Quadratic Mode Shape Components From Linear Finite Element Analysis

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Points on a vibrating structure move along curved paths rather than straight lines; however, this is largely ignored in modal analysis. Applications where the curved path of motion cannot be ignored include beamlike structures in rotating systems, e.g., helicopter rotor blades, compressor and turbine blades, and even robot arms. In most aeroelastic applications the curvature of the motion is of no consequence. The flutter analysis of T-tails is one notable exception due to the steady-state trim load on the horizontal stabilizer. Modal basis buckling analyses can also be performed when taking the curved path of motion into account. The effective application of quadratic mode shape components to capture the essential kinematics has been shown by several researchers. The usual method of computing the quadratic mode shape components for general structures employs multiple nonlinear static analyses for each component. It is shown here how the quadratic mode shape components for general structures can be obtained using linear static analysis. The derivation is based on energy principles. Only one linear static load case is required for each quadratic component. The method is illustrated for truss structures and applied to nonlinear static analyses of a linear and a geometrically nonlinear structure. The modal method results are compared to finite element nonlinear static analysis results. The proposed method for calculating quadratic mode shape components produces credible results and offers several advantages over the earlier method, viz., the use of linear analysis instead of nonlinear analysis, fewer load cases per quadratic mode shape component, and user-independence. [DOI: 10.1115/1.4004681]

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1 Introduction

The difficulties of calculating the vibration of rotating structures led Segalman, Dohrmann, and Slavin [1–3] to introduce the concept of quadratic components to capture the essential kinematics of the problem. Van Zyl [4] showed that quadratic mode shape components are also essential in the flutter analysis of T-tail aircraft. In this method the displacement of a point on the (nonrotating) structure is described by

$$\mathbf{u}(\boldsymbol{\chi},t)=\sum_i s_i(t)\mathbf{u}^i(\boldsymbol{\chi})+\sum_i\sum_j s_i(t)s_j(t)\mathbf{g}^{ij}(\boldsymbol{\chi}) \tag{1}$$

where $\mathbf{u}(\boldsymbol{\chi},t)$ is the instantaneous displacement of the point $\boldsymbol{\chi}$ on the structure, $\mathbf{u}^i(\boldsymbol{\chi})$ is the modal displacement of the point in linear mode i, and $\mathbf{g}^{ij}(\boldsymbol{\chi})$ is the modal displacement of the point in the quadratic mode i,j. The $s_i(t)$ are the generalized degrees of freedom, i.e., modal coordinates. Segalman and Dohrmann [2] and Segalman, Dohrmann, and Slavin [3] presented a method for determining the displacement fields that involves multiple nonlinear finite element static calculations. An alternative approach is presented below that starts with a linear normal modes analysis and then addresses the deficiencies of the resulting linear mode shapes from an energy perspective.

2 Derivation

In a typical finite element normal modes analysis the linear mode shape (eigenvector) is expressed as the linear translation and rotation of each node in the model. A complete parabolic displacement model would include coupling between modes, resulting in n^2 quadratic mode shapes for n linear modes. As a result of symmetry properties of the quadratic mode shape components, the number of unique quadratic components is actually $n(n+1)/2$. The quadratic mode shape components of individual modes are considered first, then the analysis is extended to include coupled quadratic modes.

2.1 Quadratic Components of Individual Modes. In a linear finite element normal modes analysis it is assumed that the elastic potential energy is proportional to the square of the generalized coordinate. However, this is not generally the case. The rotation of any element is represented by linear displacements of its nodes. This linear representation of rotation introduces stretching that is proportional to the square of the rotation angle, which in turn introduces an elastic energy term proportional to the fourth power of the generalized coordinate.

Consider a truss element k: It only supports axial loads and the only means of storing elastic energy is by compression or extension. Let the linear displacement in mode i of the first endpoint be $s_i\mathbf{u}^i_{k,1}$ and that of the second endpoint $s_i\mathbf{u}^i_{k,2}$, where s_i is the generalized coordinate. Let $\underline{\ell}_k$ be the vector from the first endpoint to the second endpoint. Only the component of the rotation vector normal to the truss element causes extension of the element. This component of the modal rotation vector, $\mathbf{R}^i_k$, of the element can be derived from the endpoint displacements as

$$\mathbf{R}^i_k=\underline{\ell}_k\times\left(\mathbf{u}^i_{k,2}-\mathbf{u}^i_{k,1}\right)\Big/|\underline{\ell}_k|^2 \tag{2}$$

The extension of the truss element in mode i, assuming a linear displacement model, is given approximately by

$$e_k=s_i\left(\mathbf{u}^i_{k,2}-\mathbf{u}^i_{k,1}\right)\cdot\underline{\ell}_k\Big/|\underline{\ell}_k|+\tfrac{1}{2}s_i^2\left|\mathbf{R}^i_k\right|^2|\underline{\ell}_k| \tag{3}$$

Only the first term is accounted for in a linear normal modes analysis. The stored elastic potential energy is proportional to the square of the extension and therefore contains terms up to s_i^4. These higher order terms are usually ignored. The alternative, which yields the quadratic mode shape component, is to add a displacement proportional to s_i^2 that minimizes the higher order terms in the expression for elastic potential energy. With the quadratic mode shape component $\mathbf{g}^{ii}$ added, the extension of the truss element is given approximately by

$$\begin{aligned}e_k&=\left\{s_i\left(\mathbf{u}^i_{k,2}-\mathbf{u}^i_{k,1}\right)+s_i^2\left(\mathbf{g}^{ii}_{k,2}-\mathbf{g}^{ii}_{k,1}\right)\right\}\cdot\underline{\ell}_k\Big/|\underline{\ell}_k|+\frac{1}{2}s_i^2\left|\mathbf{R}^i_k\right|^2|\underline{\ell}_k|\\&=s_i\left(\mathbf{u}^i_{k,2}-\mathbf{u}^i_{k,1}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+s_i^2\left\{\left(\mathbf{g}^{ii}_{k,2}-\mathbf{g}^{ii}_{k,1}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right.\\&\quad\left.+\frac{1}{2}\left|\mathbf{R}^i_k\right|^2|\underline{\ell}_k|\right\}\end{aligned} \tag{4}$$

