Collaboration toward teaching proficiency of mathematical concepts in secondary schools

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Dissertation submitted in fulfilment of the requirements for the degree Magister Educationis in Curriculum Development at the Potchefstroom Campus of the North-West University

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ANTHON MAKHAYA TSHONA (aka TOTO)

KHUMA TOWNSHIP

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Collaboration toward teaching proficiency of mathematical concepts in secondary schools

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ABSTRACT

In South Africa learner performance in mathematics is alarmingly low in comparison with other countries. Literature reveals that mathematics teachers play a prominent role toward helping learners perform well and learn mathematics in a meaningful way. As such it is important for teachers to possess adequate mathematical knowledge for teaching. Mathematics teachers are advised to take into account learning theories advocated by mathematics education research, for instance constructivist and sociocultural learning theories. Such theories place focus on the learner as well as the context in which the teaching and learning take place, describing both as significant toward facilitating meaningful learning of mathematics. This implies that mathematics teachers need to know and understand that learners learn in different ways, bring different backgrounds, ideas, and views into the mathematics classroom, and, most importantly, the mathematics teacher needs to know that mathematical tasks that learners will solve should be contextualized, e.g. posed in real-world problems related to learners' prior knowledge.

Henceforth, I carried out this study to investigate the notion of collaboration toward teaching proficiency of mathematical concepts, specifically the function concept and Euclidean geometry. In particular, this study intended to investigate the influence of a Collaborative Learning Programme (CLP) on teachers' conceptual understanding of mathematical concepts. This study was centred on the three C's: Content, Collaboration and Context. The rationale for placing this study around the three C's will be evident in the following paragraphs.

This study was entrenched in an interpretive paradigm and the research design was qualitative in nature and rooted in a multisite case study. Consequently data were collected by means of task sheets, semi-structured interviews, field notes, Self-Directed Learning Instrument as well as open-ended surveys. The population informing the central phenomenon in the study consisted of secondary school mathematics teachers as well as final-year mathematics education students. The population included both males and females. Since the study was qualitative in nature it is important to note that the findings cannot be generalised.

Crystallisation of the data emanating from numerous data gathering techniques led to the findings that the participants reacted in different ways, and positively, to the CLP. In Case 1, the findings revealed that the CLP did not have any impact on one of the participants' views and beliefs about the teaching and learning of mathematics, as well as the conceptual understanding of the function concept. Contrary to this, the participant maintained that she gained a lot from the CLP and she was interested to participate in the future CLP. In Case 2 the findings indicated that the CLP did have an influence on students' conceptual understanding Euclidean geometry, but not for all students. Findings also indicated that the CLP did not have any influence on students' self-directed learning abilities concerning mathematics. These findings were then used to make recommendations in order to address the impact of a Collaborative Learning Programme on participants' proficiency in the teaching of mathematical concepts.

KEY WORDS

Collaboration; Conceptual Knowledge; Procedural; Knowledge; Fundamental Views and Beliefs of Teachers; Mathematics Learning; Mathematics Teaching; Mathematical Knowledge for Teaching; Nature of Mathematics; Proficiency in teaching mathematics; Function Concept; Euclidean Geometry.

LIST OF ABBREVIATIONS

CAPS: Curriculum Assessment and Policy Statement.

CCK: Common Content Knowledge

CK: Content Knowledge

CLP: Collaborative Learning Programme

COP: Communities of Practice

DBE: Department of Basic Education

EMS: European Mathematical Society

KCS: Knowledge Content and Student

KCT: Knowledge Content and Teaching

MKT: Mathematical Knowledge for Teaching

NCTM: National Council of Teachers of Mathematics

PCK: Pedagogical Content Knowledge

PISA: Program for International Student Assessment

SCK: Specialised Content Knowledge

SDLI: Self-Directed Learning Instrument

TIMSS: Trends in International Mathematics and Science Study

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CHAPTER 1: STATEMENT OF THE PROBLEM AND MOTIVATION

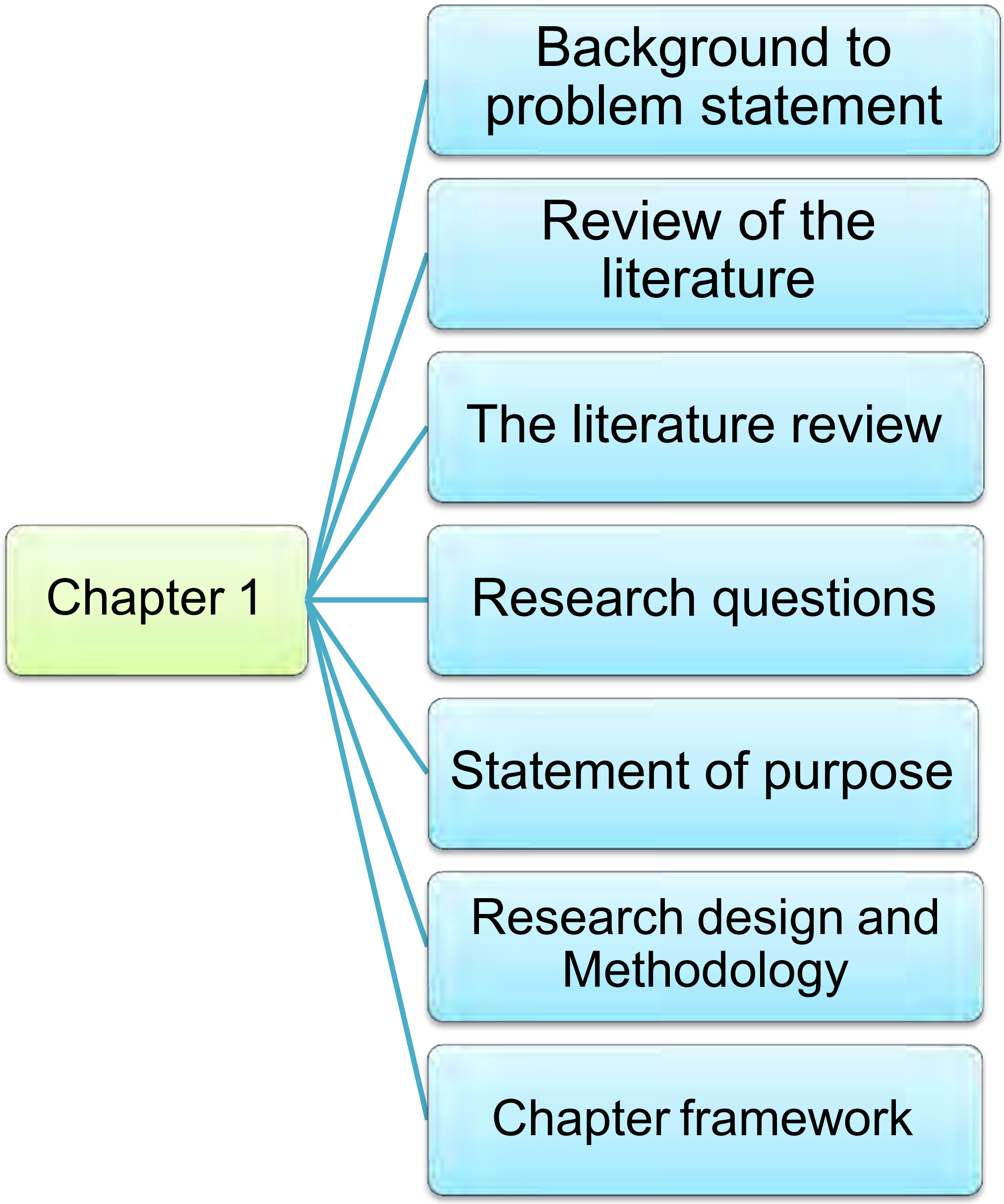


Figure 1:1 Outline of Chapter 1

1.1 Background to problem statement and intellectual conundrum

Mathematics is arguably one of the most essential school subjects for any society. It is essential for invention in Technology, Science, and Engineering, as well as competiveness in the global workforce. National Council of Teachers of Mathematics (2000:50) underscores the significance of mathematics:

> "In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competences open those doors to productive futures. A lack of mathematical competence keeps those doors closed...All students should have the opportunity and the support necessary to learn significant mathematics with depth and understanding. There is no conflict between equity and excellence." (NCTM, 2000:50)

However, in South Africa learner performance in mathematics is alarmingly low in comparison with other countries. Jaca and Padayachee (2012:169) highlight that South African learner performance in both national assessment and international comparative studies "is less than desired at all schools levels'. It has been reported that in 2003, the Trends in International Mathematics and Science Study (TIMSS) tested grade 8 learners' mathematical proficiency and South Africa came last of the 50 participating countries (Reddy, 2006). In January 2012, the Department of Basic Education released a National Diagnostic Report on Learner Performance in Selected Subjects (Grade 12), and mathematics made the list. In this report, it is stated that learners performed poorly on questions that required deeper understanding, and those which required learners to interpret, explain, provide justification or integrate concepts (DoE, 2012).

In this respect, Stols (2013) finds in his study: "An investigation into the opportunity to learn that is available to Grade 12 mathematics learners', that mathematics teachers in secondary schools "spend more time on topics that tend to be procedural, for example logarithms, sequences and series, and the remainder and factor theorems. In addition, teachers avoided the topics that required deeper understanding and problem solving" (Stols, 2013:16). The latter list of topics includes graphs, properties of graphs, transformation of graphs, application of differentiation (maximum and minimum problems, graphs of cubic functions, tangents), linear programming, and solving triangles in three dimensions (Stols, 2013:16).

According to the Curriculum and Assessment Policy Statement (DoE, 2012), secondary school mathematics learners should be exposed to mathematical experiences that give them many opportunities to develop their mathematical reasoning and creative skills in preparation for more abstract mathematics. Due to the fact that learner performance in mathematics is not as desired at all levels, it can be asserted that South African learners are not provided with opportunities from which they engage in mathematical experiences that require them (learners) to generalize, make conjectures and try to justify or prove them. For instance, the following specific aims of mathematics (DoE, 2012), amongst others, are not achieved by those (teachers) central to teaching and learning of mathematics:

- *Mathematical modelling* is an important focal point of the curriculum. Real life problems should be incorporated into all sections whenever appropriate. Examples used should be realistic and not contrived. Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues whenever possible.
- To provide the opportunity to develop in learners the ability to be methodical, *to generalize, make conjectures and try to justify or prove them*.
- To *develop problem-solving and cognitive skills.* Teaching should not be limited to *"how"* but should *rather* feature the *"when"* and *"why"* of problem types. Learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life.

According to Nieuwoudt (2006:23), mathematics is an essential and integral part of the school curriculum, and has an inescapable role to play regarding the required thinking skills and mastery of higher order learning tasks, such as conceptualisation, abstraction, generalisation, problem solving, and information-processing. Briefly, the poor learner performance (discussed in the preceding paragraphs) is a sign that those (teachers) central to teaching of mathematics do not achieve the objectives/outcomes as stipulated in the curriculum documents. For this reason, South African learners in general achieve poorly in international comparative studies and in national assessments because they are not afforded an adequate opportunity to learn mathematics in their classrooms (Chisholm *et al*., 2011; EMS, 2012; Stols, 2013).

Chisholm *et al*. (2011) highlight that South Africa has many teachers who are inadequately trained, yet are likely to remain mathematics teachers for many years. Chisholm *et al*. (2011:8) further note that a number of attempts to evolve existing teacher learning programmes in order to develop and equip un-/under-qualified teachers with the mathematical knowledge needed to facilitate mathematics meaningfully appear to have little effect on the quality of mathematics teaching in schools.

This is supported by the work of Stols (2013), who argues that learners in different classes and various schools do not have equal opportunities to learn mathematics. He emphasises that some of the learners in South Africa do not have qualified teachers to facilitate meaningful learning of mathematics with understanding (Stols, 2013:1).

With changes currently taking place in mathematics education, teachers' roles in the teaching and learning of mathematics need to be supported by all necessary means; for example, they are no longer expected to stand in front of the class and transmit knowledge to the learners (Nieuwoudt & Golightly, 2006). They (teachers) play an important role in teaching and learning of mathematics (Van der Sandt & Nieuwoudt, 2003; Stols *et al.,* 2007) for example he/she needs to pose mathematical tasks, create a classroom environment where learners are really doing mathematics, design lesson plans, interpret the curriculum, and in the process they need to ensure that learners achieve good grades (Van de Walle, 2007). Mathematics teachers are also expected to take note of emerging teaching approaches and instructional strategies, as well as various forms of mathematical knowledge (Nieuwoudt, 2006).

Based on the preceding paragraph, it is evident that mathematics teachers play a prominent role in helping learners learn mathematics in a meaningful way. As such it is important for teachers to possess adequate mathematical knowledge for teaching; they also need to take into account learning theories that are being advocated by mathematics education research, for instance constructivism and sociocultural learning theories. Such theories place focus on the learner as well as on the context in which teaching and learning take place as being significant toward facilitating meaningful learning of mathematics (Schoenfeld & Kilpatrick, 2008). This implies that teachers need to know and understand that learners learn in different ways, bring different backgrounds, ideas, and views into the mathematics classroom, and most importantly, the teacher needs to know that mathematical tasks that learners will solve should be contextualized, i.e. in real-world problems related to learners' prior knowledge (Plotz, Froneman & Nieuwoudt, 2012).

Taylor and Vinjevold (1999:159) suggest that teachers' poor grasp of the knowledge structure of mathematics, in particular with regard to the mathematical ideas they have to teach, acts as a major inhibition to teaching and learning of the subject, and that is a general problem in South Africa. This is supported by the work of Nieuwoudt and Golightly (2006), Plotz *et al*. (2012) and Stols (2013), who all contend that, to a significant extent, the lack of opportunities to learn meaningful mathematics with understanding in South African classrooms, is a result of teachers' lack of mathematical knowledge for teaching.

In light of the preceding paragraphs, it is evident that South African learners' under- performance in mathematics is closely related to their teachers' lack of mathematical competence, support and knowledge to facilitate deep and meaningful learning of mathematics; subsequently doors to productive futures for South African learners may be closed in the process. Henceforth, I undertook this study to investigate the notion of collaboration toward teaching proficiency of mathematical concepts, specifically the function concept and Euclidean geometry. In particular I intended to investigate the influence of a Collaborative Learning Programme (CLP) on teachers' conceptual understanding of mathematical concepts. This study was centred on the three Cs (Content, Collaboration and Context). The rationale for engaging this study around the three Cs will be evident in the following paragraphs.

1.2 Review of literature

Fundamental to raising learner performance in mathematics, is improving the quality of mathematics teaching (McGraner *et al.*, 2011:2). Simply, in order to improve learner performance, it is imperative to support and develop mathematics teachers in ways that will enable them (teachers) to provide learners with opportunities to learn meaningful mathematics with understanding (Van der Sandt & Nieuwoudt, 2003). In this respect, The European Mathematical Society (EMS, 2012) indicate that, in countries in which learner performance in mathematics is high, concepts such as *content, community and context* in which mathematics is taught, are considered to be of utmost importance. EMS (2012:46) contends the significance of *community, content* and *context* in the teaching and learning of mathematics:

> "Teachers need more than mathematical knowledge; they also need to communicate with and learn from each other and to get adequate internal and external support for their task. Thus, mathematics teaching, in addition to mathematical considerations, also needs to take into account a variety of *individual, social and organizational aspects.* In other words: *content* is important but so is the *community and context."* (EMS, 2012: 46)

In light of the preceding paragraphs, it is evident that content, community and context are critical factors toward the support and development of mathematics teachers. Consequently this study was executed in accordance with these three aspects. In the following section I elaborate on these aspects.

1.2.1 Community: Collaboration among mathematics teachers

EMS (2012) refers to *community* as "collaboration among teachers and such collaboration is characterized by teachers working together in continual substantive interactions as a way to further develop their profession. EMS (2012:48) emphasises the following statements with regard to collaboration among mathematics teachers:

> "When mathematics teachers (and teacher educators) share experiences, ideas, beliefs, competences, challenges and needs, they not only learn themselves but also learn to support each other's learning. The processes include working in small teams, communities of practice and loosely coupled networks. Mathematics teachers need to build identities through specialising in *students*' mathematical learning through collaborative reflection." (EMS, 2012:48)

In education, different scholars and researchers investigate collaboration for different purposes. Most of these authors have posited collaboration among teachers as the solution toward the improvement of student achievement, school management, and teacher learning. Goddard *et al*. (2007) suggest that collaboration among teachers has the potential to enhance the ability of schools to nurture teacher learning. Clark *et al.* (1996:196), assert that collaboration has the potential to provide teachers with "opportunities for reflection about practice, shared critique, and supported change." Collaborative learning processes allow teachers to converse knowledgably about their theories, methods and processes of teaching, and learning to improve upon classroom instruction. These collaborative processes have been shown to enhance learner work and performance outcomes, such as reduced learner dropout, absenteeism, and academic gains in math, science, and reading (Goddard, Goddard, & Tschannen-Moran, 2007; Strahan, 2003).

For Dallmer (2004), collaboration among teachers has the potential to move the field of teaching mathematics forward by energizing teams of teachers within schools to activate and guide teacher improvement, thereby sustaining the learning. EMS (2012:48) highlights that the notion of *contexts* as general conditions that are conducive to successfully supporting mathematics teachers, include offering learning opportunities where both the factors of content and community are regarded as essential. In particular, concerning the work of practising teachers, the context (resources, structures, commitment, etc.) plays an important role (EMS, 2012: 48).

1.2.2 Mathematical Knowledge for Teaching

Most people agree that teachers' knowledge of mathematics is essential to their ability to teach effectively (Thompson & Thompson, 1996:2). EMS (2012:47) contends that knowing mathematics is important but not enough to teach mathematics effectively. Schoenfeld and Kilpatrick (2008) highlight that proficiency in teaching mathematics concern broad and deep teachers' knowledge of school mathematics. The mathematics teachers need to know learners as both thinkers and learners, to craft and manage the learning environment, develop classroom norms and support classroom discourse as part of teaching for understanding, build relationships that support learning and reflect on their own practice. Mathematical knowledge for teaching in the broad is the backbone of proficiency in teaching mathematics (Schoenfeld & Kilpatrick, 2008). Ball *et al*. (2008:395) define mathematical knowledge for teaching as mathematical knowledge needed to facilitate meaningful learning of mathematics with understanding. EMS (2012:46), argue that MKT entails knowledge that includes high level of *content knowledge (CK)*, and in particular *pedagogical content knowledge (PCK).*

Content knowledge refers to the knowledge about the actual subject matter (mathematics) to be learned and taught - i.e. knowledge of central facts, concepts, theories, and procedures, rules of evidence and proof (Ball, *et al.,* 2008). Good teachers have insights into the mathematics content their learners will learn that allows them to identify misconceptions, set up lessons in which learners make connections, and provide learners with questions and prompts that help them access important concepts. On the one hand, *pedagogical content knowledge* refers to the specific knowledge that is needed for teaching mathematics (EMS, 2012:46). It is a unique kind of knowledge that intertwines content and aspects of teaching and learning (Ball *et. al.,* 2001:448).

Ball and McDiarmid (1990:2) highlight that it is important for teachers to possess MKT, because when teachers possess information or conceive knowledge in narrow ways, they may pass on these ideas to their learners, they may fail to challenge learners' misconceptions; they may use texts uncritically or may even alter them inappropriately. Schoenfeld and Kilpatrick (2008) point out that teachers' knowledge of school mathematics needs to be broad and deep. Philosophical arguments as well as common sense support the conviction that teachers' knowledge of subject matter influences their efforts to help learners learn subject matter (Ball & McDiarmid, 1990:2). It has been reported that teachers who possess adequate content and pedagogical content knowledge assist their learners towards learning meaningful mathematics. Research indicates, for instance, that learners taught by teachers with high content and pedagogical content knowledge perform better in the Program for International Student Assessment (PISA) as well as in the Trends in International Mathematics and Science Study (TIMSS), than learners taught by teachers with low content and pedagogical content knowledge. These teachers (with high CPCK) design their teaching so that the students are more cognitively engaged (EMS, 2012:47).

In light of preceding discussions of MKT, it is to be doubted that teachers who lack mathematical knowledge for teaching, in particular content and pedagogical content, could facilitate with understanding, meaningful learning of mathematics, that will enable learners to engage in mathematical tasks that require high order thinking skills, such as pattern-seeking, problem solving, abstraction, mathematical representations and/or models, justification and generalisation. Henceforth, it is evident that a need exists to investigate how collaboration, content and contexts may be used to support and develop mathematics teaching proficiency in the South African classes.

The preceding aspects of collaboration, content and contexts also concern the notion of mathematics teachers' dispositions (beliefs, views, and attitudes), which we consider to be relevant constructs to understand what teachers know, what teachers do and why they do it (Ponte & Chapman, 2006). Mathematics teachers' dispositions play a prominent role as a basis for understanding mathematics teacher's knowledge (Ponte & Chapman, 2006). Subtly, teachers' dispositions of knowledge shape their practice – the kinds of questions they ask, the ideas they reinforce, the types of tasks they assign (Ball & McDiarmid, 1990:2). Furthermore, the pivotal work of Thompson (1992) emphasizes the critical link between mathematics teachers' fundamental views and beliefs about mathematics and how it should be learned and then taught, and their application of their mathematical knowledge. Hence, in view of the eventual expectation of the researcher to contribute toward improved mathematics teaching in the chosen context, the study also aimed at investigating mathematics teachers' fundamental dispositions (beliefs, views, and attitudes) toward teaching and learning of mathematics in relation to their existing and acquired classroom practices.

The study also aimed at examining students' self-directed learning abilities concerning the learning of mathematics. The rationale behind this examination lies in the call made by the faculty of Education Sciences at the North West University about developing and training self-directed learning students in order to promote and encourage lifelong learning (Bullock,2013). Self-directed learning is based on the ability for the student to manage and take responsibility for their mathematical learning without having been directed by lecturers or others (Kim & Kim, 2009:110).

1.3 The literature study

An intensive review of the literature was undertaken concerning collaboration among mathematics teachers, their proficiency in teaching mathematics as well as well the learning and teaching of the function concept and Euclidean Geometry. Herein, the researcher made extensive use of the North West University library catalogue of primary and secondary nature. Databases and search engines such as EBSCOhost, Eric, Academic Search Premier, JSTOR and Google Scholar were explored. Dissertations and books were also consulted. The following key words were of assistance: collaboration, context, Euclidean geometry, function concept, learning communities, mathematical knowledge for teaching, nature of mathematics, proficiency in teaching mathematics, teaching and learning of mathematics, teacher development, teachers' dispositions, teacher learning, self-directed learning.

1.4 Research questions

1.4.1 Primary research question

The study aimed to investigate the following question:

> *To what extent can collaboration among mathematics teachers improve their proficiency in the teaching of mathematical concepts?*

Given the extent of the school mathematics curriculum and the limitations to a study at a master's level, the study intended to investigate how a collaborative learning programme can improve teachers' conceptual knowledge in relation to the specific key mathematical concepts, viz. the concept of function and Euclidean Geometry. The concept of function is central to the learning of mathematics in secondary schools and offers significant and substantial challenge to teachers in their classes in the chosen particular context of work (Knuth, 2000:48). On the other hand, the re-introduction (making it compulsory) of Euclidean Geometry requires the mathematics education research community (in particular, teachers, student teachers and mathematics educators) in South Africa to take into account the importance and implications of this big mathematical idea in relation to the teaching and learning of mathematics (DoE, 2012).

1.4.2 Secondary question

In order to investigate the primary research question, the following secondary questions need to be addressed:

- *What is the influence of a collaborative learning programme on teachers' fundamental views about the teaching and learning of mathematics?*
- *What are final-year mathematics education students' fundamental views about teaching and learning of mathematics?*
- *What is the influence of a collaborative learning programme on teachers' conceptual understanding of functions?*
- *What is the influence of a collaborative learning programme on final-year mathematics education students' conceptual understanding of Euclidean geometry?*
- *What is the influence of a collaborative learning programme on final-year mathematics education students' self-directed learning abilities concerning mathematics?*

1.5 Statement of purpose

The purpose of the study was to investigate how engaging township secondary school mathematics teachers and the students in collaborative learning may improve their proficiency in teaching key mathematical concepts. In the study mathematical concepts comprised both the function concept and Euclidean geometry. The study specifically intended to determine the influence of collaborative learning on teachers' and students' fundamental dispositions towards teaching and learning of mathematics.

To complement the research questions posed in the above section, the following objectives were set for guidance. The study aimed to:

- explore and describe mathematics teachers' conceptual understanding of the function concept;
- explore and describe fourth year mathematics BEd student's conceptual understanding of Euclidean geometry;
- find out from both teachers and students whether Collaborative Learning Programme (CLP) had an impact on their conceptual understanding of the concerned mathematical concepts;
- examine students' self-directed learning abilities concerning mathematics.

In attempting to realise these objectives, a qualitative research approach was employed.

1.6 RESEARCH DESIGN AND METHODOLOGY

1.6.1 Research design

In the study a qualitative approach was advanced, and consequently the study is in harmony with the interpretive worldview of reality. According to Borg *et al.* (1993:198) a qualitative approach seeks to explore and understand a complex phenomenon by examining it in its totality. Henning *et al.* (2004:3) argue that, in qualitative research, as researchers we seek to understand and also explain in argument by employing the evidence from multiple data sources, what the phenomenon that we are investigating is about. Consequently it is against this background that qualitative researchers may not know what to focus on until the research has commenced (Borg *et al.,* 1993:198). The philosophy that motivates this study is interpretivism. In connection to this, the interpretive philosophical assumption of reality focuses on understanding and interpreting the world in terms of people. Thus, in the interpretive perspective the assumption is that the world is made up of people with their own assumptions, intentions, attitude, beliefs, and values, and that the way of knowing reality is by exploring the experiences of others concerning a particular phenomenon (Cohen *et al.,* 2000, 181).

In connection to the aforementioned, the research team within the context of collaboration intended to investigate the deep knowledge stemming from the participants (in this case mathematics teachers and students) by personal interaction with the participants, spending extensive time with the participants, and probing them to obtain detailed meaning of understanding the concept of function as well as Euclidean Geometry. This was significant because the purpose of the study was to explore and describe collaboration toward teaching proficiency of mathematical concepts. We aimed at investigating participants' shared ideas, mathematical knowledge and experiences regarding the teaching and learning of mathematical concepts. Consequently the context for such investigation is assumed to be in harmony with the views and ideas of social constructivism on teaching and learning of mathematics.

Since the study was qualitative in nature (with its emphasis on the interpretive and subjective nature), the research design most appropriate to it, was the phenomenological research design (Nieuwenhuis, 2010a). In order to generate data to answer the research questions and accomplish the research objectives, the study portrayed this design by conducting a case study research in the form of semi-structured interviews, tasks, observation, and open- ended surveys. Creswell (2009:13) defines a case study as the investigation of a process, activity, event, programme or individual bound within a specific time and context. A case study entails an inclusive understanding of how participants relate to and interact with one another in a specific context, and how they make meaning of a phenomenon under investigation (Nieuwenhuis, 2010a:75). McMillan (2000) make a distinction between two types of case study, namely within-site and multisite. The former refers to a case study conducted within one research site while the latter is regarded as a case study conducted in more than one research site (McMillan, 2000:11). For the purpose of the study a multisite case study was conducted. This investigation takes place through detailed, in-depth techniques of data collection, focusing on multiple sources of information that are rich in contexts (Borg *et al.,* 1993). These may include interviews, observations and document analysis (Borg *et al.,* 1993; Cohen *et al.*, 2000).

Relating to the preceding paragraphs, the contemporary phenomena in the study were attempts at collaboration towards teaching proficiency of mathematical concepts, in particular the concept of function and Euclidean geometry. This study was conducted in a series of sessions for about eighteen (18) months, but at different venues. Hence in this study the context comprised such notions as the setting, collaboration and mathematical content. In the study the research team (the student and supervisor) used multiple data sources to collect data. These included tasks, interviews, observations and open-ended surveys related to several mathematical concepts.

1.6.2 Site or social network selection

The study was conducted in the North West Province. Data were generated in one school district (Dr Kenneth Kaunda) in Kanana Township and at a university. The former case study comprised two township secondary school mathematics teachers. In the latter context, the case involved a group of twelve fourth year BEd university students. These students were training at the North West University (Potchefstroom Campus) to become mathematics teachers, and their participation was voluntary.

1.6.3 Researcher's role

In the study my role was firstly to seek ethical clearance from the Ethics Committee of the North West University. Secondly, I had to get permission from Dr Kenneth Kaunda Education District Management in Potchefstroom, and carry out informed consent procedures with all the relevant stakeholders and participants before commencing the study. Thirdly, I had to design and facilitate learning activities. Thus, I was a participant observer and the primary instrument in data collection. Consequently some of my roles were to lead interviews, make notes, record, analyse, interpret and report data. This was done before, during and after the study. I met all of these responsibilities with the help of my supervisor.

1.6.4 Data generation

Since this was a qualitative multisite case study, multiple methods of data collection were used, including open-ended surveys, tasks (function and Euclidean geometry), semi- structured interviews and the Self-Directed Learning Instrument (Cheng *et al.,* 2010).

The open-ended surveys (see Appendix E) were designed in a manner that requested participants to write free responses in their own terms and to explain their responses. These questions required participants to define mathematics, and to provide their views and opinions regarding the teaching and learning of mathematics. Also, open-ended surveys addressed the views of the teaching and learning of the function concept. Our first meeting with the participants provided us (the research team) the opportunity to administer the first open-ended surveys, and this was done in the same venue (in school) but participants were encouraged to complete the questions individually. Our last meeting with the participants provided us the opportunity to administer the same open ended questions but this was done at different venues (in school) in an individual setting.

Self-constructed function tasks, based upon a thorough literature review, were used in order to determine teachers' conceptual understanding of the function concept. Tasks (see Appendix F, G, H, and I) were designed and administered before, during and after the CLP. Also, tasks comprised of the pre-assessment task, Task 1, Task 2 and post-assessment task. These tasks were based on O'Callaghan's (1998) function model (see par 3.1.3.4). These tasks were used to investigate teachers' conceptual understanding of the function concept. In addition, Euclidean Geometry tasks were also designed and administered for mathematics education students in accordance with the Mathematics Curriculum and Assessment Policy Statement for Further Education and Training Phase (Grades 10-12). Herein, the mathematical content of these tasks comprised Euclidean Geometry focusing on constructing geometric shapes (triangles, quadrilaterals and circles) and investigating the properties of such shapes (see Table 4:2). The students were encouraged to use technology (Geogebra® program) as means of learning Euclidean Geometry. Students were engaged in four different tasks (see Appendix K, L, M and N) probing their understanding with regard to Euclidean Geometry. Semi-structured interviews (see Appendix J) were conducted after the CLP, in an individual setting from which participants were required to answer a set of predetermined questions.

The interviews were conducted by the researcher at different venues (in schools) on the same day. A cell phone was used to record data because writing answers down is time consuming and coupled with distractions. Prior the execution of the interviews I requested permission from the participants to record the interviews (Nieuwenhuis, 2007a:89). These interviews were used to provide teachers with the opportunity to share their experiences about the Collaborative Learning Programme (CLP).

In order to gain insight regarding students' self-directed learning abilities, we used the Self-Directed Learning Instrument (SDLI) (Cheng *et al.*, 2010) as a means to measure the self-directed learning abilities of participating students. SDLI (see Appendix O is a model used for understanding SDL, and it consists of twenty items across the following four domains: learning motivation, planning and implementing, self-monitoring, and interpersonal, communication (Cheng *et al.,* 2010). The SDLI was adapted to suit the learning and teaching of mathematics. We administered both the pre- and post-assessments of SDLI.

1.6.5 Data analysis

In connection to the data analysis, Nieuwenhuis (2010b) distinguishes between five data analysis techniques: hermeneutics, content analysis, conversation analysis, discourse analysis and narrative analysis. In this study we (research team) used content analysis as our preferred technique of analysing data. In conducting content analysis, the research team engaged in the process of examining data from different angles with a view of identifying similarities and differences in the text that helped us to understand the data represent the data and establish an interpretation of the larger meaning of the data (Creswell, 2009:183).

On the one hand all written documents that were submitted by the participants in particular mathematical tasks were analysed by means of an adopted rubric (Dossey, *et al.,* 2002:561) which comprised five levels namely: no response, incorrect, minimal, partial and satisfactory. This rubric will be used to interpret teachers' and students' written responses, rather than to give them a mark. That is to say the tasks were scored according to a rubric. Dossey *et al.* (2002:559) developed scoring guides for teacher-constructed responses, known as rubrics; specifically, the scoring codes in the rubric assess the teachers' and students' mathematical correctness or level of understanding, their communication of the responses and the strategies engaged.

1.6.6 Ethical aspects of the research

Ethical considerations require balancing the value of advancing knowledge against the value of non-interference in the lives of others (Neumann, 2006:130). In light of this, the study was conducted with full recognition of the relevant aspects of research as embodied in individual and professional codes of conduct of the empirical research. We firstly, consulted and obtained clearance from the ethics committee of the North West University, and permission from the Dr Kenneth Kaunda Education District Management in Potchefstroom before commencing the study. Further the research team explained the purpose of the research to the participants, verbally and by means of a letter, assuring them that their participation would be voluntary and could be terminated at any time at their request. Also, participants were encouraged to grant an informed consent for their participation in the study. Pertaining to privacy, confidentiality and anonymity of participants, we assured all participants that their responses and information shared prior, during and after data collection would be kept private and that the results would be presented in an anonymous mode in order to protect their identities.

1.7 Chapter Framework

Chapter 1: Orientation and background of the study

Chapter 1 focuses on a general overview of the study, including background to the problem statement and intellectual conundrum for the study. This chapter also includes the research problem, research questions, and the purpose of the research and definition of concepts.

Chapter 2: The Nature of Mathematics

This chapter addresses the conceptual framework for the study by providing relevant literature exploration with regard to critical issues about the nature of mathematics (teaching and learning), teacher collaboration in education and teaching proficiency of school mathematics.

Chapter 3: The teaching and learning of the Function Concept and Euclidean Geometry

This chapter contains a general overview of the learning and teaching the function concept and Euclidean geometry.

Chapter 4: Research design and methodology

The focus of this chapter was to describe the research process in detail, including the research design and methodology that guided the study.

Chapter 5: Data Analysis and interpretation (Case 1)

In this chapter, the researcher highlights the raw data, analysis of data and the interpretation of the results of the study.

Chapter 6: Data analysis and interpretation (Case 2)

In this chapter, the researcher highlights the raw data, analysis of data and the interpretation of the results of the study.

Chapter 7: Summary, findings and recommendations

This chapter summarizes the findings of the study and presents conclusions drawn from the study. In addition, this chapter addresses the limitations of the study, and recommendations for future research are made.

CHAPTER 2: THE NATURE OF MATHEMATICS, PROFICIENCY IN TEACHING MATHEMATICS AND LEARNING COMMUNITIES

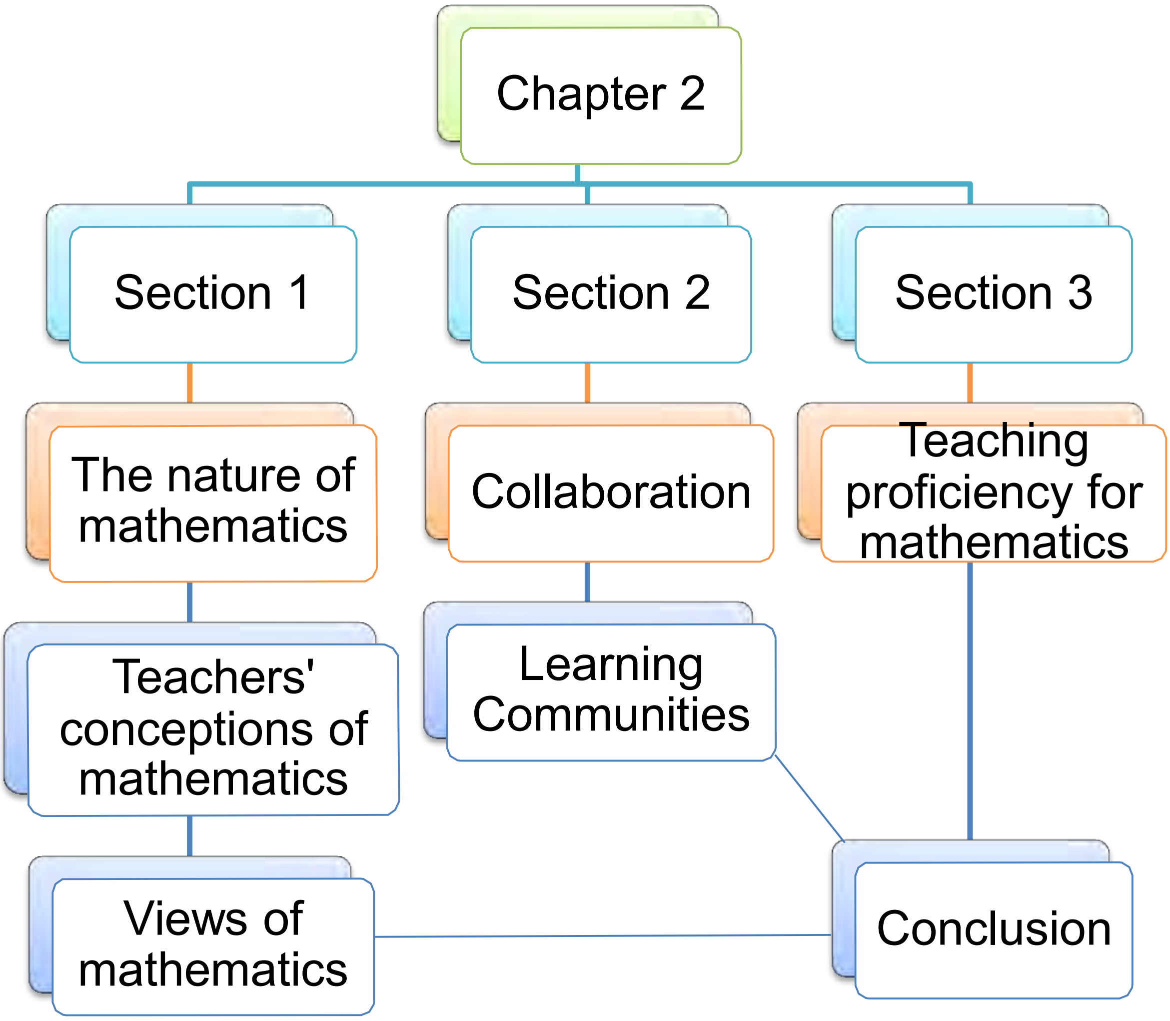


Figure 2:1 Outline of Chapter 2

2 Setting the scene

This chapter involves three sections, titled the nature of mathematics, collaboration; proficiency in mathematics teaching.

In Section 1, the researcher initially presents a decisive discussion of different views pertaining to the nature of mathematics; it provides a broad picture as to how and why mathematics is significant in the school curriculum. In light of this, the researcher will articulate and describe the nature of mathematics as far back as the fourth century during the times of Plato and Aristotle. Secondly, the researcher will examine the teaching and learning of mathematics pertaining to the different conceptions of mathematics as well as teachers' conceptions of mathematics and lastly the researcher will discuss the views of learning mathematics.

In Section 2, the researcher discusses and describes the notion of collaboration and its stance on mathematics education. Here the researcher present a definition (s) used to describe and explain collaboration and lastly the researcher will articulate collaboration with reference to the notion of learning communities.

In Section 3 of this chapter the researcher devote time discussing the issue of proficiency in teaching mathematics. In order to do this the researcher will rely on the notion of mathematical knowledge for teaching.

2.1 Section 1: The Nature of Mathematics

2.1.1 The nature of mathematics

The researcher must acknowledge the fact that different scholars within the community of mathematics education research hold different philosophical views concerning the nature of mathematics. From the literature, the researcher noted that the discourse about the nature of mathematics dates back as far as the fourth century and that different terms/notions are utilised to describe and articulate the nature of mathematics. Dossey *et al.* (2002:6) draws on the discussion of the nature of mathematics and its roles as far back as the fourth century BC, with Plato *(Platonist view)* and Aristotle (problem-solving view) as two main contributors to these differing philosophical views (see par 2.1.1 and 2.1.2). Lakatos (1976) argued for two schools of thought concerning the nature of mathematics, namely Euclidean and Quasi-experimental proponents. The latter sees the development of mathematical knowledge as the absolute truth, not open to revision or correction; on the other hand the former perceives the development of mathematical knowledge as a process that encompasses conjectures, proofs and refutations. Also the proponents of Quasi- experimental strongly reject the idea that mathematical knowledge is absolute and incorrigible; instead they accept the uncertainty of mathematical knowledge as part of the nature of mathematics (Lerman, 1990:54; Thompson, 1992:132).

Lerman (1983; 1990) distinguishes between the *absolutist* and *fallibilist* views of the nature of mathematics. The proponents of the absolutist view of the nature of mathematics regard mathematical knowledge as certain, absolute, superhuman, and unquestionable. On the other hand, proponents of the fallibilist view consider mathematical knowledge as a social construction, open to revision and correction. That means the development of mathematical knowledge is perceived as fallible and humanly created (Lerman 1990:55). Similarly, Thompson (1992) discusses issues pertaining to the nature of mathematics. She uses the work of noted scholars such as Copes (1970), Ernest (1988), Lerman (1983), Perry (1970), Skemp (1978) and several others to discuss conceptions pertaining to the nature of mathematics. From the work of these scholars, Thompson (1992:127) provides the following view of mathematics: "a kind of mental activity, a social construction involving conjectures, proofs and refutations, whose results are subject to revolutionary change and whose validity, therefore must be judged in relation to a social and cultural setting." On the one hand, Hersh (1986) argues that mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be represented or suggested by physical objects). Hersh (1986) asks the question: What are the main properties of mathematical activity, as known to all of us from daily experience? He further provides the following three answers to the above question (Dossey *et al.*, 2002:6):

- Mathematical objects are invented or created by humans.

- They are created, not arbitrarily, but arise from activity with already existing mathematical objects, and from the needs of science and daily life.
- Once created, mathematical objects have properties, which are well determined, which may have great difficulty discovering, but which are possessed independent of our knowledge of them.

In light of Hersh's view concerning the nature of mathematics, it becomes evident that knowing mathematics resonates with making mathematics (Thompson, 1992:128). Then the Curriculum and Assessment Policy Statement (CAPS) FET phase document further explains that mathematics is a human activity that encompasses higher order thinking skills such as observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves (DBE, 2011:8).

In light of the preceding paragraphs one can claim that the Euclidean and absolutist views concerning the nature of mathematics can be linked to the Platonist view of mathematics. Similarly the Quasi-experimental and fallibilist views concerning the nature of mathematics resonate with the Aristotelian view (problem-solving view) of mathematics (Thompson, 1992:132). In the following paragraphs I discuss these two views of the nature of mathematics.

2.1.2 Platonist view of mathematics

Based on the *Platonist conception*, mathematics is seen as a body of true knowledge, originating outside the individual in the external world, which human beings had to discover, not create, through rational activity (Ernest, 1991:7). Hence, the Platonist *conception* of mathematics describes distinctions between the mathematical ideas of the mind and their representations in the real world (Dossey, *et al.,* 2002:6). In terms of this view the nature of mathematics entails certain and unchallengeable truths. In other words, the mathematical knowledge is absolutely certain and represents the unique realm of certain unquestionable and objective mathematical knowledge. Henceforth the development of mathematical knowledge is associated to a unique methodology from which deductive reasoning is embraced more than anything. Herein deductive reasoning encompasses deductive reasoning, definitions and axioms as basic tenets from which to infer mathematical truth (Lerman, 1990:54).

2.1.3 Problem-solving view of mathematics

The problem-solving view encompasses interpreting a situation mathematically in terms of modelling. In the course of modelling one has to be involved in numerous iterative cycles of expressing, testing and revising mathematical interpretations, and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics (Lesh & Zawojewski, 2007:782). Nieuwoudt (2006) is of the opinion that the problem solving view encompasses aspects such as exploring, developing models and methods, proving models and methods, as well as discussing reasoning and solutions. In light of this one can state that in the problem-solving view, mathematics is perceived as a dynamic, continually expanding discipline created by mankind, a cultural product which is open to revision and construction. (Ernest, 1989:250).

It is evident from the preceding paragraphs that the nature of mathematics is multifaceted and dynamic. One notes that the problem solving view, Thompson's view, Hersh's view as well as the definition provided by the CAPS document concerning mathematics endorses the fact that mathematics is human activity (social construction) which encourages one to discover or invent mathematical concepts or solve-real life situations. In order to accomplish the research objectives set out in paragraph 1.5 this study was informed by the problem- solving view of mathematics. Herein mathematics is viewed as a human-activity that involves the solving of mathematical tasks (in the context of collaborative learning) centred on higher order thinking skills concerning the function concept and Euclidean geometry. In the course of this activity one is expected to construct concepts, discover relationships, state concepts, manipulate algorithms and proof theorems. In the following paragraph I provide a discussion concerning teachers' conceptions of mathematics which resonates with the views of the nature of mathematics discussed in the preceding paragraphs.

2.1.4 Teachers' conceptions of mathematics

The classical work of Ernest (1989; 1991), Lerman (1990) and Thompson (1984), amongst others working in the field of mathematics education, espouse the notion that in order to improve the quality of mathematics teaching and learning, it is imperative to gain first hand insight into the conceptions that the teacher holds pertaining to the nature of mathematics. The work of Thompson (1992) emphasizes the critical link between mathematics teachers' fundamental views and beliefs about mathematics and how it should be learned, and then be taught, and their application for their mathematical knowledge. One of the key reasons is that knowledge, beliefs and conceptions that teachers hold, play a significant role in shaping their thinking and behaviours, which influence their teaching practices (Ponte & Chapman, 2006). Subtly teachers' conceptions of knowledge shape their practice – the kinds of questions they ask, the ideas they reinforce, the sorts of tasks they assign (Ball & McDiarmid, 1990:2). Thompson (1992:132) asserts that teachers differ a great deal in their beliefs about the nature of mathematics. She claims that teachers' conceptions of the nature of mathematics range from viewing mathematics as an *absolute*, fixed body of knowledge to seeing mathematics as a *fallible* and expanding human knowledge. Similarly Ernest (1991) highlights three different philosophical views pertaining to teachers' conception of the nature of mathematics: *a static, unified body of knowledge (Platonist), a bag of tools (instrumentalist) and* a *dynamic problem-driven social construction (problem-solving),* views of mathematics and their implications on teachers' teaching practices. In the following paragraphs I elaborate on these views of mathematics. Ernest (1991:1) asserts that mathematics teachers who embrace the *absolutist conception of mathematics* see mathematics as objective and free of moral and human values. Mathematics teachers holding an absolutist conception about the nature of mathematics believe that mathematics is a discipline characterized by mathematical concepts, encompassing a holistic understanding of mathematics as a consistent, connected and objective structure. Herein, mathematics teaching is content-centred with an emphasis on conceptual understanding (Thompson, 1992:136).

At the other extreme, those who espouse the *fallibilist conception* connect mathematics with the rest of human knowledge through its historical and social origins (Ernest, 1991:26). Consequently, the nature of mathematics is viewed as value-laden, imbued with moral and social values, which play a prominent role in the development and applications of mathematics (Ernest, 1991). Mathematics teachers holding the *fallibilist conception* about the nature of mathematics have been found to follow a social constructivist approach concerning mathematics teaching (Thompson, 1992:136). Also, there are those mathematics teachers who espouse the instrumentalist conception of the nature of mathematics. Ernest (1989:250) considers the *instrumentalist view* of mathematics to be: "like a bag of tools, [is] made up of an accumulation of facts, rules and skills to be used by the trained artisan skilfully in the pursuance of some external end". From this view, mathematics is seen as a useful but unrelated collection of facts, rules, formulae, skills and procedures that lack deep understanding (Ernest, 1991). Mathematics teachers holding the *fallibilist conception* about the nature of mathematics have been found to follow a content-centred teaching approach with an emphasis on student performance and mastery of mathematical skills, rules and procedures (Thompson, 1992:136). In the following paragraphs I discuss the implications that teachers' conceptions about the nature of mathematics pose to the teaching and learning of mathematics.

2.1.5 Mathematics teaching-learning and the Platonist view

According to Nieuwoudt and Golightly (2006:114), the Platonist conception is in accordance with the formalist view of mathematics and its teaching and learning, and it can be connected to positivist-based traditional and behaviourist approaches to the teaching and learning of school mathematics. The positivist-based traditional teaching assumes a linear and direct relationship between teaching and learning (Nieuwoudt & Golightly, 2006:109). The learning of mathematics is perceived as a system of responses in the learner's behaviour to physical stimuli (Nieuwoudt & Golightly, 2006:117). In this light, the teaching and learning of mathematics is in accordance with the behaviour of the learner; learning occurs by absorption and it depends on individual effort, application, drill, self-discipline and self-denial (Toumasis, 1997:321). This instruction model is characterized by the teacher giving direct instructions, and it is believed that the learner will be cognitively active but physically inactive, except for note taking (Toumasis, 1997:321). From this instructional model, mathematical activities are characterized by transmission teaching and passive learning (Nieuwoudt & Golightly, 2006:107). Lastly, Toumasis (1997) maintains that the teacher teaches mathematics by communicating the mathematical knowledge as a series of concepts, theorems and proofs, correctly and clearly, to be learned and understood. Consequently, this view of mathematics teaching does not allow for much questioning, investigating or individual development of understanding mathematics (Ferguson, 2010:1).

2.1.6 Mathematics teaching-learning and the problem solving view

Nieuwoudt and Golightly (2006:115), see the problem solving view to be in accordance with the principles of post-positivist-based (social) constructivist learning views. Nieuwoudt and Golightly (2006) further highlight that the problem solving view can be connected to learner- centred teaching, in which learning is seen as a self-directed exploration of learner's own interest by the interaction of their existing knowledge and beliefs and new ideas or situations with which they come into contact in a social environment. Furthermore, learning of mathematics must be characterized by the investigation, discovery, discussion, play, cooperative work and exploration (Toumasis, 1997). Literature shows that social- constructivism sprouted from cognitive theories of learning. Nieuwoudt and Golightly (2006:117) highlight that the proponents of cognitive theories espouse the notion that, for learning to be meaningful, it has to be active, constructive, cumulative, self-controlled and goal-oriented. Van Der Walt and Fowler (2006:17) further explain that social constructivism is when the individual learner ought to discover and transform multifaceted knowledge and take responsibility for his/her own learning. Consequently, this view sees the learner as examining new information in opposition to old rules and then revising rules when they no longer work (2006). From this view, the teacher is regarded as the facilitator for learning to take place and the teacher's role is to guide, enable, challenge, and be a role model and advisor to the learners (Nieuwoudt & Golightly, 2006:111). Further, the role of the teacher in teaching and learning of mathematics is of utmost importance (Van der Sandt & Nieuwoudt, 2003, Stols *et al.,* 2007); for example he/she spends most of the classroom time creating an active learning environment where learners are really doing mathematics, designs lesson plans and interprets the curriculum (Van de Walle, 2007). Lastly, the teacher teaches mathematics through encouragement, facilitation and arrangement of carefully structured mathematical tasks for meaningful learning of mathematics with understanding (Toumasis, 1997).

2.1.7 Mathematics teaching-learning and the instrumentalist view

According to Beswick (2009), the instrumentalist view is in line with mathematics teaching in which the focus is placed on performances and mathematical learning is associated with the passive reception of knowledge. As such, mathematics teaching is about learning basic skills that learners will need in everyday life and learners" active role is omitted (Beswick, 2011:133). In light of this, the mathematical content is the central point that means the content is organized in terms of a hierarchy of skills and concepts, it is presented sequentially to the whole class, to small groups, or an individual, following a pre-assessment of learners' mastery of prerequisite skills (Thompson, 1992:136). According to this view of mathematics, the various topics that encompass the discipline are disconnected (Beswick, 2011:192). Cai (2007) posits that the implication of such a view of mathematics is that knowledge construction is omitted; instead, more emphasis is placed on the external world, and mathematics as such is not seen as human activity.

2.1.8 Views of the learning of mathematics

In his handbook chapter Cobb (2007) examines philosophical issues in mathematics education, pertaining to multiple and frequently conflicting views of the learning of mathematics. Cobb (2007) further provides us with a list of theoretical perspectives that are currently on offer in both mathematics education and the broader educational research community. From this list, he examines and compares four broad views of the learning of mathematics labelled experimental psychology, cognitive psychology, sociocultural theory and distributed cognition. These theoretical perspectives are discussed in the next sub- sections.

2.1.8.1 Experimental Psychology

Cobb (2007:16) opines that knowledge claims of experimental psychology are grounded on a particular conception of the individual and its contribution pertaining to the development of mathematics teaching and learning is accordance with administration of educational systems. The proponents of experimental psychology according to Cobb (2007:18) support the perspective that claims supremacy over the production of scientific knowledge about teaching and learning of mathematics, and that these forms of knowledge are pragmatically useful. Cobb (2007:28) explains that experimental psychology portrays the individual as statistically constructed collective individual and this individual is abstract because it need not to be corresponding to any particular student.

2.1.8.2 Cognitive Psychology

Cobb (2007:4) describes cognitive psychology as a theoretical perspective that seeks to examine teachers' and students' inferred interpretations and mathematical thinking in terms of internal cognitive structures and procedures. According to Vosniadou (1996:95), the basic assumption of cognitive psychology is that the mind is a system that constructs and manipulates mathematical concepts through cognitive processes. In light of this, cognitive psychology draws strongly on Piaget's processes of assimilation and accommodation (Van de Walle, 2007). According to Van de Walle (2007:23), assimilation takes place when the new experiences are incorporated into existing schemas, and accommodation is the process of altering existing ways of viewing things that do not fit existing schemas.

From this view, mathematical learning is characterized as a "constructive process in the course of which students successively reorganize their sensory-motor and conceptual activity" (Cobb, 2007:22). Furthermore, cognitive psychology sees mathematical learning from the perspective of the individual learner (Hodkinson, *et al*., 2008:30), and in doing so "cognitive psychology analyze[s] thought in terms of conceptual processes located in the individual" (Cobb, 1996:14).

2.1.8.3 Sociocultural theory

Cobb (2007:22) refers to sociocultural theory as a perspective that seeks to account for the process by which people develop particular forms of reasoning as they participate in established cultural practices. According to Van de Walle (2010), sociocultural theory emphasizes the idea that social interaction is a key component in the development of mathematical knowledge. Seeing learning as an inherently social phenomenon, sociocultural perspective suggest that analyses of collective learning move from individuals‟ heads (Yackel and Cobb, 1996; Cobb, 2007) to units of participation, interaction, and activity (Engeström, 1999; Lave & Wenger 1991; Rogoff 1994).

Vygotsky (1978) highlights that mental processes exist between students in social learning situations, and that from these social situations the student moves ideas into his or her own psychological domain (Van de Walle, 2007:29). In this light, sociocultural theory adheres to the view that both mathematical learning and understanding are inherently social and cultural activities (Yackel and Cobb, 1996). From this perspective, the learning and development of individuals‟ concepts and competencies cannot be separated from their participation in the interactive construction of taken-as-shared mathematical meanings (Verschaffel *et al*., 2007:600). In this perspective, mathematical learning is typically a process of enculturation into a community of practice (Jaworski, 2006:191). Briefly, sociocultural theory attempts to account for mathematical learning by focusing almost exclusively on participation in established sociocultural practices either in school or at home (Cobb, 2007).

2.1.8.4 Distributed cognition

Cobb (2007:27) argues that distributed cognition as a theoretical perspective focuses on the individual as an element of a reasoning system that also includes aspect of the immediate physical, social and symbolic environment. Hutchins *et al.* (2000:175) defined distributed cognition as a perspective that focuses "on all of cognition rather than a particular kind of cognition". From this perspective, cognition is better understood as a distributed phenomenon, one that goes beyond the boundaries of an individual to include environment, artefacts, social interaction and culture (Hollan *et al.*, 2000).

Distributed cognition according to Hollan *et al*. (2000) is committed to two related theoretical principles described as the boundaries of the unit of analysis for cognition and the range of mechanisms that may be assumed to participate in cognitive processes. In light of this, proponents of distributed cognition usually conduct a detailed analysis of reasoning processes that are stretched over people and the aspects of their immediate environment which they use as a cognitive resources (Cobb, 2007:27). Hollan *et al.* (2000:175) argue that distributed perspective accounts for cognition processes, wherever they may emerge, in terms of functional relationship of elements that participate together in the process. On the one hand, distributed cognition also accounts for a broader class of cognitive events and does not expect all such events to be encompassed by the skin or skull of an individual (Hollan *et al.*, 2000:176).

From this perspective, mathematical learning and reasoning are seen as emergent relations between people and the immediate environment in which they interact (Cobb, 2007:26). Briefly distributed cognition attempts to account for learners' mathematical learning by focusing beyond the boundaries of the learner but to include his/her physical environment, artefacts, social interaction and culture either in school or at home (Cobb, 2007).

In conclusion, to his handbook chapter Cobb (2007:29) highlights that mathematics learning should be viewed from a range of theoretical perspectives, rather than adhering to one particular theoretical perspective. To illustrate this, Cobb (2007) uses the metaphor "*bricoleur*", which according to Gravemeijer (1994b:447) "is a handyman who invents pragmatic solutions in practical situations*… [T]he bricoleur* has become adept at using whatever is available. The *bricoleur's* tools and materials are heterogeneous: Some remain from earlier jobs, others have been collected with a certain project in mind".

The natures of mathematics, and some characteristics of mathematics teaching-learning, have been articulated in this section. In light of this, there is a need to take into consideration the critical aspects of the nature of mathematics and their implications on teacher learning. For the purpose of this study it is important to note that I will base my notion of the nature of mathematics as well as mathematics teaching-learning on the problem-solving view, and I will use both cognitive psychology and sociocultural theory as means to address mathematics teaching-learning. This implies that I will employ what Cobb (2007) labels the interpretative framework to guide this study. According to Cobb, (2007:29) interpretive framework can be associated with the metaphor "*bricolage*" in that it uses both social and cognitive perspectives. The social perspective draws on sociocultural theory, whereas the cognitive perspective draws on both cognitive psychology and distributed accounts of cognition (Cobb, 2007:29). It is also important to note that there are differences and similarities between these two perspectives (see **Table 2:1** below).

Theoretical perspective	**Characterization of the individual**	**Usefulness**	**Limitations**
Experimental psychology	Statistically constructed collective individual	Administration of educational systems	Limited relevance to designs at classroom level
Cognitive psychology	Epistemic individual as reorganizer of activity	• Specification of *big ideas* • Design of instructional activities	Means of supporting learning limited to instructional tasks
Sociocultural theory	Individual as participant in cultural practices	• Designs that take account of students' out-of-school practices • Designs that take account of institutional setting of teaching and learning	Limited relevance to designs at classroom level
Distributed cognition	Individual element of the reasoning system	Design of classroom learning environments including norms, discourse, and tools.	Delegitimizes cognitive analyses of specific students' reasoning

Table 2:1 Contrasts between Four Theoretical Perspectives (Cobb, 2007:28)

2.2 Section 2: Collaboration

In this section, the researcher presents a decisive discussion of collaboration in education and mathematics education.

Lieberman (1986) suggests that schools cannot improve without teachers working together. Shah (2011:1) lambasts the traditional image of school teachers working in isolation and autonomy. Shah (2011) argues that collaboration among teachers is a critical factor toward educational change and success. In education different scholars and researchers investigate collaboration for different purposes. Most of these authors have posited collaboration among teachers as the solution toward the improvement of student achievement, school management, and teacher learning. Goddard *et al*. (2007) suggest that collaboration among teachers has the potential to enhance schools' ability to nurture teacher learning.

Clark *et al.* (1996:196) assert that collaboration has the potential to provide teachers with "opportunities for reflection about practice, shared critique, and supported change". Lastly, Strahan (2003) asserts that collaborative learning models allow teachers to converse knowledgably about their theories, methods and processes of teaching, and learning to improve upon classroom instruction.

2.2.1 What is collaboration?

Shah (2011) refers to collaboration as a dynamic framework for efforts, which supports mutuality, and similarity during interactive exchange of resources between at least two people who work together in a decision making process that is influenced by cultural and systematic factors to achieve common goals. Christiansen *et al*. (1997:9) define collaboration as "the explicit agreement among two or more persons to meet and accomplish a particular goal or goals". It is the mutual agreement of collaborating teachers in a coordinated effort to solve a problem together (Egodawatte *et al*., 2011:191).

The term that is often used to describe collaboration is teacher collaboration which "is often mentioned in the same breath together with (or even subsumed in) *collegiality*" (Kelchtermans, 2006:220). EMS (2012) links collaboration to the notion of community, and it exemplifies community as "collaboration among teachers', and such collaboration is characterized by teachers working together in continual substantive interactions as a way of further developing their profession. These interactions are characterized by shared goals, symmetry of structure, and a high degree of negotiation, interactivity and interdependence (Dillenbourg *et al*., 1999:1). In conclusion, collaboration is seen as a "means of developing knowledge about mathematics teaching and learning" (Potari *et al,* 2010, 473). For Egodawatte *et al.* (2011:190) collaboration among teachers has the potential of enhancing professional learning communities by establishing teams of teachers within schools to raise the quality of mathematics teaching and learning. Butler *et al.* (2004:348) argue that, when associated with a *cognitive perspective*, *communities of practice* among mathematics teachers or students have the potential to promote conceptual understanding, as teachers have the opportunity to share their observations/thinking, consider different ideas and conflicts and make sense of various representations or abstractions concerning mathematical ideas. In this study the notion *communities of practice* intended to offer both teachers and students to engage in the sharing of conceptual understanding concerning the function concept as well as Euclidean geometry. On the other hand, Butler *et al.* (2004:438) posit that, when associated with a *social perspective*, communities of practice among teachers or students engage teachers or students in core practices of mathematics such as conjecturing, justifying, representing, making connections and communicating about mathematical ideas.

2.2.2 Collaboration within the context of Learning Communities.

From the literature one notes that researchers working within mathematics education research criticize the traditional approach (one-day workshop) concerning the professional development of mathematics teachers (Darling-Hammond & Richardson, 2009). Researchers posit that the traditional approach lacks continuity because it is not on-going (Butler et *al.*, 2004). In light of this, researchers have argued for the professional development of mathematics teachers which is both on-going and dynamic. That means that professional development of mathematics teachers should occur within the contexts of learning communities (Butler *et al.,* 2004:436). Learning communities call for in-service teachers and mathematics students to engage with issues related to the teaching and learning of mathematics, figure things out for themselves, communicate and work with people from diverse backgrounds and view, and share what they learn with others (Bielaczyc & Collins, 1999:2).

According to Cobb (2007), researchers agree that professional development of mathematics teachers should resonate with the notion of learning communities, but there is no agreement concerning the theoretical framework that needs to be utilised in order to characterize teacher learning within communities of learning. Some researchers argue for cognitive psychology, others posit that socio-cultural theory or distributed cognition should inform teacher learning (see par 2.1.8.2, 2.1.8.3 and 2.1.8.4). In light of this, Butler *et al.* (2004:436) argue for an analytic framework that considers both the cognitive and social models without oversimplifying the contributions of either, and that explains individual and collective development in the contexts of learning communities. Whilst Cobb (2007) argues that teacher learning should be viewed from a range of theoretical perspectives, rather than adhering to one particular theoretical perspective. He encourages the community of mathematics education to consider the metaphor "*bricoleur*" (see par 2.1.8.4) when it comes to examining issues concerning mathematics learning. Since learning communities concerning professional development engage mathematics teachers in a joint activity about the teaching and learning of mathematics, a theoretical framework that has been used to examine these learning communities is *communities of practice* (COP) (Butler *et al.,* 2004:437).

Cobb (2007) argues that communities of practice were originally conceived from the work of Lave and Wenger (1991) of describing how newcomers are enculturated into a historically situated community with established traditions, roles, and practices. Herein, learning is viewed to be in accordance with the changes that occur in teachers' activities as they move from relatively peripheral participation to increasingly substantial participation in established cultural practices (Cobb, 2007:24). Wenger *et al.* (2002:4) highlight that communities of practice are established by people (*teachers, university researchers and students*) who engage in a process of collective learning in a shared domain (*mathematics*) or a passion for something they do and learn how to do it better as they interact regularly. Moreover, a community of practice entails, more than the technical knowledge or skill associated with understanding some task. Instead, it engages members in a set of relationships over time (Wenger, 1998:1). In this study the duration of the learning community was about eighteen months, comprising ten successive sessions in two different venues. Wenger (1998) argues that a community of practice needs to generate and appropriate a shared repertoire of ideas, commitments and memories. It also needs to develop diverse resources (both physical and conceptual) including tools, documents, routines, vocabulary and symbols that in some way carry the accumulated knowledge of the community. Furthermore, three elements are crucial in determining a community of practice from other groups and communities (Wenger 1998, Wenger *et al*., 2002):

- The domain (what it is about): The topic of focus that brings members together.
- The community (how it functions.): In pursuing their interest in their domain, members engage in joint activities and discussions, help each other, and share information.
- The practice (what capability it has produced): Members of a community of practice are practitioners. They develop a shared repertoire of resources: experiences, stories, tools, and ways of addressing recurring problems.

In the context of mathematics education, the construct communities of practice puts emphasis on "conceptualizing teaching as learning in practice, from which teachers work alongside mathematics educators to investigate and hence develop mathematics teaching practice" (Potari *et al*., 2010:474). In light of this, from a community of practice, teachers' learning is seen as contextually and socially situated (Lesh & Zawojewski, 2007:789). Potari *et al.* (2010:475) is of the view that teachers' learning as well as their professional development is seen as situated in types of co-participation in connected activities, for which they have identical goals and to which they bring different levels of mathematical proficiency. Staple (2008:349) is of the opinion that when teachers participate in these practices they develop proficiency in mathematics teaching.

Henceforth, to investigate the influence of *collaboration* on teachers' and students' fundamental views concerning the nature of mathematics as well as their conceptual knowledge of the function concept and Euclidean geometry, this study embraced the notion *communities of practice* (COP) from which both the cognitive and social perspectives (see par 2.1.8.2 and 2.1.8.3) were accounted for. In this study the notion *communities of practice* served to engage teachers or students in the problem solving environment that we (research team) labelled the Collaborative Learning Programme (CLP) involving mathematical activities that required justification, representing and making connections concerning conceptual understanding of the function concept as well as Euclidean geometry. Herein, teachers or students were encouraged to work together on mathematical tasks concerning mathematical concepts, reflect on these tasks and talk about their experiences and expectations concerning the teaching and learning of these concepts.

The significance, definitions and some characteristics of both collaboration and communities of practice have been articulated in this section. In light of this, communities of practice as discussed above prove to be relevant for this investigation, and hence will be used to address and describe the influence of collaboration on teachers' and students' conceptual knowledge of functions and Euclidean geometry as well as their fundamental views concerning the nature of mathematics. In section three the researcher will describe and discuss proficiency in teaching mathematics.

2.3 Section 3: Proficiency for teaching mathematics

It has been reported that teachers who possess adequate content and pedagogical content knowledge assist their learners towards learning meaningful mathematics. For instance, research indicates that learners taught by teachers with high content and pedagogical content knowledge perform better in Program for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS), than learners taught by teachers with low content and pedagogical content knowledge. These teachers design their teaching so that the learners are more cognitively engaged (EMS, 2012:47).

Most people agree that teachers' knowledge of mathematics is essential to their ability to teach effectively (Thompson & Thompson, 1996:2). EMS (2012:47) contends that knowing mathematics is important, but not enough to teach mathematics effectively. Schoenfeld and Kilpatrick (2008) highlight that teachers' knowledge of school mathematics needs to be broad and deep.

