Mean-semivariance approach for portfolio optimisation

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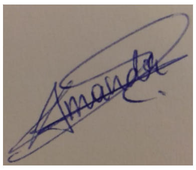
Declaration

I hereby declare that the research work `Mean-semivariance approach for portfolio optimisation` is of my own originality. All the sources consulted in this study are acknowledged and can be found in the bibliography section. This research work is submitted in partial fulfilment of the requirements for the degree Master of Science at the Centre for Business Mathematics and Informatics, North-West University (Potchefstroom campus).



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Abstract

The mean-variance method is hugely used for portfolio management. However, this approach assumes normality in the distribution of the assets' returns, which is not always observed in reality. Furthermore, using the variance as a measure of risk penalises the upside deviations of the returns, which investors consider as profit. Alternatives such as the semivariance measure has been proposed to overcome these drawbacks. This study aims to investigate the performance of the portfolios using semivariance as a measure of risk. A sample of ten companies from the Johannesburg Stocks Exchange Top 40 index is used for analysis. Using the Lagrange method for optimisation, the optimal portfolios from the mean-variance and the mean-semivariance approaches are constructed. The results show that the optimisation using the semivariance as a measure of risk produces desirable benefits: the optimal portfolios constructed achieve less risk and higher returns than those constructed using optimisation with the variance as a measure of risk. Furthermore, a tracking error analysis for portfolio performance indicates that the minimum-risk portfolio constructed by the mean-semivariance approach has less tracking error as compared to the minimum-risk portfolio constructed by the mean-variance method.

Keywords: Portfolio selection, Mean-variance model, Mean-semivariance model, Lagrange method, Portfolio performance.

Contents

List of Figures

List of Tables

1. Introduction

1.1 Background

In the financial markets, investors are continually seeking for strategies to select optimal portfolios that can achieve optimal returns. The problem investors face in selecting optimal portfolios is known as portfolio optimisation. Modern Portfolio Theory founded by Harry Markowitz in the seminal work Markowitz (1952) pioneered portfolio optimisation. In the paper, Harry Markowitz provided the mean-variance model. It is a useful tool for portfolio management developed to enable investors (financial economists, financial institutions and practitioners) to follow optimal strategies for assets selection.

To construct optimal portfolios, the mean-variance model uses the variance or the standard deviation of the returns to measure the risk. This model has quickly been integrated by practitioners and fund managers in the management of their portfolios and is regarded as the most commonly used optimisation approach for portfolio investment. However, quantifying the risk by the variance has been observed to not match with investors' perception of risk (Roy 1952). Furthermore, the mean-variance model has been criticised for assuming normality in the distribution of the assets. Indeed, the variance evaluates as a risk both the favourable and the unfavourable fluctuations of an asset or a portfolio's returns, while investors view risk as to the returns falling below their expected target returns. For this reason, using the variance to measure the risk may be inappropriate.

Given the drawbacks of the variance measure, other alternatives of risk measurement have been developed and proposed in the form of downside risk measures (Markowitz 1959). The most general is Lower Partial Moments: for a specified target return, only the nth power of the asset or the portfolio's returns deviating from this target are measured as a risk. In addition, these measures consider any possibility of asymmetry in the distribution of the assets' returns and also consider the investor's preferences towards risk. Constructing portfolios using downside risks may reduce the risk while allowing achieving higher returns than portfolios constructed under the mean-variance model.

One of the downside risk measures is the Lower Partial Moments of power two, called the semivariance. The semivariance measures the weighted sum of squared deviations of returns from the expected value of the returns. Empirical evidence has shown the superiority of the semivariance over the variance for risk measurement. Mean-semivariance model provides better portfolios than the mean-variance model (Markowitz 1959). However, computing mean-semivariance model is not easy. Unlike the mean-variance model, which uses a symmetric and exogenous matrix of covariances, the matrix of semicovariances is asymmetric and endogenous, thus creating difficulties in computation. Given this problem in the mean-semivariance model, how can one estimate the elements of the semicovariance matrix such that the model can easily be expressed and solved as the mean-variance model? The major part of the literature on semivariance is focused on developing approaches to overcome this difficulty. Hogan & Warren (1972) presented a proposal which computationally requires rigorous, intensive iterative algorithms and still ends to an endogenous semicovariance matrix. Markowitz, Todd, Xu & Yamane (1993) reformulated the mean-semivariance problem. By introducing additional variables to the mean-variance model, and applying the Critical Line Algorithm, the semivariance efficient frontier could be computed. In Estrada (2007) and Estrada (2008), a heuristic method yielding a symmetric and exogenous matrix of semicovariances was approached. This approach enables to determine optimal portfolios for the mean-semivariance by using the closed-form solution of the mean-variance model. Hogan & Warren (1972) proposed a co-Lower Partial Moments technique for the semivariance. This contribution makes the theoretical and computational utilities of the mean-semivariance model insured and has later been generalised in Nawrocki (1991) where a heuristic approach was used to convert into a positive semi-definite matrix the asymmetric matrix of semicovariances. de Athayde (2001) developed an optimal algorithm to construct a mean-downside risk portfolio efficient frontier using the analytic solution of the mean-variance model. From Sharpe's beta regression equation, Ballestero (2005) proposed a semicovariance matrix based on the semivariance below the mean return.

1.2 Problem Statement

As stated above, the mean-variance model is the one that is mostly used. However, the mean-variance model has its weaknesses, among them, assuming normality in the distribution of the assets. This assumption does not always hold in reality. Assets' distribution may exhibit skewness. The application of a measure that captures the downside part of the assets' distribution, such as the semivariance, which is the focus of the study, is more plausible.

Portfolio optimisation problems are usually expressed as quadratic problems whereby an optimisation solver is used to get an optimal solution. However, some of the optimisation problems lead to localised solutions, which might not be that optimal. As a result, the study resorts to using numerical methods to find optimal solutions to optimisation problems. In particular, the Lagrange method to optimisation proposed by Merton (1972) will be used for the study. The Lagrange method to optimisation only requires constraints in the optimisation. Moreover, optimisation solutions for downside risk frameworks are difficult to access, and the majority prefer then to use numerical methods. Hence, the Lagrange method is suitable for this.

From earlier above, the empirical evidence on semivariance is concentrated on determining the semicovariance matrix. However, the interest in this work is not on the estimation of the semicovariance matrix, but on producing optimal portfolios using the mean-semivariance approach with the semivariance approach proposed by de

Athayde (2001).

As a result, based on the above argument, the study seeks to use the mean-semivariance approach to construct portfolios using the assets/companies selected from the Johannesburg Stocks Exchange Top 40 index.

1.3 Motivation

Models in portfolio optimisation guide investors in the selection and allocation of the assets, such that their investments are exposed to minimum risk. These models have practical implications for risk management and portfolio selection. The mean-semivariance approach could be useful for such investors, allowing to control the downside risk of their investments while achieving the objectives on the return.

1.4 Aim of the Study

The study aims to review the theory on portfolio optimisation and to investigate the performance of the mean-semivariance approach for portfolio optimisation in comparison to the mean-variance approach.

1.5 Objectives of the Study

The objectives of the study are:

- To review the theory on portfolio optimisation.

- To investigate the performance of the mean-semivariance approach in the minimum-risk optimisation with comparison to the mean-variance approach.
- To investigate the tracking error of the mean-semivariance portfolio relative to the benchmark.

1.6 Method of investigation

The research is designed using the following methodology:

- Assets from the JSE Top 40 index are collected. This index is composed of the top 40 shares on the JSE market, ranked by market capitalisation. The study focuses on the daily adjusted closing prices (the last price during a trading day, adjusted for dividend) of each company, for a period ranging from May 2019 to July 2019. These data are obtained from the INET BFA website, which is Africa's leading provider of financial data feeds and analysis tools.
- Using PYTHON programming language and MICROSOFT EXCEL, the data prices are converted into returns and used for analysis. The returns of the portfolio are obtained using the respective assets' returns. The statistical moments (the mean, the variance, the semivariance, the covariance and the semicovariance) of returns are calculated. These parameters are used as inputs for both the mean-variance and the mean-semivariance models.
- The Lagrange method for optimisation is used to find optimal portfolios, and the efficient frontiers are graphically expressed.

1.7 Structure of the work

This study consists of two major parts: one (from Chapter 2 to Chapter 4) develops concepts on portfolio optimisation while the other (Chapter 5) reports on some empirical work. The remainder of the study is structured as follows:

Chapter 2 describes the concepts of risk and return in portfolio investment and describes their mathematical expressions.

Chapter 3 presents different frameworks for portfolio optimisation. These include the mean-variance model and the mean-downside risk models for the semivariance, the Value At Risk and the Conditional Value at Risk measures. In this chapter, some measures for portfolio performance are also presented. These are the Sharpe ratio, the Sortino ratio, the maximum drawdown and the tracking error measure.

Chapter 4 presents the mean-semivariance framework and discusses the difficulty related to portfolio optimisation based on the mean-semivariance approach. The algorithm proposed to overcome this difficulty is presented.

Chapter 5 provides an empirical study to support the approach presented in chapter 4 by comparing the practical results of the mean-variance and the mean-semivariance models. An analysis of portfolio performance is also provided.

Chapter 6 concludes the study and provides some insights for further researches.

2. The concept of return and risk

Models in portfolio theory are built according to investors preferences. The primary goal when investing is to make a profit or get more return. However, since the financial market is a very uncertain environment, investors should also consider the risk they may suffer from investing in any of these insecure securities available in the market. Return, risk and investor's preferences are then important concepts in portfolio optimisation that are introduced in this chapter. Before that, given that financial instruments evolve in a stochastic way, results in portfolio optimisation are not exact but estimates. Some probability concepts are thus first introduced.

2.0.1 Definition. Random variable (Shreve 2004)

Financial instruments evolving in a random way are described as a random variable. A random variable is a function that assigns a real number as value to an outcome of an experiment (e.g an investment) in a probability space. This is one of the basic concepts in probability theory.

2.0.2 Definition. Probability measure, $\sigma-$algebra (Shreve 2004)

To define a probability space one first needs three elements:

- A set $\Omega \neq \emptyset$, called the sample space. The set contains all possible outcomes ω, from a probability experiment. A subset of Ω is called an event.
- A set $\mathcal{F}$ called $\sigma-$algebra, which consists of collection of subsets $\omega \in \Omega$, or collection of all possible events. $\mathcal{F}$ is called $\sigma-$**algebra** if it satisfies the following conditions:
  - $\emptyset \in \mathcal{F}$
  - If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$; where A^c is the complement of the event A.

- If $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}$, where $A_n \in \mathcal{F}$, then $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{F}$.

- A probability measure $\mathcal{P}$ assigning probabilities to each event. Let $(\Omega, \mathcal{F})$ a measurable space (this is, $\mathcal{F}$ is $\sigma-$algebra on Ω). A probability measure is a function $\mathcal{P} : \mathcal{F} \longrightarrow [0, 1]$ such that:
  - $\mathcal{P}[\Omega] = 1 \quad$ and $\quad \mathcal{P}[\emptyset] = 0$
  - $0 \leq \mathcal{P}[A] \leq 1, \quad \forall\ A \in \mathcal{F}$
  - For any sequence of disjoint sets ($A_n \cap A_m = \emptyset$, for $n \neq m$) in $\mathcal{F}$, it holds that $\mathcal{P}[\bigcup\limits_{n=1}^{\infty} A_n] = \sum\limits_{n=1}^{\infty} \mathcal{P}[A_n]$.

2.0.3 Definition. Filtration (Carbone 2016)

Let T be an ordered set (e.g the time). A filtration is a family $\{\mathcal{F}_t\}_{t \geq 0}$ of $\sigma-$algebras on the set Ω such that $\mathcal{F}_s \subset \mathcal{F}_t, \quad \forall\ s \leqslant t$ in T. If t designates the time, $\mathcal{F}_t$ refers to the collection of all events observable until and including time t.

2.0.4 Definition. Probability space (Shreve 2004)

A probability space is defined as the set consisting of the triple $(\Omega, \mathcal{F}, \mathcal{P})$. A filtered probability space is defined as the set consisting of $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$.

A random variable X defined on $(\Omega, \mathcal{F}, \mathcal{P})$ is said to follow a Gaussian or a **normal distribution** with mean μ and variance σ^2, denoted by $X \sim N(\mu, \sigma^2)$, if for all $-\infty < a < b < +\infty$ (Weisstein 2002):

$$\mathcal{P}(a \leqslant X \leqslant b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x-\mu}{2\sigma^2}\right)^2 \mathrm{d}x\,.$$

For a given investment time interval, securities or assets will produce a sequence of random returns, whose values may all be different. A stochastic process can describe the movement of an asset's values during that period.

2.0.5 Definition. Stochastic process (Carbone 2016)

A stochastic process $\{X_t,\ t \geqslant 0\}$ is defined as a collection of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Given the set $\mathbb{R}$ of real numbers, a stochastic process can be described as a mapping $X : \mathbb{R} \times \Omega \longrightarrow \mathbb{R},\ (t, \omega) \longrightarrow X(t, \omega)$, representing the ω sample path of the process.

The stochastic process is said adapted to the filtration $\{\mathcal{F}_t\}$ if $X_t \in \mathcal{F}_t$. We say X_t is an $\mathcal{F}_t-$ measurable random variable $\forall\ t \geqslant 0$. Values of $X(t, \omega)$ can only be given by information available until time t. One commonly used stochastic process is the Brownian motion.

2.0.6 Definition. Brownian motion (Carbone 2016)

A stochastic process $\{B_t\}$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$ is called a one-dimensional Brownian motion or Weiner process if the following properties hold:

- $B_0 = 0$
- For all $0 \leqslant s < t < \infty$, $B_t - B_s$ is independent of $\mathcal{F}_s$. The process $\{B_t\}$ has independent increments
- For all $0 \leqslant s < t$, $B_t - B_s \sim N(0, t - s)$ is normally distributed
- Each sample path of the process is continuous with probability one.

One useful property of a Brownian motion is that its value increases or decreases randomly by 1 unit with equal probability.

Given the necessary probability terms, the two features of an asset: the return on a given period and the risk associated can be introduced. Note that any return is considered as a random variable defined on a probability space as in Definition 2.0.4.

2.1 Rate of return

The concept of return is described as in (Atzberger 2010). Let $P(0)$ and $P(T)$ respectively be an asset's value at time 0 and time T. The rate of return r, which also represents the yield of the asset is given from the expression $P(T) = (1+r)P(0)$ by:

$$r = \frac{P(T) - P(0)}{P(0)} \,. \tag{2.1.1}$$

The term $(1+r)$ can be seen as an interest rate required at the end T of a period, for a deposit of $P(0)$ at the beginning of the period.

However, given that assets evolve randomly over the given period $[0, T]$, the value $P(T)$ is unknown at time 0 and so will be the value of r. The mean of the returns is thus used to refer to the variability in asset's values over time. This is denoted by:

$$\mu = E[r] \,, \tag{2.1.2}$$

where E stands for the expectation of the random variable r. The expected rate of return gives an estimation of how large the returns may be on average.

Suppose now that n assets are available to construct a portfolio, and let W be the initial capital to be invested. Let's denote by W_i the amount of money to be invested in asset i. The wealth z_i invested in this asset is defined by:

$$z_i = \frac{W_i}{W} \,. \tag{2.1.3}$$

Since the total capital W is invested:

$$\sum_{i=1}^{n} W_i = W \quad \text{and this implies} \quad \sum_{i=1}^{n} z_i = 1 \,.$$

To define the return of a portfolio, denote by V the value of the portfolio, so that

$V(0) = V$ and $V(t) = \sum_{i=1}^{n} \frac{W_i}{P_i(0)} P_i(t)$. The return of the portfolio $R_p(t)$ at time t is then defined as:

$$\begin{aligned} R_p(t) &= \frac{V(t) - V(0)}{V(0)} \\ &= \frac{\sum_{i=1}^{n} \frac{W_i}{P_i(0)} P_i(t) - W}{W} \\ &= \sum_{i=1}^{n} \frac{W_i}{W} \frac{P_i(t)}{P_i(0)} - \sum_{i=1}^{n} \frac{W_i}{W} \\ &= \sum_{i=1}^{n} \frac{W_i}{W} \left(\frac{P_i(t)}{P_i(0)} - 1 \right) \\ &= \sum_{i=1}^{n} z_i \frac{P_i(t) - P_i(0)}{P_i(0)} \\ &= \sum_{i=1}^{n} z_i r_i \,. \end{aligned} \tag{2.1.4}$$

This results in a linear combination of the asset's return. The portfolio rate of return defined in Equation (2.1.4) is the weighted average of the asset's rates of return, with each asset's weight given by z_i. The portfolio's expected return μ_p, is also a linear combination of the expected rates of return of the assets, given by:

$$\begin{aligned} \mu_p &= E[\sum_{i=1}^{n} z_i r_i] \\ &= \sum_{i=1}^{n} z_i E[r_i] \\ &= \sum_{i=1}^{n} z_i \mu_i \,. \end{aligned}$$

As an asset's rate of return is considered a random variable, the resulting return from investing in such asset may be far from the return an investor is expecting to get. To quantify how the returns deviate from the expected return, the variance is used as the measure. The variance indicates then how volatile or risky an asset or a portfolio is.