The elastic potential energy of the complete truss structure in mode i is given by

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$$U=\frac{1}{2}\sum_k\frac{A_kE_k}{|\underline{\ell}_k|}e_k^2=s_i^2\sum_k\frac{A_kE_k}{2|\underline{\ell}_k|}\left\{\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right\}^2+s_i^3\sum_k\frac{A_kE_k}{|\underline{\ell}_k|}$$
$$\times\left\{\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right\}\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}$$
$$+s_i^4\sum_k\frac{A_kE_k}{2|\underline{\ell}_k|}\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}^2 \quad (5)$$

where the summation in k is over all the truss elements of the structure. It is necessary to determine the mode shape vector $\mathbf{g}^{ii}$ so that the coefficient of s_i^4 in Eq. (5) is minimized. The coefficient of s_i^3 is not explicitly minimized; however, if it were possible to zero the sum of the terms in s_i^4, the sum of the terms in s_i^3 would also be zero.

Differentiating the coefficient of s_i^4 in Eq. (5) with respect to each element of the mode shape vector $\mathbf{g}^{ii}$ and setting the result equal to zero results in a set of linear equations, which is in effect a linear static deflection problem. The load vector is constructed as follows:

(1) For each element the "stretching" is calculated from $e_k=\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|$.
(2) Forces equal to $\mathbf{f}_{k,1}^i=-\mathbf{f}_{k,2}^i=e_kA_kE_k\underline{\ell}_k\Big/|\underline{\ell}_k|^2$ are applied to the endpoints of the element.
(3) The forces for all elements are summed at the nodes.

In contrast to the elastic potential energy of the structure, the kinetic energy of a structure vibrating in a linear mode shape is proportional to the square of the modal velocity. In order to preserve this property the quadratic mode shape component should be orthogonal to the linear component with respect to the system mass matrix. It is, however, more convenient and equivalent to specify that $\boldsymbol{g}^{ii}$ should be orthogonal to $\mathbf{u}^i$ with respect to the system stiffness matrix

$$\mathbf{u}^{iT}[K]\mathbf{g}^{ii}=0 \quad (6)$$

Consequently, the set of equations from which to solve the quadratic mode shape component is different for each mode. The system of equations is; however, amenable to partial inversion. The final equation is

$$\begin{bmatrix}[K] & [K]\mathbf{u}^i\\ \mathbf{u}^{iT}[K] & 0\end{bmatrix}\begin{Bmatrix}\mathbf{g}^{ii}\\ \lambda\end{Bmatrix}=\begin{Bmatrix}\mathbf{f}^i\\ 0\end{Bmatrix} \quad (7)$$

where λ is the Lagrange multiplier and its magnitude is an indication of the influence of the orthogonality condition on the minimization problem. $\mathbf{f}^i$ is the right-hand side constructed from the elemental deformation due to rotation, multiplied by the elemental stiffness matrix, summed over all elements. The solution to this equation is

$$\lambda=\frac{\mathbf{u}^{iT}\mathbf{f}^i}{\mathbf{u}^{iT}[K]\mathbf{u}^i}$$
$$\mathbf{g}^{ii}=[K]^{-1}\left\{\mathbf{f}^i-\lambda[K]\mathbf{u}^i\right\} \quad (8)$$

The denominator in the expression for λ is simply the modal stiffness. The only matrix inversion required is that of the system stiffness matrix.

Physically, the procedure can be regarded as finding the deformation of each element due to the linearized description of rotation, i.e., the second term of Eq. (3), determining the forces that would cancel this deformation and applying these forces at the element's nodes. In addition, the quadratic mode shape component must be orthogonal to the linear component.

It is a simple exercise to calculate the elastic potential energy for a deformed truss structure by calculating the displaced nodal coordinates and the resulting extension or compression of each truss element. This procedure can be used to verify the correctness of the approximate expression in Eq. (5).

In some cases, particularly in over-restrained structures, the higher order terms in the expression for the extension of the element will not be removed completely. The remaining higher order terms are used to obtain the modal quadratic and cubic stiffness constants. The derivative of the elastic potential energy, U, with respect to the generalized coordinate s_i is given by

$$\frac{\partial U}{\partial s_i}=\sum_k\frac{A_kE_k}{|\underline{\ell}_k|}e_k^i\frac{\partial e_k^i}{\partial s_i}$$
$$=\sum_k\begin{array}{l}\frac{A_kE_k}{|\underline{\ell}_k|}\left[s_i\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+s_i^2\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}\right]\\ \times\left[\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+2s_i\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}\right]\end{array}$$
$$=\sum_k\frac{A_kE_k}{|\underline{\ell}_k|}\begin{bmatrix}s_i\left\{\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right\}^2\\ +3s_i^2\left\{\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right\}\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}\\ +2s_i^3\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}^2\end{bmatrix} \quad (9)$$

Assuming a modal stiffness model of the form

$$U=\frac{1}{2}K_1s_i^2+\frac{1}{3}K_2s_i^3+\frac{1}{4}K_3s_i^4$$
$$Q^i=\frac{\partial U}{\partial s_i}=K_1s_i+K_2s_i^2+K_3s_i^3 \quad (10)$$

where the Q^i are the generalized stiffness forces, the modal stiffness terms are given by

$$K_1=\sum_k\frac{A_kE_k}{|\underline{\ell}_k|}\left\{\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right\}^2$$
$$K_2=\sum_k\frac{3A_kE_k}{|\underline{\ell}_k|}\left\{\left(\mathbf{u}_{k,2}^i-\mathbf{u}_{k,1}^i\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|\right\}$$
$$\times\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\Big\} \quad (11)$$
$$K_3=\sum_k\frac{2A_kE_k}{|\underline{\ell}_k|}\left\{\left(\mathbf{g}_{k,2}^{ii}-\mathbf{g}_{k,1}^{ii}\right)\cdot\underline{\ell}_k/|\underline{\ell}_k|+\frac{1}{2}|\mathbf{R}_k^i|^2|\underline{\ell}_k|\right\}^2$$

It is not necessary to calculate K_1 in this way, but it can be used as a verification of the procedure.

