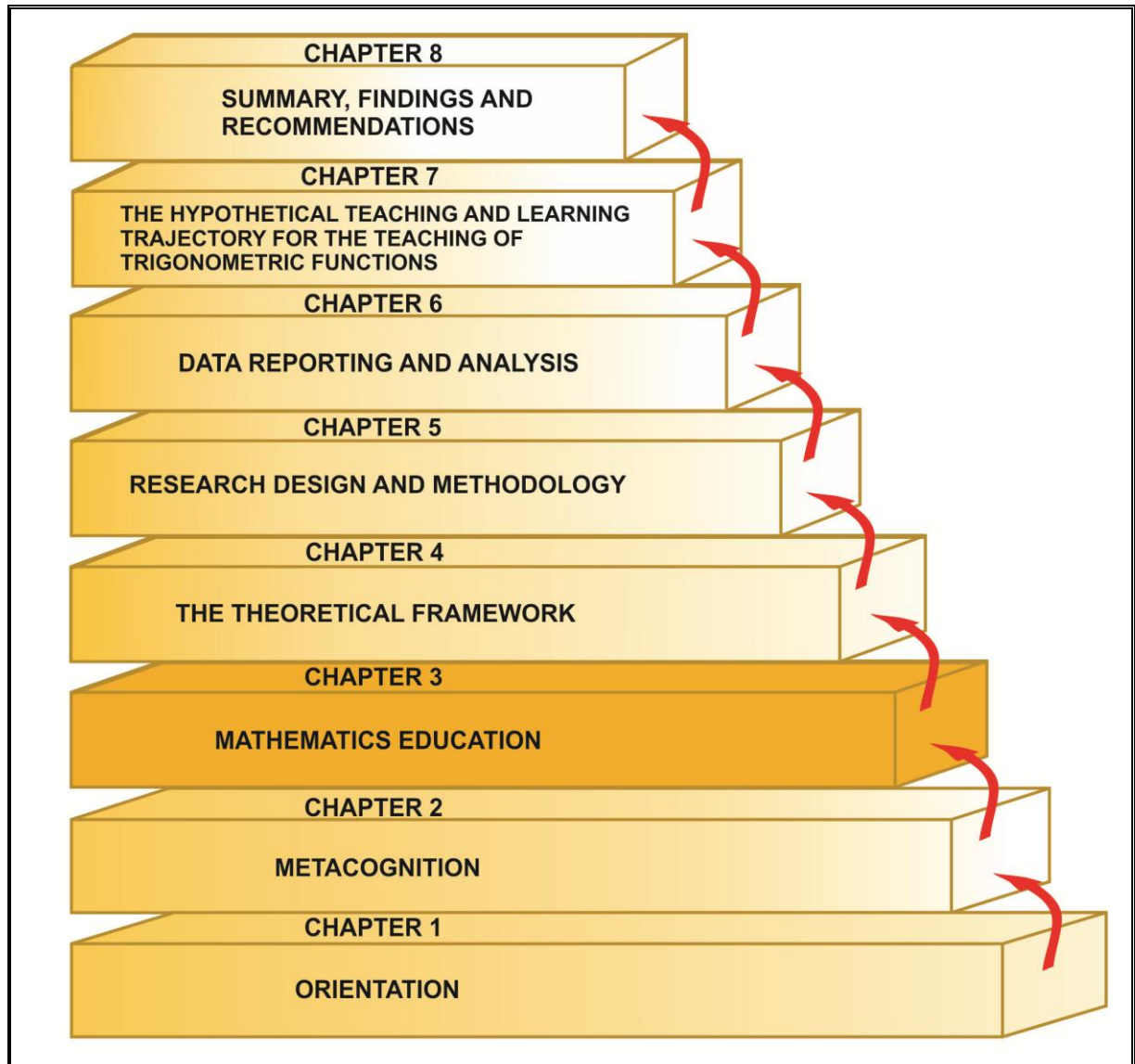
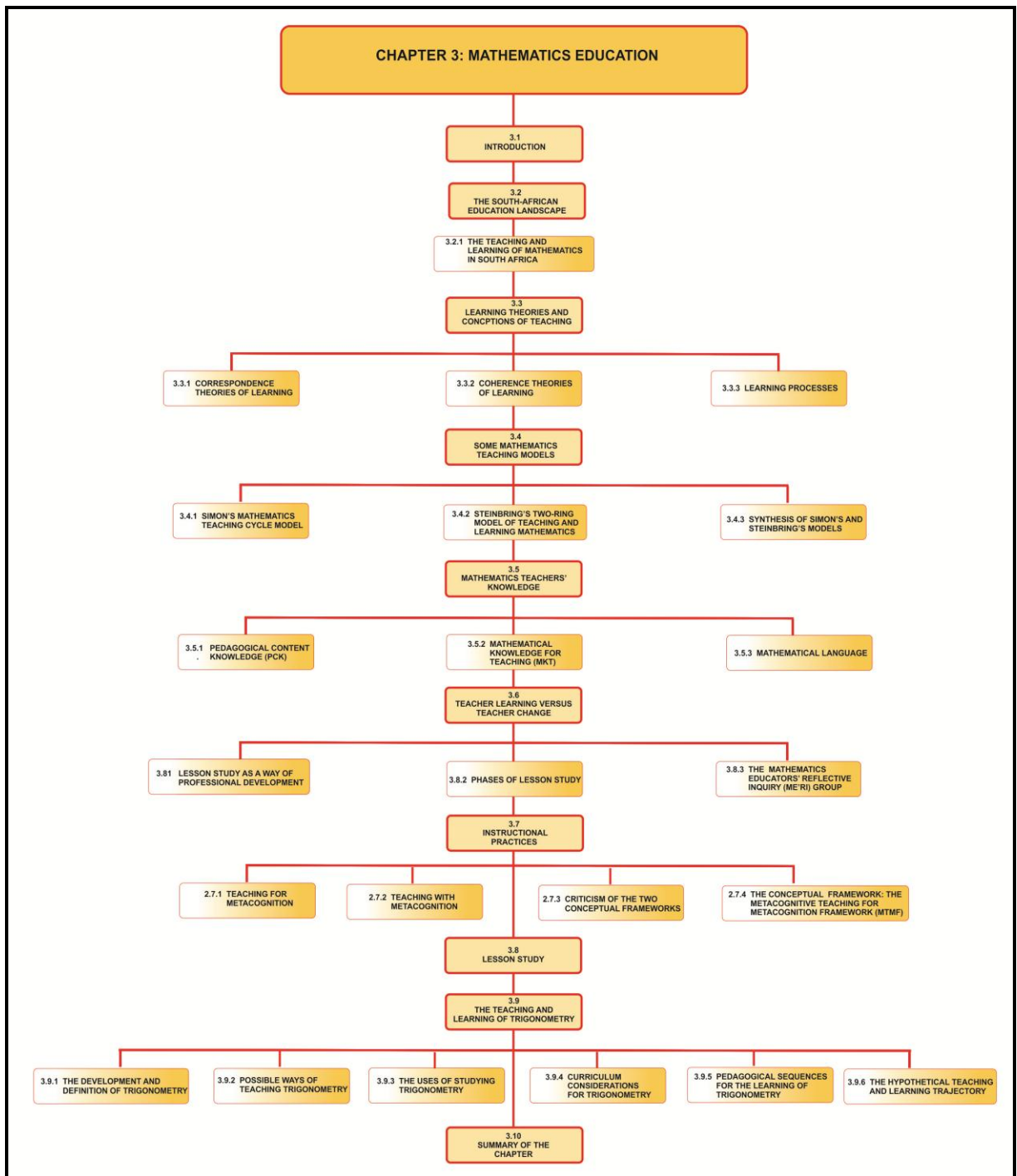


CHAPTER 3: MATHEMATICS EDUCATION





3.1 INTRODUCTION

The previous chapter outlined metacognition as context of the study. Subsequently this chapter aims to create draft guidelines for informing the design and development of the intervention from extant literature that seeks to address the problem of how to use metacognitive skills and the mathematical language of mathematics teachers for the enhanced teaching of trigonometric functions in this educational design-based study. According to Herrington, McKenney, Reeves, and Oliver (2007, p. 4094) the literature review becomes even more critical in a design-based research study because of the following reason:

It facilitates the creation of draft design guidelines to inform the design and development of the intervention that will seek to address the identified problem. In most studies, and especially in design-based research, the literature review is a continual process. Findings from an iteration of review often promulgate further literature study as well as fine-tuning of the principles guiding the design. Inherent in the literature review is the identification of the conceptual underpinnings of the problem in order to assist the researcher to understand and predict the elements of a potential solution.

