Advances in Operator Theory (2024) 9:87
https://doi.org/10.1007/s43036-024-00391-0

Tusi Mathematical Research Group

ORIGINAL PAPER

The *c*-numerical range of a quaternion skew-Hermitian matrix is convex

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Received: 22 July 2024 / Accepted: 29 September 2024 / Published online: 14 October 2024

Abstract
We show that the c-numerical range of a non-scalar skew-Hermitian quaternion matrix is convex. In fact, included in our result is that the c-numerical range of a skew-Hermitian matrix is a rotation invariant subset of the quaternions with zero real parts.

Keywords Quaternion c-numerical range · Convexity · Skew-Hermitian

Mathematics Subject Classification 15A60 · 15A63 · 15B33 · 47A12

1 Introduction

The quaternionic numerical range was introduced by Kippenhahn [10] and several basic properties were derived in [2, 3, 10, 14]. More recent papers include [1, 5, 12, 15]. The classical C-numerical range goes back to [7], which corresponds to the c-numerical range when the matrix C is Hermitian. A comprehensive account on C-numerical ranges can be found in the paper by Li [11]. In [16] it was shown that the c-numerical range of a complex matrix is always convex. It is easily seen that this does not extend to the case of quaternions in general (see, e.g., [8]). In this paper we will show that for skew-Hermitian quaternion matrices of size at least 2×2 the c-numerical

Dedicated to Ilya Spitkovsky, a wonderful colleague and a great friend.

Communicated by Natalia Bebiano.

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range is always convex. This question of convexity was earlier pursued in the paper by Aghamollaei, et al. [1], where the k-numerical range (which is a special case of the c-numerical range) of a quaternion matrix was studied. Among the results in [1] it was shown that for an even size skew-Hermitian quaternion matrix its k-numerical range is convex. It was also conjectured there that the same is true for non-scalar odd size matrices. Our main result resolves this conjecture in the positive. We present the main result in the next section.

2 The main result

For notation regarding basic quaternion linear algebra, we refer to the monograph by Rodman [13]. The real skew-field of quaternions, denoted by $\mathbb{H}$, contains elements of the form:

$$x = x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k},$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the multiplication of quaternions follows the following rules:

$$\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1, \quad \mathsf{ij} = -\mathsf{ji} = \mathsf{k}, \quad \mathsf{jk} = -\mathsf{kj} = \mathsf{i}, \quad \mathsf{ki} = -\mathsf{ik} = \mathsf{j}.$$

The real number x_0 is referred to as the *real part* of x, and we write $x_0 = \mathrm{Re}\, x$. Furthermore, $x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k}$ is called the *vector part* of x and we denote the set of quaternions that only have a vector part by $\mathbb{P} := \{x \in \mathbb{H} : \mathrm{Re}\, x = 0\}$. We denote the *conjugate quaternion* of x by $\overline{x} = x^* = x_0 - x_1\mathsf{i} - x_2\mathsf{j} - x_3\mathsf{k}$ and the *norm* of $x \in \mathbb{H}$ by $||x|| = \sqrt{x^*x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \in \mathbb{R}$. For a quaternion matrix A, let $\overline{A}$ denote the matrix where each entry is the conjugate of the corresponding entry in A, and A^* the transpose of $\overline{A}$, as usual. Also, recall that an $n \times n$ matrix A is *unitary* if $AA^* = A^*A = I_n$. We use $\mathrm{Tr}(A)$ to denote the trace of the matrix A, i.e., the sum of the diagonal entries of A. Finally, it is useful to note that $\mathsf{j}^* z \mathsf{j} = \overline{z}$ for any complex number z.

A quaternion λ is called a *right eigenvalue* of A if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$. For every right eigenvalue of A, say λ, the quaternion $a\lambda a^{-1}$, $0 \neq a \in \mathbb{H}$, is also a right eigenvalue of A. Moreover, the matrix A has exactly n right eigenvalues which are complex numbers with nonnegative imaginary parts, see for example [17, Theorem 5.4]. These complex eigenvalues are called the *standard right eigenvalues* of A.