2.2 Coupled Quadratic Modes. When the rotation of a truss element is the result of two modes, i and j, the extension of the element due to linearization of the rotation can be expressed as

$$\begin{aligned} e_k &= \frac{1}{2}\left(s_i\mathbf{R}_k^i + s_j\mathbf{R}_k^j\right)\cdot\left(s_i\mathbf{R}_k^i + s_j\mathbf{R}_k^j\right)|\underline{\ell}_k| \\ &= \frac{1}{2}s_i^2\left(\mathbf{R}_k^i\cdot\mathbf{R}_k^i\right)|\underline{\ell}_k| + \frac{1}{2}s_is_j\left(\mathbf{R}_k^i\cdot\mathbf{R}_k^j\right)|\underline{\ell}_k| \\ &\quad + \frac{1}{2}s_js_i\left(\mathbf{R}_k^j\cdot\mathbf{R}_k^i\right)|\underline{\ell}_k| + \frac{1}{2}s_j^2\left(\mathbf{R}_k^j\cdot\mathbf{R}_k^j\right)|\underline{\ell}_k| \end{aligned} \tag{12}$$

The second and third terms of this expression constitute the extension of the truss element that must be eliminated by the coupled quadratic mode shape. The first and fourth terms should be eliminated by the quadratic components of the individual modes. The displacement and velocity of a node l, due to the two modes, are given by

$$\begin{aligned} \mathbf{u}_l &= s_i\mathbf{u}_l^i + s_j\mathbf{u}_l^j + s_i^2\mathbf{g}_l^{ii} + s_is_j\mathbf{g}_l^{ij} + s_js_i\mathbf{g}_l^{ji} + s_j^2\mathbf{g}_l^{jj} \\ &= s_i\mathbf{u}_l^i + s_j\mathbf{u}_l^j + s_i^2\mathbf{g}_l^{ii} + 2s_is_j\mathbf{g}_l^{ij} + s_j^2\mathbf{g}_l^{jj} \\ \dot{\mathbf{u}}_l &= \dot{s}_i\mathbf{u}_l^i + \dot{s}_j\mathbf{u}_l^j + 2s_i\dot{s}_i\mathbf{g}_l^{ii} + 2\left(\dot{s}_is_j + s_i\dot{s}_j\right)\mathbf{g}_l^{ij} + 2s_j\dot{s}_j\mathbf{g}_l^{jj} \end{aligned} \tag{13}$$

Here the symmetry property $\mathbf{g}^{ij} = \mathbf{g}^{ji}$ was used to reduce the number of terms in the expression. The total kinetic energy of the truss structure is given by

$$\begin{aligned} T &= \frac{1}{2}\sum_l m_l|\dot{\mathbf{u}}_l|^2 = \frac{1}{2}\sum_l m_l(\dot{\mathbf{u}}_l\cdot\dot{\mathbf{u}}_l) \\ &= \frac{1}{2}\sum_l m_l\left\{\begin{array}{l}\left(\dot{s}_i\mathbf{u}_l^i + \dot{s}_j\mathbf{u}_l^j + 2s_i\dot{s}_i\mathbf{g}_l^{ii} + 2\left(\dot{s}_is_j + s_i\dot{s}_j\right)\mathbf{g}_l^{ij} + 2s_j\dot{s}_j\mathbf{g}_l^{jj}\right) \\ \cdot\left(\dot{s}_i\mathbf{u}_l^i + \dot{s}_j\mathbf{u}_l^j + 2s_i\dot{s}_i\mathbf{g}_l^{ii} + 2\left(\dot{s}_is_j + s_i\dot{s}_j\right)\mathbf{g}_l^{ij} + 2s_j\dot{s}_j\mathbf{g}_l^{jj}\right)\end{array}\right\} \\ &= \frac{1}{2}\sum_l m_l\left(\begin{array}{l}\dot{s}_i^2|\mathbf{u}_l^i|^2 + 2\dot{s}_i\dot{s}_j\left(\mathbf{u}_l^i\cdot\mathbf{u}_l^j\right) + \dot{s}_j^2\left|\mathbf{u}_l^j\right|^2 + 4s_i\dot{s}_i^2\left(\mathbf{u}_l^i\cdot\mathbf{g}_l^{ii}\right) + 4s_j\dot{s}_j^2\left(\mathbf{u}_l^j\cdot\mathbf{g}_l^{jj}\right) \\ + 4\dot{s}_i\left(\dot{s}_is_j + s_i\dot{s}_j\right)\left(\mathbf{u}_l^i\cdot\mathbf{g}_l^{ij}\right) + 4\dot{s}_j\left(\dot{s}_is_j + s_i\dot{s}_j\right)\left(\mathbf{u}_l^j\cdot\mathbf{g}_l^{ij}\right) \\ + 4\dot{s}_is_j\dot{s}_j\left(\mathbf{u}_l^i\cdot\mathbf{g}_l^{jj}\right) + 4s_i\dot{s}_i\dot{s}_j\left(\mathbf{u}_l^j\cdot\mathbf{g}_l^{ii}\right) \\ + 4s_i^2\dot{s}_i^2\left|\mathbf{g}_l^{ii}\right|^2 + 4s_j^2\dot{s}_j^2\left|\mathbf{g}_l^{jj}\right|^2 + 4\left(\dot{s}_is_j + s_i\dot{s}_j\right)^2\left|\mathbf{g}_l^{ij}\right|^2 \\ + 8s_j\dot{s}_j\left(\dot{s}_is_j + s_i\dot{s}_j\right)\left(\mathbf{g}_l^{ij}\cdot\mathbf{g}_l^{jj}\right) + 8s_i\dot{s}_is_j\dot{s}_j\left(\mathbf{g}_l^{ii}\cdot\mathbf{g}_l^{jj}\right) + 8s_i\dot{s}_i\left(\dot{s}_is_j + s_i\dot{s}_j\right)\left(\mathbf{g}_l^{ii}\cdot\mathbf{g}_l^{ij}\right)\end{array}\right) \end{aligned} \tag{14}$$

