Qualitative Theory of Dynamical Systems (2024) 23:175
https://doi.org/10.1007/s12346-024-01035-7

Traveling Wave Solutions for Two Perturbed Nonlinear Wave Equations with Distributed Delay

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Received: 15 August 2023 / Accepted: 3 April 2024 / Published online: 26 April 2024
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Abstract
Traveling wave solutions are a class of invariant solutions which are critical for shallow water wave equations. In this paper, traveling wave solutions for two perturbed KP-MEW equations with a local delay convolution kernel are examined. The model equation is reduced to a planar near-Hamiltonian system via geometric singular perturbation theorem, and the qualitative properties of the corresponding unperturbed system are analyzed by using dynamical system approach. The persistence of the bounded traveling wave solutions for the perturbed KP-MEW equations with delay is investigated. By using a criterion for the monotonicity of ratio of two Abelian integrals and Melnikov's method, the existence of kink (anti-kink) wave solutions and periodic wave solutions of the model equation are established. The result shows that the delayed KP-MEW equations with positive perturbation and the one with negative perturbation exhibit completely diverse dynamical properties. These new findings greatly enrich the understanding of dynamical properties of the traveling wave solutions of perturbed nonlinear wave equations with local delay convolution kernel. Numerical experiments further confirm and illustrate the theoretical results.

Keywords Periodic wave solution · Convolution kernel · Perturbation · Kink wave solution

Mathematics Subject Classification 35B20 · 35L05 · 34D15

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1 Introduction

Shallow water wave equations, which are always represented as partial differential equations, usually appear in nonlinear physical applications. The well-known Kadomtsev-Petviashvili (KP) equation [1, 2]

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{1}$$

that is a two-dimensional generalization of classical KdV equation, was proposed by Kadomtsev and Petviashvili in 1970 to simulate the evolution of long waves in nonlinear weakly dispersive medium.

The modified equal width (MEW) equation [3, 4], constructed for small-amplitude long waves propagating on the surface of water in a channel, is given by the normalized form

$$u_t + (u^3)_x - au_{xxt} = 0, \tag{2}$$

where a is a nonzero constant. The MEW Eq. (2) has solitary wave solutions with both positive and negative amplitudes, all of which have the same width. In 2005, Wazwaz [5] proposed the following MEW equation in the KP sense (KP-MEW)

$$(u_t + (u^3)_x - au_{xxt})_x + u_{yy} = 0 \tag{3}$$

and obtained exact traveling wave solutions by using tanh and sine-cosine method. Seadawy [6] constructed the solitary wave solutions of a generalized KP-MEW equation by utilizing modification form of extended auxiliary equation mapping method.

Bounded traveling wave solutions, such as solitary wave solution, periodic wave solution and kink wave solution, are one class of important solutions for nonlinear wave equations which have attracted extensive research attentions due to their special properties and physical applications in modeling nonlinear phenomena [7–11]. There are numerous techniques to get the exact solutions of nonlinear wave equations, such as extended sinh-Gordon equation expansion and $(\frac{G'}{G^2})$-expansion function methods [12], Hirota bilinear technique [13], extended direct algebraic method [14], ansatz approach [15], Lie symmetry method [16–19], and so on. Among which, dynamical system approach is a powerful tool for studying the traveling wave solutions of integrable nonlinear wave equations [20–25] as well as investigating other mathematical models, such as predator–prey systems in biological mathematics [26–30].

In the current work, we are interested in whether the smooth traveling waves of KP-MEW equation persist under small perturbation and distributed delay, and how it affects the wave motion. It is well known that geometric singular perturbation theory [31, 32] and Melnikov's method [33, 34] have been extensively applied to explore this phenomena. The existence of solitary and periodic waves of a generalized KP-MEW-Burgers equation with small damping and weak delay kernel

$$(u_t + a((f * u)u)_x + \tau u_{xx} + bu_{xxt})_x + ru_{yy} = 0 \tag{4}$$

is studied in [35]. The monotonicity of limit wave speed is also investigated therein by analyzing the ratio of Abelian integrals. In [36], Li et al. have established the existence of solitary wave solutions for CH-KP equation with a local delay convolution kernel. Wang et al. have considered a generalized BBM equation with distributed delay and dissipative perturbation and obtained a new type of solitary wave solutions with coexisting crest and trough [37]. Using linear chain trick and geometric singular perturbation analysis, Zhao [38] has proved that solitary wave solutions persist for a generalized KdV equation with a special convolution kernel and backward diffusion perturbation. There are some other excellent works on this topic, see for example [39–48].

