Exact solutions and conservation laws of two-wave mode Kadomtsev-Petviashvili equation and a nonintegrable nonlinear equation

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Declaration

I THABO SYLVESTER MORETLO student number 16167562, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Mr T. S. MORETLO

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Master of Science degree rules and regulations have been fulfilled.

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PROF A. R. ADEM

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Declaration of Publications

Details of contribution to publications that form part of this dissertation.

TS Moretlo, B Muatjetjeja, AR Adem, Conservation laws and exact solutions for a generalized two-wave mode Kadomtsev-Petviashvili equation. Submitted for publication to Applied Mathematics and Computation.

B Muatjetjeja, TS Moretlo, AR Adem, Conservation laws and exact solutions for a generalized nonintegrable nonlinear equation. To be submitted.

Dedication

To everyone who gave me support throughout my studies.

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List of Acronyms

DE:	Differential equation
PDE:	Partial differential equation
NLPDE:	Nonlinear partial differential equation
NLEE:	Nonlinear evolution equation
KdV:	Korteweg-de Vries
TKdV:	Two wave mode Korteweg-de Vries
KP:	Kadomtsev-Petviashvili
TKP:	Two wave mode Kadomtsev-Petviashvili

Abstract

In this dissertation we study three nonlinear partial differential equations namely the Korteweg-de Vries (KdV) equation as an illustrative example, nonintegrable nonlinear equation and two-wave mode Kadomtsev-Petviashvili (TKP) equation. We employ the multiplier method to find conservation laws. We also use Kudryashov and simplest equation methods to obtain exact solutions of a generalized two nonlinear partial differential equations.

Introduction

Nonlinear partial differential equations (NLPDEs) have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics and geochemistry. Thus, it is important to investigate the exact solutions of NLPDEs. In general it is not easy to determine exact solutions of these equations and rarely can one write down their solutions explicitly.

Conservation laws [1–3] have been used to determine the existence, uniqueness, stability of solutions and exact solutions of DEs. Steudel [3] introduced a different approach of constructing conservation laws, that involves writing a conserved vector in a characteristic form, where the characteristics are the multipliers of the differential equation. Conservation laws play a remarkable role in the study of differential equations. The mathematical idea of conservation laws comes from the formulation of well-known physical conserved quantities such as mass, momentum and energy. Finding the conservation laws of differential equations is often the first step towards finding the exact solutions. Thus, it is essential to study conservation laws of partial differential equations (PDEs).

In the last few decades, a variety of effective methods for finding exact solutions, such as homogeneous balance method [4], the ansatz method [5, 6], variable separation approach [7], inverse scattering transform method [8], Backlund transformation [9], Darboux transformation [10] and Hirota's bilinear method [11] were successfully applied to NLPDEs. The Kudryashov method was one of the methods for finding exact solutions of nonlinear partial differential equations [12].

In this dissertation we consider the Korteweg-de Vries (KdV) equation as an illustrative example, a generalized nonintegrable nonlinear equation and a two-wave mode Kadomtsev-Petviashvili equation. Firstly, we consider the Korteweg-de Vries equation of the form

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

as an illustrative example.

Secondly, we study a generalized nonintegrable nonlinear equation [13] that is given by

$$\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx} = 0, \tag{2}$$

where $u(t,x)$ is a real valued function and ρ, μ, λ, η and Q are arbitrary constants. Equation (2) models the pressure waves in a mixture of a liquid and gas bubbles by taking into account the viscosity of the liquid and the heat transfer.

Lastly, we consider a two-wave mode Kadomtsev-Petviashvili equation [14]

$$\begin{aligned} &u_{ttx} - s^2 u_{xxx} + 6(u_{tx}u_{xx} + u_x u_{xxt}) - \alpha s(6u_{xx}^2 + 6u_x u_{xxx}) + u_{xxxxt} + u_{tyy} \\ &-\beta s(u_{xxxxx} + u_{xyy}) = 0, \end{aligned} \tag{3}$$

where $u = u(t,x,y)$ denotes the wave profile.

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the multiplier method and an illustrative example using multiplier approach to construct conservation laws of the Korteweg-de Vries equation are presented.

In Chapter two, the multiplier method is used to construct conservation laws of a generalized nonintegrable nonlinear equation. Moreover, exact solutions of a generalized nonintegrable nonlinear equation are obtained with the aid of the simplest equation method [12].

In Chapter three, the conservation laws of a two-wave mode Kadomtsev-Petviashvili equation are obtained using multiplier method. Thereafter, we construct the exact solutions of a two-wave mode Kadomtsev-Petviashvili equation using Kudryashov method.

In Chapter four, a summary of the results of the dissertation is presented.

A bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter, we present some basic methods on how to obtain conservation laws and Lie point symmetries of differential equations, which will be utilized in this dissertation. We further derive the conservation laws for the Korteweg-de Vries equation via multiplier approach.

1.1 Fundamental relation on multiplier and Lie point symmetries

In this section, we present the notation that will be used to construct conservation laws of (1) by the multiplier method [15]. We will also give some basic definitions on Lie point symmetries [16, 17].

Consider a kth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the *total* *derivative operator* with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \qquad i = 1, \dots, n. \tag{1.2}$$

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \qquad \alpha = 1, \dots, m. \tag{1.3}$$

The n-tuple vector $\mathbf{T} = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (1.1) if T^i satisfies

$$D_i T^i|_{(1.1)} = 0. \tag{1.4}$$

The equation (1.4) defines a *local conservation law* of system (1.1).

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property

$$\Lambda_\alpha E_\alpha = D_i T^i, \tag{1.5}$$

that holds identically. Here we will consider multipliers of the zeroth order, i.e., $\Lambda_\alpha = \Lambda_\alpha(t, x, u)$. The right hand side of (1.5) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.6}$$

Definition 1.1 (Point symmetry) The vector field

$$\mathbf{X} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.7}$$

is said to be a *point symmetry* of the kth-order partial differential equation (1.1), if

$$\mathbf{X}^{[k]}(E_\alpha) = 0, \tag{1.8}$$

whenever $E_\alpha = 0$. This can also be written as

$$\mathbf{X}^{[k]}\, E_\alpha\big|_{E_\alpha = 0} = 0, \tag{1.9}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.2 (Generator) Consider the vector field

$$\mathbf{X} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{1.10}$$

which has the second-order prolongation

$$\mathbf{X}^{[2]} = \mathbf{X} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}, \tag{1.11}$$

where

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{tt} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),\\
\zeta_{tx} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),\\
\zeta_{xx} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi).
\end{aligned}$$

and

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots,
\end{aligned}$$

1.2 An illustrative example on finding conservation laws via multiplier approach

In this section we will employ the multiplier method to construct the first-order multiplier of the Korteweg-de Vries equation. Thereafter, we will derive the associated conservation laws of the KdV equation.

Consider the Korteweg-de Vries equation of the form

$$u_t + uu_x + u_{xxx} = 0 \tag{1.12}$$

and take the first order multiplier $\Lambda(t, x, u, u_t, u_x)$. Then from equation (1.6), we have

$$\frac{\delta}{\delta u}[\Lambda(t, x, u, u_t, u_x)(u_t + uu_x + u_{xxx})] = 0, \tag{1.13}$$

where the Eulér-Langrage operator $\delta/\delta u$ is defined by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} \\ &\quad + D_x D_t\frac{\partial}{\partial u_{xt}} + \cdots \end{aligned} \tag{1.14}$$

and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots.$$

Expanding equation (1.13) leads to

$$\begin{aligned} &\left[\frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} - D_t D_x^2\frac{\partial}{\partial u_{txx}} + D_x^4\frac{\partial}{\partial u_{xxxx}}\right] \\ &(\Lambda(u_t + uu_x + u_{xxx})) = 0, \\ &\Lambda_u(u_t + uu_x + u_{xxx}) + u_x\Lambda - D_t(\Lambda_{u_t}(u_t + uu_x + u_{xxx}) + \Lambda) \\ &-D_x(\Lambda_{u_x}(u_t + uu_x + u_{xxx}) + u\Lambda) - D_x^3(\Lambda) = 0. \end{aligned}$$

Since Λ depends only on t, x, u, u_t and u_x, the coefficients of the like derivatives can be equated to zero to yield the following system of overdetermined linear partial differential equations:

$$\Lambda_{u_t} = 0, \tag{1.15}$$

$$\Lambda_{u_x} = 0, \tag{1.16}$$

$$\Lambda_t = -u\Lambda_x, \tag{1.17}$$

$$\Lambda_{uu} = 0, \tag{1.18}$$

$$\Lambda_{ux} = 0, \tag{1.19}$$

$$\Lambda_{xx} = 0. \tag{1.20}$$

Integrating equations (1.15) and (1.16) with respect to u_t and u_x, respectively gives

$$\Lambda = b(t, x, u). \tag{1.21}$$

Differentiating equation (1.21) with respect to u twice, we obtain

$$\Lambda_{uu} = b_{uu}. \tag{1.22}$$

Then equation (1.18) implies that

$$b_{uu} = 0. \tag{1.23}$$

Integrating (1.23) twice with respect to u, we obtain

$$b = uc(t, x) + d(t, x). \tag{1.24}$$

Then equation (1.21) becomes

$$\Lambda = uc(t, x) + d(t, x). \tag{1.25}$$

Again differentiating equation (1.25) twice with respect to x, we obtain

$$\Lambda_{xx} = uc_{xx} + d_{xx}. \tag{1.26}$$

Now using equation (1.20), we get

$$uc_{xx} + d_{xx} = 0. \tag{1.27}$$

Splitting the above equation with respect to u yields

$$u \ : \ c_{xx} = 0, \tag{1.28}$$

$$1 \ : \ d_{xx} = 0. \tag{1.29}$$

Integrating equation (1.28) and (1.29) respectively, with respect to x gives

$$c(t, x) = xe(t) + f(t), \tag{1.30}$$

and

$$d(t, x) = xg(t) + h(t). \tag{1.31}$$

Therefore equation (1.25) yields the following

$$\Lambda = xue(t) + uf(t) + xg(t) + h(t). \tag{1.32}$$

Again by differentiating equation (1.32) with respect to t and x, we obtain

$$\Lambda_t = xue_t + uf_t + xg_t + h_t, \tag{1.33}$$

$$\Lambda_x = ue + g. \tag{1.34}$$

Substituting equation (1.33) and (1.34) into (1.17), yields

$$xue_t + uf_t + xg_t + h_t + u^2e + ug = 0. \tag{1.35}$$

Splitting the above equation with respect to the powers of u, we get

$$u^2 \;:\; e = 0, \tag{1.36}$$

$$u \;:\; xe_t + f_t + g = 0, \tag{1.37}$$

$$1 \;:\; xg_t + h_t = 0. \tag{1.38}$$

Again, splitting (1.37) and (1.38) respectively with respect to x, gives

$$x \;:\; e_t = 0, \tag{1.39}$$

$$1 \;:\; f_t + g = 0, \tag{1.40}$$

and

$$x \;:\; g_t = 0, \tag{1.41}$$

$$1 \;:\; h_t = 0. \tag{1.42}$$

Now, simplifying the above equations, yields

$$g = k_1, \tag{1.43}$$

$$f = -tk_1 + k_2, \tag{1.44}$$

$$h = k_3. \tag{1.45}$$

Substituting (1.43), (1.44) and (1.45) into (1.32), we get the following multiplier

$$\Lambda = (x - ut)k_1 + uk_2 + k_3. \tag{1.46}$$

We now recall equation (1.5), viz.,

$$\Lambda_\alpha E_\alpha = D_i T^i, \tag{1.47}$$

to construct the conservation laws of equation (1.12). Employing the above equation (1.47), we get

$$\begin{aligned}
&[xk_1 - tuk_1 + uk_2 + k_3](u_t + uu_x + u_{xxx})\\
&= \left[\frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{txx}\frac{\partial}{\partial u_{xx}} + u_{ttx}\frac{\partial}{\partial u_{tx}}\right] T^1(t, x, u, u_x, u_{xx})\\
&+ \left[\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{txx}\frac{\partial}{\partial u_{tx}}\right] T^2(t, x, u, u_x, u_{xx}).
\end{aligned}$$

Simplifying the above equation gives

$$\begin{aligned}
&[xk_1 - tuk_1 + uk_2 + k_3](u_t + uu_x + u_{xxx}) =\\
&T^1_t + u_t T^1_u + u_{tx} T^1_{u_x} + u_{txx} T^1_{u_{xx}} + T^2_x + u_x T^2_u + u_{xx} T^2_{u_x} + u_{xxx} T^2_{u_{xx}}.
\end{aligned} \tag{1.48}$$

Splitting equation (1.48) on u_t, u_{tx}, u_{txx} and u_{xxx} yields

$$u_t \ : \ T^1_u = xk_1 - tuk_1 + uk_2 + k_3, \tag{1.49}$$

$$u_{tx} \ : \ T^1_{u_x} = 0, \tag{1.50}$$

$$u_{txx} \ : \ T^1_{u_{xx}} = 0, \tag{1.51}$$

$$u_{xxx} \ : \ T^2_{u_{xx}} = xk_1 - tuk_1 + uk_2 + k_3, \tag{1.52}$$

$$1 \ : \ T^1_t + T^2_x + u_x T^2_u + u_{xx} T^2_{u_x} = k_1 uu_x(x - tu) + uu_x(uk_2 + k_3). \tag{1.53}$$

We can now solve the above equations for T^1 and T^2. From equation (1.49), we obtain

$$T^1(t, x, u, u_x, u_{xx}) = xuk_1 - \frac{tu^2k_1}{2} + \frac{u^2k_2}{2} + uk_3 + A(t, x, u_x, u_{xx}), \tag{1.54}$$

where $A(t, x, u_x, u_{xx})$ is an arbitrary function of t, x, u_x and u_{xx}. Using equation (1.50) and equation (1.54), we get

$$T^1(t, x, u, u_x, u_{xx}) = xuk_1 - \frac{tu^2k_1}{2} + \frac{u^2k_2}{2} + uk_3 + B(t, x, u_{xx}), \tag{1.55}$$

where $B(t, x, u_{xx})$ is an arbitrary function of t, x and u_{xx}. Differentiating equation (1.55) with respect to u_{xx}, gives

$$T^1_{u_{xx}} = B_{u_{xx}}. \tag{1.56}$$

Equating the above equation to zero using equation (1.51), we obtain

$$B_{u_{xx}} = 0. \tag{1.57}$$

Integrating equation (1.57) with respect to u_{xx}, yields

$$B = C(t, x). \tag{1.58}$$

Thus equation (1.55) becomes

$$T^1(t, x, u, u_x, u_{xx}) = xuk_1 - \frac{tu^2k_1}{2} + \frac{u^2k_2}{2} + uk_3 + C(t, x), \tag{1.59}$$

where $C(t, x)$ is an arbitrary function of t and x only. The integration of equation (1.52) with respect to u_{xx}, yields

$$T^2(t, x, u, u_x, u_{xx}) = xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 + D(t, x, u, u_x), \tag{1.60}$$

where $D(t, x, u, u_x)$ is an arbitrary function of t, x, u and u_x. Differentiating equation (1.59) with respect to t and equation (1.60) with respect to x, u and u_x respectively and substituting the resulting derivatives into equation (1.53), we obtain

$$\begin{aligned}&-\frac{u^2k_1}{2} + C_t + u_{xx}k_1 + D_x - tu_xu_{xx}k_1 + u_xu_{xx}k_2 + u_xD_u + u_{xx}D_{u_x} = xuu_xk_1 \\ &-tu^2u_xk_1 + u^2u_xk_2 + uu_xk_3.\end{aligned} \tag{1.61}$$