2.3.1 Proficiency for teaching mathematics

There is a general consensus in the mathematics education research that the effective teaching and learning of mathematics should be centred on the five components of mathematical proficiency. The work of Kilpatrick *et al.* (2001) provides a synopsis of what it means to be mathematically proficient. For them to be mathematically proficient means being able to engage in mathematical activities that comprise conceptual understanding, procedural fluency, strategic competence, adaptive reasoning as well as productive dispositions (Kilpatrick *et al.* 2001:116). Herein, conceptual understanding refers to the understanding of mathematical concepts, operations and relations. Procedural fluency appeals to the ability of manipulating procedures flexibly, accurately, efficiently and appropriately. Strategic competence is perceived as the ability to formulate, represent and solve mathematics problems. The adaptive reasoning component deals with the capacity for logical thinking, reflection, explanation and justification. And the last component which is the productive disposition is viewed as the habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy (Van de Walle *et al*., 2010:25).

Schoenfeld and Kilpatrick (2008) offer a characterization of the dimensions of proficiency for teaching mathematics consisting of eight dimensions. They argue that proficiency in teaching mathematics is prominent "to guide the selection and the use of tools for mathematics teacher education" (Schoenfeld & Kilpatrick, 2008:1). One of these dimensions claims that it is important for teachers to know the school mathematics in depth and breadth. Ball and McDiarmid (1990:2) highlight that it is indeed important for teachers to know school mathematics in depth and breadth, because when teachers possess information or conceive of knowledge in narrow ways, they may pass on these ideas to their learners; they may miss the opportunity to challenge students' misconceptions; they may use texts uncritically or may even change them incorrectly. Philosophical arguments as well as common sense support the belief that teachers' knowledge of subject matter influences their efforts to help students learn subject matter (Ball & McDiarmid, 1990:2).

Shulman (1987:9) argues that a mathematics teacher as a member of scholarly community must understand the structures of subject matter, the principles of conceptual organization, and the principles of inquiry that help to answer two kinds of questions in mathematics: What are the important ideas and skills in school mathematics? How are the new ideas added and deficient ones dropped by those who produce knowledge in mathematics? According to Shulman (1986:4), "subject matter knowledge refers to knowledge of the content of the discipline *per se*. It includes substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community)". [It] includes knowledge of the concepts, procedures and problem-solving processes. In mathematics education, subject matter knowledge is referred to as mathematical content knowledge (Plotz, 2007). This knowledge refers to the knowledge about the actual subject matter (mathematics) to be learned and taught - i.e. knowledge of central facts, concepts, theories, and procedures, rules of evidence and proof (Ball *et. al.*, 2008).

Schoenfeld and Kilpatrick (2008) also assert that it is of utmost importance for mathematics teachers to know students as both thinkers and learners. This refers to the specific knowledge (pedagogical content knowledge) that is needed for teaching mathematics (EMS, 2012:46). Pedagogical content knowledge includes understanding, which content within the function concept or Euclidean geometry learners find interesting or difficult, the common mis-conceptions that learners have, and what forms of representations are useful for teaching the function concept (Shulman, 1986:9). Pedagogical content knowledge signifies the amalgamation of content and pedagogy into an understanding of how particular aspects of mathematics are organized, adapted, and represented for instruction (Mishra & Koehler, 2006:1021). Moreover, "pedagogical content knowledge includes, but is not limited to, useful representations, unifying ideas, clarifying examples and counter examples, helpful analogies, important relationships, and connections among ideas" (Grouws & Schultz, 1996:443). This knowledge also includes knowing what teaching approaches fit the content, and likewise, knowing how elements of the content can be arranged for better teaching (Mishra & Koelher, 2006:1027).

In order to understand the complexity of proficiency in teaching mathematics, there is a need to unpack the components of mathematical knowledge for teaching in order to see what underlies expertise for teaching mathematics (Plotz 2007:7). The construct of mathematical knowledge for teaching can be traced back to the work of Shulman (1987). Informed by the work of Shulman, Ball (1990; 1991) brought to the fore the term mathematical knowledge for teaching (MKT). Mathematical knowledge for teaching is multi-faceted (see Figure 2.2; Wessels & Nieuwoudt 2010:3), includes two main components namely subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Ball *et al.* (2008) separate both SMK and PCK into three dimensions; the latter includes the common content knowledge, specialized content knowledge and the horizon content knowledge, while the former includes knowledge of content and students, knowledge of content and teaching as well as knowledge of content and curriculum. When combined, these six dimensions of knowledge signify mathematical knowledge that teachers use in classroom to produce instruction and learner growth.

Ball *et al*. (2008:395) explain mathematical knowledge for teaching as mathematical knowledge needed to carry out the work of school mathematics. This knowledge provides mathematics teachers with the opportunity "to engage in particular teaching tasks, including how to accurately present mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Verhoef & Tall, 2010:2). Mathematical knowledge for teaching in the broad is the backbone of proficiency in teaching mathematics. Hence, mathematical proficiency in some important way bridges the gap between mathematical knowledge for teaching and classroom practices (Schoenfeld & Kilpatrick, 2008). Another critical issue concerning mathematics teachers' knowledge is the notion of technological pedagogical content knowledge (TPCK). This knowledge is different from the knowledge of a disciplinary or technology expert and also from the general pedagogical knowledge shared by teachers across disciplines (Mishra & Koelher, 2006:1028). Instead technological pedagogical content knowledge is an emergent knowledge that goes beyond content, pedagogy and technology. This requires an understanding of the representation of ideas using dynamic software in a constructive way in order to teach mathematical concepts (Mishra & Koelher, 2006:1029).

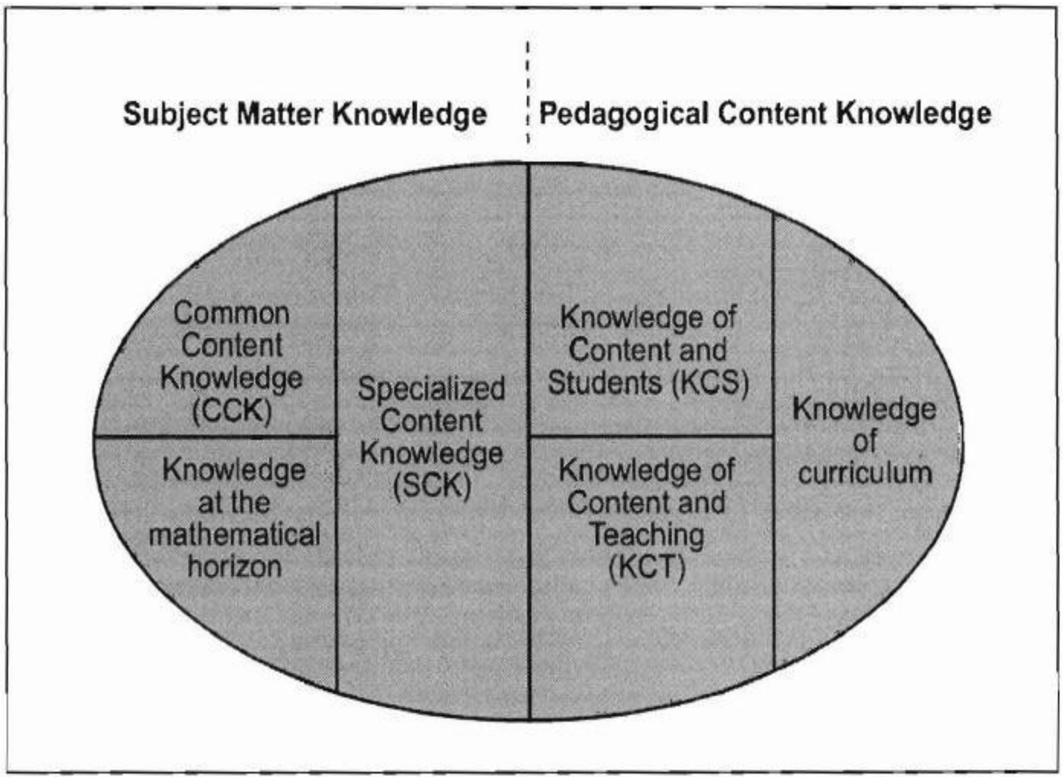


Figure 2:2 Domain map for mathematical knowledge for teaching (Ball *et al.,* 2008:377)

The importance, definitions and some characteristics of mathematical knowledge for teaching and proficiency in teaching mathematics have been articulated. In light of the preceding discussions, it is doubtful that teachers who lack mathematical knowledge for teaching could render meaningful mathematics learning with understanding that will enable learners to engage in mathematical tasks that require higher order thinking skills such as pattern-seeking, problem solving, abstraction, mathematical representations / models, justification and generalisation. In light of this, there is a need to investigate the teachers' conceptual knowledge of functions as well as their pedagogical content knowledge in relation to the function concept. Verhoef and Tall (2010:3) argue that mathematical knowledge for teaching develops in collaboration with colleagues as practical mathematics education experts as well as staff members of the university as mathematics science experts. In the study, from which one of the aims is to investigate the influence of collaboration on teacher's conceptual knowledge of functions and Euclidean geometry, I will base my notion of mathematical knowledge for teaching on both content, pedagogical content knowledge, as well as the technological pedagogical content knowledge as discussed in the preceding paragraphs.

2.4 Conclusion

The purpose of chapter 2 was to portray the nature of mathematics and the teaching and learning of mathematics. Also in this chapter the researcher established a discussion on teachers' conceptions of mathematics. In addition, this chapter provided a discussion related to the notion collaboration as well as proficiency in teaching mathematics. In the following chapter the researcher will articulate both the teaching and learning of the concept of function as well as geometry.

CHAPTER 3: THE TEACHING AND LEARNING OF THE FUNCTION CONCEPT AND GEOMETRY

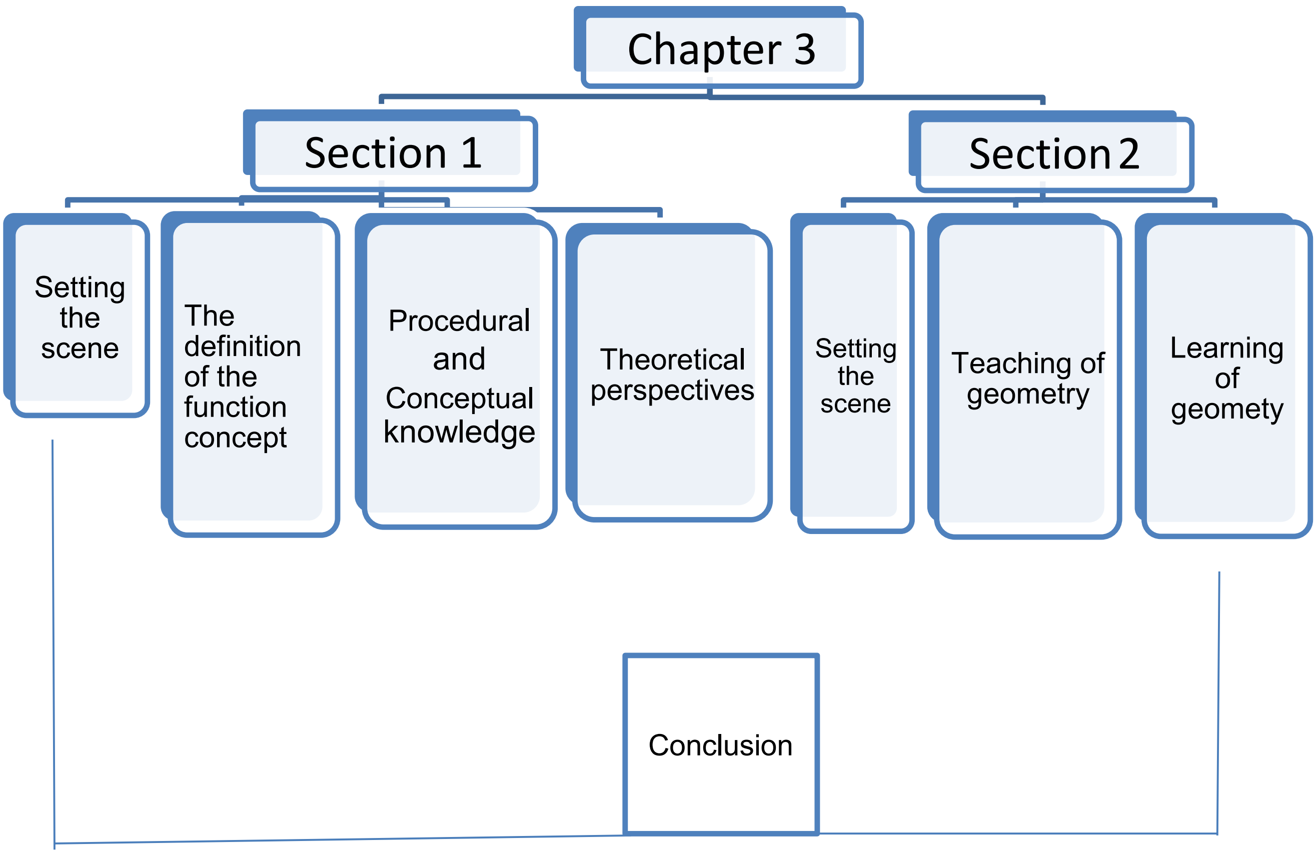


Figure 3:1 Outline of Chapter 3

3 Setting the scene

This chapter is divided into two sections. In Section 1, the researcher will articulate and describe and define the concept of function in general, and then the researcher will discuss both conceptual and procedural knowledge pertaining to the concept of function, as well as theories related to the development of the concept of function.

In Section 2, the researcher will establish a decisive discussion about the teaching and learning of geometry.

3.1 Section 1: Teaching and learning of the function concept

3.1.1 Definition of the function concept

The concept of function is rightly considered as one of the most important in all of mathematics (Ponte, 1992), which appears from primary through to university (Akkoć & Tall 2002:1), particularly in secondary and college levels where they get their maximum expressions and representations (Keller & Hirsch, 1998). Also, the function concept plays a prominent role in the development of the algebra curriculum (O'Callaghan, 1998:23) and is often considered a unifying concept that provides a framework for the study of mathematics (Cooney & Wilson, 1993; Carlson, 1998, Selden & Selden, 1992). The concept of function is essential to students' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyse graphs (Clement, 2001:745).

Although the concept of function is a central one in mathematics, many research studies of high school and college studies have shown that it is also one of the most difficult for students to understand (Dreyfus & Eisenberg, 1982, Markovits; Eylon & Bruckerheimer 1988; Sierpinska, 1992; Tall, 1996). Students face many obstacles trying to understand functions (Sajka, 2003:229). Some students' difficulties in the construction of the concept function are linked to the restriction of representations when teaching (Elia & Spyron, 2006:257).

The nature of instruction that students receive, in both the representations that are emphasized and the kinds of translation tasks that are presented, may significantly contribute to the difficulties that many students have in understanding the concept of function (Knuth, 2000:52). The following are some of the factors that make the concept of function difficult (Dreyfus & Eisenberg 1982:361):

- The function concept is not a single concept in itself, but several sub concepts associated with it, such as domain and range image, rate of change, etc.
- The function concept is used in apparently unrelated areas of mathematics, such as geometry and algebra, cord.
- The same function can be made in different ways, for example in a table, as a graph, a formula, a verbal description and a diagram image.

Sierpinska (1992) indicates that students have difficulties in making the connections between different representations of the concept (formulas, graphs, diagrams and word descriptions), in interpreting graphs, and manipulating symbols related to functions (cited by Elia & Spyron, 2006:257). Most students arrive at secondary school with many difficulties in abstract thinking, hence dealing with Cartesian graphs and an algebraic expression is not an easy task for these students (Ponte, 1990). For example, students often do not understand the concept of variable and the $f(x)$) notation, “they may not understand the distinction between $f(a)$ and finding the values of x for which $f(x)$” exists (Dreyfus & Eisenberg, 1982:363).

According to Kleiner (1989) the concept of function grew out of an attempt to capture the dynamics of motion and quantity through notation and the written word (cited by Dossey *et al.,* 2002:173). Dossey *et al.* (2002) argue that the concept of function is usually defined as a relationship between two sets of numbers that links each number in the first set with a unique number in the second set. This unique relationship is regarded as a graph where a vertical line intersects a single point. If the line intersects the graph in two or more points for some value of along the vertical axis (see Figure 3:1), then the graph is that of an arbitrary relation, not a function (Dossey *et al.,* 2002:173).

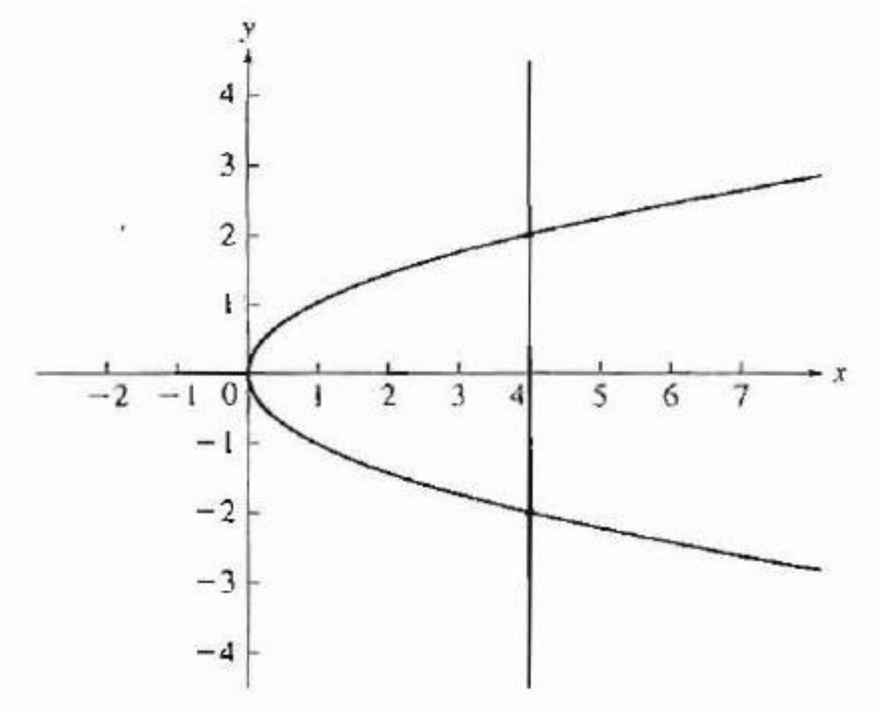


Figure 3:1 Illustration of the vertical line test (Dossey *et al.,* 2002:174)

For Vinner and Dreyfus (1989:357), the modern concept of function, which can be called the Dirichlet-Bourbaki concept, is that of a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). As a result, “uniqueness combined with domain and codomain as two arbitrary chosen nonempty sets makes the concept of function a highly general and abstract notion that proves to be demanding for students to assimilate” (Hansson, 2006:2).

> A function is a correspondence of items in one set A, called the *domain*, with items in another set B, called the *range*, such that
>
> 1. Every item in set A is corresponds with an item in set B.
> 2. No item in set A corresponds with more than one item in set B.

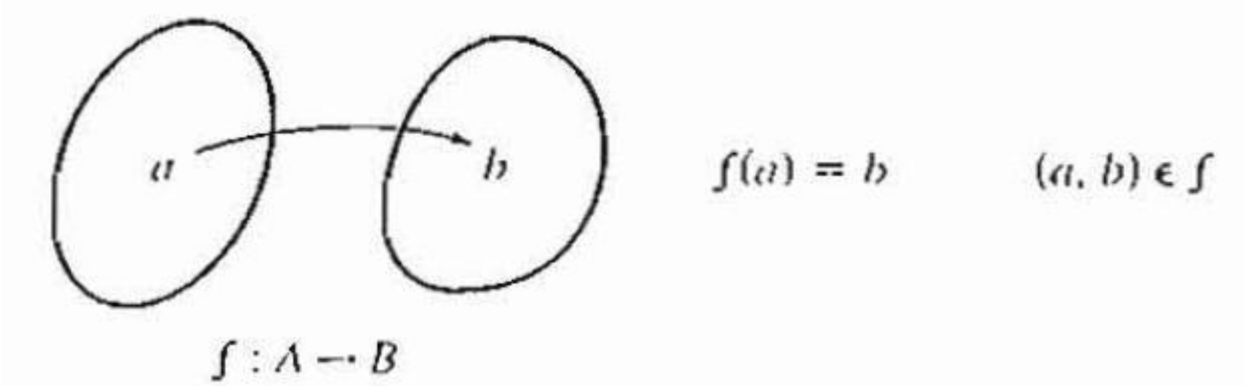


Figure 3:2 Every element in *a* in domain A corresponds to an element *b* in domain B (Dossey et al., 2002:174)

Bednarz *et al.* (1996) are of the view that the above-mentioned definition is of great importance in developing the structure of abstract algebra and establishing theorems concerning functions. However, it also offers little in terms of developing initial understanding of the concept of function (Dossey *et al*., 2002:174). Hansson (2006:7) concurs with the above statement by stating that: “A majority of researchers in the community of mathematics education...seem to agree that the concept of function should be introduced in a dynamic form, such as a type of relation, correspondence or co-variation – not favouring a static, ordered pair version of the definition, as related to Bourbaki”. According to O’Callaghan (1998:23) the concept of function can be depicted using a variety of representational systems, the three most common being equations, tables, and graphs. Knuth (2000:53) argues that developing a robust understanding of the concept of function not only means knowing which representation is most appropriate for use in different contexts, but also being able to move flexibly between different representations in different translation directions. Tall and Akkoć (2002) indicate that students are introduced to examples of the concept of function in various forms including the following:

- **A verbal representation of a function in formal or colloquial (everyday) language**

Example:
Researchers established that a population of *E.coli* bacteria doubles every 20 minutes.

- **A set diagram (representing a function by two sets and arrows between them)**

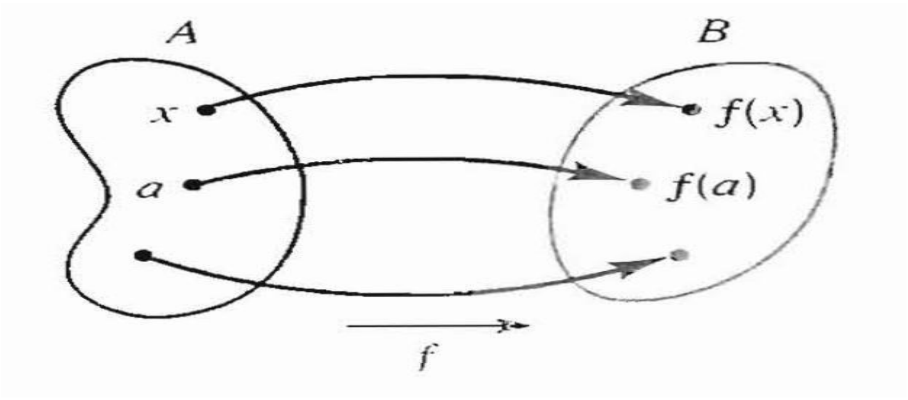


Figure 3:3 Representation of a function as a diagram (Stewart *et al.,* 2007:150)

- **A function box (representing of an input and output relationship)**

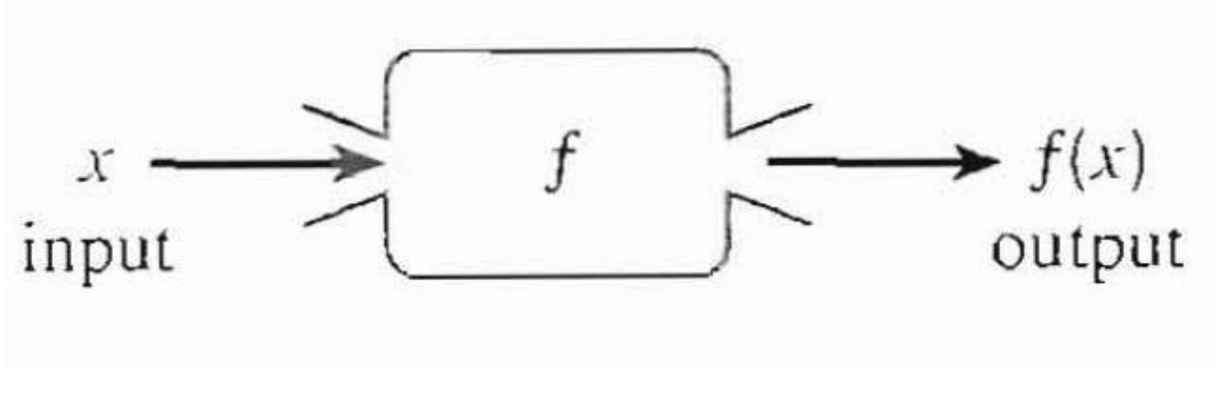


Figure 3:4 Representation of a function as a machine (Stewart *et al.,* 2007:150)

- **A set of ordered pairs**

Example:

$$\{(2;3)(4;6)(3;-1)(6;6)(2;3)\}$$

- **A graph**

Example:

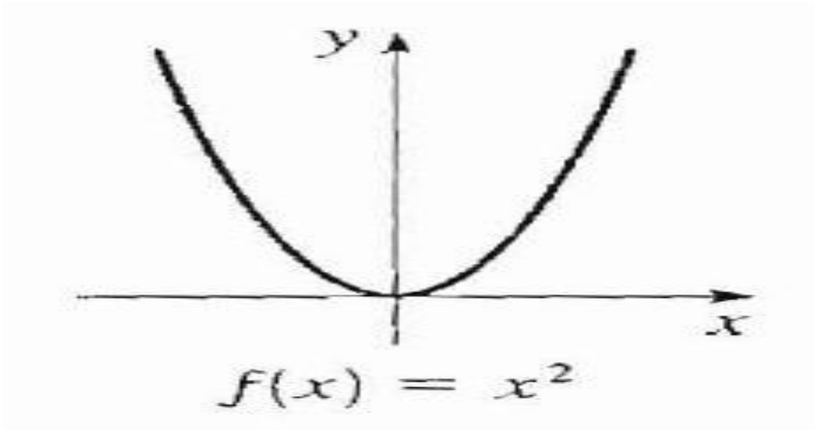


Figure 3:5 Graphical representation of a function (Stewart *et al.,* 2007:150)

- **A formula**

Example:

$$A(r) = \pi r^2$$

Area of a circle from which *r* is the independent variable and *A* is the dependent variable

Tall (1988:2) goes on to state that the concept of function "may be viewed as an action that assigns to each element x in A a corresponding element $f(x)$ in B, or as a graph, or as a table of values".

3.1.2 Procedural and conceptual knowledge of functions

The researcher needs to acknowledge the fact that in the literature one finds that there are generally two types of knowledge in mathematics education: procedural and conceptual. Also, the literature tells us that these two types of knowledge oppose each other concerning the

development of mathematical development (Schneider & Stern, 2010). Some commentators argue that mathematics procedures develop first, and on the other hand, others assert that mathematical concepts are the ones that develop first (Rittle-Johnson & Alibali, 1999:176). Furthermore the literature tells us that there are commentators who highlight the need to consider both types of knowledge because of the role they play in the teaching of meaningful learning of mathematics with understanding. In light of this, Rittle-Johnson *et al*. (2001:347) posit that conceptual and procedural knowledge concerning the development of mathematical knowledge may develop together in a handover hand process rather one type strictly preceding the other. In this study both types of knowledge were accounted for, but in the context of examining conceptual understanding of mathematical concepts, in particular the function concept and Euclidean geometry. In other words, procedural knowledge was embedded in both the function tasks and Euclidean tasks. In the following paragraphs the researcher will define and describe both the procedural and conceptual knowledge in relation to the function concept.
Van de Walle (2010:24) defines conceptual knowledge (comprehension of mathematical concepts, operations, and relations), as knowledge about the relationships or foundational ideas of a topic. Schneider and Stern (2010:179) describe conceptual knowledge of the function concept as the knowledge of the properties of functions; it is abstract in nature; it can be consciously accessed; it can be largely verbalized and flexibly transformed through processes of inference and reflection. Rittle-Johnson *et al*. (2001) are of the opinion that conceptual knowledge provides an abstract understanding of principles and connections between representational systems of the concept of the function. In relation to this study then, conceptual knowledge involves inherent or explicit understanding of the properties of the function concept. On the other hand, procedural knowledge (skills in carrying out procedures flexibly, accurately, efficiently and appropriately) is knowledge of the rules and procedures used in carrying out mathematical processes and symbolism used to represent mathematics (Van de Walle, 2010:24). Procedural knowledge involves the ability to solve problems related to the concept of function through the manipulation of mathematical skills with the help of pencil and paper, calculator and computer (Carpenter & Lehrer, 1998). Further, it allows students to solve problems quickly and efficiently, because it involves manipulation of skills (Schneider & Stern, 2010). Its mechanical nature, however, implies that procedural knowledge is not, or only partly, open to conscious inspection and can, as a result, be hardly verbalized or transformed by higher order thinking skills (Schneider & Stern, 2010:179). Hence, understanding the concept of function in one representation does not necessarily mean that a student understands it in another representation; it is thus important for students to understand the concept of function in different representations, and to be able to translate and form linkages

among and between them (Even, 1990:525). In the following section the researcher will discuss and describe learning theories concerning the function concept.

3.1.3 Theoretical perspectives underpinning the function concept

When discussing learning associated with the concept of function, there is no generally accepted theoretical framework as a basis for discussion (Eisenberg, 1991). Influential theorists have developed specific theories that state how learners come to learn the function concept. For instance, the work of noted theorists such as Gray and Tall (1994), Sfard (1991), Vinner and Tall (1981), posit that a prominent component of coming to learn the function concept is related to the encapsulation, or reification of processes.

3.1.3.1 The operational and structural form of the function concept

Sfard (1991) claims that a function is a concept which is first acquired *operationally*, and the transition to its *structural* form – mathematical object – takes place in three stages namely: *interiorization, condensation and reification* (Sajka, 2003:230). Sfard (1991:4) emphasizes that operational and structural conceptions of the concept of function are not mutually exclusive, "although ostensibly incompatible, they are in fact complementary".

Research studies describe the structural point of view of the concept of function as a set of ordered pairs; on the other hand, they describe the operational one as a computational process (Sajka, 2003:230). Transformation focuses on "processes, algorithms and actions rather than the object", i.e. it reflects an operational point of view of the concept of the function (Sfard, 1991:4).

When a learner interprets the concept of function as a process, it implies regarding it as a potential rather than actual entity, which comes to existence upon request in a sequence of actions (Sfard, 1991:4). On the other hand, when a learner sees the concept of the function as an object (structural point of view), it means being capable of referring to it as if it were a real thing; it also means being able to recognize the idea at glance and to manipulate it as a whole, without going into details (Sfard, 1991:4). In the following sections, the researcher will articulate the three stages of the learning of the function concept.

The first stage is the *interiorization.* At this stage a student becomes familiar with the processes (operations performed on low-level mathematical objects) which will eventually give rise to a new concept, (Sfard, 1991:18). For example, in the case of the concept of function, this process prevails when the idea of variable is learned and the ability of using a formula to find values of the "dependent" variable is acquired (Sfard, 1991:19).

The second stage is the phase of *condensation* (a period of "squeezing" lengthy sequences of operations into more manageable units). At this stage a student "becomes more and more capable of thinking about a given process without feeling an urge to go into details" (Sfard, 1991:19). In relation to the concept of a function, a student can investigate functions, draw their graphs, combine couples of functions, for example by composition, even to find the inverse of a given function (Sfard, 1991:19). This phase lasts as long as a new entity remains tightly connected to a certain process (Sfard, 1991:19).

The third stage is the *reification phase* (a sudden ability to see something familiar in a totally new light). Sfard (1991:20) describes this phase as "an instantaneous quantum leap: a process solidifies into object, into a static structure". In the case of the concept of function, reification may be evidenced by proficiency in solving equations in which the unknown are functions, by ability to talk general properties of different processes performed on, and by ultimate recognition that computability is not a necessary characteristic of the sets of ordered pairs that are to be regarded as functions (Sfard, 1991:20). Without reification students' conceptual understanding of the concept of function, "will remain purely operational" (Hansson, 2006:26) (see Figure 3:6).

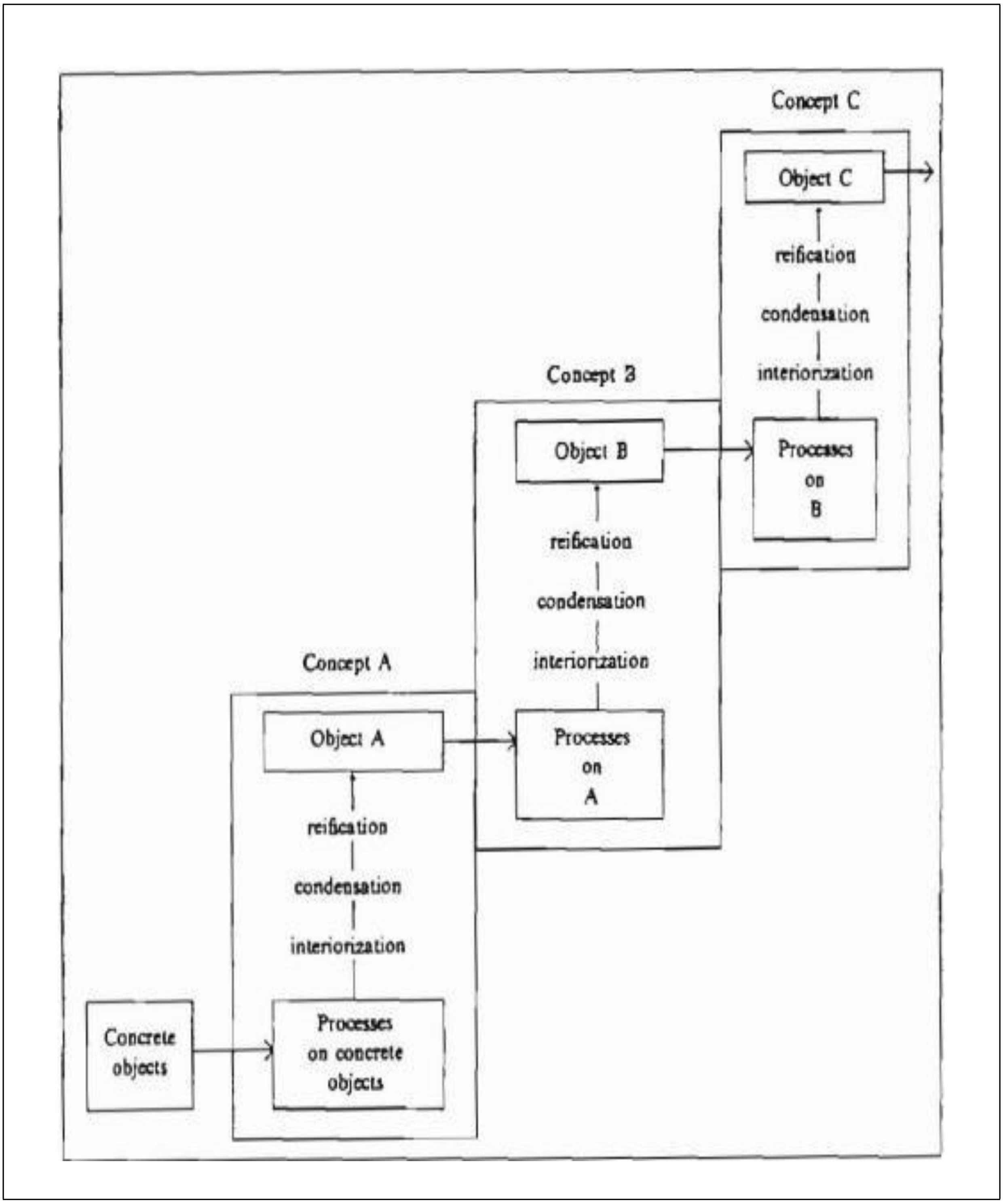


Figure 3:6 Model for concept development (Sfard, 1991:23).

Hence, this three-phase schema need to be viewed as a hierarchy, which implies that a learner should complete the first phase prior he/she can reach phase 1 and 2 respectively, (Sfard, 1991:21).

3.1.3.2 Concept images and concept definitions concerning the function concept

Vinner and Tall (1981) use the construct of *concept images* and *concept definitions* to analyse students' understandings and misconceptions of the function concept. They showed that most students use their own personal concept definition for the concept of function, giving highly idiosyncratic meanings to the term (Tall & Akkoć, 2003). Hence, the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition (Vinner & Dreyfus, 1989:356).

According to Vinner (1983:294), a concept definition is a verbal description of a mathematical concept that accurately describes the concept in a non-circular way. For Tall, (1988:2) the concept definition of the concept of function may be given in the form "relation between two sets A and B in which each element of A is related to precisely one element of B"; however, the experience of the concept has many other facets.

On the other hand, a concept image is the mental picture that is associated with the concept name in a student's mind (Vinner & Dreyfus, 1989:356). Vinner (1983:293) explains the word "picture" in a broadest sense of the word; as a result, it includes any visual representation of the concept, including symbols, diagrams, graphs, etc. A student's concept image consists of all the mental pictures that he or she associates with the concept of function (Clement, 2001:745). Clement (2001) goes a step further to note that "a student's concept image can differ greatly from a mathematically acceptable definition; and learners' concept images are often very narrow, or they may include erroneous assumptions". The student's image is a result of his or her experience with examples and non-examples of the function concept (Vinner & Dreyfus, 1989; 356). A student's concept image of function may be limited to the graph of a relation that passes the vertical-line test, or to a machine that furnishes an output when an input is supplied (Clement, 2001:745).

3.1.3.3 Elementary procept and precept concerning the function concept

Gray and Tall (1991; 1994) extend the work of Sfard (1991) by emphasising the importance of mathematical symbolism. According to Tall and Gray (1994), the notation of the concept of function, for example, $f(x) = 2x + 3$ tells us two things at the same time. Firstly, it tells us how to calculate the value of the function for particular arguments, and secondly it encapsulates the whole concept of function for any given argument.

Gray and Tall (1991; 1994) speak of an *elementary procept* and *precept* to refer to the dual use of symbolism as process and concept. Elementary procept is the amalgam of three components: “a process that produces mathematical objects and a symbol that represents either the process or objects” (Verhoef & Tall, 2010:2). On the other hand, precept consists of a collection of elementary procepts representing the same object. For instance, the notation $f(x)$ where is the name of a function entails the ideas of a process to follow to produce output from input (Gray & Tall, 1994:121).

DeMarois (1996) claims that the meaning for symbols develops by first doing procedures such as evaluating a function at a given number. Later, the procedure may then mature into processes in which the learner comprehends the idea that a function produces output from input without having to apply a specific algorithm to an input to get an input. Once the procedures mature to processes, the function concept can be thought of as object. In this case, the learner can perform operations on the function concept, such as composition or differentiation. Tall (1996:21) highlights that precepts allow the learner “not only to carry out procedures, but to regard symbols as mental objects, so they cannot do only mathematics, think about concepts”. The ability to think flexibly about the function concept as both a process and object is referred to as proceptual thinking (DeMarois, 1996:2). Verhoef and Tall (2010:2) assert that proceptual thinking gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation.

3.1.3.4 Function model concerning the function concept

Lastly, O’Callaghan (1998) used the function model to study students’ conceptual knowledge of the function concept. This model consists of four component competencies, namely modelling, interpreting, translating and reifying (O’ Callaghan, 1998:24). These four competencies are explained as follows by (O’ Callaghan, 1998:24-25):

The ability to represent a problem situation (real world situation) using functions is labelled as the first component of understanding the function concept, and it is referred to here as *modelling*. This component can be divided into a number of subcomponents depending on the representation system used to model the situation. The three most frequently used representations for functions are equations, tables, and graphs.

Example:

Two hot air balloons are being observed. A blue one is 150 metres above the ground and is descending at a constant rate of 20 metres per minute. A red balloon is only 10 metres above the ground and is rising at a constant rate of 15 metres per minute.

a) On the same set pair of axes, draw graphs to show the heights of the two balloons from the time they were first observed until the time when the blue balloon hits the ground.

The ability, labelled *interpreting*, is the second component of the model. This component can also be partitioned in to subcomponents, which would again correspond to each of the three main representational systems for functions. Problems could require students to make different types of interpretations or to focus on different aspects of a graph, for example, individual points versus more global features.

Example:

The table below provides the temperature of the water in a pan as it is set on the stove to boil.

Time (min)	0	2	4	6	8	10	12	14	16	18
Temperature (degrees)	22	29	36	44	51	58	65	72	80	8

a) Find the equation that models this data.

The ability to move from one representation of a function to another, or to *translate*, is the third component of the function model.

Example:

A business group wants to rent a meeting hall for its job fair during the week of the school holiday. The rent is R3 500, which will be divided among the businesses that agree to participate. So far only five businesses have signed up.

a) At this time, what is the cost for each business?

b) Make a table to show what happens to the cost per business as additional businesses agree to participate.

c) Write a function for the cost per business related to the number of additional businesses that agree to participate.

d) How many additional businesses must agree to participate before the cost per business is less than R150?

The final component of the model for functions is reification, defined as the creation of a mental object from what was initially perceived as a process or procedure. This mathematical object is then seen as a single entity that possesses certain properties and that can be operated on by other higher level processes, such as transformations or composition. These processes are achieved by few learners.

Example:

The temperature on a certain afternoon is modelled by the function where $C(t)\frac{1}{2}t^2+2$ represents hours after 12:00 $(0 \le t \le 6)$ and is measured in $^{\circ}C$.

a) What "shifting" and "shrinking" operations must be performed on the function defined by $y=t^2$ to obtain the function defined by $y=C(t)$?

b) Suppose you want to measure the temperature in $^{\circ}F$ instead. What transformation would you have to apply to the function $y=C(t)$ to accomplish this?

c) Write the new function $y=F(t)$ that results from this transformation.

Associated with the above four components of a conceptual knowledge is a set of procedural skills (O'Callaghan, 1998:26). These skills consist of transformations and other procedures that allow students to operate within a mathematical representational system.

Example:

Given a function determined by the equation: $y=x^2-2x+3$

Give the equation of the new graph originating if:

a) The graph of is moved ("shifted") three units to the left.

b) The x-axis is moved ("shifted") down three units relative to the graph.

O'Callaghan (1998:26) argues that the function model encompasses the process-object duality, theories regarding invariants under actions, multiple representations, and functions prototypes emphasized in some of the other models discussed in the preceding paragraphs. However, it is also important to note the differences between the function model and the other alternative frameworks (discussed above). O'Callaghan (1998) maintains that two differences can be noted, namely the distinction between a relation and a function, and that the function model is not a process model. Thus, the function model does not articulate what students do to understand functions. In contrast, this model is an explanation of various components of knowledge relevant to the function concept and is firmly rooted in a problem-solving environment (O'Callaghan, 1998:27).

The importance, definition and some theoretical perspectives underpinning the function concept have been articulated. In light of this, the function model as adopted by O'Callaghan (1998) that identifies and applies the four component competencies of a conceptual knowledge of functions and a set of procedural skills, proves to be relevant for this investigation; hence it will be used to investigate and describe teachers' conceptual knowledge of functions.

3.2 Section 2: The teaching and learning of the Euclidean Geometry

When one examines the CAPS document for secondary school mathematics (van Putten *et al.*, 2010; Naidoo, 2013) it is evident that the learning and teaching of geometry, in particular Euclidean geometry is based on two related frameworks: spatial sense and geometric reasoning, as well the specific geometric content. Spatial sense and geometric reasoning involve students' thinking and reasoning about space and shape. Herein, spatial sense is perceived as an intuition about shapes and relationships among shapes and it appeals to the ability to cognitively visualize objects and spatial relationships. On the one hand the geometric content favours the more traditional sense, that is to say knowing about symmetry, triangles, parallel and intersecting lines, and the like (Van de Walle *et al.*, 2010:399).

According to Van de Walle *et al.* (2010:400), students are capable of growing and developing in themselves the ability to think and reason in geometric thought. Nonetheless, in 2012, in the South African schools, the concept of geometry (Euclidean) was considered an optional mathematics topic; hence it was in the third paper of mathematics (van Putten *et al.*, 2010:22). In connection to this, Euclidean geometry was viewed as a challenging part of the content of school mathematics, and many teachers were not able to teach it because they had difficulty with the content knowledge of geometry (Bowie, 2013:6; Naidoo, 2013:27). Consequently, learners were not offered the opportunities to engage in mathematical activities that involved Euclidean geometry; hence they could not deal with the content of first-year University mathematics due to their lack of mathematical content knowledge in relation to the Euclidean geometry (Naidoo, 2013:27). In 2012, South African schools witnessed the re-introduction of the teaching and learning of traditional geometry, and teachers were expected to teach it (Bowie, 2013). The reason for the re-introduction might be that geometry is one of the most valuable content in school mathematics (Fujita & Jones, 2003). In the following sections the researcher portrays some of the critical issues underpinning the teaching and learning of geometry.

3.2.1 The teaching of geometry

There is a general consensus in the mathematics education research community that school mathematics, and in particular Euclidean geometry, are significant components of the school curriculum since they offer a context for developing students' higher order thinking skills (DBE, 2011). According to Jones (2002), Euclidean geometry empowers learners to comprehend geometric representations, to reason logically, make justification, construct connections among ideas, make sense of their physical environment, and develop spatial and location skills and visualize objects from different perceptions.

Nevertheless, the teaching and learning of secondary school geometry remains a challenge for mathematics teachers. One of the reasons related to this challenge is the dual nature of geometry; it is both theoretically and concretely associated to the real world (Fujita and Jones, 2002:384). Euclidean geometry comprises a complex network of interconnected geometric ideas requiring representational systems and reasoning skills in order to conceptualise and analyse not only physical but also imagined spatial environments (Battista, 2007:843). Also in school mathematics, the fundamental learning of geometry involves naming, describing, classifying and making links to measurement, position and movement (DBE, 2011; Naidoo, 2013).

In connection to this, Jones (2002) asserts that the teaching of geometry is based on teachers' dispositions towards teaching it as well as their content knowledge of geometry. Consequently, it is of utmost importance for mathematics teachers to appreciate the history and cultural context of geometry. This is so because geometry is an integral part of people's cultural experiences, and it involves the architecture as well as design, which in turn appeals to people's visual, aesthetic and intuitive senses (Jones, 2000:109). Van de Walle *et al.* (2010:ix) assert: "The fundamental core of effective teaching of mathematics combines an understanding of how children learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning on daily basis." As a result mathematics teachers are compelled to create mathematics (geometry) tasks that offer learners opportunities to engage in mathematical activities that are in harmony with the five components of mathematics proficiency (see par 2.3.1).

Another critical aspect that involves the teaching and learning of geometry is the use of technology in mathematics instruction (Van de Walle *et al.,* 2010). In the teaching and learning of geometry the term technology refers to dynamic geometry software. Within the context of mathematics education the use of dynamic geometry software has been seen as enhancing students' understanding of mathematical ideas. One can mention the Geogebra® software which allows students to construct geometrical shapes (lines, polygons and circles) on the computer screen and then manipulate and measure them by dragging the vertices (Van de Walle *et al.,* 2010:117). According to Stols (2013), Geogebra® offers mathematics teachers opportunity to reflect on their teaching practice and further develop new ways to connect, extend and enhance their instructional activities.

In the following section the researcher discusses the learning of geometry, and to do this the researcher will focus on the three geometrical paradigms, Duval's perspective and the van Hiele theory of geometric thought. Also it is important to acknowledge the fact that this study was informed by the classical work of P.M. and D. van Hiele (cf. van Hiele, 1986). Thus, the researcher did not employ the other two frameworks.

3.2.2 The learning of geometry

Kuzniak *et.al* (2010) distinguish between three theories related to the learning of geometry. The authors argue that each theory pays attention to different aspects of geometry. In doing so, these theories appeal to the different perspectives in order to examine, explore and investigate some of the critical issues pertaining to geometry education, from learning to curriculum issues. These theories comprise the famous theory of van Hiele (1986) of the levels of geometric development. The second perspective is the geometrical paradigms which are rooted in the work of Houdement and Kuzniak (2003). The third perspective is based on the work of Duval (1995) which focuses on the four apprehensions related to geometrical figures. The role of geometrical paradigms in geometry education is to assist with both the classifying of perspectives (forms of argumentation) and investigating students' difficulties and mathematical errors; in other words, it is epistemological in nature. On the one hand the theory of Duval (semiotic perspective) in geometry education is used as a means to explore and examine the different registers used in geometry as well as their implications during mathematical activities (Kuzniak *et al*., 2010:672). Lastly, the van Hiele theory focuses on students' reactions, productions and solutions to geometric tasks; in other words, it is phenomenological in nature (Kuzniak, *et al*., 2010:672).

3.2.3 Geometrical paradigms

The significance of geometry, according to Bowie (2013:49), is that it occupies a role "which is at once concrete and practical and yet highly abstract and theoretical." That is to say it is dual in nature. In connection to this, Houdement and Kuzniak (2003) propose three geometrical paradigms namely Geometry I, II and III respectively. Geometry I is named the

Natural Geometry which is based on the real and sensible world as a form of validation. On the other hand Geometry II is dubbed the Natural Axiomatic Geometry which involves a model that approaches reality, and lastly Geometry III is called the Formal Axiomatic Geometry, which is based on the system of axioms, hence it is disconnected from the real world (Braconne-Michoux, 2011).

The first geometric paradigm, which is the Natural Geometry requires the engagement of the mathematical student in activities that involve the construction of material objects. Herein, the student is expected to make deductions by means of experiment, perception and use of instruments. The context from which these mathematical activities occur is heavily dependent on reality. The second geometric paradigm, which is the Natural Axiomatic Geometry, requires a mathematics student to use deductions in correspondence to an axiomatic system. Herein, the axioms are perceived as an attempt to provide a model of the space around the student. The third geometric paradigm, which is the Formalist Geometry, requires the mathematics student to engage in mathematical activities that are in harmony with abstract Euclidean Geometry (Bowie, 2013:51). Note that, in order to comprehend how these paradigms are used, it is significant to take into account the notion named Geometrical Working Spaces (Houdement, 2007), but owing to the limitations of the study the researcher will not elaborate on this notion. Table 3.1 outlines the three geometrical paradigms.

In light of the preceding paragraph, one can note that the three paradigms of geometry are distinct to each other but at the same time there is a link among them. Bowie (2013) asserts that both Geometry I and II form part of the geometric content of the school mathematics. On the other hand, Geometry III is associated with the geometric content of university mathematics.

	Geometry I	Geometry II	Geometry III
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to the schemas of reality	Logical
Deduction	Near to real, and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms
Kind of spaces	Intuitive and physical spaces	Physical and geometric spaces	Abstract Euclidean spaces
Status of the drawing	Object of the study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and Construction	Properties and demonstration	Demonstration and links between the objects. Structure.
Objects	Physical	Theoretical	Theoretical
Validation		**Deductive**	**Deductive**

Table 3:1 Summary of the three geometrical paradigms (Braconne-Michoux, 2011:619)

3.2.4 Semiotic perspective of geometrical reasoning

In geometry three registers are utilised, namely the register of natural language, the register of symbolic language and the figurative register. In light of this, Duval (1995) approaches geometry form a cognitive and perceptual perspective and distinguishes between four cognitive apprehensions related to the learning of geometry, in particular for a geometrical figure (Kuzniak *et al.,* 2010).

These apprehensions consist of perceptual, sequential, discursive and operative (Jones, 1998:31). Perceptual apprehension entails the recognition of a geometrical figure in a plane or in depth. Herein, the perception about what the figure reveals is based on figural organization laws and pictorial cues. Also the perceptual apprehension involves the ability to name geometrical figures and the ability to recognize in the perceived geometrical figures numerous sub-figures. At the sequential apprehension the student is required to construct a geometrical figure or describe its construction. Herein, the construction of a geometrical figure is independent of the perceptual apprehension but dependent on the technical constraints and on mathematical properties. At the discursive apprehension the assumption is that mathematical properties represented in a drawing cannot be determined by means of perceptual apprehension. And lastly, the operative apprehension involves different ways of modifying a given geometrical figure and it is dependent on the other three apprehensions (Duval, 1995:145-147). In connection to the teaching of geometrical figures the semiotic perspective argues for "special and separate learning of operative as well as of discursive and sequential apprehension are required" (Jones, 1998:31). Further the semiotic perspective calls for the use of technology in teaching geometry because it enhances the development of both the sequential and operative apprehension (Jones, 1998).

3.2.5 The van Hiele theory of geometric thought

The theoretical framework that is in harmony with the learning of geometry, calls to mind the classical work of the van Hieles. This is so because I realize that most of the research that has been done about the teaching and learning of geometry involves the ideas of the van Hieles. The classical work of the van Hieles can be traced back to the mid-1950s: "*The Child's Thought and Geometry*" paved the way for the level theory of mathematical development. This theory is primarily concerned with the development of geometrical concepts on the basis of Euclid geometry. In setting the scene for their work, the van Hieles, brought to the fore the assumption that the learning of Euclid geometry is discontinuous (discrete) and is characterized by five hierarchical and qualitatively different levels of geometric understanding (Usiskin, 1982:4) This means that progress from one level to the next level is more dependent on mathematics teaching than on age or maturation. For example, the mathematics teacher may reduce the mathematical content to a lower level, leading to rote memorization; consequently learners cannot bypass levels and develop meaningful learning of geometrical concepts with understanding. This also implies that the levels are separate of each other and that transition from one level to the next one is not gradual, but sudden (Crowley, 1987). Geometrical ideas implicitly understood at one level become explicitly understood at the next level. Further, each level has its own language and its own system of relations connecting linguistic symbols. This implies that a relation which is correct at one level can reveal itself to be incorrect at another

(Van de Walle *et al.,* 2010). The following is a synopsis of these levels, inspired by the work of the van Hieles (Crowley, 1987:2-3; Van de Walle *et al*., 2010:400- 404).

3.2.5.1 Level 0: Visualisation

Students are able to identify, name and manoeuvre geometric figures according to their appearance alone. This suggests that students make decisions based on perceptions, not reasoning. For example, a learner recognizes a rectangle by its form. Consequently students recognize and operate on geometric shapes such as triangles, squares and rectangles in their visible form. At this level, properties of figures such as angles and lengths do not form part of the reasoning process.

3.2.5.2 Level 1: Descriptive/analytic

Students are able to recognize, analyse and characterize geometric figures by their properties. Consequently students do not observe geometric figures according to form. For example, a rectangle is regarded a rectangle because it has four right angles: diagonals are equal, and opposite sides are equal in length. But, at this level learners do not observe relationships within or/and between classes of geometric figures. Instead, properties are observed in isolation and they are not linked. At this level geometric figures are seen as collections of properties. For instance, students struggle to observe the relationship between a square and the rectangle.

3.2.5.3 Level 2: Abstract/relational/Informal deduction

At this level students are able to form definitions, distinguish between necessary and sufficient sets of conditions, and understand and sometimes even produce logical arguments for a geometric concept. This suggests that students are able to observe relationships between properties of classes of geometric figures by informal deduction and see that one property follows from another. This suggests that students are able to recognize, analyse and can determine relationships of properties both within and between geometric figures (see Appendix, K, L and M).

3.2.5.4 Level 3: Formal deduction

At this level students are able to construct theorems within an axiomatic system. This level is characterized by students being able to construct proofs, understand the role of axioms and definitions and know the meaning of necessary and sufficient conditions (see Appendix M).

3.2.5.5 Level 4: Rigour

Students are able to reason formally about mathematical systems. This level is characterized by students being able to comprehend formal aspects of deduction, such as constructing and comparing mathematical systems. Herein, geometry is perceived in the abstract.

3.2.6 Phases of teaching and their characteristics

The van Hieles also brought to the fore the five teaching phases that describe the objective of learning geometry. These phases are primarily characterized by the mathematics teacher's role during the teaching and learning of geometry. In each phase the teacher plays a significant role. Consequently both the geometric content and learner's role come into play and they meet the mathematics teacher in the mathematics classroom (Crowley, 1987:5-6; van Hiele, 1959/1985:243-252). Van Hiele argues that the levels of geometric thought are independent of the teaching instruction used but are essential in the explanation of thought. Also it is important to acknowledge that certain methods that are used to teach geometry hinder the development of higher levels. Consequently students do not progress to the next level. In light of this van Hiele posits that mathematics teachers need to account for the awareness of geometric properties at each level, linguistic symbols at each level, reasoning at different levels, as well as maturation, which leads to a higher level. Note that van Hiele considers maturation as a process of apprenticeship. To account for these aspects, van Hiele recommends the phases which, in the process of apprenticeship, lead to a higher level of thought (Clement, 2004:63). The following is a synopsis of these phases.

3.2.6.1 Information/inquiry

This phase is characterized by the teacher who, through discussion, identifies what the students already know about the geometrical concept to be taught, and the learners become oriented to the mathematical concept. For example, learners may be required to distinguish between examples and non-examples of polygons.

3.2.6.2 Guided instruction/directed orientation

At this phase, with the guidance of the mathematics teacher, students explore their field of investigation according to carefully structured and guided mathematical tasks. The mathematics teacher, for example, is responsible for developing tasks that will require students to be actively engaged in geometric thought. Learners may, for example, be required to construct angles and/or polygons.

3.2.6.3 Explication

This is the phase during which the mathematics teacher encourages students to communicate and use the language of Euclid geometry correctly in context. Through classroom discussion the students articulate what they have learned about the field of the investigation in their own words. For example, students may be required to communicate geometrical ideas about properties of polygons. At this phase the role of the mathematics teacher is to lead the students' discussion.

3.2.6.4 Free orientation

This is the phase during which learners apply relationships they are learning to solve mathematical tasks, and investigate more open-ended tasks. At this phase the mathematics teacher's role is to encourage students to explore the specific field of the investigation wider and deeper and to facilitate deeper understanding.

3.2.6.5 Integration

This phase is aimed at encouraging students to reflect on their investigations and thinking, and at consolidation of the learning that has taken place. This suggests that the mathematics teacher's role is to assists the students to summarize and integrate what they have learned, constructing a new network of geometric connections and relations.

In connection to the preceding paragraphs, it's been about five decades since the work of the van Hieles offered the mathematics research community the opportunity to investigate university students', teachers' and primary and secondary school learners‟ understanding of geometric ideas. The work done with respect to the van Hiele theory has indicated that the understanding of geometry of learners, students and teachers is in harmony with either level 0 or 1, in particular in the South African context (Bowie, 2013). Though it [van Hiele theory] is hailed as the hallmark of the teaching and learning of geometry, much work regarding the five teaching phases needs to be taken into consideration in order to clear the confusion surrounding the relationship between the content and levels of thinking as well as the teaching phases (Bowie, 2013:47).

3.3 Conclusion

The purpose of this chapter was to provide a discussion on the teaching and learning of both the function concept and geometry. In this discussion the researcher used the literature review as a point of departure in order to examine some of the critical issues related to the theories of both mathematical concepts. In the following chapter the researcher focuses on the research design and methodology that were employed in this study.

CHAPTER 4: RESEARCH DESIGN AND METHODOLOGY

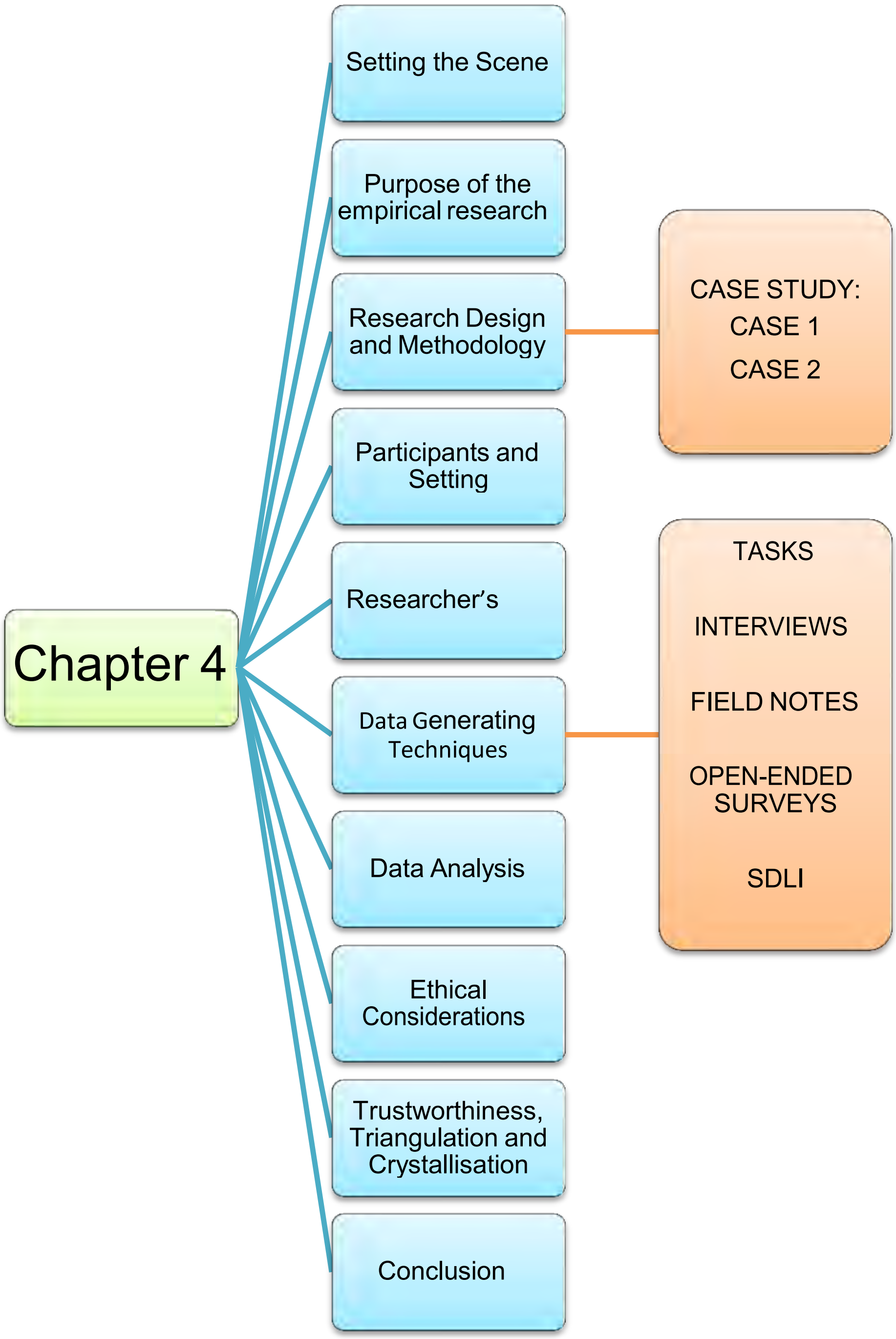


Figure 4:1 Outline of Chapter 4

4 Setting the scene

In the previous chapter the researcher provided an elaborate discussion on both the function concept and Euclidean Geometry, focusing on the nature and development of such concepts. In this chapter the researcher will focus on the research design and methodology applied in the study. In nature, the study intended to come to a better understanding of the phenomenon of collaborative professional learning, particularly in an education context. The research team (the student and supervisor) therefore undertook a qualitative research within an interpretative paradigm. Before we discuss the research design and methodology, we consider it imperative to note that there are three broad approaches to educational research: qualitative research, quantitative research and mixed methods research. Two important components in each approach to research are that it centres on philosophical assumptions as well as methodology (Creswell, 2009:5). In this study a qualitative approach was advanced in harmony with the interpretive worldview of reality.

According to Borg *et al.* (1993:198) a qualitative approach seeks to explore and understand a complex phenomenon by examining it in its totality. Henning *et al.* (2004:3) argue that, in qualitative research, as researchers we seek to understand, and also explain in argument by employing the evidence from multiple data sources, what the phenomenon that we are investigating is about. It is against this background that qualitative researchers may not know what to focus on until the research has commenced (Borg *et al.,* 1993:198). We concur, because before the commencement of the study we had not expected that this study would include Case 2.

The philosophy that motivates this study is interpretivism. The interpretive philosophical assumption of reality focuses on understanding and interpreting the world in terms of people. Thus, in the interpretive perspective, the assumption is that the world is made up of people with their own assumptions, intentions, attitudes, beliefs, and values, and that the way of knowing reality is by exploring the experiences of others concerning a particular phenomenon (Cohen *et al.,* 2000, 181). An interpretive paradigm therefore centres on "peoples experiences, on how people construct the social world by sharing meanings and how they interact with or relate to each other" (Nieuwenhuis, 2010a:57).

Pertaining to the aforementioned, the research team within the context of collaboration intended to investigate the influence of a Collaborative Learning Programme (CLP) stemming from participants (in this case in-service and student mathematics teachers) by personal interaction with the participants spending extensive time with the participants and probing them to obtain detailed meaning of understanding the concept of function as well as Euclidean Geometry. This was significant, because the purpose of the study was to explore and describe collaboration towards teaching proficiency of mathematical concepts. We aimed at investigating participants' shared ideas, mathematical knowledge and experiences regarding the teaching and learning of mathematical concepts. Consequently the context for such investigation is assumed to be in harmony with the views and ideas of social constructivism on teaching and learning of mathematics.

4.1 Purpose of the empirical research

The general aim of this study was to investigate how engaging participants in collaborative learning about central mathematical concepts influences their fundamental views and beliefs about mathematics and mathematics teaching and learning, may improve their proficiency in teaching key mathematical concepts. In this context the participants comprised township secondary school mathematics teachers (see Case 1 in 4.3.2) as well as BEd University students (see Case 2 in 4.3.2).

The study specifically intended to describe and interpret how collaborative learning influences participants' fundamental dispositions towards teaching and learning of mathematics, as well as the impact of such learning environment on their conceptual knowledge of important mathematical ideas. In the township secondary school context the focus was placed on the *concept of function* and in the university context the study focused on *Euclidean geometry*.

4.2 EMPIRICAL RESEARCH

4.2.1 Research Design

The research design, according to Punch (2006:48), comprised strategy, conceptual framework, who or what would be studied and the tools and procedures to be used both for collecting and for analysing data. Since the study was qualitative in nature (its emphasis on the interpretive and subjective nature), the research design appropriate to it, was a phenomenological research design (Nieuwenhuis, 2010a). A phenomenological research design centres on participants' lived experiences, on how they interact with or relate to one another by examining their views and opinions related to the phenomena that is under investigation (Nieuwenhuis, 2010a:59). In order to generate data to answer research questions and accomplish the research objectives, the study portrayed this design, by conducting a multisite case study research (see 4.3.2) that employed semi-structured interviews, tasks, field notes and open-ended surveys. Also in the study we used the SDLI (Self-Directed Learning Instrument) to measure student participants' self-directed learning abilities. The rationale behind the chosen research design will be evident in the paragraphs that follow.

4.2.2 Research Methodology

Case Study

Creswell (2009:13) defines a case study as the investigation of a process, activity, event, programme or individual bound within a specific time and context. Unlike historical study which focuses on past phenomena, a case study focuses on contemporary phenomena by investigating a single instance of the phenomenon (Borg *et al.*, 1993, Cohen *et al.*, 2000, Creswell, 2009; Nieuwenhuis, 2010a). In association to this, the contemporary phenomena in this study were instances of collaboration towards teaching proficiency of mathematical concepts in secondary schools, in particular the function concept and Euclidean geometry. This study was conducted in a series of sessions over a period of about eighteen months, but at different venues (sees both Table 4.1 and 4.2 below). Hence in this study the context comprised such notions as the setting, collaboration and mathematical content.

A case study entails an inclusive understanding of how participants relate and interact with each other in a specific context and how they make meaning of a phenomenon under investigation (Nieuwenhuis, 2010a:75). This investigation takes place through detailed, in- depth techniques of data collection, focusing on multiple sources of information that are rich in contexts (Borg *et al.,* 1993). These may include interviews, observations and document analysis (Borg *et al.,* 1993; Cohen *et al*., 2000). In this study the research team used multiple data sources in collecting data. These included tasks, interviews, field notes and open- ended surveys related to several mathematical concepts.

Data were collected in numerous contexts: whole-group discussions between the research team and participants, and interactions between participants as they worked on activities designed to develop their conceptual understanding of mathematical concepts. Also, data were collected from the participants in an individual setting. Data emanating from interviews and the sessions of the Collaborative Learning Programme (CLP) were recorded by means of both audio recorder and verbatim transcription, as well as field notes. Participants' task sheets and other written information related to the study also were taken into consideration.

