EXACT SOLUTIONS AND CONSERVATION LAWS OF (2+1)-DIMENSIONAL ZARKHAROV-KUZNETSOV MODIFIED EQUAL WIDTH EQUATION

by

KHADIJO RASHID ADEM (17103045)

Dissertation submitted for the degree of Master of Science in Applied Mathematics in the Department of Mathematical Sciences in the Faculty of Agriculture, Science and Technology at North-West University, Mafikeng Campus

November 2011

Supervisor: Professor C M Khalique

Contents

Declaration

I declare that the dissertation for the degree of Master of Science at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

KHADIJO RASHID ADEM

15 November 2011

Dedication

To my family my mom Mariam Isse Adem, dad Rashid Abdi Adem sisters Shukri, Saynab brothers Abdullahi, Adem, Mohomud, Ibrahim and Dr Yusuf.

Acknowledgements

I would like to thank my supervisor, Professor C.M. Khalique for his guidance, encouragement and patience in compiling this project. I would like to thank Dr B Muatjetjeja for his helpful discussions.

Thanks also to the North-West University, Mafikeng Campus and the National Research Foundation for financial support.

Finally, I would also like to say special thanks to my family and friends for their moral support, advice and encouragement while compiling my dissertation.

Abstract

In this dissertation exact solutions of the (2+1) dimensional Zakharov-Kuznetsov modified equal width equation are obtained. The Lie group analysis is used to carry out the integration of this equation. The solutions obtained include the non-topological soliton solution, cnoidal waves and the traveling wave solutions.

Also exact solutions to the Zakharov-Kuznetsov modified equal width equation with power law nonlinearity are obtained. The Lie symmetry approach along with the simplest equation method is used to obtain these solutions. Moreover, conservation laws of the generalized Zarkharov-Kuznetsov modified equal width equation with power law nonlinearity are derived using the new conservation theorem and the multiplier method. Finally, both the topological as well as non-topological soliton solutions are obtained using the solitary wave anstaz method for the underlying equation.

Introduction

Nonlinear evolution equations (NLEEs) have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics, optical fiber and geochemistry. Thus, it is important to investigate the exact explicit solutions of NLEEs. Finding solutions of such an equation is an arduous task and only in certain special cases can one write down the solutions explicitly. However, a massive amount of work has been done in the past few decades and important progress has been made in obtaining exact solutions of NLEEs. In order to obtain the exact solutions, a number of methods have been proposed in the literature. Some of the well known methods include the solitary wave ansatz method, the inverse scattering, Hirota's bilinear method, homogeneous balance method, Lie group analysis, etc. [1–11].

Among the above mentioned methods, the Lie group analysis method, also called the symmetry method, is one of the most effective methods to determine solutions of nonlinear partial differential equations. In the second half of the 19th century and about 200 years after Leibniz and Newton introduced the concept of the derivative, solving ordinary differential equations (ODEs) had become one of the most important problems in applied mathematics. Sophus Lie (1842-1899) became interested in this problem and with inspiration from Galois's theory for solving algebraic equations discovered what is known today as Lie group analysis. He showed that the majority of known methods of integration of ordinary differential equations, which until then had seemed artificial, could be derived in a unified manner using his theory of continuous transformation groups. Recently there have been considerable developments in symmetry methods for differential equations as is evident by the number of research papers, books and amount of new symbolic software devoted to the subject [11–17].

Conservation laws play a very important role in the solution process of differential equations. Finding the conservation laws of system of differential equations is often the first step towards finding the solution. In fact, the existence of a large number of conservation laws of a system of partial differential equations is a strong indication of its integrability [11].

One of the most important one-dimensional nonlinear wave equations is the Korteweg de Vries (KdV) equation [18]

$$u_t + 6uu_x + u_{xxx} = 0$$

which describes the evolution of weakly nonlinear and weakly dispersive waves used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory [19,20]. One of the best known 2-dimensional generalizations of the KdV equation is the Zakharov-Kuznetsov (ZK) equation [21] in the form

$$u_t + auu_x + (u_{xx} + u_{yy})_y = 0$$

which governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold-ions and hot isothermal electrons in the presence of a uniform magnetic field [22–27]. The ZK equation is not integrable by the inverse scattering transform method. Shivamoggi [28] showed that the ZK equation has the Painleve property. The modified equal width (MEW) equation is of the form

$$u_t + a(u^3)_x - u_{xxt} = 0$$

which appears in many physical applications [29–32]. The generalized form of the MEW equation in the ZK sense [33] is given by

$$u_t + a(u^n)_x + (bu_{xt} + ru_{yy})_x = 0. \tag{1}$$

It is natural to call this equation as the Zakharov-Kuznetsov modified equal width (ZK-MEW) equation. Wazwaz [33] studied equation (1) using the sine-cosine method and the tanh technique.

In this dissertation the ZK-MEW equation with power law nonlinearity in (2+1) dimensions, given by

$$u_t + au^n u_x + b(u_{xt} + u_{yy})_x = 0 \tag{2}$$

will be studied. Here a, b and n are real valued constants with $n > 0$.

The outline of this dissertation is as follows.

In Chapter one the basic definitions and theorems concerning the one-parameter groups of transformations are presented.

Chapter two deals with a special case of the ZK-MEW equation (2) with $n = 3$. The Lie group analysis method is employed to obtain exact solutions. Profiles of these exact solutions are plotted. The results of this Chapter have been published in [35].

In Chapter three exact solutions of the ZK-MEW equation (2) are obtained with the aid of the Lie symmetry method along with the simplest equation method [36–39]. Furthermore, conservation laws of the ZK-MEW equation (2) are derived using the new conservation theorem [40] and the multiplier method [41–44]. The results of this Chapter have been published in [45].

In Chapter four topological and non-topological soliton solutions for the ZK-MEW equation (2) are obtained using the solitary wave anstaz method [1].

Finally, in Chapter five, a summary of the results of the dissertation are presented and future work is discussed.

A bibliography is presented at the end of this dissertation.

Chapter 1

Lie symmetry methods for partial differential equations

In this chapter we give a brief introduction to the Lie group theory of partial differential equations. This will include the algorithm to determine the Lie point symmetries of partial differential equations.

1.1 Introduction

More than a hundred years ago, the Norwegian mathematician Marius Sophus Lie realized that many of the methods for solving differential equations could be unified using group theory. He developed a symmetry-based approach to obtaining exact solutions of differential equations. Symmetry methods have great power and generality. In fact, nearly all well-known techniques for solving differential equations are special cases of Lie's methods. Recently, several books have been written on this topic. We list a few of them here. Bluman and Kumei [11], Olver [12], Ovsiannikov [13], Ibragimov [14, 15], Stephani [46], Cantwell [?].
The definitions and results presented in this chapter are taken from the books mentioned above and we will not refer to them.

1.2 Local one-parameter Lie group

Here a transformation will be understood to mean an invertible transformation, i.e. a bijective map. Let x, y and t be three independent variables and u be a dependent variable. We consider a change of the variables x, y, t and u:

$$\begin{aligned} T_a \ : \ & \bar{x} = f(x,y,t,u,a),\ \bar{y} = g(x,y,t,u,a),\ \bar{t} = h(x,y,t,u,a), \\ & \bar{u} = z(x,y,t,u,a) \end{aligned} \tag{1.1}$$

with a being a real parameter, which continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and where f, g, h and z are differentiable functions.

Definition 1.1 A *continuous one-parameter (local) Lie group of transformations* is a set G of transformations (1.1) which satisfies the following three conditions:

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then T_b, $T_a = T_c \in G$, $c = \phi(a,b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

From (i) we see that the associativity property is satisfied. Also, if the identity transformation occurs at $a = a_0 \neq 0$ i.e, $T_{a_0}$is the identity, then a shift of the parameter $a = \bar{a} + a_0$ will give T_0 as above. The property (i) can be written as

$$\begin{aligned} \bar{\bar{x}} &\equiv f(\bar{x},\bar{y},\bar{t},\bar{u},b) = f(x,y,t,u,\phi(a,b)), \\ \bar{\bar{y}} &\equiv g(\bar{x},\bar{y},\bar{t},\bar{u},b) = g(x,y,t,u,\phi(a,b)), \\ \bar{\bar{t}} &\equiv h(\bar{x},\bar{y},\bar{t},\bar{u},b) = h(x,y,t,u,\phi(a,b)), \\ \bar{\bar{u}} &\equiv z(\bar{x},\bar{y},\bar{t},\bar{u},b) = z(x,y,t,u,\phi(a,b)). \end{aligned} \tag{1.2}$$

The function ϕ is termed as the *group composition law*. A group parameter a is called *canonical* if $\phi(a,b) = a + b$.

Theorem 1.1 For any $\phi(a,b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\,\phi(s,b)}{\partial b}\right|_{b=0}.$$

We now give the definition of a symmetry group for the third-order PDE

$$u_t = F(t, x, y, u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, u_{xxt}, u_{xyy}, \ldots), \quad \frac{\partial F}{\partial u_{xxt}} \neq 0. \tag{1.3}$$

Definition 1.2 (Symmetry group) A one-parameter group G of transformations (1.1) is called a *symmetry group* of (1.3) if it is form-invariant (has the same form) in the new variables $\bar{x}, \bar{y}, \bar{t}$ and $\bar{u}$, i.e. ,

$$\bar{u}_{\bar{t}} = F(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{y}}, \bar{u}_{\bar{x}\bar{t}}, \bar{u}_{\bar{x}\bar{x}\bar{t}}, \bar{u}_{\bar{x}\bar{y}\bar{y}}, \ldots), \tag{1.4}$$

where the function F is the same as in (1.3).

1.3 Infinitesimal transformations

Lie's theory tells us that the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\begin{aligned} \bar{x} &\approx x + a\,\xi^1(x,y,t,u),\ \bar{y} \approx y + a\,\xi^2(x,y,t,u),\ \bar{t} \approx t + a\,\xi^3(x,y,t,u), \\ \bar{u} &\approx u + a\,\eta(x,y,t,u) \end{aligned} \tag{1.5}$$

obtained from (1.1) by expanding the functions f, g, h and z into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f|_{a=0} = x, \quad g|_{a=0} = y, \quad h|_{a=0} = t, \quad z|_{a=0} = u\,.$$

Thus, we have

$$\begin{aligned} \xi^1(x,y,t,u) &= \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \xi^2(x,y,t,u) = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \quad \xi^3(x,y,t,u) = \left.\frac{\partial h}{\partial a}\right|_{a=0}, \\ \eta(x,y,t,u) &= \left.\frac{\partial z}{\partial a}\right|_{a=0}. \end{aligned} \tag{1.6}$$

Now one can write (1.5) as

$$\begin{aligned}\bar{x} &\approx (1+aV)x, \quad \bar{y}\approx(1+aV)y, \quad \bar{t}\approx(1+aV)t,\\ \bar{u} &\approx (1+aV)u,\end{aligned}$$

where

$$V=\xi^1(x,y,t,u)\frac{\partial}{\partial x}+\xi^2(x,y,t,u)\frac{\partial}{\partial y}+\xi^3(x,y,t,u)\frac{\partial}{\partial t}+\eta(x,y,t,u)\frac{\partial}{\partial u}. \quad (1.7)$$

This differential operator V is known as the *infinitesimal operator (generator)* of the group G. If the group G is admitted by (1.3), we say that V is an *admitted operator* of (1.3) or V is an *infinitesimal symmetry* of (1.3).