Motivated by the literature mentioned above, the current work deals with the existence of bounded traveling wave solutions for two KP-MEW equations with distributed delay and dispersive perturbation

$$(u_t + ((f * u)u^2)_x - \tau u_{xx} + au_{xxt})_x + u_{yy} = 0 \tag{5}$$

and

$$(u_t + ((f * u)u^2)_x + \tau u_{xx} + au_{xxt})_x + u_{yy} = 0, \tag{6}$$

where $0 < \tau \ll 1$, $a > 0$ is a constant, and the convolution $f * u$ denotes the distributed delay defined by

$$(f * u)(x, y, t) = \int_{-\infty}^{t} f(t - s)u(x, y, s)ds,$$

where f is the kernel function usually given in two forms: $f(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}$ (weak generic delay kernel) and $f(t) = \frac{t}{\tau^2}e^{-\frac{t}{\tau}}$ (strong generic delay kernel). Herein, we choose the strong generic delay kernel. Note that both equations reduce to MEW-KP Eq. (3) by letting $\tau \to 0$.

2 Preliminaries and Main Results

In order to investigate the existence of bounded traveling wave solutions for KP-MEW Eq. (5) with delay and perturbation, let $u(x, y, t) = \phi(\xi)$, where $\xi = x + y - ct$ is the traveling wave variable and c denotes the wave speed. Then Eq. (5) is converted into the following fourth-order ordinary differential equation

$$(-c\phi' + (\omega\phi^2)' - \tau\phi'' - ac\phi''')' + \phi'' = 0 \tag{7}$$

with

$$\omega = (f * \phi)(\xi) = \int_0^{\infty} \frac{z}{\tau^2}e^{-\frac{z}{\tau}}\phi(\xi + cz)dz,$$

where $'$ represents the derivative with respect to ξ. Integrating (7) twice and omitting the integral constant, we have

$$(1-c)\phi + \omega\phi^2 - \tau\phi' - ac\phi'' = 0. \tag{8}$$

Differentiate ω with respect to ξ to get

$$\begin{aligned}\omega'(\xi) &= \frac{1}{c}\int_0^\infty \frac{z}{\tau^2} e^{-\frac{z}{\tau}} d\phi(\xi + cz) \\ &= \frac{1}{c\tau}\left(\int_0^\infty \frac{z}{\tau^2} e^{-\frac{z}{\tau}} \phi(\xi + cz)dz - \int_0^\infty \frac{1}{\tau} e^{-\frac{z}{\tau}} \phi(\xi + cz)dz\right) \\ &= \frac{1}{c\tau}(\omega - \varphi),\end{aligned}$$

where $\varphi = \int_0^\infty \frac{1}{\tau} e^{-\frac{z}{\tau}} \phi(\xi + cz)dz$. Here, we use the integration by part and the function $\phi(\cdot)$ is bounded on $(-\infty, +\infty)$ since we only consider bounded traveling wave solutions in this context. Similarly, differentiate φ with respect to ξ to have $\varphi' = \frac{1}{c\tau}(\varphi - \phi)$. Therefore,

$$\omega' = \frac{1}{c\tau}(\omega - \varphi), \quad \varphi' = \frac{1}{c\tau}(\varphi - \phi), \tag{9}$$

and Eq. (8) is equivalent to the following four-dimensional system:

$$\phi' = \psi, \quad \psi' = \frac{1-c}{ac}\phi + \frac{1}{ac}\omega\phi^2 - \frac{\tau}{ac}\psi, \quad c\tau\omega' = \omega - \varphi, \quad c\tau\varphi' = \varphi - \phi, \tag{10}$$

which is the so-called slow system. By transformation $\xi = \tau\sigma$, it can be transformed into a fast system

