

The Finite Section Method for Infinite Vandermonde Matrices and Applications

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'To break this curse a ritual's due, I believe I'm not alone.' - James Hetfield

Abstract

In this thesis we investigate a very well known and relevant question, i.e, to solve a linear equation

$$A\mathbf{x} = \mathbf{b},$$

where A and $\mathbf{b}$ are given. In our study A denotes an infinite matrix of special form called a Vandermonde matrix and $\mathbf{b}$ will be a vector from a given sequence space. We will consider two cases of the equation above. Different constraints will be placed upon the entries of A and $\mathbf{b}$ will be chosen from different sequence spaces. We will also look at an example from the first case to show how the theory can be applied. Our approach to solving this equation will be to apply the Finite Section Method. Here we follow the exposition of [9] while clarifying and explaining their approach. In addition, we will draw on various mathematical fields to assist our investigation. These include linear algebra, functional analysis, operator theory, complex analysis and topological vector spaces.

Opsomming

In hierdie verhandeling ondersoek ons 'n bekende en relevante vraag, nl., die oplossing van 'n lineêre vergelyking

$$A\mathbf{x} = \mathbf{b},$$

waar A en $\mathbf{b}$ gegee is. In ons studie definieer A 'n oneindige matriks van spesiale vorm, genaamd 'n Vandermonde matriks, en $\mathbf{b}$ is 'n vektor uit 'n gegewe ryruimte. Ons sal twee gevalle van die vergelyking hierbo beskou. Verskillende voorwaardes sal op die inskrywings van A geplaas word, terwyl $\mathbf{b}$ uit verskillende vektorruimtes gekies sal word. Ons sal ook 'n voorbeeld van die eerste geval beskou om te illustreer hoe die teorie toegepas word. Ons benadering tot hierdie probleem sal wees om die eindige seksiemetode toe te pas. Ons volg hier die uiteensetting van [9] terwyl ons dit volledig verduidelik. Ons sal ook gebruik maak van 'n verskeidenheid wiskundige velde om ons ondersoek te ondersteun. Dit sluit in lineêre algebra, funksionaal analise, operatorsteorie, komplekse analise en topologiese vektorruimtes.

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Chapter 1

Introduction

The Finite Section Method is a scheme for approximating the solution to an infinite system of linear equations. The theory involves matrix equations which represent a system of linear equations where the associated operator for the matrix is usually continuous and defined on a Hilbert space. (See [5], [1] and [3]). In this thesis we apply the general scheme of the Finite Section Method to find a solution for an infinite system of equations where the matrix corresponding to this system is an infinite Vandermonde matrix. The finite Vandermonde matrix arises naturally from solving the unknown coefficients of a system of polynomial equations (see [2]). In this case the matrix is invertible. In the infinite case this is not true on its natural domain. Since we will work in the Banach space setting and the continuity of the operator associated with an infinite Vandermonde matrix is unknown, a new theory is developed because the standard theory is not applicable. In short, the Finite Section Method entails the use of sections or finite truncations of the original infinite matrix A and resultant vector b . The equation is then solved for the finite dimensional case. We then test if these finite solutions converge to a solution of the original equation. We can formalize our approach as follows. Consider a linear equation

$$Ax = b$$

and matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$D(A) = \{x \in \omega \mid \sum_j a_{ij}x_j < \infty \text{ for all } i\}$ denotes the natural domain of definition of matrix A and it induces a linear operator $A : D(A) \mapsto \omega$ where ω denotes the space of all complex sequences. The set $D_{abs}(A) = \{x \in \omega \mid \sum_j |a_{ij}x_j| < \infty \text{ for all } i\}$ will also be considered.

Take a $n \times n$ section of A and truncate b :

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad P_nb = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

By P_n we denote the projection onto the first n sequence entries: $P_n(b_1, b_2, b_3, \dots) = (b_1, \dots, b_n)$. We now try to solve the finite dimensional version of our linear equation,

$$A_n x_n = P_n b.$$

If we can solve this equation for all n , we can take the limit as $n \rightarrow \infty$ and test if the resulting vector exists and if it solves our original equation. This is the scheme that we will apply to our particular problem involving the Vandermonde matrix. This leads us to a fundamental definition.

Definition 1.0.1 *Let (X, τ) be a sequence space with topology τ so that it contains sequences with finite non-zero entries and $A(X \mapsto \omega)$. (The notation $A(X \mapsto \omega)$ indicates that we do not assume that A is defined everywhere on X . We use the notation $A : X \mapsto \omega$ to indicate that A is defined on the whole of X). We say that the finite section method is applicable to the equation $Ax = b$ with right hand side $b = (b_1, b_2, \dots) \in \omega$, if for any $n \in \mathbb{N}$ there is a unique solution $x_n = (x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)})$ in $\mathbb{C}^n$ to the truncated system $A_n x_n = P_n b$ and an $x \in X$ such that $Ax = b$ and $x_n \rightarrow x$ in the topology τ .*

We now give a short summary of the following chapters of this thesis.

Chapter 2: Preliminaries

Here we deal with all the necessary background theory for the results we arrive at. Some of the more important theorems and lemmas are proven while the rest of the theorems are stated and some definitions given.

Chapter 3: The Main Theorem

In this chapter we establish our main result and give an example as an application of the theory:

- **Theorem 3.0.6** Let $a_0, a_1, \dots$ be a sequence of complex numbers such that $0 < |a_0| < |a_1| < \dots$ and $\alpha = \sum_k \frac{1}{|a_k|} < \infty$. Let
$$b_k = \left| \prod_{j=0, j \neq k}^{\infty} \frac{1}{1 - \frac{a_k}{a_j}} \right|.$$

Suppose that $\sum_k |a_k^i b_k| < \infty$ for any non-negative integer i . Then the finite section method is applicable to the equation $Ax = d$ in the sense of l_1 -convergence for any $d \in l_1(\alpha)$. A represents the Vandermonde matrix generated by the sequence $a_0, a_1, \dots$. See Section 2.4 for details concerning $l_1(\alpha)$.

- **Theorem 3.1.9** For $p > 2$ and the Vandermonde matrix with entries $a_k = k^p$, there exists a real number $d > 0$ and a positive integer k_0 such that $b_k \leq e^{-dk}$ whenever $k \geq k_0$ and Theorem 3.0.6 applies.

Chapter 4: The Exponential Case

In the last chapter we prove a theorem similar to the main theorem:

Let the entries of an infinite Vandermonde matrix A be given by $a_{kj} = a^{kj}$ with $a \in \mathbb{C}$ and $|a| > 1$ for $k, j = 0, 1, 2, \dots$. Then the finite section method is applicable to the equation $Ax = d$ in the sense of l_1 -convergence for any $d \in l_{\infty}$.

Chapter 2

Preliminaries

In this chapter we cover some necessary background theory and we also revise some basic concepts while establishing notation.

2.1 Vandermonde Matrices

Lemma 2.1.1 (*Cramer's rule*) Let $A = [\mathbf{A}_0, \dots, \mathbf{A}_n]$ be an $(n+1) \times (n+1)$ matrix, and let $\mathbf{b}$ be any vector in $\mathbb{R}^{n+1}$. For each i , $0 \leq i \leq n$, let B_i be the $(n+1) \times (n+1)$ matrix:

$$B_i = [\mathbf{A}_0, \dots, \mathbf{A}_{i-1}, \mathbf{b}, \mathbf{A}_{i+1}, \dots, \mathbf{A}_n].$$

If the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent and x_i is the i -th component of a solution $\mathbf{x}$, then

$$x_i \det(A) = \det(B_i).$$

Definition 2.1.2 A matrix of the following form

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ a_0^2 & a_1^2 & \dots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{pmatrix},$$

is called a *finite Vandermonde matrix* of size $(n+1) \times (n+1)$.

Theorem 2.1.3 A $(n + 1) \times (n + 1)$ Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ a_0^2 & a_1^2 & \dots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{pmatrix},$$

is invertible if the entries of the matrix are distinct, i.e. if $a_j \neq a_i$ for $i, j = 0, \dots, n$, $i \neq j$.

Proof.

The following proof is adapted from [2]. We only have to show that the determinant of V , $\det V$, is non-zero. For integers $1 \leq k \leq n$, let $D(z_0, z_1, \dots, z_k)$ denote the determinant

$$D(z_0, z_1, \dots, z_k) = \begin{vmatrix} 1 & z_0 & \dots & z_0^k \\ 1 & z_1 & \dots & z_1^k \\ \vdots & \vdots & & \vdots \\ 1 & z_k & \dots & z_k^k \end{vmatrix}$$

and notice that $D(a_0, a_1, \dots, a_n) = \det V^T = \det V$. For each $1 \leq k \leq n$ we define a polynomial

$$V_k(z) = D(a_0, a_1, \dots, a_{k-1}, z)$$

of degree $\leq k$ (when the determinant is expanded by minors along its last row). We see that $V_n(a_n) = \det V$ and since $D(a_0, a_1, \dots, a_{k-1}, a_j)$ is the determinant of a matrix with two identical rows for $j = 0, 1, \dots, k - 1$, it is clear that $V_k(a_j) = 0$ for $j = 0, 1, \dots, k - 1$. Therefore we have

$$V_k(z) = A_k(z - a_0)(z - a_1) \dots (z - a_{k-1})$$

where A_k is the coefficient of z^k . On the other hand, expanding $V_k(z)$ by minors along its last row, shows that

$$A_k = D(a_0, a_1, \dots, a_{k-1}) = V_{k-1}(a_{k-1}).$$

This gives us the recursion formula

$$V_k(a_k) = V_{k-1}(a_{k-1})(a_k - a_0)(a_k - a_1) \dots (a_k - a_{k-1})$$

for $k = 1, 2, \dots, n$. In particular, it is clear that

$$\begin{aligned} V_2(a_2) &= D(a_0, a_1)(a_2 - a_0)(a_2 - a_1) \\ &= (a_1 - a_0)(a_2 - a_0)(a_2 - a_1), \\ V_3(a_3) &= V_2(a_2)(a_3 - a_0)(a_3 - a_1)(a_3 - a_2) \\ &= (a_1 - a_0)(a_2 - a_0)(a_2 - a_1)(a_3 - a_0)(a_3 - a_1)(a_3 - a_2) \end{aligned}$$

and so forth. Applying our recursion formula we find that

$$V_n(a_n) = D(a_0, a_1, \dots, a_n) = \prod_{\substack{i > j \\ j \in \{0, 1, \dots, n-1\}}} (a_i - a_j).$$

Since $a_j \neq a_i$ if $i \neq j$, $\det V = V_n(a_n) \neq 0$.

□

The infinite Vandermonde matrix A is not injective on its domain $D(A)$. This is verified by an application of Polya's theorem. (See [2] for a detailed discussion of this theorem). A brief discussion to demonstrate this fact follows below.

Theorem 2.1.4 (*Polya's theorem*) *Let A be an infinite matrix such that for any non-negative integers n, q the matrix*

$$\begin{pmatrix} a_{0,q} & a_{0,q+1} & \dots & a_{0,q+n} \\ a_{1,q} & a_{1,q+1} & \dots & a_{1,q+n} \\ \vdots & \vdots & & \vdots \\ a_{n,q} & a_{n,q+1} & \dots & a_{n,q+n} \end{pmatrix}$$

is invertible and for any integer $j \geq 1$

$$\lim_{k \rightarrow \infty} \frac{a_{j-1,k}}{a_{jk}} = 0.$$

Then for any $\mathbf{b} \in \omega$ there exists an $\mathbf{x} \in D_{abs}(A)$ such that $A\mathbf{x} = \mathbf{b}$.

With respect to injectivity, we have the following results which can be found in [9].

Proposition 2.1.5 *Suppose that the matrix mapping A satisfies the conditions of Polya's theorem. Then $A : D(A) \rightarrow \omega$ is not injective.*

Proof.

Let

$$A(j > n) = \begin{pmatrix} a_{0,n+1} & a_{0,n+2} & a_{0,n+3} & \cdots \\ a_{1,n+1} & a_{1,n+2} & a_{1,n+3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If A satisfies the conditions of Polya's theorem, so does $A(j > n)$ for each n . Fix $b \in \omega \setminus \{0\}$. For each n we apply Polya's theorem to find an $x^{(n)} \in D(A(j > n))$ such that $A(j > n)x^{(n)} = b$. Suppose $x^{(n)} = (x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \dots)$. Then let $y^{(n)} = (0, 0, 0, \dots, 0, x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \dots)$ where $x_0^{(n)}$ is the $(n+1)$ -th term of $y^{(n)}$. We then have

$$\begin{aligned} Ay^{(n)} &= \left(\sum_{j=0}^{\infty} a_{0,n+j+1} x_j^{(n)}, \sum_{j=0}^{\infty} a_{1,n+j+1} x_j^{(n)}, \dots \right) \\ &= A(j > n)x^{(n)} = b. \end{aligned}$$

Thus, for each $n \in \mathbb{N}$ we find $y^{(n)} \in D(A)$ such that $Ay^{(n)} = b$, where it is clear that $y^{(n)} \neq y^{(m)}$ if $m \neq n$. This shows that A is not injective. $\square$

Recall that an infinite Vandermonde matrix A is of the following form

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0^2 & a_1^2 & a_2^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In our case we will add additional constraints on the entries by choosing them so that $a_i \in \mathbb{C}$, $a_j \neq a_i \neq 0$ ($j \neq i \geq 0$) and $\sum_i \frac{1}{|a_i|} = \alpha < \infty$.

Corollary 2.1.6 *Let A be an infinite Vandermonde matrix mapping with abovementioned constraints on the entries. Then $A : D(A) \rightarrow \omega$ is surjective but not injective.*

Proof.