This chapter therefore addresses the conceptual underpinnings of the problem in terms of mathematics education in the South African context and in particular the teaching of trigonometric functions. The South African education landscape (§3.2) is therefore first sketched, followed by learning theories and conceptions of teaching (§3.3). Some mathematics teaching models (§3.4) are then discussed before the focus shifts to the knowledge of mathematics teachers (§3.5) in which mathematical language is also discussed. As this study is mainly concerned with mathematics teachers, the learning of the teacher versus teacher change (§3.6), as well as their instructional practices (§3.7) are then addressed. Adapted Lesson study as the vehicle through which the teaching of the mathematics teacher is studied is then discussed. Finally the chapter addresses the teaching and learning of trigonometry (§3.9) and ends in the identification of some principles from literature that can guide the teaching of trigonometric functions before a summary of the chapter (§3.10) is offered.

3.2 THE SOUTH AFRICAN EDUCATION LANDSCAPE

Education in South African schools is a matter of great concern. Learners are faring poorly in both international assessments (TIMSS, 2011) and the annual national assessments (ANA) (Department of Basic Education, 2012). Reports on studies conducted indicate dissatisfaction with this state of affairs:

South Africa has a problem of a lack of learning, and consequently poor learner achievement. Many students go through the system without actually learning – a problem that starts in the early grades with a failure to instil basic operational skills. The problem is not one of a lack of funding (Centre for Development and Enterprise, 2009).

The problem is not caused by accessibility, nor is it caused by a lack of money, judging from the South African budget for education or high school enrolment levels (Centre for Development and Enterprise, 2009). Not only is the teaching of poor quality, but also the management of the teaching and knowledge of the teachers are not up to standard. The report indicates that a number of qualified teachers in scarce subject specializations are not teaching what they have specialized in at the schools where they are employed (Centre for Development and Enterprise, 2011). Furthermore, the low salary and the poor image of the profession results in many qualified teachers, and especially teachers of quality, leaving the profession or going overseas, thus causing a brain-drain in the educational system which South Africa cannot afford (Centre for Development and Enterprise, 2011). Far too many schools that are still poorly resourced complete the bleak picture of a country whose educational system leaves much to be desired.

3.2.1 The teaching and learning of Mathematics in South Africa

The status quo concerning the teaching and learning of subjects like Mathematics and Science is not up to standard; without a doubt of poor quality (Sayed, 2002). In fact, the teaching of specifically Mathematics is in a sorry state (Jansen, 2011) and relies mostly on rote learning and memory (Jansen & Christie, 1999). Mathematics teachers are not confident in their classes, which results in teachers not spending enough time in the classroom (Gernetzky, 2012). Headings in papers (Jansen, 2013) such as “Why Math teachers bunk class” is a loud confirmation that the media wants society to know that Mathematics teachers do not spend time-on-task in their classrooms and do not utilize the available time they have to teach the learners effectively. Studies to this effect (Fiske & Ladd, 2004; Howie & Plomp, 2002; Macrae, Bodenhausen, Milne, & Jeten, 1994; Van der Flier, Thijis, & Zaaïman, 2003) ascribe this situation to inadequate training in the previous political dispensation.

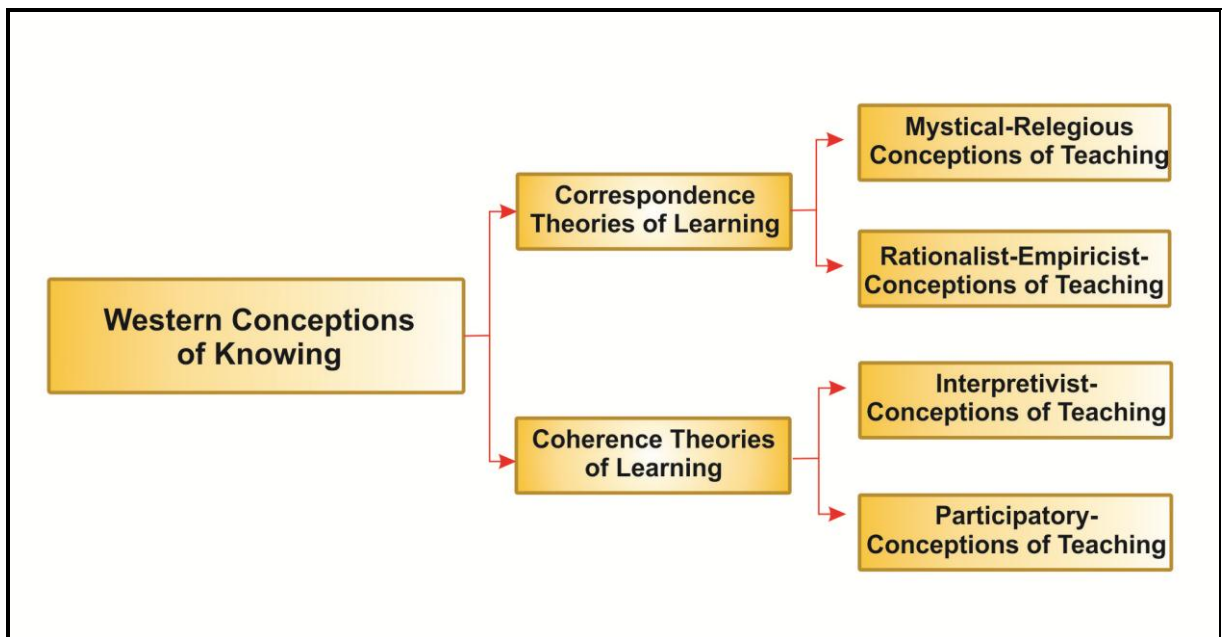
But for how long are we going to blame apartheid for the serious lack of effective mathematics teaching? The next section attempts to interrogate teaching and learning, as well as pinpoint the theories underpinning this learning and teaching.

3.3 LEARNING THEORIES AND CONCEPTIONS OF TEACHING

“To teach is to touch a life” is a very common saying in education circles. According to Davis and Sumara (2008) the word “teach” refers to the act of “signing” or “pointing” while Stigler and Hiebert (1999) assert that teaching is a form of interaction in which knowledge and skills are exchanged. Worldwide the act of teaching is not a simple issue as the following quote from Stigler and Hiebert (1999, p. 75) bears witness to:

Teaching is a system. It is not a loose mixture of individual features thrown together by the teacher. It works more like a machine, with the parts operating together and reinforcing one another, driving the vehicle forward. This is a very different way to think about teaching. It means that individual features, by themselves, are not good or bad. Their value depends on how they connect to other and fit into the lesson.

Many scholars, for example Hiebert, Gallimore, and Stigler (2002) and Davis *et al.* (2008), point out that teaching is a complex activity. It needs to be parsed in some way in order to study it and to share what is learnt. But isolating the features of teaching is not an option for Davis *et al.* (2008). The concept teaching, although very complex, is usually understood in terms of what the teacher does. However, learning, which is just as complex, informs teaching (Davis *et al.*, 2008). For Davis *et al.* (2008) teaching is about what happens to the learner, but more precisely teaching must be described in terms of the actions by which a learner is taught. This makes sense in so far as it means that whatever learning is taking place is important, but surely what the teacher does cannot be downplayed. According to Hiebert *et al.* (2002), what makes teaching meaningful is the interaction among the features of teaching and not their effects in isolation. Davis *et al.* (2008) presented the following genealogy indicating the different families of thought associated with teaching (Figure 3.1).



Source: Davis *et al.*, (2008, p. 159)

Figure 3.1: A genealogy of various conceptions of teaching

A short discussion of the relationships between knowing, learning and teaching perceptions that Davis *et al.* (2008) propose with this genealogy is given shortly. Western conceptions of knowing can be divided into two theories of learning, *viz.* Correspondence theories of learning and Coherence theories of learning. Mystical-religious conceptions of teaching and Rationalist-Empiricist conceptions of teaching resort under Correspondence theories of learning, while the Interpretivist and the Participatory conceptions of teaching resort under Coherence theories of learning.

3.3.1 Correspondence theories of learning

According to Davis and Sumara (2008) correspondence learning occurs when learning tends to be framed in terms of gathering an inner model that then gets reflected or mapped onto the outer world. The mystical–religious conceptions of teaching are understood to be dealing with issues of existence and meaning while the rationalist-empirical conceptions of teaching concern practical knowledge or know-how. These two conceptions of teaching are actually seen as complementing each other because they are both regarded as essential under the correspondence theories of learning.

Davis and Sumara (2008) list the following words as synonyms for teaching in this frame: Explaining, telling, instructing, indoctrinating, inducting, directing and setting right which are all about the act of training or conditioning individuals to respond in a certain way, and are subsequently supportive of the behaviourist theorists like Thorndike, Pavlov, Watson, Skinner and Bandura (Davis & Sumara, 2008; Goldhaber, 2000). The behaviourist theory suggests that living creatures learn by building up associations between their experience, thinking and behaviour.

The problem with these theories is that nothing merely passes from the outside to the inside during learning.

