A theory of multiplier functions and sequences and its applications to Banach spaces

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Thesis submitted for the degree of Doctor Scientiae in Mathematics of the North West University (Potchefstroom Campus)

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November 2005

POTCHEFSTROOM

Acknowledgements

I would like to express my sincerest gratitude to:

The Father. Philippians 4 : 13 : "I have the strenght to face all conditions by the power that Christ gives me."

Prof. J.H. Fourie, my supervisor and mentor. I thank you very much for your guidance, instructive comments and new ideas. I greatly appreciate everything you have done for me.

Recognition and appreciation is hereby expressed for the use in Chapter 4 of the content as contained in the following draft paper submitted for publication in J. Math. Anal. Appl. "Operator valued multipliers on operator valued sequences of multipliers and R − boundedness" (authors : O. Blasco, J.H. Fourie, I.M. Schoeman).

My husband and parents. I thank you very much for all your love and encouragement.

Abstract

In recent papers (cf. [2], [3], [5], [23]) the concept of (p,q)-summing multiplier was considered in both the general and special context. It has been shown that some geometric properties of Banach spaces and some classical theorems can be described using spaces of (p,q)-summing multipliers. This thesis is a continuation of this line of study, whereby multiplier spaces for some classical Banach spaces are considered. The scope of this research is also broadened, by studying other classes of summing multipliers.

Generally stated, a sequence of bounded linear operators $(u_n) \subset L(X,Y)$ is called a multiplier sequence from $E(X)$ to $F(Y)$ if $(u_n x_n) \in F(Y)$ for all $(x_i) \in E(X)$, whereby $E(X)$ and $F(Y)$ are two Banach spaces of which the elements are sequences of vectors in X and Y, respectively. Several cases where $E(X)$ and $F(Y)$ are different (classical) spaces of sequences, including for instance the spaces $Rad(X)$ of almost unconditionally summable sequences in X, are considered. Several examples, properties and relations among spaces of summing multipliers are discussed. Important concepts like R-bounded, semi-R-bounded and weakly-R-bounded from recent papers are also considered in this context.

Sequences in X, which are (p,q)-summing multipliers (when considered as elements of $L(X^*, \mathbb{K})$) are of considerable importance. They are called (p,q)-summing sequences in X. The role of these sequences in the study of geometrical properties of Banach spaces as well as the characterization of vector-sequence space-valued operators on Banach spaces is extensively demonstrated in paper [2]. In this thesis we develop a general theory for vector-valued multiplier sequences and functions and consider the application thereof in the study of operators on Banach spaces in general and on classical spaces (for instance,

L^p-spaces) in particular.

Another paper [14] is dedicated to an open question in the theory of tensor products of Banach spaces. From the Grothendieck Resumé [26] it follows that $\ell^1\hat{\otimes}X$ is isometrically isomorphic to the space $\ell^1(X)$ of absolutely summable sequences in X. However, $\ell^p\hat{\otimes}X \subsetneqq \ell^p(X)$ is possible for $1 < p < \infty$. In paper [17] it is stated as an open problem to find a vector sequence space characterization of the projective tensor product $\ell^p\hat{\otimes}X$. The challenge is taken up in paper [14]. Using the vector sequence space $\ell^p\langle X\rangle$ of strongly p-summable sequences (introduced by Cohen in paper [16]), the authors show that $\ell^p\hat{\otimes}X$ is indeed isometrically isomorphic to

$$\ell^p\langle X\rangle := \{(x_n) \subset X : \sum_n |x_n^*(x_n)| < \infty, \ \forall \text{ weakly } p\text{-summable sequences } (x_n^*) \text{ in } X^*\}.$$

In following the author's approach in [14], it is only possible to prove this result once a formal characterization of the sequences in $\ell^p\langle X\rangle$ is known. This is the theme of [14]. In paper [23] we prove the same result by following a different approach (using the Grothendieck theory of tensor products and nuclear operators), which does not depend on the characterization of the elements of $\ell^p\langle X\rangle$, but which in fact has this characterization as easy consequence. By letting U be a reflexive Banach space with a normalized unconditional basis (e_i), Bu [11] introduced the spaces $U_{strong}(X)$, $U_{weak}(X)$ and $U\langle X\rangle$ and considered their geometric properties, interrelationships, Köthe duals and topological duals. Based on Bu's results and following our tensor product approach in [23], we provide a characterization of $U \overset{\wedge}{\otimes} X$ in terms of the vector sequence space $U\langle X\rangle$.

In short, the purpose of our research is to:

(i) Extend the results in [5] and [23] to the more general context of "general vector sequence spaces". This entails a vector sequence space characterization of the projective tensor product $U\hat{\otimes}X$, where X is a Banach space and U is a (reflexive)Banach space with normalized unconditional basis, as well as an extensive study of U-summing and strongly U-summing multipliers. Our exposition extensively makes use of several important research articles about vector sequence spaces, mostly of Bu's work on vector sequence spaces (cf. [11], [12], [13] and [14]). Our approach in the characterization of $U\hat{\otimes}X$, however, simplifies the techniques of Bu to obtain a similar characterization.

(ii) Introduce and study classes of operators, which are defined by general vector sequence spaces, in a similar fashion as are p-summing and (p,q)-summing operators defined by the vector sequence spaces of weak p-summable and absolutely p-summable sequences of vectors in normed spaces. By doing so our idea is to embed existing theories of (p,q)-summing operators, strongly p-summing operators and others into a general framework and to consider their applications in operator and Banach space theory, also in the context of Banach lattices. The classes of strongly p-summing and strongly p-nuclear operators were introduced and studied in detail by Cohen [16] where the strongly p-nuclear operators were called p-nuclear operators. His introduction of these two classes was motivated by observations about absolutely p-summing operators, tensor products and the conjugates of absolutely p-summing operators. One of the aims of this thesis is to broaden the work of Cohen in two ways. In the first case we extend it from strongly p-summing and strongly p-nuclear operators to strongly (p,q)-summing and strongly (p,q)-nuclear operators. Secondly, we generalize the operator setting by letting U and W be reflexive Banach spaces with normalized unconditional bases (e_i) and (f_i) respectively. We then introduce the absolutely (U,W)-summing and two related classes of operators, namely the strongly (U,W)-summing operators and the strongly (U,W)-nuclear operators.

(iii) Study operator valued multipliers (of different kinds), consider examples thereof on classical Banach spaces (such as the L^p spaces) and apply our results (and recent results in the literature, for instance in [2], [3], [8] and [9]) to contribute to relevant theories and results about different types of Rademacher boundedness, the Grothendieck Theorem (G.T. spaces) and applications to the geometry of Banach spaces.

(iv) Develop a theory of operator valued multiplier functions, thereby exploring the possibility to extend our work on (p, q)-summing multipliers to the setting of function spaces. The idea here is to establish the foundation for further research after completion of the thesis. Our introduction of the (p, q)-multiplier functions is inspired by several easy examples of such functions (generated by classes of operators) and the well known fact (in literature) that a Banach space operator $u : X \to Y$ is p-summing if and only if, given any probability space (Ω, Σ, μ) and any strongly measurable $f : \Omega \to X$, which is weakly p-integrable, then $u \circ f$ is Bochner p-integrable. In our language of multiplier functions, this says that u is p-summing if and only if the constant function $\Omega \to L(X, Y) : t \mapsto u$ is a (p, p)-multiplier function.

Key terms: Banach space, sequence space, Grotendieck's theorem, type, cotype, strongly (p, q)-summing, strongly (p, q)-nuclear operators, U-summing multipliers, strongly U-summing multipliers, absolutely (U, W)-summing operators, strongly (U, W)-summing operators, strongly (U, W)-nuclear operators, positive strongly (p, q)-summing operators, positive strongly (p, q)-nuclear operators, strongly (p, q)-concave operators, strongly p-integral functions, (p, q)-integral multipliers and (p, q)-integral functions.

Samevatting

In onlangse artikels (cf. [2], [3], [5], [23]) is die konsep van 'n (p,q)-sommerende vermenigvuldiger in beide die algemene en spesiale konteks beskou. Daar is aangetoon dat sommige meetkundige eienskappe van Banachruimtes en sommige klassieke stellings beskryf kan word in terme van (p,q)-sommerende vermenigvuldigers. Hierdie proefskrif is 'n voorsetting van dié studie waar vermenigvuldigerruimtes van sekere klassieke Banachruimtes beskou word. Sodanige navorsing word uitgebrei deur die bestudering van ander klasse van sommerende vermenigvuldigers.
In die algemeen word 'n ry van begrensde lineêre operatore $(u_n) \subset L(X,Y)$ 'n vermenigvuldigerry vanaf $E(X)$ na $F(Y)$ genoem as $(u_n x_n) \in F(Y)$ vir alle $(x_i) \in E(X)$, waar $E(X)$ en $F(Y)$ beide Banachruimtes is waarvan die elemente rye van vektore in onderskeidelik X en Y is. Verskeie gevalle word ondersoek waar $E(X)$ en $F(Y)$ verskillende (klassieke) ruimtes van rye is, insluitend byvoorbeeld die ruimte $Rad(X)$ van "byna onvoorwaardelike sommerende rye" in X. Verskeie voorbeelde, eienskappe en verwantskappe tussen ruimtes van sommerende vermenigvuldigers word bespreek. Belangrike konsepte soos R-begrensheid, semi-R-begrensheid en swak-R-begrensheid uit onlangse artikels word in hierdie konteks ondersoek.

Rye in X wat (p,q)-sommerende vermenigvuldigers is (indien beskou as elemente van $L(X^*,\mathbb{K})$) speel 'n belangrike rol en word die (p,q)-sommerende rye in X genoem. Die rol wat sodanige rye in die bestudering van die meetkundige eienskappe van Banachruimtes sowel as in die karakterisering van vektorryruimtewaardige operatore op Banachruimtes speel, is omvangryk bespreek in [2]. In hierdie proefskrif ontwikkel ons 'n algemene teorie vir vektorwaardige vermenigvuldigerrye en funksies. Verder verkry ons toepassings hier-

van in die algemene teorie van operatore op Banachruimtes, sowel as in die teorie van operatore op sekere klassieke ruimtes (soos byvoorbeeld die L^p-ruimtes).

'n Onlangse artikel [14] word gewy aan 'n oop vraag in die teorie van tensorprodukte van Banachruimtes. Uit Grotendieck se Résumé [26] volg dat $\ell^1 \overset{\wedge}{\otimes} X$ isometries isomorf is aan die ruimte $\ell^1(X)$ van absoluut sommeerbare rye in X. Vir $1 < p < \infty$, is $\ell^p \hat{\otimes} X \subsetneqq \ell^p(X)$ egter moontlik. In [17] word die karakterisering van die projektiewe tensorproduk $\ell^p \overset{\wedge}{\otimes} X$ (vir $1 < p < \infty$) in terme van 'n vektorryruimte, as oop vraag gestel. Hierdie uitdaging word aanvaar in artikel [14]. Deur gebruik te maak van die vektorryruimte $\ell^p\langle X \rangle$ van sterk p-sommeerbare rye (ingevoer deur Cohen in artikel [16]) bewys die outeurs dat die ruimte $\ell^p \overset{\wedge}{\otimes} X$ isometries isomorf is aan die ruimte

$$\ell^p\langle X \rangle = \{(x_n) \subset X : \sum_n |x_n^*(x_n)| < \infty, \ \forall (x_n^*) \in \ell_w^p(X^*)\}.$$

Die skrywers in [14] bewys hierdie resultaat deur gebruik te maak van 'n formele karakterisering van die rye in $\ell^p\langle X \rangle$. In artikel [23] bewys ons dieselfde resultaat deur 'n ander benadering te volg (ons gebruik Grothendieck se stelling oor tensorprodukte en nukleêre operatore) wat onafhanklik is van die karakterisering van die elemente van $\ell^p\langle X \rangle$, maar waaruit hierdie karakterisering as 'n maklike gevolgtrekking volg. Bu [11] definieer en beskou die meetkundige eienskappe, verwantskappe, Köthe en topologiese dualiteite van die ruimtes $U_{strong}(X)$, $U_{weak}(X)$ en $U\langle X \rangle$ deur aan te neem dat U 'n refleksiewe Banachruimte is, met 'n genormaliseerde onvoorwaardelike basis (e_i). Ons gee 'n karakterisering van $U \overset{\wedge}{\otimes} X$ in terme van die vektorryruimte $U\langle X \rangle$ deur gebruik te maak van die resultate van Bu en ons tensorprodukbenadering in [23].
Kortliks kan die oogmerk van hierdie navorsing soos volg saamgevat word:

(i) Die resultate in [5] en [23] word uitgebrei na die veralgemeende konteks van "algemene vektorryruimtes". Dit bring 'n vektorryruimtekarakterisering van die projektiewe tensorproduk $U \overset{\wedge}{\otimes} X$ mee, waar X 'n Banachruimte en U 'n (refleksiewe) Banachruimte met 'n genormaliseerde onvoorwaardelike basis is. 'n Omvangryke bespreking van U-sommerende en sterk U-sommerende vermenigvuldigers word gegee. Ons uiteensetting maak gebruik van verskeie belangrike navorsingsartikels oor vektorryruimtes, veral van die werk van Bu (cf. [11], [12], [13] en [14]). Ons benadering in die karakterisering van $U\hat{\otimes}X$ is 'n vereenvoudiging van die tegnieke van Bu, hoewel ons dieselfde resultaat bewys.

(ii) Die klasse van operatore wat gedefinieer word in terme van algemene vektorryruimtes word ingevoer en bestudeer op 'n soortgelyke wyse as die p-sommerende en die (p,q)-sommerende operatore, wat gedefinieer is in terme van vektorryruimtes van swak p-sommerende en absoluut p-sommerende rye van vektore in normeerde ruimtes. Hieruit volg die idee om die bestaande teorieë van (p,q)-sommerende operatore, sterk p-sommerende operatore en ander operatore in te sluit in die algemene raamwerk en om ondersoek in te stel na toepassings in operatorteorie, Banachruimteteorie en die konteks van Banachroosters. Die klasse van sterk p-sommerende en sterk p-nukleêre operatore word ingevoer en omvangryk ondersoek deur Cohen [16] wat na die sterk p-nukleêre operatore verwys as "p-nukleêre operatore". Die invoering van hierdie twee klasse word gemotiveer deur waarnemings oor absoluut p-sommerende operatore, tensorprodukte en die toegevoegdes van absoluut p-sommerende operatore. Een van die doelwitte van hierdie proefskrif is om die werk van Cohen uit te brei op twee wyses. In die eerste plek brei ons dit uit vanaf sterk p-sommerende operatore en sterk p-nukleêre operatore na sterk (p,q)-sommerende operatore en sterk (p,q)-nukleêre operatore. Tweedens veralgemeen ons die operatorgeval deur te veronderstel dat U en W refleksiewe Banachruimtes is met die onderskeidelike genormaliseerde onvoorwaardelike basisse (e_i) en (f_i). Ons voer dan die begrip van "absoluut (U,W)-sommerende" in en definieer vervolgens twee verwante klasse van operatore naamlik die "sterk (U,W)-sommerende operatore" en die "sterk (U,W)-nukleêre operatore".

(iii) Ons bestudeer (verskillende soorte) operatorwaardige vermenigvuldigers en beskou voorbeelde daarvan op klassieke Banachruimtes (soos die L^p-ruimtes) en pas ons resultate (en onlangse resultate in die literatuur, byvoorbeeld in [2], [3], [8] en [9]) toe, om 'n bydra te lewer tot die relevante teorie en resultate in verband met die verskillende tipes Rademacher begrensdheid, Grothendieck se stelling (G.T.-ruimtes) en toepassings op sekere meetkundige eienskappe van Banachruimtes.

(iv) Ons voer 'n teorie van operatorwaardige vermenigvuldigerfunksies in, en ondersoek moontlikhede om ons werk oor (p,q)-sommerende vermenigvuldigers uit te brei na die raamwerk van funksieruimtes. Die idee hier is om 'n basis te lê vir verdere navorsing na afhandeling van hierdie proefskrif. Die invoering van die (p,q)-vermenigvuldigerfunksies is geïnspireer deur verskeie maklike voorbeelde van sulke funksies (voortgebring deur klasse van operatore) en die welbekende feit (in die literatuur) dat 'n Banachruimte operator $u : X \to Y$ p-sommerend is as en slegs as vir enige gegewe waarskynlikheidsruimte (Ω, Σ, μ) en enige sterk meetbare funksie $f : \Omega \to X$ wat swak p-integreerbaar is, geld dat $u \circ f$ in ons taal van vermenigvuldigerfunksies, Bochner p-integreerbaar is. Dus, u is p-sommerend as en slegs as die konstante funksie $\Omega \to L(X,Y) : t \mapsto u$ 'n (p,p)-vermenigvuldigerfunksie is.

Kernterme: Banachruimte, ryruimte, Grotendieck se stelling, tipe, kotipe, sterk (p,q)-sommerende vermenigvuldiger, sterk (p,q)-nukleêre operatore, U-sommerende vermenigvuldiger, sterk U-sommerende vermenigvuldiger, absoluut (U,W)-sommerende operatore, sterk (U,W)-sommerende operatore, sterk (U,W)-nukleêre operatore, sterk positiewe (p,q)-sommerende operatore, sterk positiewe (p,q)-nukleêre operatore, sterk (p,q)-konkawe operatore, sterk p-integraal funksies, (p,q)-integraal vermenigvuldigers en (p,q)-integraal funksies.

Titel: 'n Teorie van vermenigvuldigerfunksies en vermenigvuldigerrye en toepassings daarvan op Banachruimtes

Contents

Introduction

p-Summing multipliers of Banach spaces were introduced and studied in a paper of S. Aywa and J.H. Fourie (cf. [5]). In this paper the nuclearity of certain Banach space valued bounded linear operators on the classical ℓ^p-spaces (of absolutely p-summable scalar sequences) as well as geometrical properties (for instance, the Orlicz property) of Banach spaces were obtained in terms of the p-absolutely summing multipliers of the Banach space. H. Apiola (cf. [1]) and J.S. Cohen (cf. [16]) introduced $\ell^p\langle X\rangle$, the space of strongly p-summable sequences in a Banach space X, in their discussion of p-nuclear operators between Banach spaces. In [14] Q. Bu and J. Diestel considered a vector sequence space representation of the projective tensor product of ℓ^p and a Banach space X, thus obtaining that this tensor product space is the space of strongly p-summable sequences in X, i.e. $\ell^p\langle X\rangle$.

In a paper by Arregui and Blasco (cf. [2]) an extended theory of (p,q)-summing multipliers and sequences was developed. The family of p-summing multipliers introduced in [5] is a subset of the (p,p)-summing multipliers. Some surprising applications of this theory to the geometry of Banach spaces are discussed in [2], including the reformulation of important theorems (Grothendieck's Theorem, for instance) in this new context.

In [28] the authors consider some new applications of semi-R-bounded and WR-bounded sequences. They show that for each $x \in X$ and $(u_i) \in SR(X,X)$, the sequence $(u_n x)$ has a weakly Cauchy subsequence. Using this fact, they then show that if X is a weakly sequentially complete Banach space such that $L(X,X)$ contains a semi-R-bounded sequence (u_i) such that each u_i is weakly compact, $u_k u_l = u_l u_k$ for all k, $l \in \mathbb{N}$ and

$\lim_{k\to\infty} \|x - u_k x\| = 0$ for every $x \in X$, then X is isomorphic to a dual space.
In case of $L(X, X)$ containing a WR-bounded sequence with the same properties, one also needs the space X to satisfy the property (V^*) of Pelczynski to obtain the same result. Since $L^1(0,1)$ is not a dual space, it follows that $L(L^1(0,1), L^1(0,1))$ does not have a semi-R-bounded or WR-bounded sequence of operators (u_i) with the mentioned properties. It is also shown in [28] that if K is a compact metric space so that $L(C(K), C(K))$ contains an R-bounded sequence (u_n) with the above-mentioned properties, the space $C(K)$ is isomorphic to c_0. Some applications to semigroups of operators are also considered in [28].

Furthermore, in paper [15] the authors study the interplay between unconditional Schauder decompositions and the R-boundedness of collections of operators. They prove several multiplier results of the Marcinkiewicz type for L^p-spaces of functions with values in a Banach space X. In their paper the authors also show connections between R-boundedness in $L(X, X)$ and the geometric properties of the Banach space X. Fact is that a R-bounded sequence of operators is an example of a "multiplier sequence", which is the main theme of this thesis. As a matter of fact, we discuss the concepts of "multiplier sequence" and "multiplier function" in a general context and then show that different concepts that recently played important role in applications to Banach spaces, fit into this setting.

The contents of this thesis is divided into five main chapters to be summarized as follows.

Chapter 1 is a summary of basic well known facts about Banach spaces, vector sequence spaces, operators on Banach spaces, some geometrical properties of Banach spaces, tensor products of Banach spaces, vector integrals, vector valued L^p-spaces, (p, q)-summing sequences and strongly p-summable sequences, Banach lattices and bases in Banach spaces. The purpose of discussing these known facts, is to make this exposition as self contained as possible.

After introduction of the general vector sequence spaces $U_{strong}(X)$, $U_{weak}(X)$ and $U\langle X\rangle$, where U is a reflexive Banach space with normalized unconditional basis and X is a Banach space, we prove in Chapter 2 that $U\langle X\rangle$ is isometrically isomorphic to the space $\mathcal{I}(U^*, X)$ of integral operators. U being reflexive and having the m.a.p., it then follows that $U\langle X\rangle$ is isometric to the space $N(U^*, X)$ of nuclear operators and thus by Grothendieck's theory, isometric to $U \overset{\wedge}{\otimes} X$. We also discuss this result in two classical cases where $U = L^p(0,1)$ and $U = \ell^p$. The concepts U-summing multiplier and strongly U-summing multiplier are considered in Chapter 2, where we discuss the properties and relationships of the normed spaces of U-summing and strongly U-summing multipliers and consider some applications to normed space.

In Chapter 3, we introduce the absolutely (U, W)-summing and two related classes of operators, namely the strongly (U, W)-summing operators and the strongly (U, W)-nuclear operators. We investigate the relationship between these classes. In addition, we define two new classes of operators, namely the strongly (p,q)-summing operators and the strongly (p,q)-nuclear operators. The interrelationship of these operators and the (p, q)-summing operators is investigated. Properties of these spaces like inclusions and conjugate operators are also considered.

The latter part of Chapter 3 is inspired by the work of Blasco who introduced the positive (p, q)-summing operators where X denote a Banach lattice and Y a Banach space (cf. [7]). This paper of Blasco paved the way for us to extend our work by defining new classes of operators, namely the positive strongly (p, q)-summing operators and the positive strongly (p, q)-nuclear operators. We also describe the space of strongly (p, q)-concave operators in a way that is in line with the definition in ([33], p. 46) of p-concave operators.

In Chapter 4 we summarize some (recent) results on (p, q)-summing multipliers and discuss some examples of (p, q)-summing multipliers on classical Banach spaces. We extend the idea of (p,q)-summing multipliers to other families of multiplier sequences from $E(X)$ to $F(Y)$ by considering some well known and important Banach spaces of vector valued sequences in place of $E(X)$ and $F(Y)$. The work in this chapter contains largely joint work with Oscar Blasco and Jan Fourie (cf. [9]). I appreciate my co-authors' consent to use the material of our joint paper in this chapter.

In Section 4.2, we study R-bounded sequences and other variants thereof, like for instance, semi-R-bounded and weakly-R-bounded sequences in Banach spaces. Relations of several types of sequences of bounded linear operators (like R-bounded, weakly-R-bounded, semi-R-bounded, uniformly bounded, unconditionally bounded and almost summing) are studied. These relations build on well known results on type and cotype and characterizations of different families of operators. We discuss these concepts within our framework of multiplier sequences of operators, which allow us to prove new results about inclusions of sets (vector spaces) of different kinds of R-bounded sequences of operators and their connections with some geometrical properties of Banach spaces, including results about type, cotype, Orlicz property and the Grothendieck Theorem.

In Chapter 5 we lay the foundation for further research work in the general context of (p,q)-multiplier functions. The generalization that we consider here is motivated by the fact that the multiplier functions appear naturally in the sense that p-summing operators can be characterized in terms of multiplier functions. The usual duality between $L^p(\mu,X)$ and $L^{p'}(\mu,X^*)$ when X^* has RNP can be expressed as a multiplier function, easy examples of multiplier functions can be found (and are discussed in Chapter 5) and the function spaces so obtained and their properties show close resemblance to the sequence space case. We prove some inclusion theorems for spaces of multiplier functions and describe some relationships with $L^p(\mu,X)$-spaces. Hopefully, the basic results developed in our foundation work in Chapter 5 will prove to be important in further research. We hope to be able to apply the theory in situations where discrete representations are not possible.

Chapter 1

Definitions and basic facts

1.1 Some basic facts about Banach spaces, vector sequence spaces and operators on Banach spaces

If not otherwise stated, X, Y, Z, etc. will throughout this thesis be Banach spaces. Let $L(X, Y)$ denote the space of bounded linear operators from X to Y and let $K(X, Y)$ denote the space of all compact linear operators between X and Y. For given X, we denote the continuous dual space by X^*, the algebraic dual space by X' and the unit ball in X by B_X. For $1 < p < \infty$, let $p\prime$ denote its conjugate number, i.e. $1/p + 1/p\prime = 1$.

Sequences in Banach spaces will be denoted by (x_i), (y_i), etc. The "n-th section" $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ of (x_i) in X is denoted by $(x_i)(\leq n)$ and $(x_i)(> n) = (x_i) - (x_i)(\leq n)$. A vector space Λ whose elements are sequences (α_n) of numbers (real or complex) is called a *sequence space*. To each sequence space Λ we assign another sequence space $\Lambda^\times$, its *Köthe-dual*, which is the set of all sequences (β_n) for which the series $\sum_{n=1}^{\infty} \alpha_n \beta_n$ converges absolutely for all $(\alpha_n) \in \Lambda$, i.e.

$$\Lambda^\times = \{(\beta_n) \in \omega : \sum_{n=1}^{\infty} \mid \alpha_n \beta_n \mid < \infty, \ \forall \ (\alpha_n) \in \Lambda\}.$$

A Banach sequence space Λ is said to be a *BK-space* if each coordinate projection mapping $(\alpha_n) \mapsto \alpha_i$ is continuous.

Let $e_n = (\delta_{i,n})_i$, with $\delta_{i,n} = 1$ if $i = n$ and $\delta_{i,n} = 0$ if $i \neq n$. In a dual normed sequence space Λ^* the notation e_n^* for e_n will be used.

A normed scalar sequence space Λ is said to have the *AK-property* if all its elements can be approximated by their sections. That is, if each element (β_i) in the sequence space satisfies $(\beta_i) = \lim_{n\to\infty}(\beta_i)(\leq n)$, where $(\beta_i)(\leq n) = \sum_{i=1}^{n} \beta_i e_i$. A normed vector sequence space $\Lambda(X)$ is said to have the *GAK-property* if all its elements can be approximated by their sections. A BK-space Λ has the AK-property if and only if $\{e_n : n = 1, 2, \dots\}$ is a Schauder basis for Λ, that is if and only if $\lim_{n\to\infty} \|(\mu_i)(\geq n)\|_\Lambda = 0$ for all $(\mu_i) \in \Lambda$. If Λ is a normal BK-space with AK, then $\{e_n : n = 1, 2, \dots\}$ is an unconditional basis for Λ, called the *standard coordinate basis* or the unit vector basis of Λ. In this case a standard argument shows that $\Lambda^\times$ is algebraically isomorphic to the continuous dual space Λ^* with respect to the obvious duality.

If not stated otherwise all scalar sequence spaces $\Lambda \neq \ell^\infty$ will throughout be assumed to be normal BK-spaces with the AK-property. In this case we may assume that $\|e_n\|_\Lambda = 1$ for all $n \in \mathbb{N}$. For information on scalar sequence spaces we refer to [30].

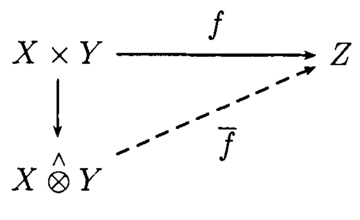
Definition

(a) The **projective or $\wedge$−norm** on $X \otimes Y$ is defined by

$$|u|_\wedge = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \right\},$$

where the infimum is taken over all representations of $u = \sum_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$. $X \overset{\wedge}{\otimes} Y$ is the completion of $(X \otimes Y, |.|_\wedge)$.

Following is the **universal mapping property for projective tensor products** (cf. [26])



For any Banach spaces X, Y and Z, the space $L(X \overset{\wedge}{\otimes} Y; Z)$ of all bounded linear operators from $X \overset{\wedge}{\otimes} Y$ to Z is isometrically isomorphic to the space $\mathcal{B}(X \times Y; Z)$ of all bounded bilinear transformations from $X \times Y$ into Z. The natural correspondence establishing this isometric isomorphism is given by

$$\overline{f} \in L(X \overset{\wedge}{\otimes} Y; Z) \Leftrightarrow f \in \mathcal{B}(X \times Y; Z)$$

via $\overline{f}(x \otimes y) = f(x, y)$.

(b) For any two Banach spaces X and Y over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ the **injective or $\vee$−norm** of $\sum_{j=1}^{n} x_j \otimes y_j \in X \otimes Y$ is

$$\mid \sum_{j=1}^{n} x_j \otimes y_j \mid_\vee = \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{Y^*}}} \mid \sum_{j=1}^{n} (x^* x_j)(y^* y_j) \mid$$

and the **injective tensor product** $X \overset{\vee}{\otimes} Y$ is the completion of $X \otimes Y$ with respect to this norm.

Let X be a Banach space. The vector sequence space $\Lambda(X) := \{(x_i) \subset X : (\|x_i\|) \in \Lambda\}$ is a complete normed space with respect to the norm

$$\|(x_i)\|_{\Lambda(X)} := \|(\|x_i\|)\|_\Lambda.$$

We put $\|(\alpha_i)\|_{\Lambda(X)} = \|(\alpha_i)\|_p$ when $\Lambda = \ell^p$, the Banach space of *p-absolutely summable* scalar sequences (with $1 \leq p < \infty$) and $X = \mathbb{K}$.

The vector sequence space $\Lambda_w(X^*) := \{(x_i^*) \subset X^* : (\langle x, x_i^* \rangle) \in \Lambda, \ \forall \ x \in X\}$ is a complete normed space with respect to the norm

$$\epsilon_\Lambda((x_i^*)) := \sup_{\|x\| \leq 1} \|(\langle x, x_i^* \rangle)\|_\Lambda.$$

We put $\epsilon_p = \epsilon_\Lambda$ when $\Lambda = \ell^p$, (with $1 \leq p < \infty$).

Let $l_w^p(X)$ denote the space of *weakly p-summable* sequences in X, i.e.

$$l_w^p(X) := \{(x_i) \subset X : (\langle x_i, x^* \rangle) \in l^p, \ \forall \ x^* \in X^*\}$$

is a complete normed space with respect to the norm

$$\epsilon_p((x_i)) := \sup_{\|x^*\| \leq 1} (\sum_{n=1}^{\infty} |x^*(x_n)|^p)^{\frac{1}{p}}.$$

If $p = \infty$, let

$$\epsilon_\infty((x_i)) := \sup_{\|x^*\| \le 1} \sup_n |x^*(x_n)|.$$

The weak Dvoretzky-Rogers Theorem (cf. [19], p. 50):

Let $1 \le p < \infty$. Then $\ell^p_w(X) = \ell^p(X)$ if and only if X is finite dimensional.

The vector sequence space

$$\begin{aligned} \Lambda_c(X) &= \{(x_i) \in \Lambda_w(X) : (x_i) = \epsilon_\Lambda - \lim_{n\to\infty}(x_1, \dots, x_n, 0, \dots)\} \\ &= \{(x_i) \in \Lambda_w(X) : \epsilon_\Lambda((x_i)(\ge n)) \to 0 \text{ if } n \to \infty\} \end{aligned}$$

is a closed subspace of $\Lambda_w(X)$. On $\Lambda_c(X)$ we consider the induced subspace norm, inherited from $\Lambda_w(X)$. The vector sequence space

$$\begin{aligned} \Lambda_c(X^*) &= \{(x_i^*) \in \Lambda_w(X^*) : (x_i^*) = \epsilon_\Lambda - \lim_{n\to\infty}(x_1^*, \dots, x_n^*, 0, \dots)\} \\ &= \{(x_i^*) \in \Lambda_w(X^*) : \epsilon_\Lambda((x_i^*)(\ge n)) \to 0 \text{ if } n \to \infty\} \end{aligned}$$

is a closed subspace of $\Lambda_w(X^*)$. On $\Lambda_c(X^*)$ the induced subspace norm, inherited from $\Lambda_w(X^*)$, will be considered.

It follows from Proposition 2 in paper [22] that the continuous dual space $\Lambda_c(X)^*$ can be identified with the vector space of all sequences (x_i^*) in X^* such that

$$\sum_{i=1}^{\infty} |\langle x_i, x_i^* \rangle| < \infty \text{ for all } (x_i) \in \Lambda_w(X).$$

Moreover, the following characterisations can also be found in [21] and in paper [24]:

Theorem 1.1 *Consider a Banach space X.*
a) Let Λ be a Banach sequence space with the AK-property. Then $\Lambda^\times_w(X)$ is isometrically isomorphic to $L(\Lambda, X)$. The isometry is given by $(x_n) \mapsto T_{(x_n)}$, where $T_{(x_n)}((\xi_i)) = \sum_{i=1}^{\infty} \xi_i x_i$.
b) Let Λ be a Banach sequence space with the AK-property such that $\Lambda^\times$ has AK. Then $\Lambda^\times_c(X)$ is isometrically isomorphic to $K(\Lambda, X)$.

From the fact that $l_w^p(X) \simeq L(l^q, X), (1 < p < \infty),\ \frac{1}{p} + \frac{1}{q} = 1$, *it follows that*

$$l_w^p(X) = \{(x_n) \subset X : \sum_{n=1}^{\infty} t_n x_n \text{ converges}, \ \forall\ (t_n)_n \in l^q\}$$

and

$$\epsilon_p((x_i)) = \|T_{(x_n)}\| = \sup_{(\lambda_i)\in B_{l^q}} \|\sum_{i=1}^{\infty} \lambda_i x_i\|,$$

where l^q *is replaced by* c_0 *if* $p = 1$.

c) Let Λ *be a Banach sequence space with the AK-property. Then* $\Lambda_w(X^*)$ *is isometrically isomorphic to* $L(X, \Lambda)$. *The isometry is given by* $(x_n^*) \mapsto T_{(x_n^*)}$, *where* $T_{(x_n^*)}x = (\langle x, x_n^*\rangle)$.

d) Let Λ *be a Banach sequence space with the AK-property. Then* $\Lambda_c(X^*)$ *is isometrically isomorphic to* $K(X, \Lambda)$. *The isometry is defined as in (c).*

Let $1 \le p \le \infty$ and let $\lambda > 1$. Then the Banach space X is a $\mathcal{L}_{p,\lambda}$*-space* if every finite dimensional subspace E of X is contained in a finite dimensional subspace F of X for which there is an isomorphism $v : F \to \ell^p_{\dim(F)}$ with $\|v\|\|v^{-1}\| < \lambda$.

Theorem 1.2 *(cf. [19], p. 61)*

(i) If (Ω, Σ, μ) *is any measure space and* $1 \le p \le \infty$, *then* $L^p(\mu)$ *is a* $\mathcal{L}_{p,\lambda}$*-space for all* $\lambda > 1$.

(ii) If K *is a compact Hausdorff space, then* $C(K)$ *is a* $\mathcal{L}_{\infty,\lambda}$*-space for all* $\lambda > 1$.

We recall the well known Radon-Nikodym property for vector valued measures:

Definition 1.3 *(cf. [20], p. 61)*
A Banach space X *has the* **Radon-Nikodym property** *(RNP in short) with respect to* (Ω, Σ, μ) *if for each* μ*-continuous vector measure* $G : \Sigma \to X$ *of bounded variation there exists* $g \in L^1(\mu, X)$ *such that* $G(E) = \int_E g\, d\mu,\ \forall\ E \in \Sigma$.

Theorem 1.4 *Reflexive Banach spaces have the Radon-Nikodym property.*

We consider the following operator ideals:

* $(\mathcal{F}, \|\cdot\|)$, where $T \in \mathcal{F}(X,Y)$ if and only if T is a *finite rank bounded linear operator* and $\|\cdot\|$ is the usual uniform operator norm. Recall that $T \in \mathcal{F}(X,Y)$ if and only if T has a representation of the form $T = \sum_{i=1}^{n} x_i^* \otimes y_i$ where $x_i^* \in X^*$ and $y_i \in Y$. Also recall that the **trace** of $S = \sum_{i=1}^{n} x_i^* \otimes x_i \in \mathcal{F}(X,X)$ is the number

$$\operatorname{tr}(S) = \sum_{i=1}^{n} \langle x_i, x_i^* \rangle,$$

which is independent of the representation of S.

The space X has the **metric approximation property** (m.a.p. in short) if for each $\epsilon > 0$ and each compact set $K \subset X$ there exists a $S \in \mathcal{F}(X,X)$ with

(1) $\|S\| \leq 1$ and

(2) $\|Sk - k\| \leq \epsilon, \ \forall\ k \in K$.

* (N, ν_1), where $T \in N(X,Y)$ if and only if T is a *nuclear operator,* i.e. T has a representation

$$Tx = \sum_{i=1}^{\infty} \lambda_i \langle x, x_i^* \rangle y_i$$

where $(\lambda_i) \in \ell^1$, (x_i^*) is bounded in X^* and (y_i) is bounded in Y. Here

$$\nu_1(T) := \inf \sum_{i=1}^{\infty} |\lambda_i|,$$

where the infimum is extended over all such representations for which $\|x_i^*\| \leq 1$ and $\|y_i\| \leq 1$ for all i.

* $(\mathcal{I}, i)$, where $T \in \mathcal{I}(X,Y)$ if and only if T is an *integral operator,* i.e. if and only if there exists $\rho \geq 0$ such that

$$|\operatorname{tr}(TS)| \leq \rho \|S\|, \qquad \forall\ S \in \mathcal{F}(Y,X).$$

The integral norm $i(T)$ equals the smallest of all numbers $\rho \geq 0$ admissible in these inequalities. Note that $(X \overset{\vee}{\otimes} Y)^*$ is identifiable with $\mathcal{I}(X, Y^*)$. From results by Grothendieck it follows that in case of either X or Y being reflexive, every $u \in \mathcal{I}(X,Y)$ is nuclear; i.e. $\mathcal{I}(X,Y)$ and $N(X,Y)$ are topological isomorphic in this case. Also, from Grothendieck's work on the metric approximation property (m.a.p. in short) it follows that in case of

X^* having the m.a.p., we have $i(u) = \nu_1(u)$ for all $u \in N(X,Y)$. Thus, if X is reflexive and X^* has m.a.p., then $N(X,Y) \overset{isometric}{=} \mathcal{I}(X,Y)$. More generally, if X^* has the m.a.p, then $N(X,Y) \overset{isometric}{=} \mathcal{I}(X,Y)$ if and only if X^* has the Radon-Nikodým property (cf. [20], Theorem 6 on p. 248).

* (Π_{as}, π_{as}), where $T \in \Pi_{as}(X,Y)$ if and only if T is an *almost summing operator,* i.e. if and only if there exists $c \geq 0$ such that

$$\int_0^1 \|\sum_{j=1}^n T(x_j)r_j(t)\| \, dt \leq c \sup_{\|x^*\|=1} (\sum_{j=1}^n |\langle x^*, x_j\rangle|^2)^{\frac{1}{2}}$$

for any finite set of vector $\{x_1, \cdots, x_n\} \subset X$ where $(r_j)_{j\in\mathbb{N}}$ are the **Rademacher functions** on $[0,1]$ defined by $r_j(t) = \text{sign}(\sin 2^j \pi t)$. The least of such constants is the *almost-summing norm* of u, denoted by $\pi_{as}(u)$.

* (cf. [35], p. 31) Let $u : X \to Y$ be an operator. Then

(i) u is of **type p**, $1 \leq p \leq 2$, if there exists a constant $c > 0$ such that for any finite subset $\{x_1, \cdots, x_n\} \subset X$ we have

$$\int_0^1 \|\sum_{j=1}^n u x_j r_j(t)\| \, dt \leq c \, (\sum_{j=1}^n \|x_j\|^p)^{\frac{1}{p}}.$$

(ii) u is of **cotype q**, $2 \leq q \leq \infty$, if there exists a constant $c > 0$ such that for any finite subset $\{x_1, \cdots, x_n\} \subset X$ we have

$$(\sum_{j=1}^n \|u x_j\|^q)^{\frac{1}{q}} \leq c \int_0^1 \|\sum_{j=1}^n x_j r_j(t)\| \, dt.$$

In case $u = id_X$ and id_X is of type p (resp. cotype q), we say that X is of type p (resp. cotype q).

Note that a Banach space X is of type 2 and cotype 2 iff it is isomorphic to a Hilbert space (cf. [35], p. 33).

1.2 Basic facts about vector integrals

The reader is referred to [20] and [4] for the following definitions. Throughout this section (Ω, Σ, μ) is a finite measure space and X is a Banach space.

Definition 1.5 *(a) A function $f : \Omega \to X$ is called* **simple** *if there exist $x_1, x_2, \cdots, x_n \in X$ and $E_1, E_2, \cdots, E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where*

$$\chi_{E_i}(\omega) = \begin{cases} 1 & \text{if } \omega \in E_i \\ 0 & \text{if } \omega \notin E_i. \end{cases}$$

(b) A function $f : \Omega \to X$ is called **μ-measurable** *if there exists a sequence of simple functions (f_n) with $\lim_n \|f_n - f\| = 0$ μ-almost everywhere, where*

$$\|f\| : \Omega \to \mathbb{R} : t \mapsto \|f(t)\|.$$

(c) A function $f : \Omega \to X$ is called **weakly μ-measurable** *if for each $x^* \in X^*$ the numerical function $x^* f$ is μ-measurable.*

(d) (cf. [20], p. 52 and p. 53)

A function f is **Dunford integrable** *if f is a weakly $\mu-$measurable $X-$valued function on Ω such that $x^* f \in L^1(\mu)$, $\forall\ x^* \in X^*$. The* **Dunford integral** *of f over $E \in \Sigma$ is defined by the element $x_E^{**} \in X^{**}$ where*

$$x_E^{**}(x^*) = \int_E x^* f \ d\mu, \ \forall\ x^* \in X^*.$$

In this case we write

$$x_E^{**} = (Dunford) - \int_E f \ d\mu.$$

The function $f : \Omega \to X$ is **Pettis integrable** *if f is weakly μ-measurable such that*

$$(Dunford) - \int_E f \ d\mu \ \in X, \ \forall\ E \in \Sigma.$$

We denote the Pettis integral of f over $E \in \Sigma$ by $(Pettis) - \int_E f \ d\mu$. Note that the Dunford and Pettis integrals coincide when X is reflexive and that a μ-measurable Dunford integrable function f is Pettis integrable if and only if the set function

$$\Omega \mapsto X^{**} : E \mapsto (Dunford) - \int_E f \ d\mu$$

is countably additive (cf. [20], p. 54).

