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THE GLOBAL GALERKIN FINITE ELEMENT METHOD AS APPLIED TO THE 1D BEAM EQUATION: A SYSTEMS APPROACH

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Abstract

The paper focuses on the finite element solution of the fourth-order differential for an Euler-Bernoulli beam. In this approach the point point loads and concentrated moments are treated in a natural way as part of the system of external forces and are included in the differential equation during the definition of the problem. The Galerkin formulation is then developed over the entire domain before the domain is discretised into a collection of elements to facilitate the evaluation of the integrals to obtain the coefficients of the global set of algebraic equations. This results in the Global Galerkin Finite Element (GGFEM) formulation. The GGFEM can be seen as a systems or holistic approach to the finite element solution of the problem.

Introduction

A number of approaches have been developed over time for the finite element solution of the differential equations encountered in engineering and science. In the traditional Galerkin approach ([1],[2],[3]) taught to many of the first time students of the finite element method, the method starts with the description of the problem consisting amongst others of the identification of the differential equation(s), parameters, domain and boundary conditions. The domain is then discretised into a collection of contiguous non-overlapping elements based on the assumption that the differential equation will be valid over any subdomain if it is valid over the entire domain. The finite element formulation over an element is then developed. This consists of the derivation of the weak formulation, the selection of the approximate solution and the associated interpolation functions over an element and the formation of the resulting system of element equations. This leads to the appearance of boundary fluxes or forces associated with each element. The global set of equations is then formed by evaluating the relevant integrals over all the elements and subsequently assembling / collecting all the element equations. After the assembly of the global equations a decision must then be made about the treatment of the inter-element fluxes (or forces) at the interfaces between adjacent elements. Traditionally it is assumed that the fluxes (forces) cancel, unless a point source (point load or concentrated moment) occurs at the interface. In such a case the point source (point load or concentrated moment) is assumed to represent the jump in flux (forces) or differences between the element fluxes (forces). After the application of the boundary conditions, the global set of equations is solved to obtain the required solution for the problem.

An alternative approach, namely the Global Galerkin Finite Element Method (GGFEM), can be followed. Aspects of the GGFEM approach can amongst others be found in Baker [4] and Gresho & Lee [5]. However, they did not develop it to its fullest extent. It is the purpose of this paper to develop the method fully using the fourth-order differential equation for an Euler-Bernoulli beam as an example.

Governing equation

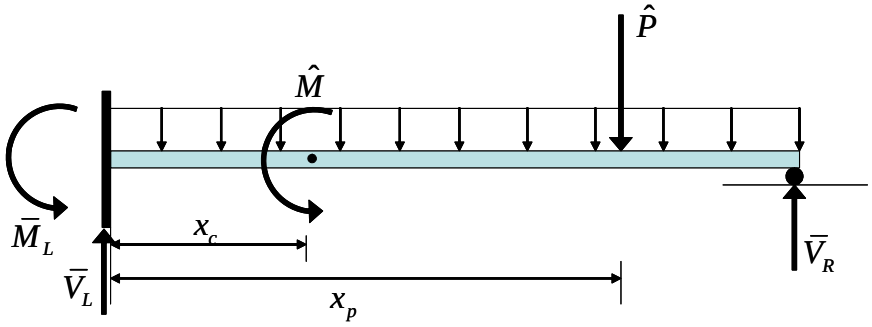

Figure 1 : Supported cantilever Euler-Bernoulli beam

Consider the fourth-order differential equation for the displacement $w(x)$ of a supported cantilever (Euler-Bernoulli) beam [6], shown in Figure 1,

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right] = f(x) + \hat{p}(x) + \hat{m}(x) \quad \text{for} \quad \Omega : 0 < x < L \tag{1}$$

subject to the boundary conditions $w(0) = dw/dx\big|_{x=0} = 0$, $w(L) = 0$ and $\bar{M}_R = M(L) = 0$. A distributed load $f(x)$, a point load $\hat{p}(x) = \hat{P}\delta(x - x_p)$ at the point $x = x_p$ and a concentrated moment $\hat{m}(x) = -\hat{M}\,D(x - x_c)$ at the point $x = x_c$ act on the beam. $EI(x)$ is the bending stiffness of the beam, $\delta(x - x_p)$ a Dirac delta function with $0 < x_p < L$, $D(x - x_c)$ a unit doublet function with $0 < x_c < L$ and Ω the domain over which the solution is sought. The bending moment and the shear force are defined respectively as [6]

$$M(x) = EI(x)\frac{d^2w}{dx^2} \qquad \text{and} \qquad V(x) = \frac{d}{dx}\left[EI(x)\frac{d^2w}{dx^2}\right] \tag{2}$$