3.3.2 Coherence theories of learning

Coherence theories differ from the correspondence theories in that learning is believed to be re-organised in terms of keeping the coherence at manifold levels of organisation within the individual and between the individual and the context, and are focused on human phenomena and more specific on individual or collective understandings as well as culture (Davis & Sumara, 2008). In fact, coherence theories assume that knowing is a dynamic, evolving and relational phenomenon in which knowledge cannot exist in isolation from the knower. Interestingly the interpretivistic and participatory conceptions of teaching hold that learning is not determined by teaching, but conditioned by and subsequently dependent on teaching. For coherence theorists it matters what teachers do, or do not do (Davis & Sumara, 2008).

For the interpretivistic conceptions of teaching, the following synonyms for teaching are listed by Davis and Sumara (2008): mediating, mentoring, modelling and initiating, which relate more to constructivism, while participatory-oriented synonyms for teaching include: improvising, occasioning, conversing, caring and engaging minds.

3.3.3 Learning processes

Pollard et al. (2008) propose three processes concerned with learning and teaching. They are Behaviourism, Constructivism and Social Cognition and some features of these in classrooms are portrayed in Figure 3.2. This table was selected because it was used to characterize the classrooms in the different lessons that were observed in this study.

Table 3.1: Some features of behaviourist, constructivist and social constructivist models of learning in school classrooms

	Behaviourism in classrooms	Constructivism in classrooms	Social constructivism in classrooms
Image of learner	• Passive • Individual • Extrinsically motivated	• Active • Individual • Intrinsically motivated	• Active • Social • Socially motivated
Images of teaching and learning	• Teacher transmits knowledge and skills • Learning depends on teaching and systematic reinforcement of correct behaviours	• Teacher gives child opportunity to construct knowledge and skills gradually • Learning can be independent of teaching	• Knowledge and skills are constructed gradually through experience, interaction and adult support • Learning comes through the interdependence of teacher and children

Source: Pollard. *et al.* (2008, p. 182)

Synthesis

The conceptions of teaching and learning by Davis and Summara (2008) show not only a transdisciplinary study of learning and learning systems, but also the pragmatics of pedagogy. It highlights the role of the teacher clearly while the distinction between the different models of learning in school classrooms made by Pollard *et al.* (2008) focus more on the role of the learner.

3.4 SOME MATHEMATICS TEACHING MODELS

3.4.1 Simon's Mathematics Teaching Cycle model

Simon (1997) proposed a Mathematics Teaching Cycle model for the teaching process, based on the theory of didactic situations in mathematics. This model prescribes that teachers design a hypothetical learning trajectory (HLT) from the different types of knowledge available to him/her. This HLT consists of three interrelated elements, viz. the learning goals, the learning tasks and the teacher's hypothesis of the learning process. New understandings come about when this HLT is implemented in the classroom after being adjusted according to classroom discourse. A hypothetical learning trajectory (HLT) is defined by Clements and Sarama (2004, p. 83) as describing the thinking and learning in:

a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through developmental progression of levels of thinking, created with the intend of supporting children's achievement of specific goals in that mathematical domain.

A hypothetical learning trajectory therefore must comprise three components. They are firstly the learning goal, the cognitive developmental progression and lastly sequenced instructional tasks. Leikin and Zazkis (2010), with regard to the three elements within the hypothetical learning trajectory, also offer learning goals and instructional tasks, but in the place of the cognitive developmental progression, they propose the teacher's hypothesis of the learning process.

3.4.2 Steinbring's two-ring model of teaching and learning mathematics

About a year later, Steinbring (1998) developed a two-ring model for the teaching-learning of mathematics as autonomous systems. Following the model, the teacher first uses his/her own knowledge together with the knowledge of his/her learners to design learning tasks for his/her learners. These tasks are then approached by the learners, who by activating their own knowledge devise, their own interpretations of the task, reflect on the process and end up constructing their own knowledge, while the teacher observes, supports the learning process, reflects and adjust the task to the situation and in the process transform his/her own knowledge. According to Steinbring's (1998) two-ring model of teaching and learning mathematics, when the teacher gives a particular task to the learners, not only does the teacher then observe and support the learning process, he/she also reflects on the learning situation and adjusts the task to the situation. Understanding the mind and actions of a

teacher has become increasingly important to such an extent that researchers have begun to build frameworks and models (Artzt & Armour-Thomas, 2002). The point here is that what the teacher does, and in this case in particular, what the mathematics teacher does, is crucially important, and to a large extent influences what happens to the learner and should not be underestimated.

3.4.3 Synthesis of Simon's and Steinbring's models

Steinbring's (1998) two-ring model of teaching and learning mathematics outlined the learning process very well and as mathematics teaching and learning is a complex phenomenon, this model was employed to make sense of the learning process, especially for the participating teachers during the adapted. On the other hand, Simon's (1997) mathematics teaching model compares favourably with design-based research in terms of the designing of HLT and therefore this study adopted this model as well and adjusted it following the empirical data gathered in the study. Having said that, I am of the opinion that the name "hypothetical learning trajectory" is somewhat contradictory to what is actually happening as it seems to focus on the learning done by the learner, whereas it mostly definitely describes the instruction by the teacher and also what the teacher hypothesizes as the cognitive process that will prevail during the trajectory. I am in agreement with Daro, Mosher, and Corcoran (2011, p. 17) who argue that:

It might have been clearer if Simon had used the term "hypothetical teaching or pedagogical trajectory," or perhaps, because of the need to anticipate the way the choices and sequence of teaching activities might interact with the development of students' thinking or understanding, they should have been called "teaching and learning trajectories," or even "instructional trajectories" (assuming "instruction" is understood to encompass both teaching and learning). There is a slight ambiguity in any case in talking about learning as having a trajectory. If learning is understood as being a process, with its own mechanisms, it isn't learning per se that develops and has a trajectory so much as the products of learning (thinking, or rather concepts, of increasing complexity or sophistication, skills, and so on) that do.

Subsequently I am changing the name "hypothetical learning trajectory" to the "hypothetical teaching and learning trajectory" for the purpose of this study.

3.5 MATHEMATICS TEACHERS' KNOWLEDGE

Understanding the structure of mathematics teachers' knowledge and how it is developed is described by many researchers as a complex endeavour (Artzt & Armour-Thomas, 2002; Leiken, 2005; Leikin & Rota, 2006; Leikin & Zazkis, 2010; Nathan & Knuth, 2003; Wilson & Bai, 2010). Therefore, after many studies done around mathematics education, our understanding about the mathematical knowledge required to teach mathematics well, is still insufficient (Ball, Lubienski, & Mewborn, 2001). However, Pollard. *et al.* (2008) mention that the subject knowledge of the teacher is a contributing factor to effective teaching, while Alexander, Rose, and Woodhead (1992) refer to subject knowledge as “a critical factor at every point in the teaching process”. Although three kinds of subject knowledge were distinguished by Shulman (1986) *viz.* content knowledge (knowledge of the subject held by the teacher), pedagogical content knowledge (knowledge for teaching purposes) and curricular knowledge (knowledge of curriculum structures and materials), only pedagogical content knowledge and the mathematical knowledge for teaching are subsequently discussed in the next section for the purpose of this study.

3.5.1 Pedagogical Content Knowledge (PCK)

The term *pedagogical content knowledge* (PCK) was coined by Shulman (1987) and came about after the realization that teachers' content knowledge is not sufficient for effective teaching. Since then researchers within the mathematics education arena refined this notion and the term *mathematical knowledge for teaching* (MKT) was developed (Silverman & Thompson, 2008). For Silverman and Thompson (2008) PCK is not merely pedagogical knowledge. Rather PCK refers to the knowledge that lies at the intersection between content knowledge, knowledge of the learner's thinking and knowledge of mathematics education and pedagogy, while MCK is more specific to mathematics teaching and answers the question: “What do teachers do when teaching mathematics?” (Ball & Bass, 2003) and thus refers to the special ways in which a teacher needs to know mathematical procedures in order to interact productively with learners in the context of mathematics teaching (Ball & Bass, 2003; Ball *et al.*, 2001; Hiebert *et al.*, 2002; Hill & Ball, 2004a; Hill, Rowan, & Ball, 2005; Hill & Ball, 2004b).

3.5.2 Mathematical Knowledge for Teaching (MKT)

Ball *et al.* (2001) maintain that the judicious actions in the mathematics classroom such as deciding on the tools, resources and curriculum material, the skillful interpretation and acting on learners' responses and the careful design of useful homework tasks are seriously hampered when the requisite MKT is lacking. Silverman and Thompson (2008) explain that a *key developmental understanding* (KDU) which was introduced by (Simon, 1997) as a way to think about understandings that are powerful springboards for learning is a key element of developing MKT. Silverman and Thompson (2008) proposed the following useful framework for mathematical knowledge for teaching.

A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she (1) has developed a KDU within which that topic exists, (2) has constructed models of the variety of ways students may understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas (p. 508).

From the above framework it is clear that MCK is not knowledge that just came about by coincidence. It seems as if mathematics teachers need to consciously and deliberately focus on developing MCK for the effective teaching of mathematics. This links very well with the effective use of metacognitive skills (§2.2.2, §2.4 and §2.6).