where the summation in l is over all the nodes of the structure. Ideally, the total kinetic energy should consist of the contribution of only the first and third terms in brackets. The contribution of the second term in brackets to the total kinetic energy is zero because of the orthogonality of the linear normal modes. The contribution of the fourth and fifth terms in brackets to the total kinetic energy is also zero because of the orthogonality condition enforced in the calculation of the quadratic components of individual modes. Forcing the contribution of the sixth and seventh terms to be zero constitutes the applicable orthogonality condition for the coupled quadratic mode $\mathbf{g}^{ij}$. The contribution of the eighth and ninth terms is an unwanted contribution that cannot be eliminated by the choice of the coupled quadratic mode $\mathbf{g}^{ij}$. It is also not practicable to force each quadratic mode shape component to be orthogonal to every linear mode shape. All the remaining terms are higher order terms.

The procedure for finding the coupled quadratic modes is therefore determining the contribution to the extension of each truss element from the second and third terms of Eq. (12), multiplying the extension (which may be negative) by the stiffness of the element, and applying this load to the two end points of the element. To the resulting static deflection problem must be added the conditions that the coupled quadratic mode shape is orthogonal to the two corresponding linear mode shapes.

2.3 Static Deflection. Although the typical applications of quadratic mode shapes are buckling problems and dynamic problems in rotating frames, static deflection problems are a necessary verification step. This is especially relevant because the present method is proposed as an alternative to using multiple nonlinear static deflection analyses to calculate the quadratic mode shape components.

In the derivation below a distinction is made between constant forces $\mathbf{F}^0$ and follower forces $\mathbf{F}^1$. The latter type rotates with the node at which it acts. In aeroelastic applications, steady-state pressure loads on translating and rotating surfaces behave like a combination of the two, and it is important to show that the essence of the structural response to the two types of forces is captured correctly.

The modal equilibrium equation, taking into account the higher order stiffness terms of individual modes only, is given by

$$[K_1]\{s\} + [K_2]\{s^2\} + [K_3]\{s^3\} = \{Q\} \tag{15}$$

The cross coupling terms in the higher order stiffness matrices are ignored here – in most aeroelastic applications the higher order stiffness would not even be considered.

The generalized forces Q^i are defined implicitly by the expression for virtual work which is also equal to the scalar product of the virtual displacement and actual force

$$\delta W = \sum_i \delta s_i Q^i = \sum_l \delta\mathbf{u}_l\cdot\mathbf{F}_l \tag{16}$$

where the summation in i is over the modes and the summation in l is over the nodes in the structure. $\mathbf{F}_l$ is the total force acting on node l, which depends on the modal coordinates. Expanding the virtual displacement and the total force yields

$$\begin{aligned} \sum_i \delta s_i Q^i = \sum_l &\left\{\sum_i \delta s_i\left(\mathbf{u}_l^i + 2\sum_j s_j\mathbf{g}_l^{ij}\right)\right. \\ &\left.\cdot\left(\mathbf{F}_l^0 + \mathbf{F}_l^1 + \sum_k s_k\mathbf{R}_l^k\times\mathbf{F}_l^1\right)\right\} \end{aligned} \tag{17}$$

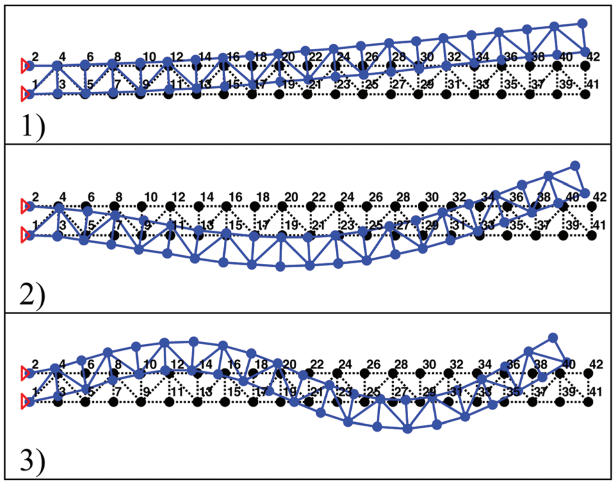

Fig. 1 First three modes of the cantilever beam

where the summations in i, j, and k are over the modes and the summation in l is over the nodes in the model. The generalized force associated with mode i is given by

$$Q^i = \sum_l \left\{ \left(\mathbf{u}_l^i + 2\sum_j s_j \mathbf{g}_l^{ij} \right) \cdot \left(\left(\mathbf{F}_l^0 + \mathbf{F}_l^1 \right) + \sum_k s_k \mathbf{R}_l^k \times \mathbf{F}_l^1 \right) \right\}$$
$$= \sum_l \left\{ \begin{array}{l} \mathbf{u}_l^i \cdot \left(\mathbf{F}_l^0 + \mathbf{F}_l^1 \right) + 2\sum_j s_j \mathbf{g}_l^{ij} \cdot \left(\mathbf{F}_l^0 + \mathbf{F}_l^1 \right) \\ + \sum_k s_k \mathbf{u}_l^i \cdot \left(\mathbf{R}_l^k \times \mathbf{F}_l^1 \right) + 2\sum_j \sum_k s_j s_k \mathbf{g}_l^{ij} \cdot \left(\mathbf{R}_l^k \times \mathbf{F}_l^1 \right) \end{array} \right\} \tag{18}$$