The matrix A satisfies the form of the matrix in Polya's theorem since each finite block

$$\begin{pmatrix} a_{0,q} & a_{0,q+1} & \dots & a_{0,q+n} \\ a_{1,q} & a_{1,q+1} & \dots & a_{1,q+n} \\ \vdots & \vdots & & \vdots \\ a_{n,q} & a_{n,q+1} & \dots & a_{n,q+n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_q & a_{q+1} & \dots & a_{q+n} \\ \vdots & \vdots & & \vdots \\ a_q^n & a_{q+1}^n & \dots & a_{q+n}^n \end{pmatrix}$$

is an invertible Vandermonde matrix. Also,

$$\lim_{k \rightarrow \infty} \frac{a_{j-1,k}}{a_{jk}} = \lim_{k \rightarrow \infty} \frac{a_k^{j-1}}{a_k^j} = \lim_{k \rightarrow \infty} \frac{1}{a_k} = 0$$

since $\sum_i \frac{1}{|a_i|} = \alpha < \infty$. □

2.2 Basic properties of Holomorphic Functions

In this section we briefly recall some concepts and facts about holomorphic functions that will play an important role in the sequel. For later reference, we recall the following test for uniform convergence of a series of complex functions, (see [11] or [7]).

Theorem 2.2.1 *Let $\{a_n\}$ be a sequence of positive real numbers and $|f_n| \leq a_n$ on $D \subseteq \mathbb{C}$ for all $n \in \mathbb{N}$. If $\sum_n a_n$ is convergent, then $\sum_n f_n$ is uniformly convergent on D .*

Definition 2.2.2 *A complex valued function of a complex variable is said to be **holomorphic** on a domain Ω if it is defined and differentiable on Ω . (By domain we mean an open connected subset).*

Definition 2.2.3 *We call a function **entire** if it is holomorphic on the whole of $\mathbb{C}$.*

Definition 2.2.4 *We write $H(\Omega)$ for the set of all holomorphic functions on the domain Ω .*

We remind the reader that an equivalent definition for a holomorphic function would be that it must have a power series representation about every point in its domain Ω , i.e. f is holomorphic on Ω if for each $z_0 \in \Omega$ there is a sequence $\{a_n\} \subset \mathbb{C}$ and a $r > 0$ such that $f(z) = \sum_{i=1}^{\infty} a_i(z - z_0)^i$ for all z satisfying $|z - z_0| < r$.

We now list several important theorems without proof. A good reference is [11].

Theorem 2.2.5 *If $f \in H(\Omega)$, then $f' \in H(\Omega)$. (By f' we mean the derivative of f).*

Theorem 2.2.6 *Suppose Ω is the domain of $f \in H(\Omega)$, and*

$$Z(f) = \{a \in \Omega : f(a) = 0\}.$$

Then either $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω . In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z - a)^m g(z) \quad (z \in \Omega),$$

where $g \in H(\Omega)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

Theorem 2.2.7 *If n is a positive integer and*

$$P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

where $a_0, \dots, a_{n-1}$ are complex numbers, then P has precisely n zeros in the complex plane, counted with multiplicity.

Definition 2.2.8 *A sequence $\{f_j\}$ of functions in Ω is said to **converge to f uniformly on compact subsets of Ω** if to every compact $K \subset \Omega$ and to every $\varepsilon > 0$ there corresponds an $N = N(K, \varepsilon)$ such that $|f_j(z) - f(z)| < \varepsilon$ for all $z \in K$ if $j > N$.*

Theorem 2.2.9 *Suppose $f_j \in H(\Omega)$, for $j = 1, 2, 3, \dots$, and $f_j \rightarrow f$ uniformly on compact subsets of Ω . Then $f \in H(\Omega)$, and $f'_j \rightarrow f'$ uniformly on compact subsets of Ω .*

2.3 Infinite products and the Gamma function $\Gamma(z)$

For the following topic the reader may consult one or more of [7], [8] and [11].

Definition 2.3.1 Given a sequence $\{a_k\}$ defined for all positive integers k , consider the finite product

$$P_n = \prod_{k=1}^n (1 + a_k) = (1 + a_1)(1 + a_2) \dots (1 + a_n).$$

If $\lim_{n \rightarrow \infty} P_n$ exists and is equal to $P \neq 0$, we say that the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ **converges** to the value P . If a finite number of the factors in the product are zero and if the infinite product with the zero factors deleted converges to a value $P \neq 0$, we say that the product **converges to zero**. If the infinite product is not convergent, it is said to be **divergent**. If that divergence is due not to the failure of $\lim_{n \rightarrow \infty} P_n$ to exist but to the fact that the limit is zero, the product is said to **diverge to zero**.

Theorem 2.3.2 If there exist positive constants M_n such that $\sum_{n=1}^{\infty} M_n$ is convergent and $|a_n(z)| < M_n$ for all z in the closed domain R , the product $\prod_{n=1}^{\infty} (1 + a_n(z))$ is uniformly convergent in R .

Definition 2.3.3 When an infinite product is uniformly convergent on all compact subsets of $\mathbb{C}$, we say it is **locally uniformly convergent** on $\mathbb{C}$.

Proposition 2.3.4 The infinite product

$$\prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right)$$

is locally uniformly convergent on $\mathbb{C}$, where $0 < |a_1| < |a_2| \dots$ and $\sum_k \frac{1}{|a_k|} < \infty$.

Proof.

Choose an arbitrary compact set $K \subset \mathbb{C}$. We know that $m := \max_{z \in K} |z|$ exists. Choose $M_j = \left|\frac{2m}{a_j}\right|$. It is clear that $M_j > \left|\frac{z}{a_j}\right|$ for all j and for all $z \in K$ with K compact. Since $\sum_j \frac{1}{|a_j|} < \infty$ we have the desired result by applying Theorem 2.3.2. $\square$

Definition 2.3.5 We define the **Weierstrass primary factors** by

$$E(w, m) = (1 - w)e^{(w + \frac{w^2}{2} + \dots + \frac{w^m}{m})}$$

for $m = 1, 2, 3, \dots$, and also $E(w, 0) = 1 - w$.

Theorem 2.3.6 For $k = 1, 2, 3, \dots$ let $\{\alpha_k\}$ be a sequence of complex numbers and let $m \geq 0$ be an integer such that

$$\sum_{k=1}^{\infty} \frac{1}{|\alpha_k|^{m+1}} < \infty.$$

Then the function

$$P(z) = \prod_{k=1}^{\infty} E\left(\frac{z}{\alpha_k}, m\right)$$

is an entire function with zeros only at α_k . The order of the zero at α_n is equal to the number of indices j such that $\alpha_j = \alpha_n$.

Definition 2.3.7 We define the gamma function $\Gamma(z)$ by

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{(-\frac{z}{n})} \right], \quad (2.1)$$

in which γ is the Euler constant.

Here γ is defined by

$$\gamma = \lim_{n \rightarrow \infty} (H_n - \log n),$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$. It can be shown that γ exists and that $0 \leq \gamma < 1$. Actually, $\gamma \approx 0.5772$.

It can also be proved that the gamma function $\Gamma(z)$ defined by (2.1) is equal to the Euler integral:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0$$

See [8] for the details. The Euler integral is the preferred point of departure in modern treatments of the gamma function with regard to measure theory and probability theory.

2.4 Brief review of sequence spaces

Most of our work only concerns sequence spaces and we will start by introducing them. Whenever necessary, we will identify a finite sequence $x = (x_0, x_1, x_2, \dots, x_{n-1}) \in \mathbb{C}^n$ with its imbedded version $x = (x_0, x_1, x_2, \dots, x_{n-1}, 0, 0, 0, \dots) \in \omega$. As usual, l_p with $1 \leq p < \infty$ denotes the Banach sequence space with well known norm, $\|x\|_p = (\sum_{i=0}^{\infty} |x_i|^p)^{\frac{1}{p}}$, and l_{∞} , the Banach space of all bounded sequences with $\|x\|_{\infty} = \sup_i |x_i|$. Hereafter we may drop the norm subscript when it is obvious to which norm we are referring to. In our main theorem to be discussed in Chapter 3, we will consider infinite Vandermonde matrices with values in a weighted $l_1(\alpha)$ space. For a given positive real number α , we define the weighted l_1 -space, $l_1(\alpha)$, by $l_1(\alpha) = \{x = (x_i) \in \omega \mid \sum_{i=0}^{\infty} |x_i| \alpha^i < \infty\}$. This is a Banach space with respect to the norm $\|x\| = \sum_{i=0}^{\infty} |x_i| \alpha^i$. In the last section of this thesis we will come across some possibly less familiar spaces and we will therefore list them formally. See [13] for more information.

Definition 2.4.1 A **Fréchet** space is a locally convex space whose topology is induced by a complete invariant metric. Therefore, every Banach space is a Fréchet space, but the converse is not always true since not all metrics can be obtained from a norm.

Definition 2.4.2 A **K-space** is a vector space of sequences which has a topology such that each Q_n is continuous, where $Q_n(x_0, \dots, x_n, x_{n+1}, \dots) = x_n$.

Definition 2.4.3 Let H be a vector space with a (not necessarily vector) Hausdorff topology (distinct points have non-intersecting neighbourhoods). An **FH-space** is a vector subspace X of H which is a Fréchet space and is continuously imbedded into H . In other words, the topology of X is larger than the relative topology of H .

Definition 2.4.4 An **FK-space** is an FH-space for which $H = \omega$. Thus it is a Fréchet sequence space which is also a K-space.

The interesting fact about FK-spaces that we will make use of in Chapter 4 and will again formulate in Theorem 4.0.13, is that if an FK-space X is a vector subspace of an FK-space Y , then the embedding of X into Y has to be continuous.

Chapter 3

The Main theorem

In this chapter we consider the finite section method for the infinite Vandermonde matrix A . Constraints on the entries of A and some additional conditions will be needed to prove the main theorem. But first, we establish a lemma necessary for our calculations.

Lemma 3.0.5 *Given the product*

$$(b - a_0)(b - a_1) \dots (b - a_{k-1})(b - a_{k+1}) \dots (b - a_n),$$

the coefficient of the term involving b^r in the polynomial resulting from this product is given by

$$\sum_{\psi \in C_{n-r,k}^n} (-1)^{n-r} a_{\psi(1)} \dots a_{\psi(n-r)}$$

for $n \geq 2$ and $0 < r < n$, where $C_{r,k}^n$ denotes all injective functions

$$\psi : \{1, 2, \dots, r\} \longrightarrow \{0, 1, \dots, k-1, k+1, \dots, n\}.$$

From this set of injective functions we exclude the ones whose range set is a permutation of another injective function.

Proof.

We will prove the lemma by induction on n as follows.
Let $n = 2$, $k = 1$ and note that

$$\begin{aligned} & (b - a_0)(b - a_2) \\ = & b^2 - b(a_0 + a_2) + a_0a_2. \end{aligned}$$

(The proof follows in the same way if other values of k are considered). On the other hand, the formula simplifies to

$$\sum_{\psi \in C_{2-r,1}^2} (-1)^r a_{\psi(1)} \dots a_{\psi(2-r)}.$$

But $n = 2$ implies $r = 1$ and thus

$$\sum_{\psi \in C_{1,1}^2} (-1)^1 a_{\psi(1)} = -a_0 - a_2.$$

We have established our first induction step. Suppose that the formula is valid for $n = p$, i.e., the coefficient of b^r equals

$$\sum_{\psi \in C_{p-r,k}^p} (-1)^{p-r} a_{\psi(1)} \dots a_{\psi(p-r)}.$$

Therefore,

$$\begin{aligned} & (b - a_0) \dots (b - a_{k-1})(b - a_{k+1}) \dots (b - a_p) \\ = & b^p + \left(\sum_{\psi \in C_{1,k}^p} (-1)^1 a_{\psi(1)} \right) b^{p-1} \\ & + \dots + \left(\sum_{\psi \in C_{p-r,k}^p} (-1)^{p-r} a_{\psi(1)} \dots a_{\psi(p-r)} \right) b^r \\ & + \dots + (-1)^p (a_0 \dots a_{k-1} a_{k+1} \dots a_p) b^0. \end{aligned}$$

For $n = p + 1$, it follows from the induction assumption that

$$\begin{aligned}
& (b - a_0) \dots (b - a_{k-1})(b - a_{k+1}) \dots (b - a_{p+1}) \\
= & b^p(b - a_{p+1}) + \left(\sum_{\psi \in C_{1,k}^p} (-1)^1 a_{\psi(1)} \right) b^{p-1}(b - a_{p+1}) \\
+ \dots + & \left(\sum_{\psi \in C_{p-r,k}^p} (-1)^{p-r} a_{\psi(1)} \dots a_{\psi(p-r)} \right) b^r(b - a_{p+1}) \\
+ \dots + & (-1)^p (a_0 \dots a_{k-1} a_{k+1} \dots a_p) b^0(b - a_{p+1}).
\end{aligned}$$

The factors multiplying b^r equal

$$(b - a_{p+1}) \left(\sum_{\psi \in C_{p-r,k}^p} (-1)^{p-r} a_{\psi(1)} \dots a_{\psi(p-r)} \right)$$

and the factors multiplying b^{r-1} equal

$$(b - a_{p+1}) \left(\sum_{\psi \in C_{p-(r-1),k}^p} (-1)^{p-(r-1)} a_{\psi(1)} \dots a_{\psi(p-(r-1))} \right).$$

If we multiply the just mentioned factors, we find the coefficient of b^r to be

$$\begin{aligned}
& -a_{p+1} \left(\sum_{\psi \in C_{p-r,k}^p} (-1)^{p-r} a_{\psi(1)} \dots a_{\psi(p-r)} \right) \\
+ & \sum_{\psi \in C_{p-(r-1),k}^p} (-1)^{p-(r-1)} a_{\psi(1)} \dots a_{\psi(p-(r-1))} \\
= & \sum_{\psi \in C_{p-r,k}^p} (-1)^{p-(r-1)} a_{\psi(1)} \dots a_{\psi(p-r)} a_{p+1} \\
+ & \sum_{\psi \in C_{p-(r-1),k}^p} (-1)^{p-(r-1)} a_{\psi(1)} \dots a_{\psi(p-(r-1))} \\
= & \sum_{\psi \in C_{p+1-r,k}^{p+1}} (-1)^{p+1-r} a_{\psi(1)} \dots a_{\psi(p+1-r)}.
\end{aligned}$$

The last equality follows from the well known formula involving combinations, namely, $C_{p-r}^p \cup C_{p+1-r}^p = C_{p+1-r}^{p+1}$. Our induction is complete and the lemma follows. $\square$

Theorem 3.0.6 *Let $a_0, a_1, \dots$ be a sequence of complex numbers such that $0 < |a_0| < |a_1| < \dots$ and $\alpha = \sum_k \frac{1}{|a_k|} < \infty$. Let $b_k = \left| \prod_{j=0, j \neq k}^{\infty} \frac{1}{1 - \frac{a_k}{a_j}} \right|$.*

Suppose that $\sum_k |a_k^i b_k| < \infty$ for any non-negative integer i . Then the finite section method is applicable to the equation $Ax = d$ in the sense of l_1 -convergence for any $d \in l_1(\alpha)$.