3.5.3 Mathematical language

It is quite understandable that people tend to think about mathematics only in terms of numbers, symbols and operations. But mathematics is more than numbers and operations. Mathematics is also a language (Department of Basic Education, 2011). However, Mathematics is not an ordinary language in the way that English, Afrikaans or Setswana are. (Halliday, 1978) equates the term mathematical language to what is called "the register of mathematics". According to Halliday (1978), the register of mathematics is the terms and grammatical structures that express mathematical purposes. Mathematics is becoming increasingly language based therefore it is essential to tie language intervention to the mathematics curriculum. An explicit focus on mastering of mathematical language through language instruction is needed because textbooks are language-dense and often lack visual representations in presenting new language in mathematics. In a study done by Swanson

(2002, p. 1475), some learners were placed in a lower stream as a result of “lack of language skills”. A lack of language skills will evidently result in a lack of mathematical language. At an education seminar hosted by the Centre for Education Policy Development, Umalusi and the University of the Witwatersrand in Johannesburg, Adler said that the use of language is crucial for learners to be able to grasp complex mathematical concepts (Gernetzky, 2011). This idea is well supported by Babineau (2010) who asserts that the language of instruction in the Mathematics classroom and the wording in the textbooks and tests includes complex language, terms, phrases and symbols. Having trouble reading and understanding the language of the problems presented to them, is according to Harmon, Hedrick, and Wood (2005) the reason why some learners do poorly in Mathematics tests. Mathematical language is also an important ingredient for communication in the mathematics classroom. The National Council of Teachers of Mathematics (NCTM) lists communication as one of the five standards for mathematics teaching (National Council of Teachers of Mathematics, 2000).

3.5 3.1 Language elements in Mathematics

The San Diego County Office of Education (San Diego County Office of Education, 2007) distinguishes between three language elements in Mathematics, viz. (i) content language (such as technical language, for example fraction, equation, degree and exponent), (ii) symbolic language (for example numbers, tables, graphs and formulas) and (iii) academic language (such as language used in the instruction of mathematics, for example simplify, evaluate and convert). Spanos, Rhodes, Dale, and Crandall (1988), however, propound that Mathematical language encompasses rules that can be classified as follows: (i) syntax (rules by which symbols and terms are put together to create and support higher units of meaning), (ii) semantics (rules by which different symbols and terms are allocated meaning), (iii) and pragmatics (rules for the register’s use and appropriate forms of writing and discourse), all of which are highly specialized. “Understanding mathematical concepts involves high cognitive demand, even for basic content” (Tchoshanov, 2010, p. 145). These elements and different kinds of rules were all considered in the mathematical language that was used in the teaching and learning of trigonometric functions and its application in solving problems in this study.

3.5.3.2 Language discourse in the mathematics classroom

Ball and Bass (2003) assert that language is a foundation of mathematical reasoning and the primary medium of instruction in mathematical classrooms. In this regard Tall, Thomas, Davis, Gray, and Simpson (2000, p. 230) propose that “a certain use of language ‘indicates’

whether individuals ‘in fact’ conceive of something a certain way”. In another project, The Learning Mathematics for Teaching Project (2010), Ball and her team developed four language codes, viz. (i) Technical language, (ii) Conventional notation, (iii) Talk about meaning and (iv) Use of general language for expressing mathematical ideas. The main aim of the project was to measure the quality of instruction in the mathematics classroom. The first two codes were used to describe whether symbols (e.g. =, 8, or x) and terms (e.g. exponent, relation, or associative) were appropriately used in instruction. The third code showed whether notation or terms were simply used during instruction or whether there is explicit talk about their meaning. The last code captured the use of general (non-technical) language for expressing mathematical ideas and concepts. Currently in mathematics teaching, statements and questions are often written in the passive (for example *twelve (is) divided by three*), and there is no one-to-one correspondence between mathematical symbols and the words they represent (Corasaniti Dale & Cuevas, 1992). For example: In the word sum: *Ten times a number is five less than five times the number*; learners must understand how key words relate to each other, that *a number* and *the number* refer to the same quantity (Corasaniti Dale & Cuevas, 1992). Questions inform the direction of the discourse, the learners’ reflection on their learning and formative assessment, and therefore the questioning techniques a teacher employs are important tools. In a sense, “teachers’ questions control learners’ learning” (Manouchehri & Lapp, 2003, p. 563). Schoenfeld (1992, p. 33) promulgate the following for successful mathematics instruction regarding mathematical language:

1. *Mathematics instruction should be geared towards precision in both written and oral presentations.*
2. *It should contribute to clear and coherent arguments reflecting the mathematical style and sophistication appropriate to the relevant mathematical level.*
3. *Mathematics instruction should lead to communication between teachers and learners and learners with their peers using the language of mathematics.*
4. *Mathematics instruction should contribute to the ability to read and use text and other mathematical materials.*

These points were taken into consideration for the development of the hypothetical teaching and learning trajectory.

The next sub-section will be a discussion on code switching as strategy in multilingual mathematics classrooms.

3.5.3.3 Code switching

In multilingual South African mathematics classrooms the problem of English as the language of learning and teaching (LOLT) remains one of the biggest challenges in effective mathematics education. As a way of providing learners access to mathematics and English, various researchers (Adler, 2001; Setati, 2005; Setati & Adler, 2000; Setati et al., 2008; Vorster, 2005) suggest that the home language is used as a resource in multilingual classrooms. This can be done by code switching, which, according to Farrugia (2009) can be seen as the practice of switching between two or more languages in a conversation or an utterance. Code switching was investigated in this study as possible strategy to enhance effective mathematical language usage by teachers and their learners.

3.5.3.4 Concept development in mathematics

Trigonometric functions have some crucial concepts embedded within which learners need a conceptual understanding of and should not rely on mnemonics. According to Cavanagh (2008) and Wongapiwatkul, Laosinchai, and Panijpan (2011) mnemonics is not an effective way to remember because the learners' understanding of trigonometric functions are then restricted to following a rule and application becomes mechanical. A **mnemonic** refers to a word, short poem, or sentence that is intended to help someone remember things such as scientific rules or spelling rules. In the context of this study it is used to remember trigonometric ratios. For concept development, Sfard (2004) theorizes that mathematical development progresses from operational understanding to conceptual understanding through interiorisation, condensation and reification. According to Challenger (2009) operational understanding refers to knowledge of a procedure, while conceptual understanding "is a static object that links together facts, processes, properties, algebraic and spatial representations simultaneously". Challenger (2009, p. 3) further explains that:

A process or representation is interiorized by an individual then condensed mentally. Reification is a cognitive re-organization that enables the individual to link the process to other facts and processes creating an object conception. This static object helps inform decisions on suitable procedures for subsequent problem solving and is itself informed by new processes and representations thus a duality of understanding is constructed with the operational feeding the reified object which in turn feeds operational know how.

This pathway for the development of a concept was used as the first guiding principle for the teaching of trigonometric functions using mathematical language. Concepts and terminology that form part of the mathematical language within trigonometric functions from the Curriculum and Assessments Policy Statement (CAPS) document (Department of Basic

Education (DBE), 2011a, p. 12) include: trigonometric functions, right-angled triangle, Sin theta, cos theta, tan theta, two-dimensional, trigonometric models, geometric models, ratios, constructing and definitions. Although only these concepts and terms could be found from the curriculum document, other terms like, hypotenuse, adjacent, opposite, reference, angle and side are all terms required in the effective teaching/learning of trigonometric functions.

3.6 TEACHER LEARNING VERSUS TEACHER CHANGE

The main body of research into mathematics teaching and learning focuses mainly on how learners learn mathematics. Little is known about how teachers can learn in their own classrooms (Leikin & Zaskis, 2010). According to Perrin-Glorian, DeBlois, and Robert (2008), what it is that changes in teachers' mathematical knowledge and how this change occurs in an authentic mathematics classroom is quite scant. However, studies that have been conducted in this area (Shulman, 1986; 1987), indicate that the subject-matter knowledge of the teacher grows through teaching. According to Brousseau (1997), the theory of didactic situations in Mathematics involves a sequence of situations that result in new knowledge construction not only by the learners but also by their teacher who, after the teaching, gains new comprehension. This process starts with the teacher's comprehension of the section of the work to be taught and the resulting transformation of the teacher's knowledge related to the topic while teaching (Brousseau, 1997; Leikin & Zaskis. 2010).

Leu (2004) compares the previous approach versus an alternative approach to teacher learning in Table 3.2.

