Azumaya algebras over schemes

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Abstract

This dissertation investigates the theory of Azumaya algebras in the context of schemes, with a particular focus on quadratic pairs and involutions. We start by giving an overview of the theory of Azumaya algebras and the Brauer group as defined by Grothendieck [Gro68a]. We discuss the injection of the Brauer group into the cohomological Brauer group $H^2_{\text{ét}}(\mathbb{G}_m, S)_{\text{tors}}$ by using an interpretation of $H^2(\mathbb{G}_m, S)$ as the set of $\mathbb{G}_m$-gerbes, and constructing the $\mathbb{G}_m$-gerbe of trivialisations of an Azumaya algebra.

Quadratic pairs were originally introduced in the context of central simple algebras by Knus, Merkurjev, Rost, and Tignol [BoI] as a generalisation of regular quadratic forms, and are particularly effective in characteristic 2. This notion was later generalised to Azumaya algebras over schemes by Calmès and Fasel [CF15]. We study the properties of quadratic pairs in this context, as well as their existence and classifications. We provide a cohomological obstruction for a specified orthogonal involution taking part in a quadratic pair, using recent results by Gille, Neher, and Ruether [GNR23].

Further, we address the existence of involutions of the first kind on Azumaya algebras. Using the proof of Knus, Parimala, and Srinivas [KPS90], we adapt the classical result by Saltman showing that a Brauer class $[A]$ supports an involution if and only if $2[A] = 0$ into the language of schemes. In contrast with the classical case of central simple algebras over a field, this is not equivalent to every representative of the Brauer class $[A]$ carrying an involution, with the exception of degree 2 algebras. We adapt an old proof by Saltman to show that every degree 2 Azumaya algebra carries an involution.

Keywords: Azumaya algebras; Brauer group; Involutions; Quadratic pairs

Contents

Introduction

Central simple algebras and the Brauer group of a field have been studied since ancient times[1] with many applications. There are numerous books written on this classical case and its applications, for example [GS17] and [BO13]. In 1951, Azumaya [Azu51] generalised the notion to what he called maximally central algebras and studied the Brauer group of a local ring. Later in 1960 Auslander and Goldman [AG60] further generalised these ideas to arbitrary commutative rings, where they used central separable algebras to define the Brauer group of a ring. Grothendieck [Gro68a] [Gro68b] [Gro68c] then finally generalised the theory to Azumaya algebras and the Brauer group $\mathrm{Br}(S)$ of a scheme S; it was he who coined the term "Azumaya algebra". Despite the rapidly increasing popularity of the theory of schemes, developments on Azumaya algebras were primarily made in the language of rings and fields for a long time. The aim of this work is to study the analogues of the classical theory in the language of Azumaya algebras over schemes. In particular, we will focus on quadratic pairs and involutions.

Quadratic pairs on central simple algebras were introduced by Knus, Merkurjev, Rost, and Tignol in The Book of Involutions [BoI] as a generalisation of regular quadratic forms. Quadratic pairs are particularly useful in the case of fields of characteristic 2, where the usual methods of studying quadratic forms struggle. This notion has proven useful in the classification of classical groups [BoI, §26]. The notion of quadratic pairs was generalised to the context of schemes by Calmès and Fasel [CF15] precisely to facilitate their classification of classical algebraic groups over an arbitary base, and connect with the existing classification in [SGA3]. We will treat the theory of quadratic pairs in this setting, generalising some of the classical theory from [BoI], such as the fact that every quadratic pair is a twisted form of a regular quadratic form. In addition, we study when quadratic pairs may exist on a given Azumaya algebra with involution of orthogonal type. This question has been a recent topic of investigation by Gille, Neher, and Ruether [GNR23], and we provide an exposition of their findings. An immediate necessary condition is that $1_{\mathcal{A}}$ is a global section of the sheaf of elements of $\mathcal{A}$ that are symmetrised under the involution. Under this hypothesis, we obtain a cohomological criterion for the existence of a quadratic pair, and classify all quadratic pairs that involve a specified orthogonal involution.

[1] roughly one hundred years ago

Our investigation of involutions (of the first kind) is chiefly concerned with their existence. In the classical theory of central simple algebras, it was Albert [Alb39] who showed that a central simple algebra A carries an involution if and only if the Brauer class $[A]$ of A is 2-torsion. However, this fails to generalise to Azumaya algebras, even in the setting of Azumaya algebras over commutative rings. In the context of rings, the best result on the existence is due to Saltman [Sal78], who showed that if the Brauer class $[A]$ is 2-torsion, then there is a representative $B \in [A]$ such that B has an involution. An alternative proof of Saltman's theorem was given by Knus, Parimala, and Srinivas [KPS90], allowing one to pick the representative B with $\deg B = 2 \deg A$. A later example by Auel, First, and Williams [AFW19] shows that we cannot expect the degree of B to be any lower in general. In particular, it is entirely possible for some representatives of the Brauer class $[A]$ to not carry an involution at all. It was known that the proof given in [KPS90] applied in the generality of schemes, in this project we adapt their proof to the language of schemes. One notable class of examples that stand in contrast to our above discussion, are quaternion algebras (Azumaya algebras of degree 2). In the same paper by Saltman [Sal78], he has shown that if A has degree 2, then A itself carries an involution (it is automatic that $2[A] = 0$). We also adapt this proof by Saltman to our context of Azumaya algebras over schemes.

We now present a brief outline of each chapter.

Chapter 1 is devoted to developing the preliminaries on sites and sheaves on sites that we shall need throughout the text. We assume that the reader already has some degree of familiarity with the theory of schemes, categories, and homological algebra.

In Chapter 2 we treat the basic theory of Azumaya algebras over schemes. We introduce the definition, and show that every Azumaya algebra is locally a matrix algebra in the étale topology using the fact that the Brauer group of a strictly henselian ring is trivial. Using this characterisation of Azumaya algebras, we interpret Azumaya algebras of constant rank n^2 as twisted forms of the matrix algebra $\mathrm{M}_n(\mathcal{O}_S)$. Through the standard equivalence of twisted forms of $\mathrm{M}_n(\mathcal{O}_S)$ and $\mathrm{PGL}_n := \mathrm{Aut}_{\mathbf{Alg}}(\mathrm{M}_n(\mathcal{O}_S))$-torsors, this allows us to classify Azumaya algebras of rank n^2 with the nonabelian cohomology group $H^1(S, \mathrm{PGL}_n)$. We also develop the necessary theory of stacks and gerbes to enable the interpretation of $H^2(S, \mathbb{G}_m)$ as the set of $\mathbb{G}_m$-gerbes, which in turn allows us to construct an injection $\mathrm{Br}(S) \to H^2(S, \mathbb{G}_m)$. Using the tools of nonabelian cohomology we determine that the Brauer group is torsion. We end the chapter with a short discussion of the comparison between the Brauer group and the cohomological Brauer group $\mathrm{Br}'(S) := H^2(S, \mathbb{G}_m)_{\mathrm{tors}}$.

Chapter 3 is dedicated to the theory of quadratic pairs. We start by discussing the essential preliminaries on Azumaya algebras with involution and types of involutions. We then move on to the definition and first properties, notably we show that every quadratic triple arises from a regular quadratic form étale-locally. The second half of this chapter is spent developing classifications and obstructions to the existence of quadratic pairs, following along with some recent theory [GNR23].

Finally, in Chapter 4 we treat existence of involutions (of the first kind). We adapt the proof of [KPS90] to the language of schemes, showing that an Azumaya $\mathcal{O}$-algebra $\mathcal{A}$ is Brauer equivalent to an algebra $\mathcal{B}$ carrying an involution if and only if $2[\mathcal{A}] = 0$ in $\mathrm{Br}(S)$. In contrast with central simple algebras, it need not be the case that $\mathcal{A}$ carries an involution itself even if the Brauer class $[\mathcal{A}]$ supports an involution. One case in which the algebra $\mathcal{A}$ carries an involution is the case of quaternion algebras (degree 2 Azumaya algebras), we show this by adapting an old proof of Saltman [Sal78, Thm 4.1] to our context.

Chapter 1

Preliminaries

This chapter serves to cover some preliminary techniques and introduce our notation in this context. We assume that the reader is familiar with the theory of schemes to the level of a one year graduate course. The reader may reference standard geometry texts such as [GW20] and [Har77] for facts about schemes and their morphisms that we do not treat in this chapter. In addition, we assume the reader has a basic familiarity with category theory and homological algebra, in particular the theory of derived functors. For categories we refer the reader to [Mac98], and [Wei94] for homological algebra.

1.1 Sites and sheaves on sites

The Zariski topology of a scheme is too coarse for our goals, in this section we place a finer topology on schemes than the Zariski topology by abandoning general topology altogether. We will now develop the theory of sites and sheaves on sites that we need for the étale and fppf topologies, while also taking the opportunity to introduce our notation. We omit all proofs and refer the unfamiliar reader to the Stacks project [Stacks, Tag 00UZ] or Vistoli's notes [Vis05] for a comprehensive treatment of the theory.

Definition 1.1. Let $\mathcal{C}$ be a category with object U. A **family of morphisms with fixed target** is a family of morphisms

$$\{U_i \to U\}_{i\in I},$$

where I is some (possibly empty) index set.

Definition 1.2. A **site** is a category $\mathcal{C}$ along with a collection $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed targets called **covers** satisfying the following axioms:

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;
2. If $\{V_i \to U\}_i \in \mathrm{Cov}(\mathcal{C})$, and $\{W_{ij} \to V_i\}_j \in \mathrm{Cov}(\mathcal{C})$, then

$$\{W_{ij} \to U\}_{i,j} \in \mathrm{Cov}(\mathcal{C}).$$

3. If $\{U_i \to U\}_i \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism, then the fibre product $U_i \times_U V$ exists for each i, and
$$\{U_i \times_U V \to V\}_i \in \mathrm{Cov}(\mathcal{C}).$$

Example 1.3. Let X be a topological space, and let $\mathcal{C}$ be the category of open embeddings into X. That is, the objects are open embeddings $U \to X$ and the morphisms are commutative triangles

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & V \arrow[dl] \\ & X, & \end{tikzcd}$$

where f is also an open embedding. The category $\mathcal{C}$ we have described is naturally equivalent to the poset category of open subsets of X. We make $\mathcal{C}$ into a site by defining a cover of an object U in $\mathcal{C}$ to be a family $\{\phi_i \colon U_i \to U\}_i$ of open embeddings that are jointly surjective, that is, $\cup_i \phi_i(U_i) = U$. In other words, the covers in $\mathcal{C}$ are precisely open covers in the usual sense. We check that this defines a site (through some abuse of notation):

1. If $V \to U$ is a homeomorphism, then $V \to U$ is an open embedding;

2. If $U = \cup_i V_i$ and $V_i = \cup_j W_{ij}$, then $U = \cup_{i,j} W_{ij}$;

3. If $U = \cup_i U_i$ and $V \to U$ is an open embedding, the fibre products $U_i \times_U V$ are simply the intersections $U_i \cap V$, and $V = \cup_i (U_i \cap V)$.

Definition 1.4. Let $\mathcal{C}$ be a site, a **presheaf of sets** on $\mathcal{C}$ is a functor
$$\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}\,.$$

We call elements of the set $\mathcal{F}(U)$ **sections** over U, the set of sections is also sometimes denoted by $\Gamma(U, \mathcal{F})$. If $\phi \colon U \to V$ in $\mathcal{C}$, we may sometimes sloppily denote the corresponding map $\mathcal{F}(\phi) \colon \mathcal{F}(V) \to \mathcal{F}(U)$ by $f \mapsto f_{|U}$ and call it *the* restriction to U, despite the fact that there may be many morphisms $U \to V$. Of course, we may similarly define presheaves to **Ab**, **Ring**, and any other category. A **morphism of presheaves** $\mathcal{F} \to \mathcal{G}$ is a natural transformation $\mathcal{F} \to \mathcal{G}$. We denote the category of presheaves on $\mathcal{C}$ by $\mathrm{Pre}(\mathcal{C})$.

Definition 1.5. A **sheaf** on $\mathcal{C}$ is a presheaf $\mathcal{F}$ satisfying the sheaf condition: for every cover $\{U_i \to U\}_{i \in I}$, the natural diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equaliser. In other words, given a collection of sections $f_i \in \mathcal{F}(U_i)$ such that $f_{i|U_i \times_U U_j} = f_{j|U_i \times_U U_j}$ for all i, j, there is a unique section $f \in \mathcal{F}(U)$ such that $f_{|U_i} = f_i$ for all i.

Definition 1.6. A **morphism of sheaves** $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves $\mathcal{F} \to \mathcal{G}$. A morphism $\mathcal{F} \to \mathcal{G}$ is **injective** if each component $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective. A morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ is **surjective** if each component $\phi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is *locally surjective* in the sense that for each $g \in \mathcal{G}(U)$, there is a cover $\{U_i \to U\}$ and sections $\phi_{U_i}(f_i) \in \mathcal{G}(U_i)$ such that $g_{|U_i} = \phi_{U_i}(f_i)$. The category of sheaves on $\mathcal{C}$ is denoted by $\mathrm{Sh}(\mathcal{C})$.

Definition 1.7. [Stacks, Tag 00W1] Given a presheaf $\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ on $\mathcal{C}$, we may *sheafify* $\mathcal{F}$ to obtain a sheaf $\mathcal{F}^\sharp$ on $\mathcal{C}$ along with a canonical map $\mathcal{F} \to \mathcal{F}^\sharp$ called the **sheafification of** $\mathcal{F}$. The sheafification $\mathcal{F} \to \mathcal{F}^\sharp$ satisfies the following universal property: if $\mathcal{F} \to \mathcal{G}$ is any morphism to a sheaf $\mathcal{G}$, there is a unique morphism of sheaves $\mathcal{F}^\sharp \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \mathcal{F}^\sharp \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

commutes. In other words, the sheafification functor $\mathrm{Pre}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ is left adjoint to the inclusion functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C})$. Moreover, if $\mathcal{F}$ is already a sheaf, then the sheafification $\mathcal{F} \to \mathcal{F}^\sharp$ is an isomorphism.

Remark 1.8. The primary use of sheafification is the computation of colimits. If $F \colon I \to \mathrm{Sh}(C)$ is any diagram, the both the limit $\lim_I F$ and the colimit $\mathrm{colim}_I F$ exists. The limit may be computed "on sections", that is, the limit is the same in $\mathrm{Sh}(\mathcal{C})$ and $\mathrm{Pre}(\mathcal{C})$. On the other hand, the colimit cannot be computed so naïvely, the colimit $\mathrm{colim}_I F$ in $\mathrm{Sh}(\mathcal{C})$ is the sheafification of the colimit in the category of presheaves [Stacks, Tag 00WI].

We now introduce introduce étale and fppf morphisms of schemes for the purpose of the étale and fppf topologies, which we will use extensively throughout this text.

Definition 1.9. A morphism of rings $B \to A$ is **finitely presented** (or **of finite presentation**) if A is finitely generated as a B-algebra with only a finite number of relations. That is, there is an isomorphism of B-algebras

$$A \simeq B[x_1, \ldots, x_n]/(f_1, \ldots, f_m).$$

Accordingly, we say a B-scheme $X \to \operatorname{Spec} B$ is **locally finitely presented** over B if there exists an affine open cover $X = \cup_i \operatorname{Spec} A_i$ such that each $B \to A_i$ is finitely presented.

Definition 1.10. A morphism $\pi\colon X \to Y$ of schemes is said to be **locally finitely presented** (or **locally of finite presentation**) if for every affine open subset $\operatorname{Spec} B \subset Y$,

$$\pi^{-1}(\operatorname{Spec} B) \to \operatorname{Spec} B$$

is locally finitely presented over B.

Definition 1.11. A map of rings $B \to A$ is **flat** if A is a flat B-module. Accordingly, a map of schemes $\pi\colon X \to Y$ is said to be **flat at** $p \in X$, if $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,\pi(p)}$-module. And $\pi\colon X \to Y$ is called **flat** if π is flat at every point $p \in X$.

Definition 1.12. A morphism $\pi\colon X \to Y$ of schemes is **fppf**, if π is flat and locally of finite presentation.

Definition 1.13. A morphism $\pi\colon X \to Y$ of schemes is **unramified**, if π is locally of finite presentation, and for each $x \in X$ and $y = f(x)$, we have that

1. The residue field $\kappa(x)$ is a separable algebraic extension of $\kappa(y)$.
2. The ideal generated by $\pi^\sharp(\mathfrak{m}_y)$ is $\mathfrak{m}_x$.

Definition 1.14. A morphism $\pi\colon X \to Y$ of schemes is **étale** if π is flat and unramified.

Example 1.15. [Stacks, Tag 020K] Fix a scheme S, we introduce a few sites associated with S that we will use throughout this text. Each of the following are examples of **ringed sites**, that is, a site equipped with a sheaf of rings called the **structure sheaf**.

1. S_{Zar}: The small Zariski site of a scheme S. This is the site associated with the topological space S as in Example 1.3. The structure sheaf of S_{Zar} is just the structure sheaf of S as a scheme: $U \mapsto \mathcal{O}_S(U)$.

2. $S_{\text{ét}}$: The small étale site of a scheme S. The objects of the underlying category are étale morphisms $X \to S$. Morphisms are commutative triangles
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S, & \end{tikzcd}$$
where f is necessarily étale. A cover of an object $X \to S$ is a jointly surjective family of étale morphisms $\{X_i \to X\}$. The topology on $S_{\text{ét}}$ is called the **étale topology** on S. The structure sheaf of $S_{\text{ét}}$ is defined by sending $X \to S$ to the ring of global sections $\Gamma(X, \mathcal{O}_X)$.

3. $\mathbf{Sch}_{/S}$: The big étale site of a scheme S. The underlying category is the category of S-schemes. An étale cover of an object $X \to S$ is again a jointly surjective family of étale morphisms $\{X_i \to X\}$, this topology is called the **étale topology** on $\mathbf{Sch}_{/S}$. In this case the structure sheaf is also defined by sending $X \to S$ to $\Gamma(X, \mathcal{O}_X)$.

4. Similarly we may define the **fppf topology** on $\mathbf{Sch}_{/S}$ by defining covers to be jointly surjective families of fppf morphisms. We will lazily denote this site by $\mathbf{Sch}_{/S}$ as well and call it the **big fppf site** of S.

We have the following important chain of implications for a morphism $\pi\colon X \to Y$ of schemes:
$$\pi \text{ is an open immersion } \Rightarrow \pi \text{ is étale } \Rightarrow \pi \text{ is fppf},$$
therefore, every Zariski cover is an étale cover and every étale cover is an fppf cover. Consequently, (the restriction of) any fppf sheaf is automatically an étale and Zariski sheaf.

Remark 1.16. There are some minor set-theoretical concerns with our current definition of the big sites. Strictly speaking, we should work with *a big étale or fppf site containing S* as in [Stacks, Tag 021L]. It can be shown, however, that the category of sheaves on the big site is independent of any choices made in the construction [Stacks, Tag 00VY] and that any choice of big site is closed under the standard "geometric operations" such as taking open subschemes and fibre products [Stacks, Tag 000R]. So we will not concern ourselves too much with this issue, and instead continue to blithely talk about *the* big sites.

The following theorem is of utmost importance.

Theorem 1.17. [Stacks, Tag 023P] Every representable functor[1] on the sites introduced in Example 1.15 is a sheaf.

Example 1.18. We now introduce some important sheaves of groups on the aforementioned sites:

- Let $\mathbb{G}_m$ be the functor defined by sending a scheme X to the group of units $\Gamma(X)^\times$ of the ring of global sections $\Gamma(X)$. Over $\operatorname{Spec}\mathbb{Z}$, $\mathbb{G}_m$ is represented by $\operatorname{Spec}\mathbb{Z}[x, x^{-1}]$, showing that $\mathbb{G}_m$ is a sheaf.
- Define GL_n by sending a scheme X to the general linear group $\mathrm{GL}_n(\Gamma(X)) = (\mathrm{M}_n(\Gamma(X)))^\times$. Observe that GL_n is represented by $\operatorname{Spec}\mathbb{Z}[a_{11}, \dots, a_{nn}, t]/(t\det(a_{ij}) - 1)$.
- Let PGL_n be the functor sending a scheme X to the group $\mathrm{PGL}_n(\mathcal{O}_X)$ of $\mathcal{O}_X$-algebra automorphisms of the sheaf $\mathrm{M}_n(\mathcal{O}_X)$. An algebra automorphism ϕ of $\mathrm{M}_n(\mathcal{O}_X)$ is defined by an $\mathcal{O}_S$-module automorphism of $\mathrm{M}_n(\mathcal{O}_X)$ satisfying the following equations given by polynomials in the entries of the $n^2 \times n^2$ matrix ϕ:

$$\phi(E_{ij}E_{kl}) = \phi(E_{ij})\phi(E_{kl}) \quad \text{for all } i, j, k, l \leq n.$$

  And so we conclude that PGL_n is represented by the closed subscheme of $\operatorname{Spec}\mathbb{Z}[a_{11}, \dots, a_{n^2n^2}]$ carved out by the above equations.
- Let μ_n be defined by sending a scheme X to the group of n-th roots of unity in $\Gamma(X)$. This defines a sheaf since μ_n is represented by $\operatorname{Spec}\mathbb{Z}[x]/(x^n - 1)$.