Proof.

First, we examine the following infinite Vandermonde system with special right-hand side

$$\begin{pmatrix} 1 & 1 & 1 & \dots \\ a_0 & a_1 & a_2 & \dots \\ a_0^2 & a_1^2 & a_2^2 & \dots \\ \vdots & \vdots & \vdots & \\ a_0^r & a_1^r & a_2^r & \dots \\ a_0^{r+1} & a_1^{r+1} & a_2^{r+1} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}.$$

Here the right-hand vector has its r -th component equal to one and all its other components are zero. The truncated system is of the form

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^r & a_1^r & \dots & a_n^r \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{pmatrix} \begin{pmatrix} x_0^{(n)} \\ x_1^{(n)} \\ \vdots \\ x_r^{(n)} \\ \vdots \\ x_n^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

for $n \geq r$. To simplify the notation, we let

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} \text{ and } D_r = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{r-1} & a_2^{r-1} & \dots & a_n^{r-1} \\ a_1^{r+1} & a_2^{r+1} & \dots & a_n^{r+1} \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix}.$$

We now apply Cramer's rule to $x_0^{(n)}$. Using the notation of Cramer's rule (see 2.1.1), we will see that B_0 in our case is equal to

$$\begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & a_1 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ 1 & a_1^r & \dots & a_n^r \\ 0 & a_1^{r+1} & \dots & a_n^{r+1} \\ \vdots & \vdots & & \vdots \\ 0 & a_1^n & \dots & a_n^n \end{pmatrix}.$$

If we use the cofactor expansion along the first column of B_0 , we see that

$$\det B_0 = (-1)^r \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{r-1} & a_2^{r-1} & \dots & a_n^{r-1} \\ a_1^{r+1} & a_2^{r+1} & \dots & a_n^{r+1} \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix} = (-1)^r D_r$$

and so

$$x_0^{(n)} = \frac{(-1)^r D_r}{D}.$$

Now, more generally, we look at the k -th component of the solution to the truncated system. By Cramer's rule we have the following expression

$$x_k^{(n)} = \frac{\det B_k}{D}$$

with

$$B_k = \begin{pmatrix} 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ a_0 & \dots & a_{k-1} & 0 & a_{k+1} & \dots & a_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_0^r & \dots & a_{k-1}^r & 1 & a_{k+1}^r & \dots & a_n^r \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_0^n & \dots & a_{k-1}^n & 0 & a_{k+1}^n & \dots & a_n^n \end{pmatrix}.$$

Then,

$$B'_k = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & a_0^r & \dots & a_{k-1}^r & a_{k+1}^r & \dots & a_n^r \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_0^n & \dots & a_{k-1}^n & a_{k+1}^n & \dots & a_n^n \end{pmatrix}$$

where B'_k is obtained from B_k by exchanging columns within B_k , k times. Therefore,

$$\det B'_k = (-1)^k \det B_k$$

and

$$\begin{aligned} (-1)^k \det B'_k &= (-1)^{2k} \det B_k \\ &= \det B_k. \end{aligned}$$

By applying Cramer's rule again we arrive at the following expression

$$x_k^{(n)} = \frac{(-1)^k \det B'_k}{D}.$$

We compute $\det B'_k$ via the cofactor expansion of its first column and for the resulting determinant, we let

$$D_{r,k} = \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_0^{r-1} & \dots & a_{k-1}^{r-1} & a_{k+1}^{r-1} & \dots & a_n^{r-1} \\ a_0^{r+1} & \dots & a_{k-1}^{r+1} & a_{k+1}^{r+1} & \dots & a_n^{r+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_0^n & \dots & a_{k-1}^n & a_{k+1}^n & \dots & a_n^n \end{vmatrix}.$$

We have now obtained the solutions $x_k^{(n)}$ of the truncated system in terms of the determinant $D_{r,k}$:

$$\begin{aligned} x_k^{(n)} &= \frac{(-1)^k (-1)^r D_{r,k}}{D} \\ &= \frac{(-1)^{k+r} D_{r,k}}{D}. \end{aligned}$$

The next step is to find an expression for $D_{r,k}$ and hence for $x_k^{(n)}$ directly in terms of the entries of the Vandermonde matrix.

We begin by considering the determinant

$$E_k = \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n & b \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_0^r & \dots & a_{k-1}^r & a_{k+1}^r & \dots & a_n^r & b^r \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_0^n & \dots & a_{k-1}^n & a_{k+1}^n & \dots & a_n^n & b^n \end{vmatrix}$$

as a polynomial of b . On the other hand, by cofactor expansion along the last column, we find that the term b^r has coefficient $(-1)^{(r+1)+(n+1)}D_{r,k} = (-1)^{n+r}D_{r,k}$. Also, from the proof of Theorem 2.1.3, it follows that

$$\begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n & b \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_0^r & \dots & a_{k-1}^r & a_{k+1}^r & \dots & a_n^r & b^r \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_0^n & \dots & a_{k-1}^n & a_{k+1}^n & \dots & a_n^n & b^n \end{vmatrix} = (b-a_0) \dots (b-a_{k-1})(b-a_{k+1}) \dots (b-a_n) \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_0^{n-1} & \dots & a_{k-1}^{n-1} & a_{k+1}^{n-1} & \dots & a_n^{n-1} \end{vmatrix}.$$

Applying Lemma 3.0.5 to the equation above, we have two expressions for the coefficient of b^r and we can compare them to find an expression for $D_{r,k}$ and hence for $x_k^{(n)}$:

$$\begin{aligned} D_{r,k} &= (-1)^{2(n+r)} D_{r,k} \\ &= (-1)^{n+r} (\text{coefficient of } b^r \text{ by expansion along the last column of } E_k) \\ &= (-1)^{n+r} \left(\sum_{\psi \in C_{n-r,k}^n} (-1)^{n-r} a_{\psi(1)} \dots a_{\psi(n-r)} \right) F_k, \end{aligned}$$

where

$$F_k = \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n \\ \vdots & & \vdots & \vdots & & \vdots \\ a_0^{n-1} & \dots & a_{k-1}^{n-1} & a_{k+1}^{n-1} & \dots & a_n^{n-1} \end{vmatrix}.$$

Also,

$$\begin{aligned} x_k^{(n)} &= \frac{(-1)^{k+r} D_{r,k}}{D} \\ &= \frac{(-1)^{k+n} (\sum_{\psi \in C_{n-r,k}^n} (-1)^{n-r} a_{\psi(1)} \dots a_{\psi(n-r)})}{D} F_k. \end{aligned}$$

Notice that

$$\begin{aligned} D &= \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^r & a_1^r & \dots & a_n^r \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} \\ &= (-1)^{n-k} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 & 1 \\ a_0 & \dots & a_{k-1} & a_{k+1} & \dots & a_n & a_k \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_0^n & \dots & a_{k-1}^n & a_{k+1}^n & \dots & a_n^n & a_k^n \end{vmatrix} \\ &= (-1)^{n-k} (a_k - a_0) \dots (a_k - a_{k-1}) (a_k - a_{k+1}) \dots (a_k - a_n) F_k. \end{aligned}$$

The former equality is obtained by exchanging the k -th column within the truncated system $(n - k)$ times. The latter equality is obtained from the well-known formula for Vandermonde determinants from Theorem 2.1.3.

Therefore,

$$\begin{aligned}
x_k^{(n)} &= \frac{(-1)^{k+n}}{(-1)^{n-k}} \frac{\sum_{\psi \in C_{n-r,k}^n} (-1)^{n-r} a_{\psi(1)} \dots a_{\psi(n-r)}}{\prod_{j=0, j \neq k}^n (a_k - a_j)} \\
&= \frac{(-1)^{k+n}}{(-1)^{n-k} (-1)^n} \frac{\sum_{\psi \in C_{n-r,k}^n} (-1)^{n-r} a_{\psi(1)} \dots a_{\psi(n-r)}}{\prod_{j=0, j \neq k}^n (a_j - a_k)} \\
&= (-1)^n \frac{\sum_{\psi \in C_{n-r,k}^n} (-1)^{n-r} a_{\psi(1)} \dots a_{\psi(n-r)}}{\prod_{j=0, j \neq k}^n (a_j - a_k)} \\
&= (-1)^r \frac{\sum_{\psi \in C_{n-r,k}^n} a_{\psi(1)} \dots a_{\psi(n-r)}}{\prod_{j=0, j \neq k}^n a_j \prod_{j=0, j \neq k}^n (1 - \frac{a_k}{a_j})} \\
&= (-1)^r \sum_{\psi \in C_{n-r,k}^n} \left(\prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \right) \frac{a_{\psi(1)} \dots a_{\psi(n-r)}}{\prod_{j=0, j \neq k}^n a_j} \\
&= (-1)^r \sum_{\psi \in C_{r,k}^n} \left(\prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \right) \frac{1}{a_{\psi(1)} \dots a_{\psi(r)}} \\
&= (-1)^r \left(\prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \right) \left(\sum_{\psi \in C_{r,k}^n} \frac{1}{a_{\psi(1)} \dots a_{\psi(r)}} \right).
\end{aligned}$$

Thus, formally,

$$x_k^{(n)} \xrightarrow[n]{\infty} (-1)^r \left(\prod_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{a_k}{a_j}} \right) \left(\sum_{\psi \in C_{r,k}} \frac{1}{a_{\psi(1)} \dots a_{\psi(r)}} \right),$$

where $C_{r,k}$ denotes all injective functions

$$\psi : \{1, 2, \dots, r\} \longrightarrow \{0, 1, \dots, k-1, k+1, \dots\}.$$

As before, we exclude injective functions whose range set is a permutation of another injective function. We will now show that both the infinite product and infinite series in the last expression are convergent. Consider the infinite product

$$\prod_{\substack{j=0 \\ j \neq k}}^{\infty} \left(1 - \frac{a_k}{a_j} \right).$$

This product is convergent since $\sum_{j=1, j \neq k}^{\infty} \left| \frac{a_k}{a_j} \right|$ is convergent. See Theorem 2.3.2 or Proposition 2.3.4. Since the number of functions in $C_{r,k}$ are countable and since for each $j \in \mathbb{N}$, $j \neq k$, there are $\psi_j \in C_{r,k}$ and $\phi_j \in C_{r-1,k}$ such that

$$\frac{1}{|a_{\psi_j(1)} \dots a_{\psi_j(r)}|} = \frac{1}{|a_j|} \frac{1}{|a_{\phi_j(1)} \dots a_{\phi_j(r-1)}|},$$

we may be sure that

$$\sum_{\psi \in C_{r,k}} \frac{1}{|a_{\psi(1)} \dots a_{\psi(r)}|} \leq \left(\sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{|a_j|} \right) \left(\sum_{\psi \in C_{r-1,k}} \frac{1}{|a_{\psi(1)} \dots a_{\psi(r-1)}|} \right).$$

We will now prove by induction on r that

$$\sum_{\psi \in C_{r,k}} \frac{1}{|a_{\psi(1)} \dots a_{\psi(r)}|} \leq \alpha^r.$$

Let $r = 1$ and notice that

$$\sum_{\psi \in C_{1,k}} \frac{1}{|a_{\psi(1)}|} \leq \sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{|a_j|} \leq \sum_{j=0}^{\infty} \frac{1}{|a_j|} = \alpha^1.$$

Assume the statement holds true for $r = p$. Then,

$$\sum_{\psi \in C_{p,k}} \frac{1}{|a_{\psi(1)} \dots a_{\psi(p)}|} \leq \alpha^p.$$

For $r = p + 1$ it follows from the induction hypothesis that

$$\begin{aligned} \sum_{\psi \in C_{p+1,k}} \frac{1}{|a_{\psi(1)} \dots a_{\psi(p+1)}|} &\leq \left(\sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{|a_j|} \right) \left(\sum_{\psi \in C_{p,k}} \frac{1}{|a_{\psi(1)} \dots a_{\psi(p)}|} \right) \\ &\leq \left(\sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{|a_j|} \right) \alpha^p \\ &= \alpha^{p+1}. \end{aligned}$$

This shows that each coordinate $x_k^{(n)}$ of the solution of the original truncated matrix equation (with special right hand side) approaches a limit as $n \rightarrow \infty$.

