades owing to their significance in the field of physics. See for example, [51–57] and the references therein. Recently, an initial boundary value problem for the (3+1)-dimensional Kawahara equation

$$u_t + u_x + uu_x + u_{xxx} + u_{xyy} + u_{zzz} - u_{xxxxx} = 0$$

posed on a channel-type domain was considered and existence and uniqueness results for global regular solutions as well as exponential decay of small solutions in the H^2 -norm were established [58].

In this chapter we study the generalized (3+1)-dimensional Kawahara equation given by

$$u_t + u_x + uu_x + au_{xxx} + bu_{xyy} + cu_{zzz} + du_{xxxx} = 0, \quad (4.2)$$

where a, b, c and d are nonzero arbitrary constants. We first find exact solutions of (4.2) using its translation symmetries in conjunction with the Kudryashov method. The conservation law multipliers are computed and then used to construct conservation laws for the generalized (3+1)-dimensional Kawahara equation.

This work has been submitted for publication [59].

4.2 Solutions of equation (4.2)

4.2.1 Lie point symmetries of (4.2)

This section aims to compute the Lie point symmetries of the generalized (3+1)-dimensional Kawahara equation (4.2) and later use the translation symmetries to transform it to an ordinary differential equation (ODE). Thereafter, we use Kudryashov's method to construct exact solutions of the ODE, which in fact are the exact solutions of the equation (4.2).

The symmetry group of the generalized (3+1)-dimensional Kawahara equation (4.2) will be generated by the vector field of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u},$$

where ξ^i , $i = 1, 2, 3, 4$ and η are functions of t, x, y, z and u . Applying the fifth prolongation $X^{[5]}$ to equation (4.2), expanding and then splitting on the derivatives

of u we obtain an overdetermined system of linear homogeneous partial differential equations. Solving this resulting system, we obtain the values of ξ^i and η . Consequently, we obtain the following six Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial z}, \\ X_5 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_6 = bz \frac{\partial}{\partial y} - cy \frac{\partial}{\partial z}. \end{aligned}$$

We now use the four translation symmetries and perform symmetry reductions. The symmetry $X = X_1 + X_2 + X_3 + \alpha X_4$, where α is a constant, yields the four invariants

$$f = t - x, \quad g = t - y, \quad h = \alpha t - z, \quad \theta = u, \quad (4.3)$$

which are obtained by solving its associated Lagrange system. Using these invariants, equation (4.2) transforms to

$$\alpha \theta_h + \theta_g - \theta \theta_f - a \theta_{fff} - b \theta_{fgg} - c \theta_{fhh} - d \theta_{fffff} = 0, \quad (4.4)$$

which is a nonlinear PDE of three independent variables f , g and h . Equation (4.4) possesses the three symmetries

$$R_1 = \frac{\partial}{\partial f}, \quad R_2 = \frac{\partial}{\partial g}, \quad R_3 = \frac{\partial}{\partial h}$$

and the linear combination $R = R_1 + R_2 + \beta R_3$, with β a constant, provides us with three invariants, namely

$$r = f - g, \quad s = \beta g - h, \quad \ell = \phi. \quad (4.5)$$

Consequently, these invariants transform equation (4.4) to the PDE

$$\begin{aligned} &(\beta - \alpha) \phi_s - \phi_r - \phi \phi_r - (a + b) \phi_{rrr} - (b\beta^2 + c) \phi_{rss} \\ &+ 2b\beta \phi_{rrs} - d \phi_{rrrrr} = 0. \end{aligned} \quad (4.6)$$

Equation (4.6) admits the two translation symmetries

$$\Gamma_1 = \frac{\partial}{\partial s}, \quad \Gamma_2 = \frac{\partial}{\partial r}$$

and $\Gamma = \Gamma_1 + \gamma\Gamma_2$, where γ is a constant, has two invariants

$$\xi = \gamma s - r, \quad \phi = \psi,$$

which give rise to the group-invariant solution $\phi = \psi(\xi)$. Thus equation (4.6) reduces to the nonlinear fifth-order ODE

$$(1 + \beta\gamma - \alpha\gamma)\psi' + \psi\psi' + (a + c\gamma^2 + b(1 + \beta\gamma)^2)\psi''' + d\psi'''' = 0. \quad (4.7)$$

4.2.2 Solutions of (4.2) using Kudryashov's method

In this subsection we apply Kudryashov's method [8, 9] and find exact solutions of the ODE (4.7), which will lead to the exact solutions of (4.2). The basic idea is to assume the solution of (4.7) in the form

$$\psi(\xi) = \sum_{j=0}^M A_j (H(\xi))^j, \quad (4.8)$$

where

$$H(\xi) = \frac{1}{1 + \cosh \xi + \sinh \xi}, \quad (4.9)$$

which satisfies the equation

$$H'(\xi) = H^2(\xi) - H(\xi). \quad (4.10)$$

Here $M, A_0, \dots, A_M$ are parameters to be determined. The application of the balancing procedure on (4.7) yields $M = 4$. Thus equation (4.8) takes the form

$$\psi(\xi) = A_0 + A_1 H(\xi) + A_2 H^2(\xi) + A_3 H^3(\xi) + A_4 H^4(\xi). \quad (4.11)$$

Substituting this value of ψ into (4.7) and taking into account equation (7.12), and thereafter setting the coefficients of like powers of H to zero, gives rise to an algebraic system of nine equations, namely

$$\begin{aligned}
& \alpha\gamma A_1 - A_1 - aA_1 - bA_1 - dA_1 - \beta\gamma A_1 - c\gamma^2 A_1 - 2b\beta\gamma A_1 - b\gamma\beta^2 A_1 - A_0 A_1 = 0, \\
& A_1 + 7aA_1 + 7bA_1 + 31dA_1 - \alpha\gamma A_1 + \beta\gamma A_1 + 7c\gamma^2 A_1 + 14b\gamma\beta A_1 + 7b\gamma\beta^2 A_1 + A_0 A_1 \\
& 2\alpha\gamma A_2 - A_1^2 - 2A_2 - 8aA_2 - 8bA_2 - 32bA_2 - 2\beta\gamma A_2 - 8c\gamma^2 A_2 - 16b\gamma\beta A_2 \\
& - 8b\gamma\beta^2 A_2 - 2A_0 A_2 = 0, \\
& 38aA_2 - 12aA_1 - 12bA_1 - 180dA_1 - 12c\gamma^2 A_1 - 24b\gamma\beta A_1 - 12b\gamma\beta^2 A_1 + A_1^2 + A_2 \\
& + 38bA_2 + 422dA_2 - 2\alpha\gamma A_2 + 2\beta\gamma A_2 + 38c\gamma^2 A_2 + 76b\gamma\beta A_2 + 38b\gamma\beta^2 A_2 + 2A_0 A_2 \\
& - 3A_1 A_2 - 3A_3 - 27aA_3 - 27bA_3 - 243dA_3 + 3\alpha\gamma A_3 - 3\beta\gamma A_3 - 27c\gamma^2 A_3 \\
& - 54b\gamma\beta A_3 - 27b\gamma\beta^2 A_3 - 3A_0 A_3 = 0, \\
& 6aA_1 + 6bA_1 + 390dA_1 + 6c\gamma^2 A_1 + 12b\gamma\beta A_1 + 6b\gamma\beta^2 A_1 - 54aA_2 - 54bA_2 - 1710dA_2 \\
& - 54c\gamma^2 A_2 - 108b\gamma\beta A_2 - 54b\gamma\beta^2 A_2 + 3A_1 A_2 - 2A_2^2 + 3A_3 + 111aA_3 + 111bA_3 \\
& + 2343dA_3 - 3\alpha\gamma A_3 + 3\beta\gamma A_3 + 111c\gamma^2 A_3 + 222b\gamma\beta A_3 + 111b\gamma\beta^2 A_3 + 3A_0 A_3 \\
& - 4A_1 A_3 - 4A_4 - 64aA_4 - 64bA_4 - 1024dA_4 + 4\alpha\gamma A_4 - 4\beta\gamma A_4 - 64\gamma^2 A_4 \\
& - 128b\gamma\beta A_4 - 64b\gamma\beta^2 A_4 - 4A_0 A_4 = 0, \\
& 24aA_2 - 360dA_1 + 24bA_2 + 3000dA_2 + 24c\gamma^2 A_2 + 48b\gamma\beta A_2 + 24b\gamma\beta^2 A_2 + 2A_2^2 \\
& - 144aA_3 - 144bA_3 - 7920dA_3 - 144c\gamma^2 A_3 - 288b\gamma\beta A_3 - 144b\gamma\beta^2 A_3 + 4A_1 A_3 \\
& - 5A_2 A_3 + 4A_4 + 244aA_4 + 244bA_4 + 8404dA_4 - 4\alpha\gamma A_4 + 4\beta\gamma A_4 + 244c\gamma^2 A_4 \\
& + 488b\gamma\beta A_4 + 244b\gamma\beta^2 A_4 + 4A_0 A_4 - 5A_1 A_4 = 0, \\
& 120dA_1 - 2400dA_2 + 60aA_3 + 60bA_3 + 12300dA_3 + 60c\gamma^2 A_3 + 120b\gamma\beta A_3 \\
& + 60b\gamma\beta^2 A_3 + 5A_2 A_3 - 3A_3^2 - 300aA_4 - 300bA_4 - 25500dA_4 - 300c\gamma^2 A_4 \\
& - 600b\gamma\beta A_4 - 300b\gamma\beta^2 A_4 + 5A_1 A_4 - 6A_2 A_4 = 0, \\
& 720dA_2 - 9000dA_3 + 3A_3^2 + 120aA_4 + 120bA_4 + 36600dA_4 + 120c\gamma^2 A_4
\end{aligned}$$

$$+ 240b\gamma\beta A_4 + 120b\gamma\beta^2 A_4 + 6A_2A_4 - 7A_3A_4 = 0,$$

$$2520dA_3 - 25200dA_4 + 7A_3A_4 - 4A_4^2 = 0,$$

$$1680d + A_4 = 0.$$

Using the computer algebra system Mathematica, the solution to the above system of algebraic equations for A_0, A_1, A_2, A_3 and A_4 is

$$A_0 = -\frac{36}{13} (a + b\beta^2\gamma^2 + 2b\beta\gamma + b + c\gamma^2) + \alpha\gamma - \beta\gamma - 1,$$

$$A_1 = 0,$$

$$A_2 = \frac{1680}{13} (a + b\beta^2\gamma^2 + 2b\beta\gamma + b + c\gamma^2),$$

$$A_3 = -\frac{3360}{13} (a + b\beta^2\gamma^2 + 2b\beta\gamma + b + c\gamma^2),$$

$$A_4 = \frac{1680}{13} (a + b\beta^2\gamma^2 + 2b\beta\gamma + b + c\gamma^2).$$

Consequently the solution of the (3+1)-dimensional Kawahara equation (4.2) is

$$u(t, x, y, z) = \sum_{j=0}^4 A_j \left(\frac{1}{1 + \cosh \xi + \sinh \xi} \right)^j, \quad (4.12)$$

where $\xi = \gamma(\beta - \alpha)t + x - (\gamma\beta + 1)y + \gamma z$.

4.3 Local conservation laws

In this section we look for the local zeroth-order conservation laws of the generalized (3+1)-dimensional Kawahara equation (4.2). We apply the algorithm described in [24] to look for the zeroth-order multiplier $\Lambda = \Lambda(t, x, y, z, u)$. The Euler-Lagrange operator in this case is

$$\begin{aligned} E_u = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_x^3 \frac{\partial}{\partial u_{xxx}} - D_t D_x^3 \frac{\partial}{\partial u_{xxx}} \\ & - D_x D_z^2 \frac{\partial}{\partial u_{zzz}} - D_x^5 \frac{\partial}{\partial u_{xxxxx}} \dots \end{aligned}$$

The determining equation for the multiplier Λ is

$$E_u \{ \Lambda(u_t + u_x + uu_x + au_{xxx} + bu_{xyy} + cu_{zzz} + du_{xxxx}) \} = 0. \quad (4.13)$$

Expanding equation (8.13) and splitting on the derivatives of u yields a system of linear homogeneous PDEs. Solving these PDEs we obtain

$$\Lambda = c_1(x - t - u_t) + C_2u + f(y, z). \quad (4.14)$$

Thus it follows from equation (4.14) that there are three nontrivial conservation law multipliers, namely

$$\Lambda_1 = x - t - u_t, \quad \Lambda_2 = u, \quad \Lambda_3 = F(y, z). \quad (4.15)$$

Corresponding to the above multipliers the associated low-order conservation laws for equation (4.2) are

$$\begin{aligned} T_1^t &= -\frac{1}{2}tu^2 - tu + xu, \\ T_1^x &= -au_x - du_{xxx} - dtu_{xxxx} - atu_{xx} + axu_{xx} - atuu_{xx} - dtuu_{xxxx} \\ &\quad + dtu_xu_{xxx} - tu^2 + xu - tu + \frac{1}{6}(3xu^2 + 3atu_x^2 - 3dtu_{xx}^2 + btu_y^2 + ctu_z^2) \\ &\quad + \frac{1}{3}(cxu_{zz} - ctu_{zz} - btu_{yy} + bxu_{yy} - btuu_{yy} - ctuu_{zz} - tu^3), \\ T_1^y &= \frac{b}{3}(tu_xu_y - 2tuu_{xy} - 2tu_{xy} + 2xu_{xy} - u_y), \\ T_1^z &= \frac{c}{3}(tu_xu_z - 2tuu_{xz} - 2tu_{xz} + 2xu_{xz} - u_z); \\ T_2^t &= \frac{1}{2}u^2, \\ T_2^x &= \frac{1}{6}(3du_{xx}^2 + 3u^2 - 3au_x^2 + 2cuu_{zz} + 2buu_{yy} + 2u^3 - bu_y^2 - cu_z^2) \\ &\quad + auu_{xx} + duu_{xxxx} - du_xu_{xxx}, \\ T_2^y &= \frac{1}{3}b(2uu_{xy} - u_xu_y), \\ T_2^z &= \frac{1}{3}c(2uu_{xz} - u_xu_z); \end{aligned}$$

$$\begin{aligned}
T_3^t &= uF, \\
T_3^x &= \frac{1}{3}(bFu_{yy} + cFu_{zz} - bF_yu_y + cuF_{zz} - cu_zF_z + buF_{yy}) + \frac{1}{2}u^2F \\
&\quad + aFu_{xx} + dFu_{xxx} + uF, \\
T_3^y &= -\frac{1}{3}b(F_yu_x - 2Fu_{xy}), \\
T_3^z &= -\frac{1}{3}c(F_zu_x - 2Fu_{xz}),
\end{aligned}$$

respectively.

4.4 Conclusion

In this chapter we studied the generalized (3+1)-dimensional Kawahara equation (4.2). We performed several symmetry reductions on (4.2) and transformed it to a fifth-order ordinary differential equation. Thereafter, Kudryashov's method was employed to obtain its exact solutions. Moreover, three conservation law multipliers were computed and corresponding to them conservation laws were constructed.

Chapter 5

Exact solutions and conservation laws of a (3+1)-dimensional KP-Boussinesq equation

5.1 Introduction

One of the generalizations of the well-known Kadomtsev-Petviashvili (KP) equation is the generalized (3+1)-dimensional KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0, \quad (5.1)$$

which has a class of Wronskian solutions and a class of Grammian solutions [60]. In [61,62] the simplified Hirota's method was employed to (6.2) and multiple soliton solutions were obtained subject to constraints on the coefficients of the spatial variables. The multiple exp-function method was used to construct multiple wave solutions to (6.2) and it was shown that the resulting solutions involved generic phase shifts [63].

In Ref. [64] a new form of (3+1)-dimensional generalized KP-Boussinesq equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tt} - u_{zz} = 0 \quad (5.2)$$

was introduced and simplified Hirota's method was used to construct one, two and three-soliton solutions were constructed for (5.2).

In this chapter the Lie symmetry method is employed to perform symmetry reduction on the (3+1)-dimensional generalized KP-Boussinesq (5.2) and the reduced ordinary differential equation is solved by direct integration. Moreover, Kudryashov's method [3] is used to obtain exact solutions of (5.2). Furthermore, conservation laws for the equation are derived using Ibragimov's conservation theorem [35]. Finally, conclusion is presented.

This work has been submitted for publication [65].

5.2 Exact solutions of (5.2)

5.2.1 Symmetry reductions of (5.2)

In this section we first calculate the Lie point symmetries of (3+1)-dimensional generalized KP-Boussinesq equation (5.2) and latter use the translation symmetries to transform it into a fourth-order nonlinear ordinary differential equation (ODE).

The symmetry group of the (3+1)-dimensional generalized KP-Boussinesq equation (5.2) will be generated by the vector field of the form

$$R = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u},$$

where ξ^i , $i = 1, 2, 3, 4$ and η depend on the variables t, x, y, z and u . Applying the fourth prolongation $X^{[4]}$ to equation (5.2) we obtain an overdetermined system

of linear homogeneous partial differential equations. Solving this resultant system one obtains the following seven Lie point symmetries:

$$\begin{aligned}
R_1 &= \frac{\partial}{\partial t}, \quad R_2 = \frac{\partial}{\partial x}, \quad R_3 = \frac{\partial}{\partial y}, \quad R_4 = \frac{\partial}{\partial z}, \\
R_5 &= 2z \frac{\partial}{\partial t} + z \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial z}, \\
R_6 &= 9t \frac{\partial}{\partial t} + 3(t+x) \frac{\partial}{\partial x} + 9y \frac{\partial}{\partial y} + 9z \frac{\partial}{\partial z} + (x+y-3u) \frac{\partial}{\partial u}, \\
R_7 &= F(z-t) \frac{\partial}{\partial u}, \quad R_8 = F(t+z) \frac{\partial}{\partial u}.
\end{aligned} \tag{5.3}$$

We now utilize the linear combination $R = R_1 + \alpha R_2 + R_3 + R_4$, where α is a constant, and perform symmetry reductions on (5.2). Solving the associated Lagrange system for R , we obtain the following four invariants:

$$f = \alpha t - x, \quad g = x - \alpha y, \quad h = y - z, \quad u(t, x, y, z) = \theta(f, g, h). \tag{5.4}$$

Now treating θ as the new dependent variable and f, g and h as new independent variables, equation (5.2) transforms to

$$\begin{aligned}
&\alpha^2 \theta_{ff} + \alpha \theta_{fgg} + \alpha (\theta_{fg} - \theta_{ff}) + \alpha \theta_{ffg} + \alpha (\theta_{fh} - \alpha \theta_{fg}) \\
&+ 3 (\theta_g - \theta_f) (\alpha \theta_{fg} - \theta_{fh} - \alpha \theta_{gg} + \theta_{gh}) + 3 (\theta_{ff} - 2\theta_{fg} + \theta_{gg}) (\theta_h - \alpha \theta_g) \\
&+ 2 (\alpha \theta_{fgg} - \alpha \theta_{ffg} - \theta_{fgh} + \theta_{ffgh}) + \theta_{ffgh} + \theta_{gggh} - \alpha \theta_{ffg} - \theta_{fgh} - \theta_{ffh} \\
&- \alpha \theta_{ggg} - \theta_{hh} = 0,
\end{aligned} \tag{5.5}$$

which is a nonlinear PDE in three independent variables. This equation (5.5) has five Lie point symmetries

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}, \quad \Gamma_3 = \frac{\partial}{\partial h}, \\
\Gamma_4 &= 3f \frac{\partial}{\partial f} + 3g \frac{\partial}{\partial g} + \frac{3}{\alpha} (3\alpha h + 2f + 2g) \frac{\partial}{\partial h} \\
&\quad + ((4f + 2g)\alpha - 4f - 2g - 3\theta) \frac{\partial}{\partial \theta}, \\
\Gamma_5 &= \frac{3}{\alpha} (\alpha h + f + g) \frac{\partial}{\partial f} - \frac{3}{\alpha} (\alpha h + f + g) \frac{\partial}{\partial g}
\end{aligned}$$

$$+ \frac{1}{\alpha}(3\alpha - (3f + 2g)\alpha + f + g)\frac{\partial}{\partial\theta}.$$

As before, we consider the combination $\Gamma = \Gamma_1 + \beta\Gamma_2 + \Gamma_3$, where β is a constant.

Solving the associated Lagrange system for Γ , we obtain the three invariants

$$r = f - h, \quad s = g - \beta h, \quad \theta(f, g, h) = \phi(r, s). \quad (5.6)$$

Using these invariants, equation (5.5) transforms to

$$\begin{aligned} & \alpha^2 \phi_{rr} - 2\alpha \phi_{rr} - \alpha^2 \phi_{rs} - \alpha\beta \phi_{rs} + 3\alpha \phi_r \phi_{ss} + 9\alpha \phi_s \phi_{rs} - 3\alpha \phi_r \phi_{rs} + \alpha \phi_{rs} \\ & - 3\alpha \phi_{rr} \phi_s - 3\alpha \phi_{rrs} + \alpha \phi_{rrr} + 3\beta \phi_r \phi_{ss} - 2\beta \phi_{rs} + 9\beta \phi_s \phi_{rs} - 3\beta \phi_r \phi_{rs} \\ & - 3\beta \phi_{rr} \phi_s - 3\beta \phi_{rrs} + \beta \phi_{rrr} - 3\phi_r \phi_{ss} - 3\phi_s \phi_{rs} + 9\phi_r \phi_{rs} - \phi_{rss} + 3\phi_{rr} \phi_s \\ & - 3\phi_{rrr} - 6\phi_r \phi_{rr} - \phi_{rr} + \phi_{rrr} - 6\alpha \phi_s \phi_{ss} + 3\alpha \phi_{rss} + 3\beta \phi_{rss} \\ & - \alpha \phi_{sss} - \beta^2 \phi_{ss} - 6\beta \phi_s \phi_{ss} - \beta \phi_{sss} + 3\phi_{rrs} = 0, \end{aligned} \quad (5.7)$$

which has the four Lie point symmetries

$$\begin{aligned} \Sigma_1 &= \frac{\partial}{\partial r}, \quad \Sigma_2 = \frac{\partial}{\partial s}, \quad \Sigma_3 = \frac{\partial}{\partial \phi}, \\ \Sigma_4 &= \left\{ \frac{((6r + 4s)\alpha^3 + ((4r + 4s)\beta - 14r - 10s - 3\phi)\alpha^2}{3(\alpha + \beta + 1)} \right. \\ & \quad \left. - \frac{(2r\beta^2 + (10r + 2s + 6\phi)\beta - 6s + 6\phi)\alpha + 3\phi(\beta + 1)^2}{3(\alpha + \beta + 1)} \right\} \frac{\partial}{\partial \phi}. \end{aligned}$$

Utilizing the first two translation symmetries, $\Sigma = \Sigma_1 + \nu\Sigma_2$, we obtain, as before, two invariants

$$\xi = s - \nu r, \quad \phi(r, s) = \psi(\xi),$$

which transform equation (5.7) into nonlinear fourth-order ODE

$$\begin{aligned} & (\alpha^2 \nu^2 + \alpha^2 \nu + \alpha\beta\nu - 2\alpha\nu^2 - \alpha\nu - \beta^2 + 2\beta\nu - \nu^2)\psi'' \\ & + (6\nu^3 - 6\alpha\nu^2 - 12\alpha\nu - 6\alpha - 6\beta - 6\beta\nu^2 - 12\beta\nu + 12\nu^2 + 6\nu)\psi'\psi'' \\ & + (\nu^4 + 3\nu^3 + 3\nu^2 + \nu - \alpha\nu^3 - 3\alpha\nu^2 - 3\alpha\nu - \alpha - \beta\nu^3 - 3\beta\nu^2 - 3\beta\nu - \beta)\psi'''' = 0, \end{aligned}$$

which we write as

$$A\psi'''' + B\psi'\psi'' + C\psi'' = 0, \quad (5.8)$$

where $A = (\nu + 1)^3(\alpha + \beta - \nu)$, $B = 6(\nu + 1)^2(\alpha + \beta - \nu)$ and $C = (\beta - \nu)^2 - \alpha^2\nu^2 - \alpha\nu(\alpha + \beta - \nu) + \alpha\nu(\nu + 1)$.

5.2.2 Exact solutions of (5.2) by direct integration

Integration of equation (5.8) yields

$$A\psi''' + \frac{B}{2}\psi'^2 + C\psi' + C_1 = 0, \quad (5.9)$$

where C_1 is an arbitrary constant of integration. Multiplying equation (5.9) by ψ'' and integrating gives

$$\frac{A}{2}\psi''^2 + \frac{B}{6}\psi'^3 + \frac{C}{2}\psi'^2 + C_1\psi' + C_2 = 0,$$

where C_2 is an arbitrary constant of integration. Putting $\Phi = \psi'$ the above equation becomes

$$\Phi'^2 = -\frac{B}{3A}\Phi^3 - \frac{C}{A}\Phi^2 - \frac{2C_1}{A}\Phi - \frac{2C_2}{A}.$$

Taking into account the transformation

$$\Phi = -\frac{12A}{B}\wp(\xi) - \frac{C}{B}, \quad (5.10)$$

we have equation for the Weierstrass elliptic function [66]

$$\wp'^2 = 4\wp^3 - g_1\wp - g_2,$$

where

$$g_1 = \frac{C^2 - 2BC_1}{12A^2}, \quad g_2 = \frac{C^3 + 3B(BC_2 - CC_1)}{216A^2}.$$

Thus integrating equation (5.10) with respect to ξ and reverting to the original variables we obtain the solution of (5.2)

$$u(t, x, y, z) = \frac{12A}{B}\zeta(\xi; g_1, g_2) - \frac{C}{B}\xi,$$

where $\zeta(\xi; g_1, g_2)$ is the Weierstrass zeta function [66] defined as $\zeta'(\xi; g_1, g_2) = -\wp(\xi; g_1, g_2)$ and $A = (\nu + 1)^3(\alpha + \beta - \nu)$, $B = 6(\nu + 1)^2(\alpha + \beta - \nu)$, $C = (\beta - \nu)^2 - \alpha^2\nu^2 - \alpha\nu(\alpha + \beta - \nu) + \alpha\nu(\nu + 1)$ and $\xi = (\nu + 1)x + (\nu - \alpha - \beta)y + (\beta - \nu)z - \alpha\nu t$.

5.2.3 Solutions of (5.2) using Kudryashov's method

We now find exact solutions of (5.2) by employing Kudryashov's method. This method is used for finding exact solutions of NLPDEs and the algorithm of this method is described in [8]. Firstly the NLPDE is reduced to an ODE. Then the exact solutions of this ODE is assumed to be written as the power series

$$\psi(\xi) = \sum_{n=0}^N a_n Q^n(\xi), \quad (5.11)$$

where the coefficients a_n ($n = 0, 1, 2, \dots, N$) are constants to be determined, such that $a_N \neq 0$, and $Q(z)$ is the solution of the first-order nonlinear ODE

$$Q'(\xi) = Q^2(\xi) - Q(\xi). \quad (5.12)$$

Equation (5.12) has the solution in terms of elementary function as

$$Q(\xi) = \frac{1}{1 + e^\xi}. \quad (5.13)$$

By substituting the value for $\psi(\xi)$ into the ODE of interest and using equation (5.12) we obtain an equation involving powers of Q . Finally, equating different powers of Q to zero, we obtain a system of algebraic equations which can be solved to give the values of coefficients $a_0, a_1, \dots, a_{N-1}, a_N$ and possibly the relations for

other parameters that may be present. As a result, we obtain exact solutions of the NLPDE owing to the group invariant solution $u(t, x, y, z) = \psi(\xi)$ which has been alluded to in the previous section.

We now implement Kudryashov's method in order to obtain exact solutions of (5.2). Considering (5.8), the balancing procedure yields $M = 1$ and thus (5.11) becomes

$$\psi(\xi) = a_0 + a_1 Q(\xi).$$

Substituting the value of $\psi(\xi)$ from the above equation into (5.8) and using (5.12) yields the algebraic equation

$$\begin{aligned} & 2a_1^2 BQ(\xi)^5 + 24a_1 A Q(\xi)^5 - 5a_1^2 BQ(\xi)^4 - 60a_1 A Q(\xi)^4 + 2a_1 C Q(\xi)^3 \\ & + 4a_1^2 BQ(\xi)^3 + 50a_1 A Q(\xi)^3 - 3a_1 C Q(\xi)^2 - a_1^2 BQ(\xi)^2 - 15a_1 A Q(\xi)^2 \\ & + a_1 C Q(\xi) + a_1 A Q(\xi) = 0, \end{aligned}$$

which splits into an algebraic system of equations

$$\begin{aligned} a_1 A + a_1 C &= 0, \\ 15a_1 A + a_1^2 B + 3a_1 C &= 0, \\ 50a_1 A + 4a_1^2 B + 2a_1 C &= 0, \\ 12a_1 A + a_1^2 B &= 0. \end{aligned}$$

Solving these equations we obtain

$$a_0 = a_0, \quad a_1 = \frac{12C}{B}, \quad C = -A, \quad (5.14)$$

thus rendering the solution of the (3+1)-dimensional generalized KP-Boussinesq equation (5.2) as

$$u(t, x, y, z) = a_0 - \frac{2(\nu + 1)}{1 + e^\xi} \quad (5.15)$$

with $\xi = (\nu + 1)x + (\nu - \alpha - \beta)y + (\beta - \nu)z - \alpha \nu t$.