The generalized force vector can therefore be expressed (ignoring terms in s^2) as

$$\{Q\} = \{Q_1\} + [Q_2]\{s\} \tag{19}$$

where

$$Q_1^i = \sum_l \mathbf{u}_l^i \cdot \left(\mathbf{F}_l^0 + \mathbf{F}_l^1 \right)$$
$$Q_2^{ij} = \sum_l \left\{ 2\mathbf{g}_l^{ij} \cdot \left(\mathbf{F}_l^0 + \mathbf{F}_l^1 \right) + \mathbf{u}_l^i \cdot \left(\mathbf{R}_l^j \times \mathbf{F}_l^1 \right) \right\} \tag{20}$$

In the application to T-tail flutter, which is the main driver for this study, the Q_2 matrix is calculated as part of the unsteady generalized aerodynamic forces. The higher order structural stiffness matrices are seldom of importance in aeroelastic applications. Substituting the expression for the generalized force, Eq. (19), into the equilibrium equation, Eq. (15), yields

$$[K_1 - Q_2]\{s\} + [K_2]\{s^2\} + [K_3]\{s^3\} = \{Q_1\} \tag{21}$$

Table 1 Cantilever beam quadratic mode tip displacements

Mode	$g^{1,1}$	$g^{1,2} = g^{2,1}$	$g^{1,3} = g^{3,1}$	$g^{2,2}$	$g^{2,3} = g^{3,2}$	$g^{3,3}$
Present	–0.0289	–0.0468	–0.0226	–0.2112	–0.01429	–0.5404
Eq. (22)	–0.0289	–0.0468	–0.0228	–0.2113	–0.01432	–0.5375
Δ%	0.01	0.01	0.85	0.04	0.21	–0.52

This equation could easily be solved using a Newton-Rhapson procedure.

3 Implementation and Examples

The procedure developed above was implemented in a direct stiffness method, finite element code for truss structures implemented in MATLAB. After the usual linear normal modes analysis, the quadratic mode shape components are solved from Eq. (8). In all the examples that follow the truss elements have a modulus of elasticity of 200 GPa, a density of 7800 kg/m^3, and a cross-section area of 7.854×10^{-5} m^2.

3.1 Calculation of the Coupled Quadratic Components of a Cantilever Beam. The procedure for calculating coupled quadratic components was verified using a two-dimensional beam as test case. The beam is 20 m long and 1 m deep and the elements along the beam axis are 1 m long. The first three modes of the beam are shown in Fig. 1. The horizontal tip displacement in each quadratic mode was calculated using the linear mode shapes from a normal modes analysis as well as the formula given by Robinett et al. [5], viz.

$$g_{ij} = -\frac{1}{2}\int_0^x \phi_i' \phi_j' d\xi \tag{22}$$

where g_{ij} is the axial displacement, positive towards the tip of the beam. These values are compared to the results from the present method in Table 1. The close correlation confirms the correctness of the present method.

3.2 Static Deflection of a Linear Structure: Two-Dimensional Tower Example. The structure consists of a tower with a cross beam at the top, modeled as a two-dimensional truss. Beam width and cross beam depth are both 1 m. The tower is 7 m high including the cross beam, and the span of the cross beam is 9 m. A static load, with a vertical component of 200 kN downwards and 20 kN to the left, is applied in the center at the top of the tower, at node 15. The MATLAB finite element code was used to calculate the static deflection and to perform the normal modes analysis. Development of the parabolic mode shape of the first mode is shown in Fig. 2.

The linear mode shape component, calculated from a normal modes analysis, is shown in Fig. 2(*a*). The quadratic mode shape component, calculated using the method described above, is shown in Fig. 2(*b*). The expected shortening of the vertical column as well as the cross beam is readily apparent. The parabolic mode shape, i.e., the combination of the linear and quadratic components, is shown in Fig. 2(*c*). Figure 2(*d*) shows the elastic potential energy as a function of the generalized coordinate for the linear and parabolic mode shapes, and the linear and cubic stiffness models. The lines represent the stiffness models and the symbols represent values calculated using the displaced node coordinates. Inclusion of the quadratic mode shape component removes the artificial cubic stiffening of the linear mode shape. Because there is virtually no physical cubic stiffening of this cantilever structure, the linear and cubic stiffness models, as well as the actual values for the parabolic mode shape, are indistinguishable.

Figure 3 shows static deflection results using linear finite element, modal basis and nonlinear finite element analyses. Figures 3(*a*) and 3(*b*) show the static deflection from a linear finite element analysis and a modal basis analysis using five linear mode shapes, respectively. The correlation is satisfactory. Figure 3(*c*) shows the result of a modal static deflection analysis using five parabolic mode shapes. The deflection is much larger than for the linear analyses. Finally, the force was modeled alternately as a fixed and a follower force and the modal basis analysis results compared to a nonlinear analysis using MSC/NASTRAN. The analyses were done for different magnitudes of the applied force, defined by a scale factor

