Portfolio optimisation approaches towards investment in the forex market

L Marudulu

orcid.org 0000-0002-1132-6343

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Supervisor: Prof S Terblanche

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Abstract

The study proposes the use of the mean-variance (M-V), semi-mean-absolute deviation (SMAD) and conditional value-at-risk (CVaR) models as measures of risk and reward in the Foreign Exchange (Forex) Market. The Forex Market is a highly volatile environment that requires a risk minimising approach to protect the investor or trader against potential big losses or unfavourable returns. The objective of the study is to select low risk but profitable currency portfolios in an optimal way. These portfolios inform the investor of how much to invest per trade. The answer to this question is generally subjective from a trader's point of view and is influenced by various factors, but this study proposes an objective solution to this problem. In this study, M-V, SMAD and CVaR optimal portfolios are generated and various properties of them are compared. Since investing is a future based activity, forecasted returns are used for constructing portfolios instead of using historic returns as has been predominantly done in the literature. Forecasted returns are generated using the Fourier series and simple exponential smoothing models. Portfolio optimisation techniques applied to the Forex Market are not popular in the literature, possibly due to the nature of the Forex data that comes in pairs. It has been shown that under certain realistic assumptions, these techniques are applicable and produce sensible results.

Keywords: Portfolio optimisation; mean-variance; semi-mean-absolute deviation; value-at-risk; conditional value-at-risk; forex market; portfolio risk; Fourier series; simple exponential smoothing, forecasting; loss function.

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Chapter 1

Introduction

With the development of financial markets throughout the years, the importance of risk management has increased, especially after the market failure in 2008. Globalization of financial markets, financial integration, technology improvement in trading systems and more complex derivative markets, result in new sources of risk. The growth in trading activity has made the environment more volatile which exposes firms and investors to more financial risk. The expansion of complex financial structures calls for better risk management techniques where risk must be accurately identified and measured. There exist numerous financial risk measures and models for portfolio optimisation and the choice of a risk measure becomes important (Ren and Bystroom, 2012). According to Marakbi (2016), investors have long been interested in ways of allocating capital amongst various assets in an efficient manner, thus creating efficient portfolios. An efficient portfolio refers to a portfolio of securities that yields the highest possible return for a given level of risk, or a portfolio that yields the lowest possible risk for a given level of return. Naturally such portfolios are appealing to portfolio managers around the world and the existing body of knowledge includes a significant amount of research on this matter, which to a large extent is dominated by quantitative models.

Out of these models the most protruding is the Mean-Variance (M-V) model, introduced in a groundbreaking paper published in 1952 by Harry Markowitz, this paper earned the American economist a Nobel prize in 1990. Markowitz's publication proved to become a cornerstone in Modern Portfolio Theory (MPT) and an important stepping stone towards the creation of further financial models such as the Capital Asset Pricing Model (CAPM), developed by Sharpe in 1964 (Marakbi, 2016). However, there has been a tremendous amount of research on improving the M-V model both computationally and theoretically. Various portfolio optimisation models have been proposed such as the Semi-Mean-Absolute deviation (SMAD) and Conditional Value-at-Risk (CVaR). The SMAD model is attractive to investors since it is a downside risk model that does not penalise above average returns, it does not depend on the normallity assumption upon which the MV model is based, and more importantly this model is more consistent with an investors true perception of risk. The CVaR model is also of interest to investors since it gives an idea of how much they can lose on average if the value at risk (VaR) amount is exceeded, hence it is also known as the mean excess loss, Whilst VaR quantifies the minimum amount that can be lost over a specified time period given a certain confidence level,it gives no indication of the size of the loss associated with the tail of the probability distribution outside of the confidence level. This drawback of VaR is the motivation behind using CVaR since it adresses this problem by quantifying losses exceeding VaR (Abdulbasah and Khalipah, 2005).

The Foreign Exchange market (Forex) is a financial market where currencies are traded against each other. It consists of banks, commercial companies, investment management firms, hedge funds, Forex brokers and individual retail traders. This market is considered to be the largest financial market in the world, trading in excess of an estimated USD 4 trillion in a single day. It should be noted that this study looks at the Forex market from a retail trader's perspective, since this group is the largest participant and yet the least profitable due to the biased nature of the market. Forex trading is defined as the financial activity of making money by speculating on the movements of currency prices of different countries, there are two ways to do this: The first is to buy expecting prices to rise and the second is to sell, expecting prices to fall.Therefore Forex trading is synonymous with buying or short selling stocks or any other financial securities with the hope of making a profit.The securities speculated upon are the currencies of various countries.The Forex market has a few trading alternatives compared to the thousands found in the stock market.The majority of Forex traders focus their efforts on seven currency pairs.These are : EUR/USD, USD/JPY, GBP/USD, USD/CHF, USD/CAD, AUD/USD, NZD/USD (Singh, 2012).

The first currency in a pair is known as a base and the second currency is known as a quote. Buying or selling a currency pair implies buying (going long) or selling (going short) the base currency. The general process of opening a trade starts with the trader anlysing the market using technical or fundamental analysis tools, the trader decides to open a trade based on the signal generated by the tools. The trader then subjectively decides on the amount or percentage of capital to invest on the chosen currency pair. The trader also sets up a stop loss and take profit level wich determine when and how the trade is closed. However, with all the tools and information available it is difficult to make consistent profits in the Forex market due to the complex dynamics that control the market, making it volatile and difficult to predict at any given time (Dolan, 2011).

1.1 Problem Statement

Investing is a financial activity that involves risk. It is the commitment of funds for a return expected to be realised in the future. Investments may be made in financial assets or physical assets. In either case there is the possibility that the actual return may vary from the expected return or result into a loss (negative return), that possibility is the risk involved in the investment (Suresh, 2013). The terms speculating (trading) and investing are used interchangeably since the study takes the view that speculating is a higher risk form of investing and this view is echoed by Graham (1949) in the book *The Intelligent Investor* in which the author states : "The most realistic distinction between the investor and the speculator is found in their attitude toward market movements. The speculator's primary interest lies in anticipating and profiting from market fluctuations. Whereas the investor's primary interest lies in acquiring and holding suitable securities at a suitable price". In a nutshell an investor is primarily concerned about the safety of investable wealth (capital) whilst making profits (losses) over a long period of time, whereas a speculator focuses on making large profits (losses) over a short period of time with no guarantee of safety of the capital. The lack of guarantee of safety of capital and the fast rate of change of currency prices in the Forex market is what makes it riskier than the stock market.

The Forex market, which has been investigated by various researchers, is a highly volatile environment which poses a problem for a risk averse investor (low apetite for risk), this is more so for high frequency trading periods such as hourly, weekly or daily which is the main focus of this study. It is important for investors to be able to predict Forex price movements in order to support trading decisions and obtain favourable returns. However, many factors, such as political events, general macro-economic conditions, influence of the market makers, and even the trader's personal beliefs and convictions may seriously influence how the Forex market behaves, which imply that forecasting the price movements of currencies is quite a difficult task (Singh, 2012). Fundamental and technical analysis are the main tools that a trader uses to make trading decisions such as when to enter and exit a trade. Fundamental analysis uses news reports, economic data and political events to make predictions about the price movements, whilst technical analysis uses charts, indicators and expert advisors to predict future price movements. Fundamental analysis takes relatively a long-term approach to analyse the market compared to technical analysis. While technical analysis can be used on a timeframe of weeks, days or even minutes, fundamental analysis is often more useful when looking at data over longer periods such as monthly, quartely, and semi annualy. However it has been shown that even with these tools and information at the trader's disposal it is difficult to make profits consistently due to the efficiency of the market (Gururaj and Kulkarni, 2013).

Motivated by the shortcomings of the analysis tools mentioned above, this study proposes a portfolio selection approach whose selection criteria is based on long term behaviour of currency prices whilst accounting for risk, the idea is that even though a currency's price may vary in an unpredictable manner, if on the average a combination of currencies offer a positive return then there's a potential for making profit whilst minimising risk. An optimisation model's objective is to then select currencies with this property. Additionally the model will also inform the investor about the proportion of capital to invest in each currency. Hence removing the subjectivity around the question of how much should be invested per trade, therefore the amount to invest per trade is chosen in an optimal way that minimises the overall risk associated with the chosen portfolio. It is important to note that the objective of the study is not to replace technical and fundamental analysis tools with portfolio optimisation models but to suggest the latter's use as an additional trading strategy that helps the trader in decision making. The idea is that an investor can run an optimisation model over the currencies to select the optimal weights and then use the technical and fundamental analysis tools to generate entry and exit signals, thereby using both approaches. Therefore one of the objectives of the study is to apply optimisation models in a highly volatile but profitable market inorder to assist the speculators to select long term profitable combinations of currencies with minimum risk.

The study achieves this by using variance, semi-mean-bsolute deviation and conditional value-at-risk as portfolio risk measures in the Forex market and compare the results generated by the three models. Markowitz (1952) showed that by using variance or standard deviation as a risk measure, an investor can reduce the portfolio risk simply by holding a large combination of securities that are not perfectly positively correlated. In other words, investors can reduce exposure to individual security risk by holding a diversified portfolio of securities. This concept of diversification is the motivation for using variance as a risk measure. However variance is a global measure of variability that considers returns below the expected value equally as bad as returns above it. Investors are interested in minimising unfavourable returns or losses and the semi-mean-absolute deviation accomplishes just that by considering only below average returns. The conditional value-at-risk is the mean of the worst 1%, 5% or 10% negative returns (losses) that occur at the left tail of the return distribution or right tail of the loss distribution. This study aims to compare the portfolios and results generated by the three risk measures given the same inputs, to see if there are any significant differences on how investors should allocate capital with regards to the different risk measures.

In the literature, historic data is often used to estimate model parameters. The assumption behind this approach is that the historic data is a good estimate of the future state of the market, in other words history repeats itself. However, this assumption does not always hold and returns on investments are realised in the future, thus using historic data for portfolio optimisation leads to logically misleading results. It is for this reason then that in this study, forecasted data will be used to estimate model parameters and the selection of efficient portfolios will be based on the forecasted data. For forecasting, the study makes use of the Fourier series (FS) and simple exponential smoothing (SES) models whose parameters will be estimated based on the historic data. The two models are compared and a model with superior predictive power is chosen for forecasting using the sum of the squared errors (SSE) as a measure of accuracy.It is the conviction of the authors that this approach is more rational and realistic from an investor's perspective. The objectives of the study are summarised below.

1.2 Objectives

- To fit FS and SES models of different orders to the Forex data and use these models to forecast future currency prices.
- To construct efficient frontiers using the M-V,SMAD and CVaR models , which will consist of possible efficient portfolios that an investor can choose from.
- To reduce the risk associated with investing in the Forex market by using the M-V, SMAD and CVaR optimisation models.
- To investigate advantages of using optimisation models for decision making compared to using the naive approach of assigning equal proportions amongst securities.
- To perform backtesting analysis on certain portfolios for the M-V, SMAD and CVaR optimisation models.

Chapter 2

Literature Review

2.1 Portfolio Risk

The previous chapter gave an introduction and overview of the concepts that are central to this study. The problem that this study is trying to solve was identified and explained. A detailed description of how the problem will be solved was also given along with the summary of objectives that the study aims to fulfill. This chapter reviews the literature of the key study concepts that were introduced in the previous chapter. Risk is a concept that denotes a potential negative impact to an asset or some-characteristic of value that may arise from some present or future event. From the economic point of view, risk is any event or action that may adversely affect an organization's ability to achieve its objectives and execute its strategies. However in finance financial risk is essentially any risk associated with any form of potential loss of money (Karadag, 2008). Risk has two components: uncertainty and exposure. Uncertainty refers to the probability of facing the risk. Exposure is the amount of the financial loss if the uncertainty risk is realised. For a portfolio of securities, the total uncertainty risk comprises of the systematic risk also known as undiversifiable risk and unsystematic risk also known as diversifiable risk. Systematic risk refers to the movement of the whole financial market, this implies that even with a perfectly diversified portfolio, there is some risk that cannot be avoided (Karadag, 2008).

Usystematic risk is the risk associated with the individual asset and it differs from asset to asset. Unlike systematic risk, it can be diversified away by including a large number of assets from different financial sectors in the portfolio. The difference between systematic and unsystematic risk can be illustrated by the following example: Suppose there are two investors A and B, investor A buys ten different currencies subject to a certain volume size that amounts to USD 100,000, and investor B buys one currency that amounts to the same USD 100,000. If the market goes against investor A (systematic risk), then one, two or more currencies may lose value but it is highly unlikely that all ten will be affected at the same time. Hence investor A will suffer a loss but still make some profit on the unaffected currencies. On the other hand, if the market goes against investor B on the single currency, then the investor suffers a loss. The reason is that investor B's portfolio has more unsystematic risk that needs to be diversified away (Karadag, 2008). The study makes use of the standard- deviation, semi-mean-absolute deviation, and conditional value-at- risk to quantify this risk.

2.2 Modern Portfolio Theory

The quote: “Divide your investments among many places for you do not know what risks might lie ahead”, is found in the book of Ecclesiastes 11:2, which to a large extent relates to today’s common proverb: “Don’t put all your eggs into one basket”. So the concept of diversification has been around for many centuries. Infact Markowitz’s paper was a reaction to previous and existing research at that time, which to a large extent employed the law of large numbers theorem by Bernoulli, leading to conclusions that all risk could be diversified away (Marakbi, 2016). Markowitz (1952) claimed that the law of large numbers was not applicable to a portfolio of securities due to the prevalent interdependency and complexity of financial markets. In other words, the inter-correlation between financial securities implies that diversification cannot entirely eliminate risk. It is from this assertion that the essence of Markowitz’s revolutionary theory stems, i.e the existence of a trade-off relationship between risk and reward, where variance is a measure of risk and mean return a measure of reward (Marakbi, 2016). According to Markowitz (1952) diversification does not depend solely on the number of securities held, a portfolio with sixty different railway securities would not be as well diversified as the same size portfolio with securities in railroad, mining, manufacturing, etc. It is more likely for firms within the same industry to do poorly at the same time than for firms in dissimilar industries. The author further explains that, the portfolio optimisation process is divided into two stages:

- Parameter estimation: from historical observations and beliefs, estimations of future perfomance are formed (in terms of return and variance) of the specified universe of securities.
- Portfolio selection: employing the estimated parameters in the first stage, an efficient portfolio of securities is chosen. The security weights of the portfolio are obtained by solving an optimisation problem that is in line with the investor’s risk preferences.

Following the seminal work by Markowitz (1952), the portfolio optimisation problem is modeled as a mean-risk bicriteria problem where the portfolio mean is maximised or some portfolio risk measure is minimised. Several other risk measures have been later considered, thus creating the entire family of mean-risk models. While the Markowitz model is classified as a quadratic programming (QP) problem, many attempts have been made to linearize the portfolio selection problem, since a linear programming (LP) problem is more flexible (can incorporate various real-life constraints such as transaction costs, cardinality and lot sizes) and is computationally tractable (Mansini et al., 2006).

Konno and Yamazaki (1991) argue that, despite the mathematical plausibility of Markowitz's model, it is not popular amongst some investors since variance fails to capture an investor's perception of risk, the normality assumption of security returns is often violated, and many non-zero weights occur in the efficient portfolio making the portfolio financially challenging to manage. The authors suggested the use of the SMAD model to try and remove the difficulties associated with the M-V model. This model replaces variance as a risk measure with a downside absolute value function that considers returns below the mean only. The rationale behind the use of this risk measure is that, if a return is below the expected value, then the investor will become more unsatisfied than if the return is above the expected value. Why then did Markowitz's formulation become so popular? Konno and Yamazaki (1991) claim that the intuitive appeal and mathematical plausibility of the model has made it to persist even though the axioms behind it are not always consistent with reality.

2.2.1 Benefits of diversification

Assuming that variance of a portfolio is a correct measure of risk (uncertainty in the price movements) that an investor would like to minimise. Under realistic assumptions, it can be shown mathematically how diversification can reduce the portfolio risk. The expression for the variance of a portfolio is given by :

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \tag{2.1}$$

where w_i , w_j are the invested proportions in security i and j and σ_{ij} is the covariance between security i and j. Equation (1) can be written as :

$$V = \sum_{i=1}^{N} w_i^2 V_i + \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} w_i w_j \sigma_{ij} \tag{2.2}$$

where V_i is the variance of security i. From equation (2) it is clear that the lower the covariance between security returns, the lower the overall risk of the portfolio. This means that the risk of a portfolio can be reduced by investing in securities whose returns are uncorrelated or, equivalently investing in independent assets. If all securities under consideration are independent, the covariance between them is zero and the formula for risk becomes :

$$V = \sum_{i=1}^{N} w_i^2 V_i \tag{2.3}$$

Assuming equal amounts are invested in each asset, then with N assets the proportion invested in each is $\frac{1}{N}$. Thus :

$$V = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 V_i = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \left(\frac{1}{N}\right) V_i = \frac{\overline{V}}{N} \tag{2.4}$$

where $\overline{V}$ represents the average variance of the securities in the portfolio. As N gets larger and larger, the risk of the portfolio approaches zero. In general if there is a sufficiently large number of independent assets, then the risk of a portfolio of these assets approaches zero. therefore a lower risk can be achieved by diversification. This is a theoretical result which hardly ever occurs in practice, in most markets the correlation coefficient or covariance between assets is positive. When watching the news it is common to hear the "experts" talk about how the market as a whole is either doing well or badly. This suggests that investment returns tend to move together, ie they are positively correlated. In these markets, the risk of the portfolio cannot be made to go to zero, but can still be much less than the variance of an individual asset. With equal investment, the proportion invested in any one security s_i is $\frac{1}{N}$ and the formula for the portfolio risk becomes :

$$V = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 V_i + \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \left(\frac{1}{N}\right) \left(\frac{1}{N}\right) \sigma_{ij} \tag{2.5}$$

Factoring out $\frac{1}{N}$ from the first summation and $\frac{(N-1)}{N}$ from the second gives :

$$V = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \left(\frac{1}{N}\right) V_i + \frac{(N-1)}{N} \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{\sigma_{ij}}{N(N-1)} \tag{2.6}$$

Replacing the variances and covariances in the summation by their averages $\overline{V}$ and $\overline{C}$ gives :

$$V = \frac{\overline{V}}{N} + \frac{N-1}{N} \overline{C} \tag{2.7}$$

The contribution to the portfolio variance of the variances of the individual securities goes to zero as N gets very large. However, the contribution of the covariance terms approaches $\overline{C}$ as N gets large. So as the number of assets in the portfolio is increased, the variance of the return on the portfolio gets closer to the average covariance of return between the pairs of assets in that portfolio. The individual risk of securities can be diversified away, but the contribution to the total risk caused by the covariance terms cannot be diversified away (ActEd Study Material: 2018 Examinations. Subject CT8, 2018).

2.3 Optimisation

Mathematical optimisation refers to the techniques involved in finding the "best" or optimal solution to a given problem, provided that it can be expressed mathematically (Snyman, 2005). This optimal solution is found using various computational techniques, and is usually subject to certain restrictions (constraints). Most decision problems require the identification of three components, namely :

- What are the decision options or choices ?
- What restrictions are present when making the decisions ?
- What value or outcome must be optimised when evaluating the different choices ?

Once the above three components are identified, the problem is ready to be solved. In this study, the class of optimisation models to be solved are the quadratic (M-V) and the linear (SMAD and CVAR) programming models respectively.

2.3.1 Optimisation Models

The Quadratic Programming Model

According to Marakbi (2016) the quadratic programming (QP) model is used to solve an optimisation problem where the objective function is a quadratic real-valued function subject to linear constraint functions. Furthermore the decision variables in a QP model are real-valued. A general n variable QP model is often formulated in the following matrix form:

$$\textit{minimise} \qquad z = \boldsymbol{x'}\, \boldsymbol{\Sigma x} \tag{2.8}$$

$$\textit{subject to} \qquad A\boldsymbol{x} = \boldsymbol{b} \tag{2.9}$$

$$\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \tag{2.10}$$

Where $\boldsymbol{x}$ is the decision vector, $\boldsymbol{l}$ and $\boldsymbol{u}$ are the lower and upper bounds of $\boldsymbol{x}$. Solving a quadratic programming problem is particularly simple when the matrix $\boldsymbol{\Sigma}$ is positive semi-definite. There are various algorithms used to solve such problems, most of which are numerical methods whose guaranteed convergence to an optimal point is subject to certain properties of the problem. most of these algorithms rely on the convexity properties of the objective function and the feasible set , this is discussed in detail by (Taha, 2002).

The Linear Programming Model

Taha (2002) explains that the linear programming (LP) model is used for optimisation problems with "strict linear objective and constraint functions". That is for a problem to be treated and solved as an LP, it has to satisfy the axioms of the linear programming theory. An example of an n variable LP model is formulated in the following matrix form:

$$\textit{maximise} \qquad y = \boldsymbol{c}'\boldsymbol{x} \tag{2.11}$$

$$\textit{subject to} \qquad A\boldsymbol{x} = \boldsymbol{b} \tag{2.12}$$

$$\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \tag{2.13}$$

Where $\boldsymbol{x}$ is the decision vector, $\boldsymbol{l}$ and $\boldsymbol{u}$ are the lower and upper bounds of $\boldsymbol{x}$. The examples below of both the LP and QP models are simple representations of these models which can be modified accordingly depending on the type of problem that is being solved. It is also important to note that from this point going forward the optimisation models are presented in an equivalent equation form rather than the matrix form that is shown here.

2.3.2 Mean-Variance Model

Markowitz (1952) introduced a parametric single-period optimisation model in a mean-variance framework which provides analytical solutions for an investor either trying to maximise the expected return for a given level of risk or trying to minimise the risk for a given level of expected return. Markowitz assumed that the future market of the securities can be correctly reflected by the historical market of the securities.

The reward (profit/loss) and risk of the portfolio is measured by the mean and variance of returns respectively. The assumption of the M-V framework is that the random vector of returns $\boldsymbol{R}$ is multivariate normally distributed with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\Sigma}$, i.e:

If

$$\boldsymbol{R} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} \tag{2.14}$$

is the random vector of returns then

$$\boldsymbol{R} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{2.15}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \tag{2.16}$$

is the mean vector of returns and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}, \tag{2.17}$$

is the covariance matrix of returns. Markowitz's M-V model is formulated as an optimisation problem over real-valued variables with a quadratic objective function and linear constraints as follows:

$$\textit{minimise} \quad \sigma^2(\boldsymbol{w}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \tag{2.18}$$

$$\textit{subject to} \quad \sum_{j=1}^{n} w_j \mu_j = \rho \tag{2.19}$$

$$\sum_{j=1}^{n} w_j = 1 \tag{2.20}$$

$$w_j \geq 0, \quad j = 1, 2, .., n \tag{2.21}$$

Where n is the number of available securities, μ_i is the expected return of security i $(i = 1, 2, ..., n)$, σ_{ij} is the covariance between security i and j $(j = 1, 2, ..., n)$ with $i \neq j$, ρ is the desired expected return, and w_i is the decision variable which represents the proportion of capital held in security i. It should be pointed out that in this study the securities of interest are currencies which will be defined later in the chapter.

Equation (18) is the variance (risk) of the portfolio whilst equation (19), the return constraint, ensures that the portfolio has a predetermined expected return ρ. Equation (20) defines the budget constraints (all the money available should be invested) for a feasible portfolio while constraint (21) requires that all investable wealth should be non-negative (no short selling is allowed). It should be pointed out that in this study the function to be minimised is the standard deviation (SD) :

$$\sigma(\boldsymbol{w}) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}} \tag{2.22}$$

This modification does not affect the optimal portfolio $\boldsymbol{w^*} = (w_1^*, w_2^*, ..., w_n^*)$ since it is the same for both variance and standard deviation.

Efficient Frontier

Lwin (2015) argues that the risk and return of an optimal portfolio are positively related, which implies that higher returns are achievable only when investors are willing to take higher risks and vice versa, i.e. the risk cannot be reduced without decreasing the return. In practice, different investors have different preferred trade-offs between risk and return. An investor who is risk-averse will choose a "safe" portfolio with a low risk and a low return. An investor who is less risk averse will choose a more risky portfolio with a higher return. Thus, the portfolio optimisation problem does not prescribe a single optimal portfolio combination that either minimises risk or maximises reward, instead, the result of the portfolio optimisation is generally a range of efficient portfolios.

Suppose

$$f_1(\boldsymbol{w}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

and

$$f_2(\boldsymbol{w}) = \sum_{i=1}^{n} w_i \mu_i .$$

According to Lwin (2015) a portfolio is said to be efficient in the context of M-V portfolio optimisation if and only if there is no other feasible portfolio that improves at least one of the two optimisation criteria without worsening the other. In a two-dimensional space of risk and return, a solution $\boldsymbol{a}$ is efficient if there does not exist any solution $\boldsymbol{b}$ such that $\boldsymbol{b}$ dominates $\boldsymbol{a}$. Solution $\boldsymbol{b}$ is considered to dominate solution $\boldsymbol{a}$ if and only if C_1 or C_2 holds :

$$C_1 : f_1(\boldsymbol{b}) \leq f_1(\boldsymbol{a}) \;\; and \;\; f_2(\boldsymbol{b}) > f_2(\boldsymbol{a}) \tag{2.23}$$

$$C_2 : f_2(\boldsymbol{b}) \geq f_2(\boldsymbol{a}) \;\; and \;\; f_1(\boldsymbol{b}) < f_1(\boldsymbol{a}) \tag{2.24}$$

The collection of these efficient portfolios forms the efficient frontier that represents the best trade-offs between risk and return. The set of efficient portfolios are traced out by solving the M-V model (11-14) repeatedly with a different value of return ρ in (12) each time. The parabola in Figure 1 below illustratres the efficient frontier where each point on the curve is an efficient portfolio as shown in Figure 2.1 .

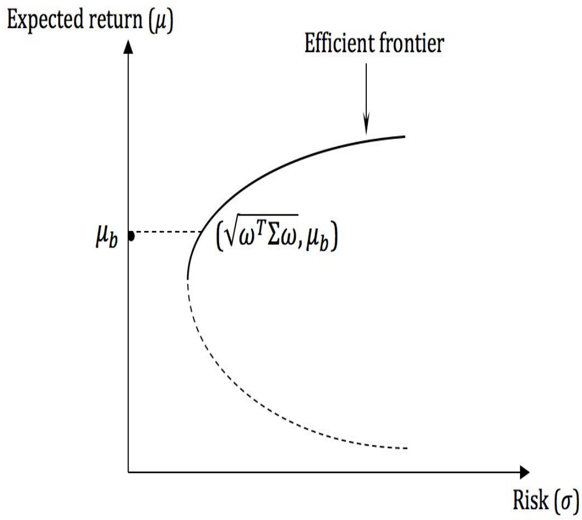


Figure 2.1: Mean-Variance Efficient Frontier.

Obtaining the efficient frontier simplifies the choice of investment for investors and the individual portfolios are selected based on the investor's risk tolerance and expectation of profit in return. Well spread distribution of portfolios along the efficient frontier provides more alternative suitable choices for investors with different risk-return profiles.

Limitations of the Mean-Variance Model

As with any model, it is crucial to understand the limitations of the M-V analysis in order to use it effectively. According to Lwin (2015) the M-V framework was developed for portfolio construction in a single period. In the single period portfolio optimisation problem, the investor is assumed to make allocations once and for all at the beginning of an investment period, based on the risk and return estimations and correlations of a universe of n investable securities.

Once the decisions are made, the decisions are not expected to change until the end of the investment period and the impact of decisions arising in subsequent periods is not considered in this case. Hence, the M-V model essentially represents a passive buy-and-hold strategy. Although Markowitz's M-V model plays a crucial role in financial theory, direct applications of this model are not of much practical use for various reasons. The model assumes the returns of securities follow a normal distribution and investors act in a risk averse manner. The model is simplified to be solvable under unrealistic assumptions. Thus, the basic Markowitz model does not reflect the restrictions (constraints) faced by real-world investors. It assumes a market without taxes or transaction costs where short sales are not allowed, and securities are infinitely divisible (they can be traded in any non-negative fraction). It is also assumed that investors do not care about different asset types in their portfolios which in general is not necessarily true. In practice however a portfolio manager often faces a number of investment constraints, such as legal restrictions, institutional features, industrial regulations, client initiated strategies and other practical matters. It is also possible that a portfolio manager may face restrictions on the maximum capital allocation to a particular industry. As a result, the basic model can be extended with a number of real world constraints that are often used in practical applications (Lwin, 2015).