5.3 Conservation laws of (5.2)

In this section we derive conservation laws of the (3+1)-dimensional generalized KP-Boussinesq equation (5.2) using the new conservation theorem due to Ibragimov [35].

We begin by determining the adjoint equation of (5.2) by using

$$F^* \equiv \frac{\delta}{\delta u} (v(u_{tt} + u_{tx} + u_{ty} - u_{zz} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxx})) = 0, \quad (5.16)$$

where $v = v(t, x, y, z)$, and $\delta/\delta u$ is the Euler-Lagrange operator defined by

$$\begin{aligned} \frac{\delta}{\delta u} = & -D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_z^2 \frac{\partial}{\partial u_{zz}} + D_t D_y \frac{\partial}{\partial u_{ty}} \\ & + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_x^3 D_y \frac{\partial}{\partial u_{xxx}}. \end{aligned} \quad (5.17)$$

The total differential operators D_t , D_x , D_y and D_z are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + u_{xz} \frac{\partial}{\partial u_z} + \dots, \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yt} \frac{\partial}{\partial u_t} + u_{yx} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_z} + \dots, \\ D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{zz} \frac{\partial}{\partial u_z} + u_{zt} \frac{\partial}{\partial u_t} + u_{zy} \frac{\partial}{\partial u_y} + u_{zx} \frac{\partial}{\partial u_x} + \dots. \end{aligned} \quad (5.18)$$

Thus (8.16) becomes

$$F^* \equiv v_{tx} + v_{ty} + v_{tt} + 6v_x u_{xy} + 3u_x v_{xy} + 3u_y v_{xx} + v_{xxx} - v_{zz} = 0. \quad (5.19)$$

The KPB equation (5.2) together with its adjoint (5.19) have the Lagrangian

$$\mathcal{L} = v(u_{tt} + u_{tx} + u_{ty} - u_{zz} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxx}), \quad (5.20)$$

which is equivalent to the second-order Lagrangian

$$\mathcal{L} = v(u_{tt} + u_{tx} + u_{ty} - u_{zz} + 3u_x u_{xy} + 3u_{xx} u_y) + u_{xy} v_{v_{xy}}. \quad (5.21)$$

A quick calculation will reveal that $\delta\mathcal{L}/\delta u = F^*$ and $\delta\mathcal{L}/\delta v = F$. We recall that (5.2) admits the eight Lie point symmetries (5.3). To obtain the conserved vectors corresponding to these eight infinitesimal generators and the second-order Lagrangian (5.21) we use the components C^i of a conservation law given by [35]

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \right] + D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}, \quad (5.22)$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2$. Here j runs from $1, \dots, 4$ in this particular case.

Let us first consider the infinitesimal generator $\partial/\partial t$. The corresponding value of Y_1 is $\partial/\partial t$ and the Lie characteristic functions W^1 and W^2 are $W^1 = -u_t$ and $W^2 = -v_t$. Thus by using (5.22) the conserved vector for the system (5.2) and (5.19) corresponding to X_1 is

$$\begin{aligned} C_1^t &= -u_{zz}v + 3u_x u_{xy}v + 3u_{xx}u_yv + \frac{1}{2}u_{ty}v + \frac{1}{2}u_{tx}v + \frac{1}{2}u_tv_x + \frac{1}{2}u_tv_y + u_tv_t \\ &\quad + u_{xx}v_{xy}, \\ C_1^x &= \frac{3}{2}u_t u_{xy}v - \frac{3}{2}u_x u_{ty}v - 3u_y u_{tx}v - \frac{1}{2}u_{tt}v + \frac{3}{2}u_t u_x v_y + 3u_t u_y v_x + u_t v_{xxy} \\ &\quad + \frac{1}{2}v_t u_{xxy} - \frac{1}{2}u_{xx}v_{ty} - u_{tx}v_{xy} + \frac{1}{2}u_tv_t, \\ C_1^y &= -\frac{3}{2}u_t u_{xx}v - \frac{3}{2}u_x u_{tx}v - \frac{1}{2}u_{tt}v + \frac{3}{2}u_t u_x v_x + \frac{1}{2}v_t u_{xxx} - \frac{1}{2}u_{xx}v_{tx} + \frac{1}{2}u_tv_t, \\ C_1^z &= u_{tz}v - u_tv_z. \end{aligned}$$

Similarly, we compute the conserved vectors for the remaining seven symmetries.

The results are given below.

$$\begin{aligned} C_2^t &= -\frac{1}{2}u_{xy}v - \frac{1}{2}u_{xx}v - u_{tx}v + v_t u_x + \frac{1}{2}u_x v_y + \frac{1}{2}u_x v_x, \\ C_2^x &= 3u_x u_{xy}v - u_{zz}v + u_{ty}v + \frac{1}{2}u_{tx}v + u_{tt}v + \frac{1}{2}v_t u_x + \frac{3}{2}u_x^2 v_y + 3u_x u_y v_x + u_x v_{xxy} \\ &\quad - \frac{1}{2}u_{xx}v_{xy} + \frac{1}{2}v_x u_{xxy}, \\ C_2^y &= -3u_{xx}u_xv - \frac{1}{2}u_{tx}v + \frac{1}{2}v_t u_x + \frac{3}{2}u_x^2 v_x - \frac{1}{2}u_{xx}v_{xx} + \frac{1}{2}u_{xxx}v_x, \end{aligned}$$

$$C_2^z = u_{xz}v - u_xv_z;$$

$$C_3^t = -\frac{1}{2}u_{yy}v - \frac{1}{2}u_{xy}v - u_{ty}v + v_tu_y + \frac{1}{2}u_yv_x + \frac{1}{2}u_yv_y,$$

$$C_3^x = -\frac{3}{2}u_yu_{xy}v - \frac{3}{2}u_xu_{yy}v - \frac{1}{2}u_{ty}v + \frac{1}{2}v_tu_y + 3u_y^2v_x + \frac{3}{2}u_xu_yv_y + u_yv_{xy} \\ - u_{xy}v_{xy} - \frac{1}{2}u_{xx}v_{yy} + \frac{1}{2}v_yu_{xxy},$$

$$C_3^y = -u_{zz}v + \frac{3}{2}u_xu_{xy}v + \frac{3}{2}u_{xx}u_yv + \frac{1}{2}u_{ty}v + u_{tx}v + u_{tt}v + \frac{1}{2}v_tu_y + \frac{3}{2}u_xu_yv_x \\ + \frac{1}{2}u_{xx}v_{xy} + \frac{1}{2}u_{xxx}v_y,$$

$$C_3^z = u_{yz}v - u_yv_z;$$

$$C_4^t = -\frac{1}{2}u_{yz}v - \frac{1}{2}u_{xz}v - u_{tz}v + v_tu_z + \frac{1}{2}u_zv_x + \frac{1}{2}u_zv_y,$$

$$C_4^x = -\frac{3}{2}u_xu_{yz}v - 3u_yu_{xz}v + \frac{3}{2}u_zu_{xy}v - \frac{1}{2}u_{tz}v + \frac{1}{2}v_tu_z + \frac{3}{2}u_xu_zv_y + 3u_yu_zv_x \\ - u_{xz}v_{xy} - \frac{1}{2}u_{xx}v_{yz} + \frac{1}{2}v_zu_{xxy} + u_zv_{xxy},$$

$$C_4^y = -\frac{3}{2}u_xu_{xz}v - \frac{3}{2}u_{xx}u_zv - \frac{1}{2}u_{tz}v + \frac{1}{2}v_tu_z + \frac{3}{2}u_xu_zv_x - \frac{1}{2}u_{xx}v_{xz} + \frac{1}{2}u_{xxx}v_z,$$

$$C_4^z = 3u_xu_{xy}v + 3u_{xx}u_yv + u_{ty}v + u_{tx}v + u_{tt}v + u_{xx}v_{xy} - u_zv_z;$$

$$C_5^t = 2vu_z + tv_yu_z + tv_xu_z + 2tv_tu_z - 2zvu_{zz} + \frac{1}{2}zu_yv_y - tvu_{yz} - \frac{1}{2}zvu_{yy} \\ + \frac{1}{2}zv_yu_x + \frac{1}{2}zu_yv_x + \frac{1}{2}zu_xv_x - tvu_{xz} - zvu_{xy} + 6zvu_xu_{xy} - \frac{1}{2}zvu_{xx} \\ + 6zvu_yu_{xx} + 2zv_{xy}u_{xx} + zv_yu_t + zv_xu_t + zu_yv_t + zu_xv_t + 2zu_tv_t - 2tvu_{tz},$$

$$C_5^x = 3zv_xu_y^2 + \frac{3}{2}zv_yu_xu_y + 6tu_zv_xu_y + 3zu_xv_xu_y - 6tvu_{xz}u_y - \frac{3}{2}zvu_{xy}u_y + zv_{xy}u_y \\ + 6zv_xu_tu_y + \frac{1}{2}zv_tv_y - 6zv_u_{tx}u_y + \frac{3}{2}zv_yu_x^2 - vu_z - zvu_{zz} + 3tu_zv_yu_x \\ - 3tvu_{yz}u_x - \frac{3}{2}zvu_{yy}u_x + 3tvu_zu_{xy} + 3zvu_xu_{xy} - 2tu_{xz}v_{xy} - zu_{xy}v_{xy} - tv_{yz}u_{xx} \\ - \frac{1}{2}zv_{yy}u_{xx} - \frac{1}{2}zv_{xy}u_{xx} + tv_zu_{xxy} + \frac{1}{2}zv_yu_{xxy} + \frac{1}{2}zv_xu_{xxy} + 2tu_zv_{xxy} + zu_xv_{xxy}$$

$$\begin{aligned}
& + 3zv_y u_x u_t + 3zvu_{xy} u_t + 2zv_{xy} u_t + tu_z v_t + \frac{1}{2} zu_x v_t + zu_{xy} v_t + zu_t v_t - tvu_{tz} \\
& + \frac{1}{2} zvu_{ty} - 3zvu_x u_{ty} - zu_{xx} v_{ty} + \frac{1}{2} zvu_{tx} - 2zv_{xy} u_{tx}, \\
C_5^y = & \frac{3}{2} zv_x u_x^2 + 3tu_z v_x u_x + \frac{3}{2} zu_y v_x u_x - 3tvu_{xz} u_x + \frac{3}{2} zvu_{xy} u_x - 3zvu_{xx} u_x \\
& + 3zv_x u_t u_x + \frac{1}{2} zv_t u_x - 3zvu_{tx} u_x - vu_z - zvu_{zz} - 3tvu_z u_{xx} + \frac{3}{2} zvu_y u_{xx} \\
& - tv_{xz} u_{xx} + \frac{1}{2} zv_{xy} u_{xx} - \frac{1}{2} zu_{xx} v_{xx} + tv_z u_{xxx} + \frac{1}{2} zv_y u_{xxx} + \frac{1}{2} zv_x u_{xxx} \\
& - 3zvu_{xx} u_t + tu_z v_t + \frac{1}{2} zu_y v_t + zu_{xx} v_t + zu_t v_t - tvu_{tz} + \frac{1}{2} zvu_{ty} + \frac{1}{2} zvu_{tx} \\
& - zu_{xx} v_{tx},
\end{aligned}$$

$$\begin{aligned}
C_5^z = & -2tu_z v_z - zu_y v_z - zu_x v_z - 2zu_t v_z + vu_y + zvu_{yz} + vu_x + zvu_{xz} \\
& + 6tvu_x u_{xy} + 6tvu_y u_{xx} + 2tv_{xy} u_{xx} + 2vu_t + 2zvu_{tz} + 2tvu_{ty} + 2tvu_{tx} + 2tvu_{tt};
\end{aligned}$$

$$\begin{aligned}
C_6^t = & -9tu_{zz} v - 6u_y v - \frac{9}{2} zu_{yz} v - \frac{9}{2} yu_{yy} v - 6u_x v - \frac{9}{2} zu_{xz} v - \frac{3}{2} tu_{xy} v - \frac{3}{2} xu_{xy} v \\
& - \frac{9}{2} yu_{xy} v + 27tu_x u_{xy} v - \frac{3}{2} tu_{xx} v - \frac{3}{2} xu_{xx} v + 27tu_y u_{xx} v - 12u_t v - 9zu_{tz} v \\
& + \frac{9}{2} tu_{ty} v - 9yu_{ty} v + \frac{3}{2} tu_{tx} v - 3xu_{tx} v + v - \frac{xv_y}{2} - \frac{yv_y}{2} + \frac{3}{2} uv_y + \frac{9}{2} zu_z v_y \\
& + \frac{9}{2} yu_y v_y + \frac{3}{2} tv_y u_x + \frac{3}{2} xv_y u_x - \frac{xv_x}{2} - \frac{yv_x}{2} + \frac{3}{2} uv_x + \frac{9}{2} zu_z v_x + \frac{9}{2} yu_y v_x \\
& + \frac{3}{2} tu_x v_x + \frac{3}{2} xu_x v_x + 9tv_{xy} u_{xx} + \frac{9}{2} tv_y u_t + \frac{9}{2} tv_x u_t - xv_t - yv_t + 3uv_t + 9zu_z v_t \\
& + 9yu_y v_t + 3tu_x v_t + 3xu_x v_t + 9tu_t v_t,
\end{aligned}$$

$$\begin{aligned}
C_6^x = & 27yv_x u_y^2 + 3vu_y - 36vu_x u_y + \frac{27}{2} yv_y u_x u_y - 3xv_x u_y - 3yv_x u_y + 9uv_x u_y \\
& + 27zu_z v_x u_y + 9tu_x v_x u_y + 9xu_x v_x u_y - 27zvu_{xz} u_y - \frac{27}{2} yvu_{xy} u_y + 9yv_{xy} u_y \\
& + 27tv_x u_t u_y + \frac{9}{2} yv_t u_y - 27tvu_{tx} u_y + \frac{9}{2} tv_y u_x^2 + \frac{9}{2} xv_y u_x^2 - 3tvu_{zz} - 3xvu_{zz} \\
& - \frac{3}{2} xv_y u_x - \frac{3}{2} yv_y u_x + \frac{9}{2} uv_y u_x + \frac{27}{2} zu_z v_y u_x - \frac{27}{2} zvu_{yz} u_x - \frac{27}{2} yvu_{yy} u_x \\
& - \frac{3}{2} xv_u u_{xy} - \frac{3}{2} yvu_{xy} + \frac{9}{2} uv_u u_{xy} + \frac{27}{2} zvu_z u_{xy} + 9tv_u u_{xy} + 9xv_u u_{xy} - 6u_x v_{xy} \\
& - 9zu_{xz} v_{xy} - 9yu_{xy} v_{xy} + v_{xy} - 9v_y u_{xx} - \frac{9}{2} zv_{yz} u_{xx} - \frac{9}{2} yv_{yy} u_{xx}
\end{aligned}$$

$$\begin{aligned}
& -\frac{3}{2}tv_{xy}u_{xx} - \frac{3}{2}xv_{xy}u_{xx} + \frac{9}{2}vu_{xxy} + \frac{9}{2}zv_zu_{xxy} + \frac{9}{2}yv_yu_{xxy} + \frac{3}{2}tv_xu_{xxy} \\
& + \frac{3}{2}xv_xu_{xxy} - xv_{xxy} - yv_{xxy} + 3uv_{xxy} + 9zu_zv_{xxy} + 3tu_xv_{xxy} + 3xu_xv_{xxy} \\
& - 6vu_t + \frac{27}{2}tv_yu_xu_t + \frac{27}{2}tvu_{xy}u_t + 9tv_{xxy}u_t - \frac{xv_t}{2} - \frac{yv_t}{2} + \frac{3}{2}uv_t + \frac{9}{2}zu_zv_t \\
& + \frac{3}{2}tu_xv_t + \frac{3}{2}xu_xv_t + \frac{9}{2}tu_{xxy}v_t + \frac{9}{2}tu_tv_t - \frac{9}{2}zv_tu_{tz} + 3tvu_{ty} + 3xvu_{ty} - \frac{9}{2}yvu_{ty} \\
& - \frac{27}{2}tvu_xu_{ty} - \frac{9}{2}tu_{xx}v_{ty} + \frac{3}{2}tvu_{tx} + \frac{3}{2}xvu_{tx} - 9tv_{xy}u_{tx} - \frac{3}{2}tvu_{tt} + 3xvu_{tt}, \\
C_6^y = & -9vu_x^2 + \frac{9}{2}tv_xu_x^2 + \frac{9}{2}xv_xu_x^2 - \frac{3}{2}xv_xu_x - \frac{3}{2}yv_xu_x + \frac{9}{2}uv_xu_x + \frac{27}{2}zu_zv_xu_x \\
& + \frac{27}{2}yu_yv_xu_x - \frac{27}{2}zv_{xx}u_x + \frac{27}{2}yvu_{xy}u_x - 9tvu_{xx}u_x - 9xvu_{xx}u_x + \frac{27}{2}tv_xu_tu_x \\
& + \frac{3}{2}tv_tu_x + \frac{3}{2}xv_tu_x - \frac{27}{2}tvu_{tx}u_x - 9yvu_{zz} + \frac{3}{2}xvu_{xx} + \frac{3}{2}yvu_{xx} - \frac{9}{2}uvu_{xx} \\
& - \frac{27}{2}zv_zu_{xx} + \frac{27}{2}yvu_yu_{xx} - 6v_xu_{xx} - \frac{9}{2}zv_{xx}u_{xx} + \frac{9}{2}yv_{xy}u_{xx} - \frac{3}{2}tu_{xx}v_{xx} \\
& - \frac{3}{2}xu_{xx}v_{xx} + \frac{9}{2}vu_{xxx} + \frac{9}{2}zv_zu_{xxx} + \frac{9}{2}yv_yu_{xxx} + \frac{3}{2}tv_xu_{xxx} + \frac{3}{2}xv_xu_{xxx} - 6vu_t \\
& - \frac{27}{2}tvu_{xx}u_t - \frac{xv_t}{2} - \frac{yv_t}{2} + \frac{3}{2}uv_t + \frac{9}{2}zu_zv_t + \frac{9}{2}yu_yv_t + \frac{9}{2}tu_{xxx}v_t + \frac{9}{2}tu_tv_t \\
& - \frac{9}{2}zv_tu_{tz} + \frac{9}{2}yvu_{ty} - \frac{3}{2}tvu_{tx} - \frac{3}{2}xvu_{tx} + 9yvu_{tx} - \frac{9}{2}tu_{xx}v_{tx} - \frac{9}{2}tvu_{tt} + 9yvu_{tt}, \\
C_6^z = & 12vu_z - 9zv_zu_z + xv_z + yv_z - 3uv_z - 9yv_zu_y + 9yvu_{y_i} - 3tv_zu_x - 3xv_zu_x \\
& + 3tvu_{xz} + 3xvu_{xz} + 27zv_uu_{xy} + 27zv_yu_{xx} + 9zv_{xy}u_{xx} - 9tv_zu_t + 9tvu_{tz} \\
& + 9zv_uu_{ty} + 9zv_uu_{tx} + 9zv_uu_{tt};
\end{aligned}$$

$$\begin{aligned}
C_t^7 = & -F_1'v - \frac{1}{2}v_xF_1(z-t) - \frac{1}{2}v_yF_1(z-t) - v_tF_1(z-t), \\
C_x^7 = & -\frac{3}{2}F_1(z-t)u_{xy}v - \frac{1}{2}F_1'v - \frac{3}{2}u_xv_yF_1(z-t) - 3u_yv_xF_1(z-t) \\
& - F_1(z-t)v_{xxy} - \frac{1}{2}v_tF_1(z-t), \\
C_y^7 = & \frac{3}{2}u_{xx}F_1(z-t)v - \frac{1}{2}F_1'v - \frac{3}{2}u_xv_xF_1(z-t) - \frac{1}{2}v_tF_1(z-t), \\
C_z^7 = & v_zF_1(z-t) - F_1'v;
\end{aligned}$$

$$\begin{aligned}
C_t^8 &= F_2' v - \frac{1}{2} v_x F_2(t+z) - \frac{1}{2} v_y F_2(t+z) - v_t F_2(t+z), \\
C_x^8 &= -\frac{3}{2} F_2(t+z) u_{xy} v + \frac{1}{2} F_2' v - \frac{3}{2} u_x v_y F_2(t+z) - 3 u_y v_x F_2(t+z) \\
&\quad - F_2(t+z) v_{xy} - \frac{1}{2} v_t F_2(t+z), \\
C_y^8 &= \frac{3}{2} u_{xx} F_2(t+z) v + \frac{1}{2} F_2' v - \frac{3}{2} u_x v_x F_2(t+z) - \frac{1}{2} v_t F_2(t+z), \\
C_z^8 &= v_z F_2(t+z) - F_2' v.
\end{aligned}$$

5.4 Conclusion

In this chapter we studied the (3+1)-dimensional generalized KP-Boussinesq equation (5.2). Firstly, Lie symmetry method was employed to perform symmetry reductions on (5.2) and the equation was reduced to a fourth-order ordinary differential equation. This ODE was solved by direct integration and thus exact solution of (5.2) was derived. We then used Kudryashov's method on the ODE and obtained more exact solutions of (5.2). Secondly, conservation laws for the equation (5.2) were constructed using Ibragimov's conservation theorem.

Chapter 6

Exact solutions and conservation laws of a (3+1)-dimensional BKP-Boussinesq equation

6.1 Introduction

The B-type of the KP equation has been investigated through a considerable size of research work. This equation has been investigated using a variety of methods, such as the Wronskian and Grammian solutions, multiple exp-function method, simplified Hirota's method, the tanh method, and the rational hyperbolic functions method by a variety of authors.

The universal completely integrable KP equation

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \quad (6.1)$$

is a nonlinear partial differential equation in two spatial and one temporal coordinate which describes the evolution of quasi-one dimensional shallow-water waves

when effects of the surface tension and the viscosity are negligible.

A variety of extended KP equations has been developed and examined in the literature. Recently, a generalized (3+1)-dimensional KP equation

$$u_{xxxx} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0, \quad (6.2)$$

was presented and examined in [61, 63, 67]. This equation was investigated in [63] where Wronskian and Grammian formulations were established for this equation. This equation was studied also in [61, 67] where the simplified Hirota's method was used to obtain multiple soliton solutions, provided specific constraints on the coefficients of the spatial variables are satisfied.

The B-type of KP equation (BKP) [61, 63, 67] is given as

$$u_{ty} - u_{xxx} - 3(u_x u_y)_x + 3u_{xz} = 0. \quad (6.3)$$

In this chapter we consider a new form of a (3+1)-dimensional BKP-Boussinesq equation, given in the form

$$u_{ty} - u_{xxx} - 3(u_x u_y)_x + u_{tt} + 3u_{xz} = 0, \quad (6.4)$$

where, like the previous extension, one extra term, namely u_{tt} is added to the generalized form of the B-type KP equation (6.3). In [64], the simplified Hirota's method was used to obtain multiple soliton solutions for equation (6.4).

In this chapter similarity reductions [19] in conjunction with the (G'/G) -expansion method [12] is employed to obtain exact solutions of the (3+1)-dimensional B-type KP-Boussinesq equation. To complete this work we construct the conservation laws of the BKP equation by utilising Ibragimov's conservation theorem [35].

This work has been submitted for publication [68].

6.2 Exact solutions of (6.4)

6.2.1 Symmetry reductions of (6.4)

In this section we first compute the Lie point symmetries of (3+1)-dimensional BKP-Boussinesq equation (6.4). The vector field

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \quad (6.5)$$

where ξ^i , $i = 1, 2, 3, 4$ and η depend on t, x, y, z and u generates the symmetry group of (6.4) if

$$X^{[4]} \Delta|_{\Delta=0} = 0, \quad (6.6)$$

where

$$\Delta \equiv u_{ty} - u_{xxy} - 3(u_x u_y)_x + u_{tt} + 3u_{xz} = 0$$

and $X^{[4]}$ is the fourth prolongation of X [18]. Expanding equation (6.6) we obtain an overdetermined system of linear homogeneous partial differential equations. Solving this resultant system one obtains the following seven Lie point symmetries:

$$\begin{aligned} X_1 &= 2F_1(z) \frac{\partial}{\partial t} - 3tF_1'(z) \frac{\partial}{\partial x} + \{-3t^2F_1''(z) + 3ytF_1''(z) + xF_1'(z)\} \frac{\partial}{\partial u}, \\ X_2 &= F_2(z) \frac{\partial}{\partial x} - yF_2'(z) \frac{\partial}{\partial u}, \\ X_3 &= 2F_3(z) \frac{\partial}{\partial y} + \{3t^2F_3''(z) - 2xF_3'(z)\} \frac{\partial}{\partial u}, \\ X_4 &= \frac{\partial}{\partial z}, \\ X_5 &= F_4(z) \frac{\partial}{\partial u}, \\ X_6 &= tF_5(z) \frac{\partial}{\partial u}, \\ X_7 &= 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 5z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}, \end{aligned} \quad (6.7)$$

where $F_1(z)$ to $F_5(z)$ are arbitrary functions of z . We now use the four translational symmetries and perform symmetry reductions of (6.4). Consider the symmetry

$X = X_1 + \alpha X_2 + X_3 + X_4$, where α is a constant, $F_1(z) = 1/2$, $F_2(z) = 1$ and $F_3(z) = 1/2$. Solving the characteristics equations of the PDE $XI = 0$ we obtain the four invariants

$$f = x - \alpha t, \quad g = x - \alpha y, \quad h = y - z, \quad u(t, x, y, z) = \theta(f, g, h), \quad (6.8)$$

which when used, reduces the (3+1)-dimensional BKP-Boussinesq equation (6.4) to

$$\begin{aligned} & \alpha^2 \theta_{ff} + \alpha^2 \theta_{fg} + 3\alpha \theta_f \theta_{gg} + 9\alpha \theta_g \theta_{fg} + 3\alpha \theta_f \theta_{fg} + 3\alpha \theta_{fgg} + 3\alpha \theta_{ffg} + 3\alpha \theta_{ffg} \\ & + \alpha \theta_{fff} - 3\theta_f \theta_{gh} - 3\theta_g \theta_{fh} - 6\theta_h \theta_{fg} - 3\theta_{fgh} - 3\theta_{fgh} - \alpha \theta_{fh} - 3\theta_f \theta_{fh} - 3\theta_{fh} \\ & - 3\theta_{ffh} - \theta_{ffh} + 6\alpha \theta_g \theta_{gg} + \alpha \theta_{ggg} - 3\theta_g \theta_{gh} - 3\theta_{gh} - 3\theta_{gg} \theta_h - \theta_{ggg} = 0. \end{aligned} \quad (6.9)$$

We now use the Lie point symmetries of equation (6.9) and perform further symmetry reductions. Equation (6.9) has six Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}, \quad \Gamma_3 = \frac{\partial}{\partial h}, \quad \Gamma_4 = \frac{\partial}{\partial \theta} \\ \Gamma_5 &= \left(\frac{\alpha h - f + g}{\alpha} \right) \frac{\partial}{\partial \theta}, \\ \Gamma_6 &= \left(\frac{3\alpha f - 2h\alpha^2 - 2\alpha g - 3\alpha h + 3f - 3g}{3\alpha} \right) \frac{\partial}{\partial f} \\ &+ \left(\frac{\alpha g - 2h\alpha^2 - 3\alpha h + 3f - 3g}{3\alpha} \right) \frac{\partial}{\partial g} + h \frac{\partial}{\partial h} \\ &- \left(\frac{-5\alpha f + 4\alpha g - \alpha(\theta) - 3f + 3g}{6} \right) \frac{\partial}{\partial \theta} \end{aligned}$$

and by considering the symmetry $\Gamma = \Gamma_1 + \beta \Gamma_2 + \Gamma_3$, where β is a constant, we see that it provides us with three invariants, namely

$$r = f - h, \quad s = g - \beta h, \quad \theta(f, g, h) = \phi(r, s). \quad (6.10)$$

As above, utilization of these invariants, reduces one independent variable of equation (6.9) and we obtain

$$\alpha^2 \phi_{rr} + \alpha \phi_{rr} + \alpha^2 \phi_{rs} + \alpha \beta \phi_{rs} + 3\alpha \phi_r \phi_{ss} + 9\alpha \phi_s \phi_{rs} + 3\alpha \phi_r \phi_{rs}$$

$$\begin{aligned}
& +3\alpha\phi_{rsss} + 3\alpha\phi_{rr}\phi_s + 3\alpha\phi_{rrss} + \alpha\phi_{rrrs} + 3\beta\phi_r\phi_{ss} + 3\beta\phi_{rs} + 9\beta\phi_s\phi_{rs} \\
& +3\beta\phi_r\phi_{rs} + 3\beta\phi_{rsss} + 3\beta\phi_{rr}\phi_s + 3\beta\phi_{rrss} + \beta\phi_{rrrs} + 3\phi_r\phi_{ss} + 3\phi_s\phi_{rs} \\
& +9\phi_r\phi_{rs} + 3\phi_{rs} + \phi_{rsss} + 3\phi_{rr}\phi_s + 3\phi_{rrss} + 3\phi_{rrrs} + 6\phi_r\phi_{rr} + 3\phi_{rr} \\
& +\phi_{rrrr} + 6\alpha\phi_s\phi_{ss} + \alpha\phi_{ssss} + 3\beta\phi_{ss} + 6\beta\phi_s\phi_{ss} + \beta\phi_{ssss} = 0, \tag{6.11}
\end{aligned}$$

which is a nonlinear PDE in two independent variables. Further symmetry reduction will transform equation (6.11) into an ordinary differential equation (ODE). Equation (6.11) has four symmetries, including the two translation symmetries $\Sigma_1 = \partial/\partial r$ and $\Sigma_2 = \partial/\partial s$ and so $\Sigma = \Sigma_1 + \nu\Sigma_2$ yields the two invariants

$$\xi = s - \nu r, \quad \phi(r, s) = \psi(\xi).$$

Using these invariants, equation (6.11) is reduces to the fourth-order nonlinear ODE

$$\begin{aligned}
& \alpha^2\nu^2\psi'' - \alpha^2\nu\psi'' - \alpha\beta\nu\psi'' - \alpha\nu^3\psi'''' + \alpha\nu^2\psi'' + 6\alpha\nu^2\psi'\psi'' + 3\alpha\nu^2\psi'''' \\
& -12\alpha\nu\psi'\psi'' - 3\alpha\nu\psi'''' + 6\alpha\psi'\psi'' + \alpha\psi'''' - \beta\nu^3\psi'''' + 6\beta\nu^2\psi'\psi'' + 3\beta\nu^2\psi'''' \\
& -3\beta\nu\psi'' - 12\beta\nu\psi'\psi'' - 3\beta\nu\psi'''' + 3\beta\psi'' + 6\beta\psi'\psi'' + \beta\psi'''' + \nu^4\psi'''' - 6\nu^3\psi'\psi'' \\
& -3\nu^3\psi'''' + 12\nu^2\psi'\psi'' + 3\nu^2\psi'' + 3\nu^2\psi'''' - 6\nu\psi'\psi'' - 3\nu\psi'' - \nu\psi'''' = 0, \tag{6.12}
\end{aligned}$$

which we write in the simplified form as

$$A\psi''''(\xi) - B\psi'(\xi)\psi''(\xi) + C\psi''(\xi) = 0, \tag{6.13}$$

where $A = (\nu + 1)^3(\nu - \alpha - \beta)$, $B = 6(\nu + 1)^2(\nu - \alpha - \beta)$, $C = (\alpha^2 + \alpha + 3)\nu^2 - (\alpha + 3)\beta\nu - (\alpha^2 + 3)\nu + 3\beta$ and $\xi = (1 - \nu)x + (\nu - \alpha - \beta)y + (\beta - \nu)z + \alpha\nu t$.