Weak formulation and approximate solution

Write Eq. (1) in residual form, multiply it with a suitable weighting function $v(x)$ and integrate it over the domain. This results in:

$$0 = \int_0^L v(x)\left\{\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right] - f(x) - \hat{p}(x) - \hat{m}(x)\right\}dx \tag{3}$$

We then integrate the product of the weighting function and the fourth-order term twice by parts and the product of the weighting function and the concentrated moment once by parts. Rearranging the result then gives:

$$\begin{aligned}\int_0^L EI(x)\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}dx &= \int_0^L vf(x)+\int_0^L v\hat{P}\delta(x-x_p)dx \\ &\quad -\left[v\hat{M}\delta(x-x_c)\right]\Big|_0^L+\int_0^L\frac{dv}{dx}\hat{M}\delta(x-x_c)dx \\ &\quad -\left\{v(L)\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\Big|_{x=L}-v(0)\frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\Big|_{x=0}\right\} \\ &\quad +\left\{\left(\frac{dv}{dx}\left[EI\frac{d^2w}{dx^2}\right]\right)\Big|_{x=L}-\left(\frac{dv}{dx}\left[EI\frac{d^2w}{dx^2}\right]\right)\Big|_{x=0}\right\}\end{aligned} \tag{4}$$

Recognising that $\delta(x-x_c)=0$ for $x=0$ and $x=L$ and incorporating the definitions (2) of the bending moment and the shear in (4) leads to the weak formulation defined over the entire domain,

$$\begin{aligned}\int_0^L EI(x)\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}dx &= \int_0^L vf(x)dx+\int_0^L v\hat{P}\delta(x-x_p)dx+\int_0^L\frac{dv}{dx}\hat{M}\delta(x-x_c)dx+ \\ &\quad v(0)\bar{V}_L+v(L)\bar{V}_R+\frac{dv}{dx}\Big|_{x=0}\bar{M}_L+\frac{dv}{dx}\Big|_{x=L}\bar{M}_R\end{aligned} \tag{5}$$

where $\bar{V}_L=V_L$, $\bar{M}_L=-M_L$, $\bar{V}_R=-V_R$ and $\bar{M}_R=M_R$. V_L, M_L, V_R and M_R are vertical forces and moments on the left hand boundary and right hand boundary of the domain according the beam convention, whilst $\bar{V}_L$, $\bar{M}_L$, $\bar{V}_R$ and $\bar{M}_R$ are the vertical forces and moments at the left hand boundary and the right hand boundary according the right hand rule. Note that the only boundary terms in the formulation are those associated with the end points of the domain. A study of the forms in which the weighting function appears in the boundary terms reveals that we have two primary variables associated with the problem, namely the vertical displacement w and the slope dw/dx of the beam. In the finite element approach we usually seek an approximate solution for the problem in the form of an interpolant. In the case of Eq. (1) the approximate solution for the vertical displacement is defined as

$$w^N(x)=\sum_{J=1}^N\left[w_J\phi_J^o(x)+\left(\frac{dw}{dx}\right)_J\phi_J^1(x)\right]=\sum_{J=1}^M W_J\Phi_J(x) \tag{6}$$

where N is the number of selected nodal points x_J at which we wish to obtain the solution and $M=2N$ the number of associated primary variables. $\Phi_{2J-1}(x)=\phi_J^o(x)$ and $\Phi_{2J}(x)=\phi_J^1(x)$ are suitable (globally defined) interpolation functions. It is informative to note that based on Eq. (6) the approximate description for the slope of the beam is given by

$$\frac{d}{dx}\left[w^N(x)\right]=\sum_{J=1}^N\left\{w_J\frac{d}{dx}\left[\phi_J^o(x)\right]+\left(\frac{dw}{dx}\right)_J\frac{d}{dx}\left[\phi_J^1(x)\right]\right\}=\sum_{J=1}^M W_J\frac{d}{dx}\left[\Phi_J(x)\right] \tag{7}$$

From Eqs. (6) and (7) it can be deduced that the interpolation functions must have the following properties:

$$\Phi_{2I-1}(x_J)=\phi_I^o(x_J) \quad =\begin{cases}1 & \text{for} \quad I=J\\ 0 & \text{for} \quad I\neq J\end{cases} \qquad \frac{d\Phi_{2I-1}}{dx}(x_J)=\frac{d\phi_I^o}{dx}(x_J)=0$$
$$\frac{d\Phi_{2I}}{dx}(x_J)=\frac{d\phi_I^1}{dx}(x_J)=\begin{cases}1 & \text{for} \quad I=J\\ 0 & \text{for} \quad I\neq J\end{cases} \qquad \Phi_{2I}(x_J)=\phi_I^1(x_J) \quad =0 \tag{8}$$

for $I,J=1,2,\ldots,N$.