For now, we have only shown that the above presheaves are sheaves on $\mathbf{Sch} = \mathbf{Sch}_{/\mathbb{Z}}$. The following lemma will fill this gap to show that we sheaves on any of the usual sites.

Lemma 1.19. *Restriction of sheaves.* Let $\mathcal{C}$ be a site and let $X \in \mathcal{C}$ be an object. Then the slice category $\mathcal{C}_{/X}$ of maps into X is also a site, where a cover of $Y \to X$ in $\mathcal{C}_{/X}$ is just a cover of Y in $\mathcal{C}$. We call the site $\mathcal{C}_{/X}$ the **localisation** of $\mathcal{C}$ at X. If $\mathcal{F}$ is a sheaf on $\mathcal{C}$, then there is a corresponding sheaf $\mathcal{F}_{|X}$ on $\mathcal{C}_{/X}$ defined by $\mathcal{F}_{|X}(Y \to X) = \mathcal{F}(Y)$. The sheaf $\mathcal{F}_{|X}$ is called the **restriction** of $\mathcal{F}$ to X.

As a particular case, we see that if S is any scheme and $\mathcal{G}$ is any of the presheaves of Example 1.18 defined on $\mathbf{Sch}_{/S}$. Then $\mathcal{G}$ defines a sheaf on $\mathbf{Sch}_{/S}$, because $\mathcal{G}$ is the restriction of a representable sheaf on $\mathbf{Sch}$.

[1] recall that a functor $F\colon \mathcal{C} \to \mathbf{Set}$ is said to be **representable** if $F \simeq \mathrm{Mor}(-, X)$ for some (necessarily unique) object $X \in \mathcal{C}$

1.2 Quasi-coherent modules

Another important category of sheaves is the category of quasi-coherent modules. We briefly introduce quasi-coherent modules, (finite) locally free modules, and treat their tensor products. A standard reference is [Stacks, Tag 03A4].

Definition 1.20. Let $\mathcal{C}$ be a ringed site with structure sheaf $\mathcal{O}$. An **$\mathcal{O}$-module** is a sheaf $\mathcal{M}$ of abelian groups on $\mathcal{C}$ along with a morphism of sheaves

$$\mathcal{M} \times \mathcal{O} \to \mathcal{M}$$

such that for each object X the map $\mathcal{M}(X) \times \mathcal{O}(X) \to \mathcal{M}(X)$ makes $\mathcal{M}(X)$ into an $\mathcal{O}(X)$-module.
An $\mathcal{O}$-module $\mathcal{M}$ is called **quasi-coherent** if for each object $T \in \mathcal{C}$, there is a cover $\{T_i \to T\}_{i \in I}$ such that for each i, $\mathcal{M}_{|T_i}$ has a global presentation. That is, there is an exact sequence of $\mathcal{O}_{|T_i}$-modules

$$\bigoplus_{j \in J_i} \mathcal{O}_{|T_i} \to \bigoplus_{k \in K_i} \mathcal{O}_{|T_i} \to \mathcal{M}_{|T_i} \to 0.$$

If the site $\mathcal{C}$ has a final object S (for example, $\mathbf{Sch}_{/S}$), then it suffices to find a cover of S on which $\mathcal{M}$ has a global presentation and there is no need to check every object.

Lemma 1.21. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}$-module, and T is any object of $\mathcal{C}$, then the restriction $\mathcal{M}_{|T}$ is a quasicoherent $\mathcal{O}_{|T}$-module.

Our interest will almost exclusively be on quasi-coherent modules. In Example 1.15 we associated four different sites with a scheme S, leading to four potentially different notions of quasi-coherent modules over a given scheme S. The following theorem shows that all four notions are the same, allowing us to continue with peace in our hearts.

Theorem 1.22. [Stacks, Tag 03DX]
The meta-theorem for quasi-coherent modules.
Let S be a scheme, the category of quasi-coherent $\mathcal{O}_S$-modules is naturally equivalent to the category of quasi-coherent $\mathcal{O}$-modules on the big fppf (or étale) site $\mathbf{Sch}_{/S}$, and is also naturally equivalent to the category of quasi-coherent modules on the small étale site $S_{\text{ét}}$. In particular, if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module, then the corresponding sheaf $\mathcal{M}^a$ on $\mathbf{Sch}_{/S}$ or $S_{\text{ét}}$ is defined by

$$\mathcal{M}^a(T \xrightarrow{f} S) = \Gamma(T, f^*\mathcal{M}),$$

where $f^*\mathcal{M}$ is the pullback of $\mathcal{M}$ along f.

We will be particularly interested in finite locally free $\mathcal{O}$-modules.

Definition 1.23. A $\mathcal{O}$-module $\mathcal{M}$ is said to be **locally free** if for every object $T \in \mathcal{C}$, there is a cover $\{T_i \to T\}$ such that for each i, there is a set J_i such that $\mathcal{M}_{|T_i} \simeq \bigoplus_{j \in J_i} \mathcal{O}_{|T_i}$. If in addition, each J_i may be chosen to be a finite set, we say $\mathcal{M}$ is a **finite locally free** $\mathcal{O}$-module. Again, if the site $\mathcal{C}$ has a final object, it suffices to show that $\mathcal{M}$ is (finite) free on a cover of the final object. Notice that every locally free $\mathcal{O}$-module is automatically quasi-coherent, and the restriction of a (finite) locally free module is again (finite) locally free.

Lemma 1.24. [Stacks, Tag 05VG]
A $\mathcal{O}_S$-module $\mathcal{F}$ is (finite) locally free if and only if the corresponding $\mathcal{O}$-module $\mathcal{F}^a$ is finite locally free.

Example 1.25. If $\mathcal{C}$ is a ringed site with sheaf of rings $\mathcal{O}$ and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}$-modules, then there is a sheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ on $\mathcal{C}$ defined by

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{M}_{|U}, \mathcal{N}_{|U}).$$

We denote $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{M})$ by $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$. In particular, the **dual sheaf** of an $\mathcal{O}$-module $\mathcal{M}$ is $\mathcal{M}^\vee := \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{O})$. Notice that if $\mathcal{M}$ and $\mathcal{N}$ are finite locally free $\mathcal{O}$-modules, then so is $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$.

Definition 1.26. [Stacks, Tag 03EK] Let $\mathcal{M}$ and $\mathcal{N}$ be quasi-coherent $\mathcal{O}$-modules. The **tensor product** $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ in the category of quasi-coherent $\mathcal{O}$-modules is given by sheafifying the tensor product presheaf

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U).$$

The tensor product $\mathcal{M} \otimes \mathcal{N}$ satisfies the usual universal property with regard to bilinear maps. If $\mathcal{M}$ and $\mathcal{N}$ are (finite) locally free, then so is $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$.

1.3 Cohomology

Fix a scheme S. The category of abelian sheaves on the big fppf site $\mathbf{Sch}_{/S}$ is abelian and has enough injectives [Stacks, Tag 01FT]. The functor $\mathcal{F} \to \Gamma(S, \mathcal{F})$ is left exact but not right exact. The right derived functors of $\Gamma(S, -)$ are called the **fppf cohomology** functors and denoted by $H^i(S, -) := R^i\Gamma(S, -)$. Similarly, $\Gamma(S, -)$ defines a left exact functor from the category of abelian sheaves on the small and big étale sites of S and we may again define the cohomology functors to be the right derived functors of $\Gamma(S, -)$. Notation: $H^i_{\text{ét}}(S, -)$. We will primarily use the fppf topology and fppf cohomology in this text, but with only a few exceptions, everything could equally as well be done in the étale topology.

The following theorem shows that for quasi-coherent modules, the cohomologies agree on the usual sites:

Theorem 1.27. [Stacks, Tag 03P2]
The meta-theorem for cohomology of quasi-coherent modules.
Let S be a scheme and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_S$-module, then the usual cohomology $H^i(S, \mathcal{M})$ agrees with the cohomology $H^i_{\text{ét}}(S, \mathcal{M}^a)$ on the small and big étale sites as well as with the cohomologies $H^i(S, \mathcal{M}^a)$ on the big fppf sites. Consequently, in the future we will talk about *the* cohomology of a quasi-coherent sheaf without specifying a topology.

The defining feature of cohomology is the following way of turning short exact sequences into long exact sequences. Given an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

of abelian sheaves, there is a natural long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(S,\mathcal{A}) \arrow[r] & \Gamma(S,\mathcal{B}) \arrow[r] & \Gamma(S,\mathcal{C}) \arrow[dll, "\delta"] \\
& H^1(S,\mathcal{A}) \arrow[r] & H^1(S,\mathcal{B}) \arrow[r] & H^1(S,\mathcal{C}) \arrow[dll, "\delta"] \\
& H^2(S,\mathcal{A}) \arrow[r] & H^2(S,\mathcal{B}) \arrow[r] & \cdots
\end{tikzcd}$$

The morphisms δ are called **connecting morphisms**. The cohomology is the obstruction to a surjective morphism of sheaves being globally surjective. By exactness, a global section $c \in \Gamma(S, \mathcal{C})$ is in the image of $\Gamma(S, \mathcal{B}) \to \Gamma(S, \mathcal{C})$ if and only if $\delta(c) = 0$ in $H^1(S, \mathcal{A})$. In particular, if $H^1(S, \mathcal{A}) = 0$, then $\Gamma(S, \mathcal{B}) \to \Gamma(S, \mathcal{C})$ is surjective.

Finally, we remind the reader that all higher cohomologies of quasi-coherent modules vanish over affine schemes.

Theorem 1.28. [Stacks, Tag 01XB] If $T \to S$ is an affine scheme and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module, then $H^i(T, \mathcal{M}) = 0$ for all i.

1.4 Stalks and étale neighborhoods

In order to talk about stalks in the étale site, we need a different notion of "point". We follow along with [Stacks, Tag 03PN].

Definition 1.29. A **geometric point** of a scheme S is a morphism

$$\bar{s}\colon \operatorname{Spec}(k) \to S,$$

where k is an algebraically closed field. Abusing the notation we shall also use $\bar{s}$ to denote the scheme $\operatorname{Spec}(k)$ and $\kappa(\bar{s})$ to denote the field k. If the point $s \in S$ is the image of $\bar{s}$ we say $\bar{s}$ lies over s.

Definition 1.30. An **étale neighborhood** of a geometric point $\bar{s}$ is a commutative diagram

$$\begin{tikzcd} & \bar{s} \arrow[dl, "\bar{u}"'] \arrow[dr, "\bar{s}"] & \\ U \arrow[rr, "\phi"'] & & S, \end{tikzcd}$$

where ϕ is an étale morphism. A **morphism** of étale neighborhoods $(U, \bar{u}) \to (U', \bar{u}')$ is a (necessarily étale) S-morphism $U \to U'$ such that

$$\begin{tikzcd} & \bar{s} \arrow[dl, "\bar{u}"'] \arrow[dr, "\bar{u}'"] & \\ U \arrow[rr] & & U' \end{tikzcd}$$

commutes.

Lemma 1.31. Let S be a scheme and let $\bar{s}$ be a geometric point of S. The category of étale neighborhoods of $\bar{s}$ is cofiltered.

Definition 1.32. Let S be a scheme and let $\bar{s}$ be a geometric point of S. Let $\mathcal{F}$ be a presheaf on $S_{\text{ét}}$, the **stalk** of $\mathcal{F}$ at $\bar{s}$ is given by

$$\mathcal{F}_{\bar{s}} = \operatorname{colim} \mathcal{F}(U),$$

where the colimit is over all étale neighborhoods $(U, \bar{u})$ of $\bar{s}$.

Remark 1.33. Since we are taking the colimit over the opposite category of étale neighborhoods of $\bar{s}$, the stalk $\mathcal{F}_{\bar{s}}$ is a filtered colimit and may be explicitly described as follows: an element of the stalk $\mathcal{F}_{\bar{s}}$ is a triple $(U, \overline{u}, f)$, where $(U, \overline{u})$ is an étale neighborhood of $\bar{s}$, and $f \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, f)$ and $(U', \overline{u}', f')$ are equivalent if there is a third étale neighborhood $(U'', \overline{u}'')$ and morphisms of étale neighborhoods $(U'', \overline{u}'') \to (U, \overline{u})$ and $(U'', \overline{u}'') \to (U, \overline{u})$ such that $f_{|U''} = f'_{|U''}$ in $\mathcal{F}(U'')$.

Lemma 1.34. Taking the stalk at a geometric point $\bar{s}$ of a scheme S defines exact functors:

$$\mathbf{Ab}_{pre}(S_{\text{ét}}) \to \mathbf{Ab}$$
$$\mathbf{Ab}(S_{\text{ét}}) \to \mathbf{Ab}\,.$$

Moreover, taking the stalk at $\bar{s}$ commutes with sheafification.

Lemma 1.35. A map of sheaves $\mathcal{F} \to \mathcal{G}$ on $S_{\text{ét}}$ is injective (resp. surjective) if and only if the induced maps on the stalks $\mathcal{F}_{\bar{s}} \to \mathcal{G}_{\bar{s}}$ are injective (resp. surjective) for all geometric points $\bar{s}$ of S. Similarly, exactness of sequences of sheaves of abelian groups may be checked at the level of stalks.

The stalk at a geometric point can also be used to compute the strict henselisations of the stalks of a scheme.

Lemma 1.36. [Stacks, Tag 04HX] Let S be a scheme and $\bar{s} \to S$ be a geometric point of S lying over $s \in S$. Then the strict henselisation $\mathcal{O}_{S,s}^{\text{sh}}$ of $\mathcal{O}_{S,s}$ may canonically be identified with the stalk $\mathcal{O}_{S,\bar{s}}$ at the geometric point $\bar{s}$.

1.5 Azumaya algebras over rings

We end this chapter with a few facts from the theory of Azumaya algebras over rings. Since our later treatment of Azumaya algebras over schemes will be largely independent from the theory over rings, we only introduce the essentials for future reference in this section. We recommend the books [For17, §7 §11], [Knu91], and [KO74] for readers that are specifically interested in Azumaya algebras over rings.

Definition 1.37. An **Azumaya** R-algebra is an R-algebra A that is a faithful, finitely generated, and projective as an R-module for which the natural map

$$\Omega_A \colon A \otimes_R A^{\text{op}} \to \text{End}_R(A)$$
$$a \otimes b \mapsto (x \mapsto axb)$$

is an isomorphism of R-algebras. A faithful, finitely generated, projective R-module is called an R**-progenerator module**.

Lemma 1.38. If A is R-Azumaya, then the centre of A is R.

Lemma 1.39. If A is R-Azumaya and $R \to S$ is a morphism of rings, then $A \otimes_R S$ is S-Azumaya.

Lemma 1.40. Let R be a local ring with residue field k, and let A be an R-algebra. Then A is Azumaya over R if and only if A is finite free as an R-module and $A \otimes k$ is a central simple k-algebra.

Lemma 1.41. If R is a local ring and A is an Azumaya algebra over R, then every algebra automorphism of A is inner. That is if ϕ is an automorphism of A, then there is a unit $u \in A$ such that $\phi = x \mapsto uxu^{-1}$.

Definition 1.42. Two Azumaya R-algebras A and A' are said to be **Brauer equivalent** (denoted $A \sim A'$) if there exist R-progenerator modules P and Q such that
$$A \otimes \mathrm{End}_R(P) \simeq A' \otimes \mathrm{End}_R(Q)$$
as R-algebras. We denote the Brauer equivalence class of an Azumaya R-algebra A by $[A]$. The set of Brauer equivalence classes of Azumaya R-algebras is denoted by $\mathrm{Br}(R)$ and form an abelian group under $\otimes$ with identity $[R]$ and where the inverse of $[A]$ is $[A^{\mathrm{op}}]$. We call $\mathrm{Br}(R)$ the **Brauer group** of R.

Lemma 1.43. Let A be an Azumaya R-algebra, $A \sim R$ if and only if there exists an R-progenerator M such that
$$A \simeq \mathrm{End}_R(M).$$

Lemma 1.44. [Mil80, Cor. IV.1.7] If R is a strictly henselian ring, then $\mathrm{Br}(R) = 0$.

Chapter 2

Azumaya algebras

Azumaya algebras over schemes and the Brauer group of a scheme were introduced by Grothendieck [Gro68a]. In this chapter we treat the basic theory of Azumaya algebras, introduce the Brauer group, and discuss the comparison with the cohomological Brauer group. Along the way, we will discuss an interpretation of Azumaya algebras as twisted forms, and develop the basic theory of torsors and gerbes we need for nonabelian cohomology.

2.1 Definition and first properties

Throughout this chapter we fix a base scheme S. This section is based on the exposition of [Mil80, §IV.2].

Definition 2.1. An $\mathcal{O}_S$**-algebra** is an $\mathcal{O}_S$-module $\mathcal{A}$ along with a multiplication map

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$$

such that for each $U \subset S$, the $\mathcal{O}_S(U)$-module $\mathcal{A}(U)$ is an (associative, unital, but not necessarily commutative) $\mathcal{O}_S(U)$-algebra. The **opposite algebra** $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$ is the algebra obtained from $\mathcal{A}$ by reversing the order of multiplication. Given an $\mathcal{O}_S$-algebra $\mathcal{A}$, there is a natural map

$$\Omega_{\mathcal{A}} \colon \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{A}),$$

defined on an open set U by sending $a \otimes b$ to the $\mathcal{O}_S$-module endomorphism on $\mathcal{A}_{|U}$ induced by $x \mapsto axb$. An **Azumaya algebra** over S is an $\mathcal{O}_S$-algebra that is finite locally free as an $\mathcal{O}_S$-module and for which the sandwich morphism $\Omega_{\mathcal{A}}$ is an isomorphism of $\mathcal{O}_S$-algebras.

Proposition 2.2. The following are equivalent for an $\mathcal{O}_S$-algebra $\mathcal{A}$ that is finite locally free as an $\mathcal{O}_S$-module:

1. $\mathcal{A}$ is Azumaya over S.

2. The Zariski stalk $\mathcal{A}_p$ is $\mathcal{O}_{S,p}$-Azumaya for each $p \in S$.

3. The fibre
$$\mathcal{A}(p) := \mathcal{A}_p \otimes_{\mathcal{O}_{S,p}} \kappa(p)$$
is a central simple algebra over $\kappa(p)$ for each $p \in S$.

4. There is an étale cover $\{U_i \to S\}$ such that for each i, there exists a positive integer r_i and an $\mathcal{O}_S$-algebra isomorphism
$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{U_i} \simeq \mathrm{M}_{r_i}(\mathcal{O}_{U_i}).$$

5. There is an fppf cover $\{U_i \to S\}$ such that for each i, there exists a positive integer r_i and an $\mathcal{O}_S$-algebra isomorphism
$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{U_i} \simeq \mathrm{M}_{r_i}(\mathcal{O}_{U_i}).$$

Proof. $(1 \iff 2)$: Since $\mathcal{A}$ is finite locally free, we have that
$$\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{A})_p = \mathrm{End}_{\mathcal{O}_{S,p}}(\mathcal{A}_p)$$
and
$$(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})_p = \mathcal{A}_p \otimes \mathcal{A}_p^{\mathrm{op}}.$$

$(2 \iff 3)$: Lemma 1.40.
$(2 \implies 4)$: For any point $s \in S$, choose a geometric point $\bar{s} \to S$ lying over s. Then $\mathcal{A}_s \otimes \mathcal{O}_{S,\bar{s}}$ is an Azumaya algebra over the strictly henselian ring $\mathcal{O}_{S,\bar{s}}$. By Lemma 1.44, there is an isomorphism $\mathcal{A}_s \otimes \mathcal{O}_{S,\bar{s}} \simeq \mathrm{M}_r(\mathcal{O}_{S,\bar{s}})$, and therefore, there exists an étale neighborhood $(U, \bar{u})$ of $\bar{s}$ such that $\mathcal{A} \otimes \mathcal{O}_U \simeq \mathrm{M}_r(\mathcal{O}_U)$.
$(4 \implies 5)$: Every étale cover is an fppf cover.
$(5 \implies 3)$: For every point $p \in S$, there is an fppf map $U_i \to S$ whose image contains p and for which $\mathcal{A} \otimes \mathcal{O}_{U_i} \simeq \mathrm{M}_{r_i}(\mathcal{O}_{U_i})$. Hence, there is a field extension $\kappa(p) \to L$ such that $\mathcal{A}(p) \otimes_{\kappa(p)} L \simeq \mathrm{M}_{r_i}(L)$. This implies that $\mathcal{A}(p)$ is a central simple $\kappa(p)$-algebra. □

Points 4 and 5 may be interpreted as saying that $\mathcal{A}$ is locally isomorphic to matrix algebras in the étale and fppf topologies. More precisely, an Azumaya algebra of constant rank n^2 is a twisted form of $\mathrm{M}_n(\mathcal{O}_S)$ (for the étale or fppf topologies). We will develop this observation a bit in Example 2.38.

Corollary 2.3. The rank of an Azumaya algebra $\mathcal{A}$ is a square on each connected component. We call the square root of the rank the **degree** of $\mathcal{A}$ and denote it by $\deg \mathcal{A}$.

Lemma 2.4. If the base scheme S is locally noetherian, it suffices to check the stalks at closed points. That is, an $\mathcal{O}_S$-algebra $\mathcal{A}$ is Azumaya over S if and only if $\mathcal{A}$ is coherent as an $\mathcal{O}_S$-module and the stalk $\mathcal{A}_p$ is Azumaya over $\mathcal{O}_{S,p}$ for every closed point $p \in S$.