More importantly as noticed by Abdulbasah and Khalipah (2005), the M-V model does not reflect an investor's true perception of risk. As such more "appropriate" risk measures are required especially within the Forex trading context. It should be noted that this study will not incorporate certain realistic constraints encountered by investors, these are only mentioned in passing. The reason behind this approach is mostly based on the fact that incorporating constraints such as number of currencies and lot sizes to be bought or sold leads to mixed integer programming problems which are often difficult and impossible to solve computationally due to a lack of an algorithm to solve this class of problems quickly (Abdulbasah and Khalipah, 2005). The study will only focus on the constraints as they appear in the basic M-V model.

2.3.3 Semi-Mean-Absolute Deviation Model

Each portfolio $\boldsymbol{w}$ defines a corresponding random variable $R_{\boldsymbol{w}} = \sum_{j=1}^{n} R_j w_j$ that represents a portfolio return. The return Y_t of a portfolio $\boldsymbol{w}$ in time period t can be computed as :

$$Y_t = \sum_{j=1}^{n} w_j R_{jt} \tag{2.25}$$

and the expected return of the portfolio $\mu_{\boldsymbol{w}}$ can be computed as a linear function of $\boldsymbol{w}$:

$$\mu_{\boldsymbol{w}} = E\{Y_t\} = \sum_{t=1}^{T} p_t Y_t = \sum_{t=1}^{T} p_t \sum_{j=1}^{n} w_j R_{jt} = \sum_{j=1}^{n} w_j \sum_{t=1}^{T} p_t R_{jt} = \sum_{j=1}^{n} w_j \mu_j \tag{2.26}$$

Due to the symmetric nature of variance as a risk measure, Konno and Yamazaki (1991) attempted to remove this undesirable property by introducing the mean absolute deviation (MAD) risk measure :

$$\delta(\boldsymbol{w}) = E\Big[\Big|R_{\boldsymbol{w}} - \mu_{\boldsymbol{w}}\Big|\Big] = E\Big[\Big|\sum_{j=1}^{n} w_j R_j - E\big[\sum_{j=1}^{n} w_j R_j\big]\Big|\Big] \tag{2.27}$$

Where R_w and μ_w are the return and mean return of the portfolio over the investment period. The MAD measures the average of the absolute value of the difference between the random variable and its expected value. With respect to the variance, the MAD considers absolute values instead of squared values. Since the expected return of the portfolio can be calculated as in equation (19), the MAD can be written as :

$$\delta(\boldsymbol{w}) = \sum_{t=1}^{T} p_t \left(\left| \sum_{j=1}^{n} w_j R_{jt} - \sum_{j=1}^{n} w_j \mu_j \right| \right) \tag{2.28}$$

The portfolio optimisation problem then becomes :

$$\textit{minimise} \qquad \delta(\boldsymbol{w}) = \sum_{t=1}^{T} p_t \left(\left| \sum_{j=1}^{n} w_j R_{jt} - \sum_{j=1}^{p} w_j \mu_j \right| \right) \tag{2.29}$$

$$\textit{subject to} \qquad \sum_{j=1}^{n} w_j \mu_j = \rho$$

$$\sum_{j=1}^{n} w_j = 1$$

$$w_j \geq 0 \ , \qquad j = 1, 2, ..., n$$

This form is not linear in the variables w_j but can be transformed into a linear form. Using equation (18) for the return of the portfolio in time t, $\delta(\boldsymbol{w})$ can be written as :

$$\delta(\boldsymbol{w}) = \sum_{t=1}^{T} p_t \left(\left| Y_t - \sum_{j=1}^{n} w_j \mu_j \right| \right) \tag{2.30}$$

The deviation D_t in time period t can be written as

$$D_t = \left| Y_t - \sum_{j=1}^{n} w_j \mu_j \right|, \qquad t = 1, 2, ..., T \tag{2.31}$$

The portfolio optimisation problem is thus given by :

$$\textit{minimise} \qquad \delta(\boldsymbol{w}) = \sum_{t=1}^{T} p_t D_t \tag{2.32}$$

$$\textit{subject to} \qquad \sum_{j=1}^{n} w_j \mu_j = \rho$$

$$D_t = \left| Y_t - \sum_{j=1}^{n} \mu_j w_j \right| \quad , \qquad t = 1, 2, ..., T$$

$$Y_t = \sum_{j=1}^{n} w_j R_{jt} \quad , \qquad t = 1, 2, ..., T$$

$$\sum_{j=1}^{n} w_j = 1$$

$$w_j \geq 0 \qquad , \qquad j = 1, 2, ..., n$$

Since ,

$$\left| Y_t - \sum_{j=1}^{n} w_j \mu_j \right| = max\left[\left(Y_t - \sum_{j=1}^{n} w_j \mu_j \right) \ , \ -\left(Y_t - \sum_{j=1}^{n} w_j \mu_j \right) \right] \tag{2.33}$$

It was shown by Mansini et al. (2006) that the above problem is equivalent to the following linear programming (LP) problem :

$$\textit{minimise} \qquad \delta(\boldsymbol{w}) = \sum_{t=1}^{T} p_t D_t$$

$$\textit{subject to} \qquad D_t \geq Y_t - \sum_{j=1}^{n} \mu_j w_j \ , \qquad t = 1, 2, ..., T \tag{2.34}$$

$$D_t \geq -\Big(Y_t - \sum_{j=1}^{n} \mu_j w_j\Big) \ , \qquad t = 1, 2, ..., T \tag{2.35}$$

$$Y_t = \sum_{j=1}^{n} w_j R_{jt} \quad , \qquad t = 1, 2, ..., T$$

$$\sum_{j=1}^{n} w_j \mu_j = \rho \quad , \qquad j = 1, 2, ..., n$$

$$\sum_{i=1}^{n} w_j = 1 \quad , \qquad j = 1, 2, ..., n$$

$$D_t \geq 0 \quad , \qquad t = 1, 2, ..., T \tag{2.36}$$

$$w_j \geq 0 \quad , \qquad j = 1, 2, ..., n$$

The equivalence comes from observing that if $Y_t - \sum_{j=1}^{n} w_j \mu_j \geq 0$, constraints (28) are redundant. In this case constraints (27) combined with the objective function (25) that pushes the value of each D_t to minimum, impose that $D_t = Y_t - \sum_{j=1}^{n} w_j \mu_j = \left| Y_t - \sum_{j=1}^{n} w_j \mu_j \right|$. If, on the contrary $Y_t - \sum_{j=1}^{n} w_j \mu_j \leq 0$, constraints (27) are redundant. In this case, constraints (28), combined with the objective function (25), impose that $D_t = -\Big(Y_t - \sum_{j=1}^{n} w_j \mu_j\Big) = \left| Y_t - \sum_{j=1}^{n} w_j \mu_j \right|$. Thus, in conclusion, the optimisation model above is a linear programming model for the optimisation of a portfolio where the risk is measured through the MAD of the return of the portfolio.

The MAD accounts for all deviations of the return of the portfolio from its expected value. However, one may sensibly think that any rational investor would consider real risk only the deviations below the expected value. In other words, the variability of the portfolio return above the mean should not be penalized since investors are concerned with under-performance rather than over-perfomance of a portfolio. Mansini et al. (2006) modified the MAD in order to consider only the deviations below the expected value. The SMAD is thus defined as :

$$\xi(\boldsymbol{w}) = E\Big[max\Big\{0, \mu_{\boldsymbol{w}} - R_{\boldsymbol{w}}\Big\}\Big] = E\Big[max\Big\{0, E\Big\{\sum_{j=1}^{n} w_j R_j\Big\} - \sum_{j=1}^{n} w_j R_j\Big\}\Big] \tag{2.37}$$

where the deviations above the expected value are not calculated. According to Mansini et al. (2006), the portfolio optimisation problem presented for the MAD can be adapted to the SMAD as follows :

$$\begin{array}{lll}
\textit{minimise} & \delta(\boldsymbol{w}) = \displaystyle\sum_{t=1}^{T} p_t D_t & \\
\textit{subject to} & D_t \geq -\Big(Y_t - \displaystyle\sum_{j=1}^{n} w_j \mu_j\Big) \ , & t = 1, 2, ..., T \\
 & Y_t = \displaystyle\sum_{j=1}^{n} w_j R_{jt} \ , & t = 1, 2, ..., T \\
 & \displaystyle\sum_{j=1}^{n} w_j \mu_j = \rho \ , & j = 1, 2, ..., n \\
 & \displaystyle\sum_{i=1}^{n} w_j = 1 \ , & j = 1, 2, ..., n \\
 & D_t \geq 0 \ , & t = 1, 2, ..., T \\
 & w_j \geq 0 \ , & j = 1, 2, ..., n
\end{array}$$

The formulation for the SMAD is the formulation for the MAD, from which constraints (27) have been dropped. If for a given time period t, if $Y_t - \sum_{j=1}^{n} w_j \mu_j \leq 0$,the return of the portfolio Y_t is below the expected value. In this case D_t in the optimum will be the difference $Y_t - \sum_{j=1}^{n} w_j \mu_j$. If instead $Y_t - \sum_{j=1}^{n} w_j \mu_j \geq 0$, constraint (28) becomes redundant and in the optimum $D_t = 0$. Thus, the deviations above the expected value are not calculated in the objective function. The SMAD is a downside risk measure which will be employed in this study. Figure 2.2 below shows the set of efficient portfolios generated by the SMAD model, where the efficient frontier is indicated by the solid line.

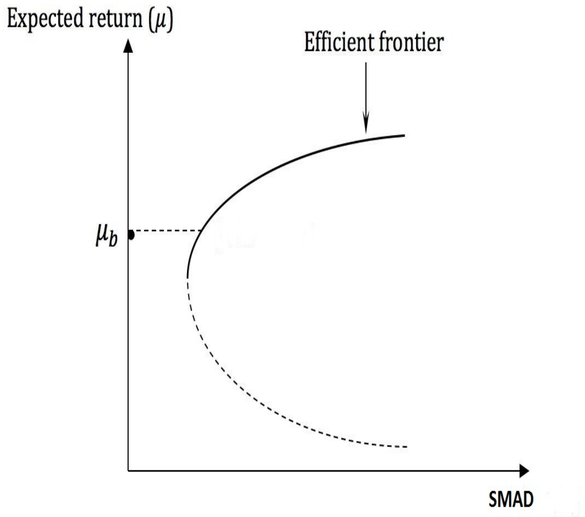


Figure 2.2: SMAD Efficient Frontier.

2.3.4 Conditional Value-at-Risk

A class of risk measures known as downside risk measures focus on the right tail of the loss distribution or the left tail of the return distribution to minimise the risk of large losses or extreme events and thus achieving the objective of risk-averse investors.The value at risk VaR at $\alpha \in (0, 1)$ confidence level of the return distribution is one of the most popular downside risk measures in mathematical finance, it was introduced in the late 1930's by financial firms indicating the amount of minimum loss at a given confidence level α. By definition with respect to a specified probability level α, the VaR_α of a portfolio is the lowest amount β such that with probability α the loss will not exceed β, whereas the CVaR_α is the conditional expectation of losses above that amount β. Three values of α are commonly used: 0.9, 0.95, and 0.99 (Rockafellar and Uryasev, 2000).

The definitions ensure that the VaR_α is never more than the CVaR_α. Although VaR is a very popular risk measure, it has undesirable mathematical characteristics such as lack of subadditivity and convexity. For example, VaR associated with a combination of two portfolios can be deemed greater than the sum of the risks of the individual portfolios. VaR can be ill-behaved as a function of portfolio weights and can exhibit multiple local extrema which can be a major handicap in trying to determine an optimal combination of weights. As an alternative measure of risk CVaR is known to have better properties than VaR (Rockafellar and Uryasev, 2000). Pflug (2000) proved that CVaR is a coherent risk measure having the following properties: transition-equivariant, positively homogeneous, convex, monotonic w.r.t stochastic dominance of order one and two. Figure 2.2 below shows the VaR and CVaR associated with the loss distribution. These two quantities occur on the right tail of the distribution where the extreme losses are.

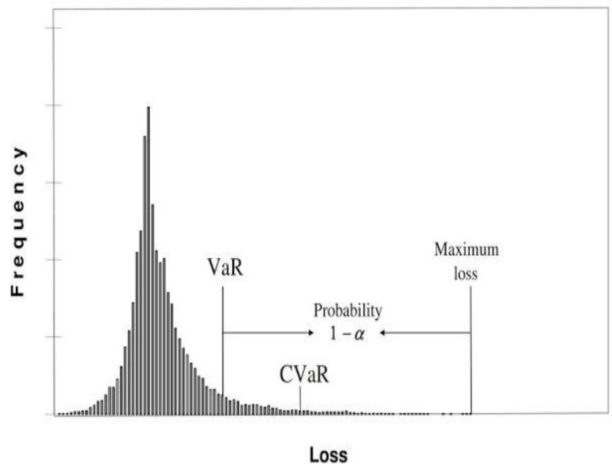


Figure 2.3: VaR_α and CVaR_α of the loss distribution

Rockafellar and Uryasev (2000) define a loss function $f(\boldsymbol{x}, \boldsymbol{y})$ to be the loss (negative portfolio return) associated with the decision vector $\boldsymbol{x}$, to be chosen from a certain subset X of $\mathbb{R}^n$ and the random vector $\boldsymbol{y} \in \mathbb{R}^m$.The vector $\boldsymbol{x}$ is interpreted as representing a portfolio, with X as the set of permissible portfolios but other interpretations are possible. The vector $\boldsymbol{y}$ represents uncertainties that affect the loss. The probability of $f(\boldsymbol{x}, \boldsymbol{y})$ not exceeding a threshold γ is given by:

$$\psi(\boldsymbol{x}, \gamma) = \int_{f(\boldsymbol{x},\boldsymbol{y}) \leq \gamma} p(\boldsymbol{y}) d\boldsymbol{y} \tag{2.38}$$

As a function of γ for a fixed $\boldsymbol{x}$, $\psi(\boldsymbol{x}, \gamma)$ is the cumulative distribution function for the loss associated with $\boldsymbol{x}$. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR. The VaR_α and CVaR_α values for the loss random variable associated with decision vector $\boldsymbol{x}$ and any specified probability level α in $(0, 1)$ are denoted by $\gamma_\alpha(\boldsymbol{x})$ and $\phi_\alpha(\boldsymbol{x})$. Mathematically they are given by :

$$\gamma_\alpha(\boldsymbol{x}) = \min\left\{\gamma \in \mathbb{R} : \psi(\boldsymbol{x}, \gamma) \geq \alpha\right\} \tag{2.39}$$

and

$$\phi_\alpha(\boldsymbol{x}) = (1-\alpha)^{-1} \int_{f(\boldsymbol{x},\boldsymbol{y}) \geq \gamma_\alpha(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}) p(\boldsymbol{y}) d\boldsymbol{y} \tag{2.40}$$

According to Rockafellar and Uryasev (2000) , $\gamma_\alpha(\boldsymbol{x})$ and $\phi_\alpha(\boldsymbol{x})$ can be characterised interms of the function F_α on $X \times \mathbb{R}$ given by :

$$F_\alpha(\boldsymbol{x}, \gamma) = \gamma + (1-\alpha)^{-1} \int_{\boldsymbol{y} \epsilon \mathbb{R}^m} \left[f(\boldsymbol{x}, \boldsymbol{y}) - \gamma\right]^+ p(\boldsymbol{y}) d\boldsymbol{y} \tag{2.41}$$

where $[t]^+ = t$ when $t > 0$ and $[t]^+ = 0$ when $t \leq 0$. The important feature of F_α in optimisation is that it is convex which is a key property that guarantees global optimality of solutions.

Theorem 1: As a function of γ, $F_\alpha(\boldsymbol{x}, \gamma)$ is convex and continously differentiable. The CVaR$_\alpha$ of the loss associated with any $\boldsymbol{x} \epsilon X$ can be determined from the formula:

$$\phi_\alpha(\boldsymbol{x}) = \min_{\gamma \epsilon \mathbb{R}} F_\alpha \ (\boldsymbol{x}, \gamma) \tag{2.42}$$

In this formula the set consisting of the values of γ for which the minimum is attained, namely :

$$\mathrm{A}_\alpha(\mathrm{x}) = \underset{\gamma \epsilon \mathbb{R}}{\mathrm{argmin}} \ F_\alpha(\boldsymbol{x}, \gamma) \tag{2.43}$$

is a nonempty, closed, bounded interval (perhaps reducing to a single point), and the VaR$_\alpha$ of the loss is given by :

$$\gamma_\alpha(\boldsymbol{x}) = \text{left endpoint of } \mathrm{A}_\alpha(\mathrm{x}) \tag{2.44}$$

In particular, one always has :

$$\gamma_\alpha(\boldsymbol{x}) \ \epsilon \ \ \mathrm{A}_\alpha(\mathrm{x}) \quad \text{and} \quad \phi_\alpha(\boldsymbol{x}) = F_\alpha(\boldsymbol{x}, \gamma_\alpha(\boldsymbol{x})) \tag{2.45}$$

Proof of Theorem 1

Before proving theorem 1, it is assumed $\psi(\boldsymbol{x}, \gamma)$ is continous continous with respect to γ, which is equivalent to knowing that, regardless of the choice of $\boldsymbol{x}$, the set of y with $f(\boldsymbol{x}, \boldsymbol{y}) = \gamma$ has probability zero, that is :

$$\int_{f(\boldsymbol{x},\boldsymbol{y})=\gamma} p(\boldsymbol{y})d\boldsymbol{y}) = 0 \tag{2.46}$$

Lemma. With $\boldsymbol{x}$ fixed, let $G(\gamma) = \int_{\boldsymbol{y}\epsilon\mathbb{R}^n} g(\gamma,\boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y}$, where $g(\gamma,\boldsymbol{y}) = [f(\boldsymbol{x},\boldsymbol{y})-\gamma]^+$. Then G is a convex continously differentiable function with derivative

$$G'(\gamma) = \psi(\boldsymbol{x},\gamma) - 1 \tag{2.47}$$

Proof: $G(\gamma) = \int_{\boldsymbol{y}\epsilon\mathbb{R}^n} g(\gamma,\boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y} = \int_{\boldsymbol{y}\epsilon\mathbb{R}^n}[f(\boldsymbol{x},\boldsymbol{y}) - \gamma]^+ p(\boldsymbol{y})d\boldsymbol{y} = \int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma}[f(\boldsymbol{x},\boldsymbol{y}) - \gamma]p(\boldsymbol{y})d\boldsymbol{y}$. Using Leibniz integral rule:

$G'(\gamma) = \int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma} \frac{\partial[p(\boldsymbol{y})f(\boldsymbol{x},\boldsymbol{y})-\gamma p(\boldsymbol{y})]d\boldsymbol{y}}{\partial\gamma} = \int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma} -p(\boldsymbol{y})d(\boldsymbol{y}) = -(1-\psi(\boldsymbol{x},\gamma)) = \psi(\boldsymbol{x},\gamma) - 1.$

In view of the defining formula for $F_\alpha(\boldsymbol{x},\gamma)$, it is immediate from the lemma that $F_\alpha(\boldsymbol{x},\gamma)$ is convex and continously differentiable with derivative:

$$\frac{\partial F_\alpha(\boldsymbol{x},\gamma)}{\partial\gamma} = 1 + \left(1-\alpha\right)^{-1}\left(\psi(\boldsymbol{x},\gamma) - 1\right) = (1-\alpha)^{-1}\left(\psi(\boldsymbol{x},\gamma) - \alpha\right) \tag{2.48}$$

and

$$\frac{\partial F_\alpha(\boldsymbol{x},\gamma)}{\partial\gamma} = 0 \iff \psi(\boldsymbol{x},\gamma) - \alpha = 0 \iff \psi(\boldsymbol{x},\gamma) = \alpha$$

Therefore the values of γ that minimise $F_\alpha(\boldsymbol{x},\gamma)$ are precisely the ones for which $\psi(\boldsymbol{x},\gamma) = \alpha$. This result is consistent with the definition of VaR_α in formula (2.39). In particular it is true that

$$\min_{\gamma\epsilon\mathbb{R}} F_\alpha(\boldsymbol{x},\gamma) = F_\alpha(\boldsymbol{x},\gamma_\alpha(\boldsymbol{x})) = \gamma_\alpha(\boldsymbol{x}) + (1-\alpha)^{-1}\int_{\boldsymbol{y}\epsilon\mathbb{R}^n}[f(\boldsymbol{x},\boldsymbol{y}) - \gamma_\alpha(\boldsymbol{x})]^+ p(\boldsymbol{y})d\boldsymbol{y}$$

but

$$\int_{\boldsymbol{y}\epsilon\mathbb{R}^n}[f(\boldsymbol{x},\boldsymbol{y})-\gamma_\alpha(\boldsymbol{x})]^+p(\boldsymbol{y})d\boldsymbol{y}=\int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma_\alpha}[f(\boldsymbol{x},\boldsymbol{y})-\gamma_\alpha(\boldsymbol{x})]p(\boldsymbol{y})d\boldsymbol{y}$$

$$=\int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma_\alpha}f(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y}-\gamma_\alpha(\boldsymbol{x})\int_{f(\boldsymbol{x},\boldsymbol{y})\geq\gamma_\alpha}p(\boldsymbol{y})d\boldsymbol{y}$$

$$=(1-\alpha)\phi_\alpha(\boldsymbol{x})-\gamma_\alpha(\boldsymbol{x})(1-\psi(\boldsymbol{x},\gamma_\alpha(\boldsymbol{x})))$$

$$\therefore\min_{\gamma\epsilon\mathbb{R}}F_\alpha(\boldsymbol{x},\gamma)=\gamma_\alpha(\boldsymbol{x})+(1-\alpha)^{-1}((1-\alpha)\phi_\alpha(\boldsymbol{x})-\gamma_\alpha(\boldsymbol{x})(1-\alpha)),\quad\text{since}\quad\psi(\boldsymbol{x},\gamma_\alpha(\boldsymbol{x}))=\alpha.$$

$$\therefore\min_{\gamma\epsilon\mathbb{R}}F_\alpha(\boldsymbol{x},\gamma)=\gamma_\alpha+\phi_\alpha(\boldsymbol{x})-\gamma_\alpha=\phi_\alpha(\boldsymbol{x}).\tag{2.49}$$

According to Rockafellar and Uryasev (2000) the power of Theorem 1 lies in the fact that continously differentiable convex functions are especially easier to optimise numerically. What is revealed also is that CVaR$_\alpha$ can be calculated without having to calculate VaR$_\alpha$ on which its definition depends and is more complicated. The VaR$_\alpha$ may be obtained instead as a by-product, but the extra effort required to do this might be omitted if VaR$_\alpha$ is not needed, since the study's focus is on CVaR$_\alpha$, the procedure of obtaining VaR$_\alpha$ from $F_\alpha(\boldsymbol{x},\gamma)$ is ommited. The function $F_\alpha(\boldsymbol{x},\gamma)$ can be approximated in various ways. One way would be to sample the probability distribution of $\boldsymbol{y}$ according to its density $p(\boldsymbol{y})$. Given a sample of set $\boldsymbol{y_1},\boldsymbol{y_2},...,\boldsymbol{y_n}$, then the corresponding approximation to $F_\alpha(\boldsymbol{x},\gamma)$ is :

$$\tilde{F}_\alpha(\boldsymbol{x},\gamma)=\gamma+\frac{1}{q(1-\alpha)}\sum_{k=1}^{n}[f(\boldsymbol{x},\boldsymbol{y_k})-\gamma]^+\tag{2.50}$$

The expression $\tilde{F}_\alpha(\boldsymbol{x},\gamma)$ is convex and piecewise linear with respect to γ. Although it is not differentiable with respect to γ, it can readily be minimised by representation as a linear programming problem. Other important advantages of viewing VaR$_\alpha$ and CVaR$_\alpha$ through the formulas in theorem 1 are captured in the next Theorem.

Theorem 2: Minimising the CVaR$_\alpha$ of the loss associated with $\boldsymbol{x}\ \epsilon$ X is equivalent to minimising $F_\alpha(\boldsymbol{x},\gamma)$ over all $(\boldsymbol{x},\gamma)\ \epsilon\ X\times\mathbb{R}$, in the sense that :

$$\min_{\boldsymbol{x} \epsilon X} \phi_{\alpha}(\boldsymbol{x}) = \min_{(\boldsymbol{x},\gamma)\ \epsilon\ X\times\mathbb{R}} F_{\alpha}(\boldsymbol{x},\gamma) \tag{2.51}$$

Furthermore, $F_{\alpha}(\boldsymbol{x},\gamma)$ is convex with respect to $(\boldsymbol{x},\gamma)$ and $\phi_{\alpha}(\boldsymbol{x})$ is convex with respect to $\boldsymbol{x}$ when $f(\boldsymbol{x},\boldsymbol{y})$ is convex with respect to $\boldsymbol{x}$, in which case if the constraints are such that X is a convex set, the joint minimisation is a case of convex programming. According to Theorem 2, it is not necessary, for the purpose of determining a portfolio $\boldsymbol{x}$ that yields minimum CVaR$_{\alpha}$, to work directly with the function $\phi_{\alpha}(\boldsymbol{x})$, which may be hard to do because of the nature of its definition in terms of the VaR$_{\alpha}$ value $\gamma_{\alpha}(\boldsymbol{x})$ and the often troublesome mathematical properties of this value. Instead one can operate on the far simpler expression $F_{\alpha}(\boldsymbol{x},\gamma)$ which is convex in the variable γ and in most cases it is also convex in $(\boldsymbol{x},\gamma)$.

Proof of theorem 2

The proof relies on the fact that the minimisation of $F_{\alpha}(\boldsymbol{x},\gamma)$ with respect to $(\boldsymbol{x},\gamma)\ \epsilon\ X\times\mathbb{R}$ can be done by first minimising over $\gamma\ \epsilon\ \mathbb{R}$ for a fixed $\boldsymbol{x}$ and then minimising the result over $\boldsymbol{x}\ \epsilon\ X$. Justification of the convexity claim starts with the observation that $F_{\alpha}(\boldsymbol{x},\gamma)$ is convex with respect to $(\boldsymbol{x},\gamma)$ whenever the integrand $[f(\boldsymbol{x},\boldsymbol{y})-\gamma]^{+}$ in the formula for $F_{\alpha}(\boldsymbol{x},\gamma)$ is itself convex with respect to $(\boldsymbol{x},\gamma)$. For each $\boldsymbol{y}$, this integrand is the composition of the function $(\boldsymbol{x},\gamma)\rightarrow f(\boldsymbol{x},\boldsymbol{y})-\gamma$ with the nondecreasing function function $t\rightarrow[t]^{+}$ and by the rules of Rockafellar and Uryasev (2000) it is convex as long as the function $(\boldsymbol{x},\gamma)\rightarrow f(\boldsymbol{x},\boldsymbol{y})-\gamma$ is convex. $f(\boldsymbol{x},\boldsymbol{y})-\gamma$ is convex when $f(\boldsymbol{x},\boldsymbol{y})$ is convex with respect to $\boldsymbol{x}$. Since in this setting the function $f(\boldsymbol{x},\boldsymbol{y})$ reprsents the portfolio loss that is covex in the variable $\boldsymbol{x}$ the result of Theorem 2 follows (Rockafellar and Uryasev, 2000).

2.3.5 Conditional Value-at-Risk Model

The perfomance function in connection with VaR$_{\alpha}$ and CVaR$_{\alpha}$ is :

$$F_{\alpha}(\boldsymbol{x},\gamma)=\gamma+(\alpha)^{-1}\int_{\boldsymbol{x}\epsilon\mathbb{R}^{m}}\left[\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{y}-\gamma\right]^{+}p(\boldsymbol{y})d\boldsymbol{y} \tag{2.52}$$

It is important to observe that in this setting, $F_{\alpha}(\boldsymbol{x},\gamma)$ is convex as a function of $(\boldsymbol{x},\gamma)$ not just γ.