6.2.2 Solutions of (6.4) by direct integration

Firstly we pursue direct integration of (6.13) with the sole objective of solving the (3+1)-dimensional BKP-Boussinesq equation (6.4). Integration of (6.13) with

respect to ξ yields the ODE

$$A\psi''' - \frac{B}{2}\psi'^2 + C\psi' + C_1 = 0, \quad (6.14)$$

where C_1 is an arbitrary integration constant. Multiplying (6.14) by ψ'' and integrating the resulting equation yields

$$\frac{1}{2}A(\psi'')^2 - \frac{1}{6}B(\psi')^3 + \frac{1}{2}C(\psi')^2 + C_1\psi' + C_2 = 0, \quad (6.15)$$

where C_2 is an arbitrary integration constant. Now equation (6.15) can be rewritten as

$$\Psi'^2 = \frac{B}{3A}\Psi^3 - \frac{C}{A}\Psi^2 - \frac{2C_1}{A}\Psi - \frac{2C_2}{A}, \quad (6.16)$$

where $\Psi = \psi'$. Suppose that λ_1 , λ_2 and λ_3 are roots of the equation

$$\Psi^3 - \frac{3C}{B}\Psi^2 - \frac{6C_1}{B}\Psi - \frac{6C_2}{B} = 0 \quad (6.17)$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then equation (6.16) can be written in the form

$$\Psi'^2 = \frac{B}{3A}(\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3). \quad (6.18)$$

Now the general solution of (6.16) can be expressed via the Jacobi elliptic function [37, 66, 69]

$$\Psi(\xi) = \lambda_2 + (\lambda_1 - \lambda_2)\text{cn}^2 \left\{ \sqrt{\frac{B(\lambda_1 - \lambda_3)}{12A}}\xi, S^2 \right\}, \quad S^2 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}, \quad (6.19)$$

where cn is the elliptic cosine function. Integrating the above equation with respect to ξ and reverting back to the original variables, we obtain the solution of (6.4) as

$$\begin{aligned} u(t, x, y, z) = & \sqrt{\frac{12A(\lambda_1 - \lambda_2)^2}{B(\lambda_1 - \lambda_3)S^8}} \left\{ \text{EllipticE} \left[\text{sn} \left(\sqrt{\frac{B(\lambda_1 - \lambda_3)}{12A}}\xi, S^2 \right), S^2 \right] \right\} \\ & + \left\{ \lambda_2 - (\lambda_1 - \lambda_2) \frac{1 - S^4}{S^4} \right\} \xi + C, \end{aligned} \quad (6.20)$$

where $\xi = (\nu + 1)x + (\nu - \alpha - \beta)y + (\beta - \nu)z - \alpha\nu t$ and C is an arbitrary constant of integration. Here $\text{EllipticE}[z, k]$ is the incomplete elliptic integral defined by [66]

$$\text{EllipticE}[z, k] = \int_0^z \sqrt{\frac{1 - k^2 w^2}{1 - w^2}} dw.$$

6.2.3 Solutions of (6.4) using the (G'/G) –expansion method

In this section we implement the (G'/G) expansion method in order to obtain exact solutions of (6.4). Assume that the solution of the ODE (6.13) can be expressed as a polynomial in (G'/G) as

$$\psi(\xi) = \sum_{i=0}^M A_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \quad (6.21)$$

where A_i , $i = 1, 2, \dots, M$ are unknown constants and $G(\xi)$ satisfies the linear ODE with constant coefficients

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (6.22)$$

where λ and μ are constants. Considering (6.13), the balancing procedure yields $M = 1$ and thus (6.21) becomes

$$\psi(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right).$$

Substituting this value of $\psi(\xi)$ into (6.13) and using (6.22) yields an algebraic equation, which on splitting with respect to powers of $(G'(\xi)/G(\xi))$ gives

$$\begin{aligned} a_1 A \lambda^3 \mu + 8a_1 A \lambda \mu^2 + a_1^2 B \lambda \mu^2 + a_1 C \lambda \mu &= 0, \\ a_1 A \lambda^4 + 22a_1 A \lambda^2 \mu + 16a_1 A \mu^2 + 2a_1^2 B \lambda^2 \mu + 2a_1^2 B \mu^2 + a_1 C \lambda^2 + 2a_1 C \mu &= 0, \\ 15a_1 A \lambda^3 + 60a_1 A \lambda \mu + a_1^2 B \lambda^3 + 6a_1^2 B \lambda \mu + 3a_1 C \lambda &= 0, \\ 25a_1 A \lambda^2 + 20a_1 A \mu + 2a_1^2 B \lambda^2 + 2a_1^2 B \mu + a_1 C &= 0, \\ 12a_1 A + a_1^2 B &= 0. \end{aligned}$$

A solution of the above system of algebraic equations is

$$A_0 \text{ arbitrary, } A_1 = 2(1 - \nu), \quad C = A(4\mu - \lambda^2).$$

Thus we obtain the following two types of travelling wave solutions of (6.4):

Case 1. When $\lambda^2 - 4\mu > 0$, we have the hyperbolic function solutions

$$u(t, x, y, z) = A_0 + 2(1 - \nu) \left(\Delta_1 \frac{C_0 \cosh(\Delta_1 \xi) + C_1 \sinh(\Delta_1 \xi)}{C_0 \sinh(\Delta_1 \xi) + C_1 \cosh(\Delta_1 \xi)} - \frac{\lambda}{2} \right),$$

where $\xi = (1 - \nu)x + (\nu - \alpha - \beta)y + (\beta - \nu)z + \alpha \nu t$, $\Delta_1 = \sqrt{\lambda^2 - 4\mu}/2$ and C_1 and C_2 are arbitrary constants.

Case 2. When $\lambda^2 - 4\mu < 0$

$$u(t, x, y, z) = A_0 + 2(1 - \nu) \left(\Delta_2 \frac{-C_0 \sin(\Delta_2 \xi) + C_1 \cos(\Delta_2 \xi)}{C_0 \cos(\Delta_2 \xi) + C_1 \sin(\Delta_2 \xi)} - \frac{\lambda}{2} \right),$$

where $\xi = (1 - \nu)x + (\nu - \alpha - \beta)y + (\beta - \nu)z + \alpha \nu t$, $\Delta_2 = \sqrt{4\mu - \lambda^2}/2$ and C_1 and C_2 are arbitrary constants.

6.3 Conservation laws of (6.4)

In this section we derive the conservation laws of (6.4) using Ibragimov's new conservation theorem [35].

We begin by determining the adjoint equation of (6.4) by using

$$F^* \equiv \frac{\delta}{\delta u} (v(u_{ty} + u_{tt} - 3u_x u_{xy} - 3u_{xx} u_y - u_{xxx} y + 3u_{xz})) = 0, \quad (6.23)$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined by

$$\begin{aligned} \frac{\delta}{\delta u} = & -D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_z^2 \frac{\partial}{\partial u_{zz}} + D_t D_y \frac{\partial}{\partial u_{ty}} \\ & + D_x D_y \frac{\partial}{\partial u_{xy}} + D_x^3 D_y \frac{\partial}{\partial u_{xxx} y}. \end{aligned} \quad (6.24)$$

In general the total differential operators D_t , D_x , D_y and D_z are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + u_{xz} \frac{\partial}{\partial u_z} + \cdots, \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yt} \frac{\partial}{\partial u_t} + u_{yx} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_z} + \cdots, \\ D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{zz} \frac{\partial}{\partial u_z} + u_{zt} \frac{\partial}{\partial u_t} + u_{zy} \frac{\partial}{\partial u_y} + u_{zx} \frac{\partial}{\partial u_x} + \cdots. \end{aligned}$$

Thus (6.23) becomes

$$F^* \equiv v_{ty} + v_{tt} - 6v_x u_{xy} - 3u_x v_{xy} - 3u_y v_{xx} - v_{xxx} + 3v_{xz} = 0. \quad (6.25)$$

The BKP equation (6.4) together with its adjoint (6.25) have the Lagrangian

$$\mathcal{L} = v(u_{ty} + u_{tt} - 3u_x u_{xy} - 3u_{xx} u_y - u_{xxx} + 3u_{xz}),$$

which is equivalent to the second-order Lagrangian

$$\mathcal{L} = v(u_{ty} + u_{tt} - 3u_x u_{xy} - 3u_{xx} u_y + 3u_{xz}) - u_{xx} v_{xy}, \quad (6.26)$$

We recall that (6.4) admits the seven point symmetries (6.7). To obtain the conserved vectors corresponding to these infinitesimal generators and the second-order Lagrangian (6.26) we use [35]

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \right] + D_k (W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}, \quad (6.27)$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2$ and j runs from $1, \dots, 4$ in this particular case.

Let us first consider the infinitesimal generator $\partial/\partial t$, which is deduced from X_1 by taking $F_1(z) = 1/2$. The corresponding operator $Y_1 = \partial/\partial t$ and the Lie characteristic functions W^1 and W^2 are $W^1 = -u_t$ and $W^2 = -v_t$. Thus by using (6.27) the conserved vector for the system (6.4) and (6.25) corresponding to $\partial/\partial t$ is

$$C_1^t = 3u_{xz}v - 3u_x u_{xy}v - 3u_{xx} u_y v + \frac{1}{2}u_{ty}v + \frac{1}{2}u_t v_y + u_t v_t - u_{xx} v_{xy},$$

$$\begin{aligned}
C_1^x &= -\frac{3}{2}u_t u_{xy} v - \frac{3}{2}u_{tx} v + \frac{3}{2}u_x u_{ty} v + 3u_y u_{tx} v - \frac{3}{2}u_t u_x v_y - 3u_t u_y v_x - u_t v_{xxy} \\
&\quad - \frac{1}{2}v_t u_{xxy} + \frac{1}{2}u_{xx} v_{ty} + u_{tx} v_{xy} + \frac{3}{2}u_t v_z, \\
C_1^y &= \frac{3}{2}u_t u_{xx} v + \frac{3}{2}u_x u_{tx} v - \frac{1}{2}u_{tt} v - \frac{3}{2}u_t u_x v_x - \frac{1}{2}v_t u_{xxx} + \frac{1}{2}u_{xx} v_{tx} + \frac{1}{2}u_t v_t, \\
C_1^z &= \frac{3}{2}u_t v_x - \frac{3}{2}u_{tx} v.
\end{aligned}$$

Likewise, for the point symmetry $\partial/\partial x$, which is deduced from X_2 by taking $F_2(z) = 1$, the conserved vector is

$$\begin{aligned}
C_2^t &= -\frac{1}{2}u_{xy} v - u_{tx} v + v_t u_x + \frac{1}{2}u_x v_y, \\
C_2^x &= -3u_x u_{xy} v + \frac{3}{2}u_{xz} v + u_{ty} v + u_{tt} v - \frac{3}{2}u_x^2 v_y - 3u_x u_y v_x - u_x v_{xxy} + \frac{1}{2}u_{xx} v_{xy} \\
&\quad - \frac{1}{2}v_x u_{xxy} + \frac{3}{2}u_x v_z, \\
C_2^y &= 3u_{xx} u_x v - \frac{1}{2}u_{tx} v + \frac{1}{2}v_t u_x - \frac{3}{2}u_x^2 v_x + \frac{1}{2}u_{xx} v_{xx} - \frac{1}{2}u_{xxx} v_x, \\
C_2^z &= \frac{3}{2}u_x v_x - \frac{3}{2}u_{xx} v.
\end{aligned}$$

Likewise, for the point symmetry $\partial/\partial y$, which is deduced from X_3 by taking $F_3(z) = 1/2$, the conserved vector is

$$\begin{aligned}
C_3^t &= -\frac{1}{2}u_{yy} v - u_{ty} v + v_t u_y + \frac{1}{2}u_y v_y, \\
C_3^x &= \frac{3}{2}u_y u_{xy} v - \frac{3}{2}u_{yz} v + \frac{3}{2}u_x u_{yy} v - 3u_y^2 v_x - \frac{3}{2}u_x u_y v_y - u_y v_{xxy} + u_{xy} v_{xy} \\
&\quad + \frac{1}{2}u_{xx} v_{yy} - \frac{1}{2}v_y u_{xxy} + \frac{3}{2}u_y v_z, \\
C_3^y &= 3u_{xz} v - \frac{3}{2}u_x u_{xy} v - \frac{3}{2}u_{xx} u_y v + \frac{1}{2}u_{ty} v + u_{tt} v + \frac{1}{2}v_t u_y - \frac{3}{2}u_x u_y v_x - \frac{1}{2}u_{xx} v_{xy} \\
&\quad - \frac{1}{2}u_{xxx} v_y, \\
C_3^z &= \frac{3}{2}u_y v_x - \frac{3}{2}u_{xy} v.
\end{aligned}$$

Likewise, for the point symmetry $\partial/\partial z$, the conserved vector is

$$C_4^t = -\frac{1}{2}u_{yz} v - u_{tz} v + v_t u_z + \frac{1}{2}u_z v_y,$$

$$\begin{aligned}
C_4^x &= -\frac{3}{2}u_{xz}v + \frac{3}{2}u_xu_{yz}v + 3u_yu_{xz}v - \frac{3}{2}u_zu_{xy}v - \frac{1}{2}v_zu_{xxy} - \frac{3}{2}u_xu_zv_y - 3u_yu_zv_x \\
&\quad + u_{xz}v_{xy} + \frac{1}{2}u_{xx}v_{yz} - u_zv_{xxy} + \frac{3}{2}u_zv_z, \\
C_4^y &= \frac{3}{2}u_xu_{xz}v + \frac{3}{2}u_{xx}u_zv - \frac{1}{2}u_{tz}v + \frac{1}{2}v_tu_z - \frac{3}{2}u_xu_zv_x + \frac{1}{2}u_{xx}v_{xz} - \frac{1}{2}u_{xxx}v_z, \\
C_4^z &= \frac{3}{2}u_{xz}v - 3u_xu_{xy}v - 3u_{xx}u_yv + u_{ty}v + u_{tt}v - u_{xx}v_{xy} + \frac{3}{2}u_zv_x.
\end{aligned}$$

Likewise, for the point symmetry $X_5 = F_4(z)\partial/\partial u$, the conserved vector is

$$\begin{aligned}
C_5^t &= -F_4(z)v_t - \frac{1}{2}F_4(z)v_y, \\
C_5^x &= \frac{3}{2}F_4(z)u_{xy}v + \frac{3}{2}F_4'(z)v + \frac{3}{2}F_4(z)u_xv_y + 3F_4(z)u_yv_x + F_4(z)v_{xxy} - \frac{3}{2}F_4(z)v_z, \\
C_5^y &= -\frac{3}{2}F_4(z)u_{xx}v - \frac{1}{2}F_4(z)v_t + \frac{3}{2}F_4(z)u_xv_x, \\
C_5^z &= -\frac{3}{2}F_4(z)v_x.
\end{aligned}$$

Likewise, for the point symmetry X_6 , the conserved vector is

$$\begin{aligned}
C_t^6 &= F_5(z) \left(v - \frac{1}{2}tv_y - tv_t \right), \\
C_x^6 &= \frac{3}{2}tF_5'(z)v + tF_5(z) \left(\frac{3}{2}u_{xy}v + \frac{3}{2}u_xv_y + 3u_yv_x + v_{xxy} - \frac{3}{2}v_z \right), \\
C_y^6 &= \frac{1}{2}F_5(z) (v + 3tu_xv_x - tv_t - 3tu_{xx}v), \\
C_z^6 &= -\frac{3}{2}tF_5(z)v_x.
\end{aligned}$$

Likewise, for the point symmetry X_7 , the conserved vector is

$$\begin{aligned}
C_7^t &= -2vu_y + \frac{3}{2}yv_yu_y - 9tvu_{xx}u_y + 3yv_tu_y + \frac{1}{2}uv_y + \frac{5}{2}zu_zv_y - \frac{5}{2}zvu_{yz} \\
&\quad - \frac{3}{2}yvu_{yy} + \frac{1}{2}xv_yu_x + 9tvu_{xz} - \frac{1}{2}xvu_{xy} - 9tvu_xu_{xy} - 3tv_{xy}u_{xx} - 4vu_t \\
&\quad + \frac{3}{2}tv_yu_t + uv_t + 5zu_zv_t + xu_xv_t + 3tu_tv_t - 5zvu_{tz} + \frac{3}{2}tvu_{ty} - 3yvu_{ty} \\
&\quad - xvu_{tx}, \\
C_7^x &= -9yv_xu_y^2 + \frac{9}{2}yv_zu_y + 12vu_xu_y - \frac{9}{2}yv_yu_xu_y - 3uv_xu_y - 15zu_zv_xu_y \\
&\quad - 3xu_xv_xu_y + 15zvu_{xz}u_y + \frac{9}{2}yvu_{xy}u_y - 3yv_{xxy}u_y - 9tv_xu_tv_y + 9tvu_{tx}u_y
\end{aligned}$$

$$\begin{aligned}
& -\frac{3}{2}xv_yu_x^2 - 9vu_z + \frac{3}{2}uv_z + \frac{15}{2}zu_zv_z - \frac{15}{2}zvu_{zz} - \frac{9}{2}yvu_{yz} + \frac{3}{2}xv_zu_x \\
& -\frac{15}{2}zu_zv_yu_x + \frac{15}{2}zvu_{yz}u_x + \frac{9}{2}yvu_{yz}u_x + \frac{3}{2}xvu_{xz} - \frac{3}{2}uvu_{xy} - \frac{15}{2}zvu_zu_{xy} \\
& -\frac{3}{2}uv_yu_x - 3xvu_xu_{xy} + 2u_xv_{xy} + 5zu_{xz}v_{xy} + 3yu_{xy}v_{xy} + 4v_yu_{xx} + \frac{5}{2}zv_{yz}u_{xx} \\
& + \frac{3}{2}yv_{yz}u_{xx} + \frac{1}{2}xv_{xy}u_{xx} - \frac{5}{2}vu_{xxy} - \frac{5}{2}zv_zu_{xxy} - \frac{3}{2}yv_yu_{xxy} - \frac{1}{2}xv_xu_{xxy} \\
& - uv_{xxy} - 5zu_zv_{xxy} - xu_xv_{xxy} + \frac{9}{2}tv_zu_t - \frac{9}{2}tv_yu_xu_t - \frac{9}{2}tvu_{xy}u_t - 3tv_{xxy}u_t \\
& - \frac{3}{2}tu_{xxy}v_t - \frac{9}{2}tvu_{tz} + xvu_{ty} + \frac{9}{2}tvu_xu_{ty} + \frac{3}{2}tu_{xx}v_{ty} + 3tv_{xy}u_{tx} + xvu_{tt}, \\
C_7^y = & 3vu_x^2 - \frac{3}{2}xv_xu_x^2 - \frac{3}{2}uv_xu_x - \frac{15}{2}zu_zv_xu_x - \frac{9}{2}yu_yv_xu_x + \frac{15}{2}zvu_{xz}u_x \\
& - \frac{9}{2}yvu_{xy}u_x + 3xvu_{xx}u_x - \frac{9}{2}tv_xu_tu_x + \frac{1}{2}xv_tu_x + \frac{9}{2}tvu_{tx}u_x + 9yvu_{xz} \\
& + \frac{3}{2}uvu_{xx} + \frac{15}{2}zvu_zu_{xx} - \frac{9}{2}yvu_yu_{xx} + 3v_xu_{xx} + \frac{5}{2}zv_{xz}u_{xx} - \frac{3}{2}yv_{xy}u_{xx} \\
& + \frac{1}{2}xu_{xx}v_{xx} - \frac{5}{2}vu_{xxx} - \frac{5}{2}zv_zu_{xxx} - \frac{3}{2}yv_yu_{xxx} - \frac{1}{2}xv_xu_{xxx} - 2vu_t + \frac{9}{2}tvu_{xx}u_t \\
& + \frac{1}{2}uv_t + \frac{5}{2}zu_zv_t + \frac{3}{2}yu_yv_t - \frac{3}{2}tu_{xxx}v_t \\
& + \frac{3}{2}tu_tv_t - \frac{5}{2}zvu_{tz} + \frac{3}{2}yvu_{ty} - \frac{1}{2}xvu_{tx} + \frac{3}{2}tu_{xx}v_{tx} - \frac{3}{2}tvu_{tt} + 3yvu_{tt}, \\
C_7^z = & -3vu_x + \frac{3}{2}xv_xu_x - 15zvu_{xy}u_x + \frac{3}{2}uv_x + \frac{15}{2}zu_zv_x + \frac{9}{2}yu_yv_x + \frac{15}{2}zvu_{xz} \\
& - \frac{9}{2}yvu_{xy} - \frac{3}{2}xvu_{xx} - 15zvu_yu_{xx} - 5zv_{xy}u_{xx} + \frac{9}{2}tv_xu_t + 5zvu_{ty} - \frac{9}{2}tvu_{tx} \\
& + 5zvu_{tt}.
\end{aligned}$$

6.4 Conclusion

In this chapter we obtained exact solutions for the (3+1)-dimensional BKP-Boussinesq equation (6.4) with the aid of Lie symmetry method, direct integration as well as the (G'/G) -expansion method. Furthermore we obtained the conservation laws of (6.4) by employing Ibragimov's conservation theorem.

Chapter 7

Exact solutions and conservation laws of the first generalized extended (3+1)-dimensional Jimbo-Miwa equation

7.1 Introduction

The (3+1)-dimensional Jimbo-Miwa equation

$$u_{xxx}y + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \quad (7.1)$$

is the second member of the well-known Kadomtsev-Petviashvili hierarchy of integrable systems. This equation has been widely studied by researchers as it is used to describe some interesting (3+1)-dimensional waves in physics. See for example [70–77] and references therein.

Recently equation (7.1) has been extended to [78]

$$u_{xxxy} + 3(u_y u_x)_x + 2u_{yt} - 3(u_{xz} + u_{yz} + u_{zz}) = 0, \quad (7.2)$$

where the term u_{xz} was extended to $u_{xz} + u_{yz} + u_{zz}$ and because of this reason it is called the first extended (3+1)-dimensional Jimbo-Miwa equation. Applying the simplified Hirota's method, multiple soliton solutions of (7.2) were derived and it was shown that the dispersion relations and the phase shifts of (7.2) were distinct compared to the dispersion and shifts of (7.1). Sun and Chen [77] obtained the lump solutions and their dynamics of (7.1) and (7.2) by using bilinear forms. In addition, the lump-kink solution which contains interaction between a lump and a kink wave were also obtained in [77].

In this chapter we consider a generalized version of the first extended (3+1)-dimensional Jimbo-Miwa equation, namely

$$u_{xxxy} + k(u_y u_x)_x + h u_{yt} - k(u_{xz} + u_{yz} + u_{zz}) = 0, \quad (7.3)$$

where h and k are constants. We obtain exact solutions of (7.3) using symmetry reductions along with the (G'/G) -expansion method. Moreover, we derive conservation laws for (7.3) using the conservation theorem due to Ibragimov.

This work has been submitted for publication [79].

7.2 Exact solutions of equation (7.3)

7.2.1 Symmetry reductions of equation (7.3)

In this section we first calculate the Lie point symmetries of extended (3+1)-dimensional Jimbo-Miwa equation (7.3) and later use the translation symmetries to transform extended (3+1)-dimensional Jimbo-Miwa equation (7.3) into the fourth-order nonlinear ODE.

The symmetry group of the extended (3+1)-dimensional Jimbo-Miwa equation (7.3) will be generated by the vector field of the form

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial z} + \xi^4 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u},$$

where ξ^i , $i = 1, 2, 3, 4$ and η depend on t, x, y, z and u . Applying the fourth prolongation $X^{[4]}$ to equation (7.3) we obtain an overdetermined system of linear partial differential equations. Solving this resultant system one obtains the following nine Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = f_1(t) \frac{\partial}{\partial u}, \\ X_5 &= z f_2(t) \frac{\partial}{\partial u}, \quad X_6 = k f_3(t) \frac{\partial}{\partial x} + h x f_3'(t) \frac{\partial}{\partial u}, \\ X_7 &= (kt + hx) \frac{\partial}{\partial y} + 2kt \frac{\partial}{\partial z} - hx \frac{\partial}{\partial u}, \\ X_8 &= hz \frac{\partial}{\partial x} + 4hy \frac{\partial}{\partial y} + (2kt + 2hz) \frac{\partial}{\partial z} - (hx + hy) \frac{\partial}{\partial u}, \\ X_9 &= (4hx + hz) \frac{\partial}{\partial x} + (6hz - 6kt) \frac{\partial}{\partial z} + 12ht \frac{\partial}{\partial t} - (hx + hy + 4hu) \frac{\partial}{\partial u}. \end{aligned}$$

We now utilize the symmetry $X = \alpha X_1 + X_2 + X_3 + X_6$ with $f_3 = 1/k$, where α is a constant, and reduce the extended (3+1)-dimensional Jimbo-Miwa equation (7.3) to a PDE in three independent variables. Solving the associated Lagrange system for X , we obtain the following four invariants:

$$w = z - y, \quad f = t - \alpha y, \quad g = x - y, \quad \theta = u. \quad (7.4)$$

Now treating θ as the new dependent variable and f, g and w as new independent variables, the extended (3+1)-dimensional Jimbo-Miwa equation (7.3) transforms to

$$\begin{aligned} &\alpha \theta_{fgg} + \theta_{ggg} + \theta_{wgg} + h(\alpha \theta_{ff} + \theta_{fg} + \theta_{wf}) + k \theta_{gg}(\alpha \theta_f + \theta_g + \theta_w) \\ &+ k \theta_g(\alpha \theta_{fg} + \theta_{gg} + \theta_{wg}) - \alpha k \theta_{wf} = 0, \end{aligned} \quad (7.5)$$

which is a nonlinear PDE in three independent variables. We now use the Lie point symmetries of equation (7.5) and reduce it to a PDE in two independent variables.