We will have to verify whether this limit approaches a solution of the infinite system and also check the general case when the right hand side $d \in l_1(\alpha)$. We relabel $x_k^{(n)}$ as $x_k^{(n),r}$ to show dependence on r , i.e.,

$$\begin{aligned} x_k^{(n),r} &= (-1)^r \left(\prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \right) \left(\sum_{\psi \in C_{r,k}^n} \frac{1}{a_{\psi(1)} \cdots a_{\psi(r)}} \right) \\ &\xrightarrow[n]{\infty} (-1)^r \left(\prod_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{a_k}{a_j}} \right) \left(\sum_{\psi \in C_{r,k}} \frac{1}{a_{\psi(1)} \cdots a_{\psi(r)}} \right) \\ &:= x_k^{[r]}. \end{aligned}$$

Hereafter we simply write $x_k^{(n)}$ and x_k instead of $x_k^{(n),0}$ and $x_k^{[0]}$ respectively. So,

$$\begin{aligned} x_k^{(n)} &= (-1)^n \left(\prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \right) \frac{\prod_{j=0, j \neq k}^n a_j}{\prod_{j=0, j \neq k}^n a_j} \\ &= (-1)^n \prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \end{aligned}$$

The first equality follows from the fact that $r = 0$ and we replaced the usual summation expression with $\prod_{j=0, j \neq k}^n a_j$, which is the expression for the coefficient of b^r in the product $(b - a_0) \cdots (b - a_{k-1})(b - a_{k+1}) \cdots (b - a_n)$. We observe that if $n \geq k$, then

$$\begin{aligned} |x_k^{(n)}| &= \prod_{\substack{j=0 \\ j \neq k}}^n \left| \frac{1}{1 - \frac{a_k}{a_j}} \right| \\ &= \prod_{j=0}^{k-1} \left| \frac{1}{1 - \frac{a_k}{a_j}} \right| \prod_{j=k+1}^n \left| \frac{1}{1 - \frac{a_k}{a_j}} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \prod_{j=0}^{k-1} \frac{1}{\left| \frac{a_k}{a_j} \right| - 1} \prod_{j=k+1}^n \frac{1}{1 - \left| \frac{a_k}{a_j} \right|} \\
&\leq \left| \prod_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{1}{1 - \left| \frac{a_k}{a_j} \right|} \right| := b_k.
\end{aligned}$$

(The first inequality is obtained by applying the well known inequality $|x - y| \geq ||x| - |y||$).

Note that if the a_j 's are positive, then $|x_k| = |x_k^{[0]}| = b_k$. If we consider the truncated system of equations for the case when the right-hand side d has its i -th component $d_i = \delta_{ir}$, we find that

$$\sum_{k=0}^n a_k^i x_k^{(n),r} = \delta_{ir}$$

where δ_{ir} is the Kronecker delta and $0 \leq i, r \leq n$, $n \in \mathbb{N}$.

We want to extend the equation above to the infinite case where we still use δ_{ir} , but keeping in mind that $0 \leq i, r$ with no upper bound.

We will prove that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k^i x_k^{(n),r}$$

exists by showing that it is dominated by some suitable convergent series. Clearly,

$$\begin{aligned}
|a_k^i x_k^{(n),r}| &= |a_k^i| |x_k^{(n),r}| = |a_k^i| \left| \prod_{\substack{j=0 \\ j \neq k}}^n \frac{1}{1 - \frac{a_k}{a_j}} \right| \left| \sum_{\psi \in C_{r,k}^n} \frac{1}{a_{\psi(1)} \cdots a_{\psi(r)}} \right| \\
&= |a_k^i| |x_k^{(n)}| \left| \sum_{\psi \in C_{r,k}^n} \frac{1}{a_{\psi(1)} \cdots a_{\psi(r)}} \right| \\
&\leq |a_k^i| |x_k^{(n)}| \alpha^r \\
&= |a_k^i x_k^{(n)}| \alpha^r.
\end{aligned}$$

Also, $|a_k^i x_k^{(n)}| \alpha^r \leq |a_k^i| b_k \alpha^r = |a_k^i b_k| \alpha^r$. Combining the inequalities above, we get $|a_k^i x_k^{(n),r}| \leq |a_k^i b_k| \alpha^r$ and also $|\chi(k \leq n) a_k^i x_k^{(n),r}| \leq |a_k^i b_k| \alpha^r$ where $\chi(k \leq$

n) denotes the characteristic function which is equal to one for $k \leq n$ and zero otherwise. Let $f_n(k) = \chi(k \leq n) a_k^i x_k^{(n),r}$ and $g(k) = |a_k^i b_k| \alpha^r \in l_1$. This gives us $|f_n(k)| \leq g(k)$ and because $\lim_{n \rightarrow \infty} f_n(k)$ exists, all the assumptions of the dominated convergence theorem of Lebesgue (see [4]) are satisfied and we conclude that

$$\begin{aligned} \delta_{ir} &= \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k^i x_k^{(n),r} = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \chi(k \leq n) a_k^i x_k^{(n),r} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{N}} f_n(k) dm = \int_{\mathbb{N}} \lim_{n \rightarrow \infty} f_n(k) dm \\ &= \sum_{k=0}^{\infty} \lim_{n \rightarrow \infty} \chi(k \leq n) a_k^i x_k^{(n),r} = \sum_{k=0}^{\infty} a_k^i x_k^{[r]} \end{aligned}$$

where we integrated with respect to the discrete measure.

Take any right hand side $d \in l_1(\alpha)$. Let $y^{(n)} = (y_0^{(n)}, \dots, y_n^{(n)}) \in \mathbb{C}^{n+1}$ be the unique solution of the truncated system $A_n y^{(n)} = P_n d$.

Considering

$$\begin{aligned} \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{pmatrix} & \left[d_0 \begin{pmatrix} x_0^{(n),0} \\ \vdots \\ x_n^{(n),0} \end{pmatrix} + \dots + d_n \begin{pmatrix} x_0^{(n),n} \\ \vdots \\ x_n^{(n),n} \end{pmatrix} \right] \\ &= d_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + d_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + d_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} d_0 \\ \vdots \\ d_n \end{pmatrix} \\ &= P_n d, \end{aligned}$$

it follows by linear combination that

$$y_k^{(n)} = \sum_{r=0}^n d_r x_k^{(n),r}.$$

We know that $|x_k^{(n),r}| \leq b_k \alpha^r$ since we arrived at $(|a_k^i x_k^{(n),r}| \leq |a_k^i b_k| \alpha^r, \text{p.29})$ and so $|\chi(r \leq n) d_r x_k^{(n),r}| \leq |d_r| b_k \alpha^r$. Let $f_n(r) = \chi(r \leq n) d_r x_k^{(n),r}$ and $g(r) = |d_r| b_k \alpha^r$. Now, $|f_n(r)| \leq g(r)$, and again we apply Lebesgue's dominated convergence theorem and find that

$$\begin{aligned} \lim_{n \rightarrow \infty} y_k^{(n)} &= \lim_{n \rightarrow \infty} \sum_{r=0}^n d_r x_k^{(n),r} = \lim_{n \rightarrow \infty} \sum_{r=0}^{\infty} \chi(r \leq n) d_r x_k^{(n),r} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{N}} f_n(r) dm = \int_{\mathbb{N}} \lim_{n \rightarrow \infty} f_n(r) dm \\ &= \sum_{r=0}^{\infty} d_r x_k^{[r]} := y_k. \end{aligned}$$

Furthermore,

$$\begin{aligned} \sum_{k=0}^{\infty} a_k^i y_k &= \sum_{k=0}^{\infty} a_k^i \sum_{r=0}^{\infty} d_r x_k^{[r]} \\ &= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} a_k^i d_r x_k^{[r]} \\ &= \sum_{r=0}^{\infty} d_r \sum_{k=0}^{\infty} a_k^i x_k^{[r]} \\ &= \sum_{r=0}^{\infty} d_r \delta_{ir} = d_i \end{aligned}$$

for any $i \geq 0$. We are allowed to change the order of summation in the second equality above because

$$\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} |a_k^i d_r x_k^{[r]}| \leq \sum_{k=0}^{\infty} |a_k^i b_k| \sum_{r=0}^{\infty} |d_r| \alpha^r < \infty.$$

The latter estimate also shows that $y = (y_0, y_1, \dots) \in D_{abs}(A)$ and in particular, $y \in l_1$, with $i = 0$. All that remains to be proved is that $\lim_{n \rightarrow \infty} y^{(n)} = y$ in the l_1 -norm. Note that $|y_k^{(n)}| \leq \sum_{r=0}^{\infty} |d_r x_k^{(n),r}| \leq \sum_{r=0}^{\infty} |d_r b_k \alpha^r|$. Thus we have,

$$|y_k^{(n)} - y_k| \leq |y_k^{(n)}| + |y_k| \leq b_k \sum_{r=0}^{\infty} |d_r| \alpha^r + |y_k|,$$

and

$$\sum_{k=0}^{\infty} \left(b_k \sum_{r=0}^{\infty} |d_r| \alpha^r + |y_k| \right) = \sum_{k=0}^{\infty} b_k \sum_{r=0}^{\infty} |d_r| \alpha^r + \sum_{k=0}^{\infty} |y_k| < \infty$$

since $y \in l_1, d \in l_1(\alpha)$ and by the assumption of Theorem 3.0.6 with $i = 0$. In these calculations we also identify $y_k^{(n)} \in \mathbb{C}^{n+1}$ with its imbedded version $y_k^{(n)} = (y_0^{(n)}, y_1^{(n)}, \dots, y_n^{(n)}, 0, 0, 0, \dots) \in \omega$. We apply the dominated convergence theorem of Lebesgue once more to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^n |y_k^{(n)} - y_k| &= \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \chi(k \leq n) |y_k^{(n)} - y_k| \\ &= \sum_{k=0}^{\infty} \lim_{n \rightarrow \infty} \chi(k \leq n) |y_k^{(n)} - y_k| = 0. \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |y_k^{(n)} - y_k| = \lim_{n \rightarrow \infty} \sum_{k=0}^n |y_k^{(n)} - y_k| + \lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} |y_k| = 0.$$

This shows that $\lim_{n \rightarrow \infty} y^{(n)} = y$ in the l_1 -norm and proves the theorem. $\square$

Until now, we do not have much information on what the solution space in l_1 looks like. We do not even know if the solutions in l_1 are unique. However, it is clear that for $d \in l_1(\alpha)$, the choice of $y \in l_1$, such that both $Ay = d$ and $y = l_1\text{-}\lim_{n \rightarrow \infty} A_n^{-1} P_n d$, is unique because of the uniqueness of the limit. Corollary 2.1.4 already gives the existence of a (not necessarily unique) solution (in $D_{abs}(A)$) of the equation $Ax = d$, without any constraints on the Vandermonde matrix A . The significance of the result in Theorem 3.0.6 of course is that it gives a method to construct such a solution, living in l_1 . Let $D_\alpha(A)$ be the subspace of l_1 given by

$$D_\alpha(A) = \{y \in l_1 : Ay \in l_1(\alpha)\}.$$

We do not know whether $A : D_\alpha(A) \rightarrow l_1(\alpha)$ is injective. It is interesting to note that since $Ae_n = (1, a_n, a_n^2, \dots)$, it is clear that with the assumptions of Theorem 3.0.6, $Ae_n \notin l_1(\alpha)$; i.e. $e_n \notin D_\alpha(A)$. Let us assume that there exists a

$z = \{\beta_i\} \in D_\alpha(A)$ with infinitely many non-zero entries, such that $P_n z \in D_\alpha(A)$ for large n . Then we would have $\beta_n e_n = P_n z - P_{n-1} z \in D_\alpha(A)$, (where n was also chosen so that $\beta_n \neq 0$) which would imply that $e_n \in D_\alpha(A)$! Thus, for all $z \in D_\alpha(A)$ which has infinitely many number of non-zero terms we have $P_n z \notin D_\alpha(A)$ for large values of n .

We have noted in the introduction that the theory of the finite section method has not really dealt with our problem in this thesis. We will elaborate on this statement by placing our situation in context with the established theory.

Let $P_n : \omega \rightarrow \omega$ be the projection defined by

$$P_n((\beta_i)_{i \geq 0}) = (\beta_0, \beta_1, \dots, \beta_n, 0, 0, \dots).$$

For an infinite matrix $A : \omega \rightarrow \omega$, we let $A_n = P_n A P_n|_{Im(P_n)}$. In the discussion of the finite section method for the equation $Ax = y$ ([1], p.14) where A is an infinite matrix on l_2 (defining a bounded linear operator), it is proved that the finite section method converges for A if and only if A is invertible and (A_n) is a stable sequence (where the sequence is stable if there exists an integer n_0 such that $\sup_{n \geq n_0} \|A_n^{-1} P_n\| < \infty$). Our situation now differs

largely from the one in Böttcher, since we consider an unbounded operator $A : l_1 \rightarrow l_1(\alpha)$ which is not injective on $D(A)$ and possibly also not injective on l_1 . However, under the conditions of the main theorem of this thesis, we still have the “stability property” for A , as is indicated in the following

Remark 3.0.7 *Given an infinite Vandermonde matrix satisfying the conditions of Theorem 3.0.6, we let A_n and P_n be as in the above discussion. Then*

$$M := \sup_n \|A_n^{-1} P_n\| < \infty,$$

whereby $\|A_n^{-1} P_n\|$ indicates the operator norm of $A_n^{-1} P_n : l_1(\alpha) \rightarrow l_1$.