Consequently, the case study offered the research team the opportunity to collaborate with participants throughout the study. The research team interviewed them rigorously, worked with them on mathematical tasks and observed them during mathematical activities (Borg *et al*., 1993:203). Consequently the research team had the opportunity to collaborate with the participants during a variety of activities, including but not limited to the solving of mathematical situations, decision making and the mathematical discourse that was taking place throughout the study.

McMillan (2000) make a distinction between two types of case study namely within-site and multisite. The former refers to a case study conducted within one research site while the latter is regarded as the case study conducted in more than one research site (McMillan, 2000:11). For the purpose of this study, a multisite case study was conducted. This case study was conducted in two distinct areas, namely Kanana Township and Potchefstroom (see case 1 and 2 below). The former case study focused on a township mathematics secondary school. In the latter context, the case comprised a group of students (see case 2 below). The students were training to become mathematics teachers at the North-West University. In the following paragraphs the researcher provides an elaborate description of each case.

Case 1

The first case focuses on township secondary school mathematics teachers. These teachers come from two different schools. In this case, two secondary school mathematics teachers collaborated with the research team in a Collaborative Learning Programme (CLP). The purpose of the study (in this context) was to explore and describe collaboration towards teaching proficiency of mathematical concepts in township secondary schools. This part of the study compromised *eight sessions (2-9),* each running for about two hours. Also, the study focused explicitly on teachers' conceptual understanding of the function concept. The following Table (4.1) outlines these sessions.

Session	Data gathering technique	Type of activity	Purpose
Session 1 *(Before CLP)*	Pre-assessment task Open-ended survey	Individual activity • Complete the pre-assessment task • Complete open-ended survey	Assess teachers' conceptual understanding of the function concept Establish teachers' disposition towards teaching and learning of mathematics
Session 2	Pre-assessment task	Teachers work together • Reflecting on the pre-assessment task • Reflecting on the function concept • Completing Task 1 • Completing Task 2	Investigate the impact of the CLP.
Session 3			
Session 4			
Session 5			
Session 6	None		
Session 7	Task1		
Session 8	None		
Session 9	Task 2		
Session 10 *(After the CLP)*	Post-assessment task *(Open-ended survey)*	Individual activity • Complete the pre-assessment task • Complete open-ended survey	Assess teachers' conceptual understanding of the function concept Establish teachers' disposition towards teaching and learning of mathematics
Session 11 *(After CLP)*	Semi-structured interview	Individual activity	Evaluate the CLP

Table 4:1 Outline of the sessions of Case 1

In Case 1, the research team implemented two open-ended surveys, four function tasks and interviews to generate data. Open-ended surveys involved the same content and were conducted before and after the commencement of the CLP. These questions required participants to provide written responses to questions probing their fundamental dispositions about the teaching and learning of mathematics as well as function concept.

There were four distinct function tasks that we used in the study. The first task we call the pre-assessment task. This task was implemented before the CLP, and this was done in an individual context at the same venue (in school).

During the CLP we designed a second and third function task respectively. Both the second and third tasks were conducted in a collaborative context at the same venue (in school). In this regard data were audio-taped and transcribed verbatim.

The fourth task was implemented after the completion of the CLP; we referred to it as the post-assessment task, and this was done in an individual setting at the venue (school) where we were working with the teachers.

Interviews were conducted in the form of semi-structured interviews and were conducted after the completion of the CLP. These interviews were conducted individually at different venues (in school). Interviews were audio-taped and transcribed verbatim.

Case 2

In the second case the focus was placed on university students. In this case there was one *group* of students, comprising twelve students taking mathematics as one of their core modules. These students were in the final year of their study (BEd) and their exposure to the learning and teaching of the Euclidean Geometry had been limited. Consequently, the purpose of the study (within the context of collaboration) was to explore and describe students' conceptual understanding of the Euclidean Geometry and their dispositions towards teaching and learning of the Euclidean Geometry in mathematics classrooms. We also measured students' self-directed learning abilities. In connection to this, the students collaborated with the research team in a Collaborative Learning Programme (CLP). The CLP comprised *ten sessions,* each running for about ninety (90) minutes. The following Table (4.2) outlines these sessions.

Session	Data gathering technique	Type of activity	Purpose
Session 1	None	Individual activity	Meeting with the student for the first time Explaining the rationale behind the project Exploring the Geogebra® program
Session 2	Task 1	Students are encouraged to work together when completing their tasks	Constructing triangles using Geogebra® (Similarity and Congruency)
Session 3	Task 2	Students are encouraged to work together when completing their task	Constructing triangles and a rectangle by using a ruler, compass and protractor Constructing angles by using a protractor (Similarity and Congruency)
Session 4	Task 3	Students are encouraged to work together when completing their task	Properties of congruency and similarity of both triangles and quadrilaterals.
Session 5	Administered the pre-assessment SDLI	Whole class discussion	Students needs/expectations
Session 6	Field notes	Feedback session Large group presentation	Properties of congruency and similarity of both triangles and quadrilaterals.
Session 7	Administer Task 4 Field notes	Large group presentation Individual, but students are encouraged to work together when completing their tasks	Circle geometry. Angle at the centre versus angle at the circumference. Properties of a Cyclic quadrilateral.
Session 8	Administer Task 4	Large group	Circle geometry.

		presentation Students are encouraged to work together when completing their tasks	Angle at the centre versus angle at the circumference. Properties of a Cyclic quadrilateral.
Session 9	Field notes	Discussion session	Comparing tasks to the CAPS document
Session 10	Administered the post-assessment SDLI	Large group presentation Consolidation session	Refection on all the tasks Proving theorems

Table 4:2 Outline of the sessions of Case 2

In this case we administered tasks that were specifically centred on Euclidean Geometry, in essence following the pattern it "unfolds" in the school curriculum. Herein, students were working individually but at the same time they were encouraged to work together as they solved geometric tasks. The use of technology was at the centre of this case, because students were requested to complete tasks by using the Geogebra® program. Tasks comprised constructing and investigating properties of geometric shapes such as triangles, quadrilaterals and circles. In total there were four tasks that students had to complete (see Table 4:2). Also in case 2, we administered the pre- and post-assessments of Self-Directed Learning Instrument (Cheng *et al.,* 2010) as well as open-ended questions. The open-ended surveys required participants to provide written responses to questions probing their fundamental dispositions about the teaching and learning of mathematics as well as Euclidean geometry.

The following two tables serve to outline both Case 1 and 2 respectively:

Case 1: Township secondary school mathematics teachers

Participants	**Two secondary school mathematics teachers**
Time	11 sessions About 2 hours per session August - November 2014
Place	North West Province, Kanana Township, at a secondary school
Mathematical Concept	The function concept
Sources of data	Semi-interviews (Audio-taped) Tasks (Written responses and audio-taped) Researcher's field notes
Phenomenon of interest	Collaboration towards teaching proficiency of mathematical concepts in secondary schools

Table 4:3 Outline of Case 1

Case 2: Students at the North West University

Participants	**15 Further Education and Training Phase (Grades 10-12) fourth year students majoring in mathematics**
Time	10 sessions 90 minutes per session May 2015 – October 2015
Place	Potchefstroom, at the North-West University
Mathematical Concepts	Geometry of Triangles. Geometry of Quadrilaterals. Geometry of Circles.
Sources of data	SDLI (Self-Directed Learning Instrument) (Cheng *et al*., 2010) Open-ended surveys Euclidean geometry tasks Researcher's field notes
Phenomenon of interest	Collaboration towards teaching proficiency of mathematical concepts in secondary schools

Table 4:4 Outline of Case 2

From Tables 4.3 and 4.4 we note that the cases were not running concurrently. The first case that we investigated was Case 1 and followed by Case 2. Also we note the differences between the cases. Here we note that the mathematical content, context, data gathering techniques as well as the participants differ significantly. The cases are similar to each other, based on the phenomenon that was under the investigation.

4.2.3 Participants and settings

The population of the study comprised township school mathematics teachers and university students. The former involved two (male and female) secondary school teachers who taught mathematics at different schools but in the same township, namely Kanana in Orkney, and their participation was voluntary. The physical setting in which the study was conducted is a classroom. The organizational environment of the classroom was equipped with a chalkboard, chairs and desks. The classroom had no physical tools such as a computer or projector, and there was no electricity. Also, it was not easy to write on the surface of the chalkboard, and it was challenging to observe what was written on it. Hence we were forced to change the location during those sessions that required us to use technology. Herein, we used about three different classrooms and in some instances we were working amidst a lot of noise coming from the outside of the classrooms. In fact it was a challenge to work in such a setting.

The latter case involved a group of twelve fourth-year BEd-students studying at the North West University. Herein the physical setting in which the study was conducted was a lecture hall with computer work stations for each student to work at. The organizational environment of the lecture room was optimally equipped with a white-board, computers (in a network and with Internet access), chairs and desks spaced out neatly; software used (e.g. Geogebra®) was installed on the network. The students were training as secondary school teachers and opted for mathematics as one of their teaching subjects. They all participated voluntarily in the study.

4.2.4 The role of the researcher

In this study, my role was first to seek ethical clearance from the Ethics Committee of the North West university. Secondly, I had to get permission from Dr Kenneth Kaunda Education District Management in Potchefstroom and carry out informed consent procedures with all the relevant participants before commencing the study. Thirdly, my role was to design and facilitate learning activities. I was a participant observer and the primary instrument in data collection; consequently some of my roles were to lead interviews, make notes, record, analyse, interpret and report data. This was done before, during and after the study. I did all of these responsibilities with the guidance of my supervisor.

4.2.5 Data generation techniques

Since this was a multisite case study, multiple methods of data collection were used, including interviews, open-ended surveys, field notes and tasks. In the following paragraphs we portray an elaborate description of these methods.

4.2.5.1 Open-ended surveys

In a case study, open-ended surveys according to Cohen *et al.* (2000: 214) can provide the researcher the opportunity to capture the authenticity, richness, and depth of response, honesty and sincerity. Open-ended surveys (see Appendix E) were administered before and after the CLP to establish participants' dispositions on the nature of mathematics teaching and learning. These open-ended surveys were designed in a manner that requested participants to write free responses in their own terms and to explain their responses. These questions required participants' to define mathematics, to provide their views and opinions regarding the teaching and learning of mathematics. Also, open-ended surveys addressed the views of the teaching and learning of the function concept. Our first meeting with the participants provided us (the research team) the opportunity to administer the first open- ended surveys and this was done in the same venue (in school), but participants were encouraged to complete the questions individually. Our last meeting with the participants provided us the opportunity to administer the same open ended questions, but this was done at different venues (in school) in an individual setting. In case 2, open-ended surveys were administered after the CLP. In this case students were offered the opportunity to reflect on the teaching and learning of mathematics as well as Euclidean geometry (see Appendix P).

4.2.5.2 Function Tasks (Case 1)

Self-constructed function tasks, based upon a thorough literature review, were used in order to determine teachers' conceptual understanding of function concept. Tasks (see Appendix F, G, H, and I) were designed and administered before, during and after the CLP, and comprised the pre-assessment task, task 1, task 2 and post-assessment task. These tasks were based on O'Callaghan's (1998) function model (see 3.1.3.4). The function model is situated in a problem-solving environment, and it consists of four component competencies (O'Callaghan, 1998:24-25), namely, modelling, interpreting, translating and reifying. Associated with these four components of a conceptual knowledge of functions, is a set of procedural skills. These skills consist of transformations and other procedures that allow students to operate within a mathematical representational system (O'Callaghan, 1998:26).

The pre-assessment task used before the CLP was comparable to the post-assessment task that was used after the CLP; however the mathematical content involved the same ideas and level of difficulty but different problems. In the following paragraphs the researcher portrays these tasks.

4.2.5.3 Pre-assessment task (Case 1)

Before the CLP, participants were requested to complete a pre-assessment task (see Appendix F). The goal of this task was to establish participants' first hand conceptual understanding of the function concept prior the implementation of the CLP. This task was administered in an individual setting, but at the same venue in our first meeting with the participants. The task comprised nine central questions related to the concept of function. These questions were designed in accordance to the function model.

4.2.5.4 Function Task 1 (Case 1)

During the course of the study we had to take a break of about six weeks before commencing with session 6 of the CLP. Consequently we had to design a revision task (see Appendix G) that we used to establish whether participants were still in the "loop" of the study. In this task participants were encouraged to work together. The task comprised seven central questions related to the concept of function. These questions were designed in accordance with the function model.

4.2.5.5 Function Task 2 (Case 1)

This task was designed and implemented in such a way that required participants to work together. The task (see Appendix H) comprised four central questions related to the concept of function. In this task the mathematical content comprised the system of equations in the form of linear and quadratic equations, the absolute value and inequalities. The task was procedural in nature, because participants were requested to solve systems of equations (linear and quadratic) and asked to represent their solutions algebraically and graphically.

4.2.5.6 Post-assessment task (Case 1)

After the CLP, participants were requested to complete a post-assessment task (see Appendix I). The goal of this task was to establish and compare whether participants' conceptual understanding of the function concept had changed over the period of the CLP. This task was administered in an individual setting, but at different venues (in school) in the last meeting with the participants. The task comprised six central questions related to the concept of function. These questions were designed in accordance with the function model.

4.2.5.7 Interviews (Case 1)

Regarding an interview, Borg *et al.* (1993:113) remark that an interview is a two way conversation of views and ideas between the researcher and participants. Consequently the researcher and participants interchange views and ideas through direct interaction. Nieuwenhuis (2007a:78) distinguishes between three types of interviews: open-ended, semi-structured and structured interviews. In this study, the semi-structured interview was used to corroborate data emerging from other data sources.

Semi-structured interviews (see Appendix J) were conducted after the CLP in an individual setting from which participants were required to answer a set of predetermined questions. The interviews were conducted by the researcher at different venues (in schools) on the same day. A cell phone was used to record data because writing answers down is time consuming and is coupled with distractions. Prior to the execution of the interviews, the researcher requested permission from the participants to record the interviews (Nieuwenhuis, 2007a:89). These interviews provided the opportunity to further explore and probe participants on other aspects of the CLP which were not on my interview schedule. The researcher conducted the interviews, audio-taped interviews and transcribed the interviews verbatim. These interviews were utilised in Case 1.

4.2.5.8 Euclidean Geometry tasks (Case 2)

In addition, Euclidean Geometry tasks were also designed and administered in accordance with the Mathematics Curriculum and Assessment Policy Statement for Further Education and Training Phase (Grades 10-12). Herein, the mathematical content of these tasks comprised Euclidean Geometry focusing on constructing geometric shapes (triangles, quadrilaterals and circles) and investigating the properties of such shapes (see Table 4:2). The students were encouraged to use technology (Geogebra®) as means of learning Euclidean Geometry. Students were engaged in four different tasks probing their knowledge with regard to Euclidean Geometry.

Task 1 (see Appendix K) required students to construct and investigate the properties of triangles by using technology. Task 2 (see Appendix L) was similar to task 1, but in task 2 students had to construct and investigate the properties of triangles by means of physical constructions, using drawing instruments (mathematical sets). Task 3 (see Appendix M) provided the students with the opportunity to construct and investigate the properties of both triangles and quadrilaterals. In task 4 (see Appendix N) students were engaged in circle geometry; they had to construct and investigate the properties of the circle geometry.

4.2.5.9 SDLI (Self-Directed Learning Instrument)

Consensus in the faculty of education in the North West University is that students who are training to become teachers should become more self-directed, that is, possess the ability to learn on their own. Consequently the research team decided to determine students' self- directed learning abilities. In order to gain insight regarding students' self-directed learning abilities we used the SDLI (Cheng *et al.*, 2010) as a means to measure the self-directed learning abilities of participating students. SDLI (see Appendix O) is a model used for understanding SDL and it consists of twenty items across the following four domains: learning motivation, planning and implementing, self-monitoring, and interpersonal, communication (Cheng *et al.,* 2010). The SDLI was adapted to suit the learning and teaching of mathematics. We administered both the pre- and post-assessments of SDLI. The pre-test assessment of SDLI was conducted in the fifth session of the CLP, and in the last session the students had to complete the post-assessment of SDLI.

4.2.5.10 Field notes

Field notes contained detailed written descriptions of what was observed, as well as my interpretations. These notes constituted two kinds of information, namely descriptive and reflective (McMillan, 2000:260).

Descriptive information involved the drawings and words that captured the details of what happened during the implementation of the CLP. In my field notes I included the date, place and time as well as a description of the mathematical activities in which participants were involved. My field notes also included a description of the organisational environment of the mathematics classrooms that were used during the implementation of the CLP. The second kind of information included in my field notes was reflective information. Herein I sought to write thoughts, ideas and interpretations related to the CLP. These reflections comprised thoughts about emerging themes and patterns related to the CLP. Field notes were utilised in all the cases.

4.2.6 Data analysis

The process of data analysis centres on making sense out of text and image data (Creswell, 2009:183). In a case study research, the process of data analysis requires the researcher to organize, transcribe, code, summarize and interpret the data, put it in text. (McMillan, 2000:269). Following the completion of data collection with participants in our three cases we (research team) began to reflect on how we might now analyse data. Consequently we had to organize, account for and explain data in terms of participants' responses of the mathematical situations they were engaged in. Further we noted patterns, themes, categories and regularities.

In connection to the aforementioned paragraph, Nieuwenhuis (2010b) distinguishes between five data analysis techniques: hermeneutics, content analysis, conversation analysis, discourse analysis and narrative analysis. In this study we (the research team) used content analysis as our preferred technique of analysing data. Because it is easy to access and it works on one level of meaning, that is the content of the data texts, Henning *et al.* (2004:102) consider it imperative to use content analysis as a tool to conduct data analysis. Nieuwenhuis (2010b:101) points out that content analysis involves analysis of such materials as documents and interviews. In conducting content analysis, the research team engaged in the process of examining data from different angles in order to identify similarities and differences in the text that would help us to understand the data, represent the data and establish an interpretation of the larger meaning of the data (Creswell, 2009:183).

In this study, content analysis was used to analyse audio-taped data in the form of transcripts, in particular data emanating from Case 1. In this context transcribed data, were data generated from semi-structured interviews and tasks which were implemented during and after the Collaborative Learning Programme (CLP). Also, open-ended surveys as well as field notes were analysed using content analysis. The following Table (4.5) outlines multiple data sources that were utilised to generate data.

Source of Data	When was it conducted?	How many?	Audio-taped?	Method of data analysis
Semi-structured interview	At the end of the study (Session 10)	Two semi-structured interviews	Yes	Content analysis
Open-ended surveys	Two similar forms. Before and after the CLP (Case 1) One after the CLP (Case 2)		No	Content analysis
Function tasks	During the study (from session 2 to session 10)	Three tasks	Yes	Content analysis
Field notes	During and after the CLP	Two Notebooks (Case 1 and 2)	No	Content analysis

Table 4:5 Outline of multiple data sources used during content analysis

In order to carry out the process of content analysis we concur with Henning *et al.* (2004:103) who assume that the process of data analysis is the "heartbeat of the research". Henning *et al.* (2004:138) asserts that content analysis centres around three phases, namely an orientation to the data, working with data, and the composition of the data.

In phase 1, our role as a research team was to comprehensively read data emanating from each form of data source. Thereafter we had to focus on phase 2, where which our role was to commence with coding and then establish categories. Herein, we aimed to seek for similarities and differences as well as relationships. We discovered patterns. Lastly, in phase 3 we had to consult our literature review. Consequently we established final themes and patterns of related themes. In the following table we represent these three phases (see Table 4:6).

Phases of Content Analysis		
Phase 1: An orientation to data	Phase 2: Working with data	Phase 3: Composition of the data
Reading or studying data sets to form overview and to apprehend the context (within the data text).	Coding segments of meaning. Categorising related codes into groups. Seeking relationships between categories to form thematic patterns.	Writing the final themes of the set of data. Presenting the pattern of related themes. The researcher will be able to make sense of the large quantity of data collected, based on the construct developed during the literature review. The research will look for similarities.

Table 4:6 Outline of three phases of content analysis (Henning et al., 2004:138)

On the one hand, all written documents that were submitted by the participants in particular mathematical tasks were analysed by means of an adopted rubric (Dossey, *et al.,* 2002:561) which comprises five levels, namely: no response, incorrect, minimal, partial and satisfactory. This rubric will be used to interpret teachers' and students' written responses, rather than to give them a mark. In other words, the tasks were scored according to a rubric (see Table 4:7) in which the scoring codes assess the teachers' and students' mathematical correctness or level of understanding, their communication of the responses and the strategies engaged.

The following rubric was used to assess understanding of the teachers' and students' mathematical tasks. The rubric was adapted and modified from the examples of rubrics as given by Dossey *et al.* (2002:556-564). This rubric's levels allude to numerous levels of understanding.

Level	Label	Meaning
0	No effort	No response or answer
1	No understanding	Incorrect answers. Incorrect reasoning. Incoherent communication. No evidence of understanding.
2	Poor understanding	Answers contain mathematical errors or are incomplete. Incomplete reasoning. Poor communication of answers and solutions. Evidence of partial understanding.
3	Moderate understanding	Correct answers. Reasoning valid, but not complete. Can communicate answers and solutions, but not coherently. Evidence of good understanding.
4	Complete understanding	Correct answers. Complete reasoning. Coherently communicates answers and solutions. Evidence of deep understanding.

Table 4:7 Rubric to assess teachers'/students' performance: Functions and Euclidean Geometry

With respect to data analysis concerning students' self-directed learning abilities we will observe data emanating from both the pre-assessment and post-assessment Self-Directed Learning Instrument (SDLI). Herein we do not intend to use quantitative methods because the study is qualitative in nature and, most significantly, the number of participants participating in this study does not allow us to do quantitative analysis. What we are going do is to compare students' responses across all the four domains of the SDLI during and after the Collaborative Learning Programme (CLP) with the aim of establishing any changes, consistencies or inconsistencies in students' learning abilities concerning mathematics. Then the findings will be used to inform results emanating from geometrical tasks as well as the open-ended surveys. Herein, our aim is to investigate the influence of a Collaborative Learning Programme (CLP) on students' self-directed learning abilities.

4.2.7 Ethical considerations

Ethical considerations require balancing the value of advancing knowledge against the value of non-interference in the lives of others (Neumann, 2006:130). Maree and van der Westhuizen (2010) highlight the significance of considering ethical considerations of the empirical research. For instance, Maree and van der Westhuizen (2010:41-42) underscore this significance of ethical considerations:

> *An essential ethical aspect is the issue of the confidentiality of the results and findings of the study and the protection of the participants' identities. This could include obtaining letters of consent, obtaining permission to be interviewed, undertaking to destroy audiotapes, and so on. It is also important for you to familiarize yourself with the ethics policy of the relevant institution.* (Maree & Van der Westhuizen, 2010:41-42)

In light of the preceding paragraphs, the study was conducted with full recognition of the relevant aspects of research as embodied in individual and professional codes of conduct of the empirical research. We firstly consulted and obtained clearance from the ethics committee of the North-West University and permission from the Dr Kenneth Kaunda Education District Management in Potchefstroom before commencing the study.

The research team explained the purpose of the research to the participants, verbally and by means of a letter, assuring them that their participation would be voluntary and could be terminated at any time at their request. In the case of privacy, confidentiality and anonymity of participants, we assured all participants that their responses and information shared prior to, during and after data collection would be kept private, and that the results would be presented in an anonymous mode in order to protect their identities.

4.2.8 Trustworthiness, triangulation and crystallisation

In a qualitative study trustworthiness is imperative because it means that the researcher examines the accuracy of the data analysis, findings and conclusions of the study (Nieuwenhuis, 2010b:113). There are several ways that a researcher can use to examine the accuracy of the study. Herein we concur with Nieuwenhuis's (2010b) suggestion that it is imperative to consider the following factors when one checks for trustworthiness in the study:

- Multiple data sources
- Raw data
- Keeping notes of the research decisions

- Coding data
- Stakeholders
- Verify and validate findings
- Controlling for bias
- Avoid generalisation
- Choose quotes carefully
- Confidentiality and anonymity
- State the limitations of the study.

Another important aspect regarding the accuracy of a qualitative study is triangulation. It is an analytical method that qualitative researchers use to enhance the credibility of qualitative data, and its purpose is to compare findings from different data sources (McMillan, 2000:272). It uses one or two methods such as observation, interviews and document analyses (Cohen et al., 2000:112). During the process of data analysis, data from the scholarly literature review, the tasks, interviews and field notes served as a basis for the triangulation that was utilised in this study.

It is evident that one or two commentators involved in qualitative research recognise the term triangulation and its essence in the methodological discourse. These commentators assert that triangulation is best used in quantitative studies (Nieuwenhuis, 2010a; Henning *et al.,* 2004; Cohen *et al.,* 2000). For instance, Nieuwenhuis (2010a:80) contends that in quantitative studies triangulation is utilised to for the purpose of confirmation and generalizing the findings of the study. This is contrary to case studies, of which the purpose is not to confirm or generalize the findings of the research. Hence, triangulation does not really fit this qualitative study, which has more to do with "interpreting and sourcing in various ways, to build a complete picture or text than locating a position from three different vantage points" (Henning, *et al.*, 2004:103).

According to Nieuwenhuis (2010a:80), it is advisable to use the term crystallisation rather than triangulation. Crystallisation is in harmony with the constructivist and phenomenological perspective, from which it is supposed that there is an emerging reality (not-fixed) which comprises multiple realities. Crystallisation therefore permits the researcher to view such multiple realities as an "infinite variety of shapes, substance, transformations, dimensions and angles of approach" Richardson (2000:394, as cited *in* Maree, 2007:81). Hence, the emerging reality Nieuwenhuis (2010a:81) surfaces from the numerous data gathering techniques and data analysis utilised, and represents our own reinterpreted understanding of the phenomena that underlie the study, in other words, findings crystallised from the data.

4.3 Conclusion

The central purpose of chapter 4 was to describe and explain the choice of research design and methodology embedded in this study. Data were generated in two different cases. We also provided an elaborate description of the role of the researcher as well as ethical aspects accompanying the study. In the chapter 5 the researcher articulates an outline of the process of the data analysis that was conducted in the study, preceded by interpretation and reflection.

CHAPTER 5: DATA ANALYSIS AND INTERPRETATION: CASE 1

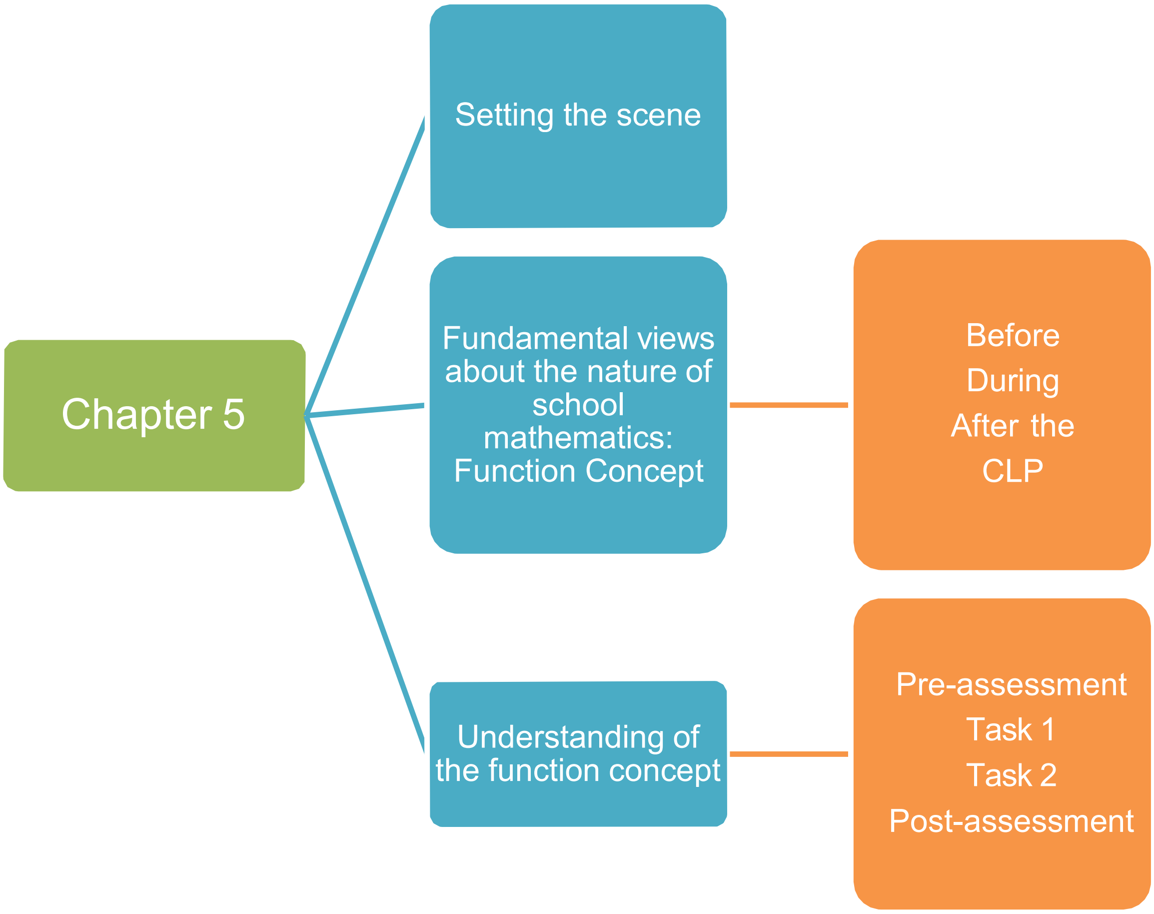


Figure 5:1 Outline of Chapter 5

5 Setting the scene

In the previous chapter I provided an elaborate discussion on the research design and methodology that were employed in this study. In this chapter my focus will be on the data analysis and interpretation applied in the study. Firstly, I discuss teachers' fundamental views about the nature of mathematics. Herein I focus on the teaching and learning of mathematics, particularly the function concept. Secondly I discuss teachers' understanding of the function concept. I then conclude this chapter by discussing teachers' learning experiences pertaining to the CLP.

The study was qualitative in nature with the intent to investigate the following question:

> *To what extent can collaboration among mathematics teachers improve their proficiency in the teaching of mathematical concepts?*

In order to investigate the primary research question, the following secondary questions need to be addressed:

- *What is the influence of a collaborative learning programme on teachers' fundamental views about the teaching and learning of mathematics?*
- *What is the influence of a collaborative learning programme on teachers' conceptual understanding of functions?*

Since this was a multisite case study, multiple methods of data collection were used, including interviews, open-ended questions, field notes and tasks. In connection to the aforementioned, content analysis (see Table 4:6) was employed to analyse data emanating from open-ended questions, semi-structured interviews and transcribed data. The rubric (see Table 4:7) was used to analyse both the function and Euclidean Geometry tasks respectively. This rubric's levels allude to numerous levels of understanding.

5.1 Fundamental views about the nature of mathematics

Thompson (1992) emphasizes the critical link between mathematics teachers' fundamental views and beliefs about mathematics and how it should be learned, and then be taught, and the application of their mathematical knowledge. She further claims that teachers' conceptions of the nature of mathematics range from viewing mathematics as an *absolute*, fixed body of knowledge to seeing mathematics as a *fallible* and expanding human knowledge (see par 2.1.2) The discussions that follow will focus on teachers' views and beliefs about the teaching and learning of mathematics as well as the teaching and learning of the function concept. This discussion will be based on the events that took place before, during and after the Collaborative Learning Programme. Herein, we look at the data from a multiple of sources and we strive to make sense of the raw data in relation to research equations as well as research objectives.

5.2 Discussion of teachers' views about the teaching and learning of mathematics: Before, during and after the Collaborative Learning Programme (CLP)

In connection to this section, an open-ended survey (see Appendix E) was administered before and after the CLP to establish teachers' fundamental views about the teaching and learning of mathematics. This open-ended survey was designed in such a way that requested participants to write free responses in their own terms and to explain their responses. These questions required teachers to define mathematics, to provide their views and opinions regarding the teaching and learning of mathematics. Also, these questions addressed the views of the teaching and learning of the function concept. In the following paragraphs I portray the process that was used in coding teachers' responses. The codes were taken directly from teachers' responses. From the codes, categories emerged (see Table 5:1, Table 5:2, Table 5:3 and Table 5:4). Also, data emanating from other sources (transcripts and function tasks) are used in association with data emanating from both the open-ended survey (before) the CLP as well as the open-ended survey (after the CLP) in order to answer the following question:

What is the influence of a collaborative learning programme on teachers' fundamental views about the teaching and learning of mathematics?

Teacher A		**Teacher B**	
Category	**Codes**	**Codes**	**Category**
What is mathematics?			
Mathematics is an important school subject because it provides learners with career opportunities.	• Is a concept • Learning area • It is important • Learners must take it • Engineering or Maths	• Is a discipline • Space • Shapes • Symbols	Mathematics is made up of a collection of mathematical ideas.
Why do we teach mathematics?			
The purpose for teaching mathematics is that it provides learners with career opportunities.	• Is very important • Subject • Scientist or Engineers • Learning area	• Leaners • Understanding • Space • Shapes • Measurement	The purpose for teaching mathematics is that the learners must understand mathematical ideas.
What does it mean to teach mathematics?			

• **No category**	• No response	• No response	No category
How should mathematics be taught?			
Mathematics must be taught from primary school to secondary school	• Lower grade [Primary school] • Secondary grade (Secondary school)	• Visualisation	Mathematics teaching should be in harmony with visualisation
What are the essential features of mathematics?			
• **No category**	• No response	• Space • Shapes • Measurement	Space, shapes and measurement form an important part of mathematics

Table 5:1 Codes and categories about fundamental views of the teaching and learning of mathematics before the CLP.

Table 5.1 is a summary of codes and categories created form teachers responses concerning their views and belies about the teaching and learning of mathematics. These codes and categories were created directly from teachers' responses. It is worth noting that both teachers provided short responses to all the questions. Also, Teacher A did not respond to questions 3 and 5. Teacher B did not respond to question 3. The open-ended survey was administered before the CLP in our first meeting with the teachers. In the following discussion I use the information from Table 5.1 together with information from Tables 5.2, 5.3, 5.4, 5.5 and 5.6 in order to answer the research question stated in section 5.1 above.

Teacher A		**Teacher B**	
Category	**Codes**	**Codes**	**Category**
What is mathematics?			
No category	• No response	• Is a discipline • Spaces • Conceptualisation	Mathematics is made up of a collection of mathematical concepts
Why do we teach mathematics?			
The purpose for teaching mathematics is that it provides learners with career opportunities.	• Count or calculate • Important • Mines • Banks	• Interaction • Everything • Earth or world	The purpose for teaching mathematics is that mathematics provides people with the opportunity to interact with everything around us.
What does it mean to teach mathematics?			
• No category	• No response	• Help • Learners • Understanding	To teach mathematics means to provide learners with learning opportunities that will promote the meaningful learning of

		• Real-life situations • Models	mathematics, that is to say conceptual understanding.
How should mathematics be taught?			
The teaching of mathematics should take into account skills, values and norms.	• Skills • Values • Norms	• Technology • Model • **Visualisation** • Important • Teaching • Mathematics	The teaching of Mathematics, in particular modelling should be taught in accordance with technology because **visualisation** is an important aspect in the teaching of mathematics.
What are essential features of mathematics?			
• No category	• No response	• Measurement • Calculation • Modelling • Spaces • Data handling	Measurements, calculation, modelling, space and data handing form an important part of mathematics

Table 5:2 Codes and categories about teachers' fundamental views of the teaching and learning of mathematics after the CLP.

Table 5.2 is a summary of codes and categories created from teachers' responses concerning their views and beliefs about the teaching and learning of mathematics. These codes and categories were created directly from teachers' responses. It is worth noting that both teachers provided short responses to all the questions. Also, Teacher A did not respond to question 1, 3 and 5. On the other hand Teacher B responded to all the questions. The open-ended survey was administered after the CLP in our last meeting with the teachers. In the following discussion I use the information from Table 5.2 together with the information from Table 5.1, 5.3, 5.4, 5.5 and 5.6 as well as the data emanating from the transcripts in order to answer the research question stated in section 5.1 above.

Teacher A		Teacher B	
Category	**Codes**	**Codes**	**Category**
How do you go about teaching the function concept?			
• No category	• No response	• Show learners • Related variables • Plotting graphs	His approach toward the teaching of mathematics is based on showing learners how to learn the content of the function concept.
What is a function?			
• No category	• No response	• Graphs	The function concept is based on a single

			representational system of the concept, in this case a graph.
How should the function concept be taught?			
• No category	• No response	• Modelling	The teaching of the function concept should be harmony with the notion of modelling.

Table 5:3 Codes and categories about teachers' fundamental views of the teaching and learning of the function concept before the CLP.

Table 5.3 is a summary of codes and categories created from teachers' responses concerning their views and beliefs about the teaching and learning of the function concept. These codes and categories were created directly from teachers' responses. It is worth noting that both Teacher A and Teacher B provided short responses to all the questions. Teacher A did not respond to questions 6, 7 and 8 of the open-ended survey. On the other hand, Teacher B responded to all the questions concerning the survey. The open-ended survey was administered before the CLP in our last meeting with the teachers. In the following discussion I use the information from Table 5.3 together with the information from Tables 5.1, 5.2, 5.4, 5.5 and 5.6 as well as the data emanating from the transcripts in order to answer the research question stated in section 5.1 above.

Teacher A		**Teacher B**	
Category	**Codes**	**Codes**	**Category**
How do you go about teaching the function concept?			
Her approach to the teaching and learning of mathematics is based on leaners being able to perform basic operations so that they can be able to sketch the graphs.	• Leaners make • Table • Identify intercepts • Plot points • Sketch graph	• Real-life situation • Represented • Graphically or equation	His approach toward the teaching of mathematics is based on real-life situations (modelling). It is important that learners perform different transitions between a graphical and algebraic representation of the concept.
What is a function?			
The function concept is based on a single representational system of the concept, in this case a graph.	• Graph	• Relationship • Between variables • Decrease, increase or stays the same	The function concept is based on the relationship between variables.

How should the function concept be taught?			
No category	• No response	• Needs technology • Geogebra® • Understanding • See clearly • Effect of parameters	The teaching of the function concept should incorporate technology. Technology in the form of Geogebra® promotes the meaningful learning of the function concept.

Table 5:4 Codes and categories about teachers' fundamental views of the teaching and learning of the function concept after the CLP

Table 5.4 is a summary of codes and categories created from teachers' responses concerning their views and beliefs about the teaching and learning of the function concept. These codes were created directly from teachers' responses. It is worth noting that both Teacher A and Teacher B provided short responses to all the questions. Teacher A responded to questions 6 and 7 respectively, but could not respond to question 8 concerning the open-ended survey. On the other hand, Teacher B responded to all questions concerning the survey. The open-ended survey was administered *after* the CLP in our last meeting with the teachers. In the following discussion I use the information from Table 5.4 and the information from Table 5.1, 5.2, 5.3, 5.5 and 5.6 as well as the data emanating from the sessions of the CLP in order to answer the research question stated in section 5.1 above.

Teacher A	Teacher B	
Codes	**Codes**	**Categories**
What did you learn from the CLP?		
• Many things • learned a lot • new topics • Finding the equation by using the factored form method (that is look at the graph)	• Methodology or approach • Graphs, equations, modelling • Little understanding on some of things • Indirect and direct proportionality not clear to me, • How do I see it (from the graph) • But now it is clear • Shifting of graphs, the what • Use of technology • Geogebra® that visualisation • can help learners without waste of time the effect of parameters in the graphs (increase and decrease)	The CLP offers numerous learning opportunities

<table>
<tr><th colspan="3">What is the meaning of the CLP?</th></tr>
<tr><td>• Meant a lot (certificate/ diploma)
• Learned a lot
• Gained a lot of things I did not know before</td><td>• Helping the teachers on how to approach functions</td><td>The CLP is a meaningful approach that enhances teacher learning and teachers reacted positively to it</td></tr>
<tr><th colspan="3">What did you like?</th></tr>
<tr><td>• The talking
• Like to listen
• Doing the problems
• Everything that we have done
• Sharing of information</td><td>• Team work
• How we tackle problems
• Being together discussing
• Towards a new approach</td><td>The CLP was stimulating and interesting and teachers enjoyed it.

The CLP provides teachers the opportunity to engage in a mathematical discourse.</td></tr>
<tr><th colspan="3">What did you dislike about the CLP?</th></tr>
<tr><td>• Test
• Writing
• Word sums
• Negative attitude
• Don't like them
• Difficult for me
• Did not teach them
• But I gained interest as we were dealing with them</td><td>• There is nothing I did not like, because I was enjoying being in the project
• Everything was profitable</td><td>Teachers appreciate the CLP but do not like that part of it that involves testing.

Teachers did not like that part of the CLP that involves modelling with real-life situations.

Teachers involved in the CLP have negative attitude toward modelling of real-life situations because they find them difficult and also they do not teach them in their classes.

The CLP has the potential to assist teachers to gain an interest in modelling with real-life situations.</td></tr>
<tr><th colspan="3">What are you going to do differently?</th></tr>
<tr><td>• Getting the equation of the graph
• Parabola
• Straight line
• Changes in method</td><td>• Reconsider whatever we have done
• Practice that in my class
• Small group
• Big groups
• Learners to be more involved
• Involved in terms of talking with each other
• A teacher teaching a learner
• Learner teaching a learner
• Facilitating</td><td>The CLP challenges teachers to change their ways of teaching mathematics in their classes.</td></tr>
</table>

Would you participate in the future CLP?		
• I will do nothing with the knowledge I gained in the CLP • We start with something else • I'm fine with functions • Something difficult for both teachers and learners • Euclidean Geometry • More teachers	• Definitely • learning new approach • Every day • Was encouraged • Help my learners to improve • Revise functions with my learners • More teachers	The CLP is enjoyable and valuable, consequent teachers would take the opportunity to participate in future CLPs from which more teachers would be involved.
The use of technology		
• Sight and saves a lot of time • Continue using it especially in geometry	• impressed • Help us understand the content • time is efficient • than drawing manually	The use of technology in the CLP enhances the meaningful learning of mathematical concepts and it saves time.
Collaboration		
• Amongst the four of us	• The collaboration was there • Four people sharing ideas • It does not have to be ten people • Share ideas amongst ourselves	The CLP involves collaboration between all parties concerned

Table 5:5 Codes and categories concerning teachers' experiences of the CLP

Table 5.5 is a summary of codes and categories concerning teachers' experiences about the CLP. These codes were created directly from teachers' responses. This information was gathered from the both Teacher A and Teacher B by means of semi-structured interviews after the CLP. Herein teachers were interviewed separately at different venues, and encouraged to talk about their experiences during the CLP. In the following sections (5.3.1 and 5.3.2) I use the information from Table 5.5 together with the information from Tables 5.1, 5.2, 5.3, 5.4 and 5.6 as well as the data emanating from the sessions of the CLP in order to answer the research question stated in section 5.1 above.

Codes: Teacher A	Codes: Teacher B	Category
The impact of the CLP		
• Positive to the learners • Pass rate	• Very positive • Teaching strategy • Explaining of content • Also positive to the learners • Preparing with other colleagues	The CLP had a positive impact on both the teachers and learners.
Have you done something differently?		
• No • I did what we did	• I was just teaching them • But not nowadays	The CLP challenged teachers to try and implement what they have

• Especially with parabolas • Tried the method I learned about finding the equation of a parabola • Did not work on real-life situations	• They must unpack the content in groups • Interpret the content • I tried to use Geogebra® program • But it was challenging • Learners did not get it, so I had to go back to the chalk and chalkboard • And the learners started to see things	learned in their classes.
Future Project		
• On geometry • Yes, other teachers will be interested • Not just geometry and probability • These two topics are difficult for teachers to teach • Yes, the way we did in the previous CLP, teachers working together on tasks, rather whereby you alone • Math is difficult/it's a killing subject • Some are not interested • No we don't talk • I am on my own • No time for us to come together	• Yes we always do that • Three teachers • Talk about different strategies • Approaching certain topics • Discuss learners" view of different aspects. • Prepare together on difficult topics • My approach has changed and I share with them • The content is not the problem • We look at the approach	Teachers would be interested in participating in the future CLP and it should focus on geometry or probability. Most teachers do not attend such learning programmes because mathematics is a difficult subject or some teachers are not interested, others claim that they do not have time. In the future CLP we should not worry about the content rather we should focus on the approach because I have changed my approach and I shared my knowledge with my colleagues. Due to the CLP I have managed to work with other teachers at my school and we prepare together, discuss learners views of different aspects and we talk about different strategies. At my school I do not work or share the knowledge I gained in the CLP with my colleagues, there is no time for us to come together.
Why did you stay in the project?		
• I knew I was going to get something • I will gain something that will help me at the end • And the expectation was fulfilled	• It is for the development • Every teacher need development • I cannot say I know all the content • Then I know all the strategies • Need for development • So the that I can present that to the learners	The CLP enhances teacher learning/development. Teachers benefited something from the CLP.

Advice for the DBE about teacher learning		
• On-going process • Every day we must give feedback • We notice our problems, share with my colleagues • One day workshop, then we meet after a month (does not work) • In our working environment	• On-going process • One day workshop is not working (not related to the previous one) • Practice it and give feedback (things are related to each other) • Team teaching • The problem is the strategy	Instead of being a one day workshop, teacher learning should be an on-going process and its content should be related to the previous knowledge/experience. Teachers need to be provided with opportunities to reflect on their learning in a manner that is consistent. Teacher learning should take into account team teaching because the problem is not necessarily content, but the instruction

Table 5:6 Codes and categories about teachers' experiences during CLP (A year later)

Table 5.6 is a summary of codes and categories concerning teachers' experiences during the CLP. These codes were directly created from teachers' responses. This information was gathered by means of a group discussion (including both Teacher A and B as well as the research team) a year after the CLP. Herein, teachers were encouraged to talk about their classroom experiences in relation to the CLP. In the following sections (5.3.1 and 5.3.2) I use the information from Table 5.6 and the information from Table 5.1, 5.2, 5.3, 5.4 and 5.5 as well as the data emanating from the sessions of the CLP in order to answer the research question stated in section 5.1 above.

5.2.1 Discussion

Teacher A

From Table 5.1 I note that Teacher A did not provide any respond to question 3 and 5 of the open ended survey; in Table 5.2 I note that Teacher A did not provide any response to questions 1, 3 and 5. These questions required her to share her views concerning the meaning behind the teaching of mathematics as well as the essential features of mathematics. Also, after the CLP Teacher A did not provide the definition of mathematics. However, before the CLP she defined mathematics as:

> *"Mathematics is a concept or a learning area whereby it is very important for each, each and everyone in the world, the learners to be Engineers or Scientists must take Mathematics"*

In connection to this Teacher A was encouraged to share her views about why do we teach mathematics and how we should teach mathematics (questions 2 and 4). Initially, Teacher A responded to question 2 by noting the following:

> *"Mathematics is the very important subject, whereby the Scientist and Engineers come from the learning area Mathematics".*

After the CLP, Teacher A provided the following respond to question 2:

> *"We teach Maths in order to count or calculate in mines Maths is very important & needed they hire only learners or people with maths. At banks Maths is needed".*

Based on these responses one can assume that Teacher A sees mathematics as an important school subject that is career related, and also thinks those who have mathematics have more opportunities than those who do not have mathematics (see par 2.1.5). In question 4, before the CLP, Teacher A was asked the question how should mathematics be taught and she noted the following: "*it should be taught from the lower grade up to secondary school*". After the CLP she noted the following in relation to question 4: *"it should be taught according to the skills, values + norms".* Herein, I note that Teacher A does not clearly share her views about the teaching of mathematics, hence it becomes a challenge to address her views about the teaching of mathematics. Also from Tables 5.3 and 5.4 I note that Teacher A did not attempt to complete questions 6, 7 and 8 respectively; hence there is nothing to discuss about her views in relation to the teaching and learning of the function concept. Initially she did not provide a response to this question, but after the CLP, Teacher A mentioned that she teaches the function concept in the following way:

> "*First, the learners must be able to make a table, they must be able to identify the intercepts* (   ) *and also to plot the points in order to be able to sketch the graph"*.

Based on this response, one can claim that Teacher A teaches the function concept in relation to skills that the leaners must learn (see par 2.1.3). For example, if a learner is given a function in the form of a formula, the learner must be able to translate from a formula to a table or graph. Before the CLP she was asked to define the function concept, but did not provide any response to this question. In session 6 of the CLP, Teacher A was asked to define the function concept, and she provided the following definition:

> *"It is a relationship between the points… Whereby say the points it will be X and Y. A function is a graph… Ya, because most of the people if you talk about the function…you are talking about something else…"*

Herein she use the term "*it is a relationship*" without providing insight and then she maintains that a function is a graph. Then she was asked if all the graphs were functions. She mentioned that we use a test line, but she was not sure which test line do we use, vertical or horizontal. The following is a discussion that transpired concerning this matter:

Researcher B	Mm… So if I may ask, if you say is a graph, does that mean all graphs are functions? Or are there conditions…
Teacher A	Ya, there are conditions, some of them they are not called functions
Researcher B	Mm
Teacher A	Although we know that they fall there
Researcher B	They fall there…
Teacher A	Ya ya
Researcher B	Err
Teacher A	Just like if you look at the graph of a circle
Researcher B	Mm
Teacher A	That one is not a function
Researcher B	Mm
Teacher A	Ya
Teacher B	The graph of…
Teacher A	Circle
Teacher B	Ooho
Teacher A	Ya, it's not a function, ya even parabola, the parabola is not a function.
Researcher B	Mm… Interesting. Why?
Teacher A	On the function, we are having that test line…the horizontal test line…we are having the horizontal test line.
Researcher B	Mm
Teacher A	And also the vertical. So if you are having a graph like this okay let's start with the circle
Researcher B	Mm
Teacher A	The circle if you draw the circle
Researcher B	Mm
Teacher A	If you use the horizontal line it will cut at two points, that is why we say is not a function
Researcher B	Can I draw…heading to the chalkboard
Teacher A	Ya
Researcher B	Which one must I draw
Teacher A	The circle

Researcher B	Mm
Teacher A	So we must use that horizontal to check
Researcher B	If it's a function
Teacher A	Ya
Researcher B	Horizontal
Teacher A	It's a horizontal test line
Researcher B	Horizontal
Teacher A	Ya, so it cuts the graph…that line is a dotted line
Researcher B	This one the horizontal
Teacher A	Ya the horizontal
Teacher A	Sorry, haa no it's a dotted line
Researcher B	Aaha
Teacher A	That line…the name for that line is called a horizontal test line
Researcher B	Is like this
Teacher A	No …horizontal this
Teacher B	This one must be vertical
Teacher A	Vertical sorry vertical
Researcher B	Vertical
Teacher A and B	Ya
Teacher A	So it cuts the graph at two points that why we say is a…is not a function

Also in session 4 when we were reflecting on question 4 of the pre-assessment task, Teacher A mentioned that in order to test if a graph is a function, we use a vertical line test. But it is worth noting that she is was not sure which line test to use horizontal or vertical. In connection to this she had this to say:

> *"It is also what we see curriculum where by Ya…, we need to check if it a function or no, sometimes because sometimes you find out what it is a function or not a function, I forgot that word… This one is not a function to check with they show their using that …horizontal or vertical? It is a vertical? Yes that one Ya….Ya….okay meaning? Meaning this one that line is going to cut the traffic to put with a cycle but it is not a function so you saying you graph a vertical line Ya… there is a test line but now it intersect the graph in two places it means…"*

But after the CLP she defined the function concept as:

> "*Function means the graph, but it will depend that the graph is a function or not".*

Herein, one can note that Teacher A sees the function concept as a single representational system, in this case a graph, and that not all of the graphs are functions. In order to see if a graph is a function, it is important to use a line test (see par. 3.1.1).

In session 6 of the CLP she indicated that she has learned many things. In light of this she mentioned that she learned a new method concerning finding the equation of a parabola from a graphical representation. The following is what she noted regarding this matter:

> *"Ya, but on the second day I was fine I was so happy I've told Teacher B that I was so happy I've gained a lot especially where we were doing the parabola graphs let's say the graph, we look at the graph and we get the equation so we are having the three methods to solve to get the equation of the graph we know I know the first one and the second one the third one I did not know about that one I've learned from [***Research A***] whereby let's say the equation you given that parabola graph let's say the X intercept here is minus two and the other one here it is three something a question like that so now I did not know I did not know I was supposed to look at the graph and I get everything there just like whereby you say if you for the equation of the graph I know that first you are supposed to write Y is equal to a into X minus X-one X..ya so that I substitute"*

In the semi-structured interviews she was asked the question "what did you learn from the project?" And she provided a response not far from the one she gave in session 6 of the CLP as well as the one she provided a year later, she had this to say:

> *"I have learned many things. How to introduce some of the…the…the…topics…just like some of the things with the functions…when you look at the parabola graph, there is a parabola graph I know only…let's say the graph is drawn and the points are there on the graph let's say the x-intercepts and what else…you answer the question find the equation of the graph I…I have known only two methods to find the…the…the…the equation of the graph when it is drawn I have learned the third method from [***Researcher A***] …ya… ya from [***Researcher A***] when let's say you are given the two x-intercepts x is 2 and x is minus 1 on the other side, so there is no need to go to whereby you are supposed to write first y is equal to a into x minus x-one and x minus x-two…we we used to that…so and err…it takes a very long time and you notice that the marks can be 2 marks let's say find the equation of the graph so there is no need to go through the things to find **a**…so to substitute just you look at the graph you get the x and the x therefore you can ntho you can get the equation."*

During the semi-structured interview, Teacher A indicated that she was going to change her teaching *“method”* with regard to finding the equation of both the parabola and the straight line graph. A year later, after the CLP, Teacher A was asked if she had tried to implement this “*method”* in her class, and she had the following to say:

> “*Yes we tried that with my learners because they noticed that this one it needs a short of time Ya...the one way before it’s when we need this and this and it need more time so this one was very easy for...them because you look and interpret in the graph therefore you get an equation*.”

Also, a year later Teacher A maintained that she had learned a lot of things that she had not known prior the CLP. In light of this she had this to say:

Researcher A	What did you say, what is the most important thing that you learn from this 10 weeks, what is the thing that stands out for you?... anything?
Teacher A	The approach of the compare to the learners and also Ya, how to approach this to the learners and also I’ve gain a lot of things I’m not yet know Ya...Ya...
Researcher A	So...
Teacher A	You are a teacher but some of the things you don’t know, you are blind, you are blind you don’t know nothing the teachers in math’s whereby they know nothing I know that, so I was blind in some of the things so at least now I’m having the different approaches whereby I can approach this to the learners Ya, Ya
Researcher A	So you have gain in terms of content
Teacher A	Ya, Ya
Researcher A	As well as how to approach...
Teacher A	Ya, Ya, Ya I have gain in terms of content again also to approach to the learners and I know that since I attended with you the 10 weeks my learners the is the improvement of pass rate of the learners, Ya..

During semi-structured interviews she indicated the CLP was important to her because of the sharing of the information between the four us, and also she noted that without the research team they (Teacher A and Teacher B) could have not managed to solve the tasks individually. In connection with this, she said:

> *“Because I know that if it was me and Teacher A my colleague our knowledge it’s less but if we get the knowledge from you and [***Researcher A***] we know that it will be more.”*

Further, she was asked the question **“**what did the project mean to you?” and her response was:

"It meant a lot to me, as if I am going to get let's say a certificate or a diploma because I was very happy with this project. Since I've said from the start I gained some of the things I did not know from the start Ya…Ya…"

A year later she was asked why she stayed in the CLP she replied by stating that:

"*Laughs...just like others, I knew that at the end I will get something, Ya... I will get something, I will gain something also that will help me at the end…Yes it was fulfilled, it was."*

Teacher A indicated that the she would be interested in participating in future projects (if any). She suggested that we should conduct another project similar to the CLP, but focusing on a topic that is difficult for both learners and teachers; she suggested that we engage in Euclidean geometry and that technology should be incorporated in the project. In order for us (research team) to improve the CLP, she indicated that we involve more teachers. A year after the CLP she was still suggesting that she would be interested to participate in the future CLP but it should focus on geometry or probability. Teacher A noted that these two topics were difficult for both teachers and learners. She went on to note that teachers would be interested to take part in it because they were struggling to teach these topics (see Table 5:6).

During the CLP, in session 6, Teacher A was asked to reflect on the CLP up to that day, and her response can be linked to the response that she provided after the CLP. In connection to this we had the following discussion concerning the writing of the pre-assessment function task:

Teacher A	On the first day I was so bored (laughing) because we write the pre-test
Researcher B	Laughs
Teacher A	Serious because we write the pre-test…
Researcher B	Mm
Teacher A	I told myself that if we are here we've come to learn something
Researcher B	Mm

After the CLP Teacher A was asked to talk about her dislikes regarding the CLP, and her response was transcribed:

"Ya…the only thing that I did…dislike…It is the test or something that would be written down it frustrates me, as if I am a student… Ya …I don"t want to write, I would like listen so that I can gain something…even if you go to [Mathematics Subject

> *Specialist]…in workshop we know that we are going to write, we are going to write…we are going to write…he will stand in front of us just only for ten minutes… Ya so that is the only thing I dislike…ya the writing.*"

Also, after the CLP Teacher A was asked why she did not provide any responses to mathematical situations from which the function concept was presented in words. Teacher A responded by saying:

> *"…when…first of all when I get the questions like that I was just having a negative attitude because I know that in maths I don't like the word questions. Even to the learners I did not waste my time to to…ya…ya to teach the word sums because I know that they are also difficult for me…"So I know that at least if they appear on the exams it would be just only four marks they are not going to lose a lot…"*

In session 6 of the CLP Teacher A made the following remark concerning modelling real-life situations with function: *"Is new to me"*. Herein we were dealing with the function concept using Dynagraphs. A year later Teacher A was asked if she had come across or worked with real-life situations tasks or problems since the previous year and her response was *"no"*.
After the CLP, when asked about collaboration Teacher A said: "*it was there amongst the four of us."* Teachers had this to say about the use of technology in the CLP: Teacher A said *"…it bring sight to me and also it saves a lot of time.*" When asked if she does work with other teachers at her school, Teacher A maintained that teachers at her school were not interested in working together. According to Teacher A, at her school they do not come together and talk about their challenges because they do not have time. Since other teachers are not interested, she is on her own.
In relation to the preceding paragraphs, I noted that even though the CLP did not have an impact on Teacher A's views and beliefs concerning the teaching and learning of mathematics, Teacher A enjoyed the CLP and as such it was interesting. I conclude by noting that the CLP did not have an impact on Teacher A's fundamental views concerning teaching and learning of mathematics. This is so because her views and beliefs concerning the nature of mathematics prior to, during and after the CLP did not change. Nevertheless, it is important to note that Teacher A appreciated the CLP, but she did not like the writing part of it. Instead she likes to listen to someone so that she can gain something. Also, she mentioned she liked the CLP because of the collaboration that transpired during the CLP.

Teacher B

From Table 5.2 I note that Teacher B attempted all the questions. This is contrary to what I note from Table 5.1, from which he did not complete question 3. I also note that Teacher B did not change his definition of mathematics before and after the CLP. Initially he defined mathematics as a: *"Discipline that deals with space, shapes and symbols".* Teacher B's definition of mathematics after the CLP was: *"Is a discipline that deals with measurement, spaces and conceptualisation".* The only change I note is that he after the CLP he added conceptualisation into his definition of mathematics. Initially Teacher B listed *"space, shapes and measurement"* as essential features of mathematics but after the CLP he listed *"measurement, calculating, modelling, spaces and data handling"* as essential features of mathematics. I note that after the CLP he listed three more essential features of mathematics, in this case calculating, modelling and data handling. Based on these responses, I suggest that Teacher B sees mathematics a collection of mathematical concepts (see par. 2.1.5). Prior to the CLP, when asked why we teach mathematics Teacher B noted that: "*because learners must understand the space, shapes and measurement".* After the CLP he changed his view concerning why we teach mathematics, and he mentioned that we teach mathematics to *"Help learners in understanding the real-life situation using models."* Once again, before the CLP, Teacher B was asked the question what it means to teach mathematics, and he did not provide any response to this question. However, after the CLP Teacher B noted the following concerning the question what it means to teach mathematics: "*so that we can be able to interact with everything on earth/world".* Also, Teacher B was asked the question how mathematics should be taught. Initially, that is before the CLP, he responded to this question by stating "*visualisation",* but after the CLP he noted the following concerning the teaching of mathematics:

> *"By using technology more especially in representing the models. Also visualisation is the most important thing in teaching mathematics".*

A year after the CLP we (research team) had the opportunity to meet Teacher B and we asked him to explain what he meant by "visualisation", and he had this to say:

> "*Yes if you…in fact if you teach learners they need to see that in their mind, they must see in their minds so that is why I use electronic maybe in terms of shifting of draw shifting though I have a chance to use a chalk to …so but again is to see if I can say draw this 5 graphs first one is so they will see the graph going up and up and up depending on that...on that…on that graph and then I will say they may see it's not about the eyes but they can visualize if I may say draw a graph now without drawing the graph without doing the table the learner knows, the learner can see is +10 and the negative 10 you see.*"

In addition to the above paragraph, he maintained that the CLP had a positive impact upon his teaching strategy; he had this to say about what had changed in his teaching practice:

"*Yes, no err...according to my view it has err...it has ah...very positive impact what we have done last year so even I...I... can say my teaching strategy or maybe the way I explained things so it was different because of that… And that it makes positive impact to the learners, eeh... for example we started with the straight line if you...if you remember... we started with gradient so in a way that I go back to the class this year explain the gradient the way we discussed it here, so I saw them being able just to get it easy without straggling infect they can some sort of ...they can get an gradient by inspection the way I define it so it is operational to me if...if is that the right way so I used to say I used the operational definition, so the way I say it then if you do it that way you can do it, then it…it really saved time because we explained it to them…"*

Still with the teaching of mathematics before the CLP, Teacher B was asked the question how the concept of function should be taught. He noted that the teaching and learning of the function concept should be taught by means of "*modelling*"**,** but when he was asked to briefly explain, he goes about the teaching of the function concept. Teacher B had this to say:

"*Show them the learners how the variables are related to each. Show the plotting of graphs"*.

After the CLP Teacher B was asked to briefly explain how he goes about the teaching of the function concept. This is what he noted concerning this matter:

"*I think, it needs technology like Geogebra, such that learners can understand and see clearly what happens as one of the parameters changes"*.

The above statement can be associated with what Teacher B meant by "visualisation" a year after the CLP. Also, after the CLP, Teacher B was encouraged to explain how he teaches the function concept and he had this to say:

"*Firstly give the real life situation…* [teacher's writing not clear]… *that the learners can represent it graphically or by equation"*.

In connection to the preceding paragraphs, I note that initially Teacher B provided responses that were not related to each other with respect to how he teachers the function concept and how he thinks it should be taught. Initially he maintained that he teaches the function concept by playing a role, for instance he shows the learners how the independent and dependent variable are related to each other. But after the CLP, I note that the responses provided by Teacher B are related to each other. Herein, Teacher B promotes the use of technology in the teaching of the function concept because he believes it will assist learners to observe the properties of functions. Also, he teaches the function model through modelling real-life situations. In light of this, Teacher B noted that he was going to reflect on everything that took place during the CLP and would try to practice it in his mathematics classroom with the learners. He indicated that he wanted his learners to be more involved in the learning and teaching of functions. He mentioned that he would like to see his learners talking to each other during the teaching and learning of the function concept and he would appreciate it if he played the role of a facilitator in this regard. In connection to this he made the following remark (see par 2.1.8.3):

> "*Ya in fact differently I will I would like my learners to be more involved, involved in terms of…talking with each other… A teacher teaching a learner, and the learner teaching a learner… Ya then if they teach themselves, they talk amongst themselves, and then as a teacher I will be facilitating"*.

A year later we asked Teacher B about what had happened in his class and he had this to say:

> "*Ya...hmm because infect what was important in fact what I do differently in most cases we just teaching them but now nowadays this year then I said okay this will be the equation they must try to unpack the equation by themselves then I will hear from them individually so sometimes when I group them but before then I will ask them to unpack the question and then just try to interpret the question for me so they will interpret it for me so I get the different interpret of the question and then there after I will just try to ...to make sure we have a different... err... the same meaning of the equation before we can start to answer the question, though in terms of electronic I...I*
>
> *... I… try to use Geogebra Ya... it was a little beat difficult now the difficulty is...eh…okay if I ...myself I can draw the graphs and show them to the sliding, the shifting but now in their case and they will just see it sliding and so and so on but now*

I've realized that though because I started with Geogebra before...before I can teach them they did not get it well..."

In connection with the preceding paragraph, I note that initially Teacher B viewed the teaching of mathematics based on showing learners how to learn the content of the function concept (see par 2.1.5). But it is important to note that after the CLP, Teacher B changed his views concerning the teaching of the function concept (see par. 2.1.4). Herein, Teacher B views the teaching of mathematics, in particular the function concept, as involving modelling and the use of technology. The role of the teacher is to facilitate the learning of mathematics instead of showing learners how to learn. Initially, that is before the CLP, Teacher B was asked to explain the concept of function and he said: A function is a "*Concept that deals with graphs"*. Herein, Teacher B sees the function concept as a single representational system, in this case a graph (see par 3.1.1). In session 6 of the CLP, Teacher B was asked to define a function and he had this to say:

"A function, basically I may say in fact err err I may say a function is a relationship between two variables…Then whereby one variable informs what…ya whereby one variable informs the output of the other one…so they are related…there must be a relationship between the two variables whereby the one has to inform the output of the other one"

After the CLP, Teacher B defined the function concept as follows:

"*The relationship between two or more variables. E.g. as one variable decreases, the other one increases or decreases or stay the same"*.

Herein, I note a change in Teacher's B views concerning the definition of the function concept. Unlike Teacher A, after the CLP Teacher B did not see the function concept as a single representational system. Instead he sees the function concept as a relationship between two variables that can be presented in at least two different representational systems. In connection to this, in session 6 of the CLP, Teacher B was asked about representational systems of the function concept, and he said the following:

"Ya a function we can we can represent it by the graph…We can represent it again by the equation".

With respect to dislike or like concerning the Collaborative Learning Programme, Teacher B revealed that there was nothing he did not like. In connection with this he noted:

“Aaah…Err in fact there is nothing I did not like because I was enjoying being in the project because I was…what I liked was in in fact the…the team work, how do we tackle problems being together discussing and then going towards the different approach or new approach…so that is why there is nothing that I disliked because everything was may I say profitable.”

Also, he was asked the question “what did the project mean to you?” and his response was:

“*So… the meaning…I must say in fact the... the project, I don’t know but now what I learned from the project, it was all about helping the teachers on how to approach functions…in an easy way.”*

In light of this, in session 4 of the CLP we asked teachers what was going to happen with our sessions because learners were going to be involved in the preparatory examinations. In connection to this Teacher B noted the following:

Researcher A	We will do that and then we need to talk what’s going to happen when your learners start writing the preparation exam, are we going to continue? Tuesday after noon?
Teacher B	Yes. Myself I don’t have a problem because I did make it clear to them I will be here in the morning session but after noon I will be here.
Researcher A	Will that be possible if we are here Tuesday afternoon so we can finish can we work like this, is this the way to learn?
Teacher B	Yes, I don’t see a problem.