Equation (7.5) has the following six symmetries:

$$\begin{aligned}\Gamma_1 &= \frac{\partial}{\partial w}, \quad \Gamma_2 = \frac{\partial}{\partial f}, \quad \Gamma_3 = \frac{\partial}{\partial g}, \quad \Gamma_4 = \frac{\partial}{\partial \theta}, \quad \Gamma_5 = (\alpha w - f) \frac{\partial}{\partial \theta}, \\ \Gamma_6 &= 3\alpha^3 k^2 w \frac{\partial}{\partial w} + 3f\alpha^3 k^2 \frac{\partial}{\partial f} + (w\alpha^3 k^2 + \alpha^3 g k^2 + \alpha^2 f k^2) \frac{\partial}{\partial g} \\ &\quad + (2w\alpha^2 h k - \alpha^3 g k^2 - \alpha^3 k^2 \theta + 2\alpha^2 f k^2 + \alpha^2 g h k - w\alpha h^2 - 3\alpha f h k + f h^2) \frac{\partial}{\partial \theta}.\end{aligned}$$

By using the symmetry $\Gamma = \beta\Gamma_1 + \Gamma_2 + \Gamma_3$, where β is a constant, and reduce the equation (7.5) to a PDE in two independent variables. Solving the associated Lagrange system for Γ , we obtain the following three invariants:

$$r = g - f, \quad s = w - \beta f, \quad \phi = \theta. \quad (7.6)$$

Now treating ϕ as the new dependent variable and r and s as new independent variables, the extended (3+1)-dimensional Jimbo-Miwa equation (7.3) transforms to

$$\begin{aligned}\phi_{rrrs} + \phi_{rrrr} - \alpha(\beta\phi_{rrrs} + \phi_{rrrr}) + k\phi_{rr}((1 - \alpha\beta)\phi_s - (\alpha - 1)\phi_r) \\ + h((\alpha - 1)\phi_{rr} + (2\alpha\beta - \beta - 1)\phi_{rs} + \beta(\alpha\beta - 1)\phi_{ss}) + \alpha k(\phi_{rs} + \beta\phi_{ss}) \\ + k\phi_r((1 - \alpha\beta)\phi_{rs} - (\alpha - 1)\phi_{rr}) = 0,\end{aligned} \quad (7.7)$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of equation (7.7) and transform it to an ordinary differential equation (ODE). Equation (7.7) gives the four symmetries, which include $\Sigma_1 = \partial/\partial r$ and $\Sigma_2 = \partial/\partial s$. The symmetry $\Sigma = \delta\Sigma_1 + \Sigma_2$, where δ is a constant yields the following two invariants

$$p = r - \delta s, \quad H = \phi,$$

which gives rise to a group invariant solution $\phi = H(p)$ and consequently using these invariants, equation (7.7) is transformed into the fourth-order nonlinear ODE

$$(\alpha + \delta - \alpha\beta\delta - 1)H'''' - 2k(1 - \alpha - \delta + \alpha\beta\delta)H'H''$$

$$+ (\beta\delta - 1)(h(\alpha + \delta - \alpha\beta\delta - 1) + \alpha\delta k)H'' = 0,$$

which can be written as

$$AH''''(p) - BH'(p)H''(p) + CH''(p) = 0, \quad (7.8)$$

where $A = \alpha + \delta - \alpha\beta\delta - 1$, $B = -2kA$, $C = (\beta\delta - 1)(hA + \alpha\delta k)$ and $p = x + (\alpha - 1 + \delta - \alpha\beta\delta)y - \delta z + (\beta\delta - 1)t$.

7.2.2 Exact solutions of (7.3) by direct integration

Integrating (7.8) twice with respect to p we obtain

$$\frac{1}{2}AH''^2 - \frac{1}{6}BH'^3 + \frac{1}{2}CH'^2 + K_1H' + K_2 = 0, \quad (7.9)$$

where K_1 and K_2 are integration constants. Letting $H' = F$, equation (7.9) becomes

$$F'^2 - \frac{B}{3A}F^3 + \frac{C}{A}F^2 + \frac{2K_1}{A}F + \frac{2K_2}{A} = 0. \quad (7.10)$$

Let us assume that λ_1 , λ_2 and λ_3 , where $\lambda_1 \geq \lambda_2 \geq \lambda_3$, are roots of the algebraic equation

$$F^3 - \frac{3C}{B}F^2 - \frac{6K_1}{B}F - \frac{6K_2}{B} = 0. \quad (7.11)$$

Equation (7.10) can now be written as

$$F'^2 = \frac{B}{3A}(F - \lambda_1)(F - \lambda_2)(F - \lambda_3). \quad (7.12)$$

The general solution of (7.10) can thus be expressed in terms of the Jacobi elliptic cosine amplitude function [69]

$$F(p) = \lambda_2 + (\lambda_1 - \lambda_2) \operatorname{cn}^2 \left(\left\{ \frac{B}{12A}(\lambda_1 - \lambda_3) \right\}^{1/2} p, S^2 \right), \quad S^2 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}, \quad (7.13)$$

where $0 \leq S^2 \leq 1$. Integrating the above equation and reverting to the original variables, the solution of the extended (3+1)-dimensional Jimbo-Miwa equation (7.3) is given by

$$u(t, x, y, z) = \lambda_2 p + \frac{2(\lambda_1 - \lambda_2)}{\sqrt{2(\lambda_1 - \lambda_3)} S^4} \left\{ \text{EllipticE} \left[\text{sn} \left(\left\{ \frac{B}{12A} (\lambda_1 - \lambda_3) \right\}^{1/2} p, S^2 \right), S^2 \right] - (1 - S^4) \left\{ \frac{B}{12A} (\lambda_1 - \lambda_3) \right\}^{1/2} p \right\}, \quad (7.14)$$

where $p = x + (\alpha - 1 + \delta - \alpha\beta\delta)y - \delta z + (\beta\delta - 1)t$ and $\text{EllipticE}[z, k]$ is the incomplete elliptic integral defined by [66]

$$\text{EllipticE}[z, k] = \int_0^z \sqrt{\frac{1 - k^2 w^2}{1 - w^2}} dw.$$

7.2.3 Solutions of (7.3) using the (G'/G) -expansion method

In this section we implement the (G'/G) expansion method in order to obtain exact solutions of (7.3). Assume that the solution of the ODE (7.8) can be expressed as a polynomial in (G'/G) as

$$F(p) = \sum_{i=0}^M A_i \left(\frac{G'(p)}{G(p)} \right)^i, \quad (7.15)$$

where A_i , $i = 1, 2, \dots, M$ are unknown constants and $G(p)$ satisfies the linear ODE with constant coefficients

$$G''(p) + \lambda G'(p) + \mu G(p) = 0, \quad (7.16)$$

where λ and μ are constants. Considering (7.8), the balancing procedure yields $M = 1$ and thus (7.15) becomes

$$F(p) = A_0 + A_1 \frac{G'(p)}{G(p)}. \quad (7.17)$$

Substituting this value of $F(p)$ into (7.8) and using (7.16) yields an algebraic equation, which on splitting with respect to powers of $(G'(p)/G(p))$ gives

$$\begin{aligned}
& \alpha\mu A_1\lambda^3 - \alpha\beta\delta\mu A_1\lambda^3 + \delta\mu A_1\lambda^3 - \mu A_1\lambda^3 + 2k\mu^2 A_1^2\lambda - 2k\alpha\mu^2 A_1^2\lambda + 2k\alpha\beta\delta\mu^2 A_1^2\lambda \\
& - 2k\delta\mu^2 A_1^2\lambda + 8\alpha\mu^2 A_1\lambda - 8\alpha\beta\delta\mu^2 A_1\lambda + 8\delta\mu^2 A_1\lambda - 8\mu^2 A_1\lambda + h\mu A_1\lambda - h\alpha\mu A_1\lambda \\
& + h\alpha\beta\delta\mu A_1\lambda - h\beta\delta\mu A_1\lambda + h\alpha\beta\delta\mu A_1\lambda - h\alpha\beta\delta\beta\delta\mu A_1\lambda - h\delta\mu A_1\lambda + k\alpha\delta\mu A_1\lambda \\
& - k\alpha\beta\delta\delta\mu A_1\lambda + h\beta\delta\delta\mu A_1\lambda = 0, \\
& \alpha A_1\lambda^4 - \alpha\beta\delta A_1\lambda^4 + \delta A_1\lambda^4 - A_1\lambda^4 + 4k\mu A_1^2\lambda^2 - 4k\alpha\mu A_1^2\lambda^2 + 4k\alpha\beta\delta\mu A_1^2\lambda^2 \\
& - 4k\delta\mu A_1^2\lambda^2 + hA_1\lambda^2 - h\alpha A_1\lambda^2 + h\alpha\beta\delta A_1\lambda^2 - h\beta\delta A_1\lambda^2 + h\alpha\beta\delta A_1\lambda^2 \\
& - h\alpha\beta\delta\beta\delta A_1\lambda^2 - h\delta A_1\lambda^2 + k\alpha\delta A_1\lambda^2 - k\alpha\beta\delta\delta A_1\lambda^2 + h\beta\delta\delta A_1\lambda^2 + 22\alpha\mu A_1\lambda^2 \\
& - 22\alpha\beta\delta\mu A_1\lambda^2 + 22\delta\mu A_1\lambda^2 - 22\mu A_1\lambda^2 + 4k\mu^2 A_1^2 - 4k\alpha\mu^2 A_1^2 + 4k\alpha\beta\delta\mu^2 A_1^2 \\
& - 4k\delta\mu^2 A_1^2 + 16\alpha\mu^2 A_1 - 16\alpha\beta\delta\mu^2 A_1 + 16\delta\mu^2 A_1 - 16\mu^2 A_1 + 2h\mu A_1 - 2h\alpha\mu A_1 \\
& + 2h\alpha\beta\delta\mu A_1 - 2h\beta\delta\mu A_1 + 2h\alpha\beta\delta\mu A_1 - 2h\alpha\beta\delta\beta\delta\mu A_1 - 2h\delta\mu A_1 + 2k\alpha\delta\mu A_1 \\
& - 2k\alpha\beta\delta\delta\mu A_1 + 2h\beta\delta\delta\mu A_1 = 0, \\
& 2kA_1^2\lambda^3 - 2k\alpha A_1^2\lambda^3 + 2k\alpha\beta\delta A_1^2\lambda^3 - 2k\delta A_1^2\lambda^3 + 15\alpha A_1\lambda^3 - 15\alpha\beta\delta A_1\lambda^3 + 15\delta A_1\lambda^3 \\
& - 15A_1\lambda^3 + 12k\mu A_1^2\lambda - 12k\alpha\mu A_1^2\lambda + 12k\alpha\beta\delta\mu A_1^2\lambda - 12k\delta\mu A_1^2\lambda + 3hA_1\lambda - 3h\alpha A_1\lambda \\
& + 3h\alpha\beta\delta A_1\lambda - 3h\beta\delta A_1\lambda + 3h\alpha\beta\delta A_1\lambda - 3h\alpha\beta\delta\beta\delta A_1\lambda - 3h\delta A_1\lambda + 3k\alpha\delta A_1\lambda \\
& - 3k\alpha\beta\delta\delta A_1\lambda + 3h\beta\delta\delta A_1\lambda + 60\alpha\mu A_1\lambda - 60\alpha\beta\delta\mu A_1\lambda + 60\delta\mu A_1\lambda - 60\mu A_1\lambda = 0, \\
& 8kA_1^2\lambda^2 - 8k\alpha A_1^2\lambda^2 + 8k\alpha\beta\delta A_1^2\lambda^2 - 8k\delta A_1^2\lambda^2 + 50\alpha A_1\lambda^2 - 50\alpha\beta\delta A_1\lambda^2 + 50\delta A_1\lambda^2 \\
& - 50A_1\lambda^2 + 8k\mu A_1^2 - 8k\alpha\mu A_1^2 + 8k\alpha\beta\delta\mu A_1^2 - 8k\delta\mu A_1^2 + 2hA_1 - 2h\alpha A_1 + 2h\alpha\beta\delta A_1 \\
& - 2h\beta\delta A_1 + 2h\alpha\beta\delta A_1 - 2h\alpha\beta\delta\beta\delta A_1 - 2h\delta A_1 + 2k\alpha\delta A_1 - 2k\alpha\beta\delta\delta A_1 + 2h\beta\delta\delta A_1 \\
& + 40\alpha\mu A_1 - 40\alpha\beta\delta\mu A_1 + 40\delta\mu A_1 - 40\mu A_1 = 0, \\
& k\lambda A_1^2 - k\alpha\lambda A_1^2 + k\alpha\beta\delta\lambda A_1^2 - k\delta\lambda A_1^2 + 6\alpha\lambda A_1 - 6\alpha\beta\delta\lambda A_1 + 6\delta\lambda A_1 - 6\lambda A_1 = 0, \\
& kA_1^2 - k\alpha A_1^2 + k\alpha\beta\delta A_1^2 - k\delta A_1^2 + 6\alpha A_1 - 6\alpha\beta\delta A_1 + 6\delta A_1 - 6A_1 = 0.
\end{aligned}$$

A solution of the above system of algebraic equations is

$$A_0 \text{ arbitrary, } A_1 = \frac{6}{k}, \quad \delta = -\frac{(\alpha - \alpha\beta\delta - 1)(\beta\delta h - h + \lambda^2 - 4\mu)}{\beta\delta h - h + \lambda^2 + \alpha k - \alpha\beta\delta k - 4\mu}. \quad (7.18)$$

Thus we obtain the following two types of travelling wave solutions of (6.4):

Case 1. When $\lambda^2 - 4\mu > 0$, we have the hyperbolic function solutions

$$u(t, x, y, z) = A_0 + A_1 \left(\Delta_1 \frac{C_0 \cosh(\Delta_1 \xi) + C_1 \sinh(\Delta_1 \xi)}{C_0 \sinh(\Delta_1 \xi) + C_1 \cosh(\Delta_1 \xi)} - \frac{\lambda}{2} \right), \quad (7.19)$$

where $\xi = (1 - \nu)x + (\nu - \alpha - \beta)y + (\beta - \nu)z + \alpha\nu t$, $\Delta_1 = \sqrt{\lambda^2 - 4\mu}/2$ and C_1 and C_2 are arbitrary constants.

Case 2. When $\lambda^2 - 4\mu < 0$

$$u(t, x, y, z) = A_0 + A_1 \left(\Delta_2 \frac{-C_0 \sin(\Delta_2 \xi) + C_1 \cos(\Delta_2 \xi)}{C_0 \cos(\Delta_2 \xi) + C_1 \sin(\Delta_2 \xi)} - \frac{\lambda}{2} \right), \quad (7.20)$$

where $\xi = (1 - \nu)x + (\nu - \alpha - \beta)y + (\beta - \nu)z + \alpha\nu t$, $\Delta_2 = \sqrt{4\mu - \lambda^2}/2$ and C_1 and C_2 are arbitrary constants.

Case 3. When $\lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u(t, x, y, z) = A_0 + A_1 \left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 p} \right), \quad (7.21)$$

where $k = x + (1 - \beta)t - 2\beta y - z$, C_1 and C_2 are arbitrary constants.

7.3 Conservation laws of (7.3) using Ibragimov's theorem

In this section we derive the conservation laws of (7.3) using Ibragimov's new conservation theorem.

We start by writing the adjoint equation of (7.3) by using

$$F^* \equiv \frac{\delta}{\delta u} [v \{ u_{xxxy} + k(u_y u_x)_x + h u_{yt} - k(u_{xz} + u_{yz} + u_{zz}) \}] = 0, \quad (7.22)$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined by

$$\begin{aligned} \frac{\delta}{\delta u} = & -D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_z^2 \frac{\partial}{\partial u_{zz}} + D_y D_t \frac{\partial}{\partial u_{yt}} + D_x D_z \frac{\partial}{\partial u_{xz}} \\ & + D_y D_z \frac{\partial}{\partial u_{yz}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_x^3 D_y \frac{\partial}{\partial u_{xxxxy}}. \end{aligned} \quad (7.23)$$

In general the total differential operators D_t , D_x , D_y and D_z are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + u_{xz} \frac{\partial}{\partial u_z} + \dots, \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yt} \frac{\partial}{\partial u_t} + u_{yx} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_z} + \dots, \\ D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{zz} \frac{\partial}{\partial u_z} + u_{zt} \frac{\partial}{\partial u_t} + u_{zy} \frac{\partial}{\partial u_y} + u_{zx} \frac{\partial}{\partial u_x} + \dots. \end{aligned} \quad (7.24)$$

Thus the adjoint equation (8.16) becomes

$$h v_{yt} + 2k v_x u_{xy} + k u_x v_{xy} + k u_y v_{xx} - k v_{xz} - k v_{yz} - k v_{zz} + v_{xxxxy} = 0 \quad (7.25)$$

The Lagrangian of (7.3) together with its adjoint (8.16) have a Lagrangian

$$\mathcal{L} = v (h u_{yt} - k(u_{xz} + u_{yz} + u_{zz}) + k u_x u_{xy} + k u_{xx} u_y + u_{xxxxy}).$$

The extended symmetries of (7.3) and the adjoint equation (7.25) are

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial y}, \quad Y_3 = \frac{\partial}{\partial z}, \quad Y_4 = f_1(t) \frac{\partial}{\partial u}, \quad Y_5 = z f_2(t) \frac{\partial}{\partial u}, \\ Y_6 &= k f_3(t) \frac{\partial}{\partial x} + h x f_3'(t) \frac{\partial}{\partial u}, \quad Y_7 = (kt + hx) \frac{\partial}{\partial y} + 2kt \frac{\partial}{\partial z} - hx \frac{\partial}{\partial u}, \\ Y_8 &= hz \frac{\partial}{\partial x} + 4hy \frac{\partial}{\partial y} + (2kt + 2hz) \frac{\partial}{\partial z} - (hx + hy) \frac{\partial}{\partial u} - 2hv \frac{\partial}{\partial v}, \\ Y_9 &= (4hx + hz) \frac{\partial}{\partial x} + (6hz - 6kt) \frac{\partial}{\partial z} + 12ht \frac{\partial}{\partial t} - (hx + hy + 4hu) \frac{\partial}{\partial u} - 6hv \frac{\partial}{\partial v}. \end{aligned}$$

To obtain the conserved vectors corresponding to these Lie point symmetries and the Lagrangian we use

$$T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} + \dots \right] + D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \dots \right] + \dots,$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2$ and j runs from $1, \dots, 4$ in this particular case. Thus the conserved vectors corresponding to the above symmetries are given by

$$\begin{aligned} T_x^1 &= \frac{1}{2} k u_{xt} v - \frac{1}{2} k u_x u_{yt} v - k u_y u_{xt} v + \frac{1}{2} k u_t u_{xy} v - \frac{3}{4} u_{xxyt} v + \frac{1}{2} k u_t u_x v_y + k u_t u_y v_x \\ &\quad - \frac{1}{2} k u_t v_z - \frac{1}{2} u_{xt} v_{xy} + \frac{1}{2} v_x u_{xyt} - \frac{1}{4} v_{xx} u_{yt} + \frac{1}{4} v_y u_{xxt} + \frac{3}{4} u_t v_{xxy}, \\ T_y^1 &= -\frac{1}{2} h u_{tt} v + \frac{1}{2} k u_{xt} v - \frac{1}{2} k u_x u_{xt} v - \frac{1}{2} k u_t u_{xx} v - \frac{1}{4} u_{xxxx} v + \frac{1}{2} h u_t v_t \\ &\quad + \frac{1}{2} k u_t u_x v_x - \frac{1}{2} k u_t v_z - \frac{1}{4} v_{xx} u_{xt} + \frac{1}{4} v_x u_{xxt} + \frac{1}{4} u_t v_{xxx}, \\ T_z^1 &= k u_{xt} v + \frac{1}{2} k u_{yt} v + \frac{1}{2} k u_{xt} v - \frac{1}{2} k u_t v_x - \frac{1}{2} k u_t v_y - k u_t v_z, \\ T_t^1 &= \frac{1}{2} h u_{yt} v - k u_{zz} v - k u_{yz} v - k u_{xz} v + k u_x u_{xy} v + k u_{xx} u_y v + u_{xxy} v + \frac{1}{2} h u_t v_y; \end{aligned}$$

$$\begin{aligned} T_x^2 &= -\frac{1}{2} k u_y u_{xy} v + \frac{1}{2} k u_{yz} v - \frac{1}{2} k u_x u_{yy} v - \frac{3}{4} u_{xxyy} v + k u_y^2 v_x + \frac{1}{2} k u_x u_y v_y \\ &\quad - \frac{1}{2} k u_y v_z + \frac{3}{4} u_y v_{xy} - \frac{1}{2} u_{xy} v_{xy} + \frac{1}{2} v_x u_{xyy} - \frac{1}{4} u_{yy} v_{xx} + \frac{1}{4} v_y u_{xxy}, \\ T_y^2 &= \frac{1}{2} h u_{yt} v - k u_{zz} v - \frac{1}{2} k u_{yz} v - k u_{xz} v + \frac{1}{2} k u_x u_{xy} v + \frac{1}{2} k u_{xx} u_y v + \frac{3}{4} u_{xxy} v \\ &\quad + \frac{1}{2} h v_t u_y + \frac{1}{2} k u_x u_y v_x - \frac{1}{2} k u_y v_x - \frac{1}{4} v_{xx} u_{xy} + \frac{1}{4} v_x u_{xxy} + \frac{1}{4} u_y v_{xxx}, \\ T_z^2 &= k u_{yz} v + \frac{1}{2} k u_{yy} v + \frac{1}{2} k u_{xy} v - \frac{1}{2} k u_y v_x - k u_y v_y - \frac{1}{2} k u_y v_y, \\ T_t^2 &= \frac{1}{2} h u_y v_y - \frac{1}{2} h u_{yy} v; \end{aligned}$$

$$\begin{aligned} T_x^3 &= \frac{1}{2} k u_{zz} v - \frac{1}{2} k u_x u_{yz} v - k u_y u_{xz} v + \frac{1}{2} k u_z u_{xy} v - \frac{3}{4} u_{xxyz} v + \frac{1}{2} k u_x u_z v_y \\ &\quad + k u_y u_z v_x - \frac{1}{2} k u_z v_z - \frac{1}{2} u_{xz} v_{xy} + \frac{1}{2} v_x u_{xyz} - \frac{1}{4} v_{xx} u_{yz} + \frac{1}{4} v_y u_{xxz} \end{aligned}$$

$$\begin{aligned}
& + \frac{3}{4}u_z v_{xy}, \\
T_y^3 &= -\frac{1}{2}hu_{zt}v - \frac{1}{2}ku_{xx}u_zv + \frac{1}{2}ku_{zz}v - \frac{1}{2}ku_xu_{xz}v - \frac{1}{4}u_{xxxx}v + \frac{1}{2}hv_tu_z \\
& + \frac{1}{2}ku_xu_zv_x - \frac{1}{2}ku_zv_z + \frac{1}{4}u_zv_{xxx} - \frac{1}{4}v_{xx}u_{xz} + \frac{1}{4}v_xu_{xxz}, \\
T_z^3 &= hu_{yt}v - \frac{1}{2}ku_{yz}v - \frac{1}{2}ku_{xz}v + ku_xu_{xy}v + ku_{xx}u_yv + u_{xxy}v - \frac{1}{2}ku_zv_x \\
& - \frac{1}{2}ku_zv_y - ku_zv_z, \\
T_t^3 &= \frac{1}{2}hu_zv_y - \frac{1}{2}hu_{yz}v;
\end{aligned}$$

$$\begin{aligned}
T_x^4 &= f_1(t) \left(\frac{1}{2}kv_z - \frac{1}{2}ku_{xy}v - \frac{1}{2}ku_xv_y - ku_yv_x - \frac{3}{4}v_{xy} \right), \\
T_y^4 &= \frac{1}{2}f_1(t) \left(ku_{xx}v - hv_t - ku_xv_x + kv_z - \frac{1}{2}v_{xxx} \right) + \frac{1}{2}hf_1'(t)v, \\
T_z^4 &= f_1(t) \left(\frac{1}{2}kv_x + \frac{1}{2}kv_y + kv_z \right), \\
T_t^4 &= -\frac{1}{2}hf_1(t)v_y;
\end{aligned}$$

$$\begin{aligned}
T_x^5 &= f_2(t) \left(\frac{1}{2}kzv_z - \frac{1}{2}kzu_{xy}v - \frac{1}{2}kv - \frac{1}{2}kzu_xv_y - kzu_yv_x - \frac{3}{4}zv_{xy} \right), \\
T_y^5 &= \frac{1}{2}f_2(t) \left(kzu_{xx}v - kv - hzv_t - kzu_xv_x + kzv_z - \frac{1}{2}zv_{xxx} \right) + \frac{1}{2}hzf_2'(t)v, \\
T_z^5 &= f_2(t) \left(\frac{1}{2}kzv_x + \frac{1}{2}kzv_y + kzv_z - kv \right), \\
T_t^5 &= -\frac{1}{2}hzf_2(t)v_y;
\end{aligned}$$

$$\begin{aligned}
T_x^6 &= hf_3'(t) \left(ku_yv - \frac{1}{2}kxu_{xy}v - \frac{1}{2}kxu_xv_y - kxu_yv_x + \frac{1}{2}kxv_z + \frac{1}{2}v_{xy} - \frac{3}{4}xv_{xy} \right) \\
& + k^2f_3(t) \left(u_xu_{xy}v - u_{zz}v - u_{yz}v - \frac{1}{2}u_{xz}v + \frac{1}{2}u_x^2v_y + u_xu_yv_x - \frac{1}{2}u_xv_z \right) \\
& + kf_3(t) \left(\frac{3}{4}u_xv_{xx} - \frac{1}{2}u_{xx}v_{xy} - \frac{1}{4}v_{xx}u_{xy} + \frac{1}{2}v_xu_{xxy} + \frac{1}{4}u_{xxx}v_y + \frac{1}{4}u_{xxy}v \right)
\end{aligned}$$

$$\begin{aligned}
& + h k f_3(t) u_{yt} v, \\
T_y^6 = & + \frac{1}{2} h^2 x f_3''(t) v + \frac{1}{2} h f_3'(t) \left(k x u_{xx} v - h x v_t - \frac{1}{2} x v_{xxx} - k x u_x v_x + k x v_z + \frac{1}{2} v_{xx} \right) \\
& + k^2 f_3(t) \left(\frac{1}{2} u_{xz} v - u_x u_{xx} v - \frac{1}{2} u_x v_z + \frac{1}{2} u_x^2 v_x \right) + \frac{1}{2} h k f_3(t) (v_t u_x - u_{xt} v) \\
& + \frac{1}{4} k f_3(t) (u_{xxx} v_x + u_x v_{xxx} - u_{xxx} v - u_{xx} v_{xx}), \\
T_z^6 = & h k f_3'(t) \left(\frac{1}{2} x v_x - \frac{1}{2} v + \frac{1}{2} x v_y + x f' v_z \right) \\
& + k^2 f_3(t) \left(u_{xz} v + \frac{1}{2} u_{xy} v + \frac{1}{2} u_{xx} v - \frac{1}{2} u_x v_y - u_x v_z - \frac{1}{2} u_x v_x \right), \\
T_t^6 = & \frac{1}{2} h k f_3(t) u_x v_y - \frac{1}{2} h k f_3(t) u_{xy} v - \frac{1}{2} h^2 x f_3'(t) v_y;
\end{aligned}$$