2.2 Volatility

Given an asset with r and μ defined respectively as in Equation (2.1.1) and Equation (2.1.2), the volatility of the return is calculated using the variance as:

$$\sigma^2(r) = E[(r - \mu)^2]\,. \tag{2.2.1}$$

Let a portfolio constructed of n assets, and denote by σ_i^2 the variance of asset i, where $i = 1, \cdots, n$. To measure how the return of the portfolio is volatile, the relationship between each different asset should be considered. Since towards risk, investors will seek to reduce or eliminate the risk if possible, a good strategy should be to combine in the portfolio assets whose returns move in opposite directions over time. That is, when a down event occurs, making the value of an asset decreasing, this asset should be mixed with one whose value increases given the down event has occurred.

To quantify how correlated the assets are, the measure of covariance or correlation is used, described as follows:

$$\sigma_{i,j} = E[(r_i - \mu_i)]E[(r_j - \mu_j)]\,,$$

where $\sigma_{i,j}$ denotes the covariance between asset i and asset j. The correlation $\rho_{i,j}$ between two assets i and j is given by the expression:

$$\rho_{i,j} = \frac{\sigma_{i,j}}{\sigma_i \sigma_j}\,. \tag{2.2.2}$$

Note that the correlation $\rho_{i,j}$ lies in the range $[-1, 1]$ and that $\sigma_{i,j} = \sigma_{j,i}$ and for

$i = j$, $\sigma_{i,i} = \sigma_i^2$. From this, all the covariances can be presented in a non-singular, positive definite and symmetric matrix $C_{i,j}$, called the covariance matrix:

$$C_{i,j} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \dots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \dots & \sigma_n^2 \end{bmatrix}. \tag{2.2.3}$$

The variance of the portfolio is given by:

$$\begin{aligned} \sigma_p^2(R_p) &= E[(R_p - \mu_p)^2] \\ &= E[(\sum_{i=1}^{n} z_i r_i - \sum_{i=1}^{n} z_i \mu_i)^2] \\ &= E[(\sum_{i=1}^{n} z_i (r_i - \mu_i))^2] \\ &= E\left[\sum_{i=1}^{n} z_i (r_i - \mu_i)\right] E\left[\sum_{j=1}^{n} z_j (r_j - \mu_j)\right] \\ &= \left(\sum_{i=1}^{n} z_i E[(r_i - \mu_i)]\right)\left(\sum_{j=1}^{n} z_j E[(r_j - \mu_j)]\right) \\ &= \sum_{i,j=1}^{n} z_i z_j E[(r_i - \mu_i)(r_j - \mu_j)] \\ &= \sum_{i,j=1}^{n} z_i z_j \sigma_{i,j} \\ &= Z^T C Z\,, \end{aligned} \tag{2.2.4}$$

where in Equation (2.2.4), $Z = (z_1, z_2, \cdots, z_n)$ represents the vector of weights, Z^T is its transpose and C is the matrix defined in Equation (2.2.3).

Note that in financial investments, the fraction of weights defined in Equation (2.1.3) can take negative values. This means an investor is allowed to trade an asset he/she doesn't own. This case is called short selling and the investor is in a short position. Indeed, the investor (short seller) borrows an asset or a stock from a broker willing to lend, in the hope of a future decline in the value of the asset. The short seller will later purchase back the stock and return to the lender at a given date, the same amount or number of shares borrowed. The short position is thus said closed. In contrast, when the seller owns the stock, he/she is in a long position (Engelberg, Reed & Ringgenberg 2018).

Figure 2.1 bellow gives an illustration of the evolution of different assets' daily returns over a specified period. This figure clearly shows how much the returns can be volatile.

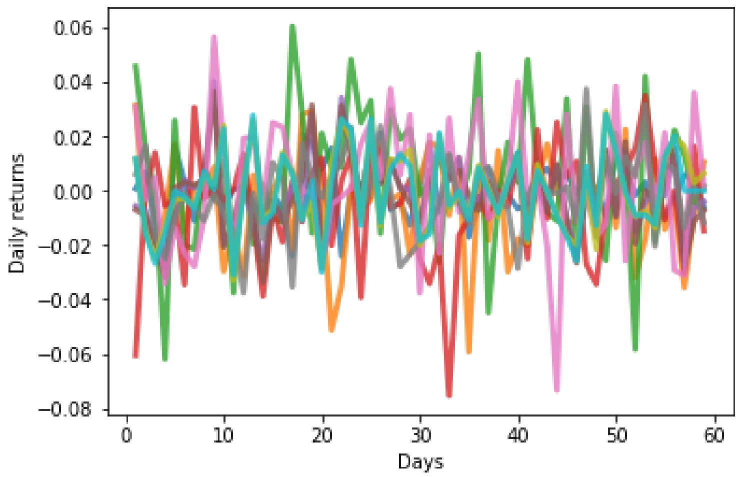

Figure 2.1: Daily asset's returns

Reducing the value of $\sigma_p^2(p)$ will result in reducing the risk of the portfolio. In finance, a common way of doing that is by diversifying the holding in the portfolio.

2.3 Diversification

Diversification is a strategy for portfolio risk management that aims to reduce the total portfolio's risk by combining a variety of financial instruments within the portfolio. The strategy is not only a matter of combining assets but especially of combining assets whose returns are not perfectly correlated. In the case of positive correlation, diversification does not hold. Indeed, assets positively correlated behave in the same way, and mixing them within the portfolio does not reduce the risk. Including non-positively correlated assets is preferred, mixing them help to reduce the portfolio's risk since, in this case, the positive performance of investing in some assets neutralizes the negative performance of investing in others.

From Equation (2.2.2), the covariance can be derived as $\sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$. This expression of $\sigma_{i,j}$ is used in the variance $\sigma_p^2(R_p)$ and present the three forms of correlation following (Roudier 2007):

- **Perfect positive correlation** ($\rho_{i,j} = 1$):

  The expression of the variance is reduced to:

$$\begin{aligned}\sigma_p^2(R_p) &= \sum_{i,j=1}^{n} z_i z_j \sigma_i \sigma_j \\ &= (\sum_{i=1}^{n} z_i \sigma_i)^2 .\end{aligned}$$

  Let's consider the case of investing in only two assets. If z_1 is the weight invested in asset 1, so $(1 - z_1)$ is invested in asset 2 and the total investor's capital is invested, is $z_1 + (1 - z_1) = 1$. The portfolio's risk and return are proportional to z_1:

$$\sigma_p^2 = (z_1\sigma_1 + (1 - z_1)\sigma_2)^2 \tag{2.3.1}$$

$$\sigma_p = \sigma_2 + z_1(\sigma_1 - \sigma_2)\,, \tag{2.3.2}$$

and the portfolio's return

$$\begin{aligned}\mu_p &= z_1\mu_1 + (1 - z_1)\mu_2 \\ &= \mu_2 + z_1(\mu_1 - \mu_2)\,.\end{aligned}$$

To get the optimal weights Z^* that will give the smallest possible risk, find the value of z_1 that will make $\sigma_p = 0$. From Equation (2.3.2):

$$z_1^* = \frac{-\sigma_2}{\sigma_1 - \sigma_2} \quad \text{and} \quad z_2^* = \frac{\sigma_1}{\sigma_1 - \sigma_2}\,.$$

The result shows that when combining two risky assets which are perfectly correlated, the minimum portfolio risk is realised by taking a short position in one of the two assets, here asset 1. This gives an optimal portfolio's return of:

$$\begin{aligned}\mu_p^* &= \mu_2 + z_1^*(\mu_1 - \mu_2) \\ &= \mu_2 + \frac{\mu_2 - \mu_1}{\sigma_1 - \sigma_2}\sigma_2\,.\end{aligned}$$

For this case, ($\rho_{i,j} = 1$), the possible portfolios to be constructed by varying the allocation are on a straight line joining a 100% investment in asset 1 to a 100% investment in asset 2.

- **No correlation** ($\rho_{i,j} = 0$):

The variance of the portfolio is given by:

$$\begin{aligned}\sigma_p^2 &= \sum_{i=1}^{n} z_i^2 \sigma_i^2 \\ &= (z_1\sigma_1)^2 + ((1-z_1)\sigma_2)^2 . \qquad (2.3.3)\end{aligned}$$

The portfolio with the minimum risk is found in the same way as above. Find the optimal weight that will make the variance in Equation (2.3.3) equals to zero. This gives:

$$z_1^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \text{and} \quad z_2^* = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} ,$$

and the optimal portfolio's return

$$\mu_p^* = \mu_1 \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} + \mu_2 \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} .$$

The possible portfolios to construct for this case will lie on a curve. Note that a correlation of zero does not mean there is no relationship between the assets, but instead, there is no linear relationship between them.

- **Negative correlation** ($\rho_{i,j} = -1$):

  The variance of the portfolio is given by:

$$\begin{aligned}\sigma_p^2 &= \sum_{i=1}^{n} z_i^2 \sigma_i^2 - 2 \sum_{i,j=1}^{n} z_i z_j \sigma_i \sigma_j \\ &= (z_1\sigma_1 - (1-z_1)\sigma_2)^2 . \qquad (2.3.4)\end{aligned}$$

  Again, the minimum portfolio's risk is realized for the optimal weight that makes equation (2.3.4) equals to zero, which weight is given by:

$$z_1^* = \frac{\sigma_2}{\sigma_2 + \sigma_1} ,$$

and the optimal portfolio's return

$$\mu_p^* = \frac{\sigma_2\mu_1 + \sigma_1\mu_2}{\sigma_2 + \sigma_1}\,.$$

When there is a negative or anti-correlation between assets, the possible portfolios to be constructed, with different combination of assets will lie on 2 segments.

Figure 2.2 below (where the correlation ρ is denoted by rho) plots possible portfolios in a $(\sigma_p - \mu_p)$ plane, with respect to the three forms of correlation:

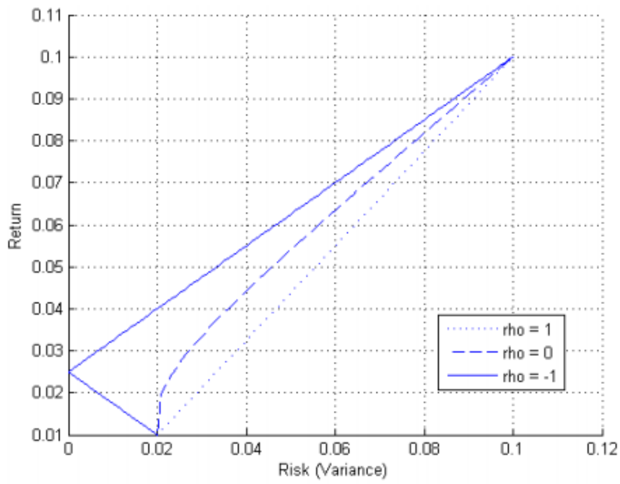

Figure 2.2: Plot of portfolios for different values of ρ

Source: Roudier (2007)

Given $-1 < \rho < 1$ and $z_1 + z_2 = 1$, a general formula for finding the minimum portfolio's risk constituted of two assets, is given by setting the derivative of the variance in Equation (2.3.1), (for $i = 1, 2$), with respect to z_1, equals to zero. This gives:

$$z_1^* = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{1,2}}{\sigma_1^2 - 2\sigma_1\sigma_2\rho_{1,2} + \sigma_1^2}\,.$$

In financial investment, some investors are observed to be risk-averse. Given the uncertainty in the financial market, they prefer to invest in a risk-less asset, even if the potential return may be lower. This behaviour of the investor's preferences is modelled by the notion of a utility function that is introduced in the next section.

2.4 Utility theory and risk aversion

Utility theory has its foundations in (Von Neumann & Morgenstern 1944). The theory is based on the assumption that investors don't choose alternatives yielding the highest return but rather choose options yielding the most top expected utility.

2.4.1 Definition. Utility functions ((Fishburn 1970))

Let A be a set of goods or alternatives. Let $a, b \in A$ and define $\geqslant$ as the relation is preferred to. To choose, investors will prefer asset a over asset b if $a \geq b$ (this superiority may be in terms of higher return or lower risk). If numbers can be assigned to a and b, these numbers are called utilities, and a utility function denoted by $u(a) \in \mathbb{R}$ will be the utility associated with each good $a \in A$. The concept of utility function measures preferences over a set of goods. The utility function $u(a)$ represents an agent's (investor) preferences if:

$$u(a) \geq u(b) \quad \text{given that } a \geq b\,.$$

In the attempt to find an accurate description of an agent willing to get a maximum of profit, Von Neumann & Morgenstern (1944) addressed the subject with the notion of a mathematical theory of games of strategy. They developed some axioms underlying utility theory, which define a rational decision-maker. The axioms can be found in Johnstone & Lindley (2013), and (Fishburn 1970).

- The completeness axiom: all outcomes are assigned a utility and can thus be compared among them. $\forall\ a, b \in A$, either $a \geq b$ or $b \geq a$.

- Transitivity axiom: for any $a, b, c \in A$ if a is preferable to b and b preferable to c then a is preferable to c. $\forall\ a, b, c \in A$, if $a \geq b$ and $b \geq c$ then $a \geq c$. Preferences are internally consistent.

- Independence axiom: for $a, b \in A$, with $a \geqslant b$. Let p the probability of the existence of a third good $c \in A$, with $p \in [0, 1]$. If $pa + (1-p)c \geqslant pb + (1-p)c$, then the choice of c is irrelevant. This means, the agent's preference of a over b will still hold independently of the existence of c.

- Continuity axiom: let $a, b, c \in A$, such that $a \geqslant b \geqslant c$. Then, there is a probability p such that the agent is indifferent between choosing the combination $pa + (1-p)c$ or the good b. The two choices are equally preferable.

However, the fundamental attribute of a utility function is that it is an increasing function, so that $u^{'}(a) > 0$ (Johnson 2007). It follows that $u^{'}(b) \neq 0$, this fact means that an agent is never delighted and will always prefer more to less.

The common known utility functions are:

- Quadratic utility: the general form is $u(a) = a - \dfrac{\alpha}{2}a^2$, with $\alpha > 0$.
- The exponential utility: $u(a) = -e^{-\alpha a}$, where $\alpha > 0$. This also called the positive utility, with $u(a) = 1 - e^{-\alpha a}$. This utility offers easiest mathematical tractability when asset returns are normally distributed.

- The logarithmic utility: $u(a) = \log a$.

- The power utility: $u(a) = \dfrac{a^{1-\alpha}}{1-\gamma}$, where $\gamma > 0,\ \gamma \neq 1$. The log utility is a particular case of the power utility with the limit of γ going to 1.

In general, it is difficult to interpret the absolute value of a utility function; instead, the utilities of wealth are ranked (Wojt 2009). Another concept discussed in the following is to describe the decision maker's preferences under risk.

2.4.2 Definition. Risk aversion

The behaviour of an agent whose preferences, when exposed to uncertainty, go to a good with the more predictable rate of return, even if lower, rather than a good with an unknown return which might be higher than expected, is described by the concept of risk aversion. An investor, for example, may choose to invest in a bank account where he knows his capital will grow at a constant known rate, rather than a risky investment as a stock, which may bring a higher return than a bank account, but associated to a very high level of risk. One of the oldest work on risk aversion can be found in (Dyer & Sarin 1982).