$$\dot{\phi} = \tau\psi, \quad \dot{\psi} = \tau\left(\frac{1-c}{ac}\phi + \frac{1}{ac}\omega\phi^2 - \frac{\tau}{ac}\psi\right), \quad \dot{\omega} = \frac{1}{c}(\omega - \varphi), \quad \dot{\varphi} = \frac{1}{c}(\varphi - \phi), \tag{11}$$

where $\cdot$ is the derivative with respect to σ. Set $\tau = 0$ in system (10), then its flow is restricted to a two-dimensional critical manifold

$$M_0 = \{(\phi, \psi, \omega, \varphi) \in \mathbb{R}^4 | \omega = \phi, \ \varphi = \phi\},$$

which is exactly the equilibrium point set of system (11) with $\tau = 0$. One can easily find that the linearized matrix of system (11) with $\tau = 0$ at each point of M_0 has eigenvalues $0, 0, \frac{1}{c}, \frac{1}{c}$, and the number of the eigenvalues with zero real part exactly equals to the dimension of M_0. Hence, M_0 is normally hyperbolic with two unstable normal directions. According to Fenichel's theorem [31, 32], there exists a locally invariant sub-manifold M_τ that is diffeomorphic to M_0 under the flow of (10). M_τ can be expressed as

$$\begin{aligned} M_\tau = \{(\phi, \psi, \omega, \varphi) \in \mathbb{R}^4 | \omega = \phi + \tau g_1(\phi, \psi) + O(\tau^2), \\ \varphi = \phi + \tau h_1(\phi, \psi) + O(\tau^2)\}, \end{aligned} \tag{12}$$

where g_1 and h_1 are smooth functions. Substituting (12) into the last two equations of the slow system (10) and equating the coefficients of τ of both sides of the two equations, one has $g_1 = 2c\psi,\ h_1 = c\psi$. Therefore, restricted to M_τ, system (10) can be written as a near-Hamiltonian planar system

$$\phi' = \psi, \quad \psi' = \frac{1-c}{ac}\phi + \frac{1}{ac}\phi^3 + \frac{\tau}{a}\left(2\phi^2 - \frac{1}{c}\right)\psi + O(\tau^2). \tag{13}$$

The main results in this paper are the following two theorems.

Theorem 1 *For $0 < \tau \ll 1$ and $a > 0$, the delayed KP-MEW Eq. (5) has*
(1) a kink and an anti-kink wave solution whose wave speed $c(\tau) \to \frac{1+\sqrt{11}}{2}$ as $\tau \to 0$;
(2) a periodic wave solution with wave speed $c > \frac{1+\sqrt{11}}{2}$;

Theorem 2 *For $0 < \tau \ll 1$ and $a > 0$, the delayed KP-MEW Eq. (6) has a periodic wave solution for an arbitrary wave speed $c < 0$;*

3 Proof of the Results

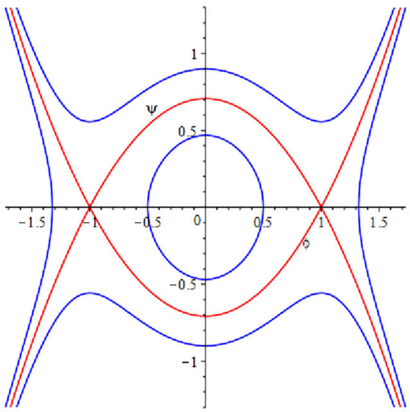
In Sect. 2, Eq. (5) is transformed into an ordinary differential equation (ODE) with strong delay convolution kernel under traveling wave framework. The ODE is further reduced to the planar system (13) on a 2-dimensional invariant manifold by geometric singular perturbation theory. We will next prove Theorem 1 by showing the existence (or nonexistence) of heteroclinic and periodic orbits of system (13). Thus, it is necessary to examine all the bounded orbits of system (13) with $\tau = 0$ at first. And then we examine the persistence of these orbits as τ is sufficiently small by using Melnikov's method.

One can easily see that when $0 < c \leq 1$, system (13) with $\tau = 0$ has a unique equilibrium point $O(0, 0)$ which is a saddle, and thus it has no bounded orbit other than the equilibrium point itself. Therefore here we only consider the case for $c > 1$ or $c < 0$.