In session 7 of the CLP, I observed that Teacher B insisted to solve a mathematical problem and by then the session was over. Here is a discussion that transpired concerning this matter:

Researcher A	Okay right I think the time is done it’s ... you already worked for two hours plus…
Teacher A	Is fine
Researcher A	I will bring along my computer again
Teacher B	No let’s let’s... let’s finish this
Researcher A	You can talk about this…think about it
Teacher A	Okay is fine
Teacher B	So is fine we can continue or
Teacher A	Mm let’s continue with this
Teacher B	Now
Teacher A	Now…ya
Researcher A	Ya
Teacher B	So I…
Teacher B	…I was saying for the equation that I got here

In light of the preceding paragraphs after the CLP Teacher B was asked the question what did you learn in the project and his response was:

> "*Ya in fact the learning was... was just a methodology. The... the approach of... of graphs… different approach of graphs and that's it mostly, although there were some little... little understanding on some of things like like…I... I may talk about the indirect and direct proportion… it was not clear to me, more especially how do I... I see it… But now……now is clear and then but in terms of the shifting of the graphs, the equations, modelling what I... I really learned is new approach".*

After the CLP Teacher B mentioned that he would appreciate to participate in future CLPs (if any), because he believed that the learners were not performing well in mathematics and that *"something is very wrong"*. He noted the following response when asked about future participation in the CLP:

> *"I may say definitely I will... I will participate because what I realized I was learning new approach…every day and then as such I was encouraged… So that will maybe help my learners to improve… So maybe if we had the time I could have revised the function… the functions with them."*

He further stated that he was very happy about the CLP:

> "*Myself…I'm just happy that I…in fact I believe that I will I will bring the changes on functions".*

A year later Teacher B maintained that he would appreciate to participate in a future CLP (see Table 5:5 and Table 5:6). He suggested that we should begin with another CLP but focusing on a different topic. In connection to this he said that it was important to engage more teachers. He indicated that he would participate in the future CLP, because in the previous CLP he had learned a new approach and as such he was encouraged to be part of it. Consequently he believed that the CLP would help his learners improve. In light of this he had this to say:

> *"Ya...the same I can say this is for the development, ya so each and every teacher need to be developed we cannot say, I cannot say I know the content I know all the*

strategy I have to be developed otherwise I would have I have 100% if I say I'm developed yes I saw it as a need for development if a have 60% it will be increase to 65 minimum its fine but as long as I'll be developed and able to unpack whatever or to take that to the learners that is why I took that 10 weeks I need to be developed so that I can present that to the learners…"

Also, After the CLP when asked about collaboration Teacher B said the following (see Table 5:5):

> "*Ya the collaboration was there because... err... four people sharing ideas so I think that is a collaboration e ya according to me err it does not necessarily have to be err maybe ten or … But if we can just try to share the ideas amongst ourselves then I think it i… it was there."*

A year later Teacher B was asked whether he had been able to work with other teachers at his school and the response was positive. He and the other two teachers at his school work together, they prepare together on difficult topics, talk about learners" views and he shares what he gained from the CLP with them. He went on to say that his approach concerning the teaching and learning of the function concept changed because of the CLP (see Table 5:6),

Herein, I conclude by noting that the Collaborative Learning Programme did not have an impact on Teacher's B definition of mathematics. But it is worth noting that the CLP did have an influence on Teacher's B views concerning the teaching and learning of mathematics, in particular the function concept. This is so because his views and beliefs concerning mathematics prior to, during and after the CLP, changed over time. Also the CLP challenged him to change his teaching strategy concerning the teaching and learning of the function concept.

Teachers had this to say about the use of technology in the CLP: Teacher B indicated that technology should be linked with the teaching and learning of mathematics, especially in modelling real-life situations. This was significant because technology could enable learners to discover relationships through observation. In addition he deemed technology as *"time efficient"* compared to drawing manually (see Table 5:6).

A year later both Teacher A and Teacher B were both asked to provide advice to the Department of Basic of Education (DBE) about professional development of mathematics teachers. They both maintained that one day workshops were not working. They state that one day workshops lack consistency and that the mathematical content is not related to each other. Also they state that one day workshops only happen after a long period of time, and as such teachers were not provided the opportunity to come back and reflect on their teaching. In connection with this, both teachers propose that the teachers' learning should be an on-going process from which teachers work together on mathematical tasks, share experiences and go back to their classroom to practice that learning. Herein, teachers believe that it is important to constantly provide feedback and talk to colleagues on a daily basis. Teacher B uses the term “team teaching” because he believes that the mathematics content is not necessarily the problem, but the teaching strategy is the challenge (see Table 5:6)

5.3 Teachers’ conceptual understanding of the function concept before, during and after the CLP.

Understanding of the function concept involves the insight of the different representational systems, namely numeric, formulae, graphic, and verbal (words) and the interplay between them. Hence, understanding of the concept of function in one representation does not necessarily mean that a student understands it in another representation. In connection to this, it is important for students to understand the concept of function in different representations and be able to perform transitions and form connections among and between them (see par 3.1.2)

In relation to the aforementioned, paragraph function tasks in different representational systems of the function concept were posed to examine teachers’ conceptual understanding of the function concept. The rubric in (Table 4:6) was used to assess teachers’ conceptual understanding of the function concept before, during and after the CLP. These include the pre-assessment task, Task 1, Task 2 and the post-assessment task. In the pre-assessment task, Task 1, Task 2 and post-assessment task, the content was based on both the procedural and conceptual knowledge of the function concept. The former involved mathematical content such as investigating the properties of the function concept by means of procedural skills. In light of this, both Teacher A and B were engaged in tasks that encouraged them to find the equations of the functions, provided that the graph of a function is given or vice versa. Also they were encouraged to determine the domain, range, shifting and point of intersection related to these tasks. The context from which these tasks appeared involved linear, quadratic, inequalities as well as absolute values and they were presented in a graph or equation. Procedural knowledge concerning the function concept was dealt with in the pre-assessment task, the post assessment task and Task 2. The latter was informed by the mathematical content that embraced the function model (O’Callaghan, 1998). Herein, teachers were investigating the properties of the function concept by means of real-life situations. Teachers were engaged in mathematical activities that encouraged them to perform transitions among different representational systems of the function concept. These representational systems included graphs, equations, tables and words. These activities were specifically dealt with in the pre-assessment task, Task 1 and post- assessment task. The following, that is Tables 5.7, 5.8, 5.9 and 5.10, represent both Teacher A and Teacher B’s performance with respect to their understanding of the function concept. The rationale behind conducting these function tasks was to answer the following question:

What is the influence of a collaborative learning programme on teachers' conceptual understanding of functions?

Question	Component	Performance	Representational System
1	Procedural	Moderate understanding (level 3)	Graphical: Interpreting, Operations and Transformations
2	Procedural	Complete understanding level 4)	Algebraic: Operations and transformations
3	Modelling	No effort (level 0)	In words: Real-life situation
4	Modelling	No effort (level 0)	In Words: Real-life situation
5	Interpreting	No effort (level 0)	Numerically (table form): Real life-situation
6	Interpreting	No effort (level 0)	In words: Real-life situation
7	Translating	No effort (level 0)	Numerically (table form): Real-life situation
8	Translating	No understanding (level 1)	Numerically (table form): Real-life situation
9	Reifying	No effort (level 0)	Algebraically: Real-life situation

Table 5:7 Task Performance of Teacher A: Pre-assessment

The above table, that is Table 5.7, is the summary of Teacher A's performance concerning the pre-assessment task which was conducted in our first meeting with the teachers. Out of nine questions she only attempted two non-real life mathematical situations (i.e. question 1 and 2), and one real-life mathematical situation (question 8). Results indicate that she did not attempt the other six mathematical situations (that is question 3, 4, 5, 6, 7, and 9) which involved modelling, interpreting, translating as well as reifying. Herein I note that she performed well in question 1.4 (see Figure 5:19) and she did not provide satisfactory responses to those questions that required her to interpret the graphical situation in the context of the point of intersection (see Figure 5:18). Whilst in question 2, which dealt with the transformation of a quadratic function (horizontal and shifting of the graph), Teacher A provided correct answers to both question 2.1 and 2.2 (see Figure 5:22). I note that the evidence suggests that she appreciated mathematical situations from which the context was procedural in nature more than real life situations. But she struggled to perform different transitions among different representational systems of the function concept, specifically those presented in words, graphs and numerically (table form).

Question	Component	Performance	Representational System
1	Procedural	Moderate understanding (level 3)	Graphical: Interpreting, Operations and Transformations
2	Procedural	Poor understanding (level 1)	Algebraic: Interpreting, Operations and Transformations
3	Modelling	Complete understanding (level 4)	In words: Real-life situation
4	Modelling	No effort (level 0)	In Words: Real-life situation
5	Interpreting	No effort (level 0)	Numerically (table form): Real-life situation
6	Interpreting	Moderate understanding (level 1)	In words: Real-life situation
7	Translating	Complete understanding (level 4)	Numerically (table form): Real-life situation
8	Translating	Moderate understanding (level 1)	Numerically (table form): Real-life situation
9	Reifying	No effort (level 0)	Algebraically: Real-life situation

Table 5:8 Task Performance of Teacher B: Pre-assessment

The above table that is Table 5.8 is the summary of Teacher B's performance concerning the pre-assessment task which was conducted in our first meeting with the teachers. Out of nine questions he only attempted 6 questions (1, 2, 3, 6, 7 and 8) and could not complete the other three questions (4, 5 and 9). I note Teacher B appreciated both question 1 and 2 but it is worth noting that he struggled with those questions (1.1-1.3) that required him to make interpretations about the graphical representation of question 1 (see Figure 5:20). Whilst in question 2, which dealt with the transformation of a quadratic function (horizontal and shifting of the graph) Teacher B did not provide correct answers to both question 2.1 and 2.2 (see Figure 5:23). In question 3 which dealt with modelling a real-life situation with functions Teacher B provided correct answers to all the questions (see Figure 5:1 and Figure 5:2). In question 6 which focused on interpreting a real-life situation within the context of an indirect relationship, Teacher B only provided correct answers to question 6.1 and 6.2 (see Figure 5:6). Also in question 7, from which the function concept was presented numerically (in a table form) he provided the correct answer (see Figure 5:11). In question 8 which comprised three questions from which the function was presented numerically (in a table form) he provided correct answers to both question 8.1 and 8.2 as well as the correct reasoning. In question 8.3 he provided a response with valid reasoning but it was not complete (see Figure 5:12).

Question	Component	Performance	Representational System
1	Interpreting	Complete understanding	Graphical: Real-life situation
2	Modelling	No effort (level 0)	In words: Real-life situation
3	Translating	No effort (level 0)	In words: Real-life situation
4	Translating	No effort (level 0)	In Words: Real-life situation
5	Modelling	Complete understanding (level 4)	In Words: Real-life situation
6	Interpreting	Complete understanding (level 4)	Graphical: Real-life situation
7	Modelling	Complete understanding (level 4)	In Words: Real-life situation

Table 5:9 Task Performance of both Teacher A and Teacher B: Task 1

The above table, that is Table 5.9, is the summary of both Teacher A and Teacher B's performance concerning Task 1 which was conducted in session 6 of the CLP with the teachers. Herein, I note that out of seven questions Teacher A and Teacher B only managed to completed four questions that is question 1, 5, 6 and 7.

In question 1 which comprised six questions, the function was presented by a graphical representation and centred on interpreting a real-life situation with functions. The questions were centred on interpreting a linear relationship and teachers provided correct answers to all questions (see Figure 5:8). In question 5 which comprised two questions the function was presented in words. The question involved linear relationship. This question was completed and correct answers with correct reasoning were provided. Herein, there is evidence of deep understanding of the function concept presented in words. They were able to model real-life situation with functions (see Figure 5:3). In question 6 which comprised five questions, the function was presented by a graphical representation. The question involved indirect proportion. Teachers showed appreciation of the situation and managed to provide corrects answers with correct reasoning to all questions. Herein, there is evidence of deep understanding about the function concept presented graphically. They were able to interpret the graphs (see Figure 5:9). In question 7 which comprised seven questions the function was presented in words. The question involved linear relationship (see Figure 5:4 and Figure 5:5). Out of seven questions they provided six correct answers with correct reasoning. Herein, there is evidence of deep understanding about the function concept presented in words. They were able to model a real-life situation with functions.

In question 2 which comprised three questions, the function was presented in words. They showed no understanding of the situation; hence they did not complete the question. In question 3, which comprised two questions, the function was presented in words. This question was based on a non-linear relationship. Teachers did not complete this question. In question 4 which comprised five questions the function was presented in words. This question involved a linear relationship. They did not complete this question. They were not able to translate among different representations of functions.

Question	Component	Performance	Representational System
1	Procedural	Complete understanding (level 4)	Two linear equations
2	Procedural	Complete understanding (level 4)	Two linear equations
3	Procedural	Complete understanding (level 4)	Linear equation and quadratic equation.
4	Procedural	Complete understanding (level 4)	Inequality and absolute value

Table 5:10 Task Performance of both Teacher A and Teacher B: Task 2

The above table (Table 5.9) is the summary of both Teacher A and Teacher B's performance concerning Task 1 which was conducted in session 7 of the CLP with the teachers. Teachers were requested to solve the system of two linear equations as well as to provide meaning for their solutions. Answers were requested both algebraically and graphically. Herein, the teachers' procedural knowledge of the function concept is at level 4 (that is complete understanding). I note that the teachers' procedural knowledge of the function concept is satisfactory. Teachers were able to provide correct answers with correct reasoning, and the communication of solutions was coherent. Teachers were able to perform operations and transformations with functions (see Figure 5:24, Figure 5:25, Figure 5:26 and Figure 5:27 respectively).

Question	Component	Performance	Representational System
1	Procedural	Moderate understanding (level 3)	Algebraic: Interpreting, Operations and Transformations
2	Procedural	Poor understanding (level 1)	Algebraic: Interpreting, Operations and Transformations
3	Modelling	No effort (level 0)	In words: Real life situation
4	Translating	Poor understanding (level 1)	Graphically: Real life situation
5	Interpreting	Moderate understanding (level 3)	Graphically: Real life situation life situation
6	Reifying	No effort (level 0)	Algebraically: Real life situation

Table 5:11 Task Performance of Teacher A: Post-assessment

The above table (Table 5.11) is the summary of Teacher A's performance concerning the post-assessment task which was conducted in our last meeting with the teachers. Out of six questions, she only attempted two non-real life mathematical situations (that is questions 1 and 2) and two real-life mathematical situations (that is questions 6 and 3). In question 1 which comprised four questions, the function was presented by an algebraic representation and the questions were centred on a hyperbola. Teacher A showed appreciation of the situation but struggled to provide correct answers to all the questions (see Figure 5:28). In question 2 which comprised four questions, the function was presented by an algebraic representation. This question was based on the system of equations (linear and quadratic), finding the point of intersection, sketching of graphs and determining the average gradient between two points, as well as explaining the meaning of the average gradient. Teacher A showed appreciation of the situation but struggled to provide correct answers to all the questions (see Figure 5:29).

Also from Table 5.11, I note that she did not attempt a situation that involved both modelling and reifying (questions 3 and 6). In question 5 from which there were two questions, the function was presented by a graphical representation in the context on interpreting a real-life situation with functions. I note that in question 5.2 and 5.3 Teacher A's responses were correct but the communication of the solutions was not coherent (see Figure 5:10). Henceforth, Teacher A struggled to interpret the information presented in different representational systems of the function concept. Consequently she could not perform different transitions among different representational systems of the function concept, specifically those presented in words, graphs and numerically (table form). Henceforth I note that there is no change in Teacher A's conceptual understanding of the function concept when comparing her performance in the pre-assessment.

Question	Component	Performance	Representational System
1	Procedural	Complete understanding (level 4)	Algebraic: Interpreting, Operations and Transformations
2	Procedural	Moderate understanding (level 3)	Algebraic: Interpreting, Operations and Transformations
3	Modelling	Moderate understanding (level 3)	In words: Real life-situation
4	Translating	Moderate understanding (level 3)	Graphically: Real-life situation
5	Interpreting	Complete understanding (level 4)	Graphically: Real-life situation
6	Reifying	Complete understanding (level 4)	Algebraically: Real-life situation

Table 5:12 Task Performance of Teacher B: Post-assessment

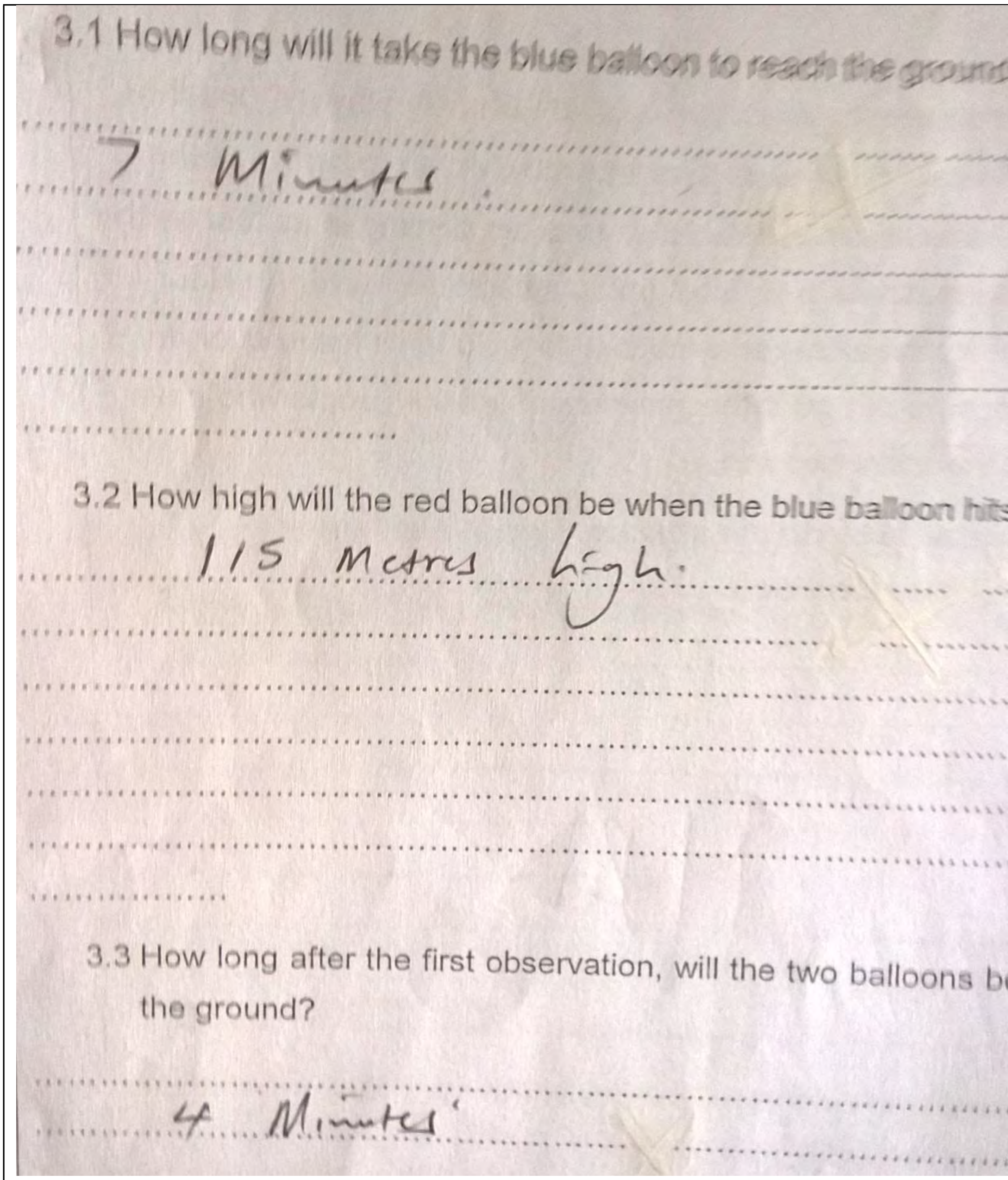
The above table that is Table 5.12 is the summary of Teacher B's performance concerning the post-assessment task which was conducted in our last meeting with the teachers. In question 1 which comprised four questions, the function was presented by an algebraic representation. The questions were centred on the equation of a hyperbola. From question 1 to question 4 Teacher B provided correct answers, correct reasoning and coherent communications concerning solutions. In question 2 which comprised four questions, the function was presented by an algebraic representation. The question involved a system of equations (linear and quadratic) finding the point of intersection, sketching of graphs and determining the average gradient between two points. Herein, I note that Teacher B provided correct answers, correct reasoning and coherent communications concerning solutions. When I compare his task performance in the pre-assessment to the post assessment, I note that there is a change in Teacher B's proficiency concerning the procedural knowledge of the function concept. In question 3, which comprised four questions, the function was presented by words. The questions were centred on modelling a real life situation with functions. He managed to provide correct answers to questions 3.1 and 3.2 respectively, but could not provide correct answers to questions 3.3 and 3.4. In question 4, from which there were two questions, the function was presented by a graphical representation. Herein, the context of the question was based on the velocity-time graph, and it consisted of three graphs showing the velocities of three girls over a given time interval. Teacher B showed appreciation of the question and provided satisfactory responses (see Figure 5:15). In question 5, from which there were two questions, the function was presented by a graphical representation. Herein, the context of the question was based on a blood pressure-time graph, and questions were involved the determining of the time intervals from which the function is decreasing or increasing. Herein, Teacher B showed appreciation of the situation and provided correct answers to all the questions (see Figure 5:3). In question 6 from which there was one question, the function was presented by an algebraic (linear relationship) representation. Herein, the context of the question was deemed reifying and it focused on combining two linear functions. I note that he had an appreciation of this situation because he provided a correct answer with correct reasoning. When I compare his task performance in the pre- assessment to the post assessment I note that there is a change in Teachers' B proficiency concerning the reifying component of the function concept. He was able to reify functions (see Figure 5:17).

In the following section I establish a decisive discussion concerning the conceptual understanding of both Teacher A and Teacher B. I establish this discussion by categorizing the four components of the function model as well as the procedural skills. Also I use data from other sources (transcripts) in order to build a coherent and clear discussion regarding both Teacher A and Teacher B's conceptual understanding of the function concept.

5.3.1 Modelling

In the pre-assessment (see Appendix F) there were two questions (3 and 4) focusing on modelling real-life situations with functions within the context of a non-linear and linear relationship. Herein, Teacher B was able to model a real-life situation within the context of a linear relationship which is question 3. That is to say Teacher B was able to a model a real- life situation presented in words whilst making different types of interpretations and translating from one representation to another. He provided satisfactory responses to question 3.1, 3.2, 3.3 and 3.4

as well as the correct graph of the situation (see Figure 5:1and Figure 5.3). Herein is evidence of deep understanding with respect to modelling a real-life situation with functions (see Table 5:8). Question 4, from which the function concept was presented in the context of a non-linear relationship, was not completed by either teacher. Herein the function concept was presented in words and teachers were required to investigate the properties of the function concept such as independent and dependent variables, the domain and range of the function as well as the graph of the function.


3.1 How long will it take the blue balloon to reach the ground

7 Minutes.

3.2 How high will the red balloon be when the blue balloon hits

115 Metres high.

3.3 How long after the first observation, will the two balloons b
the ground?

4 Minutes

Figure 5:1 Teacher B's response to question 3 of the pre-assessment

The following is a graphical solution of Teacher B's to question 3 of the pre-assessment function task.

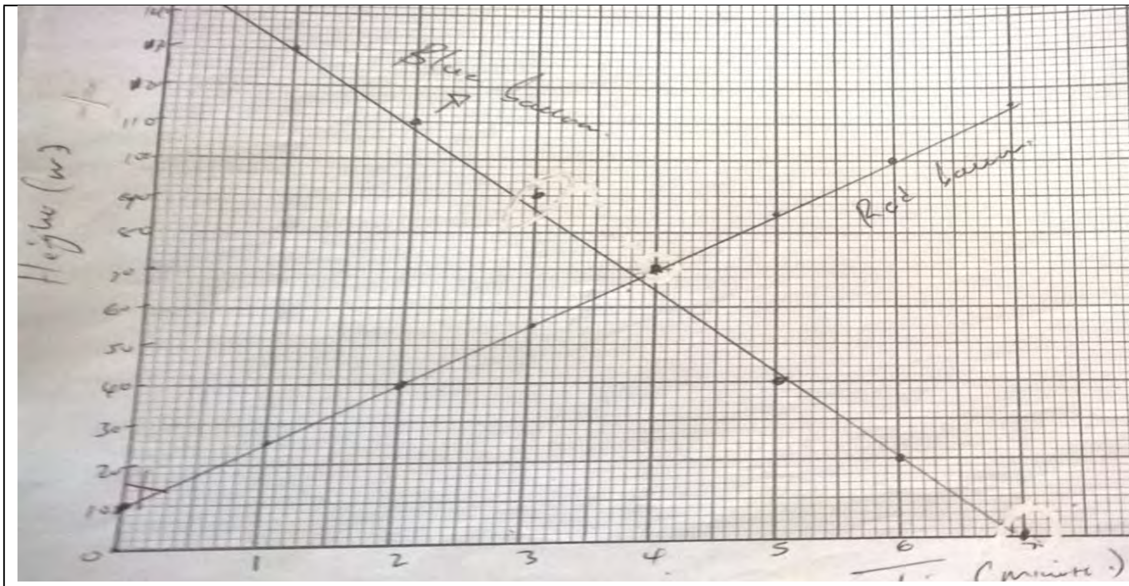


Figure 5:2 Teacher B's graphical representation of the mathematical situation in question 3

In Task 1 (see Appendix G) there were three questions (2, 5 and 7), focusing on modelling real-life situations with functions within the context of both non-linear and linear relationships (see **Table 5:9**). Herein, Teacher A and Teacher B managed to complete both question 5 and 7, but could not complete question 2. Both question 2 and 5 were situated within the context of a linear relationship. On the other hand, question 2 was situated within the context of a quadratic function or a semi-circle**.**

In question 5 of Task 1 which comprised two questions, the function concept was presented in words situated in the context of a linear relationship. Herein, teachers attempted to complete this question and they provided correct answers. They managed to determine the equation of the mathematical situation as well as the graph of the situation (see Figure 5:3). Herein, I note evidence of deep understanding with respect to modelling a real-life situation with functions (see **Table 5:9**). The following is a discussion that transpired (in session 7) between Teacher A and Teacher B while they were solving question 5:

Teacher B	A ball is thrown from a height of 150. The ball reaches the ground after 5 seconds. So represent the above situation graphically. So… so for this one…I think for this one it will be let's say 50…100…150……25….75…100…125…150…that's the distance in metres
Teacher A	Mm
Teacher B	So then, metres…

Teacher A	Mm…5 seconds
Teacher B	So *mona* is time in seconds.
Teacher A	Mm
Teacher B	But now the question is it go like this or will it go like that?
Teacher A	It will go straight
Teacher B	Go straight
Teacher A	Mm
Teacher B	This way
Teacher A	Ya it will cut also at…it means we are going to get a straight line
Teacher B	So I must draw a straight line
Teacher A	Mm, it will be a straight line
Teacher B	Ooh I don't know wow…silence…sure sure…draws the graph…*background noise+…laughs
Teacher A and Teacher B	Find the equation that models this situation
Teacher A	Y is equal to the value of C is 150, it means we must find the gradient…C…is 150 let's find the gradient
Teacher B	…the gradient
Teacher A	Over 5 so the gradient is 150 over 5 is 30
Teacher B	150
Teacher A	Ya ya is 30
Teacher B	Then we gonna move

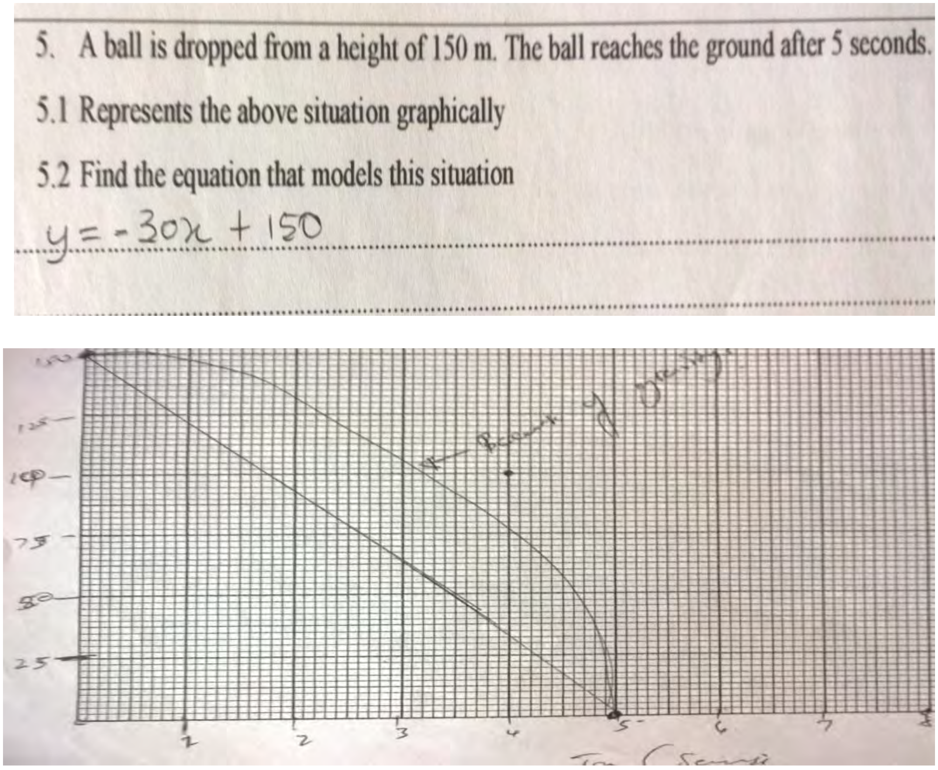


Figure 5:3 Teachers' A and B response to question 5 of Task 1

In question 7 of Task 1, which comprised two questions, the function was presented in words. The question involved modelling a real-life situation with the concept of function situated within the context of a linear relationship. This question was completed and correct answers with correct reasoning were provided, as well as the graphical representation of the situation (see Figure 5:4 and Figure 5:5). Also, when I look at the transcripts I note that both Teacher A and Teacher B appreciated the mathematical situation because they were talking to each other, they helped each, they asked each other questions and in the process provided explanations to their solutions. There was enjoyment during the completion of this question because of the laughing during the activity. Herein, I note evidence of deep understanding with respect to modelling a real-life situation with functions (see Table 5:9). What strikes me most here is the involvement of Teacher A, she is involved in a mathematical situation that involves modelling of a real-life situation. The following is the active discussion that transpired between Teacher A and Teacher B as they were attending to question 3:

Teacher B So let's take number seven

Teacher A Number seven…

Teacher B R10 per parcel

Teacher A Mm

Teacher B	Give you what? Another courier company
Teacher A	R5 per parcel
Teacher B	R5 per parcel…delivered
Teacher A	Mm
Teacher B	Plus… plus what…what is this…surplus what is that? What is the meaning of
Researcher B	It's extra
Teacher B	Extra
Researcher B	Yes
Prof	Ya
Teacher B	Extra
Researcher B	In the beginning you started with this if you cover another kilometre you just add the…
Researcher A	You pay courier and then five rand for each package
Teacher A	Parcel
Teacher B	Okay
Teacher A	…this one is a half of this…find the equation that model the above situation
Teacher B	And another courier…chargers is R5 per parcel plus extra R20
Teacher A	It means….
Teacher B	…so find the equation that models the above situation…so let's take the first equation
Teacher A	Mm
	[*Silence…*]
Teacher B	So would I say…I wouldn't say per parcel *akere*…so I may say I may say the first one y equal to is 10x
Teacher A	Mm
Teacher B	If is one it will be R10, if is two it will be R20
Teacher A	Mm
Teacher B	If is three then R30 am I right
Teacher A	What about…?
Teacher B	Per parcel delivered *akere*
Teacher A	Mm, ya per parcel delivered
Teacher B	So then Y = 10x
Teacher A	Mm
Teacher B	…but for the second company
Teacher A	Mm
Teacher B	It will be 5x + R20 *akere* it says regardless of the number of parcels delivered *le ha ba katla ka box e* one.
Teacher A	Mm
Teacher B	They will charge you R25

Teacher A	Mm
Teacher B	So 7.2 draw…draw two graphs on the same set of axes to show the total cost of using each courier company for various number of parcels
Teacher A	Mm
Teacher B	And then the graph will be…let's have that one
Teacher A	The first one…it is y = 10x
Teacher B	So which one must be
Teacher A	No let's start with the second
Teacher B	I will say say…so these are the number of parcels
Teacher A	Mm
Teacher B	So this is the cost
Teacher A	The cost
Teacher B	The cost *akere*
Teacher A	Mm
Teacher B	In rands
Teacher A	Mm
Teacher B	Then *ka mona* it will be
Teacher A	Number
Teacher B	Number of parcels…
Teacher A	Yes
Teacher B	Then but this can't I I may say 10…20…30…40…50…will see
Teacher A	Mm
Teacher B	And then but 1 for the first one *akere* one is 10
Teacher A	Mm
Teacher B	And then two will be 20 am I right and so if it's two
Teacher A	Zero start *ka* you started with the second one
Teacher B	With the first one
Teacher A	Okay…okay
Teacher B	So then if if is one parcel
Teacher A	Mm
Teacher B	You charge R10
Teacher A	Mm
Teacher B	Then if it's two…is going to be R20
Teacher A	Mm
Teacher B	Then three 30
Teacher A	Mm
Teacher B	Just like that three 30 four five 50…

Teacher A Mm…draw that one. **Drawing the graph*]

Teacher B Up to infinity

Teacher A Up to infinity, is fine

Teacher B Then

Teacher A The second one

Teacher B The second one so

Teacher A Y is 20

Teacher B 25 *akere*

Teacher A Y is 20 and x is x is 4…y is 20 and x

Teacher B Mm, y is 20

Teacher A Y is 20 and what is x…x is 24 is not

Teacher B So we going start at at 20

Teacher A Okay the y intercept to safe time at…

Teacher B But but I will saying we start at at zero

Teacher A At zero

Teacher B Eya because if if they did not deliver anything they still/cannot charge

Teacher A Okay

Teacher B Even extra….(both laughing)

Researcher A Ya

Teacher B …but if they start to deliver 1 parcel then it will be R25

Teacher A Mm

Teacher B One is one parcel is R5 plus extra of R20

Teacher A Okay so if it's two

Teacher A It will be 30…

Teacher B Ya…is 30

Teacher A Mm, if it's three, it will be 35

Teacher B Mm

Teacher A It's four, is 40, if is four, is 40

Teacher B Mm

Teacher A It's, is five

Teacher B 45 ya…five means

Teacher A it means 50

Teacher B 5 times 5 is 25 *akere*

Teacher A Ya…is 45. Is fine. Mm…ya is fine

Teacher B So but [laughs] it has to start somewhere it can't start at zero, otherwise it won't be a straight line graph…

Teacher A That is why I say

Teacher B	So you said start at 20
Teacher A	The y intercept it will be 20 and let's look at X it will be a straight line I think
Teacher B	No it won't…if you say …intercepts
Teacher A	Mm
Teacher B	…straight line
Teacher A	Okay, mm…okay the graph will intersect here right on top of the graph to the equation
Teacher B	Aaah…this one
Teacher A	y = 5x + 20
Teacher B	Mm…this one is equal to 10x
Teacher A	Mm e sharp, it's fine okay. Let's look at draw two graphs. From your graphs determine which company is cheaper if you want 3 parcels to be delivered? Okay so its company what and what write fast what and fast what?
Teacher B	*Ke* fast *mang*?
Teacher A	Which one is the fastest?
Teacher B	…the courier company the faster *akere* and then faster fetch
Teacher A	Ya, the second one fast fetch
Teacher B	So you say
Teacher A	That one of 20 it must be fast fetch
Teacher B	This one is fast *akere*
Teacher A	Err, this one is faster fetch…okay they say what…which company is cheaper if you want 3 parcels…
Teacher B	Fast courier
Teacher A	The company that is cheaper if you want 3 parcels
Teacher B	The faster *akere*
Teacher A	Ya the faster, the first one… there is no reason
Teacher B	No
Teacher A	Okay
Teacher B	Then *mona e tloba* fast fetch, fast fetch
Teacher A	Let's give also the reason so that we can something that is…
Teacher B	Faster company, so the reason for the faster company
Teacher A	Okay for the faster company**...** [*silence*+… the more the parcels will be delivered on the faster the less the…the expense, ya…look at this one…3 parcels delivered therefore now which one is cheaper? This one is not cheaper
Teacher B	The fast neh…
Teacher A	Let's look at this…the fast
Teacher B	But…but
Teacher A	The cheaper is this….
Teacher B	Is that one

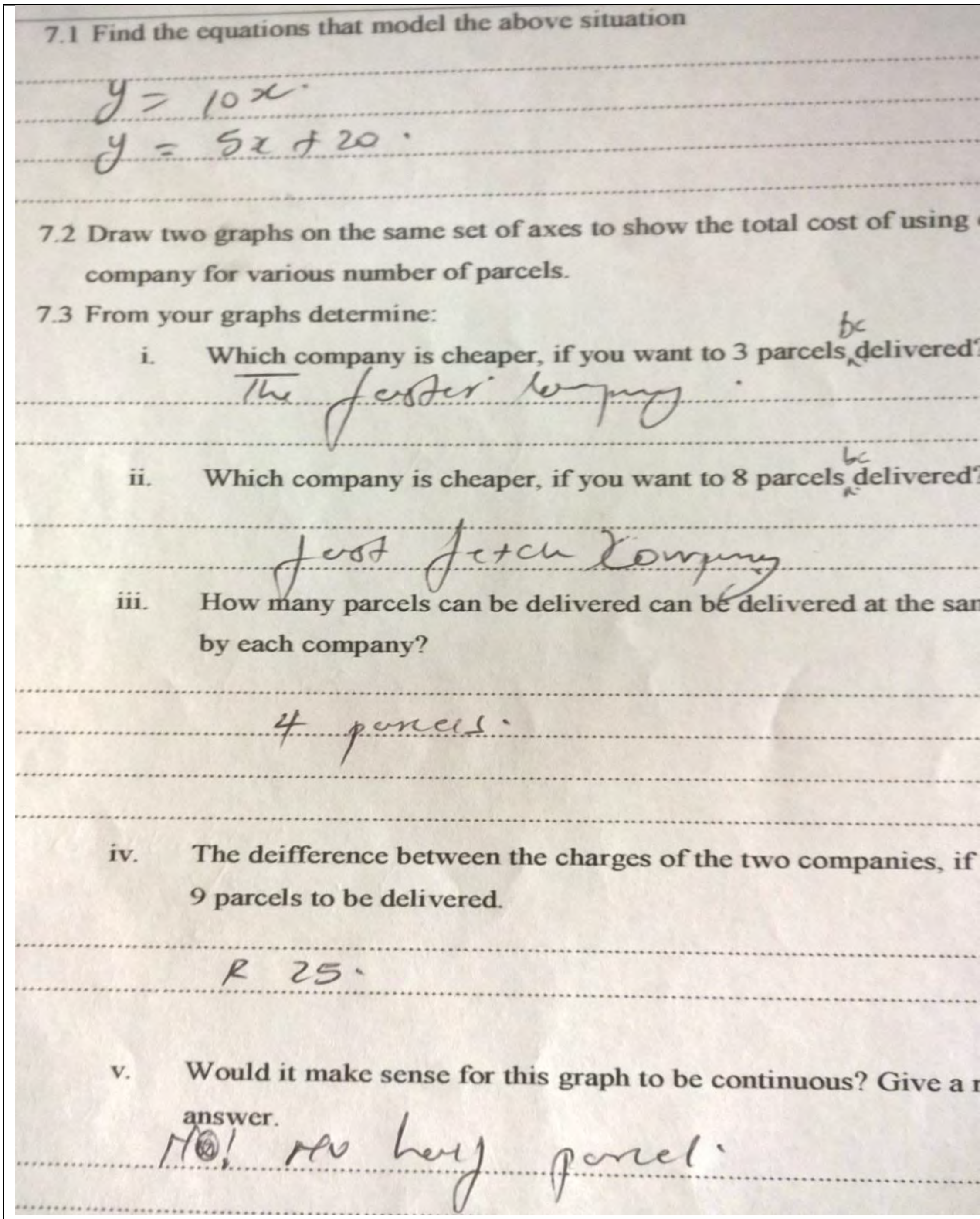
Teacher A	Ya
Teacher B	But now if this one it is going to be…it will be cheaper if we may if the number of parcels are
Teacher B	Are fewer
Teacher A	Let's let's look
Teacher B	Mm
Teacher A	*Akere* if zero there are no parcels there is nothing, it's one parcel is R10
Teacher B	Mm
Teacher A	It's more it's go up it's go up
Teacher B	Mm
Teacher A	It means this one more expensive than this one
Teacher B	…but but you see this one even if they don't deliver anything they charge you R20
Teacher A	Laughs
Researcher A	Laughs
Teacher A	Ooohoo
Teacher B	…that means is expensive
Researcher A	Laughs
Teacher A	So this one is expensive
Teacher B	…no no I'm saying not necessarily it is expensive…I'm saying err the first one the faster company
Teacher A	Err it is the one that is cheaper
Teacher B	So if if if if you order if if they deliver few parcels they are cheaper they are
Teacher A	Few parcels
Teacher B	Ya they are cheaper
Teacher A	Okay
Teacher B	mm…but now
Teacher A	This one
Teacher B	Then the first…the first the first fetch it is cheaper if it delivers
Teacher A	More
Teacher B	More parcels
Teacher A	Yes but let's look at it
Teacher B	Eya you can you can you can just check it
Teacher A	The more it goes…
Teacher B	So…will say the more…let's take our more is 5
Teacher A	Okay our more is 5
Teacher B	So they are cheaper compared to the fast but if we say let's take one…one at one this one

Teacher A	…is cheaper
Teacher B	That one
Teacher A	More expensive
Teacher B	It's R25…but as we keep on increasing the number of parcels
Teacher A	So let's look at…
Teacher B	The fast fetch their price gets less you see
Teacher A	Mm
Teacher B	So then that's why I'm saying…
Teacher A	The more the number of parcels
Teacher B	Mm
Teacher A	Mm …I'm lost
Teacher B	Laughs
Teacher A	Let's start again
Teacher B	Okay
Teacher A	You say the one that is cheaper you say it is you say what…?
Teacher B	So we on roman figure three neh
Teacher A	Eya…
Teacher B	Roman says how parcels can be or in fact we wanted the…
Teacher A	Haaa let's start with one
Teacher B	One
Teacher A	Which company is cheaper if you want 3 parcels to be delivered?
Teacher B	Eya, then this
Teacher A	Okay 3 parcels
Teacher B	This one is cheap
Teacher A	So this one is cheaper
Teacher B	Ya the fast
Teacher A	If you 3 parcels so this one is cheaper
Teacher B	Yes
Teacher A	The fast
Teacher B	Yeah
Teacher A	And then which co…the second one which company is cheaper if you want 8 parcels
Teacher B	Mm
Teacher A	Therefore now it will be the vice versa
Teacher B	Ya
Teacher A	You want 8 it means the one that will be cheaper
Teacher B	Yes

Teacher A	Vice versa
Teacher B	Mm
Teacher A	Okay, and then how many parcels can be delivered at the same total cost by each company? At the same total cost by…let's look
Teacher B	It's four…how many parcels can be delivered?
Teacher A	Mm
Teacher B	At the same total cost
Teacher A	Ya…ya it will be four…ya it's four you with that four… Okay the difference between the charges of the two companies. If you wanted 9 parcels to be delivered
Teacher B	To be delivered
Teacher A	Okay,
Teacher B	Then we say
Teacher A	We looking for the cost
Teacher B	Mm …6…7…8… but we can we we are looking for the cost can just check the pattern…*akere* this one we just keep on increasing
Teacher B	Mm
Teacher A	So then if…*akere*
Teacher B	Mm
Teacher A	But this one it will be 5 times 6 which is
Teacher B	Mm
Teacher A	30
Teacher B	Mm
Teacher A	30 + 20
Teacher B	Mm
Teacher A	Is 50…
Teacher B	E sharp and then
Teacher B	But *a re* battle difference *mona* at 5
Teacher A	Mm
Teacher B	Difference *ke bokae*?
Teacher A	Mm
Teacher B	But then *mona* difference *ke bokae? Mona* difference *ke*… is 5
Teacher A	Mm
Teacher B	Am I right
Teacher A	Mm
Teacher B	But *ko* 6 difference will be
Teacher A	Mm
Teacher B	10

Teacher A	10
Teacher B	So then *ko* 7 difference will be 15
Teacher A	15
Teacher B	Ko 8 it's going to be 20
Teacher B	And then 25
Teacher A	25…ya
Teacher B	So because here it is…so at 8
Teacher A	Mm
Teacher B	8 parcels
Teacher A	Mm
Teacher B	Is 10 times 8 ko 80 *akere*, so it's 60…70…80…and then but this one is 5 times 8 which is 40
Teacher A	Mm
Teacher B	Then 40 plus
Teacher A	Plus 20
Teacher B	20
Teacher A	Plus 20…40…60…and then the next one
Teacher B	Here is 60
Teacher A	Mm
Teacher B	So that's why I said 25
Teacher A	Is 25 ya that's right
Teacher B	So after it here increases by
Teacher A	Ii increases by
Teacher B	By 5
Teacher A	By 5
Teacher B	So…difference is 25
Teacher A	25
Teacher B	R25
Teacher A	Ya so it increases by by 5
Teacher B	Mm, would it be would it make sense…
Teacher A	For this graph
Teacher B	For this graph to be continuous? Give a reason for your answer. So what what kind of answer… **silence*+… so it don't make sense… let's say they are delivering the books here you order one book
Teacher A	Mm
Teacher B	It's either is one or nothing
Teacher A	Mm …or nothing if there is no order nothing ya

Teacher B	But if you want to deliver chairs is either you order one chair or nothing
Teacher A	Mm… sometimes you are not going to order…
Teacher B	So then our graph is wrong…laughs…*akere*…so…would it make sense
Teacher A	No
Teacher B	Mm …why? No half parcel
Teacher A	…if if there is no order it means there is no delivery
Teacher B	Mm
Teacher A	Mm
Teacher B	So you cannot buy half a parcel
Teacher A	Mm …no delivery no no order no delivery
Teacher B	No, no order delivery is there
Teacher A	Laughs
Teacher B	Delivery is there
Teacher A	Laughs
Teacher B	So…I would say what no half parcel *akere*
Teacher A	No half parcel
Teacher B	Yes, so *ya*



7.1 Find the equations that model the above situation

$y = 10x$

$y = 5x + 20$

7.2 Draw two graphs on the same set of axes to show the total cost of using

company for various number of parcels.

7.3 From your graphs determine:

i. Which company is cheaper, if you want to 3 parcels (be) delivered?

The faster company.

ii. Which company is cheaper, if you want to 8 parcels (be) delivered?

fast fetch company

iii. How many parcels can be delivered can be delivered at the sar

by each company?

4 parcels.

iv. The deifference between the charges of the two companies, if

9 parcels to be delivered.

R 25.

v. Would it make sense for this graph to be continuous? Give a r

answer.

No! no half parcel.

Figure 5:4 Teacher's A and B response to question 7 of Task 1

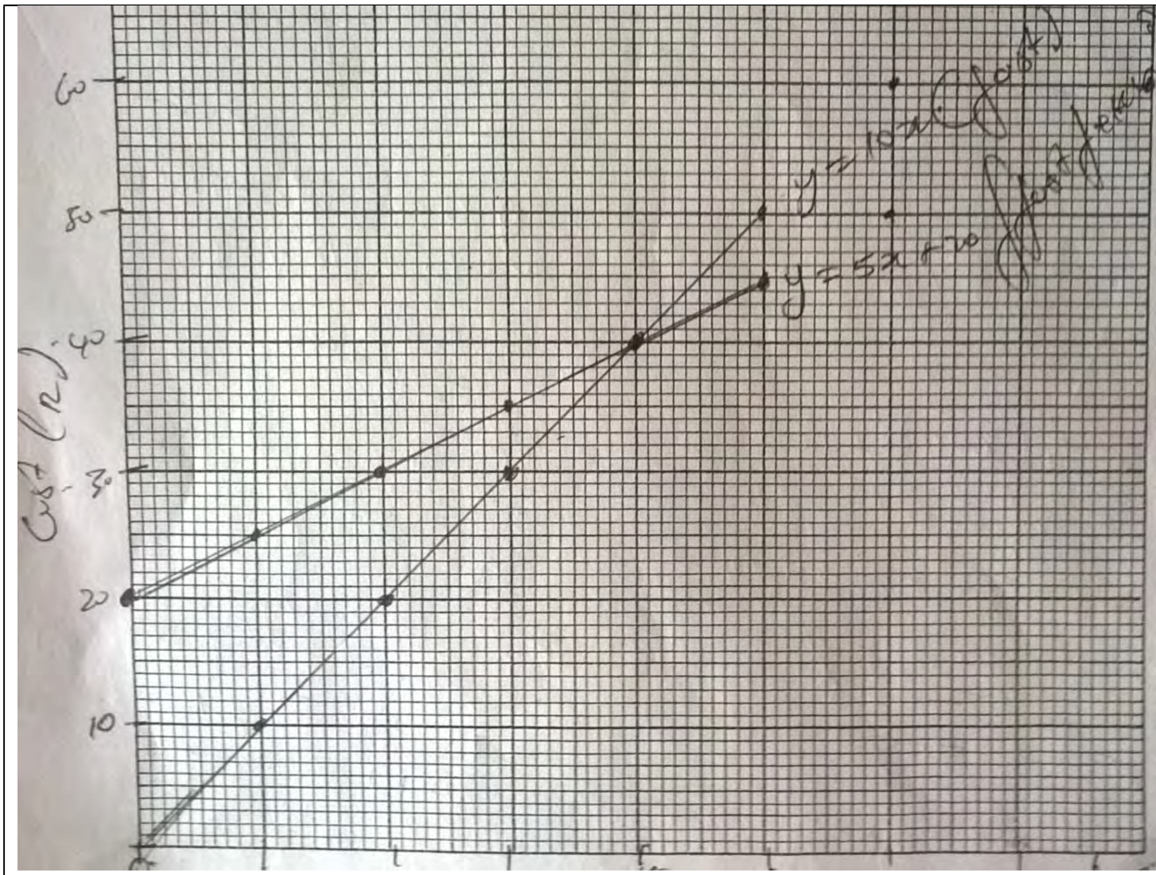
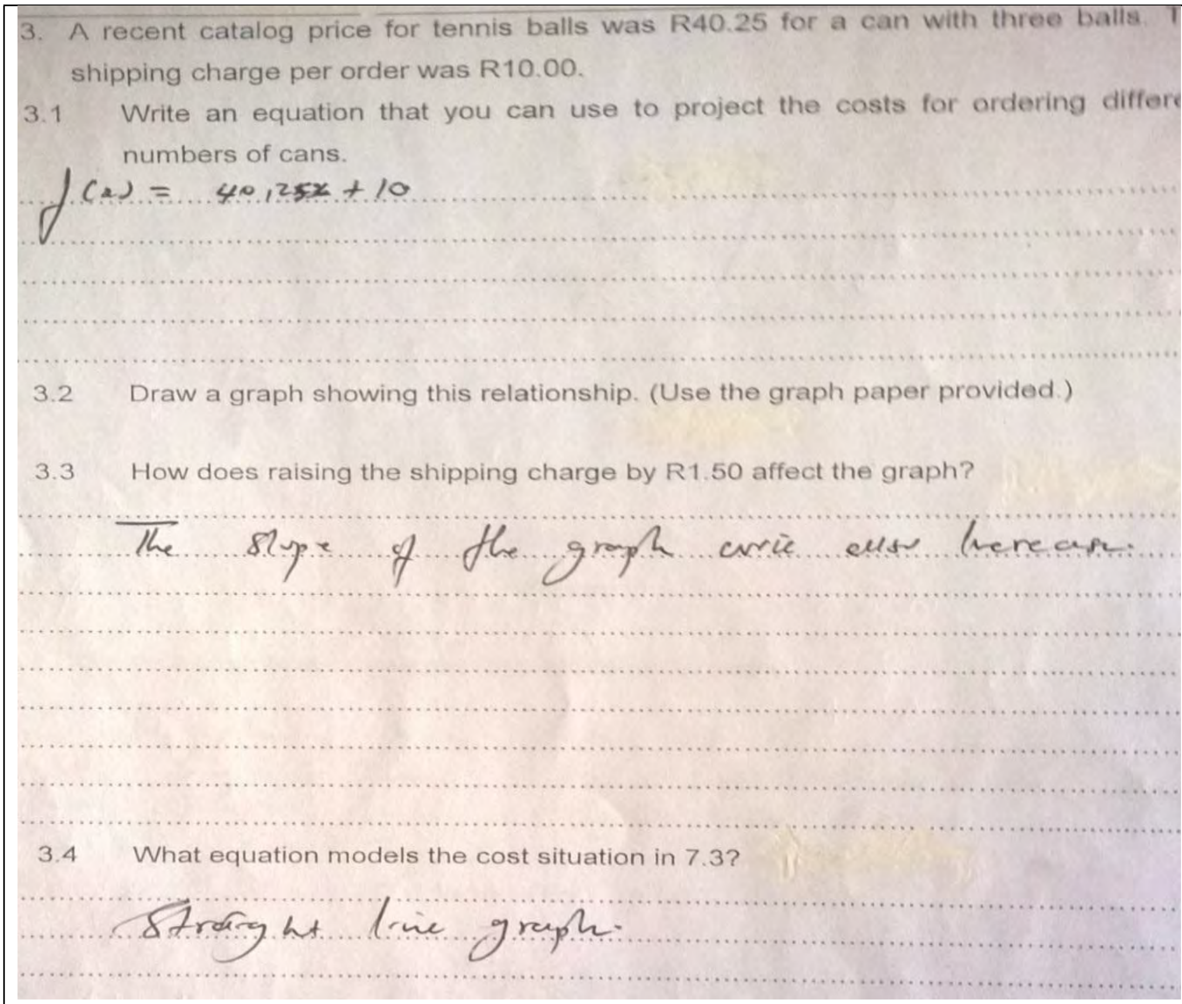


Figure 5:5 Teacher A and B's graphical representation to question 7 of Task 1

In the post-assessment (see Appendix I) there was only one question involving modelling a real-life situation. Teacher B attempted this question but could only provide correct answers to questions 3.1 and 3.2. In question 3.3 and 3.4 he provided incorrect answers. Herein, I note that he struggled to make connections between the questions, hence he could not provide satisfactory responses to questions 3.3 and 3.4. In question 3.4 he struggled to make different types of interpretations concerning the answers that he provided to question 3.1. That is to say he struggled to distinguish between the properties of the function concept such as the Y-ordinate and the slope of a linear relationship. Also, Teacher B struggled to move from question 3.3 to question 3.4. He managed to model the mathematical situation but struggled to perform transition between a graphical and algebraic representation of the function concept (see Figure 5:6).



3. A recent catalog price for tennis balls was R40.25 for a can with three balls. T
shipping charge per order was R10.00.

3.1 Write an equation that you can use to project the costs for ordering differe
numbers of cans.

$f(x) = 40{,}25x + 10$

3.2 Draw a graph showing this relationship. (Use the graph paper provided.)

3.3 How does raising the shipping charge by R1.50 affect the graph?

The slope of the graph will also increase.

3.4 What equation models the cost situation in 7.3?

Straight line graph.

Figure 5:6 Teacher B's response to question 3 of the pre-assessment

Teacher A did not complete question 3. For this question, her response was *"I know nothing"*. In session 6 of the CLP Teacher B made the following remark concerning modelling real-life situations with functions in particular using technology:

Research A	Anything new about functions…next time we will attend to…yes
Teacher B	No I was just saying this modelling thing is a new thing for me.
Researcher B	The…
Teacher B	The modelling of…
Researcher B	Of…physical
Researcher B	Yes
Research A	Of physical problems, okay
Teacher B	Mm
Teacher A	Is new to me

In question 2 of Task 1 which comprised three questions the function concept was presented in words requiring teachers to determine the equation as well as the graph of the given mathematical situation. The mathematical situation was presented in the context of a parabola or a semi-circle. They showed no appreciation of the situation and they did not complete the question (see Table 5:9). Although they did not complete question 2, they did talk about the mathematical situation and in the process they provided incorrect answers accompanied by incorrect reasoning. I note that Teacher A was the one who suggested that they should leave the mathematical situation because she was *"lost"*. In the end both Teacher A and Teacher B were not able to complete the question hence they decided to continue with the next question. Looking at the transcripts I note that both Teacher A and Teacher B were able to make different interpretations concerning this question, but their interpretations were based on a linear relationship equation instead of a quadratic or semi- circle equation. Herein, Teacher A and Teacher B were not able to perform to model a real- life situation presented in words to a graphical related to a quadratic equation or the equation of a semi-circle. The following is a discussion that transpired between Teacher A and Teacher B while they were working on question 2:

Teacher B	A boat travels under a bridge that is 8m wide and 4m high
Teacher A and Teacher B	8m wide and 4m height…sketch the graph of the bridges' arch showing…this one…sketch the graph of the bridge………….
Teacher B	So how do we do this one?
Teacher B	8m wide
Teacher A	8…8m wide…let's take that 8m wide
Teacher B	And 4m high
Teacher A	High
Teacher B	So do we need to take out the variables…
Teacher A	Mm
Teacher B	So which one is independent?
Teacher B	Can we just say height 4m Y-axis
Teacher A	Mm and the the the one it will be X-axis 8m…err it supposed to be like tha
Teacher B	But now what the question says is sketch
Teacher A	The graph of the bridges' arch's
Teacher B	…so I will say is 4 meters
Teacher A	Mm 4m
Teacher B	So is one…two…three…four
Teacher A	Two

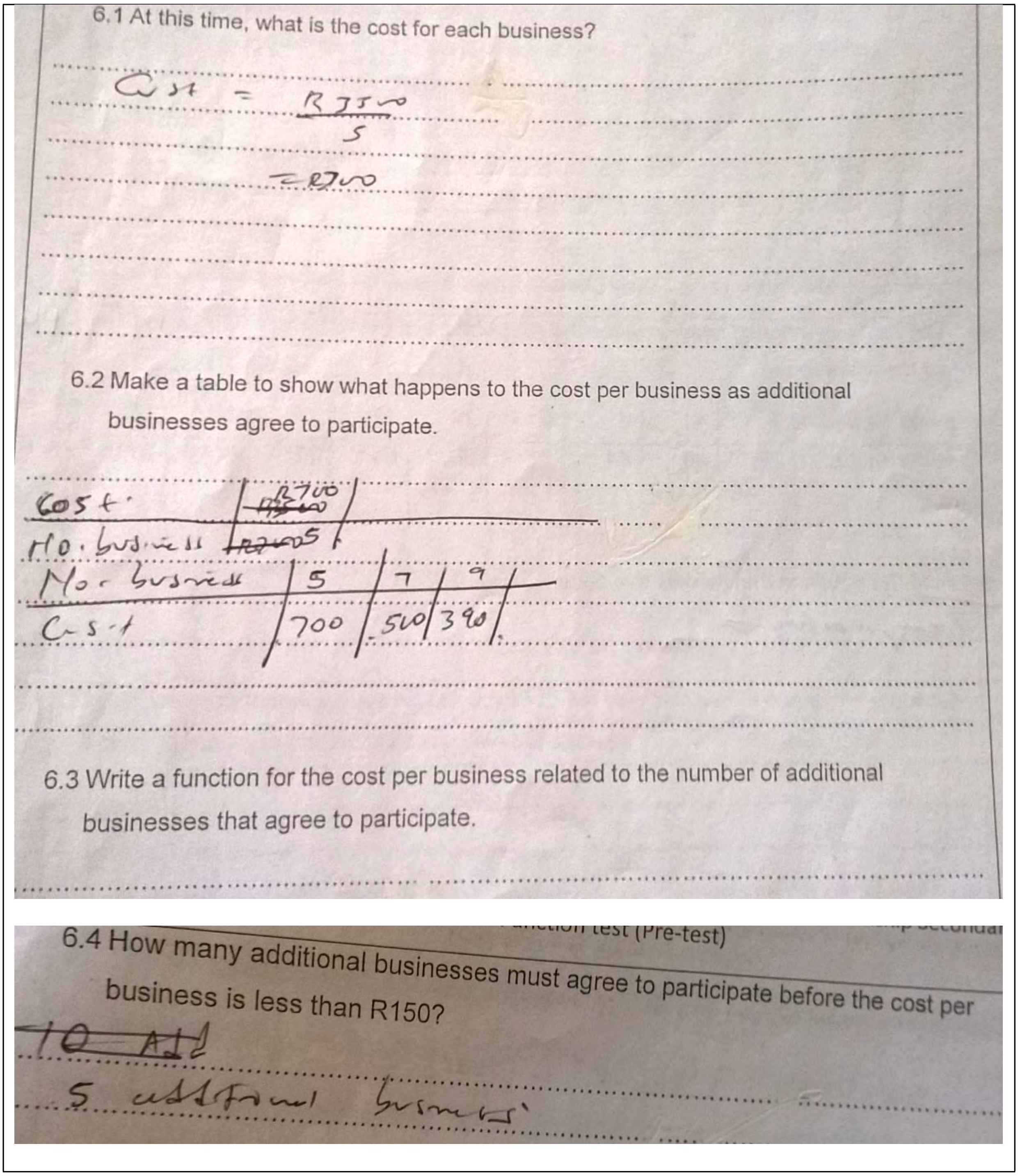
Teacher B	1…2…3…4…5…6…7…8…and then but how how… let's leave this one for a while will come back
Teacher A	When you look…..
Teacher B	So then…*silence+…so then it means something like this
Teacher A	Mm
Teacher B	Am I right
Teacher B	*silence+…there is no variable that completes…
Teacher A	There is no variable
Teacher B	That…….okay let's see, let's get to two point two
Teacher A	Two point two
Teacher B	Determine the equation that models this situation…silence
Teacher A	If I can see
Teacher B	8x plus
Teacher A	Plus four, the height of the graph is…which means y is 4
Teacher B	……….. Teacher B so then if we increase the wideness……..is also……..the height will increase
Teacher B	O iste
Teacher A	Y = 8x
Teacher B	Y= 8x
Teacher A	But it can't be like that it can't be like that
Teacher B	But but let's check
Teacher A	Because this one is not a straight line
Teacher B	Laughs
Teacher A	We are having a curve and an arch
Teacher B	Yes
Teacher A	Sho…haaa I'm lost *takes a deep breath+
Teacher B	The wideness is 2 times the height… the wideness is 2 times the height…okay mam is fine
Teacher A	Haaa…let's let's leave it
Teacher B	Let's go to electricity
Teacher A	Electricity
Teacher B	Mm…the current
Teacher A	So what are you going to do with this one? We must leave it. Let's... let's leave it
Teacher B	For now
Teacher A	For now, ya

In session 7, Teacher B noted that they encountered challenges when they completed question 2. Herein teachers noted that they could not distinguish between the independent and dependent variable. In session 7 of the CLP we reflected on this question and the following discussion transpired concerning this matter:

Researcher B	They said there there were problems…Teacher A what happened there at number two? What happened?...
Teacher B	Number 2 neh, okay a boat travels under a bridge that is 8m wide and 4m high. Then I think here our first problem we were supposed to sketch a graph
Teacher A	Mm
Teacher B	But we think that the variables are the same
Researcher B	Meaning…which variables
Teacher B	Err…height is the distance
Researcher B	Height is a
Teacher B	Is a distance
Researcher B	It's a distance
Teacher B	Wideness is also a distance
Researcher B	Aaha…
Teacher B	So is which we have one variable, so then it's difficult for us…if.. if we take that way it would be difficult for us to plot a graph with one variable

5.3.2 Interpreting

In the pre-assessment task there were two questions (5 and 6) that presented the function concept through the component of interpreting of the function model. These questions required teachers to interpret a real-life mathematical situation in the context of indirect proportion and linear relationship. In these questions, teachers were required to interpret mathematical situations presented in a table and words respectively. Teacher A did not attempt to complete both questions and Teacher B did not attend to question 5 but did manage to complete question 6. In question 6 of the pre-assessment task which comprised four questions from which the function concept was presented in words, Teacher B attempted the three questions and provided only two correct answers to question 6.1 and 6.4. In question 6.1 he managed to provide a correct answer, from which he relied on a simple procedure to arrive at the correct solution. In question 6.2 he was asked to translate the mathematical situation from words to a visual representation (in a table) (see Figure 5:7). In question (6.3) he showed no appreciation of the situation because he did not provide any response, from which he was asked to determine the equation of the mathematical situation. In question 6.4 he provided an incorrect answer accompanied with incomplete reasoning to a question that required him to determine the dependent variable when the independent variable was known or given (see Table 5:7). Concerning questions 6.3 and 6.4, I note that Teacher B was not able to interpret or move flexibly between a visual (table form) and algebraic representation of the function concept.



6.1 At this time, what is the cost for each business?

Cost = R3500 / 5

= R700

6.2 Make a table to show what happens to the cost per business as additional businesses agree to participate.

Cost	R700
No. business	5

No. business	5	7	9
Cost	700	500	390

6.3 Write a function for the cost per business related to the number of additional businesses that agree to participate.

6.4 How many additional businesses must agree to participate before the cost per business is less than R150?

~~10 Add~~

5 additional business

Figure 5:7 Teacher B's response to question 6 of the pre-assessment

Contrary to Teacher B's performance in the pre-assessment task concerning question 6, I note that in session 5 of the CLP they (Teacher A and Teacher B) showed the need to complete this question because they provided correct answers with correct reasoning to all the questions.