We denote the *numerical range* of a quaternion matrix $A \in \mathbb{H}^{n\times n}$ by

$$W_{\mathbb{H}}(A) = \left\{x^*Ax : ||x|| = 1, x \in \mathbb{H}^n\right\} \subset \mathbb{H},$$

where $\|(x_j)_{j=1}^n\| = \left(\sum_{j=1}^n \|x_j\|^2\right)^{\frac{1}{2}}$. Note that if $q \in W_{\mathbb{H}}(A)$, and $u \in \mathbb{H}$ where $||u|| = 1$, then $u^*qu \in W_{\mathbb{H}}(A)$. In general, $u^*qu \neq q$. For basic information on the quaternionic numerical range, see for instance [13, Section 3.3].

Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. The *c-numerical range* of a quaternion matrix A is defined as

$$W_{c,\mathbb{H}}(A) = \left\{ \sum_{i=1}^{n} c_i u_i^* A u_i : \{u_1, \ldots, u_n\} \text{ is an orthonormal set in } \mathbb{H}^n \right\}$$
$$= \left\{ \mathrm{Tr}(\mathrm{diag}(c_j)\, U^* A U) : U \in \mathbb{H}^{n \times n} \text{ unitary} \right\}.$$

Note that if $V \in \mathbb{H}^{n\times n}$ is unitary, then $W_{c,\mathbb{H}}(A) = W_{c,\mathbb{H}}(V^*AV)$. Consequently, if A is normal, we may assume that A is diagonal; see [6]. Basic properties of the c-numerical range of quaternion matrices have been developed in [8].

When $c = (1, 0, \ldots, 0)$, the c-numerical range reduces to the classical numerical range. When $c = \left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0\right)$, where $\frac{1}{k}$ appears k times with k a positive integer, it is the k-numerical range as defined in [1]. When A is Hermitian (i.e., $A = A^*$), then clearly $W_{c,\mathbb{H}}(A) \subseteq \mathbb{R}$ (as $\overline{\mathrm{Tr}CU^*AU} = \mathrm{Tr}U^*A^*UC = \mathrm{Tr}CU^*AU$ where in the last step we use that $C = \mathrm{diag}(c_j)$ is real and $A = A^*$), while for a skew-Hermitian A (i.e., $A = -A^*$) we have $W_{c,\mathbb{H}}(A) \subseteq \mathbb{P}$; see Proposition 2.3 below.

For a skew-Hermitian quaternion matrix A of size $n \times n$ with $n \geq 2$, it follows from [2, Theorem 4] that its numerical range $W_{\mathbb{H}}(A)$ is convex. More recently, in [1] there are several results on the k-numerical range of skew-Hermitian matrices, including its convexity in the even size case and a conjecture regarding the odd size case. In our main result below we show that the c-numerical range of any skew-Hermitian quaternion matrix A of size $n \geq 2$ is convex, which also resolves the conjecture in the last section of [1].

Theorem 2.1 *Let $n \geq 2$ and $c = (c_i)_{i=1}^n \in \mathbb{R}^n$. If $A \in \mathbb{H}^{n\times n}$ is skew-Hermitian, then $W_{c,\mathbb{H}}(A)$ is convex. In fact, if the standard right eigenvalues of A are $\mathrm{i}b_1, \ldots, \mathrm{i}b_n$, then we have*

$$W_{c,\mathbb{H}}(A) := \{x \in \mathbb{P} : \|x\| \leq \sum_{j=1}^{n} |c_j b_j|\},$$

where we order b_i, c_i so that $|c_1| \geq \cdots \geq |c_n|$ and $|b_1| \geq \cdots \geq |b_n|$.

Remark 2.2 Note that if $A = [a]$, $0 \neq a \in \mathbb{P}$, is a nonzero 1×1 skew-Hermitian matrix, then its numerical range consist of all $x \in \mathbb{P}$ with $\|x\| = \|a\|$, so then its numerical range is not convex.

We will first develop some auxiliary results before proving this main result.