Table 3.2: Teacher learning

Previous approach	Alternative approach
The goal is to have teachers who are competent in following rigid and prescribed classroom routines	The goal is to have teachers who are reflective practitioners who can make informed professional choices
Teachers are "trained" to follow patterns	Teachers are prepared to be empowered professionals
Results in passive learning	Results in active and participatory learning
Cascade model run as centralized workshops or programmes	School-based model in which all teachers participate
"Expert" driven	Teacher facilitated (with support materials)
Little inclusion of "teacher knowledge" and realities of classrooms	Central importance of "teacher knowledge" and realities of classrooms
Positivist based	Constructivist based

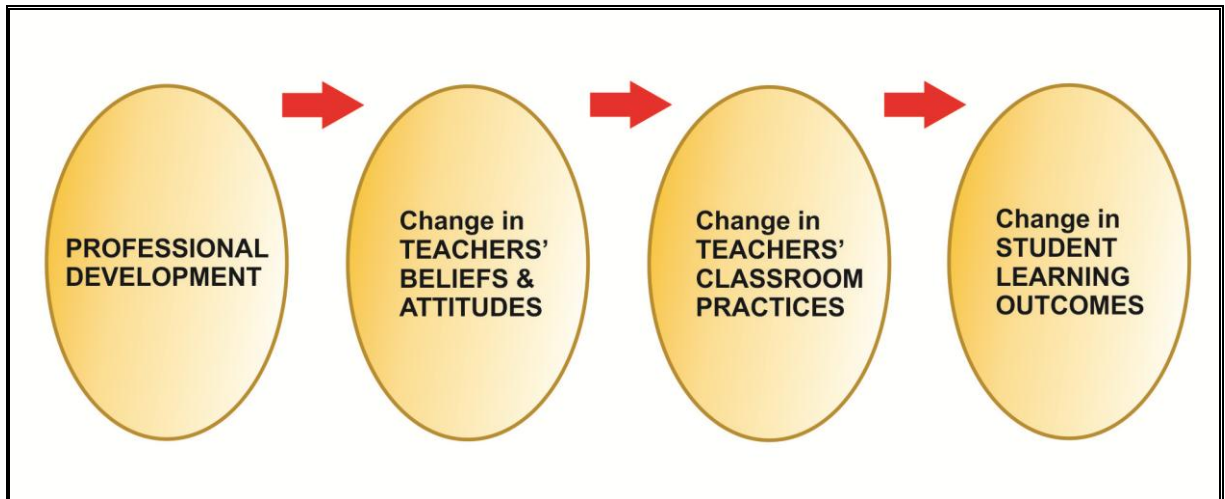
Source: Leu (2004, p. 6)

Based on the enquiry nature of this study the alternative approach was pursued and added onto, in order to address ways in which the metacognitive skills and mathematical language of teachers might be catalysts in enhancing more effective teaching of the trigonometric functions.

It was also important to review what constitutes teacher learning. Leikin and Zazkis (2010) refer to teachers' learning as the gaining of new components of knowledge as well as refining familiar ideas (Leikin, 2005) and Leikin *et al.* (2006). According to Leikin and Zazkis (2010), Learning through Teaching (LTT) is a relatively unexplored phenomenon and might be interpreted as teachers watching other experts teach, and in doing so, learning from them, or teachers learning by reflecting on the lesson afterwards. It can, however, also refer to what teachers are learning while they are teaching. However, in this study this notion that there will be some sort of learning through Mathematics teachers teaching (adapted lesson study) can be enhanced by thinking and regulating the thought processes (metacognitive skills) and should be made visible through the use of mathematical language. Stigler and Hiebert (1999) and Leikin and Zazkis (2010) suggest that, although learners are supposed to learn deliberately, teachers are learning unintentionally or indirectly through teaching. Learning through teaching was selected as lens for this study because it makes it possible to focus on what teachers can learn while they are teaching. Teacher education can never prepare the mathematics teacher for teaching mathematics like it should be taught (Tzur, 2010). Not only should the mathematics teacher know the mathematics content, but he/she should know it well enough to teach it (Ball & Bass, 2003; Mason, 2002; Tzur, 2010). Other features of teacher learning include making connections between different components of prior knowledge, achieving deeper awareness of what concepts entail (which links very well with the metacognitive skills and the mathematical knowledge of teachers), and enriching the teachers' repertoire of problems and solutions (Leikin & Zazkis, 2010).

Finally, Leikin and Zazkis (2010) noted that teachers are not always aware of their learning through teaching. Even if they are aware, they sometimes tend not to admit it. In some instances they are unsure that what they have learnt is really mathematics. Proponents of learning through teaching (LTT) claim that LTT is a means to transfer a teacher's knowledge from his/her passive repertoire to the active one (Leikin & Zaskis, 2010).

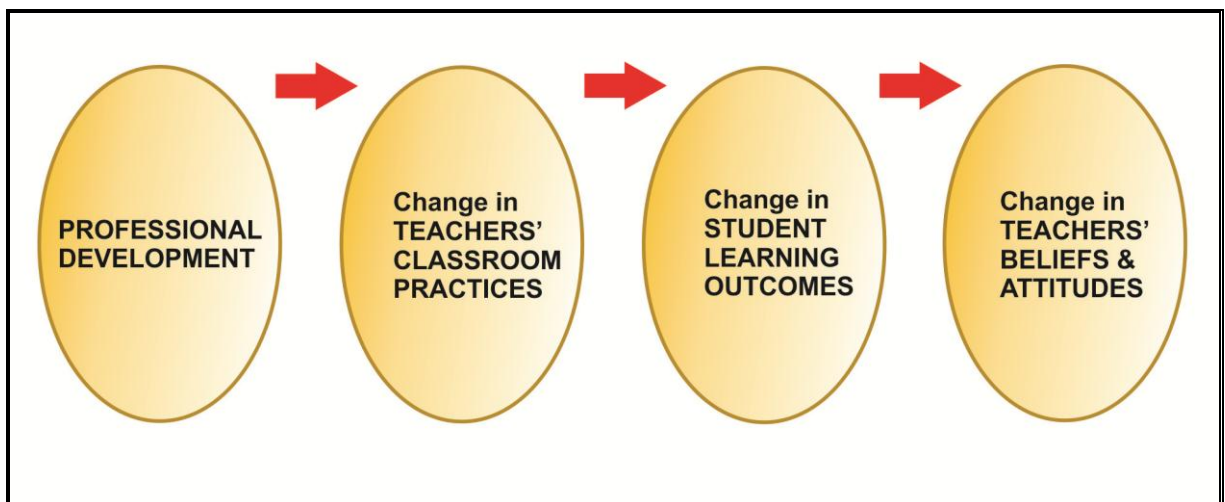
Leikin (2005) asserts that our understanding of what changes in teachers' mathematical knowledge and how these changes occur through teaching is relatively limited. Professional development activities are usually designed in the hope of changing teachers' ways of teaching to more effective ways of teaching. Lewin (1935) developed a model of teacher change which shows the order in which change occurs in teachers (Figure 3.2).



Source: Guskey, (2002, p. 383) Adjusted

Figure 3.2: Lewin's model of teacher change

Guskey (2002) differs from this perspective and proposes a model for viewing change in teachers in which the order in which the change occurs is significantly different (Figure3.3).



Source: Guskey, (2002, p. 383)

Figure 3.3: Guskey's model of teacher change

The three outcomes of professional development, *viz.* change in teachers' classroom practices, change in student learning outcomes and change in teachers' beliefs and attitudes in both the models are clearly the same, but the order differs. This begs the question to what extent the order of these three outcomes is important. Guskey (2000) argues that the order of the outcomes influences the sustainability of the change. The point he is making is that teachers tend to only change their beliefs and attitudes after they have witnessed and experienced that the desired learning outcomes were reached by their learners and not before that.

3.7 INSTRUCTIONAL PRACTICES

Instructional practice forms the heart of teaching and therefore this section embarks on an explicit focus on what it entails. Artzt and Armour-Thomas (2002) identified nine (9) critical aspects of instructional practices. These critical aspects, although deeply interrelated, were sequenced to focus firstly on the content, then on the teacher and lastly on the learner. The nine aspects are: the nature of the mathematical content, questioning, motivation and teaching strategies, homework, use of class time, verbal behaviour of learners, task orientation of learners, assessment, and teacher expectations and stereotyping. These nine aspects were used in this study as criteria for the lesson observation and part of the instructional guidelines for the development of the local instructional practice which was designed as new knowledge within the study.

The next section concentrates on lesson study as vehicle through which the hypothetical teaching and learning trajectory was designed.

3.8 LESSON STUDY

3.8.1 Lesson study as a way of professional development

Many teachers and providers of professional development globally have become interested in lesson study, a form of professional development typically used in Japanese elementary schools (Stigler & Hiebert, 1998). Japanese teachers approach lesson study as a way of professional development. The literature study reveals that lesson study in Japan has evolved into a focal point for improvement of practice and is keeping teachers grounded in everyday practice. Lesson study refers to an instructional improvement process of which the research lesson is the heart (Lewis, 2000). Yoshida (2007) is considered one of the leading researchers and educators of lesson study in the United States, and coined the term "lesson study" by translating the original Japanese term, "Jugyokenkyu", in his doctoral dissertation

research. For Japanese teachers the process is more important than the product. In China teachers are encouraged to support one another and observe one another's lessons. Records are kept and checked by the principal and inspector, who also spend time observing and monitoring the lessons (Centre for Development and Enterprise, 2009).