Splitting the above equation with respect to u_{xx}, yields

$$u_{xx} \ : \ k_1 - tu_xk_1 + u_xk_2 + D_{u_x} = 0, \tag{1.62}$$

$$1 \ : \ -\frac{u^2k_1}{2} + C_t + D_x + u_xD_u = k_1uu_x(x - tu) + uu_x(uk_2 + k_3). \tag{1.63}$$

Equation (1.62), implies that

$$D = -u_xk_1 + \frac{tu_x^2k_1}{2} - \frac{u_x^2k_2}{2} + E(t, x, u). \tag{1.64}$$

Since $D_x = E_x$ and $D_u = E_u$, then equation (1.63), implies that

$$\begin{aligned} C(t,x) &= xtuu_x k_1 - \frac{t^2u^2u_xk_1}{2} + tu^2u_xk_2 + tuu_xk_3 + \frac{tu^2k_1}{2} \\ &\quad - \int E_x dt - u_x \int E_u dt. \end{aligned} \tag{1.65}$$

Therefore equations (1.59) and (1.60) reduce to

$$\begin{aligned} T^1(t,x,u,u_x,u_{xx}) &= xuk_1 + \frac{u^2k_2}{2} + uk_3 + xtuu_xk_1 - \frac{t^2u^2u_xk_1}{2} \\ &\quad + tu^2u_xk_2 + tuu_xk_3 - \int E_x dt - u_x \int E_u dt, \qquad (1.66) \\ T^2(t,x,u,u_x,u_{xx}) &= xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 - u_xk_1 \\ &\quad + \frac{tu_x^2k_1}{2} - \frac{u_x^2k_2}{2} + E. \qquad (1.67) \end{aligned}$$

Now differentiating equation (1.66) with respect to u_x and using equation (1.50), we obtain

$$\int E_u dt = txuk_1 - \frac{t^2u^2k_1}{2} + tu^2k_2 + tuk_3. \tag{1.68}$$

Simplifying equation (1.68), yields

$$E(t,x,u) = \frac{xu^2k_1}{2} - \frac{tu^3k_1}{3} + \frac{u^3k_2}{3} + \frac{u^2k_3}{2} + F(t,x). \tag{1.69}$$

Now we have

$$\begin{aligned} E_x &= \frac{u^2k_1}{2} + F_x, \qquad (1.70) \\ \int E_u dt &= txuk_1 - \frac{t^2u^2k_1}{2} + tu^2k_2 + tuk_3, \qquad (1.71) \\ \int E_x dt &= \frac{tu^2k_1}{2} + \int F_x dt + G(x,u). \qquad (1.72) \end{aligned}$$

Back substitution of equations (1.69)-(1.72) into (1.66) and (1.67) respectively, we obtain

$$\begin{aligned} T^1(t,x,u,u_x,u_{xx}) &= xuk_1 + \frac{u^2k_2}{2} + uk_3 - \frac{tu^2k_1}{2} - \int F_x dt - G, \qquad (1.73) \\ T^2(t,x,u,u_x,u_{xx}) &= xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 - u_xk_1 + \frac{tu_x^2k_1}{2} \\ &\quad - \frac{u_x^2k_2}{2} + \frac{xu^2k_1}{2} - \frac{tu^3k_1}{3} + \frac{u^3k_2}{3} + \frac{u^2k_3}{2} + F. \qquad (1.74) \end{aligned}$$

Differentiating equation (1.73) with respect to u, gives

$$T_u^1 = xk_1 + uk_2 + k_3 - tuk_1 - G_u. \tag{1.75}$$

Using equation (1.49), equation (1.75) can be written as

$$xk_1 + uk_2 + k_3 - tuk_1 - G_u = xk_1 - tuk_1 + uk_2 + k_3. \tag{1.76}$$

Now equation (1.76), becomes

$$G_u = 0. \tag{1.77}$$

Equation (1.77) implies that

$$G = I(x). \tag{1.78}$$

Then equations (1.73) and (1.74), becomes

$$T^1 = xuk_1 + \frac{u^2k_2}{2} + uk_3 - \frac{tu^2k_1}{2} - \int F_x dt - I, \tag{1.79}$$

$$\begin{aligned} T^2 &= xu_{xx}k_1 - tuu_{xx}k_1 + uu_{xx}k_2 + u_{xx}k_3 - u_xk_1 + \frac{tu_x^2k_1}{2} \\ &\quad - \frac{u_x^2k_2}{2} + \frac{xu^2k_1}{2} - \frac{tu^3k_1}{3} + \frac{u^3k_2}{3} + \frac{u^2k_3}{2} + F. \end{aligned} \tag{1.80}$$

Therefore the components of the conserved vectors associated with the above multiplier (1.46) are

$$\begin{aligned} T_1^t &= xu - \frac{tu^2}{2}, \\ T_1^x &= xu_{xx} - tuu_{xx} - u_x + \frac{tu_x^2}{2} + \frac{xu^2}{2} - \frac{tu^3}{3}, \\ T_2^t &= \frac{u^2}{2}, \\ T_2^x &= uu_{xx} - \frac{u_x^2}{2} + \frac{u^3}{3}, \\ T_3^t &= u, \\ T_3^x &= u_{xx} + \frac{u^2}{2}, \end{aligned}$$

respectively.

Chapter 2

Conservation laws and exact solutions of a generalized nonintegrable nonlinear equation

It is known that particular kink-shaped wave solutions to nonlinear nonintegrable equations may be employed to account for important features of the kink evolution observed in numerical solutions and to check the last solutions without computations. Thus, an exact travelling wave solution may predict boundary conditions suitable for the kink realization in numerics.

Thus, in this chapter, we consider a generalized nonintegrable nonlinear equation [13] given by

$$\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx} = 0, \tag{2.1}$$

where ρ, μ, λ, η and Q are arbitrary constants. The above equation arises to account for shear strain waves $u(x,t)$ in a growth plate of a long bone. Equation (2.1) possesses nonlinear, dispersive and dissipative terms as well as the term responsible for inhomogeneity. Here we will use the multiplier method to construct conservation laws of equation (2.1). Thereafter, we will employ the simplest equation method to obtain exact solutions of equation (2.1).

2.1 Conservation laws of a generalized noninte-grable nonlinear equation (2.1)

In this section we construct the conservation laws of equation (2.1). We will consider multipliers of the second order $\Lambda(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx})$. Thus equation (1.6) becomes

$$\frac{\delta}{\delta u}[\Lambda(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx})(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx})] = 0, \tag{2.2}$$

where the Euler-Langrage Operator $\delta/\delta u$ is given by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} \\ &+ D_x D_t \frac{\partial}{\partial u_{xt}} + \cdots \end{aligned} \tag{2.3}$$

and the total differential operators are defined by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots.$$

The expansion of equation (2.2) gives

$$\begin{aligned} &\left[\frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_t D_x^2 \frac{\partial}{\partial u_{txx}} + D_x^4 \frac{\partial}{\partial u_{xxxx}}\right] \\ &\big(\Lambda(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx})\big) = 0, \\ &\Lambda_u(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx}) - 6\lambda\Lambda(u_x^2 + u u_{xx}) \\ &- D_t(\Lambda_{u_t}(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx})) \\ &- D_x(\Lambda_{u_x}(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx}) - 12\lambda u u_x \Lambda) \\ &+ D_t^2(\Lambda_{u_{tt}}(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx}) + \rho\Lambda) \\ &+ D_x^2(\Lambda_{u_{xx}}(\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx}) + \Lambda(\mu + 3\lambda u^2)) \\ &+ D_t D_x^2(\eta\Lambda) - D_x^4(\Lambda) = 0. \end{aligned}$$

Since Λ depends only on $t, x, u, u_t, u_x, u_{xx}, u_{tt}$ and u_{tx}, we then equate the coefficients of the like derivatives to zero to obtain the following system of overdetermined linear partial differential equations:

$$\Lambda_u = 0, \tag{2.4}$$

$$\Lambda_{u_t} = 0, \tag{2.5}$$

$$\Lambda_{u_{tt}} = 0, \tag{2.6}$$

$$\Lambda_{u_{tx}} = 0, \tag{2.7}$$

$$\Lambda_{u_x} = 0, \tag{2.8}$$

$$\Lambda_{u_{xx}} = 0, \tag{2.9}$$

$$\Lambda_{tt} = 0, \tag{2.10}$$

$$\Lambda_{xx} = 0. \tag{2.11}$$

Integrating equation (2.11) twice, with respect to x we obtain

$$\Lambda = b(t, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx})x + c(t, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}). \tag{2.12}$$

Differentiating equation (2.12) with respect to t twice, we obtain

$$\Lambda_{tt} = b_{tt}x + c_{tt}. \tag{2.13}$$

Then equation (2.10) implies that

$$b_{tt}x + c_{tt} = 0. \tag{2.14}$$

Splitting the above equation with respect to x yields

$$x \; : \; b_{tt} = 0, \tag{2.15}$$

$$1 \; : \; c_{tt} = 0. \tag{2.16}$$

Integrating equation (2.15) and (2.16) respectively, with respect to t gives

$$b(t, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = d(u_x, u_t, u_{xx}, u_{tt}, u_{tx})t + e(u_x, u_t, u_{xx}, u_{tt}, u_{tx}) \tag{2.17}$$

and

$$c(t, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = f(u_x, u_t, u_{xx}, u_{tt}, u_{tx})t + g(u_x, u_t, u_{xx}, u_{tt}, u_{tx}). \tag{2.18}$$

Now equation (2.12) becomes

$$\Lambda = dtx + ex + ft + g. \tag{2.19}$$

Differentiating equation (2.19) with respect to u_t and using equation (2.5), we get

$$d_{u_t}tx + e_{u_t}x + f_{u_t}t + g_{u_t} = 0. \tag{2.20}$$

Splitting the above equation with respect to x yields

$$x \ : \ d_{u_t}t + e_{u_t} = 0, \tag{2.21}$$

$$1 \ : \ f_{u_t}t + g_{u_t} = 0. \tag{2.22}$$

Again splitting equation (2.21), with respect to t gives

$$t \ : \ d_{u_t} = 0, \tag{2.23}$$

$$1 \ : \ e_{u_t} = 0. \tag{2.24}$$

Also splitting equation (2.22), with respect to t, we obtain

$$t \ : \ f_{u_t} = 0, \tag{2.25}$$

$$1 \ : \ g_{u_t} = 0. \tag{2.26}$$

Simplifying (2.23), (2.24), (2.25) and (2.26) respectively, we get

$$d = h(u_x, u_{xx}, u_{tt}, u_{tx}), \tag{2.27}$$

$$e = i(u_x, u_{xx}, u_{tt}, u_{tx}), \tag{2.28}$$

$$f = j(u_x, u_{xx}, u_{tt}, u_{tx}), \tag{2.29}$$

$$g = k(u_x, u_{xx}, u_{tt}, u_{tx}). \tag{2.30}$$

Now equation (2.19) reduces to

$$\Lambda = htx + ix + jt + k. \tag{2.31}$$

Differentiating equation (2.31) with respect to u_x and using equation (2.8) gives

$$h_{u_x}tx + i_{u_x}x + j_{u_x}t + k_{u_x} = 0. \tag{2.32}$$

Splitting the above equation with respect to x yields

$$x \; : \; h_{u_x}t + i_{u_x} = 0, \tag{2.33}$$

$$1 \; : \; j_{u_x}t + k_{u_x} = 0. \tag{2.34}$$

The separation of equation (2.33), with respect to t gives

$$t \; : \; h_{u_x} = 0, \tag{2.35}$$

$$1 \; : \; i_{u_x} = 0. \tag{2.36}$$

Again, splitting equation (2.34), with respect to t gives

$$t \; : \; j_{u_x} = 0, \tag{2.37}$$

$$1 \; : \; k_{u_x} = 0. \tag{2.38}$$

Simplifying (2.35), (2.36), (2.37) and (2.38) respectively, we obtain

$$h = l(u_{xx}, u_{tt}, u_{tx}), \tag{2.39}$$

$$i = m(u_{xx}, u_{tt}, u_{tx}), \tag{2.40}$$

$$j = n(u_{xx}, u_{tt}, u_{tx}), \tag{2.41}$$

$$k = o(u_{xx}, u_{tt}, u_{tx}). \tag{2.42}$$

Now equation (2.31) becomes

$$\Lambda = ltx + mx + nt + o. \tag{2.43}$$

Differentiating equation (2.43) with respect to u_{tt} and invoking equation (2.6) gives

$$l_{u_{tt}}tx + m_{u_{tt}}x + n_{u_{tt}}t + o_{u_{tt}} = 0. \tag{2.44}$$

Splitting the above equation with respect to x yields

$$x \; : \; l_{u_{tt}}t + m_{u_{tt}} = 0, \tag{2.45}$$

$$1 \; : \; n_{u_{tt}}t + o_{u_{tt}} = 0. \tag{2.46}$$

The separation of equation (2.45), with respect to t gives

$$t \; : \; l_{u_{tt}} = 0, \tag{2.47}$$

$$1 \; : \; m_{u_{tt}} = 0. \tag{2.48}$$

Again, splitting equation (2.46), with respect to t yields

$$t \;:\; n_{u_{tt}} = 0, \tag{2.49}$$

$$1 \;:\; o_{u_{tt}} = 0. \tag{2.50}$$

Simplifying (2.47), (2.48), (2.49) and (2.50) respectively, we obtain

$$l = q(u_{xx}, u_{tx}), \tag{2.51}$$

$$m = r(u_{xx}, u_{tx}), \tag{2.52}$$

$$n = s(u_{xx}, u_{tx}), \tag{2.53}$$

$$o = v(u_{xx}, u_{tx}). \tag{2.54}$$

Now equation (2.43) becomes

$$\Lambda = qtx + rx + st + v. \tag{2.55}$$

The differentiation of equation (2.55) with respect to u_{xx} and using equation (2.9) gives