Proposition 2.3 *Let $c \in \mathbb{R}^n$ and $A \in \mathbb{H}^{n\times n}$. If A is skew-Hermitian, then $W_{c,\mathbb{H}}(A)$ is a rotation invariant subset of $\mathbb{P}$ (i.e., if $x \in W_{c,\mathbb{H}}(A) \cap \mathbb{P}$ and $y \in \mathbb{P}$ satisfies $\|x\| = \|y\|$, then $y \in W_{c,\mathbb{H}}(A)$).*

If $c \notin \mathrm{span}\{(1, \ldots, 1)\}$ and $\sum_{j=1}^n c_j \neq 0$, then $W_{c,\mathbb{H}}(A) \subseteq \mathbb{P}$ implies that A is skew-Hermitian.

Proof For the first part, let A be skew-Hermitian. If U is unitary, we have that

$$\begin{aligned}&\overline{\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*AU)} = \mathrm{Tr}(\mathrm{diag}(c_j)\,U^*AU)^* = \mathrm{Tr}(U^*A^*U\mathrm{diag}(c_j))\\&= \mathrm{Tr}(\mathrm{diag}(c_j)\,U^*(-A)U) = -\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*AU),\end{aligned}$$

where we used that real numbers commute with quaternions. Thus the real part of $\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*AU)$ is 0. This shows that $W_{c,\mathbb{H}}(A) \subset \mathbb{P}$. Recall that if $a, b \in \mathbb{P}$ and $\|a\| = \|b\|$, then there exists a $\gamma \in \mathbb{H}$ so that $a = \gamma b \gamma^{-1}$. By [8, Theorem 2.4(a)] we have that if $x \in W_{c,\mathbb{H}}(A)$, then $\gamma x \gamma^{-1} \in W_{c,\mathbb{H}}(A)$ for all $\gamma \in \mathbb{H}$. This proves the first part.

For the second part, suppose that $W_{c,\mathbb{H}}(A) \subseteq \mathbb{P}$. Let us write $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) =: H + S$. We need to show that $H = 0$, from which it follows that A is skew-Hermitian. Suppose to the contrary that $H \neq 0$, and let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of H. We claim that there exists a unitary U so that $\mathrm{Re}\left(\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*HU)\right) \in \mathbb{R} \setminus \{0\}$. Note that we may diagonalize H, and thus we may assume without loss of generality that $H = \mathrm{diag}(\lambda_j)$. Let us also order c so that $c_1 \geq \cdots \geq c_n$. We recall from [11, Result 5.1] that

$$\left\{\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*\mathrm{diag}(\lambda_j)U) : U \in \mathbb{C}^{n\times n} \text{ unitary}\right\} = [\alpha, \beta],$$

where

$$\alpha = c_1\lambda_n + \cdots + c_n\lambda_1, \quad \beta = c_1\lambda_1 + \cdots + c_n\lambda_n.$$

We now claim that $[\alpha, \beta]$ contains a non-zero number, which is equivalent to not both α and β being 0. If $\lambda_1 = \cdots = \lambda_n$, we are done as we must have that $\lambda_1 \neq 0$ (since $H \neq 0$), and thus $\alpha = \lambda_1 \left(\sum_{j=1}^n c_j\right) \neq 0$. When λ contains two different values, we have that $\sum_{j=1}^n c_j\lambda_{\sigma(j)}$ has at least two different values when we let σ range through all permutations of $\{1, \ldots, n\}$; here we used that $c \notin \mathrm{span}\{(1, \ldots, 1)\}$. But then we have that the interval $[\alpha, \beta]$ contains two different values (as $\sum_{j=1}^n c_j\lambda_{\sigma(j)} \in [\alpha, \beta]$), proving that $[\alpha, \beta]$ contains a nonzero real number y. Thus there exists a complex unitary $U \in \mathbb{C}^{n\times n}$ so that $\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*\mathrm{diag}(\lambda_j)U) = y$. Since $\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*SU) =: z \in \mathbb{P}$, we get that $\mathrm{Tr}(\mathrm{diag}(c_j)\,U^*AU) = y + z \notin \mathbb{P}$. Thus $H \neq 0$ leads to a contradiction and therefore $H = 0$. □

It follows from Proposition 2.3 that in order to prove convexity of $W_{c,\mathbb{H}}(A)$ for A an $n \times n$ skew-Hermitian matrix ($n \geq 2$), it suffices to show that $0 \in W_{c,\mathbb{H}}(A)$.