Investors may have different attitudes toward risk Johnson (2007): Risk-averse investors are the ones avoiding risk. They are willing to accept less return than the expected return, instead of taking the risk to receive nothing. The utility function of a risk-averse investor shows diminishing marginal utility, this is, $u^{''}(a) \leq 0$ and is concave. Risk neutral investors are rather indifferent between receiving less, more than the expected return or receiving nothing. They have a level of risk equals zero and linear utility functions.

On the other hand, risk affine is investors risk-seeking. These investors are willing to undertake higher risk, as long as they earn a lot. They have convex utility functions.

The risk aversion can be measured in two ways, for an utility function $u(a)$:

- **The absolute risk aversion (ARA)**: absolute risk aversion measures risk aversion to a loss in absolute terms (Johnson 2007). It is given by:

$$A(a) = -\frac{u^{''}(a)}{u^{'}(a)}\,.$$

The allocation of capital or wealth to risky assets depends on the following ARA's characteristics:

- For agents with a constant absolute risk aversion (CARA), as the capital increases, the individual allocation weights remain the same. The unique example is the exponential utility function $u(a) = 1 - e^{-\alpha a}$, with $A(a) = \alpha$.
- For agents with a decreasing absolute risk aversion (DARA), as the capital increases, the weight allocated in each asset also increases. The following inequality holds:

$$\frac{\partial A(a)}{\partial a} = -\frac{u^{'}(a)u^{'''}(a) - (u^{''}(a))^2}{(u^{'}(a))^2} < 0\,.$$

  This only holds for $u^{'''}(a) > 0$, which allows the utility function to be positively skewed. An example of DARA is the log utility $u(a) = \log a$, with $A(a) = \dfrac{1}{a}$.
- For agents with an increasing absolute risk aversion (IARA), as the capital increases, the holding in the assets decreases. The following inequality holds:

$$\frac{\partial A(a)}{\partial a} = -\frac{u^{'}(a)u^{'''}(a) - (u^{''}(a))^2}{(u^{'}(a))^2} > 0\,.$$

  There is no restriction on $u^{'''}(a)$, however IARA can allow a negatively skewed utility function with $u^{'''}(a) < 0$.

- **The relative risk aversion (RRA)**: relative risk aversion measures aversion to a loss relative to agent's wealth (Johnson 2007). Given by:

$$R(a) = aA(a) = -\frac{au^{''}(a)}{u^{'}(a)}\,.$$

For this measure, the allocation of capital to risky assets follows the RRA's characteristics:

- For agents with a constant relative risk aversion (CRRA), as the capital increases, the assets allocation of weights remains the same.
- For agents with a decreasing relative risk aversion (DRRA), as the capital increases, the assets allocation of weights also increases.
- For agents with an increasing relative risk aversion (IRRA), as the capital increases the assets weights decrease.

CRRA is observed to be more realistic than CARA because generally, rational agents invest more significant amounts in risky assets as they become wealthier.

DARA implies CRRA, but the reverse does not always hold. As an example, the utility function $u(a) = \log a$ implies $RRA = 1$.

Given that the return and the risk on a portfolio can be quantified, how can an investor formulate models in order to manage the portfolio according to his/her preferences on risk and return? Descriptions of such models are presented in the next chapter.

3. Risk measures, portfolio optimisation frameworks and perfomance measures

Portfolio optimisation is about maximising the expected return of a portfolio for a given level of risk or minimising a portfolio's risk for a desired portfolio's return. This study focuses on optimisation for risk minimisation. For this purpose, appropriated optimisation models must be defined. The performance of a model will here depend on the risk measure used since the goal is to help the investor by determining the amount of risk he/she may face for the given return he/she expects from an investment.

As introduced in Chapter 1, the mean-variance model has been the most commonly used in the literature on portfolio optimisation. However, given its various drawbacks, which will be presented in the following sections, other measures called downside risk measures had been introduced. This chapter presents then the mean-variance model and some downside risk measures for portfolio optimisation.

3.1 Mean-variance model

The mean-variance model is based on the mean and the standard deviation or the variance of a portfolio. In the goal of risk diversification in investing, this model helps investors by selecting a group of assets as a solution, such that their collective risk is lower than any single asset on its own.

3.1.1 Mathematical formulation

Recall from Section 2.1 and Section 2.2, the expected return μ_p and the variance σ_p^2 of a portfolio:

$$\mu_p = \sum_{i=1}^{n} z_i \mu_i \,,$$

and

$$\begin{aligned} \sigma_p^2 &= \sum_{i,j=1}^{n} z_i z_j \sigma_{i,j} \\ &= Z^T C Z \,. \end{aligned}$$

Approach 1: Minimising the portfolio's risk, for a target portfolio return. The problem is formulated as:

$$\begin{aligned} \text{minimise}_Z \quad & Z^T C Z \\ \text{subject to} \quad & Z^T \mu = \mu_p \\ & Z^T \mathbf{1} = 1 \\ & z_i \geq 0 \,, \end{aligned} \tag{3.1.1}$$

where Z^T represents the transpose of the vector w, C the matrix of covariances, μ is the vector of the asset's expected returns and the vector $\mathbf{1} = \underbrace{1, 1, \cdots, 1}_{n \; times}$. The non-negativity constraint means that short position is not allowed. However this is condition is not always imposed.

Using the method of Lagrange, an analytical solution, the so-called closed-form solution, to the problem can be derived. A lemma from McLeish (2011) is first presented.

3.1.2 Lemma. Consider the following optimisation problem with p constraints:

$$\begin{aligned} \text{minimise} \;\; & \{f(z) : z \in \mathbb{R}^n\} \\ \text{s.t} \;\; & h_1(z) = 0, \cdots, h_p(z) = 0\,. \end{aligned}$$

Given that the functions $f, h_1, \cdots, h_p$ are continuously differentiable, a necessary solution to the problem is that there exists a solution in the $n+p$ variables $(z_1, \cdots, z_n, \alpha_1, \cdots, \alpha_p)$ of the equations

$$\begin{aligned} \frac{\partial}{\partial z_i}\{f(z) + \alpha_1 h_1(z) + \cdots + \alpha_p h_p(z)\} = 0, \quad & i = 1, \cdots, n \\ \frac{\partial}{\partial \alpha_j}\{f(z) + \alpha_1 h_1(z) + \cdots + \alpha_p h_p(w)\} = 0, \quad & j = 1, \cdots, p\,, \end{aligned}$$

where the constants α_j are the Lagrange multipliers and the differentiated function $\{f(z) + \alpha_1 h_1(z) + \cdots + \alpha_p h_p(z)\}$ is the Lagrangian.

Following Lemma 3.1.2 and following Merton (1972), a solution for the problem in Equation (3.1.1) can be derived as follows:

The Lagrangian function is given by:

$$L(Z, \alpha_1, \alpha_2) = Z^T C Z + \alpha_1(Z^T \mu - \mu_p) + \alpha_2(Z^T \mathbf{1} - 1)\,.$$

The first order conditions are given as:

$$\begin{cases} \dfrac{\delta L}{\delta Z} & = 2CZ - \alpha_1 \mu - \alpha_2 \mathbf{1} = 0_n \\ \\ \dfrac{\delta L}{\delta \alpha_1} & = Z^T \mu - \mu_p = 0 \\ \\ \dfrac{\delta L}{\delta \alpha_1} & = Z^T \mathbf{1} - 1 = 0 \,, \end{cases} \tag{3.1.2}$$

where 0_n is a zero-vector of n elements. Solving the first equation in the system of Equations (3.1.2) for Z:

$$Z = \frac{1}{2}\alpha_1 C^{-1}\mu + \frac{1}{2}\alpha_2 C^{-1}\mathbf{1} \,,$$

where C^{-1} represents the inverse of the covariance matrix C. Plugging the expression of Z in the two last equations in the system of Equations (3.1.2):

$$\begin{cases} \frac{1}{2}\alpha_1 \mu^T C^{-1}\mu + \frac{1}{2}\alpha_2 \mu^T C^{-1}\mathbf{1} = \mu_p \\ \\ \frac{1}{2}\alpha_1 \mu^T C^{-1}\mathbf{1} + \frac{1}{2}\alpha_2 \mathbf{1}^T C^{-1}\mathbf{1} = 1 \,. \end{cases} \tag{3.1.3}$$

Let $a = \mathbf{1}^T C^{-1}\mathbf{1}$, $b = \mu^T C^{-1}\mathbf{1}$ and $c = \mu^T C^{-1}\mu$, with a, b, c constants. The system of equations in (3.1.3) can be solved for α_1 and α_2:

$$\alpha_1 = \frac{2(a\mu_p - b)}{ac - b^2} \quad \text{and} \quad \alpha_2 = \frac{2(c - b\mu_p)}{ac - b^2} \,.$$

with α_1 and α_2 plugged in w, the expression of the optimal weight if found as:

$$Z^* = C^{-1}(\alpha_1 \mu + \alpha_2 \mathbf{1}) \,. \tag{3.1.4}$$

Note that α_1 and α_2 have dependence on μ_p, the target portfolio mean. Since C is a positive definite matrix, so is C^{-1} and $a, c > 0$ [1]and $(ac - b^2) > 0$[2]. The variance of the portfolio for a given value of μ_p is thus given by:

$$\sigma_p^2 = Z^{*T} C Z^* = \frac{a\mu_p^2 - 2b\mu_p + c}{ac - b^2}\,. \tag{3.1.5}$$

By varying the value of μ_p, this represents a parabola. To find the global minimum variance portfolio, set to zero the derivative of Equation (3.1.5) with respect to μ_p, this is:

$$\frac{a\mu_p - 2b}{ac - b^2} = 0\,.$$

This gives the portfolio with the least risk at $(\mu_p^* = \frac{b}{a}, \sigma_p^{2*} = \frac{1}{a})$ and the global minimum vector of weights:

$$Z_g = \frac{C^{-1}\mathbf{1}}{\mathbf{1}^T C^{-1}\mathbf{1}}\,. \tag{3.1.6}$$

Approach 2: Maximising the portfolio's expected return while a constraint of minimising the risk is settled. The problem is formulated as:

[1]If a matrix, in our case the covariance matrix C is a non-singular matrix, therefore positive definite, it follows that its inverse C^{-1} is also. And it also follows that the elements of the matrix inverse are such that $\sigma_{i,j} = \sigma_{j,i}$ for all i, j. Thus, a and c as defined above are quadratic forms of the matrix C^{-1}, meaning they are strictly positive, unless all $\mu_i = 0$ Merton (1972)

[2]Given that C^{-1} is positive definite, it follows by definition that $(b\mu - c)C^{-1}(b\mu^T - c) > 0 = (c^2a - 2b^2c + b^2c) > 0 = c(ac - b^2) > 0$, and since $c > 0$, hence $(ac - b^2) > 0$ (Merton 1972).

$$\begin{aligned} \text{maximise}_Z \quad & Z^T\mu \\ \text{s.t} \quad & Z^TCZ = \sigma_p^2 && (3.1.7) \\ & Z^T\mathbf{1} = 1 && (3.1.8) \\ & && (3.1.9) \end{aligned}$$

Constructing the Lagrangian again, we have:

$$L(Z, \alpha_1, \alpha_2) = Z^T\mu + \alpha_1(\sigma_p^2 - Z^TCZ) + \alpha_2(1 - Z^T\mathbf{1})\,.$$

Taking the differentiation with respect to w gives:

$$\frac{\partial L}{\partial Z} = \mu - 2\alpha_1 CZ - \alpha_2\mathbf{1} = 0_n\,.$$

Solving this equation for Z the optimal weights are derived as:

$$\begin{aligned} Z^* &= \frac{\alpha_2\mathbf{1} - \mu}{2\alpha_1 C} && (3.1.10) \\ &= C^{-1}\left(\frac{1}{2\alpha_1}\mu - \frac{\alpha_2}{2\alpha_1}\mathbf{1}\right)\,. && (3.1.11) \end{aligned}$$

To get the Lagrangian multipliers α_1 and α_2, the constraints of the problem are used.

Firstly Equation (3.1.8):

$$\begin{aligned}
Z^{*T}\mathbf{1} &= 1\\
(\mu - \alpha_2\mathbf{1})^T\left[\frac{C^{-1}}{2\alpha_1}\right]^T\mathbf{1} &= 1\\
(\mu^T - \alpha_2\mathbf{1}^T)\left[\frac{C^{-1}}{2\alpha_1}\right]\mathbf{1} &= 1\\
\frac{1}{2\alpha_1}(\mu^T C^{-1}\mathbf{1} - \alpha_2\mathbf{1}^T C^{-1}\mathbf{1}) &= 1\\
\frac{1}{2\alpha_1}(b - \alpha_2 a) &= 1\,.
\end{aligned}$$

This gives α_2 as:

$$\alpha_2 = \frac{1}{a}(b - 2\alpha_1)\,. \tag{3.1.12}$$

Now using constraint in Equation (3.1.7), the expression for α_1 is given as:

$$\alpha_1 = \sqrt{\frac{ac - b^2}{4(\sigma_p^2 a - 1)}}\,,$$

and plugging this into Equation (3.1.12):

$$\alpha_2 = \frac{1}{a}\left[b - \sqrt{\frac{ac - b^2}{(\sigma_p^2 a - 1)}}\right]\,,$$

where a, b and c are defined as for the approach 1. The calculations can be found in (Wojt 2009). with α_1 and α_2 inserted in Equation (3.1.10), the optimal weights are now given as:

$$Z^* = C^{-1}\left[\sqrt{\frac{\sigma_p^2 a - 1}{ac - b^2}}\mu + \left(\frac{1}{a}\left(1 - b\frac{\sigma_p^2 a - 1}{ac - b^2}\right)\right)\mathbf{1}\right]\,,$$

which has the form of $Z^* = C^{-1}(f\mu + h\mathbf{1})$ as in Equation (3.1.4), with $f = \dfrac{2(a\mu_p - b)}{ac - b^2}$ and $h = \dfrac{2(c - b\mu_p)}{ac - b^2}$. Using any of these two relations, let's use the first one:

$$
\begin{aligned}
\sqrt{\frac{\sigma_p^2 a - 1}{ac - b^2}} &= \frac{a\mu_p - b}{ac - b^2} \\
\frac{\sigma_p^2 a - 1}{ac - b^2} &= \frac{(a\mu_p - b)^2}{(ac - b^2)^2} \\
\sigma_p^2 &= \frac{a\mu_p^2 - 2b\mu_p + c}{ac - b^2},
\end{aligned}
$$

which is exactly the same as the minimum portfolio variance found in Equation (4.2.1).

The problem of minimising the variance of the portfolio, presented in the approach 1 and the one of maximising the portfolio return, presented in approach 2, give the same solution and are thus equivalent (Wojt 2009). In optimisation theory, this is called the feature of duality and is an essential tool, especially when looking for fewer computations.

Another approach can be to optimise the expected utility of the return on investment. Von Neumann & Morgenstern (1944)'s notion of a utility function, introduced in Section 2.4, has allowed Markowitw to interpret his mean-variance approach by the theory of rational investor's behaviour under uncertainty. Related work can be found in Kroll, Levy & Markowitz (1984), Levy & Markowitz (1979) or (Kijima & Ohnishi 1993).

Approach 3: Maximising the expected utility.

Let the return of a portfolio $R_p = Z^T R$, where R is the vector of asset's returns. If at time 0, V_0 represents the value of the portfolio, at time 1 this value will have grown from V_0 to $V_1 = V_0(1 + Z^T R)$. Given the investor is following a quadratic utility function of the form $u(z) = z - \frac{1}{2}Z^2$, where z represents the wealth which is of V_1,

$$\begin{aligned} E[u(z)] &= E[z] - \frac{1}{2}E[Z^2] \\ &= E[z] - \frac{1}{2}[\text{variance}(z) + E^2[z]]\,, \end{aligned} \tag{3.1.13}$$

where variance $= Z^TCZ$ and $E[z] = Z^T\mu$. The optimisation problem in the expected utility framework can be given by:

$$\begin{aligned} \text{maximise}_Z \quad & E[V_0(1 + Z^TR)] \\ \text{s.t} \quad & Z^T\mathbf{1} = 1\,. \end{aligned}$$

Using the expectation as in Equation (3.1.13), the problem is solved as in approaches 1 and 2 described above. A solution can be found in (Wojt 2009).

In the remaining of this research, the first approach will be used.