$$\begin{aligned}
T_x^7 = & - t u_z v_z k^2 + t v u_{zz} k^2 - \frac{1}{2} t v_z u_y k^2 + \frac{1}{2} t v u_{yz} k^2 + t u_z v_y u_x k^2 + \frac{1}{2} t u_y v_y u_x k^2 \\
& - t v u_{yz} u_x k^2 - \frac{1}{2} t v u_{yy} u_x k^2 + t u_y^2 v_x k^2 + 2 t u_z u_y v_x k^2 - 2 t v u_y u_{xz} k^2 \\
& + t v u_z u_{xy} k^2 - \frac{1}{2} t v u_y u_{xy} k^2 - \frac{1}{2} h x v_z k - \frac{1}{2} h v u_y k - \frac{1}{2} h z v_z u_y k + \frac{1}{2} h z v u_{yz} k \\
& + \frac{1}{2} h x v_y u_x k + \frac{1}{2} h z u_y v_y u_x k - \frac{1}{2} h z v u_{yy} u_x k + h z u_y^2 v_x k + h x u_y v_x k \\
& + \frac{1}{2} h x v u_{xy} k - \frac{1}{2} h z v u_y u_{xy} k - t u_{xz} v_{xy} k - \frac{1}{2} t u_{xy} v_{xy} k + t v_x u_{xyz} k + \frac{1}{2} t v_x u_{xyy} k \\
& - \frac{1}{2} t u_{yz} v_{xx} k - \frac{1}{4} t u_{yy} v_{xx} k + \frac{1}{2} t v_y u_{xxx} k + \frac{1}{4} t v_y u_{xxy} k + \frac{3}{2} t u_z v_{xxy} k + \frac{3}{4} t u_y v_{xxy} k \\
& - \frac{3}{2} t v u_{xxy} k - \frac{3}{4} t v u_{xxyy} k - \frac{1}{2} h v_{xy} - \frac{1}{2} h z u_{xy} v_{xy} + \frac{1}{2} h z v_x u_{xyy} - \frac{1}{4} h z u_{yy} v_{xx} \\
& + \frac{1}{4} h z v_y u_{xxy} + \frac{3}{4} h x v_{xxy} + \frac{3}{4} h z u_y v_{xxy} - \frac{3}{4} h z v u_{xxyy}, \\
T_y^7 = & \frac{1}{2} h^2 (x v_t + z v_t u_y + z v u_{yt}) + k h \left(t v_t u_z - v u_z - \frac{1}{2} x v_z - t v u_{zt} - z v u_{zz} \right) \\
& + \frac{1}{2} h k (t v_t u_y - z v_z u_y + t v u_{yt} - z v u_{yz} - v u_x + x u_x v_x + z u_y u_x v_x) \\
& - k z v u_{xz} h + \frac{1}{2} k z v u_x u_{xy} h - \frac{1}{2} k x v u_{xx} h + \frac{1}{2} k z v u_y u_{xx} h \\
& + \frac{1}{4} h (z v_x u_{xxy} + x v_{xxx} + z u_y v_{xxx} + 3 z v u_{xxy} - z u_{xy} v_{xx} - v_{xx}) \\
& + k^2 t (u_z u_x v_x - u_z v_z - v u_{xz} - v u_x u_{xz} - v u_z u_{xx})
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}k^2t(vu_xu_{xy} - v_zu_y - vu_{yz} + u_yu_xv_z + vu_yu_{xx}) \\
& + \frac{1}{2}kt(v_xu_{xxz} - u_{xz}v_{xx} + u_zv_{xxx} - vu_{xxxz}) \\
& + \frac{1}{4}kt(u_yv_{xxx} - u_{xy}v_{xx} + v_xu_{xxy} + 3vu_{xxxxy}), \\
T_z^7 = & hk\left(u_yv + 2tu_{yt}v + zu_{yz}v - zu_yv_z - xv_z + \frac{1}{2}zu_{yy}v + \frac{1}{2}zu_{xy}v\right) \\
& + \frac{1}{2}k^2tu_{xy}v + 2k^2tu_xu_{xy}v + 2k^2tu_{xx}u_yv + 2ktu_{xxxxy}v \\
& + \frac{1}{2}hk(v - zu_yv_x - zu_yv_y - xv_y - xv_x) \\
& - k^2t\left(u_zv_x + u_yv_z + u_zv_y + \frac{1}{2}u_yv_y + 2u_zv_z + \frac{1}{2}u_yv_x - \frac{1}{2}u_{yy}v + u_{xz}v\right), \\
T_t^7 = & hk\left(tu_zv_y - tu_{yz}v - \frac{1}{2}tu_{yy}v + \frac{1}{2}tu_yv_y\right) + \frac{1}{2}h^2(zu_yv_y + xv_y - zu_{yy}v); \\
T_x^8 = & \frac{1}{2}khz(v_yu_x^2 - v_zu_x - vu_{xx}) + \frac{1}{2}khx(v_yu_x - v_z + vu_{xy}) + khv(u_z - u_y) \\
& + \frac{1}{2}khy(v_yu_x - v_z + vu_{xy}) + kh(xu_yv_x + yu_yv_x) \\
& + khz(u_zv_yu_x - vu_{yz}u_x - u_zv_z - vu_{yz} + u_yu_xv_x + vu_zu_{xy}) \\
& + 2khy(vu_{yz} - v_zu_y + u_yv_yu_x - vu_{yy}u_x + 2u_y^2v_x) \\
& - 2kvu_yu_xh + 2kzu_zu_yv_xh - 2kzv u_yu_{xz}h - 2kyvu_yu_{xy}h + kzvu_xu_{xy}h \\
& + 2v_xu_{xy}h - zu_{xz}v_{xy}h - 2yu_{xy}v_{xy}h - \frac{1}{2}v_{xy}h + zv_xu_{xyz}h + 2yv_xu_{xy}h \\
& - \frac{1}{2}zv_{xy}u_{xx}h - u_yv_{xx}h - \frac{1}{2}zu_{yz}v_{xx}h - yu_{yy}v_{xx}h - \frac{1}{4}zu_{xy}v_{xx}h - \frac{1}{4}v_{xx}h \\
& + \frac{1}{2}zv_yu_{xxz}h - 3vu_{xxy}h + yv_yu_{xxy}h + \frac{1}{2}zv_xu_{xxy}h + \frac{3}{4}xv_{xxy}h + \frac{3}{4}yv_{xxy}h \\
& + \frac{3}{2}zu_zv_{xxy}h + 3yu_yv_{xxy}h + \frac{3}{4}zu_xv_{xxy}h - \frac{3}{2}zv u_{xxy}h - 3yv u_{xxy}h \\
& + \frac{1}{4}zv_yu_{xxx}h + \frac{1}{4}zv u_{xxy}h - k^2tu_zv_z + k^2tv u_{zz} + k^2tu_zv_yu_x - k^2tv u_{yz}u_x \\
& + 2k^2tu_zu_yv_x - 2k^2tv u_yu_{xz} + k^2tv u_zu_{xy} - ktu_{xz}v_{xy} + ktv_xu_{xyz} \\
& - \frac{1}{2}ktu_{yz}v_{xx} + \frac{1}{2}ktv_yu_{xxx} + \frac{3}{2}ktu_zv_{xxy} - \frac{3}{2}ktv u_{xxy} + zv u_{yt}h^2,
\end{aligned}$$

$$\begin{aligned}
T_y^8 = & \frac{1}{2}xv_t h^2 + \frac{1}{2}yv_t h^2 + zv_t u_z h^2 - zv u_{zt} h^2 + 2yv_t u_y h^2 + 2yv u_{yt} h^2 + \frac{1}{2}z v_t u_x h^2 \\
& - \frac{1}{2}z v u_{xt} h^2 + ktv_t u_z h - \frac{1}{2}kxv_z h - \frac{1}{2}kyv_z h - kzu_z v_z h - ktv u_{zt} h \\
& - 4kyv u_{zz} h + kzv u_{zz} h - 2kyv_z u_y h - 2kyv u_{yz} h - \frac{1}{2}kzv_z u_x h + \frac{1}{2}kzu_x^2 v_x h \\
& + \frac{1}{2}kxu_x v_x h + \frac{1}{2}kyu_x v_x h + kzu_x u_x v_x h + 2kyu_y u_x v_x h - 4kyv u_{xz} h \\
& + \frac{1}{2}kzv u_{xz} h - kzv u_x u_{xz} h + 2kyv u_x u_{xy} h - \frac{1}{2}kxv u_{xx} h - \frac{1}{2}kyv u_{xx} h \\
& - kzv u_x u_{xx} h + 2kyv u_y u_{xx} h - kzv u_x u_{xx} h - \frac{1}{2}zu_{xz} v_{xx} h - yu_{xy} v_{xx} h \\
& - \frac{1}{4}zu_{xx} v_{xx} h - \frac{1}{4}v_{xx} h + \frac{1}{2}zv_x u_{xxx} h + yv_x u_{xxy} h + \frac{1}{4}zv_x u_{xxx} h + \frac{1}{4}xv_{xxx} h \\
& + \frac{1}{4}yv_{xxx} h + \frac{1}{2}zu_z v_{xxx} h + yu_y v_{xxx} h + \frac{1}{4}zu_x v_{xxx} h - \frac{1}{2}zv u_{xxxx} h \\
& + 3yv u_{xxy} h - \frac{1}{4}zv u_{xxxx} h - k^2 t u_z v_z + k^2 t v u_{zz} + k^2 t u_z u_x v_x - k^2 t v u_x u_{xz} \\
& - k^2 t v u_z u_{xx} - \frac{1}{2}k t u_{xz} v_{xx} + \frac{1}{2}k t v_x u_{xxx} + \frac{1}{2}k t u_z v_{xxx} - \frac{1}{2}k t v u_{xxxx}, \\
T_z^8 = & 2zv u_{yt} h^2 + kvh + 2kv u_z h - kv v_z h - kyv_z h - 2kzu_z v_z h + 2kv u_y h \\
& - 4kyv_z u_y h - \frac{1}{2}kxv_y h - \frac{1}{2}kyv_y h - kzu_z v_y h - 2kyu_y v_y h + 2ktv u_{yt} h \\
& + 4kyv u_{yz} h - kzv u_{yz} h + 2kyv u_{yy} h + kv u_x h - kv v_z u_x h - \frac{1}{2}kzv_y u_x h \\
& - \frac{1}{2}kxv_x h - \frac{1}{2}kyv_x h - kzu_z v_x h - 2kyu_y v_x h - \frac{1}{2}kzu_x v_x h + 2kyv u_{xy} h \\
& + \frac{1}{2}kzv u_{xy} h + 2kzv u_x u_{xy} h + \frac{1}{2}kzv u_{xx} h + 2kzv u_y u_{xx} h + 2zv u_{xxy} h \\
& + k^2 t (2vu_y u_{xx} - 2u_z v_z - u_z v_y - vu_{yz} - u_z v_x - vu_{xz} + 2vu_x u_{xy}) + 2ktv u_{xxy}, \\
T_t^8 = & \frac{1}{2}h^2 z u_x v_y - 2h^2 u_y v - h^2 z u_{yz} v - 2h^2 y u_{yy} v - \frac{1}{2}h^2 z u_{xy} v - h k t u_{yz} v - \frac{1}{2}h^2 v \\
& + h^2 z u_z v_y + 2h^2 y u_y v_y + \frac{1}{2}h^2 x v_y + \frac{1}{2}h^2 y v_y + h k t u_z v_y; \\
T_x^9 = & 4xv u_{yt} h^2 + zv u_{yt} h^2 + 2kxv_y u_x^2 h + \frac{1}{2}kzv_y u_x^2 h + 5kv u_z h - \frac{1}{2}kxv_z h - \frac{1}{2}kyv_z h \\
& - 2kuv_z h - 6ktv_t v_z h - 3kzu_z v_z h + 6ktv u_{zt} h - 4kxv u_{zz} h + 2kzv u_{zz} h \\
& - kv u_y h - 4kxv u_{yz} h - kzv u_{yz} h - 2kxv_z u_x h - \frac{1}{2}kzv_z u_x h - 10kv u_y u_x h
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}kxv_yu_xh + \frac{1}{2}kyv_yu_xh + 2kuv_yu_xh + 6ktu_tv_yu_xh + 3kzu_zv_yu_xh \\
& - 6ktvu_{yt}u_xh - 3kzvu_{yz}u_xh + kxu_yv_xh + kyv_yv_xh + 4kuu_yv_xh \\
& + 12ktu_tv_yv_xh + 6kzu_zu_yv_xh + 4kxu_yu_xv_xh + kzu_yu_xv_xh - 12ktvu_yu_{xt}h \\
& - 2kxvu_{xz}h - \frac{1}{2}kzvu_{xz}h - 6kzvu_yu_{xz}h + \frac{1}{2}kxvu_{xy}h + \frac{1}{2}kyvu_{xy}h \\
& + 2kuvu_{xy}h + 6ktvu_tv_{xy}h + 3kzvu_zu_{xy}h + 4kxvu_xu_{xy}h + kzvu_xu_{xy}h \\
& + 4v_xu_{xy}h - 4u_xv_{xy}h - 6tu_{xt}v_{xy}h - 3zu_{xz}v_{xy}h - \frac{1}{2}v_{xy}h + 6tv_xu_{xyt}h \\
& + 3zv_xu_{xyz}h + 3v_yu_{xx}h - 2xv_{xy}u_{xx}h - \frac{1}{2}zv_{xy}u_{xx}h - u_yv_{xx}h - 3tu_{yt}v_{xx}h \\
& - \frac{3}{2}zu_{yz}v_{xx}h - xu_{xy}v_{xx}h - \frac{1}{4}zu_{xy}v_{xx}h - \frac{1}{4}v_{xx}h + 3tv_yu_{xxt}h + \frac{3}{2}zv_yu_{xxz}h \\
& - 9vu_{xxy}h + 2xv_xu_{xxy}h + \frac{1}{2}zv_xu_{xxy}h + \frac{3}{4}xv_{xxy}h + \frac{3}{4}yv_{xxy}h + 3uv_{xxy}h \\
& + 9tu_tv_{xxy}h + \frac{9}{2}zu_zv_{xxy}h + 3xu_xv_{xxy}h + \frac{3}{4}zu_xv_{xxy}h - 9tvu_{xxyt}h + xv_yu_{xxx}h \\
& - \frac{9}{2}zv_u_{xxyz}h + \frac{1}{4}zv_yu_{xxx}h + xv_u_{xxx}h + \frac{1}{4}zv_u_{xxx}h + 3k^2tu_zv_z - 3k^2tvu_{zz} \\
& - 3k^2tu_zv_yu_x + 3k^2tvu_{yz}u_x - 6k^2tu_zu_yv_x + 6k^2tvu_yu_{xz} - 3k^2tvu_zu_{xy} \\
& + 3ktu_{xz}v_{xy} - 3ktv_xu_{xyz} + \frac{3}{2}ktu_{yz}v_{xx} - \frac{3}{2}ktv_yu_{xxx} - \frac{9}{2}ktu_zv_{xxy} + \frac{9}{2}ktvu_{xxyz}, \\
T_y^9 = & - 8vu_th^2 + \frac{1}{2}xv_th^2 + \frac{1}{2}yv_th^2 + 2uv_th^2 + 6tu_tv_th^2 - 6tvu_{tt}h^2 + 3zv_tu_zh^2 \\
& - 3zvu_{xt}h^2 + 2xv_tu_xh^2 + \frac{1}{2}zv_tu_xh^2 - 2xvu_{xt}h^2 - \frac{1}{2}zvu_{xt}h^2 - 4kvu_x^2h \\
& + 8kvu_zh - 3ktv_tu_zh - \frac{1}{2}kxv_zh - \frac{1}{2}kyv_zh - 2kuv_zh - 6ktu_tv_zh - 3kzu_zv_zh \\
& + 9ktvu_{xt}h + 3kzvu_{zz}h - 2kxv_zu_xh - \frac{1}{2}kzv_zu_xh + 2kxu_x^2v_xh + \frac{1}{2}kzu_x^2v_xh \\
& + \frac{1}{2}kxu_xv_xh + \frac{1}{2}kyu_xv_xh + 2kuu_xv_xh + 6ktu_tv_xv_xh + 3kzu_zu_xv_xh \\
& - 6ktvu_xu_{xt}h + 2kxvu_{xz}h + \frac{1}{2}kzvu_{xz}h - 3kzvu_xu_{xz}h - \frac{1}{2}kxvu_{xx}h - \frac{1}{2}kyvu_{xx}h \\
& - 2kuvu_{xx}h - 6ktvu_tv_{xx}h - 3kzvu_zu_{xx}h - 4kxvu_xu_{xx}h - kzvu_xu_{xx}h \\
& + 3v_xu_{xx}h - 2u_xv_{xx}h - 3tu_{xt}v_{xx}h - \frac{3}{2}zu_{xz}v_{xx}h - xu_{xx}v_{xx}h - \frac{1}{4}zu_{xx}v_{xx}h \\
& - \frac{1}{4}v_{xx}h + 3tv_xu_{xxt}h + \frac{3}{2}zv_xu_{xxz}h - 4vu_{xxx}h + xv_xu_{xxx}h + \frac{1}{4}zv_xu_{xxx}h
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4}xv_{xxx}h + \frac{1}{4}yv_{xxx}h + uv_{xxx}h + 3tu_tv_{xxx}h + \frac{3}{2}zu_zv_{xxx}h + xu_xv_{xxx}h \\
& + \frac{1}{4}zu_xv_{xxx}h - 3tvu_{xxx}h - \frac{3}{2}zvu_{xxx}h - xv_{xxxx}h - \frac{1}{4}zvu_{xxx}h + 3k^2tu_zv_z \\
& - 3k^2tvu_{zz} - 3k^2tu_zu_xv_x + 3k^2tvu_xu_{xz} + 3k^2tvu_zu_{xx} + \frac{3}{2}ktu_{xz}v_{xx} \\
& - \frac{3}{2}ktv_xu_{xxz} - \frac{3}{2}ktu_zv_{xxx} + \frac{3}{2}ktvu_{xxx},
\end{aligned}$$

$$\begin{aligned}
T_z^9 = & 6zvu_{yt}h^2 + kvh + 10kvu_zh - kv_zh - kyv_zh - 4kuv_zh - 12ktu_tv_zh \\
& - 6kzu_zv_zh + 12ktvu_{zt}h + 2kvu_yh - \frac{1}{2}kxv_yh - \frac{1}{2}kyv_yh - 2kuv_yh \\
& - 6ktu_tv_yh - 3kzu_zv_yh - 3kzv_{yz}h + 5kvu_xh - 4kxv_zu_xh - kv_zu_xh \\
& - 2kxv_yu_xh - \frac{1}{2}kzv_yu_xh - \frac{1}{2}kxv_xh - \frac{1}{2}kyv_xh - 2kuv_xh - 6ktu_tv_xh \\
& - 3kzu_zv_xh - 2kxu_xv_xh - \frac{1}{2}kzu_xv_xh + 6ktvu_{xt}h + 4kxvu_{xz}h - 2kzv_{xz}h \\
& + 2kxvu_{xy}h + \frac{1}{2}kzv_{xy}h + 6kzv_{xu_{xy}}h + 2kxvu_{xx}h + \frac{1}{2}kzv_{xu_{xx}}h \\
& + 6kzv_{xu_{xx}}h + 6zv_{xxxy}h + 6k^2tu_zv_z + 3k^2tu_zv_y + 3k^2tvu_{yz} + 3k^2tu_zv_x \\
& + 3k^2tvu_{xz} - 6k^2tvu_xu_{xy} - 6k^2tvu_yu_{xx} - 6ktvu_{xxx},
\end{aligned}$$

$$\begin{aligned}
T_t^9 = & 2h^2v_yu - 2h^2u_yv + 6h^2tu_{yt}v - 3h^2zu_{yz}v - 2h^2xu_{xy}v - \frac{1}{2}h^2zu_{xy}v \\
& - 12hktu_{zz}v - 9hktu_{yz}v - 12hktu_{xz}v + 12hktu_xu_{xy}v + 12hktu_{xx}u_yv \\
& + 12htu_{xxx}v - \frac{1}{2}h^2v + 6h^2tu_tv_y + \frac{1}{2}h^2zu_xv_y + 2h^2xu_xv_y + 3h^2zu_zv_y \\
& + \frac{1}{2}h^2xv_y + \frac{1}{2}h^2yv_y - 3hktu_zv_y.
\end{aligned}$$

Remark. It should be noted that the above conservation laws include the energy conservation law, which corresponds to the time translation and three momentum conservation laws, which correspond to the three space translations.

7.4 Conclusion

In this chapter we studied the generalized extended $(3+1)$ -dimensional Jimbo-Miwa equation (7.3). Symmetry reductions of this equation were performed several times until it was reduced to a fourth-order ordinary differential equation. The general solution of this ordinary differential equation was obtained in terms of the incomplete elliptic integral. Travelling wave solutions of (7.3) were also constructed using the (G'/G) -expansion method. Finally, the conservation laws of (8.3) were computed by invoking the conservation theorem due to Ibragimov. These conservation laws included an energy conservation law, which corresponded to the time translation and three momentum conservation laws that corresponded to the three space translations.

Chapter 8

Solutions and conservation laws of the generalized second extended (3+1)-dimensional Jimbo-Miwa equation

8.1 Introduction

The (3+1)-dimensional Jimbo-Miwa equation

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \quad (8.1)$$

is the second member of a Kadomtsev-Petviashvili hierarchy. This equation has been studied extensively by researchers because of the fact that it can be used to describe some fascinating (3+1)-dimensional waves in physics. See for example [70–77] and references therein.

Recently equation (8.1) has been extended to equation [78]

$$u_{xxxxy} + 3(u_y u_x)_x + 2(u_{xt} + u_{yt} + u_{zt}) - 3u_{xz} = 0, \quad (8.2)$$

where the term u_{yt} was extended to $u_{xt} + u_{yt} + u_{zt}$ and because of this reason it is called the extended (3+1)-dimensional Jimbo-Miwa equation. Applying the simplified Hirota's method multiple soliton solutions of (8.2) were derived and it was shown that the dispersion relations and the phase shifts of (8.2) were distinct compared to the dispersion and shifts of (8.1). By using bilinear forms Sun and Chen [77] obtained the lump solutions and their dynamics of (8.1) and (8.2). Furthermore, the lump-kink solution which contains interaction between a lump and a kink wave were also obtained in [77].

In this chapter we consider a generalized version of the extended (3+1)-dimensional Jimbo-Miwa equation, namely

$$u_{xxxxy} + k(u_y u_x)_x + h(u_{xt} + u_{yt} + u_{zt}) - ku_{xz} = 0, \quad (8.3)$$

where h and k are constants. We obtain exact solutions of (8.3) using symmetry reductions along with simplest equation method. Furthermore, we derive conservation laws for (8.3) using the conservation theorem due to Ibragimov.

This work has been submitted for publication [80].

8.2 Exact solutions of (8.3)

In this section we present exact solution to the generalized extended (3+1)-dimensional Jimbo-Miwa equation (8.3).

8.2.1 Symmetry reductions of equation (8.3)

We apply the algorithm for computing Lie point symmetries of (8.3) and then use them to perform symmetry reductions several times until we arrive at an ordinary differential equation (ODE).

The vector field of the form

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial z} + \xi^4 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u},$$

where ξ^i , $i = 1, 2, 3, 4$ and η depend on x, y, z, t and u , will generate a symmetry group of (8.3) provided

$$X^{[4]}(u_{xxxy} + k(u_y u_x)_x + h(u_{xt} + u_{yt} + u_{zt}) - k u_{xz})|_{(8.3)} = 0, \quad (8.4)$$

where $X^{[4]}$ is the fourth prolongation of X . Expanding the determining equation (8.4) and splitting on derivatives of u , we obtain an overdetermined system of linear homogeneous partial differential equations. Solving this resultant system one obtains the values of ξ^i , $i = 1, 2, 3, 4$ and η . Consequently, we have the following nine Lie point symmetries of (8.3):

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial z}, \quad X_5 = f_1(t) \frac{\partial}{\partial u}, \quad X_6 = f_2(z) \frac{\partial}{\partial u}, \\ X_7 &= -3ht \frac{\partial}{\partial t} + (2kt - hx + hz) \frac{\partial}{\partial x} + (2hx + hy - 4kz + hu) \frac{\partial}{\partial u}, \\ X_8 &= ht \frac{\partial}{\partial t} - (kt + hz) \frac{\partial}{\partial x} - hz \frac{\partial}{\partial y} - hz \frac{\partial}{\partial z} + (kt + 2hz) \frac{\partial}{\partial u}, \\ X_9 &= ht \frac{\partial}{\partial t} - kt \frac{\partial}{\partial x} + (hy - hz) \frac{\partial}{\partial y} + (kt - hy + 2hz) \frac{\partial}{\partial u}. \end{aligned} \quad (8.5)$$

We now make use of the four translation symmetries and perform symmetry reductions. Solving the associated Lagrange system for $X = X_1 + \alpha X_2 + X_3 + X_4$, where α is a constant, we obtain four invariants

$$w = z - y, \quad f = t - y, \quad g = x - \alpha y, \quad \theta = u. \quad (8.6)$$

Using these invariants the extended (3+1)-dimensional Jimbo-Miwa equation (8.3) transforms to

$$\begin{aligned} & \theta_{fggg} + \alpha\theta_{gggg} + \theta_{gggw} + k\theta_{gg}(\theta_f + \alpha\theta_g + \theta_w) + k\theta_{gw} + k\theta_g(\theta_{fg} + \alpha\theta_{gg} + \theta_{gw}) \\ & + h((\alpha - 1)\theta_{fg} + \theta_{ff}) = 0, \end{aligned} \quad (8.7)$$

which is a nonlinear PDE in three independent variables. Equation (8.7) has the following seven Lie point symmetries:

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial w}, \quad \Gamma_2 = \frac{\partial}{\partial f}, \quad \Gamma_3 = \frac{\partial}{\partial g}, \quad \Gamma_4 = \frac{\partial}{\partial \theta}, \quad \Gamma_5 = (w - f)\frac{\partial}{\partial \theta}, \\ \Gamma_6 &= w\frac{\partial}{\partial w} + (2w - f)\frac{\partial}{\partial f} + (2\alpha w - g)\frac{\partial}{\partial g} + (2g - 2\alpha w + \theta)\frac{\partial}{\partial \theta}, \\ \Gamma_7 &= 6hkw\frac{\partial}{\partial w} + 6hkw\frac{\partial}{\partial f} + k(7\alpha ah + \alpha bh + ah + ak - bh - bk - 2dh)\frac{\partial}{\partial g} \\ & + (a\alpha^2 h^2 - 2a\alpha h^2 - 2a\alpha hk + ah^2 + 2ahk + ak^2 + 4dhk + 2h\theta k)\frac{\partial}{\partial \theta}. \end{aligned}$$

Utilizing the symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \beta\Gamma_3$, where β is a constant, we reduce equation (8.7) to a PDE in two independent variables. From the associated Lagrange system for Γ , we obtain three invariants

$$r = g - \beta f, \quad s = w - f, \quad \phi = \theta \quad (8.8)$$

and these invariants transform equation (8.7) to

$$\begin{aligned} & (\beta - \alpha)\phi_{rrrr} + (\alpha\beta h - \beta^2 h - \beta h)\phi_{rr} + \alpha h\phi_{rs} - 2\beta h\phi_{rs} - h\phi_{rs} - h\phi_{ss} - 2\alpha k\phi_r\phi_{rr} \\ & + 2\beta k\phi_r\phi_{rr} - k\phi_{rs} = 0, \end{aligned} \quad (8.9)$$

which is a nonlinear PDE in two independent variables. We perform further symmetry reduction on equation (8.9). This equation has five symmetries including the two translation symmetries $\Sigma_1 = \partial/\partial r$ and $\Sigma_2 = \partial/\partial s$. The combination $\Sigma = \nu\Sigma_1 + \Sigma_2$, yields the two invariants

$$q = r - \nu s, \quad F = \phi,$$

which give rise to a group-invariant solution $\phi = F(q)$ and consequently, equation (8.9) is transformed into the fourth-order nonlinear ODE

$$AF''''(q) + BF'(q)F'''(q) + CF''(q) = 0, \quad (8.10)$$

where $A = \alpha - \beta$, $B = 2k(\alpha - \beta)$, $C = h(\beta - \nu)(-\alpha + \beta - \nu + 1) - k\nu$ and $q = x + (\beta - \alpha)y - \nu z + (\nu - \beta)t$.

8.2.2 Exact solutions of (8.3) by direct integration

Integration of the above equation twice with respect to q gives

$$\frac{A}{2}F''^2 + \frac{B}{6}F'^3 + \frac{C}{2}F'^2 + C_1F' + C_2 = 0,$$

where C_1 and C_2 are integration constants. Letting $H = F'$, the above equation becomes

$$H'^2 = -\frac{B}{3A}H^3 - \frac{C}{A}H^2 - \frac{2C_1}{A}H - \frac{2C_2}{A}.$$

Now using the transformation

$$H = -\frac{12A}{B}\wp(q) - \frac{C}{B}, \quad (8.11)$$

we obtain equation for the Weierstrass elliptic function [66]

$$\wp'^2 = 4\wp^3 - g_1\wp - g_2,$$

where

$$g_1 = \frac{C^2 - 2BC_1}{12A^2}, \quad g_2 = \frac{C^3 + 3B(BC_2 - CC_1)}{216A^3}.$$

Thus integrating equation (8.11) and reverting to our original variables we obtain the solution of (8.3), which is given by

$$u(x, y, z, t) = \frac{12A}{B} \zeta(q; g_1, g_2) - \frac{C}{B} q,$$

where $\zeta(q; g_1, g_2)$ is the Weierstrass zeta function defined as $\zeta'(q; g_1, g_2) = -\wp(q; g_1, g_2)$ [66] and $A = \alpha - \beta$, $B = 2k(\alpha - \beta)$, $C = h(\beta - \nu)(-\alpha + \beta - \nu + 1) - k\nu$ and $q = x + (\beta - \alpha)y - \nu z + (\nu - \beta)t$.