(e) (cf. [20], p. 108)

The bounded linear operator $T : L^p(\mu) \to X$ is called a **vector integral operator** *(v.i.o.) with kernel g if $g : \Omega \to X$ is a μ-measurable function such that*

$$x^*T(f) = \int_\Omega f x^* g \, d\mu, \ \forall \ f \in L^p(\mu) \text{ and } \forall \ x^* \in X^*.$$

Equivalently, there exists a measurable $g : \Omega \to X$ such that

$$T(f) = (\text{Pettis}) - \int_\Omega fg \, d\mu, \ \forall \ f \in L^p(\mu).$$

If $p = 1$, then $T : L^1(\mu) \to X$ is a vector integral operator if and only if T is Riesz representable. According to the Riesz Representation Theorem (cf. [20], p. 63) in case of a finite measure this is so for all $T \in L(L^1(\mu), X)$ iff X has RNP.

For $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, a measurable $g : \Omega \to X$ is the **kernel** *of a vector integral operator $T : L^p(\mu) \to X$ if and only if $x^*g \in L^{p'}(\mu)$ for all $x^* \in X^*$. We see this fact as follows:*

Let $x^*g \in L^{p'}(\mu)$, $\forall \ x^* \in X^*$. Define $T : L^p(\mu) \to X^{*\prime}$ by

$$(Tf)(x^*) = \int_\Omega f(t) x^* g(t) \, d\mu(t).$$

We prove that $T(f) \in X^{**}$:

For $f \in L^p(\mu)$ fixed, define $S : X^* \to L^1(\mu)$ by $Sx^*(\cdot) = x^*(f(\cdot)g(\cdot))$. Note that S is closed. Indeed if $\lim_n x_n^* = x^*$ and $\lim_n Sx_n^* = h$ in $L^1(\mu)$, then some subsequence $x_{n_j}^*(fg) = S(x_{n_j}^*)$ tends μ-almost everywhere to h. But

$$\lim_n x_n^*((fg)(t)) = x^*((fg)(t)) \text{ everywhere.}$$

Hence, $x^*(fg) = h$ μ-almost everywhere, i.e. $Sx^* = h$ μ-almost everywhere and S is a closed linear operator. From the Closed Graph Theorem we conclude that S is continuous. Hence:

$$\|x^*(fg)\|_{L^1(\mu)} = \|Sx^*\| \leq \|S\| \|x^*\|.$$

It follows that

$$|Tf(x^*)| = |\int_\Omega f(t)x^*(g(t))\ d\mu(t)| = \|x^*(fg)\|_{L^1(\mu)} \le \|S\|\|x^*\|,\ \forall\ x^* \in X^*.$$

$\therefore\ Tf \in X^{**}$ *and* $T(f)x^* = \int_\Omega x^*(f(t)g(t))\ d\mu(t)$, *i.e.* $Tf = (Dunford) - \int_\Omega fg\ d\mu$. *If we let* $\upsilon(E) = (Dunford) - \int_E f(t)g(t)\ d\mu(t)$ *and* (E_n) *is a sequence of mutually disjoint measurable sets in* Ω, *then*

$$\begin{aligned}
\upsilon(\cup_{n=1}E_n) &= (Dunford) - \int_{\cup_{n=1}E_n} f(t)g(t)\ d\mu(t) \\
&= (Dunford) - \int_\Omega \chi_{\cup_{n=1}E_n} f(t)g(t)\ d\mu(t) \\
&= T(\chi_{\cup_{n=1}E_n} f).
\end{aligned}$$

Also,

$$\begin{aligned}
T(\chi_{\cup_{n=1}E_n} f)(x^*) &= \int_\Omega (\chi_{\cup_{n=1}E_n} f x^* g)(t)\ d\mu(t) \\
&= \int_\Omega (\sum_{n=1}^{\infty} \chi_{E_n} f x^* g)(t)\ d\mu(t) \\
&= \sum_{n=1}^{\infty} \int_\Omega (\chi_{E_n} f x^* g)(t)\ d\mu(t) \\
&= (\sum_{n=1}^{\infty} T(\chi_{E_n} f))(x^*).
\end{aligned}$$

Thus, $\upsilon(\cup_{n=1}E_n) = \sum_{n=1}^{\infty} T(\chi_{E_n} f) = \sum_{n=1}^{\infty} (Dunford) - \int_{E_n} fg\ d\mu = \sum_{n=1}^{\infty} \upsilon(E_n)$; *i.e. the set function* υ *is countably additive. By ([20], p. 54) it follows that*

$$T(f) = (Pettis) - \int_\Omega fg\ d\mu;$$

i.e. $Tf \in X$ *for all* $f \in L^p(\mu)$.

Conversely, suppose a measurable $g : \Omega \to X$ *is the kernel of a vector integral operator* $T : L^p(\mu) \to X$. *Then*

$$\begin{aligned}
(\int_\Omega |x^* g|^{p'}\ d\mu)^{\frac{1}{p'}} &= \sup_{\|f\|_{L^p(\mu)} \le 1} |\int_\Omega f(t)x^* g(t)\ d\mu(t)| \\
&= \sup_{\|f\|_{L^p(\mu)} \le 1} |x^* T(f)| \\
&\le \|x^*\|\|T\| < \infty.
\end{aligned}$$

(f) (cf. [20], p. 48) *Let f, g be μ-measurable.*

If $x^*f = x^*g$ μ-almost everywhere $\forall\ x^* \in X^*$, then $f = g$ μ-almost everywhere. Thus, the kernel of a vector integral operator is almost everywhere uniquely defined: Let g_1 and g_2 be kernels of a vector integral operator $T : L^p(\mu) \to X$, then g_1 and g_2 are measurable and $x^*g_1,\ x^*g_2 \in L^{p'}(\mu)$. Also,

$$\begin{aligned}
\int_\Omega x^*(f(t)g_1(t))\ d\mu(t) &= \int_\Omega x^*(f(t)g_2(t))\ d\mu(t),\ \forall\ x^* \in X^* \text{and} \forall\ f \in L^p(\mu).\\
\Rightarrow \quad \int_\Omega f(t)[x^*g_1(t)]\ d\mu(t) &= \int_\Omega f(t)[x^*g_2(t)]\ d\mu(t),\ \forall\ f \in L^p(\mu),\ \forall\ x^* \in X^*.\\
\Rightarrow \quad x^*g_1 &= x^*g_2 \quad \mu - a.e.,\ \forall\ x^* \in X^*.\\
\Rightarrow \quad g_1 &= g_2 \quad \mu - a.e.
\end{aligned}$$

Definition 1.6 *A function $f : \Omega \to X$ is called* **Bochner integrable** *if there exists a sequence of simple functions (f_n) such that*

$$\lim_n \int_\Omega \|f_n - f\| d\mu = 0.$$

In this case $\int_E f d\mu$ is defined for each $E \in \Sigma$ by

$$\int_E f d\mu = \lim_n \int_E f_n d\mu,$$

where $\int_E f_n d\mu$ is defined in the obvious way.

A concise characterization of Bochner integrable functions is presented next.

Theorem 1.7 *(cf. [20], p. 45) A μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\|\ d\mu < \infty$.*

Lemma 1.8 *(cf. [20], p. 172) Let $f : \Omega \to X$ be Bochner integrable. For each $\epsilon > 0$ there is a sequence (x_n) in X and a (not necessarily disjoint) sequence (E_n) in Σ such that*

(i) *the series $\sum_{n=1}^{\infty} x_n \chi_{E_n}$ converges to f absolutely μ-a.e. and*

(ii)

$$\int_{\Omega} \|f\| d\mu \leq \sum_{n=1}^{\infty} \|x_n\| \mu(E_n) \leq \int_{\Omega} \|f\| d\mu + \epsilon.$$

• If $1 \leq p < \infty$, let $L^p(\mu, X)$ denote the space of equivalence classes of X-valued Bochner integrable functions $f : \Omega \to X$ such that the norm is given by

$$\|f\|_{L^p(\mu,X)} = (\int_{\Omega} \|f\|^p \, d\mu)^{\frac{1}{p}} < \infty,$$

i.e.

$$L^p(\mu, X) = \{f : \Omega \to X \mid (\int_{\Omega} \|f\|^p \, d\mu)^{\frac{1}{p}} < \infty\}.$$

• $L^p(\mu, X)$ with this norm is a Banach space (cf. [20]).

• $L^{\infty}(\mu, X)$ will stand for all (equivalence classes of) essentially bounded μ-Bochner integrable functions $f : \Omega \to X$, where the norm is defined by $\|f\|_{L^{\infty}(\mu,X)} = \operatorname{ess\,sup}_{w \in \Omega} \|f(w)\|$.

• $L^{\infty}(\mu, X)$ with this norm is a Banach space (cf. [20]).

Remark 1.9 *(cf. [20])*

(1) For $1 \leq p < \infty$, the simple functions are dense in $L^p(\mu, X)$.

(2) The countable valued functions in $L^{\infty}(\mu, X)$ are dense in $L^{\infty}(\mu, X)$.

(3) For a finite measure space (Ω, Σ, μ) and $1 \leq p < \infty$, we have

$$L^p(\mu, X)^* = L^{p'}(\mu, X^*), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

if and only if X^* has Radon-Nikodym property with respect to μ. In this case the duality is defined by the bilinear functional

$$\langle f, g \rangle = \int_{\Omega} g(t)[f(t)] \, d\mu(t)$$

for all $f \in L^p(\mu, X)$ and $g \in L^{p'}(\mu, X^*)$. This is for instance true if X is reflexive (cf. [20], p. 76).

(4) $L^p(\mu, X)$ is reflexive if and only if $L^p(\mu)$ and X are reflexive.

••If $1 \le p < \infty$ and $(xf)(t) = f(t)(x)$ let $L^p_w(\mu, X)$ denote the space of equivalence classes of **weakly p-integral functions**, i.e.

$$\begin{aligned} L^p_w(\mu, X) &= \{f : \Omega \to X \mid f \text{ is measurable and } (x^*f)(\cdot) = f(\cdot)(x^*) \in L^p(\mu), \ \forall\, x^* \in X^*\} \\ &= \{f : \Omega \to X \mid f \text{ is the kernel of a v.i.o., } T : L^{p'}(\mu) \to X\} \end{aligned}$$

and

$$\begin{aligned} L^p_w(\mu, X^*) &= \{f : \Omega \to X^* \mid f \text{ is measurable and } xf \in L^p(\mu), \ \forall\, x \in X\} \\ &= \{f : \Omega \to X^* \mid f \text{ is the kernel of a v.i.o., } T : L^{p'}(\mu) \to X^*\}. \end{aligned}$$

Let $||g||_p^{weak} := \sup_{||x^*|| \le 1} (\int_\Omega |x^*g(t)|^p \ d\mu(t))^{\frac{1}{p}}$.
Let g_1, g_2 be kernels of vector integral operators. Notice that if

$$(\int_\Omega |x^*g_1(t) - x^*g_2(t)|^q \ d\mu(t))^{\frac{1}{q}} = 0, \ \forall\, x^* \in X^*,$$

then $x^*g_1 = x^*g_2 \quad \mu - a.e.$, $\forall\, x^* \in X^*$. Thus, it follows that $x^*g_1(w) = x^*g_2(w)$ for all $x^* \in X^*$ and $\forall\, w \in \Omega \setminus E$, where $\mu(E) = 0$; i.e

$$g_1(\omega) = g_2(\omega), \ \forall\, \omega \in \Omega \setminus E.$$

Thus, if we put

$$||g||_q^{weak} = \sup_{||x^*|| \le 1} (\int_\Omega |x^*g(t)|^q \ d\mu(t))^{\frac{1}{q}},$$

then

$$(\int_\Omega |x^*g_1(t) - x^*g_2(t)|^q d\mu(t))^{\frac{1}{q}} = 0, \quad \forall\, x^* \in X^*.$$

This implies that $g_1 = g_2 \ \mu - a.e.$ Therefore, if we denote by $L^q_w(\mu, X)$ the family of all equivalence classes of measurable $g : \Omega \to X$ such that

$$x^*g \in L^q(\mu) \quad (1 < q \le \infty), \ \forall\, x^* \in X^*$$

(i.e. all kernels of vector integral operators $T : L^{q'}(\mu) \to X$) then $(L^q_w(\mu, X), ||.||_q^{weak})$ is a normed space. It is easy to verify that $L^q(\mu, X) \subseteq L^q_w(\mu, X)$ and

$$||g||_q^{weak} \le ||g||_{L^q(\mu,X)} \text{ for all } g \in L^q_w(\mu, X) \ (cf.\ Lemma\ 5.2).$$

Also,

$$
\begin{aligned}
\|g\|_q^{weak} &= \sup_{\|x^*\|\le 1} \Big(\int_\Omega |x^* g(t)|^q \, d\mu(t) \Big)^{\frac{1}{q}} \\
&= \sup_{\|x^*\|\le 1} \sup_{\|f\|_{L^{q'}(\mu)}\le 1} \Big| \int_\Omega f(t) x^* g(t) \, d\mu(t) \Big| \\
&= \sup_{\|f\|_{L^{q'}(\mu)}\le 1} \sup_{\|x^*\|\le 1} |x^* T_g(f)| \\
&= \sup_{\|f\|_{L^{q'}(\mu)}\le 1} \|T_g(f)\| = \|T_g\|,
\end{aligned}
$$

where g is the (μ-a.e. uniquely defined) kernel of

$$T_g : L^{q'}(\mu) \to X : f \mapsto T_g f$$

and $(T_g f)(x^*) = \int_\Omega f(t)(x^* g)(t) \, d\mu(t)$. Thus $g \to T_g$ defines an isometric embedding of $L^q_w(\mu, X)$ into $L(L^{q'}(\mu), X)$.

1.3 Basics about (p, q)-summing sequences and strongly p-summable sequences

We start with a recapitulation from the theory of absolutely summing operators, which was developed mainly by Pietsch in the late sixties. The reader is referred to [19] for the following.

Definition 1.10 *(a) A* **sequence** *(x_n) in a Banach space is* **absolutely summable** *if $\sum_{n=1}^\infty \|x_n\| < \infty$.*

(b) A sequence (x_n) in a Banach space is **unconditionally summable** *if $\sum_{n=1}^\infty x_{\sigma(n)}$ converges, regardless of the permutation σ of $\mathbb{N}$.*

(c) An **operator** *$u \in L(X, Y)$ is* **absolutely summing** *if for every unconditionally convergent series $\sum_{j=1}^\infty x_j$ in X it holds that $\sum_{j=1}^\infty u x_j$ is absolutely convergent in Y.*

Theorem 1.11 Omnibus theorem on unconditional summability

(cf. [19], p. 9)

For a sequence (x_n) in a Banach space X the following are equivalent:

(i) (x_n) *is unconditionally summable.*

(ii) $(b_n) \mapsto \sum_{n=1}^{\infty} b_n x_n$ *defines a compact operator* $c_0 \to X$.

From the fact that $K(c_0, X) \simeq \ell_c^1(X)$ and the theorem above it follows that (x_n) is unconditionally summable if and only if $(x_n) \in \ell_c^1(X)$, but since $\ell_c^1(X) \subset \ell_w^1(X)$ it follows that if (x_n) is unconditionally summable then it is also weakly absolutely summable.

Given $1 \leq q \leq p < \infty$, the space $\Pi_{p,q}(X,Y)$ of (p,q)**-summing operators** is the vector space of those operators which map sequences in $\ell_w^q(X)$ onto sequences in $\ell^p(Y)$; more precisely $u \in L(X,Y)$ is in $\Pi_{p,q}(X,Y)$ if there exists a $c > 0$ such that:

$$||(ux_j)||_{\ell^p(Y)} \leq c\ \epsilon_q((x_j))$$

for any finite family of vectors x_j in X; the least of such c is the (p,q)-summing norm of u, denoted by $\pi_{p,q}(u)$. Note that (p,p)-summing is the same as p-summing and an operator is 1-summing if and only if it is absolutely summing (cf. [19], p. 34).
Apiola and Cohen were the first to introduce $\ell^p\langle X\rangle$, the space of strongly p-summable sequences in X.

Definition 1.12 *(cf. [14]) Let* $1 \leq p \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. $\ell^p\langle X\rangle$ *denotes the space of* **strongly** p**-summable sequences** *in* X, *i.e.*

$$\ell^p\langle X\rangle = \{(x_n) \in X^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n^*(x_n)| < \infty,\ \forall\ (x_n^*) \in \ell_w^q(X^*)\}$$

and

$$||(x_n)||_{\langle p\rangle} = \sup_{(x_n^*)_n \in B_{\ell_w^q(X^*)}} \sum_{n=1}^{\infty} |x_n^*(x_n)|.$$

Then $(\ell^p\langle X\rangle, ||.||_{\langle p\rangle})$ *is a Banach space (cf. [1], [16]).*

From the work of Cohen (cf. [16]) we observe that:

Theorem 1.13 *(i) For* $1 \leq p \leq \infty$,

$$\ell^p\langle X\rangle \subseteq \ell^p(X) \subseteq \ell_w^p(X) \text{ and } \epsilon_p(\cdot) \leq ||\cdot||_{\ell^p(X)} \leq ||\cdot||_{\langle p\rangle}.$$

(ii) For $p = 1$, $\ell^1\langle X\rangle = \ell^1(X)$ *and* $\|\cdot\|_{\ell^1(X)} = \|\cdot\|_{\langle 1\rangle}$.

From the work of Bu (cf. [14], p. 526) it follows that

$$\ell^p\langle Y\rangle^* \overset{isometrically}{\equiv} \ell^{p'}_w(Y^*) \text{ and } \ell^q_c(X)^* \overset{isometrically}{\equiv} \ell^{q'}\langle X^*\rangle.$$

Definition 1.14 *(cf. [2]) For any Banach space* X *we define the space* $\ell_{\pi_{p,q}}(X)$ *of* (p,q)-**summing sequences** *in* X*, as the set of all sequences* (x_j) *in* X *such that there exists a constant* $c > 0$ *for which*

$$\left(\sum_{j=1}^n |x_j^* x_j|^p\right)^{\frac{1}{p}} \leq c \sup_{x\in B_X} \left(\sum_{j=1}^n |x_j^* x|^q\right)^{\frac{1}{q}}$$

for any finite collection of vectors $x_1^*, \cdots, x_n^*$ *in* X^*.

The following theorem gives the connection between the strongly p-summable sequences and the $(1,q)$-summing sequences.

Theorem 1.15 *(cf. [23]) Let* $1 \leq p \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $\ell^p\langle X\rangle = \ell_{\pi_{1,q}}(X)$ *and* $\|(x_j)\|_{\langle p\rangle} = \pi_{1,q}((x_j))$.

Theorem 1.16 *(cf. [14] and [23]) Let* $1 < p < \infty$. *Then* $\Psi(\ell^p \overset{\wedge}{\otimes} X) = \ell^p\langle X\rangle$ *and* $\Psi(c_0 \overset{\wedge}{\otimes} X) = c_0\langle X\rangle$, *where* Ψ *is an isometry and where* $\ell^p \overset{\wedge}{\otimes} X$ *(or* $c_0 \overset{\wedge}{\otimes} X$*) is the completion of* $\ell^p \otimes X$ *(or* $c_0 \otimes X$*) with respect to the projective tensor norm* $|.|_\wedge$.

Following a similar argument as in our proof of Theorem 1.23 in [23], we prove a more general result in Chapter 2.

1.4 Basics about Banach lattices

The following definitions can be found in [33] and [35].

Recall that a Banach lattice X is an ordered vector space equipped with a lattice structure and a Banach space norm satisfying the following conditions:

$$\forall\ x,\ y \in X, \qquad |x| \leq |y| \Rightarrow \quad \|x\| \leq \|y\|, \text{ where } |x| = x \vee (-x).$$

We say h is a homomorphism between two Banach lattices X_1 and X_2 if $h : X_1 \to X_2$ is a linear operator such that

$$h(x \vee y) = h(x) \vee h(y), \qquad \forall\, x, y \in X_1.$$

Let $\widetilde{X(\ell^p)}$ be the space of all sequences $x = (x_i)$ of elements of X for which

$$\|x\|_{\widetilde{X(\ell^p)}} = \sup_n \|(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}\| < \infty.$$

Let $X(\ell^p)$ denote the closed subspace of $\widetilde{X(\ell^p)}$, spanned by the sequences (x_i), which are eventually zero.

Note that $(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \in X$ is defined by

$$(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} = \sup_{(\alpha_i)\in B_{\ell^{p'}}} \sum_{i=1}^{n} \alpha_i x_i, \text{ where } \frac{1}{p} + \frac{1}{p'} = 1.$$

Definition 1.17 *(cf. [33], p. 45)*

Let X be a Banach lattice, Y an arbitrary Banach space and let $1 \leq p < \infty$.

(i) A linear operator $T : Y \to X$ is called **p-convex** *if there exists a constant $M < \infty$ so that*

$$\|(\sum_{i=1}^{n} |Ty_i|^p)^{\frac{1}{p}}\| \leq M\ (\sum_{i=1}^{n} \|y_i\|^p)^{\frac{1}{p}}, \qquad \text{if } 1 \leq p < \infty$$

for every choice of vectors $(y_i)(\leq n)$ in Y. The smallest possible value of M is denoted by $M^p(T)$. A linear operator T from a Banach space Y to a Banach lattice X is p-convex for some $1 \leq p < \infty$ if and only if the map $\hat{T} : \ell^p(Y) \to X(\ell^p)$, defined by $\hat{T}(y_1, y_2, \cdots) = (Ty_1, Ty_2, \cdots)$, is a bounded linear operator. Moreover, $\|\hat{T}\| = M^p(T)$.

(ii) A linear operator $T : X \to Y$ is called **p-concave** *if there exists a constant $M < \infty$ so that*

$$(\sum_{i=1}^{n} \|Tx_i\|^p)^{\frac{1}{p}} \leq M\ \|(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}\|, \qquad \text{if } 1 \leq p < \infty$$

for every choice of vectors $(x_i)(\leq n)$ in X. The smallest possible value of M is denoted by $M_p(T)$. A linear operator $T : X \to Y$ is p-concave for some $1 \leq p < \infty$ if *and only if the map* $\overset{\vee}{T} : X(\ell^p) \to \ell^p(Y)$, *defined by* $\overset{\vee}{T}(x_1, x_2, \cdots) = (Tx_1, Tx_2, \cdots)$, *is a bounded linear operator. Moreover,* $\|\overset{\vee}{T}\| = M_p(T)$.

(iii) *We say that* X *is p-convex or p-concave if the identity operator* id_X *on* X *is p-convex, respectively, p-concave.*

Remark 1.18 *(1)* $L^p(\mu)$ *is both p-convex and p-concave (cf. [33], p. 45).*

(2) Let $f_1, \cdots, f_n \in L^p(\mu)$ *and* $1 \leq p < \infty$, *then there exist* $c_1, c_2 > 0$ *such that*

$$c_1\left(\int_0^1 \|\sum_{j=1}^n r_j(t) f_j\|_{L^p(\mu)}^2 \, dt\right)^{\frac{1}{2}} \leq \|(\sum_{j=1}^n |f_j|^2)^{\frac{1}{2}}\|_{L^p(\mu)} \leq c_2\left(\int_0^1 \|\sum_{j=1}^n r_j(t) f_j\|_{L^p(\mu)}^2 \, dt\right)^{\frac{1}{2}}$$

(cf. [33], p. 74).

Theorem 1.19 *(cf. [35], p. 99) A Banach lattice* X *is of cotype* 2 *iff it is* $2-$*concave. Moreover,* X^* *is of cotype* 2 *iff* X *is* $2-$*convex.*

1.5 Basics about bases in Banach spaces

From [32] and [37] we get the following definitions.

Definition 1.20 *(i)* • *A sequence* (x_n) *in a Banach space* X *is called a* **Schauder basis** *of* X *if for every* $x \in X$ *there is a unique sequence of scalars* (a_n) *so that* $x = \sum_{n=1}^{\infty} a_n x_n$.

• *A sequence* (x_n) *which is a Schauder basis of its closed linear span is called a* **basic sequence.**

• (x_i) *is an* **unconditional basic sequence** *if and only if any of the following conditions hold.*

(a) $(x_{\sigma(i)})$ *is a basic sequence for every permutation* $\sigma \in \mathbb{N}$.

(b) The convergence of $\sum_{n=1}^{\infty} a_n x_n$ *implies the convergence of* $\sum_{n=1}^{\infty} b_n x_n$ *whenever* $|b_n| \leq |a_n|$, *for all* n.

(ii) *The sequence of functions* $\{\chi_i(t)\}_1^\infty$, *defined by* $\chi_1(t) \equiv 1$, *and, for* $k = 0, 1, 2, \cdots$, *and* $j = 1, 2, \cdots, 2^k$,

$$\chi_{2^k+j}(t) = \begin{cases} 1 & \text{if} \quad t \in [\frac{(2j-2)}{2^{k+1}}, \frac{(2j-1)}{2^{k+1}}] \\ -1 & \text{if} \quad t \in [\frac{(2j-1)}{2^{k+1}}, \frac{2j}{2^{k+1}}] \\ 0 & \text{otherwise} \end{cases}$$

is called the **Haar system**.

(iii)
- *Let* (x_n) *be a basis of a Banach space* X. *The biorthogonal functionals* (x_n^*) *form a basis of* X^* *if and only if, for every* $x^* \in X^*$, *the norm of the restriction of* x^* *to the span of* (x_n) *tends to* 0 *as* $n \to \infty$. *A basis* (x_n) *which has this property is called* **shrinking** *(cf. [32], Proposition 1.b.1.).*
- *Let* (x_n) *be a shrinking basis of a Banach space* X. *Then* X^{**} *can be identified with the space of all sequences of scalars* (a_n) *such that* $\sup_n \|\sum_{i=1}^n a_i x_i\| < \infty$. *This correspondence is given by* $x^{**} \leftrightarrow (x^{**}(x_1^*), x^{**}(x_2^*), \cdots)$. *The norm of* x^{**} *is equivalent (and in case the basis constant is* 1 *even equal) to*

$$\sup_n \|\sum_{i=1}^n x^{**}(x_i^*) x_i\|$$

(cf. [32], Proposition 1.b.2.).

(iv) *A basis* (x_n) *of a Banach space is called* **boundedly complete** *if, for every sequence of scalars* (a_n) *such that* $\sup_n \|\sum_{i=1}^n a_i x_i\| < \infty$, *the series* $\sum_{n=1}^\infty a_n x_n$ *converges.*

- *The unit vector basis is boundedly complete in all the* ℓ^p *spaces.*
- *By combining the definitions of shrinking and boundedly complete we get a characterization of reflexivity in terms of bases.*
  (cf. [32], Theorem 1.b.5)
  Let X *be a Banach space with a Schauder basis* (x_n). *Then* X *is reflexive if and only if* (x_n) *is both shrinking and boundedly complete.*

(v) *Let* X *be a Banach space with a Schauder basis* (x_n).

- *Consider the projections* $P_n : X \to X$*, defined by* $P_n(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i$*, then the number* $\sup_n \|P_n\|$ *is called the* **basis constant** *of* (x_n).

- *If* (x_n^*) *is a basis sequence in* X^* *then its basis constant is identical to that of* (x_n).

- *Consider the projections* $M_\Theta : X \to X$*, defined by* $M_\Theta(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{\infty} \Theta_i a_i x_i$*, for every choice of signs* $\Theta = (\Theta_i)$*. The number* $\sup_\Theta \|M_\Theta\|$ *is the* **unconditional constant** *of* (x_n).

- *The basis constant is less or equal to the unconditional constant.*

- *If* (x_n) *is an unconditional basis of* X *we can always define an equivalent norm on* X *so that the unconditional constant becomes* 1.

- *For every integer* n *the linear functional* x_n^* *on* X *defined by*

$$x_n^*(\sum_{i=1}^{\infty} a_i x_i) = a_n$$

is a bounded linear functional. These functionals (x_n^*)*, which are characterized by the relation* $x_n^*(x_m) = \delta_{nm}$*, are the* **biorthogonal functionals** *associated to the basis* (x_n)*. If* (x_n) *is an unconditional basis sequence in* X *then the biorthogonal functionals* (x_n^*) *form an unconditional basis sequence in* X^* *whose unconditional constant is the same as that of* (x_n).

(vi) *A basis whose basis constant is* 1 *is called a* **monotone basis**, *i.e. for every choice of scalars* (a_n) *the sequence of numbers* $(\|\sum_{i=1}^{n} a_i x_i\|)$ *is increasing.*
A space with a monotone basis has the m.a.p.
Given any Schauder basis (x_n) *in* X*, we can pass to an equivalent norm in* X *for which the given basis is monotone.*

(vii) *A Banach space* X *is said to have the* **approximation property** *(A.P. in short) if, for every compact set* K *in* X *and every* $\epsilon > 0$*, there is an operator* $T : X \to X$ *of finite rank (i.e* $Tx = \sum_{i=1}^{n} x_i^*(x) x_i$*, for some* $(x_i) \subset X$ *and* $(x_i^*) \subset X^*$*) so that* $\|Tx - x\| \le \epsilon$ *for every* $x \in K$.
Every space with a Schauder basis has the A.P.

(viii) *If X^* has the approximation property then X has the approximation property. In particular, if X is reflexive then X has the approximation property if and only if X^* has the approximation property.*

(ix) **The principle of local reflexivity**

Let X be a Banach space and let E and F be finite dimensional subspaces of X^{**} and X^* respectively. Then for each $\epsilon > 0$, there is an injective operator $u : E \to X$ with the following properties:

(a) $ux = x$ *for all* $x \in E \cap X$

(b) $\|u\|\|u^{-1}\| \leq 1 + \epsilon$

(c) $\langle ux^{**}, x^* \rangle = \langle x^{**}, x^* \rangle \quad \forall\, x^{**} \in E, \ \forall\, x^* \in F.$

Chapter 2

Vector sequence spaces

2.1 General vector sequence spaces

Let U be a reflexive Banach space with a normalized unconditional basis (e_i) and let X be a Banach space. By renorming U we may assume that the unconditional basis constant is 1. Let (e_i^*) be the unconditional dual basis of U^* with the unconditional basis constant 1. By normalization we can assume that $\|e_i^*\| = 1$ for each $i \in \mathbb{N}$. Moreover, (e_i) and (e_i^*) are orthonormal, i.e. $e_i^*(e_j) = \delta_{j,i}$, where $\delta_{i,i} = 1$ and $\delta_{j,i} = 0$ if $j \neq i$.

In [11] the following vector sequence spaces are introduced:

$$U_{strong}(X) = \{\overline{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} \|x_i\| e_i \text{ converges in } U\},$$

which is a Banach space with respect to the norm

$$\|\overline{x}\|_{strong} = \left\| \sum_{i=1}^{\infty} \|x_i\| e_i \right\|_U ;$$

$$U_{weak}(X) = \{\overline{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} x^*(x_i) e_i \text{ converges in } U,\ \forall\ x^* \in X^*\},$$

which is a Banach space with respect to the norm

$$\|\overline{x}\|_{weak} = \sup_{x^* \in B_{X^*}} \|\sum_{i=1}^{\infty} x^*(x_i) e_i\|_U \ \text{ or } \|\overline{x}\|_{weak} = \sup_{u^* \in B_{U^*}} \|\sum_{i=1}^{\infty} u^*(e_i) x_i\|_X.$$

In order to avoid ambiguities the norms $\|\overline{x}\|_{U_{strong}(X)}$ and $\|\overline{x}\|_{U_{weak}(X)}$ are sometimes used.

$$U\langle X\rangle = \{\overline{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i^*(x_i)| < \infty,\ \forall\ (x_i^*) \in U_{weak}^*(X^*)\},$$

which is a Banach space with respect to the norm

$$\|\overline{x}\|_{U\langle X\rangle} = \sup_{(x_i^*)\in\ B_{U^*_{weak}(X^*)}} \sum_{i=1}^{\infty} |x_i^*(x_i)|;$$

$$U_{weak,o}(X) = \{\overline{x} \in U_{weak}(X) : \lim_n \|\overline{x}(\geq n)\|_{weak} = 0\}.$$

From the work of Bu (cf. [11], p. 29, 33 and 35) we observe that:
$U\langle X\rangle \subseteq U_{strong}(X) \subseteq U_{weak}(X)$ and

$$\frac{1}{2}\|\cdot\|_{weak} \leq \|\cdot\|_{strong} \leq 2\|\cdot\|_{U\langle X\rangle}.$$

In case U is a real Banach space, $\|.\|_{weak} \leq \|\cdot\|_{strong} \leq \|\cdot\|_{U\langle X\rangle}$.
$U\langle X\rangle^* = U^*_{weak}(X^*)$ and $U_{weak,0}(X)^* = U^*\langle X^*\rangle$ isometrically. To obtain these isometries, we identify a sequence (x_i^*) in $U^*_{weak}(X^*)$ with the linear functional $f \in U\langle X\rangle^*$ defined by $f((x_i)) = \sum_{i=1}^{\infty} x_i^*(x_i)$. It is easily seen (cf. [11], Proposition 1.5.2) that $\overline{x} = (x_i)_i \in U\langle X\rangle$ if and only if the series $\sum_{i=1}^{\infty} x_i^*(x_i)$ converges for each $(x_i^*) \in U^*_{weak}(X^*)$ and that

$$\|\overline{x}\|_{U\langle X\rangle} = \sup_{(x_i^*)\in\ B_{U^*_{weak}(X^*)}} |\sum_{i=1}^{\infty} x_i^*(x_i)|.$$

In the following lemmas and corollaries, we summarise some properties about the different vector sequence spaces.

Lemma 2.1 *Let $u \in L(X,Y)$.*
If $(x_i) \in U_{weak}(X)$, then $(ux_i) \in U_{weak}(Y)$, with $\|(ux_i)\|_{weak} \leq \|u\|\|(x_i)\|_{weak}$.

Proof Let $(x_i) \in U_{weak}(X)$ then $\sum_{i=1}^{\infty} x^*(x_i)e_i$ converges in U, for all $x^* \in X^*$ and $\|(x_i)\|_{weak} = \sup_{x^*\in B_{X^*}} \|\sum_{i=1}^{\infty} x^*(x_i)e_i\|_U$. Choose $u \in L(X,Y)$. For $y^* \in Y^*$ we have

$$\begin{aligned} \|\sum_{i=1}^{\infty} y^*(ux_i)e_i\|_U &= \|\sum_{i=1}^{\infty}(y^* \circ u)(x_i)e_i\|_U \\ &\leq \|y^* \circ u\|\|(x_i)\|_{weak} \\ &\leq \|y^*\|\|(x_i)\|_{weak}\|u\|. \end{aligned}$$

Thus $(ux_i) \in U_{weak}(Y)$ and $\|(ux_i)\|_{weak} \leq \|u\|\|(x_i)\|_{weak}$. □

Corollary 2.2 *If $(x_i^{***}) \in U_{weak}(X^{***})$ and $i_X : X \to X^{**}$ is the embedding mapping, then $(x_i^{***} \circ i_X) \in U_{weak}(X^*)$ and $\|(x_i^{***} \circ i_X)\|_{weak} \leq \|(x_i^{***})\|_{weak}$.*

Corollary 2.3 *If $(x_i^*) \in U_{weak}(X^*)$, then $(x_i^*) \in U_{weak}(X^{***})$ and*

$$\|(x_i^*)\|_{U_{weak}(X^{***})} \leq \|(x_i^*)\|_{U_{weak}(X^*)}.$$

Lemma 2.4 *A sequence (x_n) in a Banach space X is in $U\langle X\rangle$ if and only if $(i_X x_n) \in U\langle X^{**}\rangle$ and $\|(i_X x_n)\|_{U\langle X^{**}\rangle} = \|(x_n)\|_{U\langle X\rangle}$.*

Proof Let $(x_i) \in U\langle X\rangle$ and $(x_i^{***}) \in U_{weak}^*(X^{***})$. By Corollary 2.2 we have $(x_i^{***} \circ i_X) \in U_{weak}^*(X^*)$ and $\|(x_i^{***} \circ i_X)\|_{weak} \leq \|(x_i^{***})\|_{weak}$. This implies that

$$\sum_{i=1}^{\infty} |x_i^{***}(i_X x_i)| = \sum_{i=1}^{\infty} |(x_i^{***} \circ i_X)(x_i)| < \infty,$$

from which it is clear that $(i_X x_n) \in U\langle X^{**}\rangle$ and $\|(i_X x_n)\|_{U\langle X^{**}\rangle} \leq \|(x_n)\|_{U\langle X\rangle}$. Conversely, if $(i_X x_n) \in U\langle X^{**}\rangle$ and $(x_i^*) \in U_{weak}^*(X^*)$, then

$$\sum_{i=1}^{\infty} |x_i^*(x_i)| = \sum_{i=1}^{\infty} |i_X x_i(x_i^*)| < \infty$$

and $\|(i_X x_n)\|_{U\langle X^{**}\rangle} \geq \|(x_n)\|_{U\langle X\rangle}$ by Corollary 2.3. □

Lemma 2.5 *For each finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ we have*

$$\left|\sum_{i=1}^{n} e_i \otimes x_i\right|_{\vee} = \|(x_i)\|_{weak}.$$

Proof

$$\begin{aligned}\left|\sum_{j=1}^{n} e_j \otimes x_j\right|_{\vee} &= \sup_{\substack{\gamma \in B_{U^*} \\ x^* \in B_{X^*}}} \left|\sum_{j=1}^{n} \gamma(e_j) x^*(x_j)\right| \\ &= \sup_{\|x^*\| \leq 1} \left\|\sum_{j=1}^{n} x^*(x_j) e_j\right\|_U = \|(x_j)\|_{weak}.\end{aligned}$$

□

Lemma 2.5 will be instrumental in characterizing $U \overset{\wedge}{\otimes} X$ in terms of vector sequence spaces, using results from the theory of tensor products. Let us start by proving the following theorem.

Theorem 2.6 *Let X be a Banach space. Then $U\langle X\rangle \overset{isometric}{=} \mathcal{I}(U^*, X)$. The isometry is given by the mapping $(x_i) \mapsto u : U^* \to X : ue_j^* = x_j$ for all $j \in \mathbb{N}$.*

Proof We know from Grothendieck's work that $(U^* \overset{\vee}{\otimes} X)^*$ is isometrically identifiable with $\mathcal{I}(U^*, X^*)$, where each $u \in \mathcal{I}(U^*, X^*)$ is identified with ϕ_u such that $\phi_u(\sum_{j=1}^{n} e_j^* \otimes x_j) = \sum_{j=1}^{n} (ue_j^*)x_j$. The mapping $\Phi : U\langle X^*\rangle \to (U^* \otimes X)'$, defined by $\Phi((x_i^*))(e_j^* \otimes x) = x_j^* x$, satisfies

$$\left|\left\langle \Phi((x_j^*)), \sum_{j=1}^{n} e_j^* \otimes x_j \right\rangle\right| = \left|\sum_{i=1}^{n} x_i^*(x_i)\right| \le \|(x_i^*)\|_{U\langle X^*\rangle} \|(x_j)\|_{weak}.$$

Using that $\|(x_j)\|_{weak} = \left|\sum_{j=1}^{n} e_j^* \otimes x_j\right|_{\vee}$ by Lemma 2.5, it follows that

$$\Phi((x_j^*)) \in (U^* \overset{\vee}{\otimes} X)^*, \text{with } \|\Phi((x_j^*))\| \le \|(x_j^*)\|_{U\langle X^*\rangle}.$$

The bounded linear operator Φ has an inverse $\psi : (U^* \overset{\vee}{\otimes} X)^* \to U\langle X^*\rangle$, which is defined by $\xi \mapsto (x_j^*)$, where $\xi(e_j^* \otimes x) = x_j^* x$ for all $x \in X$. Using Lemma 2.5 for all finite sets $\{x_1, x_2, \ldots, x_n\} \subset X$ follows $\sum_{j=1}^{n} |x_j^* x_j| \le \|\xi\| \|(x_j)\|_{weak}$; i.e. $(x_j^*) \in U\langle X^*\rangle$ and $\|(x_j^*)\|_{U\langle X^*\rangle} \le \|\xi\|$. This shows that $\mathcal{I}(U^*, X^*) \overset{isometric}{=} U\langle X^*\rangle$, where the isometry is given by $u \mapsto (ue_j^*)$. Since this isometry holds for all Banach spaces X, then also for X^* if X is given, i.e.

$$\mathcal{I}(U^*, X^{**}) \overset{isometric}{=} U\langle X^{**}\rangle : u \mapsto (ue_j^*).$$

Finally we have

$$\begin{aligned} u \in \mathcal{I}(U^*, X) &\iff i_X \circ u \in \mathcal{I}(U^*, X^{**}) \\ &\iff ((i_X \circ u)(e_i^*)) \in U\langle X^{**}\rangle \iff (ue_i^*) \in U\langle X\rangle \end{aligned}$$

and $i(u) = \|(ue_i^*)\|_{U\langle X\rangle}$. □

Since the space U^* is reflexive and $U = U^{**}$ has the metric approximation property, it follows that:

Corollary 2.7 $U\langle X\rangle \overset{isometric}{=} N(U^*, X)$, *where the isometry is given by $(x_i) \mapsto u : U^* \to X : ue_j^* = x_j$ for all $j \in \mathbb{N}$.*

Corollary 2.8 *$(x_i) \in U\langle X\rangle$, if and only if there are $(\lambda_j) \in \ell^1$, $\{\gamma_j\}_1^\infty \subset B_U$ and $\{y_j\}_1^\infty \subset B_X$ such that $x_i = \sum\limits_{j=1}^{\infty} \lambda_j e_i^*(\gamma_j) y_j$ for all $i \in \mathbb{N}$.*

Proof $(x_i) \in U\langle X\rangle$ if and only if there exists $u \in N(U^*, X)$ such that $ue_i^* = x_i$ for all $i \in \mathbb{N}$. u being nuclear, this is so if and only if

$$x_i = ue_i^* = \sum_{j=1}^{\infty} \lambda_j \langle \gamma_j, e_i^* \rangle y_j$$

with $(\lambda_j) \in \ell^1$, $\{\gamma_j\}_1^\infty \subset B_U$ and $\{y_j\}_1^\infty \subset B_X$. □

Since $N(U^*, X) = U \overset{\wedge}{\otimes} X$ (for all Banach spaces X), when U satisfies the metric approximation property, we conclude that:

Corollary 2.9 *$U\langle X\rangle \overset{isometric}{=} U \overset{\wedge}{\otimes} X$ by the mapping $(x_j) \mapsto u$ where $ue_j^* = x_j$ for all $j \in \mathbb{N}$.*

2.1.1 U-summing multipliers and strongly U-summing multipliers

Definition 2.10 *A scalar sequence (α_i) is called a U-**summing multiplier** for a Banach space X, if*

$$(\alpha_i x_i) \in U_{strong}(X), \ \forall \ (x_i) \in U_{weak}(X).$$

Put $M_U(X) = \{(\alpha_n) \in \omega \ : \ (\alpha_n x_n) \in U_{strong}(X), \ \forall \ (x_n) \in U_{weak}(X)\}$.

Proposition 2.11 *([32], p. 19)*
Let (x_n) be an unconditional basic sequence in a Banach space, with an unconditional constant K. Then, for every choice of scalars (a_n) such that $\sum_{n=1}^{\infty} a_n x_n$ converges and every choice of bounded scalar sequences (λ_n), we have

$$\|\sum_{n=1}^{\infty} \lambda_n a_n x_n\| \leq 2K \sup_n |\lambda_n| \|\sum_{n=1}^{\infty} a_n x_n\|$$

(in the real case we can take K instead of $2K$).