Proof.

We identify the subspace $P_n(l_1)$ of l_1 with the $n + 1$ -dimensional space $\mathbb{C}^{n+1}$ which is endowed with the norm

$$\|(\alpha_0, \alpha_1, \dots, \alpha_n)\|_1 = \sum_{i=0}^n |\alpha_i|$$

and correspondingly we identify the matrix A_n with the $(n+1) \times (n+1)$ truncation $T_n(A)$ of A . This is a Vandermonde matrix, which is injective by Theorem 2.1.3. We let A_n^{-1} be the infinite matrix whose (i, j) -entries correspond with those of $T_n(A)^{-1}$ for $0 \leq i, j \leq n$ and are zero for $i, j = n+1, n+2, \dots$. The space $\mathbb{C}^{n+1}$ with the norm

$$\|(\alpha_0, \alpha_1, \dots, \alpha_n)\|_{1, \alpha} = \sum_{i=0}^n |\alpha_i| \alpha^i$$

is identified with the subspace $P_n(l_1(\alpha))$ of $l_1(\alpha)$. The linear operator $A_n^{-1}P_n : l_1(\alpha) \rightarrow l_1$ is bounded, since if $d = (\beta_i) \in l_1(\alpha)$, then

$$\begin{aligned} \|A_n^{-1}P_n d\|_{l_1} &= \|T_n(A)^{-1}(\beta_0, \beta_1, \dots, \beta_n)\|_1 \\ &\leq \|T_n(A)^{-1}\| \|(\beta_0, \beta_1, \dots, \beta_n)\|_{1, \alpha} \\ &= \|T_n(A)^{-1}\| \|(\beta_0, \beta_1, \dots, \beta_n, 0, 0, \dots)\|_{l_1(\alpha)} \\ &\leq \|T_n(A)^{-1}\| \|d\|_{l_1(\alpha)}. \end{aligned}$$

Also, for each $d \in l_1(\alpha)$, there exists (by Theorem 3.0.6) a $y \in l_1$ such that $Ay = d$ and $A_n^{-1}P_n d \rightarrow y$ when $n \rightarrow \infty$ (in l_1 -norm) if $n \rightarrow \infty$. This shows that the sequence $(A_n^{-1}P_n)$ is pointwise bounded on the Banach space $l_1(\alpha)$. By the Uniform Boundedness Theorem the sequence $(A_n^{-1}P_n)$ is uniformly bounded, i.e. it is a bounded subset of $\mathcal{L}(l_1(\alpha), l_1)$. Therefore

$$M := \sup_n \|A_n^{-1}P_n\| < \infty.$$

□

3.1 At Least Quadratic Growth

If the a_k 's appearing in the infinite Vandermonde matrix are given by some formula, it may be possible to derive a closed form of the product defining b_k . This makes it easier to check that the conditions of Theorem 3.0.6 are fulfilled. In order to provide examples, we need the following facts about the gamma function $\Gamma(z)$.

Proposition 3.1.1 *For any $z \in \mathbb{C}$ with $z \neq 0$ and non-negative integer k ,*

$$\begin{aligned} \frac{1}{\Gamma(1+z)} &= \frac{1}{z\Gamma(z)}, \quad \frac{1}{\Gamma(1+z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi z}, \quad \Gamma(k+1) = k!, \\ \Gamma(z) &= \lim_{n \rightarrow \infty} \frac{n!n^z}{z(z+1)\dots(z+n)}, \quad \lim_{z \rightarrow -k} (z+k)\Gamma(z) = \frac{(-1)^k}{k!} \\ &\quad \lim_{z \rightarrow -k} \frac{1}{(1 - \frac{z}{k})\Gamma(1-z)} = (-1)^{k+1}k! \end{aligned}$$

Proof.

We will now prove the last two equalities in the order they appear. Proofs of the other equalities can be found in [8].

$$\begin{aligned} &\lim_{z \rightarrow -k} (z+k)\Gamma(z) \\ &= \lim_{z \rightarrow -k} \frac{z+k}{\frac{1}{\Gamma(z)}} \\ &= \lim_{z \rightarrow -k} \frac{\frac{z+k}{z}}{\frac{1}{z\Gamma(z)}} \\ &= \lim_{z \rightarrow -k} \frac{\frac{z+k}{z}}{\frac{1}{\Gamma(z+1)}} \\ &= \lim_{z \rightarrow -k} \frac{1 + \frac{k}{z}}{\frac{\sin(\pi z)}{\pi z}} \lim_{z \rightarrow -k} \frac{1}{\Gamma(1-z)} \\ &= \lim_{z \rightarrow -k} \frac{\pi z + \pi k}{\sin(\pi z)} \frac{1}{\Gamma(1+k)} \end{aligned}$$

$$\begin{aligned}
&= \lim_{z \rightarrow -k} \frac{\pi}{\pi \cos(\pi z)} \frac{1}{k!} \quad (\text{l'Hôpital's rule}) \\
&= \frac{(-1)^k}{k!}
\end{aligned}$$

Now, for the second equality, it follows that

$$\begin{aligned}
&\frac{1}{(1 - \frac{z}{k})\Gamma(1 - z)} \\
&= \frac{1}{(1 - \frac{z}{k})(-z)\Gamma(-z)} \\
&= \frac{1}{(-z + \frac{z^2}{k})\Gamma(-z)} \\
&= \frac{1}{(\frac{-z}{k})(-z + k)\Gamma(-z)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{z \rightarrow k} \frac{1}{(1 - \frac{z}{k})\Gamma(1 - z)} \\
&= \lim_{z \rightarrow k} \left(\frac{1}{-\frac{z}{k}} \right) \lim_{z \rightarrow k} \frac{1}{(-z + k)\Gamma(-z)} \\
&= (-1) \lim_{w \rightarrow -k} \frac{1}{(w + k)\Gamma(w)} \quad (\text{Set } w = -z) \\
&= (-1) \left(\frac{(-1)^k}{k!} \right)^{-1} \\
&= \left(\frac{(-1)^{k+1}}{k!} \right)^{-1} \\
&= (-1)^{k+1} k!.
\end{aligned}$$

□

Proposition 3.1.2 *If $c_1, c_2, \dots, c_s \in \mathbb{C}$ and $\sum_{k=1}^s c_k = 0$, then*

$$\prod_{n=1}^{\infty} \prod_{k=1}^s \left(1 + \frac{c_k}{n} \right) = \prod_{k=1}^s \frac{1}{\Gamma(1 + c_k)}.$$

Proof.

$$\begin{aligned}
\prod_{k=1}^s \frac{1}{\Gamma(1+c_k)} &= \prod_{k=1}^s \frac{1}{c_k \Gamma(c_k)} \\
&= \prod_{k=1}^s e^{\gamma c_k} \prod_{n=1}^{\infty} \left(1 + \frac{c_k}{n}\right) \left(e^{-\frac{c_k}{n}}\right) \\
&= e^{\gamma(c_1+c_2+\dots+c_s)} \prod_{k=1}^s \prod_{n=1}^{\infty} \left(1 + \frac{c_k}{n}\right) \left(e^{-\frac{c_k}{n}}\right) \\
&= \prod_{n=1}^{\infty} e^{-\frac{c_1+c_2+\dots+c_s}{n}} \prod_{k=1}^s \left(1 + \frac{c_k}{n}\right) \\
&= \prod_{n=1}^{\infty} \prod_{k=1}^s \left(1 + \frac{c_k}{n}\right)
\end{aligned}$$

Note that we may interchange a convergent infinite product with a finite product. $\square$

We state and prove the next two elementary propositions for the sake of completeness.

Proposition 3.1.3 *If $z \in \mathbb{C}$, $\varepsilon_p = \cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p}$ and $p \geq 1$ is an integer, then*

$$1 - z^p = \prod_{j=1}^p (1 - \varepsilon_p^j z).$$

Proof.

For $z \in \mathbb{C}$, we have in general that

$$\begin{aligned}
\prod_{j=1}^p (1 - \varepsilon_p^j z) &= \prod_{j=1}^p \varepsilon_p^j (\varepsilon_p^{-j} - z) \\
&= (\varepsilon_p^1 \varepsilon_p^2 \dots \varepsilon_p^p) \prod_{j=1}^p (\varepsilon_p^{-j} - z) \\
&= \varepsilon_p^{\frac{p(p+1)}{2}} \prod_{j=1}^p (\varepsilon_p^{-j} - z)
\end{aligned}$$

$$\begin{aligned}
&= \cos \pi(p+1) \prod_{j=1}^p (\varepsilon_p^{-j} - z) && [De Moivre's theorem] \\
&= (-1)^{p+1} \prod_{j=1}^p (-1)(z - \varepsilon_p^{-j}) \\
&= (-1)^{2p+1} \prod_{j=1}^p (z - \varepsilon_p^{-j}) \\
&= - \prod_{j=1}^p (z - \varepsilon_p^{-j}).
\end{aligned}$$

Consider the function $f(z) = z^p - 1$, $z \in \mathbb{C}$. Then,

$$f(\varepsilon_p^{-j}) = (\varepsilon_p^{-j})^p - 1 = \frac{1}{(\varepsilon_p^p)^j} - 1 = 1 - 1 = 0$$

for $j = 1, 2, \dots, p$. The degree of the polynomial $f(z) = z^p - 1$ is p and so it completely decomposes into p linear factors by the fundamental theorem of algebra. Thus,

$$f(z) = z^p - 1 = \prod_{j=1}^p (z - \varepsilon_p^{-j})$$

$$\text{or } 1 - z^p = - \prod_{j=1}^p (z - \varepsilon_p^{-j}) = \prod_{j=1}^p (1 - \varepsilon_p^j z). \quad \square$$

Proposition 3.1.4 *Let $p \geq 1$ be an integer and ε_p as before. Then,*

$$\prod_{j=1}^{p-1} (1 - \varepsilon_p^j) = p.$$

Proof.

$$\begin{aligned}
\prod_{j=1}^{p-1} (1 - \varepsilon_p^j) &= \lim_{z \rightarrow 1} \prod_{j=1}^{p-1} (1 - \varepsilon_p^j z) = \lim_{z \rightarrow 1} \frac{\prod_{j=1}^p (1 - \varepsilon_p^j z)}{(1 - \varepsilon_p^p z)} \\
&= \lim_{z \rightarrow 1} \frac{1 - (z)^p}{1 - z} && (\text{by Proposition 3.1.3}) \\
&= \lim_{z \rightarrow 1} \frac{-pz^{p-1}}{-1} && (\text{l'Hôpital's rule}) \\
&= p.
\end{aligned}$$

In this and in the next two sections, it is more convenient to use the indices $1, 2, 3, \dots$ rather than $0, 1, 2, \dots$ as we did up to now. Thus, our matrix A and sequence $b_1, b_2, \dots$ are now built from the numbers $a_1, a_2, \dots$. Let $a_k = k^p$ be the entries for our matrix A where p is an integer and $p \geq 2$. Then, by uniform convergence of the infinite product b_k ,

$$\begin{aligned} \frac{1}{b_k} &= \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| 1 - \frac{k^p}{n^p} \right| = \lim_{z \rightarrow k} \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| 1 - \frac{z^p}{n^p} \right| \\ &= \left| \lim_{z \rightarrow k} \frac{1}{1 - \left(\frac{z}{k}\right)^p} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^p \right) \right|. \end{aligned}$$

We apply Propositions 3.1.1 to 3.1.4 and write

$$\begin{aligned} \lim_{z \rightarrow k} \frac{1}{1 - \left(\frac{z}{k}\right)^p} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^p \right) &= \lim_{z \rightarrow k} \frac{1}{\prod_{j=1}^p \left(1 - \varepsilon_p^j \frac{z}{k} \right)} \prod_{n=1}^{\infty} \prod_{j=1}^p \left(1 - \varepsilon_p^j \frac{z}{n} \right) \\ &= \lim_{z \rightarrow k} \frac{1}{\prod_{j=1}^p \left(1 - \varepsilon_p^j \frac{z}{k} \right) \prod_{j=1}^p \Gamma(1 - \varepsilon_p^j z)} \\ &= \frac{1}{\prod_{j=1}^{p-1} (1 - \varepsilon_p^j) \prod_{j=1}^{p-1} \Gamma(1 - \varepsilon_p^j k)} \lim_{z \rightarrow k} \frac{1}{\left(1 - \frac{z}{k} \right) \Gamma(1 - z)} \\ &= \frac{(-1)^{k+1} k!}{p} \prod_{j=1}^{p-1} \frac{1}{\Gamma(1 - \varepsilon_p^j k)}. \end{aligned}$$

Suppose that p is even, that is, $p = 2q$ where $q \geq 1$ is an integer. In this case it follows that

$$\begin{aligned} \prod_{j=1}^{p-1} \frac{1}{\Gamma(1 - \varepsilon_p^j k)} &= \frac{1}{\Gamma(1 + k)} \prod_{j=1}^{q-1} \frac{1}{\Gamma(1 - \varepsilon_p^j k)} \prod_{j=q+1}^{2q-1} \frac{1}{\Gamma(1 - \varepsilon_p^j k)} \\ &= \frac{1}{\Gamma(1 + k)} \prod_{j=1}^{q-1} \frac{1}{\Gamma(1 - \varepsilon_p^j k)} \prod_{w=1}^{q-1} \frac{1}{\Gamma(1 - \varepsilon_p^{w+q} k)} \quad (\text{Set } w = j - q) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1+k)} \prod_{j=1}^{q-1} \frac{1}{\Gamma(1-\varepsilon_p^j k)} \prod_{w=1}^{q-1} \frac{1}{\Gamma(1+\varepsilon_p^w k)} \quad (\text{Note that } \varepsilon_p^{w+q} = -\varepsilon_p^w) \\
&= \frac{1}{\Gamma(1+k)} \prod_{j=1}^{q-1} \frac{1}{\Gamma(1-\varepsilon_p^j k) \Gamma(1+\varepsilon_p^j k)} \\
&= \frac{1}{\Gamma(1+k)} \prod_{j=1}^{q-1} \frac{\sin(\pi \varepsilon_p^j k)}{\pi \varepsilon_p^j k}. \quad (\text{Proposition 3.1.1})
\end{aligned}$$