To illustrate the ideas in the upcoming proof of Theorem 2.1 we first provide an example. We will use the Schur–Horn theorem (see, e.g., [9, Chapter 4]), which states that the diagonal entries of a Hermitian matrix are majorized by its eigenvalues. Recall that $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *majorizes* $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$ if and only if

$$d \in \mathrm{conv}\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) : \sigma \in S_n\},$$

where $\mathrm{conv}\{X\}$ is the convex hull of a set X and S_n is the symmetric group of permutations on $\{1, \ldots, n\}$. Other characterizations of majorization may be found, for instance, in [9, Chapter 4] and [4, Chapter II]. We let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{R}^n$.

Example 2.4 Let

$$A = \begin{bmatrix} \mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 2\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 4\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 5\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 11\mathrm{i} \end{bmatrix}$$

and $c = (1, 1, 1, 0, 0)$. We would like to find a 5×5 unitary matrix V so that

$$\mathrm{Tr}(\mathrm{diag}(c_j)\, V^* A V) = 0.$$

This means that the vector c is orthogonal to the vector of diagonal entries of V^*AV. We will also use that $\mathrm{j}^*\mathrm{i}\mathrm{j} = -\mathrm{i}$, which provides us with a way to change signs for both the c_i's and the diagonal entries of A. To reach our objective, we will find a diagonal real unitary $\widetilde{J}$ and a vector v majorized by $(1, 2, 4, 5, 11)$ so that $v \perp \widetilde{J}c$. For instance, we take

$$\widetilde{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 1 \\ 2 \end{bmatrix},$$

giving that

$$\left\langle v, \widetilde{J}c \right\rangle = 4 + 6 - 10 = 0.$$

We next find a real unitary U so that the diagonal of U^*AU equals $\mathrm{i}v$. The existence of such a U is ensured by the Schur-Horn theorem. Here we may take

$$U = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & \sqrt{\frac{5}{6}} & \sqrt{\frac{1}{6}} \\ 0 & -\sqrt{\frac{1}{6}} & \sqrt{\frac{5}{6}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

which is unitary and the diagonal of U^*AU equals $\mathrm{i}v$. Next, we let

$$\widetilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{j} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is obtained from $\widetilde{J}$ by replacing -1 by j. Now $V = U\widetilde{Q}$ has the property that $\mathrm{Tr}(\mathrm{diag}(c_j)\, V^*AV) = 0$. □

Lemma 2.5 *Let $b, c \in \mathbb{R}^n$. Then there exist diagonal matrices $J, \widetilde{J} \in \mathbb{R}^{n\times n}$ with ± 1 on the diagonal and permutations $\sigma_1, \sigma_2 \in S_n$ so that*

$$\left\langle P_{\sigma_1}(Jb), \widetilde{J}c\right\rangle \geq 0 \quad \textit{and} \quad \left\langle P_{\sigma_2}(Jb), \widetilde{J}c\right\rangle \leq 0.$$

Here $P_{\sigma_i}(x)$ rearranges the coordinates of any tuple $x \in \mathbb{R}^n$ according to the permutation $\sigma_i \in S_n$.

Proof In the case where n is even, say $n = 2m$, choose J and $\widetilde{J}$ such that $(\widetilde{J}c)_i \geq 0$, $(Jb)_i \geq 0$, for $i = 1, \ldots, m$, and $(\widetilde{J}c)_i \leq 0$, $(Jb)_i \leq 0$, for $i = m+1, \ldots, n$. Let σ_1 be the identity permutation, then

$$\left\langle P_{\sigma_1}(Jb), \widetilde{J}c\right\rangle = \left\langle Jb, \widetilde{J}c\right\rangle \geq 0.$$

Let $\sigma_2 = (1\ m{+}1)(2\ m{+}2)\cdots(m\ 2m)$, where we use cycle notation for permutations. Then

$$\left\langle P_{\sigma_2}(Jb), \widetilde{J}c\right\rangle \leq 0.$$

For the odd case, let $n = 2m + 1$. Without loss of generality we may assume for the entries of c and b that

$$|c_1| \geq \cdots \geq |c_n| \quad \text{and} \quad |b_1| \geq \cdots \geq |b_n|.$$