Remark 2.12 *It follows in particular from Proposition 2.11 that if $\sum_{n=1}^{\infty} b_n x_n$ converges, where $b_n \geq 0$, $\forall \ n$ and if $0 \leq a_n \leq b_n$, for all n, then $\|\sum_{n=1}^{\infty} a_n x_n\| \leq K \|\sum_{n=1}^{\infty} b_n x_n\|$.*

Proof

Let $\lambda_i := \begin{cases} \frac{a_i}{b_i} & \text{if } b_i \neq 0 \\ 0 & \text{if } b_i = 0. \end{cases}$ *Then*

$$\|\sum_{n=1}^{\infty} a_n x_n\| = \|\sum_{n=1}^{\infty} \lambda_n b_n x_n\| \leq K\|\sum_{n=1}^{\infty} b_n x_n\|.$$

Theorem 2.13 *If* $\gamma = \sum_{i=1}^{\infty} \gamma_i e_i \in U$, *then* $(\gamma_i) \in M_U(X)$.

Proof Let $(x_i) \in U_{weak}(X)$. For fixed $i \in \mathbb{N}$, there exists $x^* \in X^*$, $\|x^*\| = 1$ such that

$$\|x_i\| = \langle x_i, x^* \rangle = \|\langle x_i, x^* \rangle e_i\|_U.$$

Since the basis (e_i) has unconditional constant $K = 1$, it follows from Proposition 2.11 that

$$\|x_i\| \leq 2\|\sum_{j=1}^{\infty} \langle x_j, x^* \rangle e_j\|_U \leq 2\|(x_j)\|_{weak}.$$

Thus $(\|x_i\|)$ is bounded. Proposition 2.11 also implies that

$$\left\|\sum_{i=m+1}^{n} \|\gamma_i x_i\| e_i\right\|_U \leq 2(\sup_i \|x_i\|) \left\|\sum_{i=m+1}^{n} |\gamma_i| e_i\right\|_U \overset{n,\, m}{\underset{\infty}{\longrightarrow}} 0,$$

again using that $\{e_i\}$ is an unconditional basis. Thus $(\gamma_i x_i) \in U_{strong}(X)$. □

Lemma 2.14 $(\alpha_i) \in M_U(X)$ *implies* $(\alpha_i x_i^*) \in U^*\langle X^* \rangle$ *for all* $(x_i^*) \in U^*_{strong}(X^*)$.

Proof Let $(x_i) \in U_{weak}(X)$. Then

$$\sum_{i=1}^{\infty} |\langle \alpha_i x_i^*, x_i \rangle| = \sum_{i=1}^{\infty} |\langle x_i^*, \alpha_i x_i \rangle| \leq \|(x_i^*)\|_{strong} \|(\alpha_i x_i)\|_{strong} < \infty.$$

□

On the vector space $M_U(X)$ we define a norm

$$\|(\alpha_i)\|_{M_U(X)} := \sup_{(x_i) \in B_{U_{weak}(X)}} \left\|\sum_{i=1}^{\infty} \|\alpha_i x_i\| e_i\right\|_U,$$

which is *well-defined* because for each $(\alpha_i) \in M_U(X)$ this is the operator norm of the linear operator

$$T_\alpha : U_{weak}(X) \to U_{strong}(X) :: (x_i) \mapsto (\alpha_i x_i),$$

where T_α is bounded (having closed graph):

Suppose $(x_{i,n})_i \to (x_i)$ and $T_\alpha((x_{i,n})_i) \to (y_i)$ if $n \to \infty$,

then

$$\|\alpha_i x_i - y_i\|_X \le \|(\alpha_i x_i) - (y_i)\|_{strong} \le \|(\alpha_i x_i) - (\alpha_i x_{i,n})\|_{strong} + \|(\alpha_i x_{i,n}) - (y_i)\|_{strong.}$$

Also,

$$\begin{aligned} \|(\alpha_i x_i) - (\alpha_i x_{i,n})\|_{strong} &= \sum_{i=1}^{\infty} \langle x_i^*, \alpha_i x_i - \alpha_i x_{i,n} \rangle, \quad \text{for some } \|(x_i^*)\|_{strong} = 1 \\ &= \sum_{i=1}^{\infty} \langle \alpha_i x_i^*, x_i - x_{i,n} \rangle \\ &\le \|(\alpha_i x_i^*)\|_{U^*(X^*)} \|(x_i - x_{i,n})\|_{weak} \xrightarrow[\infty]{n} 0 \end{aligned}$$

and

$$\left\| \sum_{i=1}^{\infty} \|\alpha_i x_{i,n} - y_i\| e_i \right\|_U \xrightarrow[\infty]{n} 0.$$

It follows that $\alpha_i x_i = y_i, \ \forall \ i$, i.e. that $T_\alpha((x_i)) = (y_i)$.

$M_U(X)$ is a complete normed space with respect to the above operator norm. In the exposition that follows an alternative definition of U-summing multipliers is given.

Definition 2.15 *A scalar sequence (α_i) is called a* **U-summing multiplier** *for a Banach space X, if there is a constant $c > 0$ such that regardless of the natural number m and regardless of the choice of $x_1, x_2, \cdots, x_m$ in X, we have*

$$\left\| \sum_{i=1}^{m} \|\alpha_i x_i\| e_i \right\|_U \le c \sup_{x^* \in B_{X^*}} \|\sum_{i=1}^{m} x^*(x_i) e_i\|_{U.} \tag{2.1}$$

First we prove that Definitions 2.10 and 2.15 are equivalent:

Proof Let $(\alpha_i) \in \ell^\infty$. If there is a constant $c > 0$ such that for all finite sets $\{x_1, \cdots, x_m\} \subset X$ we have

$$\left\| \sum_{i=1}^{m} \|\alpha_i x_i\| e_i \right\|_U \le c \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^{m} x^*(x_i) e_i \right\|_{U,}$$

then clearly

$$\left\| \sum_{i=1}^{m} \|\alpha_i x_i\| e_i \right\|_U < \infty, \ \forall \ (x_i) \in U_{weak}(X).$$

Conversely let $\|\sum_{i=1}^{m} \|\alpha_i x_i\| e_i\|_U < \infty, \ \forall\ (x_i) \in U_{weak}(X)$. Since the operator

$$T_\alpha : U_{weak}(X) \to U_{strong}(X) :: (x_i) \mapsto (\alpha_i x_i)$$

is bounded, we have

$$\|T_\alpha((x_i))\|_{strong} \leq \|T_\alpha\| \|(x_i)\|_{weak}, \ \forall\ (x_i) \in U_{weak}(X).$$

In particular, for all finite sets $\{x_1, \cdots, x_m\} \subset X$ it follows that

$$\left\| \sum_{i=1}^{m} \|\alpha_i x_i\| e_i \right\|_U \leq c \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^{m} x^*(x_i) e_i \right\|_{U.}$$

□

It is clear from the above discussion that on the vector space $M_U(X)$ the norm is also given by

$$\|(\alpha_i)\|_{M_U(X)} := \inf \{c > 0 : \text{ the inequality (2.1) holds}\}.$$

Proposition 2.16 $M_U(X^{**}) = M_U(X)$.

Proof Let $(\alpha_i) \in M_U(X)$ and $\epsilon > 0$.

We intend to show $(\alpha_i) \in M_U(X^{**})$.

For any finite set $\{x_1^{**}, x_2^{**}, ..., x_n^{**}\} \subset X^{**}$, let $x_k^* \in X^*$ with $\|x_k^*\| = 1$, such that

$$\|x_k^{**}\| \approx |\langle x_k^*, x_k^{**}\rangle| \quad \text{for } k = 1, 2, \cdots, n$$

" $\approx$ " meaning close enough so that $|\langle x_k^*, x_k^{**}\rangle| > 0$.

Let $E := \text{span}\{x_1^{**}, x_2^{**}, \cdots, x_n^{**}\} \subset X^{**}$ and $F := \text{span}\{x_1^*, x_2^*, \cdots, x_n^*\} \subset X^*$.

By the Principle of Local Reflexivity there exists an injective-bounded linear operator $u : E \to X$ with

(i) $\|u\| \|u^{-1}\| \leq 1 + \varepsilon$

(ii) $|\langle u x_i^{**}, x_i^*\rangle| = |\langle x_i^{**}, x_i^*\rangle| \neq 0$ for $x_i^{**} \neq 0, \ i = 1, 2, \cdots, n$.

Let $x_i := u x_i^{**} \in X, \ (i = 1, \cdots, n)$. Since $(\alpha_i) \in M_U(X)$, there is a $c > 0$ with

$$\left\| \sum_{i=1}^{n} \|\alpha_i y_i\| e_i \right\|_U \leq c \sup_{\|x^*\| \leq 1} \left\| \sum_{i=1}^{n} x^*(y_i) e_i \right\|_U, \ \forall\ \{y_1, ..., y_n\} \subset X.$$

Hence we have:

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \|\alpha_i x_i^{**}\| e_i\right\|_U &= \left\|\sum_{i=1}^{n} \|\alpha_i u^{-1} x_i\| e_i\right\|_U \\
&\leq \|u^{-1}\| \left\|\sum_{i=1}^{n} \|\alpha_i x_i\| e_i\right\|_U \\
&\leq \|u^{-1}\| \; c \sup_{\|x^*\|\leq 1} \|\sum_{i=1}^{n} \langle x^*, x_i\rangle e_i\|_U \\
&= \|u^{-1}\| \; c \sup_{\|x^*\|\leq 1} \|\sum_{i=1}^{n} \langle x^*, u x_i^{**}\rangle e_i\|_U \\
&= \|u^{-1}\| \; c \sup_{\|x^*\|\leq 1} \|\sum_{i=1}^{n} \langle u^* x^*, x_i^{**}\rangle e_i\|_U \\
&= \|u^{-1}\| \; c \sup_{\|x^*\|\leq 1} \|u^* x^*\| \left\|\sum_{i=1}^{n} \langle \frac{u^* x^*}{\|u^* x^*\|}, x_i^{**}\rangle e_i\right\|_U \\
&\leq \|u^{-1}\| \|u\| \; c \sup_{\|y^*\|\leq 1} \|\sum_{i=1}^{n} \langle y^*, x_i^{**}\rangle e_i\|_U, \quad \text{where } y^* \in X^{***} \\
&\leq (1+\varepsilon) \; c \sup_{\|y^*\|\leq 1} \|\sum_{i=1}^{n} \langle y^*, x_i^{**}\rangle e_i\|_U.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, it follows that

$$\left\|\sum_{i=1}^{n} \|\alpha_i x_i^{**}\| e_i\right\|_U \leq c \sup_{\|y^*\|\leq 1} \left\|\sum_{i=1}^{n} \langle x_i^{**}, y^*\rangle e_i\right\|_U ,$$

so that $(\alpha_i) \in M_U(X^{**})$ and

$$\|(\alpha_i)\|_{M_U(X)} \geq \|(\alpha_i)\|_{M_U(X^{**})}.$$

Conversely, suppose

$$\left\|\sum_{i=1}^{m} \|\alpha_i x_i^{**}\| e_i\right\|_U \leq c \sup_{\|x^*\|\leq 1} \left\|\sum_{i=1}^{m} \langle x_i^{**}, x^*\rangle e_i\right\|_U$$

for all finite sets $\{x_1^{**}, x_2^{**}, \cdots, x_m^{**}\} \subset X^{**}$. Then, for all $\{x_1, \cdots, x_n\} \subset X \subset X^{**}$, it follows that

$$\left\|\sum_{i=1}^{n} \|\alpha_i x_i\| e_i\right\|_U \leq c \sup_{\|x^*\|\leq 1} \left\|\sum_{i=1}^{n} \langle x_i, x^*\rangle e_i\right\|_U ,$$

i.e.

$$(\alpha_i) \in M_U(X) \quad \text{and} \quad \|(\alpha_i)\|_{M_U(X)} \leq \|(\alpha_i)\|_{M_U(X^{**})}.$$

Thus $M_U(X) = M_U(X^{**})$ as Banach spaces. □

Definition 2.17 *A scalar sequence* (α_n) *is a* **strongly** U**-summing multiplier,** *if*

$$(\alpha_n x_n) \in U\langle X\rangle,\ \forall\ (x_n) \in U_{strong}(X).$$

Put

$$M_U^{strong}(X) = \left\{ (\alpha_n) \in \omega : \sum_{n=1}^{\infty} |x_n^*(\alpha_n x_n)| < \infty,\ \forall\ (x_n^*) \in U_{weak}^*(X^*),\ \forall\ (x_n) \in U_{strong}(X) \right\}.$$

On the vector space $M_U^{strong}(X)$ we define a norm

$$\|(\alpha_i)\|_{M_U^{strong}(X)} = \sup_{\|(x_i)\|_{strong} \leq 1} \|(\alpha_i x_i)\|_{U\langle X\rangle} = \sup_{\substack{\|(x_n^*)\|_{weak} \leq 1 \\ \|(x_n)\|_{strong} \leq 1}} \sum_{n=1}^{\infty} |x_n^*(\alpha_n x_n)|.$$

Theorem 2.18

$$[U_{strong}(X)]^* = U_{strong}^*(X^*).$$

To obtain this, we identify a sequence $(x_i^*) \in U_{strong}^*(X^*)$ *with a linear functional*

$$\phi \in [U_{strong}(X)]^*,$$

defined by $\phi((x_i)) = \sum_{i=1}^{\infty} x_i^*(x_i)$.

Proof Let $(x_i^*) \in U_{strong}^*(X^*)$ and $(x_i) \in U_{strong}(X)$, then

$$\begin{aligned} |\sum_{i=m+1}^{n} x_i^*(x_i)| &\leq \sum_{i=m+1}^{n} \|x_i^*\|\|x_i\| \\ &= \langle \sum_{i=m+1}^{n} \|x_i^*\| e_i^*, \sum_{j=m+1}^{n} \|x_j\| e_j \rangle \\ &\leq \left\| \sum_{i=m+1}^{n} \|x_i^*\| e_i^* \right\|_{U^*} \left\| \sum_{j=m+1}^{n} \|x_j\| e_j \right\|_{U} \xrightarrow[\infty]{n,\ m} 0. \end{aligned}$$

Thus $\sum_{i=1}^{\infty} x_i^*(x_i)$ converges.

Define a linear functional ϕ on $U_{strong}(X)$ by $\phi((x_i)) = \sum_{i=1}^{\infty} x_i^*(x_i)$. Then ϕ is bounded

and $\|\phi\| \leq \|(x_i^*)\|_{U^*_{strong}(X^*)}$. Conversely, suppose $\phi \in [U_{strong}(X)]^*$ is given. Define a sequence (x_i^*) by $x_i^*(x) = \phi((0,0,\cdots,0,\overset{i-th}{x},0,\cdots))$. Then $x_i^* \in X^*$, with $\|x_i^*\| \leq \|\phi\|$. For $\lambda \in U$ it follows that $|\langle \sum_{i=1}^n \|x_i^*\| e_i^*, \sum_{j=1}^\infty \lambda_j e_j \rangle| \leq \sum_{i=1}^n \|x_i^*\| |\lambda_i|$.
Let $\epsilon > 0$. Let $x_i \in X$, $\|x_i\| = 1$ and $\|x_i^*\| < (1+\epsilon) x_i^*(x_i)$. Then

$$\begin{aligned}
& \sum_{i=1}^n \|x_i^*\| |\lambda_i| < (1+\epsilon) \sum_{i=1}^n x_i^*(x_i) |\lambda_i| \\
= \;& (1+\epsilon)(x_1^*(x_1)|\lambda_1| + x_2^*(x_2)|\lambda_2| + \cdots + x_n^*(x_n)|\lambda_n| + 0 + 0 + \cdots) \\
= \;& (1+\epsilon)(\phi(x_1, 0, ..)|\lambda_1| + \phi(0, x_2, 0, ..)|\lambda_2| + .. + \phi(0, .., x_n, 0, ..)|\lambda_n| + 0 + ..) \\
= \;& (1+\epsilon)[\phi(|\lambda_1| x_1, |\lambda_2| x_2, \cdots, |\lambda_n| x_n, 0, \cdots)] \\
\leq \;& (1+\epsilon)\|\phi\| \|(|\lambda_1| x_1, |\lambda_2| x_2, \cdots, |\lambda_n| x_n, 0, \cdots)\|_{strong} \\
= \;& (1+\epsilon)\|\phi\| \left\| \sum_{i=1}^n |\lambda_i| \|x_i\| e_i \right\|_U \\
\leq \;& (1+\epsilon)\|\phi\| \|\lambda\|_U, \text{ by Proposition 2.11.}
\end{aligned}$$

Since this inequality holds for each $\epsilon > 0$, it follows that

$$|\langle \sum_{i=1}^n \|x_i^*\| e_i^*, \sum_{j=1}^\infty \lambda_j e_j \rangle| \leq \|\phi\| \|\lambda\|_U, \quad \forall\, n \in \mathbb{N}$$

and therefore $\sum_{i=1}^\infty \|x_i^*\| e_i^* \in U^*$ with $\|\sum_{i=1}^\infty \|x_i^*\| e_i^*\|_{U^*} \leq \|\phi\|$, i.e.

$$\|(x_i^*)\|_{U^*_{strong}(X^*)} \leq \|\phi\|.$$

□

Proposition 2.19 *$M_{U^*}(X^*) \overset{isometric}{=} M_U^{strong}(X)$ as Banach spaces.*

Proof Let $(\alpha_i) \in M_{U^*}(X^*)$ and $(x_i) \in U_{strong}(X)$.
Let $(x_i^*) \in U^*_{weak}(X^*)$. Then $(\alpha_i x_i^*) \in U^*_{strong}(X^*) = [U_{strong}(X)]^*$.
Hence

$$\begin{aligned}
\sum_{i=1}^\infty |\langle \alpha_i x_i, x_i^* \rangle| &= \sum_{i=1}^\infty |\langle x_i, \alpha_i x_i^* \rangle| \\
&\leq \|(x_i)\|_{strong} \|(\alpha_i x_i^*)\|_{strong} \\
&\leq \|(x_i)\|_{strong} \|(\alpha_i)\|_{M_{U^*}(X^*)} \|(x_i^*)\|_{weak} < \infty
\end{aligned}$$

i.e. $(\alpha_i x_i) \in U\langle X\rangle$ and also $\|(\alpha_i)\|_{M_U^{strong}(X)} \leq \|(\alpha_i)\|_{M_{U^*}(X^*)}$.

Conversely, let $(\alpha_i) \in M_U^{strong}(X)$ and $(x_i^*) \in U^*_{weak}(X^*)$.

$\underline{(\alpha_i x_i^*) \in U^*_{strong}(X^*)}$: Define $\phi : U_{strong}(X) \to \mathbb{K}$:

$$\begin{aligned}\phi((x_i)) &= \sum_{i=1}^{\infty}\langle x_i, \alpha_i x_i^*\rangle \\ &= \sum_{i=1}^{\infty}\langle \alpha_i x_i, x_i^*\rangle.\end{aligned}$$

This converges, because $(\alpha_i x_i) \in U\langle X\rangle$.

It follows that

$$\begin{aligned}|\phi((x_i))| &= |\sum_{i=1}^{\infty}\langle \alpha_i x_i, x_i^*\rangle| \\ &\leq \|(\alpha_i x_i)\|_{U(X)}\|(x_i^*)\|_{weak} \\ &\leq \|(\alpha_i)\|_{M_U^{strong}(X)}\|(x_i)\|_{strong}\|(x_i^*)\|_{weak} < \infty.\end{aligned}$$

This shows that

$$\phi \in [U_{strong}(X)]^* = U^*_{strong}(X^*),$$

with $\|\phi\| \leq \|(\alpha_i)\|_{M_U^{strong}(X)}\|(x_i^*)\|_{weak}$. Looking at the identification in Theorem 2.18, we see that $(\alpha_i x_i^*) \in U^*_{strong}(X^*)$ such that $\|\phi\| = \|(\alpha_i x_i^*)\|_{strong}$. It follows that

$$(\alpha_i) \in M_{U^*}^{strong}(X^*) \text{ and } \|(\alpha_i)\|_{M_{U^*}(X^*)} \leq \|(\alpha_i)\|_{M_U^{strong}(X)}.$$

□

Theorem 2.20 *Let (α_i) be a bounded scalar sequence.*
Then $(\alpha_i) \in M_U(X^*)$ if and only if $T_{\overline{\alpha},\overline{x}} : U \to X :: (\beta_i) \to \sum_{i=1}^{\infty}\beta_i\alpha_i x_i$ is nuclear for all sequences $(x_i) \in U^*_{strong}(X)$.

Proof We know $(\alpha_i) \in M_U(X^*)$

$$\begin{aligned}&\overset{2.19}{\iff} (\alpha_n) \in M_{U^*}^{strong}(X) \\ &\overset{2.17}{\iff} (\alpha_n x_n) \in U^*\langle X\rangle,\ \forall\ (x_n) \in U^*_{strong}(X) \\ &\iff (T_{\overline{\alpha},\overline{x}} e_n) \in U^*\langle X\rangle,\ \forall\ (x_i) \in U^*_{strong}(X) \\ &\overset{2.7}{\iff} T_{\overline{\alpha},\overline{x}} : U \to X \text{ is nuclear},\ \forall\ (x_i) \in U^*_{strong}(X).\end{aligned}$$

□

2.2 Applications where U is replaced by classical Banach spaces

2.2.1 The case where $U = L^p(0,1)$ for $1 < p < \infty$

For $1 < p < \infty$, let $L^p(0,1)$ denote the Banach space of equivalence classes of Lebesgue measurable functions on $[0,1]$, whose p-th power is Lebesgue integrable. The norm on $L^p(0,1)$ is defined by

$$||f||_p = \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}, \quad f \in L^p(0,1).$$

Recall (cf. Section 1.5) that the sequence of functions $\{\chi_i(t)\}_1^\infty$ defined by $\chi_1(t) = 1$ for $0 \le t \le 1$ and, for $k = 0,1,2,\cdots, j = 1,2,\cdots,2^k$,

$$\chi_{2^k+j}(t) = \begin{cases} 1 & \text{if} \quad t \in [\frac{(2j-2)}{2^{k+1}}, \frac{(2j-1)}{2^{k+1}}] \\ -1 & \text{if} \quad t \in [\frac{(2j-1)}{2^{k+1}}, \frac{2j}{2^{k+1}}] \\ 0 & \text{otherwise} \end{cases}$$

is called the **Haar system**. It is well known to be a monotone, unconditional (but obviously not normalized) basis of $L^p(0,1)$, $1 < p < \infty$. From ([37], p. 268) it follows that the Haar basis of $L^p(0,1)$, $p > 1$ is a shrinking basis.
Following [13], we renorm $L^p(0,1)$ by

$$||f||_p^{new} = \sup\left\{\left\|\sum_{i=1}^\infty \theta_i a_i \chi_i\right\|_p : \theta_i = \pm 1, i = 1,2,\cdots\right\}, \quad f = \sum_{i=1}^\infty a_i \chi_i \in L^p(0,1).$$

Then

$$||\cdot||_p \le ||\cdot||_p^{new} \le K_p||\cdot||_p,$$

whereby K_p is the unconditional constant of the basis $\{\chi_i\}_1^\infty$.
If $1 < p \le q$, we have $||\sum_{i=1}^\infty \theta_i a_i \chi_i||_p \le ||\sum_{i=1}^\infty \theta_i a_i \chi_i||_q$. Thus $||f||_p^{new} \le ||f||_q^{new}$ if $f = \sum_{i=1}^\infty a_i \chi_i \in L^q(0,1)$.

With this new norm, $L^p(0,1)$ is of course also a Banach space. Furthermore, the unconditional constant of $\{\chi_i\}_1^\infty$ with respect to this new norm is 1. Now let

$$e_i = \frac{\chi_i}{||\chi_i||_p^{new}}, \qquad i = 1,2,\cdots.$$

Then $\{e_i\}_1^\infty$ is a normalized, unconditional basis of $(L^p(0,1), \|\cdot\|_p^{new})$ of which the unconditional constant is 1. The basis constant being less than or equal to the unconditional constant and $L^p(0,1)$ being reflexive, $\{e_i\}_1^\infty$ is monotone and boundedly complete.

Now let

$$e_i^* = \frac{\chi_i}{\|\chi_i\|_{p'}^{new}}, \qquad i = 1, 2, \cdots .$$

Then $\{e_i^*\}_1^\infty$ is a normalized, unconditional basis of $(L^{p'}(0,1), \|\cdot\|_{p'}^{new})$ of which the unconditional constant is 1. Moreover, $\{e_i\}_1^\infty$ and $\{e_i^*\}_1^\infty$ are orthonormal, i.e.

$$e_i^*(e_j) = \int_0^1 e_i^*(t)e_j(t)\,dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Henceforth, the norm on $L^p(0,1)$ will always be the new norm $\|.\|_p^{new}$.

Proposition 2.21 *(cf. [32], p. 18-19) Let $u = \sum_{i=1}^\infty e_i^*(u)e_i \in L^p(0,1)$ and $1 < p < \infty$. Then*

(i) $\|\sum_{i\in\sigma} e_i^*(u)e_i\|_p^{new} \leq \|u\|_p^{new}$, *for each subset σ of* $\mathbb{N}$.

(ii) $\|\sum_{i=1}^\infty \Theta_i e_i^*(u)e_i\|_p^{new} \leq \|u\|_p^{new}$, *for each choice of signs $\Theta = (\Theta_i)$.*

(iii) $\|\sum_{i=1}^\infty \lambda_i e_i^*(u)e_i\|_p^{new} \leq 2\|(\lambda_i)\|_\infty \|u\|_p^{new}$, *for each $\lambda = (\lambda_i) \in \ell^\infty$.*

We will show that the space $(L^{p'}(0,1), \|\cdot\|_{p'}^{new})$ is *topologically isomorphic* to the space $(L^p(0,1), \|\cdot\|_p^{new})^*$, where the isomorphism is defined by $\phi \mapsto \sum_{i=1}^\infty \phi(e_i)e_i^*$. In order to prove this result, we need the following lemma:

Lemma 2.22 *Let $\phi \in (L^p(0,1), \|.\|_p^{new})^*$ then $\|\phi\|^{new} \leq \|\sum_{i=1}^\infty \phi(e_i)e_i^*\|_{p'}^{new} \leq K_p\|\phi\|^{new}$, where $\|\phi\|^{new} = \|\phi\|_{(L^p(0,1),\|\cdot\|_p^{new})^*}$.*

Proof

$$\begin{aligned} \|\phi\|^{new} &= \sup_{\|\lambda\|_p^{new}\leq 1} |\phi(\lambda)| \\ &= \sup_{\|\lambda\|_p^{new}\leq 1} \Big|\sum_{i=1}^\infty \lambda_i\phi(e_i)\Big| \\ &= \sup_{\|\lambda\|_p^{new}\leq 1} \Big|\Big\langle\sum_{i=1}^\infty \lambda_i e_i, \sum_{j=1}^\infty \phi(e_j)e_j^*\Big\rangle\Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|\lambda\|_p^{new}\leq 1} \|\sum_{i=1}^{\infty}\lambda_i e_i\|_p \, \|\sum_{j=1}^{\infty}\phi(e_j)e_j^*\|_{p'} \\
&\leq \sup_{\|\lambda\|_p^{new}\leq 1} \|\sum_{i=1}^{\infty}\lambda_i e_i\|_p^{new} \|\sum_{j=1}^{\infty}\phi(e_j)e_j^*\|_{p'}^{new} \\
&\leq \|\sum_{j=1}^{\infty}\phi(e_j)e_j^*\|_{p'}^{new}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\sum_{j=1}^{\infty}\phi(e_i)e_j^*\|_{p'}^{new} &= \sup_{\epsilon_i=\pm 1}\|\sum_{j=1}^{\infty}\epsilon_j\phi(e_j)e_j^*\|_{p'} \\
&= \sup_{\epsilon_i=\pm 1}|\langle\sum_{j=1}^{\infty}\epsilon_j\phi(e_j)e_j^*, \sum_{i=1}^{\infty}\alpha_i e_i\rangle|, \quad \text{for some} \quad \|\sum_{i=1}^{\infty}\alpha_i e_i\|_p = 1 \\
&= \sup_{\epsilon_i=\pm 1}|\sum_{j=1}^{\infty}\epsilon_j\phi(e_j)\alpha_j| \\
&= \sup_{\epsilon_i=\pm 1}|\phi(\sum_{j=1}^{\infty}\epsilon_j e_j\alpha_j)| \\
&\leq \|\phi\|^{new}\sup_{\epsilon_i=\pm 1}\|\sum_{j=1}^{\infty}\epsilon_j e_j\alpha_j\|_p^{new} \\
&= \|\phi\|^{new}\|\sum_{j=1}^{\infty}\alpha_i e_j\|_p^{new} \\
&\leq K_p\|\phi\|^{new}\|\sum_{j=1}^{\infty}\alpha_j e_j\|_p \\
&= K_p\|\phi\|^{new}.
\end{aligned}
$$

□

Corollary 2.23 *From Lemma 2.22 we have* $\|\phi\|^{new} \leq \|\sum_{i=1}^{\infty}\phi(e_i)e_i^*\|_{p'}^{new} \leq K_p\|\phi\|^{new}$, *whereby* $\|\phi\|^{new} = \|\phi\|_{(L^p(0,1),\|.\|_p^{new})^*}$. *Hereby we obtain the topological isomorphism*

$$[L^p(0,1), \|.\|_p^{new}]^* = (L^{p'}(0,1), \|.\|_{p'}^{new})$$

defined by $\phi \mapsto \sum_{i=1}^{\infty}\phi(e_i)e_i^*$. *This is generally not an isometry.*

Note that $[L^p(0,1), \|.\|_p^{new}]^* \overset{isometric}{=} (L^{p'}(0,1), \|.\|^{new})$. Let X be a Banach space, and for $1 < p < \infty$, let p' denote its conjugate number, i.e. $1/p + 1/p' = 1$. In [13] the following vector sequence spaces are introduced:

$$L^p_{strong}(X) = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} \; : \; \sum_{i=1}^{\infty} \|x_i\| \, e_i \text{ converges in } L^p(0,1) \right\},$$

which is a Banach space with respect to the norm

$$\|\bar{x}\|_{strong} = \left\| \sum_{i=1}^{\infty} \|x_i\| \, e_i \right\|_p^{new} ;$$

$$L^p_{weak}(X) = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} x^*(x_i) e_i \text{ converges in } L^p(0,1) \text{ for all } x^* \in X^* \right\},$$

which is a Banach space with respect to the norm

$$\|\bar{x}\|_{weak} = \sup \left\{ \left\| \sum_{i=1}^{\infty} x^*(x_i) e_i \right\|_p^{new} : x^* \in B_{X^*} \right\};$$

$$L^p\langle X \rangle = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i^*(x_i)| < \infty, \quad \forall \, (x_i^*)_i \in L^{p'}_{weak}(X^*) \right\},$$

which is a Banach space with respect to the norm

$$\|\bar{x}\|_{L^p\langle X \rangle} = \sup \left\{ \sum_{i=1}^{\infty} |x_i^*(x_i)| : (x_i^*)_i \in B_{L^{p'}_{weak}(X^*)} \right\}.$$

We now summarize some properties about the vector sequence spaces $L^p(0,1)$ that follow directly from the general case in Section 2.1.

Theorem 2.24 *For $\frac{1}{p} + \frac{1}{p'} = 1$*

$$[L^p_{strong}(X)]^* = L^{p'}_{strong}(X^*).$$

To obtain the isomorphism, we identify a sequence $(x_i^*) \in L^{p'}_{strong}(X^*)$ with a linear functional $\phi \in [L^p_{strong}(X)]^*$, defined by $\phi((x_i)) = \sum_{i=1}^{\infty} x_i^*(x_i)$. In this case

$$\|\phi\| \leq \|(x_i^*)\|^{p'}_{strong} \leq K_p \|\phi\|.$$

Theorem 2.25 *Let X be a Banach space and let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then $L^p\langle X \rangle \overset{isometric}{=} \mathcal{I}(L^{p'}(0,1), X)$. The isometry is given by the mapping*

$$(x_i) \mapsto u : L^{p'}(0,1) \to X : ue_j^* = x_j$$

for all $j \in \mathbb{N}$.

Corollary 2.26 *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $L^p\langle X\rangle \overset{isomorphic}{=} N(L^{p'}(0,1), X)$, where the isomorphism is given by $(x_i) \mapsto u : L^{p'}(0,1) \to X : ue_j^* = x_j$ for all $j \in \mathbb{N}$.*

Corollary 2.27 *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(x_i) \in L^p\langle X\rangle$, if and only if there are $(\lambda_j) \in \ell^1$, $\{\gamma_j\}_1^\infty \subset B_{L^p(0,1)}$ and $\{y_j\}_1^\infty \subset B_X$ such that $x_i = \sum_{j=1}^{\infty} \lambda_j e_i^*(\gamma_j) y_j$ for all $i \in \mathbb{N}$.*

From our earlier discussion in connection with the projective tensor product, it is clear that $N(L^{p'}(0,1), X) \overset{isomorphic}{=} L^p_{new}(0,1) \hat{\otimes} X$ (for all Banach spaces X), when p satisfies the conditions in Corollary 2.26. We thus conclude that

Corollary 2.28 *Let $1 < p < \infty$. Then $L^p\langle X\rangle \overset{isomorphic}{=} L^p_{new}(0,1) \hat{\otimes} X$.*

2.2.2 The case where $U = \ell^p$ for $1 < p < \infty$

The unit vector basis is boundedly complete in all the ℓ^p spaces. There are normalized unconditional bases in ℓ^p, $1 < p < \infty$, $p \neq 2$, which are not equivalent to the unit vector basis. The unit vector basis is a monotone, normalized unconditional basis in ℓ^p. In this case, if we put $U = \ell^p$, then $U^* = \ell^{p'}$ isometrically. Therefore, we can now list the following isometric results directly from the corresponding results in the general setting in Section 2.1:

Corollary 2.29 *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\ell^p\langle X\rangle \overset{isometric}{=} N(\ell^{p'}, X)$, where the isometry is given by $(x_i) \mapsto u : \ell^{p'} \to X : ue_j = x_j$ for all $j \in \mathbb{N}$.*

Corollary 2.30 *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(x_i) \in \ell^p\langle X\rangle$, if and only if there are $(\lambda_j) \in \ell^1$, $(\gamma_{ij})_j \in B_{\ell^{p'}}$ for all $i \in \mathbb{N}$, and $(y_j) \subset B_X$ such that $x_i = \sum_{j=1}^{\infty} \lambda_j \gamma_{ij} y_j$ for all $i \in \mathbb{N}$.*

From our earlier discussion in connection with the projective tensor product it is clear that $N(\ell^{p'}, X) = \ell^p \hat{\otimes} X$ (for all Banach spaces X), when p satisfies the conditions in Corollary 2.29. We thus conclude that

Corollary 2.31 *Let $1 < p < \infty$. Then $\ell^p\langle X\rangle \overset{isometric}{=} \ell^p \hat{\otimes} X$.*
Also $\ell^1\langle X\rangle = \ell^1(X) = \ell^1 \hat{\otimes} X$.

Definition 2.32 *(cf. [5]) Let $1 \le p \le \infty$. A scalar sequence (α_i) is called a p-summing multiplier for a Banach space X, if $\sum_{n=1}^{\infty} \|\alpha_n x_n\|^p < \infty$ for all sequences $(x_n) \in \ell_w^p(X)$. Put*

$$m_p(X) = \{(\alpha_n) \in \omega : \sum_{n=1}^{\infty} \|\alpha_n x_n\|^p < \infty \ , \forall \ (x_n) \in \ell_w^p(X)\}.$$

Proposition 2.33 $m_p(X^{**}) = m_p(X)$.

Definition 2.34 *Let $1 \le p \le \infty$. A scalar sequence (α_n) is a strongly p-summing multiplier, if $(\alpha_n x_n) \in \ell^p\langle X\rangle$, $\forall \ (x_n) \in \ell^p(X)$. Put*

$$m_p^{strong}(X) = \left\{ (\alpha_n) \in \omega : \sum_{n=1}^{\infty} |x_n^*(\alpha_n x_n)| < \infty, \ \forall \ (x_n^*) \in \ell_w^q(X^*), \ \forall \ (x_n) \in \ell^p(X) \right\}.$$

On the vector space $m_p^{strong}(X)$ we define a norm

$$\|(\alpha_i)\|_{m_p^{strong}(X)} = \sup_{\|(x_i)\|_{\ell^p(X)} \le 1} \|(\alpha_i x_i)\|_{\langle p\rangle} = \sup_{\substack{\epsilon_q((x_n^*)) \le 1 \\ \|(x_n)\|_{\ell^p(X)} \le 1}} |\sum_{n=1}^{\infty} x_n^*(\alpha_n x_n)|.$$

There is a natural connection between $m_p^{strong}(X)$ and $m_{p'}(X^*)$, which is given by

Proposition 2.35 $(\alpha_i) \in m_{p'}(X^*) \Longleftrightarrow (\alpha_i) \in m_p^{strong}(X)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 2.36 *Let (α_i) be a bounded scalar sequence and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(\alpha_i) \in m_p(X^*)$ if and only if $T_{\overline{\alpha},\overline{x}} : \ell^p \to X :: (\beta_i) \to \sum_{i=1}^{\infty} \beta_i \alpha_i x_i$ is nuclear for all sequences $(x_i) \in \ell^{p'}(X)$.*

Corollary 2.37 *Let $(\alpha_n) \in \ell^\infty$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. The following are equivalent:*

(a) $(\alpha_n) \in m_p(X)$

(b) $\sum_n e_n^* \otimes \alpha_n x_n^* \in N(\ell^p, X^*)$, $\forall \ (x_n^*) \in \ell^{p'}(X^*)$.

Chapter 3

General operator spaces

Let U and W be reflexive Banach spaces with normalized unconditional bases (e_i) and (f_i) respectively and let X be a Banach space.

For X a compact space, $M(X)$ denotes the space of all $\mathbb{K}$ regular Borel measures on X. Let $E(X)$ and $F(Y)$ be spaces of sequences with values in X and Y respectively. Consider $T \in L(X,Y)$. The operator T induces an operator $\hat{T}$ from $E(X)$ into $F(Y)$ defined by $\hat{T}((x_i)) = (Tx_i)$.

3.1 Strongly (U, W)-summing operators and strongly (U, W)-nuclear operators

Definition 3.1 *An operator T is* **absolutely** (U, W)**-summing** *(or $T \in \Pi_{U,W}(X, Y)$) if there exists a constant $c > 0$ such that for all finite sets $\{x_1, \cdots, x_n\}$, the inequality*

$$\|(Tx_i)\|_{U_{strong}(Y)} \leq c\|(x_i)\|_{W_{weak}(X)} \tag{3.1}$$

is satisfied. The smallest number c such that the above inequality holds, is called the absolutely (U, W)-summing norm, $\pi_{U,W}(T)$ of T.

Remark 3.2 *Only the zero operator can be absolutely (U, W)-summing if $U \subsetneq W$:*
Let $0 \neq T \in \Pi_{U,W}(X, Y)$, $w = \sum_{i=1}^{\infty} \alpha_i f_i \in W$ and put $x_k = \alpha_k x$ where $x \in X$, $Tx \neq 0$.

Since

$$\begin{aligned}\|(Tx_k)\|_{strong} &= \|(\alpha_k Tx)\|_U \\ &= \|\sum_{k=1}^{\infty}|\alpha_k|\|Tx\|e_k\|_U \\ &= \|Tx\|\|\sum_{k=1}^{\infty}|\alpha_k|e_k\|_U\end{aligned}$$

and

$$\begin{aligned}\|(x_i)\|_{weak} &= \sup_{x^*\in B_{X^*}}\|\sum_{k=1}^{\infty}x^*(x_k)f_k\|_W \\ &= \sup_{x^*\in B_{X^*}}\|\sum_{k=1}^{\infty}\alpha_k x^*(x)f_k\|_W \\ &= \|x\|\|\sum_{k=1}^{\infty}\alpha_k f_k\|_W,\end{aligned}$$

it follows from (3.1) that

$$\|Tx\|\|\sum_{k=1}^{\infty}|\alpha_k|e_k\|_U \leq c\ \|x\|\|\sum_{k=1}^{\infty}\alpha_k f_k\|_W.$$

Thus $\sum_{k=1}^{\infty}\alpha_k e_k \in U$. *The mapping* $\sum_{i=1}^{\infty}\alpha_i f_i \in W \mapsto \sum_{i=1}^{\infty}\alpha_i e_i \in U$ *defines an isomorphism into, i.e.* W *is isomorphic to* U.

A discussion of two related classes of operators follows.

(1) An operator $T : X \to Y$ is **strongly (U,W)-summing** (i.e. $T \in D_{U,W}(X,Y)$) if there exists a $c > 0$ such that for all finite sets $\{x_1, x_2, \cdots, x_n\} \subset X$ we have

$$\|(Tx_i)(\leq n)\|_{U\langle Y\rangle} \leq c\ \|(x_i)(\leq n)\|_{W_{strong}(X)}, \quad \text{or}$$

equivalently, $T \in D_{U,W}(X,Y) \Leftrightarrow$

$\exists\ c > 0$ such that for any $x_1, \cdots, x_n \in X,\ y_1^*, \cdots, y_n^* \in Y^*$,

$$\sum_{i=1}^{n}|\langle y_i^*, Tx_i\rangle| \leq c\left\|\sum_{i=1}^{n}\|x_i\|f_i\right\|_W \sup_{y\in B_Y}\left\|\sum_{i=1}^{n}y(y_i^*)e_i^*\right\|_{U^*}. \qquad (3.2)$$

(2) T is **strongly (U,W)-nuclear** (or $T \in SN_{U,W}(X,Y)$) if there exists a $c > 0$ such that for all finite sets $\{x_1, x_2, \cdots, x_n\} \subset X$ we have

$$\|(Tx_i)(\leq n)\|_{U\langle Y\rangle} \leq c\ \|(x_i)(\leq n)\|_{W_{weak}(X)}. \tag{3.3}$$

Equivalently, $T \in SN_{U,W}(X,Y) \Leftrightarrow$

$\exists\ c > 0$ such that for any $x_1, \cdots, x_n \in X,\ y_1^*, \cdots, y_n^* \in Y^*$,

$$\sum_{i=1}^{n} |\langle y_i^*, Tx_i\rangle| \leq c \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^{n} x^*(x_i) f_i \right\|_W \sup_{y \in B_Y} \left\| \sum_{k=1}^{n} y(y_k^*) e_k^* \right\|_{U^*}.$$

We shall denote the strongly (U,W)-summing norm by $d_{U,W}(.)$ and the strongly (U,W)-nuclear norm by $sn_{U,W}(.)$. In this case $d_{U,W}(T) = \inf\{c > 0\ :\ (3.2) \text{ holds }\}$ and $sn_{U,W}(T) = \inf\{c > 0\ :\ (3.3) \text{ holds }\}$.

Proposition 3.3 *Let X and Y be normed linear spaces and assume that $T : X \to Y$ is a bounded linear operator. Then*

(i) $T \in D_{U,W}(X,Y)$ if and only if $\hat{T} : W_{strong}(X) \to U\langle Y\rangle$ is bounded with $d_{U,W}(T) = \|\hat{T}\|$.

(ii) $T \in SN_{U,W}(X,Y)$ if and only if $\hat{T} : W_{weak}(X) \to U\langle Y\rangle$ is bounded with $sn_{U,W}(T) = \|\hat{T}\|$.