Although there are many ways of data collection for the lesson study it seems that experience is required in order to do it well. Teachers need to be assisted with writing up more clearly what they have seen and what they have learnt. This need was also experienced by Ono and Ferreira (2010, p. 59) in a case study involving Science teachers within the Mpumalanga Secondary Science Initiative, where lesson study was used when they noted that "it was a challenge for many South African teachers to learn what to observe during lesson study and how to record their observations". Many features of daily classroom instruction that can be observed, support learners' agency and capability as mathematics learners, including regular presentation and analysis of learner solution methods, discussion that draws out key mathematical ideas from learners' solution approaches, systematic use of the blackboard to organize mathematical ideas from the learners, questioning, and summary strategies that promote learners' metacognitive reflection and learner journaling. This is where an explicit focus on the mathematical language and the usage of metacognitive skills becomes critically important.

3.8.2 Phases of lesson study

Lesson study actually involves four stages, viz (i) studying the curriculum and formulating goals for the research lesson, (ii) planning the research lesson (iii) conducting the research lesson and (iv) reflecting on the research lesson (Lewis, Perry, & Murata, 2006). For the purpose of this study *adapted* lesson study was utilized as lesson study is quite new in South Africa and lesson study requires a *culture change* (Lewis et al., 2006). It is important to note that no matter how many stages will be involved in lesson study, what is of paramount importance is that the learners and how they are learning, thinking and communicating during the lesson is the focus. To prepare for the observation it is important that the following are agreed upon prior to the observation: who to observe, what to observe, how to record data, and how to coordinate the observation (Stigler & Hiebert, 1998). On the question about the features of lesson study, Lewis et al. (2006) responded with the following questions that need to be answered by each participant in the lesson study group:

1. What did I learn about the teaching and learning that took place during the lesson?
2. Did the lesson study work affect my sense of efficacy as a teacher?

From these questions it appears as if lesson study involves more than mere observing and discussing a lesson. It forces strategic thinking and reflecting on how the lesson progressed with an explicit focus on how did the learners learn. These questions and prior preparations will also be taken into consideration for the adapted lesson study group in this study.

3.8.3 The Mathematics Educators' Reflective Inquiry (ME'RI) group

Ezra Ramasehla, president of the second largest teacher union in South Africa, NAPTOSA (National Professional Teachers Organisation of South Africa), argues that teachers are more generalists than specialists (Dale-Jones & Morgan, 2011). If teachers need to become specialists in specific learning areas or subjects, rather than generalists, they need to reflect, investigate and improve their teaching practice by establishing a tradition of reflective and critical practice and collaborative growth with other teachers in a specific learning area or subject. Furthermore, the technical report of the Integrated Strategic Planning Framework for Teacher Education and Development (TED) in South Africa 2011 – 2025 (Department of Basic Education, 2011b, p 16) lists the following recommendation for the continuous professional development of the teachers:

Time for teachers to participate in professional learning communities and engage in quality teacher development must be deliberately and formally scheduled.

In the same vein it is envisaged by the framework for TED that “time for professional development be built into the educator’s workload and the school’s timetable” (Department of Basic Education, 2011b). It is envisaged that researchers usually probe, analyse and evaluate and are in general critical of everything; therefore teaming up with the teachers could be only beneficial for any study (Barnes & Verwey, 2008; Timperley, Wilson, Barrar, & Fung, 2007). Pollard *et al.* (2008: 53) refers to this involvement as “critical friends” and “an occasion to consider any discrepancies between ‘what is’ and ‘what ought to be’”. Many teacher networks, for example the teachers’ network at www.teachersnetwork.org, mathematics knowledge network at <https://www.ncetm.org> and the Mathematics teachers’ circle at www.mathteacherscircle.org exists where teachers can network and support such enquiries electronically. The strengths and weaknesses that the researchers and teachers bring with to the group can be effectively summarized in the following table (Table 3.3) from Pollard *et al.* (2008) which was adjusted to only select the knowledge pertaining to teachers and researchers and indicate the strengths as well as the weaknesses in respect of the knowledge that each group brings to the collaboration.

Table 3.3: A comparison of teachers' and researchers' knowledge

	Strengths	Weaknesses
Teachers' knowledge	Often practically relevant and directly useful Often communicated effectively to practitioners Often concerned with the wholeness of classroom processes and experiences	May be impressionistic and can lack rigour Usually based in particular situations which limit generalization Analysis is sometimes over-influenced by existing assumptions
Researchers' knowledge	May be based on careful research with large samples and reliable methods Often provides a clear and inclusive analysis when studied Often offers novel ways of looking at situations and issues	Often uses jargon unnecessarily and communicates poorly Often seems obscure and difficult to relate to practical issues Often fragmented educational processes and experiences

Source: Pollard *et al.*, 2008: 23 table adjusted

According to Pollard *et al.* (2008) for such collaboration between teachers and researchers to be successful there must be a frank appreciation for not only each other's strengths but also each other's weaknesses. More about this collaboration follow in Chapter five.

3.9 THE TEACHING AND LEARNING OF TRIGONOMETRY

Although grade 10 learners were introduced to the triangle in earlier grades, it is only in the 10th grade when they are confronted with this new strand of Mathematics in the South African School curriculum (Department of Basic Education, 2011a) and might be overwhelmed by the new setting of Mathematics (Calzada & Scariano, 2006). The learning of trigonometry requires from learners not only to relate the triangle to numerical relationships and manipulations thereof, but also to cope with ratios. According to Hart (1981), ratios prove to be extremely difficult for children to comprehend. For Demir (2012), a robust understanding of trigonometric functions requires different algebraic, geometric, and graphical aspects due to the complex nature of the topic. Adding onto this the many new concepts and terminology that learners encountered for the first time may prove to be challenging especially in the multilingual context of South Africa where Mathematics is taught in English which for many learners is their second language and not their mother-tongue. Furthermore Davis (2005)

noted that little attention has been given to the teaching of trigonometry in the classroom. For these reasons the initial stages of the learning of trigonometry are fraught with difficulty (Blackett & Tall, 1991). (Weber, 2005) maintains that trigonometric functions tend to be difficult for learners due to the fact that trigonometry is possibly the learners' first encounter with operations that cannot be algebraically evaluated. Demir (2012, p. 1) is of the opinion that "trigonometry is complex because the evaluation of trigonometric functions requires both geometric and algebraic reasoning".

3.9.1 The development and definition of trigonometry

The term "trigonometry" was derived from the Greek word "*trigonometria*", which translates to "triangle measuring" (Gür, 2009, p. 68). Trigonometry is commonly known as the branch of mathematics in which triangles are studied...but this general definition is not doing justice to the interconnectedness and alignment of trigonometry with other areas within the school curriculum. But where did trigonometry originate from? Hipparchus, a Greek mathematician, can be seen as the founder of trigonometry because he produced the first known table of chords in about 140 BC (O' Connor & Robertson, 1996). However, although the study of triangles by the Egyptians and the Babylonians can be traced back as far as the 2nd millennium before Christ, the systematic study of trigonometric functions started in Hellenistic mathematics (310–230 BC). Hereafter trigonometric functions were studied Indian astronomy during the sixth century through the Middle Ages where it continued in Islamic mathematics until the development of modern trigonometry in the 17th-century mathematics with Isaac Newton and James Stirling. Trigonometry finally reached its modern form with Leonard Euler in 1748 (O'Connor & Robertson, 1996).

3.9.2 Possible ways of teaching trigonometry

Several studies relating to Trigonometry teaching and learning have already been done. Studies that suggest possible ways to teach trigonometry include the use of computer software (Blackett & Tall, 1991), simulations (Sokolowski & Rackley, 2012), animation (Bagheri-Toulabi, Amiripour, & Shahvarani, 2012) and even through history (Morrison, 2006). Calzada and Scariano (2006) proposed a transition from the known sections of Algebra and Geometry learnt in the previous grades to the new section of Trigonometry in grade 10. Similarly, Bhatia, Lu, and Harrison (2011) tried to link learners' prior knowledge about Algebra, in particular algebraic equations and rational equations, with trigonometric functions in a study with 19 learners that was conducted at the University of Wisconsin Marshfield/Wood County, in an attempt to establish trigonometric identities. Their study showed that, because there are no numbers involved, most learners find establishing the trigonometric identities difficult, but that more learners attempted and successfully

established the trigonometric identities in their final examinations compared to previous years. Interestingly this study was also conducted using the lesson study approach. The long term goal in their study was to get learners to think abstractly about mathematical concepts and to work logically in a multi-step mathematical problem. However, Bhatia, Lu and Harrison (2011, p. 2) made a very important statement: They reason that although “the lesson study can teach the students to solve certain types of problems, it might have limited function in reducing student misconceptions, unless the student is also simultaneously motivated as a result of their ability to solve problems”.