1.4 Group invariants

Definition 1.3 A function $F(x,y,t,u)$ is called an *invariant of the group of transformations* (1.1) if

$$\begin{aligned}F(\bar{x},\bar{y},\bar{t},\bar{u}) &\equiv F(f(x,y,t,u,a),g(x,y,t,u,a),h(x,y,t,u,a),z(x,y,t,u,a))\\ &= F(x,y,t,u), \quad (1.8)\end{aligned}$$

identically in x, y, t, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x,y,t,u)$ to be an invariant is that

$$\begin{aligned}VF &\equiv \xi^1(x,y,t,u)\frac{\partial F}{\partial x}+\xi^2(x,y,t,u)\frac{\partial F}{\partial y}+\xi^3(x,y,t,u)\frac{\partial F}{\partial t}+\eta(x,y,t,u)\frac{\partial F}{\partial u}\\ &= 0. \quad (1.9)\end{aligned}$$

From the above theorem it follows that every one-parameter group of point transformations (1.1) has three functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,y,t,u)=c_1, \quad J_2(x,y,t,u)=c_2, \quad J_3(x,y,t,u)=c_3,$$

of the characteristic equations

$$\frac{dx}{\xi^1(x,y,t,u)} = \frac{dy}{\xi^2(x,y,t,u)} = \frac{dt}{\xi^3(x,y,t,u)} = \frac{du}{\eta(x,y,t,u)}.$$

Theorem 1.3 Given the infinitesimal transformation (1.5) or its symbol X, the corresponding one-parameter group G is obtained by solving the Lie equations

$$\begin{aligned} \frac{d\bar{x}}{da} &= \xi^1(\bar{x},\bar{y},\bar{t},\bar{u}), \quad \frac{d\bar{y}}{da} = \xi^2(\bar{x},\bar{y},\bar{t},\bar{u}), \quad \frac{d\bar{t}}{da} = \xi^3(\bar{x},\bar{y},\bar{t},\bar{u}), \\ \frac{d\bar{u}}{da} &= \eta(\bar{x},\bar{y},\bar{t},\bar{u}) \end{aligned} \tag{1.10}$$

subject to the initial conditions

$$\bar{x}|_{a=0} = x, \quad \bar{y}|_{a=0} = y, \quad \bar{t}|_{a=0} = t, \quad \bar{u}|_{a=0} = u\,.$$

1.5 Construction of a symmetry group

Here we describe the algorithm to determine a symmetry group for a given PDE but first we give some definitions.

1.5.1 Prolongation of point transformations

Consider a thrid-order PDE

$$E(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{yy},u_{tt},u_{xy},u_{xt},u_{yt},u_{xxt},u_{xyy},\ldots) = 0, \tag{1.11}$$

where x, y and t are three independent variables and u is a dependent variable. Let

$$V = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}, \tag{1.12}$$

be the infinitesimal generator of the one-parameter group G of transformation (1.1). The *first prolongation* of the operator V is denoted by $V^{[1]}$ and is given by

$$\begin{aligned} V^{[1]} &= V + \zeta_1(x,y,t,u,u_x,u_y,u_t)\frac{\partial}{\partial u_x} + \zeta_2(x,y,t,u,u_x,u_y,u_t)\frac{\partial}{\partial u_y} \\ &\quad + \zeta_3(t,x,u,u_x,u_y,u_t)\frac{\partial}{\partial u_t} \end{aligned}$$

where

$$
\begin{aligned}
\zeta_1 &= D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2) - u_t D_x(\xi^3), \\
\zeta_2 &= D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2) - u_t D_y(\xi^3), \\
\zeta_3 &= D_t(\eta) - u_x D_t(\xi^1) - u_y D_t(\xi^2) - u_t D_t(\xi^3)
\end{aligned}
$$

and the total derivatives D_x, D_y and D_t are given by

$$
\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + u_{xy}\frac{\partial}{\partial u_y} + \cdots, && (1.13) \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + u_{ty}\frac{\partial}{\partial u_t} + \cdots, && (1.14) \\
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{ty}\frac{\partial}{\partial u_y} + u_{tt}\frac{\partial}{\partial u_t} + \cdots. && (1.15)
\end{aligned}
$$

The *second prolongation* of the operator V is denoted by $V^{[2]}$ and is given by

$$
V^{[2]} = V + \zeta_1 \frac{\partial}{\partial u_x} + \zeta_2 \frac{\partial}{\partial u_y} + \zeta_3 \frac{\partial}{\partial u_t} + \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{12}\frac{\partial}{\partial u_{xy}}
$$

where

$$
\begin{aligned}
\zeta_{11} &= D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xy} D_x(\xi^2) - u_{xt} D_x(\xi^3), \\
\zeta_{12} &= D_y(\zeta_1) - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{xt} D_y(\xi^3).
\end{aligned}
$$

Likewise, the the third prolongation of V is given by

$$
\begin{aligned}
V^{[3]} &= V + \zeta_1 \frac{\partial}{\partial u_x} + \zeta_2 \frac{\partial}{\partial u_y} + \zeta_3 \frac{\partial}{\partial u_t} + \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{12}\frac{\partial}{\partial u_{xy}} + \zeta_{113}\frac{\partial}{\partial u_{xxt}} && (1.16) \\
&\quad + \zeta_{122}\frac{\partial}{\partial u_{xyy}}
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_{113} &= D_x(\zeta_{11}) - u_{xxx} D_x(\xi^1) - u_{xxy} D_x(\xi^2) - u_{xxt} D_x(\xi^3), \\
\zeta_{122} &= D_y(\zeta_{12}) - u_{xxy} D_y(\xi^1) - u_{xyy} D_y(\xi^2) - u_{xty} D_x(\xi^3).
\end{aligned}
$$

Applying the definitions of D_x, D_y and D_t given above, we obtain

$$
\begin{aligned}
\zeta_1 &= \eta_x + u_x\eta_u - u_x\xi^1_x - u_x^2\xi^1_u - u_y\xi^2_x - u_x u_y \xi^2_u - u_t\xi^3_x - u_t u_x \xi^3_u, && (1.17) \\
\zeta_2 &= \eta_y + u_y\eta_u - u_x\xi^1_y - u_x u_y\xi^1_u - u_y\xi^2_y - u_y^2 \xi^2_u - u_t\xi^3_y - u_t u_y \xi^3_u, && (1.18) \\
\zeta_3 &= \eta_t + u_t\eta_u - u_x\xi^1_t - u_t u_x\xi^1_u - u_y\xi^2_t - u_t u_y \xi^2_u - u_t\xi^3_t - u_t^2 \xi^3_u, && (1.19)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{11} \;=\;& \eta_{xx} + u_x^2\eta_{uu} + 2u_x\eta_{xu} + u_{xx}\eta_u - u_t u_x^2\xi^3_{uu} - 2u_{xt}\xi^3_x - u_t u_{xx}\xi^3_u \\
& -u_t\xi^3_{xx} - u_x^2 u_y\xi^2_{uu} - 2u_x u_{xy}\xi^2_u - 2u_{xy}\xi^2_x - u_y\xi^2_{xx} - 2u_x u_y\xi^2_{xu} \\
& -u_y u_{xx}\xi^2_u - u_x^3\xi^1_{uu} - 3u_x u_{xx}\xi^1_u - 2u_t u_x\xi^3_{xu} - 2u_x^2\xi^1_{xu} \\
& -u_x\xi^1_{xx} - 2u_{xx}\xi^1_x - 2u_x u_{xt}\xi^3_u,
\end{aligned} \tag{1.20}
$$

$$
\begin{aligned}
\zeta_{12} \;=\;& \eta_{xy} + u_x u_y\eta_{uu} + u_x\eta_{yu} + u_{xy}\eta_u + u_y\eta_{xu} - u_x^2 u_y\xi^1_{uu} - u_x^2\xi^1_{yu} \\
& -2u_x u_{xy}\xi^1_u - u_x u_y\xi^1_{xu} - u_x\xi^1_{xy} - u_y u_{xx}\xi^1_u - u_{xx}\xi^1_y - u_{xy}\xi^1_x \\
& -u_x u_y^2\xi^2_{uu} - u_x u_y\xi^2_{yu} - 2u_y u_{xy}\xi^2_u - u_{xy}\xi^2_y - u_{yy}\xi^2_x - u_y^2\xi^2_{xu} \\
& -u_y\xi^2_{xy} - u_x\xi^2_u - u_x\xi^3_u u_{ty} - u_t u_x u_y\xi^3_{uu} - u_t u_x\xi^3_{uy} - u_y u_{xt}\xi^3_u \\
& -u_{xt}\xi^3_y - u_t u_{xy}\xi^3_u - u_{yt}\xi^3_x - u_t u_y\xi^3_{xu} - u_t\xi^3_{xy},
\end{aligned} \tag{1.21}
$$