Galerkin formulation

In the Galerkin formulation we take the weighting function to be of the same approximate form as the approximate solution. The weighting function is therefore defined as

$$v^N(x)=\sum_{J=1}^{M}\beta_J\Phi_J(x)=\sum_{J=1}^{N}\left[\beta_{2J-1}\phi_J^o(x)+\beta_{2J}\phi_J^1(x)\right] \tag{9}$$

where β_J are arbitrary constants. We rewrite Eq. (5) in residual form and substitute Eqs. (6) and (9) in the result. Where appropriate we drop the x for convenience. We then get:

$$\begin{aligned} I_R \quad = \quad & \int_0^L EI\frac{d^2}{dx^2}\left(\sum_{I=1}^{M}\beta_I\Phi_I\right)\frac{d^2}{dx^2}\left(\sum_{J=1}^{M}W_J\Phi_J\right)dx-\int_0^L\left(\sum_{I=1}^{M}\beta_I\Phi_I\right)f\,dx \\ & -\int_0^L\left(\sum_{I=1}^{M}\beta_I\Phi_I\right)\hat{P}\delta(x-x_p)\,dx-\int_0^L\frac{d}{dx}\left(\sum_{I=1}^{M}\beta_I\Phi_I\right)\hat{M}\,\delta(x-x_c)\,dx \\ & -\sum_{I=1}^{M}\beta_I\Phi_I(0)\bar{V}_L-\sum_{I=1}^{M}\beta_I\Phi_I(L)\bar{V}_R-\frac{d}{dx}\left(\sum_{I=1}^{M}\beta_I\Phi_I(0)\right)\bar{M}_L-\frac{d}{dx}\left(\sum_{I=1}^{M}\beta_I\Phi_I(L)\right)\bar{M}_R \\ \neq \quad & 0 \end{aligned} \tag{10}$$

Because of the presence of the approximate forms of the weighting function and the solution in Eq. (10) the integrand I_R is most likely not to be zero any more. We would in the first place like to minimize the integrand I_R with respect to the weighting function v^N by choosing the appropriate or suitable values for the β_I. A natural way to find the optimum value or extremum of the integrand I_R with respect to v^N is to take the partial derivative of I_R with respect to each β_I and set the the result equal to zero, i.e.

$$\frac{\partial I_R}{\partial\beta_I}=0 \tag{11}$$

for $I=1,2,\ldots,M$. Keeping the properties of the interpolation functions given in Eq. (8) in mind, applying Eq. (11) then leads to the following set of linearly independent algebraic equations:

$$\begin{aligned} \int_0^L EI\frac{d^2\Phi_I}{dx^2}\frac{d^2}{dx^2}\left(\sum_{J=1}^{M}W_J\Phi_J\right)dx \quad = \quad & \int_0^L\Phi_I\,f\,dx+\int_0^L\Phi_I\hat{P}\delta(x-x_p)\,dx \\ & +\Phi_I(0)\bar{V}_L+\Phi_I(L)\bar{V}_R \end{aligned} \tag{12}$$

for $I = 1,3,5,\ldots,M-1$ and

$$\begin{aligned}\int_0^L EI\frac{d^2\Phi_I}{dx^2}\frac{d^2}{dx^2}\left(\sum_{J=1}^{M} W_J\Phi_J\right)dx &= \int_0^L \Phi_I\, f\, dx + \int_0^L \frac{d\Phi_I}{dx}\hat{M}\,\delta\left(x-x_c\right)dx \\ &\quad + \frac{d\Phi_I(0)}{dx}\bar{M}_L + \frac{d\Phi_I(L)}{dx}\bar{M}_R\end{aligned} \tag{13}$$