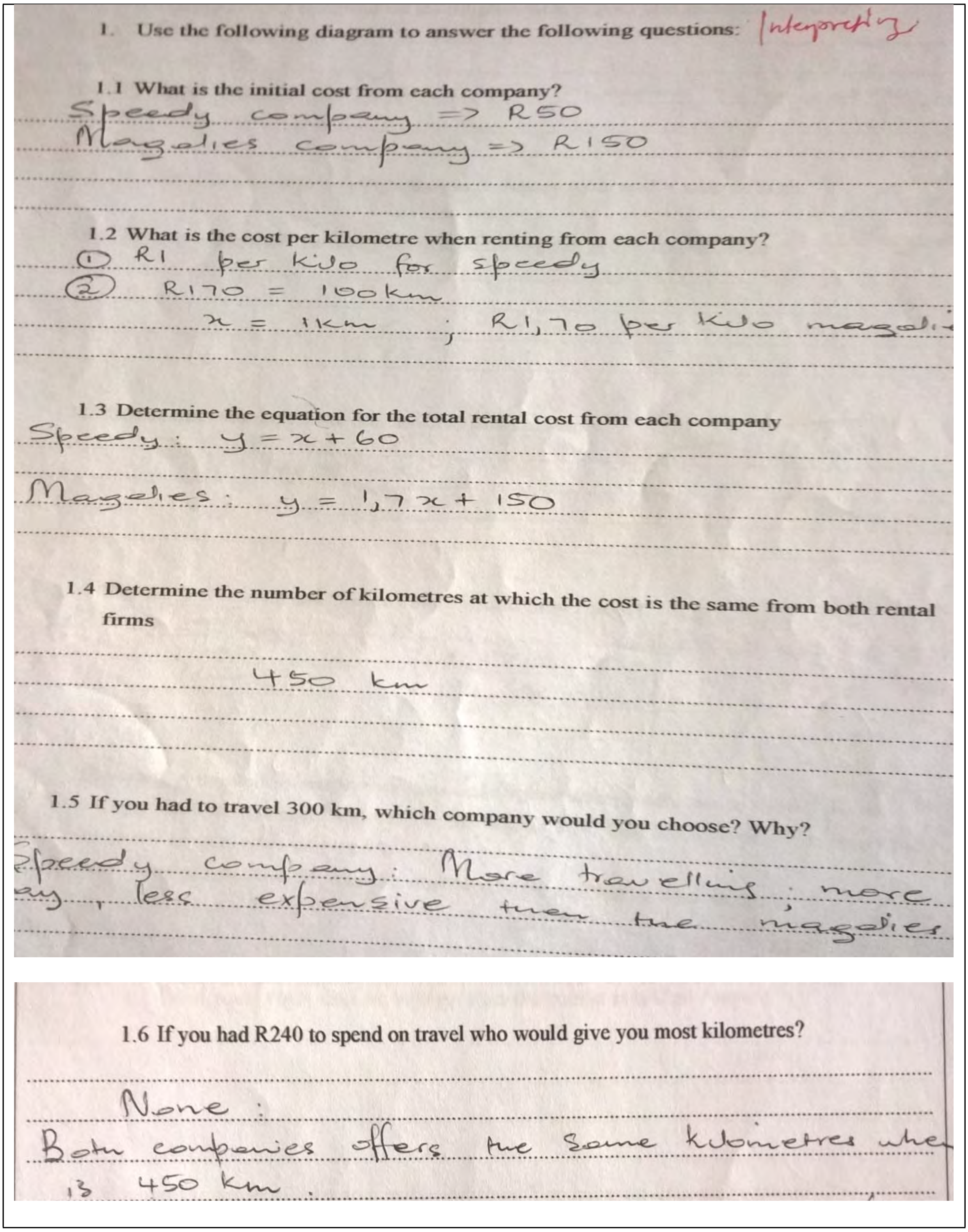
They were talking to each other, they helped each other, and they asked each other questions and in the process provided explanations to their solutions. The following is a discussion that transpired between Teacher A and Teacher B, when they working on this question:

Teacher B	So its a hyperbola…let's start a fresh *reshebe*…
Teacher A	mm
Teacher B	This is a table
Teacher A	Mm
Teacher B	…so is going to be lets say y equals to…so but the fixed amount…it will be three thousand five hundred
Teacher A	Mm
Teacher B	Divided by the number of businesses *akere*…but this time it will be is five *akere*…if we increase the function of…the number of additional businesses…to participate then…but it might not be is not only five…it will be X plus maybe plus P
Teacher A	Plus P
Teacher B	…yes
Teacher A	Mm
Teacher B	And then 6.3…6.4…how many additional businesses must be…
Teacher A	So it will be the equation of that part…
Teacher B	Yes …it will be the equation…but but because it will be or or even x…y equals three thousand five hundred divide by if we know the number of ….divided by x
Teacher A	Mm
Teacher B	So if x…then how many additional businesses must agree to participate before the…
Teacher A	…145
Teacher B	Is 145
Teacher A	Mm
Teacher B	…lets analyse the cost…is less than…150…so then 145 *re e fumane jang*
Teacher A	Go on
Teacher A	Continue continue until you get…until you get that one
Teacher B	Eya…maar we must just check our variables which is which…how many additional businesses must agree…therefore the businesses will be equals to 3500 divided by…
Teacher A	150
Teacher B	150 *akere*…and then 3500 divided by 150 so it gives us…23 point 3 *akere* so
Teacher B	So that is where we talk about extrapolation is it…ya
Research A	Mm
Teacher A	…by extrapolation…due to the fact that one cannot have 23 point 3 businesses so which is…24
Teacher B	Mm

During the CLP, that is in session 6, Teacher B was asked to reflect on the CLP until that moment. He had the following to say and it was connected to interpreting a real-life situation concerning the function concept in the context of indirect proportion. Herein, Teacher B admits that he did not realize that there were problems that involved indirect proportion in the pre-assessment function task. The following is a discussion that transpired concerning this matter:

Teacher B	In fact when when we wrote pre-test I did not see the hyperbola until on the I saw it on the 12 of August there was hyperbola
Researcher B	Mm
Teacher B	But I did not see it
Researcher B	On the 12 of August
Teacher B	*Eya* on the 12 of August…there is hyperbola
Researcher B	Okay
Teacher A	Even, myself
Researcher B	Mm
Teacher A	I did notice that it is there
Researcher B	Mm
Teacher B	And then by the way basically…I may say myself specifically err I was just I was not interpreting the the functions the way like we did it
Researcher B	Mm
Teacher B	So they were not saying anything to me…
Researcher B	They were not saying anything
Teacher B	But today I can see, if is a graph I know is talking something
Researcher B	Okay
Teacher B	If I see a graph or an equation then I know this graph speaks I can put words
Researcher B	Mm
Teacher B	Yes
Researcher B	In terms of interpretation
Teacher B	Ya…you may something so that's what I have learned so far

From the above discussion, Teacher B maintains that he learned something from the CLP in relation to making interpretations on different aspects of the function concept. The fact that he learned something regarding to making interpretations on different aspects of the function concept was evident in Task 1 as well as the post assessment function task. In Task 1, from which teachers were encouraged to work together there were two questions (1 and 6) related to interpreting functions. Teachers managed to complete both question 1 and 6, and they provided correct answers to all the sub-questions (see Figure 5:9 and Figure 5:10). In question 1 of Task 1 which comprised six questions, the function was presented by a graphical representation. This question was situated in the context of a linear relationship and teachers managed to complete the problem. Herein, teachers were able to make different interpretations concerning graphical situation that was given. In question 6 of Task 1 teachers showed appreciation of the situation because they provided correct answers to all the questions. The question was situated in the context of indirect proportion. Herein, teachers were supposed to interpret a graphical situation and determine the equation for the situation (see Table 5:9). Teachers managed to complete this question whilst providing correct responses.



1. Use the following diagram to answer the following questions: Interpreting

1.1 What is the initial cost from each company?

Speedy company => R50

Magalies company => R150

1.2 What is the cost per kilometre when renting from each company?

① R1 per kilo for speedy

② R170 = 100km

x = 1km ; R1,70 per kilo magali-

1.3 Determine the equation for the total rental cost from each company

Speedy : y = x + 60

Magalies : y = 1,7x + 150

1.4 Determine the number of kilometres at which the cost is the same from both rental firms

450 km

1.5 If you had to travel 300 km, which company would you choose? Why?

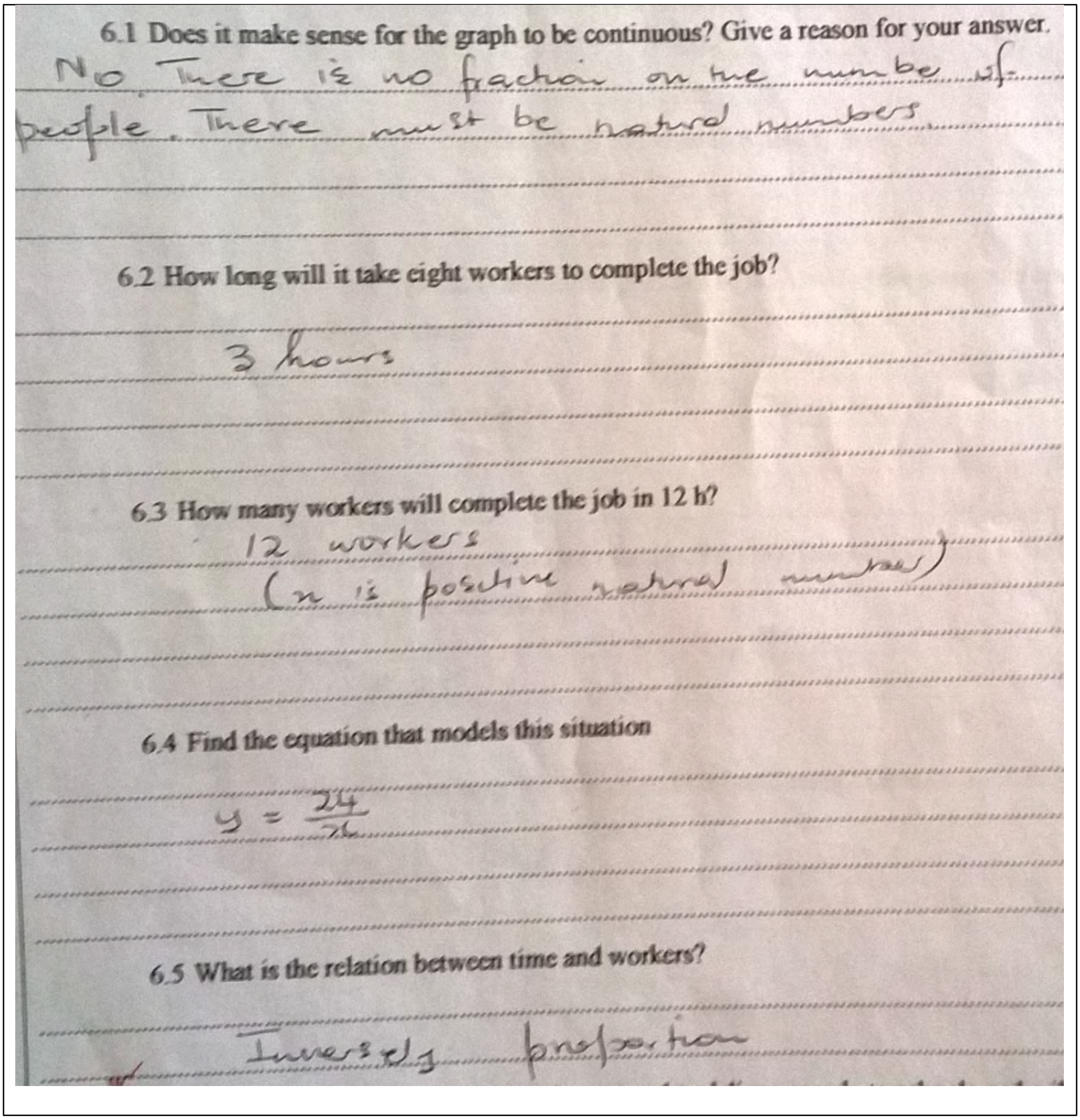
Speedy company : More travelling ; more -ay , less expensive than the magalies

1.6 If you had R240 to spend on travel who would give you most kilometres?

None :

Both companies offers the same kilometres whe- is 450 km .

Figure 5:8 Teachers' A and B response to question 1 of Task 1


6.1 Does it make sense for the graph to be continuous? Give a reason for your answer.

No. There is no fraction on the numbe of people. There must be natural numbers

6.2 How long will it take eight workers to complete the job?

3 hours

6.3 How many workers will complete the job in 12 h?

12 workers
(x is positive natural number)

6.4 Find the equation that models this situation

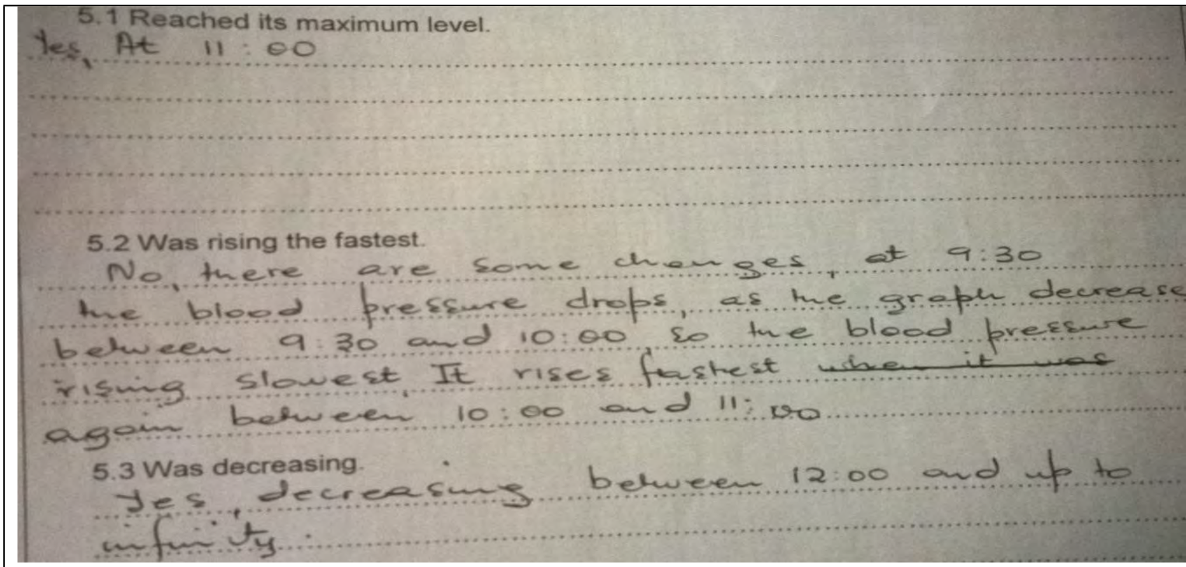
$y = \frac{24}{x}$

6.5 What is the relation between time and workers?

Inversely proportion

Figure 5:9 Teachers' A and B response to question 6 of Task 1

In question 5 of the post-assessment task, teachers were required to interpret a mathematical situation based on a graphical representation of the function concept. The question required teachers to interpret a graph in the context of a non-linear relationship (see Table 5:11 and Table 5:12). That is to work with decreasing and increasing intervals of a function. Both teachers attempted to interpret the graphical representation and they provided correct answers, but Teacher A struggled to note that the domain of the graph was restricted, this was evident in her response concerning question 5.3 (see Figure 5:10). Teacher B provided satisfactory responses to all three questions (see Figure 5:11).

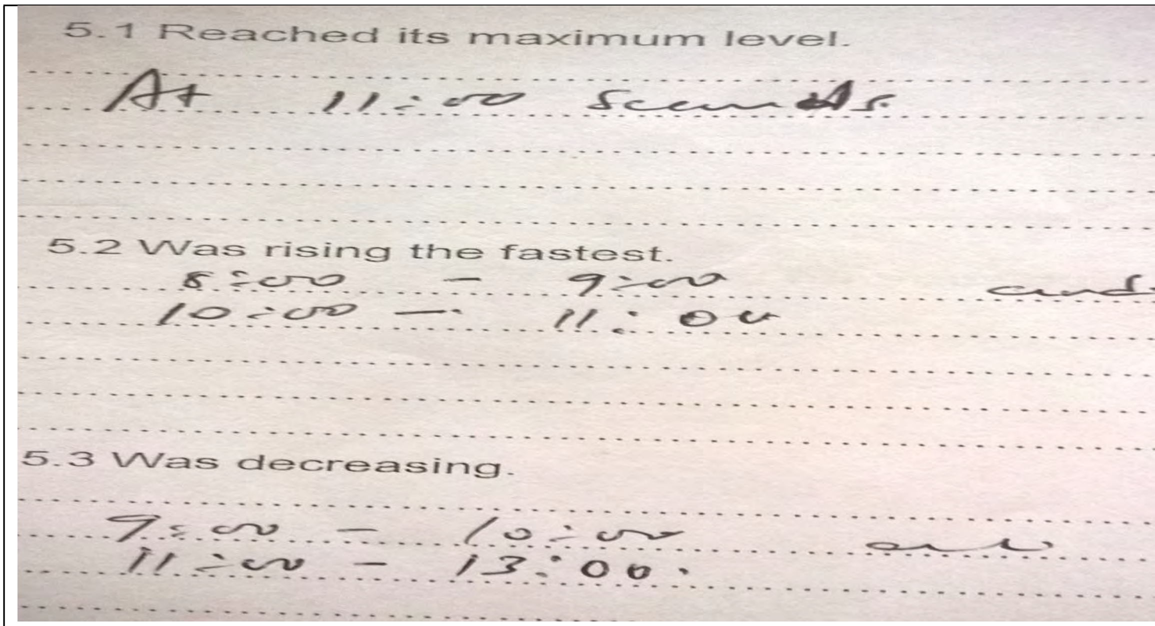


5.1 Reached its maximum level.
Yes, At 11:00

5.2 Was rising the fastest.
No, there are some changes, at 9:30 the blood pressure drops, as the graph decrease between 9:30 and 10:00, so the blood pressure rising slowest. It rises fastest ~~when it was~~ again between 10:00 and 11:00

5.3 Was decreasing.
Yes, decreasing between 12:00 and up to infinity.

Figure 5:10 Teacher A's response to question 5 of the post-assessment



5.1 Reached its maximum level.
At 11:00 seconds.

5.2 Was rising the fastest.
8:00 – 9:00 and
10:00 – 11:00

5.3 Was decreasing.
9:00 – 10:00 and
11:00 – 13:00.

Figure 5:11 Teacher B's response to question 5 of the post-assessment

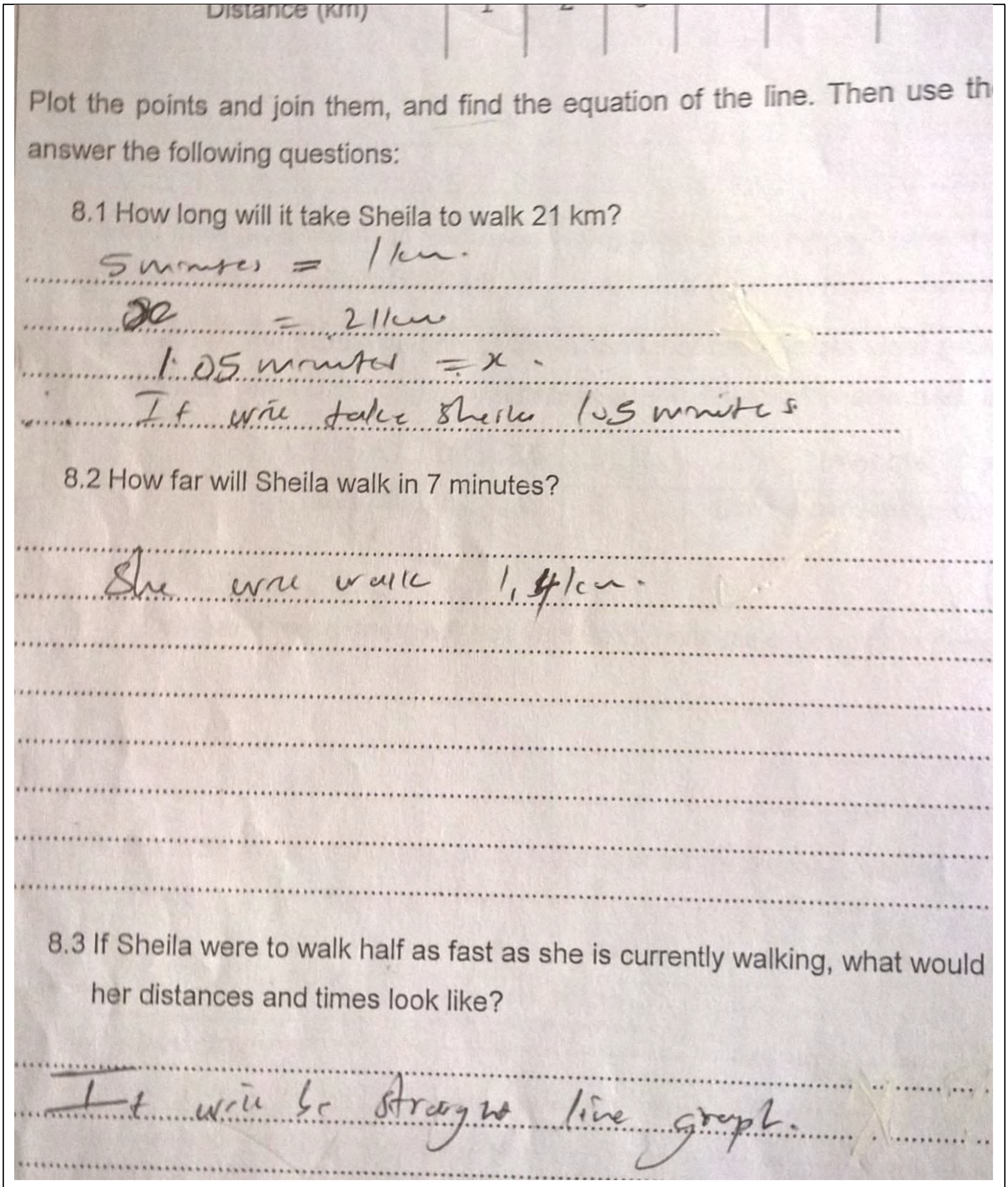
Pertaining to the preceding paragraphs, I note that Teacher B's proficiency in making different types of interpretations concerning the function concept has changed from not being able to make interpretations to being able to make different types of interpretations. This was evident in his performance in Task 1 and the post-assessment, as well as during the CLP.

Whilst Teacher A's proficiency in making different types of interpretations concerning the function concept did not change, she is still struggling to make different types of interpretations. But it is worth noting that Teacher A showed the need to interpret graphical representations of the function concept because during the CLP she was actively involved. During the CLP, when we were reflecting to such mathematical tasks Teacher A showed the appreciation to work together with Teacher B and they completed such questions and provided satisfactory responses. For instance in the post assessment she did attempt to complete both question 4 and 5 which were related to making different interpretation concerning the graphical representation of the function.

5.3.3 Translating

In the pre-assessment task there were two questions that is question 7 and 8 based on translating real-life situations with functions. Teacher B attempted both question 7 and 8 (see Table 5:8). In question 8 teachers worked with a mathematical situation from which the function concept was presented by means of a table and teachers were required to translate a mathematical situation from a table to a graphical representation. Teacher B completed this question and in doing so he provided correct answers but not to all the questions. For instance I note that he was able to plot the graphical representation but did not provide the equation for the situation. Nevertheless he provided the correct answer to question 8.1 and in question 8.2. In question 8.3 Teacher B was supposed to interpret the graphical representation of the situation by taking into account the concept of the slope. Herein, Teacher B provided a response that was not satisfactory or relevant (see Figure 5:12). Teacher A attempted question 8 but could not provide satisfactory responses to question 8.2 and 8.3 respectively. Teacher A managed to provide the correct answer to question 8.1 hence I note that Teacher A really struggled to move from one representation to the other, in this case from a table to a formula as well as the graph (see Figure 5:13). It is important to note that both Teacher A and Teacher B's graphical solutions to question 8 are not included in this discussion.

In question 7 from which there was only one question Teacher B provided the correct answer. Herein, he was able to translate a situation presented in a table to a graphical representation (see Figure 5:14). Teacher A did not complete question 7.



Distance (km)

Plot the points and join them, and find the equation of the line. Then use th
answer the following questions:

8.1 How long will it take Sheila to walk 21 km?

5 minutes = 1km.

x = 21km

1.05 minutes = x.

It will take Sheila 1,05 minutes

8.2 How far will Sheila walk in 7 minutes?

She will walk 1,4km.

8.3 If Sheila were to walk half as fast as she is currently walking, what would her distances and times look like?

It will be straight line graph.

Figure 5:12 Teacher B's response to question 8 of the pre-assessment

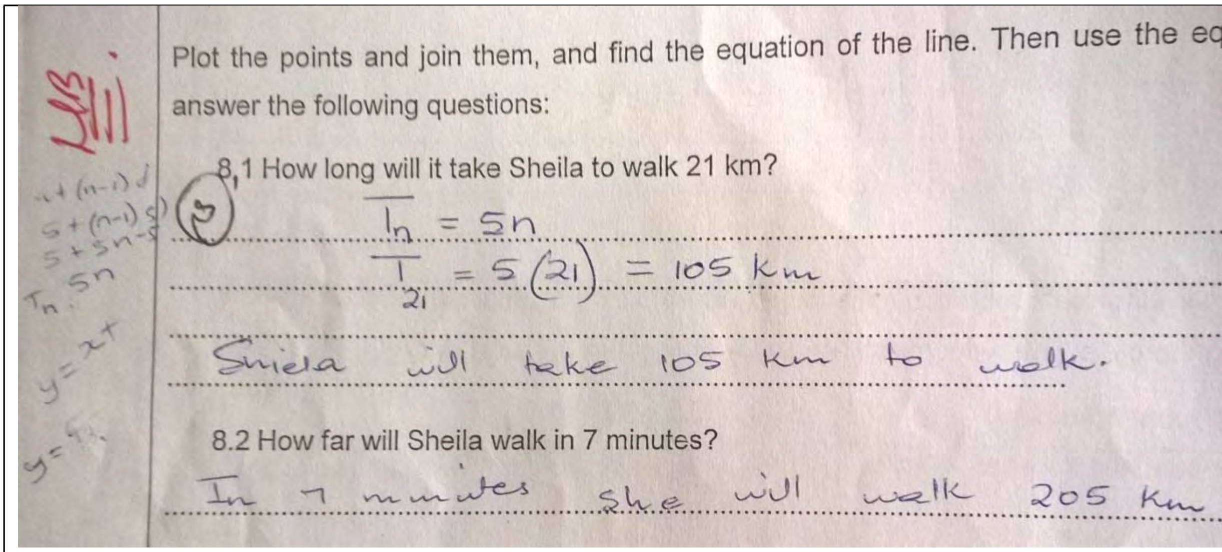


Figure 5:13 Teacher B’s response to question 8 of the pre-assessment

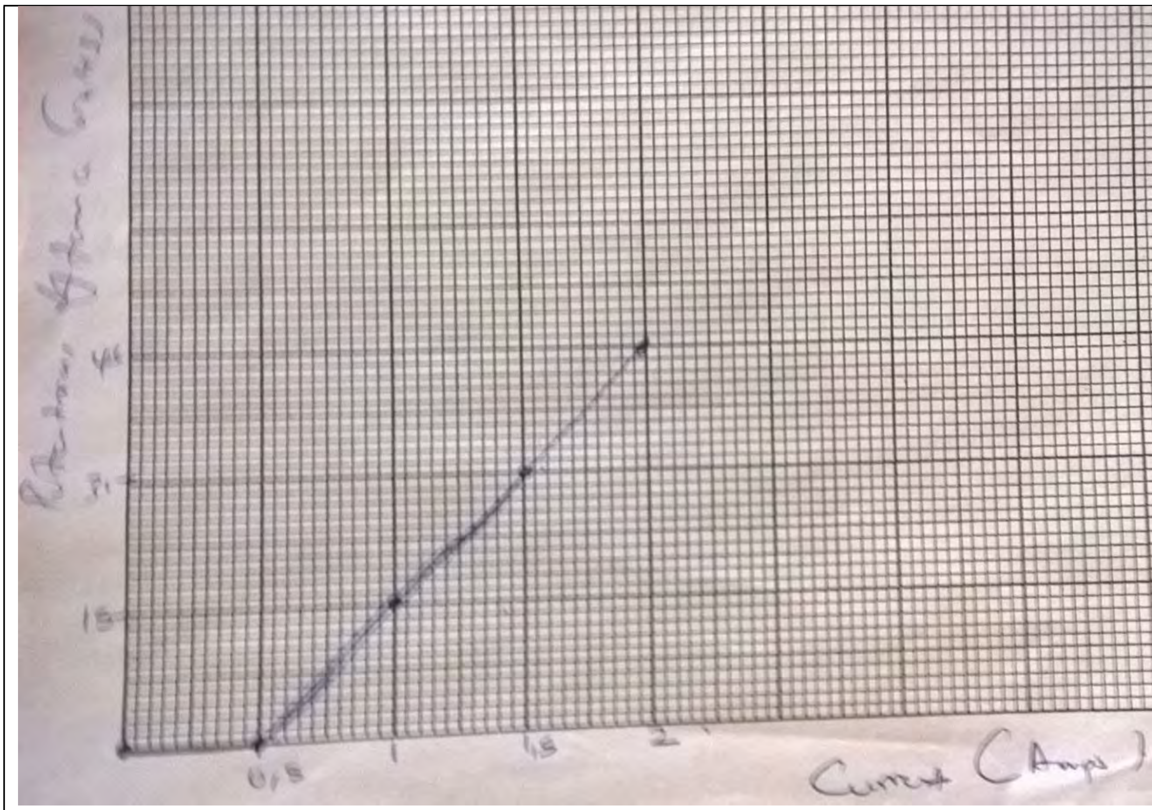
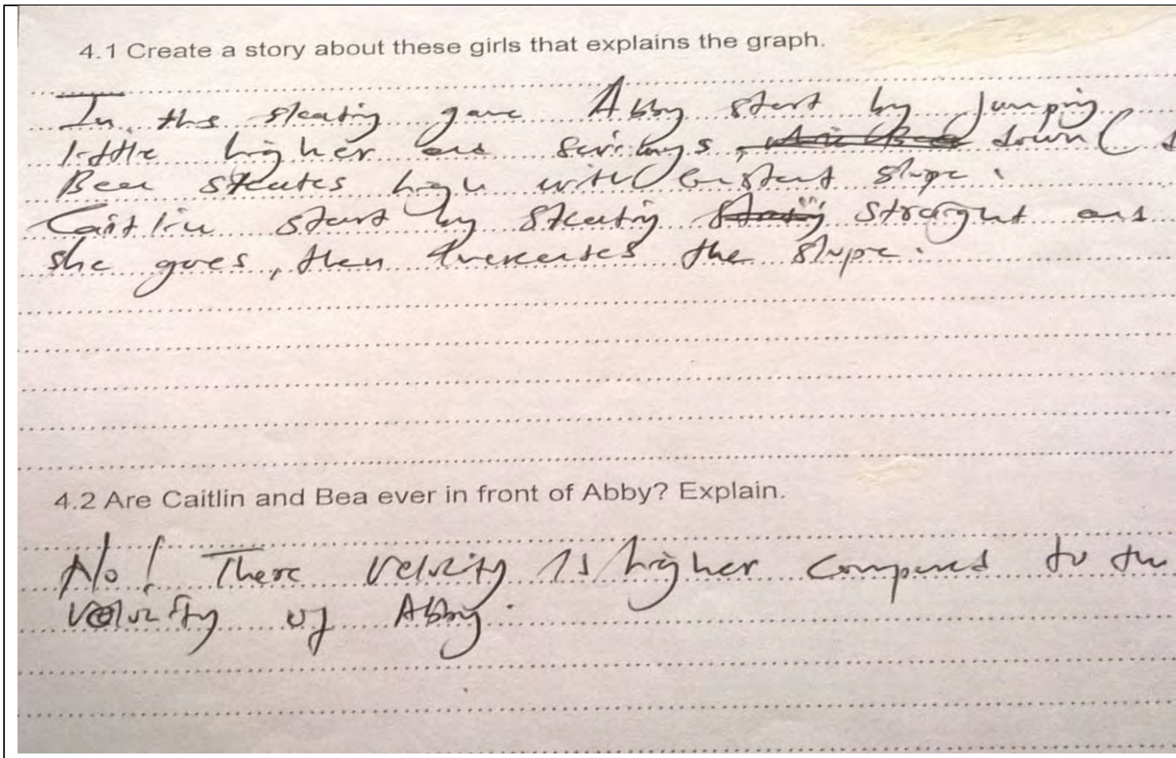


Figure 5:14 Teacher B’s response to 7 question of the pre-assessment

In the post-assessment there was only one question focusing on translating real-life situations with functions that is question 4. Both Teacher A and Teacher attempted question 4 of the post-assessment. The question required teachers to create a story about the graph in the context of both linear and non-linear relations (see Table 5:11 and Table 5:12). That is to say teachers were expected to translate from one representational system to another one (in case from a graph to words). Herein, the context of the question was based on velocity-time graph and it consisted of three graphs showing the velocities of three girls over a given time interval (relationship between velocity and time). Both Teacher A and Teacher B attempted to create the story about the graphical representation of a real-life situation. But it is important to note that the stories they created did not have enough mathematical content, in particular the story of Teacher A (see Figure 5:16). Teacher A's response to question 4.1 fails to take into account to properties of functions. For instance she does not distinguish between dependent and independent variables or mention anything about the shapes of the graphs. I also note that concerning question 4.2, Teacher A was able to interpret the situation but her reasoning was not satisfactory. That is to say teachers provided incorrect answers with incorrect reasoning. Hence Teacher A was not able to move flexibly from one representation of a function to another. Teacher B provided satisfactory responses but not in detail (see Figure 5:15). Herein Teacher B's response to question 4.1 takes into account things such as the context of the situation that is to say his responses take into account the relationship between velocity and time but not into detail. Also Teacher B's response to question 4.1 does not take into account the properties of functions. For instance Teacher B does not distinguish between dependent and independent variables or mention anything about the shapes of the graphs as well as the domain or range of the functions. Also I note that concerning question 4.2 Teacher B was able to interpret the situation and his reasoning was satisfactory.



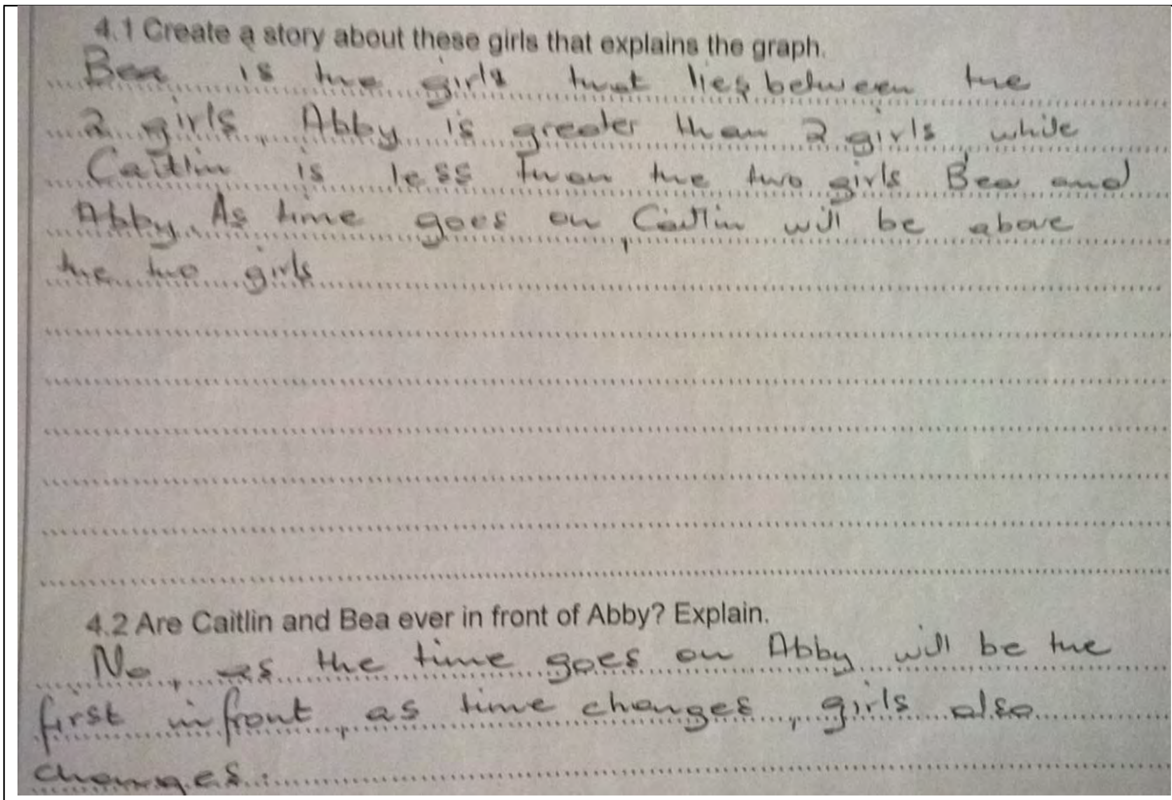
4.1 Create a story about these girls that explains the graph.

In the skating game Abby start by jumping little higher and swings, down (s Bea skates high with biggest slope. Caitlin start by skating straight as she goes, then decreases the slope.

4.2 Are Caitlin and Bea ever in front of Abby? Explain.

No! There velocity is higher compared to the velocity of Abby.

Figure 5:15 Teacher B's response to question 4 of the post assessment



4.1 Create a story about these girls that explains the graph.

Bea is the girls that lies between the 2 girls. Abby is greater than 2 girls while Caitlin is less than the two girls Bea and Abby. As time goes on, Caitlin will be above the two girls

4.2 Are Caitlin and Bea ever in front of Abby? Explain.

No, as the time goes on Abby will be the first in front as time changes, girls also changes.

Figure 5:16 Teacher A's response to question 4 of the post assessment

In Task 1 there were two questions (3 and 4) based on translating a real-life situation with functions. These questions encouraged teachers to translate between a mathematical situation presented in words to a graphical or algebraic representation (see Table 5:9). When I observe Task 1, I note that teachers did not complete both question 3 and 4 but when I look at the transcripts from session 7 of the CLP I note that they tried to complete both questions. That is to say teachers provided incorrect answers with incorrect reasoning. Hence *they were not able to move flexibly from one representation of a function to another (in this case from words to graphs or equations).* Herein, I note that initially teachers managed to interpret the situation but showed no appreciation to translate their thinking into written communication and then they decided to continue with the next question. Although they did not complete question 3, they did talk about the mathematical situation and in the process they provided incorrect answers accompanied by incorrect reasoning. Looking at the transcripts I note that teachers interpreted this question within the context of indirect proportion which was satisfactory. Also I note that Teacher B was the one who suggested that they should leave question 3. At the end both Teacher A and Teacher B decided to continue with the next question. For instance here is discussion that transpired during session 7 of the CLP between Teacher A and Teacher B while they completing question 3:

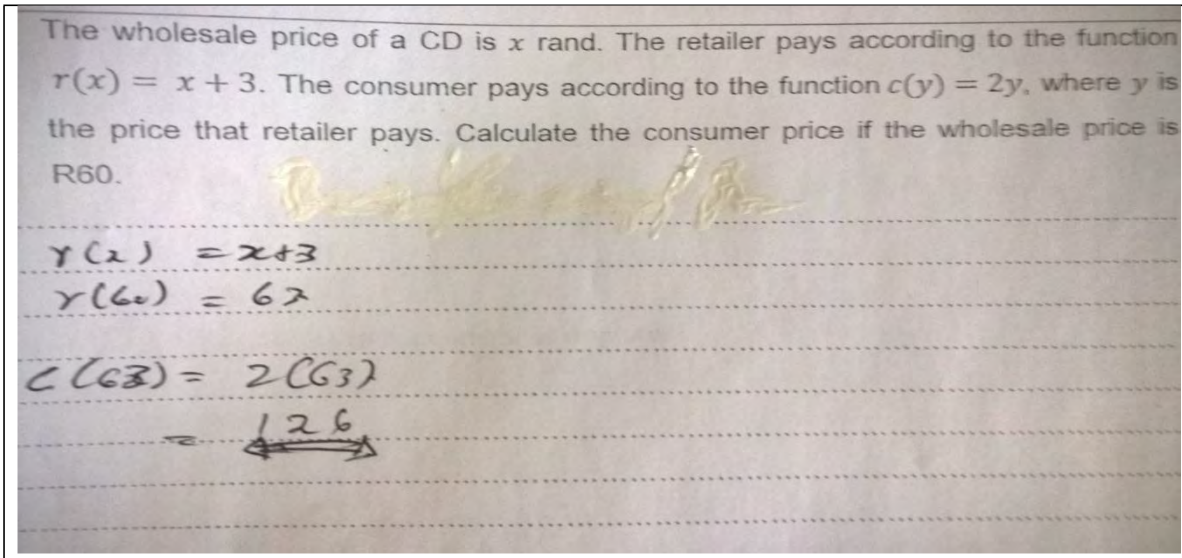
Teacher B	The current needed to work your car's headlights is inversely proportional to the voltage. So that is the current times the voltage is what one hundred four hundred and fluty four. Use a graph paper to show this situation in multiples of 60 to 360. From your graph …so we have…is this one
Teacher A	We must use this one
Teacher B	So then we have the current, we have…what voltage akere….*silence+…ha ke sa hoopla ha ntle but mona okare we have the Volts
Teacher A	Volts
Teacher B	Kapa let me say ha ke…*silence+…that is the current times voltage. The current times the voltage is 144
Teacher B	They are inversely proportional
Teacher A	Mm
Teacher B	What does that mean? If one increase the other one decreases, so wa hoopla ha re etsa the inversely proportional to re fumane graph ya…
Teacher A	Mm
Teacher B	Was it hyperbola
Teacher A	Ene e le hyperbola
Teacher B	Mm

Teacher A	Mm ene e le hyperbola
Teacher B	It was hyperbola neh
Teacher A	Mm
Teacher B	So when the one increases
Teacher A	Increases the other
Teacher B	The other one decreases
Teacher A	Decreases
Teacher B	So then but now ha re bale statement se…so that is the current times the voltage is 144…silence…let me check if R = Volts over I…silence
Teacher A	Over…mm
Teacher B	The current times the voltage is 144…silence… right find the current that would be needed at 110. So, then which comes insight….but would it be inversely proportional…because because what I've read if you increase the Volts
Teacher A	The Volts ya
Teacher B	Then the current
Teacher A	The current it will be it will go down
Teacher B	The current…the current
Teacher A	The current also will decrease
Teacher B	Mm…the current will decrease
Teacher B	So then my question is how are we going to plot that graph…silence…so then current times the voltage is 144**…**silence**…**so ka mona ke current
Teacher A	Mm, ke current and ka mona ke voltage….silence…
Teacher A and Teacher B	[*They are working… plotting the graph*] [*Silence*]
Teacher B	But this statement this statement Teacher A…
Teacher A	Mm
Teacher B	The current times voltage is 144
Teacher A	The current times voltage is 144…
Teacher B	Mm…silence
Teacher A	So which means this times this we get that 144 and use the…
Teacher B	But will it be always
Teacher B	Ha…re tla e bona…laughs…number 4

5.3.4 Reifying

In the pre-assessment task there was only one question based on reifying. Herein both Teacher A and Teacher B did not attempt to complete question 9 of the pre-assessment task see Table 5:7and Table 5:8). Hence there is nothing to say about teachers' understanding of the function concept concerning reifying. Also during the CLP, that is in session 5 both Teacher A and

Teacher B struggled to complete this question on their own; the research team had to intervene. In the post-assessment task Teacher B provided correct answers to question 6 of the post-assessment task from which emphasis was placed on reifying real-life situations with functions (see Table 5:12). Herein, he was required to combine two functions within the context of a linear relationship. The function concept was presented algebraically (see Figure 5:17). On the other hand Teacher A did not complete question 6. Teacher A's response for this question was *"I'm still having a problem."* I note that there is change concerning Teacher B's knowledge of the function concept in relation to the component of reifying because in the pre-assessment he did not complete such a question.

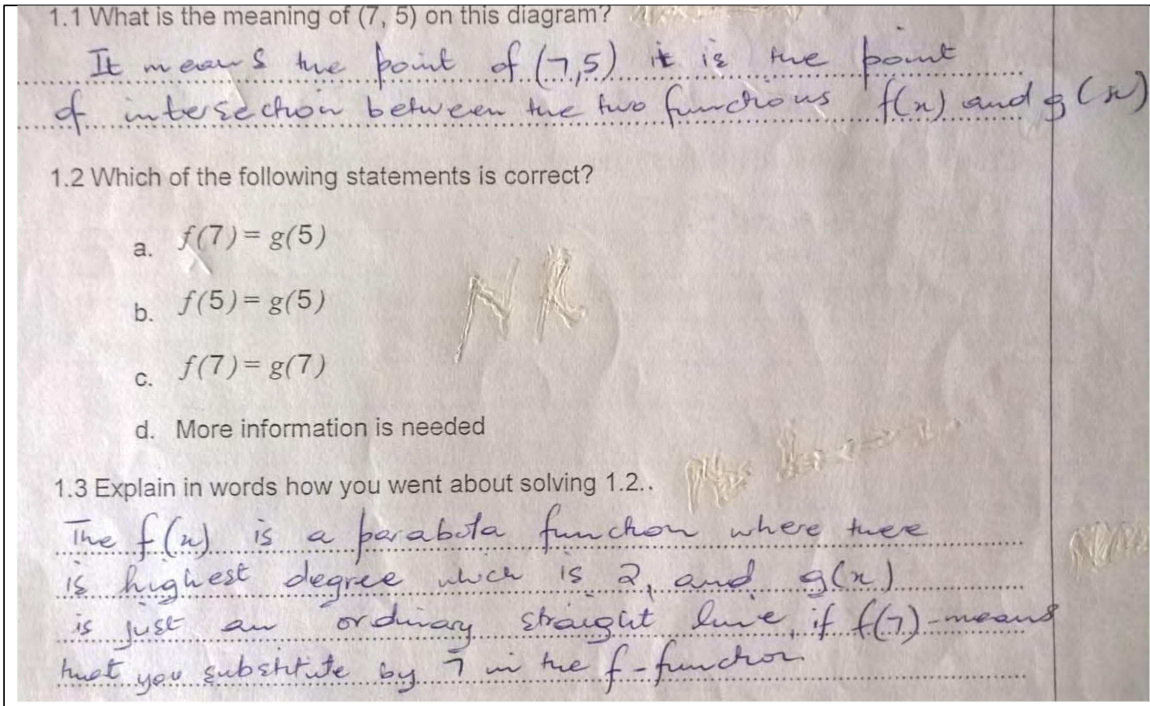

The wholesale price of a CD is x rand. The retailer pays according to the function $r(x) = x + 3$. The consumer pays according to the function $c(y) = 2y$, where y is the price that retailer pays. Calculate the consumer price if the wholesale price is R60.

r(x) = x+3
r(60) = 63
c(63) = 2(63)
= 126

Figure 5:17 Teacher B's response to question 6 of the post-assessment task

5.3.5 Procedural skills

In question 1 of the pre-assessment which comprised four sub-questions the function was presented by a graphical representation. The questions were centred on the interpretation, explanation as well as finding the equation of both the parabola and the straight line. The question was procedural in nature. Teacher A could not make interpretations concerning question 1.2 and 1.3 (see Figure 5:18). In question 1.3 she did not communicate her responses coherently. Thus Teacher A shows no evidence of understanding the function concept with respect to interpreting graphical representation which is procedural in nature (see Table 5:7). On the other hand Teacher B provided responses that were not satisfactory to question 1.1, 1.2 and 1.3 of the pre-assessment function task. Herein, Teacher B could not make different types of interpretations concerning the graphical representation (see Figure 5:19). Thus Teacher B shows no evidence of understanding with respect to interpreting the graphical representation of the function concept which is procedural in nature (see Table 5:8).


1.1 What is the meaning of (7, 5) on this diagram?

It means the point of (7,5) it is the point of intersection between the two functions f(x) and g(x)

1.2 Which of the following statements is correct?

a. $f(7)=g(5)$

b. $f(5)=g(5)$

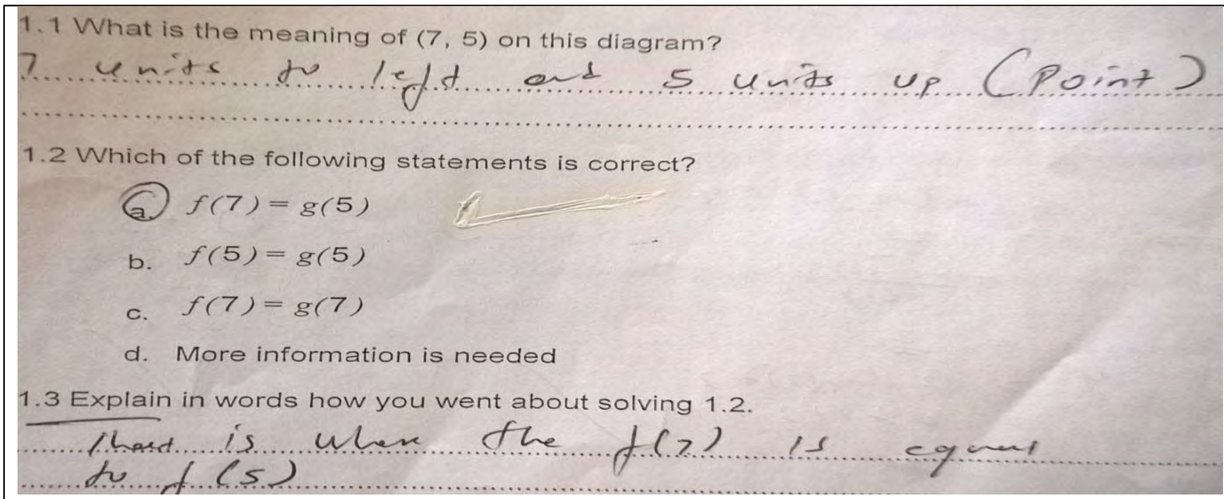
c. $f(7)=g(7)$

d. More information is needed

1.3 Explain in words how you went about solving 1.2..

The f(x) is a parabola function where there is highest degree which is 2, and g(x) is just an ordinary straight line, if f(7) means that you substitute by 7 in the f-function

Figure 5:18 Teacher A's response to question (1.1-1.3) of the pre-assessment


1.1 What is the meaning of (7, 5) on this diagram?

7 units to left and 5 units up (Point)

1.2 Which of the following statements is correct?

(a) $f(7)=g(5)$

b. $f(5)=g(5)$

c. $f(7)=g(7)$

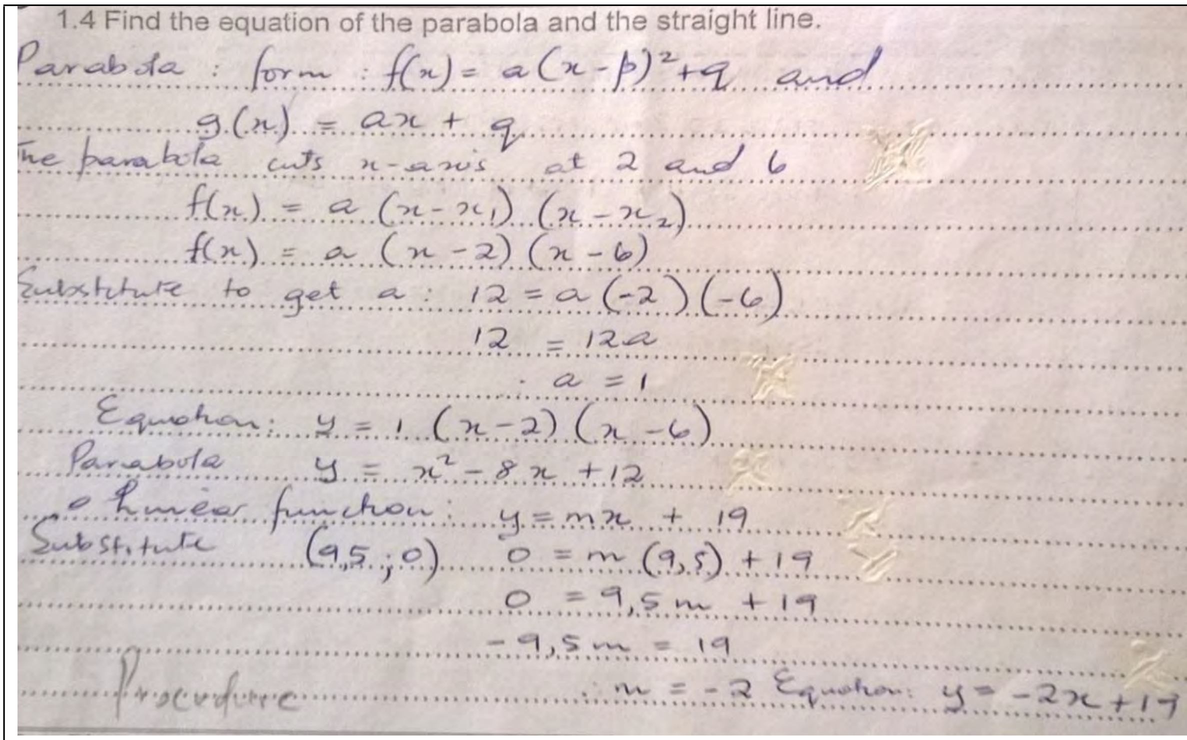
d. More information is needed

1.3 Explain in words how you went about solving 1.2.

That is when the f(7) is equal to f(5)

Figure 5:19 Teacher B's response to question (1.1-1.3) of the pre-assessment

In question 1.4, that encouraged Teacher A to find the equations of the graphs, she provided correct answers and complete reasoning. Note that in her attempt to provide the correct answers, she used a procedural skill but she had to look at the graphs, i.e. interpret the graphical representation of the situation. Herein, there is evidence that Teacher A has complete understanding of the mathematical situation which is procedural in nature. Also it is important to note that she opted to use a procedural skill to find correct answers (see Figure 5:20). In relation to question 1, I note that Teacher B was able to perform operations but struggled to interpret the graphs. In question (1.4) Teacher B showed appreciation of the situation and he relied on a procedural skill in order to provide correct answers. Surprisingly, his response is similar to that of Teacher A (see Figure 5:21). Herein there is evidence that Teacher B has complete understanding of the mathematical situation, which is procedural in nature. He was able to perform operations but struggled to interpret the graphs. I noted the same about Teacher A.


1.4 Find the equation of the parabola and the straight line.

Parabola : form : $f(x) = a(x-p)^2+q$ and

$g(x) = ax + q$

The parabola cuts x-axis at 2 and 6

$f(x) = a(x-x_1)(x-x_2)$

$f(x) = a(x-2)(x-6)$

Substitute to get a : $12 = a(-2)(-6)$

$12 = 12a$

$\therefore a = 1$

Equation: $y = 1(x-2)(x-6)$

Parabola $y = x^2 - 8x + 12$

Linear function: $y = mx + 19$

Substitute $(9,5;0)$ $0 = m(9,5) + 19$

$0 = 9,5m + 19$

$-9,5m = 19$

Procedure $\therefore m = -2$ Equation: $y = -2x + 19$

Figure 5:20 Teacher A’s response to question 1.4 of the post-assessment task

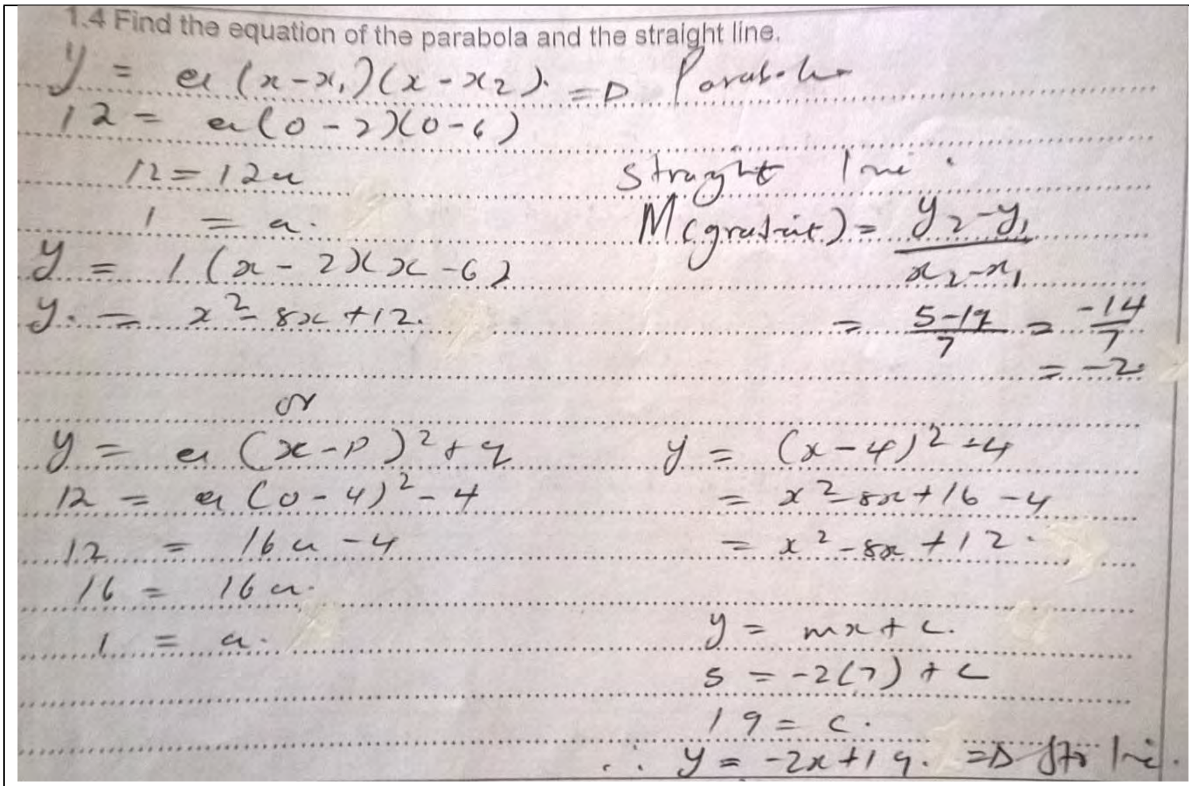
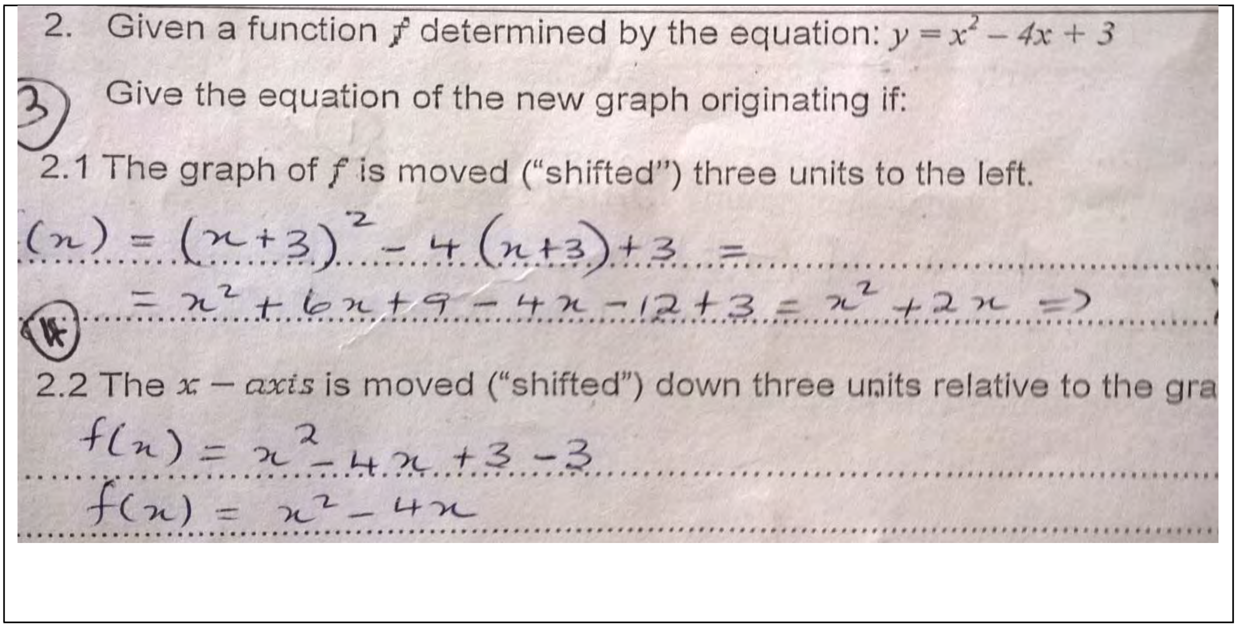


Figure 5:21 Teacher B's response to question 1.4 of the pre-assessment

In question 2 of the pre-assessment which comprised two questions the function was presented by a formula. The question involved transformations (shifting of graphs). Herein, Teacher A showed great appreciation of the question, and through a procedural skill she provided correct answers. She was able to perform operations (see Figure 5:22).



2. Given a function f determined by the equation: $y = x^2 - 4x + 3$

3) Give the equation of the new graph originating if:

2.1 The graph of f is moved ("shifted") three units to the left.

$(x) = (x+3)^2 - 4(x+3) + 3 =$

$= x^2 + 6x + 9 - 4x - 12 + 3 = x^2 + 2x \Rightarrow$

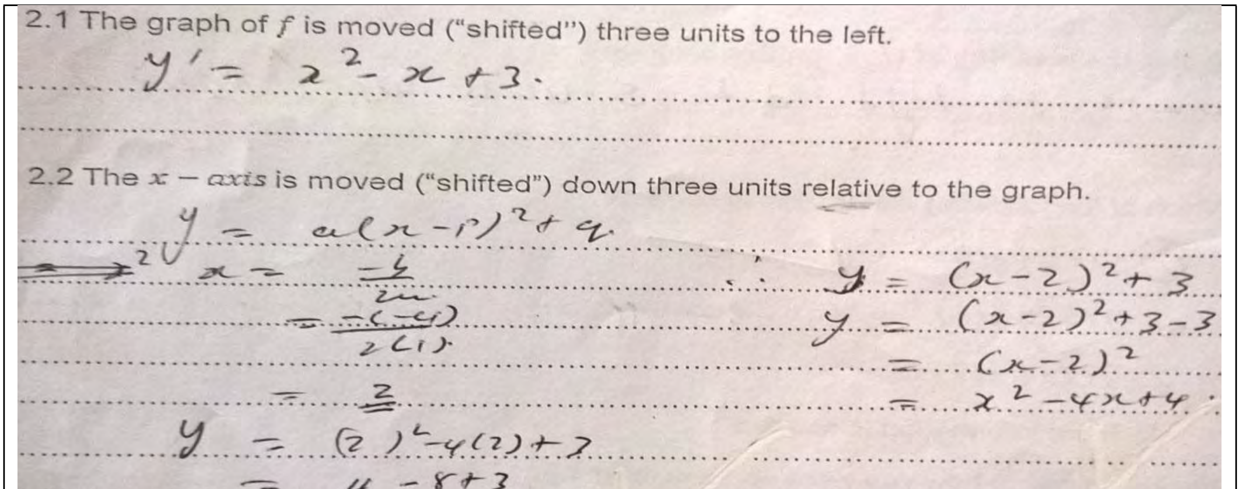
2.2 The $x - axis$ is moved ("shifted") down three units relative to the gra

$f(x) = x^2 - 4x + 3 - 3$

$f(x) = x^2 - 4x$

Figure 5:22 Teacher A's response to question 2 of the pre-assessment

On the one hand Teacher B provided incorrect answers to both question 2.1 and 2.2 of concerning the pre-assessment function task. Though he employed a procedural skill, he struggled to perform operations (see Figure 5:23).



2.1 The graph of f is moved ("shifted") three units to the left.

$y' = x^2 - x + 3.$

2.2 The $x - axis$ is moved ("shifted") down three units relative to the graph.

$y = a(x-p)^2 + q$

$x = \frac{-b}{2a}$

$= \frac{-(-4)}{2(1)}$

$= 2$

$y = (2)^2 - 4(2) + 3$

$= 4 - 8 + 3$

$= -1$

$\therefore y = (x-2)^2 + 3$

$y = (x-2)^2 + 3 - 3$

$= (x-2)^2$

$= x^2 - 4x + 4.$

Figure 5:23 Teacher B's response to question 2 of the pre-assessment

In question 1 of Task 2, (see Table 5:10) which comprised two questions, the function was presented by an algebraic representation. This question was based in the context of two parallel lines. Teachers were encouraged to work with the system of equations involving linear equations. Answers or solutions were requested both algebraically and graphically. I note that both Teacher A and Teacher B provided correct answers to this question (see Figure 5:24).

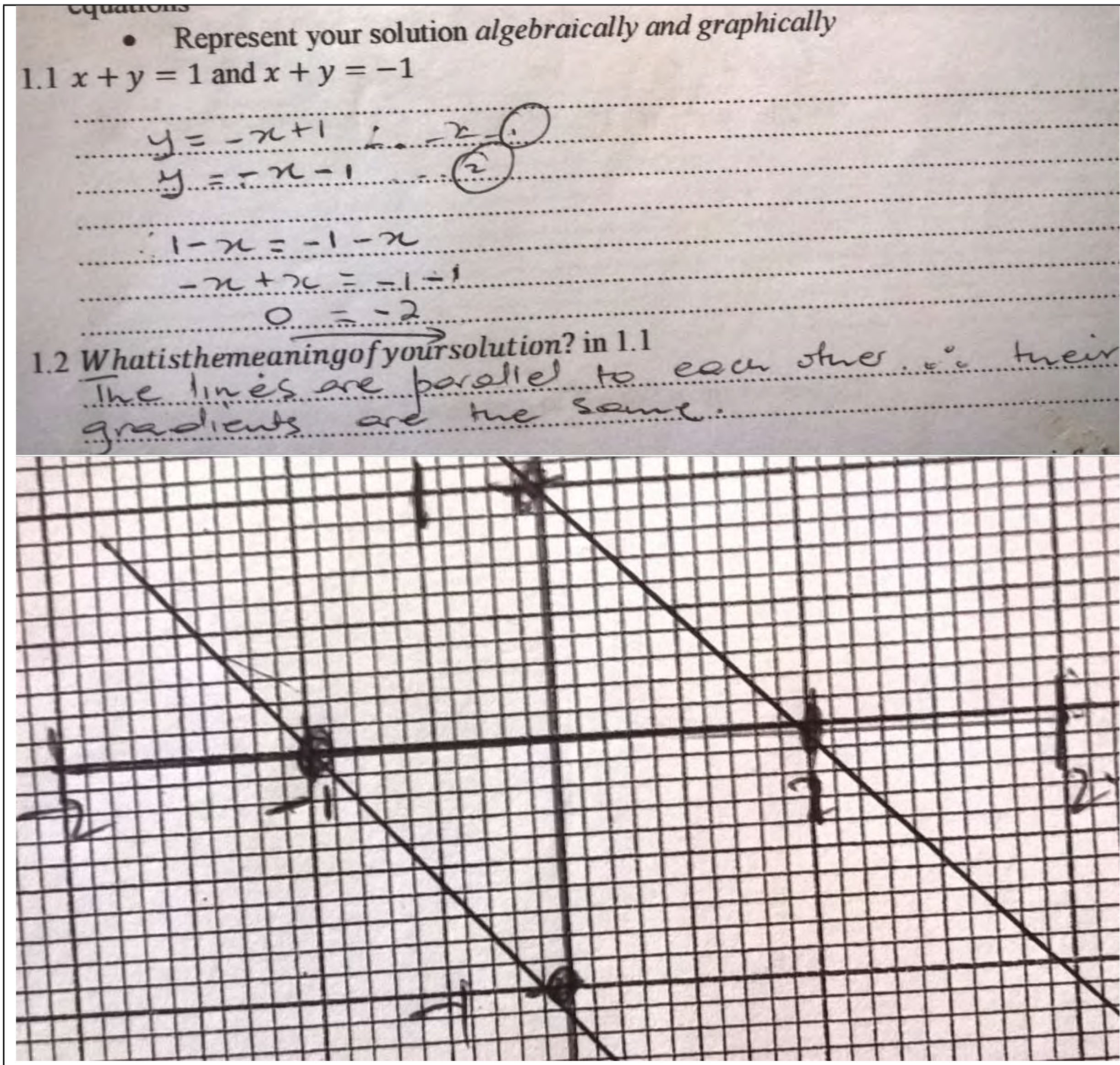


Figure 5:24 Teacher A and Teacher B's response to question 1 of Task 2

In question 2 of Task 2 which comprised two questions, the function was presented by an algebraic representation. This question was based in the context of two intersecting lines. Teachers were encouraged to work with the system of equations involving linear equations. Answers or solutions were requested both algebraically and graphically. I note that both Teacher A and Teacher B provided correct answers to this question (see Figure 5:25).