Proof We prove (i): Suppose $T \in D_{U,W}(X,Y)$. If $(x_i) \in W_{strong}(X)$, then for each fixed n and for each $(y_i^*) \in B_{U^*_{weak}(Y^*)}$, it follows that

$$\sum_{i=1}^{n} |y_i^*(Tx_i)| \leq d_{U,W}(T)\|(x_i)\|_{W_{strong}(X)}\|(y_i^*)\|_{U^*_{weak}(Y^*)}.$$

Note that the above inequality holds because of our assumption about the bases on W and U^*. Letting $n \to \infty$, we obtain $\sum_{i=1}^{\infty} |y_i^*(Tx_i)| \leq d_{U,W}(T)\|(x_i)\|_{W_{strong}(X)}\|(y_i^*)\|_{U^*_{weak}(Y^*)}$. Therefore, the series $\sum_{i=1}^{\infty} y_i^*(Tx_i)$ converges and $(Tx_i) \in U\langle Y\rangle$. Furthermore, because $\hat{T}((x_i)) = (Tx_i)$, it follows that $\hat{T}$ is continuous with $\|\hat{T}\| \leq d_{U,W}(T)$.
Conversely, suppose $\hat{T} : W_{strong}(X) \to U\langle Y\rangle$ is bounded and $T \notin D_{U,W}(X,Y)$. Then for every $n \in \mathbb{N}$, there is a finite set $\{x_{1n}, x_{2n}, \cdots, x_{m_n n}\} \subset X$ such that $(x_{in})_i \in B_{W_{strong}(X)}$

and $\|(Tx_{in})_i\|_{U\langle Y\rangle} \geq 2^n$. Let $\|(y^*_{i,n})\|_{weak} \leq 1$ such that $\sum_{i=1}^{m_n} |\langle Tx_{i,n}, y^*_{i,n}\rangle| \geq 2^n$. Consider the sequences

$$(z_i) = (x_{11}2^{-1}, x_{21}2^{-1}, \cdots, x_{m_1 1}2^{-1}, x_{12}2^{-2}, x_{22}2^{-2}, \cdots, x_{m_2 2}2^{-2}, \cdots)$$

and

$$(y^*_i) = (y^*_{11}2^{-1}, y^*_{21}2^{-1}, \cdots, y^*_{m_1 1}2^{-1}, y^*_{12}2^{-2}, y^*_{22}2^{-2}, \cdots, y^*_{m_2 2}2^{-2}, \cdots).$$

Then

$$\begin{aligned}
\|(z_i)\|_{strong} &= \left\| \sum_{n=1}^{\infty} 2^{-n} \sum_{i=1}^{m_n} \|x_{i,n}\| f_i \right\|_W \\
&\leq \sum_{n=1}^{\infty} 2^{-n} \left\| \sum_{i=1}^{m_n} \|x_{i,n}\| f_i \right\|_W \\
&\leq \sum_{n=1}^{\infty} 2^{-n} \leq 1.
\end{aligned}$$

Similary, $(y^*_i) \in B_{U^*_{weak}(Y^*)}$. However,

$$\|(Tz_i)\|_{U\langle Y\rangle} \geq \sum_{n=1}^{\infty} |\langle Tz_n, y^*_n\rangle| = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{m_n} |\langle Tx_{i,n}, y^*_{i,n}\rangle| = \infty,$$

contradicting the fact that $\hat{T}$ maps $W_{strong}(X)$ (continuously) into $U\langle Y\rangle$. Since

$$\|\hat{T}((x_i))\|_{U\langle Y\rangle} \leq \|\hat{T}\| \|(x_i)\|_{W_{strong}(X)},$$

we have $\|(T(x_i))\|_{U\langle Y\rangle} \leq \|\hat{T}\| \|(x_i)\|_{W_{strong}(X)}$. By definition of the norm on $D_{U,W}(X,Y)$, it follows that $\|\hat{T}\| \geq d_{U,W}(T)$. □

Proposition 3.4 *(i) The spaces $SN_{U,W}(X,Y)$ and $D_{U,W}(X,Y)$ are normed linear spaces.*

(ii) If $T \in SN_{U,W}(X,Y)$ (or $T \in D_{U,W}(X,Y)$), then T is continuous and $\|T\| \leq sn_{U,W}(T)$ (or $\|T\| \leq d_{U,W}(T)$).

(iii) If $T \in SN_{U,W}(X,Y)$ (or $T \in D_{U,W}(X,Y)$) and $S \in L(Y,Z)$, then $ST \in SN_{U,W}(X,Z)$ (or $ST \in D_{U,W}(X,Z)$) and $sn_{U,W}(ST) \leq \|S\| sn_{U,W}(T)$ (or $d_{U,W}(ST) \leq \|S\| d_{U,W}(T)$).

(iv) If $T \in L(X,Y)$ and $S \in SN_{U,W}(Y,Z)$ (or $S \in D_{U,W}(Y,Z)$), then $ST \in SN_{U,W}(X,Z)$ (or $ST \in D_{U,W}(X,Z)$) and $sn_{U,W}(ST) \leq sn_{U,W}(S)\|T\|$ (or $d_{U,W}(ST) \leq d_{U,W}(S)\|T\|$).

(v) If Y is complete, then $D_{U,W}(X,Y)$ and $SN_{U,W}(X,Y)$ are complete.

Proof (i) and (ii) follow directly from the definitions.

(iii) We prove this property for the space $D_{U,W}(X,Y)$. The proof for $SN_{U,W}(X,Y)$ is similar. Let $T \in D_{U,W}(X,Y)$ and $S \in L(Y,Z)$. Then, if $x_1, \cdots, x_n$ is a finite set in X, we have

$$\begin{aligned}
\|(STx_i)\|_{U(Z)} &= \sup_{(z_i^*)\in B_{U^*_{weak}(Z^*)}} \sum_{i=1}^{n} |z_i^*(STx_i)| \\
&= \sup_{(z_i^*)\in B_{U^*_{weak}(Z^*)}} \sum_{i=1}^{n} |S^*z_i^*(Tx_i)| \\
&= \|S^*\| \sup_{(z_i^*)\in B_{U^*_{weak}(Z^*)}} \sum_{i=1}^{n} \left|\frac{S^*z_i^*}{\|S^*\|}(Tx_i)\right| \\
&\leq \|S\| \sup_{(y_i^*)\in B_{U^*_{weak}(Y^*)}} \sum_{i=1}^{n} |y_i^*(Tx_i)| \\
&= \|S\|\|(Tx_i)\|_{U(Y)} \\
&\leq \|S\| d_{U,W}(T)\|(x_i)\|_{strong}.
\end{aligned}$$

From this we conclude that $ST \in D_{U,W}(X,Z)$ and $d_{U,W}(ST) \leq \|S\| d_{U,W}(T)$.

(iv) The proof is similar to part (iii).

(v) We show $D_{U,W}(X,Y)$ is complete. Let (T_n) be a Cauchy sequence in $D_{U,W}(X,Y)$. By part (ii), the sequence (T_n) is a Cauchy sequence in $L(X,Y)$; since $L(X,Y)$ is complete, (T_n) converges to an operator T in the norm topology on $L(X,Y)$. We show that (T_n) converges to T in $D_{U,W}(X,Y)$:

Fix $\epsilon > 0$. Since (T_n) is a Cauchy sequence in $D_{U,W}(X,Y)$, there is a positive integer N, such that $d_{U,W}(T_n - T_m) < \epsilon$, whenever $n,\ m \geq N$. Therefore, for each finite set $x_1, \cdots, x_k$ in X and for each $(y_i^*) \in B_{U^*_{weak}(Y^*)}$, it follows that:

$$\sum_{i=1}^{k} |y_i^*(T_n(x_i) - T_m(x_i))| \leq \epsilon \ \|(x_i)\|_{W_{strong}(X)}$$

whenever $m,\ n \geq N$. Letting $m \to \infty$, we have

$$\sum_{i=1}^{k} |y_i^*(T_n(x_i) - T(x_i))| \leq \epsilon \|(x_i)\|_{W_{strong}(X)},$$

which implies that $T_n - T \in D_{U,W}(X,Y)$ and $d_{U,W}(T - T_n) \leq \epsilon,\ \forall\ n \geq N$. Thus, $T = T - T_N + T_N \in D_{U,W}(X,Y)$ and $d_{U,W}(T - T_n) \leq \epsilon,\quad \forall\ n \geq N$. The proof for $SN_{U,W}(X,Y)$ is similar. □

Let us consider the relationship between the classes $\Pi_{U,W}(X,Y)$, $SN_{U,W}(X,Y)$ and $D_{U,W}(X,Y)$. For $U = W$ let us denote $\Pi_{U,W}(X,Y)$, $SN_{U,W}(X,Y)$ and $D_{U,W}(X,Y)$ by $\Pi_U(X,Y)$, $SN_U(X,Y)$ and $D_U(X,Y)$.

Proposition 3.5 *(i)* $SN_{U,W}(X,Y) \subseteq D_{U,W}(X,Y)$ *and* $d_{U,W}(\cdot) \leq sn_{U,W}(\cdot)$.

(ii) $SN_{U,W}(X,Y) \subseteq \Pi_{U,W}(X,Y)$ *and* $\pi_{U,W}(\cdot) \leq sn_{U,W}(\cdot)$.

(iii) If $T \in \Pi_{U,W}(X,Y)$ *and* $S \in D_U(Y,Z)$, *then* $ST \in SN_{U,W}(X,Z)$ *and* $sn_{U,W}(ST) \leq \pi_{U,W}(T)d_U(S)$.

Proof

(i) If T belongs to $SN_{U,W}(X,Y)$, then the mapping $\hat{T} : W_{weak}(X) \to U\langle Y\rangle$ is continuous with $\|\hat{T}\| = sn_{U,W}(T)$. Since $\hat{I} : W_{strong}(X) \to W_{weak}(X)$ is continuous with $\|\hat{I}\| \leq 1$, it follows that

$$\widehat{TI} := \hat{T}\hat{I} : W_{strong}(X) \to U\langle Y\rangle$$

is continuous. Thus we have $T \in D_{U,W}(X,Y)$ and

$$\begin{aligned} d_{U,W}(T) &= \|\widehat{TI}\| \\ &\leq \|\hat{T}\|\|\hat{I}\| \\ &\leq \|\hat{T}\| \\ &= sn_{U,W}(T). \end{aligned}$$

(ii) Part *(ii)* follows in a similar way.

(iii) Let $T \in \Pi_{U,W}(X,Y)$ and $S \in D_U(Y,Z)$. It follows that $\hat{T} : W_{weak}(X) \to U_{strong}(Y)$ is continuous with $\pi_{U,W}(T) = \|\hat{T}\|$ and $\hat{S} : U_{strong}(Y) \to U\langle Z\rangle$ is continuous with $\pi_U(S) = \|\hat{S}\|$. Therefore $\widehat{ST} := \hat{S}\hat{T} : W_{weak}(X) \to U\langle Z\rangle$ is continuous with $\|\hat{S}\hat{T}\| \leq \|\hat{S}\|\|\hat{T}\|$. It follows that ST belongs to $SN_{U,W}(X,Z)$ and

$$sn_{U,W}(ST) \leq d_U(S)\pi_{U,W}(T).$$

□

Next we give an exposition of the relationship between $D_{U,W}(X,Y)$ and $\Pi_{U,W}(X,Y)$.

Proposition 3.6 *(i) Let $T \in L(X,Y)$. Then $T \in \Pi_{U,W}(X,Y)$ if and only if the adjoint operator satisfies $T^* \in D_{W^*,U^*}(Y^*,X^*)$. In this case $D_{W^*,U^*}(T^*) = \pi_{U,W}(T)$.*

(ii) Similarly, $T \in D_{U,W}(X,Y)$ if and only if $T^* \in \Pi_{W^*,U^*}(Y^*,X^*)$. In this case $d_{U,W}(T) = \pi_{W^*,U^*}(T^*)$.

Proof (i) Let $T \in \Pi_{U,W}(X,Y)$; we need to show that $T^* \in D_{W^*,U^*}(Y^*,X^*)$. For any finite set $y_1^*, \cdots, y_n^*$ in Y^* and for $(x_i) \in W_{weak}(X)$ we have:

$$\begin{aligned}
|\sum_{i=1}^{n} x_i(T^*y_i^*)| &= |\sum_{i=1}^{n} Tx_i(y_i^*)| \\
&\leq \sum_{i=1}^{n} \|Tx_i\|\|y_i^*\| \\
&= \langle\sum_{i=1}^{n} \|Tx_i\|e_i, \sum_{j=1}^{n} \|y_j^*\|e_j^*\rangle \\
&\leq \pi_{U,W}(T)\|(x_i)\|_{weak}\|(y_i^*)\|_{strong}.
\end{aligned}$$

Taking the supremum over the unit ball in $W_{weak}(X)$, we obtain

$$\|(T^*y_i^*)\|_{W^*\langle X^*\rangle} \leq \pi_{U,W}(T)\|(y_i^*)\|_{strong}.$$

Therefore $T^* \in D_{W^*,U^*}(Y^*,X^*)$ and $d_{W^*,U^*}(T^*) \leq \pi_{U,W}(T)$.

Conversely, assume $T^* \in D_{W^*,U^*}(Y^*,X^*)$. Let $x_1, \cdots, x_n$ be a finite set in X and let

$(y_i^*) \in U^*_{strong}(Y^*)$. Then

$$\begin{aligned} |\sum_{i=1}^{n} y_i^*(Tx_i)| &= |\sum_{i=1}^{n} T^*y_i^*(x_i)| \\ &\leq \|(T^*y_i^*)\|_{W^*\langle X^*\rangle}\|(x_i)\|_{weak} \\ &\leq d_{W^*,U^*}(T^*)\|(y_i^*)\|_{strong}\|(x_i)\|_{weak}. \end{aligned}$$

If we take the supremum over the unit ball in $U^*_{strong}(Y^*)$, we obtain

$$\|(Tx_i)\|_{strong} \leq d_{W^*,U^*}(T^*)\|(x_i)\|_{weak}.$$

Therefore T is absolutely (U, W)-summing and $\pi_{U,W}(T) \leq d_{W^*,U^*}(T^*)$.

Part (ii) has a similar proof. □

Proposition 3.7 *An operator $T \in L(X,Y)$ is in $SN_{U,W}(X,Y)$ if and only if $T^* \in SN_{W^*,U^*}(Y^*, X^*)$ and $sn_{U,W}(T) = sn_{W^*,U^*}(T^*)$.*

Proof Choose $T \in SN_{U,W}(X,Y)$; then the operator $\hat{T} : W_{weak}(X) \to U\langle Y\rangle$ is bounded with $sn_{U,W}(T) = \|\hat{T}\|$. Since $W_{weak,0}(X)$ is a closed subspace of $W_{weak}(X)$, it follows that $\hat{T}|_{W_{weak,0}(X)} := \hat{T}_o : W_{weak,0}(X) \to U\langle Y\rangle$ is bounded with $\|\hat{T}_o\| \leq sn_{U,W}(T)$.
Consequently, the adjoint operator

$$(\hat{T}_o)^* : U^*_{weak}(Y^*) \to W^*\langle X^*\rangle$$

is bounded with $\|(\hat{T}_o)^*\| \leq sn_{U,W}(T)$. We show that $(\hat{T}_o)^* = \overset{\wedge}{T^*}$. Consider

$$\begin{aligned} \langle(\hat{T}_o)^*((y_i^*)), (x_i)\rangle &= \langle(y_i^*), \hat{T}_o((x_i))\rangle \\ &= \langle(y_i^*), (Tx_i)\rangle \\ &= \sum_{i=1}^{\infty} y_i^*(Tx_i) \\ &= \sum_{i=1}^{\infty} \langle T^*y_i^*, x_i\rangle \\ &= \langle(T^*y_i^*), (x_i)\rangle \qquad \forall\ (x_i) \in W_{weak,0}(X) \text{ and } (y_i^*) \in U^*_{weak}(Y^*). \end{aligned}$$

Convergence of $\sum_{i=1}^{\infty} y_i^*(Tx_i)$ follows from $T \in SN_{U,W}(X,Y)$. Thus

$$(T^*y_i^*) \in W_{weak,0}(X)^* = W^*\langle X^*\rangle \text{ for all } (y_i^*) \in U^*_{weak}(Y^*).$$

This shows that $(\overset{\wedge}{T_o})^* = \overset{\wedge}{T^*}$ and $T^* \in SN_{W^*,U^*}(Y^*, X^*)$.

It follows that

$$sn_{W^*,U^*}(T^*) = \|\overset{\wedge}{T^*}\| = \|(\overset{\wedge}{T_o})^*\| \leq sn_{U,W}(T).$$

Conversely, let $T^* \in SN_{W^*,U^*}(Y^*, X^*)$. We have to show $T \in SN_{U,W}(X, Y)$.

Let $x_1, \cdots, x_n$ be a finite set in X and let $(y_i^*) \in B_{U^*_{weak}(Y^*)}$. Then

$$\sum_{i=1}^{n} |\langle y_i^*, Tx_i\rangle| = \sum_{i=1}^{n} |\langle T^* y_i^*, x_i\rangle| \leq sn_{W^*,U^*}(T^*)\|(y_i^*)\|_{weak}\|(x_i)\|_{weak}.$$

Taking the supremum over all sequences $(y_i^*) \in B_{U^*_{weak}(Y^*)}$, we have

$$\|(Tx_i)\|_{U\langle X\rangle} \leq sn_{W^*,U^*}(T^*)\|(x_i)\|_{weak};$$

therefore $T \in SN_{U,W}(X, Y)$ and $sn_{U,W}(T) \leq sn_{W^*,U^*}(T^*)$. □

3.1.1 Applications where U and W are replaced by classical Banach spaces

In this section let $U = \ell^p$ and $W = \ell^q$ for $q \leq p$. Then

$$U_{strong}(X),\ U_{weak}(X),\ U\langle X\rangle,\ W_{strong}(X),\ W_{weak}(X),\ W_{weak,o}(X) \text{ and } W\langle X\rangle$$

are the spaces $\ell^p(X)$, $\ell^p_w(X)$, $\ell^p\langle X\rangle$, $\ell^q(X), \ell^q_w(X)$, $\ell^q_c(X)$ and $\ell^q\langle X\rangle$.

Let $1 \leq q \leq p \leq \infty$. A bounded linear operator $T : X \to Y$ is called a **strongly (p,q)-summing operator** (i.e. $T \in D_{p,q}(X, Y)$) if there exists a constant $c > 0$, such that for all finite sets $\{x_1, \cdots, x_n\} \subset X$ we have

$$\|(Tx_i)\|_{\langle p\rangle} \leq c\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{\frac{1}{q}}$$

or equivalently, $T \in D_{p,q}(X, Y) \iff \exists\ c > 0$ such that for any choice of $x_1, \cdots, x_n \in X,\ y_1^*, \cdots, y_n^* \in Y^*$ we have

$$\sum_{i=1}^{n} |\langle y_i^*, Tx_i\rangle| \leq c\ \|(x_i)(\leq n)\|_{\ell^q(X)}\epsilon_{p'}((y_i^*)(\leq n)).$$

The infimum $d_{p,q}(T)$ of all numbers $c > 0$ such that the above inequality holds, is called the **strongly (p,q)-summing norm** of T.

Let $1 \le q \le p \le \infty$. A bounded linear operator $T : X \to Y$ is called a **strongly (p,q)-nuclear operator**, i.e. $T \in SN_{p,q}(X,Y)$, if there exists a constant $c > 0$, such that for all finite sets $\{x_1, \cdots, x_n\} \subset X$ we have

$$\|(Tx_i)(\le n)\|_{\langle p \rangle} \le c \sup_{x^* \in B_{X^*}} (\sum_{i=1}^{n} |\langle x^*, x_i \rangle|^q)^{\frac{1}{q}}$$

or equivalently, $T \in SN_{p,q}(X,Y) \iff \exists\ c > 0$ such that for any $x_1, \cdots, x_n \in X$, $y_1^*, \cdots, y_n^* \in Y^*$ we have

$$\sum_{i=1}^{n} |\langle y_i^*, Tx_i \rangle| \le c\ \epsilon_q((x_i)(\le n))\ \epsilon_{p'}((y_i^*)(\le n)).$$

The infimum $sn_{p,q}(T)$ of all numbers $c > 0$ such that the above inequality holds, is called the strongly (p,q)-nuclear norm of T.

Note that $D_{p,q}(X,Y) = \{0\} = SN_{p,q}(X,Y)$ for $p < q$.

From [26] it follows that a continuous linear operator $u : X \to Y$ is called a **Littlewood-Orlicz operator** if u takes sequences in $l_w^1(X)$ into sequences in $l^2\langle Y \rangle$. Hereby, $SN_{2,1}(X,Y)$ is the space of Littlewood-Orlicz operators.

The classes of strongly p-summing and strongly p-nuclear operators were introduced by Cohen (cf. [16]) where the strongly p-nuclear operators were called p-nuclear operators. From this work of Cohen it follows that

$$\Pi_\infty(X,Y) = L(X,Y) = D_1(X,Y),\ \ SN_1(X,Y) = \Pi_1(X,Y)$$

and

$$SN_\infty(X,Y) = D_\infty(X,Y).$$

From the general case we get the following results.

Proposition 3.8 *Let X, Y and Z be normed linear spaces and $1 \le q \le p \le \infty$.*

(i) The spaces $(SN_{p,q}(X,Y), sn_{p,q}(\cdot))$ and $(D_{p,q}(X,Y), d_{p,q}(\cdot))$ are normed linear spaces.

(ii) *If* $T \in SN_{p,q}(X,Y)$ *(respectively,* $T \in D_{p,q}(X,Y)$*), then* T *is continuous and* $\|T\| \leq sn_{p,q}(T)$ *(respectively,* $\|T\| \leq d_{p,q}(T)$*).*

(iii) *If* $T \in SN_{p,q}(X,Y)$ *(respectively,* $T \in D_{p,q}(X,Y)$*) and* $S \in L(Y,Z)$*, then*

$$ST \in SN_{p,q}(X,Z) \text{ (respectively, } ST \in D_{p,q}(X,Z))$$

and $sn_{p,q}(ST) \leq \|S\| sn_{p,q}(T)$ *(respectively,* $d_{p,q}(ST) \leq \|S\| d_{p,q}(T)$*).*

(iv) *If* $T \in L(X,Y)$ *and* $S \in SN_{p,q}(Y,Z)$ *(respectively,* $S \in D_{p,q}(Y,Z)$*), then*

$$ST \in SN_{p,q}(X,Z) \text{ (respectively, } ST \in D_{p,q}(X,Z))$$

and $sn_{p,q}(ST) \leq sn_{p,q}(S)\|T\|$ *(respectively,* $d_{p,q}(ST) \leq d_{p,q}(S)\|T\|$*).*

(v) *If* Y *is complete, then* $D_{p,q}(X,Y)$ *and* $SN_{p,q}(X,Y)$ *are complete.*

Proposition 3.9 *Let* $1 \leq q \leq p \leq \infty$*. Then*

(1) $SN_{p,q}(X,Y) \subseteq D_{p,q}(X,Y)$ *and* $d_{p,q}(\cdot) \leq sn_{p,q}(\cdot)$*.*

(2) $SN_{p,q}(X,Y) \subseteq \Pi_{p,q}(X,Y)$ *and* $\pi_{p,q}(\cdot) \leq sn_{p,q}(\cdot)$*.*

(3) *If* $T \in \Pi_{p,q}(X,Y)$ *and* $S \in D_{p,p}(Y,Z)$*, then* $ST \in SN_{p,q}(X,Z)$ *and*

$$sn_{p,q}(ST) \leq d_p(S)\pi_{p,q}(T).$$

In the early twenties of the Twentieth Century W. Orlicz proved that the spaces L^p (for $p \leq 2$) possess a particular property, to which his name is now attached.

Definition 3.10 *We say that a space* X *has the* **Orlicz property** *if every unconditionally summable sequence* (z_n) *in* X *satisfies* $\sum_{n=1}^{\infty} \|z_n\|^2 < \infty$*. Equivalently,* X *has the Orlicz property if there is a constant* c *such that, for any finite sequence* (x_i) *in* X*,*

$$\Big(\sum_{i=1}^{n} \|x_i\|^2 \Big)^{\frac{1}{2}} \leq c \sup_{\epsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \epsilon_i x_i \Big\|.$$

Note that the spaces for which the identity operator is $(2,1)$*-summing, are said to have the* **Orlicz property***.*

Remark 3.11 *If X has the Orlicz property, then $D_2(X,Y) \subseteq SN_{2,1}(X,Y)$. In particular, $D_2(L^p(\mu),Y) \subseteq SN_{2,1}(L^p(\mu),Y)$ for $1 \le p \le 2$.*

Example 3.12 *The first two examples follow from the work of Cohen (cf. [16], p. 193). Let $1 < p \le \infty$. The imbeddings in (a), (b) and (c) are all continuous.*

(a) $\Pi_{p'}(L^p(\mu),Y) \subseteq D_p(L^p(\mu),Y)$ *and if X is a compact Hausdorff space then*

$$\Pi_{p'}(C(X),Y) \subset D_p(C(X),Y).$$

(b) $D_p(X, L^p(\mu)) \subset \Pi_{p'}(X, L^p(\mu))$.

(c) Let $1 < p < \infty$ and let H be a Hilbert space, then

$$\Pi_p(H,Y) \subset D_{p'}(H,Y)$$

(cf. [12]).

Remark 3.13 *Recall that an operator T is nuclear (i.e. $T \in N(X,Y)$) if it can be represented in the form $Tx = \sum_{i=1}^{\infty} x_i^*(x)y_i$, where $x_i^* \in X^*$, $y_i \in Y$ and*

$$\sum_{i=1}^{\infty} \|x_i^*\| \|y_i\| < \infty.$$

From the work of Cohen (cf. [16], p. 190) it follows that every nuclear operator is strongly p-nuclear and if X and Y are Hilbert spaces, an operator T is nuclear if and only if T is strongly p-nuclear.

From the general case follows:

Theorem 3.14 *Let $1 \le q \le p \le \infty$ and let $T : X \to Y$ be a bounded linear operator.*

(i) $T \in \Pi_{p,q}(X,Y)$ if and only if $T^* \in D_{q',p'}(Y^*,X^*)$. In this case $d_{q',p'}(T^*) = \pi_{p,q}(T)$.

(ii) $T \in D_{p,q}(X,Y)$ if and only if $T^* \in \Pi_{q',p'}(Y^*,X^*)$. In this case $\pi_{q',p'}(T^*) = d_{p,q}(T)$.
In particular
$T \in \Pi_1(X,Y) \Longleftrightarrow T^* \in D_\infty(Y^*,X^*)$ *or* $T \in D_\infty(X,Y) \Longleftrightarrow T^* \in \Pi_1(Y^*,X^*)$.
If $q = 1$ then $T \in \Pi_{p,1}(X,Y)$ iff $T^* \in D_{\infty,p'}(Y^*,X^*)$.
If $p = \infty$ then $T \in \Pi_{\infty,q}(X,Y)$ iff $T^* \in D_{q',1}(Y^*,X^*)$.

The following theorem gives an integral condition which is sufficient to guarantee that an operator is strongly (p,q)-summing.

Theorem 3.15 *Let $1 < q \le p < \infty$ and $T \in L(X,Y)$. Suppose there exists a positive Radon measure μ on $B_{Y^{**}}$ and a $c > 0$ such that*

$$\begin{aligned} \|Tx\|_\mu &:= \sup\left\{|y^*(Tx)| \; : \; y^* \in Y^*, \; \|y^*(\cdot)\|_{L^{p'}(B_{Y^{**}},\mu)} \le 1\right\} \\ &\le c\,\|x\| \quad \forall\, x \in X. \end{aligned}$$

Then $T \in D_{p,q}(X,Y)$ and $d_{p,q}(T) \le c$.

Proof Suppose $\|Tx\|_\mu \le c\,\|x\|$, $\forall\, x \in X$. We show that $T^* \in \Pi_{q',p'}(Y^*,X^*)$. It follows that

$$|y^*(Tx)| \le c\,\|x\|\Big(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p'}\, d\mu\Big)^{\frac{1}{p'}},$$

i.e. taking the supremum over $x \in B_X$ we obtain

$$\|T^*y^*\| \le c\,\Big(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p'}\, d\mu\Big)^{\frac{1}{p'}}, \; \forall\, y^* \in Y^*.$$

By the Pietsch Domination Theorem (cf. 2.12 in [19]) the operator T^* is p'-summing (i.e. (p',p')-summing). Since $p' \le q'$, we have $T^* \in \Pi_{q',p'}(Y^*,X^*)$ (cf. 10.4 in [19]). Also $\pi_{q',p'}(T^*) \le \pi_{p'}(T^*) \le c$. By Theorem 3.14 we have $T \in D_{p,q}(X,Y)$ and $d_{p,q}(T) \le c$. ◻

Examples 3.16 *(a) For $1 \le p \le 2$, $id_{\ell^{p'}} \in D_{\infty,2}(\ell^{p'},\ell^{p'})$ and for any $1 \le r \le q$, $q \ge 2$ with $\frac{1}{r} - \frac{1}{q} \ge \frac{1}{2}$,*

$$id_{\ell^{p'}} \in D_{r',q'}(\ell^{p'},\ell^{p'}).$$

(b) Let K be a compact Hausdorff space and $1 \le p < q < \infty$. Then $D_{\infty,p'}(Y^*,M(K)) = D_{q',p'}(Y^*,M(K))$.

(c) Let Y^* be an $\mathcal{L}_{\infty,\lambda}$-space and let X^* be an $\mathcal{L}_{p',\lambda'}$-space, with $1 < p < 2$. Then $T^* \in \Pi_{p',2}(Y^*,X^*)$ i.e. $T \in D_{2,p}(X,Y)$.

Proof

(a) Refer to ([19], p. 199) where it is mentioned that a rephrasing of Orlicz's Theorem shows that $id_{\ell^p} \in \Pi_{2,1}(\ell^p, \ell^p)$ *for* $1 \leq p \leq 2$ *and* $id_{\ell^p} \in \Pi_{q,r}(\ell^p, \ell^p)$ *for any*

$$1 \leq r \leq q,\ q \geq 2,\ \text{with } \frac{1}{r} - \frac{1}{q} \geq \frac{1}{2}.$$

Then use Theorem 3.14.

(b) This follows form ([19], Theorem 10.9), where it is proved that

$$\Pi_{q,1}(C(K), Y) = \Pi_{q,p}(C(K), Y)$$

for $1 \leq p < q < \infty$ *and from Theorem 3.14.*

(c) This follows from a result (Theorem 10.6) in ([19], p. 200) and Theorem 3.14. □

We know that every absolutely p-summing operator T is weakly compact and completely continuous (cf. [34], p. 343 - 345). However it follows that for any $p > 1$ there are $(p, 1)$-summing operators, which are not completely continuous (cf. [19], p. 209) and S. Kwapien and A. Pelczynski (cf. [31]) have shown that if $1 \leq q < p$, then the sum operator $\Sigma : \ell^1 \to \ell^\infty : (x_k) \to (\sum_{k=1}^n x_k)_n$ is (p, q)-summing but not weakly compact. Furthermore we know that if $T \in D_p(X, Y)$, then T is weakly compact and the conjugate T^* is completely continuous (cf. [16]), but from Theorem 3.14 and the above it follows that the strongly (p, q)-summing operators are not necessarily weakly compact or completely continuous.

Recall that if $T \in N(X, Y)$ then $T^* \in N(Y^*, X^*)$ (cf. [27], p. 484 and p. 164). If Y is reflexive or X^* satisfies the approximation property, then

$$T^* \in N(Y^*, X^*) \text{ if and only if } T \in N(X, Y).$$

In the case of strongly (p, q)-nuclear operators we have such a Schauder-theorem type result, without restrictions on X and Y. This was seen in Proposition 3.7 in the general case, which in this setting can be phrased as follows.

Proposition 3.17 *Let* $1 < q \leq p < \infty$ *and let* $T : X \to Y$ *be a bounded linear operator. Then* $T \in SN_{p,q}(X, Y)$ *iff* $T^* \in SN_{q',p'}(Y^*, X^*)$ *and* $sn_{p,q}(T) = sn_{q',p'}(T^*)$.

Theorem 3.18 *Let $q_1 \leq q_2$, $p_1 \leq p_2$.*

1. $D_{p_1,q_2}(X,Y) \subseteq D_{p_2,q_1}(X,Y)$ *and* $d_{p_2,q_1}(\cdot) \leq d_{p_1,q_2}(\cdot)$.

2. $SN_{p_1,q_2}(X,Y) \subseteq SN_{p_2,q_1}(X,Y)$ *and* $sn_{p_2,q_1}(\cdot) \leq sn_{p_1,q_2}(\cdot)$.

Proof We prove (1). The proof of (2) is similar. Choose $u \in D_{p_1,q_2}(X,Y)$; then for $x_1, \cdots, x_n$ in X and $y_1^*, \cdots, y_n^*$ in Y^* we have

$$\begin{aligned}
|\sum_{i=1}^{n} \langle ux_i, y_i^* \rangle| &\leq \|(ux_i)\|_{\langle p_1 \rangle}\ \epsilon_{p_1'}((y_i^*)) \\
&\leq \|(ux_i)\|_{\langle p_1 \rangle}\ \epsilon_{p_2'}((y_i^*)) \\
&\leq d_{p_1,q_2}(u)\|(x_i)\|_{\ell^{q_2}(X)}\ \epsilon_{p_2'}((y_i^*)) \\
&\leq d_{p_1,q_2}(u)\|(x_i)\|_{\ell^{q_1}(X)}\ \epsilon_{p_2'}((y_i^*)).
\end{aligned}$$

Consequently, $u \in D_{p_2,q_1}(X,Y)$ and $d_{p_2,q_1}(u) \leq d_{p_1,q_2}(u)$. □

Note that if $p_1 \leq p_2$, then $D_{p_2}(X,Y) \subseteq D_{p_1}(X,Y)$, because:

$$T \in D_{p_2}(X,Y) \overset{3.14}{\Leftrightarrow} T^* \in \Pi_{p_2'}(Y^*,X^*) \Rightarrow T^* \in \Pi_{p_1'}(Y^*,X^*)$$

by the Inclusion Theorem (cf. [19], p. 39). Again by Theorem 3.14, we have

$$T \in D_{p_1}(X,Y).$$

3.2 Positive operators

Throughout this section X will denote a Banach lattice and Y a Banach space. Given $1 \leq p < \infty$, we use the following notation:

$$B_X^+ = \{x \in B_X : x \geq 0\}$$

$$\epsilon_p^+((x_i)) = \sup_{x^* \in B_{X^*}^+} (\sum_{i=1}^{n} \langle x^*, |x_i| \rangle^p)^{\frac{1}{p}}$$

$$\ell_w^p(X)_+ = \{(x_n) \subset X : \epsilon_p^+((x_n)) < \infty\}.$$

Lemma 3.19 *Let X be a Banach lattice and $x_1, x_2, \cdots, x_n \geq 0$. Then*

$$\epsilon_p^+((x_i)(\leq n)) = \epsilon_p((x_i)(\leq n)).$$

Proof Clearly, $\epsilon_p^+((x_i)_{i\leq n}) \leq \epsilon_p((x_i)_{i\leq n})$, because

$$\epsilon_p^+((x_i)_{i\leq n}) = \sup_{x^*\in B_{X^*}^+} (\sum_{i=1}^n |\langle x^*, x_i\rangle|^p)^{\frac{1}{p}} \leq \sup_{x^*\in B_{X^*}} (\sum_{i=1}^n |\langle x^*, x_i\rangle|^p)^{\frac{1}{p}} = \epsilon_p((x_i)_{i\leq n}).$$

Now, conversely, let $x^* \in B_{X^*}$. From

$$\langle |x^*| - x^*, x\rangle = \langle (x^*)^+ + (x^*)^- - (x^*)^+ + (x^*)^-, x\rangle = 2\langle (x^*)^-, x\rangle \geq 0,$$

for all $x \geq 0$, it is clear that $x^* \leq |x^*|$. Since $x_i \geq 0$, it follows that

$$(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p)^{\frac{1}{p}} \leq (\sum_{i=1}^n |\langle |x^*|, x_i\rangle|^p)^{\frac{1}{p}} = (\sum_{i=1}^n \langle |x^*|, |x_i|\rangle^p)^{\frac{1}{p}} \leq \sup_{y^*\in B_{X^*}^+} (\sum_{i=1}^n \langle y^*, |x_i|\rangle^p)^{\frac{1}{p}}.$$

Therefore $\epsilon_p((x_i)(\leq n)) \leq \epsilon_p^+((x_i)(\leq n))$. □

Blasco (cf. [7]) introduced the positive (p,q)-summing operators as follows.

Definition 3.20 *(cf. [7], p. 14) Let $1 \leq q \leq p < \infty$.*
An operator $T : X \to Y$ is said to be **positive (p,q)-summing** *(denoted by $T \in \Lambda_{p,q}(X,Y)$) if there exists a constant $c > 0$ such that for every finite set $x_1, x_2, \cdots, x_n \geq 0$ in X we have*

$$(\sum_{i=1}^n \|Tx_i\|_Y^p)^{\frac{1}{p}} \leq c\ \epsilon_q^+((x_i)).$$

For $q < p = \infty$,

$$\sup_{1\leq i\leq n} \|Tx_i\|_Y \leq c\ \epsilon_q^+((x_i)).$$

In a similar way we define the following new class of operators.

Definition 3.21 *Let $1 \leq q \leq p < \infty$. An operator T is* **positive strongly (p,q)-nuclear** *(i.e. $T \in SN_{p,q}^+(X,Y)$) if there exists a $c > 0$ such that for all finite sets $\{x_1, \cdots, x_n\} \subset X$ of positive elements, we have*

$$\|(Tx_i)(\leq n)\|_{\langle p\rangle} \leq c\ \epsilon_q^+((x_i)(\leq n)) \overset{3.19}{=} c\ \epsilon_q((x_i)(\leq n)).$$

Equivalently, $T \in SN_{p,q}^+(X,Y) \iff$
$\exists\ c > 0$ such that for any $0 \leq x_1, \cdots, x_n \in X,\ y_1^*, \cdots, y_n^* \in Y^*$,

$$\sum_{i=1}^n |\langle y_i^*, Tx_i\rangle| \leq c \sup_{x^*\in B_{X^*}^+} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^q\right)^{\frac{1}{q}} \sup_{y\in B_Y} \left(\sum_{k=1}^n |y_k^*(y)|^{p'}\right)^{\frac{1}{p'}}.$$

$SN^+_{p,q}(X,Y)$ becomes a Banach space with norm $sn^+_{p,q}(T)$ given by the infimum of the constants $c>0$ that verify the above inequality. As before, if $p=q$, we denote $SN^+_{p,p}(X,Y)$ by $SN^+_p(X,Y)$. Note that for $p=q=1$ we have $SN^+_1(X,Y)=\Lambda_{1,1}(X,Y)$.
As in Theorem 3.18 we have the inclusion $SN^+_{p_1,q_2}(X,Y)\subseteq SN^+_{p_2,q_1}(X,Y)$ if $q_1\le q_2$ and $p_1\le p_2$.

Proposition 3.22 *Let $1<q\le p<\infty$ and assume that $T:X\to Y$ is a bounded linear operator. Then $T\in SN^+_{p,q}(X,Y)$ if and only if $\widehat{T}:\ell^q_w(X)_+\to\ell^p\langle Y\rangle$, defined by $\widehat{T}((x_n))=(Tx_n)$, is continuous.*

Proof Suppose $T\in SN^+_{p,q}(X,Y)$ and $(x_n)\in\ell^q_w(X)_+$ then

$$\begin{aligned}\|(Tx_n)\|_{\langle p\rangle} &\le \|(Tx_i^+)\|_{\langle p\rangle}+\|(Tx_i^-)\|_{\langle p\rangle}\\ &\le sn^+_{p,q}(T)\ [\epsilon_{q^+}((x_i^+))+\epsilon_{q^+}((x_i^-))]\\ &\le 2sn^+_{p,q}(T)\epsilon_{q^+}((x_i)).\end{aligned}$$

Therefore $\|\widehat{T}(x_n)\|_{\langle p\rangle}\le 2sn^+_{p,q}(T)\epsilon^+_q((x_n))$. Conversely, suppose $\widehat{T}:\ell^q_w(X)_+\to\ell^p\langle Y\rangle$ is bounded and suppose $T\notin SN^+_{p,q}(X,Y)$. Then for every $n\in\mathbb{N}$, there is a finite set $\{x_{1n},x_{2n},\cdots,x_{m_nn}\}\subset X$ such that $(x_{in})_i\in B_{\ell^q_w(X)_+}$ and $\|(Tx_{in})_{i\le m_n}\|_{\langle p\rangle}\ge 2^n$. Let $\epsilon_{p'}((y^*_{i,n}))\le 1$ such that $\sum_{i=1}^{m_n}|\langle Tx_{i,n},y^*_{i,n}\rangle|\ge 2^n$. Consider the sequence

$$(z_i)=(2^{-1}x_{11},2^{-1}x_{21},\cdots,2^{-1}x_{m_11},2^{-2}x_{12},2^{-2}x_{22},\cdots,2^{-2}x_{m_22},\cdots)$$

and

$$(y_i^*)=(2^{-1}y^*_{11},2^{-1}y^*_{21},\cdots,2^{-1}y^*_{m_11},2^{-2}y^*_{12},2^{-2}y^*_{22},\cdots,2^{-2}y^*_{m_22},\cdots).$$

Then, $z_i\ge 0$ and by Lemma 3.19

$$\begin{aligned}\epsilon^+_q((z_i))=\sup_{\|x^*\|\le 1}\Big(\sum_{i=1}^\infty|x^*(z_i)|^q\Big)^{\frac1q} &= \sup_{\|x^*\|\le 1}\Big(\sum_{n=1}^\infty\sum_{i=1}^{m_n}|x^*(2^{-n}x_{in})|^q\Big)^{\frac1q}\\ &= \sup_{\|x^*\|\le 1}\Big(\sum_{n=1}^\infty 2^{-nq}\sum_{i=1}^{m_n}|x^*(x_{in})|^q\Big)^{\frac1q}\\ &\le \Big(\sum_{n=1}^\infty\frac{1}{2^{nq}}\Big)^{\frac1q}\\ &\le 1.\end{aligned}$$

Similarly, $(y_i^*) \in B_{\ell_w^{p'}(Y^*)}$. However, $\|(Tz_i)\|_{\langle p\rangle} = \infty$, (as in the proof of Proposition 3.3), contradicting the fact that $\widehat{T}$ maps $\ell_w^q(X)_+$ (continuously) into $\ell^p\langle Y\rangle$. Since

$$\|\widehat{T}((x_i))\|_{\langle p\rangle} \le \|\widehat{T}\|\epsilon_q((x_i)),$$

we have $\|(Tx_i)\|_{\langle p\rangle} \le \|\widehat{T}\|\epsilon_q^+((x_i))$. □

Lemma 3.23 *(cf. [37] , p. 241) Let X denote a Banach space. If $(x_n) \in \ell^1(X)$, then*

$$\|(x_n)\|_{\ell^1(X)} = \sup_{(x_n^*)\in B_{c_0(X^*)}} \sum_{n=1}^{\infty} \langle x_n, x_n^*\rangle.$$

If X is a Banach lattice and (x_n) is a positive summable sequence in X, then

$$\epsilon_1((x_i)) = \|\sum_{n=1}^{\infty} x_n\|.$$

Theorem 3.24 *Let $1 < p < \infty$. Then*

(a) $SN_{p,1}^+(\ell^1, Y) = L(\ell^1, Y)$.

(b) $SN_{p,1}^+(L^1(\mu), Y) = L(L^1(\mu), Y)$.

Proof

(a) Given $T \in L(\ell^1, Y)$ and $\phi_1, \phi_2, \cdots, \phi_n \ge 0$ in ℓ^1 we have

$$\begin{aligned}
\|(T\phi_i)\|_{\langle p\rangle} &\le \|(T\phi_i)\|_{\langle 1\rangle} \\
&= \sum_{i=1}^{n} \|T\phi_i\| \\
&\le \|T\| \sum_{i=1}^{n} \|\phi_i\|_1 \\
&= \|T\| \|\sum_{i=1}^{n} \phi_i\|_1 \\
&= \|T\|\epsilon_1((\phi_i)),
\end{aligned}$$

where the last two steps in the proof follow by the positivity of each ϕ_i.