If one considers the feasible set of portfolios

$$X=\left\{\text{set of } \boldsymbol{x} \text{ satisfying (2.19), (2.20) and (2.21)}\right\} \tag{2.53}$$

This set X is convex (in fact "polyhedral" due to linearity of all the constraints). As a result the problem of minimising $F_\alpha(\boldsymbol{x}, \gamma)$ over $X \times \mathbb{R}$ subject to (2.53) is one of convex programming which guarantees globality of optimal solutions. Considering the kind of approximation of F_α that is obtained by sampling from the probability distribution of $\boldsymbol{y}$. A sample set $\boldsymbol{y_1, y_2, ..., y_n}$ yields the approximate function :

$$\tilde{F}_\alpha(\boldsymbol{x}, \gamma) = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} [\boldsymbol{x}^T \boldsymbol{y_t} - \gamma]^+$$

In terms of auxillary variables u_t for $t = 1, 2, ..., T$, minimising F_α is equivalent to minimising the linear expression:

$$z = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} u_t \tag{2.54}$$

subject to the constraints : $u_t \geq 0$ and $\boldsymbol{x}^T \boldsymbol{y_t} + \gamma + u_t \geq 0$.

Hence the CVaR$_\alpha$ minimisation problem is formulated as :

$$\begin{aligned}
&\textit{minimise} && z = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} u_t \\
&\textit{subject to} && \sum_{i=1}^{n} x_i \mu_i = \rho \\
& && \sum_{i=1}^{n} x_i = 1 \\
& && \boldsymbol{x}^T \boldsymbol{r_t} + \gamma + u_t \geq 0 \\
& && u_t \geq 0, \quad t = 1, 2, ..., T \\
& && x_i \geq 0, \quad i = 1, 2, ..., n
\end{aligned}$$

Figure 2.4 below shows the set of efficient portfolios generated by the CVaR model, where the efficient frontier is indicated by the solid line.

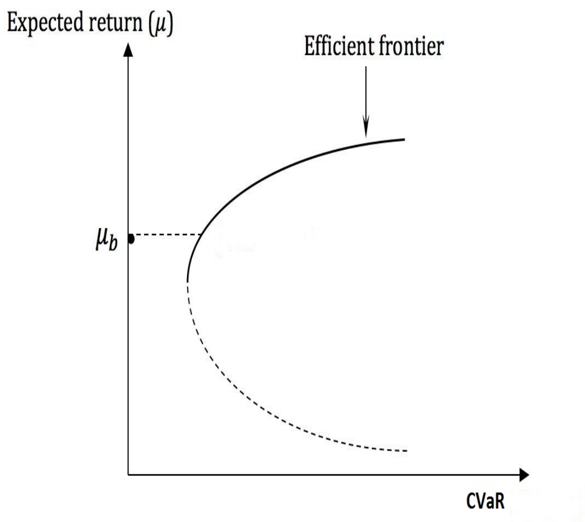


Figure 2.4: CVaR Efficient Frontier.

2.4 Forecasting Models

2.4.1 Fourier Series Model

According to Selimi (2013) the french mathematician Jean Baptiste Joseph Fourier made an astonishing invention. In 1807 he presented a paper to the Academy of Science which dealt with the problem of how heat "flows" through metallic rods and plates. In the paper Fourier claimed that any periodic function $f(x)$ that is square-integrable over the interval $(-\pi, \pi)$ can be represented by :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx)). \tag{2.55}$$

where a_n and b_n are constants given by :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \quad , \quad (n = 0, 1, 2...) \tag{2.56}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \quad , \quad (n = 0, 1, 2...). \tag{2.57}$$

This representation is valid on all of $\mathbb{R}$ since the Fourier series is guaranteed to converge, with f having a period of 2π. In this study it is assumed that the currency prices are generated by a function with the properties defined above. This function can then be approximated by an n^{th} order fourier series :

$$S_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n}(a_j cos(jx) + b_j sin(jx) \tag{2.58}$$

.

This is analogaous to the auto-regressive integrated moving average (ARIMA) model's assumption that a time series data is a realization of some linear and stationary ARIMA process (Tsai and Chen, 2016) .

Tsai and Chen (2016) aknwoledges that various models have been used in the literature to forecast time series data, these include statistical and artificial intelligence models. The author claims that these models often require relative large amounts of data for model fitting and in most cases require the estimation of a large number of variables. These models are not flexible enough to provide reasonable good forecasts when dealing with non-stationary and non-linear time series data.

Afshar and Fahmi (2012) echoe this view in saying that, there are various traditional techniques and mathematical models available for forecasting, yet there are no results demonstrating which method provides the most reliable estimation. The authors further explain that these models (ARIMA) provide reasonable accuracy but suffer from the assumptions of linearity and stationarity when the data is neither linear nor stationary, additionally certain ANNs models are equivalent to time series models, but limited only to short term forecasting.

Tsai and Chen (2016) used a FS model defined above to analyse and predict electricity consumption in buildings. Romera et al. (2008) used a hybrid artificial neural network-Fourier series model approach to forecast energy demand where the FS model was used to predict the periodic behaviour of the data. Darko (2016) employed a Fourier series model to forecast solid waste generation and compared the predictive power of this model against the traditionally used ARIMA models, the author concluded that the Fourier model was superior to the ARIMA models, and this result is a motivating factor for using the Fourier model in this study as well. Since Forex data is highly volatile and exhibits periodic behaviour, this study adopts a curve fitting approach to the data and selects the "best" n^{th} order Fourier series model for prediction using the least squares method.

2.4.2 Simple Exponential Smoothing Model

Simple exponential smoothing (SES) is probably the most widely used class of procedures for smoothing discrete time series in order to forecast the immediate future. This popularity can be attributed to its simplicity, its computational tractability, the ease of adjusting its responsiveness to changes in the time series being modelled, and its reasonable accuracy (Montgomery, 1990). The idea of exponential smoothing is to smooth the original series the way the moving average does and to use the smoothed series in forecasting future values of the variable of interest. In exponential smoothing however, it is desired to have the more rescent values of the series to have a greater influence on the forecast of future values than the more distant past observations. Exponential smoothing is a pragmatic approach to forecasting, whereby the forecast is constructed from an exponentially weighted average of past observations. The largest weight is given to the present observation, less weight to the immediately preceding observation, even less weight to the observation before that, and so on (exponential decay of influence of past data). According to Aczel (1989) this method is mostly useful when the data pattern is approximately horizontal, since the underlying assumption is that the data exhibits neither cyclic variation nor pronounced trend. Like all model assumptions this assumption is no different in the sense that it is not always valid. Let an observed time series be $y_1, y_2, ..., y_T$. Formally, the simple exponential smoothing equation takes the form of :

$$\hat{y}_{t+1} = \alpha y_t + (1 - \alpha)\hat{y}_t \tag{2.59}$$

where y_t is the actual known series value for time period t, $\hat{y}_t$ is the forecast value of the variable Y for time period t, $\hat{y}_{t+1}$ is the forecast value for time period $t+1$ and α is the smoothing parameter with $0 < \alpha < 1$. An appropriate choice of α, and initial value $\hat{y}_1$ is required by the SES model for accurate forecasts. However Gardner (2006) points out that despite a large body of research on the selection of these values, there is no consensus among forecasters and there are no consistent guidelines in the forecasting literature on how the smoothing and initial value should be selected. Gardner (2006) discusses various theoretical and empirical arguments for selecting an appropriate smoothing parameter and concludes that it is best to estimate an optimum α from the historic data. As a result the smoothing constant is commonly estimated by minimising some measure of accuracy of the model such as the mean squared error (MSE), mean absolute percentage error (MAPE). In this study the measure used for selecting α is the sum of the squared errors (SSE). There are various methods of selecting the initial forecast $\hat{y}_1$. Gardner (2006) suggests using the first observation of the data y_1 as the initial value of the smoothed series, however he points out that the choice of $\hat{y}_1$ is not really that important since it contributes very little to the forecast $\hat{y}_{t+1}$ due to its small co-efficient $(1 - \alpha)^t$.

The choice of α is important for determining the operating characteristics of exponential smoothing. The smaller the value of α, the slower the response. Larger values of α cause the smoothed value to react quickly. The major drawback of the SES model is that it does not perform well for forecasting seasonal data with trend. The SES model is used in this study for comparative purposes with the proposed Fourier series model. The "best" model is chosen based on the minimum out of sample SSE, since the goal is to provide as much accurate forecasts as possible. The selection criteria is predictive power rather than how well the model fits the sample data. The forecasting and optimisation will be performed on the Forex data, so it makes sense to review some key Forex concepts that are relevant to this study.

2.5 The Forex Market

2.5.1 Spot Market

According to Dolan (2011) *spot* refers to the price where you can buy or sell currencies *now* as in "on the spot". If one is familiar with stock trading, the price you can trade at is essentially a *spot price*. Technically, the term refers to the nearest settlement date on which a transaction can be made and is primarily meant to differentiate spot trading from futures trading or trading for some future delivery date.

The *spot* currency market is normally traded for settlement in two business days the *spot*, the trading volume in the *spot* market accounted for 37 percent of the total Forex market volume in 2007 based on a survey done by the Bank for International Settlements (Dolan, 2011). This is testimony to the popularity of the "spot" market. In this study the analysis will be purely based on trading in the *spot* Forex market.

2.5.2 Speculation

Speculating is not gambling. Gambling is about playing with money, even when you know the odds are stacked against you. Speculating is about taking calculated financial risks to seek a profitable return, usually over a short time horizon (minutes, hours, or days). The trading timeframe is not limited to day trading as the Forex market being a 24/5 market any trade can be very easily turned into an overnight trade (Dolan, 2011). However this study is restricted to day trading only.

2.5.3 A Pip

A *pip* stands for "price interest point". It is the unit of measurement to express the change in value between two currencies. Suppose that the current AUD/USD price is 1.0235, if the price rises to

1.0236 or falls to 1.0234, this is a movement of 0.0001, or 1 *pip*. If the current price of USD/JPY is 81.33, and if the price rises to 81.34 or falls to 81.32, this is a movement of 0.01, or 1 *pip*. One *pip* is thus the smallest change in value for any given Forex quote, whether it's quoted to two or four decimal places. The rule of thumb is that all pairs involving the US dollar are quoted to the fourth decimal place and all pairs involving the Japanese yen are quoted to two decimal places when calculating the number of *pips* (Investopedia, 2017). Here are more examples:

- When the EUR/USD quote moves up from 1.3255 to 1.3287, it is a movement of 32 *pips*.
- When the USD/CHF quote moves up from 0.9148 to 0.9263, it is a movement of 115 *pips*.
- When the USD/JPY quote moves up from 80.55 to 80.87, it is a movement of 32 *pips* (Investopedia, 2017).

It should be noted that in this study profits or losses will be calculated using returns instead of *pips*. But the concept of a *pip* is very important in Forex when facilitating trades, it makes calculating quantities such as a stop loss or position size much easier. However it is more convenient to use returns for analysis in this study.

2.5.4 Three points in every trade

When a trade is executed, there are essentially three price points involved: entry price, profit target, and stop loss. The entry price is defined as the price at which a trade is triggered. The profit target is defined as the price where the trade exits with a profit. The stop loss is defined as the price where the trade exits with a loss, it is a very important tool for risk management as it ensures that the trader will not lose more than the amount risked per trade which is usually 1 % of the account balance. However there is subjectivity around this percentage, hence a portfolio optimisation approach will remove this problem since the risked amount (portfolio weight) will be determined in an optimal way. Here are some examples for both a long and a short position:

2.5.5 Long Position

Suppose the current GBP/USD price is 1.5743. Because the trader expects the Pound (GBP) to appreciate against the U.S. dollar (USD), the trader decides to enter a long position (buys the GBP). The trader sets a profit target of 30 *pips* and a stop loss of 30 *pips*. Once these values are locked down in the broker's trading platform (trading software), three things can happen: The trade will hit the profit target, or it will hit the stop loss, or it will fluctuate between the throughout the trading period (day).

Consider the following example:

Entry price = 51.5743
Stop loss = 51.5713
Profit target = 51.5773

For a long position, the profit target is located above the entry price while the stop loss is located below the entry price. In this example, you take an equal amount of pips for the exit: 30 *pips* above the entry price and 30 pips below the entry price. Whenever a trade reflects an equal distance between the entry price to the profit target and between the entry price to the stop loss, the trade is said to have a risk to reward ratio of 1:1.

2.5.6 Short Position

Suppose the current NZD/USD price is 0.8138. After analysing the market, the trader expects the New Zealand dollar (NZD) to fall against the U.S dollar (USD), hence the trader enters into a short (sells the NZD) position. The trader decides to take a profit of 60 *pips* and a stop loss of 30 *pips*. Once these values are locked down in the broker 's platform, only two things can happen: The trade will hit the profit target, or will hit the stop loss, or the trade will fluctuate between these two values for the whole day. Consider the following example:

Entry price = 50.8138
Stop loss = 50.8168
Profit target = 50.8078

For a short position, the profit target is located below the entry price while the stop loss is located above the entry price. In this example a 30 *pip* stop loss and a profit target of 60 *pips* is set. This is termed a 1 : 2 risk to reward ratio (Singh, 2012).

2.5.7 Forex Broker

According to Driver (2013), online *forex brokers* function as the market-maker for retail traders, meaning that the broker is on the other side of every trade, i.e when a currency is “bought”, it is bought from the broker and when it is sold, it is sold to the broker. The broker is the one responsible for matching up buyers and sellers in the market.

2.5.8 Leverage

t23 defines *leverage* as the instrument that allows a trader to borrow and control larger amounts of trades using only small a account balance. Traders can then enter transactions using this small amount, that when leveraged gives control of a much larger stake of currency. This means that profits can be made from relatively small price movements. Singh (2012) warns that *leverage* works in both ways, therefore in the same way that it will multiply the size of the profit, it can also rapidly multiply any losses. In practice, *leverage* is used every time that a trade order is executed. When opening an account, a trader is required to select the level of leverage to use for the account, and this will determine the amount of currency that the trader will ultimately control on any given trade. The small amount that has to be put up towards the trade is known as the *margin*. This amount will have to be available in the trader's account before a broker can execute the trade. The next example illustratres how leverage works in the Forex market. Suppose a trader has a $\$1,000$ to trade in a particular currency pair and decides to enter a *buy* order. As the trader places the order, the broker will give the trader variety of *leverage* options depending on the type of account held. For this example a leverage level of 1:100 is used. This means that an investment of $\$1,000$ is multiplied by 100 giving the trader a control of $\$100,000$ worth of currency . It is this level of currency that is then applied to the trade and the trader enters the market theoretically holding that amount to trade with.

If the trade is a winning trade, meaning the trader speculated correctly and the price of the bought currency has increased over the period, then that price rise is applied to the $\$100,000$ rather than the *margin* amount of $\$1,000$, thus considerably increasing the level of profit accumulated. Conversely, if the market moves against the trader and the price closes lower over the period (day) then the same *leverage* is applied to any losses, hence *leverage* is also known as the two edged sword. This means that the trader could lose the *margin* amount if the losing trade is left open for a long time, maybe hoping for a change in the market direction. For simplicity, in this study it is assumed that a trader's account has a 1:1 *leverage* level, meaning that the trader utilises only the account balance for every trade (Driver, 2013). The concept of leverage is important in the Forex market since it allows retail traders to trade lot sizes they would normally not be able to due to the large amounts required to do so. This concept is also analogous to the one used in Physics where an object or system called a lever enables lifting up of heavy objects with litte force or effort as shown in Figure 2.5 below :

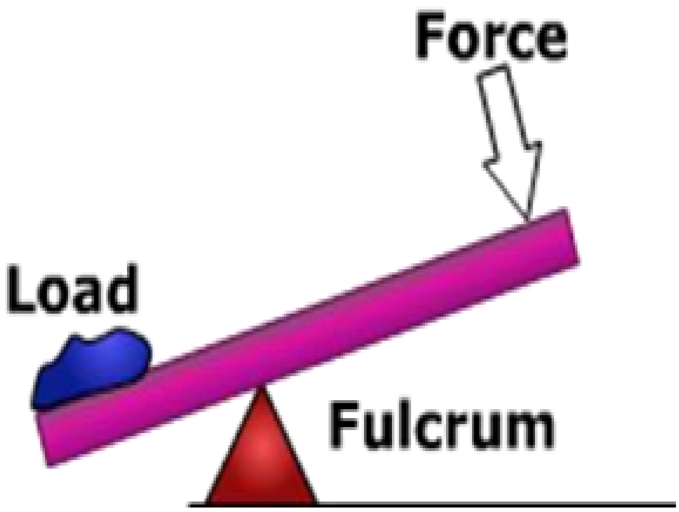


Figure 2.5: Physics: Mechanical lever

2.5.9 Lots and Account Types

When it comes to actual trade orders the Forex market is traded in volumes called *Lots*, the more lots traded the bigger the profits or losses and vice versa. *Lots* come in a variety of sizes including: *standard*, *mini*, *micro*, and *nano*, and are made up of different amounts of currency units or volume. *Lots* work according to the principles of *leverage* that have already been discussed. Each *Lot* is constructed from currency units, which simply translates into real term values, therefore a *standard lot* consisting of 100,000 units can be equivalent to controlling monetary value of $\$100,000$ (Driver, 2013). Table 2.1 below summarizes the concept of lots.

Table 2.1: Currency lots.

Lot	**units**
Standard	100,000
Mini	10,000
Micro	1,000
Nano	100

2.5.10 Account Types

The Forex market has 4 main account types:

- Demo account: This type of account is generally opened free of cost to the trader. The account is usually useful for novice traders who want to learn how the Forex market works. As it is just a demo account, novice traders can conduct risk-free trades with fake money.

- Micro account: For this account type, 1 micro lot is the equivalent of 1000 currency units with a minimum trading volume of 0.01 micro lots. For most brokers this account type does not have a restriction on the minimum deposit required, and profits/losses are very small without the use of leverage, with a single pip movement resulting in a profit/loss of $0.1 .
- Mini account: For this account type 1 mini lot is the equivalent of 10000 currency units with a minimum trading volume of 0.1 mini lots. This account type is suitable for both professional and novice traders, and the minimum deposit required varies from broker to broker. A single pip movement results in a profit/loss of $1.
- Standard account: For this account type 1 standard lot is the equivalent of 100000 currency units with a minimum trading volume of 0.01 standard lots. This account type is suitable for professional traders as it carries more risk, and the minimum deposit required varies from broker to broker but is more than that of the previous two accounts. A single pip movement results in a profit/loss of $10. In this study it is assumed that the trader makes use of the standard account with a 1:1 leverage.

2.5.11 Margin

As mentioned earlier, in Forex it is not necessary to have $\$100,000$ in an account inorder to control a lot size of that value or more. Instead, following the application of *leverage* a broker simply requires the trader to have a certain percentage amount of the trade cost upon which this *leverage* will be applied when a trade is executed. This percentage is known as the *Margin*, clearly there is an inverse relationship between *margin* and *leverage* in the sense that the more leverage available the less margin is required and vice versa. In the case of using *leverage* of $1 : 100$, the margin required would be a minimum of one percent. So if a trader wished to execute an order with a value of $\$100,000$ at $1 : 100$ leverage, then the trader would be required to have a minimum of $\$1,000$ as *margin* so that a surplus amount can be available to fund possible losses, i.e the margin amount acts as insurance for the broker should the trade suffer a loss (Sharptrader, 2017).

2.5.12 Bids, Asks and Spreads

When dealing with the pricing of a currency pair in the Forex market there will always be two prices at any one time for every currency pair. These two prices are known as the *bid* and *ask* price, and the difference between them is known as the *spread*. The *bid* price will be the price quoted at which a trader can sell a currency pair and the *ask* is the price at which a trader can buy currency pair. The *bid* price will always be lower than the *ask* price. Forex brokers profit from the market by making use of the *spread* from every trade executed in the market. Therefore the spread is the cost of trading to the retail trader, consequently the tighter the spread the lower the cost (Sharptrader, 2017).

2.5.13 Market Order

In order to execute either the purchase or sale of a currency pair, it is necessary to provide instructions as to the quantity and method with which a trade should be conducted. The combination of these instructions is known as an *order*. The two main *order* types in the Forex market are the *market order* and *stop order*. A *market order* is an instruction to buy or sell at the best available price. This means that a trade will be executed immediately at either the current *ask* or *bid* price depending on whether the trade is for a purchase or a sale. For example if the *bid* price for GBP/USD is 1.5673 and the *ask* price is 1.5675 then, with a *market order* a trader will buy at 1.5675 and sell at 1.5673 (Forexpro, 2017).

2.5.14 Stop order

A *stop order* is an instruction to buy or sell when the price reaches a predetermined level. This means that it can be used to set entry and exit targets for the purposes of limiting potential losses or locking-in profitable gains. The disadvantage of a *stop order* is the lack of absolute guarantee of getting the trade executed since the currency pair price may never reach the entry and exit points specified by the *order*. When there are sudden swings in the price then the broker may only be able to execute the *stop order* at levels lower or higher than that which was originally expected. In other words the execution of the *stop order* triggers the execution of a *market order*. In this study however, the focus will be restricted on trades executed by the *market order* (Wikipedia, 2017).

2.5.15 Fundamental Analysis

Fundamental analysis is the study of news, events and economic statistics to determine trading opportunities. The principle behind this type of analysis is based on the reasoning that when a country's economy is doing well, then this success should generally be reflected by an equally strong currency. Similarly, when a country's economy appears to be perfoming poorly, then this is likely to result in a weaker currency. The relative change that this economic strength has upon economies and currencies subsequently forms a basis for speculative trading (Investopedia, 2017).

2.5.16 Technical Analysis

Technical analysis looks at the historic price movement of currency pairs and uses this data to predict future price movements. Unlike fundamental analysts who attempt to evaluate a currency's intrinsic value, technical analysts focus on charts of price movement and various analytical tools to evaluate a currency's strength or weakness and forecast future price movements. Technical analysts believe that, past trading activity and price changes of a currency are better indicators of the currency's likely future price movements than the intrinsic value of the currency (Investopedia, 2017).

In this study, the first step of the analysis will be to make use of forecasting models to determine future price movements of the currencies but the argument is that it is not enough to use single forecasts for decision making in trading currency pairs, since they are point estimates that are prone to errors and they do not have the statistical properties that ensure the selection of a risk minimal profitable portfolio in the long run, hence a portfolio optimisation approach will be implemented thereafter, which will inform the trader of how much to invest in each trade.

2.5.17 Currency Pairs

In practice, when currencies are paired, the one that performs better over a given period will rise relative to the other. It is the movement of these currency pairings that forms the foundation for making profits or losses on the Forex market. The U.S. dollar (USD) is the central currency against which other currencies are traded. In its most recent triennial survey of the global foreign exchange market in 2010, the Bank for International Settlements (BIS) found that the USD was on one side of about 85 percent of all reported Forex market transactions (Dolan, 2011).

According to Dolan (2011), the USD's central role in the Forex market stems from a few basic factors :

- The United States of America's (USA) economy is the largest national economy in the world.
- The USD is the primary international reserve currency.
- The USD is the medium of exchange for many cross-border transactions. For example, oil is priced in USD. So even a Japanese oil importer buying crude oil from Saudi Arabia, is going to pay in USD.
- The United States is a global military superpower, with a stable political system.
- The United States has the largest and most liquid government debt markets in the world.

Major Pairs

The major currency pairs all involve the USD on one side of the pair. The designations of the major currencies are expressed using International Standardisation Organisation (ISO) codes for each currency. The table below lists the most frequently traded currency pairs, what they're called in conventional terms, and what nicknames the market has given them. It is important to note that all 8 currencies that make up the major currency pairs in Table 2.2 are also know as major currencies (Dolan, 2011).

Table 2.2: The major currency pairs.

Currency Pair	Countries	Long Name	Nickname
EUR/USD	Eurozone/United States	Euro-Dollar	Euro
USD/JPY	United States/Japan	Dollar-Yen	Ninja
GBP/USD	Britain/United States	Sterling-Dollar	Cable
USD/CHF	United States/Switzerland	Dollar-Franc	Swissy
USD/CAD	United States/Canada	Dollar-Canada	Loonie
AUD/USD	Australia/United States	Australian-dollar	Ozzie
NZD/USD	New Zealand/United States	New Zealand-dollar	Kiwi

Minor Pairs

Currency pairs that do not contain the USD are known as minors, cross-currency pairs or simply "crosses". Historically, if one wanted to convert from one currency to the other, one would have had to first convert the currency into USD and then into the currency which is desired. With the introduction of currency crosses, one no longer has to do this tedious calculation as all brokers now offer the direct exchange rates. The most active crosses are derived from the three major non-USD currencies, the Euro, the UK Pound and the Yen. These pairs are shown in Table 2.3 above (Sharptrader, 2017).

Table 2.3: The minor currency pairs.

Currency Pair	Countries	Long Name	Nickname
EUR/GBP	Eurozone/Britain	Euro-Sterling	Chunnel
EUR/CHF	Eurozone/Switzerland	Euro-swiss	Euro-Pound
EUR/CAD	Eurozone/Canada	Euro-Canada	Euro-loonie
EUR/AUD	Eurozone/Switzerland	Dollar-swiss	Euro-Ozzie
EUR/NZD	Eurozone/New Zealand	Dollar-New Zealand	Euro-Kiwi
EUR/JPY	Eurozone/Japan	Euro-yen	Yuppy
GBP/JPY	Britain/Japan	Sterling-yen	Geppy
CHF/JPY	Switzerland/Japan	swiss-yen	Swissy-Yen
CAD/JPY	Canada/Japan	Canadar-yen	Loonie-Yen
AUD/JPY	Australia/Japan	Australian-yen	Ozzie-Yen
NZD/JPY	New Zealand/Japan	New Zealand-yen	Kiwi-Yen
GBP/CHF	Britain/Switzerland	Sterling-swiss	Pound-Swissy
GBP/AUD	Britain/Australia	Sterling-Australian	Pound-Ozzie
GBP/CAD	Britain/Canada	Sterling-Canada	Pound-Loonie

Exotic Pairs

Exotic currency pairs are made up of a major currency paired with the currency of an emerging or a strong but smaller economy from a global perspective such as Hong Kong or South Africa and European countries outside of the Euro Zone. These pairs are not traded as often as the majors or minors, so often the cost (spread) of trading these pairs can be more than the majors or minors due to the lack of liquidity in these pairs (Sharptrader, 2017) .

Table 2.4: The exotic currency pairs.

Currency Pair	Countries	Long Name	Nickname
EUR/TRY	Eurozone/Turkey Lira	Euro-Lira	N/A
USD/SEK	United States/Sweden	Dollar-Swedish krona	N/A
USD/NOK	United States/Norway	Dollar-Norwegian krone	N/A
USD/DKK	United States/Denmark	Dollar-Danish krone	N/A
USD/ZAR	UnitedStates/South Africa	Dollar-Rand	N/A
USD/HKD	United States /Hong Kong	Dollar-Hong Kong dollar	N/A
USD/SGD	United States/Singapore	Dollar-Singapore dollar	N/A

2.6 Efficient Market Hypothesis

The Efficient Market Hypothesis (EMH) is defined by Titan (2015) as a financial economics “theory” whose claim is that market prices of securities incorporate all available information and it is impossible to consistently profit from the market. Essentially it means that technical and fundamental analysis can not be used to make excessive returns. The only way an investor can achieve above average returns is by taking on more risk. There are three forms of the hypothesis namely:

- **Strong EMH** : assertains that publicly and privately available information is incorporated in the market price.
- **Semi-Strong EMH** : assertains that only publicly available information is included in the market price. Hence it is impossible to use this information to gain advantage.
- **Weak EMH** : Historic stock price can not be used to predict future prices, since stock price resembles a random walk that is only governed by current information.

According to Fama, the EMH is defined as a competitive market, where the random character of the fluctuation is explained by the fact that price converges to the fundamental value. This definition is called the "Fama's EMH." According to Samuelson though, randomness of price variation, and unpredictability can be simply explained by the competition between investors, with no regard to the fundamental value. This definition is called "Samuelson's EMH". Fama reduces the random fluctuation to a deterministic relation whereas Samuelson takes the randomness of the fluctuation as a phenomenon in itself. It is fair to conclude that both interpretations rise fruitful but distinct questions. Recent oppositions in debates about EMH should be read through this interpretative issue (Delcey, 2019).