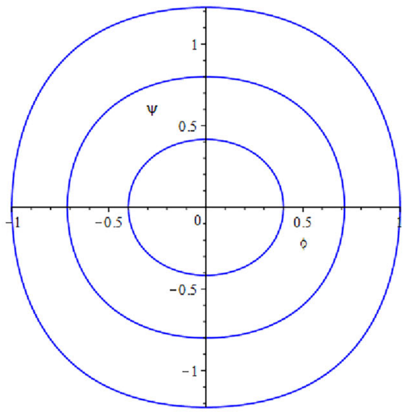
Case (I) $c > 1$.

Letting $\phi = \sqrt{c-1}\tilde{\phi}$, $\psi = \frac{c-1}{\sqrt{ac}}\tilde{\psi}$, $\xi = \sqrt{\frac{ac}{c-1}}\zeta$, $\tilde{\tau} = 2\tau\sqrt{\frac{c(c-1)}{a}}$ and $\nu = \frac{1}{2c(c-1)}$, system (13) reduces to (dropping the tiles)

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \psi, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = \phi^3 - \phi + \tau(\phi^2 - \nu)\psi + O(\tau^2). \tag{14}$$



(a) $c > 1$



(b) $c < 0$

Fig. 1 Phase portraits of the unperturbed system

Clearly, system (14)$|_{\tau=0}$ has Hamiltonian function

$$H(\phi, \psi) = \frac{1}{2}\psi^2 + \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4, \tag{15}$$

and three equilibrium points $O(0, 0)$, $E_1(1, 0)$ and $E_2(-1, 0)$. $O(0, 0)$ is a center while $E_1(1, 0)$ and $E_2(-1, 0)$ are saddles. The Hamiltonian at the three equilibrium points are $h_O = H(0, 0) = 0$, $h_{E_1} = h_{E_2} = H(\pm 1, 0) = \frac{1}{4}$. There are two heteroclinic orbits $\Gamma_\pm = \left\{(\phi, \psi)|\psi = \pm\frac{\sqrt{2}}{2}(1-\phi^2), |\phi| < 1\right\}$, both of which connect the saddles $E_1(1, 0)$ and $E_2(-1, 0)$. Inside the region enclosed by γ_+ and γ_-, there is a family of periodic orbits $\Gamma_1(h)$ surrounding the center $O(0, 0)$ (see Fig. 1a) which are given by $\Gamma_1(h) = \left\{(\phi, \psi)|\frac{1}{2}\psi^2 + \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 = h, h \in \left(0, \frac{1}{4}\right)\right\}$.

To explore the bounded traveling wave solutions of the delayed KP-MEW Eq. (5), we now examine the bounded orbits of system (14). The bifurcation has been reported in [49]. However, only the results are listed therein and a more detailed work is still open. We utilize the method introduced by Li and Zhang [50] to show the monotonicity of the ratio of two Abelian integrals, which has been well applied in [51, 52]. For convenience, we state the related criterion as the following Lemma.

Lemma 1 [50] *Suppose that a Hamiltonian function has the form $H(x, y) = \phi(x)y^2 + \Phi(x)$, where $\phi, \Phi \in C^1[\alpha, A]$ and $\phi(x) > 0$. Let Γ_h be the compact component of level curve $H(x, y) = h$, $h_1 < h < h_2$. Assume that there exists $a \in (\alpha, A)$ such that*

(i) $\Phi'(x)(x - a) > 0$ (or < 0) for $x \in (\alpha, A)\backslash\{a\}$,
(ii) $f_1(x) f_1(\tilde{x}) > 0$ for $x \in (\alpha, a)$,
(iii) $\zeta'(x) > 0$ (resp. < 0) for $x \in (\alpha, a)$,

then the ratio of two Abelian integrals $\omega(h) = \frac{\int_{\Gamma_h} f_2(x)ydx}{\int_{\Gamma_h} f_1(x)ydx}$ is decreasing (resp. increasing) for $h_1 < h < h_2$, where $\tilde{x} = \tilde{x}(x)$ is defined by $\Phi(x) = \Phi(\tilde{x})$ for $x \le a \le \tilde{x}$,

$f_1(x)$ and $f_2(x)$ are continuous, the discriminant function is given by

$$\zeta(x) = \frac{f_2(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_2(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}{f_1(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_1(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}.$$

Proposition 1 *For $0 < \tau \ll 1$,*

(1) there exists a function $\nu(\tau) = \frac{1}{5} + O(\tau)$ such that system (14) with $\nu = \nu(\tau)$ has two heteroclinic orbits;
(2) system (14) with $0 < \nu < \frac{1}{5}$ has a unique periodic orbit.