$$
\begin{aligned}
\zeta_{113} \;=\;& \eta_{xxt} + u_t\eta_{xxu} - u_x^3\xi^1_{tuu} - u_tu_x^3\xi^1_{uuu} - 3u_x^2u_{xt}\xi^1_{uu} - u_x^2u_{yt}\xi^2_{uu} \\
&+u_tu_x^2\eta_{uuu} - 2u_x^2\xi^1_{xtu} - 2u_tu_x^2\xi^1_{xuu} - u_yu_x^2\xi^2_{tuu} - u_tu_yu_x^2\xi^2_{uuu} \\
&-u_tu_x^2\xi^3_{tuu} - u_t^2u_x^2\xi^3_{uuu} + 2u_xu_{xt}\eta_{uu} - 3u_xu_{xx}\xi^1_{tu} - 3u_xu_tu_{xx}\xi^1_{uu} \\
&-4u_xu_{xt}\xi^1_{xu} - 2u_xu_{xy}\xi^2_{tu} - 2u_xu_yu_{xt}\xi^2_{uu} - 2u_tu_xu_{xy}\xi^2_{uu} - u_t^2\xi^3_{xxu} \\
&-2u_xu_{yt}\xi^2_{xu} - 2u_xu_{xt}\xi^3_{tu} - 4u_tu_xu_{xt}\xi^3_{uu} - 2u_xu_{tt}\xi^3_{xu} - 2u_xu_{xtt}\xi^3_u \\
&-3u_xu_{xxt}\xi^1_u - 2u_xu_{xyt}\xi^2_u + 2u_x\eta_{xtu} + 2u_xu_t\eta_{xuu} - 2u_tu_xu_y\xi^2_{xuu} \\
&-u_tu_x\xi^1_{xxu} - 2u_xu_y\xi^2_{xtu} - 2u_xu_t\xi^3_{xtu} - 2u_xu_t^2\xi^3_{xuu} - 2u_{xt}^2\xi^3_u - u_{tt}u_{xx}\xi^3_u \\
&-3u_{xt}u_{xx}\xi^1_u - 2u_{xt}u_{xy}\xi^2_u - u_{xx}u_{yt}\xi^2_u + u_{xx}\eta_{tu} + u_tu_{xx}\eta_{uu} + 2u_{xt}\eta_{xu} \\
&-2u_{xx}\xi^1_{xt} - 2u_tu_{xx}\xi^1_{xu} - u_{xt}\xi^1_{xx} - u_yu_{xx}\xi^2_{tu} - u_tu_yu_{xx}\xi^2_{uu} - 2u_{xy}\xi^2_{xt} \\
&-2u_yu_{xt}\xi^2_{xu} - 2u_tu_{xy}\xi^2_{xu} - u_tu_{xx}\xi^3_{tu} - u_t^2u_{xx}\xi^3_{uu} - 2u_{xt}\xi^3_{xt} - 4u_tu_{xt}\xi^3_{xu} \\
&-u_{tt}\xi^3_{xx} - 2u_{xtt}\xi^3_x + u_{xxt}\eta_u - u_{yt}\xi^2_{xx} - 2u_{xxt}\xi^1_x - u_yu_{xxt}\xi^2_u - u_{xxt}\xi^3_t \\
&-2u_tu_{xxt}\xi^3_u - u_{xxx}\xi^1_t - u_tu_{xxx}\xi^1_u - u_{xxy}\xi^2_t - u_tu_{xxy}\xi^2_u - 2\xi^2_xu_{xyt} \\
&-u_y\xi^2_{xxt} - u_x^2u_{tt}\xi^3_{uu} + u_x^2\eta_{tuu} - u_tu_y\xi^2_{xxu} - u_t\xi^3_{xxt} - u_x\xi^1_{xxt}, \qquad (1.22)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{122} \;=\;& \eta_{xyy} + u_x\eta_{yyu} - u_xu_y^3\xi^2_{uuu} - u_y^3\xi^2_{xuu} - u_y^2u_{xx}\xi^1_{uu} - 3u_y^2u_{xy}\xi^2_{uu} \\
&+u_y^2\eta_{xuu} - u_x^2u_y^2\xi^1_{uuu} - u_xu_y^2\xi^1_{xuu} - 2u_y^2\xi^2_{xyu} - 2u_xu_y^2\xi^2_{yuu} - 2u_yu_{xx}\xi^1_{yu} \\
&-u_tu_y^2\xi^3_{xuu} + 2u_yu_{xy}\eta_{uu} - 4u_xu_yu_{xy}\xi^1_{uu} - 2u_yu_{xy}\xi^1_{xu} - u_tu_xu_y^2\xi^3_{uuu} \\
&-3u_xu_yu_{yy}\xi^2_{uu} - 3u_yu_{yy}\xi^2_{xu} - 4u_yu_{xy}\xi^2_{yu} - 2u_tu_yu_{xy}\xi^3_{uu} - 2u_xu_yu_{yt}\xi^3_{uu} \\
&-2u_yu_{yt}\xi^3_{xu} - 2u_yu_{xt}\xi^3_{yu} - 2u_yu_{xxy}\xi^1_u - 2u_yu_{xyt}\xi^3_u - 3u_yu_{xyy}\xi^2_u + 2u_y\eta_{xyu} \\
&+2u_xu_y\eta_{yuu} - 2u_xu_y\xi^1_{xyu} - 2u_yu_x^2\xi^1_{yuu} - u_y\xi^2_{xyy} - u_xu_y\xi^2_{yyu} - 2u_tu_y\xi^3_{xyu} \\
&-2u_tu_xu_y\xi^3_{yuu} - 2u_{xy}^2\xi^1_u - 2u_{xy}u_{yt}\xi^3_u - u_{xt}u_{yy}\xi^3_u - u_{xx}u_{yy}\xi^1_u - 3u_{xy}u_{yy}\xi^2_u \\
&+u_xu_{yy}\eta_{uu} + u_{yy}\eta_{xu} + 2u_{xy}\eta_{yu} - u_x^2u_{yy}\xi^1_{uu} - u_xu_{yy}\xi^1_{xu} - 2u_{xy}\xi^1_{xy} \\
&-4u_xu_{xy}\xi^1_{yu} - u_{xx}\xi^1_{yy} - 2u_{yy}\xi^2_{xy} - 2u_xu_{yy}\xi^2_{yu} - u_{xy}\xi^2_{yy} + u_xu_y^2\eta_{uuu} \\
&-u_tu_{yy}\xi^3_{xu} - 2u_{yt}\xi^3_{xy} - 2u_tu_{xy}\xi^3_{yu} - 2u_xu_{yt}\xi^3_{yu} - u_{xt}\xi^3_{yy} - 2u_{xxy}\xi^1_y \\
&-2u_{xyt}\xi^3_y + u_{xyy}\eta_u - 2u_xu_{xyy}\xi^1_u - u_{xyy}\xi^1_x - 2u_{xyy}\xi^2_y - u_tu_{xyy}\xi^3_u \\
&-u_{yyt}\xi^3_x - u_xu_{yyy}\xi^2_u - u_{yyy}\xi^2_x - u_y^2u_{xt}\xi^3_{uu} - u_tu_xu_{yy}\xi^3_{uu} - u_x\xi^1_{xyy} - u_x^2\xi^1_{yyu} \\
&-u_t\xi^3_{xyy} - u_tu_x\xi^3_{yyu} - u_xu_{yyt}\xi^3_u. \qquad (1.23)
\end{aligned}
$$

1.5.2 Group admitted by a PDE

The operator

$$V = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u} \quad (1.24)$$

is a *point symmetry* of the third-order PDE (1.11) if

$$V^{[3]}(E) = 0 \quad (1.25)$$

whenever $E = 0$. This can also be written as (symmetry condition)

$$V^{[3]} E\big|_{E=0} = 0, \quad (1.26)$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.4 Equation (1.26) is called the *determining equation* of (1.11), because it determines all the infinitesimal symmetries of (1.11).

The theorem below enables us to construct some solutions of (1.11) from a known one.

Theorem 1.4 A symmetry of (1.11) transforms any solution of (1.11) into another solution of the same equation.

1.6 Lie algebras

Let us consider two operators V_1 and V_2 defined by

$$V_1 = \xi_1^1(x,y,t,u)\frac{\partial}{\partial x} + \xi_1^2(x,y,t,u)\frac{\partial}{\partial y} + \xi_1^3(x,y,t,u)\frac{\partial}{\partial t} + \eta_1(x,y,t,u)\frac{\partial}{\partial u},$$

and

$$V_2 = \xi_2^1(x,y,t,u)\frac{\partial}{\partial x} + \xi_2^2(x,y,t,u)\frac{\partial}{\partial y} + \xi_2^3(x,y,t,u)\frac{\partial}{\partial t} + \eta_2(x,y,t,u)\frac{\partial}{\partial u}.$$

Definition 1.5 (Commutator) The *commutator* of V_1 and V_2, written as $[V_1, V_2]$, is defined by $[V_1, V_2] = V_1(V_2) - V_2(V_1)$.

Definition 1.6 (Lie algebra) A Lie algebra is a vector space L of operators with the following property : For all $V_1, V_2 \in L$, the commutator $[V_1, V_2] \in L$.

The dimension of a Lie algebra is the dimension of the vector space L.

Theorem 1.5 The set of all solutions of any determining equation forms a Lie algebra.

1.7 Conclusion

In this chapter we presented briefly some basic definitions and results of the Lie group analysis of PDEs. These included the algorithm to determine the Lie point symmetries of PDEs.

Chapter 2

Solutions of (2+1)-dimensional Zarkharov-Kuznetsov modified equal width equation

In this chapter we obtain exact solutions of the (2+1)-dimensional Zarkharov-Kuznetsov modified equal width (ZK-MEW) equation, namely

$$u_t + 3au^2u_x + b(u_{xxt} + u_{xyy}) = 0, \tag{2.1}$$

where a and b are constants. Here, in (2.1), the first term represents the evolution term, the coefficient of a is the nonlinear term while the third term and fourth term together, in parentheses, are the dispersion terms.

In [34] the authors used the exp-function method to construct generalized solitary and periodic solutions of (2.1). Numerical solution was also obtained and was compared with the solution obtained by exp-function method.

Here we use the Lie group analysis to carry out the integration of this equation. The solutions obtained include the non-topological soliton solution, cnoidal waves and the traveling wave solutions. This work has been published in [35].

2.1 Symmetries of the (2+1)-dimensional ZK-MEW equation

We determine the Lie point symmetries of the (2+1)-dimensional ZK-MEW equation (2.1). This equation admits the one-parameter Lie group of transformations with infinitesimal generator

$$V = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u} \qquad (2.2)$$

if and only if

$$V^{[3]}(u_t + 3au^2u_x + b(u_{xxt} + u_{xyy}))\big|_{(2.1)} = 0. \qquad (2.3)$$

Using the definition of $V^{[3]}$ from chapter one, we get

$$\begin{aligned}&\Big[\xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}\\ &+\zeta_1\frac{\partial}{\partial u_x} + \zeta_2\frac{\partial}{\partial u_y} + \zeta_3\frac{\partial}{\partial u_t} + \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{12}\frac{\partial}{\partial u_{xy}} + \zeta_{113}\frac{\partial}{\partial u_{xxt}}\\ &+\zeta_{122}\frac{\partial}{\partial u_{xyy}}\Big]\Big(u_t + 3au^2u_x + b(u_{xxt} + u_{xyy})\Big)\Big|_{u_t=-3au^2u_x-bu_{xxt}-bu_{xyy}} = 0\end{aligned}$$

and this gives

$$6auu_x\eta + 3au^2\zeta_1 + \zeta_3 + b\zeta_{113} + b\zeta_{122}\Big|_{u_t=-3au^2u_x-bu_{xxt}-bu_{xyy}} = 0, \qquad (2.4)$$

where ζ_1, ζ_3, ζ_{113} and ζ_{122} are given by (1.17), (1.19), (1.22) and (1.23) respectively. Substituting the values of ζ_1, ζ_3, ζ_{113} and ζ_{122} in (2.4) and replacing u_t by $-3au^2u_x - bu_{xxt} - bu_{xyy}$, the following overdetermined system of linear partial differential equations are obtained:

$$\xi^3_u = 0, \tag{2.5}$$

$$\xi^1_u = 0, \tag{2.6}$$

$$\xi^2_u = 0, \tag{2.7}$$

$$\eta_{uu} = 0, \tag{2.8}$$

$$\xi^3_x = 0, \tag{2.9}$$

$$\xi^1_x = 0, \tag{2.10}$$

$$\xi^3_y = 0, \tag{2.11}$$

$$\xi^1_t = 0, \tag{2.12}$$

$$\xi^1_y = 0, \tag{2.13}$$

$$\xi^3_t - 2\xi^2_y = 0, \tag{2.14}$$

$$\xi^2_x = 0, \tag{2.15}$$

$$\eta_{tu} = 0, \tag{2.16}$$

$$\eta_{xu} = 0, \tag{2.17}$$

$$2\eta_{yu} - \xi^2_{yy} = 0, \tag{2.18}$$

$$6au\eta + 3au^2\xi^3_t + b\eta_{yyu} = 0, \tag{2.19}$$

$$\xi^2_t = 0, \tag{2.20}$$

$$\eta_t + 3au^2\eta_x + b\eta_{xxt} + b\eta_{xyy} = 0. \tag{2.21}$$

From (2.5), (2.9) and (2.11), we get

$$\xi^3 \equiv \xi^3(t) = A(t), \tag{2.22}$$

where $A(t)$ is an arbitrary function of t.