Due to the high levels of investing or trading by market partcipants, it is fair to say that most people no longer hold the strong convictions about the validity of the EMH as has been the case in the past, and this shift in thinking is attributed to some important empirical evidence that occurred at the beginning of the last decade. In other words, it is possible to make profits in the market but these profits are short lived and are justified by the risk that the investor is taking. Some researchers have tried to disprove the EMH by showing that there is some substantial correlation between successive stock price changes over given periods, but this correlation only occurs for a short period of time which makes it difficult to make conclusions about the invalidity of the EMH (Titan, 2015).

The issue of transaction costs also plays a key role in ensuring that investors who use momentum strategies do not gain an advantage in the market. What is worse is that even if the strategy enables an investor to obtain favourable returns over a given short period of time, it is difficult to use this as evidence disproving EMH since there is empirical evidence that these favouriable movements of stock price changes revert in the long run as shown by (Poterba and Summers, 1988). According to a study done by Fama and French (1988) there is strong evidence of stock price return reversal, however this evidence is not conclusive as to whether this result proves or disproves the EMH, since the returns on both sides (negative and positive) were similar and hence inconclusive as to whether an investor might obtain an advantage by assuming opposite position on both periods on which the returns were positive and negative.

Evidence to disprove the EMH based on Initial dividend yield (IDY) was found not to be conclusive as measured empirically by (Fama and French, 1988). In particular it was shown that this strategy of using IDY to predict future returns can have undesirable outcomes and lead to losses on investments. In fact it was shown that, the result which seem to indicate a positive correlation between IDY and future stock returns could be explained by the adjustment of equities to economic conditions. On the other hand the empirical evidence gathered using the Initial Price-Earnings Multiples (IPEM) was inconclusive as well, in deciding whether an increase or decrease in IPEM results in higher or lower future returns. Therefore both measures (IDY and IPEM) failed to disprove the efficient market hypothesis. There was a study known as "the size effect" which showed that small company stocks had a tendency of having higher returns than big company stocks over a long period of time. This phenomena was a potential evidence of inefficiency of the stock market until Fama and French (1993) showed this not to be the case.

It is true that here has been empirical evidence against the EMH that has been published in the literature. These exploitable patterns in the market are sensitive, do not last long, and investors are not able to make consistent excessive returns that are not proportional to the risk that is taken. Even if these patterns were insensitive there is overwhelming evidence that they would collapse in the future due to over exploitation by market participants. The market crash of 1987 is reasonable evidence that the market is not always efficient, as psychological rather than rational influences drove the market. Although it is pointed out that one of the possible reasons for the decline in stock prices is due a number of unfavourable economic events that occured before the 1987 crash, the central argument is that if markets have loopholes with deterministic patterns then the proffessional fund managers should be able to beat the market consistently. But studies have shown that these experts perform no better than your average investor with no portfolio management expertise (Titan, 2015).

In conclusion, when considering all forms of the EMH, and the abnormalities, it is clear that every now and then it is possible to observe abnormallies in the market that can be exploited for a short period of time by using technical and fundamental analysis, however these seemingly excessive returns are compensated by the costs of investing, and costs of acquiring and analysing the information available. These costs in more cases than not, exceed the advantage that the investor appears to have. Even more importantly the more investors who exploit these patterns the more corrupt the patterns become, hence it is impossible to make profits from the market consistently. Additionally, the EMH does not rule out the possibility of an investor obtaining excessive returns over a long period of time, only that this scenario is rare and attributed to only pure chance governed by the laws of probability. It should be noted that even though so much research on the EMH is solely specific to the stock market, these concepts also apply to the Forex market since both markets are driven by similar forces. Hence the conclusions about the validity of the EMH in the stock market have direct implications for the Forex market as well.

2.6.1 Empirical Evidence against the Normality Assumption of Financial Returns

The assumption of normal distribution of the stock returns is incorporated in the most popular and most used models in the theory and practice of financial economics. Among them are the mean-variance Markowitz Portfolio Theory Markowitz (1952), CAPM Sharpe (1964), and the Consumption CAPM (Lucas, 1978). Additionally, the Black-Scholes option pricing model Black and Scholes (1973) is derived based on the assumption that equity prices follow a geometric Brownian motion process, which has normally distributed increments. The bell-shaped normal distribution is completely characterised by two parameters, the mean and SD. The simple logic underlying that model is if return expectations implicit in asset prices are rational, then actual rates of return should be normally distributed around these expectations. Contrary to theoretical assumptions prevalent in the theory of financial economics, the empirical evidence strictly rejects the normal distribution of the stock returns. Today there are many studies that confirm that the long horizon returns are often found to be approximately normally distributed, and over short horizons, equity returns are far from normal. Most of the studies show that returns on stocks display significant leptokurtosis, and in many cases, skewness (negative or positive depending on the period analysed).

Among the first studies that found that the empirical distribution of the proceeds of the shares were not normal were Mandelbrot (1963) and (Fama, 1965). Mandelbrot (1963) presented evidence that distributions of returns can be well approximated by the stable Paretian distribution with a characteristic exponent less than 2 (a symmetric Levy stable law with tail index b about 1.7), thus exhibiting fat tails and an infinite variance. Fama (1965) in his research on a sample of 30 stocks from DJIA Index, confirmed Mandelbrot (1963) that the stable Paretian distribution better characterized the stock price changes. Much later, Mittnik et al. (1998) confirmed these estimates of the power tail index, as well as Mantegna and Stanley (1995), who even suggested slightly different indices of the stable law (b=1,4). Officer (1972) examined the validity of the symmetric stable class of distributions, and found that monthly returns follow normality, and the standard deviation appears to be a well behaved measure of scale. Clark (1973) found the lognormal distribution as a better fit on the sample of the data on cotton futures prices than a stable Paretian distribution proposed a couple of years previously by Mandelbrot (1963) and (Fama, 1965). Praetz (1982), analysing weekly data samples from the Sydney Stock exchange, concluded that the Student-t distribution is a better fit than the stable Paretian because the Paretian distribution has an infinite variance property and unknown density function. Blattberg and Gonedes (1977) using a daily and weekly data sample of the DJI made a comparison of the three distributions: Student-t, normal and Cauchy,and concluded that the Student-t is the better fit than the normal on the sample of the daily returns, but normal distributions apply to the monthly returns. Akgiray and Booth (1987) also found that normal distribution is a good fit for the monthly stock returns. In this line is Hagerman (1978), who rejects the normal distribution and proposes that what should be used is a mix between the normal and the Student-t distribution as an alternative.

For describing security returns, Bookstaber and McDonald (1987) introduced the generalized distribution GB2, which is an extremely flexible distribution, containing a large number of well-known distributions, such as the lognormal, log-t, and log-Cauchy distributions, as special or limiting cases and allowing large,even infinitely higher moments. The properties of the GB2 make it useful in empirical estimation of security returns and in facilitating the development of option pricing models and other models that depend on the specification and mathematical manipulation of distributions. French (1990) considered the distribution of log stock index returns of the

S&P 500 and found that log stock return distributions do not follow the normal law as is often assumed, but instead have much longer tails and more peakedness than the normal family. Three alternative distributions: the scaled-t, logistic, and exponential power distributions, demonstrate a greater ability to model log stock index returns from the S&P 500 Composite Index. Of the three alternative models considered, the EPD appears to provide a superior fit.

Since both the stock and forex markets are driven by similar forces, it is reasonable to assume that the arguments made above with regards to stock returns also hold or are applicable to the forex market as well.

Chapter 3

Research Methodology

This chapter gives a detailed explanation of the methodology that is implemented inorder to solve the problem as outlined in Chapter 1.

Due to the nature of Forex data and the dominance of the USD, all currency pairs subject to analysis in this study involve the USD on the quote side of each pair. Currency pairs of this form can be treated as individual securities with the USD denoting the price of buying one unit of currency X, with currency X being the base currency (security) and the USD being the quote currency, that is every pair will be of the form X/USD. In other words the USD will act as the money. Currency pairs which do not involve the USD are not considered in this study. Some Forex pairs like USD/JPY or USD/CHF are not in the desirable form that but can always be transformed into this form by dividing the pair by the given *bid price*. The quote currency will have a value of 1 and the base currency will have a value of $\frac{1}{bid\,price}$, this ensures that the base becomes the quote currency and vice versa. This convention is summarised in the list of assumptions stated below.

3.1 Assumptions

In the literature, portfolio optimisation models have been applied predominantly to the stock market, but in this study the focus will be purely on the Forex market, since in the Forex market you also deal with securities (currencies) that come in pars, hence in Forex when buying (going long) or selling (going short) a security implies buying or selling a currency pair. As with any study involving mathematical models various realistic assumptions are made in order to make the models applicable, it should be noted then that the applicability of these models is only limited within the framework of these assumptions.

- The trader operates on a standard Forex trading account. The rationale behind this assumption is that it is difficult to make substantial profits when trading with small lot sizes.
- The currency prices are generated by a periodic function which can be approximated by an n^{th} order Fourier series model.
- The currency prices can be approximated by a simple exponential smoothing model.
- The trader always goes *long* (buy and hold) on any given pair, short selling is not possible in the Forex market.
- The trader is risk averse, meaning when faced with two investment opportunities, one with slightly higher return (higher risk) than the other (lower risk) (not necessarily equal), the option with lower risk is prefered. That is the investor cares more about minimising losses than obtaining excessive returns.
- The trader's account balance is in USD
- The trader only takes on trades that involve the (USD) where the USD is expressed as a quote currency with currency X denoting the base currency, that is every currency pair is of the form X/USD.
- The trader uses a 1:1 *leverage*, the trader buys and holds with his own money and does not borrow from the broker.
- The currency returns follow a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. This assumption validates the applicability of the M-V model, however the SMAD and CVaR models do not depend on this assumption.

3.2 Definition of Terms

Before defining some key concepts upon which the random variables of interest are defined and consequently upon which the models' parameters are estimated. It is important to note that, security i refers to a particular currency pair with $i = 1, 2, ..., n$ and the return on security i is computed over a fixed discrete time period t (days) with $t = 1, 2, ..., T$.

- C = capital amount to be invested in the Forex market.
- A_i = amount to be invested in security i.
- $w_i = \frac{A_i}{A}$ is the proportion of capital to be invested in security i.
- P_{it} = the unit price (USD) of security i at time t.
- P_{it+1} = the unit price of security i at time $t + 1$

- $R_{it} = log(P_{it+1}) - log(P_{it})$ is the single period return (a random variable) of security i at time t.
- $\mu_i = E[R_i]$ is the mean of security i's return.
- $\sigma_i^2 = E[(R_i - \mu_i)^2]$ is the variance of security i's return.
- $\sigma_i = \sqrt{\sigma_i^2}$ is the standard deviation of security i's return.
- $\sigma_{ij} = E[(R_i - \mu_i)(R_j - \mu_j)]$ is the covariance between R_i and R_j.
- $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$ is the correlation coefficient between R_i and R_j.
- $R_p = \sum_{t=1}^{T} \sum_{i=1}^{n} w_i R_{it}$ is the return of the portfolio over a time period T.
- $M = \sum_{i=1}^{n} w_i \mu_i$ is the mean of the portfolio's return.
- $V = \sum_i \sum_j w_i w_j \sigma_{ij}$ is the variance of the portfolio return.
- $S = \sqrt{V}$ is the standard deviation of the portfolio return.
- $W = E[|R - M|]$ is the mean absolute deviation of the portfolio return.
- $F_x(z) = P\big(X \leq z\big)$
- $\text{VAR}_\alpha(X) = \min\big\{z|F_x(z) \geq \alpha\ ,\ \alpha\ \epsilon(0,1)\big\}$
- $\text{CVAR}_\alpha(X) = E[X|X \geq \text{VAR}_\alpha(X)]$
- $\hat{P}_{it}$ is the forecast of currency i at time t
- $\text{SSE} = \sum_{t=1}^{T} \big(P_{it} - \hat{P}_{it}\big)^2$ is the sum of the squared errors .
- $\text{RE} = \big|\frac{y_t - \hat{y}_t}{y_t}\big|$ is the absolute relative-error .
- $\text{MAPE} = \frac{1}{T} \sum_{t=1}^{T} \big|\frac{y_t - \hat{y}_t}{y_t}\big|$ is the mean absolute percentage error.

3.3 Fourier Series Model Formulation

This study fits and compares predictive powers of the FS and SES models. The "best" model for each currency pair is selected based on the minimum SSE as the goodness of fit and predictive power (forecast accuracy) measure. The forecasts then form the basis for computing returns and consequently the estimation of the models' parameters respectively, that is the forecasts will form the basis for portfolio selection in an optimal way. The Fourier model's parameters are estimated by the MATLAB least squares curve fitting tool. The following Fourier series approximation is used to forecast currency prices.

$$\hat{P}_{it} = S_n(t) = \frac{a_0}{2} + \sum_{j=1}^{n}(a_j cos(jt) + b_j sin(jt)) \tag{3.1}$$

Where $\hat{P}_{it}$ is the point estimate of a currency i's price at time t, t and n is the appropriate order of the model, with a_j and b_j being constants as defined in Chapter 2.

3.4 Simple Exponential Smoothing Model Formulation

In this study, since $\hat{y}_1$ is not known, it is set to the mean price. Thus $\hat{y}_1 = \frac{1}{T}\sum_{t=1}^{T} y_t$. The requirement for $0 < \alpha < 1$ comes from the fact that if $\alpha = 1$, then the original and smoothed series are identical. At the other extreme, when $\alpha = 0$, then the series is smoothed flat. Rewriting the model (37) to see one of the neat things about SES model is that, $\hat{y}_{t+1} - \hat{y}_t = \alpha(y_t - \hat{y}_t)$, that is the change in forecast value is proportional to the forecast error. In other words $\hat{y}_{t+1} = \hat{y}_t + \alpha e_t$, where $e_t = y_t - \hat{y}_t$ is the forecast error for time period t. By continuing to substitute previous forecasting values back to the starting point of the data in model (2.59), the forecasting equation becomes :

$$\hat{y}_{t+1} = \alpha y_t + (1-\alpha)[\alpha y_{t-1} + (1-\alpha)\hat{y}_{t-1}] = \alpha y_t + \alpha(1-\alpha)y_{t-1} + (1-\alpha)^2\hat{y}_{t-1},$$

$$\hat{y}_{t+1} = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + (1-\alpha)^3\hat{y}_{t-2},$$

$$\hat{y}_{t+1} = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \alpha(1-\alpha)^3\hat{y}_{t-3} + \alpha(1-\alpha)^4\hat{y}_{t-3},$$

....

The general form of the equation is :

$$\hat{y}_{t+1} = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + ...\alpha(1-\alpha)^{t-2}y_2 + \alpha(1-\alpha)^{t-1}y_1 =$$

$$\alpha\sum_{k=0}^{t-1}(1-\alpha)^k y_{t-k} \tag{3.2}$$

where $\hat{y}_{t+1}$ is the forecast value of the variable Y at time period $t + 1$ from the knowledge of the actual series values $y_t, y_{t-1}, y_{t-2}, ...y_1$. Hence $\hat{y}_{t+1}$ is the weighted moving average of all past observations. The series of weights used in producing the forecast $\hat{y}_{t+1}$ is $\alpha, \alpha(1-\alpha), \alpha(1-\alpha)^2, ...$ These weights decline towards zero in an exponential manner, thus as the historic data is traversed backwards, each value has a smaller weight in terms of its effect on the forecast. The exponential decline of the weights toward zero is evident (Aczel, 1989).

3.5 Multivariate Normal Distribution

The multivariate normal (MVN) distribution is a generalization of the univariate normal distribution which has the density function :

$$f(r) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-\mu)^2}{2\sigma^2}} \quad , \quad -\infty < r < \infty \tag{3.3}$$

where μ = mean of the distribution, σ^2 = variance. In n dimensions the density becomes

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{r}-\boldsymbol{\mu})} \tag{3.4}$$

with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. This distribution is important since the Markowitz model relies on the assumption that, asset returns follow this distribution.

3.6 Maximum-Likelihood Estimation

In statistics, maximum-likelihood estimation (MLE) is a method of estimating the parameters of a statistical model. When applied to a data set and given a statistical model, maximum-likelihood estimation provides estimates for the model's parameters. In general, for a fixed set of data and underlying statistical model, the method of maximum likelihood selects values of the model parameters that produce a distribution that gives the observed data the greatest probability of occurence, that is parameters that maximise the likelihood function. Maximum-likelihood estimation gives a unified approach to estimation, which is well-defined in the case of the normal distribution and many other distributions. However, in some complicated problems, difficulties do occur: in such problems, maximum-likelihood estimators are unsuitable or do not exist.

In this study, it is assumed that the random vector of returns $\boldsymbol{R}$ follows a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. But in practice $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown and need to be estimated from a sample data matrix using the maximum-likelihood method (Karandikar and Sinha, 2012). Since this study employs forecasts instead of historic data, the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ will be estimated from the forecasted data set. It turns out then that the maximum likelihood estimators (MLE's) for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the sample mean vector and sample covariance matrix given by :

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{r}} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{pmatrix} \tag{3.5}$$

where

$$\bar{r}_j = \frac{1}{T}\sum_{t=1}^{T} r_{jt} \quad , \quad j = 1, 2, ..., n \tag{3.6}$$

is the sample mean of return r_j and

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} (\mathbf{r}_t - \bar{\mathbf{r}})(\mathbf{r}_t - \bar{\mathbf{r}})' \tag{3.7}$$

is the biased sample covariance matrix, the unbiased estimator $\boldsymbol{S}$ is preferred, since it has the desirable statistical properties :

$$\boldsymbol{S} = \frac{1}{T-1}\sum_{t=1}^{T} (\mathbf{r}_t - \bar{\mathbf{r}})(\mathbf{r}_t - \bar{\mathbf{r}})' = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix} \tag{3.8}$$

where

$$s_{ij} = \frac{1}{T-1}\sum_{t=1}^{T} (r_{tj} - \bar{r_j})(r_{ti} - \bar{r_i}) \quad , \quad i \neq j \tag{3.9}$$

is the sample covariance between r_i and r_j with

$$s_j^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{tj} - \bar{r_j})^2 \tag{3.10}$$

being the sample variance of the random return r_j and

$$s_j = \sqrt{s_j^2} \tag{3.11}$$

is the sample standard deviation of the random return r_j , and

$$v_{ij} = \frac{s_{ij}}{s_i s_j} \tag{3.12}$$

is the sample correlation efficient between r_i and r_j respectively.

3.7 Mean-Variance Model Formulation

The optimisation models can then be expressed interms of the sample estimates derived from the sample data. The Markowitz M-V model can be formulated as :

$$minimise \quad s(\boldsymbol{w}) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j s_{ij}} \tag{3.13}$$

$$subject\ to \qquad \sum_{i=1}^{n} w_i \bar{r_i} = \rho$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0, \quad i = 1, 2, ..., n.$$

3.8 Semi-Mean-Absolute Deviation Model Formulation

Using sample estimates derived from the data, the SMAD linear programming problem is formulated as :

$$\textit{minimise} \qquad z = \sum_{t=1}^{T} p_t d_t$$

$$\textit{subject to} \qquad d_t \geq \sum_{j=1}^{p} \bar{r}_j w_j - y_t \quad , \qquad t = 1, 2, ..., T$$

$$y_t = \sum_{j=1}^{n} w_j r_{jt} \quad , \qquad t = 1, 2, ..., T$$

$$\sum_{j=1}^{n} w_j \bar{r}_j = \rho \quad , \qquad j = 1, 2, ..., n$$

$$\sum_{i=1}^{n} w_j = 1 \quad , \qquad j = 1, 2, ..., n$$

$$d_t \geq 0 \quad , \qquad t = 1, 2, ..., T$$

$$w_j \geq 0 \quad , \qquad j = 1, 2, ..., n$$

3.9 Conditional Value-at-Risk model Formulation

This study adopts the approach used by (Rockafellar and Uryasev, 2000). Central to the approach is a technique for portfolio optimisation which calculates VaR and optimises CVaR simultenously. However instead of working with the loss distribution as done by Rockafellar and Uryasev (2000), this study makes use of the return distribution, that is this study focuses on the left tail (negative returns which are interpreted as losses) and consequently some modifications need to be made with regards to the definitions and theorems stated in Section 2.6. This is neccessary since the mathematics needs to be consistent with how the VaR and CVaR are interpreted. The VaR_α of a portfolio is the largest amount γ such that with probability α the loss will not exceed γ, whereas the CVaR_α is the conditional expectation of losses that exceed γ. This study uses $\alpha = 0.95$ as the significance level. Figure (3.1) below shows the VaR_α and CVaR_α of the return distribution which is the mirror image of the loss distribution in Figure (2.2):

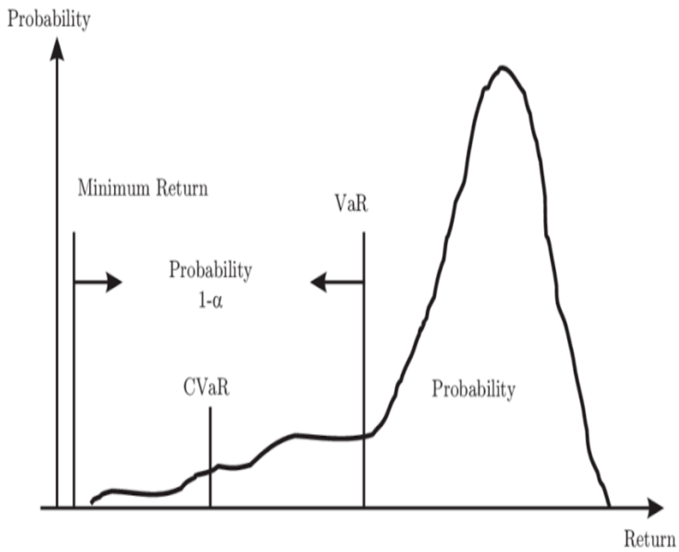


Figure 3.1: VaR_α and CVaR_α of the return distribution

Let $f(\boldsymbol{w}, \boldsymbol{r})$ be the portfolio return associated with the decision vector $\boldsymbol{w} \in \mathrm{W}$ and the random vector $\boldsymbol{r} \in \mathbb{R}^n$. $\boldsymbol{w}$ is a portfolio with W the set of permissible portfolios satisfying (2.19), (2.20), and (2.21). The vector $\boldsymbol{r}$ represents returns which maybe a loss or gain. For each $\boldsymbol{w}$, the portfolio return $f(\boldsymbol{w}, \boldsymbol{r})$ is a random variable having a distribution induced by that of $\boldsymbol{r}$. The underlying distribution of $\boldsymbol{r} \in \mathbb{R}^n$ is assumed to have density $p(\boldsymbol{r})$. However it will be shown later that an analytical expression of $p(\boldsymbol{r})$ for the implementation of the approach is not needed.

The probability of $f(\boldsymbol{w}, \gamma)$ not exceeding γ is given by :

$$\psi(\boldsymbol{w}, \boldsymbol{r}) = \int_{f\,(\boldsymbol{w},\boldsymbol{r}) \leq \gamma} p(r)dr$$

as a function of γ for a fixed $\boldsymbol{w}$, $\psi(\boldsymbol{w}, \boldsymbol{r})$ is the cumulative distribution function for the portfolio return associated with $\boldsymbol{w}$. It completely determines the behaviour of this random variable and is fundamental in defining VaR and CVaR. The VaR_α and CVaR_α for the portfolio return associated with $\boldsymbol{w}$ and $\alpha \in (0, 1)$ are denoted by $\gamma_\alpha(\boldsymbol{w})$ and $\phi_\alpha(\boldsymbol{w})$. In this setting they are given by :

$$\gamma_\alpha(\boldsymbol{w}) = \max\{\gamma \ \in \ \mathbb{R} : \psi(\boldsymbol{w}, \gamma) \leq 1 - \alpha\} \tag{3.14}$$

and

$$\phi_\alpha(\boldsymbol{w}) = (1 - \alpha)^{-1} \int_{f\,(\boldsymbol{w},\boldsymbol{r}) \leq \gamma_\alpha} f(\boldsymbol{w}, \boldsymbol{r}) p(\boldsymbol{r}) d\boldsymbol{r} \tag{3.15}$$

In the first formula, $\gamma_\alpha(\boldsymbol{w})$ comes out as the comes out as the right endpoint of the nonempty set consisting of the values of α such that $\psi(\boldsymbol{w},\gamma) = 1-\alpha$. In the second formula, the probability that $f(\boldsymbol{w},\boldsymbol{r}) \leq \gamma_\alpha(\boldsymbol{w})$ is therefore equal to $1-\alpha$. Thus $\phi_\alpha(\boldsymbol{w})$ comes out as the conditional expectation of the portfolio return being $\gamma_\alpha(\boldsymbol{w})$ or worse, and worse meaning losses larger than γ_α (or returns smaller than γ_α), that is to the left of γ_α. $\phi_\alpha(\boldsymbol{w})$ and $\gamma_\alpha(\boldsymbol{w})$ are characterised interms of the function :

$F_\alpha(\boldsymbol{w},\gamma)$ on $W \times \mathbb{R}$ defined by :

$$F_\alpha(\boldsymbol{w},\gamma) = \gamma + (1-\alpha)^{-1}\int_{\boldsymbol{r}\epsilon\mathbb{R}^m}\left[f(\boldsymbol{w},\boldsymbol{r})-\gamma\right]^{-}p(\boldsymbol{r})d\boldsymbol{r} \tag{3.16}$$

Where $[t]^- = t$ when $t \leq 0$, but $[t]^- = 0$ when $t > 0$.