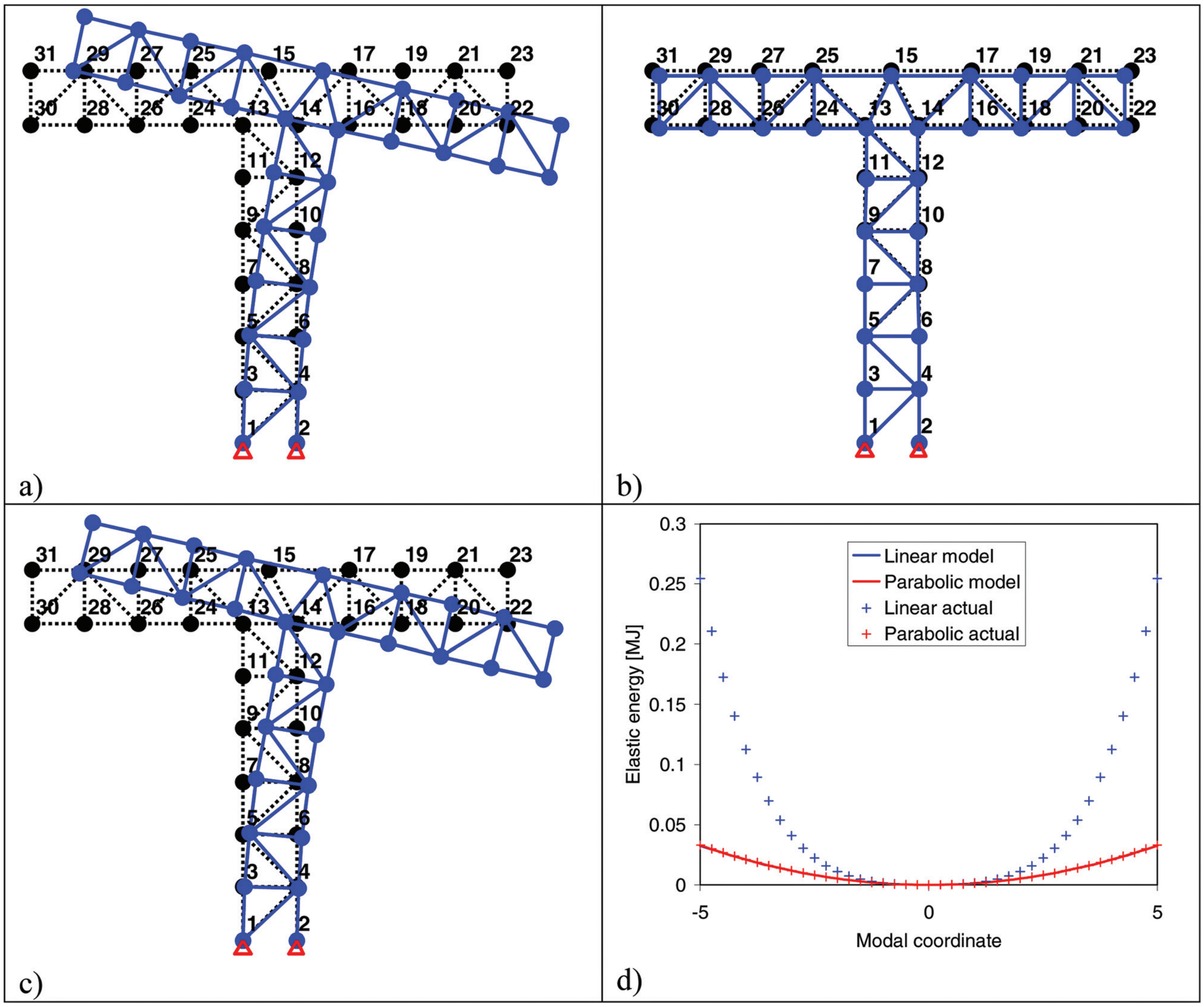

Fig. 2 Mode 1 of the two-dimensional tower: (*a*) linear mode shape, $s=5$, (*b*) quadratic mode shape component, (*c*) parabolic mode shape, (*d*) elastic energy versus generalized coordinate

applied to the nominal force of 200 kN downwards and 20 kN to the left. Comparisons between the horizontal displacement of node 15, calculated by finite element analysis and modal static deflection analysis, for these two cases are shown in Fig. 3(*d*) and are also satisfactory. Horizontal and vertical deflections of node 15, in the center of the cross beam at the top of the tower, and node 23, at the tip of the cross beam, calculated by the four methods for the nominal load are compared in Table 2.

3.3 Static Deflection of a Nonlinear, Nonsymmetric Structure: Nonsymmetric Beam Example. The structure consists of a simply supported beam modeled as a two-dimensional truss. The beam is 14 m long and 1 m deep. It is supported at both ends, at nodes 1 and 15, restrained in both directions. The over-restraining introduces physical cubic stiffening. In addition, because the supports are below the neutral axis of the beam, the force-deflection curve is nonsymmetric. A downward load of 100 kN is applied at the center node, node 8. The MATLAB finite element code was used to calculate the static deflection and to perform a normal modes analysis. Parabolic mode shapes were then calculated and used in a modal static analysis.

The development of the parabolic mode shape of the first mode is shown in Fig. 4. The linear mode shape component is shown in Fig. 4(*a*), the quadratic component in Fig. 4(*b*) and the parabolic mode shape in Fig. 4(*c*). Figure 4(*d*) shows the elastic potential energy as a function of the generalized coordinate. The inclusion of the quadratic mode shape component reduces the cubic stiffening of the linear mode shape, but the remaining cubic stiffening is still significant. In addition, the cubic stiffness model is nonsymmetric.

Figures 5(*a*) and 5(*b*) show the static deflection from a linear finite element analysis and a modal static deflection analysis using four linear mode shapes, respectively. The correlation is satisfactory. Figure 5(*c*) shows the static deflection from a modal static deflection analysis using four parabolic mode shapes. This deflection is much larger than for the linear analyses. The static analysis was repeated as a nonlinear analysis in MSC/NASTRAN. The comparison between the finite element analysis and the modal static deflection analysis results for the vertical deflection of node 8, for a range of force scale factors, is shown in Fig. 5(*d*) and is satisfactory.

4 Extension to General Elastic Elements

The proposed method for determining quadratic mode shape components from linear finite element analysis was demonstrated for truss elements. An extension to general elastic elements is straightforward.