$$q_{u_{xx}}tx + r_{u_{xx}}x + s_{u_{xx}}t + v_{u_{xx}} = 0. \tag{2.56}$$

Splitting the above equation with respect to x yields

$$x \;:\; q_{u_{xx}}t + r_{u_{xx}} = 0, \tag{2.57}$$

$$1 \;:\; s_{u_{xx}}t + v_{u_{xx}} = 0. \tag{2.58}$$

The separation of equation (2.57), with respect to t gives

$$t \;:\; q_{u_{xx}} = 0, \tag{2.59}$$

$$1 \;:\; r_{u_{xx}} = 0. \tag{2.60}$$

Again, splitting equation (2.58), with respect to t implies

$$t \;:\; s_{u_{xx}} = 0, \tag{2.61}$$

$$1 \;:\; v_{u_{xx}} = 0. \tag{2.62}$$

Simplifying (2.59), (2.60), (2.61) and (2.62) respectively, we obtain

$$q = w(u_{tx}), \quad (2.63)$$

$$r = y(u_{tx}), \quad (2.64)$$

$$s = z(u_{tx}), \quad (2.65)$$

$$v = \beta(u_{tx}). \quad (2.66)$$

Now equation (2.55) becomes

$$\Lambda = wtx + yx + zt + \beta. \quad (2.67)$$

Differentiating equation (2.67) with respect to u_{tx} and employing equation (2.7) gives

$$w_{u_{tx}}tx + y_{u_{tx}}x + z_{u_{tx}}t + \beta_{u_{xx}} = 0. \quad (2.68)$$

Splitting the above equation with respect to x yields

$$x : w_{u_{tx}}t + y_{u_{tx}} = 0, \quad (2.69)$$

$$1 : z_{u_{tx}}t + \beta_{u_{xx}} = 0. \quad (2.70)$$

The separation of equation (2.69), with respect to t gives

$$t : w_{u_{tx}} = 0, \quad (2.71)$$

$$1 : y_{u_{tx}} = 0. \quad (2.72)$$

Again, splitting equation (2.70), with respect to t yields

$$t : z_{u_{tx}} = 0, \quad (2.73)$$

$$1 : \beta_{u_{xx}} = 0. \quad (2.74)$$

Simplifying (2.71), (2.72), (2.73) and (2.74) respectively, we obtain

$$w = k_1, \quad (2.75)$$

$$y = k_3, \quad (2.76)$$

$$z = k_2, \quad (2.77)$$

$$\beta = k_4. \quad (2.78)$$

Therefore, equation (2.67) yields the following multiplier:

$$\Lambda = (k_1 x + k_2)t + k_3 x + k_4. \tag{2.79}$$

We now apply equation (1.5) to construct the conservation laws of equation (2.1)

$$\Lambda_\alpha E_\alpha = D_i T^i. \tag{2.80}$$

The insertion of equation (2.79) and equation (2.1) into equation (2.80) and expansion yields

$$\begin{aligned}
&[(k_1 x + k_2)t + k_3 x + k_4](\rho u_{tt} - \mu u_{xx} - 6\lambda u u_x^2 - 3\lambda u^2 u_{xx} - \eta u_{txx} - Q u_{xxxx}) \\
&\quad = \left[\frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{ttt}\frac{\partial}{\partial u_{tt}} + u_{txx}\frac{\partial}{\partial u_{xx}} + u_{ttx}\frac{\partial}{\partial u_{tx}}\right] T^1 \\
&\quad + \left[\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{ttx}\frac{\partial}{\partial u_{tt}} + u_{txx}\frac{\partial}{\partial u_{tx}}\right] T^2,
\end{aligned} \tag{2.81}$$

where T^1 and T^2 depend up to third-order derivatives.

Solving the above equation for T^1 and T^2, we obtain the following conservation laws corresponding to the above multiplier (2.79):

$$\begin{aligned}
T_1^t &= -\rho x u + \frac{1}{3}\eta t u_x - \frac{1}{3}\eta t x u_{xx} + \rho t x u_t, \\
T_1^x &= -\frac{2}{3}\eta u + \lambda t u^3 + \mu t u + \frac{1}{3}\eta t x u_x + Q t u_{xx} + \frac{1}{3}\eta t u_t - \mu t x u_x - Q t x u_{xxx} - \\
&\quad \frac{2}{3}\eta t x u_{tx} - 3\lambda t x u^2 u_x; \\
T_2^t &= -\rho u - \frac{1}{3}\eta t u_{xx} + \rho t u_t, \\
T_2^x &= -3\lambda t u^2 u_x - \mu t u_x - Q t u_{xxx} - \frac{2}{3}\eta t u_{tx} + \frac{1}{3}\eta u_x; \\
T_3^t &= \frac{1}{3}\eta u_x - \frac{1}{3}\eta x u_{xx} + \rho x u_t, \\
T_3^x &= -\mu x u_x - Q x u_{xxx} - \frac{2}{3}\eta\, x u_{tx} - 3\lambda x u^2 u_x + \mu u + Q u_{xx} + \frac{1}{3}\eta u_t + \lambda u^3; \\
T_4^t &= -\frac{1}{3}\eta u_{xx} + \rho u_t, \\
T_4^x &= -3\lambda u^2 u_x - \mu u_x - Q u_{xxx} - \frac{2}{3}\eta u_{tx}.
\end{aligned}$$

2.2 Point symmetries of equation (2.1)

This section aims to compute the Lie point symmetries of a generalized nonintegrable nonlinear equation (2.1) and latter use them to transform a nonintegrable nonlinear equation (2.1) into ordinary differential equations (ODEs). Thereafter we construct exact solutions of equation (2.1).

The symmetry group of the nonintegrable nonlinear equation (2.1) will be generated by the vector field of the form

$$X = \tau\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial u}, \tag{2.82}$$

where τ and ξ are functions of t, x and u. Applying the third prolongation $X^{(3)}$ to equation (2.1), we obtain an overdetermined system of linear partial differential equations, namely;

$$\begin{aligned}
&\tau_x = 0,\\
&\xi^1_u = 0,\\
&\tau_u = 0,\\
&\xi^2_{uu} = 0,\\
&2\xi^1_x - \tau_t = 0,\\
&\xi^2_u + 2\xi^1_x = 0,\\
&\rho\tau_{tt} - 2\rho\xi^2_{tu} + \eta\xi^2_{xxu} = 0,\\
&2\rho\xi^1_t + 2\eta\xi^2_{xu} - \eta\xi^1_{xx} = 0,\\
&-6Q\xi^1_{xx} + 4Q\xi^2_{xu} - \eta\xi^1_t = 0,\\
&3u^2\lambda\xi^2_{xx} - \rho\xi^2_{tt} + \mu\xi^2_{xx} + Q\xi^2_{xxxx} + \eta\xi^2_{txx} = 0,\\
&6u^2\lambda\xi^1_x + 6u\lambda\xi^2 - 4Q\xi^1_{xxx} - 2\eta\xi^1_{tx} + \eta\xi^2_{tu} + 6Q\xi^2_{xxu} + 2\mu\xi^1_x = 0,\\
&6u^2\lambda\xi^2_{xu} - 3u^2\lambda\xi^1_{xx} + 12u\lambda\xi^2_x + \rho\xi^1_{tt} - Q\xi^1_{xxxx} + 4Q\xi^2_{xxxu} - \eta\xi^1_{xxt} + 2\eta\xi^2_{txu} + 2\mu\xi^2_{xu}\\
&-\mu\xi^1_{xx} = 0.
\end{aligned}$$

Solving the resulting system of linear partial differential equations, we obtain the following two Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$

We now invoke the symmetry combination of $X = X_1 + wX_2$, to get the invariant:

$$u = \phi(z) \tag{2.83}$$

where $z = x - wt$.

Substituting equation (2.83) into equation (2.1) and simplifying yields

$$-3\lambda\phi''\phi^2 + \rho w^2\phi'' - 6\lambda u\phi'^2 + \eta w\phi''' - \mu\phi'' - Q\phi'''' = 0. \tag{2.84}$$

2.2.1 Exact solutions using simplest equation method

The basic idea in the simplest equation method is as follows. Let us consider the solutions of (2.84) in the form

$$\phi(z) = \sum_{i=0}^{M} A_i(H(z))^i \tag{2.85}$$

where $H(z)$ is a solution of Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing technique as in [12] and $A_0, \cdots, A_M$, are parameters to be computed.

The Bernoulli equation

$$H'(z) = aH(z) + bH^2(z), \tag{2.86}$$

where a and b are constants whose solution is given by

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}. \tag{2.87}$$

For the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \tag{2.88}$$

where a, b and c are constants, we shall employ the solutions

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right] \tag{2.89}$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}, \tag{2.90}$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

2.2.2 Application of the simplest equation method:Bernoulli equation

In this case the balancing procedure gives $M = 1$ so the solutions of (2.1) are of the form

$$\phi(z) = A_0 + A_1 H. \tag{2.91}$$

Inserting (2.91) into equation (2.84) and making use of equation (2.86) and then equating all coefficients of the functions H^i to zero, yields the following overdetermined system of algebraic equations in terms of A_0 and A_1:

$$-24\,Qb^4A_1 - 12\,\lambda b^2A_1^3 = 0,$$
$$-60\,Qab^3A_1 - 21\,\lambda abA_1^3 + 6\,\eta wb^3A_1 - 18\,\lambda b^2A_0A_1^2 = 0,$$
$$-Qa^4A_1 + \eta wa^3A_1 - 3\lambda a^2A_0^2A_1 + \rho w^2a^2A_1 - \mu a^2A_1 = 0,$$
$$-1.5Qa^3bA_1 + 7\eta wa^2bA_1 - 12\lambda a^2A_0A_1^2 - 9\lambda abA_0^2A_1 + 3\rho w^2abA_1 - 3\mu abA_1 = 0,$$
$$-50Qa^2b^2A_1 - 9\lambda a^2A_1^3 + 12\eta wab^2A_1 - 30\lambda abA_0A_1^2 - 6\lambda b^2A_0^2A_1 + 2\rho w^2b^2A_1$$
$$-2\mu b^2A_1 = 0.$$

Solving the resulting system of algebraic equations, we get

$$\begin{aligned}
\lambda &= -\frac{2Qb^2}{A_1^2},\\
\mu &= \frac{3Q^2a^2 + 6Q\rho w^2 + \eta^2w^2}{6Q},\\
A_0 &= \frac{A_1\left(3Qa - \eta w\right)}{6Qb}.
\end{aligned}$$

Therefore, the solution of equation (2.1) is

$$u(t,x) = A_0 + A_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\} \tag{2.92}$$

where $z = x - wt$ and C is a constant of integration. Thus, the profile of the solution is

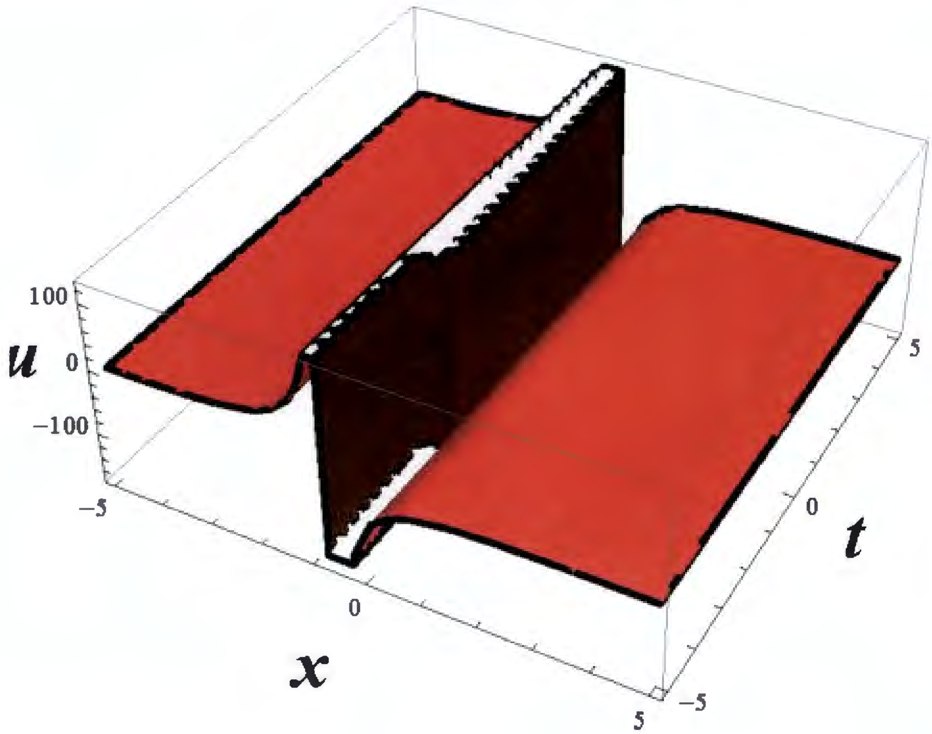

Figure 2.1: Profile of solution (2.92).