From (2.6), (2.10), (2.12) and (2.13), we get

$$\xi^1 \equiv \xi^1 = C_1, \tag{2.23}$$

where C_1 is an arbitrary constant.

Also from (2.7), (2.15) and (2.20), we get

$$\xi^2 \equiv \xi^2(y) = B(y), \tag{2.24}$$

where $B(y)$ is an arbitrary function of y.

Equation (2.8) implies that

$$\eta = C(x, y, t)u + D(x, y, t). \tag{2.25}$$

Substituting the values of ξ^3 and ξ^2 in (2.14), we obtain

$$A'(t) - 2B'(y) = 0. \tag{2.26}$$

Substituting the value of η in (2.16), we get

$$C_t = 0, \tag{2.27}$$

and in (2.17), we obtain

$$C_x = 0. \tag{2.28}$$

Substituting the values of η and ξ^2 in (2.18), we get

$$2C_y - B''(y) = 0. \tag{2.29}$$

Substituting the values of η and ξ^3 in (2.19), gives

$$6au^2C(x, y, t) + 6auD(x, y, t) + 3au^2A'(t) + bC_{yy} = 0. \tag{2.30}$$

Also substituting the value of η in (2.21), we get

$$uC_t + D_t + 3au^3C_x + 3au^2D_x + buC_{xxt} + bD_{xxt} + buC_{xyy} + D_{xyy} = 0. \tag{2.31}$$

From (2.27) and (2.28) we deduce that

$$C \equiv C(y) = C_2, \tag{2.32}$$

where C_2 is an arbitrary constant.

From (2.26) $A'(t) = 2B'(y) = K$, which implies that

$$\begin{aligned} A(t) &= C_2t + C_3, && (2.33)\\ B(y) &= \frac{1}{2}C_2y + C_4. && (2.34) \end{aligned}$$

From (2.29) applying (2.32) and (2.34) we get

$$C(y) = C_5. \tag{2.35}$$

Now from equation (2.30) substituting the values of (2.33) and (2.35) gives

$$6au^2C_5 + 6auD + 3au^2C_2 = 0, \tag{2.36}$$

separating (2.36) with respect to u, we have

$$u^2 \;:\; 6aC_5 + 3aC_2 = 0, \tag{2.37}$$

$$u \;:\; D = 0. \tag{2.38}$$

This implies that

$$C_5 = -\frac{1}{2}C_2, \tag{2.39}$$

and so equation (2.31) is satisfied.

Hence the general solution of the system (2.5)–(2.21) is

$$\begin{aligned}
\xi^1 &= C_1,\\
\xi^2 &= \frac{1}{2}C_2y + C_4,\\
\xi^3 &= C_2t + C_3,\\
\eta &= -\frac{1}{2}C_2u.
\end{aligned}$$

Thus the Lie point symmetries of the (2+1) dimensional ZK-MEW equation gives the following infinitesimal generators:

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial x},\\
V_2 &= \frac{\partial}{\partial y},\\
V_3 &= \frac{\partial}{\partial t},\\
V_4 &= y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.
\end{aligned}$$

We now compute the commutation relations for all the symmetry generators. First we compute $[V_1, V_4]$. By the definition of the Lie bracket, we have

$$
\begin{aligned}
[V_1, V_4] &= V_1V_4 - V_4V_1 \\
&= \left(\frac{1}{2}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}\right)\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left(\frac{1}{2}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}\right) \\
&= 2V_1.
\end{aligned}
$$

Likewise, one can obtain the commutation relations between other vector fields. In Table 1 below we present the commutator table of the Lie algebra of equation (2.1). Here the entry in row i and column j is represented by $[V_i, V_j]$:

Table 1. Commutator table of the Lie algebra of equation (2.1)

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	$2V_1$
V_2	0	0	0	0
V_3	0	0	0	V_3
V_4	$-2V_1$	0	$-V_3$	0

2.2 Exact solutions of ZK-MEW equation

One of the main reasons for finding symmetries of a differential equation is to use them for finding exact solutions. In this section we will utilize the symmetries calculated in the previous section to calculate exact solutions of (2.1).

One way to obtain exact solutions of (2.1) is by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by (2.1). It is well known that the reduction of a partial differential equation with respect to r-dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

First of all we utilize the symmetry $V = \alpha V_1 + \beta V_2 + \gamma V_3$, where α, β, γ are constants, and reduce the ZK-MEW equation (2.1) to a PDE in two independent variables. We calculate the invariant solution under the operator V, namely

$$V = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial t}.$$

The associated Lagrange equations yields the following three invariants:

$$f = \frac{\alpha y - \beta x}{\alpha}, \quad g = \frac{\alpha t - \gamma x}{\alpha} \quad \text{and} \quad \theta = u. \tag{2.40}$$

Now treating θ as the new dependent variable and f and g as new independent variables, the ZK-MEW equation (2.1) transforms to

$$\begin{aligned} &\alpha^2\theta_g - 3a\alpha\beta\theta^2\theta_f - 3\alpha\gamma\theta^2\theta_g - b\alpha\beta\theta_{fff} \\ &-b\alpha\gamma\theta_{ffg} + b\beta^2\theta_{ffg} + 2b\beta\gamma\theta_{fgg} + b\gamma^2\theta_{ggg} = 0, \end{aligned} \tag{2.41}$$

which is a nonlinear PDE in two independent variables. We further reduce (2.41) using its symmetries. We now determine Lie point symmetries of the equation (2.41). After some lengthy calculations we obtain the following two translational symmetries:

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial f}, \\ \Gamma_2 &= \frac{\partial}{\partial g}. \end{aligned}$$

The combination $\Gamma_1 + \delta\Gamma_2$, where δ is a constant, of the two symmetries Γ_1 and Γ_2 yields the following two invariants:

$$r = \frac{\delta f - g}{\delta} \quad \text{and} \quad \psi = \theta,$$

which gives a group invariant solution $\psi = \psi(r)$. Treating ψ as our new dependent variable and r as new independent variable the equation (2.41) is transformed into the third-order nonlinear ODE

$$\delta^2\alpha^2\psi' + 3a\alpha\delta^2(\beta\delta - \gamma)\psi^2\psi' + b(\alpha\beta\delta^3 - \alpha\gamma\delta^2 + \beta^2\delta^2 - 2\beta\gamma\delta + \gamma^2)\psi''' = 0. \tag{2.42}$$

One can integrate the above equation three times easily. Solving this equation and taking the first two constants of integration to be zero and reverting back to the original variables, we obtain the following group-invariant solution of the ZK-MEW equation (2.1):

$$u(x,y,t) = \pm\sqrt{-\frac{A_1}{A_2}}\,\csc\left(C\sqrt{A_1} \pm r\sqrt{\frac{A_1}{A_3}}\right), \tag{2.43}$$

where

$$\begin{aligned}
A_1 &= 2\delta^2\alpha^2,\\
A_2 &= a\alpha\delta^2(\beta\delta-\gamma),\\
A_3 &= 2b(\alpha\beta\delta^3 - \alpha\gamma\delta^2 + \beta^2\delta^2 - 2\beta\gamma\delta + \gamma^2),\\
r &= \left(\frac{\alpha\gamma-\beta\delta}{\alpha\delta}\right)x + y - \frac{1}{\delta}t.
\end{aligned}$$

Likewise, using Maple on the equation (2.42), one can obtain more invariant solutions of the ZK-MEW equation (2.1). However, we list here a few exact solutions.

$$u(x,y,t) = \sqrt{\frac{2\alpha}{a(\delta\beta-\gamma)}}\,\operatorname{csch}\left(\delta(\alpha r+1)\sqrt{-\frac{1}{A_4}}\right), \tag{2.44}$$

$$u(x,y,t) = \sqrt{-\frac{2\alpha}{a(\delta\beta-\gamma)}}\,\operatorname{sech}\left(\delta(\alpha r+1)\sqrt{-\frac{1}{A_4}}\right), \tag{2.45}$$

$$u(x,y,t) = \sqrt{-\frac{\alpha}{a(\delta\beta-\gamma)}}\,\tanh\left(\delta(\alpha r+1)\sqrt{\frac{1}{A_5}}\right), \tag{2.46}$$

$$u(x,y,t) = \frac{\sqrt{A_6}}{\operatorname{cn}\left(\sqrt{-1/A_7}\,\alpha\delta r+\delta, m\right)}, \tag{2.47}$$

$$u(x,y,t) = \frac{\sqrt{A_8}}{\operatorname{sn}\left(\sqrt{1/A_9}\,\alpha\delta r+\delta, m\right)}, \tag{2.48}$$

where

$$
\begin{aligned}
A_1 &= 2\delta^2\alpha^2, \\
A_2 &= \cdot\, a\alpha\delta^2(\gamma-\beta\delta), \\
A_3 &= 2b(\alpha\gamma\delta^2-\alpha\beta\delta^3-\beta^2\delta^2+2\beta\gamma\delta-\gamma^2), \\
A_4 &= b(\gamma^2+\beta^2\delta^2-2\gamma\beta\delta+\beta\alpha\delta^3-\delta^2\gamma\alpha), \\
A_5 &= 2b(\gamma^2+\beta^2\delta^2-2\gamma\beta\delta+\beta\alpha\delta^3-\delta^2\gamma\alpha), \\
A_6 &= -\frac{2\alpha(m^2-1)}{a(2\delta\beta m^2-2\gamma m^2-\delta\beta+\gamma)}, \\
A_7 &= b(-\beta^2\delta^2+\delta^2\gamma\alpha+2\gamma\beta\delta-\gamma^2+2\gamma^2m^2-\beta\alpha\delta^3 \\
&\quad -4\gamma\beta\delta m^2+2\beta^2\delta^2m^2-2\delta^2\gamma\alpha m^2+2\beta\alpha\delta^3m^2), \\
A_8 &= -\frac{2\alpha}{a(\delta\beta-\gamma m^2-\gamma+\delta\beta m^2)}, \\
A_9 &= b(\beta\alpha\delta^3-\delta^2\gamma\alpha-\delta^2\gamma\alpha m^2-2\gamma\beta\delta m^2+\gamma^2 \\
&\quad +\gamma^2m^2-2\gamma\beta\delta+\beta^2\delta^2+\beta\alpha\delta^3m^2+\beta^2\delta^2m^2), \\
r &= \left(\frac{\alpha\gamma-\beta\delta}{\alpha\delta}\right)x+y-\frac{1}{\delta}t.
\end{aligned}
$$

Here $\mathrm{cn}(Z|m)$ and $\mathrm{sn}(Z|m)$ are the Jacobian elliptic functions [48], which are defined as follows: If

$$Z=\int_0^{\phi}\frac{d\theta}{\sqrt{1-m\sin^2\theta}},$$

where the angle ϕ is called the amplitude, then the functions $\mathrm{cn}(Z|m)$ and $\mathrm{sn}(Z|m)$ are defined as $\mathrm{cn}(Z|m)=\cos\phi$ and $\mathrm{sn}(Z|m)=\sin\phi$, respectively, where m is called the modulus of the elliptic function and $0\le m\le 1$.