8.2.3 Exact solutions of (8.3) using simplest equation method

In this subsection we use the simplest equation method [3, 4] to solve the ODE (8.10) and henceforth one obtains the exact solutions of the generalized extended (3+1)-dimensional Jimbo-Miwa equation (8.3). We use the Bernoulli and Riccati equations as the simplest equations. The Bernoulli equation

$$H'(q) = cH(q) + dH^2(q), \quad (8.12)$$

where c and d are constants has solution

$$H(z) = c \left\{ \frac{\cosh[c(q + C)] + \sinh[c(q + C)]}{1 - d \cosh[c(q + C)] - d \sinh[c(q + C)]} \right\},$$

where C is a constant of integration.

The Riccati equation

$$H'(q) = cH(q) + dH^2(q) + e, \quad (8.13)$$

where c , d and e are constants, has two solutions, namely

$$H(q) = -\frac{c}{2d} - \frac{\theta}{2d} \tanh \left[\frac{1}{2} \theta (q + C) \right]$$

and

$$H(q) = -\frac{c}{2d} - \frac{\theta}{2d} \tanh \left(\frac{1}{2} \theta q \right) + \frac{\operatorname{sech} \left(\frac{\theta q}{2} \right)}{C \cosh \left(\frac{\theta q}{2} \right) - \frac{2d}{\theta} \sinh \left(\frac{\theta q}{2} \right)},$$

with $\theta^2 = c^2 - 4de > 0$ and C is a constant of integration.

The solutions of the ODE (8.10) are assumed to be of the form

$$F(q) = \sum_{i=0}^M A_i (H(q))^i, \quad (8.14)$$

where $H(z)$ solves the Bernoulli or Riccati equation, M is a positive integer which is determined by the balancing procedure and A_i , ($i = 0, 1, \dots, M$) are parameters to be determined.

Solutions of (8.3) using Bernoulli as the simplest equation

From equation (8.10) the balancing procedure yields $M = 1$, so the solutions of (8.10) can be written as

$$F(q) = A_0 + A_1 H(q). \quad (8.15)$$

Substituting (8.15) into (8.10) and invoking the Bernoulli equation (8.12) we obtain the algebraic equation

$$\begin{aligned} & \alpha A_1 c^4 H(q) - A_1 \beta c^4 H(q) + 15\alpha A_1 c^3 d H(q)^2 - 15A_1 \beta c^3 d H(q)^2 + 2\alpha A_1^2 c^3 k H(q)^2 \\ & - 2A_1^2 \beta c^3 k H(q)^2 + 50\alpha A_1 c^2 d^2 H(q)^3 - 50A_1 \beta c^2 d^2 H(q)^3 + 8\alpha A_1^2 c^2 d k H(q)^3 \\ & - 8A_1^2 \beta c^2 d k H(q)^3 - \alpha A_1 \beta c^2 h H(q) + \alpha A_1 c^2 h \nu H(q) + A_1 \beta^2 c^2 h H(q) - 2A_1 \beta c^2 h \nu H(q) \\ & + A_1 \beta c^2 h H(q) + A_1 c^2 h \nu^2 H(q) - A_1 c^2 h \nu H(q) - A_1 c^2 k \nu H(q) + 60\alpha A_1 c d^3 H(q)^4 \\ & - 60A_1 \beta c d^3 H(q)^4 + 10\alpha A_1^2 c d^2 k H(q)^4 - 10A_1^2 \beta c d^2 k H(q)^4 - 3\alpha A_1 \beta c d h H(q)^2 \\ & + 3\alpha A_1 c d h \nu H(q)^2 + 3A_1 \beta^2 c d h H(q)^2 - 6A_1 \beta c d h \nu H(q)^2 + 3A_1 \beta c d h H(q)^2 \\ & + 3A_1 c d h \nu^2 H(q)^2 - 3A_1 c d h \nu H(q)^2 - 3A_1 c d k \nu H(q)^2 + 24\alpha A_1 d^4 H(q)^5 \\ & - 24A_1 \beta d^4 H(q)^5 + 4\alpha A_1^2 d^3 k H(q)^5 - 4A_1^2 \beta d^3 k H(q)^5 - 2\alpha A_1 \beta d^2 h H(q)^3 \\ & + 2\alpha A_1 d^2 h \nu H(q)^3 + 2A_1 \beta^2 d^2 h H(q)^3 - 4A_1 \beta d^2 h \nu H(q)^3 + 2A_1 \beta d^2 h H(q)^3 \\ & + 2A_1 d^2 h \nu^2 H(q)^3 - 2A_1 d^2 h \nu H(q)^3 - 2A_1 d^2 k \nu H(q)^3 = 0. \end{aligned}$$

Equating all coefficients of the function H^i to zero, we obtain the following algebraic system of equations in terms of A_0 and A_1 :

$$\begin{aligned} & \alpha A_1 c^4 - A_1 \beta c^4 - \alpha A_1 \beta c^2 h + \alpha A_1 c^2 h \nu + A_1 \beta^2 c^2 h - 2A_1 \beta c^2 h \nu + A_1 \beta c^2 h + A_1 c^2 h \nu^2 \\ & - A_1 c^2 h \nu - A_1 c^2 k \nu = 0, \end{aligned}$$

$$\begin{aligned}
& 15\alpha A_1 c^3 d - 15A_1 \beta c^3 d + 2\alpha A_1^2 c^3 k - 2A_1^2 \beta c^3 k - 3\alpha A_1 \beta c d h + 3\alpha A_1 c d h \nu + 3A_1 \beta^2 c d h \\
& - 6A_1 \beta c d h \nu + 3A_1 \beta c d h + 3A_1 c d h \nu^2 - 3A_1 c d h \nu - 3A_1 c d k \nu = 0, \\
& 50\alpha A_1 c^2 d^2 - 50A_1 \beta c^2 d^2 + 8\alpha A_1^2 c^2 d k - 8A_1^2 \beta c^2 d k - 2\alpha A_1 \beta d^2 h + 2\alpha A_1 d^2 h \nu \\
& + 2A_1 \beta^2 d^2 h - 4A_1 \beta d^2 h \nu + 2A_1 \beta d^2 h + 2A_1 d^2 h \nu^2 - 2A_1 d^2 h \nu - 2A_1 d^2 k \nu = 0 \\
& 60\alpha A_1 c d^3 - 60A_1 \beta c d^3 + 10\alpha A_1^2 c d^2 k - 10A_1^2 \beta c d^2 k = 0 \\
& 24\alpha A_1 d^4 - 24A_1 \beta d^4 + 4\alpha A_1^2 d^3 k - 4A_1^2 \beta d^3 k = 0.
\end{aligned}$$

Solving the above system with the aid of Mathematica, we obtain

$$\alpha = \beta, \quad k = \frac{h(\nu - 1)(\nu - \beta)}{\nu}, \quad A_0 = \text{arbitrary}, \quad A_1 = -\frac{6d}{k}.$$

Thus a solution of the generalized extended (3+1)-dimensional Jimbo-Miwa equation (8.3) using the Bernoulli equation as the simplest equation is

$$u(t, x, y, z) = A_0 - \frac{6cd}{k} \left\{ \frac{\cosh[c(q + C)] + \sinh[c(q + C)]}{1 - d \cosh[c(q + C)] - d \sinh[c(q + C)]} \right\},$$

where $q = x + (\beta - \alpha)y - \nu z + (\nu - \beta)t$ and C is an arbitrary constant.

Solutions of (8.3) using Riccati as the simplest equation

Substituting (8.15) into (8.10) and using the Riccati equation (8.13) we obtain

$$\begin{aligned}
& 4d^3 k \alpha A_1^2 H(q)^5 - 4d^3 k \beta A_1^2 H(q)^5 + 24d^4 \alpha A_1 H(q)^5 - 24d^4 \beta A_1 H(q)^5 \\
& + 10cd^2 k \alpha A_1^2 H(q)^4 - 10cd^2 k \beta A_1^2 H(q)^4 + 60cd^3 \alpha A_1 H(q)^4 - 60cd^3 \beta A_1 H(q)^4 \\
& + 8c^2 dk \alpha A_1^2 H(q)^3 + 8d^2 ek \alpha A_1^2 H(q)^3 - 8c^2 dk \beta A_1^2 H(q)^3 - 8d^2 ek \beta A_1^2 H(q)^3 \\
& + 2d^2 h \beta^2 A_1 H(q)^3 + 2d^2 h \nu^2 A_1 H(q)^3 + 50c^2 d^2 \alpha A_1 H(q)^3 + 40d^3 e \alpha A_1 H(q)^3 \\
& - 50c^2 d^2 \beta A_1 H(q)^3 - 40d^3 e \beta A_1 H(q)^3 + 2d^2 h \beta A_1 H(q)^3 - 2d^2 h \alpha \beta A_1 H(q)^3 \\
& - 2d^2 h \nu A_1 H(q)^3 - 2d^2 k \nu A_1 H(q)^3 + 2d^2 h \alpha \nu A_1 H(q)^3 - 4d^2 h \beta \nu A_1 H(q)^3 \\
& + 2c^3 k \alpha A_1^2 H(q)^2 + 12cdek \alpha A_1^2 H(q)^2 - 2c^3 k \beta A_1^2 H(q)^2 - 12cdek \beta A_1^2 H(q)^2 \\
& + 3cdh \beta^2 A_1 H(q)^2 + 3cdh \nu^2 A_1 H(q)^2 + 15c^3 d \alpha A_1 H(q)^2 + 60cd^2 e \alpha A_1 H(q)^2 \\
& - 15c^3 d \beta A_1 H(q)^2 - 60cd^2 e \beta A_1 H(q)^2 + 3cdh \beta A_1 H(q)^2 - 3cdh \alpha \beta A_1 H(q)^2
\end{aligned}$$

$$\begin{aligned}
& -3cdh\nu A_1 H(q)^2 - 3cdk\nu A_1 H(q)^2 + 3cdh\alpha\nu A_1 H(q)^2 - 6cdh\beta\nu A_1 H(q)^2 \\
& + 4de^2 k\alpha A_1^2 H(q) + 4c^2 ek\alpha A_1^2 H(q) - 4de^2 k\beta A_1^2 H(q) - 4c^2 ek\beta A_1^2 H(q) \\
& + c^2 h\beta^2 A_1 H(q) + 2deh\beta^2 A_1 H(q) + c^2 h\nu^2 A_1 H(q) + 2deh\nu^2 A_1 H(q) + c^4 \alpha A_1 H(q) \\
& + 16d^2 e^2 \alpha A_1 H(q) + 22c^2 de\alpha A_1 H(q) - c^4 \beta A_1 H(q) - 16d^2 e^2 \beta A_1 H(q) \\
& - 22c^2 de\beta A_1 H(q) + c^2 h\beta A_1 H(q) + 2deh\beta A_1 H(q) - c^2 h\alpha\beta A_1 H(q) \\
& - 2deh\alpha\beta A_1 H(q) - c^2 h\nu A_1 H(q) - 2deh\nu A_1 H(q) - c^2 k\nu A_1 H(q) - 2dek\nu A_1 H(q) \\
& + c^2 h\alpha\nu A_1 H(q) + 2deh\alpha\nu A_1 H(q) - 2c^2 h\beta\nu A_1 H(q) - 4deh\beta\nu A_1 H(q) + 2ce^2 k\alpha A_1^2 \\
& - 2ce^2 k\beta A_1^2 + ceh\beta^2 A_1 + ceh\nu^2 A_1 + 8cde^2 \alpha A_1 + c^3 e\alpha A_1 - 8cde^2 \beta A_1 - c^3 e\beta A_1 \\
& + ceh\beta A_1 - ceh\alpha\beta A_1 - ceh\nu A_1 - cek\nu A_1 + ceh\alpha\nu A_1 - 2ceh\beta\nu A_1 = 0.
\end{aligned}$$

As before, equating coefficients of H^i to zero, we obtain

$$\begin{aligned}
& e\alpha A_1 c^3 - e\beta A_1 c^3 + 2e^2 k\alpha A_1^2 c - 2e^2 k\beta A_1^2 c + eh\beta^2 A_1 c + eh\nu^2 A_1 c + 8de^2 \alpha A_1 c \\
& - 8de^2 \beta A_1 c + eh\beta A_1 c - eh\alpha\beta A_1 c - eh\nu A_1 c - ek\nu A_1 c + eh\alpha\nu A_1 c - 2eh\beta\nu A_1 c = 0, \\
& \alpha A_1 c^4 - \beta A_1 c^4 + 4ek\alpha A_1^2 c^2 - 4ek\beta A_1^2 c^2 + h\beta^2 A_1 c^2 + h\nu^2 A_1 c^2 + 22de\alpha A_1 c^2 \\
& - 22de\beta A_1 c^2 + h\beta A_1 c^2 - h\alpha\beta A_1 c^2 - h\nu A_1 c^2 - k\nu A_1 c^2 + h\alpha\nu A_1 c^2 - 2h\beta\nu A_1 c^2 \\
& + 4de^2 k\alpha A_1^2 - 4de^2 k\beta A_1^2 + 2deh\beta^2 A_1 + 2deh\nu^2 A_1 + 16d^2 e^2 \alpha A_1 - 16d^2 e^2 \beta A_1 \\
& + 2deh\beta A_1 - 2deh\alpha\beta A_1 - 2deh\nu A_1 - 2dek\nu A_1 + 2deh\alpha\nu A_1 - 4deh\beta\nu A_1 = 0, \\
& 2k\alpha A_1^2 c^3 - 2k\beta A_1^2 c^3 + 15d\alpha A_1 c^3 - 15d\beta A_1 c^3 + 12dek\alpha A_1^2 c - 12dek\beta A_1^2 c \\
& + 3dh\beta^2 A_1 c + 3dh\nu^2 A_1 c + 60d^2 e\alpha A_1 c - 60d^2 e\beta A_1 c + 3dh\beta A_1 c - 3dh\alpha\beta A_1 c \\
& - 3dh\nu A_1 c - 3dk\nu A_1 c + 3dh\alpha\nu A_1 c - 6dh\beta\nu A_1 c = 0, \\
& 40e\alpha A_1 d^3 - 40e\beta A_1 d^3 + 8ek\alpha A_1^2 d^2 - 8ek\beta A_1^2 d^2 + 2h\beta^2 A_1 d^2 + 2h\nu^2 A_1 d^2 \\
& + 50c^2 \alpha A_1 d^2 - 50c^2 \beta A_1 d^2 + 2h\beta A_1 d^2 - 2h\alpha\beta A_1 d^2 - 2h\nu A_1 d^2 - 2k\nu A_1 d^2 \\
& + 2h\alpha\nu A_1 d^2 - 4h\beta\nu A_1 d^2 + 8c^2 k\alpha A_1^2 d - 8c^2 k\beta A_1^2 d = 0, \\
& 60c\alpha A_1 d^3 - 60c\beta A_1 d^3 + 10ck\alpha A_1^2 d^2 - 10ck\beta A_1^2 d^2 = 0, \\
& 24\alpha A_1 d^4 - 24\beta A_1 d^4 + 4k\alpha A_1^2 d^3 - 4k\beta A_1^2 d^3 = 0.
\end{aligned}$$

Solving the above system of algebraic equations we obtain

$$\alpha = \beta, \quad k = \frac{h(\nu - 1)(\nu - \beta)}{\nu}, \quad A_0 = \text{arbitrary}, \quad A_1 = -\frac{6d}{k}.$$

Thus solutions of the generalized extended (3+1)-dimensional Jimbo-Miwa equation (8.3) using the Riccati equation as the simplest equation are

$$u(t, x, y, z) = A_0 - \frac{6d}{k} \left\{ -\frac{c}{2d} - \frac{\theta}{2d} \tanh \left[\frac{1}{2} \theta (q + C) \right] \right\}$$

and

$$u(t, x, y, z) = A_0 - \frac{6d}{k} \left\{ -\frac{c}{2d} - \frac{\theta}{2d} \tanh \left(\frac{1}{2} \theta q \right) + \frac{\operatorname{sech} \left(\frac{\theta q}{2} \right)}{C \cosh \left(\frac{\theta q}{2} \right) - \frac{2d}{\theta} \sinh \left(\frac{\theta q}{2} \right)} \right\},$$

where $q = x + (\beta - \alpha)y - \nu z + (\nu - \beta)t$, $\theta^2 = c^2 - 4de > 0$ and C is an arbitrary constant.

8.3 Conservation laws of (8.3) using Ibragimov's theorem

In this section we derive the conservation laws of (8.3) by appealing to Ibragimov's new conservation theorem.

We begin by determining the adjoint equation of (8.3) by utilizing

$$F^* \equiv \frac{\delta}{\delta u} (v(u_{xxxy} + k(u_y u_x)_x + h(u_{xt} + u_{yt} + u_{zt}) - k u_{xz})) = 0, \quad (8.16)$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined by

$$\begin{aligned} \frac{\delta}{\delta u} = & -D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{xt}} + D_y D_t \frac{\partial}{\partial u_{yt}} + D_z D_t \frac{\partial}{\partial u_{zt}} \\ & + D_x D_z \frac{\partial}{\partial u_{xz}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_x^3 D_y \frac{\partial}{\partial u_{xxx y}} \end{aligned} \quad (8.17)$$

and the total differential operators D_t , D_x , D_y and D_z are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_y} + u_{xz} \frac{\partial}{\partial u_z} + \dots, \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yt} \frac{\partial}{\partial u_t} + u_{yx} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_z} + \dots, \\ D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{zz} \frac{\partial}{\partial u_z} + u_{zt} \frac{\partial}{\partial u_t} + u_{zy} \frac{\partial}{\partial u_y} + u_{zx} \frac{\partial}{\partial u_x} + \dots. \end{aligned} \quad (8.18)$$

Thus the adjoint equation (8.16) becomes

$$hv_{xt} + hv_{yt} + hv_{zt} + 2kv_x u_{xy} + ku_x v_{xy} + ku_y v_{xx} - kv_{xz} + v_{xxxy} = 0. \quad (8.19)$$

The Lagrangian of (8.3) and its adjoint equation (8.19) is

$$\mathcal{L} = v(h(u_{xt} + u_{yt} + u_{zt}) + ku_x u_{xy} + ku_{xx} u_y - ku_{xz} + u_{xxxy}) \quad (8.20)$$

and the extended symmetries are

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial y}, \quad Y_4 = \frac{\partial}{\partial z}, \quad Y_5 = f_1(t) \frac{\partial}{\partial u}, \quad Y_6 = f_2(z) \frac{\partial}{\partial u}, \\ Y_7 &= -3ht \frac{\partial}{\partial t} + (2kt + hz - hx) \frac{\partial}{\partial x} + (2hx + hy - 4hz + hu) \frac{\partial}{\partial u}, \\ Y_8 &= ht \frac{\partial}{\partial t} - (kt + hz) \frac{\partial}{\partial x} - hz \frac{\partial}{\partial y} - hz \frac{\partial}{\partial z} + (kt + 2hz) \frac{\partial}{\partial u}, \\ Y_9 &= ht \frac{\partial}{\partial t} - kt \frac{\partial}{\partial x} + (hy - hz) \frac{\partial}{\partial y} + (kt - hy + 2hz) \frac{\partial}{\partial u} - hv \frac{\partial}{\partial v}. \end{aligned}$$

To obtain the conserved vectors corresponding to the Lie point symmetries (8.5)

and the Lagrangian (8.20) we use

$$T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} + \dots \right] + D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} - D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} + \dots \right] + \dots,$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2$ and j runs from $1, \dots, 4$ in this particular case. Thus the conserved vectors corresponding to the nine Lie point symmetries are given by, respectively

$$T_x^1 = \frac{1}{2}ku_{xt}v - \frac{1}{2}hu_{tt}v - \frac{1}{2}ku_x u_{yt}v - ku_y u_{xt}v + \frac{1}{2}ku_t u_{xy}v - \frac{3}{4}u_{xxxy}v + \frac{1}{2}hu_t v_t$$

$$\begin{aligned}
& + \frac{1}{2}ku_tu_xv_y + ku_tu_yv_x - \frac{1}{2}ku_tv_z - \frac{1}{2}u_{xt}v_{xy} + \frac{1}{2}v_xu_{xyt} - \frac{1}{4}v_{xx}u_{yt} \\
& + \frac{1}{4}v_yu_{xxt} + \frac{3}{4}u_tv_{xxy}, \\
T_y^1 = & -\frac{1}{2}hu_{tt}v - \frac{1}{2}ku_xu_{xt}v - \frac{1}{2}ku_tu_{xx}v - \frac{1}{4}u_{xxx}v + \frac{1}{2}hu_tv_t + \frac{1}{2}ku_tu_xv_x \\
& - \frac{1}{4}v_{xx}u_{xt} + \frac{1}{4}v_xu_{xxt} + \frac{1}{4}u_tv_{xxx}, \\
T_z^1 = & -\frac{1}{2}hu_{tt}v + \frac{1}{2}ku_{xt}v + \frac{1}{2}hu_tv_t - \frac{1}{2}ku_tv_x, \\
T_t^1 = & \frac{1}{2}hu_{xt}v + \frac{1}{2}hu_{yt}v + \frac{1}{2}hu_{xt}v - ku_{xz}v + ku_xu_{xy}v + ku_{xx}u_yv + u_{xxy}v \\
& + \frac{1}{2}hu_tv_x + \frac{1}{2}hu_tv_y + \frac{1}{2}hu_tv_z;
\end{aligned}$$

$$\begin{aligned}
T_x^2 = & hu_{xt}v + hu_{yt}v + \frac{1}{2}hu_{xt}v + ku_xu_{xy}v - \frac{1}{2}ku_{xz}v + \frac{1}{4}u_{xxx}v + \frac{1}{2}hv_tu_x \\
& + \frac{1}{2}ku_x^2v_y + ku_xu_yv_x - \frac{1}{2}ku_xv_z + \frac{3}{4}u_xv_{xxy} - \frac{1}{2}u_{xx}v_{xy} - \frac{1}{4}v_{xx}u_{xy} \\
& + \frac{1}{2}v_xu_{xxy} + \frac{1}{4}u_{xxx}v_y, \\
T_y^2 = & \frac{1}{2}hv_tu_x - \frac{1}{2}hu_{xt}v - ku_{xx}u_xv - \frac{1}{4}u_{xxx}v + \frac{1}{2}ku_x^2v_x + \frac{1}{4}u_xv_{xxx} \\
& - \frac{1}{4}u_{xx}v_{xx} + \frac{1}{4}u_{xxx}v_x, \\
T_z^2 = & -\frac{1}{2}hu_{xt}v + \frac{1}{2}ku_{xx}v + \frac{1}{2}hv_tu_x - \frac{1}{2}ku_xv_x, \\
T_t^2 = & -\frac{1}{2}hu_{xz}v - \frac{1}{2}hu_{xy}v - \frac{1}{2}hu_{xx}v + \frac{1}{2}hu_xv_y + \frac{1}{2}hu_xv_z + \frac{1}{2}hu_xv_x;
\end{aligned}$$

$$\begin{aligned}
T_x^3 = & -\frac{1}{2}hu_{yt}v - \frac{1}{2}ku_yu_{xy}v + \frac{1}{2}ku_{yz}v - \frac{1}{2}ku_xu_{yy}v - \frac{3}{4}u_{xxy}v + \frac{1}{2}hv_tu_y \\
& + ku_y^2v_x + \frac{1}{2}ku_xu_yv_y - \frac{1}{2}ku_yv_z + \frac{3}{4}u_yv_{xxy} - \frac{1}{2}u_{xy}v_{xy} + \frac{1}{2}v_xu_{xxy} \\
& - \frac{1}{4}u_{yy}v_{xx} + \frac{1}{4}v_yu_{xxy}, \\
T_y^3 = & hu_{xt}v + \frac{1}{2}hu_{yt}v + hu_{xt}v - ku_{xz}v + \frac{1}{2}ku_xu_{xy}v + \frac{1}{2}ku_{xx}u_yv + \frac{3}{4}u_{xxy}v \\
& + \frac{1}{2}hv_tu_y + \frac{1}{2}ku_xu_yv_x - \frac{1}{4}v_{xx}u_{xy} + \frac{1}{4}v_xu_{xxy} + \frac{1}{4}u_yv_{xxx},
\end{aligned}$$

$$T_z^3 = -\frac{1}{2}hu_{yt}v + \frac{1}{2}ku_{xy}v + \frac{1}{2}hv_tu_y - \frac{1}{2}ku_yv_x,$$

$$T_t^3 = -\frac{1}{2}hu_{yz}v - \frac{1}{2}hu_{yy}v - \frac{1}{2}hu_{xy}v + \frac{1}{2}hu_yv_x + \frac{1}{2}hu_yv_z + \frac{1}{2}hu_yv_y;$$

$$\begin{aligned} T_x^4 = & kv_xu_y^2 + \frac{1}{2}hv_tu_y - \frac{1}{2}kv_zu_y + \frac{1}{2}kv_yu_xu_y + ku_zv_xu_y + ku_xv_xu_y - kuv_xu_y \\ & - \frac{1}{2}kvv_{xy}u_y + \frac{3}{4}v_{xxy}u_y + \frac{1}{2}kv_yu_x^2 + \frac{1}{2}hv_tu_z - \frac{1}{2}ku_zv_z + \frac{1}{2}hvu_{xt} \\ & + \frac{1}{2}kvv_{xz} + \frac{1}{2}hvu_{yt} + \frac{1}{2}kvv_{yz} + \frac{1}{2}hv_tu_x - \frac{1}{2}kv_zu_x + \frac{1}{2}ku_zv_yu_x \\ & - \frac{1}{2}kvv_{yz}u_x - \frac{1}{2}kvv_{yy}u_x + \frac{1}{2}hvu_{xt} - \frac{1}{2}kvv_{xz} + \frac{1}{2}kvv_zu_{xy} + kuv_xu_{xy} \\ & - \frac{1}{2}u_{xz}v_{xy} - \frac{1}{2}u_{xy}v_{xy} + \frac{1}{2}v_xu_{xyz} + \frac{1}{2}v_xu_{xyy} - \frac{1}{2}v_{xy}u_{xx} - \frac{1}{4}u_{yz}v_{xx} \\ & - \frac{1}{4}u_{yy}v_{xx} - \frac{1}{4}u_{xy}v_{xx} + \frac{1}{4}v_yu_{xxx} + \frac{1}{4}v_yu_{xxy} + \frac{1}{2}v_xu_{xxy} + \frac{3}{4}u_zv_{xxy} \\ & + \frac{3}{4}u_xv_{xxy} - \frac{3}{4}vu_{xxyz} - \frac{3}{4}vu_{xxyy} + \frac{1}{4}v_yu_{xxx} + \frac{1}{4}vu_{xxyy}, \end{aligned}$$

$$\begin{aligned} T_y^4 = & \frac{1}{2}hu_{xt}v + \frac{1}{2}hu_{yt}v + \frac{1}{2}hu_{xt}v - \frac{1}{2}ku_xu_{xz}v + \frac{1}{2}ku_xu_{xy}v - ku_{xx}u_{xy}v \\ & - ku_{xz}v - \frac{1}{2}ku_{xx}u_zv + \frac{1}{2}ku_{xx}u_yv - \frac{1}{4}u_{xxxz}v + \frac{3}{4}u_{xxyy}v - \frac{1}{4}u_{xxxx}v \\ & + \frac{1}{2}hv_tu_x + \frac{1}{2}hv_tu_y + \frac{1}{2}hv_tu_z + \frac{1}{2}ku_xu_yv_x + \frac{1}{2}ku_xu_zv_x + \frac{1}{2}ku_x^2v_x \\ & - \frac{1}{4}v_{xx}u_{xy} + \frac{1}{4}v_xu_{xxy} + \frac{1}{4}u_yv_{xxx} - \frac{1}{4}v_{xx}u_{xz} + \frac{1}{4}v_xu_{xxz} + \frac{1}{4}u_zv_{xxx} \\ & + \frac{1}{4}u_xv_{xxx} - \frac{1}{4}u_{xx}v_{xx} + \frac{1}{4}u_{xxx}v_x, \end{aligned}$$

$$\begin{aligned} T_z^4 = & \frac{1}{2}hu_{xt}v + \frac{1}{2}hu_{yt}v + \frac{1}{2}hu_{xt}v - \frac{1}{2}ku_{xz}v + \frac{1}{2}ku_{xy}v + ku_xu_{xy}v + \frac{1}{2}ku_{xx}v \\ & + ku_{xx}u_yv + u_{xxyy}v + \frac{1}{2}hv_tu_x + \frac{1}{2}hv_tu_y + \frac{1}{2}hv_tu_z - \frac{1}{2}ku_yv_x \\ & - \frac{1}{2}ku_zv_x - \frac{1}{2}ku_xv_x, \end{aligned}$$

$$\begin{aligned} T_t^4 = & \frac{1}{2}hu_yv_x - \frac{1}{2}hu_{zz}v - hu_{yz}v - \frac{1}{2}hu_{yy}v - hu_{xz}v - hu_{xy}v - \frac{1}{2}hu_{xx}v + \frac{1}{2}hu_xv_y \\ & + \frac{1}{2}hu_xv_z + \frac{1}{2}hu_zv_x + \frac{1}{2}hu_xv_x + \frac{1}{2}hu_yv_z + \frac{1}{2}hu_zv_y + \frac{1}{2}hu_yv_y + \frac{1}{2}hu_zv_z; \end{aligned}$$