Proof The Melnikov function for system (14) reads

$$M_1(h, \nu) = \int_{\Gamma} (\phi^2 - \nu)\psi d\phi = I_0(h)(P(h) - \nu), \tag{16}$$

where $P(h) = \frac{I_2(h)}{I_0(h)}$, $I_k(h) = \int_{\Gamma} \phi^k \psi d\phi$ $(k = 0, 2)$, Γ denotes one of the periodic orbits $\Gamma_1(h)$ $(h \in \big(0, \frac{1}{4}\big))$ or the heteroclinic orbits $\Gamma_{\pm}$.

When Γ denotes $\Gamma_1(h)$, $I_k = 2\int_{-\alpha(h)}^{\alpha(h)} \phi^k \psi d\phi$, where $\psi = \sqrt{2h - \phi^2 + \frac{1}{2}\phi^4}$, $\Gamma_1(h)$ intersects the ϕ-axis at $(\pm\alpha(h), 0)$, $\alpha(h) > 0$. We have

$$\lim_{h \to h_O} P(h) = \lim_{h \to 0} \frac{\int_{-\alpha(h)}^{\alpha(h)} \phi^2 \psi d\phi}{\int_{-\alpha(h)}^{\alpha(h)} \psi d\phi} = 0.$$

When Γ denotes γ_+ or γ_-, $I_k = \int_{-1}^{1} \phi^k \psi d\phi$, where $\psi = \frac{\sqrt{2}}{2}(1 - \phi^2)$, one has

$$\lim_{h \to h_{E_1}} P(h) = P\left(\frac{1}{4}\right) = \frac{\int_{-1}^{1} \phi^2 \psi d\phi}{\int_{-1}^{1} \psi d\phi} = \frac{1}{5}.$$

Clearly, $I_0(h) > 0$ for $h \in (0, \frac{1}{4}]$, thus, $M_1(\frac{1}{4}, \nu)$ has a simple zero $\nu = \frac{1}{5}$. From implicit function theorem, there exists a function $\nu(\tau) = \frac{1}{5} + O(\tau)$ such that system (14) has two heteroclinic orbits.

We now examine the monotonicity of $P(h)$ for $h \in (0, \frac{1}{4})$. Let $\Phi(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4$, $\tilde{\phi} = \tilde{\phi}(\phi)$ is defined by $\Phi(\phi) = \Phi(\tilde{\phi})$ for $\phi < 0 < \tilde{\phi}$, $f_1(\phi) = 1$, $f_2(\phi) = \phi^2$ and

$$\zeta(\phi) = \frac{f_2(\phi)\Phi'(\tilde{\phi}) - f_2(\tilde{\phi})\Phi'(\phi)}{f_1(\phi)\Phi'(\tilde{\phi}) - f_1(\tilde{\phi})\Phi'(\phi)}. \tag{17}$$

By Lemma 1, it suffices to prove the following three conditions:
(C1) $\Phi'(\phi)(\phi - 0) > 0$ for $\phi \in (-1, 1)\backslash\{0\}$;
(C2) $f_1(\phi) f_1(\tilde{\phi}) > 0$ for $\phi \in (-1, 0)$;
(C3) $\zeta'(\phi) < 0$ for $\phi \in (-1, 0)$.