Proof. Suppose $\mathcal{A}$ is coherent and the stalk $\mathcal{A}_p$ is $\mathcal{O}_{S,p}$-Azumaya at every closed point $p \in S$. Let $s \in S$ be arbitrary, then there is a closed point p that is a specialisation of s, that is, $p \in \overline{\{s\}}$. Through the specialisation map $\mathcal{O}_{S,p} \to \mathcal{O}_{S,s}$, the stalk $\mathcal{A}_s = \mathcal{A}_p \otimes_{\mathcal{O}_{S,p}} \mathcal{O}_{S,s}$ is $\mathcal{O}_{S,s}$-Azumaya (Lemma 1.39). Finally, since each stalk is finite free and $\mathcal{A}$ is coherent, $\mathcal{A}$ is finite locally free. □

Example 2.5. *Neutral Azumaya algebras.* Let $\mathcal{E}$ be a finite locally free $\mathcal{O}$-module, then $\mathcal{A} := \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$ is an Azumaya $\mathcal{O}_S$-algebra. Indeed, there is an étale cover $\{U_i \to S\}$ where $\mathcal{E}_{|U_i}$ is of the form $\bigoplus^n \mathcal{O}_{|U_i}$ and $\mathcal{A}_{|U_i}$ is of the form $\mathrm{M}_n(\mathcal{O}_{S|U_i})$.

Example 2.6. Let S be a scheme where $2 \in \Gamma(S)$ is invertible. Choose any two units $a, b \in \Gamma(S)^\times$. We construct the (generalised) quaternion $\mathcal{O}_S$-algebra $(a,b)_S$ as follows: for any $U \subset S$, let $(a,b)_S(U)$ be the $\mathcal{O}_S(U)$-algebra generated by two elements i, j subject to the relations

$$i^2 = a_{|U}, \quad j^2 = b_{|U}, \quad ij = -ji.$$

Notice that $\{1, i, j, ij\}$ is a basis for $(a,b)_S$ as an $\mathcal{O}_S$-module, so $(a,b)_S$ is finite free of rank 4. Moreover, for every $p \in S$, the fibre $(a,b)_S \otimes_{\mathcal{O}_{S,p}} \kappa(p)$ is just the generalised quaternion $\kappa(p)$-algebra $(a,b)_{\kappa(p)}$, which is one of the classical examples of central simple $\kappa(p)$-algebras (see [BO13, Prop. I.2.5], for example).

Theorem 2.7. Skolem-Noether
Any algebra automorphism of an Azumaya algebra is Zariski-locally an inner automorphism. That is, if ϕ is an $\mathcal{O}_S$-algebra automorphism of an Azumaya algebra $\mathcal{A}$ over S, then there is an open cover $\{U_i \to S\}$ such that each $\phi_{|U_i}$ is of the form

$$a \mapsto uau^{-1}$$

for some unit $u \in \Gamma(U_i, \mathcal{A})$.

Proof. Let ϕ be an $\mathcal{O}_S$-algebra automorphism of $\mathcal{A}$ and let $p \in S$ be any point. Then ϕ_p is an automorphism of the $\mathcal{O}_{S,p}$-Azumaya algebra $\mathcal{A}_p$. By the Skolem-Noether theorem for Azumaya algebras over local rings 1.41,

ϕ_p is an inner automorphism, and hence there is an open neighbourhood U of p and a unit $u \in \Gamma(U, \mathcal{A})$ such that $\phi_p(a) = uau^{-1}$ for all $a \in \mathcal{A}_p$. Consequently, the maps $a \mapsto uau^{-1}$ and $\phi_{|U}$ are equal on some open neighborhood $V \subset U$ of p. $\square$

Corollary 2.8. Denote by $\mathrm{GL}_{1,\mathcal{A}}$ the sheaf given by $T \mapsto \mathcal{A}(T)^{\times}$ and by $\mathrm{Aut}_{\mathcal{A}}$ the sheaf of algebra automorphisms of $\mathcal{A}$. Consider the sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_{1,\mathcal{A}} \xrightarrow{\mathrm{Inn}} \mathrm{Aut}_{\mathcal{A}} \longrightarrow 1,$$

where for any section $u \in \mathcal{A}(T)^{\times}$, $\mathrm{Inn}_T(u)$ is the algebra automorphism $a \mapsto uau^{-1}$. Since every algebra automorphism of $\mathcal{A}$ is locally an inner automorphism, the above sequence is exact. As a special case with $\mathcal{A} = \mathrm{M}_n(\mathcal{O}_S)$, the sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_n \xrightarrow{\mathrm{Inn}} \mathrm{PGL}_n \longrightarrow 1$$

is also exact.

2.2 The Brauer Group

If $\mathcal{A}$ and $\mathcal{A}'$ are Azumaya algebras over S, then so is the tensor product $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}'$, indeed, for every point $p \in S$, we have that the stalk

$$(\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}')_p = \mathcal{A}_p \otimes_{\mathcal{O}_{S,p}} \mathcal{A}'_p$$

is $\mathcal{O}_{S,p}$-Azumaya.

Definition 2.9. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ over S are said to be **Brauer equivalent** (denoted $\mathcal{A} \sim \mathcal{A}'$), if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{E}$ and $\mathcal{E}'$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}) \simeq \mathcal{A}' \otimes_{\mathcal{O}_S} \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}').$$

The **Brauer group** of S, denoted $\mathrm{Br}(S)$ is the set of Brauer equivalence classes of S-Azumaya algebras. We denote the equivalence class of an Azumaya algebra $\mathcal{A}$ by $[\mathcal{A}]$.

Proposition 2.10. The tensor product is a well-defined operation on $\mathrm{Br}(S)$, making $\mathrm{Br}(S)$ an abelian group with identity element $[\mathcal{O}_S]$, and where the inverse of $[\mathcal{A}]$ is $[\mathcal{A}^{\mathrm{op}}]$.

Proof. It is straightforward to check that the tensor product is a well-defined group operation on $\mathrm{Br}(S)$ using the fact that

$$\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}\otimes\mathcal{E}') \simeq \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})\otimes_{\mathcal{O}_S}\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}')$$

for finite locally free $\mathcal{O}_S$-modules $\mathcal{E}$ and $\mathcal{E}'$. The other statements are clear. □

Lemma 2.11. The assignment $X \mapsto \mathrm{Br}(X)$ defines a presheaf of abelian groups on $\mathbf{Sch}_{/S}$.

Proof. Let $f\colon X \to Y$ be a morphism of S-schemes, and let $\mathcal{A}$ be an Azumaya algebra over Y. Then the pullback $f^*\mathcal{A}$ is an Azumaya algebra over X. Indeed, the pullback of a finite locally free module is finite locally free, and condition 2. of Proposition 2.2 may be checked on an affine open cover using Lemma 1.39. Finally, we note that $[\mathcal{A}] \mapsto [f^*\mathcal{A}]$ is a well defined map $\mathrm{Br}(Y) \to \mathrm{Br}(X)$, since if

$$\mathcal{A}\otimes\mathrm{End}(\mathcal{E}) \simeq \mathcal{A}'\otimes\mathrm{End}(\mathcal{E}')$$

for Azumaya algebras $\mathcal{A}'$ and $\mathcal{A}'$ over Y and finite locally free $\mathcal{O}_Y$-modules $\mathcal{E}$ and $\mathcal{E}'$, then

$$f^*\mathcal{A}\otimes\mathrm{End}(f^*\mathcal{E}) \simeq f^*\mathcal{A}'\otimes\mathrm{End}(f^*\mathcal{E}').$$

□

Remark 2.12. The presheaf $\mathrm{Br}(-)\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Ab}$ is not a sheaf in the Zariski topology[1]. A first example of a nontrivial Azumaya algebra that is Zariski-locally trivial was constructed by Ojanguren [Oja74], who gave an Azumaya algebra of order 3 on a cubic surface with three singular points. DeMeyer and Ford [DF92] expanded on Ojanguren's example, by giving a class of examples of nontrivial, locally trivial Azumaya algebras over certain toric surfaces.

For an affine scheme $\mathrm{Spec}\,R$, we recover the definitions in Section 1.5 with $\mathrm{Br}(\mathrm{Spec}\,R) = \mathrm{Br}(R)$. A particularly interesting case is when $S = \mathrm{Spec}\,k$, where k is a field. There is a natural isomorphism between $\mathrm{Br}(k)$ and the Galois cohomology group $H^2(G, k_s^\times)$, where k_s is a separable closure of k and $G := \mathrm{Gal}(k_s/k)$ is the absolute Galois group (see [GS17, §4] for details). But we may recover the Galois cohomology group from étale cohomology using the canonical isomorphism

[1] since every Azumaya algebra is locally trivial in the étale topology, we already knew $\mathrm{Br}(-)$ was not an étale or fppf sheaf

$H^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m) = H^2(G, k_s^\times)$ (see [Stacks, Tag 03QQ]). This leads us to the natural question: for an arbitrary scheme S, do we have that $\operatorname{Br}(S) = H^2_{\text{ét}}(S, \mathbb{G}_m)$? We spend the remainder of this chapter comparing these two groups.

2.3 Intermission: Stacks

In this section we develop the theory of stacks we need for Giraud's non-abelian cohomology [Gir71] in order to facilitate our comparison of $\operatorname{Br}(S)$ with $H^2_{\text{ét}}(S, \mathbb{G}_m)$, as well as our discussion on twisted forms. Roughly speaking, a stack over S is a sort of 2-sheaf of groupoids $\mathbf{Sch}_{/S}^{\text{op}} \to \mathbf{Groupoids}$. We construct this through the notion of a category fibred in groupoids satisfying descent. For references on stacks, torsors, and gerbes we recommend the book [Ols16] by Olsson and the recent notes [Alp24] by Alper.

Definition 2.13. Let $\mathcal{C}$ be a category. A **category over** $\mathcal{C}$ is a functor $p\colon \mathcal{F} \to \mathcal{C}$. An object $u \in \mathcal{F}$ **lies above** an object $U \in \mathcal{C}$ if $U = p(u)$. Similarly, a morphism $\phi\colon u \to v$ in $\mathcal{F}$ **lies above** a morphism $f\colon U \to V$ in $\mathcal{C}$ if $f = p(\phi)$. Given a fixed object $U \in \mathcal{C}$, we form the category $\mathcal{F}(U)$ whose objects are the objects in $\mathcal{F}$ lying above U and whose morphisms are the morphisms lying above the identity $U \to U$. We call the category $\mathcal{F}(U)$ the **fibre** of $\mathcal{F}$ at U.

Definition 2.14. Let $p\colon \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$. A morphism $\phi\colon u \to v$ in $\mathcal{F}$ is said to be **cartesian** if for any other morphism $\psi\colon w \to v$ and factorisation $p(\phi) \circ h$ through $p(u)$ of $p(\psi)$, there is a unique morphism $\sigma\colon w \to u$ lying above h such that $\psi = \phi \circ \sigma$.

$$\begin{tikzcd}
w \arrow[rr, bend left, "\psi"] \arrow[d, squiggly] \arrow[dr, dashed, "\exists!\sigma"] & & v \arrow[d, squiggly] \\
p(w) \arrow[rr, bend left] \arrow[dr, "h"'] & u \arrow[ur, "\phi"'] \arrow[d, squiggly] & p(v) \\
& p(u) \arrow[ur] &
\end{tikzcd}$$

Definition 2.15. A **fibred category** over $\mathcal{C}$ is a functor $p\colon \mathcal{F} \to \mathcal{C}$ such that for each morphism $f\colon U \to V$ in $\mathcal{C}$ and $v \in \mathcal{F}(V)$, there is an object $u \in \mathcal{F}(U)$ and a cartesian morphism $\phi\colon u \to v$ lying above f. In this case, it is straightforward to check that u is unique up to canonical isomorphism, and so we denote u by $f^*(v)$, and call $f^*(v)$ the **pullback of v along f.**

Definition 2.16. A **morphism of fibred categories** $\phi\colon \mathcal{F} \to \mathcal{F}'$ is a commutative triangle

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \mathcal{F}' \arrow[dl, "p'"] \\ & \mathcal{C}, & \end{tikzcd}$$

where ϕ preserves cartesian morphisms and $p = p' \circ \phi$ is equality of functors, and not merely a natural isomorphism of functors.

Example 2.17. Let $\mathcal{C}$ be a category and let X be an object in $\mathcal{C}$, denote by $\mathcal{C}_{/X}$ the slice category of maps into X. The natural functor $\mathcal{C}_{/X} \to \mathcal{C}$ defined by $(Y \to X) \mapsto Y$ is a fibred category over $\mathcal{C}$.

Definition 2.18. A set X can naturally be viewed as a category where the objects of the category are the elements of the set X and the morphisms are just the identities. A fibred category $\mathcal{F} \to \mathcal{C}$ is said to be **fibred in sets** if for each object U in $\mathcal{C}$, the fibre category $\mathcal{F}(U)$ is a set.

Example 2.19. Let $\mathcal{C}$ be a category and let $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf. We turn $\mathcal{F}$ into a fibred category as follows: overloading the notation, let $\mathcal{F}$ be the category whose objects are pairs (U, u), where U is an object in $\mathcal{C}$ and $u \in \mathcal{F}(U)$. A morphism $(U, u) \to (V, v)$ in $\mathcal{F}$ is a morphism $f\colon U \to V$ such that $\mathcal{F}f(v) = u$. Let $\mathcal{F} \to \mathcal{C}$ be the functor given by $(U, x) \mapsto U$, then $\mathcal{F} \to \mathcal{C}$ is a fibred category and each fibre $\mathcal{F}(U)$ is just the set of sections $\mathcal{F}(U)$. Thus, any presheaf $\mathcal{F}$ on $\mathcal{C}$ gives a category fibred in sets over $\mathcal{C}$.

Example 2.20. Conversely, if $\mathcal{F} \to \mathcal{C}$ is a category fibred in sets, we may turn this fibred category into a presheaf $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. For each object $U \in \mathcal{C}$, let $\mathcal{F}(U)$ be the fibre $\mathcal{F}(U)$. Given a morphism $f\colon U \to V$ in $\mathcal{C}$, define the corresponding set function $\mathcal{F}(V) \to \mathcal{F}(U)$ by $v \mapsto f^*(v)$. Since $\mathcal{F} \to \mathcal{C}$ is fibred in sets, there is only one choice for $f^*(v)$, so this assignment is well defined and the canonical isomorphism $(gf)^*(v) = f^*g^*(v)$ is necessarily equality. We have successfully turned $\mathcal{F}$ into a presheaf. It is straightforward to verify that this example and the inverse process given in Example 2.19 establishes an equivalence between

1. the category of presheaves on $\mathcal{C}$, and
2. the category of categories fibred in sets over $\mathcal{C}$.

Definition 2.21. A fibred category $\mathcal{F} \to \mathcal{C}$ is said to be **fibred in groupoids** if each fiber $\mathcal{F}(U)$ is a groupoid, that is, a category where every morphism is an isomorphism.

Every set is a groupoid, indeed, the only morphisms are identities, and therefore, every presheaf is a category fibred in groupoids.

Example 2.22. Let G, H be groups, viewed as one-object categories, and $p\colon G \to H$ be a group homomorphism viewed as a functor. Then $p\colon G \to H$ is a fibred category if and only if p is surjective, in this case, the fibre is just the kernel of p, a groupoid.

Lemma 2.23. If $\mathcal{F} \to \mathcal{C}$ is fibred in groupoids then every morphism in $\mathcal{F}$ is cartesian.

Proof. Let $\phi\colon u \to v$ be any morphism in $\mathcal{F}$. We will show that ϕ is cartesian, to do so let $\psi\colon w \to v$ be any morphism such that ϕ and ψ lie over a commutative triangle as in the diagram below:

$$\begin{tikzcd}
w \arrow[rr, bend left, "\psi"] \arrow[d, squiggly] & & v \arrow[d, squiggly] \\
p(w) \arrow[rr, "g"] \arrow[dr, "h"'] & u \arrow[ur, "\phi"] \arrow[d, squiggly] & p(v) \\
& p(u) \arrow[ur, "f"'] &
\end{tikzcd}$$

We construct the unique morphism $\sigma\colon w \to u$ as follows: choose a cartesian lift $\alpha\colon f^*v \to v$ of f, then there is a unique isomorphism $\beta\colon u \to f^*v$ such that β lifts $\mathrm{id}_{p(u)}$ and $\phi = \alpha \circ \beta$. Similarly, there is a unique map $\gamma\colon w \to f^*v$ such that γ lifts h and $\psi = \alpha \circ \gamma$. Set $\sigma = \beta^{-1} \circ \gamma$ and we are done:

$$\begin{tikzcd}
w \arrow[rr, bend left, "\psi"] \arrow[d, dashed, "\exists!\gamma"'] & & v \\
f^*v \arrow[urr, "\alpha"] \arrow[r, dashed, "\exists!\beta^{-1}"'] & u. \arrow[ur, "\phi"'] &
\end{tikzcd}$$

$\square$

Let $\mathcal{F} \to \mathcal{C}$ be fibred in groupoids, let X be an object in $\mathcal{C}$ and suppose $x, x' \in \mathcal{F}(X)$ are two objects lying over X. We define a presheaf $\underline{\mathrm{Iso}}_X(x, x') \colon (\mathcal{C}_{/X})^{\mathsf{op}} \to \mathbf{Set}$ by

$$(Y \xrightarrow{f} X) \mapsto \mathrm{Iso}_{\mathcal{F}(Y)}(f^*x, f^*x'),$$

where $\mathrm{Iso}_{\mathcal{F}(Y)}(f^*x, f^*x')$ is the set of isomorphisms $f^*x \to f^*x'$ in the fibre category $\mathcal{F}(Y)$, for a given choice[2] of f^*x and f^*x'. If

$$\begin{tikzcd}
Z \arrow[rr, "g"] \arrow[dr, "fg"'] & & Y \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

is a morphism in $\mathcal{C}_{/X}$, the corresponding morphism

$$\mathrm{Iso}_{\mathcal{F}(Y)}(f^*x, f^*x') \to \mathrm{Iso}_{\mathcal{F}(Z)}((fg)^*x, (fg)^*x')$$

is given by mapping an isomorphism $\phi \colon f^*x \to f^*x'$ to the lift

$$g^*\phi \colon g^*f^*x \to g^*f^*x'$$

using the cartesian morphism $g^*f^*x' \to f^*x'$ as in the diagram below:

$$\begin{tikzcd}
& & g^*f^*x \arrow[dll, dashed, "\exists ! g^*\phi"'] \arrow[d, squiggly] \arrow[drr] & & \\
g^*f^*x' \arrow[d, squiggly] \arrow[drr] & & Z \arrow[dll, "\mathrm{id}"'] \arrow[drr, "g"] & & f^*x \arrow[dll, "\phi"'] \arrow[d, squiggly] \\
Z \arrow[drr, "g"'] & & f^*x' \arrow[d, squiggly] & & Y \arrow[dll, "\mathrm{id}"] \\
& & Y. & &
\end{tikzcd}$$

As a particular case of this construction, when $x = x'$, we obtain a presheaf of groups $\underline{\mathrm{Iso}}_X(x, x)$ that we denote by

$$\underline{\mathrm{Aut}}_X(x) \colon (\mathcal{C}_{/X})^{\mathsf{op}} \to \mathbf{Group}\,.$$

We are finally in the position to define the stack condition. For the remainder of this section let $p \colon \mathcal{F} \to \mathcal{C}$ be a category fibred in groupoids, where $\mathcal{C}$ is a site with finite fibre products. Let $\{f_i \colon X_i \to X\}_{i \in I}$ be a cover in $\mathcal{C}$.

[2] Different choices of pullback yield presheaves that are canonically isomorphic to $\underline{\mathrm{Iso}}_X(x, x')$.

Definition 2.24. The **category of descent data** $\mathcal{F}(\{X_i \to X\})$ relative to the cover $\{f_i \colon X_i \to X\}_{i \in I}$ is defined as follows:

- Objects are pairs $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$ where $E_i \in \mathcal{F}(X_i)$ lies above X_i, and where the
$$\sigma_{ij} \colon \pi_1^* E_i \to \pi_2^* E_j$$
are isomorphisms in $\mathcal{F}(X_i \times_X X_j)$ satisfying the usual compatibility condition on the triple overlaps $X_i \times_X X_j \times_X X_k$ in the form of the following commutative diagram:
$$\begin{tikzcd}
\pi_{12}^*\pi_1^*E_i \arrow[r, "\pi_{12}^*\sigma_{ij}"] \arrow[d, equal] & \pi_{12}^*\pi_2^*E_j \arrow[r, equal] & \pi_{23}^*\pi_1^*E_j \arrow[d, "\pi_{23}^*\sigma_{jk}"] \\
\pi_{13}^*\pi_1^*E_i \arrow[r, "\pi_{13}^*\sigma_{ik}"] & \pi_{13}^*\pi_2^*E_k \arrow[r, equal] & \pi_{23}^*\pi_2^*E_k.
\end{tikzcd}$$
In the above, we wrote the canonical isomorphisms as equality, π_a for the projection from $X_i \times_X X_j$ to the a^{th} factor, and π_{ab} for the projection from $X_i \times_X X_j \times_X X_k$ to the product of the a^{th} factor with the b^{th} factor. This compatibility condition is often called the **cocycle condition**.