for $I = 2,4,6,\ldots,M$. In the GGFEM approach we endeavour to place nodal points at the positions where point loads and concentrated moments act. Let us assume that at the node $x_I = x_P$ where the point load acts, $I = P$ and that at the node $x_I = x_c$ where the concentrated moment acts, $I = C$. We can summarize Eqs. (12) and (13) in matrix form. This gives

$$[\mathbf{K}]\{\mathbf{W}\} = \{\mathbf{F}\} + \{\hat{\mathbf{P}}\} + \{\hat{\mathbf{M}}\} + \{\mathbf{Q}\} \tag{14}$$

with

$$K_{IJ} = \int_0^L EI\frac{d^2\Phi_I}{dx^2}\frac{d^2\Phi_J}{dx^2}dx \quad \text{for} \quad I,J = 1,2,\ldots,M \tag{15}$$

$$F_I = \int_0^L \Phi_I\, f\, dx \quad \text{for} \quad I = 1,2,\ldots,M \tag{16}$$

$$\hat{P}_I = \begin{cases}\hat{P} & \text{for} \quad I = P \\ 0 & \text{for} \quad I \neq P\end{cases} \tag{17}$$

$$\hat{M}_I = \begin{cases}\hat{M} & \text{for} \quad I = C \\ 0 & \text{for} \quad I \neq C\end{cases} \tag{18}$$

$$Q_I = \begin{cases}\bar{V}_L & \text{for} \quad I = 1 \\ \bar{M}_L & \text{for} \quad I = 2 \\ 0 & \text{for} \quad 3 \le I \le M-2 \\ \bar{V}_R & \text{for} \quad I = M-1 \\ \bar{M}_R & \text{for} \quad I = M\end{cases} \tag{19}$$

$[\mathbf{K}]$ is the global coefficient matrix and $\{\mathbf{F}\}$ the global distributed force vector. Note that the correct placement of the point load, concentrated moment and the forces at the boundaries follows naturally from the consistent treatment of the (weighting) interpolation functions.

Global Galerkin finite element formulation

The finite element approach provides a systematic and general method for constructing the interpolation functions, and therefore the solution, and to evaluate the coefficients of the global coefficient matrix and the global distributed force vector. This is based on the property that the operation of integration is additive. Any integral over the domain Ω can therefore be defined as the sum of the integrals over any number of non-overlapping sub-domains (elements), say Ω_e, into which the domain can be sub-divided, i.e.

$$\Omega = \sum_{e=1}^{E}\Omega_e \tag{20}$$

where E is the number of elements. The interpolation functions $\Phi_I(x)$ can also be defined piecewise over the elements as:

$$\Phi_I(x) = \sum_{e=1}^{E} \Psi_I^e(x) \tag{21}$$

where $\Psi_I^e(x)$ is the part of $\Phi_I(x)$ defined over the element e. Eq. (14) can therefore be rewritten to become

$$\left[\sum_{e=1}^{E}\left[K^e\right]\right]\{\mathbf{W}\} = \left\{\sum_{e=1}^{E}\left\{F^e\right\}\right\} + \left\{\hat{\mathbf{P}}\right\} + \left\{\hat{\mathbf{M}}\right\} + \{\mathbf{Q}\} \tag{22}$$

with

$$K_{ij}^e = \int_{\Omega_e} EI \frac{d^2\Psi_i^e}{dx^2}\frac{d^2\Psi_j^e}{dx^2} dx \quad \text{for} \quad i, j = 1, 2, \ldots, m \tag{23}$$

$$F_i^e = \int_{\Omega_e} \Psi_i^e f\, dx \quad \text{for} \quad i = 1, 2, \ldots, m \tag{24}$$

where n is number of nodal points associated with each element and m the number of primary variables associated with each element. Note that in Eq. (22) only the contributions to the global coefficient matrix and the global force vector are evaluated at element level, whilst the elements of the point load vector (Eq. (17)), concentrated moment vector (Eq. (18)) and the boundary force vector (Eq. (19)) are still determined at global level. (There is nothing that dictates that all the terms must be evaluated at element level.)

GGFEM procedure

The global Galerkin finite element method, therefore, consists of the following main steps.