(b) The proof is similar to the one in (a). □

Theorem 3.25 *Let* $1 \leq p,\ q \leq \infty$.

(a) If $r \leq q$ *then* $SN^+_{p,q}(X,Y) \subseteq SN^+_{p,r}(X,Y)$.

(b) For X_1 *a subspace of* X_2 *and* $\overline{X_1} = X_2$ *it follows that* $SN^+_{p,q}(X_2,Y)$ *is isometrically embedded into* $\subseteq SN^+_{p,q}(X_1,Y)$.

Proof

(a) It is obvious since $\epsilon^+_q((x_i)(\leq n)) \leq \epsilon^+_r((x_i)(\leq n))$.

(b) The inclusion $SN^+_{p,q}(X_2,Y) \subseteq SN^+_{p,q}(X_1,Y)$ is clear, since the mapping $T \mapsto T|_{X_1}$ is injective and $T|_{X_1} \in SN^+_{p,q}(X_1,Y)$. Moreover, since each $x^* \in B^+_{X_1^*}$ extends uniquely to $x^* \in B^+_{X_2^*}$, $\ell^q_w(X_1)_+$ is isometrically embedded into $\ell^q_w(X_2)_+$. Therefore the inclusion of $SN^+_{p,q}(X_2,Y)$ into $SN^+_{p,q}(X_1,Y)$ is also an isometry. □

Recall from ([33], p. 42) that for $x_1, x_2, \dots x_n$ in a Banach lattice X and for $p > 1$ the vector $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ in X can be considered as

$$(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} = \sup_{\|(\alpha_i)\|_{\ell^{p'}} \leq 1} \sum_{i=1}^n \alpha_i x_i,$$

where $\ell^{p'}$ is replaced by c_0 if $p = 1$.

Definition 3.26 *Let* $1 \leq p, q < \infty$. *An operator* $T \in L(X,Y)$ *is said to be* **strongly (p,q)-concave** *(i.e.* $T \in SC_{p,q}(X,Y)$*) if there exists a constant* $c > 0$ *such that for every* $x_1, x_2, \cdots, x_n$ *in* X *we have*

$$\|(Tx_i)(\leq n)\|_{\langle p\rangle} \leq c\|(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}\|_X.$$

Equivalently, $T \in SC_{p,q}(X,Y) \iff \exists\ c > 0$ *such that* $\forall\ \{x_1, \cdots, x_n\} \subseteq X$ *we have*

$$\sum_{i=1}^n |\langle y_i^*, Tx_i\rangle| \leq c \sup_{y \in B_Y} (\sum_{k=1}^n |y_k^*(y)|^{p'})^{\frac{1}{p'}} \|(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}\|_X, \tag{3.4}$$

with norm $sc_{p,q}(\cdot) = \inf\{c > 0 : c$ *verifies* $(3.4)\}$. *Note that strongly (p,p)-concave is the same as strongly p-concave.*

Remark 3.27 *Since* $\|(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}\|_{L^q(\mu)} = (\sum_{i=1}^n \|x_i\|^q_{L^q(\mu)})^{\frac{1}{q}}$ *(cf. Remark 1.18), it follows that if* $u \in L(L^q(\mu),Y)$ *then* $u \in SC_{p,q}(L^q(\mu),Y)$ *if and only if* $u \in D_{p,q}(L^q(\mu),Y)$.

Proposition 3.28 *For $1 \leq q \leq p \leq \infty$.*

$$SN_{p,q}(X,Y) \overset{3.21}{\subseteq} SN^+_{p,q}(X,Y) \subseteq SC_{p,q}(X,Y).$$

Proof The first inclusion is obvious. Let us have a look at the second inclusion. Choose $T \in SN^+_{p,q}(X,Y)$. There exists $c > 0$ such that for any $x_1, \cdots, x_n \in X$, it follows that

$$\begin{aligned}
\|(Tx_i)(\leq n)\|_{\langle p\rangle} &\leq \|(Tx_i^+)(\leq n)\|_{\langle p\rangle} + \|(Tx_i^-)(\leq n)\|_{\langle p\rangle} \\
&\leq c[\sup_{(\alpha_i)\in B^+_{\ell^{q'}}} \|\sum_{i=1}^n \alpha_i x_i^+\|_X + \sup_{(\alpha_i)\in B^+_{\ell^{q'}}} \|\sum_{i=1}^n \alpha_i x_i^-\|_X] \\
&\leq 2c \sup_{(\alpha_i)\in B^+_{\ell^{q'}}} \|\sum_{i=1}^n \alpha_i |x_i|\|_X \\
&\leq 2c\|(\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}\|_X.
\end{aligned}$$

Therefore $T \in SC_{p,q}(X,Y)$ □

Note that $SN_{p,p}(\ell^2, \ell^\infty) \neq D_{p,p}(\ell^2, \ell^\infty)$ (cf. [16], Theorem 2.4.2.) but from Remark 3.27 it follows that $SC_{p,p}(\ell^2, \ell^\infty) = D_{p,p}(\ell^2, \ell^\infty)$. Therefore

$$SC_{p,p}(\ell^2, \ell^\infty) \neq SN_{p,p}(\ell^2, \ell^\infty).$$

Proposition 3.29 *Let $1 \leq p, q < \infty$.*

(a) $SN^+_{p,1}(X,Y) = SC_{p,1}(X,Y)$.

(b) $SN^+_{p,1}(X,Y) \subseteq SC_{q,1}(X,Y)$ *for all $p < q$.*

Proof

(a) By Proposition 3.28 we only need to show that $SC_{p,1}(X,Y) \subseteq SN^+_{p,1}(X,Y)$. Let $T \in SC_{p,1}(X,Y)$. There exists a $c > 0$ such that for any finite set $\{x_1, \cdots, x_n\} \subset X$ of positive elements we have

$$\begin{aligned}
\|(Tx_i)(\leq n)\|_{\langle p\rangle} &\leq sc_{p,1}(T)\ \|\sum_{i=1}^n |x_i|\|_X \\
&= sc_{p,1}(T)\ \|\sum_{i=1}^n x_i\|_X \\
&\overset{3.23}{=} sc_{p,1}(T)\ \epsilon_1((x_i)).
\end{aligned}$$

(b) For $T \in SN^+_{p,1}(X,Y)$, there exists a $c > 0$ such that for all $x_1, \cdots, x_n$ in X we have

$$\|(Tx_i)(\le n)\|_{\langle q\rangle} \le \|(Tx_i)(\le n)\|_{\langle p\rangle} \overset{(a)}{\le} c\|\sum_{i=1}^{n} |x_i|\|_X.$$

□

Theorem 3.30 *Let Ω be a compact topological space with $1 \le p < \infty$. Then*

$$SN_p(C(\Omega), Y) = SN^+_p(C(\Omega), Y) = SC_p(C(\Omega), Y).$$

Proof Choose $T \in SC_p(C(\Omega), Y)$. For $\varphi_1, \varphi_2, \cdots, \varphi_n$ belonging to $C(\Omega)$ we have

$$\begin{aligned}
\|(T\varphi_i)(\le n)\|_{\langle p\rangle} &\le sc_p(T)\|(\sum_{i=1}^{n} |\varphi_i|^p)^{\frac{1}{p}}\|_{C(\Omega)} \\
&= sc_p(T) \sup_{t\in\Omega}(\sum_{i=1}^{n} |\varphi_i(t)|^p)^{\frac{1}{p}} \\
&= sc_p(T) \sup_{t\in\Omega} \sup_{(\alpha_i)\in B_{\ell^{p'}}} |\sum_{i=1}^{n} \varphi_i(t)\alpha_i| \\
&= sc_p(T) \sup_{(\alpha_i)\in B_{\ell^{p'}}} \|\sum_{i=1}^{n} \varphi_i\alpha_i\|_{C(\Omega)} \\
&= sc_p(T)\epsilon_p((\varphi_i)).
\end{aligned}$$

□

Proposition 3.31 *Let $1 < p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$SN^+_p(\ell^{p'}, Y) \subseteq \ell^p\langle Y\rangle \subseteq SN^+_1(\ell^{p'}, Y)$$

the inclusions being norm (≤ 1) embeddings.

Proof For $T \in SN^+_p(\ell^{p'}, Y)$ and $\{e_n : n \in \mathbb{N}\}$ the canonical basis in $\ell^{p'}$ (which satisfies $(e_n) \in \ell^p_w(\ell^{p'})$), we let $x_n := T(e_n)$ for all $n \in \mathbb{N}$. Then $(x_n) \in \ell^p\langle Y\rangle$ and

$$\|(x_n)\|_{\langle p\rangle} = \|(T(e_n))\|_{\langle p\rangle} \le sn^+_p(T)\epsilon_p((e_n)) = sn^+_p(T).$$

This proves that $SN^+_p(\ell^{p'}, Y) \subseteq \ell^p\langle Y\rangle$, the norm ($\le 1$) embedding being $T \mapsto ((Te_n))$.

Let $(x_n) \in \ell^p\langle Y\rangle$, and consider the operator $T : \ell^{p'} \to Y$ defined by

$$T((\alpha_n)) = \sum_{n\in\mathbb{N}} \alpha_n x_n.$$

Choose $0 \leq \xi_1, \xi_2, \cdots, \xi_m \in \ell^{p'}$, where $\xi_i = (\xi_{i,n})_n$ with $\xi_{i,n} \geq 0$ for all $i = 1, 2, \ldots, m$ and all $n \in \mathbb{N}$. Then since $\ell^1\langle Y\rangle = \ell^1(Y)$, we can do the following calculation:

$$\begin{aligned}
\|(T(\xi_i)(\leq m))\|_{\langle 1\rangle} &= \sum_{i=1}^{m} \|T(\xi_i)\| \\
&\leq \sum_{i=1}^{m} \sum_{n\in\mathbb{N}} \xi_{i,n}\|x_n\| \\
&= \sum_{i=1}^{m} \langle \xi_i, (\|x_j\|)_j\rangle \\
&= \|(x_n)\|_{\ell^p\langle Y\rangle} \sum_{i=1}^{m} \langle \xi_i, (\|(x_n)\|^{-1}_{\ell^p\langle Y\rangle}\|x_j\|)_j\rangle \\
&\leq \|(x_n)\|_{\langle p\rangle} \sup_{\|(\lambda_j)\|_{\ell^p}\leq 1} \sum_{i=1}^{m} |\langle \xi_i, (\lambda_j)\rangle| \\
&= \|(x_n)\|_{\langle p\rangle}\epsilon_1((\xi_i)(\leq m)).
\end{aligned}$$

It follows that $T \in SN_1^+(\ell^{p'}, Y)$ and $sn_1^+(T) \leq \|(x_n)\|_{\langle p\rangle}$. □

Remark 3.32 *(i) Suppose Z is a Banach lattice, $S \in L(Z, X)$ is positive and*

$$T \in SC_{p,q}(X, Y).$$

For a finite set $z_1, z_2, \ldots, z_n$ of positive vectors in Z, we know that the corresponding set $Sz_1, Sz_2, \ldots, Sz_n$ is a finite set of positive vectors in X. Thus,

$$\begin{aligned}
\|(TS)(z_i)(\leq n)\|_{\langle p\rangle} &\leq sc_{p,q}(T)\|(\sum_{i=1}^{n} |Sz_i|^q)^{1/q}\|_X \\
&\leq sc_{p,q}(T)\|S\|\|(\sum_{i=1}^{n} |z_i|^q)^{1/q}\|_Z.
\end{aligned}$$

This shows that $TS \in SC_{p,q}(Z, Y)$ and $sc_{p,q}(TS) \leq sc_{p,q}(T)\|S\|$.

(ii) The reader is referred to [33], see (p. 56 – 57) where it is explained that:

(a) If $1 \le p < \infty$ and $\{f_i\}_{i=1}^n$ is a finite set in a $C(K)$ space, then

$$\sup\{(\sum_{i=1}^n |\mu(f_i)|^p)^{1/p} : \mu \in C(K)^*, \|\mu\| = 1\} = \sup\{(\sum_{i=1}^n |f_i(k)|^p)^{1/p} : k \in K\}.$$

(b) If X is a Banach lattice and $\{x_j\}_{j=1}^m$ is a finite set in X, then the completion of the normed ideal $I(x_0)$ generated by the element $x_0 = (\sum_{i=1}^m |x_i|^q)^{1/q}$ and with norm

$$\|x\|_\infty := \inf\{\lambda > 0 : |x| \le \frac{\lambda x_0}{\|x_0\|}\},$$

is order isometric to a $C(K)$ space. Let J denote the formal identity mapping from $I(x_0)$ into X, i.e. we may consider $J : C(K) \to X$ as a (positive) norm (≤ 1) embedding. Clearly, $x_i \in I(x_0)$ for all $i \in \{1, \cdots, m\}$.

Proposition 3.33 *Let $1 \le p, q < \infty$ and $0 < c < \infty$. For $T \in L(X, Y)$, we have:*

(a) $T \in SN_{p,1}^+(X,Y)$ with $sn_{p,1}^+(T) \le c$, if and only if, for every positive operator $S \in L(c_0, X)$, $TS \in SN_{p,1}^+(c_0, Y)$ and $sn_{p,1}^+(TS) \le c\ \|S\|$.

(b) $T \in SC_{p,q}(X,Y)$ with $sc_{p,q}(T) \le c$, if and only if, for every positive bounded linear operator S from a $C(\Omega)$ space into X, the composition TS belongs to $SN_{p,q}(C(\Omega), Y)$ and $sn_{p,q}(TS) \le c\ \|S\|$.

Proof

(a) Let $T \in SN_{p,1}^+(X,Y)$ with $sn_{p,1}^+(T) \le c$. Let $S \in L(c_0, X)$ be positive. Now let $z_1, z_2, \cdots, z_n \ge 0$ in c_0. Then

$$\begin{aligned}
\|TS((z_i)(\le n))\|_{\langle p\rangle} &= \|(T(Sz_i))_{i\le n}\|_{\langle p\rangle} \text{ and } Sz_i \ge 0 \text{ in } X \\
&\le sn_{p,1}^+(T)\epsilon_1^+((Sz_i)(\le n)) \\
&= sn_{p,1}^+(T) \sup_{x^* \in B_{X^*}^+} \sum_{i=1}^n |\langle x^*, Sz_i\rangle| \\
&= sn_{p,1}^+(T) \sup_{x^* \in B_{X^*}^+} \sum_{i=1}^n |\langle S^* x^*, z_i\rangle| \\
&= sn_{p,1}^+(T)\|S\| \sup_{x^* \in B_{X^*}^+} \sum_{i=1}^n |\langle \frac{S^*}{\|S^*\|} x^*, z_i\rangle| \\
&\le sn_{p,1}^+(T)\|S\| \sup_{y^* \in B_{\ell 1}^+} \sum_{i=1}^n |\langle y^*, z_i\rangle|,
\end{aligned}$$

because $S^* \geq 0$. Thus, $\|((TS)(z_i))_{i\leq n}\|_{\langle p\rangle} \leq sn^+_{p,1}(T)\|S\|\epsilon^+_1((z_i)(\leq n))$. This shows that $TS \in SN^+_{p,1}(c_0, Y)$ and $sn^+_{p,1}(TS) \leq sn^+_{p,1}(T)\|S\|$.

Conversely, let $\{x_1, x_2, \cdots, x_n\} \subset X,\ x_i \geq 0$. Consider

$$S : c_0 \to X, \text{ such that } S((\xi_n)) = \sum_{i=1}^{n} \xi_i x_i.$$

Then S is bounded, with $\|S\| = \epsilon_1((x_i)(\leq n))$. Also, since $x_i \geq 0$, we have for all $(\xi_i) \in c_0,\ \xi_i \geq 0,\ \forall\ i$, that $S((\xi_i)) = \sum_{i=1}^n \xi_i x_i \geq 0$; i.e. S is a positive bounded linear operator. By assumption, $TS \in SN^+_{p,1}(c_0, Y)$ and $sn^+_{p,1}(TS) \leq c\ \|S\|$. Thus

$$\begin{aligned}
\|(Tx_i)(\leq n)\|_{\langle p\rangle} &= \|(TSe_i)(\leq n)\|_{\langle p\rangle} \\
&\leq c\ \|S\|\epsilon^+_1((e_i)(\leq n)) \\
&= c\ \|S\| \sup_{\lambda \in B^+_{\ell 1}} \sum_{i=1}^{n} |\langle e_i, \lambda\rangle| \\
&= c\ \epsilon_1((x_i)(\leq n)) \\
&= c\epsilon^+_1((x_i)(\leq n)) \text{ since } x_i \geq 0.
\end{aligned}$$

(b) Let $T \in SC_{p,q}(X, Y)$ and $S \in L(C(\Omega), X)$ be positive. For every finite set $\{f_1, \cdots, f_n\}$ in $C(\Omega)$ we have

$$\begin{aligned}
\|(TS)((f_i)(\leq n))\|_{\langle p\rangle} &= \|T((Sf_i)(\leq n))\|_{\langle p\rangle} \\
&\leq c\|(\sum_{i=1}^{n} |Sf_i|^q)^{\frac{1}{q}}\|_X \\
&\leq c\|S\| \sup_{k\in\Omega}(\sum_{i=1}^{n} |f_i(k)|^q)^{\frac{1}{q}} \quad (\textit{cf. Remark } 3.32(ii)) \\
&= c\ \|S\| \sup_{\mu \in B_{C(\Omega)^*}} (\sum_{i=1}^{n} |\mu(f_i)|^q)^{\frac{1}{q}}.
\end{aligned}$$

Hence $TS \in SN_{p,q}(C(\Omega), Y)$ and $sn_{p,q}(TS) \leq c\|S\|$.

Conversely, assume that for every $C(\Omega)$ space and every positive $S \in L(C(\Omega), X)$ we have $TS \in SN_{p,q}(C(\Omega), Y)$ and $sn_{p,q}(TS) \leq c\ \|S\|$. Let $\{x_1, x_2, \ldots, x_n\} \subset X$ and consider the $C(K)$ space and norm ≤ 1 embedding $J : C(K) \to X$, which are mentioned in Remark 3.32. Let $f_i \in C(K)$ such that

$$Jf_i = x_i \text{ for } i = 1, 2, \ldots, n.$$

We have

$$\begin{aligned}
\|(Tx_i)_{i\leq n}\|_{\langle p\rangle} &= \|(TJf_i)_{i\leq n}\|_{\langle p\rangle} \\
\leq \quad & sn^+_{p,q}(TJ)\sup\{(\sum_{i=1}^{n}|\mu(f_i)|^q)^{1/q} : \mu\in C(K)^*, \|\mu\|=1\} \\
= \quad & sn^+_{p,q}(TJ)\sup\{(\sum_{i=1}^{n}|f_i(k)|^q)^{1/q} : k\in K\} \quad (\textit{cf. Remark } 3.32) \\
= \quad & sn^+_{p,q}(TJ)\|(\sum_{i=1}^{n}|f_i|^q)^{1/q}\|_{C(K)} \\
= \quad & sn^+_{p,q}(TJ)\|(\sum_{i=1}^{n}|x_i|^q)^{1/q}\|_{\infty} \quad (\textit{cf. Remark } 3.32) \\
= \quad & sn^+_{p,q}(TJ)\inf\{\lambda>0 : (\sum_{i=1}^{n}|x_i|^q)^{1/q}\leq\lambda\|(\sum_{i=1}^{n}|x_i|^q)^{1/q}\|_X^{-1}(\sum_{i=1}^{n}|x_i|^q)^{1/q}\} \\
= \quad & sn^+_{p,q}(TJ)\|(\sum_{i=1}^{n}|x_i|^q)^{1/q}\|_X.
\end{aligned}$$

This shows that $T\in\mathcal{SC}_{p,q}(X,Y)$ and $sc_{p,q}(T)\leq c\|J\|\leq c$. □

Chapter 4

Operator valued multipliers

4.1 Strongly (p,q)-summing and strongly (p,q)-nuclear multipliers

Definition 4.1 *(cf. [2], p. 3) A sequence of operators $(u_n) \subseteq L(X,Y)$ is a called* **multiplier sequence** *from $E(X)$ to $F(Y)$ if there exists a constant $c > 0$ such that*

$$\|(u_j x_j)_{j=1}^n\|_{F(Y)} \leq c \ \|(x_j)_{j=1}^n\|_{E(X)},$$

for all finite families $x_1, \cdots, x_n$ in X. The infimum of all the numbers $c > 0$, which satisfy this condition, is denoted by $\|(u_j)\|_{(E(X),F(Y))}$. The set of all **multiplier sequences** *from $E(X)$ to $F(Y)$ is denoted by $(E(X), F(Y))$.*

Definition 4.2 *(cf. [2]) Let X and Y be Banach spaces and let $1 \leq p, q \leq \infty$. A sequence $(u_j)_{j\in\mathbb{N}}$ of operators in $L(X,Y)$ is called a* **(p,q)-summing multiplier** *for the pair (X,Y), in short $(u_j) \in \ell_{\pi_{p,q}}(X,Y)$, if there exists a constant $c > 0$ such that, for any finite collection of vectors $x_1, x_2, \cdots, x_n \in X$, it holds that*

$$\left(\sum_{j=1}^{n} \|u_j x_j\|^p\right)^{\frac{1}{p}} \leq c \sup_{x^* \in B_{X^*}} \left(\sum_{j=1}^{n} |x^* x_j|^q\right)^{\frac{1}{q}}.$$

We use $\ell_{\pi_{p,q}}(X,Y)$ to denote the vector space of all (p,q)-summing multipliers from X into Y and $\pi_{p,q}((u_j))$ is the least constant c for which (u_j) verifies the inequality in the definition. If $q = p$ we simply say that the sequence (u_j) is a p-summing multiplier.

Note that $\ell_{\pi_{p,q}}(X,Y) = (\ell_w^q(X), \ell^p(Y))$. A constant sequence (u_j), $u_j = u \in L(X,Y)$ for all $j \in \mathbb{N}$, belongs to $\ell_{\pi_{p,q}}(X,Y)$ if and only if $u \in \Pi_{p,q}(X,Y)$. Also the case

$(u_j) = (\lambda_j u) \in (\ell^q_w(X), \ell^1(Y))$ for all $(\lambda_j) \in \ell^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, corresponds to $u \in \Pi_{p,q}(X,Y)$. These facts motivate the use of the notation $\ell_{\pi_{p,q}}(X,Y)$ instead of $(\ell^q_w(X), \ell^p(Y))$ and $\ell_{\pi_p}(X,Y)$ for the case $q = p$.

Blasco and Arregui (cf. [3]) constructed the following examples by taking tensor products of some classical spaces.

Example 4.3 *(cf. [3], Examples 3.1) Let X and Y be Banach spaces, and $1 \le p, q \le \infty$.*
(1) $\ell_{\pi_{r,q}}(X, \mathbb{K}) \hat{\otimes} \ell^s(Y) \subset \ell_{\pi_{p,q}}(X,Y)$ for $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.
(2) $\ell^s \hat{\otimes} \Pi_{r,q}(X,Y) \subset \ell_{\pi_{p,q}}(X,Y)$ for $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.
In particular $\ell^p \hat{\otimes} X \subset \ell_{\pi_{1,p'}}(X) = \ell^p\langle X \rangle$.
It is proved in [14] and [23] that indeed $\ell^p \hat{\otimes} X = \ell^p\langle X \rangle$ isometrically.
(3) $\ell^s(Y) \hat{\otimes} X^* \subset \ell_{\pi_{p,q}}(X,Y)$ for $p < q$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$.

We consider some examples.

Example 4.4 *Let K be a compact set and μ a probability measure on the Borel sets of K. Let $1 \le p < q < \infty$, $1/r = 1/p - 1/q$ and (ϕ_j) a sequence of continuous functions on K. Consider $u_j : C(K) \to L^p(\mu)$ given by $u_j(\psi) = \phi_j \psi$. Then $(u_j) \in \ell_{\pi_{p,q}}(C(K), L^p(\mu))$ if and only if*

$$(\sum_{j=1}^{\infty} |\phi_j|^r)^{1/r} \in L^p(\mu).$$

Example 4.5 *Let (Ω, Σ, μ) and (Ω', Σ', μ') be finite measure spaces.*
Let $1 \le p \le q < \infty$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. For each $n \in \mathbb{N}$ and $(f_j) \subset L^p(\mu, X)$ with $X = L^1(\mu')$, consider the operator $u_n : L^\infty(\mu') \to L^p(\mu)$ defined by

$$u_n(\phi)(\cdot) = \int_{\Omega'} \phi(\omega') f_n(\cdot)(\omega') \; d\mu'(\omega').$$

Put $f_n(.,\omega') = f_n(.)(\omega')$. If

$$(\sum_{k=1}^{n} |f_k|^r)^{\frac{1}{r}} \in L^p(\mu, L^1(\mu')) \; (\textit{where } (\sum_{k=1}^{n} |f_k|^r)^{\frac{1}{r}}(\omega)(.) = (\sum_{k=1}^{n} |f_k(w,.)|^r)^{\frac{1}{r}}),$$

then $(u_n) \in \ell_{\pi_{p,q}}(L^\infty(\mu'), L^p(\mu))$.

Proof Given $n \in \mathbb{N}$ and $\phi_1, \phi_2, \cdots, \phi_n \in L^\infty(\mu')$, then

$$\begin{aligned}
\sum_{k=1}^n \|u_k(\phi_k)\|_{L^p(\mu)}^p &= \sum_{k=1}^n \int_\Omega |u_k(\phi_k)|^p \, d\mu \\
&= \int_\Omega \sum_{k=1}^n |\int_{\Omega'} \phi_k(\omega') f_k(\omega, \omega') d\mu'(\omega')|^p \, d\mu(\omega) \\
&= \int_\Omega \|(\int_{\Omega'} \phi_k(\omega') f_k(\omega, \omega') d\mu'(\omega'))_{k\le n}\|_p^p \, d\mu(\omega) \\
&= \int_\Omega \| \int_{\Omega'} (\phi_k(\omega') f_k(\omega, \omega'))_{k\le n} d\mu'(\omega')\|_p^p \, d\mu(\omega) \\
&\le \int_\Omega (\int_{\Omega'} \|(\phi_k(\omega') f_k(\omega, \omega'))_{k\le n}\|_p \, d\mu'(\omega'))^p \, d\mu(\omega) \\
&= \int_\Omega (\int_{\Omega'} (\sum_{k=1}^n |\phi_k(\omega')|^p |f_k(\omega, \omega')|^p)^{\frac{1}{p}} \, d\mu'(\omega'))^p \, d\mu(\omega) \\
&\le \int_\Omega \{\int_{\Omega'} (\sum_{k=1}^n |f_k(\omega, \omega')|^r)^{\frac{1}{r}} (\sum_{k=1}^n |\phi_k(\omega')|^q)^{\frac{1}{q}} \, d\mu'(\omega')\}^p \, d\mu(\omega) \\
&\le \|(\sum_{k=1}^n |\phi_k(\cdot)|^q)^{\frac{1}{q}}\|_{L^\infty(\mu')}^p \int_\Omega (\int_{\Omega'} (\sum_{k=1}^n |f_k(\omega, \omega')|^r)^{\frac{1}{r}} d\mu'(\omega'))^p \, d\mu(\omega).
\end{aligned}$$

Hence, since $\epsilon_q((\phi_n)) = \|(\sum_{k=1}^n |\phi_k|^q)^{\frac{1}{q}}\|_{L^\infty(\mu')}$, it follows that

$$\pi_{p,q}((u_k)) \le \|(\sum_{k=1}^n |f_k(\omega, \omega')|^r)^{\frac{1}{r}}\|_{L^p(\mu, L^1(\mu'))}.$$

□

Example 4.6 *Let* $1 \le p \le q < \infty$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ *and* (A_n) *be a sequence of infinite matrices. Consider* $T_n \in L(c_0, \ell^p)$ *given by* $T_n((\lambda_k)) = (\sum_{k=1}^\infty A_n(k,j)\lambda_k)_j$. *If*

$$\sum_{k=1}^\infty \{\sum_{n=1}^\infty (\sum_{j=1}^\infty |A_n(k,j)|^p)^{\frac{r}{p}}\}^{\frac{1}{r}} < \infty, \text{ then } (T_n) \in \ell_{\pi_{p,q}}(c_0, \ell^p).$$

Proof Note that $T_n = \sum_{k=1}^\infty e_k^* \otimes y_{n,k}$, where $y_{n,k} \in \ell^p$ is given by $y_{n,k} = (A_n(k,j))_j$. For $x_n = (\lambda_{n,k})_k$ it follows that

$$\begin{aligned}
\sum_{n=1}^\infty \|T_n(x_n)\|^p &= \|(\|T_n x_n\|)\|_p^p \\
&= (\sum_{n=1}^\infty \|T_n x_n\| \alpha_n)^p \text{ for some positive } (\alpha_i) \in \ell^{p'}, \ \|(\alpha_i)\|_{p'} = 1.
\end{aligned}$$

$$
\begin{aligned}
&= (\sum_{n=1}^{\infty}\alpha_n\|\sum_{k=1}^{\infty}\langle e_k^*, x_n\rangle y_{n,k}\|)^p \\
&\leq (\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\langle e_k^*, x_n\rangle|\|y_{n,k}\|\alpha_n)^p \\
&\leq [\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}|\langle e_k^*, x_n\rangle|^q)^{\frac{1}{q}}(\sum_{n=1}^{\infty}\|y_{n,k}\|^{q'}\alpha_n^{q'})^{\frac{1}{q'}}]^p \\
&\leq \|(x_n)\|^p_{\ell^q_w(c_0)}[\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}(\|y_{n,k}\|\alpha_n)^{q'})^{\frac{1}{q'}}]^p \\
&\leq \|(x_n)\|^p_{\ell^q_w(c_0)}[\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}\|y_{n,k}\|^r)^{\frac{1}{r}}(\sum_{n=1}^{\infty}\alpha_n^{p'})^{\frac{1}{p'}}]^p \text{ since } \frac{1}{q'} = \frac{1}{r} + \frac{1}{p'} \\
&= \|(x_n)\|^p_{\ell^q_w(c_0)}[\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}\|y_{n,k}\|^r)^{\frac{1}{r}}]^p.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\sum_{n=1}^{\infty}\|T_n(x_n)\|^p)^{\frac{1}{p}} &\leq \epsilon_q((x_n))\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}\|y_{n,k}\|^r)^{\frac{1}{r}} \\
&= \epsilon_q((x_n))\sum_{k=1}^{\infty}\{\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty}|A_n(k,j)|^p)^{\frac{r}{p}}\}^{\frac{1}{r}}.
\end{aligned}
$$

□

Next, the case $E(X) = \ell^p(X)$ (or $E(X) = \ell^q_w(X)$) and $F(Y) = \ell^q\langle Y\rangle$ will be considered.

Definition 4.7 *Let $1 \leq p, q \leq \infty$. A sequence $(u_j)_{j\in\mathbb{N}}$ of operators in $L(X,Y)$ is called a* **strongly (p,q)-summing multiplier** *(i.e. $(u_j) \in (\ell^q(X), \ell^p\langle Y\rangle)$, if there exists a $c \geq 0$ such that, for any finite set $\{x_1, \cdots, x_n\} \subset X$ it holds that*

$$\|(u_i x_i)(\leq n)\|_{\langle p\rangle} \leq c\ \|(x_i)(\leq n)\|_{\ell^q(X)}, \quad i.e.$$

equivalently, $(u_j) \in (\ell^q(X),\ \ell^p\langle Y\rangle) \Leftrightarrow \exists\ c > 0$ such that for all $x_1, \cdots, x_n \in X,\ y_1^*, \cdots, y_n^* \in Y^*$, we have

$$\sum_{i=1}^{n}|\langle y_i^*, u_i x_i\rangle| \leq c\left(\sum_{i=1}^{n}\|x_i\|^q\right)^{\frac{1}{q}} \sup_{y\in B_Y}\left(\sum_{k=1}^{n}|y_k^*(y)|^{p'}\right)^{\frac{1}{p'}}. \tag{4.1}$$

Let $\|(u_j)\|_{(\ell^q(X),\ell^p\langle Y\rangle)}$ be the least constant c for which (u_j) verifies the inequality in the definition. Note that a constant sequence (u_j), $u_j = u \in L(X,Y)$ for all $j \in \mathbb{N}$, belongs to $(\ell^q(X), \ell^p\langle Y\rangle)$ if and only if $u \in D_{p,q}(X,Y)$.

Proposition 4.8 *Let X and Y be Banach spaces, $1 \leq p, q \leq \infty$ and let $(u_j)_{j\in\mathbb{N}}$ be a sequence of operators in $L(X,Y)$. Then*

$$(u_j) \in (\ell^q(X), \ell^p\langle Y\rangle) \Leftrightarrow (u_j^*) \in \ell_{\pi_{q',p'}}(Y^*, X^*).$$

In this case $\|(u_i)\|_{(\ell^q(X),\ \ell^p\langle Y\rangle)} = \pi_{q',p'}((u_i^*))$.

Proof Let $(u_j^*) \in \ell_{\pi_{q',p'}}(Y^*, X^*)$. If $x_1, \cdots, x_n$ is a finite set in X and if $(y_i^*) \in \ell_w^{p'}(Y^*)$, we have

$$\begin{aligned}
\sum_{i=1}^n |\langle u_i x_i, y_i^*\rangle| &= \sum_{i=1}^n |\langle x_i, u_i^*(y_i^*)\rangle| \\
&\leq \sum_{i=1}^n \|u_i^*(y_i^*)\|\|x_i\| \\
&\leq (\sum_{i=1}^n \|u_i^*(y_i^*)\|^{q'})^{\frac{1}{q'}} (\sum_{i=1}^n \|x_i\|^q)^{\frac{1}{q}} \\
&= \|(u_i^* y_i^*)(\leq n)\|_{\ell^{q'}(X^*)} \|(x_i)(\leq n)\|_{\ell^q(X)} \\
&\leq \pi_{q',p'}((u_i^*)) \epsilon_{p'}((y_i^*)) \|(x_i)\|_{\ell^q(X)}.
\end{aligned}$$

Taking the supremum over the unit ball in $\ell_w^{p'}(Y^*)$, we obtain

$$\|(u_i x_i)\|_{\langle p\rangle} \leq \pi_{q',p'}((u_i^*)) \|(x_i)\|_{\ell^q(X)}.$$

Therefore $(u_j) \in (\ell^q(X), \ell^p\langle Y\rangle)$ and $\|(u_i)\|_{(\ell^q(X),\ \ell^p\langle Y\rangle)} \leq \pi_{q',p'}((u_i^*))$.
Conversely, assume $(u_j) \in (\ell^q(X), \ell^p\langle Y\rangle)$. Let $y_1^*, \cdots, y_n^*$ be a finite set in Y^* and let $(x_i) \in \ell^q(X)$. It follows that

$$\begin{aligned}
\sum_{i=1}^n |\langle u_i^* y_i^*, x_i\rangle| &= \sum_{i=1}^n |\langle y_i^*, u_i x_i\rangle| \\
&\leq \|(u_i x_i)\|_{\langle p\rangle} \epsilon_{p'}((y_i^*)) \\
&\leq \|(u_i)\|_{(\ell^q(X),\ \ell^p\langle Y\rangle)} \|(x_i)\|_{\ell^q(X)} \epsilon_{p'}((y_i^*)).
\end{aligned}$$

If we take the supremum over the unit ball in $\ell^q(X)$, we obtain

$$\|(u_i^* y_i^*)\|_{\ell^{q'}(X^*)} \leq \|(u_i)\|_{(\ell^q(X),\ \ell^p\langle Y\rangle)} \epsilon_{p'}((y_i^*)).$$

Therefore $(u_i^*) \in \ell_{\pi_{q',p'}}(Y^*, X^*)$ and $\pi_{q',p'}((u_i^*)) \leq \|(u_i)\|_{(\ell^q(X),\ \ell^p\langle Y\rangle)}$. □

Example 4.9 *Let μ be a probability measure on Ω. Let $1 \leq p < q < \infty$, $1/r = 1/p - 1/q$ and (ϕ_j) a sequence of functions in $L^{q'}(\mu)$. Consider $u_j : L^q(\mu) \to L^1(\mu)$ given by $u_j(\psi) = \phi_j \psi$. Then*

$$(\sum_j |\phi_j|^r)^{1/r} \in L^{q'}(\mu) \Longrightarrow (u_j) \in (\ell^q(L^q(\mu)), \ell^p\langle L^1(\mu)\rangle).$$

Proof Let $\psi_1, \psi_2, ..., \psi_n \in L^q(\mu)$. Then, taking into account that

$$\ell^p\langle L^1(\mu)\rangle \overset{2.31}{=} \ell^p \hat{\otimes} L^1(\mu) \overset{(cf. [36])}{=} L^1(\mu, \ell^p),$$

we have

$$\begin{aligned}
\|(u_j\psi_j)\|_{\ell^p\langle L^1(\mu)\rangle} &= \|(\sum_{j=1}^n |\phi_j\psi_j|^p)^{1/p}\|_{L^1(\mu)} \\
&\leq \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}(\sum_{j=1}^n |\psi_j|^q)^{1/q}\|_{L^1(\mu)} \\
&\leq \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^{q'}(\mu)}\|(\sum_{j=1}^n |\psi_j|^q)^{1/q}\|_{L^q(\mu)} \\
&= \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^{q'}(\mu)}(\sum_{j=1}^n \|\psi_j\|^q_{L^q(\mu)})^{1/q}.
\end{aligned}$$

□

Remarks 4.10 *(1) Suppose, as in Example 4.9 that (ϕ_j) is a sequence of functions in $L^{q'}(\mu)$ such that $(\sum_{j=1}^\infty |\phi_j|^r)^{\frac{1}{r}} \in L^{q'}(\mu)$. Here $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and $1 \leq p < q < \infty$. Consider $\nu_j : L^\infty(\mu) \to L^{q'}(\mu)$ defined by $\nu_j(\chi) = \phi_j\chi$. It follows from*

$$\begin{aligned}
\langle \phi_j\chi, \theta\rangle &= \int_\Omega [\chi(t)\phi_j(t)]\theta(t)\ d\mu(t) \\
&= \int_\Omega \chi(t)[\phi_j(t)\theta(t)]\ d\mu(t) \\
&= \langle \chi, \phi_j\theta\rangle \\
&= \langle \chi, u_j(\theta)\rangle, \text{ where } u_j : L^q(\mu) \to L^1(\mu) : \theta \to \theta\phi_j
\end{aligned}$$

for all $\chi \in L^\infty(\mu)$, $\theta \in L^q(\mu)$, that $\nu_j = u_j^*$, $\forall j$. It follows from Example 4.9 and Proposition 4.8 that $(\nu_j) \in \ell_{\pi_{q',p'}}(L^\infty(\mu), L^{q'}(\mu))$.

(2) Let $1 \leq p, q < \infty$. If X is a Banach lattice and Y a Banach space, recall from Chapter 3 that we call an operator $u \in L(X,Y)$ strongly $(p,q)-$concave (and write $u \in SC_{p,q}(X,Y)$) if there exists a $c > 0$ such that for all $x_1, \cdots, x_n$ in X we have

$$\|(ux_i)(\leq n)\|_{\langle p\rangle} \leq c\ \|(\sum_{i=1}^{n}|x_i|^q)^{\frac{1}{q}}\|_X.$$

Now, consider the case $X = L^q(\mu)$. Then $u \in L(L^q(\mu), Y)$ is strongly (p,q)-concave iff there exists a $c > 0$ such that for all finite sets $\{\chi_1, \chi_2, \cdots, \chi_n\}$ in $L^q(\mu)$, we have

$$\begin{aligned}\|(u(\chi_i))(\leq n)\|_{\langle p\rangle} &\leq c\ \|(\sum_{i=1}^{n}|\chi_i|^q)^{\frac{1}{q}}\|_{L^q(\mu)}\\ &= c\ (\int_\Omega[(\sum_{i=1}^{n}|\chi_i(t)|^q)^{\frac{1}{q}}]^q\ d\mu(t))^{\frac{1}{q}}\\ &= c\ (\int_\Omega\sum_{i=1}^{n}|\chi_i(t)|^q\ d\mu(t))^{\frac{1}{q}}\\ &= c\ (\sum_{i=1}^{n}\|\chi_i\|^q_{L^q(\mu)})^{\frac{1}{q}}.\end{aligned}$$

Thus it follows that $u \in L(L^q(\mu), Y)$ is strongly $(p,q)-$concave iff the constant sequence $(u, u, \cdots)$ belongs to $(\ell^q(L^q(\mu)), \ell^p\langle Y\rangle)$.
Also, $sc_{p,q}(u) = \|(u, u, \cdots)\|_{(\ell^q(L^q(\mu)),\ \ell^p\langle Y\rangle)}$. From Proposition 4.8 it follows that

$$(u^*, u^*, \cdots) \in \ell_{\pi_{q',p'}}(Y^*, L^{q'}(\mu)) = (\ell_w^{p'}(Y^*), \ell^{q'}(L^{q'}(\mu))).$$

This corresponds to $u^* \in \Pi_{q',p'}(Y^*, L^{q'}(\mu))$.
Thus, it follows that $u : L^q(\mu) \to Y$ is strongly $(p,q)-$concave iff $u^* : Y^* \to L^{q'}(\mu)$ is (q',p')-summing and $sc_{p,q}(u) = \pi_{q',p'}(u^*)$.

The following examples follow from Proposition 4.8 and by [8], (Example 2.2, 2.3) for $S_n = T_n^*$, the conjugate operator of T_n.

Example 4.11 *Let (Ω, Σ, μ) and (Ω', Σ', μ') be finite measure spaces and $1 \leq p < \infty$. Let $(f_n) \subset L^p(\mu, L^1(\mu'))$ and consider the operator $S_n : L^{p'}(\mu) \to L^1(\mu')$ defined by*

$$S_n(g)(.) = \int_\Omega g(w)f_n(w,.)\ d\mu(w),$$

where, as before, we let $f_n(w,.) := f_n(w)(.)$. *If*

$$\sup_n |f_n| \in L^p(\mu, L^1(\mu')) \text{ (where, } \sup_n |f_n|(w)(\cdot) = \sup_n |f_n(w,\cdot)|),$$

then $(S_n) \in (\ell^{p'}(L^{p'}(\mu)), \ell^{p'}\langle L^1(\mu')\rangle)$.

Example 4.12 *Let* $1 \leq p < \infty$ *and* (A_n) *be a sequence of matrices. Consider the bounded operator* $S_n : \ell^{p'} \to \ell^1$ *given by*

$$S_n((\xi_j)) = (\sum_{j=1}^{\infty} A_n(k,j)\xi_j)_k.$$

Then $(S_n) \in (\ell^\infty(\ell^{p'}), \ell^\infty\langle \ell^1\rangle)$ *if* $\sum_{k=1}^{\infty} \sup_n (\sum_{j=1}^{\infty} |A_n(k,j)|^p)^{\frac{1}{p}} < \infty$.