3.1.3 The efficient frontier

Solving the problem of whether maximising the return or minimising the risk of a portfolio, results in a set of portfolios that are constructed from different ways of combining assets. The collection of these portfolios constitutes the feasible region. Among them are the optimal portfolios, that offer less risk for a given target return or high return for a specific level of risk. The combination of the optimal portfolios traces out a convex curve on the $\sigma - \mu_p$ plane. Markowitz calls this the *efficient frontier*. Portfolios that lie from the minimum-variance portfolio to the upper right of the curve are those from which there is no other portfolio that offers a higher return for the same risk. These portfolios are sub-optimal and preferred to investors. Figure 3.1 below describes the efficient frontier.

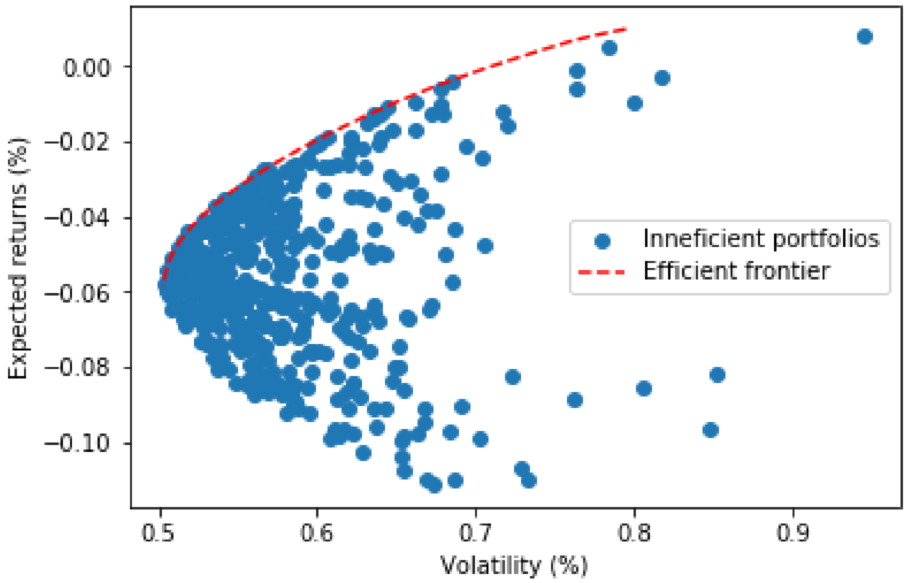

Figure 3.1: Markowitw efficient frontier

The circles represent the different portfolios, and the curve represents the efficient frontier. The collection of all possible points constitutes the feasible set or feasible region.

The equation of μ_p along the frontier is given in Merton (1972) by:

$$\mu_p = \mu_p^* \pm \sigma_p^{2*} \sqrt{(ac - b^2)(a\sigma_p^2 - 1)}\,.$$

Up to here, the models and the Markowitw's efficient frontier described above were for cases when investing in only risky assets. Now, consider there are also risk-less or risk-free assets available on the market. The inclusion of such an asset can affect the efficient frontier and thus, the investor's decision.

3.1.4 Including a risk-free asset

Let r_f the return of the risk-free asset. Since it is known with certainty, $r_f = \mu_f$, where μ_f represents the expected risk-free return. Let α the weight invested in the risky asset and so $1-\alpha$ the weight in the risk-free asset. The expected portfolio return is expressed as:

$$\mu_p = \alpha\mu + (1-\alpha)\mu_f \,.$$

The covariance $\sigma_{i,j}$ between any risky asset i and the risk-free asset j will be equal to zero since:

$$\sigma_{i,j} = E[r-\mu]\underbrace{E[r_f-\mu_f]}_{zero} \,.$$

The variance of the portfolio is expressed as:

$$\begin{aligned}\sigma_p^2 &= \alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\underbrace{\sigma_{i,j}}_{zero} + (1-\alpha)^2\underbrace{\sigma_j^2}_{zero}\\ &= \alpha^2\sigma_i^2 \,,\end{aligned}$$

so that the standard deviation is given as:

$$\sigma_p = \alpha\sigma_i \,.$$

The portfolios represented by (σ_p, μ_p) for varying values of α will lie on a straight line joining the points $(0, r_f)$ and (σ_i, r). This tangent line will touch the efficient frontier composed of risky assets at a point let us say F, where F lies on the efficient frontier.

3.1.5 Definition. One fund theorem (Merton 1972)

This theorem stipulates that any efficient portfolio, any point on the new feasible region, can be constructed by combining the risk-free asset and the portfolio of risky assets represented by F. Then, every investor will purchase a single portfolio, which is the market portfolio.

The mean-variance model is now formulated as follows:

$$\begin{aligned} \text{minimise}_Z \quad & Z^T C Z \\ \text{s.t} \quad & Z^T \mu + (1 - Z^T \mathbf{1}) r_f = \mu_p \,. \end{aligned}$$

As in the case of only risky assets, an analytical solution of the problem, as in Ekern (2007), Haugh (2006) or Engels (2004), is derived using the Lagrange method. The Lagrangian function is constructed as:

$$L(w, \alpha) = Z^T C Z + \alpha(\mu_p - r_f - (\mu - r_f \mathbf{1})^T Z) \,.$$

Taking the derivatives, we have:

$$\begin{cases} \dfrac{\partial L}{\partial Z} & = ZC - \alpha(\mu - r_f \mathbf{1}) = 0_n \\ \\ \dfrac{\partial L}{\partial \alpha} & = \mu_p - (\mu - r_f \mathbf{1})^T Z - r_f = 0 \,. \end{cases} \tag{3.1.14}$$

Solving the first equation in (3.1.14) for Z, the optimal weights are given as:

$$Z^* = \alpha C^{-1}(\mu - r_f \mathbf{1}) \,,$$

plugging this into the second equation gives:

$$\begin{aligned} \mu_p - r_f &= \alpha(\mu - r_f \mathbf{1})^T C^{-1} (\mu - r_f \mathbf{1}) \\ &= \alpha(c - 2 r_f b + r_f^2 a) \,. \end{aligned} \tag{3.1.15}$$

The expression for the portfolio's risk is by:

$$\begin{aligned}
\sigma_p^2 &= Z^{*T} C Z^* \\
&= \alpha(Z^{*T}\mu - r_f Z^{*T}\mathbf{1}) \\
&= \alpha(\mu_p - r_f) \quad \text{from (3.1.15)}\,. \qquad (3.1.16)
\end{aligned}$$

Using expressions in Equation (3.1.15) and Equation (3.1.16), the variance can be written as:

$$\sigma_p^2 = \frac{(\mu_p - r_f)^2}{(c - 2r_f b + r_f^2 a)}\,,$$

such that

$$\mu_p = r \pm \sigma_p \sqrt{(c - 2r_f b + r_f^2 a)}\,.$$

Recall that from the case where the portfolio is managed for only risky assets, the set of minimum-variance portfolios lie on the parabola (or hyperbola when working with the standard deviation σ_p) given in Equation (4.2.1). As the investor includes a risk-free asset, the minimum-variance portfolios will lie on two lines given by the expressions:

$$\begin{aligned}
\text{Upper-line} &= r + \sigma_p \sqrt{(c - 2r_f b + r_f^2 a)} \\
\text{Lower-line} &= r - \sigma_p \sqrt{(c - 2r_f b + r_f^2 a)}\,.
\end{aligned}$$

Since investors are interested in portfolios on the upper-right of the efficient frontier, for which $\mu_p \geq \dfrac{b}{a}$, the Upper-line tangent to the hyperbola will provide the optimal portfolio or, the tangency portfolio (tangent point to the efficient frontier through the point $(0, r_f)$). Figure 3.2 shows of the new efficient frontier.

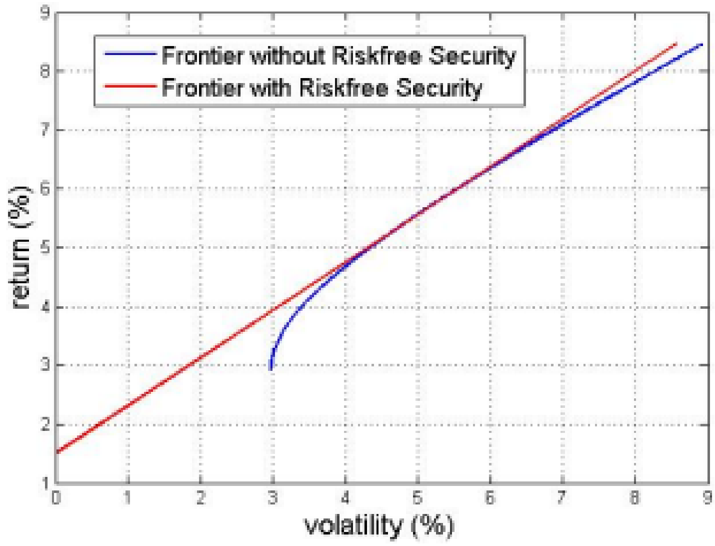

Figure 3.2: Markowitw efficient frontier with a risk-free asset

Source: Haugh (2006)

By solving simultaneously equations:

$$\sigma_p^2 = \frac{a\mu_p^2 - 2b\mu_p + c}{ac - b^2}$$
$$\mu_p = r + \sigma_p\sqrt{(c - 2r_f b + r_f^2 a)}\,,$$

the coordinates (σ_p^F, μ_p^F) of the tangency portfolio F are found to be:

$$\sigma_p^{2F} = \frac{c - 2r_f b + ar_f^2}{(b - ar_f)^2}$$
$$\mu_p^F = \frac{c - br_f}{b - ar_f}\,,$$

and the value $\alpha = \dfrac{\mu_p^F - r_f}{c - 2r_f b + ar_f^2}$, so that the tangency optimal vector of weights is given by:

$$Z_F^* = \frac{C^{-1}(\mu - r_f \mathbf{1})}{b - ar_f}\,.$$

Even though the mean-variance model is easily applied and gives an excellent presentation on the risk-return trade-off, researches after Markowitw have shown that using the variance for risk measurement is not efficient. Indeed, the model rests on assumptions that do not always hold in reality and so impact on its performance.

3.1.6 Criticisms on the mean-variance model

Some of the assumptions in the mean-variance model are discussed:

- There are no transaction costs. Taxes and brokerage commissions are not considered. The only factor that accounts in the selection of assets is the risk.
- Investors are rational and risk-averse. Between two portfolios that offer the same return, investors will prefer least riskier one.
- Variance as risk measure considers both upper and lower returns deviations as risk, while in reality, investors care about losing, and are more interested in quantifying the magnitude of the loss they may suffer from lower returns.
- Investors are constrained to observe only the two first moments, the mean and the variance (even if higher moments like skewness or kurtosis are observable) which perform well when returns are assumed multivariate normally distributed and the utility function quadratic. However, assuming normality in the returns of assets is a fact that does not always hold in reality. If the underlying distribution of returns is not normal, the variance is likely to provide misleading asset

allocation decisions. Researchers such as Jansen & De Vries (1991), Karoglou (2010) investigated on the nonnormality of asset's returns.

- There is no uncertainty considered, the mean returns and the covariances are assumed known and estimated using historical information. This produces portfolios sensitive to estimation errors. Indeed, future uncertainty must be included in the estimation of these parameters. In Best & Grauer (1991) and Chopra & Ziemba (2013), it is shown that for small changes in the inputs parameters, the resulting assets allocation is affected, producing extreme portfolio weights and a lack of diversification

- Variance as risk measure tends to overweight assets with the higher expected return. The benefit of diversification is then broken.

Markowitz himself recognised the inefficiency of the variance as a measure of risk and proposed the use of a more realistic measure, the semivariance, which is one of the downside risk measures that are presented below.

3.2 Downside risk measures

As mentioned above, the variance measure rests on some unrealistic assumptions. Other measures called downside risk were then introduced by (Markowitz 1959).

These measures, as the name suggests, unlike the variance, only consider downside deviations of the returns. They all focused on the left-hand tail of the returns' distribution, however, each with its specific given minimum acceptable level of return, from which the left-hand tail begins. In this section, some of those measures are presented. Before, let's introduce the characteristics that a sufficient risk measure must possess.

3.2.1 Coherent risk measure

A measure of risk, denoted by φ, is a mapping:

$$\varphi : \mathcal{G} \to \mathbb{R} \,,$$

where $\mathcal{G}$ represents the set of possible risks. For a stochastic random variable $A \in \mathcal{G}$, the measure $\varphi(A)$ represents then the risk of A.

Artzner, Delbaen, Eber & Heath (1999), introduced some properties that must satisfy a good risk measure, a coherent risk:

- **Positive homogeneity**. For any number $\lambda \geq 0$, and for any $A \in \mathcal{G}$,

  $$\varphi(\lambda A) = \lambda \varphi(A) \,.$$

  The amount of risk depends on the size of the position. If the amount of λ increases the size of a portfolio, the risk will be scaled by the same amount. This makes sense if risks are measured in different currencies.

- **Sub-additivity**. For all risky assets $A\,, B \in \mathcal{G}$,

  $$\varphi(A + B) \leq \varphi(A) + \varphi(B) \,. \qquad (3.2.1)$$

  Two assets should achieve a risk lesser than or equal to the sum of the risks of the individual asset. This property presents a sense of diversification. Indeed, combining assets reduces the overall risk of the portfolio. Furthermore, it gives an upper bound for the combined risk, which is the summation of the risks of the individual assets.

From the first property:

$$\varphi(A + A) = \varphi(2A) = 2\varphi(A)\,.$$

This changes the inequality in Equation (3.2.1) into equality. A convex measure is a risk measure satisfying both the positive homogeneity and the sub-additivity properties. It can be shown that:

$$\begin{aligned}\forall\ \lambda \in (0, 1)\,,&\\ \varphi(\lambda A + (1-\lambda)B) &\leq \varphi(\lambda A) + \varphi((1-\lambda)B)\\ &= \lambda\varphi(A) + (1-\lambda)\varphi(B)\,.\end{aligned}$$

- **Monotonicity**. For any risky asset $A\,, B \in \mathcal{G}$, such that $A \leq B$, it implies that

$$\varphi(B) \leq \varphi(A)\,.$$

  If asset A is worth less than asset B, then the risk of asset B is always less than the risk of asset A.

- **Translation invariance**. For any $A \in \mathcal{G}$, and for any number λ,

$$\varphi(A + \lambda) = \varphi(A) - \lambda\,.$$

  This property means, when adding (or subtracting) cash of an amount λ, to the portfolio, the risk is reduced (or added) by the same amount λ. A particular case is when adding an amount of $\lambda = \varphi(A)$, the risk is reduced, since:

$$\varphi(A + \varphi(A)) = \varphi(A) - \varphi(A) = 0\,.$$

Other measures of risk, presented against the variance are now introduced.

3.2.2 The Lower Partial Moments

The concept of moments represents a set of parameters used to measure a distribution. A general mathematical formulation for moments can be given as follows:

3.2.3 Definition. Moments (Walck 1996)

Let A a random variable with Cumulative Density Function $F_A(a)$, and let a given target β. The moment of degree n is given by:

$$\begin{aligned} \mu_n(F_A(a)) &= E((A-\beta)^n) \\ &= \int_{-\infty}^{+\infty} (a-\beta)^n \mathrm{d}F_A(a)\,, \end{aligned}$$

where E represents the expectation and the degree $n = 1, 2, 3, \cdots$. When β is the mean of the distribution, the moments are called central moments.

Four moments are the most commonly used:

- The first order, with $n = 1$, which represents the mean, denoted $\mu = E[A]$.
- The central moment of the second order $n = 2$, the variance, $E[(A-\mu)^2]$.
- The third order, $n = 3$ called the skewness,

  $$\frac{E[(A-\mu)^3]}{\sigma^3}\,,$$

  where σ^3 is the standard deviation of the degree of the moment. This parameter, let us denote by γ, measures the asymmetry of a distribution. Any symmetric distribution has the third moment equals to zero. When the right tail of the distribution is longer than the left tail, there is a positive skewness, and when the left tail is longer than the right tail, a negative skewness occurs.

- The kurtosis, central moment of the fourth order, $n = 4$,

$$\frac{E[(A-\mu)^4]}{\sigma^4}.$$

It is a measure of the heaviness of the tail of a distribution. For distribution with a heavy tail, the kurtosis is high and called the leptokurtic. As well, a thin-tail distribution has a low kurtosis called platykurtic.