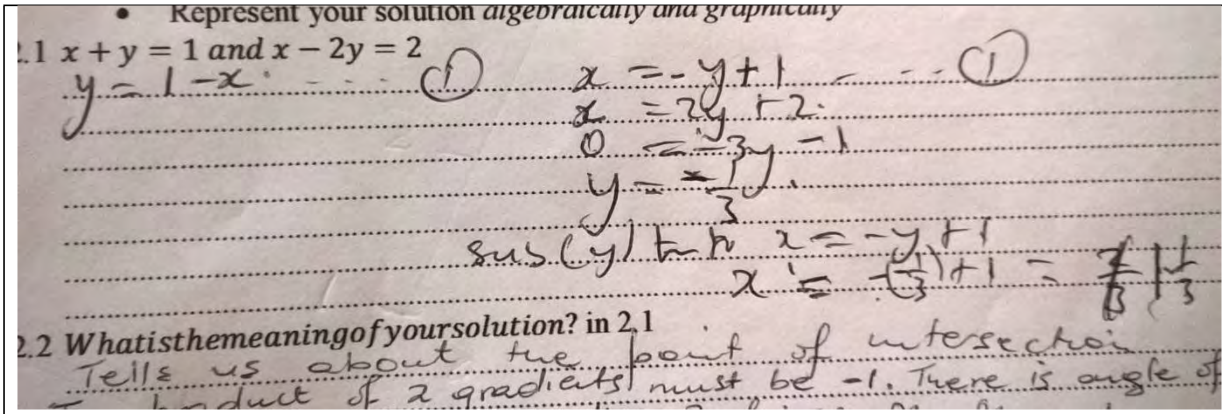
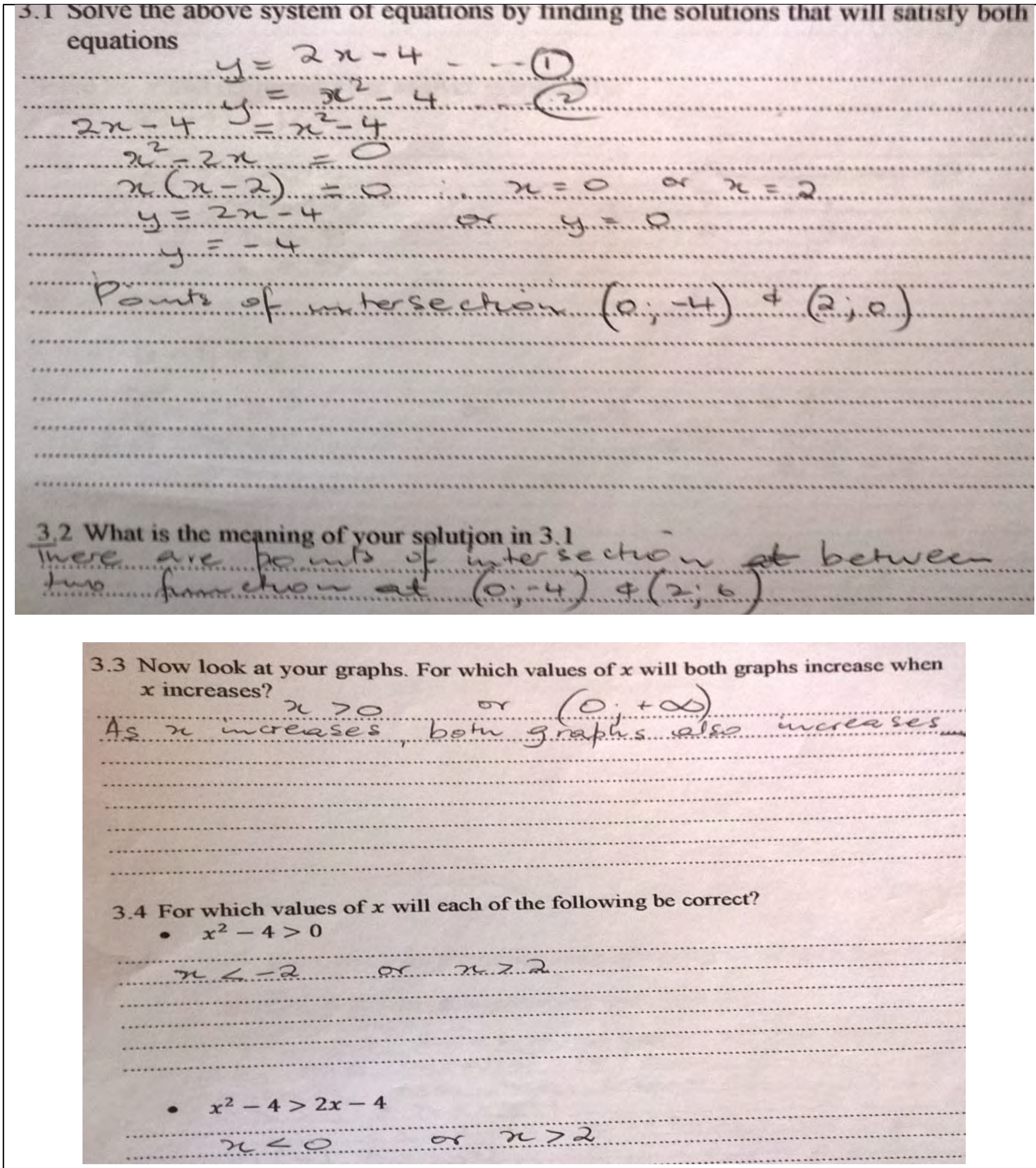


Figure 5:25 Teacher A and Teacher B's response to question 2 of Task 2

In question 3 of Task 2 which comprised four questions, the function was presented by an algebraic representation. Teachers were encouraged to work with the system of equations involving a linear equation as well as the quadratic equation. Answers or solutions were requested both algebraically and graphically. I note that both Teacher A and Teacher B provided correct answers to this question (see Figure 5:26). It is important to note that I did not include graphical solution for this question.


3.1 Solve the above system of equations by finding the solutions that will satisfy both equations

$y = 2x - 4$ ... (1)

$y = x^2 - 4$ ... (2)

$2x - 4 = x^2 - 4$

$x^2 - 2x = 0$

$x(x-2) = 0$ ∴ $x = 0$ or $x = 2$

$y = 2x - 4$ or $y = 0$

$y = -4$

Points of intersection $(0;-4)$ & $(2;0)$

3.2 What is the meaning of your solution in 3.1

There are points of intersection ~~at~~ between two function at $(0;-4)$ & $(2;0)$

3.3 Now look at your graphs. For which values of x will both graphs increase when x increases?

$x > 0$ or $(0;+\infty)$

As x increases, both graphs also increases

3.4 For which values of x will each of the following be correct?

- $x^2 - 4 > 0$

$x < -2$ or $x > 2$

- $x^2 - 4 > 2x - 4$

$x < 0$ or $x > 2$

Figure 5:26 Teachers' A and B response to question 3 of Task 2

In question 4 of Task 2 which comprised two questions, the function was presented by an algebraic representation. Teachers were encouraged to work with an inequality as well as the absolute value. Answers were requested both algebraically and graphically. Teachers were able to provide correct answers with correct reasoning and the communication of solutions was coherent. Teachers were able to perform operations and transformations concerning the function concept (see Figure 5:27).

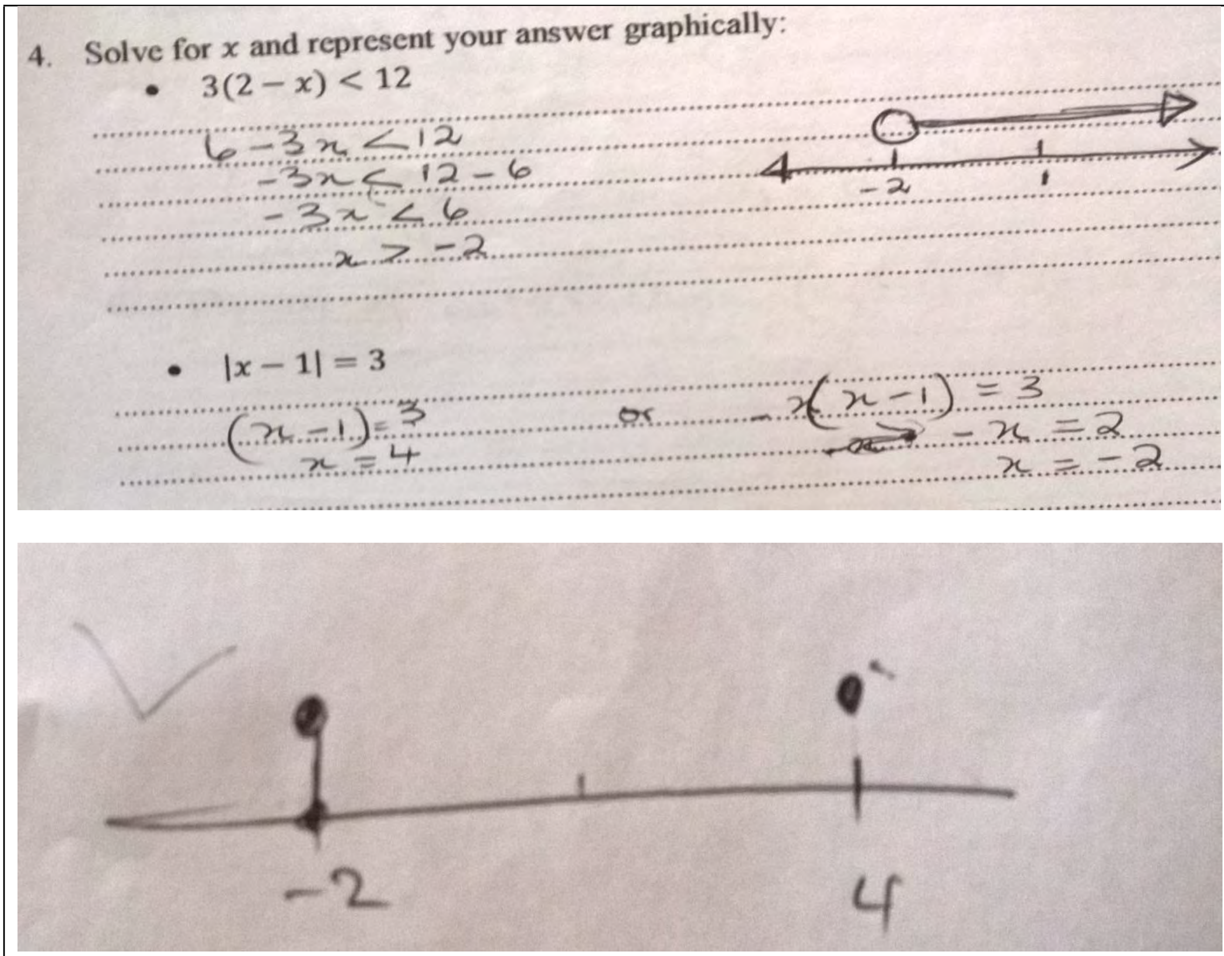
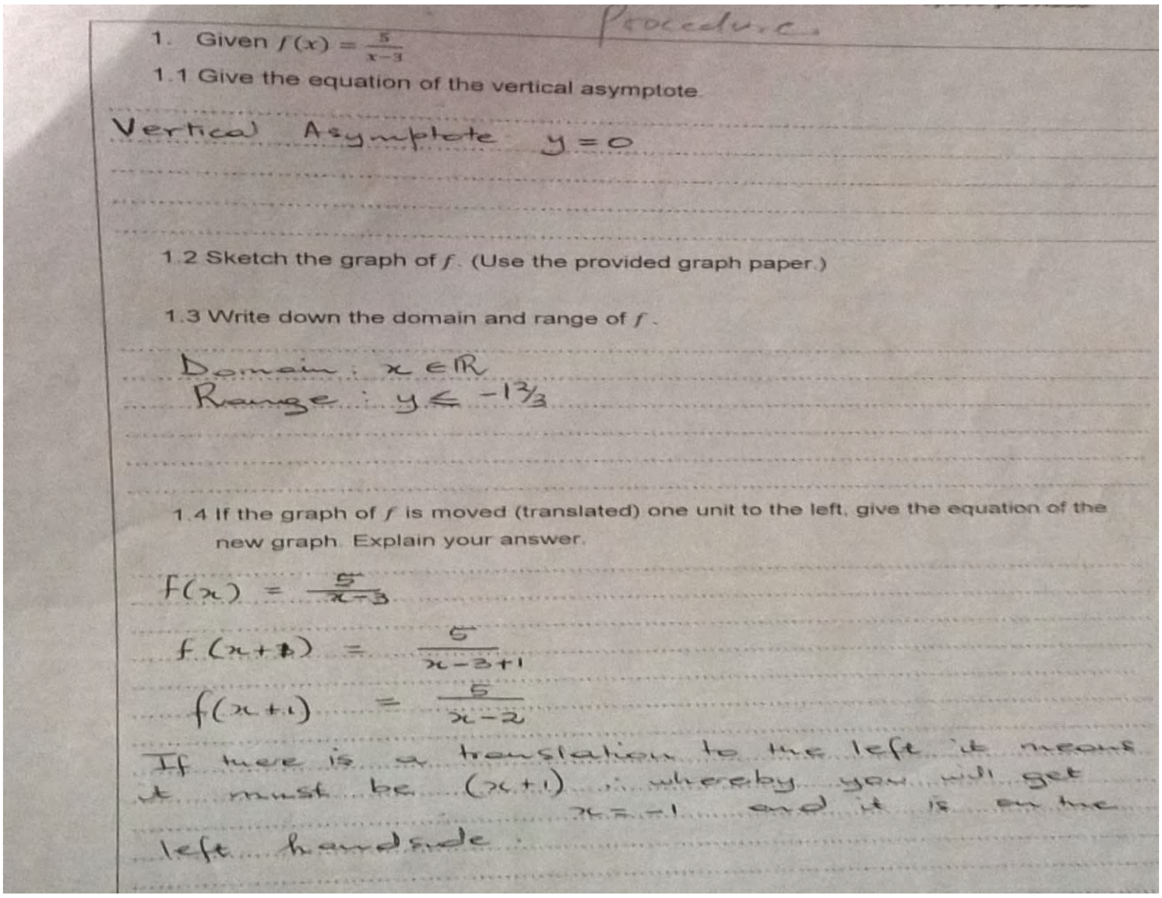


Figure 5:27 Teachers' A and B response to question 4 of Task 2

In question 1 of the post-assessment, which comprised four questions, the function was presented by a formula. The context of this was based on sketching the graph of the given equation. In addition, focus was on the asymptote, domain, range and transformations (shifting) of the equation. Herein Teacher A only managed to graph the situation, but struggled to interpret the situation that is to say she provided incorrect answers to question 1.1, 1.2 and 1.3. But in question 1.4 which involved the horizontal shifting of the function concept, she provided a correct answer (see Figure 5:28).



Procedure

1. Given $f(x) = \frac{5}{x-3}$

1.1 Give the equation of the vertical asymptote.

Vertical Asymptote $y = 0$

1.2 Sketch the graph of f. (Use the provided graph paper.)

1.3 Write down the domain and range of f.

Domain: $x \in \mathbb{R}$

Range: $y \leq -1\frac{2}{3}$

1.4 If the graph of f is moved (translated) one unit to the left, give the equation of the new graph. Explain your answer.

$f(x) = \frac{5}{x-3}$

$f(x+1) = \frac{5}{x-3+1}$

$f(x+1) = \frac{5}{x-2}$

If there is a translation to the left it means it must be $(x+1)$ ∴ whereby you will get $x = -1$ and it is on the left handside.

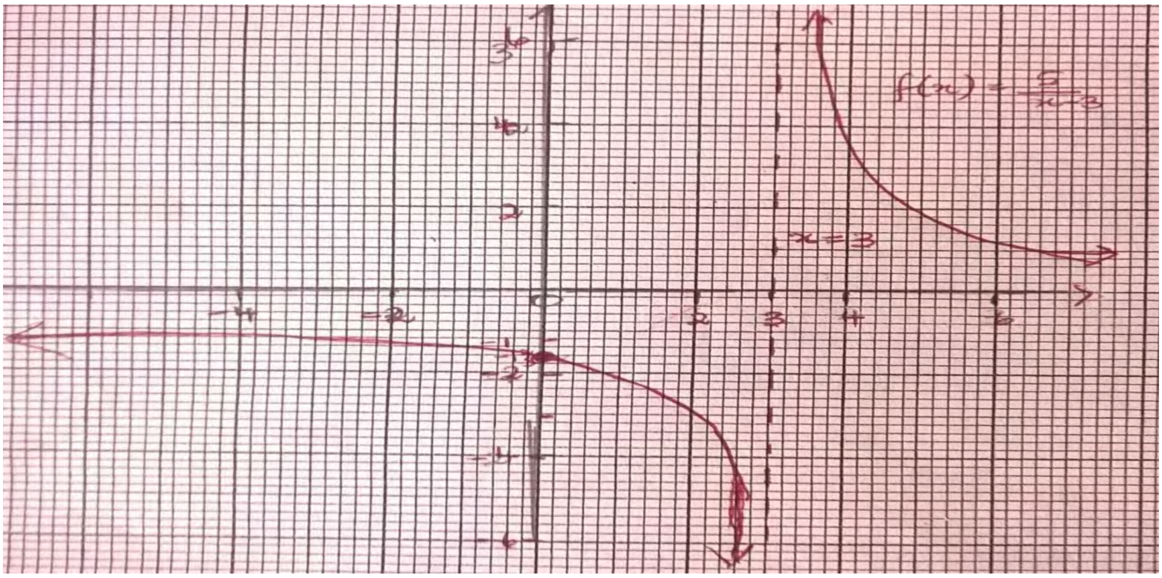
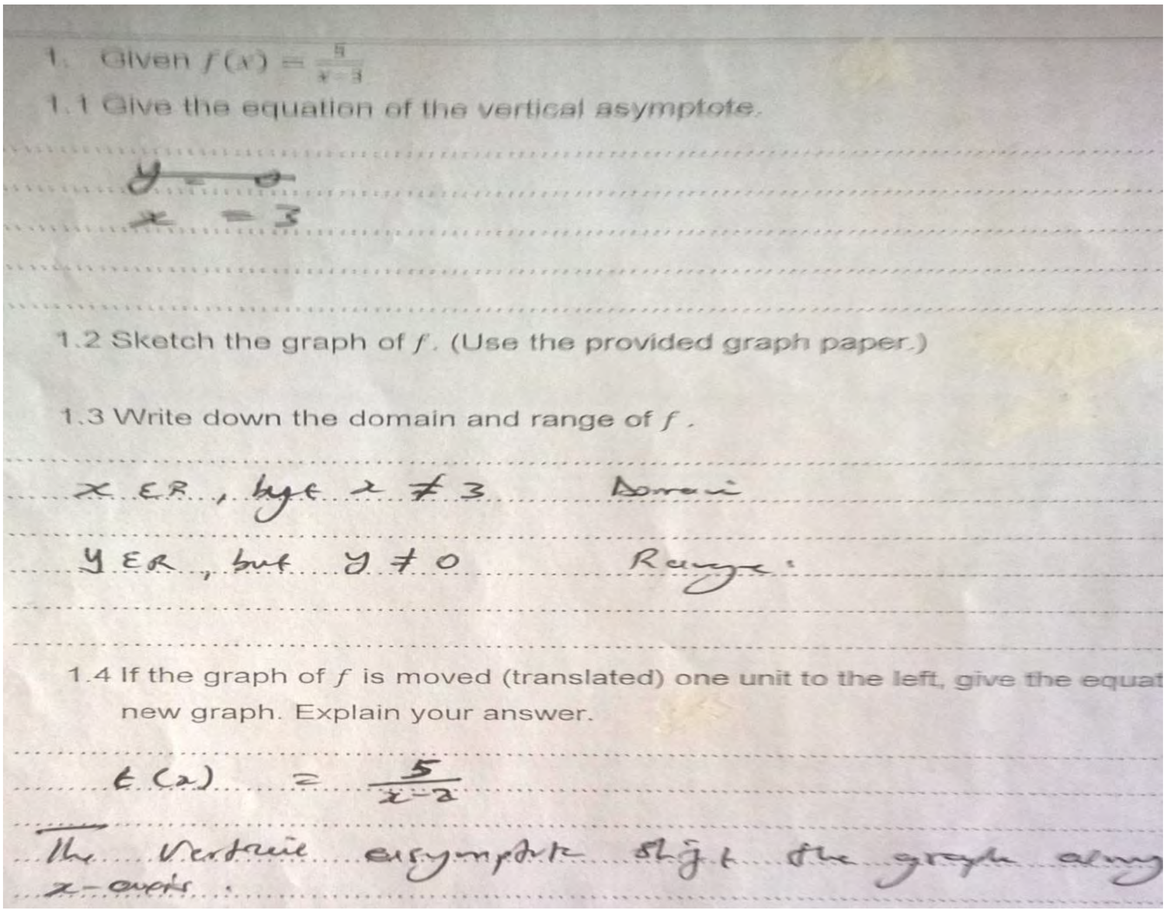


Figure 5:28 Teacher A's response to question 1 of the post-assessment

In sketching the graph, Teacher B showed appreciation of the situation and provided the correct graph to question 1.2 of the post-assessment. He managed to determine the domain, range and the asymptote of the function (see question 1.1, 1.3 and 1.4 in Figure 5.30). Herein, there is evidence of complete understanding with reference to graphing and transformations of the function concept which is procedural in nature. He was able to perform operations, something he struggled with before the CLP. Graphical solution not included.



1. Given $f(x) = \frac{5}{x-3}$

1.1 Give the equation of the vertical asymptote.

~~$y = 0$~~

$x = 3$

1.2 Sketch the graph of f. (Use the provided graph paper.)

1.3 Write down the domain and range of f.

$x \in R$, but $x \neq 3$ Domain

$y \in R$, but $y \neq 0$ Range:

1.4 If the graph of f is moved (translated) one unit to the left, give the equat

new graph. Explain your answer.

$f(x) = \frac{5}{x-2}$

The vertical asymptote shift the graph along x-axis.

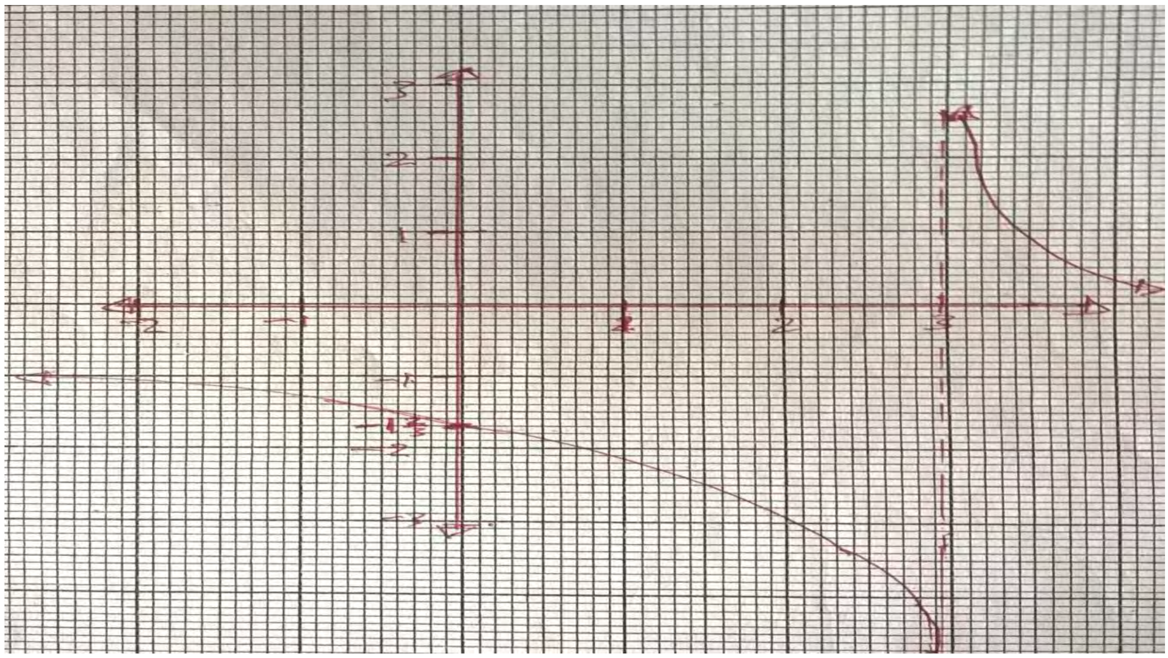


Figure 5:29 Teacher B's response to question 1 of the post-assessment

In question 2 of the post-assessment which comprised four questions, the function was presented by a formula. The question centred on a system of equations (linear and quadratic) finding the point of intersection, sketching of graphs and determining the average gradient between two points, as well as explaining the meaning of the average gradient. Teacher A attempted to complete question 2 but it is worth noting that she only provided satisfactory response to question 2.1where she was able to translate from an algebraic representation to a graphical representation concerning the function concept (see Figure 5:30). In determining the average gradient between two points she also relied on an algorithmic skill, but she made a substitution error in her formula, hence she provided an incorrect answer. Consequently, she could not provide a satisfactory response to question 2.3 (see Figure 5:30). Herein, there is evidence of complete understanding with reference to graphing and solving of system of equations of the function concept which is procedural in nature. She was able to perform operations, but struggled to interpret the information; this was also evident in the pre-assessment task.

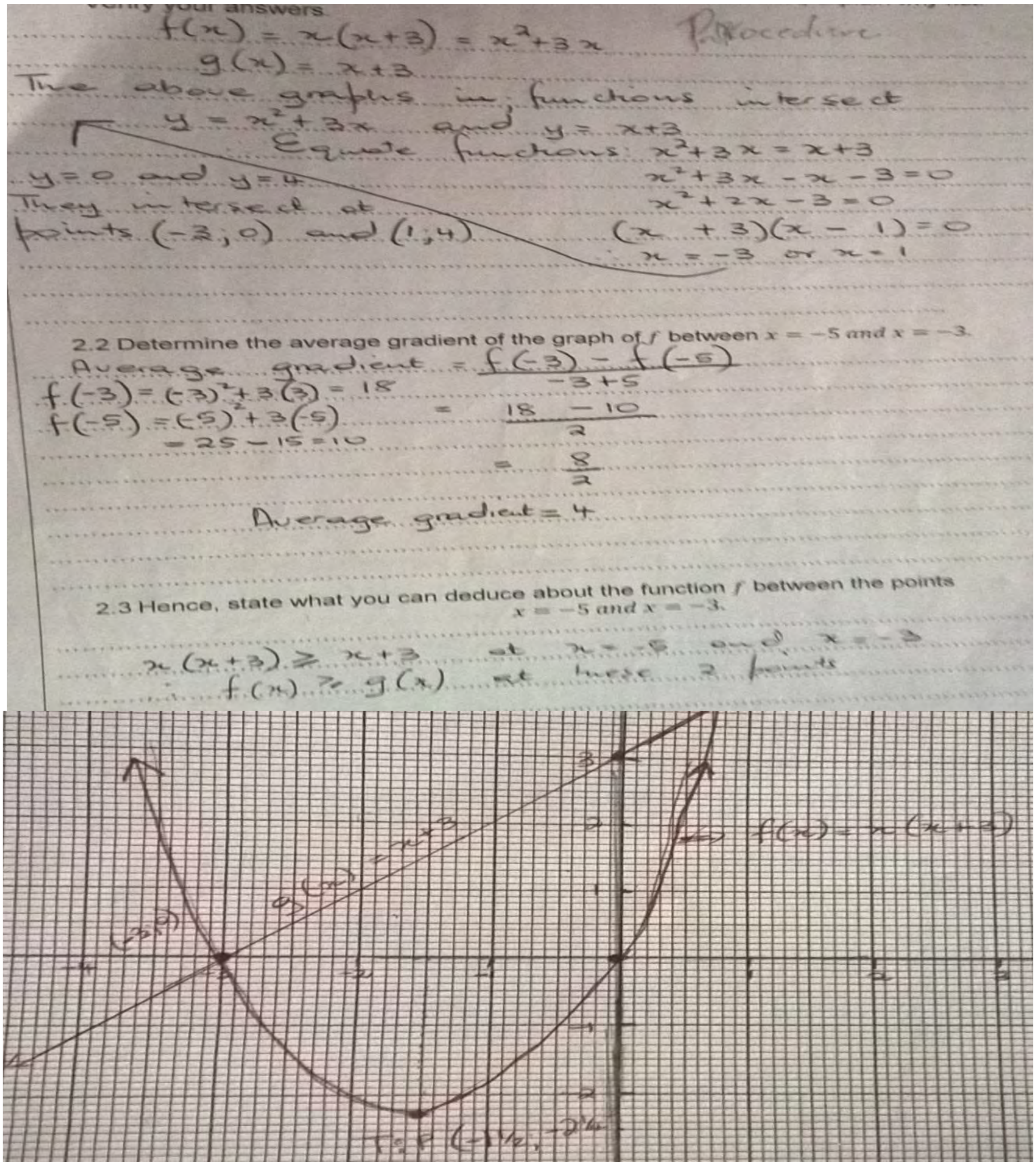


Figure 5:30 Teacher A's response to question 2 of the post-assessment

Teacher B completed question 2 of the post-assessment function task and he provided correct answers to all the questions. With regard to questions 2.1, 2.2 and 2.3 Teacher B relied on an algorithmic skill in order to provide satisfactory responses as well as the relevant meaning of the average gradient concerning question 2.3. Also Teacher B managed to translate the algebraic representation into a graphical one. Herein, there is evidence of complete understanding with reference to the graphing and solving of the system of equations of the function concept, which is procedural in nature. He was able to perform operations and managed to make interpretations, something he struggled with before the CLP (see Figure 5:31), concerning the function concept as well as sketching the graph of the function (see Figure 5:32).

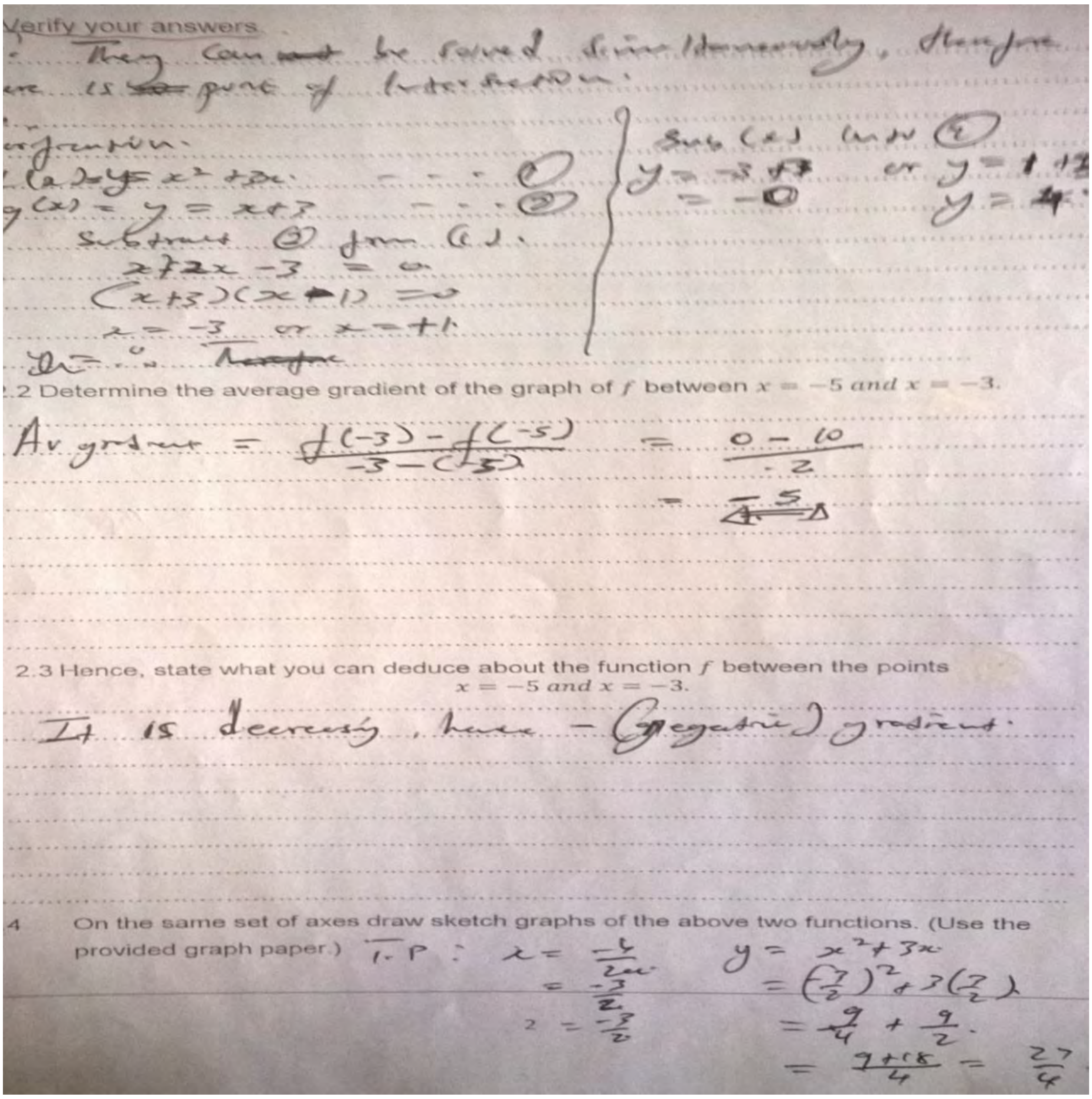


Figure 5:31 Teacher B’s response to question 2 of the post-assessment

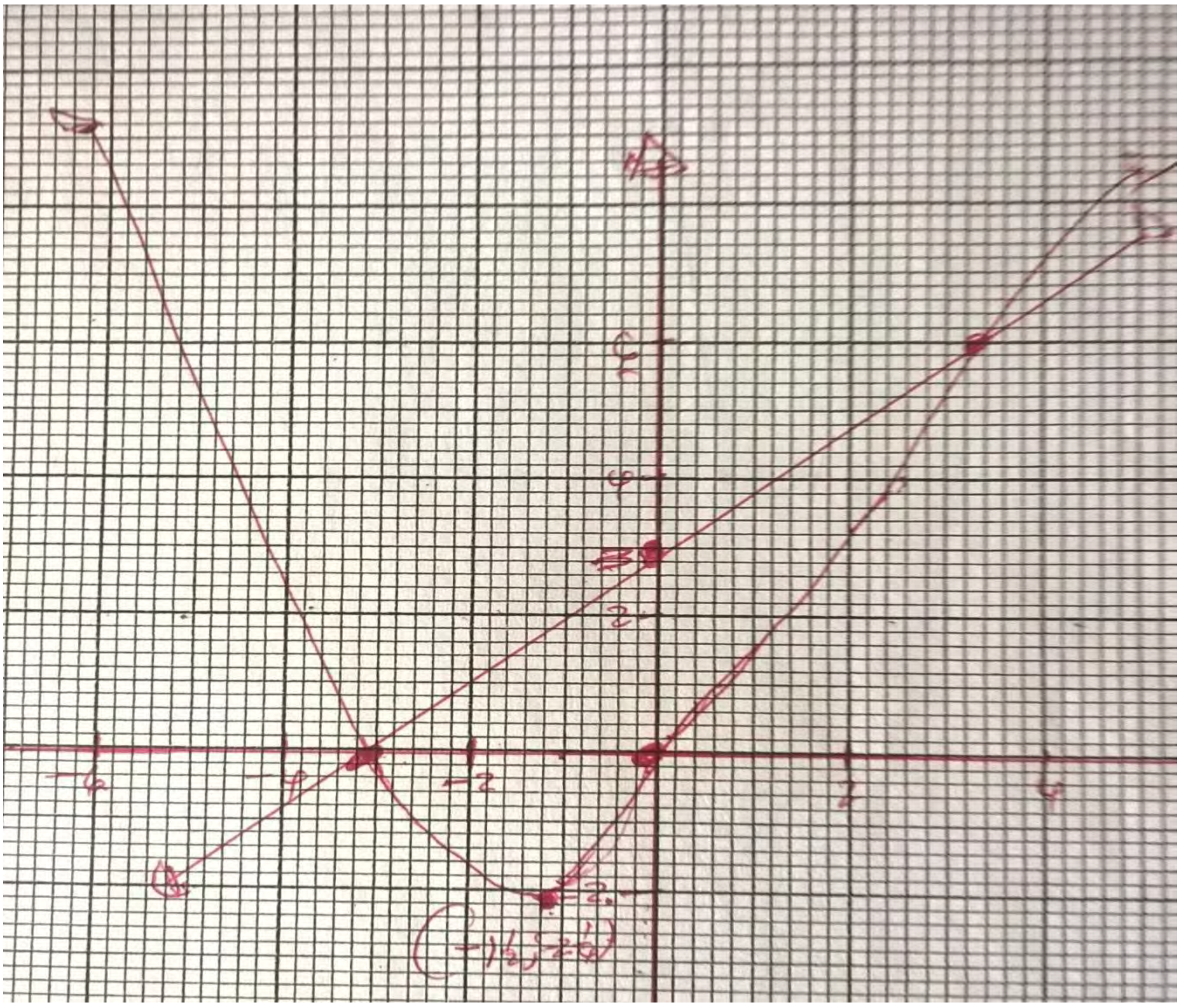


Figure 5:32 Teacher B's graphical solution to question 2 of the post-assessment

Following from the above sections, when I compare Teacher A's performance in the pre-assessment to the post assessment I note that there is no change in her proficiency concerning the procedural knowledge of the function concept. Herein, the evidence suggests that Teacher A appreciates to engage in mathematical situations from which the context is based on using procedural skills instead of modelling real-life situations within the four components of the function model. In other words, she did well in questions that required her to find the equation of a given graph or to sketch the graph of a given equation (see Table 5:7, Table 5:10 and Table 5:11). Also, it is important to note Teacher A struggled with some of the questions that required her to interpret a graphical or algebraic representation of a function concept. This is supported by the fact that Teacher A did not attempt to make different types of interpretations concerning real-life situations involving the function concept in particular in the pre-assessment task (see Table 5:7). But, contrary to this, I noted during the CLP that Teacher A and Teacher

B worked together in completing the same mathematical situations that formed the pre-assessment task and they were able to make different interpretations concerning the function concept presented in words and by means of a table respectively. Once, again during the CLP, Teacher A worked with Teacher B in completing similar mathematical situations and they were able to make different representations (see Table 5:9). In connection to this, one can note that Teacher A appreciates to engage in such mathematical situations in a way that encourages collaborative learning. After the CLP, I note that Teacher A attempted to make different types of interpretations concerning a mathematical situation presented graphically; she provided satisfactory responses in this regard, and she was working individually (see Table 5:11).

In relation to modelling real-life situations with functions, Teacher A showed no appreciation to engage in such mathematical situations individually, because in both the pre- and post-assessment she did not attempt to complete them (see Table 5:7and Table 5:11). In light of this, in the post-assessment task Teacher A mentioned that she knew nothing about such mathematical situations (linear relationship). Teacher A maintained that before the CLP she had a negative attitude toward such situations, hence she did not see the need to teach them in her classroom. But during the CLP when she was working with Teacher B, they provided satisfactory responses to such situations (linear relationship), except in one situation from which the function concept was presented within the context of a quadratic or semi-circle function (see Table 5:9). Translating is the third component of the function concept, and herein I note that Teacher A showed no appreciation to work with mathematical situations that required her to move from one representation of the function to another one. Before the CLP she did not complete such mathematical situations from which the function concept was presented in words (see Table 5:7). After the CLP, Teacher A worked individually on one such mathematical situation that required her to translate the function concept from a graph to words. She attempted to complete this mathematical situation, but her responses were not satisfactory (see Table 5:11). Reifying is the fourth and last component of the function model, and herein Teacher A showed no appreciation to engage in such questions because both in the pre- and post-assessment tasks she did not attempt to complete such questions (see Table 5:7and Table 5:11). On the one hand when I compare Teacher B's performance in the pre-assessment to the post assessment I note that there is change in his proficiency concerning the procedural knowledge of the function concept. That is to say, Teacher B performed well in questions that required him to use procedural skills with respect to finding the horizontal shifting of the graph, interpreting a graph or a formula, compared to the pre-assessment task. In the pre- assessment task he struggled to provide satisfactory responses to questions that required him to interpret the graph as well as to perform both vertical and horizontal transformations with respect to functions presented in a formula. But it is worth noting to note that he performed well in those questions that required him to find the equations of a given graph (see Table 5:8). In the post-assessment, I note that Teacher B managed to provide satisfactory responses to all the questions that were based on procedural knowledge of the function concept (see Table 5:12). Also, during the CLP, Teacher B worked with Teacher A on Task 2 that involved working with the system of equation within the context of linear and non-linear systems. Herein, teachers were engaged with problems that were focusing on parallel and intersecting lines, absolute value as well as the inequalities. In connection with this, I note that both Teacher A and Teacher B performed well (see Table 5:10). Herein one can assume that the CLP did have a positive influence on Teacher B's procedural knowledge of the function concept. Overall the CLP exposed both Teacher A and Teacher B to six questions that were deemed as modelling real-life situations with the function concept. Two of these questions were based on non-linear relationships, whereas the other four questions were situated in a linear relationship. Herein, Teacher B appreciated all the questions that were focusing on the function concept within the context of a linear relationship. That is to say before, during and after the CLP, Teacher B provided satisfactory responses to such questions (see Table 5:8, Table 5:9 and Table 5:12). However, before and during the CLP he did not complete the questions that were based on non-linear relationships such as quadratic or semi-circle (see Table 5:9). Nevertheless, Teacher B showed complete conceptual understanding of the function concept. In relation to interpreting, which is the second component of the function model, Teacher B's performance before, during and after the CLP reveals that his conceptual understanding of the function concept with respect to making different types of interpretations concerning real-life situations has improved. In the pre-assessment he did not attend to one such question, and completed only one, but without providing correct answers to all the questions (see Table 5:8). During the CLP Teacher A and Teacher B worked together on such questions and they provided correct answers to all the questions (see Table 5:9). Also in the post-assessment, Teacher B provided correct answers to all the questions (see Table 5:12). Translating is the third component of the function model, and overall there were five mathematical situations that were centred on translating the function concept from one representation to another. In the pre-assessment, Teacher B attempted to complete two such mathematical situations. Herein he was able to translate from a table to a graphical representation, as well as from words to a formula and a graph respectively. But it is worth noting that he struggled to interpret the graphical representation in one situation (see Table 5:8). Also, during the CLP, Teacher A and Teacher B worked together in completing such mathematical situations that encouraged them to move from a words to a graph and a formula respectively, and they performed well in this regard (see Table 5:9). In the post-assessment Teacher B managed to complete one such mathematical situation in which he was encouraged to move from a graphical representation of the function to words. Herein, he showed appreciation of the situation but he did not produce the mathematical content of the situation; in other words, he overlooked aspects of the graphical representations such as the domain, range or the shape. Herein, I note that Teacher B's conceptual understating of the function concept with respect to component of translating of the function did not change over the course of the CLP; it was the same before, during and after the CLP (see Table 5:8, Table 5:9 and Table 5:12). Two questions involved reifying with functions, and herein I note that Teacher B's conceptual understanding with respect to reifying real-life situations (linear relationship) had changed. This is so because in the pre-assessment he did not complete one such mathematical situation and in the post-assessment he completed one such situation (see Table 5:8). Herein, he managed to provide a satisfactory response to the situation (see Table 5:12). Henceforth the CLP did have a positive influence on Teacher B's conceptual understanding of the function concept with respect to the component of reifying of the function concept.

5.4 Conclusion

The central purpose of this was to establish the analysis and interpretation of the data gathered for the study in particular Case 1. The data analysis appealed to the research questions as well as the research objectives of the study. The results showed that the CLP did not have an impact on teachers' definitions of mathematics. However, it is important to note that Teacher B's views about the teaching and learning of function concept were influenced positively by the CLP. Teacher B's understanding of the function concept was positively influenced by the CLP as well, because his understanding of the function concept ranged from no effort to complete understanding with moderate understanding in between. Whilst the CLP did not have any influence on Teacher A's conceptual understanding of the function concept, it is worth noting that Teacher A did learn an algorithmic skill during the CLP. Both Teacher A and Teacher B maintained that they liked the CLP, but Teacher A stated that the only thing she did not like about the CLP was the writing part. In the following chapter (6) I establish a discussion concerning the process of data analysis that prevailed in Case 2 of this study.

CHAPTER 6: DATA ANALYSIS AND INTERPRETATION: CASE 2

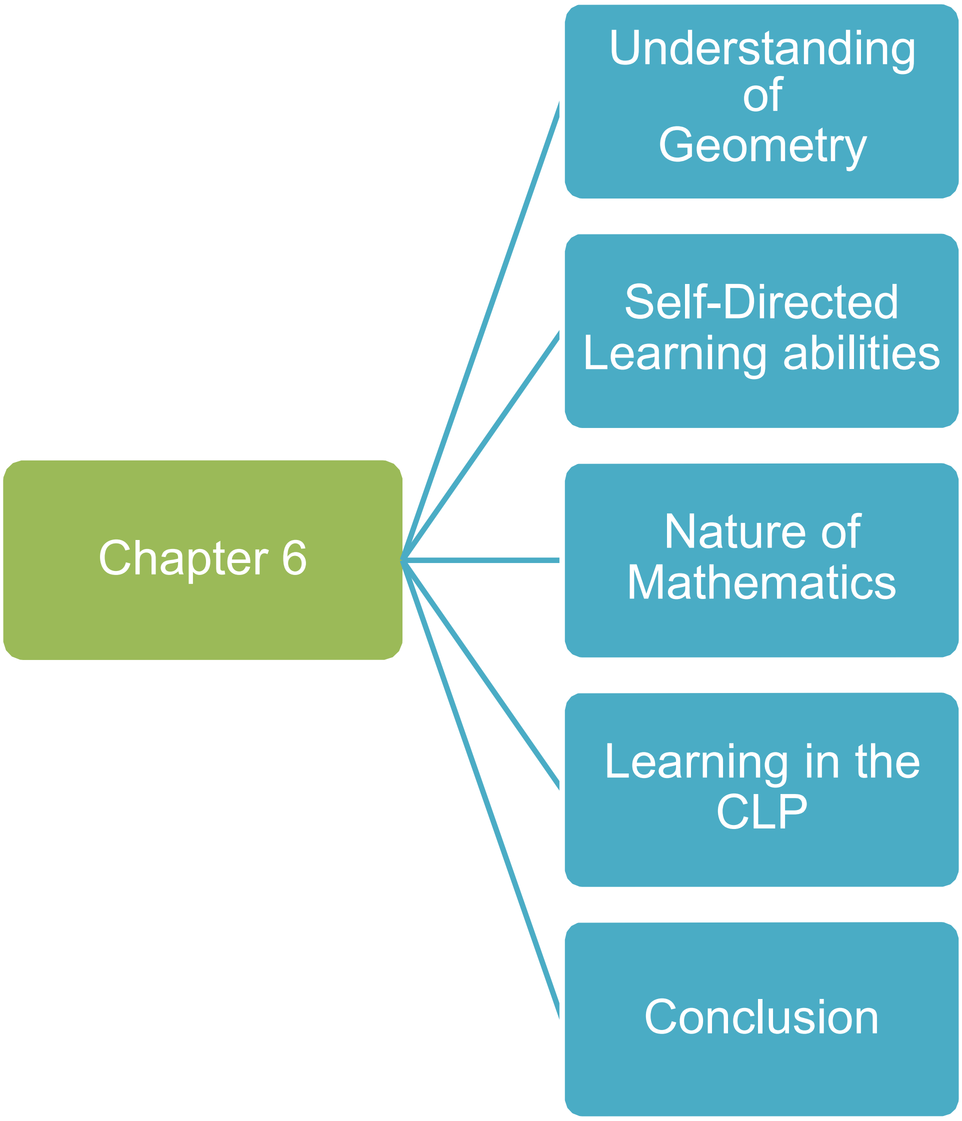


Figure 6:1 Outline of Chapter 6

6 Setting the scene

It is important to indicate that these students were never exposed to the teaching and learning of Euclidean geometry by means of the Geogebra® program. However, two step- by-step practical manuals for the use of Geogebra® (Stols, 2009, 2013) were made available to them on the official university learning platform (*eFundi*) they all had access to. All these students had had experience of using another dynamic software package Geometer's SketchPad® during their mathematics content modules (first to third year). Students' conceptual understanding of Euclidean geometry was investigated over the ten sessions of the CLP. Each session took about ninety (90) minutes to complete. Euclidean Geometry tasks involving the properties and relationships between triangles, quadrilaterals and circles were posed to examine students' understanding of Euclidean Geometry. Herein, students solved the problems (for exploration on computer with dynamic Geogebra® program) with the research team serving as the facilitators. Students were encouraged to make conjectures, discover, formulate, classify, define, refute and prove theorems (where necessary) based on the observed regularities and then testing them on multiple situations.

In totality students were engaged in four geometric tasks (see Appendix K, L, M and N). A rubric (see Table 4:6) was used to assess students' understanding of Euclidean Geometry. Also I used the van Hiele theory of geometric thought as a means of making sense of students' understanding of geometry. In these sessions students were engaged in numerous mathematical activities and tasks. These tasks were administered in session 1, 2, 3 and 8 respectively. The rationale behind conducting these Euclidean geometry tasks was to answer the following question:

> *What is the influence of a collaborative learning programme on final-year mathematics education students' conceptual understanding of Euclidean geometry?*

The other remaining sessions were focused on the discussions or reflections about the project. In the following paragraphs I assess students' understanding and I portray the mathematical activities that occurred in the CLP. Henceforth in this chapter I discussed students' understanding of geometry, their self-directed learning abilities, and their fundamental views of the nature of mathematics.

6.1 Discussion of students' conceptual understanding geometry

First meeting

In session 1, we met with the students for the first time. In this session we came to know that the students were never exposed to the learning of Euclidean geometry and that they have never used the Geogebra ® program. We took this opportunity to explain and provide students with the background of the study. Also, the Researcher B exposed students to the use of Geogebra® the program and he explained and showed them how some of the fundamental menus of the program work. On the other hand students were encouraged to open the program and try some few things out and they did exactly that.

Task 1

In session 2, we administered Task 1 (see Appendix K). This task was used to investigate students' understanding of the properties of triangles and it comprised of four questions. In this task students worked together. Herein, the students were engaged in questions that required them to use the Geogebra® program and construct any triangle. They were investigating the segments joining the mid-points of two sides of a triangle. Henceforth, students were encouraged to make observations and make conjectures, as well as to explain or justify their conjectures.

Seven students completed Task 1 and they all provided correct answers to all the questions. In completing this task students had to construct any triangle and join the mid-points of two sides of that triangle (see Figure 6:1, Figure 6:2 and Figure 6:3).

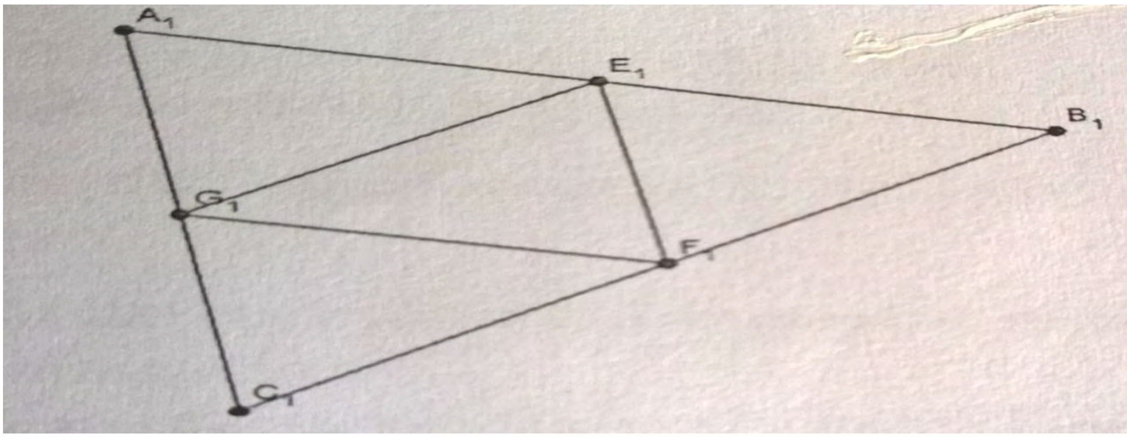


Figure 6:1 Student 2's construction of any triangle to Task 1

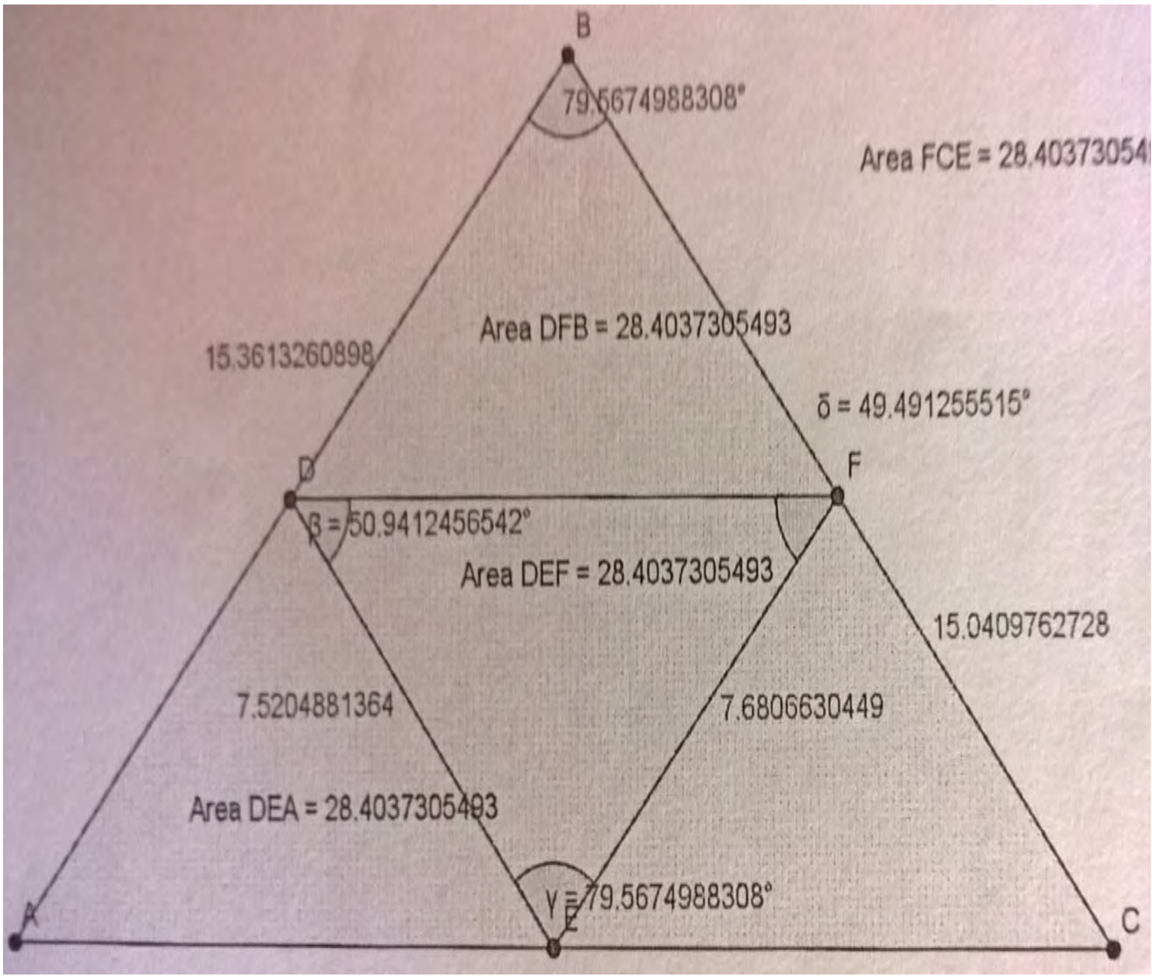


Figure 6:2 Student 3's construction of any triangle to Task 1

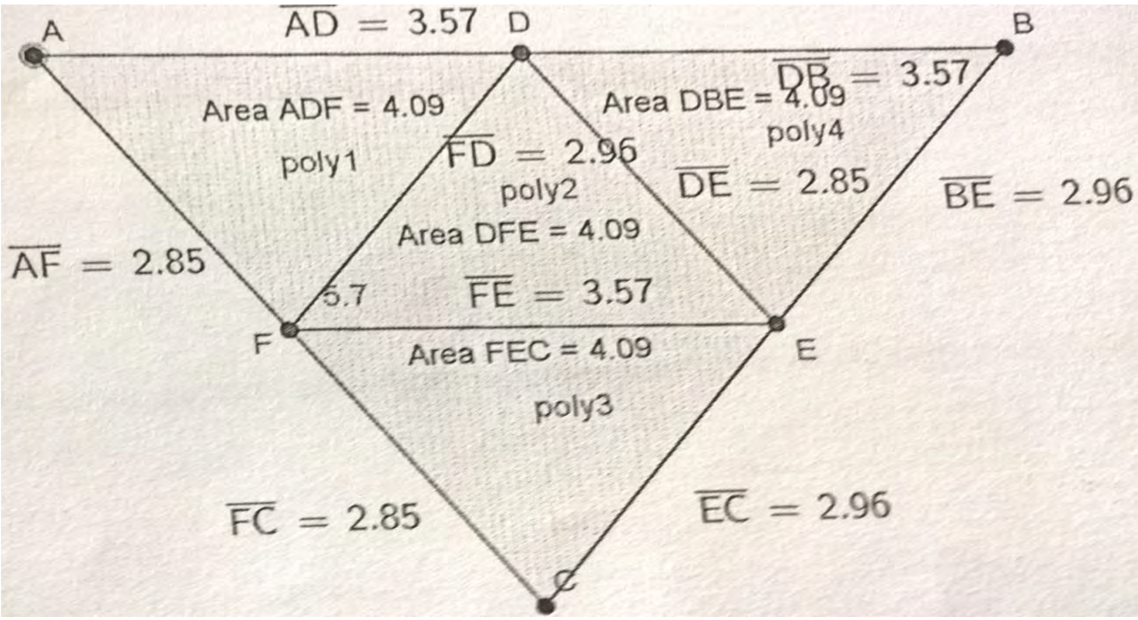
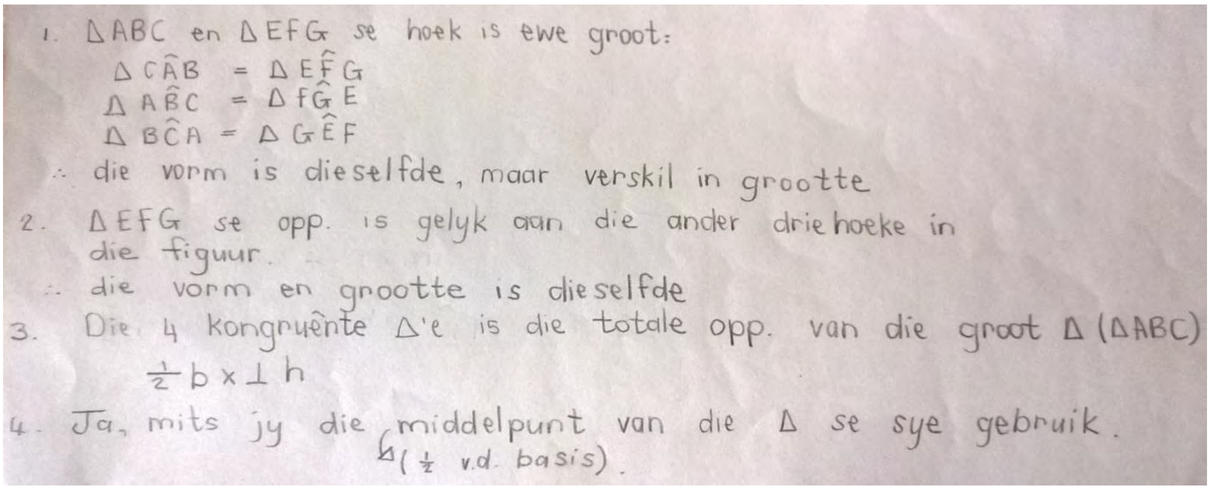


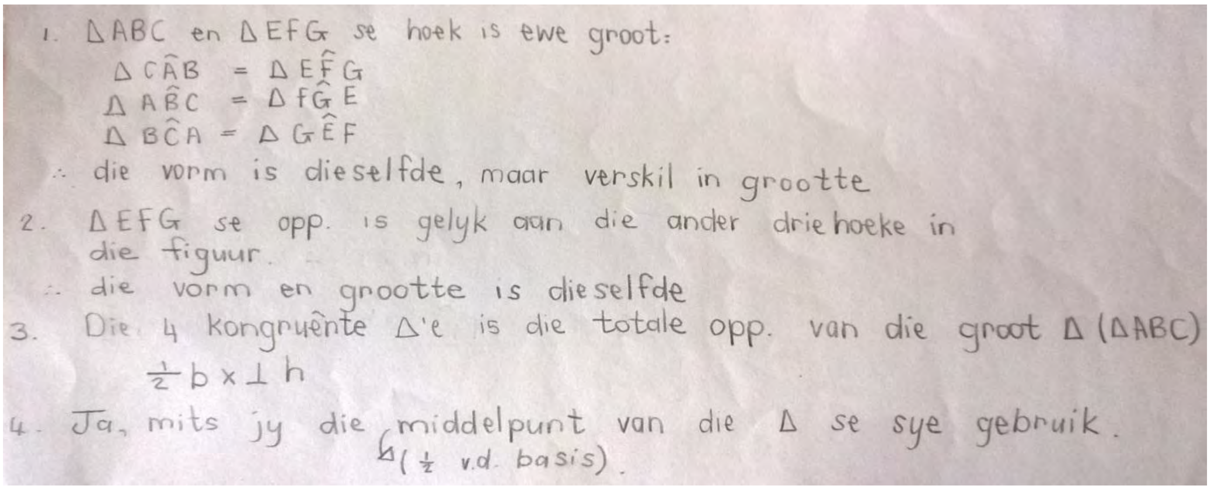
Figure 6:3 Student 12's construction of any triangle to Task 1

Consequently students discovered the relationships between the triangles that they constructed. These relationships led them to discover the idea of congruency and similarity of the triangles (question 1-4). Herein, students' responses resonate with level 2 of the Van Hiele theory concerning geometric thought within the context of observations, making conjectures and formulating theorems instead of proving theorems. The following figures (6.5-6.9) represent five students' (that is Student 1, 2, 3, 7 and 12) responses concerning Task (1, 2, 3, and 4), and the rationale for selecting these students' responses lies with the fact that they took part in all of the tasks that formed this study. But it is important to note that Student 7 missed session 3, hence she could not complete Task 2. Using the rubric (see Table 4:6) I noted that students' understanding of the properties of a triangle and its mid-points was complete.



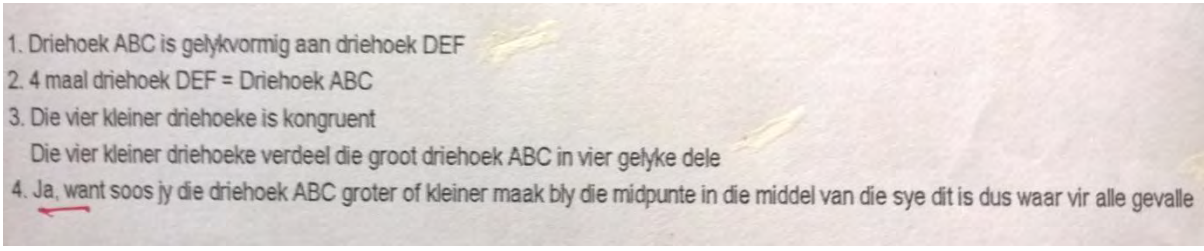
1. ΔABC en ΔEFG se hoek is ewe groot:
$\Delta C\hat{A}B = \Delta E\hat{F}G$
$\Delta A\hat{B}C = \Delta F\hat{G}E$
$\Delta B\hat{C}A = \Delta G\hat{E}F$
$\therefore$ die vorm is dieselfde, maar verskil in grootte
2. ΔEFG se opp. is gelyk aan die ander driehoeke in die figuur.
$\therefore$ die vorm en grootte is dieselfde
3. Die 4 kongruênte Δ'e is die totale opp. van die groot Δ (ΔABC)
$\frac{1}{2} b \times \perp h$
4. Ja, mits jy die middelpunt van die Δ se sye gebruik.
($\frac{1}{2}$ v.d. basis).

Figure 6:4 Student 2's responses to Task 1



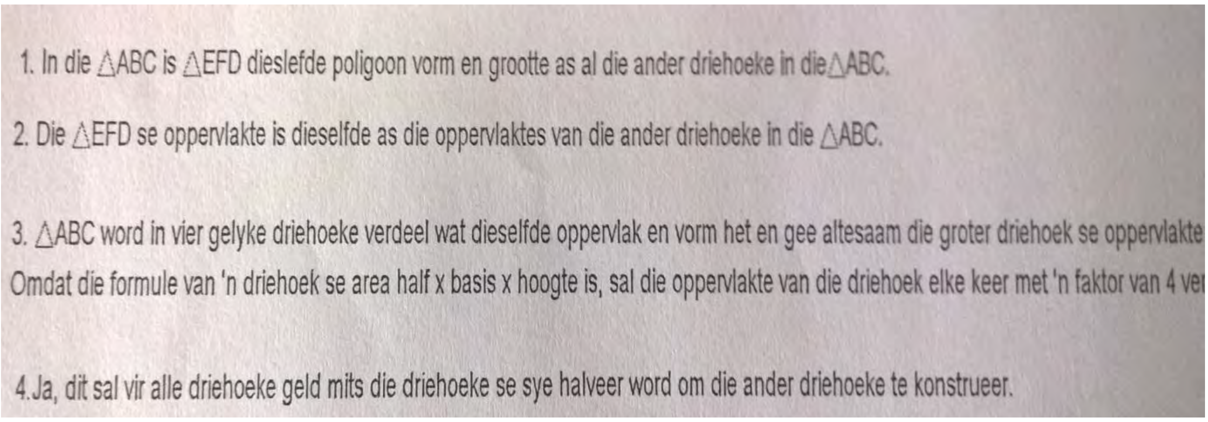
1. ΔABC en ΔEFG se hoek is ewe groot:
$\Delta C\hat{A}B = \Delta E\hat{F}G$
$\Delta A\hat{B}C = \Delta F\hat{G}E$
$\Delta B\hat{C}A = \Delta G\hat{E}F$
$\therefore$ die vorm is dieselfde, maar verskil in grootte
2. ΔEFG se opp. is gelyk aan die ander driehoeke in die figuur.
$\therefore$ die vorm en grootte is dieselfde
3. Die 4 kongruênte Δ'e is die totale opp. van die groot Δ (ΔABC)
$\frac{1}{2} b \times \perp h$
4. Ja, mits jy die middelpunt van die Δ se sye gebruik.
($\frac{1}{2}$ v.d. basis).

Figure 6:5 Student 7's responses to Task 1



1. Driehoek ABC is gelykvormig aan driehoek DEF
2. 4 maal driehoek DEF = Driehoek ABC
3. Die vier kleiner driehoeke is kongruent
Die vier kleiner driehoeke verdeel die groot driehoek ABC in vier gelyke dele
4. Ja, want soos jy die driehoek ABC groter of kleiner maak bly die midpunte in die middel van die sye dit is dus waar vir alle gevalle

Figure 6:6 Student 3's responses to Task 1

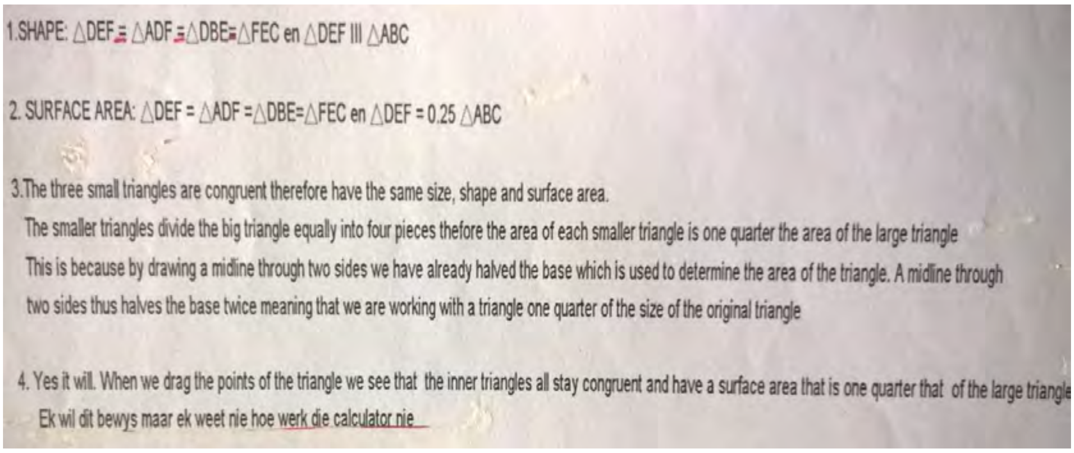

1. In die △ABC is △EFD dieslefde poligoon vorm en grootte as al die ander driehoeke in die△ABC.

2. Die △EFD se oppervlakte is dieselfde as die oppervlaktes van die ander driehoeke in die △ABC.

3. △ABC word in vier gelyke driehoeke verdeel wat dieselfde oppervlak en vorm het en gee altesaam die groter driehoek se oppervlakte
Omdat die formule van 'n driehoek se area half x basis x hoogte is, sal die oppervlakte van die driehoek elke keer met 'n faktor van 4 ver

4.Ja, dit sal vir alle driehoeke geld mits die driehoeke se sye halveer word om die ander driehoeke te konstrueer.