- A morphism $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I}) \to (\{D_i\}_{i \in I}, \{\tau_{ij}\}_{i,j \in I})$ is the data of morphisms $E_i \to D_i$ such that
$$\begin{tikzcd}
\pi_1^*E_i \arrow[r] \arrow[d, "\sigma_{ij}"'] & \pi_1^*D_i \arrow[d, "\tau_{ij}"] \\
\pi_2^*E_j \arrow[r] & \pi_2^*D_j
\end{tikzcd}$$
commutes for all $i, j \in I$.

There is a natural functor $\mathcal{F}(X) \to \mathcal{F}(\{X_i \to X\})$ defined as follows: Given $E \in \mathcal{F}(X)$, set $E_i := f_i^* E$, then $\pi_1 E_i$ and $\pi_2 E_j$ are both pullbacks of E along
$$\begin{tikzcd}
X_i \times_X X_j \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_j \arrow[d, "f_j"] \\
X_i \arrow[r, "f_i"'] & X,
\end{tikzcd}$$
so there exist canonical isomorphisms $\sigma_{ij} \colon \pi_1^* E_i \to \pi_2^* E_j$ that define a descent datum. We call the descent datum obtained in this way the **standard descent datum** of E.

Definition 2.25. A cover $\{X_i \to X\}$ is of **effective descent** if the functor

$$\mathcal{F}(X) \to \mathcal{F}(\{X_i \to X\})$$

is an equivalence of categories. In other words, if for each descent datum, there is a unique object above X that is obtained by glueing along the descent datum.

Definition 2.26. A category fibred in groupoids $\mathcal{F} \to \mathcal{C}$ is a **stack** if every cover $\{X_i \to X\}$ in $\mathcal{C}$ is of effective descent. A **morphism of stacks** is just a morphism of fibered categories.

Remark 2.27. One can define stacks without the assumption that $\mathcal{F} \to \mathcal{C}$ is fibred in groupoids (see [Stacks, Tag 0268]), but stacks in groupoids are both easier to work with and the more common definition in the literature. Since we won't need more general stacks, we stick to Definition 2.26.

Lemma 2.28. If $\mathcal{F} \to \mathcal{C}$ is a stack, then for any object X in $\mathcal{C}$ and $x, x' \in \mathcal{F}(X)$, the presheaf $\underline{\mathrm{Iso}}_X(x, x')$ is a sheaf on $\mathcal{C}_{/X}$.

Proof. Let $\{f_i \colon X_i \to X\}$ be a cover in $\mathcal{C}_{/X}$, and let $\phi_i \in \underline{\mathrm{Iso}}_X(x, y)(X_i)$ be isomorphisms $f_i^* x \to f_i^* y$ that agree on the intersections $X_i \times X_j$, i.e. isomorphisms ϕ_i such that the diagram

$$\begin{tikzcd}
\pi_1^* f_i^* x \arrow[r, "\phi_i"] \arrow[d, equal] & \pi_1^* f_j^* y \arrow[d, equal] \\
\pi_2^* f_j^* x \arrow[r, "\phi_j"'] & \pi_2^* f_j^* y
\end{tikzcd}$$

commutes, where we denote the canonical isomorphisms by the vertical equalities. This defines an isomorphism of descent data from the standard descent datum of x to the standard descent datum of y. By the stack condition this corresponds to a unique global isomorphism $x \to y$, i.e. a unique global section in $\underline{\mathrm{Iso}}_X(x, y)(X)$. $\square$

We now discuss some important examples of stacks.

Example 2.29. *Every sheaf is a stack.* Let $\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a sheaf, and turn $\mathcal{F}$ into a fibred category $\mathcal{F} \to \mathcal{C}$ as in Example 2.19, then $\mathcal{F}$ is a stack. Indeed, let $\{f_i \colon X_i \to X\}$ be any cover in $\mathcal{C}$ and let $(\{X_i, x_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$ be a descent datum for this cover. Then the $x_i \in \mathcal{F}(X_i)$ are sections that agree on the overlaps and glue up to a unique section $x \in \mathcal{F}(X)$ by the sheaf condition. The unique glued up object of the descent datum is then (X, x). We may proceed as in Example 2.20 to obtain an equivalence of categories between

1. the category of sheaves on $\mathcal{C}$, and
2. the category of stacks over $\mathcal{C}$ fibred in sets.

Example 2.30. *Every scheme is a stack.* Fix a base scheme S and let $X \to S$ be an S-scheme. We identify X and S with their respective fppf (or étale sites) $\mathbf{Sch}_{/X}$ and $\mathbf{Sch}_{/S}$. Then $\mathbf{Sch}_{/X} \to \mathbf{Sch}_{/S}$ is a stack. We encourage the reader to verify that the stack associated with the sheaf h_X on $\mathbf{Sch}_{/S}$ is the same as the stack $\mathbf{Sch}_{/X} \to \mathbf{Sch}_{/S}$.

Example 2.31. *Descent for quasi-coherent modules.* Let $\mathbf{Sch}_{/S}$ be the big fppf site of a scheme S and let $\mathbf{QCOH}_S$ be the category of pairs $(T, \mathcal{M})$, where T is an S-scheme and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_T$-module. A morphism $(T, \mathcal{M}) \to (T', \mathcal{M}')$ is a pair (f, ϕ), where $f\colon T \to T'$ is a morphism of S-schemes and $\phi\colon \mathcal{M} \to f^*\mathcal{M}'$ is a morphism of $\mathcal{O}_T$-modules. The natural functor $\mathbf{QCOH}_S \to \mathbf{Sch}_{/S}$ is a fibred category with fibres $\mathbf{Qcoh}(T)$. By fppf descent for quasi-coherent modules, $\mathbf{QCOH}_S \to \mathbf{Sch}_{/S}$ is a stack [Vis05, §4.2.2] (in the more general sense of stack, Remark 2.27 - the fibres are not groupoids).

Example 2.32. *Stack of Azumaya algebras.* Let $\mathbf{Az}_n$ be the category where:

- Objects are pairs $(T, \mathcal{A})$, where $T \to S$ is an S-scheme and $\mathcal{A}$ is an Azumaya $\mathcal{O}_T$-algebra of constant rank n^2.
- A morphism $(T, \mathcal{A}) \to (T', \mathcal{A}')$ is a morphism $f\colon T \to T'$ of S-schemes along with an isomorphism $\mathcal{A} \to f^*\mathcal{A}'$ of Azumaya $\mathcal{O}_T$-algebras.

Since we may glue sheaves of algebras in the fppf topology, and verify that an $\mathcal{O}_S$-algebra is Azumaya fppf-locally using Proposition 2.2 5., the natural functor $\mathbf{Az}_n \to \mathbf{Sch}_{/S}$ defines a stack.

2.4 Torsors, gerbes, and cohomology

For the remainder of this chapter we fix a base scheme S and work over the big fppf site $\mathbf{Sch}_{/S}$.

Definition 2.33. Let $\mathcal{G}$ be a sheaf of (possibly nonabelian) groups on $\mathbf{Sch}_{/S}$. A $\mathcal{G}$**-torsor** is a sheaf $\mathcal{P}$ on $\mathbf{Sch}_{/S}$ with a left $\mathcal{G}$-action $\mathcal{G} \times \mathcal{P} \to \mathcal{P}$ satisfying the following properties:

1. For each S-scheme X, there is a cover $\{X_i \to X\}$ such that each $\mathcal{P}(X_i)$ is nonempty.

2. The map
$$\begin{aligned} \mathcal{G} \times \mathcal{P} &\to \mathcal{P} \times \mathcal{P} \\ (g, p) &\mapsto (p, gp) \end{aligned}$$
is an isomorphism. In other words if $\mathcal{P}(T)$ is nonempty, the $\mathcal{G}(T)$-action on $\mathcal{P}(T)$ is *simply transitive*.

A **morphism of $\mathcal{G}$-torsors** $\mathcal{P} \to \mathcal{P}'$ is a $\mathcal{G}$-equivariant map $\mathcal{P} \to \mathcal{P}'$. Every morphism of $\mathcal{G}$-torsors is automatically an isomorphism. If a $\mathcal{G}$-torsor $\mathcal{P}$ has a global section $p \in \mathcal{P}(S)$, then the G-equivariant map $\mathcal{G} \to \mathcal{P}$ given by $g \mapsto gp$ is an isomorphism. In this case we say $\mathcal{P}$ is a **trivial torsor** . Since torsors are locally nonempty, they are locally trivial, and a $\mathcal{G}$-torsor is locally isomorphic to $\mathcal{G}$ acting on itself.

There is a relation between $\mathcal{G}$-torsors and the classical notion of principal G-bundles. If $\mathcal{G}$ is representable by an fppf affine group scheme G, then any principal G-bundle $P \to X$ over X gives a $\mathcal{G}$-torsor $\mathcal{P}$ by the Yoneda embeddings of P and $G \times_S P \to P$. In fact, this defines an equivalence of categories between the category of $\mathcal{G}$-torsors and principal G-bundles [Ols16, Prop. 4.5.6.].

Proposition 2.34. Let $\mathcal{G}$ be a sheaf of abelian groups on $\mathbf{Sch}_{/S}$, then $H^1(S, \mathcal{G})$ is the set of isomorphism classes of $\mathcal{G}$-torsors, and the neutral element of $H^1(S, \mathcal{G})$ corresponds to the isomorphism class of the trivial $\mathcal{G}$-torsor.

Proof. Denote by $\mathbb{Z}$ the constant sheaf associated to $\mathbb{Z}$ and by $\mathbf{EXT}(\mathbb{Z}, \mathcal{G})$ the category of extensions of $\mathbb{Z}$ by $\mathcal{G}$. There is a functor from $\mathbf{EXT}(\mathbb{Z}, \mathcal{G})$ to the category of $\mathcal{G}$-torsors defined by sending an extension
$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$
to the $\mathcal{G}$-torsor $\pi^{-1}(1)$ of liftings of the global section $1 \in \Gamma(S, \mathbb{Z})$ to $\mathcal{E}$:
$$\begin{aligned} \pi^{-1}(1) \colon (\mathbf{Sch}_{/S})^{\mathsf{op}} &\to \mathbf{Set} \\ (T \to S) &\mapsto \left\{ e \in \mathcal{E}(T) \mid \pi(e) = 1_{|T} \right\} \end{aligned}$$
with the $\mathcal{G}$-action given by $(g, e) \mapsto g + e$ in $\mathcal{E}$. This defines an equivalence of categories between $\mathbf{EXT}(\mathbb{Z}, \mathcal{G})$ and the category of $\mathcal{G}$-torsors (see [Ols16, Prop. 12.1.4]). Consequently, an isomorphism class of a $\mathcal{G}$-torsor corresponds uniquely to an isomorphism class of an extension of $\mathbb{Z}$ by $\mathcal{G}$. Under this correspondence the trivial extension
$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G} \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

corresponds to the trivial torsor since $\mathcal{G} \oplus \mathbb{Z} \to \mathbb{Z}$ is globally surjective, so $\pi^{-1}(1)$ admits a global section. We finish the proof by noting that the set $\mathrm{Ext}^1(\mathbb{Z}, \mathcal{G})$ of isomorphism classes of extensions of $\mathbb{Z}$ by $\mathcal{G}$ is canonically in bijection with $H^1(S, \mathcal{G})$ since $\Gamma(S, -) = \mathrm{Hom}(\mathbb{Z}, -)$. □

Remark 2.35. The proposition above suggests that for a nonabelian sheaf of group $\mathcal{G}$, we may define the **nonabelian cohomology** $H^1(S, \mathcal{G})$ as the pointed set of $\mathcal{G}$-torsors, where the distinguished element is the trivial torsor. Similar to abelian cohomology, if

$$1 \longrightarrow \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \longrightarrow 1$$

is an exact sequence of sheaves of (possibly nonabelian) groups, then there is a natural exact sequence of pointed sets

$$\mathcal{G}(S) \longrightarrow \mathcal{H}(S) \longrightarrow \mathcal{K}(S) \xrightarrow{\delta} H^1(S, \mathcal{G}) \xrightarrow{\alpha_*} H^1(S, \mathcal{H}) \xrightarrow{\beta_*} H^1(S, \mathcal{K}),$$

where the connecting morphism δ is given by sending $k \in \mathcal{K}(S)$ to the $\mathcal{G}$-torsor $\beta^{-1}(k)$. Given a morphism $\alpha\colon \mathcal{G} \to \mathcal{H}$ of sheaves of groups, the corresponding morphism $\alpha_*\colon H^1(S, \mathcal{G}) \to H^1(S, \mathcal{H})$ is given by sending a $\mathcal{G}$-torsor $\mathcal{P}$ to the $\mathcal{H}$-torsor $\alpha_*\mathcal{P}$ defined as follows. As a sheaf, $\alpha_*\mathcal{P}$ is the sheafification of the quotient of $\mathcal{H} \times \mathcal{P}$ by the $\mathcal{G}$-action given on sections $g \in \mathcal{G}(T), h \in \mathcal{H}(T), p \in \mathcal{P}(T)$ by

$$g * (h, p) := (h\alpha(g)^{-1}, gp).$$

The $\mathcal{H}$-action on α_* is given by descending the $\mathcal{H}$-action

$$h \cdot (h', p) := (hh', p)$$

on $\mathcal{H} \times \mathcal{P}$ to the quotient. In the abelian case, the above long exact sequence with the morphisms we have just described is compatible with the usual long exact cohomology sequence. See [Gir71, III§3] for more details on this nonabelian cohomology.

An important class of examples of torsors are twisted forms. The author learned about the correspondence that we will present here from the notes [You19, §5], a more sophisticated treatment of the material is [CF15, §2.2.4].

Definition 2.36. Let $\mathcal{F} \to \mathbf{Sch}_{/S}$ be a stack. Let $x \in \mathcal{F}(S)$ be an object lying above S. A **twisted form** of x is an object $y \in \mathcal{F}(S)$ lying above S that is locally isomorphic to x in the sense that there is a cover $\{f_i\colon U_i \to S\}$ such that $f_i^*(x) \simeq f_i^*(y)$ in each fibre $\mathcal{F}(U_i)$. We denote the set of (isomorphism classes of) twisted forms of x by $\mathbf{Twist}(x)$.

Example 2.37. A line bundle over S is just a twisted form of $\mathcal{O}_S$, and a vector bundle of rank n is just a twisted form of $\mathcal{O}_S^n$.

Example 2.38. An Azumaya $\mathcal{O}_S$-algebra of constant rank n is precisely a twisted form of $\mathrm{M}_n(\mathcal{O}_S)$ (for the stack of $\mathcal{O}_S$-algebras $\mathbf{Alg}_S \to \mathbf{Sch}_{/S}$).

If $\mathcal{F} \to \mathbf{Sch}_{/S}$ is a stack, $x \in \mathcal{F}(S)$, and $y \in \mathbf{Twist}(x)$ is a twisted form of x, then the sheaf $\underline{\mathrm{Iso}}_S(x,y)$ is an $\underline{\mathrm{Aut}}_S(x)$-torsor, where the $\underline{\mathrm{Aut}}_S(x)$-action is given by precomposition. Indeed, if $\phi, \psi \colon f^*(x) \to f^*(y)$ are isomorphisms lying above $f \colon T \to S$, then $\sigma := \psi^{-1} \circ \phi$ is the unique automorphism of $f^*(x)$ such that $\phi = \psi \circ \sigma$, so the action is simply transitive, and $\underline{\mathrm{Iso}}_S(x,y)$ is locally nonempty by the assumption that y is locally isomorphic to x.

Proposition 2.39. The assignment $y \mapsto \underline{\mathrm{Iso}}_S(x,y)$ defines a bijection between $\mathbf{Twist}(x)$ and the set of (isomorphism classes of) $\underline{\mathrm{Aut}}_S(x)$-torsors $H^1(S, \underline{\mathrm{Aut}}_S(x))$.

Proof. We define the inverse function as follows. Given an $\underline{\mathrm{Aut}}_S(x)$-torsor $\mathcal{P}$, choose a cover $\{f_i \colon U_i \to S\}_{i \in I}$ where each $\mathcal{P}(U_i)$ is nonempty and sections $p_i \in \mathcal{P}(U_i)$. Then since the action of $\underline{\mathrm{Aut}}_S(x)$ is simply transitive, there are unique automorphisms $\sigma_{ij} \in \underline{\mathrm{Aut}}_S(x)(U_i \times_S U_j)$ such that

$$p_{i|U_i \times_S U_j} = \sigma_{ij} \cdot p_{j|U_i \times_S U_j}.$$

We interpret the automorphisms σ_{ij} as isomorphisms $\pi_1^* f_i^* x \to \pi_2^* f_j^* x$ satisfying the cocycle condition. In other words, we have a descent datum $(\{f_i^* x\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$, and since $\mathcal{F}$ is a stack, this descent datum is effective and we glue to obtain an object $y \in \mathcal{F}(S)$ lying above x. By construction, y is locally isomorphic to x, so $y \in \mathbf{Twist}(x)$. Note that this descent datum is independent of the choice of sections p_i. Indeed, if we chose different sections $q_i \in \mathcal{P}(U_i)$, there are unique automorphisms ϕ_i on $f_i^*(x)$ satisfying $p_i = \phi_i \cdot q_i$, thereby providing an isomorphism of descent data from the descent datum obtained from the p_i to the descent datum obtained from the q_i.

We quickly verify that these functions are mutually inverse. On the one hand, if we start with an $\underline{\mathrm{Aut}}_S(x)$-torsor $\mathcal{P}$, we follow the procedure above, picking a cover $\{f_i \colon U_i \to S\}$ admitting elements $p_i \in \mathcal{P}(U_i)$ and use this to construct the descent datum for a twisted form $y \in \mathbf{Twist}(x)$. Next we may recover $\mathcal{P}$ as $\underline{\mathrm{Iso}}_S(x,y)$ by constructing a morphism of $\underline{\mathrm{Aut}}_S(x)$-torsors $\alpha \colon \underline{\mathrm{Iso}}_S(x,y) \to \mathcal{P}$. Let $\phi \in \mathrm{Iso}(f^*x, f^*y)$ be a section on $f \colon U \to S$, denote by ϕ_i the restriction of ϕ to $U \times_S U_i$. By construction $f_i^* x = f_i^* y$, so

ϕ_i defines an automorphism on $\pi_i^* f_i^* x$ lying above $U \times_S U_i$. The sections $\phi_i \cdot p_{i|} \in \mathcal{P}(U \times_S U_i)$ are compatible on the intersections $U \times_S \times U_i \times_S U_j$ since

$$\phi_{i|} \cdot p_{i|} = \phi_{i|} \cdot (\sigma_{ij} \cdot p_{j|}) = \phi_{j|} \cdot p_{j|}.$$

We glue the sections $\phi_i \cdot p_{i|}$ to obtain $\alpha(\phi) \in \mathcal{P}(U)$. Since every morphism of torsors is an isomorphism, we are done.

On the other hand, if we start with a twisted form $y \in \mathbf{Twist}(x)$, we get the $\underline{\mathrm{Aut}}_S(x)$-torsor $\underline{\mathrm{Iso}}_S(x, y)$, from this we pick a cover $\{f_i : U_i \to S\}$ admitting isomorphisms $p_i : f_i^* x \to f_i^* y$ lying above U_i. Following the procedure above, we obtain a descent datum $(\{f_i^* x\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$ for a twisted form y' of x. But the isomorphisms p_i define an isomorphism of descent data from this descent datum for y' to the canonical descent datum of y, and therefore, $y' \simeq y$. □

We have that $\mathbb{G}_m = \mathrm{Aut}(\mathcal{O}_S)$, $\mathrm{GL}_n = \mathrm{Aut}(\bigoplus^n \mathcal{O})$, and $\mathrm{PGL}_n = \mathrm{Aut}_{\mathrm{Alg}}(M_n(\mathcal{O}))$. So as an application of the proposition and examples above, we obtain the following descriptions: A line bundle is just a $\mathbb{G}_m$-torsor, a rank n vector bundle is a GL_n-torsor, and a degree n Azumaya $\mathcal{O}_S$-algebra is a PGL_n-torsor. It is well known that line bundles are classified by $H^1(-, \mathbb{G}_m)$, so it is quite satisfying to note that this may be generalised to the nonabelian case where rank n vector bundles are classified by $H^1(-, \mathrm{GL}_n)$ and degree n Azumaya algebras are classified by $H^1(-, \mathrm{PGL}_n)$.

Definition 2.40. Let $\mathcal{G}$ be a sheaf of groups on $\mathbf{Sch}_{/S}$. The **classifying stack** $\mathcal{BG}$ is the category where:

- Objects are pairs $(T, \mathcal{P})$ where T is an S-scheme and $\mathcal{P}$ is a $\mathcal{G}_{|T}$ torsor on $\mathbf{Sch}_{/T}$.
- A morphism $(T, \mathcal{P}) \to (T', \mathcal{P}')$ is a morphism $f : T \to T'$ of S-schemes along with an isomorphism $\mathcal{P} \to \mathcal{P}'_{|T}$ of $\mathcal{G}_{|T}$-torsors.

Refer to [Stacks, Tag 04UK] to see that we may glue $\mathcal{G}$-torsors so that $\mathcal{BG}$ defines a stack.