The nth normalised moment of the random variable A can also be introduced as:

$$\mu_n = \frac{E[(A-\mu)^n]}{\sigma^n},$$

where σ^n is the standard deviation of degree of the moment.

3.2.4 Definition. Lower Partial Moments, LPM (Fishburn 1977)

One of the drivers of LPM in portfolio theory is Fishburn (1977). The nth LPM is a family of measures of downside risk. It calculates the moment of degree n, of observations a_i that fall below a given threshold β fixed according to investor's preference. Given A, β and n as defined above, the mathematical formulation is given by:

$$\begin{aligned}\mathrm{LPM}_{n,\beta}(F_A(a)) &= E[\min(A-\beta,0)^n] \\ &= \int_{-\infty}^{+\infty}(a-\beta)^n \mathrm{d}F_A(a).\end{aligned}$$

In practice, the discrete case calculation is used to estimate the LPM. For T observations from a random variable A, this is:

$$\mathrm{LPM}_{n,\beta} = \sum_{t=1}^{T} \min\ P_t(a_t-\beta,0)^n, \tag{3.2.2}$$

where P_t is the probability that observation a_t occurs. If all the observations can occur with the same probability, then $P_t = \frac{1}{T}$, and:

$$\mathrm{LPM}_{n,\beta} = \frac{1}{T}\sum_{t=1}^{T} \min(a_t - \beta, 0)^n \,. \tag{3.2.3}$$

A formulation of Markowitw's model using the LPM to measure the risk can be found in Wojt (2009) as:

$$\begin{aligned} \text{minimise}_Z \quad & Z^T L Z \\ \text{s.t} \quad & Z^T \mu = \mu_p \\ & Z^T \mathbf{1} = 1 \,, \end{aligned}$$

where L will be a symmetric matrix composed of co-lower partial moments given as:

$$\mathrm{CLPM}_{n-1,\beta,i,j} = \frac{1}{T}\sum_{t=1}^{T} (\min(a_{i,t} - \beta))^{n-1}(a_{j,t} - \beta) \,,$$

so that the matrix L is:

$$L = \begin{bmatrix} \mathrm{CLPM}_{n-1,\beta,1,1} & \mathrm{CLPM}_{n-1,\beta,1,2} & \dots & \mathrm{CLPM}_{n-1,\beta,1,T} \\ \mathrm{CLPM}_{n-1,\beta,2,1} & \mathrm{CLPM}_{n-1,\beta,2,2} & \dots & \mathrm{CLPM}_{n-1,\beta,2,T} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{CLPM}_{n-1,\beta,T,1} & \mathrm{CLPM}_{n-1,\beta,T,2} & \dots & \mathrm{CLPM}_{n-1,\beta,T,T} \end{bmatrix} .$$

LPM are measures specified by β and n which captures an investor's preference. In (Harlow 1991):

When $n = 0$, the risk measure is of the 0-th order moment. It measures the probability of failure below the target of β. And if $\beta = 0\%$, this is just a measure of the likelihood of a loss.

When $n = 1$, the risk measure is of the first-order moment. It calculates the expected deviation of observations below the target β. If observations are assets returns, then β will represent a given target rate of return. This measure is called the target shortfall.

When $n = 2$, the risk measure is of the second-order moment, analogous to the variance. It is a measure of the probability weighting of squared deviations. And if the observations represent assets returns and β = mean return, the risk measure is called the semivariance. However, if the target is fixed to the risk-free rate, $\beta = r_f$ and normality is assumed in the distribution of the assets, the measure is equivalent to the variance.

These measures are exceptional cases of the generalised LPM. As so, optimisation approaches using the target shortfall or the semivariance as risk measure can be defined as in the following.

3.2.5 Definition. Semivariance (Jin, Markowitz & Yu Zhou 2006)

The principle for the semivariance model is the same as for the variance described in Section 3.1.1.

Let n different risky assets, and T observations. Denote by r_i and μ_i respectively the return and expected return of asset i. Denote by μ_p the portfolio target return and Z the vector of the weights of the assets. The semivariance is described as:

$$\text{Semi} = \frac{1}{T}\sum_{t=1}^{T} \min(R_{pt} - E[R_p], 0)^2 \,,$$

and the problem is to find the portfolio $Z = (z_1, z_2, \cdots, z_n)$ that will

$$\begin{aligned}
\text{minimise} \quad & \frac{1}{T}\sum_{t=1}^{T} \min\,(R_{pt} - E[R_p], 0)^2 \\
\text{s.t} \quad & \sum_{i=1}^{n} z_i \mu_i = \mu_p \\
& \sum_{i=1}^{n} z_i = 1 \\
& z_i \geq 0\,, \qquad (3.2.4)
\end{aligned}$$

where R_p is defined as in Equation (2.1.4). More about the semivariance model is discussed in the next chapter.

Using target shortfall risk measure, the investor's problem is described as:

$$\begin{aligned}
\text{minimise} \quad & \frac{1}{T}\sum_{t=1}^{T} \min\,(R_{pt} - \beta, 0) \\
\text{s.t} \quad & \sum_{i=1}^{n} z_i \mu_i = \mu_p \\
& \sum_{i=1}^{n} z_i = 1 \\
& z_i \geq 0\,,
\end{aligned}$$

for any given target β, like for example the risk-free rate.

3.2.6 Value at Risk (VaR)

Another downside risk measure is the so-called Value at Risk, abbreviated VaR. This measure determines the potential loss and the probability of occurrence for the defined loss. VaR is composed of three main components to know: a time horizon, a given confidence level and a loss amount expressed in money or percentage. Using the VaR as the risk measure, the problem will be to find an optimal portfolio such that the highest expected loss does not exceed the VaR for a given investment period, at a given confidence level. More explanations on this measure can also be found in (Linsmeier & Pearson 2000).

3.2.7 Example. A portfolio has a VaR equals to \$100 with a 99% weekly confidence. It means, there is a 1% probability that the value of the portfolio will fall by more than \$100 over one week. Or that the probability that the loss of the portfolio will exceed \$100 in a week is less than 99%.

The measure is formulated as follows: let the value of a portfolio A over a time horizon from 0 to T, with a Cumulative Density Function F_A. Let a given confidence level of $\lambda \in (0, 1)$ and let us define a VaR level of a,

$$\begin{aligned}\mathrm{VaR}_\lambda(A) &= \inf\{a \in \mathbb{R} : P(A(0) - A(T)) > a) \leq (1-\lambda)\}\\ &= \inf\{a \in \mathbb{R} : 1 - F_A(a) \leq (1-\lambda)\}\\ &= \inf\{a \in \mathbb{R} : F_A(a) \geq \lambda\}\ .\end{aligned}$$

where *inf* stands for infimum and P for the probability. Again, following Markowitw's model, the mean-VaR portfolio optimisation problem is formulated. A related study can be found in (Lwin, Qu & MacCarthy 2017).

$$\begin{aligned}\text{minimise} \quad & VaR_\lambda(z)\\ \text{s.t} \quad & \sum_{i=1}^{n} z_i\mu_i = \mu_p\\ & \sum_{i=1}^{n} z_i = 1\,.\end{aligned}$$

Value at Risk is easy to interpret since it is a number expressed in monetary units or percentage, and applicable to all types of assets. However, this measure presents some drawbacks, the two most frequent ones:

- VaR does not measure the loss in the worst case. In the example above, at 1% of cases, the loss is expected to exceed the amount of the VAR. But VaR gives no information about the size of the loss within that 1% if a tail event occurs, neither the maximum possible loss. It might, unfortunately, comes that trading days within that 1% are the worst ones, that may liquidate a company.
- VaR as a measure of risk does not always satisfy the sub-additivity property described in 3.2.1. which property ensures that diversification on a portfolio holds and always generates lower risk for diversified portfolios.

As VaR lacks the sub-additivity property for coherent risk, investors seeking to reduce their portfolio's risk may be less encouraged to use the VaR measure. Figure 3.3 gives and illustration on the VaR measure.

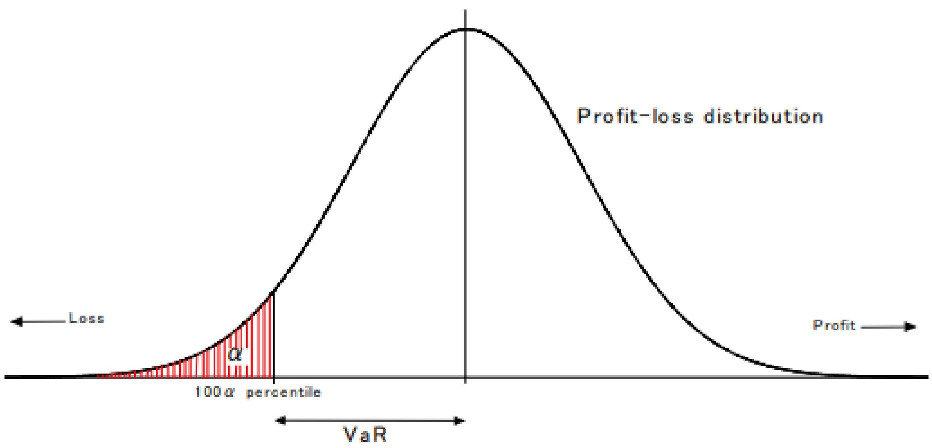

Figure 3.3: Profit-loss distribution and VaR

Source: Yamai & Yoshiba (2002)

As an alternative for VaR, a coherent measure, that captivates the lose even in the worst cases is introduced in Rockafellar & Uryasev (2000), the Conditional Value at Risk.

3.2.8 Conditional Value at Risk (CVaR)

The Conditional Value at Risk has some advantages over the VaR as a measure of risk, given that it is a coherent risk measure. CVaR measures the conditional expectation of loss given that the loss is at the tail of or beyond the VaR level, and it also calculates the size of the loss to be expected in the worst cases. Sarykalin, Serraino & Uryasev (2008) explained the strong and weak features of the VaR and the CVaR measures for application in risk management and portfolio optimisation. Once VaR has been calculated, the CVaR can then be calculated as:

$$CVaR_{\lambda}(A) = \int_{-\infty}^{+\infty} a \mathrm{d}F_A(a)\,,$$

where

$$F_A(a) = \begin{cases} 0 & \text{when} \quad a < VaR_{\lambda}(A) \\ \dfrac{F_A(a) - \lambda}{1 - \lambda} & \text{when} \quad a \geq VaR_{\lambda}(A). \end{cases}$$

The CVaR can be understood in two views Sarykalin et al. (2008) as:

- CVaR$^+$ (upper CVaR): This is called the Expected Shortfall or the Mean Excess Loss. In this case, the CVaR calculates the expected value of the loss A strictly exceeding the VaR,

$$CVaR^+ = E[A|A > VaR_{\lambda}(A)]\,.$$

- CVaR$^-$ (lower CVaR): This is called the Tail VaR. The lower CVaR calculates the expected value of A exceeding the VaR,

$$CVaR^- = E[A|A \geq VaR_\lambda(A)]\,.$$

Comparative studies between VaR and CVaR measures can be found in Yamai & Yoshiba (2002) or (Yamai & Yoshiba 2005). Figure 3.4 below gives an illustration on both the VaR and the CVaR measures. However, for fat-tailed distribution, errors in the estimation by the ES become larger than estimation errors by the VaR, given that losses beyond the VaR are not regular and a lack of accuracy may occur when estimating the lose. To overcome this, more massive data are generally required when using the ES as a risk measure, which is a weakness of the ES, since it makes it less effective than the VaR when only a few data are available.

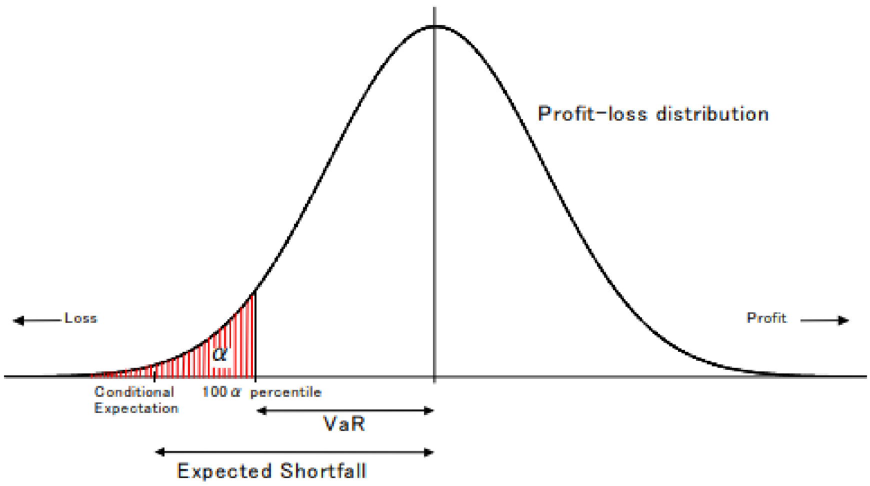

Figure 3.4: Profit-loss distribution, VaR and CVaR representation

Source: Yamai & Yoshiba (2002)

3.3 Performance measures

There exists more than one way of quantifying the risk of a portfolio, the performance of the portfolios constructed under the different measures of risk can be evaluated through performance analysis. Many measures are available to realise a performance analysis. In this section, five of them are presented.

3.3.1 The Sharpe ratio

Introduced in Sharpe (1994) by the American economist William Sharpe, the Sharpe ratio is a measure evaluating the performance of a portfolio by comparing the return of the portfolio relative to its risk. It calculates then the risk-adjusted return, a handy tool to determine how much risk is being taken to achieve a certain level of return.

Taking into account the fact that an investor may wish to evaluate the past performance of a portfolio, or to estimate the future performance of a portfolio, Sharpe (1994) defines both the ex-ante and the ex-post versions of the Sharpe Ratio as follows:

- **The ex-ante Sharpe ratio**: this is the forecasting Shape ratio. Its calculates an estimation of the future performance of a portfolio, formulated as:

$$S_{\text{ante}} = \frac{E[R_p - R_b]}{\sigma_p} \,,$$

  where R_p represents the unknown future return of the portfolio, R_b represents the future return of a benchmark (generally the Sharpe ratio uses the risk-free rate of return as the benchmark). The numerator of this expression defines the portfolio's excess return, and the denominator is the prediction of the standard deviation of the portfolio's excess return. The ex-ante Sharpe ratio calculates then the expected excess return per unit of total risk.

- **The ex-post Sharpe ratio**: this version of the Sharpe ratio evaluates the past performance of a portfolio. Some applications relate futures realisations to the past ones. This means if evaluated yearly; for example, investment A had a higher return than investment B; it is assumed that the same scenario will happen the following year. To express this measure, let the following notations: R_{pt} the return of the portfolio at the past period t, R_{bt} the return of the benchmark at the same period t, and let $D_t = R_{pt} - R_{bt}$, so that $\bar{D}$ represents the average value, calculated as $\bar{D} = \frac{1}{T}\sum_{t=1}^{T}(R_{pt} - R_{bt})$ for every past period t over $[1, T]$. The ex-post Sharpe ratio is expressed as:

$$S_{\text{post}} = \frac{\bar{D}}{\sigma_d}, \quad \text{where} \quad \sigma_d = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(D - \bar{D})^2}\,.$$

  The ratio here indicates the historical average excess return per unit of historical total risk.

Adding diversification, this is adding the number of non-correlated assets in a portfolio decreases the risk of the portfolio and proportionally increases the value of the Sharpe ratio. The higher the ratio, the better the performance of the portfolio. Nevertheless, this performance only accounts if it does not cost any additional risk.

The standard deviation, however, has been criticised as mentioned in Section 3.1.1, to evaluate as a risk the deviations from both the upper and lower fluctuations of the returns, which may not be the investor's view on risk, and this measure performs well when the underlying data are assumed normally distributed. In the following, another performance measure is presented, using the downside deviation for evaluating the risk instead.