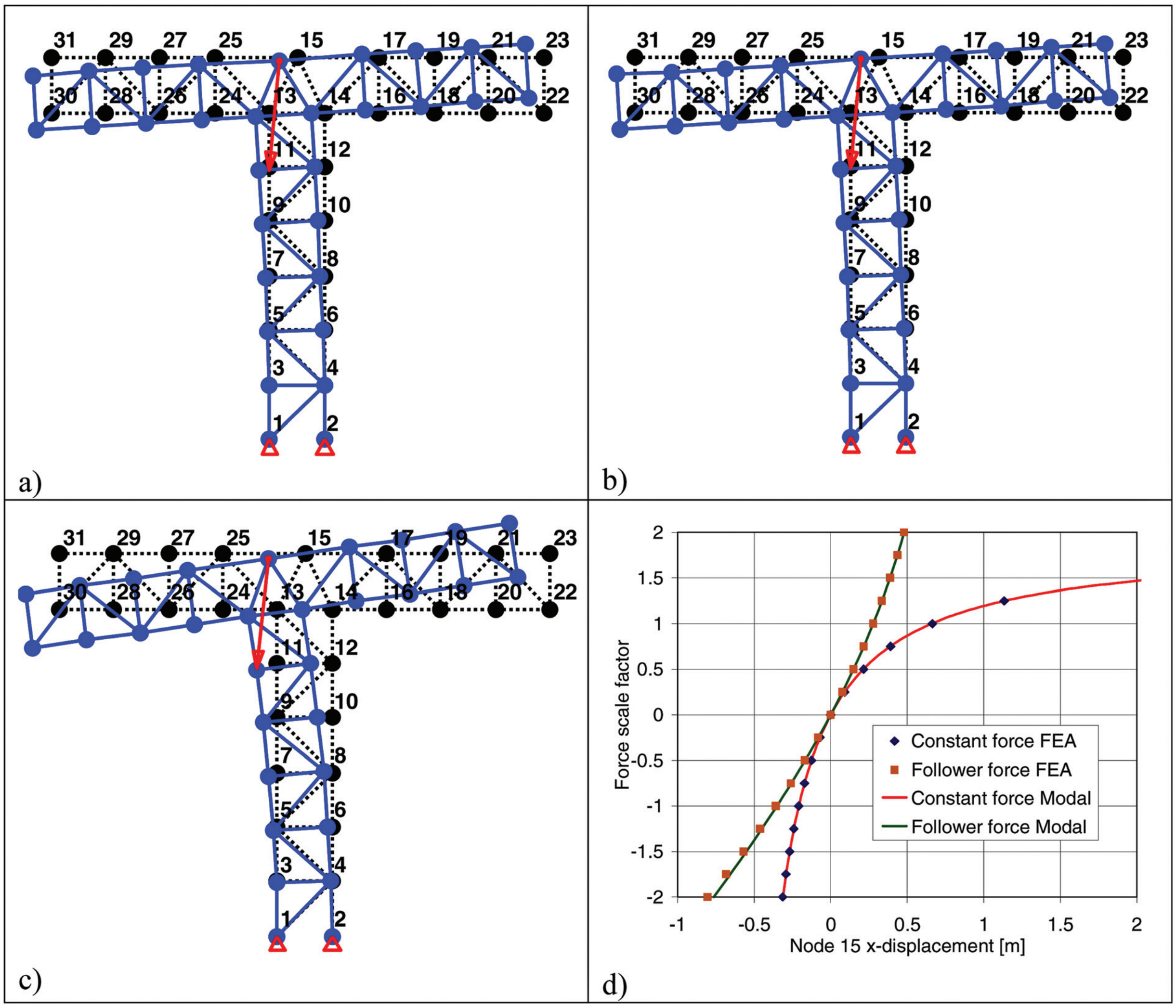

Fig. 3 Linear static analysis: (*a*) linear finite element analysis (MATLAB code), (*b*) linear modal basis analysis, (*c*) quadratic modal basis analysis, (*d*) node 15 horizontal displacement comparison with MSC/NASTRAN nonlinear analysis

The procedure is as follows:

(1) For each element a reference point and the relative position vector from the reference point to each node of the element is determined. The reference point does not have to be the centroid of the element and can be taken as the average of the element's node coordinates.
(2) For each mode, the modal rotation of the element is determined. The modal rotation of the element must be determined from the relative displacements of the nodes rather than the rotation of the nodes.
(3) For each pair of modes i, j the deformation of the element due to the linearized representation of rotation is determined. The analogous expression to the second and third terms in Eq. (12), i.e., the coefficient of $s_i s_j$, is

$$\mathbf{u}_l = \frac{1}{4}\left(\mathbf{R}^i \times \left(\mathbf{R}^j \times \mathbf{p}_l\right) + \mathbf{R}^j \times \left(\mathbf{R}^i \times \mathbf{p}_l\right)\right) \tag{23}$$

where $\mathbf{u}_l$ is the displacement of node l and $\mathbf{p}_l$ is the relative position vector of node l.

(4) The displacement components are arranged in the order of the elemental degrees of freedom and multiplied by the elemental stiffness matrix. All nodal rotations should be zero because it is only the difference between physical rigid rotation and the linear representation of rigid rotation that is of interest.
(5) The resulting force vectors are summed over all elements to obtain the static loading.
(6) The applicable orthogonality conditions are added to the static deflection problem and solved to obtain $\mathbf{g}^{ij}$. Depending on whether $i = j$, there will be either one or two orthogonality conditions.

Table 2 Displacements at two nodes from different analysis methods

Analysis method	Node 15 x	Node 15 y	Node 23 x	Node 23 y
Linear FEA	–0.317	–0.048	–0.317	0.256
Linear modal	–0.316	–0.033	–0.319	0.262
Parabolic modal	–0.680	–0.078	–0.738	0.569
Non-linear FEA	–0.664	–0.086	–0.709	0.554

It should be noted that truss elements do not bend and that the extension of the element that is proportional to s_i^2, where s_i is the modal coordinate, is entirely due to its rigid rotation. General elastic elements also experience strain proportional to s_i^2 due to the deformation of the element. The procedure outlined above only accounts for the extension due to rigid rotation. To illustrate the