Remark 2.41. In Proposition 2.39 we showed that every twisted form is naturally a torsor, but on the other hand, every $\mathcal{G}-$torsor is just a twisted form of $\mathcal{G}$ acting on itself. Applying Proposition 2.39 one more time reveals the deep fact that a $\mathcal{G}$-torsor is just a $\mathrm{Aut}_{\mathcal{G}}(\mathcal{G}) = \mathcal{G}$-torsor.

Definition 2.42. Let $\mathcal{G}$ be a sheaf of *abelian* groups on $\mathbf{Sch}_{/S}$. A $\mathcal{G}$**-gerbe** over S is a stack $\mathcal{X}$ satisfying the following properties:

1. Every S-scheme T has a cover $\{T_i \to T\}$ such that the fibre $\mathcal{X}(T_i)$ is nonempty for each T_i.

2. For any two objects $x, y \in \mathcal{X}(T)$ lying above an S-scheme T, there is a cover $\{f_i \colon T_i \to T\}$ such that $f_i^* x \simeq f_i^* y$ in $\mathcal{X}(T_i)$ for each i.

3. For every S-scheme T and object $x \in \mathcal{X}(T)$ lying above T, there is an isomorphism $\psi_x \colon \mathcal{G}_{|T} \to \underline{\mathrm{Aut}}_T(x)$ that satisfies the following compatibility condition: for any isomorphism $\alpha \colon x \to y$ in $\mathcal{X}(T)$, the triangle

$$\begin{tikzcd} & \mathcal{G}_{|T} \arrow[dl, "\psi_x"'] \arrow[dr, "\psi_y"] & \\ \underline{\mathrm{Aut}}_T(x) \arrow[rr, "\tau \mapsto \alpha\tau\alpha^{-1}"'] & & \underline{\mathrm{Aut}}_T(y) \end{tikzcd}$$

commutes.

The collection of isomorphisms ψ_x is called the **band** of $\mathcal{X}$, and a $\mathcal{G}$-gerbe is sometimes called a **gerbe banded by $\mathcal{G}$**. A **morphism of $\mathcal{G}$-gerbes** $f \colon (\mathcal{X}, \{\psi_x\}) \to (\mathcal{X}', \{\psi'_{x'}\})$ is a morphism $f \colon \mathcal{X} \to \mathcal{X}'$ of stacks that is compatible with the bands in the sense that

$$\begin{tikzcd} & \mathcal{G}_{|T} \arrow[dl, "\psi_x"'] \arrow[dr, "\psi'_{f(x)}"] & \\ \underline{\mathrm{Aut}}_T(x) \arrow[rr, "f"'] & & \underline{\mathrm{Aut}}_T(f(x)) \end{tikzcd}$$

commutes. By [Ols16, Lemma 12.2.4], every morphism of $\mathcal{G}$-gerbes is an isomorphism.

Example 2.43. *The trivial $\mathcal{G}$-gerbe.* Let $\mathcal{G}$ be a sheaf of abelian groups on $\mathbf{Sch}_{/S}$. The classifying stack $\mathcal{B}\mathcal{G}$ is a $\mathcal{G}$-gerbe. Indeed, any two $\mathcal{G}$-torsors are locally isomorphic to $\mathcal{G}$, so they are isomorphic to each other on a sufficiently fine cover. For any S-scheme T and $\mathcal{G}_{|T}$-torsor $\mathcal{P} \in \mathcal{B}\mathcal{G}(T)$ the band isomorphism $\psi_{\mathcal{P}} \colon \mathcal{G}_{|T} \to \underline{\mathrm{Aut}}_T(x)$ is given on sections by mapping g to the automorphism $p \mapsto gp$.

Example 2.44. If $\mathcal{F} \to \mathbf{Sch}_{/S}$ is a $\mathcal{G}$-gerbe, and $x, y \in \mathcal{F}(X)$ are objects of $\mathcal{F}$ lying above an S-scheme X, then the sheaf $\underline{\mathrm{Iso}}_X(x, y)$ is an $\underline{\mathrm{Aut}}_X(x)$-torsor, where the $\underline{\mathrm{Aut}}_X(x)$-action is given by precomposition.

Example 2.45. Gerbe lifting a torsor. Let $\mathcal{G}$ be a sheaf of abelian groups on $\mathbf{Sch}_{/S}$ and let

$$1 \longrightarrow \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \longrightarrow 1$$

be a short exact sequence of sheaves of groups, with $\mathcal{H}$ and $\mathcal{K}$ possibly nonabelian. Let $\mathcal{P}$ be a $\mathcal{K}$-torsor. We construct a $\mathcal{G}$-gerbe $\mathcal{G}_{\mathcal{P}} \to \mathbf{Sch}_{/S}$ as follows:

- An object is a triple $(X, \mathcal{R}, \epsilon)$, where X is an S-scheme, $\mathcal{R}$ is an $\mathcal{H}$-torsor on $\mathbf{Sch}_{/X}$, and $\epsilon\colon \beta_*\mathcal{R} \to \mathcal{P}_{|X}$ is an isomorphism of $\mathcal{K}$-torsors on $\mathbf{Sch}_{/X}$.
- A morphism $(X, \mathcal{R}, \epsilon) \to (X', \mathcal{R}', \epsilon')$ is a pair $(f, f^\sharp)$ where $f\colon X \to X'$ is a morphism of schemes and $f^\sharp\colon f^*\mathcal{R} \to \mathcal{R}'$ is an isomorphism of $\mathcal{H}$-torsors over X such that the natural diagram

$$\begin{tikzcd} \beta_* f^*\mathcal{R}' \arrow[r, "b_* f^\sharp"] \arrow[d, "\sim"'] & \beta_*\mathcal{R} \arrow[drr, "\epsilon"] & & \\ f^*\beta_*\mathcal{R}' \arrow[r, "f^*\epsilon'"] & f^*(\mathcal{P}_{|X'}) \arrow[rr, "\sim"] & & \mathcal{P}_{|X} \end{tikzcd}$$

  commutes.

For an object $x = (X', \mathcal{R}, \epsilon) \in \mathcal{G}_{\mathcal{P}}$, the automorphism presheaf $\underline{\mathrm{Aut}}_X(x)$ is given by sending an X-scheme X' to the group of $\mathcal{H}_{|X'}$-torsor automorphisms ϕ on $\mathcal{R}_{|X'}$ such that $\beta_*\phi = \mathrm{id}$. This group is canonically isomorphic to $\mathcal{G}(X')$, so we obtain a natural isomorphism $\psi_x\colon \mathcal{G}_{|X} \to \underline{\mathrm{Aut}}_X(x)$ that gives the band for our gerbe. For the verification that $\mathcal{G}_{\mathcal{P}}$ is a $\mathcal{G}$-gerbe, we refer the reader to [Ols16, Prop. 12.2.6].

Proposition 2.46. Let $\mathcal{G}$ be a sheaf of abelian groups, then $H^2(S, \mathcal{G})$ is the set of isomorphism classes of $\mathcal{G}$-gerbes.

Proof. Choose an inclusion of abelian sheaves $i\colon \mathcal{G} \to \mathcal{I}$, with $\mathcal{I}$ injective and set $\mathcal{K} = \operatorname{Coker} i$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{i} \mathcal{I} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Since $\mathcal{I}$ is injective, $H^k(S, \mathcal{I}) = 0$ and the connecting morphism

$$\delta\colon H^1(S, \mathcal{K}) \to H^2(S, \mathcal{G})$$

is an isomorphism. Given an element $\epsilon \in H^2(S,\mathcal{G})$, the corresponding element $\delta^{-1}(\epsilon) \in H^1(S,\mathcal{K})$ is a $\mathcal{K}$-torsor. With the procedure described in Example 2.45 we may lift $\mathcal{K}$-torsor $\delta^{-1}(\epsilon)$ to a $\mathcal{G}$-gerbe $\mathcal{G}_{\delta^{-1}(\epsilon)}$. The assignment $\epsilon \mapsto \mathcal{G}_\epsilon$ defines a bijection between $H^2(S,\mathcal{G})$ and the set of $\mathcal{G}$-gerbes (see [Ols16, Theorem 12.2.8.]). $\square$

Using this characterisation of $H^2(S,\mathcal{G})$, we are finally in the position to compare the Brauer group $\mathrm{Br}(S)$ to the cohomology group $H^2(S,\mathbb{G}_m)$.

2.5 The cohomological Brauer Group

Theorem 2.47. Let S be a scheme. There is an injection of the Brauer group $\mathrm{Br}(S)$ of S into $H^2(S,\mathbb{G}_m)$.

Proof. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}_S$-algebra over S. We construct the **gerbe of trivialisations** $\mathcal{G}_\mathcal{A}$ of $\mathcal{A}$ as follows:

- Objects are triples $(T,\mathcal{E},\sigma)$, where $f\colon T \to S$ is an S-scheme, $\mathcal{E}$ is a finite locally free $\mathcal{O}_T$-module, and $\sigma\colon \mathcal{E}nd(\mathcal{E}) \to f^*\mathcal{A}$ is an isomorphism of $\mathcal{O}_T$-algebras.
- A morphism $(f\colon T \to S,\mathcal{E},\sigma) \to (f'\colon T' \to S,\mathcal{E}',\sigma')$ is a pair $(\phi,\phi^\sharp)$, where $\phi\colon T \to T'$ is a morphism of S-schemes and $\phi^\sharp\colon \mathcal{E} \to \phi^*\mathcal{E}'$ is an isomorphism of $\mathcal{O}_T$-modules such that

$$\begin{tikzcd} \mathcal{E}nd(\mathcal{E}) \arrow[r, "\overline{\phi}"] \arrow[d, "\sigma"'] & \mathcal{E}nd(\phi^*\mathcal{E}') \arrow[d, "\phi^*\sigma'"] \\ f^*\mathcal{A} \arrow[r, equal] & \phi^* f'^*\mathcal{A} \end{tikzcd}$$

  commutes, where $\overline{\phi}$ is the isomorphism induced by $\phi^\sharp$.

- The band isomorphisms $\mathbb{G}_m \to \underline{\mathrm{Aut}}_T(f\colon T \to S,\mathcal{E},\sigma)$ are given on sections by $u \mapsto (\mathrm{id}, L_u)$, where L_u is scalar multiplication by u.

We first verify that the stack $\mathcal{G}_\mathcal{A}$ is a $\mathbb{G}_m$-gerbe [Ols16, p. 12.3.6]. Since every Azumaya algebra is locally trivial, the fibres of $\mathcal{G}_\mathcal{A}$ are locally nonempty. To show that any two objects in $\mathcal{G}_\mathcal{A}$ are locally isomorphic, we may first pass to a cover where $\mathcal{A}$ is trivial and assume that $\mathcal{A} = M_n(\mathcal{O}_S)$. Next, given any object $(T,\mathcal{E},\sigma)$, we may refine our cover to assume $\mathcal{E} = \bigoplus^n \mathcal{O}_S$ and σ is an algebra automorphism of $\mathrm{M}_n(\mathcal{O}_S)$. By further refining our cover and Skolem-Noether 2.7, we may assume that σ is an inner automorphism determined by a section $g \in \mathrm{GL}_n(S)$. Considering g as an isomorphism $\bigoplus^n \mathcal{O}_s \to \bigoplus^n \mathcal{O}_s$, this induces an isomorphism

$$(T, \oplus^n \mathcal{O}_s, \sigma) \to (T, \oplus^n \mathcal{O}_s, \mathrm{id}).$$

If we have any other object $(T', \mathcal{E}', \sigma')$, we may similarly set up an isomorphism to $(T, \bigoplus^n, \mathrm{id})$ on a cover, and therefore, any two objects are isomorphic on a sufficiently fine cover.
The assignment $\mathcal{A} \mapsto \mathcal{G}_{\mathcal{A}}$ is a well-defined injection $\mathrm{Br}(S) \to H^2(S, \mathbb{G}_m)$ because $\mathcal{A}$ is trivial if and only if the fibre $\mathcal{G}_{\mathcal{A}}(S)$ is nonempty, that is, if and only if $\mathcal{G}_{\mathcal{A}}$ is the trivial $\mathbb{G}_m$-gerbe. $\square$

Remark 2.48. We are working in the fppf topology, but this is no problem since $H^2_{\text{ét}}(S, \mathbb{G}_m) = H^2(S, \mathbb{G}_m)$ (see [Mil80, Thm. 3.9]).

Remark 2.49. Since $\mathbb{G}_m$ is commutative, the long exact nonabelian cohomology sequence

$$\cdots \longrightarrow \mathrm{PGL}_n(S) \xrightarrow{\delta} H^1(S, \mathbb{G}_m) \longrightarrow H^1(S, \mathrm{GL}_n) \longrightarrow H^1(S, \mathrm{PGL}_n)$$

may be extended with $\delta \colon H^1(S, \mathrm{PGL}_n) \to H^2(S, \mathbb{G}_m)$ given by the construction described in Example 2.45 (see [Gir71, Rem. IV.4.2.10]). Let us now suppose that our scheme S is connected so that all Azumaya algebras have constant ranks. By the discussion following Proposition 2.39, an Azumaya algebras $\mathcal{A}$ of degree n corresponds to the PGL_n-torsor $\underline{\mathrm{Iso}}_S(M_n(\mathcal{O}_S), \mathcal{A})$. The $\mathbb{G}_m$-gerbe lifting this PGL_n-torsor is isomorphic to the gerbe of trivialisations $\mathcal{G}_{\mathcal{A}}$ of $\mathcal{A}$ [Ols16, Lem. 12.3.9]. In other words, $\mathcal{G}_{\mathcal{A}}$ is the image of the torsor corresponding to $\mathcal{A}$ under the connecting morphism $\delta \colon H^1(S, \mathrm{PGL}_n) \to H^2(S, \mathbb{G}_m)$. Therefore, the map $\mathrm{Br}(X) \to H^2(S, \mathbb{G}_m)$ may be described as sending a Brauer class $[\mathcal{A}]$ to the PGL_n-torsor corresponding to a representative $\mathcal{A}'$ of $[\mathcal{A}]$, and then through the connecting morphism δ.

Corollary 2.50. Suppose the scheme S has only finitely many connected components, then the Brauer group $\mathrm{Br}(S)$ is a torsion group.

Proof. By the finiteness condition, it suffices to prove that every Brauer class represented by an Azumaya algebra of constant rank is torsion. Consider the following commutative diagram of sheaves of groups on the big fppf site of S:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r, "x \mapsto x^n"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & SL_n \arrow[r] \arrow[d] & \mathrm{GL}_n \arrow[r, "\det"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{PGL}_n \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{PGL}_n \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1. &
\end{tikzcd}$$

The top row (Kummer sequence) is exact since we are working in the fppf topology[3] [Stacks, Tag 040N], from this we obtain that det is surjective, and the middle row is also exact since $\mathrm{SL}_n = \operatorname{Ker} \det$. The middle column is exact by Skolem-Noether 2.7. And finally, the left column is exact by the same diagram chase one uses to prove the nine lemma. Applying the nonabelian cohomology to the left two columns, we obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(S, \mathrm{PGL}_n) \arrow[r] \arrow[d, equal] & H^2(S, \mu_n) \arrow[d] \\
\cdots \arrow[r] & H^1(S, \mathrm{PGL}_n) \arrow[r] & H^2(S, \mathbb{G}_m).
\end{tikzcd}$$

Since $H^2(S, \mu_n)$ is n-torsion, and the map $H^1(S, \mathrm{PGL}_n) \to H^2(S, \mathbb{G}_m)$ factors through $H^2(S, \mu_n)$, every Azumaya algebra of constant rank n^2 represents a Brauer class that is n-torsion. □

Remark 2.51. As an alternative to the above cohomological proof, in the context of rings, Saltman [Sal81] has shown that if $\mathcal{A}$ has constant rank n^2, then $[\mathcal{A}]$ is n-torsion by explicitly constructing a finite locally free module $\mathcal{E}$ with $\otimes^n \mathcal{A} \simeq \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$. Saltman's argument was adapted to the language of schemes in [Stacks, Tag 0A2L].

Remark 2.52. Returning to the discussion at the end of Section 2.2, we may now answer that $\mathrm{Br}(S) \neq H^2(S, \mathbb{G}_m)$, since $H^2(S, \mathbb{G}_m)$ is in general not a

[3]This is the first instance where we really need the fppf topology, everything else up to this point could have been done equally as well in the étale topology.

torsion group, as was shown by Grothendieck [Gro68b, Rem. 1.11.b] using a normal surface S constructed by Mumford.

We will refine the question by defining the **cohomological Brauer group** $\mathrm{Br}'(S) := H^2(S, \mathbb{G}_m)_{\mathrm{tors}}$ to be the torsion subgroup of $H^2(S, \mathbb{G}_m)$. The question now becomes: if the scheme S has only finitely many connected components, is the injection $\mathrm{Br}(S) \to \mathrm{Br}'(S)$ also surjective? This remains an open question that has been widely studied. For example, Hoobler [Hoo82] has produced some classes of schemes for which the Brauer map is an isomorphism. The current best result is due to Gabber who found that if S admits an ample line bundle, then $\mathrm{Br}(S) = \mathrm{Br}'(S)$. Gabber never published his proof, and eventually a proof was written down by de Jong [deJ03]. De Jong's proof was reproduced in the book [CS21, §4.2] by Colliot-Thélène and Sokorobogatov. Notice that Gabber's hypothesis imply that S is separated [Stacks, Tag 09MP]. The separatedness assumption cannot be relaxed. A counterexample using a nonseparated scheme was constructed in [EHKV, Cor. 3.11], following their result that a $\mathbb{G}_m$-gerbe is a quotient stack if and only if it represents the image of a Brauer class in $H^2(S, \mathbb{G}_m)$.

Chapter 3

Quadratic pairs

Quadratic pairs were introduced in the Book of Involutions [BoI] as a generalisation of quadratic forms to provide a more flexible theory over fields of arbitrary characteristic. In particular for the case of characteristic 2, where the traditional theory of quadratic forms are full of exceptions and holes. This notion of a quadratic pair was extended to Azumaya algebras over schemes by Calmès and Fasel [CF15] to aid in the classification of classical algebraic groups, and to establish the connection with the classification in [SGA3]. In this chapter give an overview of the theory of the theory of quadratic pairs over schemes, and explain how quadratic pairs are twisted forms of quadratic forms. We also develop a cohomological obstruction to a given orthogonal involution admitting a quadratic pair, following along with some recent theory established by Gille, Neher, and Reuther [GNR23]. This allows us to classify all quadratic pairs on a fixed Azumaya algebra with orthogonal involution.

In this chapter we will work over the big fppf (or étale) sites, so we take a moment to present our conventions in this context. Once and for all we fix a base scheme S and work over the big fppf site $\mathbf{Sch}_{/S}$ as in Example 1.15. The structure sheaf of $\mathbf{Sch}_{/S}$ is defined by mapping $T \to S$ to $\Gamma(T)$ and denoted by $\mathcal{O}$.

Definition 3.1. An $\mathcal{O}$-algebra $\mathcal{A}$ is called **Azumaya** if $\mathcal{A}$ is finite locally free as an $\mathcal{O}$-module and the sandwich morphism

$$\begin{aligned} \Omega_{\mathcal{A}} \colon \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\mathrm{op}} &\to \mathcal{E}nd_{\mathcal{O}}(\mathcal{A}) \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism.

We may connect with our original definition of Azumaya $\mathcal{O}_S$-algebras as follows:

Lemma 3.2. An $\mathcal{O}$-algebra $\mathcal{A}$ is Azumaya if and only if $\mathcal{A}$ corresponds to an Azumaya $\mathcal{O}_S$-algebra under the correspondence of Theorem 1.22.

Proof. Let $\mathcal{B}$ be the $\mathcal{O}_S$-algebra corresponding to $\mathcal{A}$ with $\mathcal{A} = \mathcal{B}^a$. Then since $\mathcal{B}$ is finite locally free, we have that

$$(\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{B}^{\mathrm{op}})^a = \mathcal{B}^a \otimes_{\mathcal{O}} (\mathcal{B}^{\mathrm{op}})^a = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\mathrm{op}}$$

and
$$(\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{B}))^a = \mathcal{E}nd_{\mathcal{O}}(\mathcal{B}^a) = \mathcal{E}nd_{\mathcal{O}}(\mathcal{A})$$
(see [Stacks, Tag 0GN9]). □

Lemma 3.3. An $\mathcal{O}$-algebra $\mathcal{A}$ is Azumaya if and only if for each affine scheme $T \to S$, the algebra $\mathcal{A}(T)$ is Azumaya over the ring $\Gamma(T)$.

Proof. The isomorphism in Definition 3.1 may be checked on an affine cover. □

3.1 Involutions

We now introduce the theory of orthogonal involutions that we will need to discuss quadratic pairs. The theory in this section was adapted from [CF15, §2.7] and [GNR23, §3].

Definition 3.4. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$-algebra. An **involution** on $\mathcal{A}$ is an anti-isomorphism of order at most 2. In other words, an $\mathcal{O}$-module map $\sigma\colon \mathcal{A} \to \mathcal{A}$ satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ on sections and $\sigma^2 = \mathrm{id}_{\mathcal{A}}$.