Recall that

$$\frac{1}{b_k} = \left| \lim_{z \rightarrow k} \frac{1}{1 - \left(\frac{z}{k}\right)^p} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^p\right) \right| = \left| \frac{(-1)^{k+1} k!}{p} \prod_{j=1}^{p-1} \frac{1}{\Gamma(1 - \varepsilon_p^j k)} \right|.$$

Then we have

$$b_k = \left| \frac{p}{(-1)^{k+1} k!} \prod_{j=1}^{p-1} \Gamma(1 - \varepsilon_p^j k) \right| = \frac{p}{k!} \prod_{j=1}^{p-1} |\Gamma(1 - \varepsilon_p^j k)| \quad (3.1)$$

$$= \frac{p |\Gamma(1+k)| \prod_{j=1}^{q-1} |\pi \varepsilon_p^j k|}{k! \prod_{j=1}^{q-1} |\sin(\pi \varepsilon_p^j k)|} \quad (3.2)$$

$$= \frac{p(\pi k)^{q-1} \prod_{j=1}^{q-1} |\varepsilon_p^j|}{\prod_{j=1}^{q-1} |\sin(\pi \varepsilon_p^j k)|} \quad (3.3)$$

$$= \frac{p(\pi k)^{q-1}}{\prod_{j=1}^{q-1} |\sin(\pi \varepsilon_p^j k)|}. \quad (3.4)$$

The first equality is valid for all integers $p \geq 2$, while the second equality is only valid when p is even. In particular, for $p = 2$, we have $b_k = 2$.

Proposition 3.1.5 *For $p = 3$ we have*

$$b_k = \frac{3}{k!} \left(\prod_{j=0}^k |j + \varepsilon_3 k| \right) \left| \frac{\pi}{\sin \pi(1 - \varepsilon_3 k)} \right|,$$

and there exists a constant $d > 0$ such that $b_k < e^{-dk}$ for any k large enough.

Proof.

Since $1 - \varepsilon_3 k \in \mathbb{C} \setminus \mathbb{Z}$ for all k , we assume in the following calculations that $z \in \mathbb{C} \setminus \mathbb{Z}$. In this case it follows from Proposition 3.1.1 that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ and $\Gamma(1+z) = z\Gamma(z)$ hold for all non-integral $z \in \mathbb{C}$. By induction, we will prove that

$$\Gamma(z)\Gamma(k+2-z) = \left(\prod_{j=1}^{k+1} (j-z) \right) \frac{\pi}{\sin \pi z} \quad (3.5)$$

for $k \geq 1$. Let $k = 1$ and note that

$$\begin{aligned} \Gamma(z)\Gamma[1+(2-z)] &= \Gamma(z)(2-z)\Gamma(2-z) \\ &= \Gamma(z)(2-z)\Gamma[1+(1-z)] \\ &= \Gamma(z)(2-z)(1-z)\Gamma(1-z) \\ &= \prod_{j=1}^2 (j-z) \frac{\pi}{\sin \pi z}. \end{aligned}$$

Let $k = n$ and assume

$$\Gamma(z)\Gamma(n+2-z) = \left(\prod_{j=1}^{n+1} (j-z) \right) \frac{\pi}{\sin \pi z}.$$

For $k = n+1$, note that

$$\begin{aligned} \Gamma(z)\Gamma(n+1+2-z) &= \Gamma(z)\Gamma[1+(n+2-z)] \\ &= \Gamma(z)(n+2-z)\Gamma(n+2-z) \\ &= \left(\prod_{j=1}^{n+1} (j-z) \right) \frac{\pi}{\sin \pi z} (n+2-z) \quad (\text{assumption}) \\ &= \left(\prod_{j=1}^{n+2} (j-z) \right) \frac{\pi}{\sin \pi z}. \end{aligned}$$

This proves equation (3.5). Notice that $\varepsilon_3 + \varepsilon_3^2 = -1$; i.e. $1 - \varepsilon_3 k + 1 - \varepsilon_3^2 k = k + 2$. Take $z = 1 - \varepsilon_3 k$. Then it follows from equations

(3.1) and (3.5) that

$$b_k = \frac{3}{k!} |\Gamma(1 - \varepsilon_3 k) \Gamma(1 - \varepsilon_3^2 k)| = \frac{3}{k!} \left(\prod_{j=1}^{k+1} |j - 1 + \varepsilon_3 k| \right) \left| \frac{\pi}{\sin \pi(1 - \varepsilon_3 k)} \right|$$

since $z = 1 - \varepsilon_3 k$ is not integral. This proves the first part of the proposition.

For the estimate, we suppose that k is as large as required for the calculations to be valid. Recall that

$$|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right| \geq \frac{1}{2} ||e^{iz}| - |e^{-iz}|| = \frac{1}{2} |e^{-Imz} - e^{Imz}|$$

for any $z \in \mathbb{C}$. If $z = \pi(1 - \varepsilon_3 k)$, then $Imz = -\frac{\sqrt{3}\pi}{2}k$, thus, for an arbitrary constant $1 < c < \frac{\sqrt{3}\pi}{2}$,

$$|\sin \pi(1 - \varepsilon_3 k)| \geq \frac{1}{2} |e^{\frac{\sqrt{3}\pi}{2}k} - e^{-\frac{\sqrt{3}\pi}{2}k}| \geq \frac{1}{2} e^{ck}.$$

To verify that the last inequality holds for large k , suppose that

$$e^{ck} > |e^{\frac{\sqrt{3}\pi}{2}k} - e^{-\frac{\sqrt{3}\pi}{2}k}| > e^{\frac{\sqrt{3}\pi}{2}k} - e^{-\frac{\sqrt{3}\pi}{2}k}$$

and hence

$$e^{-\frac{\sqrt{3}\pi}{2}k} > e^{ck} (e^{(\frac{\sqrt{3}\pi}{2}-c)k} - 1).$$

This leads to a contradiction since the left hand side will go to zero and the right-hand side to infinity for large k . We also have that

$$\begin{aligned} |j + \varepsilon_3 k|^2 &= \left| j + k \cos \frac{2\pi}{3} + ik \sin \frac{2\pi}{3} \right|^2 \\ &= \left(j + k \cos \frac{2\pi}{3} \right)^2 + \left(k \sin \frac{2\pi}{3} \right)^2 \\ &= j^2 + 2jk \cos \frac{2\pi}{3} + k^2 \\ &= j^2 - jk + k^2 \\ &\leq k^2 \end{aligned}$$

for $0 \leq j \leq k$. Therefore, $|j + \varepsilon_3 k| \leq k$ and $\prod_{j=0}^k |j + \varepsilon_3 k| \leq k^{k+1}$ for $0 \leq j \leq k$.

According to Stirling's formula, $\lim_{k \rightarrow \infty} k! - k^k e^{-k} \sqrt{2\pi k} = 0$. See [12]. Because $\sqrt{2\pi k}$ is always greater than 1 for all $k > 0$, there exists a $n \in \mathbb{N}$ so that for all $k \geq n$, $k! \geq \left(\frac{k}{e}\right)^k$. Thus,

$$\begin{aligned} b_k &= \frac{3}{k!} \left(\prod_{j=0}^k |j + \varepsilon_3 k| \right) \left| \frac{\pi}{\sin \pi(1 - \varepsilon_3 k)} \right| \\ &\leq \frac{3}{k!} k^{k+1} \pi 2e^{-ck} \\ &\leq 6\pi \frac{k \cdot k^k e^k}{k^k} e^{-ck} \\ &= 6\pi k e^{(1-c)k} \\ &\leq 6\pi k e^{-dk}, \end{aligned}$$

where $0 < d < |1 - c|$, and $1 < c < \frac{\sqrt{3}\pi}{2}$ as before. Furthermore, by a suitable transformation, it is clear that we can get a $d > 0$ so that $b_k \leq e^{-dk}$ for sufficiently large k . $\square$

Lemma 3.1.6 *Let $\{e_k\}$ and $\{f_k\}$ be strictly increasing sequences of positive numbers with $k \geq 1$. Suppose that there exists a positive integer k_0 such that $\frac{f_k}{e_k} \geq \max_{1 \leq j < k} \frac{f_j}{e_j}$ for all $k \geq k_0$. Then,*

$$\left| \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{f_k}{f_j}} \right| \leq \left| \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{e_k}{e_j}} \right|$$

whenever $k \geq k_0$.

Proof.

By assumption we have that $\frac{f_k}{e_k} \geq \max_{1 \leq j < k} \frac{f_j}{e_j}$ for $k \geq k_0$. This means that $\frac{f_k}{f_j} \geq \frac{e_k}{e_j} > 1$ for $k \geq k_0$ and $1 \leq j < k$. In other words, $\left| 1 - \frac{f_k}{f_j} \right| \geq \left| 1 - \frac{e_k}{e_j} \right|$ for $k \geq k_0$ and $1 \leq j < k$. This gives,

$$\left| \prod_{j=1}^{k-1} \frac{1}{1 - \frac{f_k}{f_j}} \right| \leq \left| \prod_{j=1}^{k-1} \frac{1}{1 - \frac{e_k}{e_j}} \right|.$$

For $j > k$, we have by assumption $\frac{f_j}{e_j} \geq \max_{1 \leq k < j} \frac{f_k}{e_k}$ for $j \geq k_0$. Using the same argument as before, we have $|1 - \frac{f_k}{f_j}| \geq |1 - \frac{e_k}{e_j}|$ for $j \geq k_0$ and $1 \leq k < j$. Therefore,

$$\left| \prod_{j=k+1}^{\infty} \frac{1}{1 - \frac{f_k}{f_j}} \right| \leq \left| \prod_{j=k+1}^{\infty} \frac{1}{1 - \frac{e_k}{e_j}} \right|.$$

Together, we arrive at

$$\left| \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{f_k}{f_j}} \right| \leq \left| \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{e_k}{e_j}} \right|$$

whenever $k \geq k_0$. □

Corollary 3.1.7 *If $p \geq 3$ is any real number and $|a_k| = k^p$, then $b_k \leq e^{-dk}$ for $k \geq k_0$ with constant $d > 0$ by Proposition 3.1.5 and therefore $\sum_{k=1}^{\infty} |a_k^j b_k| < \infty$ for all non-negative integers j . All the assumptions of Theorem 3.0.6 are satisfied and therefore Theorem 3.0.6 applies.*

Proof.

Let $p = 3$. Now, $|a_k^j b_k| = (k^p)^j |b_k| \leq k^{3j} e^{-dk}$. Consider the series $\sum_{k=1}^{\infty} k^{3j} e^{-dk}$. We test for convergence of this series using the ratio test. It follows from the ratio test that

$$\lim_{k \rightarrow \infty} \frac{(k+1)^{3j} e^{-d(k+1)}}{k^{3j} e^{-dk}} = \lim_{k \rightarrow \infty} \left(\frac{k+1}{k} \right)^{3j} e^{-d} = e^{-d} < 1$$

and therefore the series is convergent. Let $p > 3$; $\frac{k^p}{k^3} = k^{p-3} > j^{p-3} = \frac{j^p}{j^3}$ for all k such that $1 \leq j < k$. This shows that $\frac{k^p}{k^3} \geq \max_{1 \leq j < k} \frac{j^p}{j^3}$. Since k^p is strictly increasing for $k > 1$, we have satisfied the assumptions for Lemma 3.1.6 and therefore

$$\left| \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{k^p}{j^p}} \right| \leq \left| \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{k^3}{j^3}} \right| = b_{k(p=3)} < e^{-dk}.$$

Notice that the first expression in the inequality above is exactly $b_{k(p>3)}$. Thus, $b_{k(p>3)} \leq e^{-dk}$, and we prove that $\sum_{k=1}^{\infty} k^{pj} e^{-dk}$ for $p > 3$ is convergent

in the same way as for $p = 3$. We have established the assumptions of Theorem 3.0.6 for Vandermonde matrices with special entries given by $a_k = k^p$ where $p \geq 3$. $\square$

We have seen that for $p = 2$ ($a_k = k^2$) the sequence $\{b_k\}$ is constant, but for $p \geq 3$ it decays exponentially. We shall prove a lemma which will enable us to show exponential decay of $\{b_k\}$ for $p > 2$.

Lemma 3.1.8 *Let $p > 2$ be a real number. Then there exists a constant $c > 1$ and an integer k_0 such that*

$$\left(\frac{k}{j}\right)^p - 1 \geq c \left[\left(\frac{k}{j}\right)^2 - 1 \right],$$

for all integers $k \geq k_0$ and $1 \leq j < k$.

Proof.