Solution (2.45) has no singularities. By taking $\delta=1$, $\alpha=1$, $a=-1$, $b=-1$, $\beta=2$, $\gamma=1$ and $t=0$ in (2.45) we have the following profile.

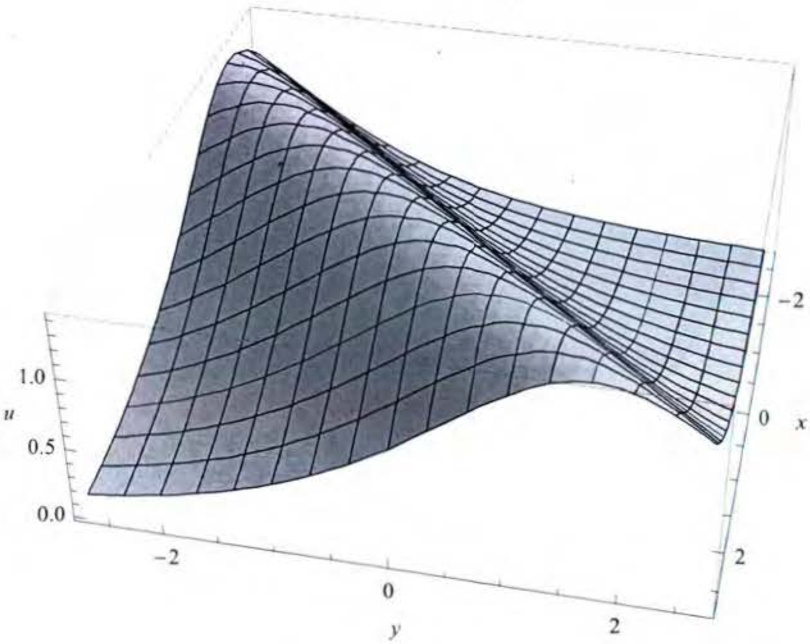

Figure 2.1: Profile of solution (2.45)

Solution (2.48) has infinitely many singularities. These singularities occur when the denominator of (2.48) is zero.

By taking $m = 0.8$, $\delta = 2$, $\alpha = 0.1$, $a = 1$, $b = 1$, $\beta = -3$, $\gamma = 2$ and $t = 3$ in (2.48) we have the following profile of solution (2.48):

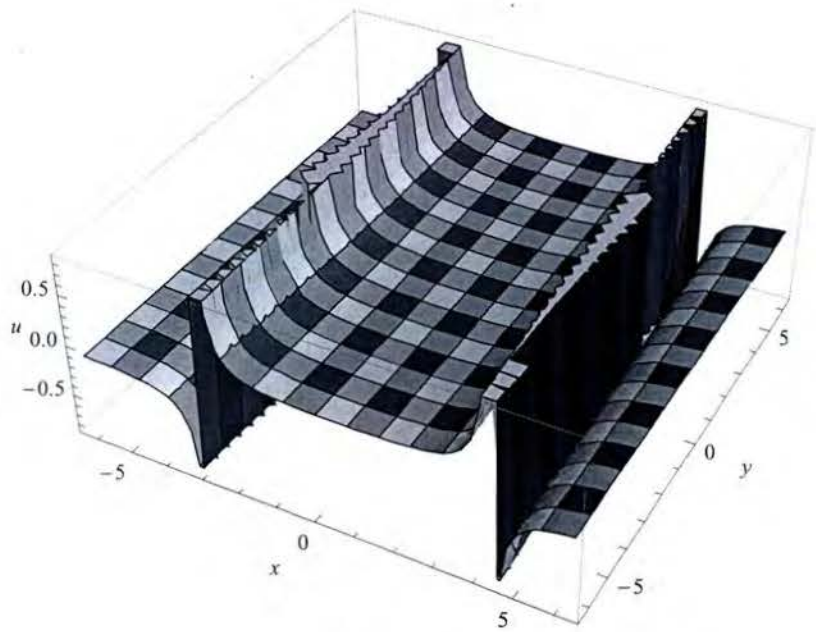

Figure 2.2: Profile of solution (2.48)

We now make use of the symmetry $\Gamma = V_1 + \alpha V_2$ to obtain another group-invariant solution of the ZK-MEW equation (2.1). Using the associated Lagrange equations for the symmetry Γ, we obtain the following three invariants:

$$f = t, \quad g = \frac{\alpha x - y}{\alpha} \quad \text{and} \quad \theta = u. \tag{2.49}$$

Treating θ as the new dependent variable and f and g as new independent variables, the ZK-MEW equation (2.1) transforms to a nonlinear third-order PDE in two independent variables, namely

$$\alpha^2\theta_f + 3a\alpha^2\theta^2\theta_g + b\alpha^2\theta_{fgg} + b\theta_{ggg} = 0. \tag{2.50}$$

Using the standard method of finding Lie point symmetries of a PDE, we see that the equation (2.50) has two Lie point symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial f} \quad \text{and} \quad \Gamma_2 = \frac{\partial}{\partial g}$$

and their combination $\Gamma_1 + \beta\Gamma_2$ yields the following two invariants

$$r = \frac{f\beta - g}{\beta} \quad \text{and} \quad \psi = \theta,$$

which gives a group invariant solution $\psi = \psi(r)$ and (2.50) is then transformed to

$$\alpha^2\beta^3\psi' - 3a\beta^2\alpha^2\psi^2\psi' + b(\beta\alpha^2 - 1)\psi''' = 0, \tag{2.51}$$

which is a third-order nonlinear ODE. The equation (2.51) can be directly integrated three time. Now solving this equation and taking the first two constants of integration to be zero and reverting back to the original variables, we obtain the following solutions of the ZK-MEW equation (2.1):

$$u(x, y, t) = \sqrt{\frac{2\beta}{a}} \sec\left\{\sqrt{2}\alpha\beta^{\frac{3}{2}}\left(\pm\frac{\alpha\beta t - \alpha x + y}{\alpha\beta\sqrt{2b(\beta\alpha^2 - 1)}} + C\right)\right\}, \tag{2.52}$$

where α, β and C are arbitrary constants.

Solution (2.52) has infinitely many singularities.

By taking $a = 1$, $b = 1$, $\beta = 2$, $\alpha = 1$, $y = 0$, $C = 1$ and positive sign in the solution of (2.52) we have its following profile:

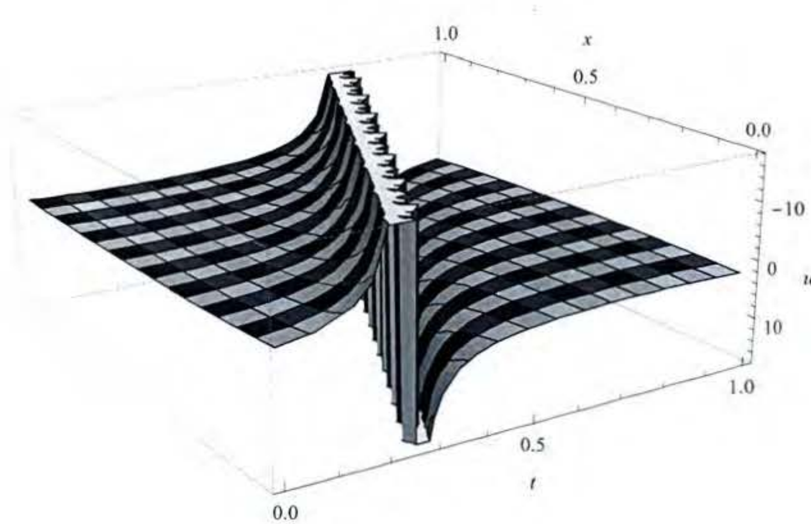

Figure 2.3: Profile of solution (2.52)

2.3 Conclusion

In this chapter we obtained some exact solutions of (2+1)-dimensional ZK-MEW equation (2.1). The integration of the ZK-MEW equation was performed using the Lie group analysis method. The solutions obtained include the non-topological soliton solution, cnoidal waves and the traveling wave solutions. This approach is different from the regular methods of integrability that includes Hirota's bilinear method, Exp-function method and others. The obtained exact solutions can be used as benchmarks against numerical integrators. We have also verified that the solutions we have found are indeed solutions to the original equations and that we have reduced the number of arbitrary constants to a minimum.

Chapter 3

Solutions and conservation laws of ZK-MEW equation with power law nonlinearity

In this chapter we use Lie symmetry method along with the simplest equation method [36, 37], to obtain exact solutions of Zakharov-Kuznetsov modified equal width (ZK-MEW) equation with power law nonlinearity that is given by

$$u_t + au^n u_x + b\left(u_{xt} + u_{yy}\right)_x = 0, \tag{3.1}$$

where a, b and n are real valued constants with $n > 0$. Here, in (3.1), the first term represents the evolution term, the coefficient of a is the nonlinear term while the third term and fourth term together, in parentheses, are the dispersion terms. The solitons are a result of a delicate balance between dispersion and nonlinearity.

We also derive the conservation laws for the ZK-MEW (3.1) using the new conservation theorem [40] and the multiplier method [42, 43]. This work has been published in [45].

A similar equation to our (3.1) was studied in [3]. Solitary wave ansatz method was used to carry out the integration of the equation and 1-soliton solution was obtained. Also, a couple of conserved quantities were calculated.