Remark 3.5. What we have defined as an involution is typically called an **involution of the first kind**. In general, an involution need not be $\mathcal{O}$-linear, in which case the involution restricts to a possibly nontrivial involution on $\mathcal{O}$. Involutions that are not $\mathcal{O}$-linear are called **involutions of the second kind**. However, we will only concern ourselves with involutions of the first kind, so we will just call them **involutions** without the suffix.

Involutions on Azumaya algebras are deeply related to regular bilinear forms on neutral Azumaya algebras. We recall some basic definitions and properties of bilinear forms in our context before connecting them to involutions.

Definition 3.6. A **bilinear form** $(\mathcal{M}, b)$ is a finite locally free $\mathcal{O}$-module $\mathcal{M}$ together with a bilinear map $\mathcal{M} \times \mathcal{M} \to \mathcal{O}$. Given a bilinear form b, there is an associated map $\phi_b\colon \mathcal{M} \to \mathcal{M}^\vee$ defined on sections by sending $m \in \mathcal{M}(T)$ to the linear form $\phi_b(m) := b(m, -)\colon \mathcal{M}_{|T} \to \mathcal{O}$. A bilinear form b is called **regular**[1] if ϕ_b is an isomorphism. A bilinear form b is called **symmetric** if $b(m, n) = b(n, m)$ for all sections $m, n \in \mathcal{M}(T)$, similarly b is called **skew-symmetric** if $b(m, n) = -b(n, m)$, and **alternating** if $b(m, m) = 0$.

[1] or **nondegenerate**, or **nonsingular**

Given a regular bilinear form $(\mathcal{M}, b)$, there is an associated anti-automorphism η_b on the Azumaya algebra $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ given on sections $f \in \mathrm{End}_{\mathcal{O}}(\mathcal{M}_{|T})$ by defining $\eta_b(f)$ to be the composite

$$\mathcal{M}_{|T} \xrightarrow{\phi_{b|T}} \mathcal{M}_{|T}^{\vee} \xrightarrow{f^{\vee}} \mathcal{M}_{|T}^{\vee} \xrightarrow{\phi_{b|T}^{-1}} \mathcal{M}_{|T},$$

where $f^{\vee} \colon \mathcal{M}_{|T}^{\vee} \to \mathcal{M}_{|T}^{\vee}$ is the usual transpose of f. Then on sections $m, n \in \mathcal{M}_{|T}(V)$ we have that

$$b(\eta_b(f)(m), n) = b(m, f(n)),$$

and hence we call η_b the **adjoint anti-automorphism** of b. If there exists a scalar $\delta \in \mu_2(S) = \{u \in \Gamma(S) \mid u^2 = 1\}$ such that $b(m, n) = \delta_{|T} \cdot b(n, m)$ for every S-scheme T and sections $m, n \in \mathcal{M}(T)$ (i.e. b is δ-symmetric), then η_b is an involution since for any $f \in \mathcal{E}nd_{\mathcal{O}}(\mathcal{M})(T)$,

$$\begin{aligned} b(\eta_b(\eta_b(f))(m), n) &= b(m, \eta_b(f)(n)) \\ &= \delta \cdot b(\eta_b(f)(n), m) \\ &= \delta \cdot b(n, f(m)) \\ &= \delta^2 \cdot b(f(m), n), \end{aligned}$$

and b is regular, so $\eta_b^2(f) = f$.

Definition 3.7. An **isomorphism** of Azumaya algebras with involution $(\mathcal{A}, \sigma) \to (\mathcal{B}, \tau)$ is an isomorphism of $\mathcal{O}$-algebras $f \colon \mathcal{A} \to \mathcal{B}$ such that $f \circ \sigma = \tau \circ f$.

Proposition 3.8. Every Azumaya algebra with involution $(\mathcal{A}, \sigma)$ is étale-locally isomorphic to $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b)$ for some regular δ-symmetric bilinear form $(\mathcal{M}, b)$. Moreover, the scalar $\delta \in \mu_2(S)$ depends only on the involution σ, and we call δ the **type** of σ.

Proof. Since this is a local property, we may reduce to the affine case with $S = \operatorname{Spec} R$, $\mathcal{M} = R^n$, and $A = \mathrm{End}_R(R^n) = \mathrm{M}_n(R)$. The composite $\sigma \circ (-)^{\mathsf{T}}$ is an automorphism of $\mathrm{M}_n(R)$, and we may assume that it is an inner automorphism by localising further as necessary to use Skolem-Noether 2.7. Hence, there is some $P \in \mathrm{GL}_n(R)$ such that $\sigma(M^{\mathsf{T}}) = PMP^{-1}$ for all $M \in \mathrm{M}_n(R)$. Since $\sigma^2 = \mathrm{id}$, it must be that

$$M = \sigma^2(M) = P(P^{\mathsf{T}})^{-1} M P^{\mathsf{T}} P^{-1}$$

for all $M \in \mathrm{M}_n(R)$, and so $P^\mathsf{T}P^{-1} \in Z(\mathrm{M}_n(R)) = R$. Set $\delta = P^\mathsf{T}P^{-1}$, then $P = \delta P^\mathsf{T} = \delta^2 P$, so $\delta \in \mu_2(R)$. Now, P defines a regular δ-symmetric bilinear form b by $b(m,n) = m^\mathsf{T}Pn$ such that the adjoint anti-involution η_b is equal to σ. Since b is determined up to multiplication by a unit $\lambda \in R^\times$, the element $\delta = P^\mathsf{T}P^{-1}$ is fixed, and we may glue to obtain a global section $\delta \in \mu_2(S)$ depending only on σ. □

Definition 3.9. Let $\mathcal{A}$ be an Azumaya algebra with involution σ. In light of Proposition 3.8, we say σ is an **orthogonal involution** if $\delta = 1$. Notice that we may check that an involution is orthogonal on a cover, and orthogonal involutions are stable under base change.

Remark 3.10. If the scheme S is connected and $2 \in \mathbb{G}_m(S)$ is invertible, then $\mu_2(S) = \{\pm 1\}$ and an involution is either of orthogonal type or of symplectic type. But if either of the conditions are not met, then there may be arbitrarily many elements in $\mu_2(S)$, and each $\delta \in \mu_2(S)$ is the type of an involution: for example, on $\mathrm{M}_2(\mathcal{O})$ we may define an involution σ of type δ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & \delta b \\ \delta c & a \end{pmatrix}.$$

Given an Azumaya algebra $\mathcal{A}$ with involution σ, we introduce the following submodules of $\mathcal{A}$ for future use:

- Symmetric elements: $\mathcal{S}ym_{\mathcal{A},\sigma} = \mathrm{Ker}(\mathrm{id} - \sigma)$;
- Alternating elements: $\mathcal{A}lt_{\mathcal{A},\sigma} = \mathrm{Im}(\mathrm{id} - \sigma)$;
- Skew-symmetric elements: $\mathcal{S}kew_{\mathcal{A},\sigma} = \mathrm{Ker}(\mathrm{id} + \sigma)$;
- Symmetrised elements: $\mathcal{S}ymd_{\mathcal{A},\sigma} = \mathrm{Im}(\mathrm{id} + \sigma)$,

where the image is the sheaf-theoretic image in the category of $\mathcal{O}$-modules. These modules fit into the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{A}lt_{\mathcal{A},\sigma} \arrow[r, equal] \arrow[d] & \mathcal{A}lt_{\mathcal{A},\sigma} \arrow[d] & & \\
0 \arrow[r] & \mathcal{S}kew_{\mathcal{A},\sigma} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r, "\mathrm{id}+\sigma"] \arrow[d] & \mathcal{S}ymd_{\mathcal{A},\sigma} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{S}kew_{\mathcal{A},\sigma} / \mathcal{A}lt_{\mathcal{A},\sigma} \arrow[r] \arrow[d] & \mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma} \arrow[r, "\xi"] \arrow[d] & \mathcal{S}ymd_{\mathcal{A},\sigma} \arrow[r] & 0 \\
 & 0 & 0, & &
\end{tikzcd}$$

where $\xi\colon \mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma} \to \mathcal{S}ymd_{\mathcal{A},\sigma}$ is the map induced by $\mathrm{id}+\sigma$. Note that ξ is well defined since $\mathcal{A}lt_{\mathcal{A},\sigma}$ is a submodule of $\mathcal{S}kew_{\mathcal{A},\sigma} = \mathrm{Ker}(\mathrm{id}+\sigma)$.

Our interest will primarily be on Azumaya algebras with orthogonal involutions, so we take a moment to note down some special properties of the above submodules in this case.

Lemma 3.11. [GNR23, Lem. 3.3] If $\mathcal{A}$ is an Azumaya $\mathcal{O}$-algebra with orthogonal involution σ, then $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are affine-locally direct summands of $\mathcal{A}$. That is, if $T \to S$ is affine, then $\mathcal{S}ym_{\mathcal{A},\sigma}(T)$ and $\mathcal{A}lt_{\mathcal{A},\sigma}(T)$ are direct summands of $\mathcal{A}(T)$. In particular, $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are also finite locally free modules. Moreover, if $\mathcal{A}$ has constant rank n^2, then the ranks of $\mathcal{S}ym_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}$ are $n(n+1)/2$ and $n(n-1)/2$ respectively.

3.2 Definition and Examples

Definition 3.12. Any Azumaya algebra $\mathcal{A}$ is equipped with a linear form $\mathrm{Trd}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{O}$ called the **reduced trace of $\mathcal{A}$** that is étale-locally the trace of matrix algebras [CF15, Def. 2.5.3.15].

We summarise some important properties of the reduced trace from [CF15, Lem. 2.5.3.16] and [GNR23, Lem. 3.6] for future use:

Lemma 3.13. Let $\mathcal{A}$ be an Azumaya algebra. Then

1. $\mathrm{Trd}_{\mathcal{A}}(ab) = \mathrm{Trd}_{\mathcal{A}}(ba)$ for every S-scheme T and sections $a, b \in \mathcal{A}(T)$.
2. The reduced trace defines a regular symmetric bilinear form on $\mathcal{A}$ by $(a,b) \mapsto \mathrm{Trd}_{\mathcal{A}}(ab)$.
3. If σ is an involution $\mathcal{A}$, then $\mathrm{Trd}_{\mathcal{A}} \circ \sigma = \mathrm{Trd}_{\mathcal{A}}$.
4. If σ is an involution on $\mathcal{A}$, then
$$\mathrm{Trd}_{\mathcal{A}}(\mathcal{S}ym_{\mathcal{A},\sigma} \cdot \mathcal{A}lt_{\mathcal{A},\sigma}) = 0$$
$$\mathrm{Trd}_{\mathcal{A}}(\mathcal{S}kew_{\mathcal{A},\sigma} \cdot \mathcal{S}ymd_{\mathcal{A},\sigma}) = 0.$$
5. If $\mathcal{M}$ is a submodule of $\mathcal{A}$, we denote by $\mathcal{M}^{\perp}$ the **orthogonal complement of $\mathcal{M}$ relative to** $\mathrm{Trd}_{\mathcal{A}}$ defined on $T \to S$ as
$$\mathcal{M}^{\perp}(T) = \left\{ a \in \mathcal{A}(T) \mid \mathrm{Trd}(a_{|V} \cdot m) = 0, \text{ for all } m \in \mathcal{M}(V \to T) \right\}.$$
If σ is an orthogonal involution, then $\mathcal{S}ym_{\mathcal{A},\sigma}^{\perp} = \mathcal{A}lt_{\mathcal{A},\sigma}$ and $\mathcal{A}lt_{\mathcal{A},\sigma}^{\perp} = \mathcal{S}ym_{\mathcal{A},\sigma}$.

Definition 3.14. Let $\mathcal{A}$ be an Azumaya algebra. A **quadratic pair** (σ, f) on $\mathcal{A}$ is an orthogonal involution σ on $\mathcal{A}$ along with a linear form

$$f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$$

satisfying

$$f(a+\sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all sections $a \in \mathcal{A}(T)$. We call the triple $(\mathcal{A}, \sigma, f)$ a **quadratic triple**. An **isomorphism of quadratic triples** $(\mathcal{A}, \sigma, f) \to (\mathcal{B}, \tau, g)$ is an isomorphism $\phi\colon (\mathcal{A}, \sigma) \to (\mathcal{B}, \tau)$ of Azumaya $\mathcal{O}$-algebras with involution such that

$$\phi \circ f = g.$$

Example 3.15. Let $(\mathcal{A}, \sigma, f)$ be a quadratic triple. Then for any S-scheme T and $s \in \mathcal{S}ym_{\mathcal{A},\sigma}(T)$ we have that

$$2 \cdot f(s) = f(s+\sigma(s)) = \mathrm{Trd}_{\mathcal{A}}(s).$$

Therefore, if 2 is invertible in $\Gamma(S)$, then any Azumaya algebra with orthogonal involution $(\mathcal{A}, \sigma)$ has exactly one quadratic pair (σ, f), and f is given by $\frac{1}{2} \cdot \mathrm{Trd}_{\mathcal{A}}$. The real power of quadratic pairs is realised exactly when 2 is not a unit.

Example 3.16. If 2 is not invertible in $\Gamma(S)$, then it may be that an Azumaya algebra with orthogonal involution does not admit a quadratic pair. Consider, for example $S = \operatorname{Spec} \mathbb{Z}/2\mathbb{Z}$, $\mathcal{A} = \mathrm{M}_2(\mathcal{O})$ equipped with the transpose involution. Consider the element $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{A}$, if the transpose partakes in a quadratic pair $((-)^{\mathsf{T}}, f)$, then it would be that

$$0 = f(0) = f(a + a^{\mathsf{T}}) = \mathrm{tr}(a) = 1.$$

As we will see later, if $2 \in \Gamma(S)$ is not a unit, it is also possible for a given orthogonal involution to partake in more than one quadratic pair. In sections 3.3 and 3.4 we will fully answer the existence and classification questions for any fixed orthogonal involution.

We start by surveying the relation between regular quadratic forms and quadratic pairs on neutral Azumaya algebras.

Definition 3.17. A **quadratic form** (M, q) is a finite locally free $\mathcal{O}$-module $\mathcal{M}$ along with a map of sheaves $q\colon \mathcal{M} \to \mathcal{O}$ such that

1. $q(\lambda \cdot m) = \lambda^2 \cdot q(m)$ for all sections $\lambda \in \mathcal{O}(T), m \in \mathcal{M}(T)$;

2. the associated **polar form** $b_q \colon \mathcal{M} \times \mathcal{M} \to \mathcal{O}$ defined on sections $m, n \in \mathcal{M}(T)$ by
$$b_q(m,n) = q(m+n) - q(m) - q(n)$$
is bilinear.

The quadratic form q is said to be **regular** if the polar form b_q is regular.

Proposition 3.18. Let $(\mathcal{M}, b)$ be a regular symmetric bilinear form, then $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b)$ is an Azumaya $\mathcal{O}$-algebra, where η_b is the adjoint involution of b. The involution η_b is part of a quadratic pair on $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ if and only if there is a regular quadratic form $q \colon \mathcal{M} \to \mathcal{O}$ such that $b = b_q$.

Proof. First suppose $b = b_q$ for a regular quadratic form q. Denote by ϕ_b the composite isomorphism

$$\begin{array}{ccccc} \mathcal{M} \otimes \mathcal{M} & \longrightarrow & \mathcal{M} \otimes \mathcal{M}^\vee & \longrightarrow & \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}) \\ m_1 \otimes m_2 & \longmapsto & m_1 \otimes b(m_2, -) & \longmapsto & m_1 \cdot b(m_2, -), \end{array}$$

then ϕ_b is an *$\mathcal{O}$-isomorphism of modules with involution*, where we give $\mathcal{M} \otimes \mathcal{M}$ the *switch involution* τ defined by $m \otimes n \mapsto n \otimes m$. Therefore, $\mathcal{S}ym_{\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b}$ corresponds to the symmetric elements $\mathcal{S}ym_{\mathcal{M} \otimes \mathcal{M}, \tau}$ of $\mathcal{M} \otimes \mathcal{M}$ under ϕ_b. Now, $\mathcal{S}ym_{\mathcal{M} \otimes \mathcal{M}, \tau}$ is the submodule of $\mathcal{M} \otimes \mathcal{M}$ generated by elements of the form $m \otimes m$. Hence, we may define the linear form $f_q \colon \mathcal{S}ym_{\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b} \to \mathcal{O}$ by the formula $f_q(\phi_b(m \otimes m)) = q(m)$. Since we may verify everything locally, reduce to an affine cover where $\mathcal{M}$ is free and apply the proof in the Book of Involutions [BoI, Prop. 5.11] without change.

On the other hand if (η_b, f) is a quadratic pair on $\mathcal{A}$, define the quadratic form $q \colon \mathcal{M} \to \mathcal{O}$ on sections $m \in \mathcal{M}(T)$ by
$$q_f(m) = f(\phi_b(m \otimes m)).$$

To see that the associated bilinear form b_{q_f} is the same as b, we compute that for any sections $m, n \in \mathcal{M}(T)$, we have
$$\begin{aligned} b_{q_f}(m,n) &= q(m+n) - q(m) - q(n) \\ &= f(\phi_b(m \otimes n) + \phi_b(n \otimes m)) \\ &= f(\phi_b(m \otimes n) + \eta_b(\phi_b(m \otimes n))) \\ &= \mathrm{Trd}_{\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})}(\phi_b(m \otimes n)) \quad \text{since } (\eta_b, f) \text{ is a quadratic pair} \\ &= \mathrm{Trd}_{\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})}(m \cdot b(n, -)) = b(m,n). \end{aligned}$$

The last equality may be checked on a cover where $\mathcal{M}$ is finite free with constant rank r by computing the matrix trace of the product of the $r \times 1$ matrix m with the $1 \times r$ matrix $b(n, -)$. $\square$

Remark 3.19. In the context of Proposition 3.18, a quadratic pair (η_b, f) necessarily comes from a regular quadratic form $q\colon \mathcal{M} \to \mathcal{O}$. If the rank of $\mathcal{M}$ is odd, then this forces 2 to be invertible in $\mathcal{O}(S)$. To see this, reduce to an affine cover where $\mathcal{M}$ is finite free. The matrix of the polar form b_q of q is symmetrised, and the determinant of a symmetrised matrix is 0 modulo 2, indeed, if $2 = 0$, then symmetrised implies alternating, and the determinant of an alternating matrix is 0 in odd rank. Since the determinant is both 2-divisible and a unit, this forces 2 to be a unit. By glueing, we conclude that $2 \in \mathbb{G}_m(S) = \mathcal{O}(S)^\times$.

With the above remark in mind, we present the **split examples** in odd and even degree separately.

Example 3.20. *Split quadratic triple in even degree.* Let q be the regular quadratic form on $\mathcal{M} = \bigoplus^{2n} \mathcal{O}$ defined

$$q(x_1, \ldots, x_{2n}) = \sum_{i=1}^{n} x_{2i-1}x_{2i}$$

on sections. We call q the **hyperbolic quadratic form** in rank $2n$, and the quadratic triple $(\mathrm{M}_{2n}(\mathcal{O}), \eta_{b_q}, f_q)$ corresponding to q we call the **split quadratic triple** in degree $2n$. We write the quadratic triple down explicity, following along with the calculation after [CF15, Prop. 2.7.0.31]. The involution η_q is given by $A \mapsto HA^\mathsf{T}H^{-1}$, where

$$H = \sum_{i=1}^{n}(E_{2i-1,2i} + E_{2i,2-i})$$

and $E_{i,j}$ is the elementary matrix with 1 in position i, j and 0 in every other entry. The linear form $f_q\colon \mathcal{S}ym \to \mathcal{O}$ is given by extending

$$f_q(E_{i,i} + E_{i+1,i+1}) := 1$$

linearly for $i = 1, \ldots, n$, and by sending all other elements to 0.

Example 3.21. *Split quadratic triple in odd degree.* In light of our earlier discussion, we will require that 2 is invertible in $\Gamma(S)$. Let $\mathcal{M} = \bigoplus^{2n+1} \mathcal{O}$ and define q by

$$q(x_1, \ldots, x_{2n+1}) = x_{2n+1}^2 + \sum_{i=1}^{n} x_{2i-1}x_{2i}.$$

We call q the **hyperbolic quadratic form** in rank $2n+1$, and the resulting quadratic triple $(\mathrm{M}_{2n+1}(\mathcal{O}), \eta_{b_q}, f_q)$ is called the **split quadratic triple** in degree $2n+1$. In this case f_q is uniquely determined by

$$f_q = \frac{1}{2}\,\mathrm{Trd} = \frac{1}{2}\,\mathrm{tr}\,.$$

Proposition 3.22. *Every quadratic triple is split étale-locally.*
If $(\mathcal{A}, \sigma, f)$ is a quadratic triple, then there is an étale cover $\{T_i \to S\}$ such that for each i, the restriction of $(\mathcal{A}, \sigma, f)$ to T_i is isomorphic to a split quadratic triple $(\mathrm{M}_{n_i}(\mathcal{O}), \eta_q, f_q)$.