Consider the function

$$f(x) = \frac{x^p - 1}{x^2 - 1},$$

where $p > 2$. It is quite clear that if $f(x) > 1$ for all $x > 1$, then the proposition would follow as a special case. We can rewrite $f(x)$ as follows:

$$f(x) = \frac{x^p - 1}{x^2 - 1} = \frac{x^{p-2} - \frac{1}{x^2}}{1 - \frac{1}{x^2}}.$$

Since $x^{p-2} > 1$ for all $x > 1$, it follows immediately that $f(x) > 1$ for all $x > 1$. $\square$

For $p > 2$ and $a_k = k^p$, Lemma 3.1.8 yields the following:

$$\prod_{j=1}^{k-1} \left[\left(\frac{k}{j}\right)^p - 1 \right] \geq c^{k-1} \prod_{j=1}^{k-1} \left[\left(\frac{k}{j}\right)^2 - 1 \right]$$

and

$$\begin{aligned}
\frac{1}{b_k} &= \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| 1 - \frac{k^p}{j^p} \right| = \prod_{j=1}^{k-1} \left[\left(\frac{k}{j} \right)^p - 1 \right] \prod_{j=k+1}^{\infty} \left[1 - \left(\frac{k}{j} \right)^p \right] \\
&\geq c^{k-1} \prod_{j=1}^{k-1} \left[\left(\frac{k}{j} \right)^2 - 1 \right] \prod_{j=k+1}^{\infty} \left[1 - \left(\frac{k}{j} \right)^2 \right] \\
&= c^{k-1} \frac{1}{b_{k(p=2)}} = \frac{1}{2} c^{k-1},
\end{aligned}$$

since $b_k = 2$ if $p = 2$. In other words, $b_k \leq 2c^{1-k}$. Fix $y > 0$ such that $e^y = c(> 1)$; i.e. $b_k \leq 2e^{y(1-k)}$. Now let $0 < d < y$. Clearly, $\lim_{k \rightarrow \infty} 2e^{y-k(y-d)} = 0$; thus $2e^{y(1-k)}e^{dk} = 2e^{y-k(y-d)} \geq 1$ for large k is impossible. Hence, $b_k \leq 2e^{y(1-k)} < e^{-dk}$ for sufficiently large k .

We have filled the gap where $2 < p < 3$ and thus we conclude with a summary of these results in the following theorem.

Theorem 3.1.9 *For $p > 2$ and the Vandermonde matrix with entries $a_k = k^p$, there exists a real number $d > 0$ and a positive integer k_0 such that $b_k \leq e^{-dk}$ whenever $k \geq k_0$ and Theorem 3.0.6 applies.*

Remark 3.1.10 *Let $u = (1, 0, 0, \dots)$. In the case of $a_k = k^2$, according to Definition 1.0.1, the finite section method is not applicable to the system $Ax = u$, simply because it yields $x_k = 2(-1)^{k+1}$, and $(x_1, x_2, \dots) = x \notin D(A)$.*

Proof.

Let $a_k = k^2$ and u as stated in the remark. If we follow the exposition of the previous chapter with the entries of the original equation, $(Ax = u)$, starting at 1 instead of 0, we find that

$$x_k^{(n),r} = (-1)^{1-r} \left(\prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{1}{1 - \frac{a_k}{a_j}} \right) \left(\sum_{\varphi \in C_{n-r+1,k}^n} \frac{a_{\varphi(1)} \cdots a_{\varphi(n-r+1)}}{\prod_{j=1, j \neq k}^{n+1} a_j} \right)$$

and

$$x_k^{(n),1} = \left(\prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{1}{1 - \frac{a_k}{a_j}} \right).$$

It follows that

$$x_k^{[1]} = \lim_{n \rightarrow \infty} x_k^{(n),1} = \lim_{n \rightarrow \infty} \left(\prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{1}{1 - \frac{a_k}{a_j}} \right) = \left(\prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{a_k}{a_j}} \right).$$

With $a_k = k^2$ we have

$$\begin{aligned} x_k^{[1]} &= \left(\prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1}{1 - \frac{k^2}{j^2}} \right) \\ &= \prod_{j=1}^{k-1} \left(\frac{1}{1 - \frac{k^2}{j^2}} \right) \prod_{j=k+1}^{\infty} \left(\frac{1}{1 - \frac{k^2}{j^2}} \right) \\ &= (-1)^{k-1} \prod_{j=1}^{k-1} \left(\frac{1}{\frac{k^2}{j^2} - 1} \right) \prod_{j=k+1}^{\infty} \left(\frac{1}{1 - \frac{k^2}{j^2}} \right) \\ &= (-1)^{k+1} \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{1}{1 - \frac{k^2}{j^2}} \right| \\ &= (-1)^{k+1} b_{k(p=2)} = 2(-1)^{k+1}. \end{aligned}$$

Finally, $x \notin D(A)$, since $\sum_k x_k$ is divergent.

Chapter 4

The Exponential Case

In this chapter we consider the special infinite Vandermonde matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & a & a^2 & \dots \\ 1 & a^2 & a^4 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where the entries of the matrix are given by $a_{kj} = a^{kj}$ and $a \in \mathbb{C}$ with $|a| > 1$. Because it is a Vandermonde matrix we can write $a_j = a^j$ for $j = 0, 1, 2, \dots$. We will show that the finite section method applies to A in the sense of l_1 convergence for every right-hand side which is in l_∞ . Recall that in this case $\alpha = \sum_i \frac{1}{|a|^i} > 1$ since $|a| > 1$. If $d \in l_1(\alpha)$, then $\sum_r |d_r| \alpha^r < \infty$ implies that $|d_r| \alpha^r \xrightarrow[r \rightarrow \infty]{} 0$; i.e. $\{|d_r| \alpha^r\} \in c_0$. In particular, $|d_r| \leq |d_r| \alpha^r$ for all r , i.e. $\{d_r\} \in c_0 \subset l^\infty$. Thus we see that $l_1(\alpha) \subset l^\infty$ in this case and it is clear that it does not follow as a consequence of Theorem 3.0.6 that the finite section method applies to A in the case of l_1 -convergence for every $d \in l^\infty$.

Before we start applying the finite section method to this case, we first state three important results from [13] and prove a key lemma.

Theorem 4.0.11 *A basis for a Fréchet space must be a Schauder basis.*

Proposition 4.0.12 *Let X be a sequence space with basis $B = \{\delta_n\}$. Then X is a K -space if and only if B is a Schauder basis.*

Theorem 4.0.13 *Let X and Y be FH-spaces with $X \subset Y$. Then the inclusion map is continuous, that is, X has a larger topology than that of Y (on X). In particular, the topology of a FH-space is unique.*

Lemma 4.0.14 *Let $f_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j$ be a sequence of entire complex functions such that $f_n \rightarrow f$ locally uniformly on the complex plane. Then f is entire and if $f(z) = \sum_{j=0}^{\infty} c_j z^j$, $c^{(n)} = (c_0^{(n)}, c_1^{(n)}, \dots)$, $c = (c_0, c_1, \dots)$, then $c^{(n)} \rightarrow c$ in l_1 sense.*

Proof.

We know that f is entire from [10], page 32. Consider the space of all entire functions on which locally uniform convergence induces a completely metrizable vector space topology, which is also Fréchet. See [10], Section I.45. Let E denote the space of Taylor coefficients of entire functions, that is, $E = \{(c_0, c_1, \dots) \in \omega \mid \sum_j c_j z^j \text{ is entire}\}$. The one-to-one linear mapping $(c_0, c_1, \dots) \mapsto \sum_j c_j z^j$ equips E with a completely metrizable topology τ , in a way that for $c^{(n)}$ and c in E we have $c^{(n)} \rightarrow c$ in τ -sense if and only if $\sum_j c_j^{(n)} z^j \rightarrow \sum_j c_j z^j$ locally uniformly in $\mathbb{C}$ since the coefficients of entire functions are unique. Hence, E and the space of entire functions are isometric vector spaces and therefore E is also a Fréchet space. Now, $E \subset l_1$: Take an arbitrary $c \in E$. This means that $z \mapsto \sum_j c_j z^j$ is an entire function. Choose $z = 1$. Since a power series is also absolutely convergent, $\sum_j |c_j| < \infty$ and thus $c \in l_1$.

Consider (E, τ) . For each $e_n = (0, 0, \dots, 1, 0, 0, \dots) \in l_1$ we have that

$$\begin{aligned} f(z) &= 0 \cdot z^0 + 0 \cdot z^1 + \dots + 1 \cdot z^n + 0 + \dots \\ &= z^n \end{aligned}$$

is an entire function. Thus, $e_n \in E$ for each $n \in \mathbb{N}$. We show that $\{e_n : n \in \mathbb{N}\}$ is a basis for E : Let $x = (x_0, x_1, \dots) \in E$. We know that the power series $\sum_{i=0}^{\infty} x_i z^i$ converges for all $z \in \mathbb{C}$. Moreover, since we know that this convergence is locally uniform at every $z \in \mathbb{C}$, (see [7]), it follows that $\sum_{i=0}^n x_i z^i$ converges to $\sum_{i=0}^{\infty} x_i z^i$ locally uniformly as $n \rightarrow \infty$. Put $x^{(n)} = (x_0, x_1, x_2, \dots, x_n, 0, 0, \dots)$ and recall that $x = (x_0, x_1, x_2, \dots)$. By definition of the topology τ on E , it follows that $x = \tau\text{-}\lim_{n \rightarrow \infty} x^{(n)}$. In

other words,

$$x = \sum_{i=0}^{\infty} x_i e_i \text{ in } (E, \tau).$$

This shows that $\{e_n | n \in \mathbb{N}\}$ is a basis for E . Since E is a Fréchet space, its basis is Schauder by Theorem 4.0.11. Proposition 4.0.12 shows that E is a K -space since its basis is Schauder.

But, it is well known that l_1 is also an F -space as well as a K -space. Recall that the space ω of all scalar sequences is endowed with the product topology, i.e. the smallest topology such that the coordinate projections $P_n(x) = x_n$, $x = (x_j) \in \omega$, are continuous. This topology, which is defined by the paranorm $\|x\| = \sum_i \frac{1}{2^i} \frac{|x_i|}{1+|x_i|}$, (see [13]), is a Hausdorff topology. Clearly we now have two Fréchet K -spaces (E, τ) and l_1 , which are subspaces of ω ; in fact they are both continuously imbedded into ω since the projections P_n are continuous with respect to both the topologies of l_1 and E . This means that (E, τ) and $(l_1, \|\cdot\|_1)$ are both FK-spaces, where E is a subspace of l_1 . By Theorem 4.0.13 the inclusion is continuous. In other words, if $c^{(n)} \rightarrow c$ in (E, τ) as $n \rightarrow \infty$, then $c^{(n)} \rightarrow c$ in $(l_1, \|\cdot\|_1)$ as $n \rightarrow \infty$. □

Consider the non-homogeneous system $Ag = d$,

$$\begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & a & a^2 & \dots \\ 1 & a^2 & a^4 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \end{pmatrix}$$

and its truncated version $A_n g^{(n)} = P_n d$. To solve this system we have to find a polynomial $q_n(z) = \sum_{k=0}^n g_k^{(n)} z^k$ such that $q_n(a^j) = d_j$ for $0 \leq j \leq n$; that is, the unique solution of the truncated equation is given by the coefficients of the Lagrange interpolating polynomial (see [6])

$$q_n(z) = \sum_{k=0}^n \left(d_k \prod_{\substack{j=0 \\ j \neq k}}^n \frac{z - a^j}{a^k - a^j} \right).$$

If $p_n(z) = \prod_{j=0}^n (1 - \frac{z}{a^j})$, then $p_n(z) \xrightarrow[n \rightarrow \infty]{} e(z) = \prod_{j=0}^{\infty} (1 - \frac{z}{a^j}) = \sum_{j=0}^{\infty} e_j z^j$ locally uniformly on the complex plane. Also, by applying Theorem 2.3.6

with $m = 0$ we know that $e(z)$ is an entire function admitting a simple root at each a^j . We now rewrite q_n as follows. Consider the expression $\prod_{j=0, j \neq k}^n \frac{z-a^j}{a^k-a^j}$. Then,

$$\begin{aligned}
\prod_{\substack{j=0 \\ j \neq k}}^n \left(\frac{z-a^j}{a^k-a^j} \right) &= (-1)^n \prod_{\substack{j=0 \\ j \neq k}}^n \left(\frac{z}{a^j} - 1 \right) (-1)^n \frac{1}{\prod_{j=0, j \neq k}^n \left(\frac{a^k}{a^j} - 1 \right)} \\
&= \prod_{\substack{j=0 \\ j \neq k}}^n \left(1 - \frac{z}{a^j} \right) \frac{1}{\prod_{j=0, j \neq k}^n \left(1 - \frac{a^k}{a^j} \right)} \\
&= \frac{1}{1 - \frac{z}{a^k}} \prod_{j=0}^n \left(1 - \frac{z}{a^j} \right) \lim_{w \rightarrow a^k} \left(1 - \frac{w}{a^k} \right) \prod_{j=0}^n \left(\frac{1}{1 - \frac{w}{a^j}} \right) \\
&= \frac{p_n(z)}{\left(1 - \frac{z}{a^k} \right)} \frac{1}{\lim_{w \rightarrow a^k} \frac{p_n(w)}{1 - \frac{w}{a^k}}}
\end{aligned}$$

and then $q_n(z) = \sum_{k=0}^n d_k \left(\frac{p_n(z)}{1 - \frac{z}{a^k}} \right) \frac{1}{\lim_{w \rightarrow a^k} \frac{p_n(w)}{1 - \frac{w}{a^k}}}$. Also,

$$\begin{aligned}
\frac{p_n(z)}{\left(1 - \frac{z}{a^k} \right)} \frac{1}{\lim_{w \rightarrow a^k} \frac{p_n(w)}{1 - \frac{w}{a^k}}} &= \frac{p_n(z)}{1 - \frac{z}{a^k}} \lim_{w \rightarrow a^k} \frac{1 - \frac{w}{a^k}}{p_n(w)} \\
&= \frac{p_n(z)}{\frac{z}{a^k} - 1} \lim_{w \rightarrow a^k} \frac{\frac{w}{a^k} - 1}{p_n(w)} \\
&= \frac{p_n(z)}{z - a^k} \lim_{w \rightarrow a^k} \frac{w - a^k}{p_n(w)} \\
&= \frac{p_n(z)}{z - a^k} \lim_{w \rightarrow a^k} \frac{w - a^k}{p_n(w) - p_n(a^k)} \\
&= \frac{p_n(z)}{z - a^k} \frac{1}{p'_n(a^k)}.
\end{aligned}$$

Again replacing expressions for $q_n(z)$ we see that $q_n(z) = \sum_{k=0}^n d_k \frac{p_n(z)}{z - a^k} \frac{1}{p'_n(a^k)}$.