3.3.2 The Sortino ratio

Unlike the Sharpe ratio which uses the standard deviation to evaluate the risk taken associated to the portfolio's excess return, the Sortino ratio uses the downside risk which only focuses on the distribution of the returns that are below a required target return, and uses as a benchmark the target return. Since positive deviations from a target are considered to benefit for investors, the Sortino ratio is assumed to give a better portfolio's performance than the Sharpe ratio. From RollingeR & Hoffman (2013), the expression of the Sortino ratio is expressed similarly to the one of the Sharpe ratio, but with the parameters calculated differently. The target downside risk TDR is given as:

$$\text{TDR} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\min(R_{pt} - \beta))^2}\,,$$

where min denotes the minimum, and β the target return. The Sortino ratio is calculated as:

$$\text{Sortino} = \frac{(R_p - \beta)}{\text{TDR}}\,.$$

In Figure 3.5 below, an illustration of the distribution of returns as considered by the standard deviation and by the downside risk measures is shown.

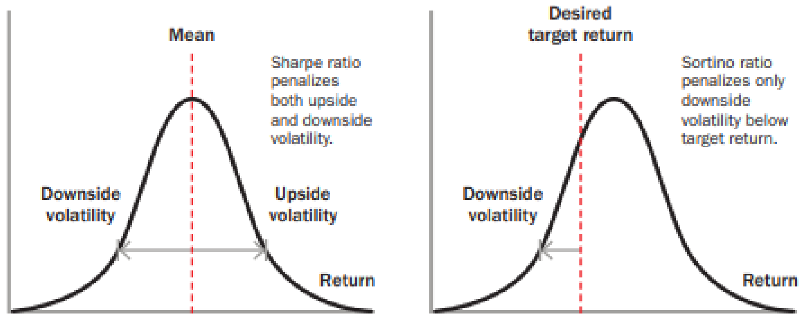

Figure 3.5: Sharpe ratio vs Sortino ratio

Source: RollingeR & Hoffman (2013)

3.3.3 The maximum drawdown

To define this measure, the notion of drawdown is first introduced.

3.3.4 Definition. Drawdown measure, DD (Goldberg & Mahmoud 2017)

A drawdown measures in percentage the decline of an investment (or an asset, trading) during a given period. The fall occurs when the value of the investment drops from a peak to a trough. The peak can be considered as the highest return of an investment or a portfolio, and the trough the lowest return, during a given period. The decline remains until the value of the investment recovers back to the peak. The time it takes to recover a drawdown is then a significant factor because as long as the value of the investment is below its previous peak, a lower trough could occur, increasing then the amount of the drawdown. This measure is expressed as:

$$\text{DD} = \frac{\text{peak value} - \text{trough value}}{\text{peak value}}.$$

Note that, a drawdown is not necessarily equivalent to a loss as viewed by investors. The drawdown measures the loss from a peak to a trough. Let's look at an example:

3.3.5 Example. Suppose an investor decides to buy an asset available for a value of \$1000. The value of the asset after rises to \$1200 (the asset reaches a peak) and at a later period drops to \$700 (a trough) and rises back to \$1200. By the time the asset dropped in value, the investor may have declared a loss of 30%, relative to the initial value the asset has been purchased. However, this investment has known a drawdown of $\frac{1200-700}{1200} \simeq 42\%$, relative to the highest value or peak the investment has reached.

If again the value of the asset rises to \$1400, drops to \$1250 and rises to \$1450. According to the initial cost of the asset, which is of \$700, the investor may not claim any loss; however, still, the investment has known a drawdown of 11%. This confusion usually arises because investors view failures according to the initial value of a trade.

The drawdown measure is beneficial for measuring the historical risk of different investments or portfolios and comparing their performances — the smaller the amount of the drawdown, the better the performance of the investment.

Assume an investment witnessed more than one drawdown during a specific period. The maximum drawdown of this investment will be the greatest period from a peak to a trough before recovering back to the peak, during the total investment period.

3.3.6 Definition. Maximum drawdown, MDD (Goldberg & Mahmoud 2017)

Maximum drawdown is a specific calculation of drawdown. It measures the size of the most significant loss or the largest drawdown an investment has achieved. However, it does not indicate how frequently this largest loss occurred, neither the time it took the investor to recover from this most significant drawdown.

The smaller the amount of the MDD, the better the performance of the investment. For potential forecasting strategy, investors may look at the historical drawdowns of their finances and consider their maximum drawdowns such that they can avoid if possible in the future the strategy used that led to a maximum drawdown.

3.3.7 The tracking error, TE

The TE optimisation described in this section can be found in Bertrand (2010), Jiao (2003), Jorion (2003) or (Maxwell & Vuuren 2019).

This measure evaluates in percentage the performance of a portfolio relative to a benchmark portfolio. The comparison is studying concerning the deviation of a portfolio's return from the benchmark return. In the sense that, investors seek the return of their portfolios to be as close as possible (or more) to the return of the benchmark, and would like to calculate then how far is their portfolio from the benchmark portfolio. Alternatively, such investors would like to quantify the risk of their portfolios relative to the benchmark. The smaller the risk, the better the portfolio tracks the benchmark, and the closer the return of the portfolio to the benchmark return.

In this section, the tracking error is then presented for measuring the risk of a portfolio relative to a benchmark portfolio.

Mathematically, the TE measure is expressed as the standard deviation of the active return (portfolio return minus benchmark return). Let R_{pt} the return of a portfolio at period t and R_{bt} the return of the benchmark the portfolio tracks, for $t \in [1, T]$. The TE is given by:

$$\text{TE} = \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (R_{at} - \bar{R_{at}})^2}\,, \tag{3.3.1}$$

where $R_{at} = R_{pt} - R_{bt}$ and $\bar{R_{at}}$ is its mean.

However, a tracking error efficient frontier can also be obtained. Since the investor would prefer his/her portfolio to achieve lower TE value, an optimisation problem for minimising the variance of the TE can be settled.

3.3.8 Definition. Tracking error optimisation (Jiao 2003)

This optimisation is expressed as Problem (3.1.1), developed by Markowitz (1952). However here, the vector of the assets weights is replaced by a vector of the assets active weights (portfolio's assets weights minus the benchmark portfolio's assets weights), denoted $Z_a = Z_p - Z_b$. The function to minimise is the variance of the TE, $Z_a^T C Z_a$, where C still represents the matrix of the assets' returns covariances as described in Section 2.2. The model is formulated as follows:

$$\begin{aligned} \text{minimise} \quad & Z_a^T C Z_a \\ \text{s.t} \quad & Z_a^T \mathbf{1} = \mu_b \\ & Z_a^T \mathbf{1} = 0\,. \end{aligned} \tag{3.3.2}$$

The second constraint is explained by the fact that since the weights of the portfolio Z_p must sum to 1, and as well the benchmark weights Z_b must sum to 1, then the active weights should sum to 0. Here μ_b represents the target portfolio return relative to the benchmark return.

Using the Lagrange method as in Section 3.1, an analytical solution to the problem is found:

Define the Lagrangian function as:

$$L(Z_a, \alpha_1, \alpha_2) = Z_a^T C Z_a + \alpha_1(\mu_p - Z_a^T \mu) + \alpha_2(-Z_a^T \mathbf{1})\,.$$

The first order conditions are given as:

$$\begin{cases} \dfrac{\delta L}{\delta Z_a} & = 2CZ_a - \alpha_1 \mu - \alpha_2 \mathbf{1} = 0_n \\ \\ \dfrac{\delta L}{\delta \alpha_1} & = Z_a^T \mu - \mu_p = 0 \\ \\ \dfrac{\delta L}{\delta \alpha_1} & = Z_a^T \mathbf{1} = 0\,. \end{cases}$$

From the first equation :

$$Z_a = \frac{1}{2}\alpha_1 C^{-1}\mu + \frac{1}{2}\alpha_2 C^{-1}\mathbf{1}\,.$$

Plugging the expression of w_a in the two last equations in the system of Equations :

$$\alpha_1 = \frac{2a\mu_p}{ac - b^2} \quad \text{and} \quad \alpha_2 = -\frac{2b\mu_p}{ac - b^2}\,,$$

where $a = \mathbf{1}^T C^{-1}\mathbf{1}$, $b = \mu^T C^{-1}\mathbf{1}$ and $c = \mu^T C^{-1}\mu$, with a, b, c constants. The optimal active weights are then expressed as:

$$Z_a^* = \frac{\mu_p(aC^{-1}\mu - bC^{-1}\mathbf{1})}{ac - b^2}\,.$$

And the optimised TE, or the variance of the TE, let's denote by σ_{TE}^2:

$$\begin{aligned} \sigma_{TE}^2 &= Z_a^{*T} C Z_a^* \\ &= \frac{a\mu_p^2}{ac - b^2}\,. \end{aligned}$$

Solving Problem (3.3.2) for different values of μ_b will result in construct a TE efficient frontier.

In the next chapter, the study focuses on the mean-semivariance model and the estimation of the semicovariance matrix. For the performance analysis, the Sharpe ratio, the Sortino ratio and the tracking error measures will be considered in this study.

4. Mean-semivariance framework

The mean-variance method for portfolio optimisation has been widely criticised. Markowitz proposed then in Markowitz (1959) the use of the semivariance as a measure of risk in portfolio management. The measure is recognised more plausible than the variance since it is applicable even when assets return distribution shows fatter tails. Mao (1970) provided supports on the fact that investors are only interested in downside risks and that the semivariance measure is more appropriate to use. As well Harlow (1991) gave support for measuring risk according to dispersions below a specific target return to achieve a more attractive risk-return tradeoff. This chapter is entirely consecrated to portfolio optimisation under the semivariance risk measure. Recall from Chapter 3:

Let n risky assets available in the market, observed for a period of T. Denote by r_i and μ_i the return and expected return of asset i, respectively, μ_p the portfolio expected return and Z_i the capital fraction allocated in asset i. The semivariance is defined as:

$$\text{Semi} = \frac{1}{T}\sum_{t=1}^{T} \min(R_{pt} - E[R_p], 0)^2 \,, \tag{4.0.1}$$

and the mean-semivariance model:

$$\begin{aligned}
\text{minimise} \quad & \frac{1}{T}\sum_{t=1}^{T} \min\,(R_{pt} - E[R_p], 0)^2 \\
\text{s.t} \quad & \sum_{i=1}^{n} z_i \mu_i = \mu_p \\
& \sum_{i=1}^{n} z_i = 1 \\
& z_i \geq 0\,.
\end{aligned} \tag{4.0.2}$$

Markowitz (1959) suggested to approach the problem as:

$$\begin{aligned}
\text{minimise}_{z_i} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j \Sigma_{i,j} \\
\text{s.t} \quad & \sum_{i=1}^{n} z_i \mu_i = \mu_p \\
& \sum_{i=1}^{n} z_i = 1 \\
& z_i \geq 0\,,
\end{aligned} \tag{4.0.3}$$

where $\Sigma_{i,j}$ represents the semicovariance between assets i and j, formulated as:

$$\Sigma_{i,j} = \frac{1}{T}\sum_{t=1}^{K}(r_{i,t} - B)(r_{j,t} - B)\,, \tag{4.0.4}$$

where B represents the investor's target, which is the mean of the assets returns, and K is the set of the period in which the portfolio's return underperforms the target return B. The problem is formulated as for the mean-variance model and provides an exact estimation of the portfolio semivariance. However, although it provides an exact solution to the portfolio semivariance, the solution is not as straightforward as for the mean-variance case. Indeed, unlike the covariance matrix, for which each parameter are externally estimated (exogenous) and then used as input in the matrix, it comes that the matrix Σ of semicovariances is endogenous. Since semivariance only analyses the downside risk, elements of the semicovariance matrix will depend on the portfolio returns that underperform the investor's target, and a change in the weights will affect the periods in which the portfolio underperforms the target, which in turn affects the elements of the semicovariance matrix.

4.1 The endogeneity of the semicovariance matrix

Let's repeat the same example found in Estrada (2008), to give more light on the endogeneity of the semicovariance matrix.

4.1.1 Example. Let two assets A and B with annual returns described in columns two and three in Table 4.1 below, for a period from year 1 to 10. Let also a portfolio of 80% investment in the asset A and 20% in asset B, call it portfolio 80-20. A second portfolio invested 10% in asset A and 90% in asset B, called portfolio 10-90. The returns of the portfolios are calculated as in Equation (2.1.4), described in columns four and five of the table.

Table 4.1: Example on the endogeneity of the semicovariance matrix

					Portfolio 80-20			Portfolio 10-90		
Year	A(%)	B(%)	$80-20$	$10-90$	A	B	Product	A	B	Product
1	31.0	−21.2	20.6	−16.0	0.0	0.0	0.0	31.0	−21.2	−6.6
2	26.7	−9.3	19.5	−5.7	0.0	0.0	0.0	26.7	−9.3	−2.5
3	19.5	36.8	23.0	35.1	0.0	0.0	0.0	0.0	0.0	0.0
4	−10.1	−27.2	−13.5	−25.5	−10.1	−27.2	2.8	−10.1	−27.2	2.8
5	−13.0	−23.5	−15.1	−22.5	−13.0	−23.5	3.1	−13.0	−23.5	3.1
6	−23.4	−18.6	−22.4	−19.1	−23.4	−18.6	4.4	−23.4	−18.6	4.4
7	26.4	24.5	26.0	24.7	0.0	0.0	0.0	0.0	0.0	0.0
8	9.0	7.6	8.7	7.7	0.0	0.0	0.0	0.0	0.0	0.0
9	3.0	40.2	10.4	36.5	0.0	0.0	0.0	0.0	0.0	0.0
10	13.6	6.9	12.3	7.6	0.0	0.0	0.0	0.0	0.0	0.0

Consider portfolio 80-20. Using the square root of equation in Equation (2.2.4) the standard deviation can be calculated. For that, the standard deviation of the assets, let denote σ_A and σ_B, need to first be obtained using Equation (2.2.1), and also their covariance. We have the mean of the two assets $\mu_A = 8.27$ and $\mu_B = 1.62$, so that $\sigma_A = 17.8\%$ and $\sigma_B = 24.1\%$. The covariance is found to be 0.0163, giving the portfolio's standard deviation as:

$$\sigma_{80-20} = \sqrt{(0.8)^2(0.178)^2 + (0.2)^2(0.241)^2 + 2(0.8)(0.2)(0.0163)} = 16.7\%\,.$$

Lets consider an expected return or a target benchmark of $B = 0\%$. Using Equation (4.0.1) and returns in the third column of the table, the semi deviation, (semivariance) of the portfolio 80-20 can be calculated and found as $\text{semi}_{80-20} = 9.6\%$, (92.16%).

Now applying the approach proposed by Markowitz. Using Equation (4.0.4), the term $(r_{i,t} - B)$, for $t \in K$, returns a value of 0% when the portfolio outperforms the benchmark and the value of the asset's return when the portfolio underperforms the benchmark. Results are shown in the sixth, seventh columns of the table, and the eighth column contains their products. The elements of the semicovariance matrix can then be calculated, and this gives:

$$\Sigma_{A,A} = 0.0082$$
$$\Sigma_{B,B} = 0.0164$$
$$\Sigma_{A,B} = 0.0102\,,$$

giving a semideviation of:

$$\text{semi}_{80-20} = \sqrt{(0.8)^2(0.0082) + (0.2)^2(0.0164) + 2(0.8)(0.2)(0.0102)} = 9.6\%\,,$$

which gives an exact estimation of the semivariance defined in Equation (4.0.1). Now changing the allocation, consider the other portfolio 10-90 and do the same calculations. The portfolio's returns are shown in the fifth column, and the elements of the semicovariance matrix will be calculated using data in the ninth, tenth and eleventh columns:

$$\Sigma_{A,A} = 0.0249$$
$$\Sigma_{B,B} = 0.0217$$
$$\Sigma_{A,B} = 0.0011\,,$$

values totally different from the ones gotten for portfolio 80-20, giving a semideviation of:

$$\text{semi}_{10-90} = \sqrt{(0.8)^2(0.0249) + (0.2)^2(0.0217) + 2(0.8)(0.2)(0.0011)} = 13.4\%\,.$$

This example shows how endogenous the semicovariance matrix is because its elements depend on the weights of the assets.

Recently, Jin et al. (2006) demonstrated the possible existence of a closed-form solution for the mean-semivariance model, presented in Theorem 4.1.3 below.

Let the optimisation problem:

$$\min_{Z\in\mathbb{R}^n} \quad E[(A+B^{'}Z)^{-}]^2\,, \tag{4.1.1}$$

where $B^{'}$ is the transpose of the matrix $B = (B_1, \cdots, B_n)$. The term $x^- = \min(x, 0)$, the variable Z is the vector of the assets weights and A, B_i, $i = 1, \cdots, n$ are random variables with $E[A^2] < +\infty$, $E[B_i^2] < +\infty$.