Definition 4.13 *Let* $1 \leq p, q \leq \infty$. *A sequence* $(u_j)_{j\in\mathbb{N}}$ *of operators in* $L(X,Y)$ *is called a* **strongly (p,q)-nuclear multiplier** *(i.e.* $(u_j) \in (\ell^q_w(X), \ell^p\langle Y\rangle)$*) if there exists a constant* $c > 0$ *such that, for any finite set* $\{x_1, \cdots, x_n\} \subset X$, *it holds that*

$$\|(u_i x_i)(\leq n)\|_{\langle p\rangle} \leq c\ \epsilon_q((x_i)(\leq n)). \tag{4.2}$$

Equivalently, $(u_j) \in (\ell^q_w(X), \ell^p\langle Y\rangle) \Leftrightarrow \exists\ c > 0$ *such that, for all finite collections of vectors* $x_1, \cdots, x_n \in X$, $y_1^*, \cdots, y_n^* \in Y^*$,

$$\sum_{i=1}^{n} |\langle y_i^*, u_i x_i\rangle| \leq c \sup_{x^*\in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x^*, x_i\rangle|^q\right)^{\frac{1}{q}} \sup_{y\in B_Y} \left(\sum_{k=1}^{n} |y_k^*(y)|^{p'}\right)^{\frac{1}{p'}}.$$

Let $sn_{p,q}((u_j))$ *be the least constant* c *for which* (u_j) *verifies the inequality in the definition. To avoid ambiguities the norm* $\|(u_j)\|_{(\ell^q_w(X),\ell^p\langle Y\rangle)}$ *is sometimes used.*

Proposition 4.14 *Let* $1 \leq p, q < \infty$.

(i) Then $((\ell^q_w(X), \ell^p\langle Y\rangle), sn_{p,q}((.)))$ *is a Banach space.*

(ii) $((\ell^q(X),\ \ell^p\langle Y\rangle), \|(.)\|_{(\ell^q(X),\ \ell^p\langle Y\rangle)})$ *is a Banach space.*

Proof (i) Take a Cauchy sequence $(u^{(n)})$ in $(\ell^q_w(X), \ell^p\langle Y\rangle)$ where $u^{(n)} = (u_j^{(n)})_j$. Then it is also a Cauchy sequence in the Banach space $\ell^\infty(L(X,Y))$ and so convergent. Let $u^{(n)} \xrightarrow[\infty]{n} (u_j) \in \ell^\infty(L(X,Y))$, i.e.

$$\sup_j \|u_j^{(n)} - u_j\| \xrightarrow[\infty]{n} 0.$$

On the other hand

$$u^{(n)} \longleftrightarrow T_n \in L(\ell_w^q(X), \ell^p\langle Y\rangle) :: (x_j) \mapsto (u_j^{(n)} x_j)_j,$$

with

$$\begin{aligned} \|T_n - T_m\| &= \sup_{\epsilon_q((x_j))\leq 1} \|(u_j^{(n)} x_j - u_j^{(m)} x_j)\|_{\langle p\rangle} \\ &= sn_{p,q}(\ (u_j^{(n)} - u_j^{(m)})_j\ ) \xrightarrow[\infty]{n,\ m} 0. \end{aligned}$$

Hence, $T_n \xrightarrow[\infty]{n} T \in L(\ell_w^q(X), \ell^p\langle Y\rangle)$ in the operator norm.

For $\{x_1, \cdots, x_n\} \subset X$ and $\{y_1^*, \cdots, y_n^*\} \subset Y^*$ it follows that

$$\begin{aligned} \sum_{j=1}^{n} |\langle u_j x_j, y_j^*\rangle| &= \lim_{k\to\infty} \sum_{j=1}^{n} |\langle u_j^{(k)} x_j, y_j^*\rangle| \\ &\leq \limsup_{k\to\infty} sn_{p,q}((u^{(k)})) \sup_{x^*\in B_{X^*}} (\sum_{j=1}^{n} |x^* x_j|^q)^{\frac{1}{q}} \sup_{y\in B_Y} (\sum_{j=1}^{n} |\langle y_k^*, y\rangle|^{p'})^{\frac{1}{p'}} \\ &= \lim_{k\to\infty} \|T_k\| \sup_{x^*\in B_{X^*}} (\sum_{j=1}^{n} |x^* x_j|^q)^{\frac{1}{q}} \sup_{y\in B_Y} (\sum_{j=1}^{n} |\langle y_k^*, y\rangle|^{p'})^{\frac{1}{p'}} \\ &= \|T\| \sup_{x^*\in B_{X^*}} (\sum_{j=1}^{n} |x^* x_j|^q)^{\frac{1}{q}} \sup_{y\in B_Y} (\sum_{j=1}^{n} |\langle y_k^*, y\rangle|^{p'})^{\frac{1}{p'}}. \end{aligned}$$

Hence, $(u_j) \in (\ell_w^q(X), \ell^p\langle Y\rangle)$ and $sn_{p,q}((u_j)) \leq \|T\|$. Also,

$$\begin{aligned} T((x_j)) &= \lim_n T_n((x_j)) \\ &= \lim_n (u_j^{(n)} x_j)_j \\ &= (u_j x_j)_j, \quad \text{pointwise.} \end{aligned}$$

Hence $\|T((x_j))\|_{\langle p\rangle} = \|(u_j x_j)_j\|_{\langle p\rangle} \leq sn_{p,q}((u_j))\epsilon_q((x_j))$. Thus it follows that $sn_{p,q}((u_j)) = \|T\|$ and $sn_{p,q}(\ (u_j) - (u_j^{(n)})\ ) = \|T - T_n\| \xrightarrow[\infty]{n} 0$.

(ii) The proof is similar to the proof of part (i). □

Proposition 4.15 *Let X and Y be Banach spaces, $1 \leq p, q \leq \infty$ and let $(u_j)_{j\in\mathbb{N}}$ be a sequence of operators in $L(X, Y)$. Then*

$$(u_j) \in (\ell_w^q(X), \ell^p\langle Y\rangle) \ \Leftrightarrow\ (u_j^*) \in (\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)$$

and $\|(u_i)\|_{(\ell_w^q(X), \ell^p\langle Y\rangle)} = \|(u_i^*)\|_{(\ell_w^{p'}\langle Y^*), \ell^{q'}\langle X^*\rangle)}$.

Proof Choose $(u_j^*) \in (\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)$; let $x_1, x_2, \cdots, x_n \in X$. Then

$$\begin{aligned}\sum_{i=1}^n |\langle u_i x_i, z_i^*\rangle| &= \sum_{i=1}^n |\langle x_i, u_i^* z_i^*\rangle|, \text{ where } (z_i^*) \in \ell_w^{p'}(Y^*)\\ &\le \|(u_i^* z_i^*)(\le n)\|_{\langle q'\rangle} \epsilon_q((x_i)(\le n))\\ &\le \|(u_i^*)\|_{(\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)} \epsilon_{p'}((z_i^*)(\le n)) \epsilon_q((x_i)(\le n)).\end{aligned}$$

Thus $\|(u_i x_i)(\le n)\|_{\langle p\rangle} \le \|(u_i^*)\|_{(\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)} \epsilon_q((x_i)(\le n))$ and therefore $\|(u_i)\|_{(\ell_w^q(X), \ell^p\langle Y\rangle)} \le \|(u_i^*)\|_{(\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)}$.

Conversely, let $(u_i) \in (\ell_w^q(X), \ell^p\langle Y\rangle)$. Let $y_1^*, \cdots, y_n^*$ be a finite set in Y^* and let $(x_i) \in B_{\ell_w^q(X)}$. Then

$$\sum_{i=1}^n |\langle x_i, u_i^* y_i^*\rangle| = \sum_{i=1}^n |\langle u_i x_i, y_i^*\rangle| \le \|(u_i)\|_{(\ell_w^q(X), \ell^p\langle Y\rangle)} \epsilon_q((x_i)) \epsilon_{p'}((y_i^*)).$$

Taking the supremum over all sequences $(x_i) \in B_{\ell_w^q(X),}$ we have

$$\|(u_i^* y_i^*)(\le n)\|_{\langle q'\rangle} \le \|(u_i)\|_{(\ell_w^q(X), \ell^p\langle Y\rangle)} \epsilon_{p'}((y_i^*));$$

therefore, $(u_i^*) \in (\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)$ and $\|(u_i^*)\|_{(\ell_w^{p'}(Y^*), \ell^{q'}\langle X^*\rangle)} \le \|(u_i)\|_{(\ell_w^q(X), \ell^p\langle Y\rangle)}$. □

Example 4.16 *Let K be a compact set and μ a probability measure on the Borel sets of K. Let $1 \le p < q < \infty$, $1/r = 1/p - 1/q$ and (ϕ_j) a sequence of continuous functions on K. Consider $u_j : C(K) \to L^1(\mu)$, given by $u_j(\psi) = \phi_j \psi$.*
Then $(u_j) \in (\ell_w^q(C(K)), \ell^p\langle L^1(\mu)\rangle)$ if

$$(\sum_j |\phi_j|^r)^{1/r} \in L^{q'}(\mu).$$

Proof As in Example 4.9, if $\psi_1, \psi_2, ..., \psi_n \in C(K)$, we have

$$\begin{aligned}\|(u_j\psi_j)\|_{\ell^p\langle L^1(\mu)\rangle} &\le \|(\sum_{j=1}^n |\phi_j|^r)^{1/r} (\sum_{j=1}^n |\psi_j|^q)^{1/q}\|_{L^1(\mu)}\\ &\le \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^1(\mu)} \sup_{t\in K} (\sum_{j=1}^n |\psi_j(t)|^q)^{1/q}\\ &= \|(\sum_{j=1}^n |\phi_j|^r)^{1/r}\|_{L^1(\mu)} \sup_{\|\nu\|_{M(K)}=1} (\sum_{j=1}^n | < \psi_j, \nu > |^q)^{1/q},\end{aligned}$$

again using Remark 3.32(ii). □

4.2 R-bounded sequences

In this section we consider notions that have been shown to be relevant in some recent problems.

The sequence $(x_n) \subset X$ is **almost unconditionally summable** if $\sum_{n=1}^{\infty} r_n(t)x_n$ converges for **almost all** $t \in [0,1]$ in the Lebesque sense.

$\mathrm{Rad}(X)$ can be identified with the space of the **almost unconditionally summable** sequences (x_j), corresponding to functions given by $t \mapsto \sum_{j=1}^{\infty} r_j(t)x_j$, where the series converges for almost every t with respect to the Lebesque measure.

The following definition can be found in [19].

Definition 4.17 *Let* $1 \leq p \leq \infty$.

Then $Rad_p(X)$ *is the closure in* $L^p([0,1], X)$ *of the set of functions of the form*

$$\sum_{j=1}^{n} r_j x_j, \ x_j \in X,$$

where $(r_j)_{j\in\mathbb{N}}$ *are the* **Rademacher functions** *on* $[0,1]$ *defined by* $r_j(t) = \mathrm{sign}(\sin 2^j \pi t)$ *i.e.*

$$Rad_p(X) = \{(x_n) \in X^{\mathbb{N}} : \sup_{n\in\mathbb{N}} (\int_0^1 \| \sum_{j=1}^{n} r_j(t)x_j\|^p \, dt)^{\frac{1}{p}} < \infty\}$$

and for each $(x_n) \in Rad_p(X)$, *define*

$$\|(x_n)\|_{R_p} = \sup_{n\in\mathbb{N}} (\int_0^1 \| \sum_{j=1}^{n} r_j(t)x_j\|^p \, dt)^{\frac{1}{p}}.$$

We recall the main result about the Rademacher functions.

Theorem 4.18 Khintchine's inequality *(cf. [32], p. 66)*

Let $r_n(t) = \mathrm{sign}(\sin 2^n \pi t)$, $n = 0, 1, 2\ldots$ *be the Rademacher functions on* $[0,1]$. *For every* $1 \leq p < \infty$ *there exist positive constants* A_p *and* B_p *with*

$$\begin{aligned} A_p \left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}} &\leq \left(\int_0^1 | \sum_{n=1}^{m} a_n r_n(t)|^p \, dt \right)^{\frac{1}{p}} \\ &\leq B_p \left(\sum_{n=1}^{m} |a_n|^2 \right)^{\frac{1}{2}} \end{aligned}$$

for every choice of scalars $\{a_n\}_{n=1}^m$.

Remark 4.19 *(i)* $A_2 = B_2 = 1$, $A_p = 1$ *if* $p \geq 2$ *and* $B_p = 1$ *if* $1 \leq p < 2$ *(cf. [32], p. 66).*

(ii) From Khintchine's inequality it follows that a Rademacher sequence is equivalent, in the L^p *norm,* $0 < p < \infty$*, to the unit vector basis of* ℓ^2.

Many results concerning p-summing operators as well as several applications of these operators are based on the following inequality due to Grothendieck.

Theorem 4.20 *(cf. [32], p. 68)*
Let $(\alpha_{i,j})_{i,j=1}^n$ *be a matrix of scalars such that* $|\sum_{i,j=1}^n \alpha_{i,j} t_j s_j| \leq 1$ *for every choice of scalars* $\{t_i\}_{i=1}^n$ *and* $\{s_j\}_{j=1}^n$ *satisfying* $|t_i| \leq 1, |s_j| \leq 1$.
Then, for any choice of vectors $\{x_i\}_{i=1}^n$ *and* $\{y_j\}_{j=1}^n$ *in a Hilbert space,*

$$\left| \sum_{i=1}^n \sum_{j=1}^n \alpha_{i,j} \langle x_i, y_j \rangle \right| \leq K_G \max_i \|x_i\| \max_j \|y_j\|,$$

where K_G *is Grothendieck's universal constant (in case the scalars are real,* $K_G \leq (\frac{e^{\frac{\pi}{2}} - e^{\frac{-\pi}{2}}}{2})$).

The best possible value of K_G seems to be unknown.

Definition 4.21 *(***Kahane's Inequality** *(cf. [19], p. 211))*
If $0 < p, q < \infty$*, then there is a constant* $K_{p,q} > 0$ *so that, for any Banach space* X *and every finite subset* $\{x_1, \cdots, x_n\} \subset X$*, we have*

$$\left(\int_0^1 \|\sum_{k=1}^n r_k(t) x_k\|^q dt \right)^{\frac{1}{q}} \leq K_{p,q} \left(\int_0^1 \|\sum_{k=1}^n r_k(t) x_k\|^p \, dt \right)^{\frac{1}{p}}.$$

Kahane's Inequality ensures that the norms on $\text{Rad}_p(X)$ are equivalent for all $0 < p < \infty$. Thus, put $\text{Rad}_p(X) = \text{Rad}(X)$, $\forall\ 0 < p < \infty$ and the norm

$$\|(x_j)\|_{R_2} = \|\sum_{j=1}^n r_j x_j\|_{L^2([0,1],X)} = \left(\int_0^1 \|\sum_{k=1}^n r_k(t) x_k\|^2 \, dt \right)^{\frac{1}{2}}.$$

We agree to (mostly) use the norm $\|.\|_{R_2}$ on $Rad(X)$. It is easy to realize that $\mathrm{Rad}_\infty(X)$ corresponds to the space of compact operators $K(c_0, X)$ or also the space of unconditionally convergent series, since for any finite family $(x_j)_{j\leq n}$, we have

$$\|(x_j)\|_{R_\infty} \sim \epsilon_1((x_j)) = \|T\colon e_j \mapsto x_j\|_{K(c_0,X)}.$$

Definition 4.22 *(cf. [35], p. 71) A Banach space X is a G.T. space if it satisfies* **Grotendieck's theorem**, *i.e.*

$$L(X, \ell^2) = \Pi_1(X, \ell^2).$$

Therefore X is a **G.T. space** *iff for all $u : X \to \ell_n^2$ and for all n, there exists a constant λ such that $\pi_1(u) \leq \lambda\|u\|$.*

The next Proposition characterizes the duals or preduals of *G.T.* spaces.

Proposition 4.23 *(cf. [35], p. 71) A Banach space X is a G.T. space iff its bidual X^{**} is also a G.T. space, and this is equivalent to $L(X^*, L^1) = \Pi_2(X^*, L^1)$.*

Note that X^* is a G.T. space iff $L(X, L^1) = \Pi_2(X, L^1)$ (cf. [35], p. 71).

Remark 4.24 *(cf. [35], p. 73) All the known examples of G.T. spaces are of cotype* 2.

The next result clarifies the meaning of the notion of a G.T. space of cotype 2.

Theorem 4.25 *(cf. [35], p. 75) Let X be a Banach space. The following assertions are equivalent.*

(i) X^* is a G.T. space of cotype 2.

(ii) There is a constant c' such that, for all $n \in \mathbb{N}$ and all subsets $\{x_1, \cdots, x_n\} \subset X$ and $\{x_1^*, \cdots, x_n^*\} \subset X^*$, we have

$$|\sum_{i=1}^{\infty}\langle x_i^*, x_i\rangle| \leq c' \; \|\sum_{i=1}^{\infty} r_i(.)x_i^*\|_{L^1([0,1],X^*)} \; \epsilon_2((x_i)).$$

Theorem 4.26 *(cf. [35], Corollary 6.7) A Banach space X is a G.T. space of cotype 2 iff there is a constant c such that for all x_i^* in X^* and all x_i in X, we have*

$$|\sum_{i=1}^{\infty}\langle x_i^*, x_i\rangle| \leq \; c \; \|\sum_{i=1}^{\infty} r_i(.)x_i\|_{L^1([0,1],X)} \; \epsilon_2((x_i^*)).$$

Proposition 4.27 *(cf. [19], p. 220)*

If a Banach space X has type p, then its dual X^* has cotype p'.

Remark 4.28 *(cf. [19], p. 219 and p. 234)*

If $1 \leq p \leq 2$, then ℓ^p has type p and cotype 2.

If $2 \leq q < \infty$, then ℓ^q has type 2 and cotype q.

$Rad(X) \cong \ell^2(X)$ when X is isomorphic to a Hilbert space.

Theorem 4.29 *(cf. [2], Theorem 8) Let X be a Banach space. Then*

$$\ell_{\pi_{1,2}}(X) \subset \mathrm{Rad}(X) \subset \ell_{\pi_1}(X).$$

Theorem 4.30 *(cf. [2], Theorem 9) $Rad(X) = \ell_{\pi_{1,2}}(X)$ if and only if X is a G.T. space of cotype 2.*

Remark 4.31 *(i) It follows from [23] (p. 637) that $\ell^p\langle X\rangle = \ell_{\pi_{1,p'}}(X)$. Therefore*

$$\ell^2\langle X\rangle \subset Rad(X) \subset \ell^\infty\langle X\rangle$$

and if X is a G.T. space of cotype 2, then $Rad(X) = \ell^2\langle X\rangle$.

(ii) We recall from Chapter 2 that $\ell^p\langle X\rangle \cong \ell^p \overset{\wedge}{\otimes} X$, such that for each $(x_i) \in \ell^p\langle X\rangle$ we have

$$\|(x_i)(\leq n)\|_{\langle p\rangle} = |\sum_{i=1}^{n} e_i \otimes x_i|_\wedge, \; \forall \; n.$$

Definition 4.32 *(cf. [15] and [25]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \subset L(X,Y)$ is said to be* **Rademacher bounded**, *i.e.* **R-bounded**, *if there exists a constant $C > 0$ such that*

$$(\int_0^1 \|\sum_{j=1}^{n} u_j(x_j)r_j(t)\|^2 dt)^{\frac{1}{2}} \leq C \; (\int_0^1 \|\sum_{j=1}^{n} x_j r_j(t)\|^2 dt)^{\frac{1}{2}}$$

for all finite collections $x_1, x_2, ..., x_n \in X$.

The space of R-bounded sequences of operators from X into Y is denoted by $R(X,Y)$ and $\|(u_j)\|_R$ denotes the infimum of the constants satisfying the previous inequality for all finite subsets of X. It is easy to see that $(\mathrm{Rad}(X,Y), \|(u_j)\|_R)$ is a Banach space, which coincides with the multiplier space $(\mathrm{Rad}(X), \mathrm{Rad}(Y))$.

Definition 4.33 *(cf. [29]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \subset L(X,Y)$ is called* **Weakly Rademacher bounded, shortly WR-bounded** *if there exists a constant $C > 0$ such that for all finite collections $x_1, \cdots, x_n \in X$ and $y_1^*, \cdots, y_n^* \in Y^*$ we have*

$$\sum_{k=1}^{n} |\langle u_k x_k, y_k^* \rangle| \leq C\left(\int_0^1 \|\sum_{j=1}^{n} x_j r_j(t)\|^2 \, dt\right)^{\frac{1}{2}} \left(\int_0^1 \|\sum_{j=1}^{n} y_j^* r_j(t)\|^2 \, dt\right)^{\frac{1}{2}}.$$

The space of WR-bounded sequences in $L(X,Y)$ is denoted by $WR(X,Y)$ and $\|(u_n)\|_{WR}$ is the infimum of the constants in the previous inequality, taken over all finite subsets of X and Y^*. Then $\|(u_n)\|_{WR}$ is a norm on $WR(X,Y)$, which is exactly the norm of the bilinear map $\mathrm{Rad}(X) \times \mathrm{Rad}(Y^*) \to \ell^1$, defined by $((x_k), (y_k^*)) \to (\langle u_k x_k, y_k^* \rangle)$.

Definition 4.34 *(cf. [8]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \subset L(X,Y)$ is said to be* **almost summing** *if there exists $C > 0$ such that for any finite set of vectors $\{x_1, \cdots, x_n\} \subset X$ we have*

$$\left(\int_0^1 \|\sum_{j=1}^{n} u_j(x_j) r_j(t)\|^2 \, dt\right)^{1/2} \leq C \sup_{\|x^*\|=1} \left(\sum_{j=1}^{n} |\langle x^*, x_j \rangle|^2\right)^{\frac{1}{2}} \tag{4.3}$$

(or, equivalently, $(u_j) \subset L(X,Y)$ is almost summing if there exists a constant $C' > 0$ such that for any finite set of vectors $\{x_1, \cdots, x_n\} \subset X$ we have

$$\int_0^1 \|\sum_{j=1}^{n} u_j(x_j) r_j(t)\| \, dt \leq C' \sup_{\|x^*\|=1} \left(\sum_{j=1}^{n} |\langle x^*, x_j \rangle|^2\right)^{\frac{1}{2}}).$$

We write $\ell_{\pi_{as}}(X,Y)$ for the space of almost summing sequences, which is endowed with the norm

$$\|(u_i)\|_{as} := \inf\{C > 0 \mid \text{such that (4.3) holds}\}.$$

Notice that $\ell_{\pi_{as}}(X,Y) = (\ell_w^2(X), Rad(Y))$. If the constant sequence $(u, u, u, \ldots)$ is in $\ell_{\pi_{as}}(X,Y)$, then the operator u is called almost summing (cf. [19], p. 234). The space of almost summing operators is denoted by $\Pi_{as}(X,Y)$ and the norm on this space is given by

$$\pi_{as}(u) = \|(u,u,u\dots)\|_{as} = \|\hat{u}\|,$$

where in this case $\hat{u} : \ell^2_w(X) \to Rad(Y)$ is given by $\hat{u}((x_j)) = (ux_j)$.

Definition 4.35 *(cf. [29]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \subset L(X,Y)$ is called* **unconditionally bounded or U-bounded** *if there exists a constant $C > 0$ such that for all finite collections $x_1, \cdots, x_n \in X$ and $y_1^*, \cdots, y_n^* \in Y^*$ we have*

$$\sum_{k=1}^n |\langle u_k x_k, y_k^*\rangle| \leq C \max_{\epsilon_k=\pm1} \|\sum_{k=1}^n \epsilon_k x_k\| \max_{\epsilon_k=\pm1} \|\sum_{k=1}^n \epsilon_k y_k^*\|.$$

We write $UR(X,Y)$ for the space of U-bounded sequences in $L(X,Y)$. The space $UR(X,Y)$ is endowed with the norm $\|(u_n)\|_{UR}$, which is given by the infimum (taken over all finite subsets of X and Y^*) of the constants in the previous inequality.

Proposition 4.36 *Let X and Y be Banach spaces. The following inclusions hold*

$$\ell_{\pi_{as}}(X,Y) \subseteq R(X,Y) \subseteq WR(X,Y) \subseteq UR(X,Y) \subseteq \ell^\infty(L(X,Y)).$$

Proof The inclusion $\ell_{\pi_{as}}(X,Y) \subseteq R(X,Y)$ is a trivial consequence of the embedding $Rad(X) \subseteq \ell^2_w(X)$. Suppose $(u_i) \in R(X,Y)$. Orthogonality of the Rademacher variables, duality and the contraction principle, allow us to write

$$\begin{aligned}
&\sum_{k=1}^n |\langle u_k x_k, y_k^*\rangle| = \sum_{k=1}^n \langle u_k x_k, \epsilon_k y_k^*\rangle \text{ where } \epsilon_k = \frac{|\langle u_k x_k, y_k^*\rangle|}{\langle u_k x_k, y_k^*\rangle} \\
= &\int_0^1 \langle \sum_{k\leq n} r_k(t) u_k x_k, \sum_{k\leq n} r_k(t)\epsilon_k y_k^*\rangle\, dt \\
\leq &\left(\int_0^1 \|\sum_{k=1}^n u_k x_k r_k(t)\|^2 dt\right)^{1/2} \left(\int_0^1 \|\sum_{k=1}^n \epsilon_k y_k^* r_k(t)\|^2 dt\right)^{1/2} \\
\leq &\|(u_j)\|_R \left(\int_0^1 \|\sum_{k=1}^n x_k r_k(t)\|^2 dt\right)^{1/2} \left(\int_0^1 \|\sum_{k=1}^n y_k^* r_k(t)\|^2 dt\right)^{1/2}.
\end{aligned}$$

This proves the inclusion $R(X,Y) \subseteq WR(X,Y)$. The inclusion $WR(X,Y) \subseteq UR(X,Y)$ is clear from the definitions.

If $(u_n) \in UR(X,Y)$, then it is clear from the definition of unconditional boundedness that there exists $C > 0$ such that for $x \in X, y^* \in Y^*$, we have

$$|\langle u_k x, y^* \rangle| \leq C\|x\|\|y^*\|$$

for all $k \in \mathbb{N}$. Thus the inclusion $UR(X,Y) \subseteq \ell^\infty(L(X,Y))$ also follows. $\square$

Remark 4.37 *If $u \in L(X,Y)$, then $(u,u,\dots) \in R(X,Y)$ and $\|(u,u,\dots)\|_R = \|u\|$. However, $(u,u,\dots) \in \ell_{\pi_{as}}(X,Y)$ if and only if $u \in \Pi_{as}(X,Y)$. This shows that $\ell_{\pi_{as}}(X,Y) \subset R(X,Y)$ is strict.*

Recall (cf. Remark 1.18) that for $1 \leq p < \infty$, the p-convexity and p-concavity of $L^p(\mu)$ imply the following equivalence of norms:

$$\|(\phi_j)\|_{Rad(L^p(\mu))} \approx \|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{L^p(\mu)}$$

for any collection $\phi_1, \phi_2, ..., \phi_n$ in $L^p(\mu)$.

If $X = C(K)$ for any compact set K or if $X = \ell^\infty$, then

$$\epsilon_p((\phi_j)) \approx \|(\sum_{j=1}^n |\phi_j|^p)^{1/p}\|_X$$

for all finite subsets $\phi_1, \phi_2, ..., \phi_n$ of X.

Therefore we have the following versions of Definitions 3.1, 3.2, 3.3 and 3.4 in some special cases.

Proposition 4.38 *(i) Let $X = C(K)$ and $Y = L^q(\nu)$ for $1 \leq q < \infty$. Then $(u_j) \in \ell_{\pi_{as}}(X,Y)$ if and only if there exists $C > 0$ such that*

$$\|(\sum_{j=1}^n |u_j(\phi_j)|^2)^{1/2}\|_{L^q(\nu)} \leq C\|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{C(K)}$$

for any finite collection $\phi_1, \phi_2, ..., \phi_n$ in $C(K)$.

(ii) Let $X = L^p(\mu)$ and $Y = L^q(\nu)$ for $1 \leq p, q < \infty$. Then $(u_j) \in R(X,Y)$ if and only if there exists $C > 0$ such that

$$\|(\sum_{j=1}^n |u_j(\phi_j)|^2)^{1/2}\|_{L^q(\nu)} \leq C\|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{L^p(\mu)}$$

for any finite collection $\phi_1, \phi_2, ..., \phi_n$ *in* $L^p(\mu)$.

(iii) Let $X = \ell^p$ *and* $Y = c_0$ *for* $1 \leq p < \infty$. *Then* $(u_j) \in WR(X, Y)$ *if and only if there exists* $C > 0$ *such that*

$$\sum_{j=1}^{n} |\langle u_j(\phi_j), \varphi_j \rangle| \leq C \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_p \; \|(\sum_{j=1}^{n} |\varphi_j|^2)^{1/2}\|_1$$

for all finite collections $\phi_1, \phi_2, ..., \phi_n$ *in* ℓ^p *and* $\varphi_1, \varphi_2, ..., \varphi_n$ *in* ℓ^1.

(iv) Let $X = \ell^\infty$ *and* $Y = \ell^1$. *Then* $(u_j) \in UR(X, Y)$ *if and only if there exists* $C > 0$ *such that*

$$\sum_{j=1}^{n} |\langle u_j(\phi_j), \varphi_j \rangle| \leq C \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_\infty \; \|(\sum_{j=1}^{n} |\varphi_j|^2)^{1/2}\|_\infty$$

for all finite collections $\phi_1, \phi_2, ..., \phi_n$ *and* $\varphi_1, \varphi_2, ..., \varphi_n$ *in* ℓ^∞.

Proposition 4.39 *Let* $2 \leq r \leq \infty$. *If* $u_j = \lambda_j u$ *for* $u \in \Pi_{as}(X, Y)$ *and* $(\lambda_j) \in \ell^r$ *then* $(u_j) \in (\ell^q_w(X), Rad(Y))$ *for* $1/q = 1/2 - 1/r$.

In particular, if $u \in \Pi_{as}(X, Y)$ *and* $(\lambda_j) \in \ell^\infty$, *then* $(u_j) = (\lambda_j u) \in \ell_{\pi_{as}}(X, Y)$.

Proof From $u \in \Pi_{as}(X, Y)$, we have

$$\begin{aligned} (\int_0^1 \|\sum_{j=1}^{n} u_j(x_j) r_j(t)\|^2 \, dt)^{1/2} &= (\int_0^1 \|\sum_{j=1}^{n} u(\lambda_j x_j) r_j(t)\|^2 dt)^{1/2} \\ &\leq \pi_{as}(u) \sup_{\|x^*\|=1} \Big(\sum_{j=1}^{n} |\lambda_j|^2 |x^* x_j|^2 \Big)^{1/2} \\ &\leq \pi_{as}(u) \|(\lambda_j)\|_r \sup_{\|x^*\|=1} \Big(\sum_{j=1}^{n} |x^* x_j|^q \Big)^{1/q}. \end{aligned}$$

□

Remark 4.40 *We would like to point out that* $\cup_p \Pi_{p,p}(X, Y) \subset \Pi_{as}(X, Y)$ *(cf. [19], 12.5). Nevertheless this is not the case for sequences of operators. Indeed, it suffices to take* $u_n = x^* \otimes y_n$ *for fixed* $x^* \in X^*$ *and* $(y_n) \in \ell^\infty(Y)$. *In this case,* (u_n) *belongs to* $\ell_{\pi_{2,2}}(X, Y)$, *but not to* $\ell_{\pi_{as}}(X, Y)$ *(consider for example* $Y = c_0$ *and* $y_n = e_n$, *the canonical basis).*

Proposition 4.41 *Let Y be a Banach space of*

$$\text{type } p = p(Y) \geq 1 \text{ and cotype } q = q(Y) \leq \infty.$$

Then $\ell_{\pi_{p,2}}(X,Y) \subset \ell_{\pi_{as}}(X,Y) \subset \ell_{\pi_{q,2}}(X,Y)$.

In particular if Y is a Hilbert space, then $\ell_{\pi_{2,2}}(X,Y) = \ell_{\pi_{as}}(X,Y)$.

Proof It follows from the fact $\ell^p(Y) \subset Rad(Y) \subset \ell^q(Y)$ in this case. □

Let us mention that it was pointed out in [29] that if X has nontrivial type then $WR(X,X) = R(X,X)$. Actually the assumption only needs to be taken in the second space.

Recall that the notion of nontrivial type is equivalent to K-convexity (cf. [19], p. 260). X is said to be K-convex if $f \to (\int_0^1 f(t)r_n(t)dt)_n$ defines a bounded linear operator from $L^p([0,1])$ onto $Rad_p(X)$ for some (equivalently for all) $1 < p < \infty$.

For K-convex spaces one has $Rad(X^*) = Rad(X)^*$ (cf. [38], or [10] for more general systems), the duality being defined (as usual) by the bilinear functional

$$\langle (x_i), (x_i^*) \rangle = \sum_{i=1}^{\infty} x_i^* x_i.$$

Let us point out that this shows that there are no infinite dimensional K-convex G.T. spaces of cotype 2. Indeed, assume X is K-convex and a G.T. space of cotype 2. On the one hand $Rad(X) = \ell^2\langle X \rangle$ and on the other hand $Rad(X)^* = Rad(X^*)$, with equivalent norms. Therefore $Rad(X^*) = (\ell^2\langle X \rangle)^* = \ell^2_w(X^*)$. Hence the identity on X^* is almost summing and then X^* is finite dimensional.

It is well known that, in general, one can only expect $Rad(X^*)$ to be continuously embedded in $Rad(X)^*$, but that the embedding needs not even be isomorphic. Take, for instance, $X = \ell^1$. Then since ℓ^1 satisfies Grothendieck's Theorem,

$$Rad(\ell^1) = \ell^2\langle \ell^1 \rangle = \ell^2 \hat{\otimes} \ell^1,$$

that is to say $(x_n)_n \subset \ell^1$ (with $x_n = (x_n(k))_k$) belongs to $Rad(\ell^1)$ if and only if

$$\sum_{k \in \mathbb{N}} (\sum_{n \in \mathbb{N}} |x_n(k)|^2)^{1/2} < \infty.$$

As a matter of fact, it follows from earlier discussions that

$$Rad(\ell^1) = \ell^2\langle\ell^1\rangle = \ell^2\hat{\otimes}\ell^1 = \ell^1\hat{\otimes}\ell^2 = \ell^1\langle\ell^2\rangle = \ell^1(\ell^2).$$

Therefore, in case of $X = \ell^1$, $Rad(X)^*$ can be identified with $L(\ell^2, \ell^\infty)$ or with $\ell^\infty(\ell^2)$, and

$$\|(x_n^*)\|_{Rad(X)^*} = \sup_{k\in\mathbb{N}}(\sum_{n\in\mathbb{N}} |x_n^*(k)|^2)^{1/2}.$$

However,

$$\|(x_n^*)\|_{Rad(X^*)} = \int_0^1 \sup_{k\in\mathbb{N}} |\sum_{n\in\mathbb{N}} x_n^*(k) r_n(t)| dt.$$

These two norms are not equivalent on $Rad(X^*)$.

Proposition 4.42 *If Y is a K-convex space then $WR(X,Y) = R(X,Y)$.*

Proof Let $(u_n) \in WR(X,Y)$ and let $x_i \in X$ for $i = 1, ..., n$. Using that $Rad(Y)^* = Rad(Y^*)$, we have

$$\begin{aligned}(\int_0^1 \|\sum_{j=1}^n u_j(x_j) r_j(t)\|^2 \, dt)^{1/2} &\leq \|(u_j(x_j)(\leq n))\|_{R_2} \\ &= \sup\{|\sum_{j=1}^n \langle u_j(x_j), y_j^*\rangle| : \|\sum_{j=1}^n y_j^* r_j\|_{L^2([0,1],Y^*)} \leq 1\}.\end{aligned}$$

□

It is clear from the proof of Proposition 4.42 that $WR(X,Y) = R(X,Y)$ for all Banach spaces Y such that $Rad(Y)^* = Rad(Y^*)$. For later use, we point out that

Lemma 4.43 *Let $1 \leq p, q \leq \infty$. For a sequence (u_j) in $L(X,Y)$ we have $(u_j) \in \ell_{\pi_{p,q}}(X,Y)$ if and only if $F : \ell_w^q(X) \times \ell^{p'}(Y^*) \to \ell^1$ defined by $F((x_n),(y_n^*)) = (\langle u_n x_n, y_n^*\rangle)$ is a bounded bilinear operator. In this case $\|F\| = \pi_{p,q}((u_j))$.*

Proof Let $(u_j) \in \ell_{\pi_{p,q}}(X,Y)$. We have

$$\begin{aligned}\|F((x_i),(y_i^*))\|_1 &= \sum_{n=1}^\infty |\langle u_n x_n, y_n^*\rangle| \\ &\leq \|(u_n x_n)\|_{\ell^p(Y)} \|(y_n^*)\|_{\ell^{p'}(Y^*)} \\ &\leq \pi_{p,q}((u_i)) \epsilon_q((x_i)) \|(y_i^*)\|_{\ell^{p'}(Y^*)}.\end{aligned}$$

Therefore F is bounded, with $\|F\| \leq \pi_{p,q}((u_i))$.

Conversely, suppose F is bounded. Then

$$\sum_{n=1}^{\infty} |\langle u_n x_n, y_n^* \rangle| = \|F((x_i), (y_i^*))\|_1 \leq \|F\| \epsilon_q((x_i)) \|(y_i^*)\|_{\ell^{p'}(Y^*)}.$$

This shows that $(u_j) \in \ell_{\pi_{p,q}}(X, Y)$ and $\pi_{p,q}((u_i)) \leq \|F\|$. $\square$

Theorem 4.44 *Let* $1 \leq p \leq 2$.

(i) If Y *has type* p*, then* $\ell_{\pi_{p,2}}(X, Y) \subset \ell_{\pi_{as}}(X, Y)$.

(ii) If Y^* *has cotype* p'*, then* $\ell_{\pi_{p,2}}(X, Y) \subset WR(X, Y)$.

(iii) If Y^* *has cotype* p'*, then* $\ell_{\pi_{p,1}}(X, Y) \subset UR(X, Y)$.

(iv) If Y^* *has the Orlicz property, then* $\ell_{\pi_{2,1}}(X, Y) \subset UR(X, Y)$.

Proof (i) This follows from $\ell^p(Y) \subset Rad(Y)$.

(ii) Assume Y^* has cotype p'. Then $Rad(Y^*) \subset \ell^{p'}(Y^*)$ continuously, i.e. there exists $C > 0$ such that $\|(y_i^*)\|_{\ell^{p'}(Y^*)} \leq C\|(y_i^*)\|_{R_2}$. Also,

$$Rad(X) \subset \ell_w^2(X), \text{ with } \epsilon_2((x_i)) \leq \|(x_i)\|_{R_2} \ (cf.\ [19],\ p.\ 234).$$

Suppose $(u_j) \in \ell_{\pi_{p,2}}(X, Y)$. Then

$$F : \ell_w^2(X) \times \ell^{p'}(Y^*) \to \ell^1 : ((x_n), (y_n^*)) \mapsto (\langle u_n x_n, y_n^* \rangle)$$

is bounded with $\|F\| = \pi_{p,2}((u_i))$. Thus for all finite sets of elements $x_1, x_2, \cdots, x_n$ in X and $y_i^*, \cdots, y_n^*$ in Y^*, we have

$$\begin{aligned} \sum_{k=1}^{n} |\langle u_k x_k, y_k^* \rangle| &= \|F((x_i),\ (y_i^*))\| \\ &\leq \pi_{p,2}((u_i)) \epsilon_2((x_i)) \|(y_i^*)\|_{\ell^{p'}(Y^*)} \\ &\leq \pi_{p,2}((u_i)) C \|(x_i)\|_{R_2} \|(y_i^*)\|_{R_2} \\ &= K \|(x_i)\|_{R_2} \|(y_i^*)\|_{R_2}, \text{ where } K = \pi_{p,2}((u_i))C. \end{aligned}$$

Thus we have $(u_i) \in WR(X, Y)$. We may of course put $C = C_{p'}(Y^*)$, the cotype p' constant of Y^* (cf. [19]).

(iii) Let $(u_n) \in \ell_{\pi_{p,1}}(X,Y)$. For $x_1, \cdots, x_n \in X$ and $y_1^*, \cdots y_n^* \in Y^*$ we have by Lemma 4.43 that

$$\sum_{i=1}^{n} |\langle u_i x_i, y_i^* \rangle| \leq \pi_{p,1}((u_i))\epsilon_1((x_i))\|(y_n^*)\|_{\ell^{p'}(Y^*)} \leq c\ \pi_{p,1}((u_i))\epsilon_1((x_i))\epsilon_1((y_i^*)).$$

(iv) The same argument applies as in the proof of (iii). Now using that by the Orlicz property of Y^* we have $\ell_w^1(Y^*) \subset \ell^2(Y^*)$. □

Similar arguments yield the following:

Theorem 4.45 *Let $1 \leq p \leq 2$.*

(i) If Y has cotype p', then $\ell_{\pi_{as}}(X,Y) \subset \ell_{\pi_{p',2}}(X,Y)$.

(ii) If Y has cotype p', then $R(X,Y) \subset \ell_{\pi_{p',1}}(X,Y)$.

(iii) If Y^* has type p, then $WR(X,Y) \subset \ell_{\pi_{p',1}}(X,Y)$.

Remark 4.46 *Let $1 \leq p \leq 2 \leq q \leq \infty$ and denote by $C_q(X,Y)$ and $T_p(X,Y)$ the spaces of operators of cotype q and type p, that is*

$$C_q(X,Y) = \{u : X \to Y : (u_j)_j \in (Rad(X), \ell^q(Y)), u_j = u, j \in \mathbb{N}\}$$

and

$$T_p(X,Y) = \{u : X \to Y : (u_j)_j \in (\ell^p(X), Rad(Y)), u_j = u, j \in \mathbb{N}\}.$$

Let X and Y be Banach spaces.

(1) If $(u_j) \in Rad(X,Y)$ and $u \in C_q(Y,Z)$, then $(uu_j) \in \ell_{\pi_{q,1}}(X,Z)$.

(2) If $(u_j) \in Rad(X,Y)$ and $u \in T_p(Z,X)$, then $(u_j u) \in (\ell^p(Z), \ell_w^2(Y))$.

(3) If $(u_j) \in Rad(X,Y)$, $v \in C_q(Y,U)$ and $u \in \Pi_{as}(Z,X)$, then $(vu_j u) \in \ell_{\pi_{q,2}}(Z,U)$.

Theorem 4.47 *Let $1 \leq p \leq 2$ and let X be a Banach space such that X has cotype p', let Y be a G.T. space of cotype 2 and let $u_j : X \to Y$ be bounded linear operators for all $j \in \mathbb{N}$. Then*

$$(u_j^*) \in \ell_{\pi_{p,2}}(Y^*, X^*) \implies (u_j) \in R(X,Y).$$

Proof Recall from Proposition 4.8 that $(u_j) \in (\ell^{p'}(X), \ell^2\langle Y\rangle)$. Since we can identify $Rad(Y)$ with $\ell^2\langle Y\rangle$ (see Theorem 4.30), it follows that there exists a $C > 0$ such that

$$\begin{aligned} & \int_0^1 \|\sum_{j=1}^n u_j(x_j)r_j(t)\|dt \leq \|(u_j(x_j))\|_{\langle 2\rangle} \\ \leq\ & C\|(u_i)\|_{(\ell^{p'}(X),\ell^2\langle Y\rangle)}\|(x_j)\|_{\ell^{p'}(X)} \\ \leq\ & K\int_0^1 \|\sum_{j=1}^n x_j r_j(t)\|dt, \text{ where, } K = C\|(u_i)\|_{(\ell^{p'}(X),\ell^2\langle Y\rangle)}C_{p'}(X). \end{aligned}$$

□

Corollary 4.48 *Let $1 \leq r \leq \infty$ and $u_j : L^r(\mu) \to L^1(\nu)$ be bounded linear operators. If $(u_j^*) \in \ell_{\pi_{p,2}}(L^\infty(\nu), L^{r'}(\mu))$ for $p = \min\{r', 2\}$, then there exists $C > 0$ such that*

$$\|(\sum_{j=1}^n |u_j(\phi_j)|^2)^{1/2}\|_{L^1(\nu)} \leq C\|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{L^r(\mu)}$$

for any collection $\phi_1, \phi_2, ..., \phi_n$ in $L^r(\mu)$.