Note that $\Phi'(\phi)(\phi - 0) = \phi^2(1-\phi^2) > 0$ for $\phi \in (-1, 1)\backslash\{0\}$ and $f_1(\phi) \equiv 1$, thus (C1) and (C2) hold. From $\Phi(\phi) = \Phi(\tilde{\phi})$, one has

$$(\phi + \tilde{\phi})(\phi - \tilde{\phi})(\phi^2 + \tilde{\phi}^2 - 2) = 0, \tag{18}$$

which implies that $\tilde{\phi} = -\phi$ for $-1 < \phi < 0 < \tilde{\phi} < 1$. Thus, we have $\zeta'(\phi) = 2\phi < 0$ for $\phi \in (-1, 0)$. By Lemma 1, one sees that $P(h)$ is increasing on $(0, \frac{1}{4})$ and $0 < P(h) < \frac{1}{5}$. The monotonicity of $P(h)$ implies that $M_1(h, \nu)$ has a unique simple zero $h^* \in (0, \frac{1}{4})$ for $0 < \nu < \frac{1}{5}$. According to Melnikov's theorem [33, 34], we know that system (14) has a unique periodic orbit for $0 < \nu < \frac{1}{5}$. □

Substituting $\nu = \frac{1}{2c(c-1)}$ into Proposition 1 and noting that $c > 1$, we get the existence of traveling wave solutions in terms of the wave speed c for Eq. (5). This completes the proof of Theorem 1.

Case (II) $c < 0$.

Letting $\phi = \sqrt{1-c}\tilde{\phi}$, $\psi = \frac{1-c}{\sqrt{-ac}}\tilde{\psi}$, $\xi = \sqrt{\frac{-ac}{1-c}}\zeta$, $\tilde{\tau} = 2\tau\sqrt{\frac{c(c-1)}{a}}$ and $\delta = \frac{1}{2c(1-c)}$, system (13) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \psi, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = -\phi - \phi^3 + \tau(\phi^2 - \delta)\psi + O(\tau^2). \tag{19}$$

System $(19)|_{\tau=0}$ has the Hamiltonian

$$H(\phi, \psi) = \frac{1}{2}\psi^2 + \frac{1}{2}\phi^2 + \frac{1}{4}\phi^4, \tag{20}$$

and a unique equilibrium point $O(0, 0)$, which is a center. There exists a family of periodic orbits

$$\Gamma_2(h) = \left\{(\phi, \psi)|\frac{1}{2}\psi^2 + \frac{1}{2}\phi^2 + \frac{1}{4}\phi^4 = h, h \in (0, +\infty)\right\},$$

which tends to the center $O(0, 0)$ as $h \to 0$. See Fig. 1b.

The Melnikov function for system (19) is

$$M_2(h, \delta) = \oint_{\Gamma_2(h)} (\phi^2 - \delta)\psi d\phi = I_0(h)(Q(h) - \delta), \tag{21}$$

where $Q(h) = \frac{I_2(h)}{I_0(h)}$, $I_k(h) = \oint_{\Gamma_2(h)} \phi^k \psi d\phi$ $(k = 0, 2)$.

By applying the same method as in Proposition 1, we can easily prove that $Q(h)$ is increasing on $(0, +\infty)$ and $Q(h) > 0$. Note that $\delta = \frac{1}{2c(1-c)} < 0$. Thus, $M_2(h, \delta) > 0$ for $h \in (0, +\infty)$. It indicates that system (19) has no periodic orbit if $0 < \tau \ll 1$. However, recall that the integral constants are omitted in Eq. (8), so we still can not determine whether the delayed KP-MEW Eq. (5) has periodic wave solution. Theorem 2 can be proved similarly, so we ignore it.