3.1 Solutions of ZK-MEW equation with power law nonlinearity

3.1.1 Solutions of (3.1) using Lie point symmetries

The symmetry group of ZK-MEW equation with power law nonlinearity (3.1) will be generated by the vector field of the form

$$\Gamma = \xi^1(x, y, t, u)\frac{\partial}{\partial x} + \xi^2(x, y, t, u)\frac{\partial}{\partial y} + \xi^3(x, y, t, u)\frac{\partial}{\partial t} + \eta(x, y, t, u)\frac{\partial}{\partial u}. \qquad (3.2)$$

Applying the third prolongation $\mathrm{pr}^{(3)}\Gamma$ to (3.1) we obtain the overdetermined system of linear partial differential equations (PDEs)

$$\begin{aligned}
\xi^1_u &= 0, \\
\xi^2_u &= 0, \\
\xi^3_u &= 0, \\
\eta_{uu} &= 0, \\
\xi^1_t &= 0, \\
\xi^2_t &= 0, \\
\eta_{tu} &= 0, \\
\xi^1_y &= 0, \\
\xi^3_y &= 0, \\
\eta_{yyu} &= 0,
\end{aligned}$$

$$\eta_x = 0,$$
$$\xi^2_x = 0,$$
$$\xi^3_x = 0,$$
$$\xi^1_{xx} = 0,$$
$$-n\eta u_t + u\xi^1_z u_t - u\xi^3_t u_t + u\eta_t = 0,$$
$$n\eta + u\xi^1_x + u\xi^3_t = 0,$$
$$2\eta_{yu} - \xi 2_{yy} = 0,$$
$$n\eta + 2u\xi^2_y = 0.$$

Solving the above system for ξ^i, $i = 1, 2, 3$ and η we deduce that the symmetry algebra of ZK-MEW equation with power law nonlinearity (3.1) is spanned by the four vector fields

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial x},\\
\Gamma_2 &= \frac{\partial}{\partial y},\\
\Gamma_3 &= \frac{\partial}{\partial t},\\
\text{and} \quad \Gamma_4 &= ny\frac{\partial}{\partial y} + 2nt\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.
\end{aligned}$$

The commutation relations between these vector fields is given in Table 1, where the entry in row i and column j is represented by $[\Gamma_i, \Gamma_j]$:

Table 1. Commutator table of the Lie algebra of equation (1)

$[\Gamma_i, \Gamma_j]$	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	0	0	0	0
Γ_2	0	0	0	$n\Gamma_2$
Γ_3	0	0	0	$2n\Gamma_3$
Γ_4	0	$-n\Gamma_2$	$-2n\Gamma_3$	0

We now construct group-invariant solutions of ZK-MEW equation with power law nonlinearity (3.1) by using the symmetries calculated above. One way to derive such solutions of (3.1) is by reducing it to ordinary differential equations (ODEs). The form of the ODE, in general, depends on the particular subgroup chosen. The reduction of a partial differential equation with respect to r-dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r [12].

We consider a linear combination of space translations $\partial/\partial x$, $\partial/\partial y$ and time translation $\partial/\partial t$, namely the symmetry $\Gamma = \alpha\Gamma_1 + \beta\Gamma_2 + \gamma\Gamma_3$ and reduce the ZK-MEW equation (3.1) to a PDE in two independent variables. The symmetry Γ gives the following three invariants:

$$f = \frac{\beta t - \gamma y}{\beta}, \quad g = \frac{\beta x - \alpha y}{\beta} \quad \text{and} \quad \theta = u. \tag{3.3}$$

Now treating θ as the new dependent variable and f and g as the new independent variables, the ZK-MEW equation (3.1) transforms to

$$\beta^2\theta_f + a\beta^2\theta^n\theta_g + b\beta^2\theta_{fgg} + b\gamma^2\theta_{ffg} + 2b\alpha\gamma\theta_{fgg} + b\alpha^2\theta_{ggg} = 0, \tag{3.4}$$

which is a nonlinear PDE in two independent variables. We will now further reduce (3.4) using its symmetries. Equation (3.4) has the two translation symmetries

$$\Upsilon_1 = \frac{\partial}{\partial f} \quad \text{and} \quad \Upsilon_2 = \frac{\partial}{\partial g}.$$

The combination $\Upsilon_1 + \delta\Upsilon_2$ of the two symmetries Υ_1 and Υ_2 yields the invariants

$$q = \frac{\delta f - g}{\delta} \quad \text{and} \quad \psi = \theta,$$

which gives a group invariant solution $\psi = \psi(q)$ and consequently using these invariants, (3.4) is transformed to the third-order nonlinear ODE

$$B_1\psi'''(q) + B_2\psi^n(q)\psi'(q) + B_3\psi'(q) = 0, \tag{3.5}$$

where

$$B_1 = -b\delta\beta^2 + b\gamma^2\delta^2 - 2b\alpha\gamma\delta + b\alpha^2, \quad B_2 = a\delta^2\beta^2, \quad B_3 = --\beta^2\delta^3. \tag{3.6}$$

Integrating equation (3.5) twice with respect to q and taking the constants to be zero we obtain

$$\frac{B_1}{2}\psi'^2(q) + \frac{B_2}{(n+2)(n+1)}\psi^{n+2}(q) + \frac{B_3}{2}\psi^2(q) = 0. \tag{3.7}$$

Integrating the above equation and reverting back to the original variables, we obtain the following solution of the ZK-MEW equation (3.1) for arbitrary values of n in the form

$$u(x,y,t) = \left(-\frac{A_3}{A_2}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}(\chi), \tag{3.8}$$

where

$$\begin{aligned}
&\chi = \frac{n}{2\sqrt{A_1}}\left(\pm\sqrt{-A_3}q - \sqrt{A_1 A_3}\right),\\
&q = -x/\delta + y(\alpha-\gamma\delta)/(\beta\delta) + t,\\
&A_1 = \frac{1}{2}b\left((\alpha-\gamma\delta)^2 - \beta^2\delta\right),\ A_2 = \frac{a\beta^2\delta^2}{(n+1)(n+2)},\ A_3 = -\frac{1}{2}\beta^2\delta^3.
\end{aligned}$$

The solution (3.8) represents a non-topological 1-soliton solution.

3.1.2 Solutions of (3.1) for n=1,2 using simplest equation method

We now use the simplest equation method which was introduced by Kudryashov [36, 37] to solve (3.5) for $n = 1, 2$. The simplest equations that will be used are the Bernoulli and Ricatti equations.

Let us consider the solution of (3.5) in the form

$$\psi(q) = \sum_{i=0}^{M} A_i (G(q))^i, \tag{3.9}$$

where $G(q)$ satisfies the Bernoulli and Ricatti equations, M is a positive integer that can be determined by balancing procedure as in [38] and $A_0, \cdots, A_M$ are parameters to be determined. We note that the Bernoulli and Ricatti equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

For the Bernoulli equation

$$G'(q) = cG(q) + dG^k(q), \tag{3.10}$$

where c and d are constants and k is an integer and $k > 1$, we shall use the solutions [39]

$$G(q) = \left(\frac{c \exp[c(k-1)(q+C)]}{1 - d\exp[c(k-1)(q+C)]} \right)^{\frac{1}{k-1}} \tag{3.11}$$

for the case $c > 0$, $d < 0$ and

$$G(q) = \left(- \frac{c \exp[c(k-1)(q+C)]}{1 + d\exp[c(k-1)(q+C)]} \right)^{\frac{1}{k-1}} \tag{3.12}$$

for the case $c < 0$ and $d > 0$. Here C is a constant of integration. For the Ricatti equation

$$G'(q) = cG^2(q) + dG(q) + e \tag{3.13}$$

where c, d and e are constants, we shall use the solutions [39]

$$G(q) = -\frac{d}{2c} - \frac{\theta}{2c} \tanh\left[\frac{1}{2}\theta(q+C)\right] \tag{3.14}$$

and

$$G(q) = -\frac{d}{2c} - \frac{\theta}{2c} \tanh\left(\frac{q\theta}{2}\right) + \frac{\exp(\frac{q\theta}{2})}{2\cosh(\frac{q\theta}{2})[\frac{c}{\theta} + 2C\exp(\frac{q\theta}{2})\cosh(\frac{q\theta}{2})]}, \tag{3.15}$$

with $\theta^2 = d^2 - 4ce > 0$ and C is a constant of integration.

Solutions of (3.1) using the Bernoulli equation as the simplest equation

Solutions of (3.1) for $n = 1$

The balancing procedure with $k = 2$ [38] yields $M = 2$ so the solutions in (3.9) are of the form

$$\psi(q) = A_0 + A_1 G + A_2 G^2. \tag{3.16}$$

Substituting (3.16) into (3.5) and equating all coefficients of the functions G^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 and A_2. These algebraic equations are

$$A_1B_1c^3 + A_0A_1B_2c + A_1B_3c = 0,$$

$$8A_2B_1c^3 + 7A_1B_1c^2d + A_1^2B_2c + 2A_0A_2B_2c + 2A_2B_3c + A_0A_1B_2d + A_1B_3d = 0,$$

$$38A_2B_1c^2d + 12A_1B_1cd^2 + 3A_1A_2B_2c + A_1^2B_2d + 2A_0A_2B_2d + 2A_2B_3d = 0,$$

$$54A_2B_1cd^2 + 2A_2^2B_2c + 6A_1B_1d^3 + 3A_1A_2B_2d = 0,$$

$$24A_2B_1d^3 + 2A_2^2B_2d = 0,$$

where B_1, B_2 and B_3 are given by (3.6). Solving the above system of algebraic equations with the aid of Mathematica, we obtain the following values of A_0, A_1 and A_2:

$$A_0 = -(B_1c^2 + B_3)/B_2, \quad A_1 = -12B_1cd/B_2, \quad A_2 = A_1d/c.$$

Therefore when $c > 0$ and $d < 0$ the solution of (3.1) with $n = 1$ is given by

$$u(x,y,t) = A_0 + A_1\frac{c\exp[c(q+C)]}{1 - d\exp[c(q+C)]} + A_2\left(\frac{c\exp[c(q+C)]}{1 - d\exp[c(q+C)]}\right)^2 \tag{3.17}$$

and when $c < 0$ and $d > 0$ the solution of (3.1) for $n = 1$ is

$$u(x,y,t) = A_0 - A_1\frac{c\exp[c(q+C)]}{1 + d\exp[c(q+C)]} + A_2\left(-\frac{c\exp[c(q+C)]}{1 + d\exp[c(q+C)]}\right)^2, \tag{3.18}$$

where $q = -x/\delta + y(\alpha - \gamma\delta)/(\beta\delta) + t$ and C is a constant of integration.

Solutions of (3.1) for $n = 2$

The balancing procedure with $k = 2$ yields $M = 1$ so by (3.9) the solutions are of the form

$$\psi(q) = A_0 + A_1G. \tag{3.19}$$

As before, substituting (3.19) into (3.5), we obtain the following algebraic system of equations

$$A_1B_1c^3 + A_0^2A_1B_2c + A_1B_3c = 0,$$

$$7A_1B_1c^2d + 2A_0A_1^2B_2c + A_0^2A_1B_2d + A_1B_3d = 0,$$

$$12A_1B_1cd^2 + A_1^3B_2c + 2A_0A_1^2B_2d = 0,$$

$$6A_1B_1d^3 + A_1^3B_2d = 0,$$

which on solving yields

$$c = \mp\sqrt{2B_3/B_1}, \quad A_0 = \mp\sqrt{-3B_3/B_2}, \quad A_1 = 2A_0 d/c.$$

Hence when $c > 0$ and $d < 0$ the solution of (3.1) for $n = 2$ is given by

$$u(x,y,t) = A_0 + A_1 \frac{c\exp[c(q+C)]}{1 - d\exp[c(q+C)]} \tag{3.20}$$

and when $c < 0$ and $d > 0$ the solution of (3.1) for the $n = 2$ case is

$$u(x,y,t) = A_0 - A_1 \frac{c\exp[c(q+C)]}{1 + d\exp[c(q+C)]}, \tag{3.21}$$

where $q = -x/\delta + y(\alpha - \gamma\delta)/(\beta\delta) + t$ and C is a constant of integration.