2.2.3 Application of the simplest equation method:Ricatti equation

The balancing procedure yields $M = 1$, so the solutions of (2.84) are of the form

$$\phi(z) = A_0 + A_1 H. \tag{2.93}$$

Substituting (2.93) into equation (2.84) and making use of equation (2.88) and then equating all coefficients of the functions H^i to zero, we get the following overdetermined system of algebraic equations in terms of A_0 and A_1:

$$
\begin{aligned}
&-24Qa^4A_1 - 12\lambda a^2A_1^3 = 0,\\
&-60Qa^3bA_1 + 6\eta wa^3A_1 - 18\lambda a^2A_0A_1^2 - 21\lambda abA_1^3 = 0,\\
&-8Qabc^2A_1 - Qb^3cA_1 + 2\eta wac^2A_1 + \eta wb^2cA_1 - 3\lambda bcA_0^2A_1 + \rho w^2bcA_1\\
&-6\lambda c^2A_0A_1^2 - \mu bcA_1 = 0,\\
&-40Qa^3cA_1 - 50Qa^2b^2A_1 + 12\eta wa^2bA_1 - 6\lambda a^2A_0^2A_1 + 2\rho w^2a^2A_1 - 30\lambda abA_0A_1^2\\
&-18\lambda acA_1^3 - 9\lambda b^2A_1^3 - 2\mu a^2A_1 = 0,\\
&-60Qa^2bcA_1 - 15Qab^3A_1 + 8\eta wa^2cA_1 + 7\eta wab^2A_1 - 9\lambda abA_0^2A_1 + 3\rho w^2abA_1\\
&-24\lambda acA_0A_1^2 - 12\lambda b^2A_0A_1^2 - 15\lambda bcA_1^3 - 3\mu abA_1 = 0,\\
&-16Qa^2c^2A_1 - 22Qab^2cA_1 - Qb^4A_1 + 8\eta wabcA_1 - 6\lambda acA_0^2A_1 + 2\rho w^2acA_1\\
&+\eta wb^3A_1 - 3\lambda b^2A_0^2A_1 + \rho w^2b^2A_1 - 18\lambda bcA_0A_1^2 - 6\lambda c^2A_1^3 - 2\mu acA_1 - \mu b^2A_1 = 0.
\end{aligned}
$$

Thus the solutions of the above system of algebraic equations are

$$
\begin{aligned}
Q &= -\frac{\lambda A_1^2}{2a^2},\\
\mu &= -\frac{4\eta^2w^2a^4 - 12\lambda\rho w^2a^2A_1^2 - 12\lambda^2acA_1^4 + 3\lambda^2b^2A_1^4}{12\lambda a^2A_1^2},\\
A_0 &= \frac{2\eta wa^2 + 3\lambda bA_1^2}{6\lambda aA_1}.
\end{aligned}
$$

Consequently a solution of equation (2.1) is

$$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \tag{2.94}$$

and

$$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \tag{2.95}$$

where $z = x - \nu t$ and C is a constant of integration.

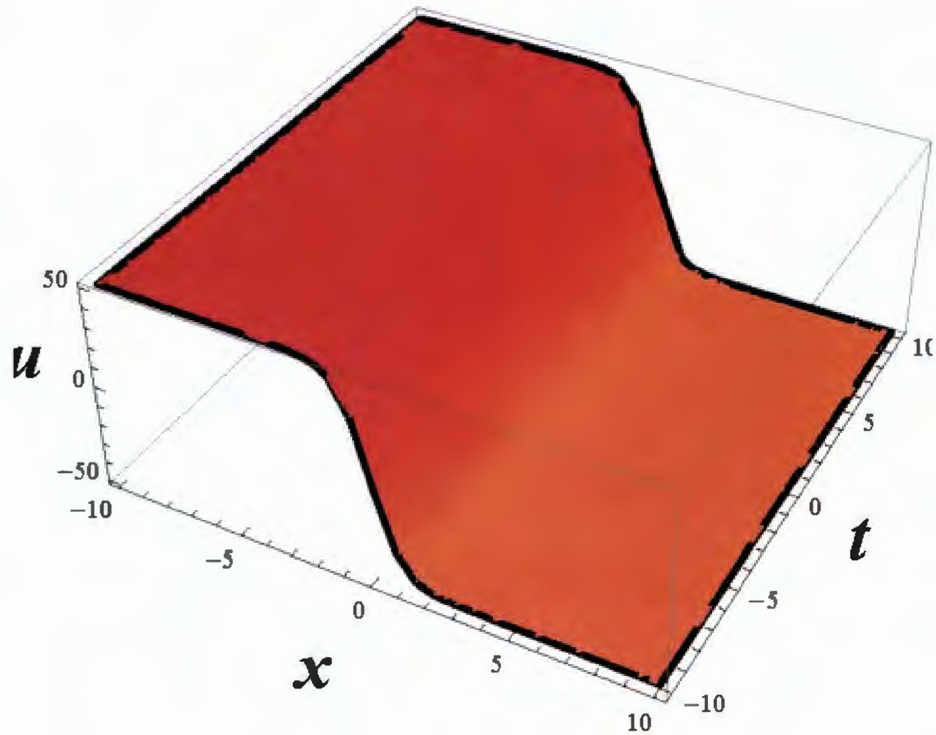

Figure 2.2: Profile of solution (2.94).

2.3 Concluding remarks

Exact solutions and conservation laws of a generalized nonintegrable nonlinear equation were computed. Simplest equation method was employed to find exact solutions of the underlying equation, while the conservation laws were computed via the multiplier approach.

Chapter 3

Conservation laws and exact solutions of a generalized two-wave mode Kadomtsev-Petvishvili equation

Lie symmetry analysis is performed on a two-wave mode equation for the integrable Kadomtsev-Petviashvili (TKP) equation which describes the propagation of two different wave modes in the same direction simultaneously. The similarity reductions and an exact solution are computed. In addition to this, we derive the conservation laws for the underlying equation.

The scholarship of nonlinear evolution equations (NLEEs) has made an extensive advancement during the last few decades [1-15]. A plethora of methods has been used to carry out the integration of such equations. Some of the most frequently used techniques are the tanh method, Hirota's bilinear method and Lie symmetry methods.

The Korteweg-de Vries (KdV) equation [20]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3.1}$$

describes the dynamics of solitary waves. It is an significant equation in the theory of integrable systems since it has an infinite number of conservation laws, multiple-soliton solutions, and many other physical properties. A natural generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation [21]

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0 \tag{3.2}$$

which is a model for shallow long waves in the $x-$direction with some mild dispersion in the $y-$direction. It is entirely integrable by the inverse scattering transform method and provides multiple-soliton solutions. Korsunsky [22] introduced the non-linear dispersive two-wave mode Korteweg-de Vries equation (TKdV) for the integrable KdV equation

$$u_{tt} - s^2 u_{xx} + u_x u_t + u u_{xt} - \alpha s(u_x^2 + u u_{xx}) + u_{xxxt} - \beta s u_{xxxx} = 0. \tag{3.3}$$

The TKdV equation (3.3) models the propagation of two distinct wave modes in the same direction concurrently with the same dispersion relation but different phase velocities, nonlinearity and dispersion parameters. It should be noted that for $s = 0$, the TKdV equation (3.3) reduces to the classic KdV equation (3.1) after integrating with respect to time t. A two-wave mode Kadomtsev-Petviashvili (TKP) equation [14] is

$$u_{ttx} - s^2 u_{xxx} + 6(u_{tx}u_{xx} + u_x u_{xxt}) - \alpha s(6u_{xx}^2 + 6u_x u_{xxx}) + u_{xxxxt} + u_{tyy} - \beta s(u_{xxxxx} + u_{xyy}) = 0. \tag{3.4}$$

It describes the propagation of two different wave modes in the same direction simultaneously. It should be noted that for $s = 0$, the TKP reduces to the classical KP equation (3.2).

In this chapter the TKP (3.4) is studied. Lie symmetry method along with the Kudryashov method are employed. Moreover, we use the multiplier method to derive the conservation laws for (3.4).

3.1 Lie point symmetries of (3.4)

The vector field operator

$$\mathbf{X} \;=\; \xi^1(t,x,y,u)\frac{\partial}{\partial t}+\xi^2(t,x,y,u)\frac{\partial}{\partial x}+\xi^3(t,x,y,u)\frac{\partial}{\partial y}+\eta(t,x,y,u)\frac{\partial}{\partial u} \quad (3.5)$$

is a Lie point symmetry of (3.4) if

$$\mathbf{X}^{[5]}\Big\{ \quad u_{ttx}-s^2u_{xxx}+6(u_{tx}u_{xx}+u_xu_{xxt})-\alpha s(6u_{xx}^2+6u_xu_{xxx})+$$
$$u_{xxxxt}+u_{tyy}-\beta s(u_{xxxxx}+u_{xyy})=0\Big\}\Big|_{(3.4)}=0,$$

where $\mathbf{X}^{[5]}$ is the fifth extension of (3.5). Expanding the above equation and splitting on the derivatives of u leads to an overdetermined system of linear partial differential equations:

$$\xi^3_t=0,$$
$$\xi^3_u=0,$$
$$\xi^1_y=0,$$
$$\xi^2_y=0,$$
$$\xi^3_x=0,$$
$$\xi^1_x=0,$$
$$\xi^2_u=0,$$
$$\xi^1_u=0,$$
$$\xi^2_{tx}=0,$$
$$\eta_{tu}=0,$$
$$\xi^2_{xx}=0,$$
$$\eta_{xu}=0,$$
$$\eta_{uu}=0,$$
$$\eta_{yyu}=0,$$
$$-\xi^3_{yy}+2\eta_{yu}=0,$$

$$-\eta_{xxx}\alpha s + \eta_{txx} = 0,$$

$$-\xi^1_t + 3\xi^2_x = 0,$$

$$-\xi^1_{tt} + 6\eta_{xx} = 0,$$

$$\xi^2_x + \eta_u = 0,$$

$$2\xi^2_x - \xi^3_y = 0,$$

$$-\xi^2_t + 3\eta_x = 0,$$

$$-\xi^3_{yy} + 2\eta_{yu} = 0,$$

$$6\eta_x\alpha + \xi^1_t s + \xi^2_x s = 0,$$

$$-12\eta_{xx}\alpha s + 6\eta_{tx} - \xi^2_{tt} = 0,$$

$$-\eta_u\alpha s - \xi^1_t\alpha s - \xi^2_t = 0,$$

$$-\eta_u\alpha s - \xi^1_t\alpha s - \xi^2_t = 0,$$

$$-\xi^1_t\beta s + \beta s\xi^2_x - \xi^2_t = 0,$$

$$-\xi^1_t\beta s - 3\beta s\xi^2_x + 2\beta s\xi^3_y - \xi^2_t = 0,$$

$$-\beta s\eta_{xyy} - \beta s\eta_{xxxxx} - \eta_{xxx}s^2 + \eta_{ttx} + \eta_{tyy} + \eta_{txxxx} = 0.$$

Solving the above system with the aid of Maple yields the following symmetries of equation (3.4):

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{X}_2 &= \frac{\partial}{\partial y}, \\
\mathbf{X}_3 &= \frac{\partial}{\partial x}, \\
\mathbf{X}_4 &= p(y)\frac{\partial}{\partial u}, \\
\mathbf{X}_5 &= q(t)\frac{\partial}{\partial u}, \\
\mathbf{X}_6 &= yr(t)\frac{\partial}{\partial u}.
\end{aligned}$$

3.1.1 Symmetry reductions of (3.4)

Taking the linear combination of the translation symmetries, $\mathbf{\Gamma} = a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_3\mathbf{X_3}$ converts (3.4) to a partial differential equation (PDE) in two independent variables. The symmetry $\mathbf{\Gamma}$ yields the following three invariants:

$$f = a_2 t - a_1 y, \quad g = a_2 x - a_3 y, \quad \phi = u.$$

By employing the above invariants, we construct the transformed equation as

$$\begin{aligned}
&\phi_{ffg}{a_2}^3 - s^2\phi_{ggg}{a_2}^3 + 6\phi_{fg}{a_2}^4\phi_{gg} + 6\phi_g{a_2}^4\phi_{fgg} \\
&\quad - \alpha s\left(6\phi_g{a_2}^4\phi_{ggg} + 6{\phi_{gg}}^2{a_2}^4\right) + \phi_{fgggg}{a_2}^5 \\
&\quad + \left(-\left(-\phi_{fff}a_1 - \phi_{ffg}a_3\right)a_1 - \left(-\phi_{ffg}a_1 - \phi_{fgg}a_3\right)a_3\right)a_2 - \beta s\left(\phi_{ggggg}{a_2}^5\right. \\
&\quad \left.+ \left(-\left(-\phi_{ffg}a_1 - \phi_{fgg}a_3\right)a_1 - \left(-\phi_{fgg}a_1 - \phi_{ggg}a_3\right)a_3\right)a_2\right) = 0.
\end{aligned} \tag{3.6}$$

The Lie point symmetries of the above equation are as follows:

$$\mathbf{\Upsilon_1} = \frac{\partial}{\partial f}, \quad \mathbf{\Upsilon_2} = \frac{\partial}{\partial g}, \quad \mathbf{\Upsilon_3} = \frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_4} = f\frac{\partial}{\partial \phi}, \quad \mathbf{\Upsilon_5} = f^2\frac{\partial}{\partial \phi}.$$

Considering a linear combination $b_1\mathbf{\Upsilon_1} + b_2\mathbf{\Upsilon_2}$, one obtains the invariants

$$z = b_2 f - b_1 g, \quad \phi = F$$

and this leads to following nonlinear ordinary differential equation

$$\begin{aligned}
&F_{zzz}\left(a_2b_2 - a_3b_1\right){a_1}^2{b_1}^2 - s^2F_{zzz}\left(a_2b_2 - a_3b_1\right)^3 \\
&\quad + 6{F_{zz}}^2\left(a_2b_2 - a_3b_1\right)^3 a_1b_1 + 6F_z\left(a_2b_2 - a_3b_1\right)^3 F_{zzz}a_1b_1 \\
&\quad - \alpha s\left(6{F_{zz}}^2\left(a_2b_2 - a_3b_1\right)^4 + 6F_z\left(a_2b_2 - a_3b_1\right)^4 F_{zzz}\right) \\
&\quad + F_{zzzzz}\left(a_2b_2 - a_3b_1\right)^4 a_1b_1 + F_{zzz}{a_1}^3{b_2}^2b_1 \\
&\quad - \beta s\left(F_{zzzzz}\left(a_2b_2 - a_3b_1\right)^5 + F_{zzz}{a_1}^2{b_2}^2\left(a_2b_2 - a_3b_1\right)\right) = 0.
\end{aligned} \tag{3.7}$$