Notice that $p_{n+1}(z) = \left(1 - \frac{z}{a^{n+1}} \right) p_n(z)$. Thus,

$$p'_{n+1}(z) = p'_n(z) \left(1 - \frac{z}{a^{n+1}} \right) - \frac{1}{a^{n+1}} p_n(z) = p'_n(z) - \frac{1}{a^{n+1}} \left(p'_n(z) z + p_n(z) \right),$$

and therefore,

$$\begin{aligned}
p'_{n+1}(a^k) &= p'_n(a^k) - \frac{1}{a^{n+1}}(p'_n(a^k)a^k + p_n(a^k)) = p'_n(a^k) - \frac{1}{a^{n+1}}(p'_n(a^k)a^k) \\
&= p'_n(a^k) \left(1 - \frac{a^k}{a^{n+1}}\right)
\end{aligned}$$

for $0 \leq k \leq n$. This gives

$$\begin{aligned}
|p'_n(a^k)| &= \left| p'_{n-1}(a^k) \left(1 - \frac{a^k}{a^n}\right) \right| \\
&= \left| p'_{n-2}(a^k) \left(1 - \frac{a^k}{a^n}\right) \left(1 - \frac{a^k}{a^{n-1}}\right) \right| \\
&= \dots
\end{aligned}$$

and applying the above formula recursively we arrive at

$$\begin{aligned}
|p'_n(a^k)| &= \left| p'_k(a^k) \left(1 - \frac{a^k}{a^n}\right) \dots \left(1 - \frac{a^k}{a^{k+1}}\right) \right| \\
&= \left| p'_k(a^k) \left(1 - \frac{1}{a^{n-k}}\right) \dots \left(1 - \frac{1}{a}\right) \right| \\
&= |p'_k(a^k)| \left| \prod_{j=1}^{n-k} \left(1 - \frac{1}{a^j}\right) \right| \\
&\geq |p'_k(a^k)| \left| \prod_{j=1}^{n-k} \left(1 - \frac{1}{|a^j|}\right) \right| \\
&= |p'_k(a^k)| \prod_{j=1}^{n-k} \left(1 - \frac{1}{|a^j|}\right) \\
&\geq |p'_k(a^k)| \prod_{j=1}^{\infty} \left(1 - \frac{1}{|a^j|}\right) \\
&= C |p'_k(a^k)|
\end{aligned}$$

for $n \geq k + 1$ because $\prod_{j=1}^{\infty} \left(1 - \frac{z}{a^j}\right)$ is locally uniformly convergent on $\mathbb{C}$. The constant C only depends on a . Since $|a| > 1$, we see that $\{a^k\}$ is an unbounded sequence with no accumulation points. Therefore, each $z \in \mathbb{C} \setminus \{1, a, a^2, \dots\}$ has a neighbourhood whose intersection with $\{a^k\}$ is

empty.

Suppose $|z - a^k| > \delta$ for all k , then for $k \leq n - 1$

$$\begin{aligned} \left| \frac{d_k}{(z - a^k)p'_n(a^k)} \right| &\leq \frac{|d_k|}{\delta C |p'_k(a^k)|} \\ &= \frac{|d_k|}{\delta C |p'_{k-1}(a^k) - \frac{1}{a^k}(p'_{k-1}(a^k)a^k + p_{k-1}(a^k))|} \\ &= \frac{|d_k a^k|}{\delta C |p_{k-1}(a^k)|}. \end{aligned}$$

Fix $z \in \mathbb{C} \setminus \{1, a, a^2, \dots\}$ and let $\delta > 0$ be as before. Then let $f_z(k) = \frac{d_k}{(z - a^k)e'(a^k)}$ and $f_{n,z}(k) = \chi(k \leq n) \frac{d_k}{(z - a^k)p'_n(a^k)}$. Then $\lim_{n \rightarrow \infty} f_{n,z}(k) = f_z(k)$, because $p_n(w) \xrightarrow[n \rightarrow \infty]{} e(w)$ locally uniformly on the complex plane implies that $p'_n(w) \xrightarrow[n \rightarrow \infty]{} e'(w)$ locally uniformly on the complex plane ([11]). Also, $e'(k) \neq 0$, $P'_n(a^k) \neq 0$ for $k \leq n - 1$. Now, let $h(k) = \frac{|d_k a^k|}{\delta C |p_{k-1}(a^k)|}$. We see that $|f_{n,z}(k)| \leq h(k)$ for all n by the inequality above. Thus the conditions for Lebesgue's dominated convergence theorem are satisfied and so

$$\begin{aligned} \lim_{n \rightarrow \infty} q_n(z) &= \lim_{n \rightarrow \infty} \left(p_n(z) \sum_{k=0}^{\infty} \chi(k \leq n) \frac{d_k}{(z - a^k)p'_n(a^k)} \right) \\ &= \lim_{n \rightarrow \infty} \left(p_n(z) \sum_{k=0}^{\infty} f_{n,z}(k) \right) \\ &= e(z) \sum_{k=0}^{\infty} \lim_{n \rightarrow \infty} f_{n,z}(k) \\ &= e(z) \sum_{k=0}^{\infty} f_z(k) \\ &= e(z) \sum_{k=0}^{\infty} \frac{d_k}{(z - a^k)e'(a^k)} := q(z). \end{aligned}$$

Consider the disc $D_{\delta/2} = \{w \in \mathbb{C} \mid |w - z| < \frac{\delta}{2}\}$. For each $w \in D_{\delta/2}$ we have $|w - a^k| > \frac{\delta}{2}$ for all k ; thus, as before, we have $\left| \frac{d_k}{(w - a^k)p'_{n'}(a^k)} \right| \leq \frac{2|d_k a^k|}{\delta C |p_{k-1}(a^k)|}$ for

all $k \leq n-1$ and for all $w \in D_{\delta/2}$. As before, $\sum_k \frac{|d_k a^k|}{|p_{k-1}(a^k)|} < \infty$ by assumption. Therefore, using the Weierstrass M-test, the series $\sum_{k=0}^{\infty} \frac{d_k}{(w-a^k)p'_n(a^k)}$ converges uniformly on $D_{\delta/2}$. This implies that $q_n(w) \xrightarrow[n \rightarrow \infty]{} q(w)$ uniformly on the disc $D_{\delta/2}$ about z . Since this is true for any $z \in \mathbb{C} \setminus \{1, a, a^2, \dots\}$, it follows that the convergence is locally uniformly on this set. Remember that $q_n(a^j) = d_j$ for $0 \leq j \leq n$. Let $n_j \geq j$ for any fixed $j \geq 0$. Thus, $q_n(a^j) = d_j \forall n \geq n_j$ and therefore $\lim_{n \rightarrow \infty} q_n(a^j) = d_j$. Define $q(a^j) = d_j \forall j \geq 0$. Combining these results show that

$$\lim_{n \rightarrow \infty} q_n(a^j) = d_j = q(a^j)$$

for all $j \geq 0$. It follows that $q_n(z) \xrightarrow[n \rightarrow \infty]{} q(z)$ locally uniformly on $\mathbb{C}$. Each $q_n(z)$ is a polynomial of degree $\leq n$ and therefore entire. We have just shown that $q_n(z) \rightarrow q(z)$ as $n \rightarrow \infty$ locally uniformly on $\mathbb{C}$. With this in mind, [11] shows that $q(z)$ is also entire. We can now write $q(z) = \sum_{k=0}^{\infty} g_k z^k$ to obtain a solution $g = (g_0, g_1, \dots)$ of the original system $Ag = d$. By the previous lemma we see that $g^{(n)} \rightarrow g$ in l_1 sense.

Remember we assumed that $\sum_k \left| \frac{d_k a^k}{p_{k-1}(a^k)} \right| < \infty$. Is this assumption farfetched or not? An answer will be given in the proof of the following.

Theorem 4.0.15 *Let the matrix A and $a \in \mathbb{C}$ satisfy the conditions of the beginning of this section. The finite section method is applicable in the sense of l_1 convergence to the system $Ag = d$, whenever $d = (d_0, d_1, \dots) \in l_{\infty}$.*

Proof.

All that remains is to prove that for any $d \in l_{\infty}$ we have

$$\sum_k \left| \frac{d_k a^k}{p_{k-1}(a^k)} \right| < \infty.$$

Because $d \in l_{\infty}$ we only have to see that

$$\sum_k \left| \frac{a^k}{p_{k-1}(a^k)} \right| < \infty.$$

We calculate $p_{k-1}(a^k)$ and find that

$$\begin{aligned}
|p_{k-1}(a^k)| &= \left| \left(1 - \frac{a^k}{a^{k-1}}\right) \left(1 - \frac{a^k}{a^{k-2}}\right) \cdots \left(1 - \frac{a^k}{a^0}\right) \right| \\
&= \prod_{j=1}^k |(1 - a^j)| \\
&\geq \prod_{j=1}^k (|a^j| - 1) \\
&\geq (|a|^{k-1} - 1)(|a|^k - 1)
\end{aligned}$$

for k large enough (since we can choose k so that $\prod_{j=1}^{k-2} (|a|^j - 1) \geq 1$) and where we used the inequality $(|x - y| \geq ||x| - |y||)$. This gives

$$\left| \frac{a^k}{p_{k-1}(a^k)} \right| \leq \left(\frac{1}{|a|^{k-1} - 1} \right) \left(\frac{|a|^k}{|a|^k - 1} \right) = \left(\frac{1}{|a|^{k-1} - 1} \right) \left(\frac{1}{1 - \frac{1}{|a|^k}} \right)$$

for k large enough. But, $\lim_{k \rightarrow \infty} (1 - \frac{1}{|a|^k}) = 1$ and therefore we can choose k so that $1 - \frac{1}{|a|^k} \geq \frac{1}{2}$. Putting it together it follows that

$$\left| \frac{a^k}{p_{k-1}(a^k)} \right| \leq \left(\frac{1}{|a|^{k-1} - 1} \right) \left(\frac{|a|^k}{|a|^k - 1} \right) = \left(\frac{1}{|a|^{k-1} - 1} \right) \left(\frac{1}{1 - \frac{1}{|a|^k}} \right) \leq \frac{2}{|a|^{k-1} - 1}.$$

We now show that $\sum_k \frac{1}{|a|^{k-1} - 1}$ is convergent by the ratio test and thus complete the proof.

Indeed,

$$\begin{aligned}
&\lim_{k \rightarrow \infty} \left| \frac{\frac{1}{|a|^k - 1}}{\frac{1}{|a|^{k-1} - 1}} \right| \\
&= \lim_{k \rightarrow \infty} \left| \frac{|a|^{k-1} - 1}{|a|^k - 1} \right| \\
&= \lim_{k \rightarrow \infty} \frac{|a|^{k-1}}{|a|^k - 1} + \lim_{k \rightarrow \infty} \frac{-1}{|a|^k - 1} \\
&= \lim_{k \rightarrow \infty} \frac{1}{|a| - \frac{1}{|a|^{k-1}}} + 0 \\
&= \frac{1}{|a|} < 1.
\end{aligned}$$

□

Bibliography

- [1] Böttcher, Albrecht, Dijksma, Aad, Langer, Heinz, Ditschel, Michael A., Rovnyak, James and Kaashoek, M.A. 1996. Lectures on Operator Theory and Its Applications. Fields Institute Monographs. American Mathematical Society.
- [2] Davis, Philip J. 1975. Interpolation and Approximation, Dover Publications, Inc., New York.
- [3] Gohberg, I.C. and Fel'dman, I.A. 1974. Convolution Equations and Projection Methods for their Solution. Translations of Mathematical Monographs Vol.41, American Mathematical Society, Providence, Rhode Island.
- [4] Grobler, J.J. 2003. Real Analysis and Measure and Integration theory, Potchefstroom.
- [5] Gohberg, Israel, Goldberg, Seymour, and Kaashoek, Marinus A. 2003. Basic Classes of Linear Operators. Birkhäuser Verlag. Germany.
- [6] Kincaid, David and Cheney, Ward. 1996. Numerical Analysis. Brooks/Cole Publishing Company. United States of America.
- [7] Levinson, Norman and Redheffer, Raymond M. 1970. Complex Variables. Holden-Day, Inc. United States of America.
- [8] Rainville, D. 1960. Special Functions, Chelsea Publishing Company, Bronx, New York.
- [9] Ran, A.C.M. and Serény, A. 2006. The Finite Section Method for Infinite Vandermonde Matrices, preprint.
- [10] Rudin, Walter. 1973. Functional Analysis, McGraw-Hill, New York.

- [11] Rudin, Walter. 1970. Real and Complex analysis, McGraw-Hill, London.
- [12] Spiegel, Murray, R. 1968. Mathematical Handbook of Formulas and Tables, McGraw-Hill, Inc., United States of America.
- [13] Wilansky, Albert. 1978. Modern Methods in Topological Vector Spaces, McGraw-Hill, Inc., United States of America.