Figure 6:7 Student 1's responses to Task 1


1.SHAPE: △DEF ≡ △ADF ≡△DBE≡△FEC en △DEF III △ABC

2. SURFACE AREA: △DEF = △ADF =△DBE=△FEC en △DEF = 0.25 △ABC

3.The three small triangles are congruent therefore have the same size, shape and surface area.
The smaller triangles divide the big triangle equally into four pieces thefore the area of each smaller triangle is one quarter the area of the large triangle
This is because by drawing a midline through two sides we have already halved the base which is used to determine the area of the triangle. A midline through two sides thus halves the base twice meaning that we are working with a triangle one quarter of the size of the original triangle

4. Yes it will. When we drag the points of the triangle we see that the inner triangles all stay congruent and have a surface area that is one quarter that of the large triangle
Ek wil dit bewys maar ek weet nie hoe werk die calculator nie

Figure 6:8 Student 12's responses to Task 1

Task 2

In session 3, we administered Task 2 (see Appendix L), which was used to investigate students' understanding of the properties of triangles and it comprised of four questions. Herein, the students were engaged in questions that required them to construct any triangle using physical tools. That is to say they were investigating the segments joining the mid- points of two sides of a triangle. Also, students were required to construct angles and other types of triangles. In this task students were encouraged to work together. Essentially the students had to do the same tasks they had done during the previous session (Task 1), but by means of physical constructions, using drawing instruments (and not Geogebra®).

Six students completed the task and they all provided correct answers to all the questions. In completing this task, students had to construct any triangle and join the mid-points of two sides of that triangle. From my filed notes I noted that four of these students did not possess the experience required to use the physical instruments (compass and protractor) but the other two students had the experience. In order to complete the task the four students were constantly in communication with the other two students. They were asking them to show them how to use these instruments to construct the triangles, quadrilaterals and angles. The following figures represent students' responses to questions of Task 2. Consequently they discovered the relationships between the triangles that they constructed. These relationships led them to discover the idea of congruency and similarity of the triangles. There were numerous similarities in their responses, and this might be due the fact that they were working together on some of the questions. Their responses indicated that they reacted positively to the task. This, according to the rubric, indicates complete understanding of the properties of the segments joining the mid-points of two sides of a triangle

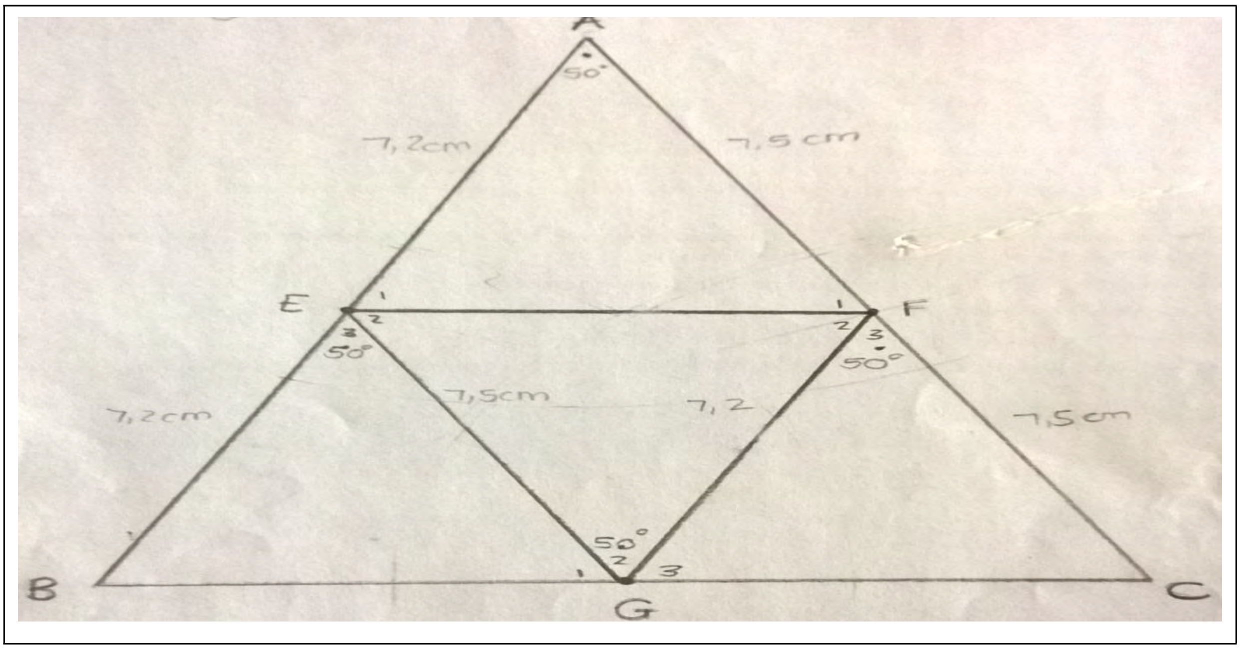


Figure 6:9 Student 1's response to question 1 of Task 2

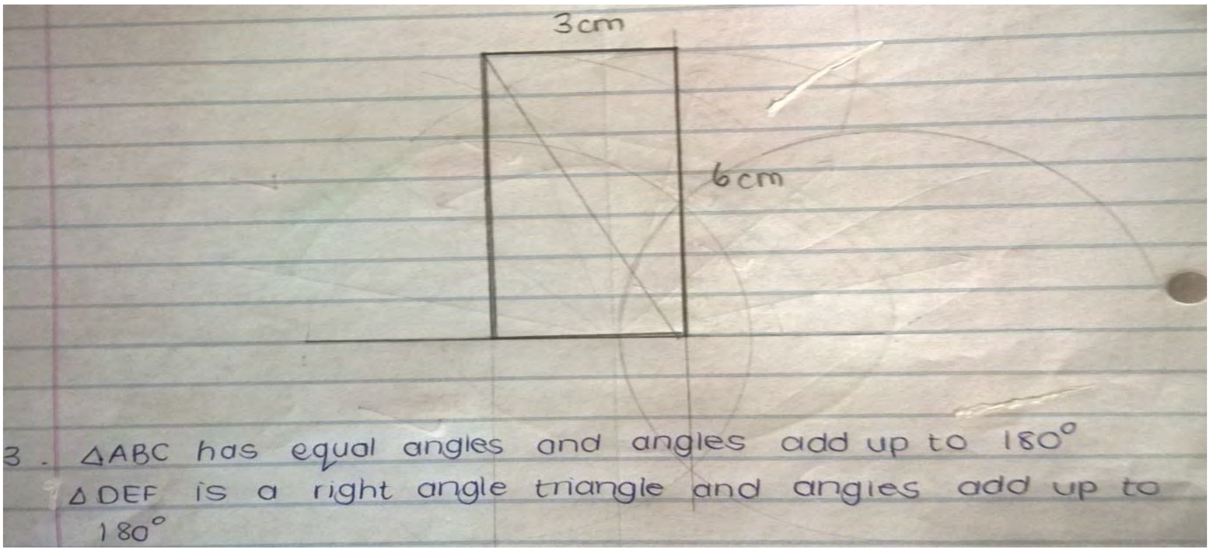


Figure 6:10 Student 12 response to question 2d of Task 2

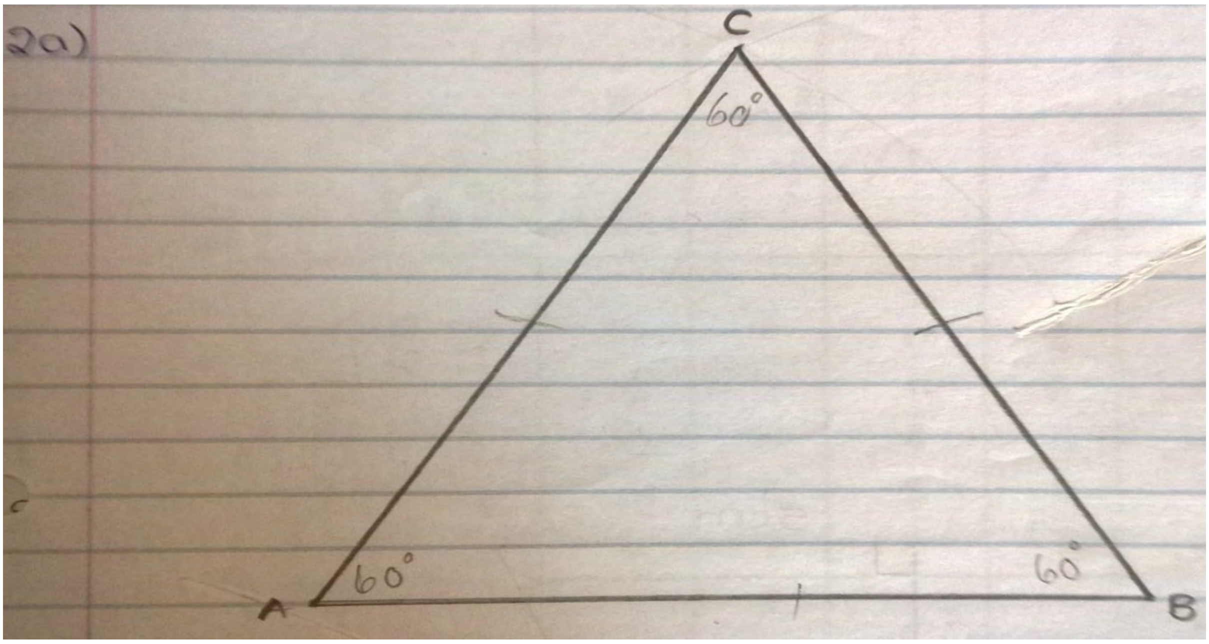


Figure 6:11 Student 2 response to question 2a of Task 2

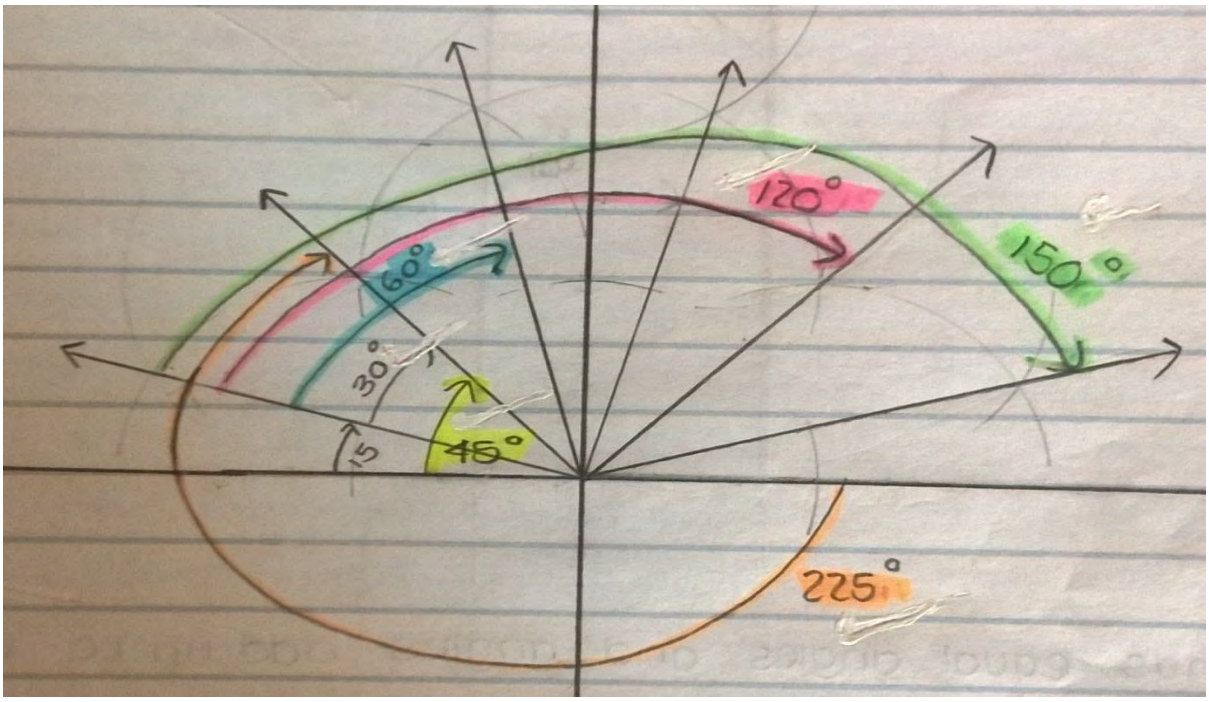


Figure 6:12 Student 12 response to question 2c of Task 2

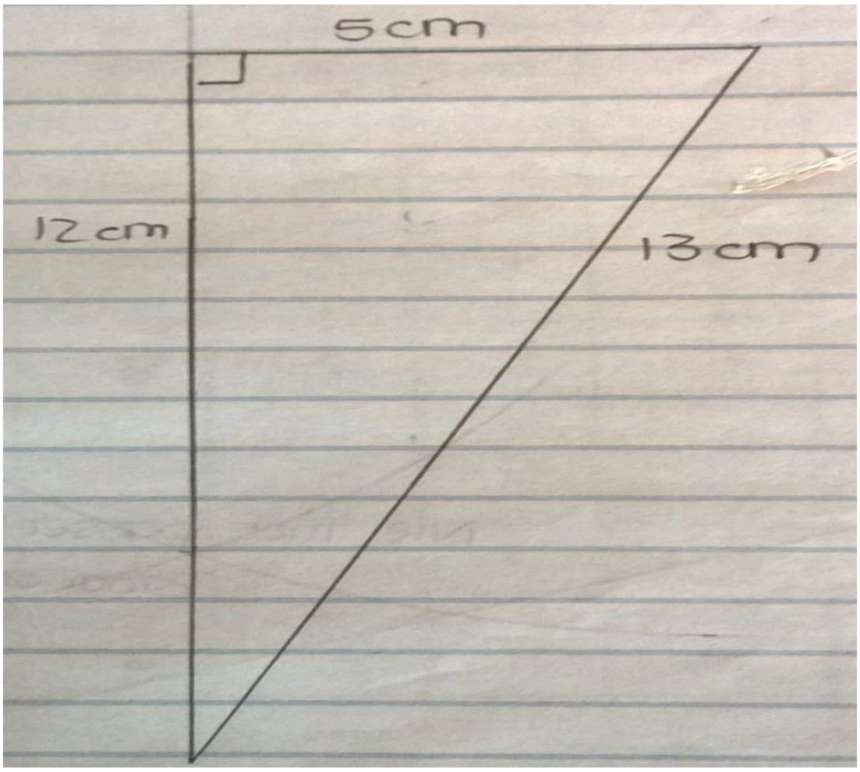


Figure 6:13 Student 3 response to question 2b of Task 2

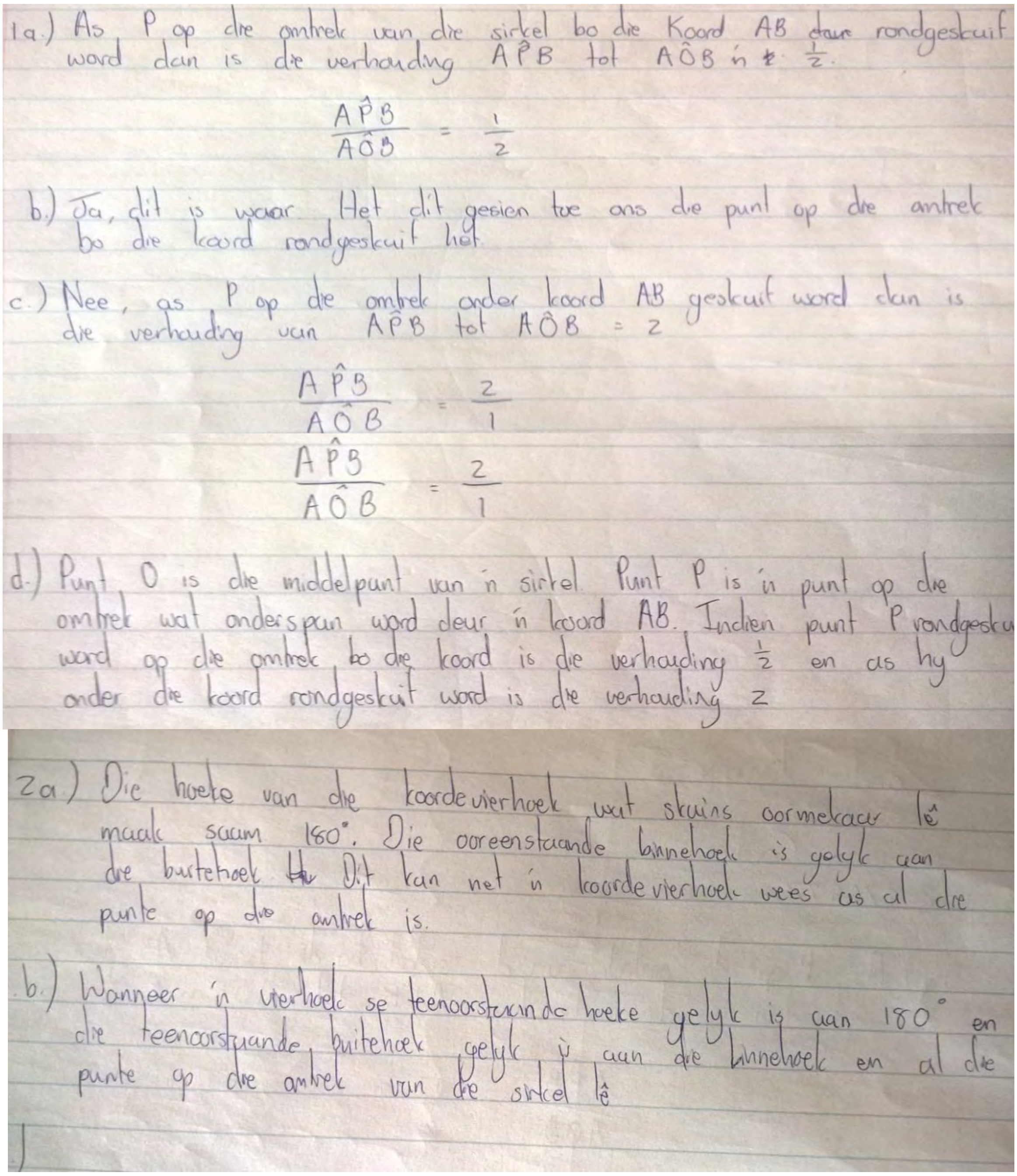
Task 3

In session 4, we administered Task 3 (see Appendix M). This task was used to investigate students' understanding of the properties of triangles and it comprised of three questions. In this task students worked together but each student had his or her own task sheet. Herein, the students were engaged in questions that required them to construct any triangle by using the Geogebra® program. That means they were investigating the congruency and similarity of triangles. Also, students were engaged in tasks that involved the congruency and similarity of quadrilaterals. This, according to the rubric, indicates complete understanding of the properties of the segments joining the mid-points of two sides of a triangle.

In conclusion, Tasks 1, 2 and 3 were particularly centred on level 2 (informal deduction) of the van Hiele theory of geometric thought. However Task 2 involved the use of physical tools such as ruler, protractor, pencil, paper and compass. Both Task 1 and 3 favoured the use of a software tool such as the Geogebra® program. The geometric content of these tasks required students to discover relationships between and among the properties of triangles and quadrilaterals. In connection to the preceding paragraphs, the students were able to discover a relationship about the similarity and congruency of triangles as well as quadrilaterals. Students' performance concerning Task 1, 2 and 3, indicates that their understating of the relationships between and among the properties of triangles, as well as quadrilaterals, clearly fits into Level 2 (informal deduction) of the van Hiele theory of geometric thought (Van de Walle *et al.,* 2010:403). This indicates that there is an appreciation to create conjectures concerning relationships between and among properties of geometric shapes.

Task 4

In session 8, we administered Task 4 (see Appendix N). This task was used to investigate students' understanding of the properties of circle geometry and it comprised two questions. In this task students were encouraged to work together. Herein they engaged in questions that required them to use the Geogebra® program. They investigated the properties of circle geometry and proving theorems. In question 1 of the task, students were requested to investigate and prove that the angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the circle). On the other hand, question 2 required students to investigate the properties of a cyclic quadrilateral. Ten students completed this task and they all provided correct answers to all the questions. But it is important to note that they were not able to provide satisfactory responses to the question that required them to prove their conjectures. The following figures (6.15-6.18) represent students' responses concerning Task 4. It is worth noting that I only use responses from Students 1, 2, 3, 7, and 12.



1a.) As P op die omtrek van die sirkel bo die Koord AB daar rondgeskuif word dan is die verhouding $A\hat{P}B$ tot $A\hat{O}B$ 'n ~~t~~ $\frac{1}{2}$.

$$\frac{A\hat{P}B}{A\hat{O}B} = \frac{1}{2}$$

b.) Ja, dit is waar. Het dit gesien toe ons die punt op die omtrek bo die koord rondgeskuif het.

c.) Nee, as P op die omtrek onder koord AB geskuif word dan is die verhouding van $A\hat{P}B$ tot $A\hat{O}B$ = 2

$$\frac{A\hat{P}B}{A\hat{O}B} = \frac{2}{1}$$

$$\frac{A\hat{P}B}{A\hat{O}B} = \frac{2}{1}$$

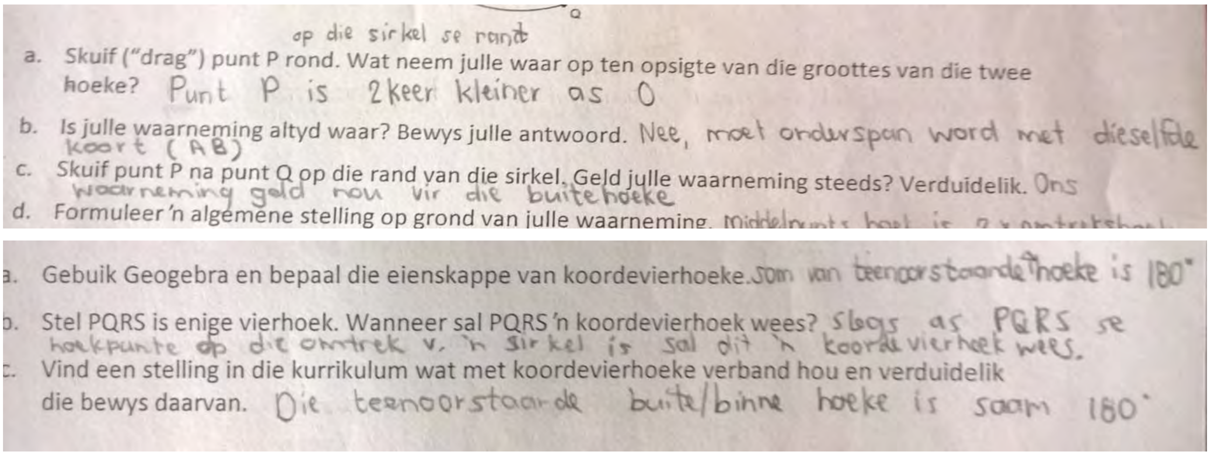
d.) Punt O is die middelpunt van 'n sirkel. Punt P is 'n punt op die omtrek wat onderspan word deur 'n koord AB. Indien punt P rondgeskuif word op die omtrek bo die koord is die verhouding $\frac{1}{2}$ en as hy onder die koord rondgeskuif word is die verhouding 2

2a.) Die hoeke van die koordevierhoek wat skuins oormekaar lê maak saam 180°. Die ooreenstaande binnehoek is gelyk aan die buitehoek ~~H~~ Dit kan net 'n koordevierhoek wees as al die punte op die omtrek is.

b.) Wanneer 'n vierhoek se teenoorstaande hoeke gelyk is aan 180° en die teenoorstaande buitehoek gelyk is aan die binnehoek en al die punte op die omtrek van die sirkel lê

.)

Figure 6:14 Student 9 responses to Task 4



op die sirkel se rand

a. Skuif ("drag") punt P rond. Wat neem julle waar op ten opsigte van die groottes van die twee hoeke? Punt P is 2keer kleiner as O

b. Is julle waarneming altyd waar? Bewys julle antwoord. Nee, moet onderspan word met dieselfde koort (AB)

c. Skuif punt P na punt Q op die rand van die sirkel. Geld julle waarneming steeds? Verduidelik. Ons waarneming geld nou vir die buitehoeke

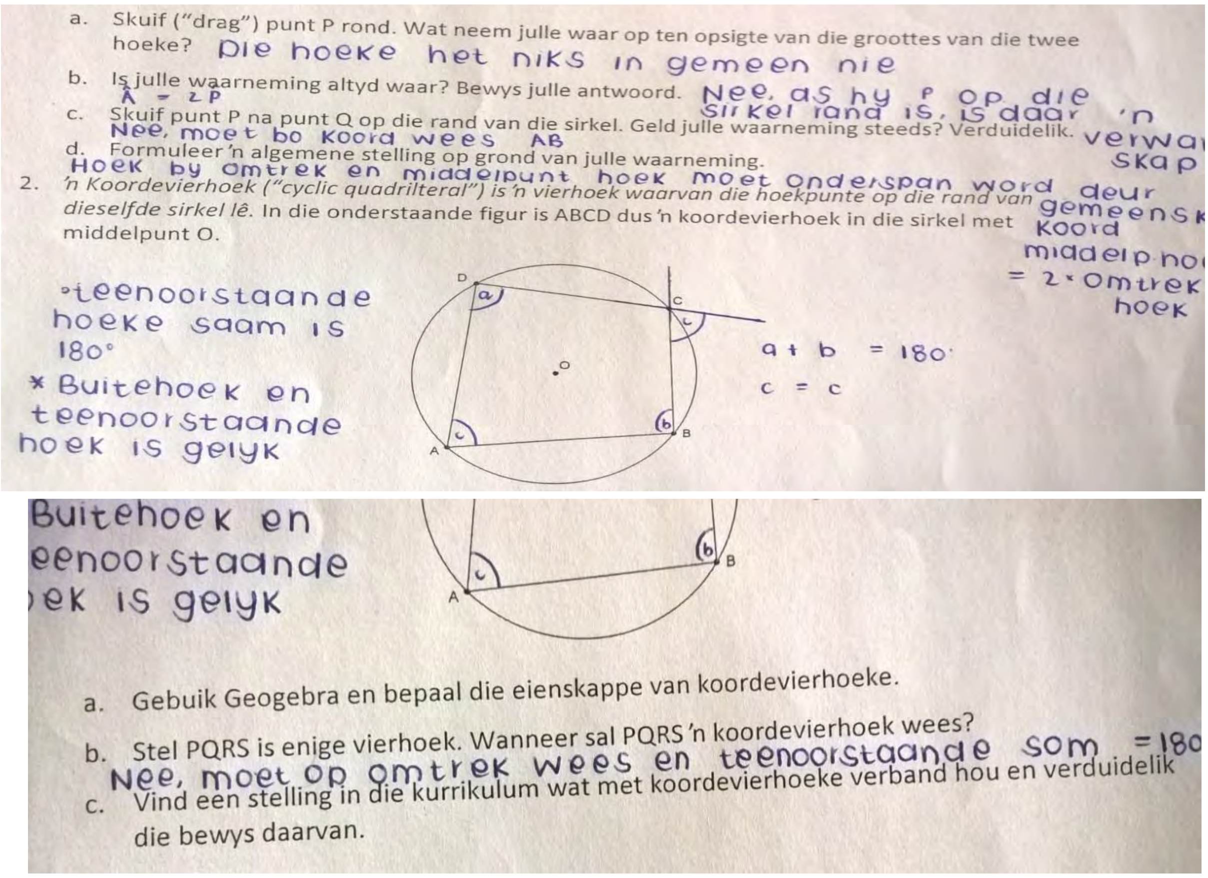
d. Formuleer 'n algemene stelling op grond van julle waarneming.

a. Gebuik Geogebra en bepaal die eienskappe van koordevierhoeke. Som van teenoorstaande hoeke is 180°

b. Stel PQRS is enige vierhoek. Wanneer sal PQRS 'n koordevierhoek wees? Slegs as PQRS se hoekpunte op die omtrek v. 'n sirkel is sal dit 'n koordevierhoek wees.

c. Vind een stelling in die kurrikulum wat met koordevierhoeke verband hou en verduidelik die bewys daarvan. Die teenoorstaande buite/binne hoeke is saam 180°

Figure 6:15 Student 7's responses to Task 4



a. Skuif ("drag") punt P rond. Wat neem julle waar op ten opsigte van die groottes van die twee hoeke? Die hoeke het niks in gemeen nie

b. Is julle waarneming altyd waar? Bewys julle antwoord. Nee, as hy P op die sirkel rand is, is daar 'n verwantskap
Â = 2P̂

c. Skuif punt P na punt Q op die rand van die sirkel. Geld julle waarneming steeds? Verduidelik. Nee, moet bo koord wees AB

d. Formuleer 'n algemene stelling op grond van julle waarneming. Hoek by omtrek en middelpunt hoek moet onderspan word deur gemeensk koord
middelp hoek = 2 × omtrek hoek

2. *'n Koordevierhoek ("cyclic quadrilteral") is 'n vierhoek waarvan die hoekpunte op die rand van dieselfde sirkel lê.* In die onderstaande figur is ABCD dus 'n koordevierhoek in die sirkel met middelpunt O.

• teenoorstaande hoeke saam is 180°
* Buitehoek en teenoorstaande hoek is gelyk

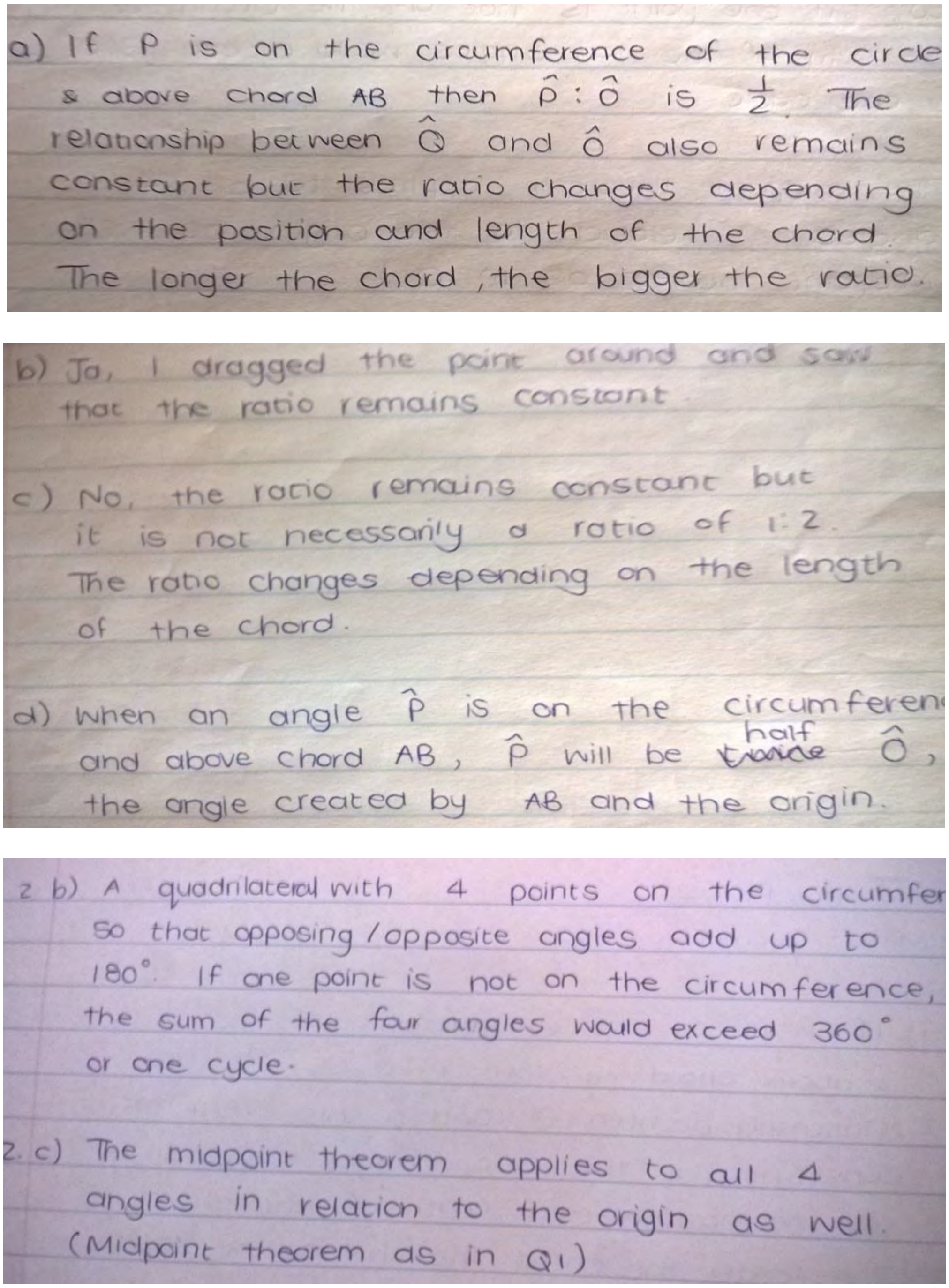
a + b = 180°
c = c

a. Gebuik Geogebra en bepaal die eienskappe van koordevierhoeke.

b. Stel PQRS is enige vierhoek. Wanneer sal PQRS 'n koordevierhoek wees? Nee, moet op omtrek wees en teenoorstaande som = 180

c. Vind een stelling in die kurrikulum wat met koordevierhoeke verband hou en verduidelik die bewys daarvan.

Figure 6:16 Student 2 responses to Task 4



a) If P is on the circumference of the circle & above chord AB then $\hat{P} : \hat{O}$ is $\frac{1}{2}$. The relationship between $\hat{Q}$ and $\hat{O}$ also remains constant but the ratio changes depending on the position and length of the chord. The longer the chord, the bigger the ratio.

b) Ja, I dragged the point around and saw that the ratio remains constant.

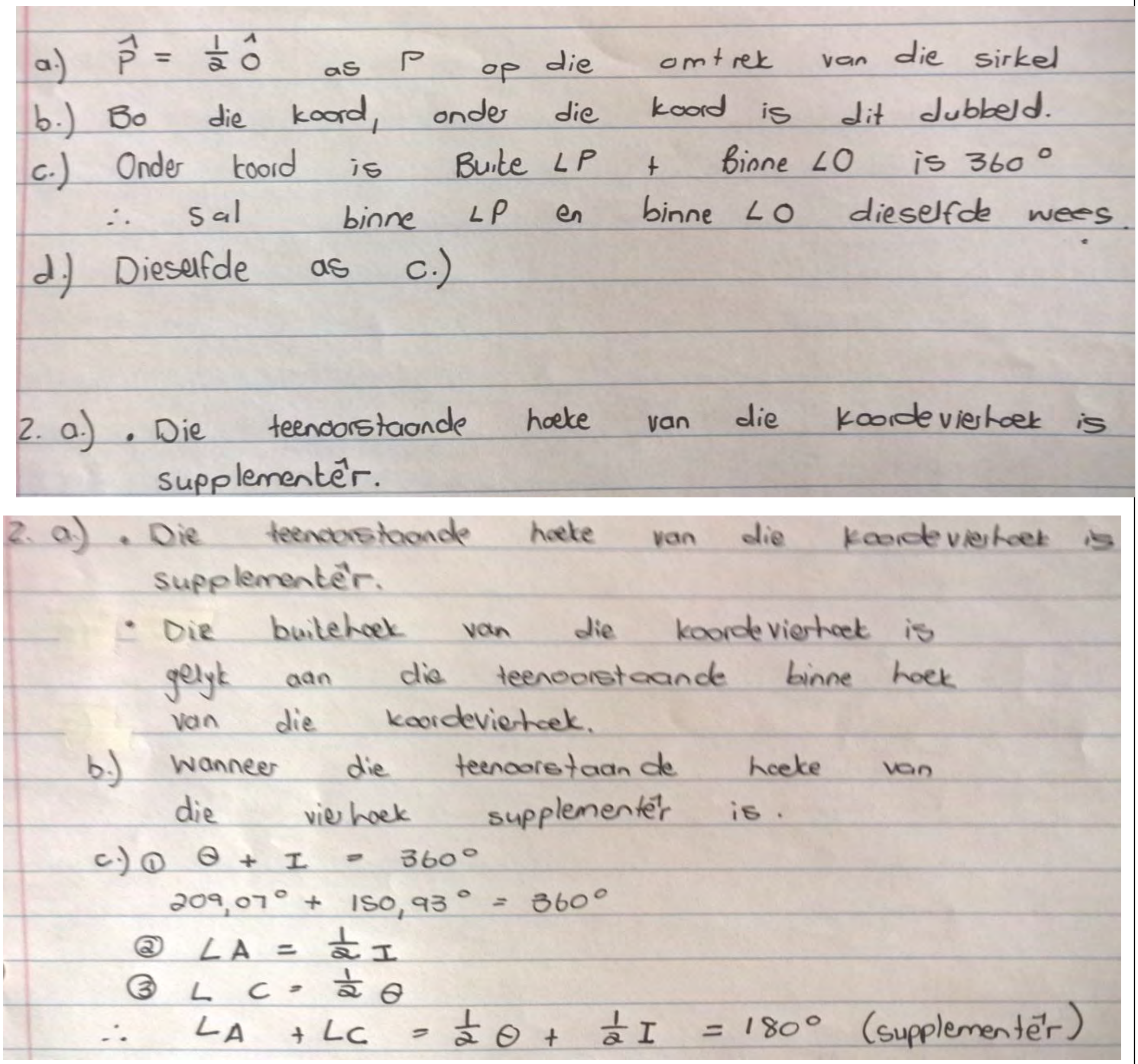
c) No, the ratio remains constant but it is not necessarily a ratio of 1:2. The ratio changes depending on the length of the chord.

d) When an angle $\hat{P}$ is on the circumference and above chord AB, $\hat{P}$ will be ~~twice~~ half $\hat{O}$, the angle created by AB and the origin.

2 b) A quadrilateral with 4 points on the circumference so that opposing/opposite angles add up to 180°. If one point is not on the circumference, the sum of the four angles would exceed 360° or one cycle.

2. c) The midpoint theorem applies to all 4 angles in relation to the origin as well. (Midpoint theorem as in Q1)

Figure 6:17 Student 12 responses to Task 4


a.) $\hat{P} = \frac{1}{2}\hat{O}$ as P op die omtrek van die sirkel
b.) Bo die koord, onder die koord is dit dubbeld.
c.) Onder koord is Buite ∠P + binne ∠O is 360°
∴ sal binne ∠P en binne ∠O dieselfde wees.
d.) Dieselfde as c.)

2. a.) • Die teenoorstaande hoeke van die koordevierhoek is supplementêr.

2. a.) • Die teenoorstaande hoeke van die koordevierhoek is supplementêr.
• Die buitehoek van die koordevierhoek is gelyk aan die teenoorstaande binne hoek van die koordevierhoek.
b.) Wanneer die teenoorstaande hoeke van die vierhoek supplementêr is.
c.) ① $\Theta + I = 360°$
$209{,}07° + 150{,}93° = 360°$
② $\angle A = \frac{1}{2} I$
③ $\angle C = \frac{1}{2}\Theta$
$\therefore \angle A + \angle C = \frac{1}{2}\Theta + \frac{1}{2} I = 180°$ (supplementêr)

Figure 6:18 Student 1's responses to Task 4

In light of the preceding paragraphs, all five students did not provide any formal proof concerning their conjectures with respect to both questions 1 and 2 of Task 4. The responses that they provided were in harmony with level 2 (informal deduction) of the van Hiele theory of geometric thought. Their responses were rooted in what they observed while they were completing the task. In connection to this, one student (Student 7) indicated that she did not appreciate the fact that she had to struggle on her own. Instead she preferred that someone understanding proving theorems should first show her how to prove theorems. Her view was echoed by other students who felt the same. Throughout the CLP she believed that it was important that someone showed her the basics about circle geometry, and then she would be able to engage on it individually. From my field notes I noted that this student did not appreciate to work with other students, instead she preferred to work alone. In question 2, students were requested to construct a cyclic quadrilateral and investigate the properties of it. Herein, students provided satisfactory responses based on their observations and discovered the properties of a cyclic quadrilateral that were acceptable (see Figure 6:14, Figure 6:15, Figure 6:16and Figure 6:17). This indicates that there is an appreciation of the need to create conjectures concerning relationships between and among properties of geometric shapes. This is in association with level 2 (informal deduction) of the van Hiele theory of geometric thought. In totality, Task 4 was centred on both level 2 (informal deduction) and 3 (deduction) of the van Hiele theory of geometric thought (see section 3.2.2.3.4). Based on students' performance in Task 4, their understanding of circle geometry (proof) clearly fits into level 2 (informal deduction). In other words, students were not able to formally prove their conjectures by means of an axiomatic system. This is an indication that there is no appreciation of the need to prove or engage in the abstract statements about geometric properties and make conclusions based on logic rather than intuition (Van de Walle *et al*., 2010:403).

- **Session 5**

In session 5, we administered the pre-assessment of the Self-Directed Learning Instrument (SDLI). Students completed the instrument individually at the same venue. We also conducted a whole-group discussion about the CLP. During the whole-group discussion some of the students maintained the fact that they were struggling to discover the properties of Euclidean geometry on their own, and that they needed someone to stand in front of them and show them how to find those properties.

- **Session 6**

Connecting to session 5, in session 6 we reflected on Tasks 1, 2, and 3. Firstly the session began with Researcher A providing the background of Euclidean geometry within the school context. In this background he talked about geometric concepts such triangles, quadrilaterals and circle geometry. In the previous session they maintained that they were struggling to work with the Geogebra program and needed someone to provide them with basics. They maintained again in this session that they don't know how to use the Geogebra program. When asked whether they had tried to look at the two manuals, no one answered. Then Researcher A guides the students on how to use the Geogebra® program. As he was doing this, they were also busy on their individual computers, attempting to do what Researcher A was demonstrating. As soon as Researcher A had completed the large group presentation, students were encouraged to work on task 1, task 2 and task 3 on their own. Two students come to the front of the class, explained and showed the entire class how they solved these tasks (1 and 3). Some of the students were actively involved; they shared ideas as well as the skills on how to use Geogebra® program.

- **Session 7**

Researcher B continued with the large group presentation. In this session he introduced the falling ladder and bucket problem. This involved a demonstration from which students were encouraged to make observations and provide justifications for their observations. As way to develop their technical skills for the use of the Geogebra® program, the students were encouraged to construct or model this situation by using the Geogebra® program. They attempted to, but could not provide satisfactory responses and Researcher A had to consolidate.

- **Session 9**

In session 9, we conducted the second whole-group discussion. The discussion was based on comparing the Curriculum Assessment Policy Statement with Task 1, 2, 3, and 4. Herein, students had to download the CAPS document for Further Education and Training phase from the Internet and they looked at page 25-40. Students were encouraged to go through the documents, analyse and provide verbal feedback regarding that comparison. In light of this, students noticed that the tasks that they were working with were similar to the geometric content found in the CAPS document.

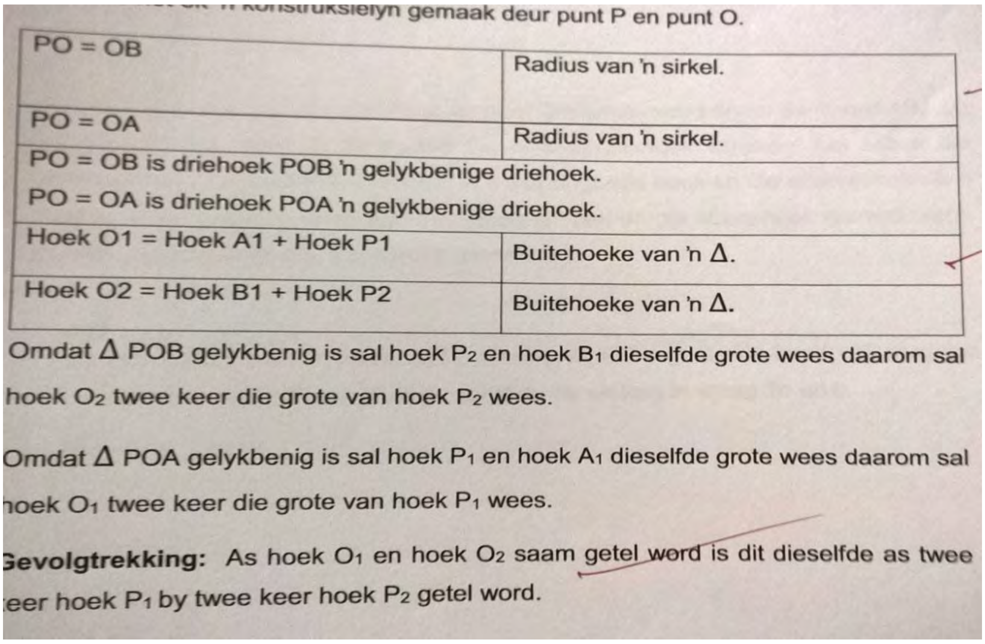
- **Session 10**

In session 10, we were engaged in a large-group presentation during which students were exposed to the properties of Euclidean geometry, in particular the proving of theorems. Researcher A presented the session, and he provided a general overview of Euclidean geometry as an axiomatic deductive system. The presentation entailed a discussion based on axioms, postulates and theorems. We looked at the issue of why the sum of interior angles of a plane triangle is 180 degrees, parallel lines, alternating angles, and corresponding angles, exterior angle of a triangle as well as the circle geometry.
Students listened attentively, asking questions as well as answering where necessary. Here, the Geogebra® program was not used; instead the whiteboard was utilised as a writing surface. Also, in session 10 we administered the post-assessment of the SDLI. Students completed the instrument individually.

- **Module assignment 2 (Practical)**

After the ten sessions of the Collaborative Learning Programme, I had an opportunity to assess (mark) students' module assignment 2 (see Appendix Q). In this assignment a question which was similar to one of the questions in Task 4. While I was busy marking students' assignments I came to realize that 9 students out of 11 were able to engage in the abstract statements about geometric properties and make conclusions based on an axiomatic system.

The following figures (6.19-16.23) represent the students' responses to question 1a of the practical assignment 2 (see Appendix Q). It is worth noting that the following figures only represent responses from Students 1, 2, 3, 7 and 12 concerning the practical assignment 2.



...uksielyn gemaak deur punt P en punt O.

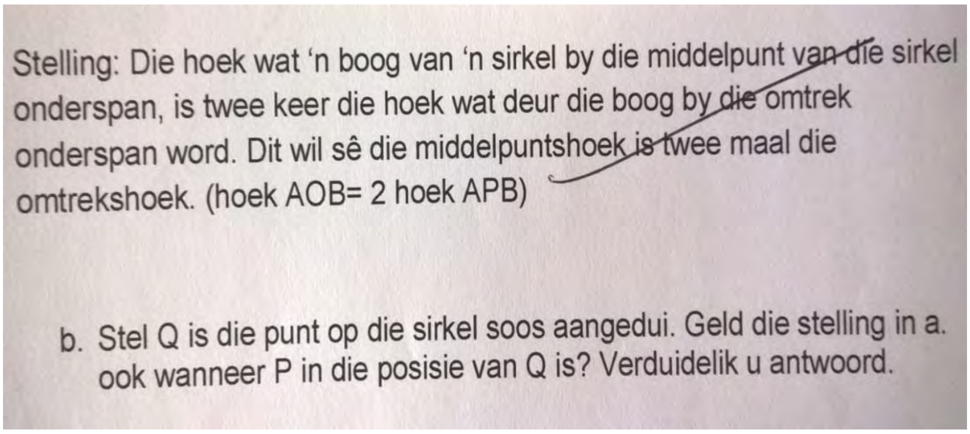
PO = OB	Radius van 'n sirkel.
PO = OA	Radius van 'n sirkel.
PO = OB is driehoek POB 'n gelykbenige driehoek.	
PO = OA is driehoek POA 'n gelykbenige driehoek.	
Hoek O1 = Hoek A1 + Hoek P1	Buitehoeke van 'n Δ.
Hoek O2 = Hoek B1 + Hoek P2	Buitehoeke van 'n Δ.

Omdat Δ POB gelykbenig is sal hoek P_2 en hoek B_1 dieselfde grote wees daarom sal hoek O_2 twee keer die grote van hoek P_2 wees.

Omdat Δ POA gelykbenig is sal hoek P_1 en hoek A_1 dieselfde grote wees daarom sal hoek O_1 twee keer die grote van hoek P_1 wees.

Gevolgtrekking: As hoek O_1 en hoek O_2 saam getel word is dit dieselfde as twee keer hoek P_1 by twee keer hoek P_2 getel word.

Figure 6:19 Student 1's responses to practical assignment



Stelling: Die hoek wat 'n boog van 'n sirkel by die middelpunt van die sirkel onderspan, is twee keer die hoek wat deur die boog by die omtrek onderspan word. Dit wil sê die middelpuntshoek is twee maal die omtrekshoek. (hoek AOB= 2 hoek APB)

b. Stel Q is die punt op die sirkel soos aangedui. Geld die stelling in a. ook wanneer P in die posisie van Q is? Verduidelik u antwoord.

Figure 6:20 Student 2's responses to practical assignment 2

It is worth noting that Student 2 is still at Level 2 of the van Hiele theory concerning geometric thought. This is so because from Figure 6.21 I noted that the responses provided by Student 2 still resonate with the properties of geometric thought, within the context of making observations and conjectures instead of the formal axiomatic system. Hence I note that the CLP did not have an impact on Student 2‟s conceptual understanding of Euclidean geometry.

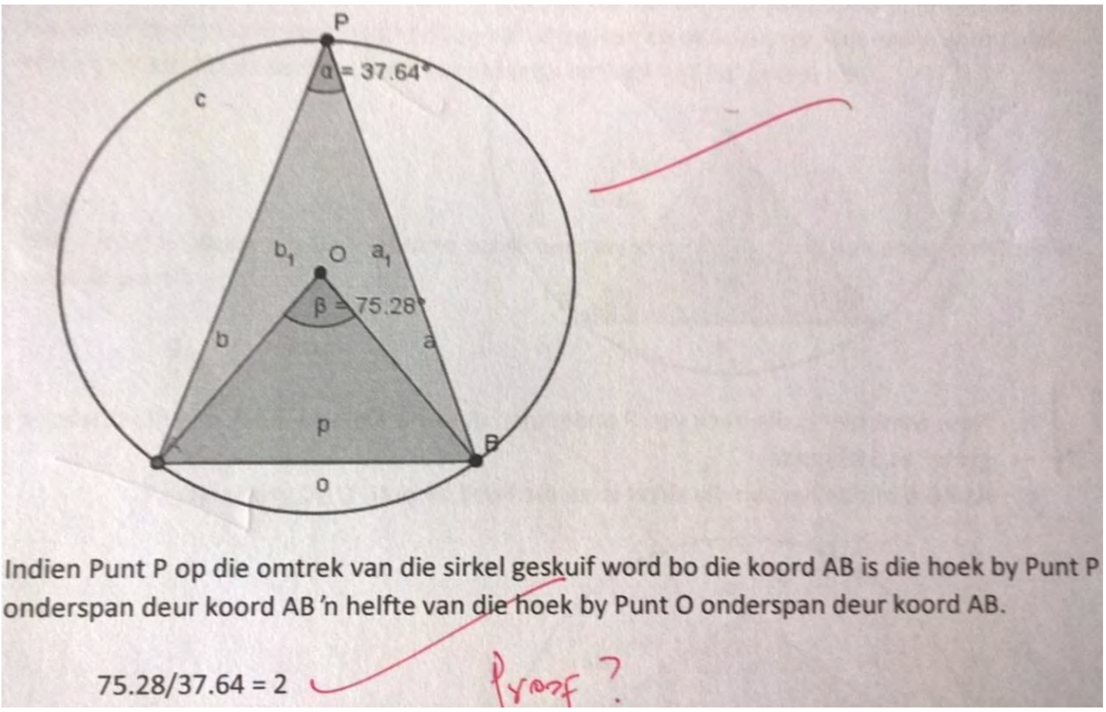


Figure 6:21 Student 3's responses to practical assignment 2

It is worth noting that Student 3 is still at Level 2 of the Van Hiele theory concerning geometric thought. This is so because from Figure 6.22 I noted that the responses provided by Student 3 still resonate with the properties of geometric thought, within the context of making observations and conjectures instead of the formal axiomatic system. Hence the CLP did not have an impact on Student 3's conceptual understanding of Euclidean geometry.

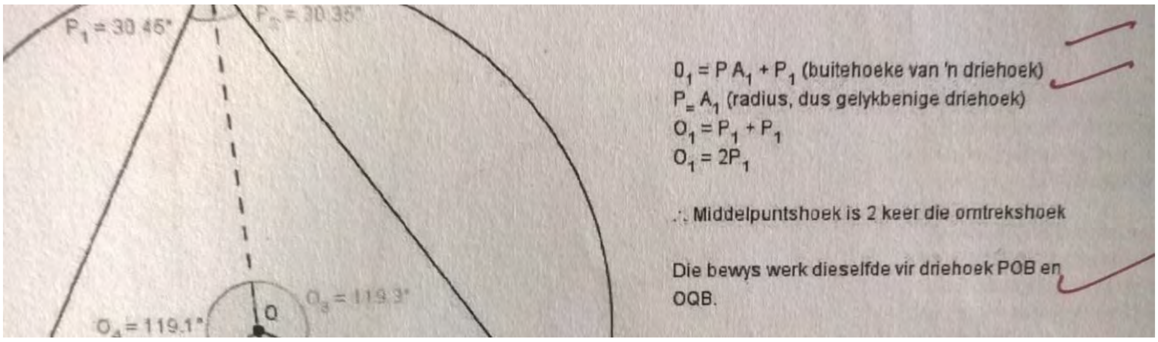
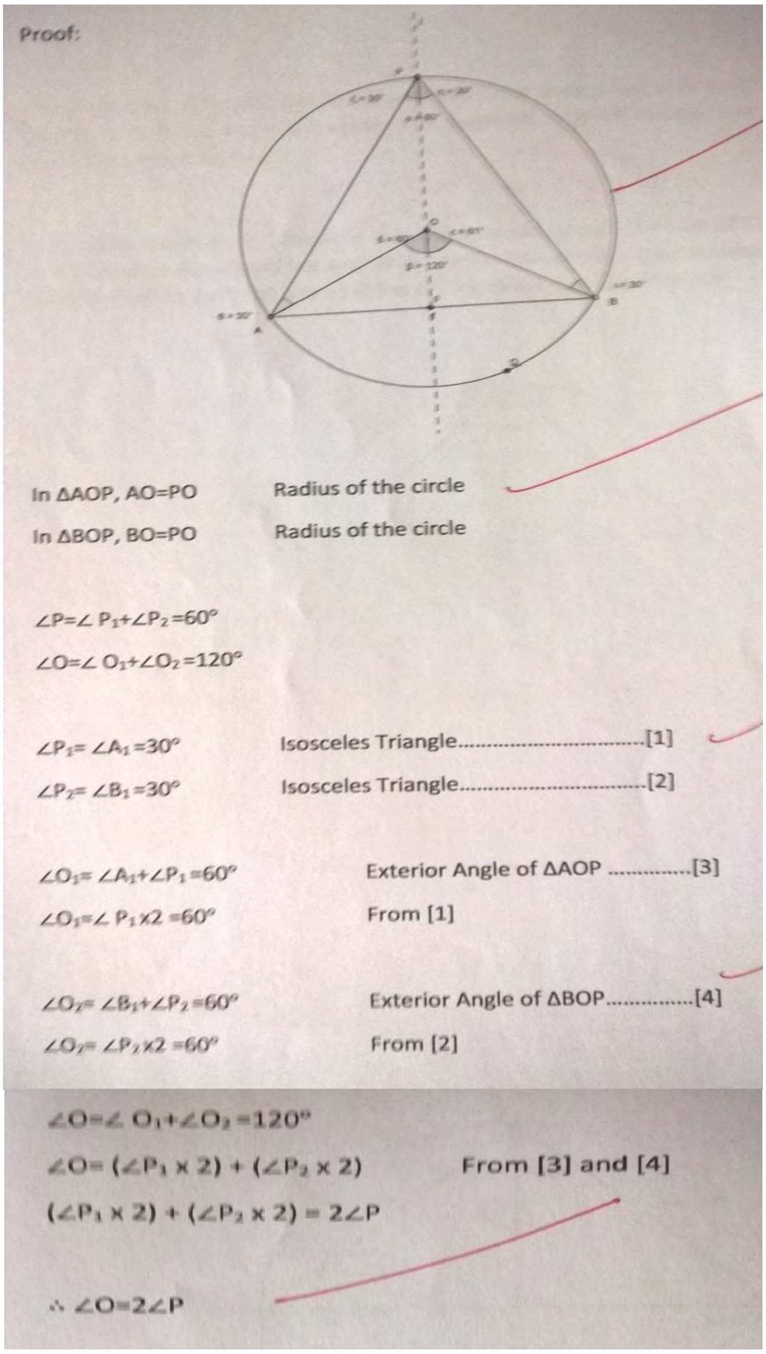


Figure 6:22 Student 7's responses to practical assignment 2

It is worth noting that Student 7 (the student who argued that someone had to stand in front of and show her basics) has moved from level 2 to level 3 of the Van Hiele theory concerning geometric thought. This is so because from Figure 6.23, it is clear that the responses provided by Student 3 resonate with the properties of geometric thought, within the context of making observations and conjectures as well as the formal axiomatic system. Hence the CLP did have an impact on Student 3's conceptual understanding of Euclidean geometry.



Proof:

In ΔAOP, AO=PO Radius of the circle

In ΔBOP, BO=PO Radius of the circle

∠P=∠ P₁+∠P₂=60°

∠O=∠ O₁+∠O₂=120°

∠P₁= ∠A₁=30° Isosceles Triangle................................[1]

∠P₂= ∠B₁=30° Isosceles Triangle................................[2]

∠O₁= ∠A₁+∠P₁=60° Exterior Angle of ΔAOP ....[3]

∠O₁=∠ P₁x2 =60° From [1]

∠O₂= ∠B₁+∠P₂=60° Exterior Angle of ΔBOP...............[4]

∠O₂= ∠P₂x2 =60° From [2]

∠O=∠ O₁+∠O₂=120°

∠O= (∠P₁ x 2) + (∠P₂ x 2) From [3] and [4]

(∠P₁ x 2) + (∠P₂ x 2) = 2∠P

∴ ∠O=2∠P

Figure 6:23 Student 12's responses to practical assignment 2

In Task 4 results indicated that Student 1, 2, and 12's understanding of circle geometry (proof) was at level 2. Contrary to their performance in Task 4, their performance in the practical assignment 2 indicates that their understanding of Euclidean geometry has moved to level 3 of the van Hiele theory of geometric thought. This is an indication that the CLP did have an impact on these students' understanding of the axiomatic system concerning proof.

In the following sections I establish a discussion on session 5, 6, 7, and 9. In connection to this, students' understanding of Euclidean geometry (proof) seemed to be at level 3 (deductive) of the van Hiele theory of geometric thought. This means that the students' understanding of Euclidean geometry ranged from moderate understanding to complete understanding. Henceforth the CLP had a positive impact on their understanding of Euclidean geometry, particularly in mathematical activities that involve constructing geometric shapes, discovering relationships, making conjectures and proving theorems.

6.1.1 Discussion

In light of the preceding paragraphs it can be seen that the CLP did have a positive impact on Students' 1, 2, 3, 7 and 12's conceptual understanding of Euclidean Geometry. But, it is important to note that towards the end, one of these students (Student 7) was adamant that she could learn about Euclidean geometry only if someone had shown her fundamental basics. She liked to work alone, but she had the courage to ask questions or provide direction when necessary. What happened during the sessions is that each student worked individually on his/her own computer, but they were encouraged to work together throughout the CLP, hence most of their responses are similar. Herein students ask questions, compare solutions, and share technical skills where necessary.

6.2 Students' Self-Directed Learning abilities: During and after the Collaborative Learning Programme (CLP)

The rationale behind conducting these self-directed learning instruments was to answer the following question:

> *What is the influence of a collaborative learning programme on final-year mathematics education students' self-directed learning abilities concerning mathematics?*

Main domain of the SDLI	Sub-domain 1	Sub-domain 2	Sub-domain 3	Sub-domain 4	Sub-domain 5
Learning Motivation	2 Down	2 Down	4 Down	2 Down	0 Down
	7 No change	7 No change	4 No change	4 No change	8 No change
	2 Up	3 Up	3 Up	5 Up	3 Up
Planning and Implementing	2 Down	3 Down	3 Down	2 Down	3 Down
	3 No change	5 No change	5 No change	8 No change	5 No change
	6 Up	3 Up	3 Up	1 Up	3 Up
Self-monitoring and Interpersonal	2 Down	2 Down	1 Down	0 Down	2 Down
	6 No change	7 Up	5 No change	9 No change	4 No change
	3 Up	2 Up	5 Up	2 Up	5 Up
Communication	0 Down	3 Down	3 Down	2 Down	5 Down
	5 No change	5 No change	1 No change	7 No change	6 No change
	6 Up	3 Up	7 Up	2 Up	1 Up

Table 6:1 A representation of students' responses about their Self-Directed Learning abilities

6.2.1 Discussion of students' self-directed learning abilities

Learning motivation is the first domain of the Self-Directed Learning Instrument, and herein there are five sub-domains, based on learning motivation. It is evident that for sub-domains 1, 2, 3, 4 and 5 (see Appendix R), students' responses indicate a need to learning motivation. Herein the learning abilities of the students regarding the issue of learning motivation range from seldom to always, with sometimes in between. From Table 6.1 I note that most students (seven, seven and eight) recorded consistent responses concerning the sub-domains (1, 2, and 5) respectively. That is to say the responses that the students recorded in the pre-SDLI did not change when compared to the responses they provided in the post-SDLI. But it is important to note that with regard to sub-domain 4 (success and failure), five students recorded responses that went up by either 1 or 2 units in the post-SDLI scale. With regard to the sub-domain 3 (inner drive), four students recorded responses that went down by either 1 or 2 units in the post-SDLI scale. With regard to the sub-domain 4 (success and failure), four students recorded responses that went down by either 1 or 2 units in the post-SDLI scale respectively.

Planning and implementing is the second domain of the Self-Directed Learning Instrument, and herein there are five sub-domains. It is evident that for sub-domains 1, 2, 3, 4 and 5 (see Appendix

R) students' responses indicate a need to take responsibility for their own learning. Herein the learning abilities of the students regarding the issue of planning and implementation range from seldom to always, with sometimes in between. From Table 6.1 it can be seen that most of the students (five, five, eight and five) recorded consistent responses concerning the sub-domains (2, 3, 4 and 5) respectively. The responses that the students recorded in the pre-SDLI did not change compared to the responses they provided in the post-SDLI. But, it is important to note that, with regard to sub-domain 1 (not give up), six students recorded responses that went up by either 1 or 2 units in the post-SDLI scale. With regard to sub-domain 3 (learning goals, learning strategies and the selection of learning methods) three students recorded responses that went down by either 1 or 2 units in the post-SDLI scale respectively. On the one hand, in relation to sub-domain 1 and 4 (not give up and responsibility for learning), two students recorded responses that went down by either 1 or 2 units in the post-SDLI scale respectively.

Self-monitoring is the third domain of the Self-Directed Learning Instrument, and herein there are five sub-domains. It is evident that for sub-domains 1, 2, 3, 4 and 5 (see Appendix R), students' responses indicate a need to monitor their own learning. Herein the learning abilities of the students regarding the issue of monitoring their own learning range from seldom to always, with sometimes in between. Table 6.1 shows that most of the students (six, five and nine) recorded consistent responses concerning the sub-domains (1, 3, and 4) respectively. Thus the responses that the students recorded in the pre-SDLI did not change compared to the responses they provided in the post-SDLI. Herein the learning abilities of the students regarding the issue of monitoring their own learning range from seldom to always, with sometimes in between. But, it is important to note that with regard to sub- domain 2 (finding resources), seven students recorded responses that went up by either 1 or 2 units in the post-SDLI scale. With regard to the sub-domain 1, 2 and 5 (planning and managing, finding resources and learning progress) two students recorded responses that went down by either 1 or 2 units in the post-SDLI scale respectively.

Communication is the fourth and last domain of the Self-Directed Learning Instrument and herein there are five sub-domains. For sub-domains 1, 2, 3, 4 and 5 (see Appendix R), students' responses indicate a need to communicate their own learning. Herein the learning abilities of the students regarding the issue of communicating their own learning range from seldom to always with sometimes in between. Table 6.1 shows that most of the students (five, five, seven and six) recorded consistent responses concerning the sub-domains (1, 2, 4 and 5) respectively. The responses that the students recorded in the pre-SDLI did not change compared to the responses they provided in the post-SDLI. However, with regard to sub-domain 1 and 3 (learning progress as well as the culture and other languages), students recorded responses that went up by either 1 or 2 units in the post-SDLI scale respectively. With regard to the sub-domain 2 and 3 (interaction as well as the cultures and languages of others) two students recorded responses that went down by either 1 or 2 units in the post-SDLI scale respectively. In the sub-domain 5 (written communication), five students recorded responses that went down by either 1 or 2 units in the post-SDLI scale respectively.

In connection to the preceding paragraphs I note that most of the students' learning abilities toward mathematics range from seldom to always with sometimes in between. However it is important to note that over the course of the CLP students were constantly maintaining that it was important for someone to stand in front of them and show them how to use the Geogebra® program and provide them with basics about geometric content. Seven students maintained that they seldom or always located resources for learning mathematics, but it was not the case when it came to the ten lessons that they were supposed to go through in order to learn more about the Geogebra® program. During the CLP, students were asked if they did go through the lessons, and the response was negative. Another interesting aspect about students' learning abilities has to do with the learning about other cultures as well as the languages of other people. In the pre-assessment eight students maintained that they never or sometimes wanted to learn about other cultures and languages, but in the post- assessment the same group of students noted that they seldom or often, with sometimes in between, wanted to learn about other cultures and languages. Again, it is important to note that, contrary to what students claim about their learning abilities toward school mathematics, out of 11 students who responded to the SDLI, only six managed to respond to the open-ended questions.

6.3 Discussion of students' views about the teaching of mathematics:

In order to investigate students' fundamental views about teaching and learning of mathematics, the research team administered an open-ended survey which comprised four questions. Out of twelve students only six responded to the open-ended survey. Three of these students, Students 1, 7 and 12's understanding of Euclidean geometry was discussed in the preceding paragraphs. This implies that the other three students, (6, 9 and 11) did not form part of the discussion because they did not complete all of the tasks that were implemented in the CLP. Student 2 and Student 3 did not complete the open-ended survey concerning the teaching of mathematics. Hence, in the following paragraphs I only discuss the views of Students 1, 7, 12, 6, 9 and 12 concerning the teaching of mathematics. The students provided responses in Afrikaans, hence the researcher had to translate these responses to English. In conducting data analysis, I heavily depended on content analysis (see 4.7). The aim of an open-survey was to answer the following question:

What are final-year mathematics education students' fundamental views about teaching and learning of mathematics?

The following Table (6.1) outlines the approach that was undertaken during the process of the analysis. Herein I created codes directly from students' responses, hence categories were established.

Codes: Student 9	**Codes: Student 1**	**Codes: Student 11**	**Codes: Student 12**	**Codes: Student 6**	**Codes: Student 7**	**Categories**
	What is mathematics?					
Important role In everyday life Not just abstract Discover concepts Play games Maths is fun Everyone uses maths	Formula Rules Symbols Language Talk and learn Integrated Problem-solving Application Make life easy Helps me in the workplace	Language People must learn World of numbers Solve real-life problems Develops the brain of the learner Learn how to solve problems Not just memorization	Way of thinking Develops cognitive skills Skills to think analytically Solve problems See relations and patterns Not just formulae and rules	Human activity Make sense of life A way to solve problems It appears in everyday life Assist people in life New thinking for discovery and design	Mathematics deals with real-life situations Different techniques, methods, skills and critical thinking	Mathematics plays an important role in our daily lives and it helps us. Mathematics is a language that must be spoken and learned. Mathematics helps us to develop cognitive skills Mathematics is not just about formulas, procedures and rules.