Weber’s study (2005) shows that the learners who were taught using the lecture-based instruction developed a very limited understanding of trigonometric functions while the learners who received the experimental instruction developed a deep understanding of the trigonometric functions.

Gür (2009) describes particular types of errors and underlying misconceptions that occur during trigonometry lessons. In his study Gür (2009) involved 140 grade 10 learners and six grade 10 teachers who had to perform a diagnostic test consisting of seven trigonometric questions. The following problem areas were identified in the study: Order of operations, value and place of the sine and cosine functions, improper use of equation, distorted definitions, misused data, misinterpreted language, logically invalid inferences and technical mechanical errors. A study by Brown (2005) in which a framework called “the trigonometric connections” was used to investigate the learners understanding of the Sine and the Cosine function. Fi (2003) focused on the trigonometric content and pedagogical knowledge of pre-service secondary school mathematics teachers. Orhun (2002) studied the challenges experienced by learners in using trigonometry for solving problems in trigonometry.

From the above study it is evident that ways of teaching trigonometry may vary greatly. Also, trigonometric functions show a grey area in respect to “what works” in mathematics classrooms in that learners seem to have only a fragmented understanding of trigonometry most of the time (§1.2.3).

3.9.3 The uses of studying trigonometry

Trigonometry is one of those areas in Mathematics in which learners constantly ask the question why they need to learn about trigonometry. This question is fair because they do not normally in their daily life encounter words like sine, cosine and tangent. Corral (2009) for example is of the opinion that no learners will ever have to calculate the height of a tree. The situation is further exacerbated by the fact that teachers, in an attempt to enhance memorization of the trigonometric ratios, tend to teach mnemonics such as SOHCAHTOA or “All Learners Take Calculus” to remember the signs of these functions in the different

quadrants (Brown, 2005). This kind of teaching then leads learners to stop trying to make sense of trigonometry because they have a simple rule to follow (Cavanagh, 2008; Wongapiwatkul et al., 2011). Demir (2012) finds trigonometry useful as foundational knowledge for the more advanced courses, like calculus for example.

There are mainly two functionalities of teaching trigonometry: The use of trigonometry in everyday life, or otherwise the use of trigonometry as foundation for other courses. Brown (2005), Cavanagh (2008) and Wongapiwatkul, Laosinchai and Paniipan (2011) all argue for trigonometry teaching to be utilised in everyday life, but Corral (2009) differs from this and argues that trigonometry must be taught to aid the student in other courses like engineering, physics, astronomy and architecture. He is of the opinion that mathematics instructors tend to teach trigonometry without considering this important fact. This notion is supported by Beyers (2010) who is of the opinion that knowledge of trigonometry is crucial for success in technology programs, such as electrical engineering, mechanical engineering, and architectural design, offered at most colleges.

3.9.4 Curriculum considerations for trigonometry

The content of the South African Further Education and Training (FET) curriculum was refigured in the new Curriculum and Assessment Policy Statements (CAPS) (Department of Basic Education, 2011a) document and greater emphasis is placed on the mathematical content which compares well internationally (Adler, 2011). In this document trigonometry is divided into four sections of which trigonometric functions form the basis of all four sections. The topic trigonometric functions in grade 10 includes understanding trigonometric ratios and their relationships and solving problems in two dimensions by constructing geometric and trigonometric models (Department of Basic Education, 2011: 13).

3.9.5 Pedagogical sequences for the learning of trigonometry

From extant literature it emanates that the very first time learners are introduced to trigonometry as new area of mathematics, should be handled cautiously. For Blackett and Tall (1991) the initial stages of learning about trigonometric functions are challenging because of the following reasons: Firstly learners from previous grades are not used to the kind of reasoning required for relating numerical relationships to diagrams of triangles, and as such learners then also find it difficult to manipulate the symbols involved in these relationships. Another reason why learning about trigonometry for the first time is problematic is given by (Breidenbach, Dubinsky, Hawks, & Nichols, 1992) who argue that learners have trouble with trigonometric functions because these functions are operations that cannot be expressed as algebraic formulae which involve arithmetical procedures. One way of introducing trigonometry is to teach learners the different trigonometric functions and their

ratios, making use of SOHCAHTOA, a mnemonic used to remember the functions and their ratios, followed by applying the mnemonic to find various unknown angles and sides in triangles with no real life context. Mitchelmore (2000) refers to this approach where abstract definitions are taught to learners before they are given concrete examples, as the ABC approach. Quinlan (2004) is of the opinion that concrete examples of a concept should be explored first before formal definitions and terminology can be given to the learners. Most textbooks indeed traditionally start with naming the sides of right-angled triangles, but modern texts are introducing the sine of an angle not as a ratio, but as the opposite side length in a right-angled triangle, with the hypotenuse as one unit which must be recognized with the triangle rotated into any position in the unit circle on the Cartesian plane. The sine of an angle can then be defined as the rise of the angle, and the cosine of the angle as the run thereof, while the tan of the angle can be defined as the inclination of the angle in the unit circle as shown in Figure 3.4.

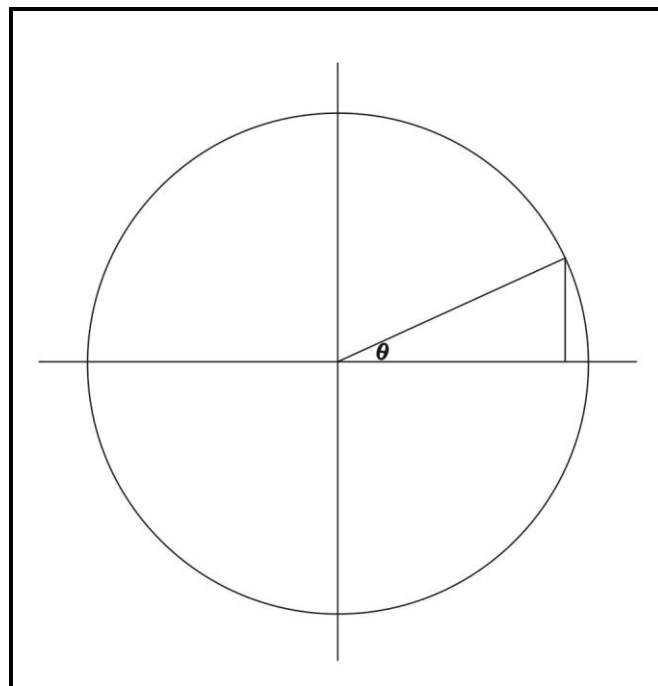


Figure 3.4; Trigonometric functions within the unit circle

In a study focusing on the right triangle approach versus the unit circle approach, Kendall and Stacey (1997) found that learners who had learnt trigonometric functions in the context of a right triangle model performed better on a post-test than those who had learnt about the subject using a unit circle model. Corral (2009) supports this notion and is of the opinion that approaching trigonometric functions with too much of an analytic emphasis is confusing to learners and make much of the material appearing unmotivated. He starts his book which is

simply called “Trigonometry” (Corral, 2009, p. 2) with the “old-fashioned” right-angled triangle.

Studies on trigonometry (Blackett & Tall, 1991; Weber, 2005) and some mathematics textbooks indicate that certain prior knowledge is required when introducing trigonometry in grade 10 for the first time. This prior knowledge should include the names and sizes of the different angles; the names and characteristics of the different triangles; certain geometric concepts in particular: adjacent, opposite, perpendicular, complementary (sum of the angles equals 90 degrees), supplementary (sum of the angles equals 180 degrees), explementary (sum of the angles equals 360 degrees), similar triangles, congruent triangles and the particular relationships of their angles and their sides in each case, certain symbols as well as some letters of the Greek alphabet for example α which is pronounced as alpha, β pronounced as beta, γ pronounced as gamma and θ which is pronounced as theta, the theorem of Pythagoras, also called the Pythagorean Theorem, Pythagorean triplets such as 3, 4 and 5 or 5, 12 and 13. Cavanagh (2008) tried out the method found in many textbooks of drawing one diagram with various lengths of the different sides as can be seen in Figure 3.5.