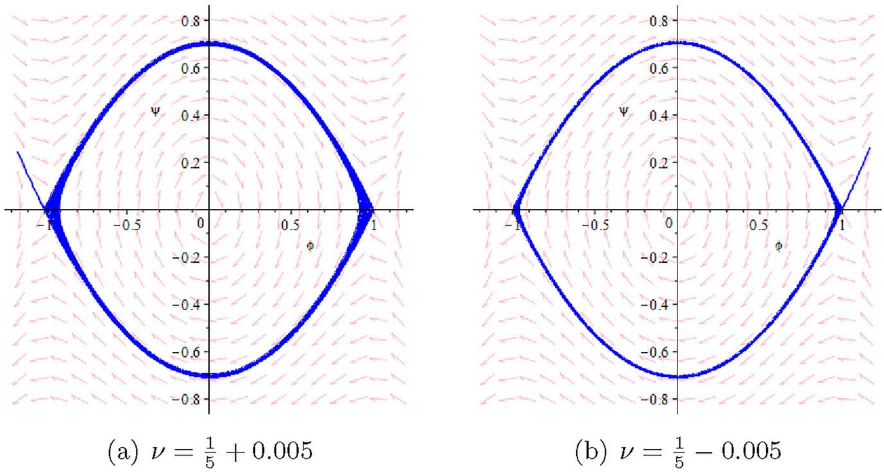


Fig. 2 Trajectories of system (14) with $\tau = 0.01$ and $(\phi(0), \psi(0)) = (0, \frac{\sqrt{2}}{2})$

4 Numerical Experiments

We now clarify the derived theoretical results via numerical analysis and computations. The existence of heteroclinic and periodic orbits for system (14) is shown numerically to illustrate the existence of kink and periodic wave solution of the original model equation, respectively.

For system (14), let $\tau = 0.01$ since τ is small, and the initial state $(\phi(0), \psi(0)) = (0, \frac{\sqrt{2}}{2})$, which is the intersection point of the heteroclinic orbit of the unperturbed system $(14)|_{\tau=0}$ with the positive half axis of the $\psi-$axis. Then in terms of the condition of Proposition 1, we plot its trajectories by taking $\nu = \frac{1}{5} + 0.005$ and $\nu = \frac{1}{5} - 0.005$, respectively (See Fig. 2). From the relative position of the unstable manifold at $(-1, 0)$ and the stable manifold at $(1, 0)$, one will find that there exists a heteroclinic orbit in the upper half plane and the bifurcation value $\nu \in (0.195, 0.205)$. The existence of heteroclinic orbit in the lower half plane can be simulated similarly.

To present the existence of periodic orbit, the profile of function $P(h)$ for $h \in (0, \frac{1}{4})$ is shown in Fig. 3a. It is clearly consistent with our theoretical analysis on $P(h)$. Let $\tau = 0.1$, $h = \frac{4}{25}$, $\nu = P(\frac{4}{25}) \approx 0.0966$ in system (14), then we can plot two trajectories by taking $(\phi(0), \psi(0)) = (0.8, 0)$ and $(\phi(0), \psi(0)) = (0.5, 0)$, respectively. In Fig. 3b, the blue curve represents the trajectory passing through $(0.8, 0)$ and the red one represents the trajectory passing through $(0.5, 0)$. Figure 3c and d show the corresponding time traveling curve of $\phi(\zeta)$. We can easily observe that both trajectories tend to a periodic orbit as $\zeta \to -\infty$, that indicates the existence of an unstable limit cycle.

In short, Fig. 2 demonstrates the existence of heteroclinic orbits by determining the relative positions between stable and unstable manifolds at the two saddle points, respectively. Figure 3 shows that two trajectories tend to the same periodic orbit as