	What does it mean to teach mathematics effectively?						
Teach learners to survive In a world full of number Logical reasoning Teach must use Everyday examples Stay updated with current changes	Know what is mathematics Learners must feel math Living subject and new language Knowledge of teaching Teach learners skills Problem solving Critical thinking Calculation skills Offer learners with the opportunity to engage in discovery of concepts	Not just memorize Leaners must discovery concepts Use real-life examples Everyday situations Learner needs	Leaners‟ strength and strategies Self-discovery/ construct their own learning Do not give learners rules/ ask them to practice Mathematics in the real and relevant contexts The “why” and “how” questions are important.	So that learners develop necessary skills To solve problem Not teaching just rules and procedures Opportunity and space for learners to create their own knowledge Experimenting Teacher is the facilitator	Teacher must inspire: Learners to be curious and love mathematics Learners must think for themselves Learners must engage in self-discovery Mathematics must be practical Skills and concepts Enough exercises	Teacher should play the role of a facilitator. The teacher should not show the learners how to learn mathematics The teacher should set an example The teacher should possess content knowledge and know how learners learn. Teaching and learning of mathematics should begin with the learner not the teacher (promote self-discovery).
	How should geometry be taught?						
Assist with skills in the world Problem solving development Important in other fields Used in other professions Pretty and fun	Absolutely investigation Learners must be involved in their own learning Not just memorization Teacher is the facilitator Learner does all the thinking	Learner discover theorems for themselves	Proper foundation should be laid at the primary school Practically doing geometry	Experimenting and self-discovery Teacher must not just teach learners theorems and rules Learners discover theorems themselves Many exercises and real-life	Teacher must introduce basic concepts Before introducing a program like Geogebra Students must introduce statements Then students discover and make senses	Learning of geometry should be based on investigations /experiments. Learners need to construct and discover relationships and theorems for themselves but not learn how to memorize theorems and

				examples	But the teacher must act as a facilitator	rules. Teacher should play the role of a facilitator. Firstly proper foundation about geometry should be laid in the primary school. The teacher should not show the learners how to solve and proof theorems.

Table 6:2 Codes and categories about the nature of mathematics and geometry

From Table 6.1 it can be seen that students' views about the nature of mathematics are firmly rooted in the problem-solving perspective of mathematics (see par 2.1.4). Herein, mathematics is viewed as a human activity as well as the language which people engage with when they want to solve real-life situations. Also, students see mathematics as a school subject that is fun and contributes to the development of higher thinking order skills. Students went on to associate mathematics with formulas, skills and procedures, but they caution that these skills should not be given priority over real-life problems.

In relation to the teaching and learning of mathematics, students seem to claim that the role of the teacher should be that of a facilitator, and that the teaching and learning of mathematics should begin with the learner. The teacher should possess knowledge of mathematics, as well as know how learners learn mathematics. Herein, students argue that the teaching of mathematics should be in harmony with the construction and discovery of mathematical ideas, instead of a teacher showing learners how to do mathematics. With respect to the teaching and learning of geometry, students maintain that geometry teaching should be investigative in nature. It should promote the construction and development of theorems, and offer the learners the opportunity to develop their own mathematical knowledge (see par. 2.1.4). In light of this I note that students' views about the nature as well as the teaching of mathematics (geometry) are rooted in the problem solving view of mathematics.

Referring to Table 6.1, students were asked the question “what is the most important thing that you think you have learned from the CLP?” According to Student 1, the CLP helped her to note many ways where she can use mathematics as well as the concepts she did not know before the CLP. This is in line with her views concerning the teaching of mathematics (see Table 6:1), and also in accordance with her conceptual understanding of Euclidean geometry. Initially Student 1’s conceptual understanding of Euclidean geometry was at level 2 of the Van Hiele theory of geometric thought, but after the CLP it moved to level 3. In light of this, Student 1 maintained**:**

> *“This project helped me to see many ways I can use math. I definitely learned a lot and it learned me math concepts I didn’t know.”*

According to Student 9, investigating the properties of Euclidean geometry is a challenging activity, but the results are worthwhile and useful. Student 10 maintains that investigating the properties of Euclidean geometry deepens one’s understanding. This is in line with her views concerning the teaching of Euclidean geometry (see Table 6:1). In connection to this she noted the following statement with respect to what she learned from the CLP:

> *"Self-discovery is not always easy, but it works !!!!! In the process of discovery one is often discouraged, but the results are worth it. Another thing I learned is that discoveries deepen your understanding.”*

Student 11 maintains that she learned an approach on how to offer learners learning opportunities concerning the properties of Euclidean geometry, and that Geogebra® can assist students to solve Euclidean geometry tasks. This is in line with her views concerning the teaching of Euclidean geometry (see Table 6:1).

> *"The way to offer geometry for the learners an opportunity where they can discover the composition and more understand why a certain theorem. And how to effectively Geogebra can use in the classroom to help students with geometric problems."*

Student 12 only responded to the post-assessment of the SDLI, she was not part of that session from which the pre-assessment of the SDLI was administered and she also missed two other sessions; that means she attended seven of the ten sessions of the project. Student 12 really appreciated the CLP because she also moved from level 2 to level 3 of the Van Hiele theory of geometric thought. This is in line with her views concerning the teaching of Euclidean geometry (see Table 6:1). After the CLP she noted the following response concerning what she learned:

"I learned about how assumptions form the basis of geometry and that all assumptions and proofs are in some way linked and build on one another (for a particular shape or topic). Before this project all proofs were separate and random entities I had to try and remember but now I can make a relationship evident to my learners one day...with some practice and help from more exprienced teachers."

Regarding the teaching and learning of geometry she is not certain that the CLP offered her an opportunity to learn about how to teach geometry. She made the following remark:

"I don't have an efficient answer to this because I have not really learned much about how to present geometry at schools and I didn't experience effective education in geometry when I was at school myself. All I am willing to say is that if a proper foundation in geometry is not laid down in primary school, it is very hard for a high school student to keep up and for a high school teacher to get that learner on track. From the project I have noticed that practically doing geometry was the aim as well as building on assumptions and proofs but I did not experienced that this helped me much with the geometry at hand so I cannot say (until I have applied it in a school situation) that it would be an effective way to teach my future learners."

Student 6 did not learn a lot from the CLP. In fact she does not say anything about what she learned. Her main concern is that Researcher B was of no assistance because he could not answer some of the questions. Instead of helping them, Researcher B noted that the tasks were based on investigations. Student 6 says that she struggled with the use of Geogebra program because it was not discussed in detail with them from the beginning of the CLP. Also there was no textbook for them to use regarding the learning of the geometric content. Student 13 prefers direct instruction from which someone teaches her the content and provides her with learning material. This is contrary to her views about the teaching of Euclidean geometry (see Table 6:1). In connection to this, Student 6 noted the following concerning what she learned in the CLP.

"To be honest I do not really learned a lot from this project. It was to a large extent very frustrating for me, [Researcher B] few of our questions in class could not answer. We really struggled to use Geogebra because the basic functions of the program have not been discussed with us in detail from the beginning. Also, we usually get the same answer to a question, saying, "this self-examination, I cannot help you." We have not had a book in which we could read

up and it's good that I've long done back."

Student 7 really appreciated the CLP because she also moved from level 2 to level 3 of the Van Hiele theory of geometric thought. This implies she did learn something about geometry, but her main concern is that it is important for one to have basic rules and insight, because she thinks that discovering geometric ideas is difficult. Over the course of the CLP, Student 7 was constantly raising such views. She did appreciate to discover geometric properties on her own. When asked about how geometry should be taught, she asserted that the teacher should not play the role of the facilitator; instead, the teacher should first provide learners with basics of geometry before beginning with investigations. To her, mathematics involves the solving of real-life situations by means of different methods, techniques and skills that require critical thinking skills. Student 7(the student who liked to work individually over the course of the CLP) had this to say about her learning experiences:

> *"Circle geometry is very interesting and a harder but still is a nice topic, but you only need the basic rules and understanding of it to interpret and use the statements. Discovery can only take place if you have a basic to know in which direction one should think".*

Considering the preceding responses I can claim that the students reacted positively to the CLP. Henceforth they argue that they have come to appreciate the notion of discovering geometric properties for themselves. The students appreciated the fact that they had developed the understanding of Euclidean geometry through guided investigations. According to Student 9, geometric investigations are not always easy; in fact they are challenging, but at the same time worthwhile because they develop a deep understanding of Euclidean geometry. It is of paramount importance to note that Student 12 really learned a lot about the proofs of geometry because she noted that before the CLP she only viewed geometric proofs as separate entities, but at the end of the CLP she maintained that the proofs were linked to each other Student 11 mentioned that in the CLP she had learned about the approach of how to provide opportunities to learners to enable them to discover geometric properties for themselves by means of the Geogebra® program. However, not all of them appreciated the CLP, because one of them indicated that that Researcher B was not helpful and that there were no textbooks for them to use. She also mentioned that the use of the Geogebra® program over the course of the CLP was a bit challenging for her, considering the fact that she had not used it before. Though some of the students did not attend regularly, the data reveals that the students did learn a lot from the CLP (see par 6.2 and 6.4). Students really appreciated the CLP and reacted positively to it because they were part of it since the first session until the last session. Their performance on module assignment 2 clearly supports this claim. Because they were able to use the knowledge that they gained in the CLP to complete their assignment, they completed it exceptionally well (see par 6.2).

6.4 Conclusion

The central purpose of this chapter was to establish the analysis and interpretation of the data gathered for the study (Case 2). The data analysis appealed to the research questions as well as the research objectives of the study. The results showed that the CLP did have an impact on students' understanding of geometry. Considering the above, one can claim that students reacted positively to the CLP. They appreciated the mathematical tasks that they were engaged in, and in doing so they were able to appreciate that part of the geometry structure that deals with axiomatic systems. Also, the results show that the students' fundamental views about the nature of mathematics are firmly rooted in the problem-solving view of mathematics. Case I refers to the students who managed to complete all of the tasks that informed this study: Students 1, 2, 3, 7 and 12 respectively. It is important to note that Student 1 did not complete Task 2, and Student 2 as well as Student 3 did not complete the open-ended survey. Students' Self-Directed Learning abilities seem to range from 3 (that is sometimes) to 5 (that is always) with 4 (that is agree) in between. Despite indications of a positive influence from the CLP, students' reluctance to find ways and information (e.g. from the manuals, the library) for themselves as to be able to engage with and resolve the tasks seem to indicate that those students still tend to be dependent on an external authority in their learning (contrary to SDL expectations), and that their participation in the CLP did not change their disposition.

In chapter 7 I shall address the research questions, and put forward the limitations of the study and recommendations for future research.

CHAPTER 7: SUMMARY AND RECOMMENDATIONS

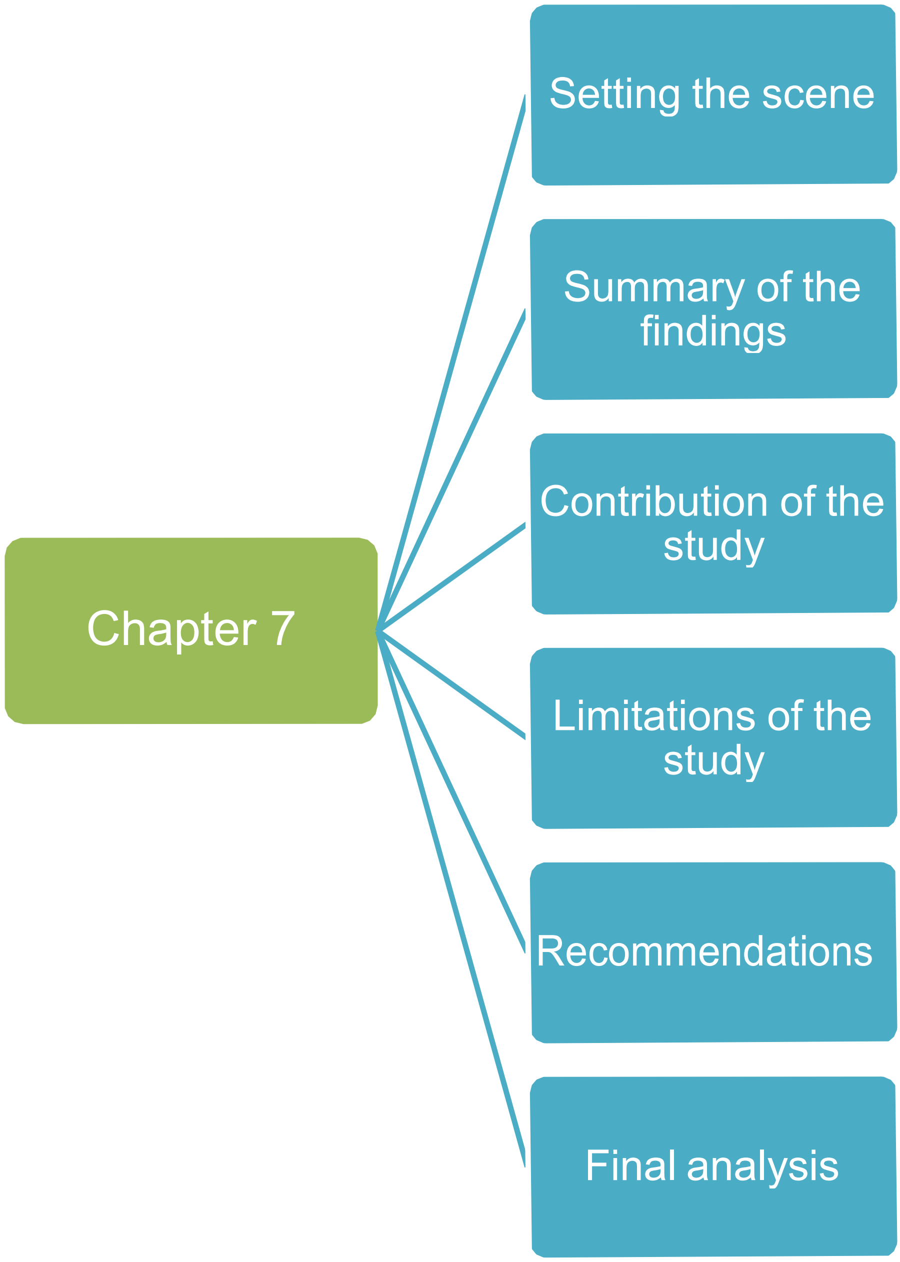


Figure 7:1 Outline of Chapter

7 Setting the scene

What was the investigation all about? This section aims to answer that question by firstly providing an overview of the contents of each chapter. Secondly this chapter will answer this question by addressing the findings emanating from the study. Thirdly, this chapter deals with a discussion of the contributions that this study might have on the community of mathematics education research. Lastly, this chapter ends with a discussion of the limitations of the study and the recommendations are put forward.

7.1 Summary of chapters

Chapter 1 focused on a general overview of the study, including background to the problem statement and intellectual conundrum for the study. This chapter also included the research problem, research questions, and the purpose of the research and definition of concepts.

Chapter 2 addressed the conceptual framework for the study by providing relevant literature exploration with regard to information on the nature of mathematics (teaching and learning), teacher collaboration in education and teaching proficiency for school mathematics.

Chapter 3 was based on the learning and teaching of the function concept and Euclidean geometry. Herein, philosophical issues pertaining to the learning of both the function concept and geometry were addressed.

The focus of chapter 4 was to describe the research process in detail, including the research design and methodology that guided the study.

Chapter 5 highlighted the analysis of data and the results of the study. Herein, the intention was to address issues emanating from Case 1 of the study, and chapter 6 addressed the analysis of data and the results of the study pertaining to Case 2 of the study.

Chapter 7 summarized the findings of the study and present conclusions drawn from the study. Also, this chapter addressed the limitations of the study and recommendations for future research are made.

7.2 Summary of findings

7.2.1 Addressing the research questions

This section reports the results of the study. I discuss the results in relation to teachers' and students' views about the nature of mathematics as well as the teaching and learning of both the function concept and Euclidean geometry. Further I elaborate on teachers' understanding of the function concept as well as the students' understanding of Euclidean geometry. Also, I discuss the implications of the results for mathematics education research.

7.2.1.1 What is the influence of a collaborative learning programme on teachers' fundamental views about the teaching and learning of mathematics?

Data regarding teachers' fundamental views about the teaching and learning of mathematics were gathered by means of an open-ended survey that was implemented before and after the CLP. The CLP did not have any impact on Teacher A's views about the teaching and learning of mathematics or the teaching and learning of the function concept. Before, during and after the CLP her views about the nature of school mathematics are still associated with the traditional mathematics teaching rooted informed by the instrumentalist view of mathematics. These views were observed throughout the CLP. Though the CLP did not have an impact on her views about the teaching and learning of mathematics, it is evident from the above analysis that Teacher A reacted positively to the CLP. Her willingness to participate in it from the first session to the last session speaks volumes about how she appreciated the CLP. She had the courage to ask and answer questions during the entire course of the CLP. She embraced the sharing of information between the research team and teacher. Consequently she was willing to participate in the future project.

The CLP did not have an impact on Teacher B's views about mathematics. After the CLP his definition of mathematics was similar to the one he provided before the CLP. He still holds the instrumentalist view about mathematics. But, it is important to note that the CLP did have an impact on his view regarding the teaching of mathematics. Before the commencement of the CLP he believed that the teaching of mathematics should be teacher centred. However, after the CLP he had a different view regarding the teaching and learning of the function concept, which was informed by the constructivist mathematics teaching. According to him, modelling real-life situations with functions should be the focus of the teaching and learning of the function concept. Connecting to this, the teaching and learning of the function concept should be learner centred, from which learners are encouraged to participate in class discussions and problem solving. Furthermore he believes that technology should be incorporated into the teaching and learning of the function concept. Also, before the CLP he believed that the function concept entailed graphs, but after the CLP he had a different view about the function concept; he believed it was based on the relationship between variables. His desire to attempt, communicate and share the information during the CLP speaks volumes in this regard. This is an indication that Teacher B reacted positively to CLP. Consequently he was willing to participate in the future project.

7.2.1.2 What are students' fundamental views about teaching and learning of mathematics?

To answer this question I rely on the findings of the open-ended survey that were administered at the end of the CLP. It is evident from the findings that students' fundamental views about the teaching and learning of mathematics are rooted in the problem-solving view of mathematics. Herein, students caution that the teaching and learning of mathematics should not only be confined to rules and procedures. They maintain that it is of paramount importance for the teacher to play the role of the facilitator and that the learner should do mathematics. I also noted that in their views about the teaching and learning of mathematical concepts, in this case geometry, they did not mention the use of technology. Compare this to the results of Case 1 from which technology was viewed as tool that enhances the learning of mathematics. This view can be compared to the views of Teacher B in Case 1 of the study.

7.2.1.3 What is the influence of a collaborative learning programme on teachers' conceptual understanding of functions?

In order to investigate teachers' conceptual understanding of the function the researcher used four function tasks. Also the CLP did not have an impact on Teacher A's understanding of the function concept. Before the CLP her conceptual knowledge of the function concept was at level 0 (that is no effort); she completely struggled to perform different transitions among different representational systems of the function concept, specifically those presented in words, graphs and numerically (table form). After the CLP, Teacher A did not appreciate that part of the function concept which involves real-life situations, in particular presented in words. Her understanding of the function concept has moved from level 0 (that is no effort) to level 1 (that is poor understanding). Teacher A indicated that before the CLP she had a negative attitude toward modelling with functions; to her the function concept deals with graphs and procedural skills. One positive thing was that at the end of the CLP she mentioned that she had gained interest regarding modelling real-life situations with functions. Though the CLP did not have an impact on her conceptual understanding of the function concept it did have an impact on her procedural knowledge of the function concept. This is so because she reacted positively in those activities that involved procedural skills and she learned an algorithm that she did not know about prior the CLP.

With regard to the procedural knowledge of the function concept, the CLP did have an impact on Teacher B's procedural knowledge of the function concept. His procedural knowledge about the function concept did change. Initially it was at level (3) but after the CLP it moved to level 4 (complete understanding). Prior to the commencement of the CLP his understanding of the function concept was at level 2 (that is poor understanding) but at the end of the CLP his conceptual understanding of the function concept was at level 3 (moderate understanding). He was able to perform different transitions among different representational systems of functions, in particular those that involved all four components of the function model, namely modelling, interpreting, translating and reifying with functions. It is evident that the CLP provided both the teachers and research team the opportunity to engage in learning opportunities pertaining to the function concept. These learning opportunities were offered in the context that favoured both the procedural and conceptual knowledge of the function concept. In this context teachers were encouraged to reflect on all the activities that they were dealing with. And they achieved this through communication because throughout the CLP they were either talking to each other or to the research team. Also, they were encouraged to complete mathematical tasks together. In addition, the context from which the CLP occurred involved the use of technology in the learning of the function concept. In short, the CLP occurred in the environment that was in harmony with the problem solving view of the teaching and learning of mathematics. And, most notably, the CLP occurred in the working place of teachers, in this case a township secondary school. In short, the CLP involved the content, collaboration and context.

7.2.1.4 What is the influence of collaborative learning programme on students' conceptual understanding of Euclidean geometry?

With regard to students' understanding of geometry, I use the findings emanating from geometry tasks that the students completed during the course of the CLP. The results indicated that students' understanding of geometry had improved from the level of informal deduction (2) to the level of deduction (3) of the van Hiele theory of geometric thought. An important observation is that the students' understanding of geometry only came evident in the practical assignment that they had to complete. Over the course of the CLP their understanding of geometry was at the level of informal deduction. In other words, there was no more need to work with the axiomatic system of geometry. They only showed appreciation for making conjectures and providing justifications, but were not able to prove their conjectures. In connection to this, one can say that the CLP had a positive impact on students' understanding of geometry.

7.2.1.5 What is the influence of a collaborative learning programme on students' self- directed learning abilities?

Data indicate that the CLP did not have an impact on students learning abilities because there were too many inconsistences on students' responses and also most of them provided responses that were consistent. Those who provided inconsistent responses were those students who went down by 1, 2 or 3 on the scale of the SDLI. On the other side, those students who provided consistent responses were those who did not change their responses on the scale of the CLP. Another group of students provided responses that went up either by 1 or 3 on the scale of the SDLI. Even though students maintained that they sometimes or always or often take responsibility for their own learning of mathematics, this was not apparent over the CLP because most the students were constantly saying that they needed someone to stand in front of them and teach (show) them geometry. I noted that there was no need for them to learn things on their own. For instance, they did not take time and go through the ten lessons about how to use the Geogebra® program; Researcher A had to spend time on two successive sessions demonstrating and showing them how the program works.

7.3 Contributions of this study

The study indicates how a collaborative learning programme can improve teachers' and students' conceptual knowledge in relation to the specific key mathematical concepts, viz *the concept of* function and Euclidean Geometry. This study may lead to the design and implementation of a collaborative learning programme for secondary school mathematics teachers as well as primary teachers. This could then be ultimately used as a means to strengthen teaching and learning of mathematical concepts in South African schools. Hence, this collaborative learning might enhance collaborating teachers' mathematical knowledge for teaching. In addition, this study might serve as a research tool that can inform the community of mathematics education research about teachers' existing and acquired practices. Lastly, this study serves as a means to enhance the researchers' understanding of the complexity of teacher learning as well as the nature of school mathematics. However, it is important to note that the results of this study will be valid for the selected participants only and cannot be generalised, and that further studies will be necessary before it can be used as a means to strengthen teaching and learning of mathematical concepts in South African schools.

7.4 Limitations of this study

7.4.1 Generalizability

Since the study was qualitative in nature. It is important to acknowledge that the results of the study cannot be generalised because it was a multi-site case study involving two different areas with different participants. In Case 1, we only worked with two township secondary school mathematics teachers who came from two different schools. On the other case we worked with twelve fourth year BEd students who were training to become mathematics teachers. Also in Case 1, we were working in a township at a school and the mathematical content involved the function concept. While in Case 2 we worked at the university and the mathematical content was based on Euclidean geometry. Consequent the aim of the study was not to generalize. Instead the aim of the study was to investigate how engaging mathematics teachers in a collaborative learning may improve their teaching proficiency in teaching mathematics. In connection to this, the results of this study were discussed and compared in association with what the literature tells us regarding the objectives of the study as set out in paragraph 1.5.

7.4.2 Succession of sessions

In both Case 1 and 2 we encountered challenges about completing the ten sessions successively. In case 1 we had to wait for about six weeks to attend to session 6. In case 2 we had to wait for about eight weeks in order to attend to session 5 and another three weeks to attend to session 6. These unintended interruptions might have had an impact on the findings of the study.

7.4.3 Teacher-participation

When we started the project we expected the study to involve township secondary school mathematics teachers coming from two different areas. We consulted all the relevant stakeholders, and important meetings were scheduled. In one case 25 teachers agreed to participate and in the other case fourteen teachers said that they would take part. In the area whereby 25 teachers agreed to take part in the project, the research team struggled to get hold of teachers. Teachers did not pitch for the first time. Another effort was made to meet with the teachers, and once again teachers did not pitch. Then there was a school principal from the same township who contacted us and offered us the opportunity to work with the mathematics teachers working at her school. We were offered this opportunity based on the fact that the school wanted help from us; they expected us to help their grade 12 learners with geometric content. We agreed and we stood by our agreement; we did indeed assist their learners. Consequently, we managed to hold the first session with five teachers. In our second session, only three teachers pitched and in our third session no teachers pitched. In Case 2 we ended up working with two teachers out of the five teachers who attended the first session. Herein, we started working with two teachers from the third session onwards. Then we got the opportunity to work with university students, but only six students were part of it from the first until the last session. Other students were not consistent with respect to their attendance.

7.5 Recommendations

In light of the literature study and the empirical results the following recommendations can be useful concerning a Collaborative Learning Programme among mathematics teachers or students:

- A Collaborative Learning Programme among teachers or students involves the context, content and collaboration. That is to say a collaborative learning programme must offer the teachers or students the opportunity to engage in mathematical activities within the context of problem solving from which teachers or students are encouraged to work together, talk and share experiences concerning mathematical concepts as well as the teaching and learning of these concepts.
- A Collaborative Learning Programme should offer teachers or students the opportunity to make decisions with respect to the selection of the mathematical content as well as the physical setting of the CLP.
- Teachers reacted positively to the CLP and they find it interesting. In light of this teachers indicated that a collaborative learning programme should be an on-going process that promotes reflection among teachers concerning their own learning.
- A collaborative learning programme should incorporate the use of technology (for example, the Geogebra® program) in the teaching and learning of mathematical concepts.
- A collaborative learning programme has the potential to develop both teachers' conceptual and procedural knowledge concerning mathematical concepts hence the CLP offers teachers numerous learning opportunities and it challenges them to reflect on their own teaching practices.
- Future research should investigate the impact of the CLP within the context of other school subjects as well as with primary school teachers.
- Future research should ensure that more teachers are encouraged to participate in the CLP.

7.6 Final analysis

The purpose of this study which emanated from aspects (see section 1.1) pertaining to the teaching and learning of mathematics in the South African schools was to investigate how engaging mathematics teachers in a collaborative learning programme can improve their proficiency in teaching mathematics. The CLP had a positive impact on both teachers' and students' conceptual understanding of mathematical concepts, in this case the function concept and geometry.

With regard to the views about the teaching and learning of mathematics, the CLP only had a positive impact on one teacher, the CLP did not have an impact on another teacher's views, because no change was noted over the project. The latter teacher maintained that he attempted to do things differently in his classroom and that it was helpful to the learners. On the other hand, the former teacher indicated that she continued to teach the learners the way she did before the CLP, yet she maintained that she gained a lot from the CLP. It is also important to note that this is the same teacher who appreciated that part of the function concept that dealt with procedural knowledge.

In relation to students' views about the teaching and learning of mathematics, one cannot claim that the CLP did have an impact, because we only administered open-ended questions after the CLP. But it is important to note that according to students' responses their views about the nature of mathematics are in line with the constructivist views about the teaching and learning of mathematics. Also, results indicate that students take responsibility for their own learning of mathematics. Contrary to their responses, I noted that over the course of the CLP they were constantly arguing that they wanted someone to stand in front of them, and show them, teach them. They did not attempt to go through the manual that was supposed to help them about how to use the Geogebra® program. Henceforth the CLP did not have any significant change on students' self-directed learning abilities regarding mathematics.

Considering the preceding paragraphs, one can argue that the Collaborative Learning Programme (CLP) offered both the teachers and students numerous learning opportunities, because one way or the other, they maintain that they have learned a lot from the CLP. In conclusion collaboration among teachers has the potential to move the field of teaching mathematics forward by energizing teams of teachers within schools to activate and guide teacher improvement, thereby sustaining learning (see par. 1.2.1).

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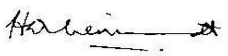
APPENDIX A: LETTER TO DR KENNETH KAUNDA DISTRICT MANAGEMENT IN POTCHEFSTROOM

3476 EXT 5
Khuma Township
Stilfontein
2251

Mr H Motara
Executive Manager
Kenneth Kaunda District
North West Education Department
Teemane Building
Potchefstroom

The initial research project will be undertaken by Mr Anthon Makhaya (Toto) Tshona, under supervision and with active involvement of the undersigned (Prof Hercules Nieuwoudt). Mr Tshona is currently enrolled with the North West University (Potchefstroom Campus) for the M.Ed. degree with specialisation in Mathematics Education. The Masters‟ and Doctoral Programme Committee of the Faculty of Education Sciences has already approved his proposed research project and the Ethical Committee of the NWU has already granted ethical clearance to do so. The academic and research merits of the project and the pertinent ethical conditions it has to comply with, hence, have properly been assessed and clearly been determined.

In the study, Mr Tshona intends to focus his research on *"Collaboration toward teaching proficiency of mathematical concepts in secondary schools*". The study aims to investigate the following primary question:



Prof Hercules Nieuwoudt (MSc, PhD)
Chair: Mathematics Education
Education Supervisor

Mr Toto Tshona
M.Ed. student in Mathematics
NWU: Potchefstroom Campus

APPENDIX B: LETTER OF APPROVAL

27. May. 2014 11:18 No. 5915 P. 1/1

education
Lefapha la Thuto
Onderwys Departement
Department of Education
NORTH WEST PROVINCE

Teemane Building
8 O R Tambo Street
Private Bag X1256
POTCHEFSTROOM 2520
Tel.: (018) 299-8216
Fax: (018) 294-8234
Enquiries: MR H MOTARA
e-mail: hmotara @nwpg.gov.za

DR KENNETH KAUNDA DISTRICT

PROFESSIONAL SERVICES

26 May 2014

Mr M.A. Tshona
University of the North West
Potchefstroom Campus
Private Bag X6001
POTCHEFSTROOM

PERMISSION TO CONDUCT RESEARCH ABOUT COLLABORATION TOWARD TEACHING PROFICIENCY OF MATHEMATICAL CONCEPTS IN TOWNSHIP SECONDARY SCHOOLS

The above matter refers.

Permission is hereby granted to you to conduct research in the Dr Kenneth Kaunda District under the following provisions:

- **the activities you undertake at school should not tamper with the normal process of learning and teaching and will take place after school hours;**
- **you inform the principals of your identified schools of your impending visit and activity;**
- **you provide my office with a report in respect of your findings from the research; and**
- **you obtain prior permission from this office before availing your findings for public or media consumption.**

Wishing you well in your endeavour.

Thanking you

MR H. MOTARA
DISTRICT DIRECTOR
DR KENNETH KAUNDA DISTRICT
CC *MR S S Mogotsi- Area Manager: Matlosana*
MS S. Yssel- Area Manager: Tlokwe

APPENDIX C: LETTER TO SCHOOL PRINCIPAL

Mr M. A. Tshona
University of the North West
Potchefstroom
Private Bag X6001
POTCHEFSTROOM

Request: support in engaging mathematics teachers from your area in a project

The Executive Manager, Mr H Motara, has kindly granted us permission to conduct the research project outlined below in the Kenneth Kaunda district. Dr Palesa Bungane has been appointed as coordinator from the regional office side. Furthermore, the project has already been approved by the Masters' and Doctoral Programme Committee of the Faculty of Educational Sciences at the Potchefstroom Campus of the NWU, and ethical clearance for the research has been granted by the Ethics Committee of the NWU.

Firstly we want to emphasise that the project idea stems from our deep concern about the quality of mathematics learning and teaching in South African schools, particularly in the mentioned schools in our region, and our commitment to engage with the department and schools to make a meaningful contribution towards devising a workable and sustainable approach that can enhance the performance in and outcome of mathematics education in the schools.

Secondly we want to emphasise that we are of the opinion that we need to start on a limited scale (in a research project) to effectively work on and refine the design of such an approach that could assist teachers in collaboratively experiencing and exploring powerful practices regarding the further and deeper develop of their conceptual understanding of the mathematical ideas they have to teach in their classes, as well as to engage their learners in meaningful and successful learning of those ideas. In particular we intend to create an opportunity for the teachers to embark on a route of forming communities of learning and eventually communities of practice – a situation we deem of utmost importance in order to begin to resolve the problem of under-performance in school mathematics in a sustainable way.

Thirdly we want to emphasise that once we have succeeded in designing such an approach and properly assessed its impact and value for teachers, we would want to also engage more teachers on a larger scale in benefitting from it (hence, a professional development programme).

Prof Hercules Nieuwoudt (MSc, PhD)
Chair: Mathematics Education
Supervisor

Mr Toto Tshona
M.Ed. student in Mathematics Education
NWU: Potchefstroom Campus

APPENDIX D: CONSENT FORM

Informed Consent

Herewith I, the undersigned, declare that I am fully informed about the purpose of *Mr Anthon Makhaya Tshona* investigation titled: *Collaboration toward teaching proficiency of mathematical concepts in secondary schools*.

I further declare that I am fully informed about the following ethical guidelines according to which the project will be executed:

- That my participation is voluntary
- That I can end my participation at any stage without any implication for me
- That all information I provide will be treated confidentially
- That my identity or the identity of the school/institution will not be revealed
- That I will receive oral feedback about the findings of the investigation should I prefer so?

I also declare that, in view of the above, I consent to participate voluntarily in the project and that I am willing to as best as possible share the information required from me by *Mr Anthon Makhaya Tshona*.

Participant (Name and Surname)……………………………………………….
Participant's signature: ....
Date: ....

APPENDIX E: OPEN ENDED SURVEY CASE 1

Survey Number					

ALL INFORMATION SUPPLIED HERE WILL BE HANDLED CONFIDENTIALLY

You are cordially requested to respond to the following questions in the space provided.

1. What is mathematics?

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2. Why do we teach mathematics? Why is mathematics an integral part of the Curriculum and Assessment Policy Statement.

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3. What does it means to teach mathematics?

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4. How should mathematics be taught?

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5. What are the essential features of mathematics?

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6. Briefly, explain how you go about teaching the concept of function?

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7. What do you understand about the concept of function?

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8. How should the concept of function be taught?

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9. A learner is working with the function $f(x) = -x + 7$. She claims that $f(2 + 3) = f(2) + f(3)$for this function.

a. Is she correct? Explain.

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b. Can this property be generalized to any group of functions?

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c. Is it always true for functions? Explain.

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APPENDIX F: PRE-ASSESSMENT

Survey Number					

ALL INFORMATION SUPPLIED HERE WILL BE HANDLED CONFIDENTIALLY

SECTION A

This section concerns biographical information.
Please mark the applicable box with X or supply the required information there.

1. Gender:

Male	
Female	

2. Experience (in years):

As teacher (any subject) Specify:	
As Mathematics teacher	
As Mathematical Literacy teacher	
In any other capacity Specify:	

3. What is your highest qualification?

Teacher Diploma or certificate Specify:	
B degree	
B degree and Teacher diploma or certificate	
Post-graduate qualification Specify:	
Other Specify:	

4. Are currently involved with further study?

No	
Yes Specify:	

5. I am currently teaching...

Mathematics	
Mathematical Literacy	

6. What grade(s) do you currently teach?

Mathematics	8	9	10	11	12
Mathematical Literacy	8	9	10	11	12

7. Have you been specifically trained for your current position as Mathematics / Mathematical Literacy teacher?

No	
Yes Specify:	

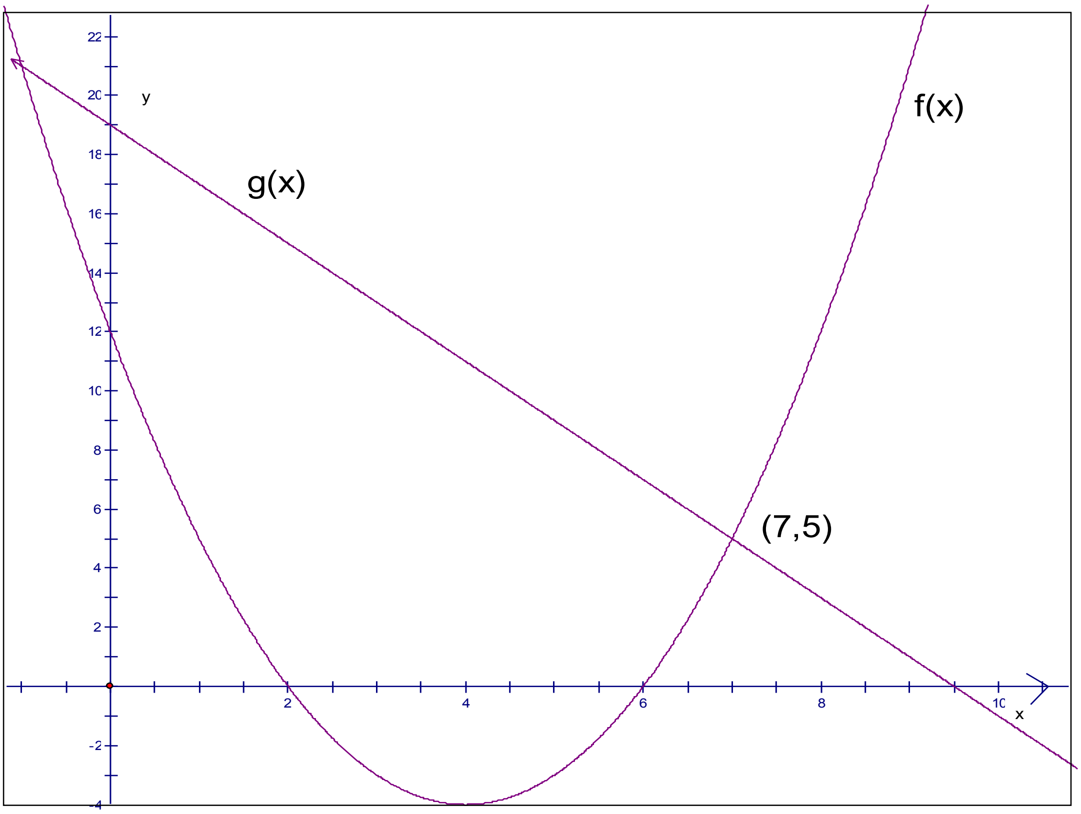
8. Do you regularly participate in organized professional subject activities (e.g. AMESA)?

No	
Yes Specify:	

SECTION B

Section B commences on the next page. You are cordially requested to complete the mathematical tasks in the provided space.

1. Consider the following diagram and answer the following questions.



1.1 What is the meaning of (7, 5) on this diagram?

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1.2 Which of the following statements is correct?

a. $f(7) = g(5)$

b. $f(5) = g(5)$

c. $f(7) = g(7)$

d. More information is needed

1.3 Explain in words how you went about solving 1.2.

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1.4 Find the equation of the parabola and the straight line.

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2. Given a function f determined by the equation: $y = x^2 - 2x + 3$.

   Give the equation of the new graph originating if:

2.1 The graph of f is moved (“shifted'') three units to the left.

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2.2 The $x - axis$ is moved (“shifted”) down three units relative to the graph.

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3. Two hot air balloons are being observed. A blue one is 150 meters above the ground and is descending at a constant rate of 20 meters per minute.

A red balloon is only 10 meters above the ground and is rising at a constant rate of 15 meters per minute.

On the same set pair of axes, draw graphs to show the heights of the two balloons from the time they were first observed until the time when the blue balloon hits the ground.

Use your graphs to answer the following questions:

3.1 How long will it take the blue balloon to reach the ground?

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3.2 How high will the red balloon be when the blue balloon hits the ground?

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3.3 How long after the first observation, will the two balloons be at the same height from the ground?

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3.4 How high will the balloons be when they are the same height above the ground?

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4. Before a volleyball match, the school gymnasium is filled with only players, coaches and the people working at the event. Fans and parents slowly start to arrive as the time nears the first match. Minutes before the match fans are coming in as fast as the tickets can be sold. After the match is over, most parents and fans leave. An Hour later students start to arrive for the after-game dance. Most students leave after an hour though and the people who remain in the gymnasium are the people who have been working at the event the entire evening.

4.1 Define an independent variable to model this situation. Decide what units of measurement you will use for this variable.

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4.2 What are reasonable values for the domain? Are they positive or negative numbers? Whole numbers or decimals? Motivate your answer.

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4.3 Define the dependent variable for this situation. Decide what units of measurement you will use for this variable.

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4.4 Draw and label a graph for this situation. (You may use the graph paper provided.)

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5. The below mentioned table provides the temperature of the water in a pan as it is set on the stove to boil.

Time (min)	0	2	4	6	8	10	12	14	16	18
Temperature $(°C)$	22	29	36	44	51	58	65	72	80	87

5.1 Find the equation that models this data.

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....

5.2 How long will it take the water to boil $(100°C)$?

....

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6. A business group wants to rent a meeting hall for its job fair during the week of the school holiday. The rent is R3 500, which will be divided among the businesses that agree to participate. So far only five businesses have signed up.

6.1 At this time, what is the cost for each business?

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6.2 Make a table to show what happens to the cost per business as additional businesses agree to participate.

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6.3 Write a function for the cost per business related to the number of additional businesses that agree to participate.

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6.4 How many additional businesses must agree to participate before the cost per business is less than R150?

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7. The following table represents measured values during an experiment of Ohm's Law (using the case where the resistance is 30 ohm):

Current *I* (ampere)	0	0.5	1.0	1.5	2
Potential difference *V* (volt)	0	0	15	30	45

7.1 Draw a graph of the above situation. (You may use the graph paper provided.)

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8. The table below lists the times that Sheila takes to walk the given distances.

Plot the points and join them, and find the equation of the line. Then use the equation to answer the following questions:

Time (minutes)		5	10	15	20	25	30
Distance (km)		1	2	3	4	5	6

8.1 How long will it take Sheila to walk 21 km?

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8.2 How far will Sheila walk in 7 minutes?

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8.3 If Sheila were to walk half as fast as she is currently walking, what would the graph of her distances and times look like?

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9. The temperature on a certain afternoon is modelled by the function $C(t) = \frac{1}{2}t^2 + 2$ where t represents hours after 12:00 $(0 \leq t \leq 6)$, and C is measured in $°C$.

9.1 What transformations ("shifting" and "shrinking" operations) must be performed on the function defined by $y = t^2$ to obtain the function defined by $y = C(t)$?

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9.2 Suppose you want to measure the temperature $°F$ instead. What transformation would you have to apply to the function $y = C(t)$ to accomplish this?

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....

9.3 Write the new function $y = F(t)$ that results from this transformation

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APPENDIX G: TASK 1 (Case 1)

1. Use the provided diagram to answer the following questions:

1.1 What is the initial cost from each company?

....

1.2 What is the cost per kilometre when renting from each company

....

....

1.3 Determine the equation for the total rental cost from each company?

....

....

1.4 Determine the number of kilometres at which the cost is the same from both rental firms/company

....

....

1.5 If you had to travel 300 km, which company would you choose? Why?

....

....

1.6 If you had R240 to spend on travel who would give you most kilometres?

....

....

2. A boat travels under a bridge that is 8 meters wide and 4 meters high

2.1 Sketch a graph of the bridge, showing the given information.

....

....

2.2 Determine the equation that models this situation

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2.3 How close to the edge of the bridge a 3 metre high boat may travel?

....

....

3. The current needed to work your car's headlights is inversely proportional to the voltage. That is, the current time the voltage is 144. Use a graph to show this situation, in multiples of 60 to 360 Volts.

3.1 From your graph, find the current that would be needed at 110 Volts

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3.2 From four graph find the voltage when the current is half an Ampere.

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4. You need to travel 100 km to a math workshop. You have the following options:
   - Take a taxi and travel at 100km/h
   - Take a bus and travel at 50 km/h
   - Cycle and travel at 25 km/h
   - Walk and travel at 5 km/h
   - Represent the above data in a table

4.1 draw a graph representing the above data

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4.2 How fast would you travel if you had to cover the distance in 1.5h?

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4.3 How long will the journey take if you travelled at speed of 75 km/h?

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4.4 What is the relationship between speed and time?

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4.5 Determine the equation that models this situation?

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5. A ball is dropped from a height of 150 m. the ball reaches the ground after 5 seconds

5.1 Represents the above situation graphically

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5.2 Find the equation that models this situation

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6. The use the provided diagram and answer the following questions

6.1 Does it make sense for the graph to be continuous? Give a reason for your answer.

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6.2 How long will it take eight workers to complete the job?

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6.3 How many workers will complete the job in 12 hours?

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6.4 Find the equation that models this situation?

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7. A courier company (The Faster) is prepared to deliver parcels to schools and will charge 10 per parcel delivered. Another courier company (Fast Fetch) will also deliver, but charges R5 per parcel delivered, a surcharge of R20 (regardless of the Number of parcels delivered).

7.1 Find the equation that models the given situations

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7.2 Draw two graphs on the same set of axes to show the total cost of using each courier company for various number of parcels

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7.3 From your graphs determine the following:

7.3.1 Which company is cheaper, if you want three parcels to be delivered?

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....

7.3.2 Which company is cheaper if you want eight parcels to be delivered?

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7.3.3 The difference between the charges of the two companies, if you wanted nine parcels to be delivered.

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7.3.4 Would it make sense for this graph to be continuous? Give a reason for your answer.

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APPENDIX H: TASK 2 (Case 1)

1.1 Solve the following systems of linear equations by finding the solutions that satisfy both equations

$$x + y = 1 \text{ and } x - y = 1$$

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1.2 Represent your solution *algebraically and graphically* in 1.1

………
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2.3 Solve the following systems of linear equations by finding the solutions that satisfy both equations. Represent your solution *algebraically and graphically:*

$$x + y = 1 \; and \; x - 2y = 2$$

………
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2.4 What is the meaning of your answer in 2.1?

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3.1 Given: $x^2 - 4 \; and - 2x + y = 4$. Represent your solution *algebraically and graphically*

Solve the above system of equations by finding the solutions that will satisfy both equations

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3.2 What is the meaning of your solution in 3.1?

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3.3 Now look at your graphs. For which values of will both graphs increase when increases?

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3.4 For which values of x will each of the following be correct

- $x^2 - 4 > 0$

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- $x^2 - 4 > 2x - 4$

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- $x^2 - 4 \leq 2x - 4$

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4.1 Solve for X and represent your answer graphically

- $3(2 - x) < 3$

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- $|x - 1| = 3$

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APPENDIX I: POST ASSESSMENT

1. Given $f(x) = \frac{5}{x-3}$

1.1 Give the equation of the vertical asymptote

1.2 Sketch the graph of f. Clearly indicates the asymptote of the graph and intercepts on the axes

1.3 Write down the domain and range of f .

1.4 If the graph of f moves one unit to the left, give the equation of the new graph.

2. The graph of $f(x) = x(x+3)$ is given.

2.1 Determine the average gradient of the curve f between $x = -5$ and $x = -3$.

2.2 Hence, state what you can deduce about the function of f between the points $x = -5$ and $x = -3$.

3. A recent catalog price for tennis balls was R40.25 for a can with three balls. The shipping order charge per order was R10.00.

3.1 Write an equation that you can use to project the costs for ordering different numbers of cans.

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3.2 showing this relationship (Use the graph paper provided)

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3.3 How does raising the shipping charge by R1.50 affect the graph?

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3.4 What equation models the situation in 7.3?

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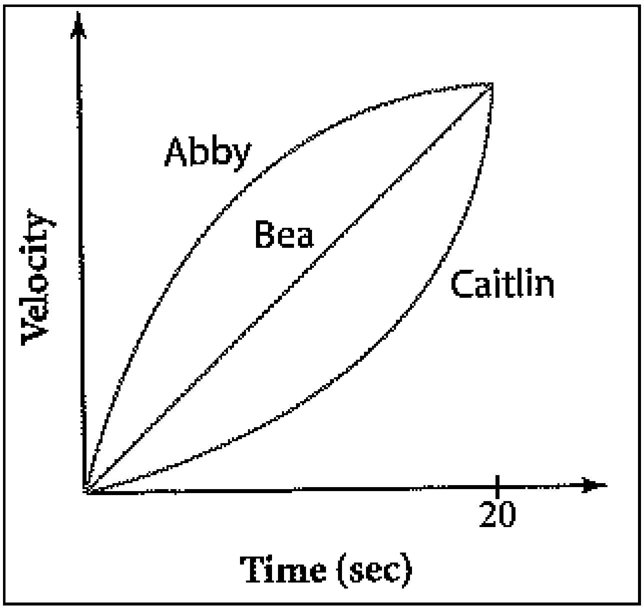
4. The graph below shows the velocities of three girls inline skating over a given time interval. Assume they start at the same place at the same time.



4.1 Create a story about these girls that explains the graph

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4.2 Are Caitlin and Bea ever in front of Abby? Explain.

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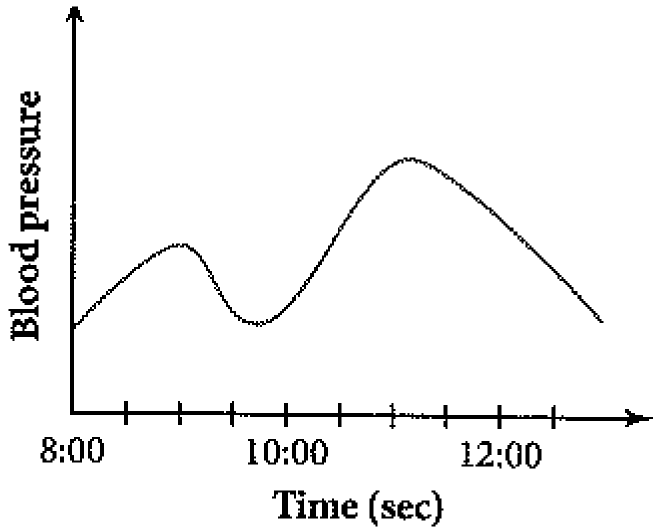
…………………………………………………………………………………………………

5. This graph shows Anne's blood pressure level during the morning at school. Use the graph to show the points or intervals when her blood pressure



5.1 Reached its maximum

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

5.2 Was raising the fastest.

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

5.3 Was decreasing

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………………………………………

6. The whole price of a CD is x rand. The retailer pays accrding to the function $c(y) = 2y$, where y is the price that the retailer pays. Calculate the consumer price if the wholesale price is $R60$.

…………………………………………………………………………………………………

…………………………………………………………………………………………………

APPENDIX J: SEMI-STRUCTURED INTERVIEW (CASE 1)

- What did you learn in the project?
- So, what does the project mean to you?
- What did you like best about the project?
- What did you like least about the project
- What if anything, will you do differently at school (mathematics classroom) because of the experiences you had while participating in the project?
- Now, what are you going to do about (or with) what you have learned or experienced in the project?
- Would like to participate in the project again? Why or why not?
- If we did this project again, what could we do to make it better?
- What is you impression regarding the use of a technological tool such as Geogebra® program in the project?

APPENDIX K: EUCLIDEAN GEOMETRY TASK 1 (CASE 2)

Open 'n Geogebra-skerm op jou rekenaar. Teken enige driehoek ABC op die skerm. Konstrueer die middelpunte E, F en G van die drie sye van ∆ABC. Konstrueer ∆EFG.

Beantwoord nou die volgende vrae na aanleiding van die figuur wat jy gekry het op hierdie vel papier en sonder om met iemand daaroor te praat.

- Wat neem jy waar ten opsigte van ∆EFG sover dit die vorm daarvan en van die ander driehoeke in die figuur betref? Skryf jou waarneming neer.
- Hoe vergelyk die oppervlakte van ∆EFG met dié van die ander driehoeke in die figuur? Skryf jou bevinding neer.
- Kan jy jou waargenome eienskappe verduidelik? Skryf jou verduidelikings neer.
- Sal dit wat jy waargeneem het vir alle driehoeke geld? Hoe seker is jy? Motiveer jou antwoord.

Handig jou werk in. Dit word weer volgende keer aan jou terugbesorg, want Taak 2 gaan hiermee te make hê.

APPENDIX L: EUCLIDEAN GEOMETRY TASK 2 (CASE 2)

Anders as met Taak 1, word daar nou verwag dat julle konstruksies fisies met passer, potlood, lineaal en gradeboog uitvoer.

2.1 Gebruik eers net jou passer, potlood en lineaal en kontrueer agter-op hierdie blad enige driehoek ABC, die middelpunte E, F en G van die drie sye daarvan en die “binne”-driehoek EFG. Gebruik dan ook jou gradeboog en vergelyk die vorms en groottes van die vier driehoeke wat gevorm is.

Jy het bostaande ook as deel van Taak 1 gedoen en wel op ‘n Geogebra-skerm. Hoe vergelyk die twee konstruksie- en leerervarings? Verduidelik.

2.2 Konstrueer die volgende met net jou passer, potlood en lineaal:

a. ‘n Gelyksydige driehoek

b. ‘n Driehoek met sylengtes 5 cm, 12 cm en 13 cm

c. Hoeke met groottes: 30°, 45°, 120° en 150°

d. ‘n Reghoek waarvan die lang sye dubbel die lengte van die kort sye is.

2.3 Gebuik nou jou gradeboog ook en bepaal al die eienskappe van die driehoeke in a.en b. hierbo.

2.4 Waarom dink jy vereis die “CAPS”-kurrikulum dat leerders fisiese konstruksies in meetkunde moet doen? Verduidelik.

Heg al die blaaie waarop jy gewerk het aanmekaar en handig die pakkie in; dit word volgende keer aan jou terugbesorg.

APPENDIX M: EUCLIDEAN GEOMETRY TASK 3 (CASE 2)

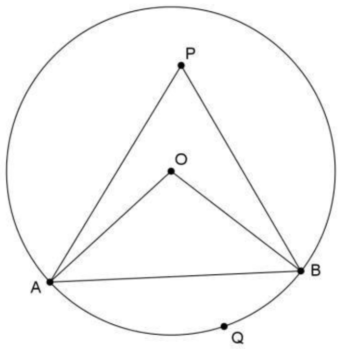
Werk saam om die volgende take uit te voer en die vrae te beantwoord. Skryf julle response op „n vel papier neer; onthou om die betrokke vraagnommers by te voeg en ook die studentenommers van die lede van die groep wat saamgewerk het.

- Teken enige driehoek ABC op „n GGB-skerm. Verleng AB na D sodat AB:AD = 2:3 en verleng dan AC na E sodat AC:AE = 2:3. Beantwoord die volgende vrae tov driehoeke ABC en ADE:
- Hoe vergelyk die driehoeke tov hulle vorm? Verduidelik.
- Hoe vergelyk die driehoeke tov hulle grootte (oppervlakte)? Verklaar waarom dit so is.
- Hoe vergelyk die lengtes van BC en DE? Verklaar waarom dit so is.
- Hoe vergelyk die driehoeke tov hulle omtrekke? Verklaar waarom dit so is.
- Verbind B en E en dan C en D. Hoe vergelyk die driehoeke ACD en ABE tov vorm, oppervlakte en omtrek? Verklaar waarom dit so is.
- Vir driehoeke PQR en STU geld dat: PQ = ST; PR = SU; ∠PQR = ∠STU. Die driehoeke voldoen dus aan „n SSH-voorwaarde.
- Is die driehoeke kongruent? Verduidelik waarom (nie).
- Onder watter voorwaardes sal ∆PQR en ∆STU kongruent wees? Verduidelik waarom dit so is.
- *Twee vierhoeke is kongruent as hulle dieselfde grootte (area) en vorm het*. Bepaal een voorwaarde waaronder twee vierhoeke ABCD en PQRS kongruent sal wees, *sonder om dit iewers op te soek*. Verduidelik waarom die voorwaarde geldig is.

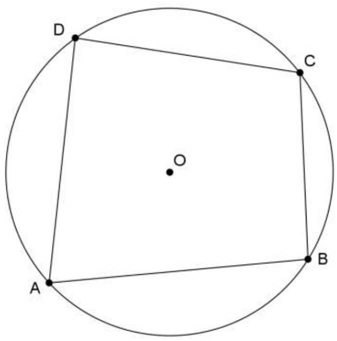
APPENDIX N: EUCLIDEAN GEOMETRY TASK 4 (CASE 2)

Werk saam om die volgende take uit te voer en die vrae te beantwoord. Skryf julle response op n vel papier neer; onthou om die betrokke vraagnommers by te voeg en ook die studentenommers van die lede van die groep wat saamgewerk het.

- Konstrueer die volgende figuur op ’n Geogebra-skerm. Meet die hoeke wat by die middelpunt O van die sirkel en punt P onderspan word deur die koord AB:



- Skuif (“drag”) punt P rond. Wat neem julle waar op ten opsigte van die groottes van die twee hoeke?
- Is julle waarneming altyd waar? Bewys julle antwoord.
- Skuif punt P na punt Q op die rand van die sirkel. Geld julle waarneming steeds? Verduidelik.
- Formuleer ’n algemene stelling op grond van julle waarneming.
- *’n Koordevierhoek (“cyclic quadrilteral”) is ’n vierhoek waarvan die hoekpunte op die rand van dieselfde sirkel lê.* In die onderstaande figur is ABCD dus ’n koordevierhoek in die sirkel met middelpunt O.



- Gebuik Geogebra en bepaal die eienskappe van koordevierhoeke.
- Stel PQRS is enige vierhoek. Wanneer sal PQRS ’n koordevierhoek wees?
- Vind een stelling in die kurrikulum wat met koordevierhoeke verband hou en verduidelik die bewys daarvan.

APPENDIX O: SELF-DIRECTED LEARNING INSTRUMENT (SDLI)

The SELF-DIRECTED LEARNING INSTRUMENT (SDLI) is developed as a self-rating instrument to measure the self-directed learning ability of students.

Student Number : | | | | | | | | |

Directions:
Please read each statement and make an (x) in the block that best describes how you think and feel about your own learning. There is no right or wrong answer.
.

1. Never	2. Seldom	3. Sometimes	4. Often	5. Always

Statements	1	2	3	4	5
1. I identify my own learning needs.					
2. I am able to stay self-motivated.					
3. My inner drive directs me towards further development and improvement in my learning.					
4. I find that both success and failure inspire me to further learning.					
5. I regard problems as challenges.					
6. I will not give up learning because I face some difficulties.					
7. I am able to set and plan my own learning goals.					
8. I can decide my own learning strategies.					
9. I am responsible for my own learning.					
10. I am able to select the best method for my own learning.					
11. I am good at planning and managing my own learning time.					
12. I know how to find resources for my learning.					
13. I am able to connect new knowledge with my own personal experience.					
14. I am able to identify the strengths and weaknesses of my learning.					
15. I am able to monitor my learning progress.					
16. I monitor whether I have accomplished my learning goals.					
17. My interaction with others helps me to plan for further learning.					
18. I intend to learn more about other cultures and languages I am frequently exposed to.					
19. I am successful in communicating verbally.					
20. I am able to express my ideas effectively in writing.					

APPENDIX P: OPEN-ENDED SURVEY

Wat is u antwoord op die vraag wat wiskunde is? /What is your answer to the question what is mathematics?
………
………
………
………

Wat dink u beteken dit om skoolwiskunde *goed te onderrig*?/ What does it mean to teach school mathematics effectively?

………
………
………
………
………
………

Hoe dink u behoort meetkunde op skool onderrig te word?/ How should Eucledian geometry be tauhgt?
………
………
………
………
………

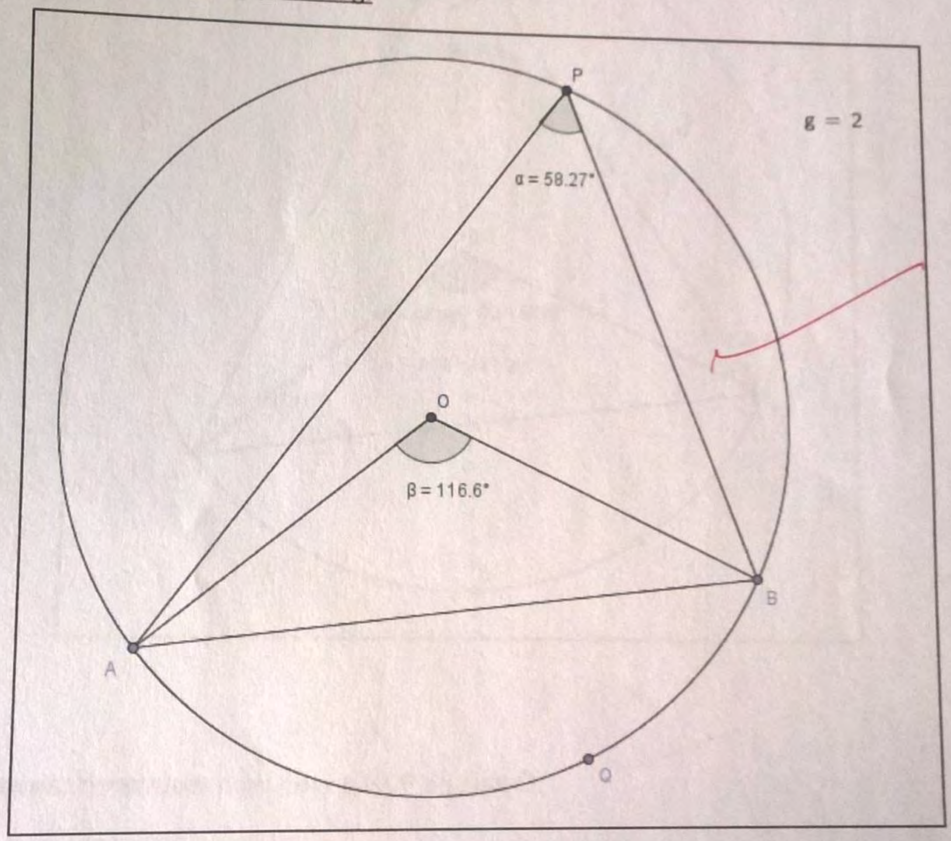
Wat is die belangrikste wat u uit die projek geleer het?/ What is the most important thing that you have learned in the project?

………
………
………
………

APPENDIX Q: PRACTICAL ASSIGNMENT

Vraag 1: Figuur 1 stel 'n sirkel met middelpunt O en koord AB en enige ander punt P voor.

a. Ondersoek die groottes van die hoeke onderspan deur AB by O en by P op 'n GeoGebra-skerm deur P oor die skerm te sleep. Wat merk julle op? Formuleer julle waarneming as 'n stelling. *Bewys* hierdie stelling.



APPENDIX R: STUDENTS' RESPONSES TO PRE- AND POST- ASSESSMENT OF THE SDLI

	Learning Motivation	Planning and implementing	Self-monitoring	Communication
1	4	5	4	4
	3	5	2	5
	Down by 1	No change	Down by 2	Down by 1
2	4	5	2	4
	3	5	3	4
	Down by 1	No change	Up by 1	No change
3	4	4	2	4
	3	4	2	3
	Down by 1	No change	No change	Down by 1
4	4	3	3	5
	3	4	2	5
	Down by 1	Up by 1	Down by 1	No change
5	3	4	3	3
	3	4	3	3
	No change	No change	No change	No change

6	3	4	4	3
	3	5	4	4
	No change	Up by 1	No change	Up by 1
7	3	4	3	5
	3	4	2	5
	Down by 1	No change	Down by 1	No change
8	4	5	4	5
	2	5	2	5
	Down by 2	No change	Down by 2	No change
9	4	5	3	4
	4	5	2	4
	No change	No change	Down by 1	No change
10	4	5	3	5
	3	5	3	4
	Down by 1	No change	No change	Down by 1
11	3	5	4	5
	3	4	4	5
	No change	Down by 1	No change	No change
12	3	4	2	3
	3	4	2	4
	No change	No change	No change	Up by 1

13	2	3	1	4
	4	5	3	3
	Up by 2	Up by 3	Up by 2	Up by 1
14	4	5	4	5
	4	5	4	5
	No change	No change	No change	No change
15	4	4	3	4
	4	5	3	5
	No change	Up by 1	Up by one	Up by one
16	3	4	2	4
	3	5	3	5
	No change	Up by 1	Up by 1	Up by 1
17	4	5	4	5
	4	5	4	4
	No change	No change	No change	Down by 1
18	3	3	1	2
	4	4	2	3
	Up by one	Up by 1	Up by 1	Up by 1
19	4	5	3	4
	4	5	3	4
	No change	No change	No change	No change

20	4	5	3	4
	4	5	2	4
	No change	No change	Down by 1	No change
	Learning Motivation	Planning and implementing	Self-monitoring	Communication
1	4	3	4	4
	4	3	4	4
	No change	No change	No change	No change
2	3	3	4	5
	4	3	4	3
	Up by 1	No change	No change	Down by 2
3	4	4	4	5
	3	3	4	4
	Down by 1	Down by 1	No change	Down by 1
4	4	3	3	3
	4	4	3	5
	No change	Up by 1	No change	Up by 2

5	3	4	4	5
	4	4	5	5
	Up by 1	No change	Up by 1	No change
6	4	3	3	5
	5	4	4	4
	Up by 1	Up by 1	Up by 1	Down by 1
7	5	4	2	3
	5	3	5	3
	No change	Down by 1	Up by 3	No change
8	4	5	5	3
	5	5	5	3
	Up by 1	No change	No change	No change
9	5	5	5	4
	4	5	5	4
	Down by 1	No change	No change	No change
10	4	5	4	3
	4	4	5	4
	No change	Down by 1	Up by 1	Up by 1
11	2	3	4	5
	2	4	5	3
	No change	Up by 1	Up by 1	Down by 2

12	4	4	5	4
	3	4	4	4
	Down by 1	No change	Down by 1	No change
13	4	4	4	5
	4	4	4	4
	No change	No change	No change	Down by 1
14	4	4	4	
	4	4	4	3
	No change	No change	No change	
15	5	5	4	3
	3	3	5	4
	Down by 2	Down by 2	Up by 1	Up by 1
16	3	3	3	4
	4	3	4	5
	Up by 1	No change	Up by 1	Down by 1
17	2	5	3	5
	5	4	5	4
	Up by 3	Down by 1	Up by 2	Down by 1
18	3	3	4	4
	4	3	5	3
	Up by 1	No change	Up by 1	Down by 1

19	4	5	5	3
	3	3	5	3
	Down by 1	Down by 2	No change	No change
20	4	4	5	3
	3	3	5	3
	Down by 1	Down by 1	No change	No change

	Learning Motivation	Planning and implementing	Self-monitoring
1	4	4	5
	5	4	5
	Up by 1	No change	No change
2	4	5	4
	5	5	4
	Up by 1	No change	No change
3	3	3	4
	4	4	4
	Up by 1	Up by 1	No change

4	3	4	4
	4	5	4
	Up by 1	Up by 1	No change
5	4	4	4
	4	4	5
	No change	No change	Up by 1
6	3	5	4
	5	4	4
	Up by 2	Down by 1	No change
7	3	3	4
	5	4	4
	Up by 2	Up by 1	No change
8	4	4	3
	5	3	5
	Up by 1	Down by 1	Up by 2
9	4	5	4
	5	5	4
	Up by 1	No change	No change
10	5	3	4
	5	3	5
	No change	No change	Up by 1

11	5	3	3
	5	3	4
	No change	No change	Up by 1
12	4	2	4
	4	2	5
	No change	No change	Up by 1
13	4	2	4
	4	3	4
	No change	Up by 1	No change
14	3	4	4
	3	5	4
	No change	Up by 1	No change
15	3	2	4
	5	4	4
	Up by 2	Up by 2	No change
16	3	3	4
	3	4	4
	No change	Up by 1	No change
17	5	1	5
	5	4	4
	No change	Up by 3	Down by 1

18	2	3	4
	3	2	3
	Up by 1	Down by 1	Down by 1
19	3	3	4
	4	4	4
	Up by 1	Up by 1	No change
20	3	3	5
	4	3	4
	Up by 1	No change	Down by 1