Theorem 3:

As a function of γ, $F_\alpha(\boldsymbol{w},\gamma)$ is convex and continously differentiable. The CVaR$_\alpha$ of the portfolio return associated with any decision vector $\boldsymbol{w}$ can be determined from the formula

$$\text{CVaR}_\alpha(\boldsymbol{w}) = \min_{\gamma\epsilon\mathbb{R}} F_\alpha\ (\boldsymbol{w},\gamma) \tag{3.17}$$

In this formula the set consisting of the values of α for which the minimum is attained, namely

$$\text{A}_\alpha(\boldsymbol{w}) = \operatorname*{argmin}_{\gamma\epsilon\mathbb{R}} F_\alpha\ (\boldsymbol{w},\gamma) \tag{3.18}$$

is a nonempty, closed, bounded interval (perhaps reducing to a single point) and the VaR$_\alpha$ of the portfolio return is given by

$$\gamma_\alpha(\boldsymbol{w}) = \text{right endpoint of } \text{A}_\alpha(\boldsymbol{w}) \tag{3.19}$$

In particular, one always has

$$\gamma_\alpha(\boldsymbol{w})\ \epsilon\ \text{A}_\alpha(\boldsymbol{w}) \ \text{ and } \ \phi_\alpha(\boldsymbol{w}) = F_\alpha(\boldsymbol{w},\gamma_\alpha(\boldsymbol{w})) \tag{3.20}$$

Proof of Theorem 3

Before proving Theorem 3, it is assumed that $\psi(\boldsymbol{w},\gamma)$ is continous continous with respect to γ, which is equivalent to knowing that, regardless of the choice of $\boldsymbol{w}$, the set of r with $f(\boldsymbol{w},\boldsymbol{r}) = \gamma$ has probability zero, that is :

$$\int_{f(\boldsymbol{w},\boldsymbol{r})=\gamma} p(\boldsymbol{r})d\boldsymbol{r}) = 0$$

Lemma. With $\boldsymbol{w}$ fixed, let $G(\gamma) = \int_{\boldsymbol{r}\epsilon\mathbb{R}^p} g(\gamma,\boldsymbol{r})p(\boldsymbol{r})d\boldsymbol{r}$, where $g(\gamma,\boldsymbol{r}) = [f(\boldsymbol{w},\boldsymbol{r}) - \gamma]^-$. Then G is a convex continously differentiable function with derivative:

$$G'(\gamma) = -\psi(\boldsymbol{w},\gamma) \tag{3.21}$$

Proof: $G(\gamma) = \int_{\boldsymbol{r}\epsilon\mathbb{R}^p} g(\gamma,\boldsymbol{r})p(\boldsymbol{r})d\boldsymbol{r} = \int_{\boldsymbol{r}\epsilon\mathbb{R}^p}[f(\boldsymbol{w},\boldsymbol{r}) - \gamma]^- p(\boldsymbol{r})d\boldsymbol{r} = \int_{f(\boldsymbol{w},\boldsymbol{r})\leq\gamma}[f(\boldsymbol{w},\boldsymbol{r}) - \gamma]p(\boldsymbol{r})d\boldsymbol{r}$. Using Leibniz integral rule :

$G'(\gamma) = \int_{f(\boldsymbol{w},\boldsymbol{r})\leq\gamma} \frac{\partial[p(\boldsymbol{r})f(\boldsymbol{w},\boldsymbol{r})-\gamma p(\boldsymbol{r})]d\boldsymbol{r}}{\partial\gamma} = \int_{f(\boldsymbol{w},\boldsymbol{r})\leq\gamma} -p(\boldsymbol{r})d(\boldsymbol{r}) = -\psi(\boldsymbol{w},\gamma).$

In view of the defining formula formula for $F_\alpha(\boldsymbol{w},\gamma)$, it is immediate from the lemma that $F_\alpha(\boldsymbol{w},\gamma)$ is convex and continously differentiable with derivative:

$$\frac{\partial F_\alpha(\boldsymbol{w},\gamma)}{\partial\gamma} = 1 + (1-\alpha)^{-1}(-\psi(\boldsymbol{w},\gamma)) = 1 - (1-\alpha)^{-1}\psi(\boldsymbol{w},\gamma) = \frac{1-\psi(\boldsymbol{w},\gamma)-\alpha}{1-\alpha} \tag{3.22}$$

and

$$\frac{\partial F_\alpha(\boldsymbol{w},\gamma)}{\partial\gamma} = 0 \iff 1-\psi(\boldsymbol{w},\gamma)-\alpha = 0 \iff -\psi(\boldsymbol{w},\gamma)-\alpha = -1 \iff \psi(\boldsymbol{w},\gamma) = 1-\alpha$$

Therefore the values of γ that minimise $F_\alpha(\boldsymbol{w},\gamma)$ are precisely the ones for which $\psi(\boldsymbol{w},\gamma) = 1-\alpha$. This result is consistent with the definition of VaR_α in formula (3.14). In particular it is true that:

$$\min_{\gamma\epsilon\mathbb{R}} F_\alpha(\boldsymbol{w},\gamma) = F_\alpha(\boldsymbol{w},\gamma_\alpha(\boldsymbol{w})) = \gamma_\alpha(\boldsymbol{w}) + (\alpha)^{-1}\int_{\boldsymbol{r}\epsilon\mathbb{R}^p}[f(\boldsymbol{w},\boldsymbol{r}) - \gamma_\alpha(\boldsymbol{w})]^- p(\boldsymbol{r})d\boldsymbol{r}$$

but

$$\int_{\boldsymbol{r} \epsilon \mathbb{R}^p} [f(\boldsymbol{w}, \boldsymbol{r}) - \gamma_\alpha(\boldsymbol{w})]^- p(\boldsymbol{r}) d\boldsymbol{r} = \int_{f(\boldsymbol{w}, \boldsymbol{r}) \leq \gamma_\alpha} [f(\boldsymbol{w}, \boldsymbol{r}) - \gamma] p(\boldsymbol{r}) d\boldsymbol{r}$$

$$= \int_{f(\boldsymbol{w}, \boldsymbol{r}) \leq \gamma_\alpha} f(\boldsymbol{w}, \boldsymbol{r}) p(\boldsymbol{r}) d\boldsymbol{r} - \gamma_\alpha(\boldsymbol{w}) \int_{f(\boldsymbol{w}, \boldsymbol{r}) \leq \gamma_\alpha} p(\boldsymbol{r}) d\boldsymbol{r}$$

$$= (1 - \alpha)\phi_\alpha(\boldsymbol{w}) - \gamma_\alpha(\boldsymbol{w})\psi(\boldsymbol{w}, \gamma_\alpha(\boldsymbol{w}))$$

$$\therefore \min_{\gamma \epsilon \mathbb{R}} F_\alpha(\boldsymbol{w}, \gamma) = \gamma_\alpha(\boldsymbol{w}) + (1 - \alpha)^{-1}((1 - \alpha)\phi_\alpha(\boldsymbol{w}) - \gamma_\alpha(1 - \alpha)), \text{ since } \psi(\boldsymbol{w}, \gamma_\alpha(\boldsymbol{w})) = 1 - \alpha.$$

$$\therefore \min_{\gamma \epsilon \mathbb{R}} F_\alpha(\boldsymbol{w}, \gamma) = \gamma_\alpha + \phi_\alpha(\boldsymbol{w}) - \gamma_\alpha = \phi_\alpha(\boldsymbol{w}) \tag{3.23}$$

An approximation to $F_\alpha(\boldsymbol{w}, \gamma)$ is given by :

$$\tilde{F}_\alpha(\boldsymbol{w}, \gamma) = \gamma + \frac{1}{q(1 - \alpha)} \sum_{k=1}^{n} [f(\boldsymbol{w}, \boldsymbol{r_k}) - \gamma]^- \tag{3.24}$$

Theorem 4:

Minimising CVaR$_\alpha$ of the portfolio return associated with decision vector $\boldsymbol{w} \epsilon W$ is equivalent to minimising $F_\alpha(\boldsymbol{w}, \gamma)$ over $(\boldsymbol{w}, \gamma) \epsilon W \times \mathbb{R}$, in the sense that

$$\min_{\boldsymbol{w} \epsilon W} \phi_\alpha(\boldsymbol{w}) = \min_{(\boldsymbol{w}, \gamma) \epsilon W \times \mathbb{R}} F_\alpha(\boldsymbol{w}, \gamma) \tag{3.25}$$

Furthermore, $F_\alpha(\boldsymbol{w}, \gamma)$ is convex with respect to $(\boldsymbol{w}, \gamma)$ and $\phi_\alpha(\boldsymbol{w})$ is convex with respect to $\boldsymbol{w}$, when $f(\boldsymbol{w}, \boldsymbol{r})$ is convex with respect to $\boldsymbol{w}$, in which case if the constraints are such that W is a convex set, the joint minimisation is an instance of convex programming. According to Theorem 2, it is not necessary, for the purpose of determining a portfolio $\boldsymbol{w}$ that yields minimum CVaR$_\alpha$, to work directly with the function $\phi_\alpha(\boldsymbol{w})$, which may be hard to do because of the nature of its definition in terms of the VaR$_\alpha$ value $\gamma_\alpha(\boldsymbol{w})$ and the often troublesome mathematical properties of that value. Instead one can operate on the far simpler expression $F_\alpha(\boldsymbol{w},\gamma)$ which is convex in the variable γ and in most cases it is also convex in $(\boldsymbol{w},\gamma)$.

Proof of Theorem 4

The proof relies on the fact that the minimisation of $F_\alpha(\boldsymbol{w},\gamma)$ with respect to $(\boldsymbol{w},\gamma) \in W \times \mathbb{R}$ can be done by first minimising over $\gamma \in \mathbb{R}$ for a fixed $\boldsymbol{w}$ and then minimising the result over $\boldsymbol{w} \in W$. Justification of the convexity claim starts with the observation that $F_\alpha(\boldsymbol{w},\gamma)$ is convex with respect to $(\boldsymbol{w},\gamma)$ whenever the integrand $[f(\boldsymbol{w},\boldsymbol{r})-\gamma]^-$ in the formula for $F_\alpha(\boldsymbol{w},\gamma)$ is itself convex with respect to $(\boldsymbol{w},\gamma)$. For each $\boldsymbol{r}$, this integrand is the composition of the function $(\boldsymbol{w},\gamma) \rightarrow f(\boldsymbol{w},\boldsymbol{r})-\gamma$ with the nondecreasing function function $t \rightarrow [t]^-$ and by the rules of Rockafellar and Uryasev (2000), it is convex as long as the function $(\boldsymbol{w},\gamma) \rightarrow f(\boldsymbol{w},\boldsymbol{r})-\gamma$ is convex. $f(\boldsymbol{w},\boldsymbol{r})-\gamma$ is convex when $f(\boldsymbol{w},\boldsymbol{r})$ is convex with respect to $\boldsymbol{w}$. Since in this setting the function $f(\boldsymbol{w},\boldsymbol{r})$ reprsents the portfolio return that is covex in the variable $\boldsymbol{w}$, the result of Theorem 4 follows.

Given that the decision vector $\boldsymbol{w}$ represents a portfolio of currencies in the sense that $\boldsymbol{w} = (w_1, w_2, ..., w_n)'$ with w_i being the position in currency i and $w_i \geq 0$ for $i = 1, 2, ..., n$ with $\sum_{i=1}^n w_i = 1$. Denoting by r_i the return on instrument i, the random vector of returns is given by $\boldsymbol{r} = (r_1, r_2, ..., r_n)$. The distribution of $\boldsymbol{r}$ constitutes a joint distribution of the returns that is independent of $\boldsymbol{w}$ and has density $p(\boldsymbol{r})$. The return on a portfolio $\boldsymbol{w}$ is the sum of the returns on the individual currencies in the portfolio, scaled by the proportions w_i and is given by:

$f(\boldsymbol{w},\boldsymbol{r}) = \boldsymbol{w}^T\boldsymbol{r} = w_1r_1 + w_2r_2 + ... + w_pr_p = \sum_{i=1}^n w_ir_i.$

The perfomance function on which the focus is on in connection with VaR$_\alpha$ and CVaR$_\alpha$ is :

$$F_\alpha(\boldsymbol{w},\gamma) = \gamma + (1-\alpha)^{-1}\int_{\boldsymbol{w}\in\mathbb{R}^m} \left[\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{r}-\gamma\right]^- p(\boldsymbol{r})d\boldsymbol{r} \tag{3.26}$$

It is important to observe that in this setting, $F_\alpha(\boldsymbol{w},\gamma)$ is convex as a function of $(\boldsymbol{w},\gamma)$ not just γ.

If one considers the feasible set of portfolios

$$W = \left\{\text{set of } \boldsymbol{w} \text{ satisfying } (19), (20) \text{ and } (21)\right\} \tag{3.27}$$

This set W is convex (in fact “polyhedral” due to linearity of all the constraints). As a result the problem of minimising $F_\alpha(\boldsymbol{w},\gamma)$ over $W \times \mathbb{R}$ subject to (64) is one of convex programming which guarantees globality of optimal solutions. Considering the kind of approximation of F_α that is obtained by sampling from the underlying distribution of $\boldsymbol{r}$. A sample set $\boldsymbol{r_1, r_2, ..., r_p}$ yields the approximate function:

$$\tilde{F}_\alpha(\boldsymbol{w}, \gamma) = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} [\boldsymbol{w}^T \boldsymbol{r_t} - \gamma]^-$$

In terms of auxillary variables u_t for $t = 1, 2, ..., T$, minimising F_α is equivalent to minimising the linear expression:

$$\zeta = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} u_t \tag{3.28}$$

subject to the constraints $u_t \geq 0$ and $\boldsymbol{w}^T \boldsymbol{r_t} + \gamma + u_t \geq 0$.

Hence the CVaR$_\alpha$ minimisation problem is formulated as :

$$\begin{aligned}
&\textit{minimise} \quad && \zeta = \gamma + \frac{1}{T(1-\alpha)} \sum_{t=1}^{T} u_t \\
&\textit{subject to} && \sum_{i=1}^{n} w_i \bar{r}_i = \rho \\
& && \sum_{i=1}^{n} w_i = 1 \\
& && w^T r_t + \gamma + u_t \geq 0 \\
& && u_t \geq 0, \quad t = 1, 2, ..., T \\
& && w_i \geq 0, \quad i = 1, 2, ..., n
\end{aligned}$$

It should be noted that Theorem 3 and 4 are modifications of Theorem 1 and 2 as stated in Chapter 2. The neccessity of the modifications come from the fact that the study focuses on the distribution of the portfolio return: $f(\boldsymbol{w}, \boldsymbol{r}) = \boldsymbol{w}^T \boldsymbol{r} = w_1 r_1 + w_2 r_2 + ... + w_p r_p = \sum_{i=1}^{n} w_i r_i$, not the portfolio loss: $f(\boldsymbol{w}, \boldsymbol{r}) = -\boldsymbol{w}^T \boldsymbol{r}$ as discussed by (Rockafellar and Uryasev, 2000). One may wonder then, why not just use the loss distribution? Well since a comparison is made across three risk measures namely: variance, semi-mean-absolute deviation and conditional value-at-risk, and the first two risk measures employ the portfolio return it only makes sense to use the portfolio return for the last risk measure as well. The next chapter focuses on the implementation of the methodologies discussed on this chapter.

Chapter 4

Results and Discussions

4.1 Currency pairs in the study

The study looks at 18 currency pairs with the USD as the quote currency (money) in every pair. The data consists of 16 years of daily closing prices which makes a total of 105156 data points. The data was extracted from (OFX website, 2018). Table 4.1 gives the list of currency pairs focused on in this study.

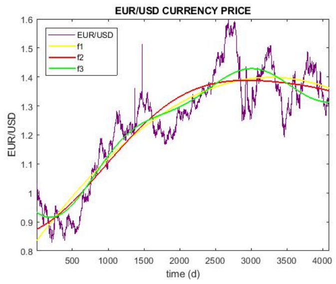
Table 4.1: Currency pairs in the study.

Currency Pair	Countries	Long Name	Nickname
EUR/USD	Eurozone/United States	Euro-Dollar	Euro
AUD/USD	Australia/United States	Australian-Dollar	Ozzie
CAD/USD	Canada/United States	Canadian-Dollar	Loonie
CHF/USD	Switzerland/United States	Franc-United States	Swissy
JPY/USD	Japan/United States	Yen-Dollar	Ninja
NZD/USD	New Zealand/United States	New Zealand-Dollar	Kiwi
ZAR/USD	South Africa/United States	Rand-Dollar	Kaching
GBP/USD	Britain/United States	Sterling-Dollar	Cable
SEK/USD	Sweden/United States	Swedish Krona-Dollar	Stokkie
NOK/USD	Norway/United States	Norwegian Krona-Dollar	Nokkie
DKK/USD	Denmark/United States	Danish Krona-Dollar	N/A
SGD/USD	Singapore/United States	Singapore Dollar-Dollar	N/A
MXN/USD	Mexico/United States	Mexican Peso-Dollar	N/A
TWD/USD	Taiwan/United States	Taiwan Dollar-Dollar	N/A
HKD/USD	Hong Kong/United States	Hong Kong Dollar-Dollar	N/A
INR/USD	India/United States	Indian Rupee-Dollar	N/A
TRY/USD	Turkey/United States	Turkish Lira-Dollar	N/A
THB/USD	Thailand/United States	Thailand Baht-Dollar	N/A

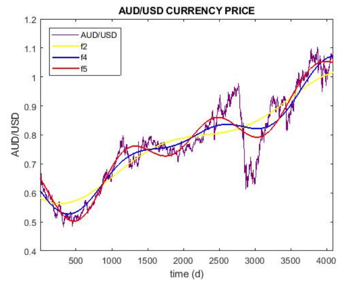
4.2 Forecasting Results

4.2.1 Fourier Series in-sample Model Fitting

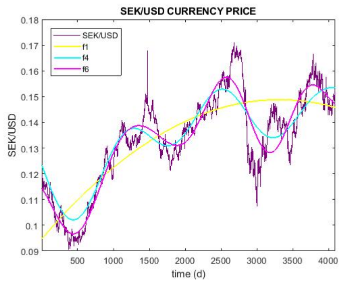
The Fourier series model defined in chapter 3 was fitted for each currency pair X|USD. Models of different orders were fitted using the MATLAB curve fitting tool. In the graphs the notation f_k is used to indicate a Fourier series model of order k. The historic data was divided into two parts: The training and testing parts, the training part is 70 % and the testing part is 30 % of the data. The 70 % refers to the first or oldest data points and the 30 % refers to the recent data points. These 18 pairs were chosen because they fit within the framework of the assumptions made and the data for them was available for the specified time period. According to Siegfried (1995) there is no general rule as to how sample data should be split, only that the training part is large enough to estimate the model parameters. Figure 4.1 below shows how well the FS models fit the training data.



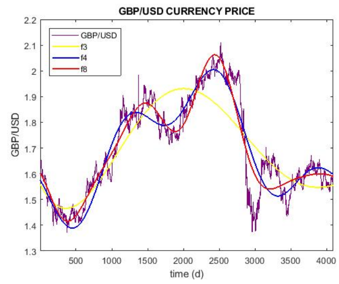
4.1.1: EUR/USD Fourier model fitting.



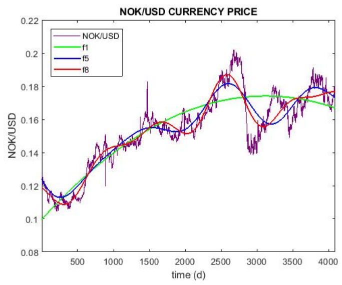
4.1.2: AUD/USD Fourier model fitting.



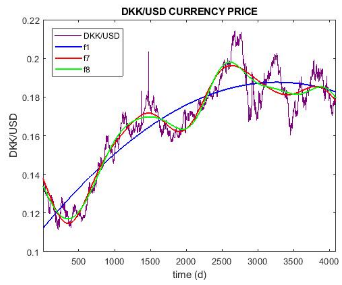
4.1.3: SEK/USD Fourier model fitting.



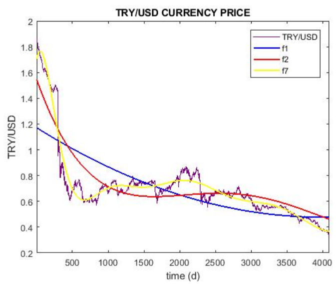
4.1.4: GBP/USD Fourier model fitting.



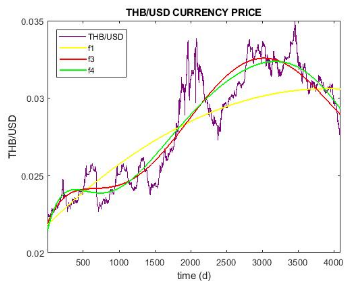
4.1.5: NOK/USD Fourier model fitting.



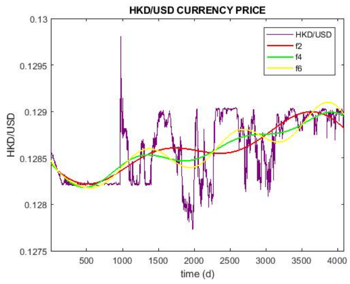
4.1.6: DKK/USD Fourier model fitting.



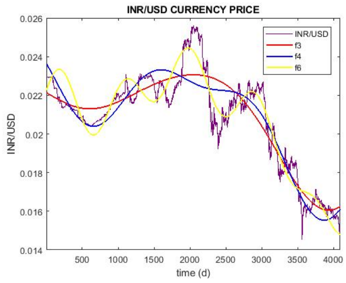
4.1.7: TRY/USD Fourier model fitting.



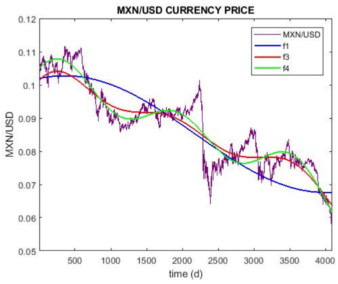
4.1.8: THB/USD fourier model fitting.



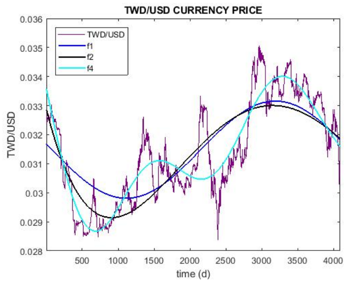
4.1.9: HKD/USD Fourier model fitting.



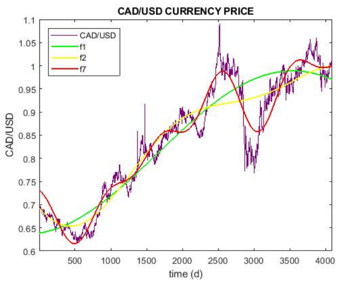
4.1.10: INR/USD Fourier model fitting.



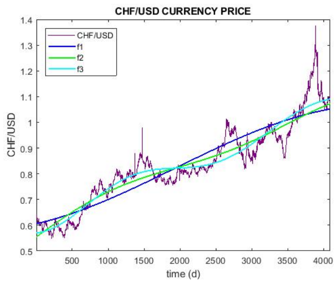
4.1.11: MXN/USD fourier model fitting.



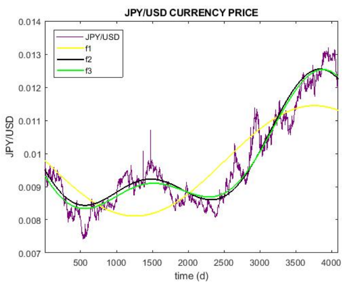
4.1.12: TWD/USD fourier model fitting.



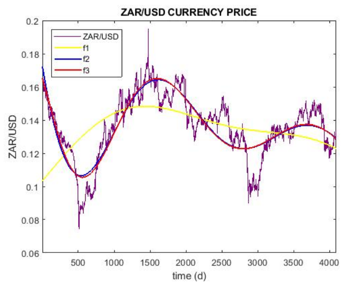
4.1.13: CAD/USD Fourier model fitting.



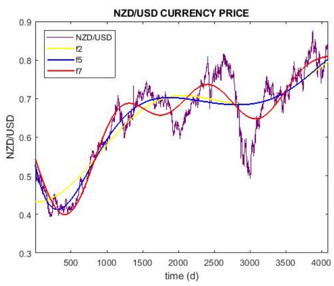
4.1.14: CHF/USD Fourier model fitting.



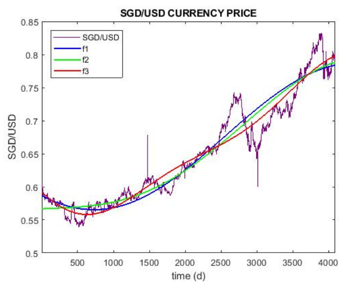
4.1.15: JPY/USD Fourier model fitting.



4.1.16: ZAR/USD Fourier model fitting



4.1.17: NZD/USD Fourier model fitting.

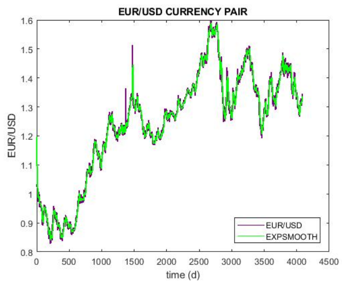


4.1.18: SGD/USD Fourier model fitting.

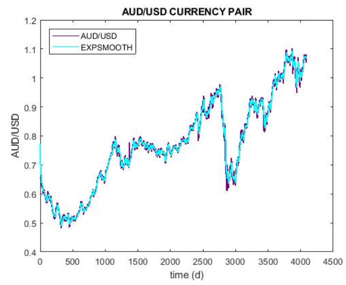
Figure 4.1: FS in-sample model fitting of currency pairs.

4.2.2 Simple Exponential Smoothing in-sample Model Fitting

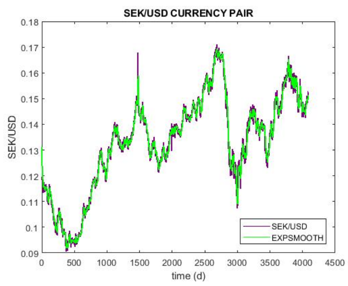
The simple exponential smoothing (SES) model defined in chapter 3 was fitted for each currency pair X/USD. The SES model made use of a constant alpha value of 0.2, the rationale behind this value comes from the observation that, a small value of α ensures that the random variation (noise) in the data set is smoothed and the resulting predictions are stable (Ostertagova and Ostertag, 2011). The model fitting process was implemented MATLAB, Figure 4.2 below shows the in-sample SES model fitting and the smoothing effect of the model is evident.



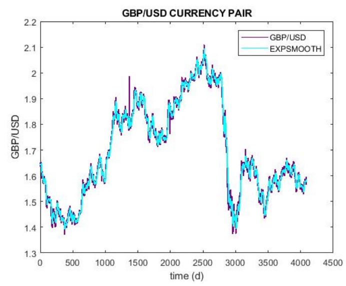
4.2.1: EUR/USD SES model fitting.



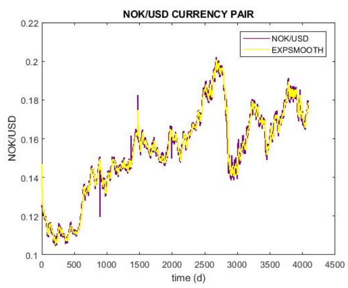
4.2.2: AUD/USD SES model fitting.



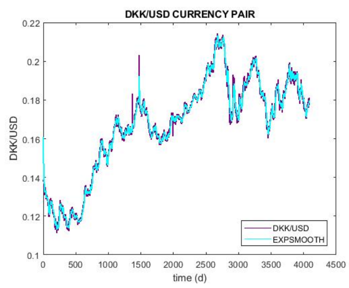
4.2.3: SEK/USD SES model fitting.



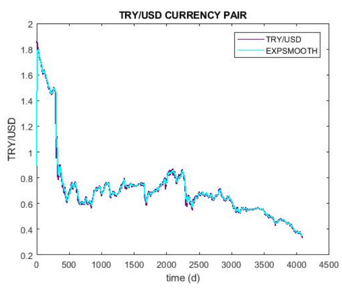
4.2.4: GBP/USD SES model fitting.



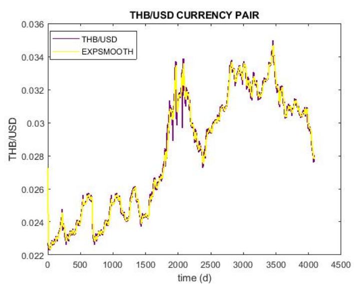
4.2.5: NOK/USD SES model fitting.



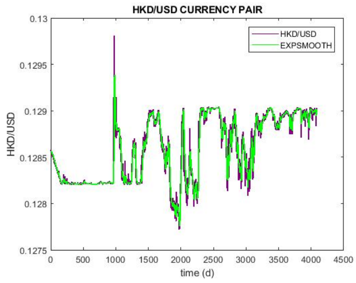
4.2.6: DKK/USD SES model fitting.



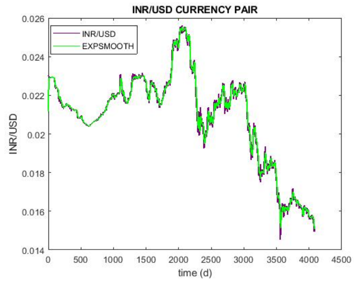
4.2.7: TRY/USD SES model fitting.



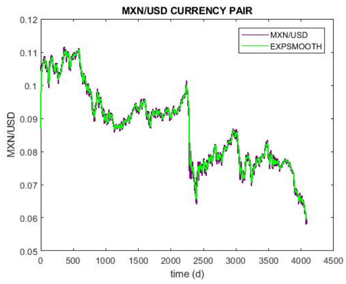
4.2.8: THB/USD SES model fitting.



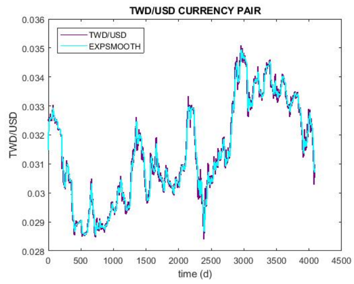
4.2.9: HKD/USD SES model fitting.



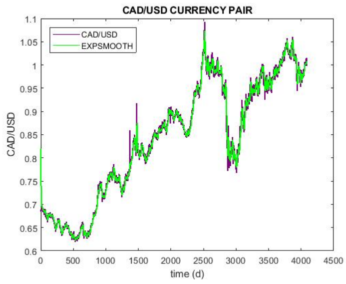
4.2.10: INR/USD SES model fitting.



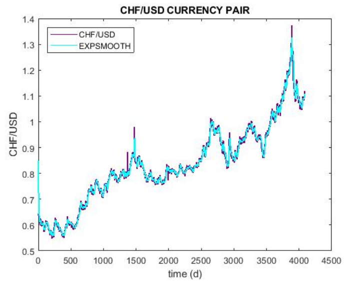
4.2.11: MXN/USD SES model fitting.



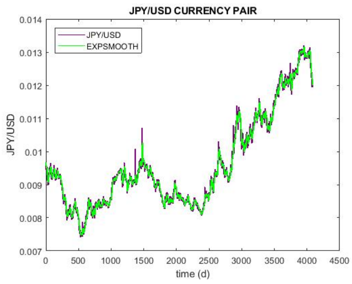
4.2.12: TWD/USD SES model fitting.