3.1.2 Kudryashov method

In this subsection we use the Kudryashov method [12, 23, 24]. To the best of our knowledge, it is for the first time that this method has been applied to (3.4). The solutions of (3.7) are of the form

$$F(z) = A_0 + A_1 H. \tag{3.8}$$

Inserting (3.8) into (3.7) and utilizing Riccati equation [12, 23, 24], and thereafter setting the coefficients of powers of H^i to zero, gives rise to the following system:

$$
\begin{aligned}
&-\beta s A_1 a_2^5 b_1^5 - A_1 a_2^5 b_1^4 b_2 - s^2 A_1 a_2^3 b_1^3 + A_1 a_2^3 b_1 b_2^2 - \beta s A_1 a_2 a_3^2 b_1^3 - A_1 a_1^2 a_2 b_2^3 - \\
&\beta s A_1 a_1^2 a_2 b_1 b_2^2 + 2A_1 a_1 a_2 a_3 b_1 b_2^2 - A_1 a_2 a_3^2 b_1^2 b_2 + 2\beta s A_1 a_1 a_2 a_3 b_1^2 b_2 = 0, \\
&31\beta s A_1 a_2^5 b_1^5 + 31 A_1 a_2^5 b_1^4 b_2 - 12\alpha s A_1^2 a_2^4 b_1^4 - 12 A_1^2 a_2^4 b_1^3 b_2 + 7s^2 A_1 a_2^3 b_1^3 - \\
&7A_1 a_2^3 b_1 b_2^2 + 7\beta s A_1 a_2 a_3^2 b_1^3 + 7A_1 a_1^2 a_2 b_2^3 + 7\beta s A_1 a_1^2 a_2 b_1 b_2^2 - 14 A_1 a_1 a_2 a_3 b_1 b_2^2 + \\
&7A_1 a_2 a_3^2 b_1^2 b_2 - 14\beta s A_1 a_1 a_2 a_3 b_1^2 b_2 = 0, \\
&-180\beta s A_1 a_2^5 b_1^5 - 180 A_1 a_2^5 b_1^4 b_2 + 84\alpha s A_1^2 a_2^4 b_1^4 + 84 A_1^2 a_2^4 b_1^3 b_2 - 12 s^2 A_1 a_2^3 b_1^3 + \\
&12A_1 a_2^3 b_1 b_2^2 - 12\beta s A_1 a_2 a_3^2 b_1^3 - 12 A_1 a_1^2 a_2 b_2^3 - 12\beta s A_1 a_1^2 a_2 b_1 b_2^2 + 24 A_1 a_1 a_2 a_3 b_1 b_2^2 - \\
&12A_1 a_2 a_3^2 b_1^2 b_2 + 24\beta s A_1 a_1 a_2 a_3 b_1^2 b_2 = 0, \\
&390\beta s A_1 a_2^5 b_1^5 + 390 A_1 a_2^5 b_1^4 b_2 - 192\alpha s A_1^2 a_2^4 b_1^4 - 192 A_1^2 a_2^4 b_1^3 b_2 + 6s^2 A_1 a_2^3 b_1^3 - \\
&6A_1 a_2^3 b_1 b_2^2 + 6\beta s A_1 a_2 a_3^2 b_1^3 + 6A_1 a_1^2 a_2 b_2^3 + 6\beta s A_1 a_1^2 a_2 b_1 b_2^2 - 12 A_1 a_1 a_2 a_3 b_1 b_2^2 + \\
&6A_1 a_2 a_3^2 b_1^2 b_2 - 12\beta s A_1 a_1 a_2 a_3 b_1^2 b_2 = 0, \\
&-360\beta s A_1 a_2^5 b_1^5 - 360 A_1 a_2^5 b_1^4 b_2 + 180\alpha s A_1^2 a_2^4 b_1^4 + 180 A_1^2 a_2^4 b_1^3 b_2 = 0, \\
&120\beta s A_1 a_2^5 b_1^5 + 120 A_1 a_2^5 b_1^4 b_2 - 60\alpha s A_1^2 a_2^4 b_1^4 - 60 A_1^2 a_2^4 b_1^3 b_2 = 0.
\end{aligned}
$$

The solution for the above system with the aid of Maple is

$$
\begin{aligned}
s &= \frac{(-\beta) a_2^4 b_1^5 - \beta a_3^2 b_1^3 + 2\beta a_1 a_3 b_2 b_1^2 - \beta a_1^2 b_2^2 b_1 + L}{2a_2^2 b_1^3}, \\
A_1 &= \frac{2\left(\beta s a_2 b_1^2 + a_2 b_2 b_1\right)}{\alpha s b_1 + b_2}, \\
L &= \big[\left(\beta a_2^4 b_1^5 + \beta a_3^2 b_1^3 - 2\beta a_1 a_3 b_2 b_1^2 + \beta a_1^2 b_2^2 b_1\right)^2 \\
&\quad -4a_2^2 b_1^3 \left(a_2^4 b_2 b_1^4 + a_3^2 b_2 b_1^2 - a_2^2 b_2^2 b_1 - 2a_1 a_3 b_2^2 b_1 + a_1^2 b_2^3\right)\big]^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, the solution of equation (3.4) is

$$
u(t,x,y) = \frac{A_0 + 2\left(a_2 b_1 b_2 + \frac{1}{2a_2 b_1}\beta\left(-\beta a_3^2 b_1^3 - \beta a_2^4 b_1^5 + 2\beta a_1 a_3 b_1^2 b_2 - \beta a_1^2 b_1 b_2^2 + L\right)\right)}{\left(1 + \cosh(z) - \sinh(z)\right)\left(b_2 + \frac{1}{2a_2^2 b_1^2}\alpha\left(-\beta a_3^2 b_1^3 - \beta a_2^4 b_1^5 + 2\beta a_1 a_3 b_1^2 b_2 - \beta a_1^2 b_1 b_2^2 + L\right)\right)},
$$

where $z = b_2\left(a_2 t - a_1 y\right) - b_1\left(a_2 x - a_3 y\right)$.

3.2 Conservation laws

In this section we compute the conservation law of the (3.4) using the multiplier method. Here we will consider multipliers of the zeroth order,
i.e., $\Lambda_\alpha = \Lambda_\alpha(t, x, u)$. The right hand side of (1.6) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0.$$

Expanding the above equation with help of Maple computer algebra system prompts the following Lemmas:

Lemma 1. *The TKP equation (3.4) establishes the conservation law multiplier of the form*

$$\begin{aligned}\Lambda_1 &= k_1 u + \frac{1}{6}(\beta st + x)^3 k_2'' + \frac{1}{6}\left(-6\beta^2 + 6\right) s^2 t k_2(y) + \frac{1}{2}(st + x)^2 k_3(y) \\ &\quad + \frac{1}{6}(6\beta st + 6x) k_4(y) + k_5(y) + y k_6(t) + k_7(t),\end{aligned}$$

whenever $\alpha = \beta$.

Lemma 2. *The TKP (3.4) constitutes the conservation law multiplier of the form*

$$\Lambda_2 = f_1 u + f_2(y, -st - x) + f_3(t) y + f_4(t) + f_5(y),$$

whenever $\alpha = \beta = 1$.

Lemma 3. *The TKP (3.4) admits the conservation law multiplier of the form*

$$\Lambda_3 = g_1 u + g_2(y, -st + x) + g_3(t) y + g_4(t) + g_5(y),$$

whenever $\alpha = \beta = -1$.

Proof of the above Lemmas. One can deduce this from (1.6). Consequentially we conclude that the conservation laws corresponding to the above $\Lambda_1, \Lambda_2, \Lambda_3$ are respectively given by the following:

$$\begin{aligned}T_1^t &= -\frac{1}{3}u_{xxx}u^2 - \frac{1}{3}u_x u_t + \frac{2}{3}u u_{tx} + \frac{1}{5}u u_{xxxx} - \frac{1}{5}u_{xxx}u_x + 2u_x u_{xx} u - \frac{2}{3}u_x^3 + \frac{1}{10}u_{xx}^2 \\ &\quad - \frac{1}{6}u_y^2 + \frac{1}{3}u u_{yy},\end{aligned}$$

$$
\begin{aligned}
T_1^x &= \beta s u_x u_{xxx} - 6\beta s u u_x u_{xx} - \frac{1}{2}\beta s u_{xx}^2 + \frac{2}{5}u_{tx}u_{xx} - \frac{3}{5}u_x u_{txx} - \frac{1}{3}\beta s u u_{yy} - \beta s u u_{xxxx} \\
&\quad + 2\beta s u_x^3 - s^2 u u_{xx} + \frac{2}{3}u u_{xx} u_t + \frac{1}{6}\beta s u_y^2 + \frac{10}{3}u u_x u_{tx} - \frac{1}{6}u_t^2 + \frac{4}{5}u u_{txxx} + \frac{1}{3}u u_{tt} \\
&\quad - \frac{1}{5}u_t u_{xxx} + \frac{1}{2}s^2 u_x^2 - \frac{4}{3}u_x^2 u_t + \frac{1}{3}u^2 u_{txx},
\end{aligned}
$$

$$
T_1^y = -\frac{2}{3}\beta s u u_{xy} + \frac{1}{3}\beta s u_x u_y + \frac{2}{3}u u_{ty} - \frac{1}{3}u_t u_y,
$$

$$
\begin{aligned}
T_2^t &= \frac{1}{4}x^2 u_{xx} u k_2'' + x\, u_x k_2'' + \frac{1}{4}x^3 u_x u_{xx} k_2'' - \frac{1}{12}x^3 u_{xxx} u k_2'' + \frac{1}{3}\beta^2 s^2 u_x k_2(y) + \\
&\quad \frac{2}{3}s^2 t u_{tx} k_2(y) + \frac{1}{5}s^2 t u_{xxxx} k_2(y) + \frac{1}{3}s^2 t u_{yy} k_2(y) + \frac{1}{3}s^2 u_t k_2'' - \frac{1}{3}s^2 t u_y c_2' + \\
&\quad \frac{1}{10}\beta^2 s^2 t^2 x u_{xxxx} k_2'' + \frac{1}{10}\beta s t x^2 u_{xxxx} k_2'' + \frac{1}{6}\beta^2 s^2 t^2 x u_{yy} k_2'' + \frac{1}{6}\beta x^2 s t u_{yy} k_2'' - \\
&\quad \frac{3}{2}\beta^2 s^2 t u_x u_{xx} k_2(y) + \frac{1}{2}\beta^2 s^2 u_{xxx} u_t k_2(y) + \frac{3}{4}\beta x^2 s t u_x u_{xx} k_2'' + \frac{3}{4}\beta^2 s^2 t^2 x u_x u_{xx} k_2'' \\
&\quad - \frac{1}{4}\beta^2 s^2 t^2 x u_{xxx} u k_2'' - \frac{1}{4}\beta s t x^2 u_{xxx} u k_2'' + \frac{1}{2}\beta s t x u_{xx} u k_2'' + \beta s t u u_x k_2'' \\
&\quad + \frac{1}{4}\beta^3 s^3 t^3 u_x u_{xx} k_2'' - \beta s t x {u_x}^2 k_2'' - \frac{1}{12}\beta^3 s^3 t^3 u_{xxx} u k_2'' + \frac{1}{4}\beta^2 s^2 t^2 u_{xx} u k_2'' + \\
&\quad \frac{1}{3}\beta s t x^2 u_{tx} k_2'' + \frac{1}{3}\beta^2 s^2 t^2 x u_{tx} k_2'' - \frac{1}{3}\beta^2 s^2 t x u_x k_2'' - \frac{1}{5}\beta s t x u_{xxx} k_2'' - \frac{1}{6}\beta s t x^2 u_y k_2'' \\
&\quad - \frac{1}{6}\beta^2 s^2 t^2 x u_y k_2''' - \frac{1}{3}\beta s t x u_t k_2'' + \frac{1}{6}\beta s t x^2 u k_2'''' + \frac{1}{6}\beta^2 s^2 t^2 x u k_2'''' - \frac{1}{18}x^3 u_y k_2''' \\
&\quad + \frac{1}{5}x u_{xx} k_2'' + \frac{1}{18}x^3 u k_2'''' - \frac{1}{6}x^2 u_t k_2'' - \frac{1}{5}u_x k_2'' + \frac{3}{2}s^2 t u_x u_{xx} k_2(y)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{5}\beta^2 s^2 t u_{xxxx} k_2(y) - \frac{2}{3}\beta^2 s^2 t u_{tx} k_2(y) - \frac{1}{2}s^2 u_{xxx} u_t k_2(y) - \frac{1}{3}\beta^2 s^2 t u_{yy} k_2(y) \\
&+ \frac{1}{18}\beta^3 s^3 t^3 u_{yy} k_2'' - \frac{1}{2}\beta^2 s^2 t^2 {u_x}^2 k_2'' + \frac{1}{30}\beta^3 s^3 t^3 u_{xxxx} k_2'' + \frac{1}{9}\beta^3 s^3 t^3 u_{tx} k_2'' \\
&- \frac{1}{6}\beta^3 s^3 t^2 u_x k_2'' - \frac{1}{6}\beta s x^2 u_x k_2'' + \frac{1}{3}\beta^2 s^2 t u_y k_2' - \frac{1}{10}\beta^2 s^2 t^2 u_{xxx} k_2'' + \\
&\frac{1}{5}\beta s t u_{xx} k_2'' - \frac{1}{18}\beta^3 s^3 t^3 u_y k_2''' + \frac{2}{3}\beta s x u k_2'' - \frac{1}{6}\beta^2 s^2 t^2 u_t k_2'' - \frac{1}{3}s^2 u_x k_2(y) \\
&+ \frac{1}{3}\beta^2 s^2 t u k_2'' + \frac{1}{18}\beta^3 s^3 t^3 u k_2'''' - \frac{1}{2}x^2 {u_x}^2 k_2'' + \frac{1}{30}x^3 u_{xxxx} k_2'' + \frac{1}{9}x^3 u_{tx} k_2'' \\
&+ \frac{1}{18}x^3 u_{yy} k_2'' - \frac{1}{10}x^2 u_{xxx} k_2'',
\end{aligned}
$$