Proof. Since this is a local property, we may assume that $\mathcal{A}$ has constant degree n^2 and reduce to the neutral case where our quadratic triple is $(\mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \eta_b, f)$ for some regular bilinear form b. By Proposition 3.18, $b = b_q$ for a regular quadratic form $q\colon \mathcal{M} \to \mathcal{O}$ and $\eta_b = \eta_{b_q}$. If n is even, then by [CF15, Lem. 2.6.1.13], q is étale-locally hyperbolic, so by further refining our cover, we may assume that $(\mathcal{M}, q)$ is hyperbolic, and then our quadratic triple becomes the split example of Example 3.20. On the other hand, if n is odd, then 2 is invertible and $f = \frac{1}{2}\,\mathrm{Trd}_{\mathcal{A}} = f_q$ is forced. □

Remark 3.23. We have just established an equivalence between quadratic triples of constant rank n and twisted forms of the split quadratic triple $(M_n(\mathcal{O}), \eta_q, f_q)$ in degree n. The automorphism group of the quadratic triple $(M_n(\mathcal{O}), \eta_q, f_q)$ is called the **projective orthogonal group** and denoted by $\mathbf{PGO}_n$. By Proposition 2.39 a quadratic pair of degree n is just a $\mathbf{PGO}_n$-torsor. For more details (such as representability) about the sheaf of groups $\mathbf{PGO}_n$ we refer the reader to [CF15, §4.4].

Corollary 3.24. If $(\mathcal{A}, \sigma, f)$ is a quadratic triple and $\mathcal{A}$ has constant rank n^2, then $f(1) = \frac{1}{2}n$.

Proof. There is a cover $\{T_i \to S\}$ that splits $\mathcal{A}$, so we may assume that our quadratic triple is $(\mathrm{M}_n(\mathcal{O}), \eta_{b_q}, f_q)$ for the hyperbolic quadratic form q and perform our calculations with the identity matrix 1 to see that

$$2f(1) = f(1 + \sigma(1)) = \mathrm{Trd}(1) = n.$$

Now, if n is odd, then 2 is invertible and $f(1) = \frac{1}{2}n$. If $n = 2m$, then by the calculation of f_q done in Example 3.20,

$$f(1) = f\left(\sum_{i=1}^{m} (E_{i,i} + E_{i+1,i+1})\right) = m.$$

□

3.3 Classification

For the duration of this section, let $\mathcal{A}$ be an Azumaya $\mathcal{O}$-algebra with orthogonal involution σ. In this section we seek to understand when σ can partake in a quadratic pair (σ, f) and if it can, we classify all such quadratic pairs. Our treatment of the classification and obstruction in the next two sections is based on [GNR23].

Lemma 3.25. If $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$, in other words if there is a global section $l \in \mathcal{A}(S)$ satisfying $1_{\mathcal{A}} = l + \sigma(l)$, then the linear form $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ defined on sections $s \in \mathcal{S}ym_{\mathcal{A},\sigma}(T)$ by $f(s) = \mathrm{Trd}(l_{|T} \cdot s)$ gives a quadratic pair (σ, f). Moreover, f is uniquely determined by l modulo an alternating difference, more precisely, $\mathrm{Trd}(l_1-) = \mathrm{Trd}(l_2-)$ on $\mathcal{S}ym_{\mathcal{A},\sigma}$ if and only if $l_1 - l_2 \in \mathcal{A}lt_{\mathcal{A},\sigma}(S)$.

Proof. By using Lemma 3.13, we easily compute that for any $a \in \mathcal{A}(T)$,

$$\begin{aligned} f(a+\sigma(a)) &= \mathrm{Trd}(l_{|T} \cdot a) + \mathrm{Trd}(l_{|T} \cdot \sigma(a)) \\ &= \mathrm{Trd}(l_{|T} \cdot a) + \mathrm{Trd}(\sigma(l_T) \cdot a) \\ &= \mathrm{Trd}((l_{_T} + \sigma(l_{|T})) \cdot a) = \mathrm{Trd}(a). \end{aligned}$$

Therefore, (σ, f) is a quadratic pair. Now, suppose $\mathrm{Trd}(l_1-) = \mathrm{Trd}(l_2-)$ on $\mathcal{S}ym_{\mathcal{A},\sigma}$, then $\mathrm{Trd}((l_1 - l_2)-) = 0$ on $\mathcal{S}ym_{\mathcal{A},\sigma}$, and by Lemma 3.13(5), this implies that $l_1 - l_2 \in \mathcal{A}lt_{\mathcal{A},\sigma}(S)$. Conversely, if $l_1 - l_2 \in \mathcal{A}lt_{\mathcal{A},\sigma}(S)$, say with $l_1 = l_2 + a$ for some $a \in \mathcal{A}lt_{\mathcal{A},\sigma}(S)$, then for any section $b \in \mathcal{S}ym_{\mathcal{A},\sigma}(T)$,

$$\mathrm{Trd}(l_{1|T} \cdot b) = \mathrm{Trd}(l_{2|T} \cdot b) + \mathrm{Trd}(a_{|T} \cdot b),$$

but $\mathrm{Trd}(a_{|T} \cdot b) = 0$ by Lemma 3.13 (5). □

We will see that global sections l satisfying the hypothesis of Lemma 3.25 are key in classifying quadratic pairs on $(\mathcal{A}, \sigma)$.

Lemma 3.26. Let (σ, f) be a quadratic pair on $\mathcal{A}$. Then f extends to a linear form f' on all of $\mathcal{A}$ if and only if there is a global section $l \in \mathcal{A}(S)$ such that $1 = l + \sigma(l)$ and $f = \mathrm{Trd}(l-)$

Proof. If $f = \mathrm{Trd}(l-)$, then f may trivially be extended to all of $\mathcal{A}$. Conversely, suppose f extends to some $f' \in \mathcal{A}^\vee$. By regularity of the trace bilinear form (Lemma 3.13(2)), the map

$$\begin{aligned} \phi_t\colon \mathcal{A} &\to \mathcal{A}^\vee \\ l &\mapsto \mathrm{Trd}(l-) \end{aligned}$$

is an isomorphism. Therefore, f' is of the form $\mathrm{Trd}(l-)$ for some $l \in \mathcal{A}(S)$. Now for any section $a \in \mathcal{A}(S)$

$$\begin{aligned}\mathrm{Trd}(1 \cdot a) &= f(a + \sigma(a)) = \mathrm{Trd}(l \cdot (a + \sigma(a)) \\ &= \mathrm{Trd}(l \cdot a) + \mathrm{Trd}(\sigma(l) \cdot a) = \mathrm{Trd}((l + \sigma(l)) \cdot a),\end{aligned}$$

so $1 = l + \sigma(l)$, by regularity of of the trace bilinear form. □

We first specialise the classification to the affine case, before translating to the global case.

Lemma 3.27. Suppose S is an affine scheme and let $(\mathcal{A}, \sigma, f)$ be a quadratic triple. Then there is a global section l_f such that $l_f + \sigma(l_f) = 1$ and

$$f = \mathrm{Trd}(l_f -).$$

Proof. By Lemma 3.11, $\mathcal{S}ym_{\mathcal{A},\sigma}$ is a direct summand of $\mathcal{A}$, so f automatically extends to all of $\mathcal{A}$, now apply Lemma 3.26. □

The above lemma shows that $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ is a necessary condition for σ to partake in a quadratic pair over affine schemes. We immediately notice that it is also a necessary condition in the global case. Indeed, if $(\mathcal{A}, \sigma, f)$ is a quadratic pair, then $1_{\mathcal{A}|U_i} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(U_i)$ on an affine cover $\{U_i \to S\}$, and since $\mathcal{S}ymd_{\mathcal{A},\sigma}$ is a sheaf, it must be that $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$. However, this does not mean that $1_{\mathcal{A}}$ is of the form $l + \sigma(l)$ for a global section $l \in \mathcal{A}(S)$ - remember $\mathcal{S}ymd_{\mathcal{A},\sigma}$ is the *sheafification* of the image presheaf $\mathrm{Im}(id + \sigma)$.

We are now able to completely classify all quadratic pairs on $\mathcal{A}$ in the affine case. Recall that there is a map $\xi \colon \mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma} \to \mathcal{S}ymd_{\mathcal{A},\sigma}$ induced by $\mathrm{id} + \sigma$. If S is affine and $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$, then the inverse image $\xi_S^{-1}(1_{\mathcal{A}})$ of $\{1_{\mathcal{A}}\}$ under

$$(\mathcal{A}/\mathcal{A}lt_{\mathcal{A},\sigma})(S) \xrightarrow{\xi_S} \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$$

is nonempty, since $\mathcal{S}kew_{\mathcal{A},\sigma}$ is quasi-coherent so $H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}) = 0$ and

$$0 \longrightarrow \mathcal{S}kew_{\mathcal{A},\sigma}(S) \longrightarrow \mathcal{A}(S) \xrightarrow{\mathrm{id} + \sigma} \mathcal{S}ymd_{\mathcal{A},\sigma}(S) \longrightarrow 0$$

is exact.

Theorem 3.28. *The affine classification.*
Suppose S is an affine scheme and $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$. Denote by F the set

of linear forms $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ such that (σ, f) is a quadratic pair. Then there is a bijection

$$\begin{aligned} F &\to \xi_S^{-1}(1_{\mathcal{A}}) \\ f &\mapsto \overline{l_f}, \end{aligned}$$

where $\overline{l_f}$ is the image of the element l_f from Lemma 3.27 in $(\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S)$.

Proof. Notice that the map is well defined, since by Lemma 3.25 l_f is uniquely determined up to addition by an alternating element. We first show the surjectivity of the map. Since S is affine and $\mathcal{A}lt_{\mathcal{A},\sigma}$ is quasi-coherent, we have that

$$0 \longrightarrow \mathcal{A}lt_{\mathcal{A},\sigma}(S) \longrightarrow \mathcal{A}(S) \longrightarrow (\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S) \longrightarrow 0$$

is exact and that $(\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S) = \mathcal{A}(S)/\,\mathcal{A}lt_{\mathcal{A},\sigma}(S)$. Combining this fact with the preceding discussion, we obtain a commutative square

$$\begin{tikzcd} \mathcal{A}(S) \arrow[r, "\mathrm{id}+\sigma"] \arrow[d] & \mathcal{S}ymd_{\mathcal{A},\sigma}(S) \arrow[d, "\mathrm{id}"] \\ \mathcal{A}(S)/\,\mathcal{A}lt_{\mathcal{A},\sigma}(S) \arrow[r, "\xi"'] & \mathcal{S}ymd_{\mathcal{A},\sigma}(S), \end{tikzcd}$$

where every arrow is surjective. Therefore, every element of $\xi_S^{-1}(1_{\mathcal{A}})$ is of the form $\bar{l}$ for some $l \in \mathcal{A}(S)$ satisfyng $l + \sigma(l) = 1$. Now $f := \mathrm{Trd}(l-)$ defines a quadratic pair (σ, f) on $\mathcal{A}$ (Lemma 3.25), so by Lemma 3.27, there is some section $l_f \in \mathcal{A}(S)$ such that $\mathrm{Trd}(l_f-) = f = \mathrm{Trd}(l-)$ on $\mathcal{S}ym_{\mathcal{A},\sigma}$. Finally, applying Lemma 3.25 one more time gives $l_f - l \in \mathcal{A}lt_{\mathcal{A},\sigma}$, and $\overline{l_f} = \bar{l}$.

Next, if (σ, f_1) and (σ, f_2) are two quadratic pairs on $\mathcal{A}$, then by the preceding Lemma 3.27, there are global sections l_{f_1} and $l_{f_2} \in \mathcal{A}(S)$ determining f_i by $f_i = \mathrm{Trd}(l_{f_i}-)$. If $\overline{l_{f_1}} = \overline{l_{f_2}}$, then $l_{f_1} - l_{f_2} \in \mathcal{A}lt_{\mathcal{A},\sigma}$, and so $\mathrm{Trd}(l_{f_1}-) = \mathrm{Trd}(l_{f_2}-)$ on $\mathcal{S}ym_{\mathcal{A},\sigma}$ by Lemma 3.25. This shows that $f_1 = f_2$ and that the map in question is injective. □

We have now established that over an affine scheme S, $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ is a necessary and sufficient condition for σ to partake in a quadratic pair. For the global case with an arbitrary scheme S, it is still a necessary condition by our discussion following Lemma 3.27, but it is no longer a sufficient condition by the next example.

Example 3.29. We give a short overview of an example of an orthogonal involution with $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ that does not admit a quadratic pair. This example is computed with full detail in [GNR23, §7.4]. Let k be an algebraically closed field of characteristic 2. Set $G := \mathrm{PGL}_2(\mathbb{F}_4)$ and denote by G_k the associated group scheme in $\mathbf{Sch}_{/k}$. There is a Galois G-cover $Y \to S$ with Y and S projective k-varieties. We consider S as our base scheme, and define the quaternion[2] algebra $\mathcal{Q}$ over S as the fixed points of $\mathrm{M}_2(\mathcal{O}_{|Y})$ under the action of G_k, where G_k acts through PGL_2. More precisely, if T is an S-scheme, then

$$\mathcal{Q}(T) = \{m \in \mathrm{M}_2(\mathcal{O}(T \times_S Y)) \mid \phi(m) = m, \text{ for all } \phi \in G\}.$$

There is an orthogonal involution σ on $\mathcal{Q}$ obtained by restricting the split involution η_{b_q} on $\mathrm{M}_2(\mathcal{O})$ (Example 3.20). The involution σ has $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$, but does not admit a quadratic pair [GNR23, Lem. 7.5].

Remark 3.30. The problem is that $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ does not guarantee that there is a global section $l \in \mathcal{A}(S)$ with $1_{\mathcal{A}} = l + \sigma(l)$, merely that $1_{\mathcal{A}}$ is locally of the form $l + \sigma(l)$. We will see that the failure of this local-to-global translation is part of the obstruction to quadratic pairs on $(\mathcal{A}, \sigma)$.

Lemma 3.31. Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$-algebra with orthogonal involution σ. The following equivalent:

1. $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$;
2. For every affine scheme $U \to S$, there is a linear form
$$f_U \colon \mathcal{S}ymd_{\mathcal{A}_{|U},\sigma_{|U}} \to \mathcal{O}_{|U}$$
such that $(\sigma_{|U}, f_U)$ is a quadratic pair on $\mathcal{A}_{|U}$.
3. For every affine open subscheme $U \to S$, there is a linear form
$$f_U \colon \mathcal{S}ymd_{\mathcal{A}_{|U},\sigma_{|U}} \to \mathcal{O}_{|U}$$
such that $(\sigma_{|U}, f_U)$ is a quadratic pair on $\mathcal{A}_{|U}$.
4. There is a cover $\{T_i \to S\}$ such that for each i, there is a linear form
$$f_U \colon \mathcal{S}ymd_{\mathcal{A}_{|U},\sigma_{|U}} \to \mathcal{O}_{|U}$$
such that $(\sigma_{|U}, f_U)$ is a quadratic pair on $\mathcal{A}_{|U}$.

[2] an Azumaya algebra of constant rank 4

5. There is a cover $\{T_i \to S\}$ such that for each i, there is a regular quadratic form $(\mathcal{M}_i, q_i)$ over $\mathcal{O}_{|T_i}$ with isomorphism

$$(\mathcal{A}_{|T_i}, \sigma_{|T_i}) \simeq (\mathcal{E}nd_{\mathcal{O}_{|T_i}}(\mathcal{M}_i), \eta_{q_i})$$

of Azumaya $\mathcal{O}_{|T_i}$-algebras with involution, where η_{q_i} denotes the adjoint involution of the polar bilinear form of q_i.

In this case we say the involution σ is **locally-quadratic**.

Proof. (1) $\Rightarrow$ (2) : If $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ then for any affine scheme $U \to S$, we have that $1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(U)$ by restriction. Since U is affine, there is a section $l \in \mathcal{A}(U)$ such that $1 = l + \sigma(l)$ by our discussion leading up to the affine classification theorem 3.28. Now applying the theorem proves (2).
(2) $\Rightarrow$ (3) : Clear.
(3) $\Rightarrow$ (4) : Every affine open cover is an fppf cover.
(4) $\Rightarrow$ (1) : Suppose $\{T_i \to S\}$ is an fppf cover satisfying the hypothesis of (4), then for each i, $1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(T_i)$, and so by glueing, $1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$.
(4) $\Rightarrow$ (5) : Since every quadratic triple is locally split (Proposition 3.22), we only need to refine the given cover to prove (5).
(5) $\Rightarrow$ (4) : Immediate from Proposition 3.18. □

The question now becomes: when does an Azumaya algebra $\mathcal{A}$ with locally quadratic involution σ carry a quadratic pair.?

Example 3.32. *A local answer.* Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$-algebra with orthogonal involution. Suppose $\{T_i \to S\}$ is an fppf cover such that for each i, there is a section $l_i \in \mathcal{A}(T_i)$ satisfying $l_i + \sigma(l_i) = 1$, and such that on each overlap $T_{ij} := T_i \times_S T_j$ we have that $l_{i|T_{ij}} - l_{j|T_{ij}} \in \mathcal{A}lt_{\mathcal{A},\sigma}(T_{ij})$. Then locally, we have quadratic triples $(\mathcal{A}_{|T_i}, \sigma_{|T_i}, f_i)$, where $f_i = \mathrm{Trd}_{\mathcal{A}_{T_i}}(l_i -)$. By the compatibility condition on the overlaps and Lemma 3.25, the linear forms f_i and f_j agree on the restriction of $\mathcal{S}ymd_{\mathcal{A},\sigma}$ to T_{ij}, for each i and j, therefore, we may glue the f_i to obtain a global linear form $f\colon \mathcal{S}ymd_{\mathcal{A},\sigma} \to \mathcal{O}$. This process yields a quadratic triple $(\mathcal{A}, \sigma, f)$, since we may check the quadratic pair condition locally.

Lemma 3.33. *A global answer.* Let $(\mathcal{A}, \sigma)$ be an Azumaya $\mathcal{O}$-algebra with orthogonal involution and suppose σ is locally quadratic. Let $\xi_S\colon (\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S) \to \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ be our usual map induced by $\mathrm{id} + \sigma$. Suppose there is an element $\lambda \in \xi_S^{-1}(1_{\mathcal{A}})$, then λ induces a quadratic triple $(\mathcal{A}, \sigma, f_\lambda)$.

Proof. Since

$$\mathcal{A} \longrightarrow \mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma} \longrightarrow 0$$

is exact, there is a cover $\{T_i \to S\}$ with elements $l_i \in \mathcal{A}(T_i)$ such that $\overline{l_i} = \lambda_{|T_i}$ in $(\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(T_i)$. Now, since

$$\begin{tikzcd}
\mathcal{A}(T_i) \arrow[r, "\mathrm{id}+\sigma"] \arrow[d] & \mathcal{S}ymd_{\mathcal{A},\sigma}(T_i) \arrow[d, "\mathrm{id}"] \\
(\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(T_i) \arrow[r, "\xi_{T_i}"'] & \mathcal{S}ymd_{\mathcal{A},\sigma}(T_i)
\end{tikzcd}$$

commutes and $\xi_{T_i}(\lambda_{|T_i}) = 1$, each l_i satisfies $l_i + \sigma(l_i) = 1$. Moreover, on the overlaps $T_{ij} := T_i \times_S T_j$, we have that $l_{i|T_{ij}} - l_{j|T_{ij}} = 0 \in \mathcal{A}lt_{\mathcal{A},\sigma}(T_ij)$, since both are equal to $\lambda_{|T_{ij}}$. Proceeding as in Example 3.32, we obtain our desired quadratic triple. □

Remark 3.34. The linear form f_λ is independent of the cover we chose. Indeed, if we choose another cover $\{T_i' \to S\}$ and obtain a linear form f_λ' from sections $l_i' \in \mathcal{A}(T_i')$, we may compare f_λ and f_λ' on a common refinement $\{U_i \to S\}$ of the two covers to see that $l_{i|U_i} = \lambda_{|U_i} = l'_{i|U_i}$, so $f_\lambda = f_\lambda'$.

We will now see that every quadratic triple is of the form of Lemma 3.33.

Lemma 3.35. Let $(\mathcal{A}, \sigma, f)$ be a quadratic triple, then there is some $\lambda_f \in \xi_S^{-1}(1_\mathcal{A})$ such that $f = f_{\lambda_f}$ is the linear form constructed in Lemma 3.33.

Proof. This immediately follows from the fact that $\xi^{-1}(1_\mathcal{A})$ is a sheaf by applying Lemma 3.27 to obtain local sections λ_f that are compatible to be glued by the uniqueness result in Lemma 3.25. □

Theorem 3.36. *The global classification.*
Let $\mathcal{F}$ be the sheaf given by

$$\mathcal{F}(T) := \left\{ f\colon \mathcal{S}ymd_{\mathcal{A}_{|T},\sigma_{|T}} \to \mathcal{O}_{|T} \mid (\sigma_{|T}, f) \text{ is a quadratic pair on } \mathcal{A}_{|T} \right\}.$$

The map

$$\mathcal{F} \to \xi^{-1}(1_\mathcal{A})$$
$$f \mapsto \lambda_f$$

is an isomorphism of sheaves.