4.1.2 Lemma. If $E[B_i] = 0$, for all i, then problem in Lemma (4.1.1) admits a solution. The proof of this lemma can be found in (Jin et al. 2006).

4.1.3 Theorem. *For any initial budget, lets denote $a \in \mathbb{R}$, such that*

$$\sum_{i=1}^{n} z_i = a\,, \tag{4.1.2}$$

and for any expected portfolio's return $\mu_p \in \mathbb{R}$, problem in Equation (4.0.2) admits optimal solutions if and only if it admits feasible solutions.

Proof: The objective function in the model described in Equation (4.0.2) can be rewritten as:

$$E\left[\left(\sum_{i=1}^{n} z_i r_i - \sum_{i=1}^{n} z_i \mu_i\right)^{-}\right]^2$$
$$= E\left[\left(\sum_{i=1}^{n} z_i (r_i - \mu_i)\right)^{-}\right]^2 .$$

Let us denote $R_i = r_i - \mu_i$, and with a as the invested budget, if we extract asset 1 from Equation (4.1.2), we have $z_1 = a - \sum_{i=2}^{n} z_i$. The objective function becomes:

$$
\begin{aligned}
& E\left[\left((aR_1 - \sum_{i=2}^{n} z_i R_1) + \sum_{i=2}^{n} z_i R_i\right)^{-}\right]^2 \\
&= E\left[\left(aR_1 + \sum_{i=2}^{n} z_i (R_i - R_1)\right)^{-}\right]^2 .
\end{aligned}
$$

Problem described in Equation (4.0.2) is now written as:

$$
\begin{aligned}
\text{minimise}_{z_i \in \mathbb{R}^{n-1}} \quad & E\left[\left(aR_1 + \sum_{i=2}^{n} z_i (R_i - R_1)\right)^{-}\right]^2 \\
\text{s.t} \quad & \sum_{i=2}^{n} z_i (\mu_i - \mu_1) = \mu_p - a\mu_1 .
\end{aligned}
$$

Two cases are considered to solve the new problem. First, consider $\mu_i = \mu_1$ for all i, so that it is assumed $\mu_p = a\mu_1$. The problem is then reduced to:

$$
\text{minimise}_{z_i \in \mathbb{R}^{n-1}} \quad E\left[\left(aR_1 + \sum_{i=2}^{n} z_i (R_i - R_1)\right)^{-}\right]^2 , \tag{4.1.3}
$$

which has the same form as the optimisation problem in Equation (4.1.1) and admits optimal solutions by Lemma 4.1.2. The second case is to consider $\mu_i \neq \mu_1$, for any i, let us take the case $i = 2$, so we have:

$$
w_2 = \frac{\mu_p - a\mu_1}{\mu_2 - \mu_1} - \sum_{i=3}^{n} z_i \frac{\mu_i - a\mu_1}{\mu_2 - \mu_1} .
$$

Plugging this last equation into Equation (4.1.3), the problem becomes:

$$\text{minimise}_{z_i \in \mathbb{R}^{n-2}} \quad E\left[\left(aR_1 + \frac{\mu_p - a\mu_1}{\mu_2 - \mu_1}(R_2 - R_1) + \sum_{i=3}^{n} z_i((R_i - R_1) - \frac{\mu_i - a\mu_1}{\mu_2 - \mu_1}(R_2 - R_1))\right)^{-}\right]^2,$$

which also admits optimal solutions by Lemma 4.1.2. □

The result of the theorem is that the optimisation problem in Equations (4.0.2) is equivalent to a problem with a bounded closed feasible region (recalling the definition from Section 3.1.3), that leads to the existence of optimal solutions. Therefore, even with additional constraints, this result still holds as long as the sets of constraints are closed (Jin et al. 2006).

Given that the mean-semivariance model indeed admits optimal solutions, some researchers proposed techniques to treat the difficulty of dealing with the semicovariance matrix, such that the model can be easily solved as the mean-variance model.

4.2 A solution to the endogeneity problem

Several attempts have been developed to overcome the difficulty on the semicovariance matrix and to find a closed-form solution that makes the mean-semivariance approach to be addressed as the Markowitz's mean-variance model. In addition to the literature mentioned in Chapter 1, in Ang (1975), using linear programming an approximation of the semivariance proposed by Hogan & Warren (1972) is developed, the semi-linear deviation. This measure calculates in percentage terms for each observation, the potential loss from the investor's target return. Unlike the other proposed semivariances, which are either positive or zero, the semi-linear deviation is either negative or zero. Hogan & Warren (1974) proposed a solution which unfortunately allows only for the risk-free rate as the benchmark and still results in an asymmetric semicovariance matrix. Cumova & Nawrocki (2011) provided a proof that converts the exogenous asymmetric semicovariance matrix to a symmetric matrix allowing the existence of the closed-form solution. The approach is shown to produce better results than the proposed approaches by Markowitz and Estrada. The approach of Ballestero (2005) is used in Boasson, Boasson & Zhou (2011), with a comparative analysis study showing the performance of the mean-semivariance over the mean-variance model.

In this chapter, the approach proposed in de Athayde (2001) for the estimation of the semicovariances is presented. Athayde proposed an algorithm that solves the problem in Equations (4.0.3) using an iterative procedure presented as follows:

The algorithm

Using the same notations, to know: B for the benchmark return, $r_{i,t}$ and $R_{p,t}$ for the return of asset i and the portfolio's return respectively at period t, Z for the portfolio vector weights and μ for the vector of asset's expected returns. Let us denote by $P_{i,t} = r_{i,t} - B$ and by M the matrix with coefficients as in Equation (4.0.4), this is $\Sigma_{i,j} = \frac{1}{T}\sum_{t=1}^{K} P_{i,t}P_{j,t}$. The idea is to construct after a finite number of iterations, a semicovariance matrix M invariant to the portfolio. The iteration will be repeated on the optimisation problem:

$$\begin{aligned} \text{minimise}_Z \quad & Z^T M Z \\ \text{s.t} \quad & Z^T \mathbf{1} = 1\,, \end{aligned}$$

where Z^T is the vector transpose and $\mathbf{1}$ is a vector of 1 as defined in Section 3.1.1. The algorithm is described as follows:

Step 1: Start with an initial portfolio $Z_0 = (z_{0,1}, \cdots, z_{0,n})$, usually taken as an equally weighted portfolio. Collect in a set S_0 the periods when the returns of the portfolio Z_0 are under the target return B, and define the matrix M_0 for this portfolio as:

$$M_0 = \frac{1}{T}\sum_{t\in S_0}\begin{bmatrix} P_{1,t}^2 & P_{1,t}P_{2,t} & \dots & P_{1,t}P_{n,t} \\ P_{2,t}P_{1,t} & P_{2,t}^2 & \dots & P_{2,t}P_{n,t} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,t}^2P_{1,t} & P_{n,t}P_{2,t} & \dots & P_{n,t}^2 \end{bmatrix}.$$

Step 2: Find the portfolio Z_1 solution to the problem:

$$\begin{aligned} \text{minimise}_Z \quad & Z^T M_0 Z \\ \text{s.t} \quad & Z^T\mathbf{1} = 1\,. \end{aligned}$$

Using Lagrange method, we recall from in Equation (3.1.6), the solution to this problem is:

$$Z_1 = \frac{M_0^{-1}\mathbf{1}}{\mathbf{1}^T M_0^{-1}\mathbf{1}}\,.$$

As in the first step, select in a set S_1 the periods in which the portfolio Z_1 underperforms the benchmark and construct the matrix:

$$M_1 = \frac{1}{T}\sum_{t\in S_1}\begin{bmatrix} P_{1,t}^2 & P_{1,t}P_{2,t} & \dots & P_{1,t}P_{n,t} \\ P_{2,t}P_{1,t} & P_{2,t}^2 & \dots & P_{2,t}P_{n,t} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,t}^2P_{1,t} & P_{n,t}P_{2,t} & \dots & P_{n,t}^2 \end{bmatrix}.$$

Step 3: We repeat again the all process. Find the portfolio Z_2 that solves the problem:

$$\begin{aligned} \text{minimise}_Z \quad & Z^T M_1 Z \\ \text{s.t} \quad & Z^T\mathbf{1} = 1\,. \end{aligned}$$

Using Lagrange method, the solution to this problem is:

$$Z_2 = \frac{M_1^{-1}\mathbf{1}}{\mathbf{1}^T M_1^{-1}\mathbf{1}} \,.$$

Select the elements of S_2 and construct the matrix M_2 for $t \in S_2$ in the same way as done above.

The previous calculations are repeated in iterations. A sequence of matrices are constructed until the first matrix M_m such that $M_m = M_{m+1}$ is gotten. The optimal portfolio that will provide the global minimum downside risk portfolio is then given by:

$$Z_m^g = \frac{M_m^{-1}\mathbf{1}}{\mathbf{1}^T M_m^{-1}\mathbf{1}} \,.$$

Construct the efficient frontier

For a given portfolio's target return μ_p, the optimisation problem to solve is:

$$\begin{aligned} \text{minimise} \quad & \frac{1}{T}\sum_{t=1}^{T} \min\,(R_{pt} - B, 0)^2 = Z^T M_m Z \\ \text{s.t} \quad & Z^T \mu = \mu_p \\ & Z^T \mathbf{1} = 1 \,. \end{aligned}$$

Recall again that for the mean-semivariance model, the benchmark $B = E[R_p]$, and that the non-negativity constraint may or may not be required.

Using the Lagrange method, processing exactly as in Section 3.1.1, with the semi-define constructed matrix M_m after m iterations. For a fixed μ_p, the expression of the optimal portfolio is given by:

$$Z_m = M_m^{-1} \left(\frac{a\mu_p - b}{ac - b^2} \mu + \frac{c - b\mu_p}{ac - b^2} \mathbf{1} \right) ,$$

where $a = \mathbf{1}^T M_m^{-1} \mathbf{1}$, $b = \mu^T M_m^{-1} \mathbf{1}$, and $c = \mu^T M_m^{-1} \mu$, giving the minimum portfolio downside risk as:

$$Z_m^T M_m Z_m = \frac{a\mu_p^2 - 2b\mu_p + c}{ac - b^2} . \tag{4.2.1}$$

Varying the value of μ_p will construct a parabolic curve, which is the mean-semivariance efficient frontier.

Given the algorithm described for solving the problem on the endogeneity of the semicovariance matrix, the mean-semivariance model can now easily be applied for portfolio management. In the next chapter, using real financial data, we present a practical application of the theory explained above.

5. Empirical analysis and results

This chapter presents a comparative study between the mean-variance and the mean-semivariance models for portfolio optimisation. The analysis assumes no transaction costs; no taxes and short selling is allowed. The investor is assumed risk-averse and therefore seeks the portfolio with the least risk. Furthermore, the investor has the objective to manage the risk of his/her portfolio, such that the return on the investment is kept as close as possible to the return of a market index fixed as the benchmark. That is, the portfolio tracks the benchmark. The minimum-variance portfolio is solved as proposed in Markowitz (1952), and the minimum-semivariance portfolio is solved as proposed in de Athayde (2001) see Section 4.2.

5.1 Data

The analysis is based on real data obtained from the Johannesburg Stock Exchange (JSE) Top 40 index. This market is composed of the 40 biggest companies on the JSE, ranked by market capitalisation (number of shares times the current share's price). Even though among the 400 companies on the JSE only 40 are included in the Top 40 index, it represents over 80% of the total market capitalisation of all JSE's companies and thus gives an overall view of what happens to the South African stock market as a whole. In the remaining of the work, the companies are called assets.

The data collected are the close of day assets' prices over the May 2019 - July 2019 period and expressed in cent South African currency, giving a total of 600 observations. However, throughout the analysis, the assets' returns are instead used, calculated using the formula described in Equation (2.1.1). A universe of 10 companies of the JSE Top 40 is selected as sample. This selection is made randomly.

5.1.1 Summary statistics

Table 5.1 shows the statistics of the selected assets. The values of the third and the fourth moment, which are respectively the skewness and the kurtosis, can easily give an idea on the shape of a distribution. A skewness value different from zero will indicate the presence of more peaked tail than for the normal distribution.

Table 5.1: Summary statistics of the daily assets returns

Assets	Abbreviation	Min	Max	Mean	Std	Skewness	Kurtosis
Growthpoint Prop Ltd	GRT	−0.028246	0.021241	−0.000235	0.010382	−0.273926	0.551221
Imperial Logistics Ltd	IPM	−0.059232	0.031410	−0.004927	0.019072	−0.454898	0.285586
Impala Platinum Hlgs Ltd	IMP	−0.061887	0.060491	0.004915	0.025830	−0.262326	0.226407
Intu Properties Plc	ITU	−0.075360	0.034944	−0.005559	0.020788	−0.894064	1.548389
Investec Plc	INP	−0.027659	0.040258	−0.001408	0.013737	0.563527	1.130256
Investec Ltd	INL	−0.033708	0.036584	−0.001521	0.014365	0.218677	0.632366
Kumba Iron Ore Ltd	KIO	−0.073357	0.056523	0.002078	0.023890	−0.315303	0.522947
Life Healthc Grp Hldgs Ltd	LHC	−0.037490	0.037431	−0.002329	0.015311	0.048051	0.249872
Mondi Plc	MNP	−0.032913	0.029199	0.000231	0.014901	−0.012301	−0.685630
Mondi Ltd	MND	−0.030944	0.028286	−0.000251	0.014825	0.074218	−0.526131

The third and the fourth columns show respectively, the minimum and the maximum returns for each stock. The fifth column shows the mean returns of each asset. The standard deviations are shown in the sixth column, and the seventh and eighth columns show the values of the skewness and the kurtosis, respectively. The summary statistics are calculated over the sample period under analysis. It can be observed that:

$$-0.894064 \leq \text{Skewness} \leq 0.563527$$
$$-0.685630 \leq \text{Kurtosis} \leq 1.548389\,.$$

The boundaries of the skewness and the kurtosis (calculated using their respective expressions as described in Section 3.2.2) indicate that the shape of the distribution is skewed to the left. (See 5.1). This fact encourages the use of a downside risk measure in portfolio risk management.

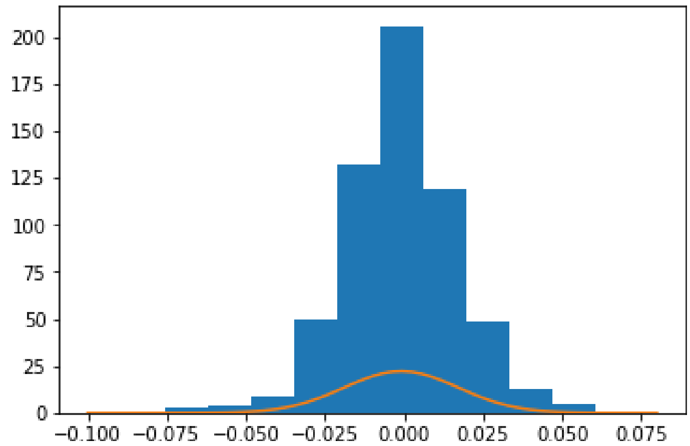

Figure 5.1: Histogram of the selected JSE Top 40 stocks

The possible correlations between the 10 assets are calculated, and the results are shown in Table 5.2 below:

Table 5.2: Matrix of correlation between assets

	GRT	**IPM**	**IMP**	**ITU**	**INP**	**INL**	**KIO**	**LHC**	**MNP**	**MND**
GRT	1									
IPM	0.357137	1								
IMP	0.080947	−0.029411	1							
ITU	−0.098249	−0.086547	−0.084453	1						
INP	0.006868	0.273102	0.104482	0.151029	1					
INL	0.113797	0.296715	0.121193	0.159769	0.970578	1				
KIO	−0.030523	−0.085350	0.353893	−0.055426	0.171763	0.147360	1			
LHC	0.385404	0.246839	0.134744	0.017395	0.251294	0.303149	0.071327	1		
MNP	−0.227135	−0.041787	0.213756	−0.001380	0.409198	0.377051	0.286389	0.150963	1	
MND	−0.200741	0.052363	0.211368	−0.000402	0.491535	0.464909	0.283200	0.191078	0.972536	1

This is a symmetric matrix such that, the correlation between assets GRT and IMP, for example, is the same as the correlation between IMP and GRT. These values indicate the relationship of the assets on the selected assets/companies. Recall from Section 2.3, a correlation of 1 means that two assets are correctly moving together. For example, INP-INL are almost perfectly correlated (correlation value closed to 1) to each other. If one drops in value, so will the other, and inversely if one rises in value, the other will also increase. As well as MNP-MND are also highly correlated.