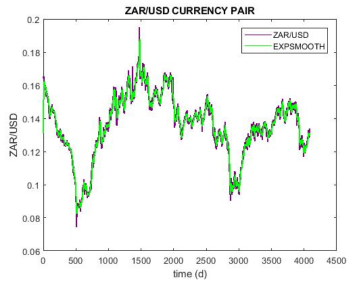
4.2.13: CAD/USD SES model fitting.



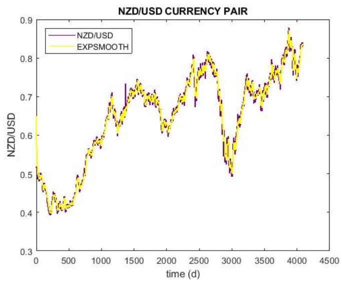
4.2.14: CHF/USD SES model fitting.



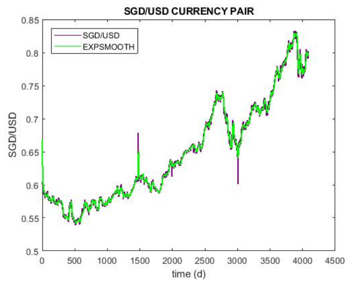
4.2.15: JPY/USD SES model fitting.



4.2.16: ZAR/USD SES model fitting.



4.2.17: NZD/USD SES model fitting.



4.2.18: SGD/USD SES model fitting.

Figure 4.2: SES in-sample model fitting of currency pairs.

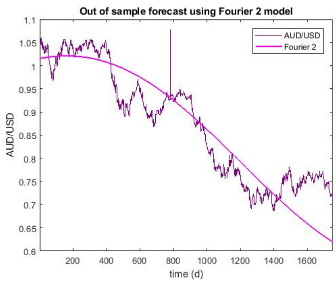
The motivation behind using a deterministic FS model is to capture the cyclical behaviour that is evident in the Forex market. However the SES model fits the historic data better (based on the average SSE in Table 4.2 below) which is an indication that it may be a better choice for forecasting. Table 4.2 below illustrates the currency pairs and the chosen best models (FS and SES) based on the in-sample minimum SSE.

Table 4.2: FS and SES in-sample model fitting accuracy.

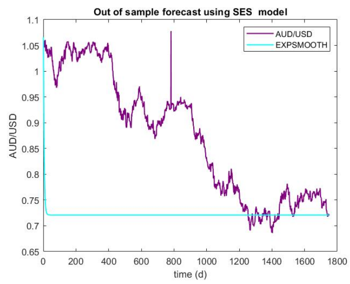
currency pair	**FS model**	**SES (model αvalue)**	**SSE (FS Model)**	**SSE (SES Model)**
EUR/USD	Fourier2	0.2	2.5374	0.7970
AUD/USD	Fourier2	0.2	0.6518	0.4283
CAD/USD	Fourier1	0.2	0.8454	0.3273
CHF/USD	Fourier2	0.2	1.1990	0.5312
JPY/USD	Fourier2	0.2	0.0004	0.0000
NZD/USD	Fourier2	0.2	0.9372	0.3467
ZAR/USD	Fourier2	0.2	0.0135	0.0233
GBP/USD	Fourier8	0.2	1.4596	0.9384
SEK/USD	Fourier4	0.2	0.0125	0.0054
NOK/USD	Fourier5	0.2	0.0157	0.0165
DKK/USD	Fourier7	0.2	0.0078	0.0141
SGD/USD	Fourier1	0.2	0.3940	0.0769
MXN/USD	Fourier4	0.2	0.0198	0.0041
TWD/USD	Fourier1	0.2	0.0006	0.0000
HKD/USD	Fourier6	0.2	0.0000	0.0000
INR/USD	Fourier6	0.2	0.0013	0.0000
TRY/USD	Fourier1	0.2	3.9087	3.6740
THB/USD	Fourier1	0.2	0.0114	0.0002
Average	-	-	0.6676	0.3992

4.2.3 FS and SES out-of-sample Model Fitting

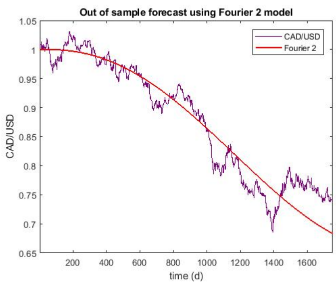
During the in-sample model fitting, the SES model performed better than the FS model, however when the two models were compared on 6 years of out-of-sample data. The FS outperformed the SES model based on the average SSE. Figure and Table 4.3 below consists of all the currencies and the fitted models and show the comparison between the two models. The FS model is chosen for forecasting currency prices, as predicting accuracy of a model is considered more important than fitting accuracy. It should be noted that the in-sample data refers to the 70 % of the historic data and out-sample data refers to the remaining 30 % of the historic data.



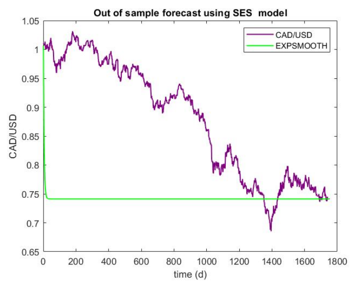
4.3.1: AUD/USD FS model forecast.



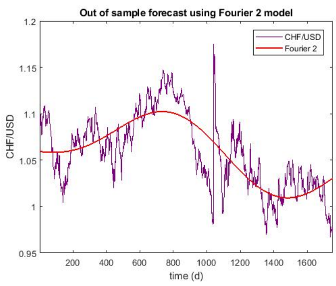
4.3.2: AUD/USD SES model forecast.



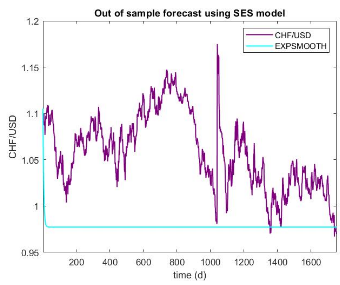
4.3.3: CAD/USD FS model forecast.



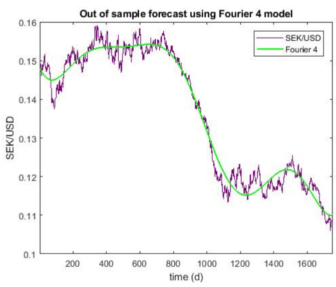
4.3.4: CAD/USD SES model forecast.



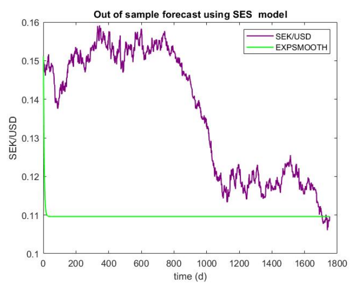
4.3.5: CHF/USD FS model forecast.



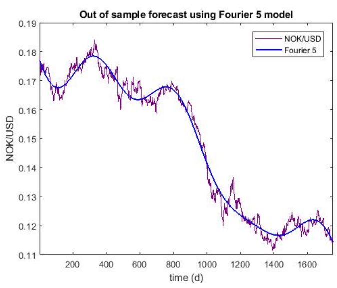
4.3.6: CHF/USD SES model forecast.



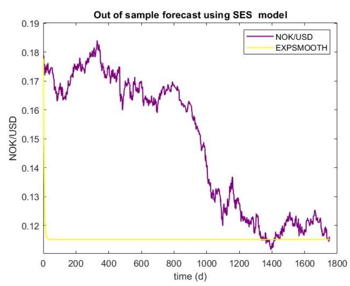
4.3.7: SEK/USD FS model forecast.



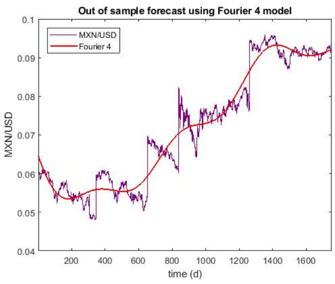
4.3.8: SEK/USD SES model forecast.



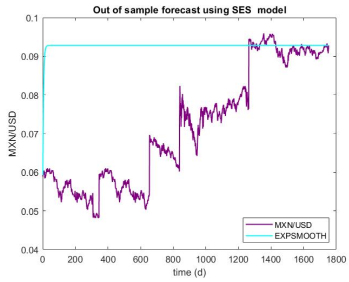
4.3.9: NOK/USD FS model forecast.



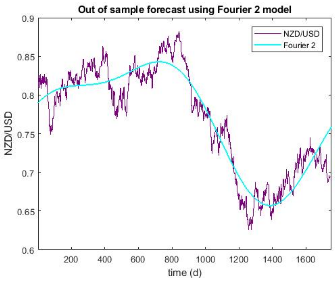
4.3.10: NOK/USD SES model forecast.



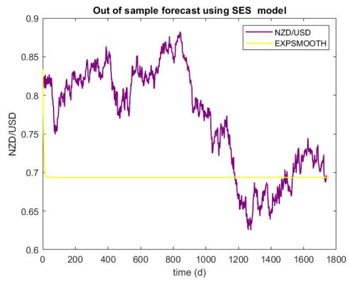
4.3.11: MXN/USD FS model forecast.



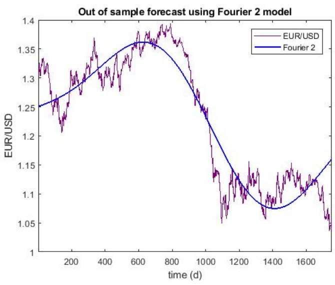
4.3.12: MXN/USD SES model forecast.



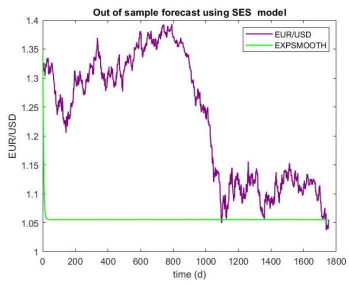
4.3.13: NZD/USD FS model forecast.



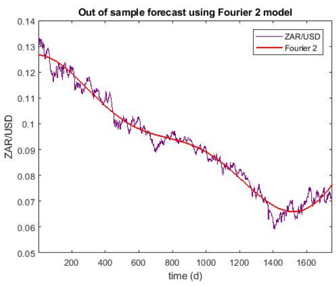
4.3.14: NZD/USD SES model forecast.



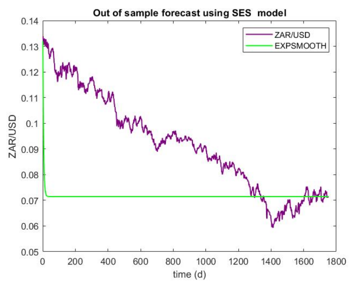
4.3.15: EUR/USD FS model forecast.



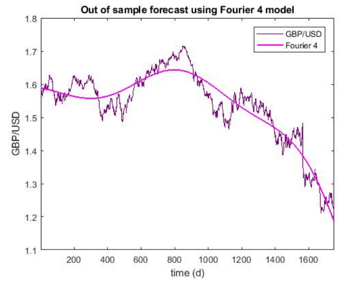
4.3.16: EUR/USD SES model forecast.



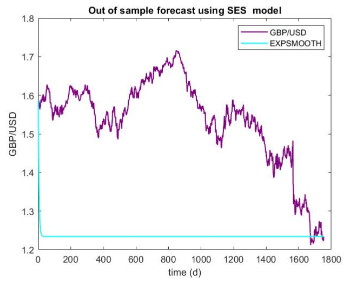
4.3.17: ZAR/USD FS model forecast.



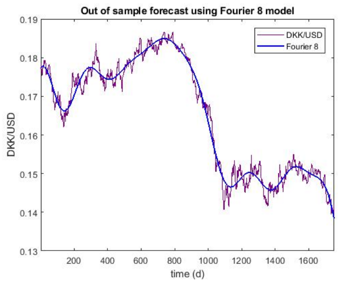
4.3.18: ZAR/USD SES model forecast.



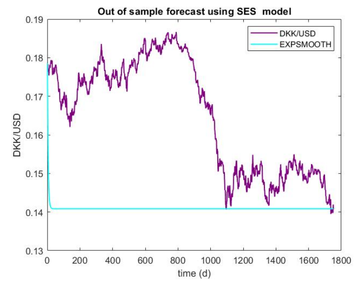
4.3.19: GBP/USD FS model forecast.



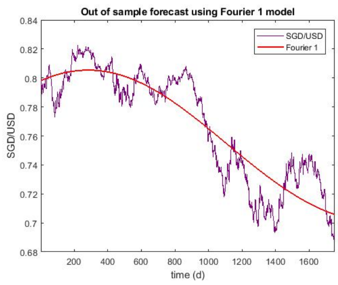
4.3.20: GBP/USD SES model forecast.



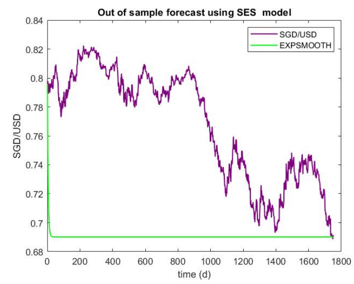
4.3.21: DKK/USD FS model forecast.



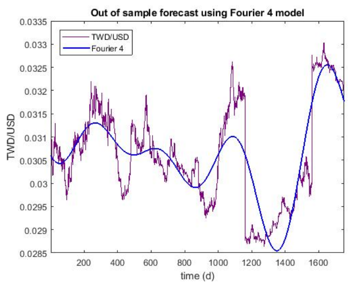
4.3.22: DKK/USD SES model forecast.



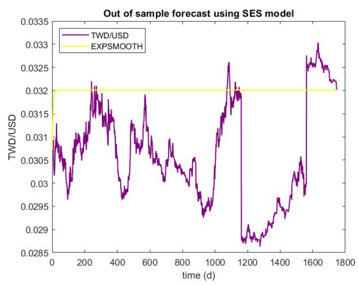
4.3.23: SGD/USD FS model forecast.



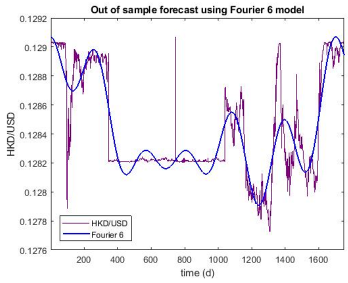
4.3.24: SGD/USD SES model forecast.



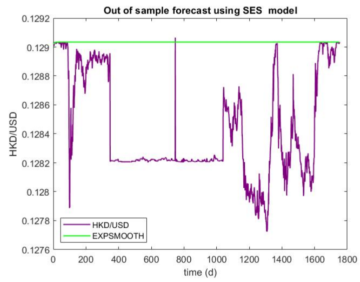
4.3.25: TWD/USD FS model forecast.



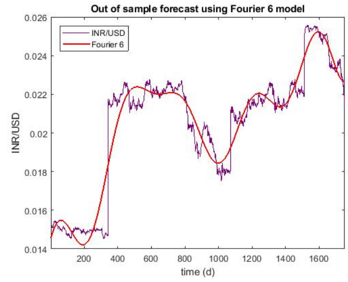
4.3.26: TWD/USD SES model forecast



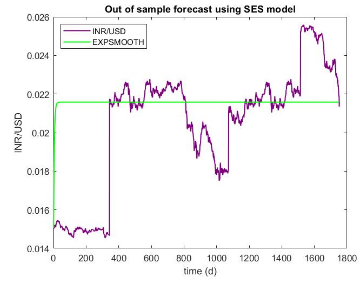
4.3.27: HKD/USD FS model forecast.



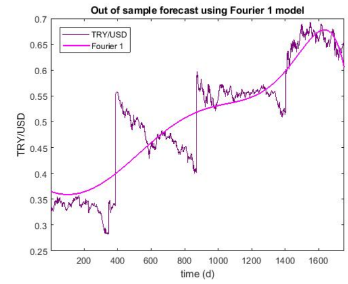
4.3.28: HKD/USD SES model forecast.



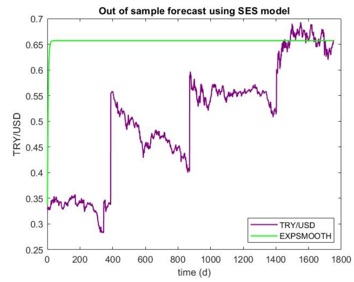
4.3.29: INR/USD FS model forecast.



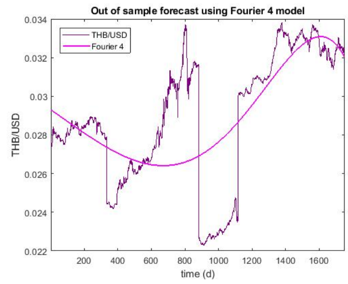
4.3.30: INR/USD SES model forecast.



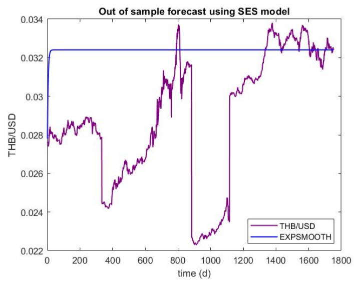
4.3.31: TRY/USD FS model forecast.



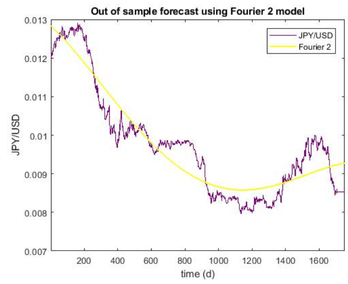
4.3.32: TRY/USD SES model forecast.



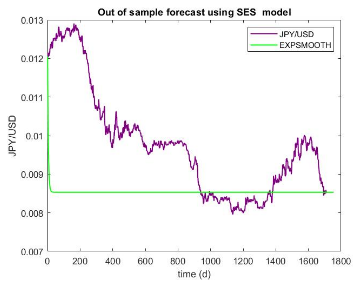
4.3.33: THB/USD FS model forecast.



4.3.34: THB/USD SES model forecast.



4.3.35: JPY/USD FS model forecast.



4.3.36: JPY/USD SES model forecast.

Figure 4.3: FS and SES out-of-sample model fitting.

The objective is to forecast future currency prices with the SES model beyond a single point $\hat{y}_{t+1}$. The SES model in its current form is incapable of doing this due to the lack of historical data or observations needed to compute the forecasts. However according to Handbook (2020) this drawback can be adressed by fixing the last data point and compute the next forecast based on the formula:

$$\hat{y}_{t+2} = \alpha y_T + (1 - \alpha)\hat{y}_{t+1}. \tag{4.1}$$

Since the first term αy_T is a constant, one can write the above equation as :

$$\hat{y}_{t+2} = c_1 + (1 - \alpha)\hat{y}_{t+1}, \tag{4.2}$$

$$\hat{y}_{t+3} = c_2 + (1 - \alpha)^2\hat{y}_{t+1}, \tag{4.3}$$

$$\hat{y}_{t+4} = c_3 + (1 - \alpha)^3\hat{y}_{t+1}, \tag{4.4}$$

$$\ldots \tag{4.5}$$

The general equation is :

$$\hat{y}_{t+k} = C + (1 - \alpha)^{k-1}\hat{y}_{t+1}, \quad k \geq 2 \tag{4.6}$$

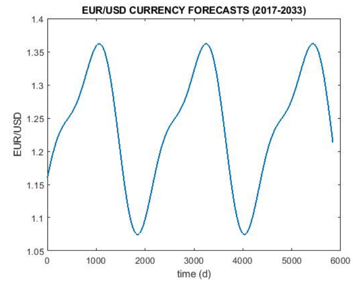
However this modification proves to be useful when forecasting a few data points beyond $\hat{y}_{t+1}$ since the forecasts stabilise to a constant value when k is large, that is $\lim_{k \longrightarrow \infty} \hat{y}_{t+k} = C$. This result is confirmed by Figure 4.3. Based on the results of Figure and Table 4.3, it is clear that the SES model is inferior to the FS model and is not a suitable choice for forecasting.

Table 4.3: Out-of-sample FS and SES model accuracy.

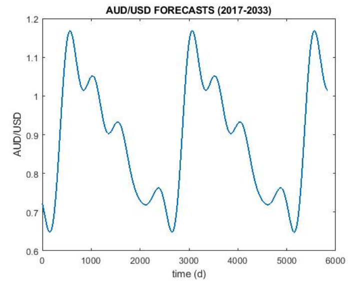
Currency pair	FS Model	SES Model (α value)	SSE (FS Model)	SSE (SES Model)
EUR/USD	Fourier2	0.2	2.5374	73.1301
AUD/USD	Fourier2	0.2	5.0593	64.3796
CAD/USD	Fourier1	0.2	1.6554	49.3528
CHF/USD	Fourier2	0.2	1.1990	14.3822
JPY/USD	Fourier2	0.2	0.0004	0.0058
NZD/USD	Fourier2	0.2	0.9372	18.0278
ZAR/USD	Fourier2	0.2	0.0135	1.3245
GBP/USD	Fourier8	0.2	2.6541	179.6821
SEK/USD	Fourier4	0.2	0.0125	1.6992
NOK/USD	Fourier5	0.2	0.0157	2.8367
DKK/USD	Fourier7	0.2	0.0078	1.3732
SGD/USD	Fourier1	0.2	0.4662	12.8873
MXN/USD	Fourier4	0.2	0.0198	1.1745
TWD/USD	Fourier1	0.2	0.0006	0.0055
HKD/USD	Fourier6	0.2	0.0000	0.0009
INR/USD	Fourier6	0.2	0.0013	0.0190
TRY/USD	Fourier1	0.2	3.7533	62.3321
THB/USD	Fourier1	0.2	0.0110	0.0414
Average	-	-	1.0191	26.8142

4.2.4 Fourier Series Model Forecasting

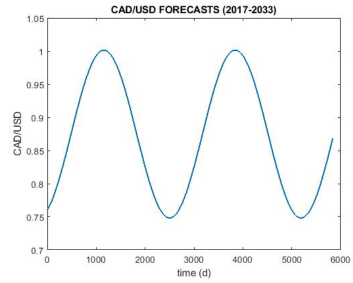
Forecasts were made using the “best” Fourier series model for each currency for the period (2017-2033). The figures below illustrate the forecasting results. These forecasts are used to calculate returns which in turn form the basis for the portfolio optimisation process. These forecasts reflect the future (not historic) state of the Forex market.



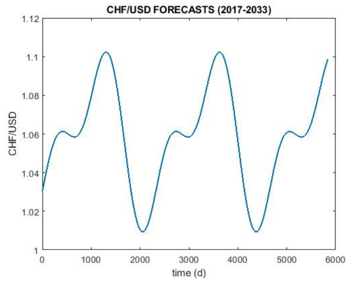
4.4.1: EUR/USD Fourier 2 forecasts.



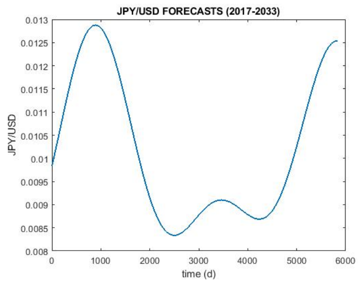
4.4.2: AUD/USD Fourier 2 forecasts.



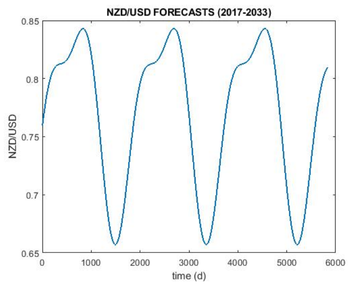
4.4.3: CAD/USD Fourier 1 forecasts.



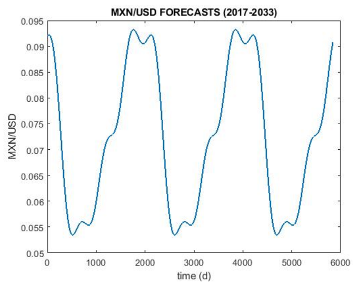
4.4.4: CHF/USD Fourier 2 forecasts.



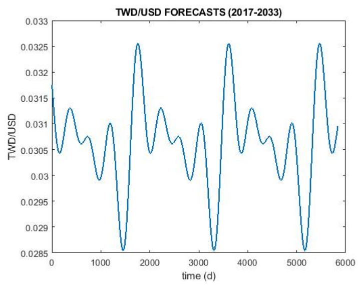
4.4.5: JPY/USD Fourier 2 forecasts.



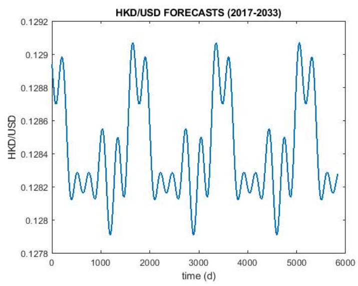
4.4.6: NZD/USD Fourier 2 forecasts.



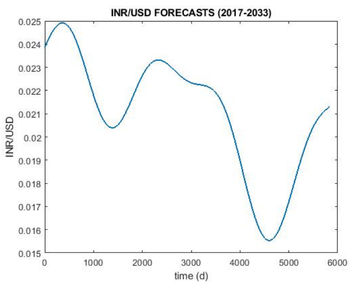
4.4.7: MXN/USD Fourier 2 forecasts.



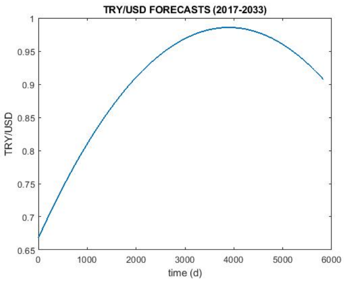
4.4.8: TWD/USD Fourier 1 forecasts.



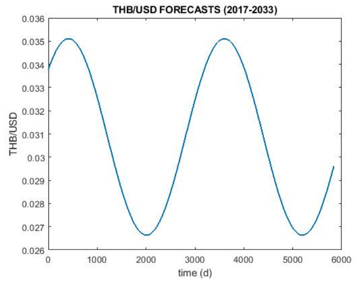
4.4.9: HKD/USD Fourier 6 forecasts



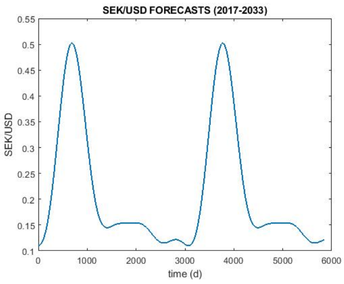
4.4.10: INR/USD Fourier 6 forecasts.



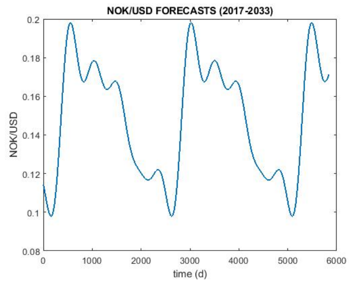
4.4.11: TRY/USD Fourier 1 forecasts.



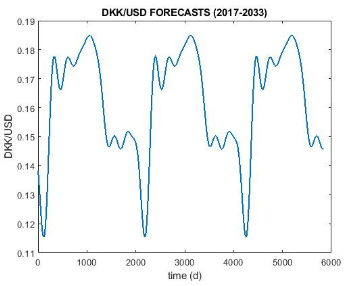
4.4.12: THB/USD Fourier 1 forecasts.



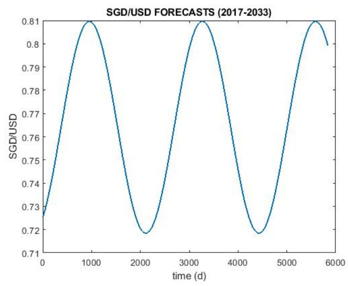
4.4.13: SEK/USD Fourier 4 forecasts.



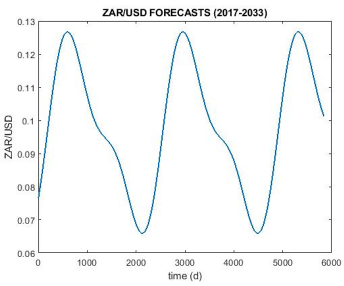
4.4.14: NOK/USD Fourier 5 forecasts.



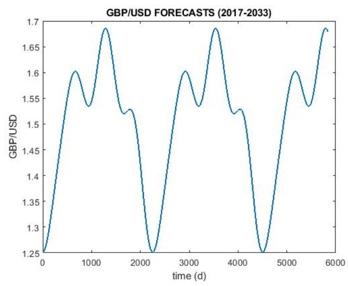
4.4.15: DKK/USD Fourier 7 forecasts.