Proof. May be checked on an affine cover using the preceding lemmas and the affine classification 3.28. □

3.4 The Cohomological Obstructions

In this section we study the obstruction to a locally quadratic involution giving a global quadratic pair. For this section let $\mathcal{A}$ be an Azumaya $\mathcal{O}$-algebra with locally quadratic involution σ, that is, $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathcal{A}(S) & \xrightarrow{\mathrm{id}+\sigma} & \mathcal{S}ymd_{\mathcal{A},\sigma}(S) & \xrightarrow{\delta} & H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}) \\ \downarrow & & \| & & \downarrow \\ (\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S) & \xrightarrow{\xi_S} & \mathcal{S}ymd_{\mathcal{A},\sigma}(S) & \xrightarrow{\delta'} & H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}\,/\,\mathcal{A}lt_{\mathcal{A},\sigma}). \end{array} \tag{3.37}$$

Definition 3.38. The **weak obstruction** is the element

$$\omega(\mathcal{A},\sigma) := \delta'(1_{\mathcal{A}}) \in H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}\,/\,\mathcal{A}lt_{\mathcal{A},\sigma}).$$

Theorem 3.39. *Weak obstruction theorem.* There is a linear map

$$f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$$

such that $(\mathcal{A},\sigma,f)$ is a quadratic triple if and only if $\omega(\mathcal{A},\sigma) = 0$.

Proof. By the exactness of diagram 3.37, $0 = \omega(A,\sigma)$ if and only if $\xi_S^{-1}(1_{\mathcal{A}})$ is not empty, by the global classification theorem 3.36, this happens if and only if there is a linear form

$$f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$$

such that $(\mathcal{A},\sigma,f)$ is a quadratic triple. □

Definition 3.40. The **strong obstruction** is the element

$$\Omega(\mathcal{A},\sigma) := \delta(1_{\mathcal{A}}) \in H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}).$$

We immediately note that if the strong obstruction is trivial, then the weak obstruction is also trivial. The weak obstruction measures the failure of σ to partake in a quadratic pair (σ, f). The strong obstruction measures the failure of σ to partake in a quadratic pair (σ, f) where f extends to a linear form on all of $\mathcal{A}$.

Theorem 3.41. *Strong obstruction theorem.* There is a linear map $f\colon \mathcal{A} \to \mathcal{O}$ such that $(\mathcal{A},\sigma,f_{|\,\mathcal{S}ym_{\mathcal{A},\sigma}})$ is a quadratic triple if and only if $\Omega(\mathcal{A},\sigma) = 0$. In this case, there is a global section $l \in \mathcal{A}(S)$ such that $1 = l + \sigma(l)$ and $f = \mathrm{Trd}(l-)$.

Proof. If $\Omega(\mathcal{A},\sigma) = 0$, then by the exactness of diagram 3.37, there is a section $l \in \mathcal{A}(S)$ satisfying $1 = l + \sigma(l)$. Applying Lemma 3.25 gives us our quadratic triple. On the other hand, if $f\colon \mathcal{A} \to \sigma$ is a linear form such that $(\mathcal{A},\sigma,f_{|\mathcal{S}ym_{\mathcal{A},\sigma}})$ is a quadratic triple, then $f_{|\mathcal{S}ym_{\mathcal{A},\sigma}}$ trivially extends to a linear form on all of $\mathcal{A}$, so Lemma 3.26 concludes our proof. □

Connecting with Remark 3.30, if we interpret $H^1(S,\mathcal{S}kew_{\mathcal{A},\sigma})$ as the set of $\mathcal{S}kew_{\mathcal{A},\sigma}$-torsors (see Remark 2.35), the strong obstruction $\delta(1_{\mathcal{A}})$ is exactly the isomorphism class of the $\mathcal{S}kew_{\mathcal{A},\sigma}$-torsor $(\mathrm{id}+\sigma)^{-1}(1_{\mathcal{A}})$. So the strong obstruction is trivial if and only if we can write $1_{\mathcal{A}} = l + \sigma(l)$ for a global section $l \in \mathcal{A}(S)$. In comparison with the weak obstruction, it suffices for $\xi^{-1}(1_{\mathcal{A}})$ to be the trivial $\mathcal{S}kew_{\mathcal{A},\sigma} / \mathcal{A}lt_{\mathcal{A},\sigma}$-torsor for σ to partake in a quadratic pair, even if we cannot write $1_{\mathcal{A}} = l + \sigma(l)$ globally. We now consider an explicit example of this phenomenon.

Example 3.42. In [GNR23, §7.1] the authors give an example of a quaternion $\mathcal{O}$-algebra $\mathcal{Q}$ over an ordinary elliptic curve S (taken as the base scheme) defined over a field k of characteristic 2. The algebra $\mathcal{Q}$ partakes in a quadratic pair $(\mathcal{Q},\sigma,f)$, and its global sections are $\mathcal{Q}(S) = k$. Under these conditions the strong obstruction is nontrivial. To see this, consider the exact sequence

$$0 \longrightarrow \mathcal{S}kew_{\mathcal{Q},\sigma}(S) \longrightarrow \mathcal{Q}(S) \longrightarrow \mathcal{S}ymd_{\mathcal{Q},\sigma}(S) \xrightarrow{\delta} H^1(S,\mathcal{S}kew_{\mathcal{Q},\sigma}). \tag{3.43}$$

Since $\mathcal{Q}(S) = k$ and $2k = 0$, every element of $\mathcal{Q}(S)$ is skew-symmetric and $\mathcal{S}kew_{\mathcal{Q},\sigma}(S) \to \mathcal{Q}(S)$ is the identity on k. Therefore, sequence 3.43 becomes

$$0 \longrightarrow k \xrightarrow{\mathrm{id}} k \xrightarrow{0} \mathcal{S}ymd_{\mathcal{Q},\sigma}(S) \xrightarrow{\delta} H^1(S,\mathcal{S}kew_{\mathcal{Q},\sigma}),$$

and the strong obstruction $\delta(1)$ is nontrivial, since $\operatorname{Ker}\delta = 0$.

Finally, we can also give a cohomological obstruction to the linear form f in a given quadratic triple $(\mathcal{A},\sigma,f)$ extending to a form on all of $\mathcal{A}$. We have an exact sequence

$$\mathcal{A}(S) \longrightarrow (\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S) \xrightarrow{\gamma} H^1(S,\mathcal{A}lt_{\mathcal{A},\sigma})$$

By Lemma 3.35, f corresponds to an element

$$\lambda_f \in \xi_S(1_{\mathcal{A}})^{-1} \subset (\mathcal{A}/\,\mathcal{A}lt_{\mathcal{A},\sigma})(S).$$

We define the **extension obstruction** of $(\mathcal{A}, \sigma, f)$ to be the element

$$\gamma(\lambda_f) \in H^1(S, \mathcal{A}lt_{\mathcal{A},\sigma}).$$

Now, $\gamma(\lambda_f) = 0$ if and only if λ_f comes from a global section $l \in \mathcal{A}(S)$ satisfying $1 = l + \sigma(l)$, which happens if and only if f can be extended to all of $\mathcal{A}$ by $f' = \mathrm{Trd}(l-)$ (Lemma 3.26). In summary:

Proposition 3.44. The linear form f in a quadratic pair $(\mathcal{A}, \sigma, f)$ extends to a linear form $f' \colon \mathcal{A} \to \mathcal{O}$ if and only if $\gamma(\lambda_f) = 0$.

Chapter 4

Existence of Involutions

We finish this text with a short discussion about the existence of involutions (of the first kind). If an Azumaya $\mathcal{O}$-algebra $\mathcal{A}$ has an involution σ, then σ is an isomorphism $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$, so $2[\mathcal{A}] = 0$ in $\mathrm{Br}(S)$. If in addition, $S = \operatorname{Spec} k$ is a field, the converse also holds, that is, $\mathcal{A}$ has an involution of the first kind if and only if $2[\mathcal{A}] = 0$. This remarkable theorem was first proven by Albert [Alb39]. A partial converse is due to Saltman [Sal78], who showed that over a ring R, there is an Azumaya R-algebra $\mathcal{B}$ that carries an involution and is Brauer equivalent to $\mathcal{A}$ if and only if $2[\mathcal{A}] = 0$ in $\mathrm{Br}(R)$. In this case we say the Brauer class $[\mathcal{A}]$ supports an involution. Another proof of Saltman's theorem was later given by Knus, Parimala, and Srinivas [KPS90, Thm. 4.1.], who provided an explicit construction allowing one to pick $\mathcal{B}$ such that $\deg \mathcal{B} = 2 \deg \mathcal{A}$. It was known that the proof of [KPS90] holds in the generality of schemes, this is the current best result on the existence of involutions. In this chapter we rewrite their argument to the language of schemes, making the necessary adaptations.

Theorem 4.1. Let S be a scheme and $\mathcal{A}$ be an Azumaya $\mathcal{O}$-algebra with $2[\mathcal{A}] = 0$ in $\mathrm{Br}(S)$. Then there is an Azumaya $\mathcal{O}$-algebra $\mathcal{B}$ with involution such that $\mathcal{B}$ is Brauer equivalent to $\mathcal{A}$ and $\deg \mathcal{B} = 2 \deg \mathcal{A}$.

Proof. Since $2[\mathcal{A}] = 0$, there is a finite locally free $\mathcal{O}$-module $\mathcal{E}$ and isomorphism of $\mathcal{O}$-algebras

$$\phi \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{E}).$$

We consider $\mathcal{E}$ as a left $\mathcal{A}$-module by the action given by

$$\begin{aligned} \mathcal{A}(T) \times \mathcal{E}(T) &\to \mathcal{E}(T) \\ (a, x) &\mapsto \phi_T(a \otimes 1)(x) \end{aligned}$$

on points $T \to S$.

Lemma 4.2. ϕ induces an isomorphism of $\mathcal{O}$-algebras

$$\overline{\phi} \colon \mathcal{A} \to \mathcal{E}nd_{\mathcal{A}}(\mathcal{E})$$

given on sections by $a \mapsto \phi(1 \otimes a)$.

Proof of lemma. We may check that $\overline{\phi}$ is an isomorphism on an affine cover. Suppose $S = \operatorname{Spec} R$ is affine, and write $A = \mathcal{A}(S)$. An R-endomorphism $f\colon A \to A$ is A-linear if and only if

$$\phi(a \otimes 1) \circ f = f \circ \phi(a \otimes 1)$$

for all a in A. Since ϕ is an R-algebra isomorphism, we may write $f = \phi(e)$. Then f is A-linear if and only if $e \cdot a \otimes 1 = a \otimes 1 \cdot e$ for all a, and since A is a central algebra, this happens if and only if e is of the form $1 \otimes b$ for some $b \in A$. $\square$

Set $\mathcal{B} = \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \oplus \mathcal{E})$, then $\mathcal{B}$ is an Azumaya $\mathcal{O}$-algebra. We will construct an involution on $\mathcal{B}$ and show that $\mathcal{B}$ is Brauer equivalent to $\mathcal{A}$.

Lemma 4.3. There is a global section $e \in \Gamma(\mathcal{A} \otimes \mathcal{A})$ such that the switch map $s\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the inner automorphism induced by e and with $e^2 = 1$. The element e is commonly called the **Goldman element** of $\mathcal{A}$.

Proof of lemma. Consider the reduced trace $\operatorname{Trd}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{O} \hookrightarrow \mathcal{A}$ as a global section of $\mathcal{E}nd_{\mathcal{O}}(\mathcal{A})$, then $\operatorname{Trd}_{\mathcal{A}}$ corresponds to a global section e of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ under the sandwich isomorphism $\Omega_{\mathcal{A}}\colon \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{A})$. We consider e as a global section of $\mathcal{A} \otimes \mathcal{A}$. We may verify the identities $s = \operatorname{Inn}(e)$ and $e^2 = 1$ on an affine cover using [KO74, Prop. 4.1]. $\square$

Set $\psi = \phi(e) \in \mathcal{E}nd_{\mathcal{O}}(\mathcal{E})$, then $\psi^2 = \mathrm{id}$. We compute that for any section $a \in \mathcal{A}(T)$,

$$\overline{\phi}(a) = \phi(1 \otimes a) = \phi(e.a \otimes 1.e) = \phi(e) \circ \phi(a \otimes 1) \circ \phi(e) = \psi(a \cdot \psi(-)).$$

The following properties of ψ now follow:

Lemma 4.4. For any sections $a \in \mathcal{A}(T), x \in \mathcal{E}(T), \lambda \in \mathcal{E}nd_{\mathcal{A}}(\mathcal{E})(T)$:

1. $\psi(a \cdot x) = \overline{\phi}(a)(\psi(x))$;
2. $(\psi \circ \lambda)(x) = \overline{\phi}^{-1}(\lambda) \cdot \psi(x)$.

Proof of lemma. The first identity follows immediately from $\psi^2 = \mathrm{id}$ and $\overline{\phi}(a) = \psi(a \cdot \psi(-))$. For the second identity we write $\lambda = \overline{\phi}(a) = \phi(1 \otimes a)$ for some $a \in \mathcal{A}$, and compute

$$(\psi \circ \lambda)(x) = \phi(e \cdot 1 \otimes a)(x) = \phi(a \otimes 1 \cdot e)(x) = a \cdot \psi(x) = \overline{\phi}^{-1}(\lambda) \cdot \psi(x).$$

$\square$

Denote by $\mathcal{E}^*$ the $\mathcal{A}$-dual $\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{A})$ of $\mathcal{E}$. We define the map $\psi^*\colon \mathcal{E}^* \to \mathcal{E}^*$ as follows: for any $f \in \mathcal{E}^*(T)$, define $\psi^*(f)$ on sections $x \in \mathcal{E}(U \to T)$ by
$$\psi^*(f)(x) = \overline{\phi}^{-1}(f(-)\cdot\psi(x)).$$
Note that $\psi^*(f)$ is indeed $\mathcal{A}$-linear since for any $a \in \mathcal{A}(U)$,
$$\begin{aligned}\overline{\phi}(\psi^*(f)(a\cdot x)) &= f(-)\cdot\psi(a\cdot x)\\ &= f(-)\cdot\overline{\phi}(a)(\psi(x))\\ &= \overline{\phi}(a)\circ(f(-)\cdot\psi(x)),\end{aligned}$$
but also
$$\overline{\phi}(a\cdot\psi^*(f)(x)) = \overline{\phi}(a)\circ\overline{\phi}(\psi^*(f)(x)) = \overline{\phi}(a)\circ(f(-)\cdot\psi(x)),$$
so by cancelling $\overline{\phi}$ on the left, $\psi^*(f)(a\cdot x) = a\cdot\psi^*(f)(x)$. Now for some important properties of ψ^*:

Lemma 4.5. For any $a \in \mathcal{A}(T), x \in \mathcal{E}(U \to T), f \in \mathcal{E}^*(T), \lambda \in \mathcal{E}nd_{\mathcal{A}}(\mathcal{E})$ we have that:

1. $\psi^*(R_a \circ f) = \psi^*(f)\circ\overline{\phi}(a)$, where $R_a\colon \mathcal{A}_{|T} \to \mathcal{A}_{|T}$ is right multiplication by a;
2. $\psi^*(f\circ\lambda) = R_{\overline{\phi}^{-1}(\lambda)}\circ\psi^*(f)$;
3. $\psi^*(f)(-)\cdot\psi(x) = \overline{\phi}(f(x))$.

Proof of lemma. For the first identity, we compute:
$$\begin{aligned}\psi^*(R_a\circ f)(x) &= \overline{\phi}^{-1}((R_a\circ f)(-)\cdot\psi(x))\\ &= \overline{\phi}^{-1}(f(-)a\cdot\psi(x))\\ &= \overline{\phi}^{-1}(f(-)\cdot(\phi(a\otimes 1)\circ\phi(e))(x))\\ &= \overline{\phi}^{-1}(f(-)\cdot(\phi(e)\circ\phi(1\otimes a))(x))\\ &= \overline{\phi}^{-1}(f(-)\cdot\psi(\overline{\phi}(a)(x)))\\ &= (\psi^*(f)\circ\overline{\phi}(a))(x).\end{aligned}$$
Now for the second identity:
$$\begin{aligned}\psi^*(f\circ\lambda)(x) &= \overline{\phi}^{-1}(f(\lambda(-))\cdot\psi(x))\\ &= \overline{\phi}^{-1}(f(-)\cdot\psi(x)\circ\lambda)\\ &= \overline{\phi}^{-1}(f(-)\cdot\psi(x))\overline{\phi}^{-1}(\lambda)\\ &= (R_{\overline{\phi}^{-1}(\lambda)}\circ\psi^*(f))(x).\end{aligned}$$

Let $y \in \mathcal{E}(U \to T)$, then the final identity follows from

$$\begin{aligned}\psi(\psi^*(f)(y) \cdot \psi(x)) &= \overline{\phi}(\psi^*(f)(y))(\psi^2(x)) \\ &= f(x) \cdot \psi(y) \\ &= \phi(e \cdot 1 \otimes f(x) \cdot e)(\psi(y)) \\ &= \psi(\overline{\phi}(f(x))(y))\end{aligned}$$

by cancelling ψ on the left. □

Next, we note that $(\psi^*)^2 = \mathrm{id}$, since for any $f \in \mathcal{E}^*(T)$ and $x \in \mathcal{E}(U \to T)$, we have that

$$\overline{\phi}((\psi^*)^2(f)(x)) = \psi^*(f)(-) \cdot \psi(x) = \overline{\phi}(f(x)),$$

so by cancelling $\overline{\phi}$ on the left, $(\psi^*)^2(f)(x) = f(x)$. Now by identifying $\mathcal{A}^{\mathrm{op}} \simeq \mathcal{E}nd_{\mathcal{A}}(\mathcal{A})$ by $a \mapsto R_a$ and $\mathcal{E} \simeq \mathcal{H}om_{\mathcal{A}}(\mathcal{A}, \mathcal{E})$ we may write

$$\mathcal{B} = \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \oplus \mathcal{E}) \simeq \begin{pmatrix} \mathcal{A}^{\mathrm{op}} & \mathcal{E} \\ \mathcal{E}^* & \mathcal{E}nd_{\mathcal{A}}(\mathcal{E}) \end{pmatrix},$$

where multiplication is given on sections by

$$\begin{pmatrix} a & x \\ f & \lambda \end{pmatrix} \begin{pmatrix} a' & x' \\ f' & \lambda' \end{pmatrix} = \begin{pmatrix} aa' + f'(x) & a \cdot x' + \lambda'(x) \\ R_{a'} \circ f + f' \circ \lambda & f(-) \cdot x + \lambda' \circ \lambda \end{pmatrix}.$$

Using this characterisation of $\mathcal{B}$ we see that $\mathcal{B}$ is Brauer equivalent to $\mathcal{A}$. Indeed, $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \Gamma(S, \mathcal{B})$ is a torsion free idempotent, so by [Sal99, Lem. 3.4],

$$\mathcal{A} \sim \mathcal{A}^{\mathrm{op}} \simeq e\mathcal{B}e \sim \mathcal{B}.$$

Finally, we define the involution $\sigma\colon \mathcal{B} \to \mathcal{B}$ by

$$\sigma \begin{pmatrix} a & x \\ f & \lambda \end{pmatrix} = \begin{pmatrix} \overline{\phi}^{-1}(\lambda) & \psi(x) \\ \psi^*(f) & \overline{\phi}(a) \end{pmatrix}$$

It is immediate that σ is $\mathcal{O}$-linear and satisfies $\sigma^2 = \mathrm{id}$. Finally, a straightforward calculation using the identities developed in Lemmas 4.4 and 4.5 shows that $\sigma(ab) = \sigma(b)\sigma(a)$. □

In the proof above the algebra $\mathcal{B}$ satisfied $\deg \mathcal{B} = 2 \deg \mathcal{A}$. In general, we may not expect a representative with smaller degree to carry an involution. Auel, First, and Williams [AFW19] have explicitly constructed an example of an Azumaya algebra $\mathcal{A}$ of degree 4 such that $2[\mathcal{A}] = 0$, and where any representative $\mathcal{B} \in [\mathcal{A}]$ carrying an involution has degree at least 8. Establishing conditions under which a particular algebra in $[\mathcal{A}]$ carries an involution is an open problem.

One notable class of such algebras are quaternion algebras (Azumaya algebras of degree 2). The following theorem is due to Saltman [Sal78, Thm. 4.1], we rewrite his proof in the language of schemes.

Theorem 4.6. Suppose $\mathcal{A}$ is an Azumaya $\mathcal{O}$-algebra of constant rank 4, then $\mathcal{A}$ carries an involution.

Proof. Define $\sigma\colon \mathcal{A} \to \mathcal{A}$ on sections $a \in \mathcal{A}(U)$ by $\sigma(a) = \mathrm{Trd}_{\mathcal{A}}(a) \cdot 1_{\mathcal{A}} - a$. We may verify that σ is an involution on an fppf cover $\{T_i \to S\}$ where $\mathcal{A}_{|T_i} \simeq \mathrm{M}_2(\mathcal{O}_{T_i})$. On this cover $\mathrm{Trd}_{\mathcal{A}}$ is just the usual matrix trace, and hence we may compute $\sigma_{|T_i}$ by

$$\sigma_{|T_i}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is just the usual involution of type -1 on $\mathrm{M}_2(\mathcal{O})$ (see Remark 3.10). $\square$

Given our earlier work on quadratic pairs, one may also ask when an involution of a specified type exists. Some work in this direction has been done by First and Williams [FW20] who provided a cohomological criterion in the setting of a connected locally ringed topos. For involutions of the first kind over a scheme they required that 2 be invertible, and found that in this context if a Brauer class $[\mathcal{A}]$ supports an involution, then $[\mathcal{A}]$ supports involutions of both orthogonal type and symplectic type [FW20, Cor. 6.3.5]. Unfortunately, as of this time, it is not known if the condition that 2 is invertible may be relaxed.

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