Another related notion is the following:

Definition 4.49 *(cf. [28]) Let X and Y be Banach spaces. A sequence of operators $(u_j) \in L(X,Y)$ is said to be* **semi-R-bounded** *(i.e. $(u_n) \in SR(X,Y)$) if there exists $C > 0$ such that for every $x \in X$ and $a_1, \cdots, a_n \in \mathbb{C}$ we have*

$$(\int_0^1 \|\sum_{j=1}^n u_j(x)r_j(t)a_j\|^2\, dt)^{1/2} \leq C\, (\sum_{j=1}^n |a_j|^2)^{\frac{1}{2}}\|x\|. \tag{4.4}$$

$\|(u_i)\|_{SR} := \inf\{C > 0 \mid$ *such that (4.4) holds*$\}$ *is the norm on $SR(X,Y)$.*

It was observed (cf. [28], Proposition 2.1) that $SR(X,X) = \ell^\infty(L(X,X))$ if and only if X is of type 2. Note that R-boundedness of sequences in $L(X,Y)$ implies semi-R-boundedness of the same. It is known that if X is separable and is either a Hilbert space (i.e. $X = \ell^2$) or a $G.T.$ space of cotype 2, then $SR(X,X) = R(X,X)$ (cf. [28], Theorem 2.2 for proof). Moreover, the result in [28] actually characterizes the separable Banach spaces X for which $SR(X,X) = R(X,X)$. The proof of (a more general version of) the result in [28] can be greatly simplified in the context of multiplier sequences and basically follows via the following characterization of $SR(X,Y)$.

Theorem 4.50 *The space* $(SR(X,Y), \|.\|_{SR})$ *is isometrically isomorphic to the space* $(\ell^2\langle X\rangle, \mathrm{Rad}(Y))$.

Proof Suppose $(u_n) \in SR(X,Y)$ and $\{x_1,\cdots,x_n\} \subset X$. From Chapter 2 we know that

$$\|(x_i)\|_{\langle 2\rangle} = \left|\sum_{i=1}^n e_i \otimes x_i\right|_\wedge$$

in $\ell^2\hat{\otimes}X$.

It is clear that if $(\lambda_i) \in \ell^2$ and $x \in X$, we have that $(\lambda_j u_j x) \in \mathrm{Rad}(Y)$ and

$$\|(\lambda_j u_j x)\|_{R_2} \leq \|(u_i)\|_{SR}\|(\lambda_i)\|_2\|x\|.$$

Hence

$$(0,0,\cdots,0,u_ix_i,0,\cdots) = (\delta_{ij}u_jx_i)_j \in Rad(Y)$$

and

$$\|(\delta_{ij}u_jx_i)_j\|_{R_2} \leq \|(u_i)\|_{SR}\|(\delta_{ij})_j\|_2\|x_i\| = \|(u_i)\|_{SR}\|x_i\|\|e_i\|_2.$$

Therefore, it follows that $(u_ix_i) = \sum_{i=1}^n(\delta_{ij}u_jx_i)_j \in \mathrm{Rad}(Y)$ and

$$\|(u_ix_i)\|_{R_2} \leq (\sum_{i=1}^n \|e_i\|_2\|x_i\|)\|(u_i)\|_{SR}.$$

Since $\sum_{i=1}^n e_i \otimes x_i$ is just one of the representations of this element of $\ell^2\hat{\otimes}X$ and by the definition of the projective norm, it follows that

$$\|(u_ix_i)\|_{R_2} \leq \left|\sum_{i=1}^n e_i \otimes x_i\right|_\wedge \|(u_i)\|_{SR} = \|(x_i)\|_{\langle 2\rangle}\|(u_i)\|_{SR}.$$

This holds for all finite sets $\{x_1,\cdots,x_n\} \subset X$, showing that $(u_i) \in (\ell^2\langle X\rangle, \mathrm{Rad}(Y))$ and $\|(u_i)\|_{(\langle 2\rangle,R_2)} \leq \|(u_i)\|_{SR}$.

Conversely, suppose $(u_i) \in (\ell^2\langle X\rangle, \mathrm{Rad}(Y))$ and let $\alpha_1,\cdots,\alpha_n \in \mathbb{C}$ and $x \in X$. Then we have

$$\begin{aligned}
(\int_0^1 \|\sum_{i=1}^n r_i(t)\alpha_iu_ix\|^2\,dt)^{\frac{1}{2}} &\leq \|(u_i)\|_{(\langle 2\rangle,R_2)}\|(\alpha_ix)\|_{\langle 2\rangle} \\
&= \|(u_i)\|_{(\langle 2\rangle,R_2)} \sup_{\epsilon_2((x_i^*))\leq 1} \sum_{i=1}^n |\alpha_i x_i^*(x)| \\
&\leq \|(u_i)\|_{(\langle 2\rangle,R_2)}(\sum_{i=1}^n|\alpha_i|^2)^{\frac{1}{2}} \sup_{\epsilon_2((x_i^*))\leq 1}(\sum_{i=1}^n|x_i^*(x)|^2)^{\frac{1}{2}}
\end{aligned}$$

$$= \|(u_i)\|_{(\langle 2\rangle,R_2)}(\sum_{i=1}^{n}|\alpha_i|^2)^{\frac{1}{2}}\|x\|.$$

Since this is true for all $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ and $x \in X$, it follows that $(u_i) \in SR(X,Y)$ and $\|(u_i)\|_{SR} \leq \|(u_i)\|_{(\langle 2\rangle,R_2)}$. □

It follows from the continuous inclusion $\ell^2\langle X\rangle \subset Rad(X)$ and Theorem 4.50, that $R(X,Y) \subseteq SR(X,Y)$ for all Banach spaces X and Y. The reader is referred to [28] (p. 380) for an example of a sequence of operators which is semi-R-bounded, but not R-bounded; indeed, the authors in [28] show that if (e_k^*) is the standard basis of $\ell^{q'}$ (where, $2 < q < \infty$) and $w = (\xi_i) \in \ell^q$ is fixed, then the uniformly bounded sequence of operators $(S_k) := (e_k^* \otimes w)$ in $L(\ell^q, \ell^q)$ is not WR-bounded. Because of ℓ^q having type 2, the inequality (4.4) can be obtained as follows:

$$\begin{aligned}(\int_0^1 \|\sum_{j=1}^{n} S_j(x)r_j(t)a_j\|^2\, dt)^{\frac{1}{2}} &\leq c\, \|(S_j(x)a_j)(\leq n)\|_{\ell^2(\ell^q)} \\ &\leq c\, (\sum_{j=1}^{n}(|a_j|\|S_j\|\|x\|)^2)^{\frac{1}{2}} \\ &\leq c\, \sup_j \|S_j\|(\sum_{j=1}^{n}|a_j|^2)^{\frac{1}{2}}\|x\|,\end{aligned}$$

where $\|S_j\| \leq \|w\|$ for all j. Thus $(S_k) \in SR(\ell^q, \ell^q)$.

The following proposition sheds more light on the question of when the equality $SR(X,Y) = R(X,Y)$ holds.

Proposition 4.51

(i) If X is a Grothendieck space of cotype 2, then $SR(X,Y) = R(X,Y)$ for all Banach spaces Y.

(ii) If for some Banach space Y (thus also for $Y = X$) the equality $SR(X,Y) = R(X,Y)$ holds, then X has cotype 2.

(iii) If X is a Hilbert space and Y is a Banach space of type 2, then $SR(X,Y) = R(X,Y)$.

Proof (i) This follows from Theorem 4.50 and the characterization of Grothendieck spaces of cotype 2 by $\ell^2\langle X\rangle = \ell^2\hat{\otimes}X = Rad(X)$.

(ii) We show that $SR(X,Y) = R(X,Y)$ implies that $Rad(X)$ is a linear subspace of $\ell^2(X)$. Let $(x_i) \in Rad(X)$ and let $x_i^* \in X^*$, with $\|x_i^*\| = 1$ and $x_i^*(x_i) = \|x_i\|$. Put $u_i = x_i^* \otimes y$, where $y \in Y$ is fixed, with $\|y\| = 1$. Then

$$(u_i) \in SR(X,Y) = (\ell^2\langle X\rangle, Rad(Y)),$$

because of

$$\begin{aligned}
\int_0^1 \|\sum_{i=1}^n u_i(z_i)r_i(t)\|^2\, dt &= \int_0^1 \|\sum_{i=1}^n r_i(t)x_i^*(z_i)y\|^2\, dt \\
&= \int_0^1 |\sum_{i=1}^n r_i(t)x_i^*(z_i)|^2\, dt \\
&\overset{Remark\ 4.19}{=} \sum_{i=1}^n |x_i^*(z_i)|^2 \\
&\leq \sum_{i=1}^\infty \|z_i\|^2 \leq \|(z_i)\|^2_{\langle 2\rangle},
\end{aligned}$$

for all $(z_i) \in \ell^2\langle X\rangle \subset \ell^2(X)$. Hence, $(u_i) \in (Rad(X), Rad(Y))$. However, for all $n \in \mathbb{N}$, we also have

$$\begin{aligned}
\sum_{i=1}^n \|x_i\|^2 &= \int_0^1 |\sum_{i=1}^n r_i(t)\|x_i\||^2\, dt \\
&= \int_0^1 |\sum_{i=1}^n r_i(t)x_i^*(x_i)|^2\, dt \\
&= \int_0^1 \|\sum_{i=1}^n r_i(t)x_i^*(x_i)y\|^2\, dt \\
&= \int_0^1 \|\sum_{i=1}^n r_i(t)u_i(x_i)\|^2\, dt\,.
\end{aligned}$$

Therefore, it follows that

$$\sum_{i=1}^\infty \|x_i\|^2 \leq \sup_n \int_0^1 \|\sum_{i=1}^n r_i(t)u_i(x_i)\|^2\, dt < \infty\,,$$

showing that $Rad(X) \hookrightarrow \ell^2(X)$ is a norm ≤ 1 embedding.

(iii) We refer to Remark 4.52 below, where a more general case is discussed. □

In the following few remarks, we analyse the relationship of $\ell^\infty(L(X,Y))$ to the other families of multiplier sequences.

Remark 4.52 *(see for instance [8]) Let X be a Banach space of cotype q , Y be a Banach space of type p for some $1 \le p \le q \le \infty$ and let $1/r = 1/p - 1/q$. Then*

$$\ell^r(L(X,Y)) \subset R(X,Y) \subset \ell^\infty(L(X,Y)).$$

In particular, if X has cotype 2 and Y has type 2, then $R(X,Y) = \ell^\infty(L(X,Y))$.

Remark 4.53 *If X and Y^* have the Orlicz property then $\ell^\infty(L(X,Y)) = UR(X,Y)$.*

Proof By Proposition 4.36 we only need to show that $\ell^\infty(L(X,Y)) \subseteq UR(X,Y)$. Notice that the continuous inclusions $\ell^1_w(X) \subseteq \ell^2(X)$ and $\ell^1_w(Y^*) \subseteq \ell^2(Y^*)$ correspond to the Orlicz properties of X and Y^* respectively. Then, for $(u_n) \in \ell^\infty(L(X,Y))$, we have

$$\begin{aligned}
\sum_{k=1}^{n} |\langle u_k x_k, y_k^* \rangle| &\le \sum_{k=1}^{n} \|u_k\| \|x_k\| \|y_k^*\| \\
&\le (\sup_k \|u_k\|) (\sum_{k=1}^{n} \|x_k\|^2)^{1/2} (\sum_{k=1}^{n} \|y_k^*\|^2)^{1/2} \\
&\le C (\sup_k \|u_k\|) \max_{\epsilon_k = \pm 1} \| \sum_{k=1}^{n} \epsilon_k x_k \| \max_{\epsilon_k = \pm 1} \| \sum_{k=1}^{n} \epsilon_k y_k^* \|,
\end{aligned}$$

where in the last step of the proof the existence of $C > 0$ such that the inequality holds, is a direct consequence of the inclusions mentioned in the second line of this proof. □

Remark 4.54 *Let Y be a Banach space of type p for some $1 \le p \le 2$ and let $r \ge 1$ satisfy $1/r = 1/p - 1/2$. Then*

$$\ell^r(L(X,Y)) \subset SR(X,Y) \subset \ell^\infty(L(X,Y)).$$

In particular if Y has type 2, then $SR(X,Y) = \ell^\infty(L(X,Y))$.

Proof We prove the inclusion $\ell^r(L(X,Y)) \subset SR(X,Y)$. Let $(u_j) \in \ell^r(L(X,Y))$. There exists $C > 0$ such that

$$(\int_0^1 \| \sum_{j=1}^{n} u_j(x) r_j(t) a_j \|^2 \, dt)^{1/2} \le C (\sum_{j=1}^{n} \|u_j(a_j x)\|^p)^{1/p}$$

$$\leq C\|x\|(\sum_{j=1}^{n}\|u_j\|^p|a_j|^p)^{1/p}$$
$$\leq C\|x\|\|(u_j)\|_r(\sum_{j=1}^{n}|a_j|^2)^{1/2}.$$

The other inclusion is immediate. □

Remark 4.55 *Neither $SR(X,Y) \subset WR(X,Y)$ nor $WR(X,Y) \subset SR(X,Y)$ is generally true. For instance, if Y has type 2, then $SR(X,Y) = \ell^\infty(L(X,Y))$ and $WR(X,Y) = R(X,Y)$. So, $WR(X,Y) \subset SR(X,Y)$ for all X in this case. On the other hand, if we consider a G.T. space X having cotype 2, then $SR(X,Y) = R(X,Y)$ (see Proposition 4.51) for all Y. So, in this case, $SR(X,Y) \subset WR(X,Y)$ for all Y.*

Remark 4.56 *In [28] the authors consider some new applications of semi-R-bounded and WR-bounded sequences. They show that for each $x \in X$ and $(u_i) \in SR(X,X)$, the sequence $(u_n x)$ has a weakly Cauchy subsequence. Using this fact, they then show that if X is a weakly sequentially complete Banach space such that $L(X,X)$ contains a semi-R-bounded sequence (u_i) such that each u_i is weakly compact, $u_k u_l = u_l u_k$ for all k, $l \in \mathbb{N}$ and $\lim_{k\to\infty}\|x - u_k x\| = 0$ for every $x \in X$, then X is isomorphic to a dual space.*
In case of $L(X,X)$ containing a WR-bounded sequence with the same properties, one also needs the space X to satisfy the property (V^*) of Pelczynski to obtain the same result. Since $L^1(0,1)$ is not a dual space, it follows that $L(L^1(0,1), L^1(0,1))$ does not have a semi-R-bounded or WR-bounded sequence of operators (u_i) with the mentioned properties. It is also shown in [28] that if K is a compact metric space so that $L(C(K), C(K))$ contains an R-bounded sequence (u_n) with the above mentioned properties, then $C(K)$ is isomorphic to c_0. Some applications to semigroups of operators are also considered in [28].

4.3 Connection of R-boundedness and Schauder decompositions

The authors of the paper [15] highlighted the importance of the concept R-boundedness (randomized boundedness) of collections of operators in multiplier results of Marcinkiewicz type for L^p-spaces of functions with values in a Banach space, by showing the interplay between unconditional decompositions and R-boundedness of collections of operators. Also, in the same paper, the authors show connections between R-boundedness and geometric properties of the underlying Banach space.

Some important operators in analysis may be represented in the form

$$T_\lambda(x) = \sum_k \lambda_k D_k x,$$

where $\lambda = (\lambda_k)$ is a sequence in $\mathbb{C}$ and $D = \{D_n\}_{n=1}^{\infty}$ is a given Schauder decomposition of the Banach space X. Characterization of sequences λ for which T_λ is bounded on X (i.e. $T \in L(X,X)$) is an interesting problem.

Recall that a collection $D = \{D_n\}_{n=1}^{\infty}$ of bounded linear projections in a complex Banach space X is called a *Schauder decomposition* of X, if

(i) $D_k D_\ell = 0$ whenever $k \neq \ell$,

(ii) $x = \sum_{k=0}^{\infty} D_k x$ for all $x \in X$.

If the series $\sum_{k=0}^{\infty} D_k x$ is unconditionally convergent for all $x \in X$, then D is called an *unconditional decomposition.* If $D = \{D_n\}_{n=1}^{\infty}$ is an unconditional decomposition of X, then the smallest constant C_D such that

$$\|\sum_{k=0}^{N} \epsilon_k D_k x\|_X \leq C_D \|\sum_{k=0}^{N} D_k x\|_X$$

holds for all $\epsilon_k = \pm 1$, $k = 0, 1, \ldots, N$, all $N \in \mathbb{N}$, and all $x \in X$, is called the unconditional constant of the decomposition.

Let $(\Omega, \mathcal{F}, P)$ be some probability space. Using the formulation of unconditional summability given in Lemma 1.4 of [19], it can be shown that if $D = \{D_n\}_{n=1}^{\infty}$ is an unconditional Schauder decomposition of the Banach space X, then for all $1 \leq p < \infty$ we have that

$$C_D^{-1} \|\sum_{k=0}^{N} D_k x\|_X \leq \|\sum_{k=0}^{N} \epsilon_k D_k x\|_{L^p(\Omega, X)} \leq C_D \|\sum_{k=0}^{N} D_k x\|_X$$

holds for all $\epsilon_k = \pm 1$, $k = 0, 1, \ldots, N$, all $N \in \mathbb{N}$, and all $x \in X$. If, on the other hand, for some $1 \leq p < \infty$ there is a constant C such that the above inequalities hold, with C in place of C_D, then the decomposition D is unconditional.

By $\{\epsilon_k\}_{k=0}^{\infty}$ we shall denote a sequence of independent symmetric $\{-1, 1\}$-valued random variables on some probability space $(\Omega, \mathcal{F}, P)$. In the previous section we considered R-bounded sequences of operators. We can actually define R-bounded families of operators (with respect to the given probability space) as in the following definition.

Definition 4.57 *A collection of operators $\Gamma \subset L(X, X)$ is called R-bounded if there exists $M > 0$ such that*

$$\|\sum_{k=0}^{N} \epsilon_k T_k x_k\|_{L^2(\Omega,X)} \leq M \|\sum_{k=0}^{N} \epsilon_k x_k\|_{L^2(\Omega,X)}$$

for all $\{T_k\}_{k=0}^{N} \subset \Gamma$, all $\{x_k\}_{k=0}^{N} \subset X$ and alle $N \in \mathbb{N}$. The smallest constant M such that the inequality holds, is called the R-bound of Γ.

Note that by Kahane's inequality we can replace the $L^2(\Omega, X)$ in Definition 4.57 by $L^p(\Omega, X)$, $1 \leq p < \infty$, adjusting the constant M appropriately.

Although the collections $\{T_k\}_{k=0}^{N}$ in the above Definition 4.57 need not be mutually distinct, it is proved in [15] that we may replace the phrase "for all $\{T_k\}_{k=0}^{N} \subset \Gamma$" in the definition by "for all $\{T_k\}_{k=0}^{N} \subset \Gamma$ for which $T_i \neq T_j$ if $i \neq j$".

The results (and their proofs) in the following short discussion in which we aim to introduce the reader to yet another area of application of R-boundedness, can be found in paper [15], which is a beautiful display of the natural occurrence of R-bounded collections of operators. We have no claim to fame, but only mention the results here to illustrate how a special type of "multiplier boundedness" comes into play in the study of Schauder decompositions.

Given a strictly increasing sequence $\{q_k\}_{k=0}^{\infty}$ in $\mathbb{N}$ and a Schauder decomposition $D = \{D_n\}_{n=1}^{\infty}$ of X, put

$$\Delta_k = \sum_{\ell=q_{k-1}+1}^{q_k} D_\ell \qquad (k = 0, 1, \ldots),$$

where $q_{-1} = -1$. Then $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ is also a Schauder decomposition, called a *blocking of D*.

The following two theorems show the relevance of R-boundedness in the context of unconditional decompositions.

Theorem 4.58 *([15], Theorem 3.4) Let $\{\Delta_k\}_{k=0}^{\infty}$ be an unconditional Schauder decomposition of the Banach space X. Suppose $\Gamma \subset L(X,X)$ is R-bounded (with R-bound M). If $\{T_k\}_{k=0}^{\infty} \subset \Gamma$ such that $\Delta_k T_k = T_k \Delta_k$ for all k, then the series $Sx := \sum_{k=0}^{\infty} T_k \Delta_k x$ is convergent in X for all $x \in X$ and defines a bounded linear operator $S : X \to X$ with $\|S\| \leq K$ (where K only depends on M and the unconditional constant of $\{\Delta_k\}_{k=0}^{\infty}$).*

Theorem 4.59 *([15], Theorem 3.5) Let $D = \{D_k\}_{k=0}^{\infty}$ be a Schauder decomposition of the Banach space X. Let $\{q_k\}_{k=0}^{\infty}$ be a strictly increasing sequence in $\mathbb{N}$ and let $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ be the corresponding blocking of D. Let $K > 0$ and let Λ_K be the set of all complex sequences $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ such that*

- $|\lambda_k| \leq K$ *for all* $k \in \mathbb{N}$,
- $\sum_{\ell=q_{k-1}+1}^{q_k-1} |\lambda_{\ell+1} - \lambda_\ell| \leq K$ *for all* $k \in \mathbb{N}$.

Then the following are equivalent:

(i) $\{T_\lambda : \lambda \in \Lambda_K\} \subset L(X,X)$ *with* $\|T_\lambda\| \leq CK$ *for all* $\lambda \in \Lambda_K$ *and some constant* $C > 0$.

(ii) The blocking Δ is unconditional and there is a constant $M > 0$ such that

$$\|\sum_{k=0}^{N} \epsilon_k P_{m_k} x_k\|_{L^2(\Omega,X)} \leq M \|\sum_{k=0}^{N} \epsilon_k x_k\|_{L^2(\Omega,X)}$$

for all $N \in \mathbb{N}$, all $\{x_k\}_{k=0}^{\infty} \subset X$ with $x_k \in \mathcal{R}(\Delta_k)$ and all $\{m_k\}_{k=0}^{\infty}$ such that $q_{k-1} + 1 \leq m_k \leq q_k$ for $k \in \mathbb{N}$.

Here P_m and T_λ are as before.

Given a probability space (Ω, Σ, μ) and an increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \ldots$ of sub-σ-algebras of Ω, we denote by $\mathbb{E}(\cdot|\mathcal{A}_j)$ and $\mathbb{E}^X(\cdot|\mathcal{A}_j)$ the conditional expectation operators with respect to $\mathcal{A}_j$ in $L^p(\mu)$ and $L^p(\mu, X)$ respectively ($1 < p < \infty$), where X is a Banach space.

Definition 4.60 *A Banach space X is called a UMD-space if there exists a constant $C_2(X)$ (the UMD-constant of X) such that*

$$\|\alpha_0 \mathbb{E}^X(f|\mathcal{A}_0) + \sum_{j=1}^{n} \alpha_j \{\mathbb{E}^X(f|\mathcal{A}_j) - \mathbb{E}^X(f|\mathcal{A}_{j-1})\}\|_{L^2(\mu,X)} \le C_2(X)\|f\|_{L^2(\mu,X)}$$

for all choices of $\alpha_j = \pm 1$, for all $f \in L^2(\mu, X)$, for all $n = 1, 2, \ldots$ and for all (Ω, Σ, μ) and $\{\mathcal{A}_j\}_{j=0}^{\infty}$ as above.

The condition in (ii) of Theorem 4.59 is in general strictly weaker than the R-boundedness of the collection $\{P_m\}_{m=0}^{\infty}$. However, the following theorem gives a condition which guarantees the R-boundedness of the same.

Theorem 4.61 *([15], Theorem 3.9) Let X be a UMD space and let $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ be an unconditional Schauder decomposition with unconditional constant C_Δ. Let $P_n = \sum_{k=0}^{n} \Delta_k$. Then*

$$\|\sum_{k=0}^{n} \epsilon_k P_k x_k\|_{L^2(\mu,X)} \le C_2(X) C_\Delta^2 \|\sum_{k=0}^{n} \epsilon_k x_k\|_{L^2(\mu,X)}$$

for all $x_0, x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. Thus, $\{P_n\}_{n\in\mathbb{N}}$ is R-bounded.

From the work in [15] we have the following nice example of an unconditional Schauder decomposition $\Delta = \{\Delta_k\}_{k=0}^{\infty}$ for which the collection

$$S := \{\sum_{k\in F} \Delta_k : F \subset \mathbb{N}, F \text{ is finite}\}$$

is not R-bounded (even if X is a UMD-space).

Example 4.62 *Let H be a separable Hilbert space and let $X = C_p$, $1 \le p < \infty$, be the Schatten p-class of compact operators on H. Take a fixed orthonormal basis $\{e_n\}_{n=0}^{\infty}$ in H. For $m, n \in \mathbb{N}$ we define $E_{mn} \in C_p$ by $E_{mn}(x) = \langle x, e_n\rangle e_m$ for all $x \in H$. For $m \in \mathbb{N}$ we define the (row) projections $R_m : C_p \to C_p$ and (column) projections $C_m : C_p \to C_p$ by $R_m(A) = E_{mm}A$ and $C_m(A) = AE_{mm}$, $A \in C_p$, respectively. Then $\{R_m\}_{m=0}^{\infty}$ and $\{C_m\}_{m=0}^{\infty}$ are both unconditional decompositions of C_p and $C_n R_m = R_m C_n$ for all $m, n \in \mathbb{N}$. The authors in [15] (Example 3.10) show that the collection*

$$\mathcal{R} = \left\{\sum_{k\in F} R_k : F \subset \mathbb{N}, F \textit{ finite}\right\}$$

is not R-bounded if $p \neq 2$.

In case of $p = 1$, *i.e.* $X = \mathcal{C}_1$, *the collection* $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$, *with* $P_m = \sum_{k=0}^{m} R_k$, *is not R-bounded. This shows that the result of Theorem 4.61 does not hold in general, if* X *is not a UMD-space.*

Chapter 5

(p,q)-multiplier functions

Throughout this chapter (Ω, Σ, μ) is a finite measure space and X is a Banach space.

Definition 5.1 *We call a function $h : \Omega \to X$ a* **strongly p-integral function** *if for each weak p'-integral function $g : \Omega \to X^*$, the function $\Omega \to \mathbb{K} :: t \mapsto \langle h(t), g(t) \rangle$ is a 1-integrable scalar function.*

Let $L^p\langle \mu, X \rangle$ be the vector space of (equivalence classes of) strongly p-integral functions $h : \Omega \to X$ such that $t \mapsto \langle h(t), g(t) \rangle$ is in $L^1(\mu)$ for all $g \in L^{p'}_w(\mu, X^*)$ and

$$\|h\|_{L^p\langle \mu, X \rangle} := \sup_{\|g\|^{weak}_{p'} \leq 1} \int_\Omega |\langle h(t), g(t) \rangle| \, d\mu(t) < \infty.$$

For the moment we assume that $\|h\|_{L^p\langle \mu, X \rangle}$ is a norm. We will prove this fact later on.

Lemma 5.2 *(a) Let $1 \leq p < \infty$ then $L^p(\mu, X) \subseteq L^p_w(\mu, X)$, $\forall\ X$. The embedding is continuous with norm ≤ 1.*

(b) Let $1 \leq p < \infty$. Then $L^p\langle \mu, X \rangle \subseteq L^p(\mu, X)$ where the embedding is continuous with norm ≤ 1. In particular $L^1\langle \mu, X \rangle = L^1(\mu, X)$ with $\|f\|_{L^1\langle \mu, X \rangle} = \|f\|_{L^1(\mu, X)}$.

Proof

(a) If $f \in L^p(\mu, X)$, then for each $x^* \in X^*$ we have $x^* f$ is μ-measurable and

$$|(x^* \circ f)(t)| = |x^*(f(t))| \leq \|x^*\| \|f(t)\|, \qquad \forall\ t \in \Omega.$$

Thus $|(x^* \circ f)(t)|^p \leq \|x^*\|^p \|f(t)\|^p$, so that

$$\left(\int_\Omega |(x^* \circ f)(t)|^p \, d\mu(t)\right)^{\frac{1}{p}} \leq \|x^*\| \left(\int_\Omega \|f(t)\|^p \, d\mu(t)\right)^{\frac{1}{p}} < \infty.$$

Therefore $f \in L^p_w(\mu, X)$ and $\|f\|_p^{weak} \leq \|f\|_{L^p(\mu,X)}$. (This is true for $1 \leq p < \infty$).

(b) Let $h \in L^1(\mu, X)$. For each $g \in L^\infty_w(\mu, X^*)$ there is $E \in \Sigma$, $\mu(E) = 0$, such that $\{g(t) \mid t \notin E\}$ is *weak** bounded (thus, norm bounded) in X^*, i.e. $g \in L^\infty(\mu, X^*)$. Thus we have

$$\int_\Omega |\langle h(t), g(t)\rangle| \, d\mu(t) \leq (\operatorname{ess\,sup}_{t\in\Omega} \|g(t)\|) \int_\Omega \|h(t)\| \, d\mu(t).$$

This shows that $h \in L^1\langle\mu, X\rangle$ and $\|h\|_{L^1\langle\mu,X\rangle} \leq \|h\|_{L^1(\mu,X)}$. The inclusion

$$L^p\langle\mu, X\rangle \subseteq L^p(\mu, X) \text{ with } \|h\|_{L^p(\mu,X)} \leq \|h\|_{L^p\langle\mu,X\rangle}$$

for $1 \leq p < \infty$, will follow from a more general result, Theorem 5.15, later on in this chapter.

□

Theorem 5.3 *Let (Ω, Σ, μ) be a finite measure space. Then $L^p\langle\mu, X\rangle$ is a Banach space.*

Proof Let (h_n) be a Cauchy sequence in $L^p\langle\mu, X\rangle$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{g\in B_{L^{p'}_w(\mu,X^*)}} \int_\Omega |\langle h_m(t) - h_n(t), g(t)\rangle| \, d\mu(t) < \epsilon, \ \forall \ m, n \geq N.$$

In particular, for each $g \in B_{L^{p'}_w(\mu,X^*)}$ we have

$$\int_\Omega |\langle h_m(t) - h_n(t), g(t)\rangle| \, d\mu(t) < \epsilon, \ \forall \ m, n \geq N;$$

i.e. $(\langle h_n(\cdot), g(\cdot)\rangle)_n$ is a mean Cauchy sequence (Cauchy sequence in $L^1(\mu)$). Therefore, there exists $f_g \in L^1(\mu)$ such that $\langle h_n(\cdot), g(\cdot)\rangle \underset{\infty}{\overset{n}{\to}} f_g$ in L^1-norm. By Lemma 5.2, (h_n) is also a Cauchy sequence in $L^p(\mu, X)$, i.e. there exists $h \in L^p(\mu, X)$ such that $h_n(\cdot) \underset{\infty}{\overset{n}{\to}} h(\cdot)$ in L^p-norm (i.e. also in mean). By 2.5.1, 2.5.3 in [4] (see p. 93) there is a subsequence (h_{n_k}) so that $\|h_{n_k}(\cdot) - h(\cdot)\| \underset{\infty}{\overset{k}{\to}} 0$ a.e. Since for each $t \in \Omega$, $g(t) \in X^*$, it follows that

$$\langle h_{n_k}(t), g(t)\rangle \underset{\infty}{\overset{k}{\to}} \langle h(t), g(t)\rangle \text{ a.e.}$$

Thus we see that

(i) $(\langle h_{n_k}(\cdot), g(\cdot)\rangle)$ is a mean Cauchy sequence.

(ii) $(\langle h_{n_k}(\cdot), g(\cdot)\rangle)$ converges to $\langle h(\cdot), g(\cdot)\rangle$ a.e.

(iii) $(\langle h_{n_k}(\cdot), g(\cdot)\rangle)$ is integrable, since $h_{n_k}(\cdot) \in L^p\langle\mu, X\rangle$ and $g \in L_w^{p'}(\mu, X^*)$.

Therefore we may conclude that $\langle h(\cdot), g(\cdot)\rangle$ is integrable and $\langle h_{n_k}(\cdot), g(\cdot)\rangle \overset{k}{\underset{\infty}{\to}} \langle h(\cdot), g(\cdot)\rangle$ in mean (cf. [6], p. 104). Since g was arbitrary chosen, we see that $h \in L^p\langle\mu, X\rangle$. Also,

$$\int_\Omega |\langle h(t), g(t)\rangle| \, d\mu(t) = \lim_k \int_\Omega |\langle h_{n_k}(t), g(t)\rangle| \, d\mu(t) = \int_\Omega |f_g(t)| \, d\mu(t).$$

It follows that $f_g(\cdot) = \langle h(\cdot), g(\cdot)\rangle$ a.e., i.e.

$$\begin{aligned}
\int_\Omega |\langle h(t) - h_m(t), g(t)\rangle| \, d\mu(t) &= \int_\Omega |\langle h(t), g(t)\rangle - \langle h_m(t), g(t)\rangle| \, d\mu(t) \\
&= \int_\Omega |f_g(t) - \langle h_m(t), g(t)\rangle| \, d\mu(t) \\
&= \lim_{n\to\infty} \int_\Omega |\langle h_n(t), g(t)\rangle - \langle h_m(t), g(t)\rangle| \, d\mu(t) \\
&< \epsilon, \qquad \forall\, m \geq N \text{ and all } g \in B_{L_w^{p'}(\mu, X^*)}. \\
\therefore \qquad h_n \to h &\in L^p\langle\mu, X\rangle \text{ in norm.}
\end{aligned}$$

□

Remark 5.4 *We refer to [20] (Example 10 on p. 228) or [18] for the fact that*

$$J : L^1(\mu) \otimes X \longrightarrow L^1(\mu, X) \text{ such that } J\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i\right) = \sum_{i=1}^n x_i \chi_{A_i}$$

is a norm ≤ 1 bounded linear operator, which maps the dense subspace of $L^1(\mu) \hat{\otimes} X$ consisting of elements of the form $\sum_{i=1}^n \chi_{A_i} \otimes x_i$, where $A_1, \cdots, A_n$ are disjoint sets in Σ and $x_1, \cdots, x_n \in X$, onto the dense subspace of simple functions in $L^1(\mu, X)$. Moreover, the unique extension of J to $L^1(\mu) \hat{\otimes} X$ is an isometry onto $L^1(\mu, X)$.

By the Universal Mapping Property it follows that

Theorem 5.5

$$L^p(\mu) \hat{\otimes} X \subseteq L^p\langle\mu, X\rangle$$

by an injective embedding of norm ≤ 1.

Proof

$$\begin{array}{ccc} L^p(\mu) \times X & \xrightarrow{\quad\beta\quad} & L^p\langle\mu, X\rangle \\ \downarrow & \nearrow_{\phi} & \\ L^p(\mu) \overset{\wedge}{\otimes} X & & \end{array}$$

Let $f \in L^p(\mu)$ and $x \in X$ and define a bilinear mapping $\beta : L^p(\mu) \times X \to L^p\langle\mu, X\rangle$ as follows:

$\beta(f, x) = h_{f,x}$ with $h_{f,x}(t) = f(t)x$. First we show that $h_{f,x} \in L^p\langle\mu, X\rangle$. Choose $g \in L_w^{p'}(\mu, X^*)$ and consider $F : \Omega \to \mathbb{K} :: t \to \langle h_{f,x}(t), g(t)\rangle$; then

$$F(t) = \langle f(t)x, g(t)\rangle = f(t)\langle x, g(t)\rangle,$$

showing that $F \in L^1(\mu)$. β is bounded, since

$$\begin{aligned} \|\beta(f,x)\| &= \sup_{\|g\|_{p'}^{weak} \le 1} \int_\Omega |\langle h_{f,x}(t), g(t)\rangle| \, d\mu(t) \\ &= \sup_{\|g\|_{p'}^{weak} \le 1} \int_\Omega |\langle f(t)x, g(t)\rangle| \, d\mu(t) \\ &= \sup_{\|g\|_{p'}^{weak} \le 1} \int_\Omega |f(t)\langle x, g(t)\rangle| \, d\mu(t) \\ &\le \sup_{\|g\|_{p'}^{weak} \le 1} \left(\int_\Omega |f(t)|^p \, d\mu(t)\right)^{\frac{1}{p}} \left(\int_\Omega |\langle x, g(t)\rangle|^{p'} \, d\mu(t)\right)^{\frac{1}{p'}} \\ &\le \|f\|_{L^p(\mu)} \|x\|, \text{ i.e. } \|\beta\| \le 1. \end{aligned}$$

Therefore, by the Universal Mapping Property it follows that there exists a bounded linear operator $\phi : L^p(\mu) \overset{\wedge}{\otimes} X \longrightarrow L^p\langle\mu, X\rangle$ such that $\phi(f \otimes x) = h_{f,x}$ with $\|\phi\| = \|\beta\|$. Note that ϕ corresponds with J in Remark 5.4, but now we only work with the subspaces $L^p(\mu) \overset{\wedge}{\otimes} X$ of $L^1(\mu) \otimes X$ and $L^p\langle\mu, X\rangle$ of $L^1(\mu, X)$. Thus, $L^p(\mu) \overset{\wedge}{\otimes} X$ is a norm ≤ 1 injective embedding into $L^p\langle\mu, X\rangle$. □

If $g : \Omega \to X$ and $h : \Omega \to L(X, Y)$ are μ-measurable functions, then $F_{h,g} : \Omega \to Y$ such that $F_{h,g}(t) = h(t)(g(t))$ is also μ-measurable:

Let $\|h(t) - h_n(t)\| \xrightarrow[\infty]{n} 0$, μ-a.e., where each h_n is a simple function,

$$h_n = \sum_{j=1}^{k_n} T_{j,n} \chi_{E_{j,n}}, \quad T_{j,n} \in L(X, Y).$$

Then

$$\begin{aligned} F_{h,g}(t) &= h(t)(g(t)) \\ &= \lim_{n\to\infty} h_n(t)(g(t)) \quad \mu - a.e \\ &= \lim_{n\to\infty} \sum_{j=1}^{k_n} T_{j,n}(g(t))\chi_{E_{j,n}}(t). \end{aligned}$$

Since g is μ-measurable, there exists a sequence

$$g_m : \Omega \to X \ : \ g_m = \sum_{i=1}^{l_m} x_{i,m}\chi_{E_{i,m}}$$

such that $g_m \to g, \quad \mu - a.e.$ where $E_{i,m} \in \Sigma$ and $E_{i,m} \cap E_{j,m} = \emptyset, \quad \forall\, i \neq j$. Now

$$\sum_{j=1}^{k_n} T_{j,n}(g(t))\chi_{E_{j,n}}(t) = \lim_{m\to\infty} \sum_{j=1}^{k_n}\sum_{i=1}^{l_m} T_{j,n}(x_{i,m})\chi_{E_{j,n}\cap E_{i,m}}(t).$$

Clearly $\sum_{j=1}^{k_n}\sum_{i=1}^{l_m} T_{j,n}(x_{i,m})\chi_{E_{j,n}\cap E_{i,m}} : \Omega \to Y$ is a simple function for each $m \in \mathbb{N}$. Thus $(\sum_{j=1}^{k_n} T_{j,n}(g(t))\chi_{E_{j,n}}(t))_n$ is a sequence of μ-measurable functions, which converges to the limit function $F_{h,g}$ pointwise μ-a.e. This shows that $F_{h,g}$ is μ-measurable.

Definition 5.6

*(1) A measurable function $h : \Omega \to L(X,Y)$ is called a $(p,q)-$***integral multiplier** *for the pair (X,Y) if for each $g \in L^q_w(\mu, X)$, we have:*
The function $F_{h,g} : \Omega \to Y :: t \mapsto h(t)(g(t))$ is in $L^p(\mu, Y)$ and the linear operator $\hat{h} : L^q_w(\mu, X) \to L^p(\mu, Y)$, given by $\hat{h}(g) = F_{h,g}$, is bounded.
In this case, we let

$$\pi_{p,q}(h) := \|\hat{h}\| = \sup_{\|g\|_q^{weak}\leq 1} (\int_\Omega |F_{h,g}(t)|^p d\mu(t))^{\frac{1}{p}} < \infty.$$

(2) More generally, if $E(\Omega, X)$ and $F(\Omega, Y)$ are normed spaces of μ - measurable functions from Ω into X and from Ω into Y respectively, containing the constant functions, then a $(E(\Omega, X), F(\Omega, Y))-$ **integral multiplier** *for the pair (X,Y) is a measurable function $h : \Omega \to L(X,Y)$ such that for each $g \in E(\Omega, X)$, the function*

$$F_{h,g} : \Omega \to Y : t \to h(t)(g(t))$$

is in $F(\Omega, Y)$ *and the corresponding* $\hat{h} : E(\Omega, X) \to F(\Omega, Y)$ *is bounded. In this case we let*

$$\|h\|_{(E,F)} := \|\hat{h}\| = \sup_{\|g\|_{E(\Omega,X)}\leq 1} \|F_{h,g}\|_{F(\Omega,Y)}.$$

We agree to denote the vector space of equivalence classes of (p, q)-integral multipliers by $L_{\pi_{p,q}}(X, Y)$; i.e. for $h_1, h_2 \in L_{\pi_{p,q}}(X, Y)$, we have $h_1 = h_2$ if and only if $h_1(t) = h_2(t)$ μ-almost everywhere. In the general case, the vector space of equivalence classes of $(E(\Omega, X), F(\Omega, Y))$ - integral multipliers, will be denoted by $(E(\Omega, X), F(\Omega, Y))$.

Let $L^\infty(\mu, X) \subseteq E(\Omega, X)$ and $F(\Omega, Y) \subseteq L^1(\mu, Y)$ and $h \in (E(\Omega, X), F(\Omega, Y))$ be given. Since $h : \Omega \to L(X, Y)$ is measurable, it follows for each $\epsilon > 0$ that there exists a simple function $\Omega \to L(X, Y) : t \mapsto \sum_{i=1}^{k} S_i \chi_{A_i}(t)$, with $A_i \in \Sigma$ and $A_i \cap A_j = \emptyset$ if $i \neq j$, such that $\|h(t) - \sum_{i=1}^{k} S_i \chi_{A_i}(t)\| < \epsilon$, almost everywhere. Let $h_{\|\cdot\|} : \Omega \to \mathbb{R}$ be given by $h_{\|\cdot\|}(t) = \|h(t)\|$. We have

$$\left| \|h(t)\| - \sum_{i=1}^{k} \|S_i\| \chi_{A_i}(t) \right| \leq \|h(t) - \sum_{i=1}^{k} S_i \chi_{A_i}(t)\| < \epsilon, \text{ almost everywhere.}$$

It is thus clear that $h_{\|\cdot\|}$ is a measurable real valued function. The fact that $h : \Omega \to L(X, Y)$ is measurable, also implies by the Pettis Measurability Theorem (cf. [20], Theorem 2, p. 42) that there exists $A \in \Sigma$ such that $\mu(A) = 0$ and $h(\Omega \backslash A)$ is a (norm) separable subset of $L(X, Y)$. Let $(T_n) \subset L(X, Y)$ be a dense countable subset (sequence) of $h(\Omega \backslash A)$. Let $\epsilon > 0$ be given. The sets $A_n = h_{\|\cdot\|}^{-1}((\|T_n\| - \epsilon/3, \|T_n\| + \epsilon/3))$ are measurable, therefore the sets $B_n = A_n \cap (\Omega \backslash A)$ are measurable too. For each $t \in \Omega \backslash A$, there is $n \in \mathbb{N}$ such that $\|h(t) - T_n\| < \epsilon/3$, i.e. $t \in B_n$. Thus we see that $\Omega \backslash A = \cup_n B_n$. Then let $x_n \in X$ with $\|x_n\| = 1$, such that $\|T_n\| < \|T_n(x_n)\| + \epsilon/3$. The function $g : \Omega \to X$ such that $g(t) = \sum_n x_n \chi_{B_n}(t)$ is measurable. Since the range of g is bounded, it follows that $g \in L^\infty(\mu, X)$, i.e. $g \in E(\Omega, X)$. It is now clear that the function $t \mapsto h(t)(g(t))$ is in $L^1(\mu, Y)$. Also, if $t \in B_n$, then

$$\begin{aligned} \|h(t)\| - \|h(t)(g(t))\| &< \|h(t) - T_n\| + \|T_n(x_n)\| + \epsilon/3 - \|h(t)(x_n)\| \\ &\leq \epsilon/3 + 2\|T_n - h(t)\| < \epsilon. \end{aligned}$$

This shows that $\|h(t)\| < \|h(t)(g(t))\| + \epsilon$ almost everywhere. We have thus verified that if $L^\infty(\mu, X) \subseteq E(\Omega, X)$ and $F(\Omega, Y) \subseteq L^1(\mu, Y)$, then

$$(E(\Omega, X), F(\Omega, Y)) \subset L^1(\mu, L(X, Y)).$$

In particular, we conclude that $L_{\pi_{p,q}}(X, Y) \subset L^1(\mu, L(X, Y))$.