$$
\begin{aligned}
T_2^x \;=\;& \beta^2 s^2 {u_x}^2 k_2(y) + \frac{1}{2}\, s^2 u_{xx} u k_2(y) + \frac{1}{5}\,\beta^2 s^2 u_{xxx} k_2(y) - s^4 t u_{xx} k_2(y) + \frac{4}{5}\, s^2 t u_{txxx} k_2(y) \\
& + \frac{1}{3}\,\beta^2 s^2 u_t k_2(y) + \frac{1}{3}\, s^2 t u_{tt} k_2(y) - 6\,\beta s^3 t u_x u_{xx} k_2(y) + 6\,\beta^3 s^3 t u_x u_{xx} k_2(y) - \\
& \frac{1}{2}\,\beta^2 s^2 t u u_{txx} k_2(y) - 4\,\beta^2 s^2 t u_{tx} u_x k_2(y) - \frac{1}{2}\,\beta^2 s^2 t u_{xx} u_t k_2(y) + \frac{2}{5}\, x u_{tx} k_2'' - \\
& \frac{3}{10}\, x^2 u_{txx} k_2'' + \frac{2}{15}\, x^3 u_{txxx} k_2'' + \beta^3 s^3 t^2 {u_x}^2 k_2'' + \beta s x^2 u_x^2 k_2'' + \frac{2}{5}\,\beta s t u_{tx} k_2'' - \\
& \frac{3}{5}\,\beta^2 s^2 t u_{xx} k_2'' + \frac{1}{4}\,\beta^2 s^2 t^2 x u u_{txx} k_2'' + 2\beta^2 s^2 t^2 x u_{tx} u_x k_2'' + \frac{1}{4}\,\beta s t x^2 u u_{txx} k_2'' + \\
& \frac{1}{2}\,\beta^2 s^2 t x u_{xx} u k_2'' - \beta s t x u_t u_x k_2'' + 2\,\beta s t x^2 u_{tx} u_x k_2'' - 3\beta^3 s^3 t^2 x u_x u_{xx} k_2'' - \\
& \beta s t x u_{tx} k_2'' + \frac{1}{4}\,\beta^2 s^2 t^2 x u_{xx} u_t k_2'' + \frac{1}{4}\,\beta s t x^2 u_{xx} u_t k_2'' - 3\beta^2 s^2 t x^2 u_x u_{xx} k_2'' + \\
& \frac{1}{18}\, x^3 u_{tt} k_2'' - \frac{1}{2}\,\beta^2 s^2 t x^2 u_{xxxx} k_2'' - \frac{1}{2}\,\beta^3 s^3 t^2 x u_{xxxx} k_2'' + \beta s^3 t x u_x k_2'' - \beta s u_x u_{xx} k_2'' \\
& + \frac{1}{4}\,\beta^3 s^3 t^2 u_{xx} u k_2'' + \frac{1}{12}\,\beta^3 s^3 t^3 u_{xx} u_t k_2'' - \frac{1}{2}\,\beta^2 s^2 t^2 u_t u_x k_2'' - \beta s x u u_x k_2'' \\
& - \beta^2 s^2 t u u_x k_2'' + 2\beta^2 s^2 t x {u_x}^2 k_2'' - \frac{1}{2}\,\beta^2 s^2 t^2 u u_{tx} k_2'' + \frac{1}{12}\,\beta^3 s^3 t^3 u u_{txx} k_2'' + \\
& \frac{1}{4}\,\beta s x^2 u u_{xx} k_2'' + \frac{2}{3}\,\beta^3 s^3 t^3 u_{tx} u_x k_2'' - \beta^4 s^4 t^3 u_x u_{xx} k_2'' + \frac{2}{5}\,\beta s t x^2 u_{txxx} k_2'' + \\
& \frac{2}{5}\,\beta^2 s^2 t^2 x u_{txxx} k_2'' - \frac{3}{5}\,\beta s t x u_{txx} k_2'' - \frac{1}{6}\,\beta^2 s^2 t x^2 u k_2'''' - \frac{1}{6}\,\beta^3 s^3 t^2 x u k_2'''' -
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2}\,\beta s^3 t x^2 u_{xx} k_2'' - \frac{1}{2}\,\beta^2 s^4 t^2 x u_{xx} k_2'' + \frac{1}{6}\,\beta^2 s^2 t x^2 u_y k_2''' + \frac{1}{6}\,\beta^3 s^3 t^2 x u_y k_2''' + \frac{4}{5}\,\beta^2 s^2 t x u_{xxx} k_2'' + \\
& \frac{1}{6}\,\beta s t x^2 u_{tt} k_2'' + \frac{1}{6}\,\beta^2 s^2 t^2 x u_{tt} k_2'' - \frac{1}{3}\,\beta^2 s^2 t x u_t k_2'' - \frac{1}{6}\,\beta^2 s^2 t x^2 u_{yy} k_2'' - \\
& \frac{1}{6}\,\beta^3 s^3 t^2 x u_{yy} k_2'' - \frac{1}{3}\,\beta^2 s^2 t u_{tt} k_2(y) + \frac{1}{3}\,\beta^3 s^3 t u_{yy} k_2(y) - \frac{1}{3}\,\beta s^3 t u_{yy} k_2(y) - \\
& \frac{4}{5}\,\beta^2 s^2 t u_{txxx} k_2(y) + \beta^2 s^4 t u_{xx} k_2(y) + \beta^3 s^3 t u_{xxxx} k_2(y) - \beta s^3 t u_{xxxx} k_2(y) + \\
& \frac{1}{2}\, s^2 t u_{xx} u_t k_2(y) + \frac{1}{2}\, s^2 t u u_{txx} k_2(y) - \frac{1}{2}\,\beta^2 s^2 u u_{xx} k_2(y) + 4 s^2 t u_{tx} u_x k_2(y) + \\
& \frac{1}{18}\,\beta^3 s^3 t^3 u_{tt} k_2'' - \frac{1}{6}\,\beta s x^2 u_t k_2'' - \frac{1}{6}\,\beta^3 s^3 t^2 u_t k_2'' - \frac{1}{18}\,\beta s x^3 u_{yy} k_2'' - \\
& \frac{1}{18}\,\beta^4 s^4 t^3 u_{yy} k_2'' + \frac{2}{15}\,\beta^3 s^3 t^3 u_{txxx} k_2'' - \frac{3}{10}\,\beta^2 s^2 t^2 u_{txx} k_2'' - \frac{1}{18}\,\beta s x^3 u k_2'''' - \\
& \frac{1}{18}\,\beta^4 s^4 t^3 u k_2'''' + \frac{1}{3}\,\beta^2 s^2 x u k_2'' + \frac{2}{3}\,\beta^3 s^3 t u k_2'' - \frac{4}{3}\,\beta s^3 t u k_2'' - \frac{1}{6}\,\beta s x^3 u_{xxxx} k_2'' - \\
& \frac{1}{6}\,\beta^3 s^5 t^3 u_{xx} k_2'' - \frac{1}{6}\,\beta^4 s^4 t^3 u_{xxxx} k_2'' + \frac{1}{2}\,\beta^2 s^4 t^2 u_x k_2'' + \frac{1}{3}\,\beta s^3 t u_y k_2' - \frac{1}{3}\,\beta^3 s^3 t u_y k_2' + \\
& \frac{1}{18}\,\beta^4 s^4 t^3 u_y k_2''' + \frac{1}{18}\,\beta s x^3 u_y k_2''' + \frac{2}{5}\beta s x^2 u_{xxx} k_2'' - \frac{3}{5}\,\beta s x u_{xx} k_2'' + \frac{2}{5}\beta^3 s^3 t^2 u_{xxx} k_2''
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{5}\,u_tk_2''+\frac{2}{3}\,x^3u_{tx}u_xk_2''+\frac{2}{5}\,\beta su_xk_2''-\frac{1}{2}\,x^2u_tu_xk_2''-\frac{1}{6}\,s^2x^3u_{xx}k_2''-s^2xuk_2''-\\
&\frac{1}{2}\,x^2uu_{tx}k_2''+\frac{1}{12}\,x^3uu_{txx}k_2''+\frac{1}{2}\,s^2x^2u_xk_2''+\frac{1}{12}\,x^3u_{xx}u_tk_2''-s^2{u_x}^2k_2(y)-\\
&\frac{1}{5}\,s^2u_{xxx}k_2(y)-\frac{1}{3}\,s^2u_tk_2(y),
\end{aligned}
$$

$$
\begin{aligned}
T_2^y \;=\;& \frac{1}{6}\,\beta^2s^2tx^2u_xk_2'''-\frac{1}{3}\,s^2tu_tk_2'+\frac{1}{3}\,\beta^2s^2u_yk_2(y)+\frac{2}{3}\,s^2tu_{ty}k_2(y)-\frac{2}{3}\,\beta^2s^2uk_2'-\\
&\frac{1}{3}\,\beta^3s^3t^2xu_{xy}k_2''-\frac{1}{3}\,\beta^2s^2tx^2u_{xy}k_2''-\frac{1}{6}\,\beta^2s^2t^2xu_tk_2'''-\frac{1}{6}\,\beta stx^2u_tk_2'''+\\
&\frac{1}{3}\,\beta^2s^2t^2xu_{ty}k_2''+\frac{1}{3}\,\beta stx^2u_{ty}k_2''+\frac{1}{6}\,\beta^3s^3t^2xu_xk_2'''-\frac{2}{3}\,\beta s^3tu_{xy}k_2'(y)+\frac{2}{3}\,\beta^3s^3tu_{xy}k_2(y)\\
&-\frac{2}{3}\,\beta^2s^2tu_{ty}k_2(y)-\frac{1}{9}\,\beta^4s^4t^3u_{xy}k_2''-\frac{1}{9}\,\beta sx^3u_{xy}k_2''+\frac{1}{3}\,\beta s^3tu_xk_2'-\frac{1}{3}\beta^3s^3tu_xk_2'+\\
&\frac{1}{9}\,\beta^3s^3t^3u_{ty}k_2''+\frac{1}{18}\,\beta^4s^4t^3u_xk_2'''+\frac{1}{18}\,\beta sx^3u_xk_2'''-\frac{1}{18}\,\beta^3s^3t^3u_tk_2'''+\frac{1}{3}\,\beta^2s^2tu_tk_2'+\\
&\frac{1}{9}\,x^3u_{ty}k_2''+\frac{2}{3}\,s^2uk_2'-\frac{1}{3}\,s^2u_yk_2(y)-\frac{1}{18}\,x^3u_tk_2''',
\end{aligned}
$$

$$
\begin{aligned}
T_3^t \;=\;& \frac{3}{2}\,\beta stxu_xu_{xx}k_3(y)-\frac{1}{4}\,x^2uu_{xxx}k_3(y)+\frac{1}{2}\,xuu_{xx}k_3(y)+\frac{3}{4}\,x^2u_xu_{xx}k_3(y)\\
&+\frac{2}{3}\,\beta suk_3(y)+\frac{1}{3}\,\beta stxuk_3''-\frac{1}{3}\,\beta stxu_yk_3'+\frac{2}{3}\,\beta stxu_{tx}k_3(y)-\\
&\frac{1}{4}\,\beta^2s^2t^2uu_{xxx}k_3(y)+\frac{1}{2}\,\beta stuu_{xx}k_3(y)+\frac{3}{4}\,\beta^2s^2t^2u_xu_{xx}k_3(y)+\\
&\frac{1}{5}\,\beta stxu_{xxxx}k_3(y)+\frac{1}{3}\,\beta stxu_{yy}k_3(y)+\frac{1}{10}\,x^2u_{xxxx}k_3(y)+\frac{1}{3}\,x^2u_{tx}k_3(y)-\\
&\frac{1}{5}\,xu_{xxx}k_3(y)+uu_xk_3(y)+\frac{1}{6}\,x^2u_{yy}k_3(y)-xu_x^2k_3(y)-\frac{1}{3}\,xu_tk_3(y)-\\
&\frac{1}{2}\,\beta stxuu_{xxx}k_3(y)+\frac{1}{6}\,\beta^2s^2t^2uk_3''-\frac{1}{6}\,\beta^2s^2t^2u_yk_3'+\frac{1}{3}\,\beta^2s^2t^2u_{tx}k_3(y)+\\
&\frac{1}{10}\,\beta^2s^2t^2u_{xxxx}k_3(y)-\beta stu_x^2k_3(y)-\frac{1}{6}\,x^2u_yk_3'+\frac{1}{6}\,x^2uk_3''-\frac{1}{3}\,\beta^2s^2tu_xk_3(y)-\\
&\frac{1}{3}\,\beta sxu_xk_3(y)+\frac{1}{6}\,\beta^2s^2t^2u_{yy}k_3(y)-\frac{1}{3}\,\beta stu_tk_3(y)-\frac{1}{5}\,\beta stu_{xxx}k_3(y)+\frac{1}{5}\,u_{xx}k_3(y),
\end{aligned}
$$

$$
\begin{aligned}
T_3^x \;=\;& \frac{1}{3}\,\beta^2s^2txu_yk_3'-\frac{1}{3}\,\beta^2s^2txuk_3''-\frac{1}{3}\,\beta^2s^2txu_{yy}k_3(y)+\frac{1}{3}\,\beta stxu_{tt}k_3(y)-\\
&\beta^2s^2txu_{xxxx}k_3(y)-\beta s^3txu_{xx}k_3(y)+\frac{4}{5}\,\beta stxu_{txxx}k_3(y)-\beta stuu_{tx}k_3(y)+\\
&\frac{1}{4}\,\beta^2s^2t^2uu_{txx}k_3(y)-3\beta^3s^3t^2u_xu_{xx}k_3(y)-3\beta sx^2u_xu_{xx}k_3(y)+\\
&\frac{1}{4}\,\beta^2s^2t^2u_{xx}u_tk_3(y)+2\beta^2s^2t^2u_{tx}u_xk_3(y)-\beta stu_tu_xk_3(y)+\frac{1}{2}\,\beta sxu_{xx}uk_3(y)+
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\beta^2 s^2 tuu_{xx}k_3(y) + \frac{1}{6}x^2u_{tt}k_3(y) + \frac{2}{5}x^2u_{txxx}k_3(y) - \frac{3}{5}xu_{txx}k_3(y) - s^2uk_3(y) + \\
&\frac{1}{6}\beta^2 s^2 t^2 u_{tt}k_3(y) - \frac{1}{3}\beta sxu_t k_3(y) - \frac{1}{3}\beta^2 s^2 tu_t k_3(y) - \frac{1}{6}\beta sx^2 u_{yy}k_3(y) - \\
&\frac{1}{6}\beta^3 s^3 t^2 u_{yy}k_3(y) + \frac{2}{5}\beta^2 s^2 t^2 u_{txxx}k_3(y) - \frac{3}{5}\beta stu_{txx}k_3(y) - \frac{1}{2}\beta sx^2 u_{xxxx}k_3(y) - \\
&\frac{1}{2}\beta^2 s^4 t^2 u_{xx}k_3(y) - \frac{1}{2}\beta^3 s^3 t^2 u_{xxxx}k_3(y) + \beta s^3 tu_x k_3(y) + \frac{4}{5}\beta^2 s^2 tu_{xxx}k_3(y) + \\
&\frac{4}{5}\beta sxu_{xxx}k_3(y) + 2\beta^2 s^2 tu_x^2 k_3(y) - \beta suu_x k_3(y) + 2\beta sxu_x^2 k_3(y) - \\
&\frac{1}{6}\beta^3 s^3 t^2 uk_3'' - \frac{1}{6}\beta sx^2 uk_3'' + \frac{1}{6}\beta^3 s^3 t^2 u_y k_3' + \frac{1}{6}\beta sx^2 u_y k_3' - \frac{3}{5}\beta su_{xx}k_3(y) \\
&+s^2 xu_x k_3(y) - \frac{1}{2}s^2 x^2 u_{xx}k_3(y) + \frac{1}{3}\beta^2 s^2 uk_3(y) - xu_t u_x k_3(y) + 2x^2 u_{tx}u_x k_3(y) \\
&+\frac{1}{4}x^2 u_{xx}u_t k_3(y) - xuu_{tx}k_3(y) + \frac{1}{4}x^2 uu_{txx}k_3(y) - 6\beta^2 s^2 txu_x u_{xx}k_3(y) + \\
&\frac{1}{2}\beta stxu_{xx}u_t k_3(y) + \frac{1}{2}\beta stxuu_{txx}k_3(y) + 4\beta stxu_{tx}u_x k_3(y) + \frac{2}{5}u_{tx}k_3(y),
\end{aligned}
$$