$$\begin{aligned}
T_x^5 &= \frac{1}{2}hf_1'(t)v + f_1(t) \left(\frac{1}{2}kv_z - \frac{1}{2}hv_t - \frac{1}{2}ku_xv_y - ku_yv_x - \frac{3}{4}v_{xxy} - \frac{1}{2}ku_{xy}v \right), \\
T_y^5 &= \frac{1}{2}kf_1(t)u_{xx}v + \frac{1}{2}hf_1'(t)v - \frac{1}{2}hf(t)v_t - \frac{1}{2}kf(t)u_xv_x - \frac{1}{4}f(t)v_{xxx}, \\
T_z^5 &= \frac{1}{2}hf_1'(t)v - \frac{1}{2}hf_1(t)v_t + \frac{1}{2}kf_1(t)v_x, \\
T_t^5 &= -\frac{1}{2}hf_1(t)v_x - \frac{1}{2}hf_1(t)v_y - \frac{1}{2}hf_1(t)v_z;
\end{aligned}$$

$$\begin{aligned}
T_x^6 &= -\frac{1}{2}kf_2'(z)v + f_2(z) \left(\frac{1}{2}kv_z - \frac{1}{2}hv_t - \frac{1}{2}ku_xv_y - ku_yv_x - \frac{3}{4}v_{xxy} - \frac{1}{2}ku_{xy}v \right), \\
T_y^6 &= \frac{1}{2}kf_2(z)u_{xx}v - \frac{1}{2}hf_2(z)v_t - \frac{1}{2}kf_2(z)u_xv_x - \frac{1}{4}f_2(z)v_{xxx}, \\
T_z^6 &= \frac{1}{2}kf_2(z)v_x - \frac{1}{2}hf_2(z)v_t, \\
T_t^6 &= \frac{1}{2}hf_2'(z)v - \frac{1}{2}hf_2(z)v_x - \frac{1}{2}hf_2(z)v_y - \frac{1}{2}hf_2(z)v_z;
\end{aligned}$$

$$\begin{aligned}
T_x^7 &= 2vu_th^2 - xv_th^2 - \frac{1}{2}yv_th^2 + 2zv_th^2 - \frac{1}{2}uv_th^2 - \frac{3}{2}tu_tv_th^2 + \frac{3}{2}tvu_{tt}h^2 \\
&\quad - xvu_{xt}h^2 + zvu_{xt}h^2 - xvu_{yt}h^2 + zvu_{yt}h^2 - \frac{1}{2}xv_tu_xh^2 + \frac{1}{2}zv_tu_xh^2 \\
&\quad - \frac{1}{2}xvu_{xt}h^2 + \frac{1}{2}zvu_{xt}h^2 - \frac{1}{2}kxv_yu_x^2h + \frac{1}{2}kzv_yu_x^2h + 2kvh \\
&\quad - \frac{1}{2}kvu_zh + kxv_zh + \frac{1}{2}kyv_zh - 2kzv_zh + \frac{1}{2}kuv_zh + \frac{3}{2}ktu_tv_zh \\
&\quad + \frac{1}{2}ktvu_{xt}h + 2kvu_yh + 2ktvu_{yt}h + ktv_tu_xh + \frac{1}{2}kxv_zu_xh \\
&\quad - \frac{1}{2}kzv_zu_xh + \frac{5}{2}kvu_yu_xh - kxv_yu_xh - \frac{1}{2}kyv_yu_xh + 2kzv_yu_xh \\
&\quad - \frac{1}{2}kuv_yu_xh - \frac{3}{2}ktu_tv_yu_xh + \frac{3}{2}ktvu_{yt}u_xh - 2kxu_yv_xh - kyv_yv_xh \\
&\quad + 4kzu_yv_xh - ku_{yy}v_xh - 3ktu_tu_yv_xh - kxu_yu_xv_xh + kzu_yu_xv_xh \\
&\quad + ktvu_{xt}h + 3ktvu_yu_{xt}h + \frac{1}{2}kxvu_{xz}h - \frac{1}{2}kzvu_{xz}h - kxvu_{xy}h \\
&\quad - \frac{1}{2}kyvu_{xy}h + 2kzvu_{xy}h - \frac{1}{2}kuvu_{xy}h - \frac{3}{2}ktvu_tu_{xy}h - kxvu_xu_{xy}h \\
&\quad + kzvu_xu_{xy}h - v_xu_{xy}h + u_xv_{xy}h + \frac{3}{2}tu_{xt}v_{xy}h + v_{xy}h - \frac{3}{2}tv_xu_{xyt}h
\end{aligned}$$

$$\begin{aligned}
& -\frac{3}{4}v_y u_{xx}h + \frac{1}{2}xv_{xy}u_{xx}h - \frac{1}{2}zv_{xy}u_{xx}h + \frac{1}{4}u_y v_{xx}h + \frac{3}{4}tu_{yt}v_{xx}h \\
& + \frac{1}{4}xu_{xy}v_{xx}h - \frac{1}{4}zu_{xy}v_{xx}h + \frac{1}{4}v_{xx}h - \frac{3}{4}tv_y u_{xxt}h + \frac{9}{4}vu_{xxy}h \\
& - \frac{1}{2}xv_x u_{xxy}h + \frac{1}{2}zv_x u_{xxy}h - \frac{3}{2}xv_{xxy}h - \frac{3}{4}yv_{xxy}h + 3zv_{xxy}h \\
& - \frac{3}{4}uv_{xxy}h - \frac{9}{4}tu_tv_{xxy}h - \frac{3}{4}xu_x v_{xxy}h + \frac{3}{4}zu_x v_{xxy}h + \frac{9}{4}tvu_{xxyt}h \\
& - \frac{1}{4}xv_y u_{xxx}h + \frac{1}{4}zv_y u_{xxx}h - \frac{1}{4}xvu_{xxy}h + \frac{1}{4}zvu_{xxy}h + k^2tv_y u_x^2 \\
& - k^2tv_z u_x + 2k^2tu_y u_x v_x - k^2tvu_{xx} + 2k^2tvu_x u_{xy} - ktv_{xy}u_{xx} \\
& - \frac{1}{2}ktu_{xy}v_{xx} + ktv_x u_{xxy} + \frac{3}{2}ktu_x v_{xxy} + \frac{1}{2}ktv_y u_{xxx} + \frac{1}{2}ktvu_{xxy}, \\
T_y^7 = & 2vu_t h^2 - xv_t h^2 - \frac{1}{2}yv_t h^2 + 2zv_t h^2 - \frac{1}{2}uv_t h^2 - \frac{3}{2}tu_t v_t h^2 + \frac{3}{2}tvu_{tt} h^2 \\
& - \frac{1}{2}xv_t u_x h^2 + \frac{1}{2}zv_t u_x h^2 + \frac{1}{2}xvu_{xt} h^2 - \frac{1}{2}zvu_{xt} h^2 + kvu_x^2 h + ktv_t u_x h \\
& - \frac{1}{2}kxu_x^2 v_x h + \frac{1}{2}kzu_x^2 v_x h - kxu_x v_x h - \frac{1}{2}kyu_x v_x h + 2kzu_x v_x h \\
& - \frac{1}{2}kuu_x v_x h - \frac{3}{2}ktu_t u_x v_x h - ktv_{xt} h + \frac{3}{2}ktvu_x u_{xt} h + kxvu_{xx} h \\
& + \frac{1}{2}kyvu_{xx} h - 2kzvu_{xx} h + \frac{1}{2}kuvu_{xx} h + \frac{3}{2}ktvu_t u_{xx} h + kxvu_x u_{xx} h \\
& - kzvu_x u_{xx} h - \frac{3}{4}v_x u_{xx} h + \frac{1}{2}u_x v_{xx} h + \frac{3}{4}tu_{xt} v_{xx} h + \frac{1}{4}xu_{xx} v_{xx} h \\
& - \frac{1}{4}zu_{xx} v_{xx} h + \frac{1}{2}v_{xx} h - \frac{3}{4}tv_x u_{xxt} h + vu_{xxx} h - \frac{1}{4}xv_x u_{xxx} h + \frac{1}{4}zv_x u_{xxx} h \\
& - \frac{1}{2}xv_{xxx} h - \frac{1}{4}yv_{xxx} h + zv_{xxx} h - \frac{1}{4}uv_{xxx} h - \frac{3}{4}tu_t v_{xxx} h - \frac{1}{4}xu_x v_{xxx} h \\
& + \frac{1}{4}zu_x v_{xxx} h + \frac{3}{4}tvu_{xxt} h + \frac{1}{4}xvu_{xxx} h - \frac{1}{4}zvu_{xxx} h + k^2tu_x^2 v_x \\
& - 2k^2tvu_x u_{xx} - \frac{1}{2}ktu_{xx} v_{xx} + \frac{1}{2}ktv_x u_{xxx} + \frac{1}{2}ktu_x v_{xxx} - \frac{1}{2}ktvu_{xxx}, \\
T_z^7 = & 2h^2 u_t v - \frac{1}{2}h^2 v_t u + \frac{3}{2}h^2 tu_{tt} v + \frac{1}{2}h^2 xu_{xt} v - \frac{1}{2}h^2 zu_{xt} v - 2hku_x v \\
& + \frac{1}{2}hkv_x u - \frac{5}{2}hktu_{xt} v - \frac{1}{2}hkkxu_{xx} v + \frac{1}{2}hkkzu_{xx} v + k^2tu_{xx} v - hkv \\
& + \frac{1}{2}h^2 zv_t u_x - \frac{1}{2}h^2 xv_t u_x - \frac{3}{2}h^2 tu_t v_t - h^2 xv_t - \frac{1}{2}h^2 yv_t + 2h^2 zv_t + hktv_t u_x \\
& + \frac{3}{2}hktu_t v_x - \frac{1}{2}hkkzu_x v_x + \frac{1}{2}hkkxu_x v_x + \frac{1}{2}hkyv_x - 2hkkz v_x + hkv_x - k^2tu_x v_x,
\end{aligned}$$

$$\begin{aligned}
T_t^7 = & -\frac{1}{2}vh^2 + \frac{1}{2}vu_zh^2 - xv_zh^2 - \frac{1}{2}yv_zh^2 + 2zv_zh^2 - \frac{1}{2}uv_zh^2 - \frac{3}{2}tu_tv_zh^2 \\
& - \frac{3}{2}tvu_{xt}h^2 + \frac{1}{2}vu_yh^2 - xv_yh^2 - \frac{1}{2}yv_yh^2 + 2zv_yh^2 - \frac{1}{2}uv_yh^2 - \frac{3}{2}tu_tv_yh^2 \\
& - \frac{3}{2}tvu_{yt}h^2 + \frac{1}{2}vu_xh^2 - \frac{1}{2}xv_xh^2 + \frac{1}{2}zv_xh^2 - \frac{1}{2}uv_xh^2 + \frac{1}{2}zv_yu_xh^2 \\
& - xv_xh^2 - \frac{1}{2}yv_xh^2 + 2zv_xh^2 - \frac{1}{2}uv_xh^2 - \frac{3}{2}tu_tv_xh^2 - \frac{1}{2}xu_xv_xh^2 \\
& + \frac{1}{2}zu_xv_xh^2 - \frac{3}{2}tvu_{xt}h^2 + \frac{1}{2}xvu_{xz}h^2 - \frac{1}{2}zvu_{xz}h^2 + \frac{1}{2}xvu_{xy}h^2 \\
& - \frac{1}{2}zvu_{xy}h^2 + \frac{1}{2}xvu_{xx}h^2 - \frac{1}{2}zvu_{xx}h^2 + ktv_zu_xh + ktv_yu_xh + ktu_xv_xh \\
& + kht(2vu_{xz} - vu_{xy} - 3vu_{xy} - vu_{xx} - 3vu_yu_{xx}) - 3htvu_{xxxxy};
\end{aligned}$$

$$\begin{aligned}
T_x^8 = & \frac{1}{2}tu_tv_th^2 - \frac{1}{2}vu_th^2 - \frac{1}{2}tvu_{tt}h^2 - \frac{1}{2}zv_tu_zh^2 - \frac{1}{2}zvu_{xt}h^2 - \frac{1}{2}zv_tu_yh^2 \\
& - \frac{1}{2}zvu_{yt}h^2 - \frac{1}{2}zv_tu_xh^2 - \frac{1}{2}zvu_{xt}h^2 - \frac{1}{2}kzv_yu_x^2h - \frac{1}{2}kvh - \frac{1}{2}ktv_th \\
& - \frac{1}{2}kvu_zh + kzv_zh - \frac{1}{2}ktu_tv_zh + \frac{1}{2}kzu_zv_zh - \frac{1}{2}ktvu_{xt}h - \frac{1}{2}kzvu_{zz}h \\
& - \frac{1}{2}kvu_yh + \frac{1}{2}kzv_zu_yh - ktv_{yt}h - \frac{1}{2}kzvu_{yz}h - \frac{1}{2}ktv_tu_xh + \frac{1}{2}kzv_zu_xh \\
& - kzv_yu_xh + \frac{1}{2}ktu_tv_yu_xh - \frac{1}{2}kzu_zv_yu_xh - \frac{1}{2}kzu_yv_yu_xh - \frac{1}{2}ktv_{yt}u_xh \\
& + \frac{1}{2}kzvu_{yz}u_xh + \frac{1}{2}kzvu_{yy}u_xh - kzu_y^2v_xh - 2kzu_yv_xh + ktu_tv_yu_xh \\
& - kzu_zu_yv_xh - kzu_yu_xv_xh - \frac{1}{2}ktvu_{xt}h - ktv_{uy}u_{xt}h + \frac{1}{2}kzvu_{xz}h \\
& + kzv_{uy}u_{xz}h - kzv_{uxy}h + \frac{1}{2}ktv_{ut}u_{xy}h - \frac{1}{2}kzvu_zu_{xy}h + \frac{1}{2}kzvu_yu_{xy}h \\
& - kzv_{ux}u_{xy}h - \frac{1}{2}tu_{xt}v_{xy}h + \frac{1}{2}zu_{xz}v_{xy}h + \frac{1}{2}zu_{xy}v_{xy}h + \frac{1}{2}tv_xu_{xyt}h \\
& - \frac{1}{2}zv_xu_{xyz}h - \frac{1}{2}zv_xu_{xyy}h + \frac{1}{2}zv_{xy}u_{xx}h - \frac{1}{4}tu_{yt}v_{xx}h + \frac{1}{4}zu_{yz}v_{xx}h \\
& + \frac{1}{4}zu_{yy}v_{xx}h + \frac{1}{4}zu_{xy}v_{xx}h + \frac{1}{4}tv_yu_{xxt}h - \frac{1}{4}zv_yu_{xxz}h - \frac{1}{4}zv_yu_{xxy}h \\
& - \frac{1}{2}zv_xu_{xxy}h - \frac{3}{2}zv_{xxy}h + \frac{3}{4}tu_tv_{xxy}h - \frac{3}{4}zu_zv_{xxy}h - \frac{3}{4}zu_yv_{xxy}h \\
& - \frac{3}{4}zu_xv_{xxy}h - \frac{3}{4}tv_{u_{xyt}}h + \frac{3}{4}zvu_{xxy}h + \frac{3}{4}zvu_{xxy}h - \frac{1}{4}zv_yu_{xxx}h
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}zvu_{xxx}h - \frac{1}{2}k^2tv_yu_x^2 + \frac{1}{2}k^2tv_z + \frac{1}{2}k^2tv_zu_x - \frac{1}{2}k^2tv_yu_x - k^2tu_yv_x \\
& - k^2tu_yu_xv_x + \frac{1}{2}k^2tvu_{xz} - \frac{1}{2}k^2tvu_{xy} - k^2tvu_{xu_{xy}} + \frac{1}{2}ktv_{xy}u_{xx} + \frac{1}{4}ktu_{xy}v_{xx} \\
& - \frac{1}{2}ktv_xu_{xy} - \frac{3}{4}ktv_{xy} - \frac{3}{4}ktu_xv_{xy} - \frac{1}{4}ktv_yu_{xxx} - \frac{1}{4}ktvu_{xxx} - zv_t h^2, \\
T_y^8 = & -\frac{1}{2}vu_t h^2 - zv_t h^2 + \frac{1}{2}tu_tv_t h^2 - \frac{1}{2}tvu_{tt} h^2 - \frac{1}{2}zv_tu_z h^2 - \frac{1}{2}zvu_{xt} h^2 - \frac{1}{2}zv_tu_y h^2 \\
& - \frac{1}{2}zvu_{yt} h^2 - \frac{1}{2}zv_tu_x h^2 - \frac{1}{2}zvu_{xt} h^2 + \frac{1}{2}kvh - \frac{1}{2}ktv_t h + \frac{1}{2}kvu_x h - \frac{1}{2}ktv_tu_x h \\
& - \frac{1}{2}kzu_x^2v_x h - kzu_xv_x h + \frac{1}{2}ktu_tu_xv_x h - \frac{1}{2}kzu_zu_xv_x h - \frac{1}{2}kzu_yu_xv_x h \\
& + \frac{1}{2}ktvu_{xt} h - \frac{1}{2}ktvu_xu_{xt} h + kzvu_{xz} h + \frac{1}{2}kzvu_xu_{xz} h - \frac{1}{2}kzvu_xu_{xy} h + kzvu_{xx} h \\
& - \frac{1}{2}ktvu_tu_{xx} h + \frac{1}{2}kzvu_zu_{xx} h - \frac{1}{2}kzvu_yu_{xx} h + kzvu_xu_{xx} h - \frac{1}{4}tu_{xt}v_{xx} h \\
& + \frac{1}{4}zu_{xz}v_{xx} h + \frac{1}{4}zu_{xy}v_{xx} h + \frac{1}{4}zu_{xx}v_{xx} h + \frac{1}{4}tv_xu_{xt} h - \frac{1}{4}zv_xu_{xx} h - \frac{1}{4}zv_xu_{xy} h \\
& - \frac{1}{4}zv_xu_{xxx} h - \frac{1}{2}zv_{xxx} h + \frac{1}{4}tu_tv_{xxx} h - \frac{1}{4}zu_zv_{xxx} h - \frac{1}{4}zu_yv_{xxx} h - \frac{1}{4}zu_xv_{xxx} h \\
& - \frac{1}{4}tvu_{xxx} h + \frac{1}{4}zvu_{xxx} h - \frac{3}{4}zvu_{xxy} h + \frac{1}{4}zvu_{xxx} h - \frac{1}{2}k^2tu_x^2v_x - \frac{1}{2}k^2tu_xv_x \\
& + \frac{1}{2}k^2tvu_{xx} + k^2tvu_xu_{xx} + \frac{1}{4}kt(u_{xx}v_{xx} - v_xu_{xxx} - v_{xxx} - u_xv_{xxx} + vu_{xxx}), \\
T_z^8 = & -\frac{1}{2}h^2u_tv - \frac{1}{2}h^2tu_{tt}v - \frac{1}{2}h^2zu_{xt}v - \frac{1}{2}h^2zu_{yt}v - \frac{1}{2}h^2zu_{xt}v + \frac{1}{2}hku_xv + hktu_{xt}v \\
& + \frac{1}{2}hkHzu_{xz}v - \frac{1}{2}hkHzu_{xy}v - hkHzu_xu_{xy}v - \frac{1}{2}hkHzu_{xx}v - hkHzu_{xx}u_yv - hzu_{xxy}v \\
& - \frac{1}{2}k^2tu_{xx}v + \frac{1}{2}hkv - \frac{1}{2}h^2zv_tu_x - \frac{1}{2}h^2zv_tu_y - \frac{1}{2}h^2zv_tu_z + \frac{1}{2}h^2tu_tv_t - h^2zv_t \\
& - \frac{1}{2}hktv_tu_x - \frac{1}{2}hktu_tv_x - \frac{1}{2}hktv_t + \frac{1}{2}hkHzu_yv_x + \frac{1}{2}hkHzu_zv_x + \frac{1}{2}hkHzu_xv_x \\
& + hkHzv_x + \frac{1}{2}k^2tu_xv_x + \frac{1}{2}k^2tv_x, \\
T_t^8 = & vh^2 + \frac{1}{2}vu_zh^2 - zv_zh^2 + \frac{1}{2}tu_tv_zh^2 - \frac{1}{2}zu_zv_zh^2 + \frac{1}{2}tvu_{zt}h^2 + \frac{1}{2}zvu_{zz}h^2 + \frac{1}{2}vu_yh^2 \\
& - \frac{1}{2}zv_zu_yh^2 - zv_yh^2 + \frac{1}{2}tu_tv_yh^2 - \frac{1}{2}zu_zv_yh^2 - \frac{1}{2}zu_yv_yh^2 + \frac{1}{2}tvu_{yt}h^2 + zvu_{yz}h^2 \\
& + \frac{1}{2}zvu_{yy}h^2 + \frac{1}{2}vu_xh^2 - \frac{1}{2}zv_zu_xh^2 - \frac{1}{2}zv_yu_xh^2 - zv_xh^2 + \frac{1}{2}tu_tv_xh^2 - \frac{1}{2}zu_zv_xh^2 \\
& - \frac{1}{2}zu_yv_xh^2 - \frac{1}{2}zu_xv_xh^2 + \frac{1}{2}tvu_{xt}h^2 + zvu_{xz}h^2 + zvu_{xy}h^2 + \frac{1}{2}zvu_{xx}h^2 - \frac{1}{2}ktv_zh
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}ktv_yh - \frac{1}{2}ktv_zu_xh - \frac{1}{2}ktv_yu_xh - \frac{1}{2}ktv_xh - \frac{1}{2}ktu_xv_xh - \frac{1}{2}ktvu_{xz}h \\
& + \frac{1}{2}ktvu_{xy}h + ktvu_xu_{xy}h + \frac{1}{2}ktvu_{xx}h + ktvu_yu_{xx}h + tvu_{xxx}h;
\end{aligned}$$

$$\begin{aligned}
T_x^9 = & -\frac{1}{2}vu_th^2 + \frac{1}{2}yv_th^2 - zv_th^2 + \frac{1}{2}tu_tv_th^2 - \frac{1}{2}tvu_{tt}h^2 + \frac{1}{2}yv_tu_yh^2 - \frac{1}{2}zv_tu_yh^2 \\
& - \frac{1}{2}yvu_{yt}h^2 + \frac{1}{2}zvu_{yt}h^2 - \frac{1}{2}kvh - \frac{1}{2}ktv_th - \frac{1}{2}kyv_zh + kzv_zh - \frac{1}{2}ktu_tv_zh \\
& - \frac{1}{2}ktvu_{zt}h - \frac{1}{2}kvu_yh - \frac{1}{2}kyv_zu_yh + \frac{1}{2}kzv_zu_yh - ktvu_{yt}h + \frac{1}{2}kyvu_{yz}h \\
& - \frac{1}{2}kzv_v_zh - \frac{1}{2}ktv_tu_xh - \frac{1}{2}kvu_yu_xh + \frac{1}{2}kyv_yu_xh - kzv_yu_xh + \frac{1}{2}ktu_tv_yu_xh \\
& + \frac{1}{2}kyu_yv_yu_xh - \frac{1}{2}kzu_yv_yu_xh - \frac{1}{2}ktvu_{yt}u_xh - \frac{1}{2}kyvu_{yy}u_xh + \frac{1}{2}kzv_v_yu_xh \\
& + ky_u^2v_xh - kz_u^2v_xh + ky_uv_xh - 2kzu_yv_xh + ktu_tu_yv_xh - \frac{1}{2}ktvu_{xt}h \\
& - ktvu_yu_{xt}h + \frac{1}{2}kyvu_{xy}h - kzvu_{xy}h + \frac{1}{2}ktvu_tu_{xy}h - \frac{1}{2}kyvu_yu_{xy}h \\
& + \frac{1}{2}kzv_vu_{xy}h + \frac{1}{2}v_xu_{xy}h - \frac{1}{2}tu_{xt}v_{xy}h - \frac{1}{2}yu_{xy}v_{xy}h + \frac{1}{2}zu_{xy}v_{xy}h + \frac{1}{2}tv_xu_{xyt}h \\
& + \frac{1}{2}yv_xu_{xyy}h - \frac{1}{2}zv_xu_{xyy}h - \frac{1}{4}u_yv_{xx}h - \frac{1}{4}tu_{yt}v_{xx}h - \frac{1}{4}yu_{yy}v_{xx}h + \frac{1}{4}zu_{yy}v_{xx}h \\
& - \frac{1}{4}v_{xx}h + \frac{1}{4}tv_yu_{xxt}h - \frac{3}{4}vu_{xxy}h + \frac{1}{4}yv_yu_{xxy}h - \frac{1}{4}zv_yu_{xxy}h + \frac{3}{4}yv_{xxy}h \\
& - \frac{3}{2}zv_{xxy}h + \frac{3}{4}tu_tv_{xxy}h + \frac{3}{4}yu_yv_{xxy}h - \frac{3}{4}zu_yv_{xxy}h - \frac{3}{4}tvu_{xxyt}h - \frac{3}{4}yvu_{xxyy}h \\
& + \frac{3}{4}zv_u_{xxyy}h - \frac{1}{2}k^2tv_yu_x^2 + \frac{1}{2}k^2tv_z + \frac{1}{2}k^2tv_zu_x - \frac{1}{2}k^2tv_yu_x - k^2tu_yv_x \\
& - k^2tu_yu_xv_x + \frac{1}{2}k^2tvu_{xz} - \frac{1}{2}k^2tvu_{xy} - k^2tvu_xu_{xy} + \frac{1}{2}ktv_{xy}u_{xx} + \frac{1}{4}ktu_{xy}v_{xx} \\
& - \frac{1}{2}ktv_xu_{xxy} - \frac{3}{4}ktv_{xxy} - \frac{3}{4}ktu_xv_{xxy} - \frac{1}{4}ktv_yu_{xxx} - \frac{1}{4}ktvu_{xxx}, \\
T_y^9 = & -\frac{1}{2}vu_th^2 + \frac{1}{2}yv_th^2 - zv_th^2 + \frac{1}{2}tu_tv_th^2 - \frac{1}{2}tvu_{tt}h^2 + yvu_{xt}h^2 - zv_u_{xt}h^2 \\
& + \frac{1}{2}yv_tu_yh^2 - \frac{1}{2}zv_tu_yh^2 + \frac{1}{2}yvu_{yt}h^2 - \frac{1}{2}zvu_{yt}h^2 + yvu_{xt}h^2 - zv_u_{xt}h^2 \\
& + \frac{1}{2}kvh - \frac{1}{2}ktv_th + \frac{1}{2}kvu_xh - \frac{1}{2}ktv_tu_xh + \frac{1}{2}kyu_xv_xh - kzv_xv_xh \\
& + \frac{1}{2}ktu_tu_xv_xh + \frac{1}{2}kyu_yu_xv_xh - \frac{1}{2}kzu_yu_xv_xh + \frac{1}{2}ktvu_{xt}h - \frac{1}{2}ktvu_xu_{xt}h
\end{aligned}$$

$$\begin{aligned}
& -kyvu_{xz}h + kzvu_{xz}h + \frac{1}{2}kyvu_xu_{xy}h - \frac{1}{2}kzvu_xu_{xy}h - \frac{1}{2}kyvu_{xx}h + kzvu_{xx}h \\
& - \frac{1}{2}ktvu_tu_{xx}h + \frac{1}{2}kyvu_yu_{xx}h - \frac{1}{2}kzvu_yu_{xx}h - \frac{1}{4}tu_{xt}v_{xx}h - \frac{1}{4}yu_{xy}v_{xx}h \\
& + \frac{1}{4}zu_{xy}v_{xx}h + \frac{1}{4}tv_xu_{xxt}h + \frac{1}{4}yv_xu_{xxy}h - \frac{1}{4}zv_xu_{xxy}h + \frac{1}{4}yv_{xxx}h - \frac{1}{2}zv_{xxx}h \\
& + \frac{1}{4}tu_tv_{xxx}h + \frac{1}{4}yu_yv_{xxx}h - \frac{1}{4}zu_yv_{xxx}h - \frac{1}{4}tvu_{xxx}h + \frac{3}{4}yvu_{xxx}h \\
& - \frac{3}{4}zvu_{xxx}h - \frac{1}{2}k^2tu_x^2v_x - \frac{1}{2}k^2tu_xv_x + \frac{1}{2}k^2tvu_{xx} + k^2tvu_xu_{xx} + \frac{1}{4}ktu_{xx}v_{xx} \\
& - \frac{1}{4}ktv_xu_{xxx} - \frac{1}{4}ktv_{xxx} - \frac{1}{4}ktu_xv_{xxx} + \frac{1}{4}ktvu_{xxxx}, \\
T_z^9 = & -\frac{1}{2}h^2u_tv - \frac{1}{2}h^2tu_{tt}v - \frac{1}{2}h^2yu_{yt}v + \frac{1}{2}h^2zu_{yt}v + \frac{1}{2}hku_xv + hktu_{xt}v \\
& + \frac{1}{2}hkyu_{xy}v - \frac{1}{2}hkv - \frac{1}{2}h^2zv_tu_y + \frac{1}{2}h^2yv_tu_y \\
& + \frac{1}{2}h^2tu_tv_t + \frac{1}{2}h^2yv_t - h^2zv_t - \frac{1}{2}hktv_tu_x - \frac{1}{2}hktu_tv_x - \frac{1}{2}hktv_t + \frac{1}{2}hkv_{xx} \\
& - \frac{1}{2}hkyu_yv_x - \frac{1}{2}hkyv_x + hkv_x + \frac{1}{2}k^2tu_xv_x + \frac{1}{2}k^2tv_x, \\
T_t^9 = & \frac{1}{2}vh^2 + \frac{1}{2}yv_zh^2 - zv_zh^2 + \frac{1}{2}tu_tv_zh^2 + \frac{1}{2}tvu_{zt}h^2 + \frac{1}{2}yv_zu_yh^2 - \frac{1}{2}zv_zu_yh^2 \\
& + \frac{1}{2}yv_yh^2 - zv_yh^2 + \frac{1}{2}tu_tv_yh^2 + \frac{1}{2}yu_yv_yh^2 - \frac{1}{2}zu_yv_yh^2 + \frac{1}{2}tvu_{yt}h^2 - \frac{1}{2}yv_yu_zh^2 \\
& + \frac{1}{2}zvu_{yz}h^2 - \frac{1}{2}yvu_{yy}h^2 + \frac{1}{2}zvu_{yy}h^2 + \frac{1}{2}yv_xh^2 - zv_xh^2 + \frac{1}{2}tu_tv_xh^2 \\
& + \frac{1}{2}yu_yv_xh^2 - \frac{1}{2}zu_yv_xh^2 + \frac{1}{2}tvu_{xt}h^2 - \frac{1}{2}yvu_{xy}h^2 + \frac{1}{2}zvu_{xy}h^2 - \frac{1}{2}ktv_zh \\
& - \frac{1}{2}ktv_yh - \frac{1}{2}ktv_zu_xh - \frac{1}{2}ktv_yu_xh - \frac{1}{2}ktv_xh - \frac{1}{2}ktu_xv_xh - \frac{1}{2}ktv_{xx}h \\
& + \frac{1}{2}ktvu_{xy}h + ktvu_xu_{xy}h + \frac{1}{2}ktvu_{xx}h + ktvu_yu_{xx}h + tvu_{xxy}h.
\end{aligned}$$

Remark. It should be noted that the above conservation laws include the energy conservation law, which corresponds to the time translation and three momentum conservation laws, which correspond to the three space translations.