Also, if $1 \leq p < \infty$, it follows that

$$(L^{p'}(\mu, X^*), L^1(\mu)) \subset L^1(\mu, X^{**}).$$

Now, suppose X^{**} has RNP (and thus, X has RNP) and $p > 1$.

If $h \in (L^{p'}(\mu, X^*), L^1(\mu))$, then the function $t \mapsto h(t)[g(t)]$ belongs to $L^1(\mu)$ for all $g \in L^{p'}(\mu, X^*)$, where $h \in L^1(\mu, X^{**})$. Therefore

$$h \in \{f \in L^1(\mu, X^{**}) \mid f(\cdot)[g(\cdot)] \in L^1(\mu),\ \forall g \in L^{p'}(\mu, X^*)\} = (L^{p'}(\mu, X^*))^* = L^p(\mu, X^{**})$$

because X^{**} has Radon-Nikodym property. Also, if $\tilde{h} \in L^p(\mu, X^{**})$ is given and $g \in L^{p'}(\mu, X^*)$, then $\tilde{h} \in L^{p'}(\mu, X^*)^*$ and $t \mapsto \tilde{h}(t)[g(t)]$ is in $L^1(\mu)$; i.e. $\tilde{h} \in (L^{p'}(\mu, X^*), L^1(\mu))$. Thus we see that

$$(L^{p'}(\mu, X^*), L^1(\mu)) = L^p(\mu, X^{**})$$

if X^{**} has RNP.

In particular, if X is any reflexive Banach space it follows that

$$(L^{p'}(\mu, X^*), L^1(\mu)) = L^p(\mu, X) = (L^{p'}(\mu, X^*))^*.$$

For $h \in (E(\Omega, X), F(\Omega, Y))$ such that $\|h\|_{(E,F)} = 0$, it follows that $\hat{h} = 0$, i.e.

$$F_{h,g}(t) = 0, \mu - a.e. \text{ for all } g \in E(\Omega, X).$$

Again using the Pettis Measurability Theorem, there exists $A \in \Sigma$ such that $\mu(A) = 0$ and $h(\Omega \backslash A)$ is a (norm) separable subset of $L(X, Y)$. Let $(T_n) \subset L(X, Y)$ be a dense sequence in $h(\Omega \backslash A)$. We consider the following possibilities:

1. $T_n = 0$ for all $n \in \mathbb{N}$. Then clearly, $h(t) = 0$ for all $t \in \Omega \backslash A$, i.e. $h = 0$ μ- a.e.

2. There is a subsequence (or, possibly a finite set) (T_{n_k}) such that $T_{n_k} \neq 0$ for all k and $T_n = 0$ for all $n \neq n_k$. In this case, for the fixed $A \in \Sigma$ and sequence (T_{n_k}), we may have:

   (i) $T_{n_k} \to 0$ if $k \to \infty$. Then let $x_{n_k} \in X$, such that

   $$\|x_{n_k}\| = 1 \text{ and } \|T_{n_k} x_{n_k}\| \approx \|T_{n_k}\|.$$

   Put $g_k(t) = x_{n_k}$ for all $t \in \Omega$ and for all k. By assumption, $g_k \in E(\Omega, X)$ for all k. Therefore, for each k there exists $A_k \in \Omega$, such that $\mu(A_k) = 0$ and $h(t)(x_{n_k}) = 0, \forall t \in \Omega \backslash A_k$. Let $B = (\cup_k A_k) \cup A$; i.e. B is a μ-null set. For $t \notin B$, we have $h(t) \in h(\Omega \backslash A)$ and $h(t)(x_{n_k}) = 0$ for all k. Clearly, $h(t) \neq T_{n_k}$ for all k. Thus, if $h(t) \in \overline{\{T_{n_k} : k \in \mathbb{N}\}}$, then $h(t) = 0$. If $h(t) \notin \overline{\{T_{n_k} : k \in \mathbb{N}\}}$, then there exists a subsequence T_{m_j} of (T_n), such that $T_{m_j} \neq T_{n_k}$ for all j and all k and $T_{m_j} \to h(t)$ if $j \to \infty$. Now $T_{m_j} = 0$ for all j. Thus we have $h(t) = 0$. Since $t \in \Omega \backslash B$ was arbitrary, we conclude that $h(t) = 0$ μ - a.e. in the case when $T_{n_k} \to 0$ if $k \to \infty$.

   (ii) $T_{n_k} \not\to 0$ if $k \to \infty$. Then there exists $\epsilon > 0$ such that $\|T_{n_k}\| \geq \epsilon$ for all k. We follow a similar argument (as in (i)), now choosing $x_{n_k} \in X$ such that $\|x_{n_k}\| = 1$ and $\|T_{n_k} x_{n_k}\| \geq \epsilon/2$. We choose g_k and $A_k \in \Sigma$ such that

   $$h(t)(x_{n_k}) = 0, \ \forall t \in \Omega \backslash A_k$$

   as before and again let $B = (\cup_k A_k) \cup A$. Then $t \notin B$ implies that $h(t) \in h(\Omega \backslash A)$ and $h(t)(x_{n_k}) = 0$ for all k. In this case we have

   $$\|h(t) - T_{n_k}\| \geq \|h(t)(x_{n_k}) - T_{n_k} x_{n_k}\| = \|T_{n_k} x_{n_k}\| \geq \epsilon/2$$

   for all k. This shows that $h(t) \notin \overline{\{T_{n_k} : k \in \mathbb{N}\}}$, from which it follows as in (i) that $h(t)$ is the limit of a null sequence. Thus we see that $h(t) = 0, \forall t \in \Omega \backslash B$. Again we have $h(t) = 0$ μ- a.e.

Our conclusion is that $\|h\|_{(E,F)} = 0$ implies that $h = 0$. Thus,

$$((E(\Omega, X), F(\Omega, Y)), \|h\|_{(E,F)})$$

is a normed space.

Both $L^q_w(\mu, X)$ and $L^p(\mu, Y)$ contain the constant functions from Ω to X and from Ω to Y respectively, so that we see from the previous general result that $(L_{\pi_{p,q}}(X,Y), \pi_{p,q}(\cdot))$ is a normed space.

Definition 5.7 *For any Banach space X the space $L_{\pi_{p,q}}(X)$ of* (p,q)**-integral functions in X,** *is defined by*

$L_{\pi_{p,q}}(X) := L_{\pi_{p,q}}(X^*, \mathbb{K}) \cap L^1(\mu, X)$

$$= \left\{ \; h \in L^1(\mu, X) \mid \sup_{\|g\|_q^{weak} \le 1} (\textstyle\int_\Omega |\langle h(t), g(t)\rangle|^p \, d\mu(t))^{\frac{1}{p}} < \infty \; \right\}.$$

We put $\pi_{p,q}(h) := \sup_{\|g\|_q^{weak} \le 1} (\int_\Omega |\langle h(t), g(t)\rangle|^p \, d\mu(t))^{\frac{1}{p}}$ and observe from the general case that $\pi_{p,q}(\cdot)$ is a norm on $L_{\pi_{p,q}}(X)$.

In this case the fact that $\pi_{p,q}(.)$ is a norm on $L_{\pi_{p,q}}(X)$ also follows by a straightforward argument: Suppose $h_1, \; h_2 \in L_{\pi_{p,q}}(X)$ and $\pi_{p,q}(h_1 - h_2) = 0$. For $0 \neq x^* \in X^*$, we put

$$g(t) = \frac{x^*}{\mu(\Omega)\|x^*\|} \text{ for all } t \in \Omega.$$

Then $g \in L^q_w(\mu, X^*)$ and $\|g\|_q^{weak} \le 1$. Thus, $(\int_\Omega |\langle h_1(t) - h_2(t), \frac{x^*}{\mu(\Omega)\|x^*\|}\rangle|^p \, d\mu(t))^{\frac{1}{p}} = 0$, i.e.

$$\frac{1}{\mu(\Omega)\|x^*\|}(\int_\Omega |(x^* \circ h_1 - x^* \circ h_2)(t)|^p \, d\mu(t))^{\frac{1}{p}} = 0.$$

It follows that $x^* \circ h_1 = x^* \circ h_2 \;\; \mu - a.e.$ This is true for all choices of $x^* \in X^*$. As we mentioned before (refer to Corollary 7, p. 48 of [20]) this implies that $h_1 = h_2 \;\; \mu - a.e.$

Following are some elementary examples of integral multipliers and integral functions.

Example 5.8 *Let $1 \le p \le q$.*

(1) Choose $x \in X$. Define $h : \Omega \to X$ by $h(t) = x, \; \forall \; t \in \Omega$. Let $g \in L^q_w(\mu, X^*)$, then the function $F_{h,g}(\cdot) = h(\cdot)[g(\cdot)] \in L^q(\mu) \subset L^p(\mu)$ and therefore $h \in L_{\pi_{p,q}}(X)$.

(2) Consider $T = \sum_{i=1}^n x_i^* \otimes f_i, f_i \in L^\infty(\mu, Y)$ and $x_i^* \in X^*, \; i = 1, \cdots, n$. Define

$$h : \Omega \to L(X,Y) \text{ by } h(t) = \sum_{i=1}^n x_i^* \otimes f_i(t).$$

Let $F_{h,g}(t) = h(t)[g(t)]$ *with* $g \in L^q_w(\mu, X)$. *We show that* $F_{h,g} \in L^q(\mu, Y)$:

$$
\begin{aligned}
(\int_\Omega \|F_{h,g}(t)\|^q \, d\mu(t))^{\frac{1}{q}} &= (\int_\Omega \|\sum_{i=1}^n \langle x_i^*, g(t)\rangle f_i(t)\|^q \, d\mu(t))^{\frac{1}{q}} \\
&\leq \operatorname*{ess\,sup}_{t\in\Omega}(\max_{1\leq i\leq n} \|f_i(t)\|) \left(\int_\Omega (\sum_{i=1}^n |\langle x_i^*, g(t)\rangle|)^q \, d\mu(t)\right)^{\frac{1}{q}} < \infty.
\end{aligned}
$$

Hence $F_{h,g} \in L^q(\mu, Y) \subseteq L^p(\mu, Y)$, *therefore* $h \in L_{\pi_{p,q}}(X, Y)$.

(3) Let $1 \leq p \leq q$, $f \in L^\infty(\mu, X)$ *and* $x \in X$, $\|x\| = 1$. *Define*

$$h_{f,x} : \Omega \to L(X^*, X) \text{ by } h_{f,x}(t) = x \otimes f(t).$$

For $F_{h,g}(t) = h_{f,x}(t)[g(t)]$, *where* $g \in L^q_w(\mu, X^*)$, *we have*

$$
\begin{aligned}
(\int_\Omega \|F_{h,g}(t)\|_X^p \, d\mu(t))^{\frac{1}{p}} &= (\int_\Omega |\langle g(t), x\rangle|^p \|f(t)\|^p \, d\mu(t))^{\frac{1}{p}} \\
&\leq \|f\|_{L^\infty(\mu,X)} (\int_\Omega |\langle g(t), x\rangle|^q \, d\mu(t))^{\frac{1}{q}} \\
&\leq \|f\|_{L^\infty(\mu,X)} \|g\|_q^{weak} < \infty.
\end{aligned}
$$

Thus, $h_{f,x} \in L_{\pi_{p,q}}(X^*, X)$ *and* $\pi_{p,q}(h_{f,x}) \leq \|f\|_{L^\infty(\mu,X)}$.

Remark 5.9 *If* $f \in L^\infty(\mu)$ *and* $x \in X$, $\|x\| = 1$, *we let* $h : \Omega \to X$ *be defined by* $h(t) = f(t)x$. *Then* h *is measurable and for each* $g \in L^q_w(\mu, X^*)$ *and* $1 \leq p \leq q$ *we have:*

$$
\begin{aligned}
(\int_\Omega |\langle h(t), g(t)\rangle|^p \, d\mu(t))^{\frac{1}{p}} &\leq (\int_\Omega |f(t)|^p |\langle x, g(t)\rangle|^p \, d\mu(t))^{\frac{1}{p}} \\
&\leq \|f\|_{L^\infty(\mu)} (\int_\Omega |\langle x, g(t)\rangle|^p \, d\mu(t))^{\frac{1}{p}} \\
&\leq \|f\|_{L^\infty(\mu)} \|g\|_q^{weak} < \infty.
\end{aligned}
$$

Thus, with each $f \in L^\infty(\mu)$ *we associate* $h \in L_{\pi_{p,q}}(X)$ *such that* $\pi_{p,q}(h) \leq \|f\|_{L^\infty(\mu)}$. *Clearly, each* $f \in L^\infty(\mu)$ *gives rise to many* $h \in L_{\pi_{p,q}}(X)$ *by just choosing different* $x \in X$ *fixed. This, and the previous examples, indicate that if* $p \leq q$, *then* $L_{\pi_{p,q}}(X)$ *is big.*

(4) Let $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$; $f \in L^r(\mu, X)$; $g \in L^q_w(\mu, X^*)$ *and* $x \in X$, $\|x\| = 1$. *Define* $h_{f,x} : \Omega \to L(X^*, X)$ *by* $h_{f,x}(t) = x \otimes f(t)$, *with*

$$F_{h,g}(t) = h_{f,x}(t)[g(t)] = \langle g(t), x\rangle f(t).$$

Then

$$\left(\int_\Omega \|F_{h,g}(t)\|_X^p \, d\mu(t)\right)^{\frac{1}{p}} \le \left(\int_\Omega |\langle g(t), x\rangle|^q \, d\mu(t)\right)^{\frac{1}{q}} \left(\int_\Omega \|f(t)\|^r \, d\mu(t)\right)^{\frac{1}{r}} < \infty$$

i.e. $h \in L_{\pi_{p,q}}(X^*, X)$.

(5) Let $T \in N(X, L^\infty(\mu, Y))$, *then*

$$T = \sum_{i=1}^{\infty} \lambda_i x_i^* \otimes f_i \text{ with } \|x_i^*\| \le 1, \ \|f_i\|_{L^\infty(\mu,Y)} \le 1 \text{ and } (\lambda_i) \in \ell^1.$$

Define $h_T : \Omega \to L(X, Y)$ *by* $h_T(t) = \sum_{i=1}^{\infty} \lambda_i x_i^* \otimes f_i(t)$. *Notice that* $h_T(t) \in N(X, Y)$ *for each* $t \in \Omega$. *Consider*

$$F_{T,g}(t) = h_T(t)[g(t)] = \sum_{i=1}^{\infty} \lambda_i \langle x_i^*, g(t)\rangle f_i(t) \text{ for all } g \in L^q_w(\mu, X).$$

We have

$$\begin{aligned}
\left(\int_\Omega \|F_{T,g}(t)\|^q \, d\mu(t)\right)^{\frac{1}{q}} &= \left(\int_\Omega \|\sum_{i=1}^{\infty} \lambda_i \langle x_i^*, g(t)\rangle f_i(t)\|^q \, d\mu(t)\right)^{\frac{1}{q}} \\
&\le \left(\int_\Omega \left(\sum_{i=1}^{\infty} |\lambda_i| |\langle x_i^*, g(t)\rangle| \|f_i(t)\|\right)^q d\mu(t)\right)^{\frac{1}{q}} \\
&\le \left(\int_\Omega \left(\sum_{i=1}^{\infty} |\lambda_i| |\langle x_i^*, g(t)\rangle|\right)^q d\mu(t)\right)^{\frac{1}{q}} \\
&\le \sum_{i=1}^{\infty} \left(\int_\Omega |\lambda_i|^q |\langle x_i^*, g(t)\rangle|^q \, d\mu(t)\right)^{\frac{1}{q}} \\
&= \sum_{i=1}^{\infty} |\lambda_i| \left(\int_\Omega |\langle x_i^*, g(t)\rangle|^q \, d\mu(t)\right)^{\frac{1}{q}} \\
&\le \sum_{i=1}^{\infty} |\lambda_i| \|g\|_q^{weak} \\
&< \infty.
\end{aligned}$$

Hence, it follows that with each $T \in N(X, L^\infty(\mu, Y))$ *we assosiate* $h_T \in L_{\pi_{p,q}}(X, Y)$ *for all* $q \ge p$.

(6) *In [19] (p. 56) it is said that* $T : X \to Y$ *is p-summing iff given any probability space* (Ω, Σ, μ) *and any strongly measurable function* $g : \Omega \to X$ *such that*

$$g \in L^p_w(\mu, X), \text{ then } T \circ g \in L^p(\mu, Y).$$

Now, for $T \in \Pi_p(X, Y)$ *given, let* $h_T : \Omega \to L(X, Y)$ *be the constant function* $h_T(t) = T$ *for all* $t \in \Omega$. *For any* $g \in L^p_w(\mu, X)$ *it follows that*

$$t \mapsto h_T(t)[g(t)] = T(g(t)) = (T \circ g)(t)$$

is a function in $L^p(\mu, Y)$. *Therefore,* $h_T \in L_{\pi_{p,p}}(X, Y)$. *Thus, with each* $T \in \Pi_p(X, Y)$ *we associate* $h_T \in L_{\pi_{p,p}}(X, Y)$.

(7) *Suppose* $h : \Omega \to L(X, Y)$ *has the form:*
$h = \sum_{i=1}^n (x_i^* \otimes y_i)\chi_{E_i}$ *with* $E_i \in \Sigma,\ E_i \cap E_j = \emptyset,\ \forall\ i \neq j$. *Then*

$$h(t)(x) = \sum_{i=1}^n x_i^*(x) y_i \chi_{E_i}(t),\ \forall\ t \in \Omega.$$

For $g \in L^q_w(\mu, X)$ *we have*

$$F_{h,g}(t) = h(t)(g(t)) = \sum_{i=1}^n x_i^*(g(t)) y_i \chi_{E_i}(t).$$

For each $f \in L^p(\mu),\ 1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
\|f(t)F_{h,g}(t)\| &= \|f(t)\sum_{i=1}^n x_i^*(g(t)) y_i\ \chi_{E_i}(t)\| \\
&\leq \sum_{i=1}^n |f(t)x_i^*(g(t))| \|y_i\|\ \chi_{E_i}(t) \\
\therefore \quad \int_\Omega \|f(t)F_{h,g}(t)\|\ d\mu(t) &\leq \sum_{i=1}^n \int_\Omega |f(t)x_i^*(g(t))| \|y_i\|\ \chi_{E_i}(t)\ d\mu(t) \\
&\leq (\max_{1 \leq i \leq n} \|y_i\|) \sum_{i=1}^n \int_\Omega |f(t)x_i^* g(t)|\ d\mu(t) < \infty,
\end{aligned}$$

since $x_i^* \circ g \in L^q(\mu)$ *for each* x_i^*.

Thus, $F_{h,g} \in L^q(\mu, Y)$. *This shows that* $h \in L_{\pi_{q,q}}(X, Y)$.

Lemma 5.10 *Let* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$; $k \in L^r(\mu)$ *and* $x \in X$. *If* $(kx)(t) = k(t)x\ \forall t \in \Omega$, *then* $kx \in L_{\pi_{p,q}}(X)$ *and* $\pi_{p,q}(kx) \leq \|k\|_{L^r(\mu)}\|x\|$.

Proof Given $g \in L^q_w(\mu, X^*)$, it follows that

$$\begin{aligned}
\|\langle kx, g\rangle\|_{L^p(\mu)} &= \left(\int_\Omega |\langle k(t)x, g(t)\rangle|^p \, d\mu(t)\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega |k(t)|^r \, d\mu(t)\right)^{\frac{1}{r}} \left(\int_\Omega |g(t)[x]|^q \, d\mu(t)\right)^{\frac{1}{q}} \\
&\leq \|k\|_{L^r(\mu)} \|x\| \|g\|_q^{weak}.
\end{aligned}$$

Hence $\pi_{p,q}(kx) \leq \|k\|_{L^r(\mu)}\|x\|$. □

Theorem 5.11 *If $p \leq q$ then $(L_{\pi_{p,q}}(X), L^p(\mu, Y))$ is continuously imbedded into $L(X, L^q(\mu, Y))$.*

Proof Choose $h \in (L_{\pi_{p,q}}(X), L^p(\mu, Y))$ and define $U_h : X \to L^q(\mu, Y)$ by

$$U_h(x) = g_x \text{ with } g_x(t) = h(t)[x].$$

Let $\frac{1}{r} = \left(\frac{1}{p} - \frac{1}{q}\right)^+$. By Hahn-Banach it follows that for a given $x \in X$ there exists $f \in L^{\frac{r}{p}}(\mu)$ with $\|f\|_{L^{\frac{r}{p}}(\mu)} = 1$ such that $\left(\int_\Omega (\|h(t)[x]\|^p)^{\frac{q}{p}} \, d\mu(t)\right)^{\frac{p}{q}} = \int_\Omega |f(t)|\|h(t)[x]\|^p \, d\mu(t)$. Put $k = |f|^{\frac{1}{p}}$; then

$$\int_\Omega |k(t)|^r d\mu(t) = \int_\Omega |f(t)|^{\frac{r}{p}} d\mu(t) = 1$$

and

$$\begin{aligned}
\|U_h x\|_{L^q(\mu,Y)} &= \left(\int_\Omega \|h(t)x\|^q \, d\mu(t)\right)^{\frac{1}{q}} \\
&= \left[\left(\int_\Omega (\|h(t)[x]\|^p)^{\frac{q}{p}} \, d\mu(t)\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} \\
&= \left[\int_\Omega |f(t)|\|h(t)[x]\|^p \, d\mu(t)\right]^{\frac{1}{p}} \\
&= \left[\int_\Omega k(t)^p \|h(t)[x]\|^p \, d\mu(t)\right]^{\frac{1}{p}} \\
&= \left(\int_\Omega \|k(t)h(t)[x]\|^p \, d\mu(t)\right)^{\frac{1}{p}} \\
&\leq \|h\|_{(L_{\pi_{p,q}}(X),\ L^p(\mu,Y))} \ \pi_{p,q}[kx] \\
&\leq \|h\|_{(L_{\pi_{p,q}}(X),\ L^p(\mu,Y))} \ \|k\|_{L^r(\mu)} \ \|x\| \text{ (by Lemma 5.10)} \\
&= \|h\|_{(L_{\pi_{p,q}}(X),\ L^p(\mu,Y))} \ \|x\| \\
&< \infty.
\end{aligned}$$

Hereby $||U_h x||_{L^q(\mu,Y)} \leq ||h||_{(L_{\pi p,q}(X),\ L^p(\mu,Y))}||x||$, which implies U_h is bounded and

$$||U_h|| \leq ||h||_{(L_{\pi p,q}(X),\ L^p(\mu,Y))}.$$

□

Proposition 5.12 *(i) Let $1 \leq p < q$ and put $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then,*

$$(L^q(\mu, X), L^p(\mu, Y)) = L^r(\mu, L(X, Y)).$$

In particular, $L^\infty(\mu, L(X, Y)) \subseteq (L^q(\mu, X), L^p(\mu, Y))$.

(ii) $L^\infty(\mu, L(X, Y)) \subseteq (L^p(\mu, X), L^p(\mu, Y))$ for all $1 \leq p \leq \infty$.

(iii) Let $1 \leq q < p$. Then $(L^q(\mu, X), L^p(\mu, Y)) \subseteq L^\infty(\mu, L(X, Y))$.

Proof

(i) For a measurable $h : \Omega \to L(X, Y)$ and a measurable $g : \Omega \to X$ we let $F_{h,g} : \Omega \to Y$ be the function $F_{h,g}(t) = h(t)[g(t)]$ as before. If $h \in (L^q(\mu, X), L^p(\mu, Y))$, then $h \in L^1(\mu, L(X, Y))$ as was showed just after Definition 5.6.

Suppose $\boldsymbol{r = 1}$: (i.e. $p = 1$ and $q = \infty$)
If $h \in L^1(\mu, L(X, Y))$ and $g \in L^q(\mu, X) = L^\infty(\mu, X)$, then

$$\begin{aligned}
(\int_\Omega ||F_{h,g}(t)||\ d\mu(t)) &= \int_\Omega ||h(t)(g(t))||\ d\mu(t) \\
&\leq (\operatorname*{ess\,sup}_{t\in\Omega} ||g(t)||) \int_\Omega ||h(t)||\ d\mu(t) \\
&= c\ ||g||_{L^\infty(\mu,X)}, \quad c = \int_\Omega ||h(t)||\ d\mu(t).
\end{aligned}$$

Therefore $L^1(\mu, L(X, Y)) = (L^\infty(\mu, X), L^1(\mu, Y))$.

Suppose $\boldsymbol{p < q}$ and let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$; $r \neq 1$. Thus

$$\begin{aligned}
(\int_\Omega ||F_{h,g}||^p\ d\mu)^{\frac{1}{p}} &= (\int_\Omega ||h(t)(g(t))||^p\ d\mu)^{\frac{1}{p}} \\
&\leq (\int_\Omega ||h(t)||^p ||g(t)||^p\ d\mu)^{\frac{1}{p}} \\
&\leq (\int_\Omega ||h(t)||^r\ d\mu)^{\frac{1}{r}} (\int_\Omega ||g(t)||^q\ d\mu)^{\frac{1}{q}}.
\end{aligned}$$

Thus if $h \in L^r(\mu, L(X,Y))$, then $c := (\int_\Omega \|h(t)\|^r \, d\mu(t))^{\frac{1}{r}}$ gives us

$$(\int_\Omega \|F_{h,g}\|^p \, d\mu)^{\frac{1}{p}} \le c \, \|g\|_{L^q(\mu,X)} \text{ for all } g \in L^q(\mu, X).$$

Hence,

$$L^r(\mu, L(X,Y)) \subseteq (L^q(\mu, X), L^p(\mu, Y))$$

and the embedding has norm ≤ 1.

Conversely, let $h \in (L^q(\mu, X), L^p(\mu, Y))$. Since

$$L^\infty(\mu, X) \subseteq L^q(\mu, X) \text{ and } L^p(\mu, Y) \subseteq L^1(\mu, Y)$$

it follows from our previous observation that $h \in L^1(\mu, L(X,Y))$. We show that $h \in L^r(\mu, L(X,Y))$:

Suppose $f \in L^{r'}(\mu)$. Put $f_1(t) = f(t)^{\frac{r'}{q}}$ and $f_2(t) = f(t)^{\frac{r'}{p'}}$. From $\frac{r'}{q} + \frac{r'}{p'} = 1$ it follows that $f(t) = f_1(t) f_2(t)$. Also, $f_1 \in L^q(\mu)$ and $f_2 \in L^{p'}(\mu)$. For $\epsilon > 0$ fixed, let $\|x_t\| = 1$ such that $\|h(t)\| < \|h(t)(x_t)\| + \epsilon$ for all $t \in \Omega$. Then $t \mapsto f_1(t)x_t$ is in $L^q(\mu, X)$. Thus, $t \mapsto h(t)(f_1(t)x_t)$ is in $L^p(\mu, Y)$. Since $f_2 \in L^{p'}(\mu)$, it follows that $t \mapsto \|h(t)(f_1(t)x_t)\| |f_2(t)|$ is in $L^1(\mu)$. Also,

$$\begin{aligned} \|h(t)\| |f(t)| &\le \|h(t)(x_t)\| |f(t)| + \epsilon |f(t)| \\ &= \|h(t)(f_1(t)x_t)\| |f_2(t)| + \epsilon |f(t)|, \ \forall \, t \in \Omega. \end{aligned}$$

This implies that $t \mapsto \|h(t)\| |f(t)|$ is in $L^1(\mu)$ and

$$\int_\Omega \|h(t)\| |f(t)| \, d\mu(t) \le \int_\Omega \|h(t)[f_1(t)x_t]\| |f_2(t)| \, d\mu(t) + \epsilon \int_\Omega |f(t)| \, d\mu(t) < \infty.$$

This shows that $h \in L^r(\mu, L(X,Y))$. Since $\epsilon > 0$ was arbitrary, we also see that $(\int_\Omega \|h(t)\|^r \, d\mu(t))^{\frac{1}{r}} \le \pi_{p,q}(h)$, because if $\|f\|_{L^{r'}(\mu)} \le 1$, then $\int_\Omega \|f_1(t)x_t\|^q \, d\mu(t) \le 1$.

(ii) $h \in L^\infty(\mu, L(X,Y))$ and $g \in L^p(\mu, X)$ imply

$$(\int_\Omega \|h(t)[g(t)]\|^p \, d\mu(t))^{\frac{1}{p}} \le \|h\|_{L^\infty(\mu, L(X,Y))} (\int_\Omega \|g(t)\|^p \, d\mu(t))^{\frac{1}{p}}.$$

(iii) Let $1 \le q < p$ and let $h \in (L^q(\mu, X), L^p(\mu, Y))$. We need to show that $t \mapsto \|h(t)\|$ is in $L^\infty(\mu)$. Take any $f \in L^1(\mu)$. We may write f as $f = f_1 \cdot f_2$, where $f_1 =$

$f^{\frac{1}{q}} \in L^q(\mu)$ and $f_2 = f^{\frac{1}{q'}} \in L^{q'}(\mu)$. For $\epsilon > 0$ given, let $\|x_t\| = 1$ such that $\|h(t)\| < \|h(t)x_t\| + \epsilon$. The function $t \mapsto f_1(t)x_t$ is in $L^q(\mu, X)$. Therefore, the function $t \mapsto h(t)[f_1(t)x_t]$ is in $L^p(\mu, Y)$. However, $f_2 \in L^{q'}(\mu) \subset L^{p'}(\mu)$, i.e. the function $t \mapsto \|h(t)[f_1(t)x_t]\||f_2(t)|$ is in $L^1(\mu)$. Also,

$$\|h(t)\||f(t)| < \|h(t)x_t\||f(t)| + \epsilon|f(t)| = \|h(t)[f_1(t)x_t]\||f_2(t)| + \epsilon|f(t)|$$

for all $t \in \Omega$. It is therefore clear that $t \mapsto \|h(t)\||f(t)|$ is in $L^1(\mu)$ and

$$\int_\Omega \|h(t)\||f(t)|\ d\mu(t) \leq \int_\Omega \|h(t)[f_1(t)x_t]\||f_2(t)|\ d\mu(t) + \epsilon \int_\Omega |f(t)|\ d\mu(t) < \infty.$$

Since this holds for all $f \in L^1(\mu)$, it follows that $h \in L^\infty(\mu, L(X, Y))$.

□

In Proposition 5.12 we saw that

$$L^r(\mu, L(X, Y)) = (L^q(\mu, X), L^p(\mu, Y)) \text{ when } \frac{1}{r} = (\frac{1}{p} - \frac{1}{q})^+.$$

Also, $L_{\pi_{p,q}}(X, Y) = (L^q_w(\mu, X), L^p(\mu, Y))$. Now, $L^\infty_w(\mu, X) = L^\infty(\mu, X)$. So $L_{\pi_{p,\infty}}(X, Y) = L^p(\mu, L(X, Y))$ and

$$L_{\pi_{\infty,q}}(X, Y) = (L^q_w(\mu, X), L^\infty(\mu, Y)) \subseteq L^\infty(\mu, L(X, Y)),$$

where the last inclusion follows from:

$h \in (L^q_w(\mu, X), L^\infty(\mu, Y)) \Rightarrow F_{h,g}(t) = h(t)(g(t)) \in L^\infty(\mu, Y)$ for all $g \in L^q_w(\mu, X)$, and

$$\sup_{\|g\|_q^{weak} \leq 1} \operatorname{ess\,sup}_{t \in \Omega} \|h(t)(g(t))\| < \infty.$$

Now, for each $x \in X$ with $\|x\| \leq 1$ the mapping $g(t) = x$, $\forall t \in \Omega$, is in $L^q_w(\mu, X)$. Hence $\|h(t)(g(t))\| = \|h(t)(x)\|$ and

$$\begin{aligned} \operatorname{ess\,sup}_{t \in \Omega} \|h(t)\| &= \operatorname{ess\,sup}_{t \in \Omega} \sup_{\|x\| \leq 1} \|h(t)(x)\| \\ &\leq \operatorname{ess\,sup}_{t \in \Omega} \sup_{\|g\|_q^{weak} \leq \mu(\Omega)} \|h(t)(g(t))\| \\ &< \infty. \end{aligned}$$

5.1 Inclusions among the spaces $L_{\pi_{p,q}}(X)$

Theorem 5.13 *Let* $1 \leq r, s < \infty, 1 \leq p_1 \leq p_2$ *and* $1 \leq q_1 \leq q_2$.

Then

(i) $L_{\pi_{r,q_1}}(X,Y) \subseteq L_{\pi_{r,q_2}}(X,Y)$;

(ii) $L_{\pi_{p_2,s}}(X,Y) \subseteq L_{\pi_{p_1,s}}(X,Y)$;

with continuous inclusions of norm ≤ 1.

In particular, for $1 \leq p, q < \infty$; $L_{\pi_{p,1}}(X) \subset L_{\pi_p}(X)$ *and* $L_{\pi_1}(X) \subset L_{\pi_{1,q}}(X)$.

Proof

(i) Let $h \in L_{\pi_{r,q_1}}(X,Y)$; i.e. for $g \in L_w^{q_2}(\mu, X) \subset L_w^{q_1}(\mu, X)$ we have

$$\begin{aligned}
(\int_\Omega \|h(t)(g(t))\|^r \, d\mu(t))^{\frac{1}{r}} &\leq \pi_{r,q_1}(h) \sup_{\|x^*\|\leq 1} (\int_\Omega |x^*g(t)|^{q_1} \, d\mu(t))^{\frac{1}{q_1}} \\
&\leq \pi_{r,q_1}(h) \sup_{\|x^*\|\leq 1} (\int_\Omega |x^*g(t)|^{q_2} \, d\mu(t))^{\frac{1}{q_2}}.
\end{aligned}$$

Consider the embedding $I : L_{\pi_{r,q_1}}(X,Y) \to L_{\pi_{r,q_2}}(X,Y)$. It follows from the above inequality that $\pi_{r,q_2}(h) \leq \pi_{r,q_1}(h)$, i.e. $\|I\| \leq 1$.

(ii) Let $h \in L_{\pi_{p_2,s}}(X,Y)$. Then, for $g \in L_w^s(\mu, X)$ we have $x^*g \in L^s(\mu)$ for all $x^* \in X^*$ and

$$\begin{aligned}
(\int_\Omega \|h(t)(g(t))\|^{p_1} \, d\mu(t))^{\frac{1}{p_1}} &\leq (\int_\Omega \|h(t)(g(t))\|^{p_2} \, d\mu(t))^{\frac{1}{p_2}} \\
&\leq \pi_{p_2,s}(h) \sup_{\|x^*\|\leq 1} (\int_\Omega |x^*g(t)|^s \, d\mu(t))^{\frac{1}{s}}.
\end{aligned}$$

Hence, $h \in L_{\pi_{p_1,s}}(X,Y)$. As in (i) the inclusion $I : L_{\pi_{p_2,s}}(X,Y) \to L_{\pi_{p_1,s}}(X,Y)$ has norm ≤ 1.

□

The following theorem gives the connection between the strongly p-integral functions and the $(1,p')$-integral functions in X. From this connection it follows that $(L^p\langle\mu, X\rangle, \|h\|_{L^p\langle\mu,X\rangle})$ is a normed space.

Theorem 5.14 *Let* $1 \leq p \leq \infty$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then* $L^p\langle\mu, X\rangle = L_{\pi_{1,p'}}(X)$ *and* $\|h\|_{L^p\langle\mu,X\rangle} = \pi_{1,p'}(h)$.

Proof By definition we have

$$\begin{aligned} L^p\langle\mu, X\rangle &= L_{\pi_{1,p'}}(X^*, \mathbb{K}) \cap L^1(\mu, X) \\ &= L_{\pi_{1,p'}}(X). \end{aligned}$$

For $h \in L^p\langle\mu, X\rangle$ it follows that

$$\int_\Omega |\langle h(t), g(t)\rangle| \, d\mu(t) \leq \|g\|_{p'}^{weak} \|h\|_{L^p\langle\mu,X\rangle}.$$

$$\therefore \quad \pi_{1,p'}(h) \leq \|h\|_{L^p\langle\mu,X\rangle}.$$

Let $\pi_{1,p'}(h) \leq 1$. Then

$$\|h\|_{L^p\langle\mu,X\rangle} = \sup_{\|g\|_{p'}^{weak} \leq 1} \int_\Omega |\langle h(t), g(t)\rangle| \, d\mu(t) \leq \sup_{\|g\|_{p'}^{weak} \leq 1} \|g\|_{p'}^{weak} \leq 1.$$

$$\therefore \quad \|h\|_{L^p\langle\mu,X\rangle} \leq \pi_{1,p'}(h).$$

□

Theorem 5.15 *Let* X *be a Banach space,* $1 \leq p < q$ *and* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. *Then we have the following norm* ≤ 1 *inclusions:*

$$L^r\langle\mu, X\rangle \overset{(1)}{\subseteq} L_{\pi_{p,q}}(X) \overset{(2)}{\subseteq} L^r(\mu, X).$$

Proof

(1) We prove the inclusion $L^r\langle\mu, X\rangle \subseteq L_{\pi_{p,q}}(X)$:
Let $h \in L^r\langle\mu, X\rangle = L_{\pi_{1,r'}}(X)$ and let $g \in L^q_w(\mu, X^*)$ be given. For all $f \in L^{p'}(\mu)$, consider the functions $t \mapsto f(t)g(t)$. We show that they belong to $L^{r'}_w(\mu, X^*)$. To do so, let $x \in X$ and consider

$$\left(\int_\Omega |\langle x, f(t)g(t)\rangle|^{r'} \, d\mu(t)\right)^{\frac{1}{r'}} \leq \left(\int_\Omega |f(t)|^{p'} \, d\mu(t)\right)^{\frac{1}{p'}} \left(\int_\Omega |\langle x, g(t)\rangle|^q \, d\mu\right)^{\frac{1}{q}} < \infty,$$

because $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q}$. From our assumption it follows that $t \mapsto h(t)[f(t)g(t)]$ is in $L^1(\mu)$. Since this is true for all $f \in L^{p'}(\mu)$, it follows that the function $t \mapsto h(t)[g(t)]$ is in $L^p(\mu)$. This holds for all $g \in L^q_w(\mu, X^*)$, so that $h \in L_{\pi_{p,q}}(X)$ is our conclusion.

(2) The inclusion $L_{\pi_{p,q}}(X) \subseteq L^r(\mu, X)$ follows from

$$\begin{aligned}
L_{\pi_{p,q}}(X) &= L_{\pi_{p,q}}(X^*, \mathbb{K}) \cap L^1(\mu, X) \\
&= (L^q_w(\mu, X^*), L^p(\mu)) \cap L^1(\mu, X) \\
&\subseteq (L^q(\mu, X^*), L^p(\mu)) \cap L^1(\mu, X) \\
&= L^r(\mu, L(X^*, \mathbb{K})) \cap L^1(\mu, X) \\
&= L^r(\mu, X^{**}) \cap L^1(\mu, X) \\
&= L^r(\mu, X).
\end{aligned}$$

□

Remark 5.16 *It follows that $L^p\langle\mu, X\rangle = L_{\pi_{1,p'}}(X) \subseteq L^p(\mu, X)$, from which the converse inequality in Lemma 5.2 follows.*

Notation

X, Y	Banach spaces.
B_X	The closed unit ball in X.
$\mathcal{B}(X;Y)$	The space of bounded bilinear maps between X and Y.
c_0	The space of all null sequences.
$C_p(X,Y)$	The space of operators of cotype p.
$\mathcal{F}(X,Y)$	The space of all finite rank bounded linear operators.
$\mathcal{I}(X,Y)$	The space of all integral operators between X and Y.
$\mathbb{K}$	Denotes $\mathbb{R}$ or $\mathbb{C}$ if no difference is relevant.
$K(X,Y)$	The space of all compact linear operators between X and Y.
$L(X,Y)$	The space of all bounded linear operators between X and Y.
$L^p(\mu, X)$	The space of equivalence classes of X-valued Bochner integrable functions.
$L^p(\mu)$	The space $L^p(\mu, \mathbb{K})$.
$L^p(0,1)$	The space of equivalence classes of Lebesgue integrable functions on $[0,1]$.
$L^p((0,1), X)$	The space of equivalence classes of Lebesgue measurable X-valued functions.
ℓ^∞	The space of all bounded sequences.
ℓ^p	The space of p-absolutely summable scalar sequences.

ℓ_n^p	The n-dimensional Euclidean (real or complex) space.
$\ell^\infty(L(X,Y))$	$\{(u_n) \subset L(X,Y) \ : \ \sup_n \|u_n\| < \infty\}$
$\ell_w^p(X)$	The space of weakly p-summable sequences in X.
$\ell_c^1(X)$	The space of unconditionally summable sequences in X.
$\ell^p\langle X\rangle$	The space of strongly p-summable sequences in X.
$\ell^p(X)$	The space of absolutely p-summable sequences in X.
$\ell_{\pi_{p,q}}(X,Y)$	The space of (p,q)-summing multipliers.
$\ell_{\pi_{p,q}}(X)$	The space of (p,q)-summing sequences in X.
$m_p(X)$	The space of all p-summing multipliers in X.
$m_p^{strong}(X)$	The space of all strongly p-summing multipliers in X.
$M(X)$	The space of all $\mathbb{K}$ regular Borel measures on X.
$N_p(X,Y)$	The space of all p-nuclear operators between X and Y.
$Rad(X)$	Almost unconditionally summable sequences in X.
$E(X)$	The space of all sequences with values in X.
$T_p(X,Y)$	The space of operators of type p.
U	Reflexive Banach space with a normalized unconditional basis.
$\Pi_{as}(X,Y)$	The space of all almost summing operators between X and Y.
$\Pi_{p,q}(X,Y)$	The space of all (p,q)-summing operators between X and Y.
$(E(X),F(Y))$	The set of all $(E(X),F(Y))$-multiplier sequences.
Λ	A vector sequence space whose elements are sequences (α_n) of numbers.
$\Lambda(X)$	A vector sequence space whose elements are sequences $(x_n) \subset X$.

$\Lambda^{\times}$	The Köthe dual of the sequence space Λ.
X'	The algebraic dual space of X.
X^*	The continuous dual space of X.
w	The vector space of all (complex and real) scalar sequences.
$X^{\mathbb{N}}$	The set of all functions from $\mathbb{N}$ into X; i.e. all sequences in X.

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