Solutions of (3.1) using the Ricatti equation as the simplest equation

Solutions of (3.1) for $n = 1$

The balancing procedure yields $M = 2$ so the solutions in (3.9) are of the form

$$\psi(q) = A_0 + A_1 G + A_2 G^2. \tag{3.22}$$

Substituting (3.22) into (3.5), we obtain an algebraic system of equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions G^i to zero. The corresponding algebraic equations are

$$\begin{aligned}
&2A_1B_1ce^2 + A_1B_1d^2e + 6A_2B_1de^2 + A_0A_1B_2e + A_1B_3e = 0,\\
&8A_1B_1cde + 16A_2B_1ce^2 + A_1B_1d^3 + 14A_2B_1d^2e + A_0A_1B_2d\\
&+A_1B_3d + A_1^2B_2e + 2A_0A_2B_2e + 2A_2B_3e = 0,\\
&8A_1B_1c^2e + 7A_1B_1cd^2 + 52A_2B_1cde + A_0A_1B_2c + A_1B_3c\\
&+8A_2B_1d^3 + A_1^2B_2d + 2A_0A_2B_2d + 2A_2B_3d + 3A_1A_2B_2e = 0,\\
&12A_1B_1c^2d + 40A_2B_1c^2e + 38A_2B_1cd^2 + A_1^2B_2c\\
&+2A_0A_2B_2c + 2A_2B_3c + 3A_1A_2B_2d + 2A_2^2B_2e = 0,\\
&6A_1B_1c^3 + 54A_2B_1c^2d + 3A_1A_2B_2c + 2A_2^2B_2d = 0,\\
&24A_2B_1c^3 + 2A_2^2B_2c = 0,
\end{aligned}$$

where B_1, B_2 and B_3 are given by (3.6). Solving the above equations yields

$$A_0 = -(8B_1ce + B_1d^2 + B_3)/B_2, \quad A_1 = -12B_1cd/B_2, \quad A_2 = A_1c/d$$

and hence the solutions of (3.1) for $n = 1$ are

$$\begin{aligned} u(x,y,t) &= A_0 + A_1\left\{-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(q+C)\right]\right\} \\ &\quad + A_2\left\{-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(q+C)\right]\right\}^2 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} u(x,y,t) &= A_0 + A_1\left\{-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{q\theta}{2}\right) + \frac{\exp(\frac{q\theta}{2})}{2\cosh(\frac{q\theta}{2})[\frac{c}{\theta} + 2C\exp(\frac{q\theta}{2})\cosh(\frac{q\theta}{2})]}\right\} \\ &\quad + A_2\left\{-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{q\theta}{2}\right)\right. \\ &\quad \left. + \frac{\exp(\frac{q\theta}{2})}{2\cosh(\frac{q\theta}{2})[\frac{c}{\theta} + 2C\exp(\frac{q\theta}{2})\cosh(\frac{q\theta}{2})]}\right\}^2, \end{aligned} \tag{3.24}$$

where $q = -x/\delta + y(\alpha - \gamma\delta)/(\beta\delta) + t$ and C is a constant of integration.

Solutions of (3.1) for $n = 2$

The balancing procedure yields $M = 1$ so the solutions in (3.9) are of the form

$$\psi(q) = A_0 + A_1G. \tag{3.25}$$

Substituting (3.25) into (3.5), we obtain the algebraic system of equations

$$\begin{aligned} 2A_1B_1ce^2 + A_1B_1d^2e + A_0^2A_1B_2e + A_1B_3e &= 0, \\ 8A_1B_1cde + A_1B_1d^3 + A_0^2A_1B_2d + A_1B_3d + 2A_0A_1^2B_2e &= 0, \\ 8A_1B_1c^2e + 7A_1B_1cd^2 + A_0^2A_1B_2c + A_1B_3c + 2A_0A_1^2B_2d + A_1^3B_2e &= 0, \\ 12A_1B_1c^2d + 2A_0A_1^2B_2c + A_1^3B_2d &= 0, \\ 6A_1B_1c^3 + A_1^3B_2c &= 0. \end{aligned}$$

Solving the above equations with the aid of Mathematica, we obtain

$$c = \frac{B_1d^2 - 2B_3}{4B_1e}, \quad A_0 = \mp\sqrt{-3B_1/2B_2}\,d, \quad A_1 = \frac{2A_0c}{d}.$$

Hence we have the following solutions of (3.1) for $n = 2$:

$$u(x,y,t) \;=\; A_0 + A_1\left\{-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(q+C)\right]\right\}, \qquad (3.26)$$

$$\begin{aligned} u(x,y,t) \;=\;& A_0 + A_1\left\{-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{q\theta}{2}\right)\right. \\ &\left. + \frac{\exp(\frac{q\theta}{2})}{2\cosh(\frac{q\theta}{2})[\frac{c}{\theta} + 2C\exp(\frac{q\theta}{2})\cosh(\frac{q\theta}{2})]}\right\}, \end{aligned} \qquad (3.27)$$

where $q = -x/\delta + y(\alpha - \gamma\delta)/(\beta\delta) + t$ and C is a constant of integration.

Solution (3.20) has no singularities. By taking $a = -1$, $b = 1$, $\alpha = 1$, $\beta = 2$, $\delta = 3$, $\gamma = 4$, $c = 1$, $d = 2$, $e = -3$, $y = 0$ and $C = 0$ in the negative solution of (3.20) we have the following profile.

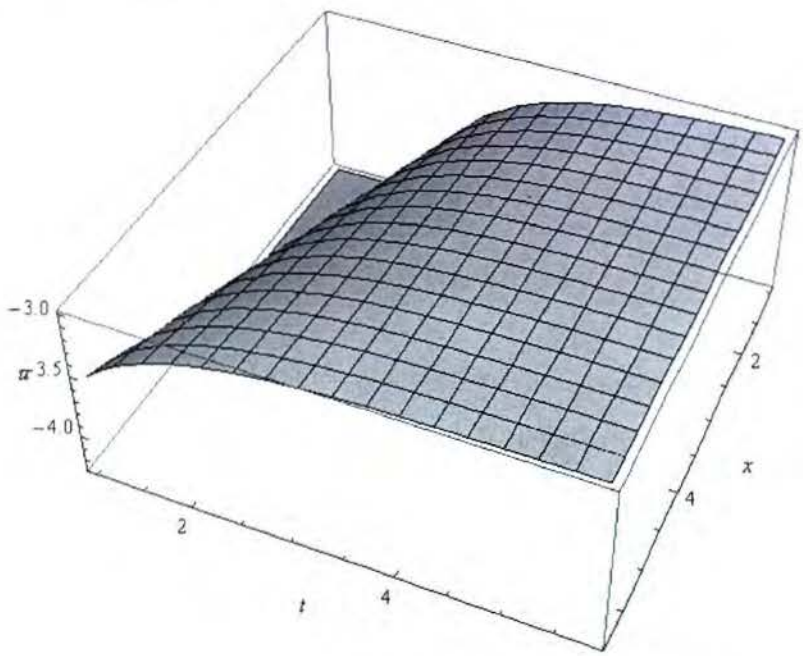

Figure 3.1: Profile of solution (3.20)

Solution (3.23) has no singularities. By taking $a = 1$, $b = 1$, $\alpha = 1$, $\beta = 2$, $\delta = 3$, $\gamma = 4$, $c = 1$, $d = 2$, $e = -3$, $y = 0$ and $C = 0$ in (3.23) we have the following profile.

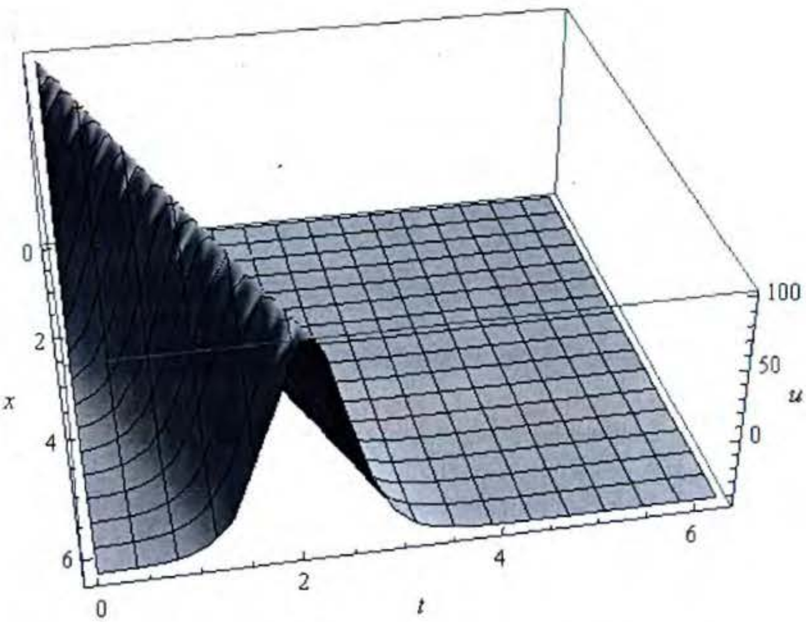

Figure 3.2: Profile of solution (3.23)

3.2 Conservation laws of ZK-MEW equation with power law nonlinearity

In this section we obtain conservation laws for the Zarkharov-Kuznetsov modified equal width (ZK-MEW) equation with power law nonlinearity (3.1) using two different approaches; the new conservation theorem due to Ibragimov [40] and the multiplier method [42, 43]. First we present some preliminaries which we will need later in this section.

Preliminaries

We briefly present the notation and pertinent results which we utilize below. For details the reader is referred to [15, 40–43].

Fundamental operators and their relationship

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$

and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.28}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the *total derivative operator* with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n, \tag{3.29}$$

where the summation convention is used whenever appropriate.

The following are known (see for example, [15] and the references therein).