8.4 Conclusion

In this chapter we studied the generalized extended $(3+1)$ -dimensional Jimbo-Miwa equation (8.3). Symmetry reductions of this equation were performed several times until it was reduced to a fourth-order ordinary differential equation. The general solution of this ordinary differential equation was obtained in terms of the Weierstrass zeta function. Travelling wave solutions of (8.3) were also derived using the simplest equation method. Finally, the conservation laws of (8.3) were computed by invoking the conservation theorem due to Ibragimov. These conservation laws included an energy conservation law, which corresponded to the time translation and three momentum conservation laws that corresponded to the three space translations.

Chapter 9

Solutions and conservation laws of the combined KdV–negative-order KdV equation

9.1 Introduction

In a recent paper [81] a new integrable equation, called the combined KdV–negative-order KdV equation (KdV-nKdV)

$$u_{xt} + 2u_t u_{xx} + 4u_x u_{xt} + 6u_x u_{xx} + u_{xxx} + u_{xxxx} = 0. \quad (9.1)$$

was developed by combining the Korteweg-de Vries (KdV) equation and the negative-order KdV equation. The KdV recursion operator and the inverse KdV recursion operator were used concurrently to construct this new integrable equation. It was shown in [81] that this newly constructed equation (9.1) passes the Painlevé test and thus it possesses the complete integrability phenomenon. Also a variety of travelling wave solutions, singular solutions, and rational hyperbolic and rational trigonometric solutions were obtained [81].

In this chapter, we use the (G'/G) -expansion method to find the travelling wave solutions of the KdV-nKdV equation (9.1). Furthermore, the multiplier method is implemented to construct conservation laws of this equation.

This work has been submitted for publication [82].

9.2 Solution of (9.1) using (G'/G) -expansion method

The use of travelling wave variable substitution

$$u(x, t) = F(\xi), \quad \xi = x - ct, \quad (9.2)$$

transforms the KdV-nKdV (9.2) into the nonlinear ordinary differential equation

$$F''''(\xi) - cF''''(\xi) - cF'''(\xi) - 6cF'(\xi)F''(\xi) + 6F'(\xi)F''(\xi) = 0. \quad (9.3)$$

The balancing procedure gives $M = 1$ so the solutions of (9.3) are of the form

$$F(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \quad (9.4)$$

Substituting (9.4) into (9.3) and making use of (1.57), and thereafter setting the coefficients of the functions $(G'/G)^i$ to zero, give rise to the following algebraic systems of six equations:

$$\begin{aligned} 6A_1^2c\lambda\mu^2 - A_1c\lambda^3\mu - 8A_1c\lambda\mu^2 - A_1c\lambda\mu + A_1\lambda^3\mu - 6A_1^2\lambda\mu^2 + 8A_1\lambda\mu^2 &= 0, \\ 12A_1^2c\lambda^2\mu - A_1c\lambda^4 - 22A_1c\lambda^2\mu - A_1c\lambda^2 + 12A_1^2c\mu^2 - 16A_1c\mu^2 - 2A_1c\mu \\ + A_1\lambda^4 - 12A_1^2\lambda^2\mu + 22A_1\lambda^2\mu - 12A_1^2\mu^2 + 16A_1\mu^2 &= 0, \\ 2A_1^2c\lambda^3 - 5A_1c\lambda^3 + 12A_1^2c\lambda\mu - 20A_1c\lambda\mu - A_1c\lambda - 2A_1^2\lambda^3 + 5A_1\lambda^3 - 12A_1^2\lambda\mu \\ + 20A_1\lambda\mu &= 0, \\ 12A_1^2c\lambda^2 - 25A_1c\lambda^2 + 12A_1^2c\mu - 20A_1c\mu - A_1c - 12A_1^2\lambda^2 + 25A_1\lambda^2 - 12A_1^2\mu \\ + 20A_1\mu &= 0, \end{aligned}$$

$$A_1^2 c \lambda - 2A_1 c \lambda - A_1^2 \lambda + 2A_1 \lambda = 0,$$

$$A_1^2 c - 2A_1 c - A_1^2 + 2A_1 = 0.$$

Solving the above system of algebraic equations, with the aid of Mathematica, we obtain

$$c = \frac{\lambda^2 - 4\mu}{\lambda^2 - 4\mu + 1}, \quad A_0 = \text{arbitrary}, \quad A_1 = 2.$$

Thus the travelling wave solutions of equation (9.1) are:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \alpha \left(\frac{C_1 \sinh(\alpha\xi) + C_2 \cosh(\alpha\xi)}{C_1 \cosh(\alpha\xi) + C_2 \sinh(\alpha\xi)} \right) \right\},$$

where $\xi = x - ct$, $\alpha = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

By giving particular values to the constants C_1 and C_2 we may obtain the following special solutions. For example, if $C_1 = 0$, $C_2 \neq 0$, we obtain

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \alpha \coth(\alpha\xi) \right\}.$$

If $C_2 = 0$, $C_1 \neq 0$, we obtain

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \alpha \tanh(\alpha\xi) \right\}.$$

If $C_1 \neq 0$, $C_1 > C_2$, we obtain

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \alpha \tanh(\alpha\xi + \alpha_1) \right\},$$

where $\alpha_1 = \tan^{-1}(C_2/C_1)$.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \alpha \left(\frac{-C_1 \sin(\alpha\xi) + C_2 \cos(\alpha\xi)}{C_1 \cos(\alpha\xi) + C_2 \sin(\alpha\xi)} \right) \right\},$$

where $\xi = x - ct$, $\alpha = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

Again by giving special values to C_1 and C_2 we may obtain the following special solutions. For example, if $C_1 = 0$, $C_2 \neq 0$, we obtain

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \alpha \cot(\alpha \xi) \right\}.$$

If $C_2 = 0$, $C_1 \neq 0$, we obtain

$$u(t, x) = A_0 + A_1 \left\{ \frac{\lambda}{2} - \alpha \tan(\alpha \xi) \right\}.$$

If $C_1 \neq 0$, $C_1 > C_2$, we obtain

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} - \alpha \tan(\alpha \xi - \alpha_1) \right\},$$

where $\alpha_1 = \tan^{-1}(C_2/C_1)$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u(t, x) = A_0 + A_1 \left\{ -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right\},$$

where $\xi = x - ct$, C_1 and C_2 are arbitrary constants.

9.3 Conservation laws of equation (9.1)

In this section we construct conservation laws of the combined KdV-negative-order KdV equation (9.1) by employing the multiplier method [24]. The third-order multiplier $\Lambda(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{ttt}, u_{xxx})$ for (9.1) is given by

$$\Lambda = \frac{1}{2}(2u_t + 2u_x - 1)F(t) + C_1(3u_x^2 + u_{xxx}) + C_2u_x + C_3, \quad (9.5)$$

where C_1 , C_2 and C_3 are constants and $F(t)$ is an arbitrary function of variable t . Corresponding to the above multiplier the associated conserved vectors for equation (9.1) are

$$T_1^t = \frac{3}{2}(u_x)^4 + \frac{1}{2}(u_x)^3 - \frac{3}{2}uu_{xx}u_{xxx} + \frac{1}{8}(u_{xxx})^2 - 3u(u_x)^2u_{xx}$$

$$\begin{aligned}
& -uu_xu_{xx} - \frac{7}{6}uu_xu_{xxx} + \frac{1}{8}u_xu_{xxxx} - \frac{1}{8}uu_{xxxx} - \frac{1}{4}uu_{xxx} \\
& + \frac{1}{4}u_xu_{xxx} + \frac{17}{12}(u_x)^2u_{xxx} - \frac{1}{8}u_{xx}u_{xxxx}, \\
T_1^x = & uu_{tx}u_x - \frac{7}{6}u_xu_{tx}u_{xx} + \frac{7}{6}u_tu_xu_{xxx} + \frac{9}{2}(u_x)^4 + \frac{3}{8}u_{xx}u_{txxx} \\
& + \frac{5}{12}(u_x)^2u_{txx} + \frac{7}{6}(u_{xx})^2u_t + \frac{1}{3}uu_{txx}u_{xx} + \frac{7}{6}uu_{tx}u_{xxx} \\
& + 3u(u_x)^2u_{tx} + \frac{7}{6}uu_{txx}u_x + 3u_t(u_x)^3 + \frac{1}{4}u_tu_{xxx} + \frac{1}{2}u_t(u_x)^2 \\
& + \frac{1}{2}(u_{xxx})^2 + \frac{1}{8}uu_{txxxx} + \frac{1}{8}u_tu_{xxxx} + \frac{3}{8}u_{txx}u_{xxx} - \frac{1}{4}u_{tx}u_{xxxx} \\
& - \frac{1}{4}u_xu_{txxxx} + 3(u_x)^2u_{xxx} + \frac{1}{4}uu_{txxx} - \frac{1}{2}u_xu_{txx} + \frac{1}{2}u_{xx}u_{tx};
\end{aligned}$$

$$\begin{aligned}
T_2^t = & \frac{2}{3}(u_x)^3 - \frac{1}{8}(u_{xx})^2 + \frac{1}{4}(u_x)^2 - \frac{2}{3}uu_xu_{xx} - \frac{1}{8}uu_{xxxx} \\
& + \frac{1}{4}u_xu_{xxx} - \frac{1}{4}uu_{xx},
\end{aligned}$$

$$\begin{aligned}
T_2^x = & \frac{2}{3}uu_{tx}u_x + 2(u_x)^3 - \frac{1}{2}(u_{xx})^2 + \frac{1}{4}uu_{tx} + \frac{1}{8}u_tu_{xxx} \\
& + \frac{4}{3}u_t(u_x)^2 + u_xu_{xxx} + \frac{1}{8}uu_{txxx} + \frac{5}{8}u_xu_{txx} - \frac{3}{8}u_{xx}u_{tx} + \frac{1}{4}u_tu_x;
\end{aligned}$$

$$T_3^t = \frac{1}{2}u_x + \frac{1}{4}u_{xxx} + (u_x)^2,$$

$$T_3^x = 3(u_x)^2 + u_{xxx} + \frac{3}{4}u_{txx} + \frac{1}{2}u_t + 2u_tu_x;$$

$$\begin{aligned}
T_4^t = & \frac{1}{4}u_tu_xF + \frac{3}{8}uu_{xxxx}F + \frac{2}{3}uu_tu_{xx}F + \frac{2}{3}uu_xu_{tx}F \\
& + \frac{4}{3}uu_xu_{xx}F + \frac{2}{3}(u_x)^3F - \frac{1}{8}(u_{xx})^2F + \frac{2}{3}u_t(u_x)^2F \\
& + \frac{1}{8}u_tu_{xxx}F + \frac{1}{8}u_xu_{txx}F + \frac{1}{4}uu_{tx}F + \frac{3}{8}uu_{txxx}F - \frac{1}{8}u_{xx}u_{tx}F \\
& - \frac{1}{4}uu_{xx}F - \frac{1}{4}(u_x)^2F - \frac{1}{8}u_{xxx}F + \frac{1}{4}u_xu_{xxx} - \frac{1}{4}u_xF,
\end{aligned}$$

$$\begin{aligned}
T_4^x = & \frac{1}{8}u_{xx}F_t + \frac{1}{4}uF_t - \frac{1}{2}u_{xxx}F - \frac{2}{3}uu_tu_{tx}F - \frac{2}{3}uu_xu_{tt}F \\
& - \frac{2}{3}u(u_x)^2F_t + \frac{1}{4}uu_xF_t - \frac{1}{8}u_tu_{xx}F_t + \frac{1}{8}u_xu_{xx}F_t - \frac{3}{8}uu_{txx}F_t
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}uu_tF_t + \frac{1}{4}u_xu_{tx}F_t - \frac{3}{8}uu_{xxx}F_t + \frac{1}{2}u_tu_{txx}F - \frac{1}{8}u_{tt}u_{xx}F \\
& + \frac{1}{4}u_{ttt}u_xF + \frac{4}{3}u_x(u_t)^2F - \frac{1}{4}uu_{tt}F - \frac{3}{8}uu_{ttxx}F + 2(u_x)^3F \\
& - \frac{1}{2}(u_{xx})^2F - \frac{3}{2}(u_x)^2F - \frac{3}{4}u_tu_xF + \frac{10}{3}(u_x)^2u_tF + \frac{9}{8}u_{txx}u_xF \\
& \frac{1}{4}uu_{tx}F - \frac{3}{8}uu_{txxx}F - \frac{7}{8}u_{xx}u_{tx}F - \frac{1}{4}u_tF - \frac{1}{4}(u_{tx})^2F \\
& + \frac{1}{4}(u_t)^2F - \frac{3}{8}u_{txx}F + u_xu_{xxx}F + \frac{5}{8}u_tu_{xxx}F \\
& - \frac{2}{3}uu_tu_xF - \frac{4}{3}uu_{tx}u_xF.
\end{aligned}$$

9.4 Conclusion

In this chapter we used the travelling wave variable to transform the combined KdV-negative-order KdV equation (9.1) into the fourth-order ODE (9.3). Thereafter we used the (G'/G) -expansion method to find its travelling wave solutions. The obtained solutions were in terms of hyperbolic, trigonometric and rational functions, which include solitons and periodic solutions. Moreover, the conservation laws were derived using the multiplier method.

Chapter 10

Lagrangian formulation of the Calogero-Bogoyavlenskii-Schiff equation

In this chapter we construct conservation laws of the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation

$$u_{xt} + 4u_x u_{xz} + 2u_{xx} u_z + u_{xxx} = 0 \quad (10.1)$$

by applying Noether's theorem.

This work has been accepted for publication [83].

10.1 Introduction

The CBS equation was first constructed by Bogoyavlenskii and Schiff in different ways. Bogoyavlenskii used the modified Lax formalism [84–86], whereas Schiff derived the same equation by reducing the self-dual Yang-Mills equation [87]. Equations

tion (10.1) has been of interest to several researchers. For example, in [88] the authors used classical and nonclassical methods to obtain symmetry reductions and exact solutions of (10.1). Wazwaz [89] employed the tanh-coth method and derived travelling wave solutions for (10.1). Symmetry method was applied to (10.1) and symmetry reductions and exact solutions were obtained in [90]. Equation (10.1) was also studied in [91] where exact solutions and conservation laws were obtained by the application of non-local conservation theorem. Exact solutions of (10.1) using the singular manifold method after Lie reductions were derived in [92].

10.2 Conservation laws of (10.1)

We employ Noether's theorem to construct conservation laws of the Calogero-Bogoyavlenskii-Schiff equation (10.1). In order to use Noether's theorem we need a Lagrangian of equation (10.1).

It can be easily verified that a second-order Lagrangian of (10.1) is given by

$$\mathcal{L} = \frac{1}{2}u_{xx}u_{xz} - \frac{1}{2}u_t u_x - u_x^2 u_z. \quad (10.2)$$

This is because $\mathcal{L}$ satisfies the Euler-Lagrange equation $\delta\mathcal{L}/\delta u = 0$ on the solutions of (10.1), where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_z \frac{\partial}{\partial u_z} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_z \frac{\partial}{\partial u_{xz}}. \quad (10.3)$$

The vector field of the form

$$X = \xi^1(t, x, z, u) \frac{\partial}{\partial t} + \xi^2(t, x, z, u) \frac{\partial}{\partial x} + \xi^3(t, x, z, u) \frac{\partial}{\partial z} + \eta(t, x, z, u) \frac{\partial}{\partial u} \quad (10.4)$$

is called a Noether point symmetry corresponding to a second-order Lagrangian $\mathcal{L}$ of (10.1) if

$$X^{[2]}(\mathcal{L}) + \{D_t(\xi^1) + D_x(\xi^2) + D_z(\xi^3)\}\mathcal{L} = D_t(B^1) + D_x(B^2) + D_z(B^3) \quad (10.5)$$

for some gauge functions $B^1(t, x, z, u)$, $B^2(t, x, z, u)$ and $B^3(t, x, z, u)$. Here $X^{[2]}$ is the second-order prolongation defined by

$$X^{[2]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_z \frac{\partial}{\partial u_z} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xz} \frac{\partial}{\partial u_{xz}} + \dots,$$

where the expressions for ζ_t , ζ_x , ζ_z , ζ_{xx} and ζ_{xz} are given by

$$\begin{aligned}\zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_z D_t(\xi^3), \\ \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_z D_x(\xi^3), \\ \zeta_z &= D_z(\eta) - u_t D_z(\xi^1) - u_x D_z(\xi^2) - u_z D_z(\xi^3), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xz} D_x(\xi^3), \\ \zeta_{xz} &= D_z(\zeta_x) - u_{tx} D_z(\xi^1) - u_{xx} D_z(\xi^2) - u_{xz} D_z(\xi^3)\end{aligned}$$

and

$$\begin{aligned}D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \dots, \\ D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{xz} \frac{\partial}{\partial u_x} + u_{tz} \frac{\partial}{\partial u_t} + \dots.\end{aligned}$$

are the total differential operators.

Inserting the value of $\mathcal{L}$ from (10.2) into the determining equation (10.5), expanding and then splitting on the derivatives of u , we obtain the values of ξ^1 , ξ^2 , ξ^3 and η . Thus the Noether point symmetries corresponding to the Lagrangian $\mathcal{L}$ of (10.1) along with the gauge functions are

$$\begin{aligned}X_1 &= \frac{\partial}{\partial t}, \quad B^1 = B^2 = B^3 = 0, \\ X_2 &= \frac{\partial}{\partial z}, \quad B^1 = B^2 = B^3 = 0, \\ X_3 &= 4t \frac{\partial}{\partial z} + x \frac{\partial}{\partial u}, \quad B^1 = -\frac{1}{2}u, \quad B^2 = 0, \quad B^3 = 0, \\ X_4 &= 4t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}, \quad B^1 = B^2 = B^3 = 0,\end{aligned}$$

$$\begin{aligned}
X_5 &= q(t) \frac{\partial}{\partial u}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}q'(t)u, \quad B^3 = 0, \\
X_6 &= 2h(t) \frac{\partial}{\partial x} + zh'(t) \frac{\partial}{\partial u}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}zh''(t)u, \quad B^3 = 0,
\end{aligned}$$

where $q(t)$ and $h(t)$ are arbitrary functions of the variable t .

We now invoke the conserved vector components [38]

$$T^k = \mathcal{L}\xi^k + (\eta - u_{x^j}\xi^j) \left(\frac{\partial \mathcal{L}}{\partial u_{x^k}} - \sum_{l=1}^k D_{x^l} \left(\frac{\partial \mathcal{L}}{\partial u_{x^l x^k}} \right) \right) + \sum_{l=k}^n (\zeta_l - u_{x^l x^j}\xi^j) \frac{\partial \mathcal{L}}{\partial u_{x^k x^l}} - B^k$$

corresponding to Noether point symmetries and obtain the following conserved vectors:

$$\begin{aligned}
T_1^t &= \frac{1}{2}u_{xx}u_{xz} - u_x^2u_z, \\
T_1^x &= 2u_tu_xu_z + \frac{3}{4}u_tu_{xxx} - \frac{1}{4}u_{xx}u_{tz} - \frac{1}{2}u_{tx}u_{xz} + \frac{u_t^2}{2}, \\
T_1^z &= u_tu_x^2 + \frac{1}{4}u_tu_{xxx} - \frac{1}{4}u_{xx}u_{tx};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{2}u_xu_z, \\
T_2^x &= \frac{1}{2}u_tu_z + 2u_xu_z^2 + \frac{3}{4}u_zu_{xxx} - \frac{1}{2}u_{xz}^2 - \frac{1}{4}u_{xx}u_{zz}, \\
T_2^z &= -\frac{1}{2}u_tu_x + \frac{1}{4}u_{xx}u_{xz} + \frac{1}{4}u_{xxx}u_z;
\end{aligned}$$

$$\begin{aligned}
T_3^t &= 2tu_xu_z - \frac{1}{2}xu_x + \frac{1}{2}u, \\
T_3^x &= 8tu_xu_z^2 + 3tu_zu_{xxx} - 2tu_{xz}^2 - tu_{xx}u_{zz} - \frac{1}{2}xu_t + 2tu_tu_z - 2xu_xu_z \\
&\quad + \frac{1}{2}u_{xz} - \frac{3}{4}xu_{xxx}, \\
T_3^z &= tu_{xx}u_{xz} + tu_{xxx}u_z - 2tu_tu_x - xu_x^2 + \frac{1}{4}u_{xx} - \frac{1}{4}xu_{xxx};
\end{aligned}$$

$$T_4^t = \frac{1}{2}u_xu - 4tu_x^2u_z + 2tu_{xx}u_{xz} + zu_xu_z + \frac{1}{2}xu_x^2,$$

$$\begin{aligned}
T_4^x &= 2u_x u_z u + \frac{3}{4} u_{xxz} u + \frac{1}{2} u_t u + 8t u_t u_x u_z + 3t u_t u_{xxz} - t u_{xx} u_{tz} - 2t u_{tx} u_{xz} \\
&\quad + z u_t u_z + 2t u_t^2 + 4z u_x u_z^2 + x u_x^2 u_z - \frac{3}{4} u_{xx} u_z + \frac{3}{2} z u_z u_{xxz} - z u_{xz}^2 - u_x u_{xz} \\
&\quad - \frac{1}{2} z u_{xx} u_{zz} - \frac{1}{4} x u_{xx} u_{xz} + \frac{3}{4} x u_x u_{xxz}, \\
T_4^z &= u_x^2 u + \frac{1}{4} u_{xxx} u - z u_t u_x + 4t u_t u_x^2 + t u_t u_{xxx} - t u_{xx} u_{tx} + \frac{1}{2} z u_{xx} u_{xz} \\
&\quad + \frac{1}{2} z u_{xxx} u_z + x u_x^3 - \frac{1}{2} u_{xx} u_x + \frac{1}{4} x u_{xxx} u_x - \frac{1}{4} x u_{xx}^2;
\end{aligned}$$

$$\begin{aligned}
T_5^t &= -\frac{1}{2} q(t) u_x, \\
T_5^x &= -2q(t) u_x u_z - \frac{3}{4} q(t) u_{xxz} - \frac{1}{2} q(t) u_t + \frac{1}{2} q'(t) u, \\
T_5^z &= -q(t) u_x^2 - \frac{1}{4} q(t) u_{xxx};
\end{aligned}$$

$$\begin{aligned}
T_6^t &= h(t) u_x^2 - \frac{1}{2} z h'(t) u_x, \\
T_6^x &= -\frac{1}{2} z h' u_t - 2z h' u_x u_z - \frac{3}{4} z h' u_{xxz} + \frac{1}{4} h' u_{xx} + 2h(t) u_x^2 u_z + \frac{3}{2} h(t) u_x u_{xxz} \\
&\quad - \frac{1}{2} h(t) u_{xx} u_{xz} + \frac{1}{2} z h''(t) u, \\
T_6^z &= -z h' u_x^2 - \frac{1}{4} z h' u_{xxx} + 2h(t) u_x^3 + \frac{1}{2} h(t) u_{xxx} u_x - \frac{1}{2} h(t) u_{xx}^2.
\end{aligned}$$

10.3 Conclusion

In this chapter we obtained the conservation laws for the Calogero-Bogoyavlenskii-Schiff equation (10.1) using Noether's theorem. Noether point symmetries were first computed and then using Noether's theorem the associated conserved vectors were derived. The conservation laws obtained here contain the energy and momentum conservation laws and the dilational energy conservation law.

Chapter 11

Concluding remarks

The aim of this work was to obtain closed-form solutions and conservation laws for certain nonlinear multi-dimensional partial differential equations which govern many physical phenomena of the real world.

In Chapter one we presented a brief introduction to the Lie symmetry analysis and conservation laws of PDEs. This included the algorithms to determine the Lie point symmetries and conservation laws of PDEs. We also provided certain methods that determine the exact solutions of nonlinear partial differential equations.

In Chapter two Lie symmetries as well as the simplest equation method were used to obtain exact solutions of the $(2+1)$ -dimensional Boussinesq equation. The solutions obtained were solitary waves and non-topological soliton. Moreover, the conservation laws for the $(2+1)$ -dimensional Boussinesq equation were also derived using the new conservation theorem due to Ibragimov.

The $(3+1)$ -dimensional Boussinesq equation was studied in Chapter three. Exact solutions of this equation were obtained with the aid of Lie point symmetries along with the simplest equation method. Furthermore, conservation laws were also constructed utilizing the new conservation theorem due to Ibragimov.

In Chapter four exact solutions of the nonlinear evolution partial differential equation, namely the generalized (3+1)-dimensional Kawahara equation were obtained using Lie symmetries in conjunction with the Kudryashov's method. Moreover, the conservation laws for this equation were derived using the multiplier method.

Chapter five studied the exact solutions and conservation laws of a (3+1)-dimensional generalized KP-Boussinesq equation. Exact solutions were obtained using the Lie symmetries and Kudryashov's method. Furthermore, conservation laws for the equation were derived using Ibragimov's conservation theorem.

In Chapter six, we obtained exact solutions for the (3+1)-dimensional BKP-Boussinesq equation using Lie symmetry reductions, direct integration as well as the (G'/G) -expansion method. Thereafter, we obtained the conservation laws of (6.4) by employing the conservation theorem due to Ibragimov.

Chapter seven studied a generalized extended (3+1)-dimensional Jimbo-Miwa equation. Travelling wave solutions were constructed using Lie symmetry method and the (G'/G) -expansion method. Conservation laws were computed by invoking the conservation theorem due to Ibragimov.

In chapter eight the exact solutions and conservation laws of a generalized extended (3+1)-dimensional Jimbo-Miwa equation. Travelling wave solutions were derived using the Lie symmetry method and simplest equation method. Conservation laws were derived by using the conservation theorem due to Ibragimov.

Chapter nine presented the travelling wave solutions and conservation laws of the combined KdV-negative-order KdV equation. The (G'/G) -expansion method was used to find its travelling wave solutions. The obtained solutions were in terms of hyperbolic, trigonometric and rational functions, which include solitons and periodic solutions. Conservation laws were derived using the multiplier approach.

In Chapter ten, we obtained the conservation laws for the Calogero-Bogoyavlenskii-

Schiff equation using Noether's theorem. First we computed the Noether point symmetries and then using Noether's theorem we derived the associated conserved vectors.

Future studies would focus on the use of conservation laws to obtain exact solutions of the multi-dimensional nonlinear differential equations studied in this thesis.

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