$$
\begin{aligned}
T_3^y \;=\;& -\frac{1}{6}x^2 u_t k_3' + \frac{1}{3}x^2 u_{ty}k_3(y) - \frac{2}{3}\beta^2 s^2 txu_{xy}k_3(y) - \frac{1}{3}\beta stxu_t k_3' + \frac{2}{3}\beta stxu_{ty}k_3(y) \\
&+\frac{1}{3}\beta^2 s^2 txu_x k_3' - \frac{1}{6}\beta^2 s^2 t^2 u_t k_3' + \frac{1}{3}\beta^2 s^2 t^2 u_{ty}k_3(y) + \frac{1}{6}\beta^3 s^3 t^2 u_x k_3' + \\
&\frac{1}{6}\beta sx^2 u_x k_3' - \frac{1}{3}\beta^3 s^3 t^2 u_{xy}k_3(y) - \frac{1}{3}\beta sx^2 u_{xy}k_3(y),
\end{aligned}
$$

$$
\begin{aligned}
T_4^t \;=\;& -u_x^2 k_4(y) - \frac{1}{3}u_t k_4(y) - \frac{1}{5}u_{xxx}k_4(y) + \frac{2}{3}xu_{tx}k_4(y) - \frac{1}{3}xu_y k_4' + \frac{1}{3}xu_{yy}k_4(y) + \\
&\frac{1}{2}uu_{xx}k_4(y) + \frac{1}{3}xuk_4'' + \frac{1}{5}xu_{xxxx}k_4(y) - \frac{1}{2}\beta stuu_{xxx}k_4(y) + \frac{3}{2}\beta stu_x u_{xx}k_4(y) \\
&-\frac{1}{2}xuu_{xxx}k_4(y) + \frac{3}{2}xu_x u_{xx}k_4(y) - \frac{1}{3}\beta su_x k_4(y) + \frac{1}{3}\beta stu_{yy}k_4(y) + \frac{1}{3}\beta stuk_4'' \\
&-\frac{1}{3}\beta stu_y k_4' + \frac{2}{3}\beta stu_{tx}k_4(y) + \frac{1}{5}\beta stu_{xxxx}k_4(y),
\end{aligned}
$$

$$
\begin{aligned}
T_4^x \;=\;& \frac{1}{2}xu_{xx}u_t k_4(y) + \frac{1}{2}xuu_{txx}k_4(y) + 4xu_{tx}u_x k_4(y) + 2\beta su_x^2 k_4(y) + \frac{4}{5}\beta su_{xxx}k_4(y) \\
&-s^2 xu_{xx}k_4(y) - \frac{1}{3}\beta su_t k_4(y) + \frac{1}{2}\beta stuu_{txx}k_4(y) - 6\beta^2 s^2 tu_x u_{xx}k_4(y) - \\
&6\beta sxu_x u_{xx}k_4(y) + 4\beta stu_{tx}u_x k_4(y) + \frac{1}{2}\beta stu_{xx}u_t k_4(y) + \frac{1}{3}\beta stu_{tt}k_4(y) + \\
&\frac{4}{5}\beta stu_{txxx}k_4(y) - \frac{1}{3}\beta sxu_{yy}k_4(y) - \beta s^3 tu_{xx}k_4(y) + \frac{1}{2}\beta suu_{xx}k_4(y) - \beta sxu_{xxxx}k_4(y)
\end{aligned}
$$

$$-\frac{1}{3}\beta^2 s^2 t u_{yy} k_4(y) - \beta^2 s^2 t u_{xxxx} k_4(y) + \frac{1}{3}\beta^2 s^2 t u_y k_4' + \frac{1}{3}\beta s x u_y k_4' - \frac{1}{3}\beta^2 s^2 u_t k_4'' -$$
$$\frac{1}{3}\beta s x u k_4'' - \frac{3}{5} u_{txx} k_4(y) + \frac{4}{5} x u_{txxx} k_4(y) - u_t u_x k_4(y) - u u_{tx} k_4(y) + \frac{1}{3} x u_{tt} k_4(y)$$
$$+ s^2 u_x k_4(y),$$

$$T_4^y = -\frac{1}{3} x u_t k_4' + \frac{2}{3} x u_{ty} k_4(y) - \frac{2}{3}\beta s x u_{xy} k_4(y) - \frac{2}{3}\beta^2 s^2 t u_{xy} k_4(y) - \frac{1}{3}\beta s t u_t k_4' +$$
$$\frac{2}{3}\beta u_{ty} k_4(y) + \frac{1}{3}\beta s x u_x k_4' + \frac{1}{3}\beta^2 s^2 t u_x k_4',$$

$$T_5^t = \frac{2}{3} u_{tx} k_5(y) + \frac{1}{5} u_{xxxx} k_5(y) + \frac{1}{3} u_{yy} k_5(y) + \frac{1}{3} u k_5'' - \frac{1}{3} u_y k_5' - \frac{1}{2} u u_{xxx} k_5(y)$$
$$+ \frac{3}{2} u_x u_{xx} k_5(y),$$

$$T_5^x = -6\beta s u_x u_{xx} k_5(y) + \frac{4}{5} u_{txxx} k_5(y) + \frac{1}{3} u_{tt} k_5(y) + \frac{1}{2} u_{xx} u_t k_5(y) + \frac{1}{2} u u_{txx} k_5(y)$$
$$+ 4 u_{tx} u_x k_5(y) - s^2 u_{xx} k_5(y) + \frac{1}{3}\beta s u_y k_4' - \beta s u_{xxxx} k_5(y) - \frac{1}{3}\beta s u k_5'',$$
$$-\frac{1}{3}\beta s u_{yy} k_5(y)$$

$$T_5^y = \frac{2}{3} u_{ty} k_5(y) - \frac{1}{3} u_t k_5' + \frac{1}{3}\beta s u_x k_5' - \frac{2}{3}\beta s u_{xy} k_5(y),$$

$$T_6^t = -\frac{1}{2} y u u_{xxx} k_6(t) + \frac{3}{2} y u_x u_{xx} k_6(t) + \frac{1}{3} y u_{yy} k_6(t) + \frac{1}{5} y u_{xxxx} k_6(t) + \frac{2}{3} y u_{tx} k_6(t) -$$
$$\frac{1}{3} y u_x k_6' - \frac{1}{3} u_y k_6(t),$$

$$T_6^x = -6\beta s y u_x u_{xx} k_6(t) - y u_x^2 k_6' + \frac{1}{3} y u k_6'' - \frac{1}{5} y u_{xxx} k_6' + \frac{4}{5} y u_{txxx} k_6(t) - \frac{1}{3} y u_t k_6'$$
$$-\frac{1}{3} y u_{tt} k_6(t) - \beta s y u_{xxxx} k_6(t) - \frac{1}{3}\beta s y u_{yy} k_6(t) + \frac{1}{2} y u u_{xx} k_6' + \frac{1}{2} y u_{xx} u_t k_6(t)$$
$$+ \frac{1}{2} y u u_{txx} k_6(t) + 4 y u_{tx} u_x k_6(t) + \frac{1}{3}\beta s u_y k_6(t) - s^2 y u_{xx} k_6(t),$$

$$T_6^y = -\frac{2}{3}\beta s y u_{xy} k_6(t) + \frac{2}{3} u k_6' - \frac{1}{3} u_t k_6(t) - \frac{1}{3} y u_y k_6' + \frac{2}{3} y u_{ty} k_6(t) + \frac{1}{3}\beta s u_x k_6(t),$$

$$T_7^t = \frac{1}{3}u_{yy}k_7(t) - \frac{1}{3}u_x k_7' + \frac{2}{3}u_{tx}k_7(t) + \frac{1}{5}u_{xxxx}k_7(t) + \frac{3}{2}u_x u_{xx}k_7(t) - \frac{1}{2}uu_{xxx}k_7(t),$$

$$\begin{aligned} T_7^x = {} & -6\beta s u_x u_{xx}k_7(t) - u_x^2 k_7' - \frac{1}{5}u_{xxx}k_7' + \frac{4}{5}u_{txxx}k_7(t) - \frac{1}{3}u_t k_7' + \frac{1}{3}u_{tt}k_7(t) + \\ & \frac{1}{3}uk_7'' + 4u_{tx}u_x k_7(t) + \frac{1}{2}uu_{txx}k_7(t) + \frac{1}{2}u_{xx}u_t k_7(t) + \frac{1}{2}uu_{xx}k_7' - s^2 u_{xx}k_7(t) \\ & -\beta s u_{xxxx}k_7(t) - \frac{1}{3}\beta s u_{yy}k_7(t), \end{aligned}$$

$$T_7^y = -\frac{2}{3}\beta s u_{xy}k_7(t) + \frac{2}{3}u_{ty}k_7(t) - \frac{1}{3}u_y k_7',$$

$$\begin{aligned} T_1^t = {} & -\frac{1}{6}u_y^2 - \frac{2}{3}u_x^3 + 2uu_x u_{xx} + \frac{1}{5}uu_{xxxx} - \frac{1}{3}u_x u_t + \frac{1}{3}uu_{yy} + \frac{2}{3}uu_{tx} - \frac{1}{5}u_x u_{xxx} \\ & -\frac{1}{3}u^2 u_{xxx} + \frac{1}{10}u_{xx}^2, \end{aligned}$$

$$\begin{aligned} T_1^x = {} & su_x u_{xxx} - 6suu_x u_{xx} + \frac{2}{5}u_{tx}u_{xx} - \frac{3}{5}u_x u_{txx} - \frac{1}{2}su_{xx}^2 + \frac{1}{6}su_y^2 + \frac{1}{3}u^2 u_{txx} + \\ & \frac{1}{2}s^2 u_x^2 + 2su_x^3 - \frac{4}{3}u_x^2 u_t - \frac{1}{5}u_t u_{xxx} + \frac{4}{5}uu_{txxx} + \frac{1}{3}uu_{tt} - \frac{1}{3}suu_{yy} + \\ & \frac{10}{3}uu_x u_{tx} - suu_{xxxx} - s^2 uu_{xx} + \frac{2}{3}uu_{xx}u_t - \frac{1}{6}u_t^2, \end{aligned}$$

$$T_1^y = -\frac{2}{3}suu_{xy} + \frac{1}{3}su_x u_y + \frac{2}{3}uu_{ty} - \frac{1}{3}u_y u_t;$$

$$\begin{aligned} T_2^t = {} & \frac{1}{3}u_t f_{2(z)}(y,z) + \frac{1}{5}uf_{2(zzzz)}(y,z) + \frac{1}{5}u_x f_{2(zzz)}(y,z) + \frac{1}{5}u_{xx}f_{2(zz)}(y,z) + \\ & \frac{1}{5}u_{xxx}f_{2(z)}(y,z) - \frac{1}{3}u_y f_{2(y)}(y,z) + \frac{1}{3}uf_{2(yy)}(y,z) + \frac{2}{3}u_{tx}f_2(y,z) + \\ & \frac{1}{5}u_{xxxx}f_2(y,z) + \frac{1}{3}u_{yy}f_2(y,z) + u_x^2 f_{2(z)}(y,z) - \frac{1}{2}uu_{xxx}f_2(y,z) + \\ & \frac{3}{2}u_x u_{xx}f_2(y,z) + \frac{1}{3}su_x f_{2(z)}(y,z) - \frac{1}{2}uu_{xx}f_{2(z)}(y,z) + \frac{2}{3}suf_{2(zz)}(y,z) + \\ & uu_x f_{2(zz)}(y,z), \end{aligned}$$

$$\begin{aligned} T_2^x = {} & \frac{1}{5}u_t f_{2(zzz)}(y,z) + \frac{2}{5}u_{tx}f_{2(zz)}(y,z) + \frac{1}{3}u_{tt}f_2(y,z) + \frac{4}{5}u_{txxx}f_2(y,z) + \\ & \frac{3}{5}utxxf_{2(z)}(y,z) - \frac{3}{5}su_{xx}f_{2(zz)}(y,z) - \frac{2}{5}su_x f_{2(zzz)}(y,z) - \frac{1}{5}suf_{2(zzzz)}(y,z) \end{aligned}$$

$$+u_xu_tf_{2(z)}(y,z) - \frac{4}{5}\,su_{xxx}f_{2(z)}(y,z) + uu_{tx}f_{2(z)}(y,z) + \frac{1}{2}\,uu_{txx}f_2(y,z) +$$
$$\frac{1}{2}\,u_{xx}u_tf_2(y,z) + \frac{1}{3}\,su_yf_{2(y)}(y,z) - \frac{1}{3}\,suf_{2(yy)}(y,z) - 2su_x^2f_{2(z)}(y,z) +$$
$$\frac{1}{3}\,su_tf_{2(z)}(y,z) - s^2u_{xx}f_2(y,z) - su_{xxxx}f_2(y,z) - \frac{1}{3}\,su_{yy}f_2(y,z) +$$
$$4u_xu_{tx}f_2(y,z) - s^2u_xf_{2(z)}(y,z) - \frac{2}{3}\,s^2uf_{2(zz)}(y,z) - \frac{1}{2}\,suu_{xx}f_{2(z)}(y,z) -$$
$$suu_xf_{2(zz)}(y,z) - 6su_xuxxf_2(y,z),$$

$$T_2^y = \frac{1}{3}\,su_xf_{2(y)}(y,z) - \frac{2}{3}\,su_{xy}f_2(y,z) - \frac{1}{3}\,u_tf_{2(y)}(y,z) + \frac{2}{3}\,u_{ty}f_2(y,z),$$

$$T_3^t = -\frac{1}{2}\,yuu_{xxx}f_3(t) + \frac{3}{2}\,yu_xu_{xx}f_3(t) - \frac{1}{3}\,yu_xf_3'(t) - \frac{1}{3}\,u_yf_3(t) + \frac{1}{3}\,yu_{yy}f_3(t) +$$
$$\frac{1}{5}\,yu_{xxxx}f_3(t) + \frac{2}{3}\,yu_{tx}f_3(t),$$