4.4.16: SGD/USD Fourier 1 forecasts.



4.4.17: ZAR/USD Fourier 2 forecasts.



4.4.18: GBP/USD Fourier 8 forecasts.

Figure 4.4: Fourier series model forecasting of currency prices.

4.2.5 Risks and returns of individual currencies

Without any loss of information, since the USD acts as the quote currency in each pair meaning it acts as the money to buy the corresponding base currency which acts as an asset that appreciates and depreciates in value. It is omitted from any pair which makes the notation simpler and easier to interpret. From Table 4.4 below it is clear that if a trader was not educated about portfolio optimisation theory, then the investor would invest all the money on the Swedish Krona (SEK) since it has the highest mean return of 0.3274. According to the SD, SMAD and CVaR risk measures, the Swedish Krona (SEK) is also the riskiest asset with an average risk of 0.7505 percent, where the average is computed across the 3 risk measures (by row). This result confirms the old saying : "With high risk comes high reward". These values are found in the mean risk column. According to the analysis, the CVaR risk measure is smaller than the SD and SMAD on average, which is to be expected since the latter are deviation risk measures whilst the former is a loss (negative returns) risk measure . The value at risk is computed empirically using the data and a significance level $\alpha = 0.95$, and the CVaR is simply computed as the mean of the worst 5% of cases.

Table 4.4: Risks and returns of individual currencies.

Currency	**Return**	**SD**	**SMAD**	**VaR**	**CVaR**	**Mean risk**
EUR	0.03153	0.2243	0.09133	-0.4575	-0.02399	0.0972
AUD	0.1069	0.8908	0.3361	-0.8132	-0.04504	0.394
CAD	0.06771	0.2453	0.1108	-0.3346	-0.01694	0.113
CHF	0.01804	0.08551	0.03377	-0.1752	-0.0092	0.03669
JPY	-0.0222	0.2706	0.114	-0.4279	-0.02159	0.121
NZD	-0.03059	0.3203	0.1308	-0.6065	-0.03081	0.1401
ZAR	0.07675	0.6785	0.2996	-0.7079	-0.03763	0.3135
GBP	0.08439	0.3449	0.148	-0.572	-0.03367	0.1531
SEK	0.3274	1.734	0.6589	-2.653	-0.1419	0.7505
NOK	0.1271	1.088	0.4042	-1.117	-0.06159	0.4768
DKK	0.01717	1.003	0.328	-1.426	-0.09276	0.4128
SGD	0.02812	0.1107	0.04927	-0.1575	-0.0080	0.0506
MXN	-0.05954	0.8037	0.3028	-1.846	-0.09596	0.3368
TWD	-0.0023	0.2765	0.106	-0.4294	-0.02371	0.1196
HKD	-0.0004	0.0282	0.01151	-0.04777	-0.0025	0.01237
INR	-0.02464	0.1523	0.06039	-0.3006	-0.01541	0.06577
TRY	0.11	0.06305	0.0269	0.02413	0.001015	0.03032
THB	0.01032	0.1868	0.08347	-0.2686	-0.01355	0.08557
Average	0.0481	0.4726	0.1831	-0.6842	-0.0374	-

4.3 Portfolio Optimisation

The three optimisation models (M-V, SMAD and CVaR) were applied and the target return set was chosen to be the positive returns found in the second column of Table 4.4. These returns correspond with the currencies in the first column of Table 4.4. The reason behind selecting only positive returns is sensible since an investor would prefer a profitable portfolio over an unprofitable one. A comparison was made between the risk levels of the generated portfolios and the individual currencies, to see if it is possible to have portfolios that outperform the individual currencies in-terms of smaller risk for the same return. The results are shown in Table 4.5 and the optimal portfolios outperform the individual currencies on average. Indeed the results show that the investor is better off using optimisation models when making investment decisions rather than using a naive approach with no mathematical basis. The SMAD portfolios offer less risk than the other two risk measures on average. The last three columns show the return to risk ratios of portfolios, the bigger the ratio the better the portfolio since the ratio represents the return per unit risk exposure. This is the so called sharpe ratio with the risk-free currency return $R_f = 0$, since all currencies in the study are considered to be risky. It helps the investors to understand the return of an investment compared to its risk (Gatfaoui, 2009). Risk-port refers to the portfolio risk associated with the corresponding return in column 2 where the risk is measured as the standard deviation (SD), semi-mean-absolute deviation (SMAD) and conditional value-at-risk (CVaR). It is clear from Table 4.5 that the CVaR portfolios outperform the other two risk measures in this regard. By convention values smaller than 0.0001 are assigned a value of zero which explains the infinity sign in the last column, since the divisor is a very small number this leads to a very large ratio not necessarily infinity.

Table 4.5: Risks and returns of individual currencies and portfolios.

Currency	**Return**	**SD**	**SD-Port**	**SMAD**	**SMAD-Port**	**CVaR**	**CVaR-port**	$\frac{\textbf{Return}}{\textbf{SD-Port}}$	$\frac{\textbf{Return}}{\textbf{SMAD-Port}}$	$\frac{\textbf{Return}}{\textbf{CVaR-Port}}$
EUR	0.010	0.224	0.019	0.091	0.007	-0.023	0.017	0.521	1.298	0.582
AUD	0.017	0.890	0.019	0.336	0.007	-0.045	0.009	0.893	2.289	1.839
CAD	0.018	0.245	0.019	0.110	0.007	-0.016	0.008	0.938	2.417	2.174
CHF	0.028	0.085	0.019	0.033	0.007	-0.009	0	1.445	3.839	∞
ZAR	0.0315	0.678	0.019	0.299	0.007	-0.037	0	1.596	4.28	∞
GBP	0.067	0.344	0.029	0.148	0.010	-0.033	0	2.305	6.562	∞
SEK	0.076	1.734	0.033	0.658	0.011	-0.141	0	2.32	6.629	∞
NOK	0.084	1.088	0.036	0.404	0.012	-0.061	0	2.315	6.504	∞
DKK	0.106	1.003	0.057	0.328	0.023	-0.092	0	1.85	4.635	∞
SGD	0.11	0.110	0.062	0.049	0.024	-0.008	0	1.772	4.45	∞
TRY	0.127	0.063	0.150	0.026	0.060	0.001	0.089	0.842	2.097	1.418
THB	0.327	0.186	1.734	0.083	0.658	-0.013	2.842	0.188	0.496	0.115
Average	0.083	0.554	0.183	0.214	0.070	-0.040	0.247	1.415	3.791	∞

Table 4.6, 4.7, 4.8, and 4.9 show the optimal allocation of capital (portfolio weights) for the 12 positive target returns in Table 4.5 (second column). These are the weights that produce the minimum risk (column 4,6, and 8). It's interesting to note the similarities between the three models on how they allocate capital especially the last three columns of Table 4.9 where all three models suggest putting all your money on the Swedish Krona (SEK) to achieve minimum risk. The three models (M-V,SMAD, CVaR) were solved again for 100 different target return levels (in an increasing order), hence generating 100 portfolios. What was observed from these results was that as the return increases so does the risk, in other words if an investor wants to achieve higher returns, the investor has to take on more risk as suggested by optimisation theory (efficient frontier). It is also clear from Table A.1 in appendix A that as the target return increases, the portfolios (M-V, SMAD, CVaR) generated become less diversified, but the MV model is still the most diversified when compared with the other two models. This is based on the average number of non-zero weights found in Table A.1. On average the MV model takes less time to be solved than the other two models. This is evident based on the last three columns of Table A.1.

Table 4.6: Optimal allocation of capital (**return=(0.0103,0.0172,0.0180)**).

Currency	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**
Return	**0.0103**	**0.0103**	**0.0103**	**0.0172**	**0.0172**	**0.0172**	**0.0180**	**0.0180**	**0.0180**
EUR	0	0	0	0	0	0	0	0	0
AUD	0	0	0.0081	0	0	0.0046	0	0	0.0045
CAD	0	0	0	0	0	0	0	0	0
CHF	0.1556	0.1692	0.1834	0.148	0.1857	0.1361	0.147	0.1869	0.1336
JPY	0	0	0	0	0	0	0	0	0
NZD	0.0137	0.0184	0.0170	0.0078	0.0152	0.0085	0.0071	0.0147	0.0079
ZAR	0	0	0.0010	0	0	0.0038	0	0	0.0038
GBP	0	0	0.0048	0	0	0.0034	0	0	0.0034
SEK	0	0	0	0	0	0	0	0	0
NOK	0.0056	0.0020	0	0.0050	0.0007	0	0.0049	0.0006	0
DKK	0	0	0	0	0	0	0	0	0
SGD	0.0343	0.0568	0	0.0356	0.0467	0.051	0.0355	0.0452	0.0534
MXN	0.0126	0.0122	0.0189	0.0126	0.0114	0.0213	0.0126	0.0112	0.0210
TWD	0.0104	0.0096	0.0016	0.0128	0.0095	0	0.0131	0.0096	0
HKD	0.6027	0.5636	0.5836	0.55	0.4859	0.5207	0.5436	0.4775	0.5131
INR	0.0803	0.0851	0.0918	0.0810	0.0973	0.1007	0.08103	0.0985	0.1017
TRY	0.0845	0.0827	0.0894	0.1468	0.1474	0.1497	0.1547	0.1556	0.1572
THB	0	0	0	0	0	0	0.0001	0	0

Table 4.7: Optimal allocation of capital (**return=(0.0281,0.0315,0.0677)**).

Currency	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**
Return	**0.0281**	**0.0281**	**0.0281**	**0.0315**	**0.0315**	**0.0315**	**0.0677**	**0.0677**	**0.0677**
EUR	0	0	0	0	0	0	0	0	0
AUD	0	0	0	0	0	0	0	0	0
CAD	0	0	0	0	0	0	0	0	0
CHF	0.1100	0.1785	0.0417	0.0968	0.1788	0	0	0.0508	0
JPY	0	0	0	0	0	0	0	0	0
NZD	0.0014	0.0116	0	0	0.0076	0	0	0	0
ZAR	0	0	0	0	0	0	0	0	0
GBP	0	0	0	0	0	0	0	0	0
SEK	0	0	0	0	0	0	0	0	0
NOK	0.0036	0	0.0022	0.0030	0	0.0016	0	0	0
DKK	0	0	0	0	0	0	0	0	0
SGD	0	0	0	0	0	0	0	0	0
MXN	0.0122	0.0101	0.0143	0.0126	0.0116	0.0077	0.0211	0.0229	0
TWD	0.0156	0.0086	0.0101	0.0166	0.0087	0.0198	0.0220	0	0.0006
HKD	0.5052	0.4186	0.5564	0.4832	0.3799	0.5774	0.2015	0.1081	0.3824
INR	0.0597	0.0861	0.0579	0.0546	0.0898	0.0449	0.0018	0.069821	0
TRY	0.2524	0.2531	0.2645	0.2841	0.2839	0.2970	0.615548	0.6237	0.6169
THB	0.0395	0.0330	0.0524	0.0487	0.0393	0.0513	0.1378	0.1244	0

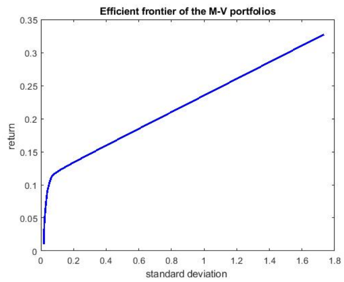
Table 4.8: Optimal allocation of capital (**return=(0.0768,0.0844,0.1069)**).

Currency	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**
Return	**0.0768**	**0.0768**	**0.0768**	**0.0844**	**0.0844**	**0.0844**	**0.1069**	**0.1069**	**0.1069**
EUR	0	0	0	0	0	0	0	0	0
AUD	0	0	0	0	0	0	0	0	0
CAD	0	0	0	0	0	0	0	0	0
CHF	0	0.0243	0	0	0	0	0	0	0
JPY	0	0	0	0	0	0	0	0	0
NZD	0	0	0	0	0	0	0	0	0
ZAR	0	0	0	0	0	0	0	0	0
GBP	0	0	0	0	0	0	0	0	0
SEK	0	0	0	0	0	0	0.0050	0.0085	0
NOK	0	0	0	0	0	0	0	0	0
DKK	0	0	0	0	0	0	0	0	0
SGD	0	0	0	0	0	0	0	0	0
MXN	0.0242	0.0263	0	0.0269	0.0268	0	0.0095	0.0055	0
TWD	0.023	0	0	0.0237	0	0	0	0	0
HKD	0.1021	0.0278	0.3012	0.0162	0	0.2321	0	0	0.0284
INR	0	0.0669	0	0	0.0444	0	0	0	0
TRY	0.6971	0.7092	0.6988	0.7664	0.7771	0.7679	0.9593	0.9454	0.9716
THB	0.1535	0.1453	0	0.1666	0.1515	0	0.0261	0.0405	0

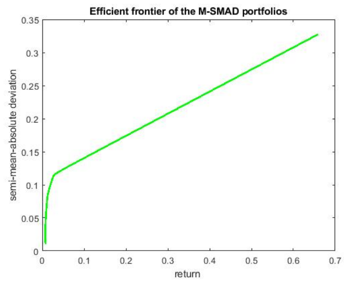
Table 4.9: Optimal allocation of capital (**return=(0.1100,0.1271,0.3274)**).

Currency	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**
Return	**0.1100**	**0.1100**	**0.1100**	**0.1271**	**0.1271**	**0.1271**	**0.3274**	**0.3274**	**0.3274**
EUR	0	0	0	0	0	0	0	0	0
AUD	0	0	0	0	0	0	0	0	0
CAD	0	0	0	0	0	0	0	0	0
CHF	0	0	0	0	0	0	0	0	0
JPY	0	0	0	0	0	0	0	0	0
NZD	0	0	0	0	0	0	0	0	0
ZAR	0	0	0	0	0	0	0	0	0
GBP	0	0	0	0	0	0	0	0	0
SEK	0.0062	0.0094	0	0.0783	0.0783	0.0783	1	1	1
NOK	0	0	0	0	0	0	0	0	0
DKK	0	0	0	0	0	0	0	0	0
SGD	0	0	0	0	0	0	0	0	0
MXN	0.0068	0	0	0	0	0	0	0	0
TWD	0	0	0	0	0	0	0	0	0
HKD	0	0	0	0	0	0	0	0	0
INR	0	0	0	0	0	0	0	0	0
TRY	0.9848	0.97	1.0	0.9217	0.9217	0.9217	0	0	0
THB	0.0021	0.0205	0	0	0	0	0	0	0

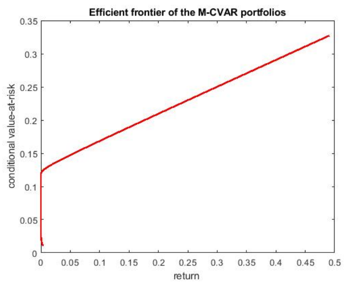
Figure 4.5 below shows the efficient frontiers of the M-V, SMAD and CVaR models consisting of the 100 portfolios discussed above. Subfigure 4.5.4 confirms the results of Table A.1 in that the M-V model has a wider range of risk, meaning it offers higher risks for the same return than the SMAD and CVaR models.



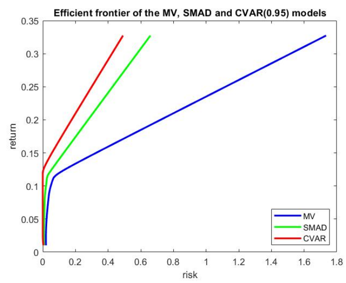
4.5.1: Efficient frontier of M-V portfolios



4.5.2: Efficient frontier of SMAD portfolios



4.5.3: Efficient frontier of CVaR portfolios



4.5.4: Efficient frontiers of portfolios

Figure 4.5: Efficient frontiers of M-V, SMAD, and CVaR portfolios

The time taken to generate each portfolio, and for each model to be solved was investigated. The results of Table A.1 show that on average, the M-V model is quicker to solve than the SMAD and CVaR models. This result is consistent with Figure 4.6 below, and it is against the assertion made by Konno and Yamazaki (1991) that due to the dense covariance matrix calculations involved in the M-V model, the computing time for an optimal portfolio is longer.

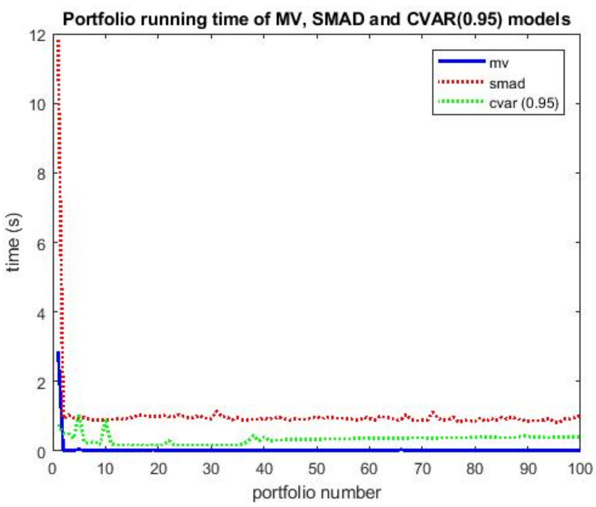


Figure 4.6: Computing time and efficient frontiers of portfolios

4.4 Backtesting analysis

Portfolio risks and returns

Table 4.10 below shows the three “extreme” portfolios associated with the 1^{st}, 50^{th} and 100^{th} mean return in Table A.1. These three portfolios are reffered to as “extreme” since they correspond to the lowest, medium and highest return levels and they will be used when performing backtesting analysis on the out-of-sample data in the next section.

Table 4.10: Optimal allocation of capital.

Currency	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**	**Optimal weights (M-V)**	**Optimal weights (SMAD)**	**Optimal weights (CVaR)**
Return	**0.0103**	**0.0103**	**0.0103**	**0.1672**	**0.1672**	**0.1672**	**0.3274**	**0.3274**	**0.3274**
EUR	0	0	0	0	0	0	0	0	0
AUD	0	0	0	0	0	0	0	0	0
CAD	0	0	0	0	0	0	0	0	0
CHF	0.1556	0.1692	0.1715	0	0	0	0	0	0
JPY	0	0	0	0	0	0	0	0	0
NZD	0.01376	0.01848	0.01215	0	0	0	0	0	0
ZAR	0	0	0	0	0	0	0	0	0
GBP	0	0	0	0	0	0	0	0	0
SEK	0	0	0	0.2632	0.2632	0.262	1	1	1
NOK	0.005626	0.00202	0.00377	0	0	0.01481	0	0	0
DKK	0	0	0	0	0	0	0	0	0
SGD	0.03436	0.05687	0.05794	0	0	0	0	0	0
MXN	0.01262	0.01229	0.01602	0	0	0	0	0	0
TWD	0.0104	0.009617	0.008187	0	0	0	0	0	0
HKD	0.6027	0.5636	0.5298	0	0	0	0	0	0
INR	0.08037	0.08516	0.1141	0	0	0	0	0	0
TRY	0.08455	0.08275	0.08664	0.7368	0.7368	0.7232	0	0	0
THB	0	0	0	0	0	0	0	0	0

It should be noted that in this study, it is assumed that there is a trading strategy in place that generates buying (long position) signals, and whenever a long position signal is available a trade is opened and currencies are bought based on the optimal weights that result from this study. Portfolio selection was performed on 56 % of the forecasted returns and the remaining 44 % of the data is used for backtesting analysis on the “extreme” portfolios discussed in the previous section. The 56 % refers to the first or oldest return and the 44 % refers to the recent data points. The reasoning behind using the 56-44 % split is due to the fact that in general the training data set should be bigger than the testing set as suggested by (Siegfried, 1995). However since the objective is to evaluate the perfomance of portfolios in a “real” trading setting. The selection (training) and testing scenarios need to be similar in size so as to give meaningful interpretation of comparative results, hence the close percentages between the two data-sets. The efficient portfolios only inform the investor which currencies to select and how much to invest, in principle the optimal weights “ensure” that the risk exposed to is as small as possible, but how is it known if these portfolios do actually protect the investor during a particular trading period, that is how is it known if the risk exposure during a trading period is the same, smaller, or bigger than the risk exposure during the portfolio selection period? Additionally the main objective of investing is to make money, the optimisation models in this study make use of the mean as the target return, but in a real trading setting profits and losses are observed over shorter time periods such as daily. How is it known if these portfolios (“extreme”) are profitable in a “real” trading period?

An attempt to answer these two questions is provided by perfoming backtesting analysis on some portion of the data which is assumed to be what the market looks like if one were to invest during that period. The objective is to check whether the selected portfolios do actually reduce the risk exposure during the testing period to the levels as determined by the models during the optimisation period. To also compare the profitability of the three models’ portfolios to see which is less risky and/or more profitable during this period. In particular it will be interesting to see how these portfolios compare with the equal weight portfolio which is used for comparison. Three optimal portfolios were selected: the low, medium, and high risk portfolios. The backtesting was done on approximately 2300 trading days which amounts to 9 years. In this study, it is assumed that the trader employs the buy and hold strategy, and that there is no rebalancing of the portfolios for the whole “trading” period.

Figure 4.6.1 below shows the extreme M-V, SMAD and CVaR upon which the backtesting analysis is performed.

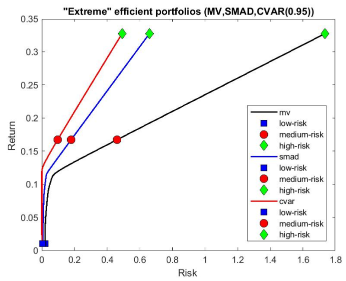


Figure 4.7: M-V,SMAD, and CVaR “extreme” portfolios

Table 4.11,4.12 and 4.13 show the maximum, minimum and average return in monetary values (USD) for the M-V, SMAD and CVaR models. The last two columns of each table compares the in-sample and out-of-sample risk measures. The risk measure or model that has the least deviation on average is considered superior since this is an indication of consistency of the corresponding model. The M-V and CVaR models have high mean-absolute percentage error (MAPE) values compared to the lower MAPE value (284.0541) of the SMAD model. This MAPE value as defined in Chapter 3 is calculated between the the in-sample and out-of-sample risk measure values. This comparison is important for investors since it offers insight as to which optimisation model or risk measure produces realistic or consistent values. In this case the SMAD model outperforms the M-V and CVaR models. When considering the mean return column for each table (risk-measure), the high-risk portfolio dominates the other portfolios which is testament to the notion that an investor has to take on more risk to achieve higher returns. Based on these results this notion is true in the Forex market as well. Using MAPE and mean return as measures of consistency and profitability of portfolios or models, the SMAD model dominates the M-V and CVaR models as evident from Tables 4.11, 4.12 and 4.13. Consequently the “best” portfolio is selected from Table 4.12. Using mean return and relative-error (defined in Chapter 3) of each portfolio in Table 4.12, the “best” portfolio is chosen between the medium and hih-risk SMAD portfolios since they have the smallest relative-error (R-E) and largest mean return values. In this context it reasonable to treat the R-E as a risk-measure since a small R-E value is an indication of consistency or reliability of a portfolio and an investor will have more confidence in selecting a portfolio with these properties since the in-sample results do not differ significantly from the out-of-sample results. Using the ratio $\frac{\text{mean}-\text{return}}{\text{relative}-\text{error}}$ to select between the medium and high-risk SMAD portfolios, the high-risk SMAD portfolio has a larger ratio of 225.65 compared to the medium-risk ratio of 144.2.

Table 4.11: M-V “extreme” portfolios

Portfolio	**Min-ret**	**Max-ret**	**Mean-ret**	**SD-in**	**SD-out**	**Rel-error**	$\frac{\text{mean}-\text{return}}{\text{relative}-\text{error}}$
Low-risk	-8.042	7.929	0.2821	0.01981	0.0000	427.2	0.0006
Med-risk	-73.74	78.17	14.42	0.4616	0.0003	1493.6	0.0096
High-risk	-275.6	293.9	45.13	1.734	0.0011	1473.4	0.0306
Equal-weight	-38.33	49.61	2.157	0.2238	0.0001	1135.8	0.0018
Average	-98.9205	107.3955	15.4993	0.6099	0.0004	1132.5	-

Table 4.12: SMAD “extreme” portfolios

Portfolio	**Min-ret**	**Max-ret**	**Mean-ret**	**SMAD-in**	**SMAD-out**	**Rel-error**	$\frac{\text{mean}-\text{return}}{\text{relative}-\text{error}}$
Low-risk	-7.304	8.552	0.2381	0.0079	0.0089	0.1	2.381
Med-risk	-73.74	78.17	14.42	0.1798	0.2083	0.1	144.2
High-risk	-275.6	293.9	45.13	0.6589	0.8084	0.2	225.65
Equal-weight	-38.33	49.61	2.157	0.2238	0.0002	1135.8	0.0018
average	-98.7361	107.5512	15.4883	0.2676	0.2565	284.0541	-

Table 4.13: CVaR “extreme” portfolios

Portfolio	Min-ret	Max-ret	Mean-ret	CVaR-in	CVaR-out	Rel-error	$\frac{\text{mean-return}}{\text{relative-error}}$
Low-risk	-8.97	10.04	0.2972	0.0035	0.0040	0.1	2.972
Med-risk	-73.37	79.41	14.35	0.0970	0.1202	0.2	71.75
High risk	-275.6	293.9	45.13	0.4912	0.6011	0.2	225.65
Equal-weight	-38.33	49.61	2.157	0.2238	0.0001	1135.8	0.0018
Average	-99.0581	108.2318	15.4847	0.2039	0.1814	284.0726	-

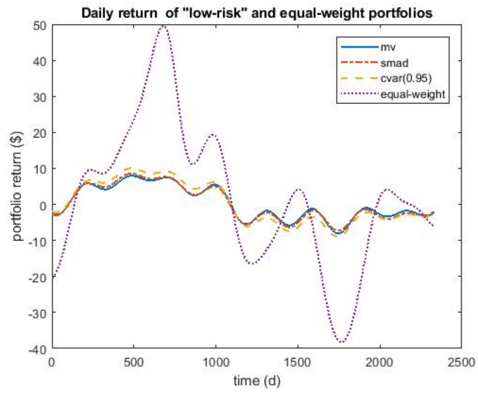
It is assumed that the investor has an initial trading balance C = \$100000 in the account and buys and holds currency positions according to the MV, SMAD and CVaR “extreme” and equal weight portfolios. Using the formulas:

$$\mathrm{R}_p = \mathrm{C}\sum_{j=1}^{n} w_j R_{tj} \quad , \qquad t = 1, 2, ..., T. \tag{4.7}$$

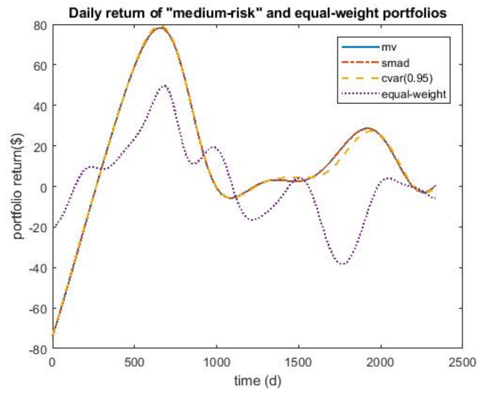
and

$$\mathrm{V}_p = \mathrm{C}\sum_{j=1}^{n} w_j \left(1 + R_{tj}\right) \quad , \qquad t = 1, 2, ..., T. \tag{4.8}$$

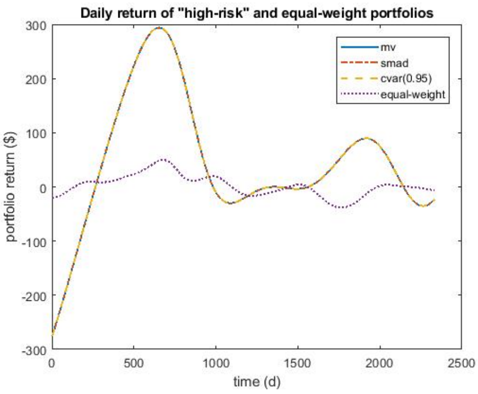
to denote the daily portfolio return and value, the results are shown in Figures 4.7 and 4.8. Figure 4.7 shows the daily returns (in monetary value) of the 4 portfolios whilst Figure 4.8 shows the portfolio values.