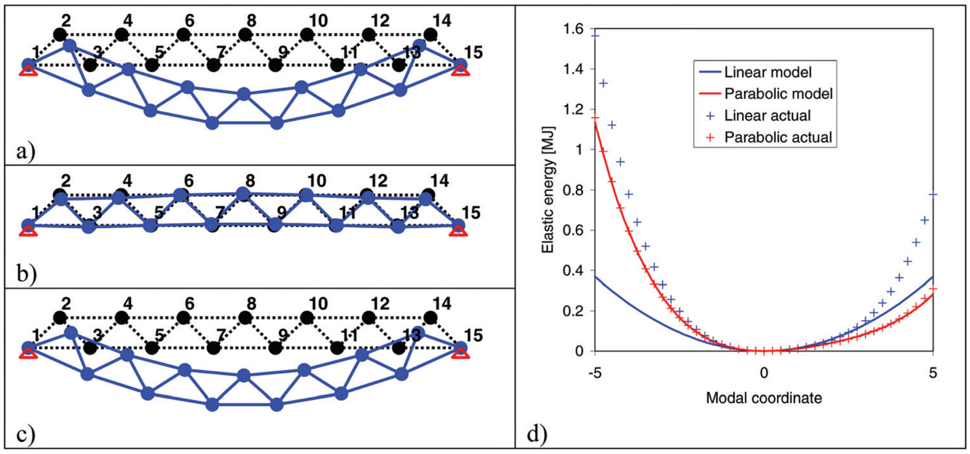

Fig. 4 Mode 1 of the nonsymmetric beam: (*a*) linear mode shape component, *s* = 5 (MATLAB finite element code), (*b*) quadratic mode shape component, (*c*) parabolic mode shape, (*d*) elastic energy versus generalized coordinate

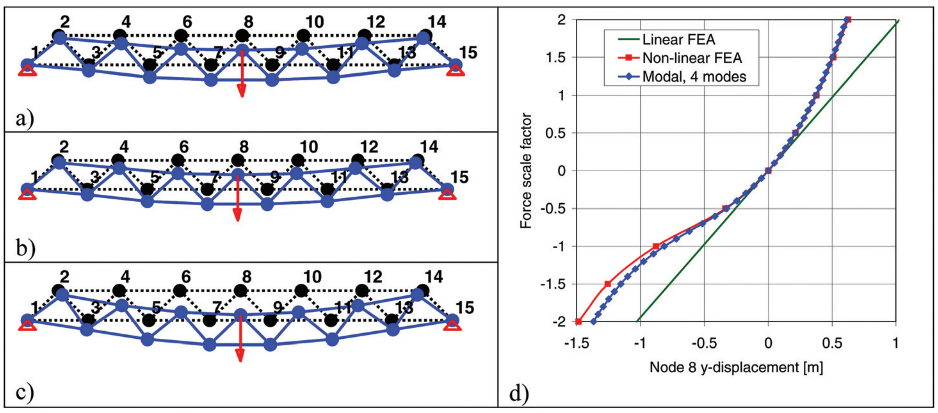

Fig. 5 Static deflection of the nonsymmetric beam: (*a*) linear finite element analysis (MATLAB code), (*b*) linear modal basis analysis, (*c*) quadratic modal basis analysis, (*d*) node 8 vertical displacement comparison with MSC/NASTRAN nonlinear analysis

effect of neglecting the stretching of beam elements due to deformation, the foreshortening of a cantilever beam in its first four bending modes was calculated for different numbers of beam elements. The percentage of the total foreshortening of the beam that is due to the element deformation is presented in Table 3. The contribution of the element deformation to the foreshortening of the beam is not insignificant, but reduces rapidly with increasing element numbers. In general, finite element models consisting of simple elements (e.g., 2-node beams, 4-node quadrilateral plates, and 8-node hexagonal solid elements) that produce accurate normal modes analysis results can be expected to produce reasonably accurate quadratic mode shape components without accounting for element deformation.

Table 3 Percentage of foreshortening of a cantilever beam due to element deformation for different numbers of beam elements

Mode	10 Elements	20 Elements	40 Elements
1	0.22	0.06	0.01
2	1.22	0.31	0.08
3	3.90	1.01	0.26
4	7.77	2.08	0.53

5 Conclusion

A method for determining quadratic mode shape components as well as higher order modal stiffness terms from linear finite element analysis, using energy considerations, was developed for truss structures. Several examples showed that the method produces credible results for static deflection problems. The procedure for extending the method of calculating quadratic mode shape components to general elastic elements was outlined. In the case of general elastic elements; however, the method is only approximate as it neglects the effect of element deformation.

Compared to the alternative method of calculating the quadratic mode shapes from multiple nonlinear static analyses, the present method has the following advantages:

(1) The determination of the quadratic mode shapes is done at infinitesimal amplitude, thereby eliminating the effect of geometric nonlinearities on the quadratic mode shapes.
(2) There is no need for a user to select amplitudes for the nonlinear static deflection analyses, making the method user-independent.
(3) The computational effort is reduced.

An MSC/NASTRAN DMAP Alter that implements the method for general elastic elements is available from the authors.

References

[1] Dohrmann, C. R., and Segalman, D. J., 1996, "Use of Quadratic Components for Buckling Calculations," Sandia National Laboratories, Albuquerque, NM, Technical Report No. SAND-96-2367C.

[2] Segalman, D. J., and Dohrmann, C.R., 1996, "A Method for Calculating the Dynamics of Rotating Flexible Structures, Part 1: Derivation," ASME J. Vibr. Acoust., **118**, pp. 313–317.

[3] Segalman, D. J. Dohrmann, C. R. and Slavin, A.M., 1996, "A Method for Calculating the Dynamics of Rotating Flexible Structures, Part 2: Example Calculations," ASME J. Vibr. Acoust., **118**, pp. 318–322.

[4] Van Zyl, L. H., and Mathews, E. H., 2011, "Aeroelastic Analysis of T-tails Using an Enhanced Doublet Lattice Method," J. Aircr., **48**(3), pp. 823–831

[5] Robinett, R. D., III, Wilson, D. G., Eisler, G. R., and Hurtado, J. E., 2005, *Applied Dynamic Programming for Optimization of Dynamical Systems*, Society for Industrial and Applied Mathematics, Philadelphia, PA, pp. 134–150