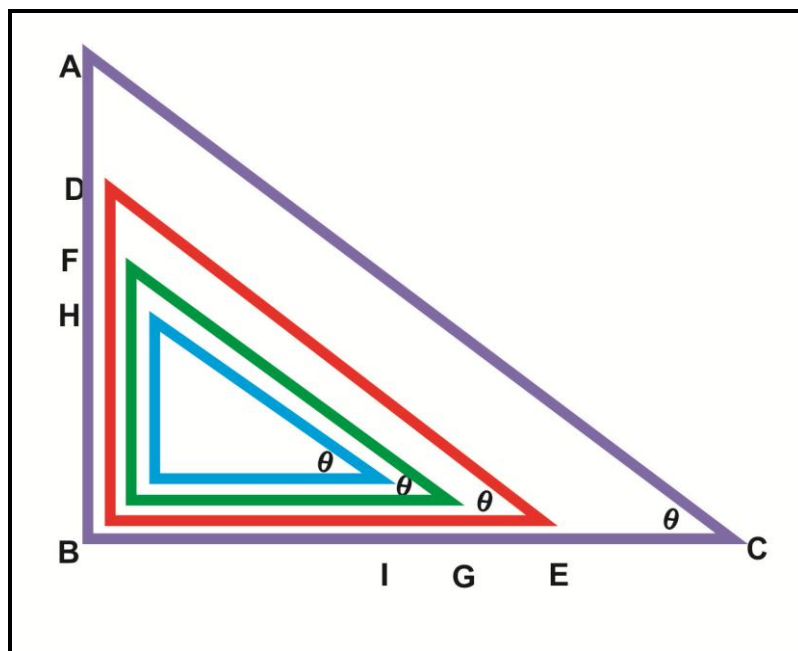


Figure 3.5: Discovering the Sine, Cosine and Tan ratios

He then asked learners to measure the two opposite sides with the corresponding hypotenuse side each time and divides it by each other respectively. Noticing that the same value is obtained leads learners to discover the sine ratio. After deciding that this method is still too abstract for the learners, he tried linking the tangent ratio to the gradient of a straight line which proved to be a useful starting point for the teaching of trigonometry. Beyers (2010) is of the opinion that a possible hypothetical learning trajectory for the learning/teaching of trigonometry must include trigonometric representations.

3.9.6 The hypothetical teaching and learning trajectory

What Weber and Demir are saying, poses somewhat of a paradox because Weber (2005) mentions that the operations within trigonometry cannot be algebraically evaluated while Demir (2012) argues that the evaluation of trigonometric functions indeed requires algebraic reasoning. According to Breidenbach et al. (1992) trigonometric functions are indeed operations that cannot be expressed as algebraic formulae involving procedures which are arithmetical. Therefore, although trigonometric functions cannot be expressed as algebraic formulae, applying algebraic reasoning is possible. Furthermore, all the afore-mentioned studies looked at trigonometry or some section thereof, but none of the studies focused specifically on how the metacognitive skills and mathematical language of the teacher can be factored in to enhance the teaching and learning in Trigonometry by using adapted lesson study. Quinlan (2004, p. 20) cites the following basic principles of pedagogy when trigonometry is introduced in grade 10:

1. *Go from concrete to abstract. Avoid starting with definitions.*
2. *Go from particular to general.*
3. *Immerse students in the context of any new concept before explicating its technicalities and intricacies and mathematical jargon.*
4. *The lesson introducing a new concept should be one that results in favourable reactions from the students.*

In order to develop a hypothetical teaching learning trajectory for the teaching of trigonometry focusing on the metacognitive skills and mathematical language usage by the teacher, these four basic principles needed to be applied in a real time classroom and then needed to be tweaked. Therefore the four principles, together with the key concepts within the conceptual framework for metacognitive instruction (§2.9) and the key guidelines for mathematics instruction focusing on mathematical language (§3.5.3) formed the main categories in the lesson observation schedule (see addendum D2).

Mathematical language and metacognitive skills are the two main concepts which are problematized through the lens of the phenomenon learning through Teaching (LTT) focusing on rural township mathematics education and specifically on the teaching of trigonometric functions. Leikin and Zazkis (2010) proposed a model for the teaching process in mathematics which they refer to as the Mathematics Teaching Cycle (MTC) model.

As already mentioned (§3.4.3) the term “hypothetical learning trajectory” is problematic for me. In this study I am using the term “hypothetical teaching and learning trajectory”. A prototype for the hypothetical teaching and learning trajectory (Figure 3.6) was developed which includes, on this stage, three interrelated elements which are: (1) the learning goal, (2) the plan for the tasks (principles of pedagogy for the teaching of trigonometric functions and the metacognitive skills) and (3) the hypothesis of the learning process (mathematical language). According to Van den Akker, Bannan, Kelly, Nieveen, and Plomp (2010) a prototype can be seen as a version of an intervention developed in collaboration between researchers and practitioners. They further explain that teachers need to adjust this trajectory based on their interactions with their learners each time they implement it in order to gain new understandings that should inform consecutive hypotheses. However, proponents of this view (Artzt & Armour-Thomas, 2002; Leikin & Zazkin, 2010; Simon, 1997) agree that this cyclic view of teaching demonstrates that teaching shows great potential for teachers’ learning rather than claiming that teachers do learn through teaching.

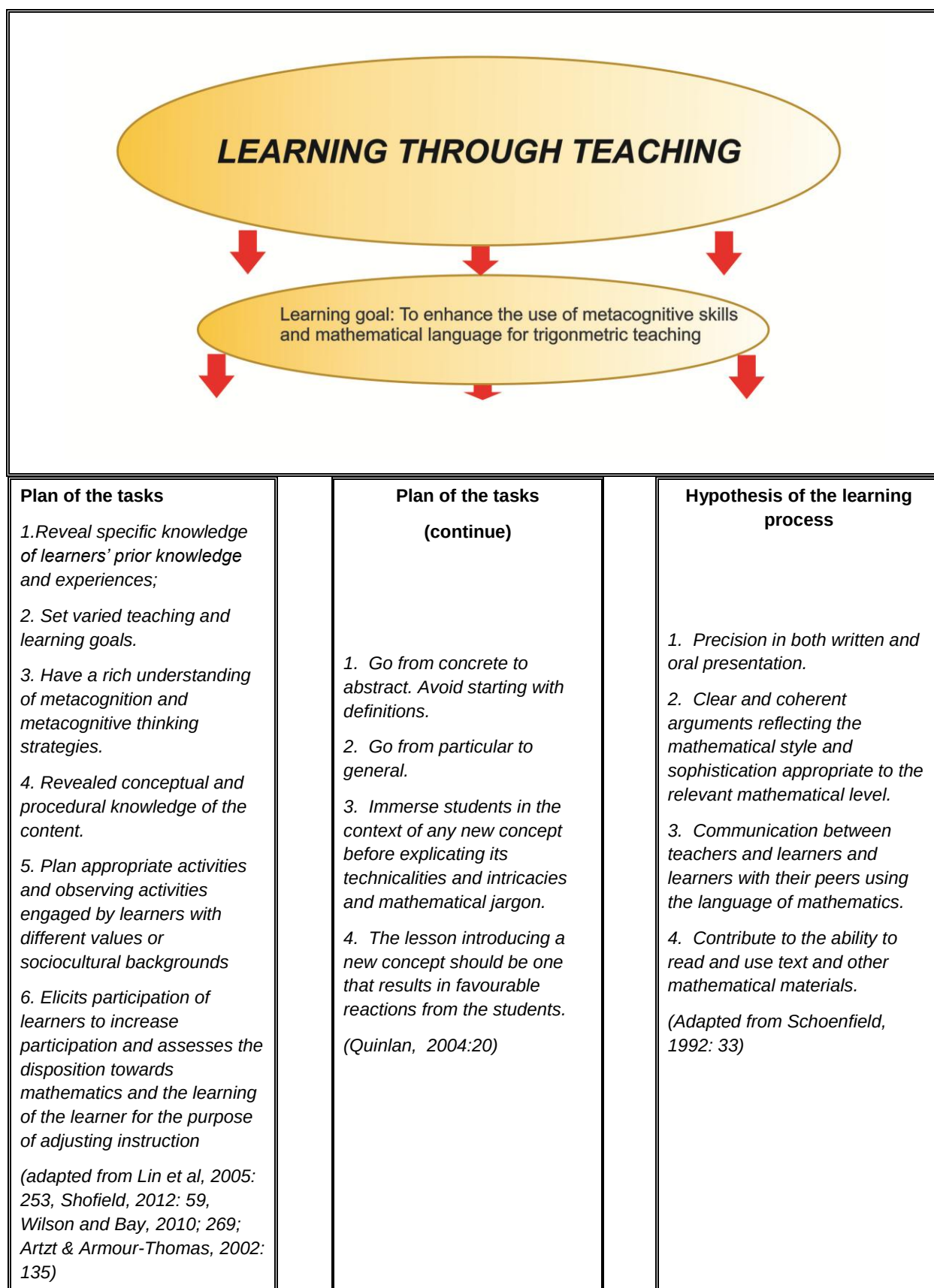


Figure 3.6: Prototype for the hypothetical teaching and learning trajectory

3.10 SUMMARY OF THE CHAPTER

In this chapter, instructional guidelines for the teaching of trigonometric functions and in particular for the introduction thereof were identified. It was made possible firstly by a sketch of the South African education landscape, the discussion of the teaching and learning within mathematics education, focusing especially on the particular knowledge of mathematics teachers, the mathematical language used by them as well as their instructional practices. Furthermore, adapted lesson study was discussed and finally the hypothetical teaching and learning trajectory for the teaching of trigonometric functions was covered extensively to conclude this second chapter of the literature review.

The next chapter outlines the theoretical framework employed in the study.