- It starts with the description of the problem consisting amongst others of the identification of the differential equation(s), parameters, domain and boundary conditions. The point sources (point loads or concentrated moments) are included in the the source term(s) of the differential equation(s) during the definition of the problem (see Eq. (1)).
- Subsequently the Galerkin formulation over the entire domain is developed. This entails the derivation of the weak formulation, the selection of the approximate solution over the domain and the formation of the resulting system of global equations through the Galerkin method.
- In order to evaluate the integrals, we make use of the property that an integral can be written as the sum of a number of integrals. We therefore discretise the domain into a collection of sub-domains (elements) which leads to the GGFEM formulation.
- After definition of the nodal points and the selection of the appropriate associated interpolation functions we evaluate the relevant integrals over each element (or sub-domain) to obtain the element contributions to the coefficients of the global equations.
- To complete the equations the relevant contributions evaluated at global level are added.
- We then apply the boundary conditions in the usual way, and the system of global equations is solved to obtain the required solution for the problem.

Something which is also appropriate to discussed here and which is seldomly, if at all, treated in textbooks, is that the secondary variables (in this case reaction forces) associated with the

essential boundary conditions (in this case vertical displacement and slope) can be be obtained directly as part of the primary solution (Du Toit, 2002). In the example under consideration, Eq. (1), we have the essential boundary conditions $W_1 = W_2 = W_{M-1} = 0$. Eq. (14) (= Eq.(22)) must be adapted as follows to be able to obtain the reaction forces $\bar{V}_L$, $\bar{M}_L$ and $\bar{V}_R$ directly as part of the primary solution.

$$F_I = F_I - K_{I1}W_1 - K_{I2}W_2 - K_{I,M-1}W_{M-1} \tag{25}$$

$$K_{I1} = \begin{cases} -1 & \text{for} \quad I = 1 \\ 0 & \text{for} \quad I \neq 1 \end{cases} \tag{26}$$

$$K_{I2} = \begin{cases} -1 & \text{for} \quad I = 2 \\ 0 & \text{for} \quad I \neq 2 \end{cases} \tag{27}$$

$$K_{I,2N-1} = \begin{cases} -1 & \text{for} \quad I = M-1 \\ 0 & \text{for} \quad I \neq M-1 \end{cases} \tag{28}$$

for $1 \leq I \leq M$. The solution vector then becomes $\left\{\bar{V}_L, \bar{M}_L, W_3, W_4, \ldots, W_{M-3}, W_{M-2}, \bar{V}_R, W_M\right\}^T$.

The final solution for the vertical displacement of the beam is then given as

$$w^N(x) = \sum_{J=1}^{M} W_J \left[\sum_{e=1}^{E} \Psi_J^e \right] \tag{29}$$

A point that is often not emphasised enough is that in the finite element method the only discretisation that takes place is the discretisation of the domain. In the finite difference method the equations are also discretised and the solution is a true discrete solution producing only the values at the selected nodal points. In the finite element formulation the equations are never discretised and the solution (29) is not a discrete solution, but represents a continuous solution over the entire domain.

Conclusion

In this paper the focus was on the global Galerkin finite element solution of the fourth-order differential for an Euler-Bernoulli beam.

In the traditional Galerkin finite element approach of the problem under consideration the point loads and concentrated moments are not included in the system of external forces during the definition of the governing differential equation. The domain is also first of all discretised into elements and the Galerkin finite element formulation is then developed at element level. This leads to the occurance of boundary terms (or in this case forces) associated with each element. The integrals are evaluated over each element to form the element equations. The global equations are then formed by assembling all the element equations. If no point load or concentrated moment act at a particular nodal point the inter-element boundary terms are assumed to cancel. However, at any nodal point associated with a point load or concentrated moment, the point load or concentrated moment is assumed to represent the difference in the inter-element boundary terms.

In contrast in the GGFEM approach the point loads and concentrated moments are treated in a natural way as part of the system of external forces and are included in the differential

equation during the definition of the problem. The Galerkin formulation is then developed over the entire domain. Subsequently the domain is discretised into a collection of elements to facilitate the evaluation of the integrals to obtain the coefficients of the global set of algebraic equations. The integrals over the elements therefore provide the element contributions to the coefficients of the global stiffness matrix and load vector. The inter-element boundary terms do not appear in the GGFEM formulation. It should also be noted that there is no requirement that all the terms of the global equations must be evaluated at element level. Where appropriate terms may be evaluated at the global level, such as is the case with the coefficients of the point load and concentrated moment vectors in this case. The GGFEM can be seen as a systems or holistic approach to the finite element solution of the problem.

Although the differences between the traditional GFEM and the GGFEM approaches at a first glance may appear to be subtle, from a philosophical, physical and educational point of view they are significant.

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