Choose

$$J = \mathrm{diag}\,(-1, I_m, -I_m)\,\mathrm{diag}\,(\mathrm{sign}(b_1), \ldots, \mathrm{sign}(b_n))$$

and

$$\widetilde{J} = \mathrm{diag}\,(I_m, -I_{m+1})\,\mathrm{diag}\,(\mathrm{sign}(c_1), \ldots, \mathrm{sign}(c_n))\,.$$

Here we set $\mathrm{sign}(0) = 1$.Then

$$\begin{aligned}&\left(\widetilde{J}c\right)_1 (Jb)_2 + \cdots + \left(\widetilde{J}c\right)_m (Jb)_{m+1} + \left(\widetilde{J}c\right)_{m+1} (Jb)_{m+2}\\ &\quad + \cdots + \left(\widetilde{J}c\right)_{2m} (Jb)_{2m+1} + \left(\widetilde{J}c\right)_{2m+1} (Jb)_1 \geq 0.\end{aligned}$$

For the other permutation, note that

$$\underbrace{\left(\widetilde{J}c\right)_1 (Jb)_1 + \left(\widetilde{J}c\right)_{2m+1} (Jb)_{2m+1}}_{\leq 0} + \underbrace{\left(\widetilde{J}c\right)_{m+1} (Jb)_2 + \cdots + \left(\widetilde{J}c\right)_{2m} (Jb)_{m+1}}_{\leq 0}$$
$$+ \underbrace{\left(\widetilde{J}c\right)_2 (Jb)_{m+2} + \cdots + \left(\widetilde{J}c\right)_m (Jb)_{2m}}_{\leq 0} \leq 0.$$

Then we have opposite signs in the two dot products as required. □

As usual, for $x \in \mathbb{R}^n$ we let

$$x^{\perp} = \{y \in \mathbb{R}^n : y \perp x\} = \{y \in \mathbb{R}^n : \langle x, y\rangle = 0\}.$$

Corollary 2.6 *Let $b, c \in \mathbb{R}^n$. Then there exist diagonal unitary matrices $J, \widetilde{J} \in \mathbb{R}^{n\times n}$ such that*

$$conv\,\{P_\sigma(Jb) : \sigma \in S_n\} \cap \left(\widetilde{J}c\right)^{\perp} \neq \emptyset. \tag{2.1}$$

Proof Using Lemma 2.5 we know we can find matrices $J, \widetilde{J} \in \mathbb{R}^{n\times n}$ and permutations $\sigma_1, \sigma_2 \in S_n$ such that

$$\left\langle P_{\sigma_1}(Jb), \widetilde{J}c\right\rangle \geq 0 \quad \text{and} \quad \left\langle P_{\sigma_2}(Jb), \widetilde{J}c\right\rangle \leq 0.$$

Then there exists an $x \in \operatorname{conv}\left\{P_{\sigma_i}(Jb) : i = 1, 2\right\} \subseteq \operatorname{conv}\{P_\sigma(Jb) : \sigma \in S_n\}$ so that $\left\langle x, \widetilde{J}c\right\rangle = 0$, yielding (2.1). □

We will use the following notation in the proof. Let Δ be the map $\mathbb{H}^{n\times n} \mapsto \mathbb{H}^n$ that extracts the diagonal entries from a matrix; that is, $\Delta(M) = (M_{jj})_{j=1}^n$.

Proof of Theorem 2.1 Let A be skew-Hermitian with eigenvalues $b_1\mathsf{i}, \ldots, b_n\mathsf{i}$, where $b_j \geq 0$ for all j. Without loss of generality, we may assume that b_i, c_i are ordered so that $|c_1| \geq \cdots \geq |c_n|$ and $|b_1| \geq \cdots \geq |b_n|$. Let $D = \operatorname{diag}(b_j)_{j=1}^n$. Due to the invariance of the c-numerical range under unitary similarities, we may assume that $A = \mathsf{i}D$. Choose $J, \widetilde{J}$ such that (2.1) holds. Take a vector v so that