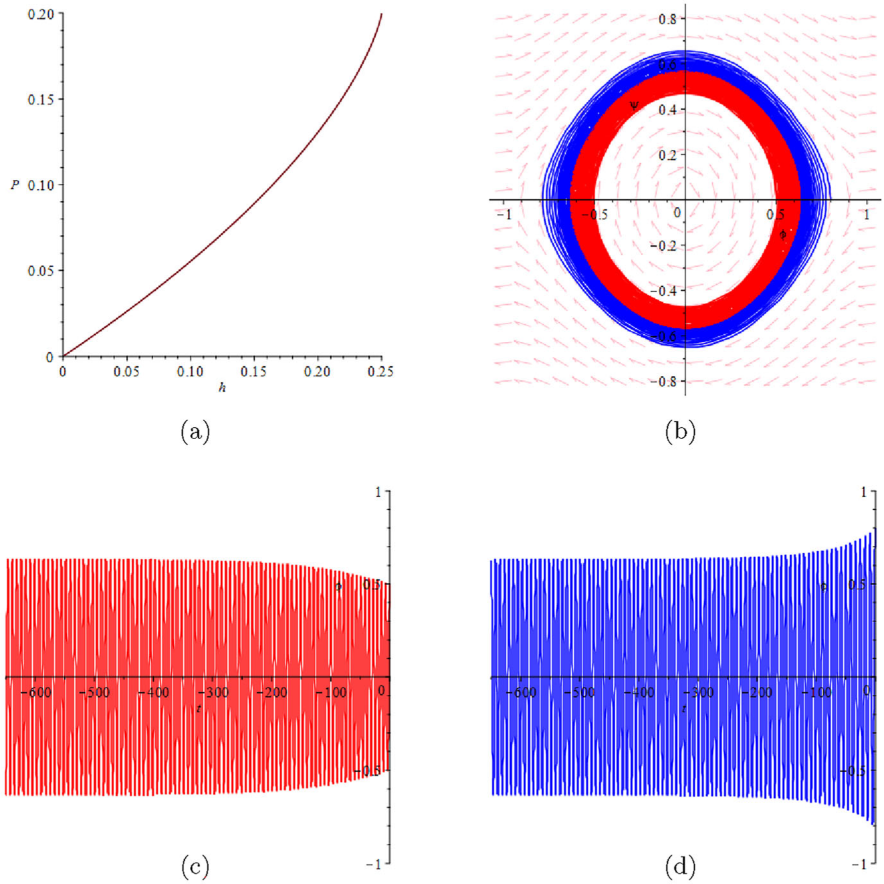


Fig. 3 Trajectories and time traveling curves of system (14) with $\tau = 0.1$, $\nu = P(\frac{4}{25}) \approx 0.0966$. **a** curve of function $P(h)$ for $h \in (0, \frac{1}{4})$; **b** two trajectories of system (14) passing through $(\phi(0), \psi(0)) = (0.8, 0)$ and $(\phi(0), \psi(0)) = (0.5, 0)$, respectively; **c** time traveling curve of $\phi(\zeta)$ satisfying $\phi(0) = 0.5$, $\phi'(0) = 0$; **d** time traveling curve of $\phi(\zeta)$ satisfying $\phi(0) = 0.8$, $\phi'(0) = 0$

$\zeta \to -\infty$, which indicates the existence of a limit cycle. These graphs further confirm and illustrate our theoretical results.

5 Conclusion

In this paper, the sufficient conditions to guarantee the existence of some bounded traveling wave solutions for two perturbed KP-MEW equation with distributed delay are presented. We mainly focus on the persistence of kink (anti-kink) wave solutions and periodic wave solutions, and prove that the delayed KP-MEW Eq. (5) has a kink (anti-kink) wave solution and a periodic wave solution while the model Eq. (6) has a periodic wave solution when the wave speed satisfies certain conditions. Compared with the recent existing methods that is used in [35, 36], the proof of monotonicity of the ratio of two Abelian integrals is much more efficient and concise since it avoids complicated algebraic computations. The disadvantage is that the upper and lower bounds of the limit wave speed cannot be obtained. However, As far as the existence of traveling wave solutions are concerned, the method is sufficient.

The existence of kink (anti-kink) and periodic wave solution for Eq. (5) theoretically demonstrates that when there exist distributed delay and dispersive perturbation in the KP-MEW Eq. (3), there still exist kink (anti-kink) and periodic wave propagating on the surface of fluid in a channel if the wave speed satisfies certain conditions. So does the solution of Eq. (6). it is very interesting that the delayed KP-MEW equation exhibits completely diverse dynamical properties with positive perturbation and with negative perturbation. This shows, to some extent, that the perturbation is critical for nonlinear wave equations. A minor variation of perturbation term may lead to significant changes in their dynamical behavior. The finding of this phenomena greatly enriches the understanding of traveling wave solutions of nonlinear wave equations.

Acknowledgements The authors would like to thank the anonymous referees' valuable suggestions.

Author Contributions JW wrote the original draft. LZ and XH performed formal analysis and validation. NM and CMK examined English spelling & grammar. All authors reviewed the manuscript.

Funding This work is supported by the National Natural Science Foundation of China (No. 12172199, 12011530062).

Data Availability Not applicable.

Declarations

Conflict of interest All authors have declared that no Conflict of interest exists.

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