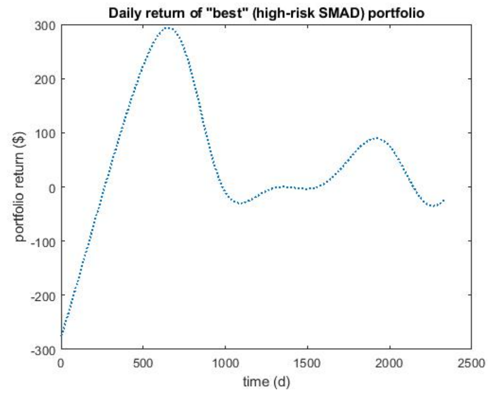
4.8.1: low risk portfolio daily return



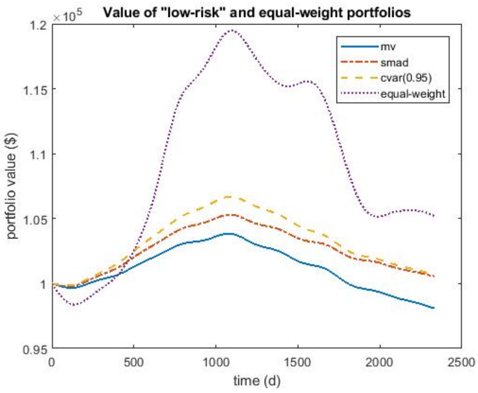
4.8.2: medium risk portfolio daily return



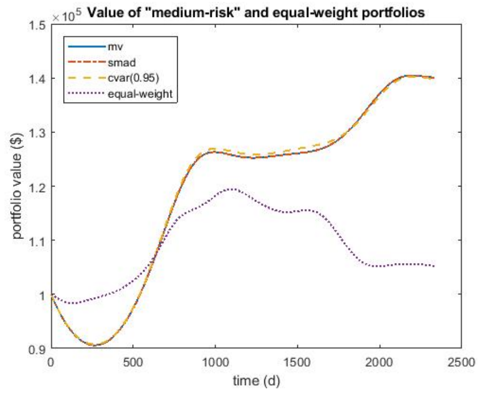
4.8.3: high risk portfolio daily return



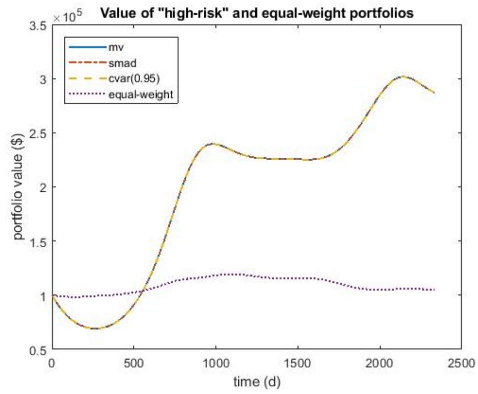
4.8.4: daily return of "best" portfolio



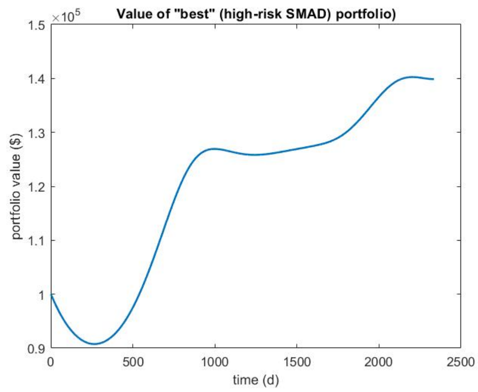
4.8.5: low risk portfolio value



4.8.6: medium risk portfolio value



4.8.7: high risk portfolio value

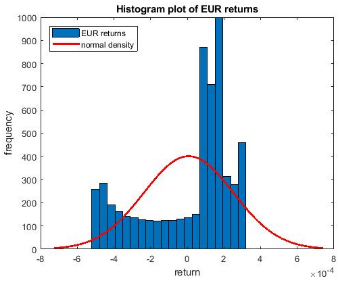


4.8.8: Value of "best" portfolio

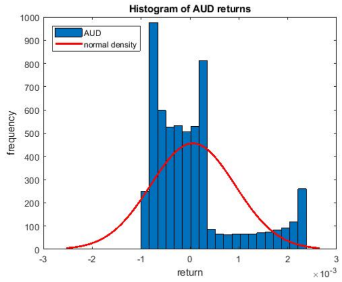
Figure 4.8: portfolio return and value of low, medium, high, equal weight and "best" portfolios

Normality assumption

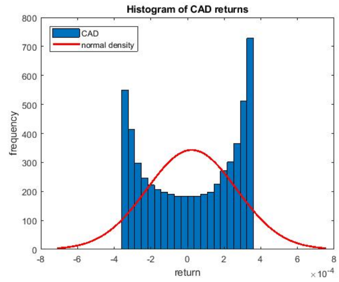
The M-V model rests on the normality assumption of returns, one of the properties used to measure normality are skewness and kurtosis. Skewness is a measure of the assymetry of the probability distribution of a real-valued random variable about its mean. The skewness value can be positive,zero,negative, or undefined. The normal distribution has a skewness of 0 and kurtosis of 3. Kurtosis is a measure of the “tailednes” of the probability distribution of a real-valued random variable. A higher kurtosis value (> 3) corresponds to greater extremity of outliers (Wikipedia website, 2020). Figure 4.7 above shows histogram plots of returns which appear to deviate from the normality assumption, this is further supported by the skewness and kurtosis values in Table B.1 in the appendix. The author is being cautious here by using term appear since the conclusions are based on one sample of data with no guarantee of immunity to scrutiny and critique. The currency returns appear to be skewed either positively or negatively and with more extreme tails than the normal distribution. The normality assumption appears to be violated and this is consistent with the results of Table 4.11, where it was shown that M-V approach produced unstable portfolios with a lot of uncertainty compared to its counterparts.



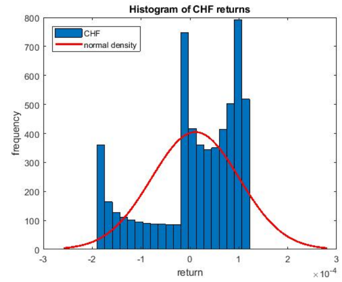
4.8.9: Histogram of EUR returns



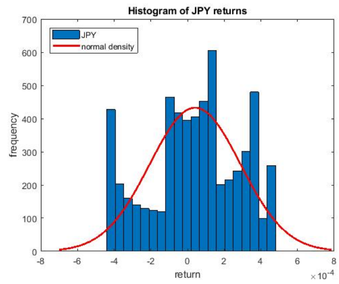
4.8.10: Histogram of AUD returns



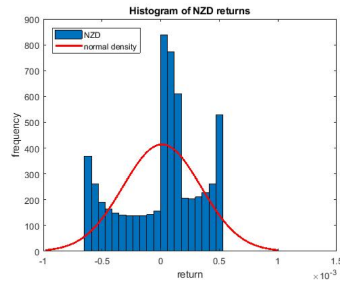
4.8.11: Histogram of CAD returns



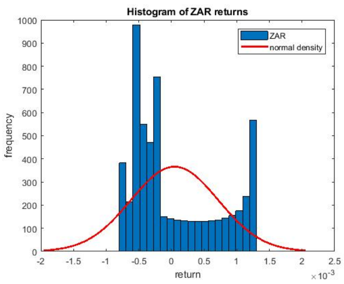
4.8.12: Histogram of CHF returns



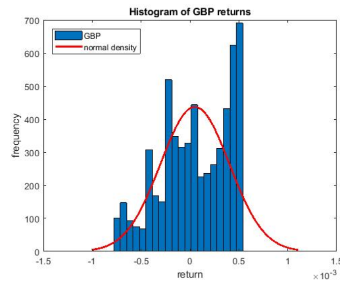
4.8.13: Histogram of JPY returns



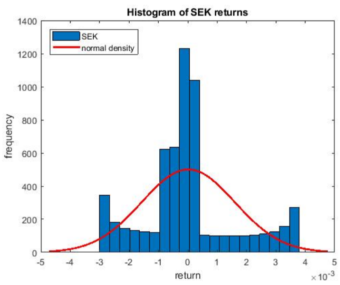
4.8.14: Histogram of NZD returns



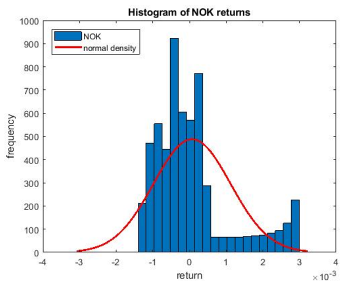
4.8.15: Histogram of ZAR returns



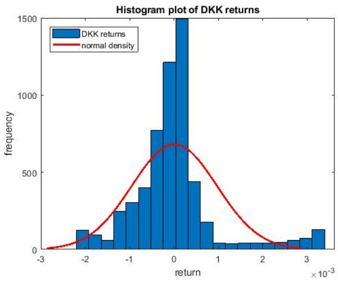
4.8.16: Histogram of GBP returns



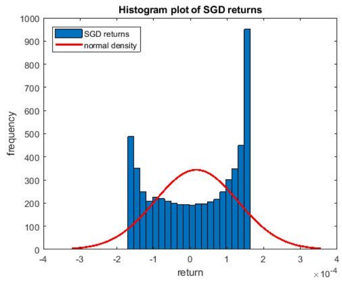
4.8.17: Histogram of SEK returns



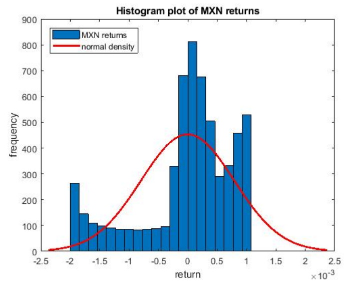
4.8.18: Histogram of NOK returns



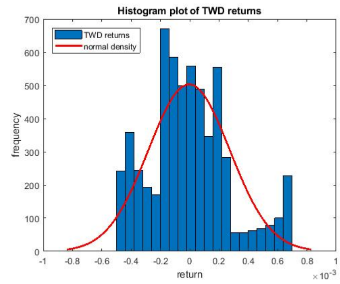
4.8.19: Histogram of DKK returns



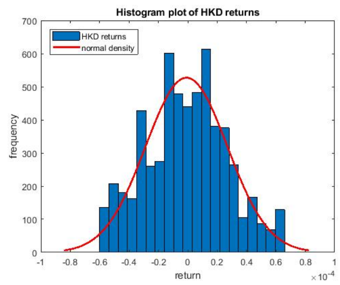
4.8.20: Histogram of SGD returns



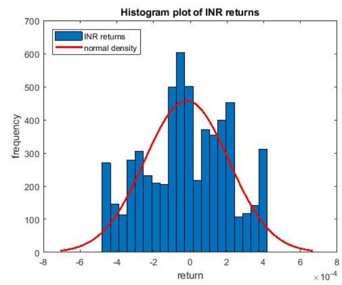
4.8.21: Histogram of MXN returns



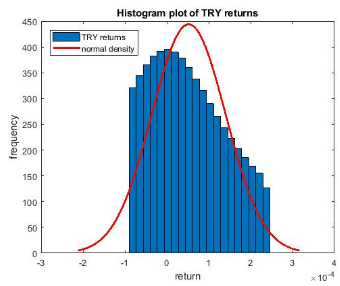
4.8.22: Histogram of TWD returns



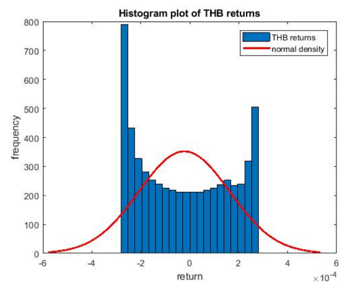
4.8.23: Histogram of HKD returns



4.8.24: Histogram of INR returns



4.8.25: Histogram of TRY returns



4.8.26: Histogram of THB returns

Figure 4.8: Histograms of currency returns

Chapter 5

Conclusion and Future Research

The objective of this study was to use forecast based optimisation models as a means of minimising risk in a highly volatile but profitable Forex market, additionaly these models provide an objective way of selecting the amount to risk per trade (portfolio weights), thus removing the subjectivity of how much a trader should invest per trade. These models offer the Forex trader with an additional trading strategy that can be used in the Forex market. This objective has been achieved and various properties of both models were compared. As far as the author can tell portfolio optimisation models in the Forex market have not been covered in the literature possibly due to the nature of the Forex data, but with the approach, notation and assumptions used in this study, it has been shown that portfolio optimisation is applicable to the Forex market. When security returns are not multivariate normally distributed as required by the M-V model, the SMAD and CVaR model are more suitable candidates, in particular the high-risk SMAD portfolio. Even if the returns are normally distributed, various authors in the literature such as Mansini et al. (2006) have claimed "advantages " of the SMAD and CVaR models over the M-V model. It is these "advantages" that were put into question in this study.

Additionally, the success or failure of a model can only be judged by how well or poor it performs against another. In this study the M-V model has been used as a benchmark since it paved the way for modern portfolio theory, and it has an intuitive appeal to it even though the assumptions (normality assumption) that make it work are often violated (Konno and Yamazaki, 1991). It is the opinion of the author that a Fourier series forecasting model applied to Forex data was to a certain degree applied successfully. Backtesting analysis was performed on the "extreme" portfolios of the three models, the low, medium and high risk in comparison with the equal weight portfolio which was used as a benchmark. The analysis showed that even though the MV model has smaller risk values compared to the other portfolios during the portfolio selection stage, however it is the SMAD portfolio that proved not only to reduce risk but also is profitable in the long run. Hence the equal-weight, M-V and CVaR portfolios prove to be inferior to the SMAD portfolio.It has been shown that an investor can use optimisation models to select profitable currencies and minimise (diversify) risk at the same time, this strategy is somewhat different from those traditionally used in the Forex market. This strategy can be classified as a passive strategy since it only informs the investor of profitable currencies to select and invest in. Unlike most trading strategies found on the Forex market, this strategy does not generate trading signals.

The limitations of these models when applied to Forex data is that on any given pair there must be a common quote currency which denotes the price (money) of the base currency. This limits the trader to only pairs of this nature, and pairs of this nature are not neccessarily the most liquid or profitable in the market. As an addition to optimisation models, once optimal portfolios are determined, it makes sense to have a signal generating trading strategy that will inform the trader of when to enter and exit a trade. As a direction for future research, a trading strategy that makes use of time series analysis techniques such as statistical arbitrage which is a mean-reverting trading strategy might be employed.

At the beginning of Chapter 3. Various assumptions were made in-order to make the models applicable. The rationale behind the choice of some of these assumptions was explained as the need to do so emerged. However some were not explained such as why only the long position is assumed on any given pair, well the reason is that one cannot really short sell in the Forex market since when one is buying (selling) a certain currency they are automatically selling (buying) the other. The investor's account balance is assumed to be in USD since this is consistent with the notion that the USD acts as the quote currency or money in any pair. However this assumption is not necessary but convenient. The study assumed that the investor is risk averse. This assumption is consistent with the main objective of this study which is to minimise risk. The study was limited to only pairs of the form X/USD as this made it easy to keep track of profits and losses made from the investments since the account balance is also assumed to be in USD. The account was assumed to be a standard account, meaning an investor needs a substantial amount to open a trade which results in substantial profits or losses. The reason behind this assumption is that it is difficult to make large profits with small capital in the Forex market unless the investor makes use of leverage which is borrowed money from the broker. Since the study assumes no leverage, it makes sense then that the trading account be a standard account.

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Data collection

OFX Website. [www.ofx.com/en-au/forex-news/historical-exchange-rates/]

Software used :

- Microsoft Excel
- MATLAB R2016b.

Appendix A

Table of Optimal Portfolios

Table A.1: Risks and returns of mv,smad and cvar portfolios

port #	ret	sd	smad	cvar	#-sd	#-smad	#-cvar	t-sd	t-smad	t-cvar
1	0.01032	0.01981	0.007953	0.003514	9.0	9.0	9.0	0.2323	2.306	0.5863
2	0.01353	0.01943	0.007711	0.002245	9.0	9.0	9.0	0.01599	0.912	0.5142
3	0.01673	0.01924	0.007523	0.001294	9.0	9.0	9.0	0.009456	0.9086	0.4539
4	0.01993	0.01923	0.007404	0.0006197	10.0	9.0	10.0	0.009698	0.8866	0.3745
5	0.02313	0.01926	0.007359	0.000151	10.0	9.0	10.0	0.01048	0.9234	0.299
6	0.02633	0.01936	0.007325	0	9.0	8.0	9.0	0.006018	0.946	0.1852
7	0.02954	0.01957	0.007336	0	9.0	8.0	7.0	0.006627	0.9207	0.1822
8	0.03274	0.01989	0.007395	0	8.0	8.0	7.0	0.006282	0.8945	0.1831
9	0.03594	0.02033	0.00751	0	8.0	8.0	6.0	0.006503	0.8954	0.1817
10	0.03914	0.02088	0.007669	0	8.0	8.0	5.0	0.006079	0.8872	0.334
11	0.04235	0.02153	0.007864	0	8.0	7.0	4.0	0.006061	0.919	0.2024
12	0.04555	0.02227	0.008097	0	8.0	7.0	4.0	0.007272	0.9285	0.4736
13	0.04875	0.0231	0.008349	0	8.0	7.0	5.0	0.007535	1.03	0.1793
14	0.05195	0.024	0.008617	0	8.0	7.0	4.0	0.006289	0.9425	0.4012
15	0.05516	0.02497	0.008905	0	7.0	7.0	4.0	0.006164	0.9813	0.2705
16	0.05836	0.026	0.009218	0	7.0	7.0	4.0	0.00629	0.9699	0.3079
17	0.06156	0.02709	0.009561	0	6.0	7.0	3.0	0.0064	1.025	0.1826
18	0.06476	0.02826	0.009938	0	6.0	7.0	3.0	0.006282	1.006	0.1837
19	0.06797	0.02948	0.01035	0	6.0	6.0	3.0	0.006273	1.006	0.1782
20	0.07117	0.03075	0.01079	0	6.0	6.0	2.0	0.006559	0.9892	0.1763
21	0.07437	0.03207	0.01124	0	5.0	6.0	2.0	0.006412	1.042	0.1771
22	0.07757	0.03343	0.0117	0	5.0	6.0	2.0	0.006616	0.9927	0.176
23	0.08077	0.03483	0.01217	0	5.0	5.0	2.0	0.006292	1.011	0.1764
24	0.08398	0.03627	0.01285	0	5.0	4.0	2.0	0.006699	0.9698	0.1766
25	0.08718	0.03779	0.014	0	4.0	5.0	2.0	0.007258	0.9461	0.1762
26	0.09038	0.03985	0.0152	0	4.0	5.0	2.0	0.006244	0.9526	0.1769
27	0.09358	0.04246	0.01648	0	4.0	4.0	2.0	0.007608	1.005	0.179
28	0.09679	0.04556	0.01805	0	4.0	4.0	2.0	0.006464	0.9976	0.914
29	0.09999	0.04908	0.01963	0	4.0	4.0	2.0	0.00636	0.9593	0.2747
30	0.1032	0.05296	0.02121	0	4.0	4.0	2.0	0.04695	0.9264	0.2269
31	0.1064	0.05711	0.02281	0	4.0	4.0	2.0	0.006129	0.9861	0.1793
32	0.1096	0.06148	0.02448	0	4.0	3.0	2.0	0.006619	0.9304	1.034
33	0.1128	0.06725	0.02635	0	2.0	3.0	2.0	0.04516	0.9509	0.2098
34	0.116	0.07982	0.03019	0	2.0	2.0	2.0	0.0152	0.9069	0.1835

35	0.1192	0.09758	0.03821	0	2.0	2.0	2.0	0.01095	0.972	0.1921
36	0.1224	0.1182	0.04718	0.001508	2.0	2.0	2.0	0.01091	0.9622	0.1926
37	0.1256	0.1405	0.05638	0.005302	2.0	2.0	3.0	0.01067	0.9941	0.2215
38	0.1288	0.1637	0.06571	0.01055	2.0	2.0	3.0	0.00632	1.061	0.2646
39	0.132	0.1875	0.0751	0.01664	2.0	2.0	3.0	0.006542	0.9195	0.3028
40	0.1352	0.2117	0.08454	0.02319	2.0	2.0	3.0	0.007233	0.9488	0.3058
41	0.1384	0.2362	0.09401	0.03005	2.0	2.0	3.0	0.006068	0.9491	0.3136
42	0.1416	0.2608	0.1035	0.03712	2.0	2.0	3.0	0.006877	0.96	0.3201
43	0.1448	0.2857	0.113	0.04435	2.0	2.0	3.0	0.006408	0.973	0.3246
44	0.148	0.3106	0.1225	0.05168	2.0	2.0	3.0	0.006202	0.9418	0.3263
45	0.1512	0.3356	0.1321	0.05911	2.0	2.0	3.0	0.007529	0.8952	0.3372
46.0	0.1544	0.3607	0.1416	0.0666	2.0	2.0	3.0	0.006434	0.9367	0.3725
47	0.1576	0.3859	0.1511	0.07416	2.0	2.0	3.0	0.006552	0.9156	0.3449
48	0.1608	0.4111	0.1607	0.08175	2.0	2.0	3.0	0.00635	0.9225	0.3336
49	0.164	0.4363	0.1702	0.08939	2.0	2.0	3.0	0.006693	0.9725	0.3723
50	0.1672	0.4616	0.1798	0.09706	2.0	2.0	3.0	0.006998	1.002	0.3507
51	0.1704	0.4869	0.1893	0.1048	2.0	2.0	3.0	0.00667	0.9513	0.3498
52	0.1736	0.5122	0.1989	0.1125	2.0	2.0	3.0	0.00695	0.9864	0.3583
53.0	0.1768	0.5375	0.2085	0.1202	2.0	2.0	3.0	0.006201	0.9716	0.3533
54	0.18	0.5629	0.218	0.128	2.0	2.0	3.0	0.006874	0.9612	0.3536
55	0.1832	0.5882	0.2276	0.1357	2.0	2.0	3.0	0.007055	0.9598	0.3484
56	0.1865	0.6136	0.2372	0.1435	2.0	2.0	3.0	0.006437	0.9397	0.3453
57	0.1897	0.639	0.2467	0.1513	2.0	2.0	3.0	0.007339	0.9216	0.3517
58	0.1929	0.6644	0.2563	0.1591	2.0	2.0	3.0	0.006624	0.8938	0.369
59.0	0.1961	0.6898	0.2659	0.167	2.0	2.0	3.0	0.006383	0.9856	0.3715
60	0.1993	0.7152	0.2754	0.1748	2.0	2.0	3.0	0.006817	0.9512	0.3669
61	0.2025	0.7407	0.285	0.1826	2.0	2.0	3.0	0.006773	1.099	0.3649
62	0.2057	0.7661	0.2946	0.1905	2.0	2.0	3.0	0.006847	0.9594	0.3672
63	0.2089	0.7915	0.3041	0.1983	2.0	2.0	3.0	0.006551	0.9425	0.3736
64	0.2121	0.817	0.3137	0.2062	2.0	2.0	3.0	0.006754	0.9501	0.3687
65	0.2153	0.8424	0.3233	0.214	2.0	2.0	3.0	0.01147	0.8843	0.3769
66	0.2185	0.8679	0.3328	0.2219	2.0	2.0	3.0	0.006397	0.9234	0.3747
67	0.2217	0.8933	0.3424	0.2298	2.0	2.0	3.0	0.006205	1.008	0.3684
68.	0.2249	0.9188	0.352	0.2377	2.0	2.0	3.0	0.006064	0.889	0.3771
69	0.2281	0.9442	0.3616	0.2455	2.0	2.0	3.0	0.006614	0.8837	0.3777
70	0.2313	0.9697	0.3711	0.2534	2.0	2.0	3.0	0.006534	0.899	0.3812
71	0.2345	0.9952	0.3807	0.2613	2.0	2.0	3.0	0.006932	0.9117	0.371
72	0.2377	1.021	0.3903	0.2692	2.0	2.0	3.0	0.00642	0.9523	0.3746
73	0.2409	1.046	0.3999	0.2771	2.0	2.0	3.0	0.006377	0.9511	0.3746
74	0.2441	1.072	0.4095	0.285	2.0	2.0	3.0	0.007199	0.8889	0.3844
75	0.2473	1.097	0.419	0.2929	2.0	2.0	3.0	0.006657	0.9193	0.3822
76	0.2505	1.123	0.4286	0.3008	2.0	2.0	3.0	0.008947	0.8531	0.3889
77	0.2537	1.148	0.4382	0.3087	2.0	2.0	3.0	0.006177	0.8323	0.3809
78	0.2569	1.173	0.4478	0.3166	2.0	2.0	3.0	0.006748	0.8924	0.5035
79	0.2601	1.199	0.4574	0.3246	2.0	2.0	3.0	0.006913	0.8817	0.3896
80	0.2633	1.224	0.4669	0.3325	2.0	2.0	3.0	0.00619	0.861	0.3856
81	0.2665	1.25	0.4765	0.3404	2.0	2.0	3.0	0.007191	0.9738	0.387
82	0.2697	1.275	0.4861	0.3483	2.0	2.0	3.0	0.006143	0.8603	0.3867
83	0.2729	1.301	0.4957	0.3562	2.0	2.0	3.0	0.006048	0.9367	0.3921
84	0.2761	1.326	0.5053	0.3641	2.0	2.0	3.0	0.008009	0.8883	0.3859
85	0.2793	1.352	0.5149	0.3721	2.0	2.0	3.0	0.006449	0.9726	0.3871
86	0.2825	1.377	0.5245	0.38	2.0	2.0	3.0	0.007103	0.9405	0.4673
87	0.2857	1.403	0.5341	0.3879	2.0	2.0	3.0	0.006486	0.9006	0.4311

88	0.2889	1.428	0.5437	0.3958	2.0	2.0	3.0	0.008929	0.9202	0.4085
89	0.2921	1.454	0.5533	0.4038	2.0	2.0	3.0	0.008056	0.9053	0.3842
90	0.2953	1.479	0.5629	0.4117	2.0	2.0	3.0	0.00659	0.9708	0.3854
91	0.2985	1.505	0.5725	0.4196	2.0	2.0	3.0	0.006388	0.8747	0.3929
92	0.3017	1.53	0.5821	0.4276	2.0	2.0	3.0	0.006286	0.8836	0.3963
93	0.3049	1.556	0.5917	0.4355	2.0	2.0	3.0	0.007056	0.8935	0.391
94	0.3081	1.581	0.6013	0.4435	2.0	2.0	3.0	0.007004	0.9041	0.3864
95	0.3113	1.607	0.6109	0.4514	2.0	2.0	3.0	0.006771	0.85	0.3948
96	0.3145	1.632	0.6205	0.4594	2.0	2.0	3.0	0.006302	0.9111	0.3935
97	0.3177	1.658	0.6301	0.4673	2.0	2.0	3.0	0.006108	0.8909	0.4005
98	0.321	1.683	0.6397	0.4753	2.0	2.0	3.0	0.007328	0.9274	0.3991
99	0.3242	1.709	0.6493	0.4832	2.0	2.0	3.0	0.009013	0.8489	0.3988
100	0.3274	1.734	0.6589	0.4912	1.0	1.0	1.0	0.009886	0.8019	0.3956
average	0.1688	0.6103	0.2336	0.1556	3.4600	3.4300	3.3900	0.0102	0.9514	0.3418

Appendix B

Skewness and Kurtosis

Table B.1: Skewnes and kurtosis

Currency	**Skewness**	**Kurtosis**
EUR	-0.8445	2.345
AUD	1.308	3.92
CAD	-0.1393	1.49
CHF	-0.8583	2.698
JPY	-0.2607	2.257
NZD	-0.416	2.275
ZAR	0.683	1.993
GBP	-0.3839	2.126
SEK	0.5014	3.28
NOK	1.332	4.127
DKK	1.285	6.303
SGD	-0.2077	1.562
MXN	-1.018	3.349
TWD	0.5465	3.153
HKD	0.08403	2.537
INR	-0.0712	2.283
TRY	0.3344	2.11
THB	0.146	1.591