$$T_3^x = \frac{1}{2}\,yuu_{txx}f_3(t) + \frac{1}{2}\,yu_{xx}u_tf_3(t) + 4yu_xu_{tx}f_3(t) + \frac{1}{2}\,yuu_{xx}f_3'(t) - s^2yu_{xx}f_3(t)$$
$$-syu_{xxxx}f_3(t) - \frac{1}{3}\,syu_{yy}f_3(t) + \frac{1}{3}\,yuf_3''(t) + \frac{1}{3}\,yu_{tt}f_3(t) - \frac{1}{3}\,yu_tf_3'(t) +$$
$$\frac{1}{3}\,su_yf_3(t) + \frac{4}{5}\,yu_{txxx}f_3(t) - \frac{1}{5}\,yu_{xxx}f_3'(t) - yu_x^2f_3'(t) - 6syu_xu_{xx}f_3(t),$$

$$T_3^y = -\frac{1}{3}\,yu_yf_3'(t) + \frac{2}{3}\,yu_{ty}f_3(t) + \frac{1}{3}\,su_xf_3(t) - \frac{2}{3}\,syu_{xy}f_3(t) + \frac{2}{3}\,uf_3'(t) - \frac{1}{3}\,u_tf_3(t),$$

$$T_4^t = -\frac{1}{2}\,uu_{xxx}f_4(t) + \frac{3}{2}\,u_xu_{xx}f_4(t) + \frac{2}{3}\,u_{tx}f_4(t) - \frac{1}{3}\,u_xf_4'(t) + \frac{1}{5}\,u_{xxxx}f_4(t) + \frac{1}{3}\,u_{yy}f_4(t),$$

$$T_4^x = 4u_xu_{tx}f_4(t) - s^2u_{xx}f_4(t) - su_{xxxx}f_4(t) - \frac{1}{3}\,su_{yy}f_4(t) + \frac{1}{2}\,uu_{xx}f_4'(t) - u_x^2f_4'(t)$$
$$+\frac{1}{3}\,uf_4''(t) + \frac{1}{3}\,u_{tt}f_4(t) - \frac{1}{3}\,u_tf_4'(t) + \frac{4}{5}\,u_{txxx}f_4(t) - \frac{1}{5}\,u_{xxx}f_4'(t) - 6su_xu_{xx}f_4(t)$$
$$+\frac{1}{2}\,uu_{txx}f_4(t) + \frac{1}{2}\,u_{xx}u_tf_4(t),$$

$$T_4^y = -\frac{2}{3}\,su_{xy}f_4(t) - \frac{1}{3}\,u_yf_4'(t) + \frac{2}{3}\,u_{ty}f_4(t),$$

$$
\begin{aligned}
T_5^t &= -\frac{1}{3}u_y f_5'(y) + \frac{2}{3}u_{tx} f_5(y) + \frac{1}{5}u_{xxxx} f_5(y) + \frac{1}{3}u f_5'(y) + \frac{1}{3}u_{yy} f_5(y) - \frac{1}{2}u u_{xxx} f_5(y) \\
&+\frac{3}{2}u_x u_{xx} f_5(y),
\end{aligned}
$$

$$
\begin{aligned}
T_5^x &= \frac{1}{3}u_{tt} f_5(y) + \frac{4}{5}u_{txxx} f_5(y) - 6s u_x u_{xx} f_5(y) + \frac{1}{2}u_{xx}u_t f_5(y) + \frac{1}{3}s u_y f_5'(y) - \\
&s^2 u_{xx} f_5(y) - s u_{xxxx} f_5(y) - \frac{1}{3}s u_{yy} f_5(y) - \frac{1}{3}s u f_5''(y) + \frac{1}{2}u u_{txx} f_5(y) \\
&+4u_x u_{tx} f_5(y),
\end{aligned}
$$

$$
T_5^y = \frac{1}{3}u_x f_5'(y) - \frac{2}{3}s u_{xy} f_5(y) - \frac{1}{3}u_t f_5'(y) + \frac{2}{3}u_{ty} f_5(y),
$$

where $z = -x - st$,

$$
\begin{aligned}
T_1^t &= \frac{1}{10}u_{xx}^2 + 2u u_x u_{xx} - \frac{2}{3}u_x^3 + \frac{1}{5}u u_{xxxx} + \frac{1}{3}u u_{yy} - \frac{1}{3}u_t u_x - \frac{1}{5}u_x u_{xxx} - \\
&\frac{1}{3}u^2 u_{xxx} + \frac{2}{3}u u_{tx} - \frac{1}{6}u_y^2,
\end{aligned}
$$

$$
\begin{aligned}
T_1^x &= -s u_x u_{xxx} + \frac{1}{2}s u_{xx}^2 - \frac{3}{5}u_x u_{txx} + \frac{2}{5}u_{tx}u_{xx} + \frac{10}{3}u u_x u_{tx} + s u u_{xxxx} + \frac{1}{3}s u u_{yy} \\
&-s^2 u u_{xx} + \frac{2}{3}u u_{xx} u_t + \frac{4}{5}u u_{txxx} - \frac{1}{5}u_t u_{xxx} - \frac{1}{6}s u_y^2 + \frac{1}{3}u^2 u_{txx} - \frac{4}{3}u_t u_x^2 - \\
&2s u_x^3 + \frac{1}{2}s^2 u_x^2 + \frac{1}{3}u u_{tt} - \frac{1}{6}u_t^2 + 6s u u_x u_{xx},
\end{aligned}
$$

$$
T_1^y = \frac{2}{3}s u u_{xy} - \frac{1}{3}s u_x u_y + \frac{2}{3}u u_{ty} - \frac{1}{3}u_t u_y;
$$

$$
\begin{aligned}
T_2^t &= \frac{1}{5}u g_{2(rrrr)}(y,r) - \frac{1}{5}u_x g_{2(rrr)}(y,r) + \frac{1}{5}u_{xx} g_{2(rr)}(y,r) - \frac{1}{5}u_{xxx} g_{2(r)}(y,r) - \\
&\frac{1}{3}u_t g_{2(r)}(y,r) + \frac{2}{3}u_{tx} g_2(y,r) + \frac{1}{5}u_{xxxx} g_2(y,r) + \frac{1}{3}u_{yy} g_2(y,r) + \frac{1}{3}u g_{2(yy)}(y,r) \\
&-\frac{1}{3}u_y g_{2(y)}(y,r) - g_{2(r)}(y,r)u_x^2 + \frac{1}{2}u u_{xx} g_{2(r)}(y,r) + u u_x g_{2(rr)}(y,r) \\
&+\frac{3}{2}u_x u_{xx} g_2(y,r) - \frac{1}{2}u u_{xxx} g_2(y,r) + \frac{1}{3}s u_x g_{2(r)}(y,r) - \frac{2}{3}s u g_{2(rr)}(y,r),
\end{aligned}
$$

$$
\begin{aligned}
T_2^x \;=\;& s u u_x g_{2(rr)}(y,r) - \frac{3}{5}\, u_{txx} g_{2(r)}(y,r) - \frac{1}{5}\, u_t g_{2(rrr)}(y,r) + \frac{2}{5}\, u_{tx} g_{2(rr)}(y,r) + \\
& \frac{4}{5}\, u_{txxx} g_2(y,r) + \frac{1}{3}\, u_{tt} g_2(y,r) + s u_{xxxx} g_2(y,r) - s^2 u_{xx} g_2(y,r) + s^2 u_x g_{2(r)}(y,r) \\
& + \frac{1}{3}\, s u_{yy} g_2(y,r) - \frac{2}{3}\, s^2 u g_{2(rr)}(y,r) + \frac{1}{3}\, s u g_{2(yy)}(y,r) - \frac{1}{3}\, s u_y g_{2(y)}(y,r) + \\
& \frac{1}{3}\, s u_t g_{2(r)}(y,r) - \frac{4}{5}\, s u_{xxx} g_{2(r)}(y,r) + \frac{1}{5}\, s u g_{2(rrrr)}(y,r) - \frac{2}{5}\, s u_x g_{2(rrr)}(y,r) + \\
& \frac{3}{5}\, s u_{xx} g_{2(rr)}(y,r) - u_x u_t g_{2(r)}(y,r) - 2 s u_x^2 g_{2(r)}(y,r) + \frac{1}{2}\, u u_{txx} g_2(y,r) + \\
& \frac{1}{2}\, u_t u_{xx} g_2(y,r) + 4 u_x u_{tx} g_2(y,r) - u u_{tx} g_{2(r)}(y,r) - \frac{1}{2}\, s u u_{xx} g_{2(r)}(y,r) + \\
& 6 s u_x u_{xx} g_2(y,r),
\end{aligned}
$$

$$
T_2^y = \frac{2}{3}\, u_{ty} g_2(y,r) - \frac{1}{3}\, u_t g_{2(y)}(y,r) - \frac{1}{3}\, s u_x g_{2(y)}(y,r) + \frac{2}{3}\, s u_{xy} g_2(y,r),
$$

$$
\begin{aligned}
T_3^t \;=\;& \frac{3}{2}\, y u_x u_{xx} g_3(t) - \frac{1}{2}\, y u u_{xxx} g_3(t) + \frac{2}{3}\, y u_{tx} g_3(t) + \frac{1}{5}\, y u_{xxxx} g_3(t) + \frac{1}{3}\, y u_{yy} g_3(t) - \\
& \frac{1}{3}\, y u_x g_3'(t) - \frac{1}{3}\, u_y g_3(t),
\end{aligned}
$$

$$
\begin{aligned}
T_3^x \;=\;& 6 s y u_x u_{xx} g_3(t) + \frac{1}{3}\, s y u_{yy} g_3(t) - s^2 y u_{xx} g_3(t) + s y u_{xxxx} g_3(t) + \frac{1}{2}\, y u u_{txx} g_3(t) + \\
& \frac{1}{2}\, y u_t u_{xx} g_3(t) + 4 y u_x u_{tx} g_3(t) + \frac{1}{2}\, y u u_{xx} g_3'(t) - y u_x^2 g_3'(t) - \frac{1}{3}\, s u_y g_3(t) - \\
& \frac{1}{5}\, y u_{xxx} g_3'(t) + \frac{4}{5}\, y u_{txxx} g_3(t) - \frac{1}{3}\, y u_t g_3'(t) + \frac{1}{3}\, y u_{tt} g_3(t) + \frac{1}{3}\, u_y g_3''(t),
\end{aligned}
$$

$$
T_3^y = \frac{2}{3}\, u g_3'(t) - \frac{1}{3}\, y u_y g_3'(t) - \frac{1}{3}\, u_t g_3(t) - \frac{1}{3}\, s u_x g_3(t) + \frac{2}{3}\, y u_{ty} g_3(t) + \frac{2}{3}\, s y u_{xy} g_3(t),
$$

$$
\begin{aligned}
T_4^t \;=\;& \frac{2}{3}\, u_{tx} g_4(t) + \frac{1}{5}\, u_{xxxx} g_4(t) + \frac{1}{3}\, u_{yy} g_4(t) - \frac{1}{3}\, u_x g_4'(t) - \frac{1}{2}\, u u_{xxx} g_4(t) + \\
& \frac{3}{2}\, u_x u_{xx} g_4(t),
\end{aligned}
$$

$$
\begin{aligned}
T_4^x \;=\;& -u_x^2 g_4'(t) - \frac{1}{5}\, u_{xxx} g_4'(t) + \frac{4}{5}\, u_{txxx} g_4(t) - \frac{1}{3}\, u_t g_4'(t) + \frac{1}{3}\, u_{tt} g_4(t) + \frac{1}{3}\, u g_4''(t) \\
& + 6 s u_x u_{xx} g_4(t) + \frac{1}{2}\, u u_{txx} g_4(t) + \frac{1}{2}\, u_t u_{xx} g_4(t) + 4 u_x u_{tx} g_4(t) + \frac{1}{3}\, s u_{yy} g_4(t) \\
& - s^2 u_{xx} g_4(t) + s u_{xxxx} g_4(t) + \frac{1}{2}\, u u_{xx} g_4'(t),
\end{aligned}
$$

$$
T_4^y = -\frac{1}{3}u_y g_4'(t) + \frac{2}{3}u_{ty}g_4(t) + \frac{2}{3}su_{xy}g_4(t),
$$

$$
\begin{aligned}
T_5^t =\ & \frac{2}{3}u_{tx}g_5(y) + \frac{1}{5}u_{xxxx}g_5(y) + \frac{1}{3}u_{yy}g_5(y) + \frac{1}{3}ug_5''(y) - \frac{1}{3}u_y g_5'(y) + \frac{3}{2}u_x u_{xx}g_5(y) \\
& -\frac{1}{2}uu_{xxx}g_5(y),
\end{aligned}
$$

$$
\begin{aligned}
T_5^x =\ & \frac{4}{5}u_{txxx}g_5(y) + \frac{1}{3}u_{tt}g_5(y) + \frac{1}{3}sug_5''(y) + \frac{1}{3}su_{yy}g_5(y) - s^2u_{xx}g_5(y) + \\
& su_{xxxx}g_5(y) - \frac{1}{3}su_y g_5'(y) + 6su_x u_{xx}g_5(y) + \frac{1}{2}u_t u_{xx}g_5(y) + 4u_x u_{tx}g_5(y) \\
& +\frac{1}{2}uu_{txx}g_5(y),
\end{aligned}
$$

$$
T_5^y = -\frac{1}{3}su_x g_5'(y) - \frac{1}{3}u_t g_5'(y) + \frac{2}{3}u_{ty}g_5(y) + \frac{2}{3}su_{xy}g_5(y),
$$

where $r = -st + x$.

Here we notice that due to the presence of the arbitrary functions in the conservation laws, one can generate an infinite number of conservation laws for the two-wave mode Kadomtsev-Petviashvili (TKP).

3.3 Concluding Remarks

In this chapter, we computed the Lie point symmetries of a two-wave mode equation for the integrable Kadomtsev-Petviashvili equation. An exact solution with the aid of the Kudryashov method was obtained. It was shown that the equation admitted an infinitely large number of conservation laws.

Chapter 4

Conclusions and Discussions

In this dissertation we first briefly reviewed some basic concepts and definitions of symmetry analysis which were later used through out the dissertation. In Chapter two we constructed the conservation laws of a generalized nonintegrable nonlinear equation (2.1) by applying the multiplier method. We also derived the symmetries of the underlying equation. Thereafter, simplest equation method was employed to compute exact solutions of a generalized nonintegrable nonlinear equation(2.1).

In Chapter three the multiplier method was used to derive the conservation laws of a two-wave mode Kadomtsev-Petviashvili (TKP)equation (3.4). We then employed the Lie symmetry method in conjunction with Kudryashov method to find the exact solutions of a two-wave mode Kadomtsev-Petviashvili (TKP)equation (3.4). Finally, in Chapter four we summarized the work done in the dissertation.

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