$$v \in \operatorname{conv}\left\{P_{\sigma_i}(Jb) : \sigma_i \in S_n\right\} \cap \left(\widetilde{J}c\right)^{\perp},$$

the existence of which is guaranteed by Corollary 2.6. It follows from the Schur-Horn theorem that there exists a unitary matrix $U \in \mathbb{R}^{n\times n}$ such that

$$\Delta\left(U^*JDU\right) = v.$$

Let Q be a diagonal matrix where $Q_{rr} = 1$ if $J_{rr} = 1$ and $Q_{rr} = \mathsf{j}$ if $J_{rr} = -1$. Similarly, let $\widetilde{Q}$ be a diagonal matrix where $\widetilde{Q}_{rr} = 1$ if $\widetilde{J}_{rr} = 1$ and $\widetilde{Q}_{rr} = \mathsf{j}$ if $\widetilde{J}_{rr} = -1$. Then $Q^*AQ = \mathsf{i}JD$ and

$$\operatorname{Tr}\left(\operatorname{diag}\left(c_j\right)\widetilde{Q}^*U^*Q^*AQU\widetilde{Q}\right) = \operatorname{Tr}\left(\operatorname{diag}\left(c_j\right)\widetilde{Q}^*\mathsf{i}\operatorname{diag}\left(v_j\right)\widetilde{Q}\right)$$

$$= \mathrm{i}\sum_{r=1}^{n} \left(\widetilde{J}c\right)_r v_r = \mathrm{i}\left\langle v, \widetilde{J}c\right\rangle = 0.$$

Thus $0 \in W_{c,\mathbb{H}}(A)$ and therefore by Proposition 2.3 the convexity of $W_{c,\mathbb{H}}(A)$ follows.

Finally, notice that one may choose $\widetilde{Q}$, Q and U a permutation, so that

$$\operatorname{Tr}\left(\operatorname{diag}\left(c_j\right)\widetilde{Q}^* U^* Q^* A Q U \widetilde{Q}\right) = \mathrm{i}\sum_{r=1}^{n} |c_r b_r|,$$

showing that $\mathrm{i}\sum_{r=1}^{n} |c_r b_r| \in W_{c,\mathbb{H}}(A)$.

Finally, let $x \in W_{c,\mathbb{H}}(A)$. We need to show that $\|x\| \leq \sum_{r=1}^{n} |c_r b_r|$. For this, let $V = (v_{kl})_{k,l=1}^{n} \in \mathbb{H}^{n\times n}$ be unitary so that $x = \operatorname{Tr}(\operatorname{diag}(c_j)V^* A V)$ and let $(d_k)_{k=1}^{n}$ denote the diagonal entries of $V^* A V$. Then, for $k = 1, \ldots, n$,

$$\|d_k\| = \left\|\sum_{l=1}^{n} v_{lk}^* \mathrm{i} b_l v_{lk}\right\| \leq \sum_{l=1}^{n} \|v_{lk}^*\||b_l|\|v_{lk}\| = \sum_{l=1}^{n} \|v_{lk}\|^2 |b_l| =: \beta_k.$$

Consequently,

$$(\beta_k)_{k=1}^{n} = (\|v_{lk}\|^2)_{k,l=1}^{n} (|b_l|)_{l=1}^{n},$$

where the matrix $(\|v_{lk}\|^2)_{k,l=1}^{n}$ is doubly stochastic (since V is unitary). Thus the vector $(\beta_k)_{k=1}^{n}$ is majorized by the vector $(|b_l|)_{l=1}^{n}$. But then (e.g., by using [4, Section II.4]) it follows that

$$\|x\| = \left\|\sum_{k=1}^{n} c_k d_k\right\| \leq \sum_{k=1}^{n} |c_k|\|d_k\| \leq \sum_{k=1}^{n} |c_k|\beta_k \leq \sum_{k=1}^{n} |c_k||b_k|.$$

□

Acknowledgements We gratefully acknowledge funding support from the Knowledge, Interchange and Collaboration (KIC) program of South Africa's National Research Foundation (NRF). We thank the referee for carefully reading our manuscript and their helpful feedback. The research of HJW is supported by National Science Foundation grant DMS 2000037.

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