5.2 Analysis of mean-variance and mean-semivariance approaches

In this section, results of portfolio optimisations from the mean-variance and the mean-semivariance approaches are presented.

The input to the mean-variance approach is the covariance matrix (as constructed in Section 2.2) presented in Table 5.3.

Table 5.3: Covariance matrix

	GRT	**IPM**	**IMP**	**ITU**	**INP**	**INL**	**KIO**	**LHC**	**MNP**	**MND**
GRT	0.0001078									
IPM	7.07E-05	0.000364								
IMP	2.17E-05	-1.44E-05	0.000667							
ITU	-2.12E-05	-3.43E-05	-4.53E-05	0.000432						
INP	9.80E-07	7.15E-05	3.71E-05	4.31E-05	0.000189					
INL	1.69724E-05	8.13E-05	4.45E-05	4.77E-05	0.000192	0.000206				
KIO	-7.57E-06	-3.89E-05	0.000218	-2.75E-05	5.63E-05	5.05E-05	0.000571			
LHC	6.13E-05	7.21E-05	5.32E-05	5.53E-06	5.28E-05	6.67E-05	2.61E-05	0.000234		
MNP	-3.51E-05	-1.19E-05	8.23E-05	-4.28E-07	8.37E-05	8.07E-05	0.000102	3.44E-05	0.000222	
MND	-3.09E-05	1.48E-05	8.09E-05	-1.24E-07	0.000100	9.90E-05	0.000100	4.34E-05	0.000215	0.000219

The table shows that asset INP, INL and LHC have positive covariances with each other assets. The degree of the different covariances is calculated by the correlations shown in Table 5.2, where the assets INP and INL, assets MNP and MND are very strongly correlated.

The input to the mean-semivariance approach is the semicovariance matrix (as constructed in Section 4.2) presented in Table 5.4.

Table 5.4: Semicovariance matrix

	GRT	**IPM**	**IMP**	**ITU**	**INP**	**INL**	**KIO**	**LHC**	**MNP**	**MND**
GRT	0.013458									
IPM	0.0104500	0.049349								
IMP	0.002676	−0.003260	0.096602							
ITU	−0.003510	0.000368	−0.026423	0.086421						
INP	−0.001043	0.007355	0.006588	0.007745	0.026114					
INL	0.0011800	0.008288	0.006502	0.009178	0.026928	0.029196				
KIO	−0.000697	−0.000858	0.032622	−0.019796	0.006401	0.006177	0.0743800			
LHC	0.008333	0.010346	−0.001875	0.000139	0.012133	0.014955	0.012269	0.023856		
MNP	−0.00421	0.005176	0.0098321	−0.003107	0.017976	0.017345	0.014834	0.0109685	0.034601	
MND	−0.003553	0.008777	0.008573	−0.002858	0.020800	0.020484	0.015901	0.012809	0.032533	0.032621

The results in Table 5.4 show the relationship between the assets as calculated by the downside deviations. Asset INL is the only one which is positively correlated to the other assets.

Given the matrices of covariances and semicovariances above, next is to solve the mean-variance and the mean-semivariance models as described in Equations (3.1.1) and (4.0.3). The performance measures in the analysis are the Sharpe and the Sortino ratios. Lets denote the minimum-variance portfolio by MV, and the minimum-semivariance portfolio by MS. The results of the optimisations from the two approaches - MV and MS are presented in Table 5.5.

Table 5.5: Minimum-risk portfolios

	Optimisation Approaches	
Portfolio Performance	**MV portfolio**	**MS portfolio**
Annual Return	−0.19	0.02
Annual Risk	0.09	0.06
Sharpe Ratio	−0.12	0.01
Sortino Ratio	−0.15	0.02
Portfolio Allocation		
GRT	0.52	0.72
IPM	0.04	−0.1
IMP	0.002	0.01
ITU	0.12	0.11
INP	0.81	1.07
INL	−0.73	−1.03
KIO	0.02	0.05
LHC	0.01	−0.04
MNP	0.34	0.25
MND	−0.15	−0.04

- The results show that the MS portfolio achieves a high annual return of 0.02 with an annual risk of 0.06, as compared to the MV portfolio which had an annual return of −0.19 with an annual risk of 0.09. The annual return from the MS portfolio is higher for less annual risk as compared to the MV portfolio, which had a higher annual risk. In terms of the performance measures, the MS portfolio has higher Sharpe and Sortino ratios..

- Asset INP constitutes the most significant component in the two portfolios with 0.81 in the MV portfolio and 1.07 in the MS portfolio, followed by asset GRT.

Asset MNP also constitutes an essential contribution to the optimal portfolios with a composition of 0.34 in the MV portfolio and 0.25 in the MS portfolio. These high allocations result from the fact that by selling some assets, the investor gets more fund than initially.

- The results show that there is more short selling in the MS portfolio, as compared to the MV portfolio. Asset INL is shorted the most in both portfolios.

The respective optimal portfolios can be shown on the efficient frontiers presented in Figure 5.2 and Figure 5.3.

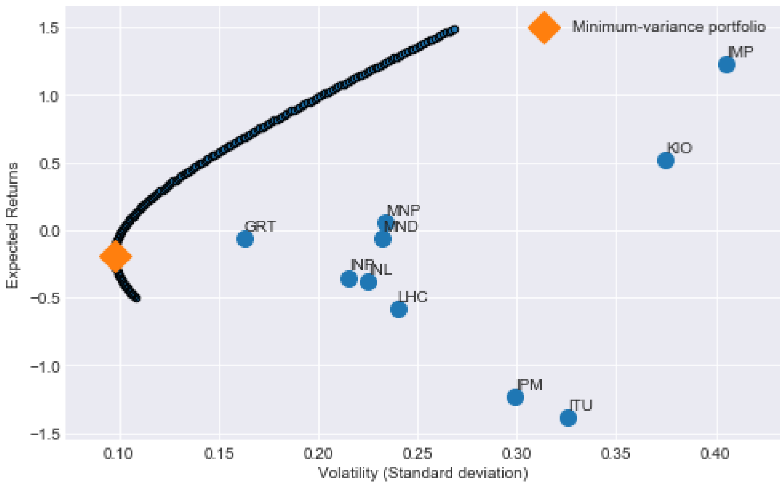

Figure 5.2: Mean-variance efficient frontier

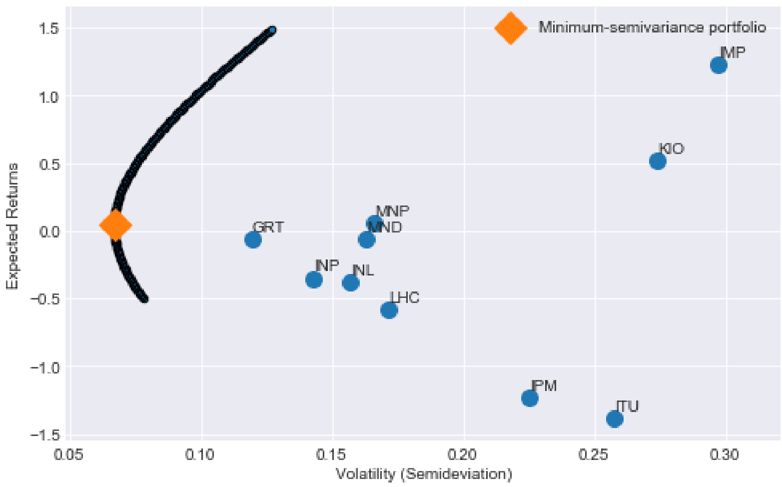

Figure 5.3: Mean-semivariance efficient frontier

The circles in the figures show each asset positioned according to its annual return and annual level of risk. The diamond on the different curves represents the minimum-variance and the minimum-semivariance portfolio, respectively, which are combinations of all the ten assets. The expected portfolios' returns on the mean-semivariance efficient frontier are achieved with less risk. The expected portfolios' returns on the mean-variance efficient frontier are achieved with more risk.

5.3 Tracking error analysis

Consider now that the investor wants his/her minimum-risk portfolio to track the JSE Top 40 benchmark portfolio. The goal is to construct a new portfolio that will have the same return as the benchmark and the same risk as to the minimum-risk portfolio. For this analysis, the mean JSE Top 40 for the period May 2019 - July 2019 is used as the benchmark portfolio return. The achieved mean return is of 0.09. A tracking error analysis, denoted by TE, is then investigated. From the formula in Equation

(3.3.1), the respective values of the TE are calculated. Lets call TE-MV and TE-MS respectively the minimum-variance and minimum-semivariance portfolios relative to the benchmark portfolio. The compositions of the new portfolios are shown in Table 5.6.

Table 5.6: Minimum-risk portfolios relative to the benchmark portfolio

	Tracking Error Analysis	
Portfolio Performance	**TE-MV portfolio**	**TE-MS portfolio**
TE	0.17	0.12
Portfolio Allocation		
GRT	0.63	0.77
IPM	−0.04	−0.12
IMP	0.06	0.01
ITU	0.05	0.1
INP	0.89	1.07
INL	−0.77	−1.03
KIO	0.03	0.04
LHC	−0.05	−0.06
MNP	0.47	0.36
MND	−0.27	−0.12

- The tracking error performance measure is less for the TE-MS portfolio (0.12) as compared to the TE-MV portfolio (0.17). This result indicates that the mean-semivariance approach tracks the benchmark better than the mean-variance method.
- The allocations show that the two optimal portfolios should go short in assets IPM, INL, LHC and MND to track the benchmark portfolio.

- Asset INP still constitutes the largest allocation in TE-MS portfolio (1.07) and TE-MV portfolio (0.89).
- The results show that there is not much difference in the allocations of the MSB portfolio as compared to the MS portfolio.

The respective efficient frontiers are shown in Figure 5.4 and Figure 5.5.

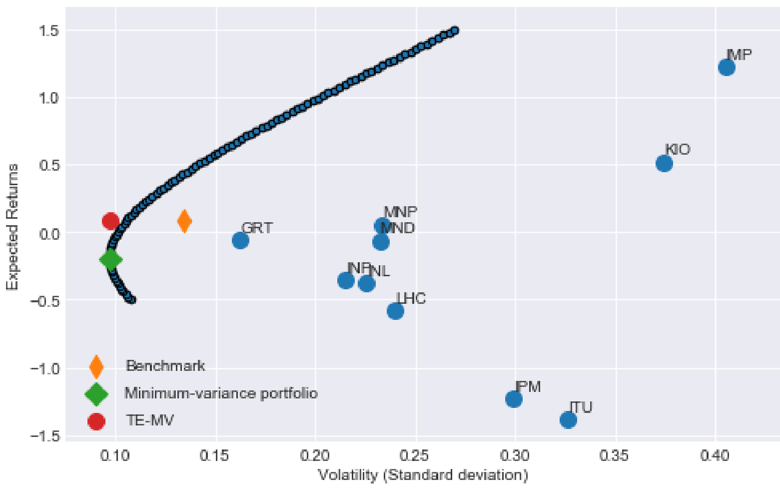

Figure 5.4: Minimum-variance portfolio relative to the benchmark

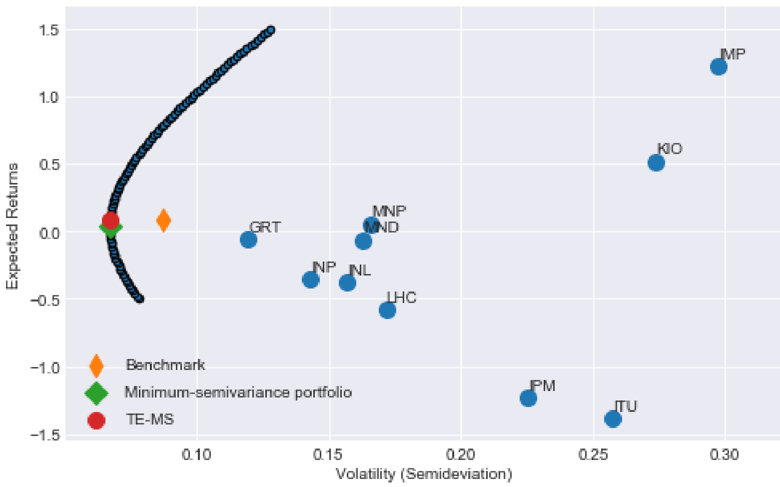

Figure 5.5: Minimum-semivariance portfolio relative to the benchmark

- The results show that the TE-MS portfolio is confused as a portfolio on the mean-semivariance efficient frontier, with a higher annual return of 0.09, as compared to the return of the MS portfolio which is of 0.02. In comparison to the benchmark portfolio, the TE-MS portfolio achieves the same annual return (0.09) however at lower risk (0.06) as compared to the risk of the benchmark portfolio (0.08).

- The TE-MV portfolio, however, with an annual return of 0.09 is clearly shown to be far from the MV portfolio, which has a lesser annual return of −0.19. In comparison to the benchmark portfolio, the TE-MV achieves the same annual return (0.09) for a lower risk of 0.09 as compared to the risk of the benchmark portfolio (0.13).

- The TE-MS portfolio achieves the same annual return than the benchmark portfolio with a smaller risk than does the TE-MV portfolio.

6. Conclusions and recommendations

The mean-variance model is the most popular used approach for portfolio selection because it possesses a closed-form solution, and it is mathematically simple to express. However, the model is also widely criticised because of its assumptions, specifically on normality in the distribution of the assets' returns and on using the variance as the measure of risk. Many alternatives have been proposed as downside risk approaches to consider the asymmetry in the analysis of assets' returns. However, the computation of a downside risk approach may not be as natural as for the mean-variance method. In this study, the mean-semivariance model was considered. Thus the study aimed to investigate the performance of the mean-semivariance approach in portfolio optimisation, in comparison to the mean-variance model. The Lagrange method for optimisation was used to find solutions to optimisation problems. Furthermore, the difficulty in the calculation of the semicovariance matrix could be resolved, resulting in a symmetric exogenous matrix, which could make the mean-semivariance model mathematically expressed as the mean-variance model. A sample of ten companies selected from the JSE Top 40 index was used for analysis and optimal portfolios constructed for both the mean-variance and the mean-semivariance approaches. The summary statistics showed that the distribution of the returns of the selected assets was skewed. This result hinted that the mean-semivariance approach performed much better than the mean-variance method. The following are the significant findings from the empirical analysis:

- The results from the minimum-risk portfolio optimisations showed that the mean-semivariance approach produced better annualised return than the mean-variance model. The mean-semivariance efficient frontier presented optimal portfolios at lower risk, as compared to the portfolios on the mean-variance efficient frontier. The differences in the allocations are due because semivariance as a risk measure only evaluates the downside deviations of the returns from a specified target return, unlike the variance measure which assesses both the upper and the downside deviations. Mean-semivariance optimisation provides lower portfolio risk than mean-variance optimisation, and this for the same level of return. These findings are in line with previous researches that also focused on comparing the results from the two models (Ballestero (2005), Boasson et al. (2011), (de Athayde 2001)). Alternatively, investors following a mean-semivariance strategy can reduce the risk of their portfolios while achieving the same or higher returns comparing to mean-variance portfolios.

- The results from the tracking error analysis showed that the mean-semivariance approach produced allocations that tracked better the benchmark as compared to the mean-variance method. Again, the differences in the allocations are because the mean-semivariance approach uses the semi deviation in the tracking error calculation, unlike the mean-variance method, which uses the standard deviation. As a result, the tracking error on the mean-semivariance penalises the downside deviations only which can reduce the tracking error on the benchmark.

In conclusion, this study only focused on portfolio selection for equities. Other assets, such as bonds or derivatives, can be considered for future research. Besides, the method used to construct the optimal portfolios and the efficient frontiers is limited for constraints in the equality form. However, investors may always need to put some restrictions on their investments, such as upper and lower bounds for the capital to be invested in each asset. Other measures of risk than the semivariance such as the VAR and the CVAR can be used for comparison relative to the variance. These shortcomings are factors that may affect the findings explained above and can constitute areas for future research.

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