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \qquad \alpha = 1, \ldots, m, \tag{3.30}$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \ \xi^i, \eta^\alpha \in \mathcal{A}, \tag{3.31}$$

where $\mathcal{A}$ is the space of *differential functions.* The operator (3.31) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \tag{3.32}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha \\ \zeta_{i_1 \ldots i_s}^\alpha &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha, \quad s > 1, \end{aligned} \tag{3.33}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{3.34}$$

One can write the Lie-Bäcklund operator (3.32) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}. \tag{3.35}$$

The *Noether operators* associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{ii_1i_2\ldots i_s}}, \quad i=1,\ldots,n, \tag{3.36}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (3.30) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\ldots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1j_2\ldots j_s}}, \quad i=1,\ldots,n, \quad \alpha=1,\ldots,m, \tag{3.37}$$

and the Euler-Lagrange , Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{3.38}$$

The n-tuple vector $T=(T^1,T^2,\ldots,T^n)$, $T^j\in\mathcal{A}$, $j=1,\ldots,n$, is a *conserved vector* of (3.28) if T^i satisfies

$$D_i T^i|_{(3.28)} = 0. \tag{3.39}$$

The equation (3.39) defines a *local conservation law* of system (3.28).

Multiplier Method

A multiplier $\Lambda_\alpha(x,u,u_{(1)},\ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{3.40}$$

hold identically. Here we will consider multipliers of the second order, that is $\Lambda_\alpha = \Lambda_\alpha(t,x,u,v,u_x,v_x,u_{xx},v_{xx})$. The right hand side of (3.40) is a divergence expression. The determining equations for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{3.41}$$

Once the multipliers are obtained the conserved vectors are calculated via a homotopy formula [42–44].

Variational method for a system and its adjoint

The system of *adjoint equations* for the system of kth-order differential equations (3.28) is defined by [41]

$$E^*_\alpha(x,u,v,\ldots,u_{(k)},v_{(k)}) = 0, \quad \alpha=1,\ldots,m, \tag{3.42}$$

where

$$E_\alpha^*(x,u,v,\dots,u_{(k)},v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1,\dots,m,\ v = v(x) \tag{3.43}$$

and $v = (v^1, v^2, \dots, v^m)$ are new dependent variables.

We recall here the following results as given in Ibragimov [40].

Assume that the system of equations (3.28) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{3.44}$$

Then the system of adjoint equations (3.42) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}, \quad \eta_*^\alpha = -[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)], \tag{3.45}$$

where the operator (3.45) is an extension of (3.44) to the variable v^α and the λ_β^α are obtainable from

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \tag{3.46}$$

Theorem 3.1. [40] Every Lie point, Lie-Bäcklund and non local symmetry (3.44) admitted by the system of equations (3.28) gives rise to a conservation law for the system consisting of the equation (3.28) and the adjoint equation (3.42), where the components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\dots D_{i_s}(W^\alpha)\frac{\delta L}{\delta u^\alpha_{ii_1 i_2\dots i_s}},\ i = 1,\dots,n, \tag{3.47}$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x,u,\dots,u_{(k)}). \tag{3.48}$$

3.2.1 Construction of conservation laws for ZK-MEW equation with power law nonlinearity

We now construct conservation laws for the ZK-MEW with power law nonlinearity using the two approaches described above.

Application of the new Conservation Theorem

The ZK-MEW equation with power law nonlinearity together with its adjoint equation are given by

$$E_\alpha \equiv u_t + au^n u_x + bu_{xxt} + bu_{xyy} = 0, \tag{3.49a}$$

$$E^*_\alpha \equiv v_t + au^n v_x + bv_{xxt} + bv_{xyy} = 0. \tag{3.49b}$$

By recalling (3.48), we get the following third-order Lagrangian for the system of equations (3.49a) and (3.49b):

$$L = v(u_t + au^n u_x + bu_{xxt} + bu_{xyy}). \tag{3.50}$$

We reduce the third-order Lagrangian to second-order Lagrangian which is given by

$$L = v(u_t + au^n u_x) - bv_t u_{xx} - bv_x u_{yy}. \tag{3.51}$$

We recall that the ZK-MEW equation with power law nonlinearity

$$u_t + au^n u_x + b\left(u_{xt} + u_{yy}\right)_x = 0,$$

admits the following Lie point symmetry generators:

$$\begin{aligned} \Gamma_1 &= \partial_x, \\ \Gamma_2 &= \partial_y, \\ \Gamma_3 &= \partial_t, \\ \text{and}\quad \Gamma_4 &= ny\partial_y + 2nt\partial_t - 2u\partial_u. \end{aligned}$$

We have the following four cases:

(i) We first consider the Lie point symmetry generator $\Gamma_1 = \partial_x$ of the ZK-MEW equation with power law nonlinearity. Corresponding to this symmetry the Lie characteristic functions are $W^1 = -u_x$ and $W^2 = -v_x$. Thus by using (3.47), the components of the conserved vector are given by

$$\begin{aligned}
T_1^t &= -vu_x + bv_x u_{xx}, \\
T_1^x &= vu_t - bu_x v_{tx}, \\
T_1^y &= -bu_x v_{xy} + bv_x u_{xx}.
\end{aligned}$$

(ii) The Lie point symmetry generator $\Gamma_2 = \partial_y$ has Lie characteristic functions given by $W^1 = -u_y$ and $W^2 = -v_y$. Hence using (3.47), one can obtain the conserved vector whose components are

$$\begin{aligned}
T_2^t &= -vu_y + bv_y u_{xx}, \\
T_2^x &= -avu^n u_y - bu_y v_{tx} + bv_y u_{yy} + bv_t u_{xy}, \\
T_2^y &= vu_t + avu^n u_x - bv_t u_{xx} - bu_y v_{xy}.
\end{aligned}$$

(iii) The Lie point symmetry generator $\Gamma_3 = \partial_t$ has Lie characteristic functions given by $W^1 = -u_t$ and $W^2 = -v_t$. Hence using (3.47), one can obtain the conserved vector whose components are

$$\begin{aligned}
T_3^t &= avu^n u_x - bv_x u_{yy}, \\
T_3^x &= -avu^n u_t - bu_t v_{tx} + bv_t u_{yy} + bv_t u_{tx}, \\
T_3^y &= -bu_t v_{xy} + bv_x u_{ty}.
\end{aligned}$$

(iv) Finally, we consider the symmetry generator $\Gamma_4 = ny\partial_y + 2nt\partial_t - 2u\partial_u$, admitted by the ZK-MEW equation with power law nonlinearity and extend it to the variable v. The pronlogation of Γ_4 to the derivatives involved in the ZK-MEW has the form

$$\begin{aligned}
\Gamma_4 &= 2nt\frac{\partial}{\partial t} + ny\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} + (-2u_t - 2nu_t)\frac{\partial}{\partial u_t} - 2u_x\frac{\partial}{\partial x} \\
&+ (-2u_{xxt} - 2nu_{xxt})\frac{\partial}{\partial u_{xxt}} + (-2u_{xyy} - 2nu_{xyy})\frac{\partial}{\partial u_{xyy}}.
\end{aligned}$$

This shows that (3.46) is written as $\Gamma(u_t + au^n u_x + bu_{xxt} + bu_{xyy}) = -2 - 2n(u_t + au^n u_x + bu_{xxt} + bu_{xyy})$, hence $\lambda = -2 - 2n$. Noting that in our case $D_i(\xi^i) \neq 0$ and using (3.45) we obtain $\eta^2 = v(2 - n)$. Hence, the extension (3.45) of the symmetry generator Γ_4 to v has the form

$$\Upsilon = ny\partial_y + 2nt\partial_t - 2u\partial_u + (2-n)v\partial_v$$

Thus the Lie characteristic functions of the extended symmetry are $W^1 = -2u - 2ntu_t - nyu_y$ and $W^2 = 2v - nv - 2ntv_t - nyv_y$. Hence using (3.47), one can obtain the conserved vector whose components are

$$\begin{aligned}
T_4^t &= 2antu_xvu^n + bnu_{xx}v - 2bu_{xx}v - nyu_yv - 2uv - 2bntu_{yy}v_x + bnyu_{xx}v_y,\\
T_4^x &= -anyu_yvu^n - 2antu_tvu^n + bnu_{yy}v - 2bv_{tx}u - 2bu_{yy}v - 2avu^{n+1} + bnyv_tu_{xy}\\
&\quad -bnyu_yv_{tx} + 2bntv_tu_{tx} - 2bntu_tv_{tx} + 2bntv_tu_{yy} + bnyu_{yy}v_y + 2bv_tu_x,\\
T_4^y &= 2anyu_xvu^n - 2bv_{xy}u + nyu_tv - 2bntu_tv_{xy} - bnyv_tu_{xx} + 2bntv_xu_{ty} + bnu_yv_x\\
&\quad -bnyu_yv_{xy} + 2bu_yv_x.
\end{aligned}$$

Remark 1. The components of the conserved vectors contain the arbitrary solutions v of the adjoint equation (3.49b) and hence one can obtain an infinite number of conservation laws.

Application of the Multiplier Method

For the ZK-MEW equation with power law nonlinearity, we obtain the zeroth-order multiplier, $\Lambda(t, x, y, u)$ that is given by

$$\Lambda = Cu + f(y) \tag{3.52}$$

where C is a constant and $f(y)$ is an arbitrary function.

Corresponding to the above multiplier we have the following conserved vectors of (3.1):

$$\begin{aligned}
\Phi_1^t &= \frac{1}{6}\left\{2bu_{xx}u + 3u^2 - bu_x^2\right\}, \\
\Phi_1^x &= \frac{1}{6(n+2)}\left\{2bnu_{yy}u + 4bnu_{tx}u + 4bu_{yy}u + 8bu_{tx}u\right. \\
&\quad \left. +6au^{n+2} - 2bnu_tu_x - bnu_y{}^2 - 4bu_tu_x - 2bu_y{}^2\right\}, \\
\Phi_1^y &= \frac{1}{3}\left\{2bu_{xy}u - bu_xu_y\right\},
\end{aligned}$$

$$\begin{aligned}
\Phi_2^t &= \frac{1}{3}\left\{3f(y)u + bf(y)u_{xx}\right\}, \\
\Phi_2^x &= \frac{1}{3(n+1)}\left\{3af(y)u^{n+1} + bnf''(y)u + bf''(y)u - bnf'(y)u_y\right. \\
&\quad \left. -bf'(y)u_y + 2bnf(y)u_{tx} + bnf(y)u_{yy} + 2bf(y)u_{tx} + bf(y)u_{yy}\right\}, \\
\Phi_2^y &= \frac{1}{3}\left\{2bf(y)u_{xy} - bf'(y)u_x\right\}.
\end{aligned}$$

Remark 2. Due to the presence of the arbitrary function in the multiplier, one can obtain infinitely many conservation laws for the ZK-MEW equation with power law nonlinearity.

3.3 Conclusion

In this chapter we obtained the solutions of the ZK-MEW equation with power law nonlinearity by using its Lie point symmetries along with the simplest equation method. Solutions obtained were solitary wave solutions. We also obtained conservation laws for the ZK-MEW equation with power law nonlinearity by applying the new conservation theorem [40] and the multiplier method [42, 43]. These solutions obtained could be used as benchmarks against numerical integrators.

Chapter 4

Topological and non-topological solitons for Zakharov-Kuznetsov MEW equation with power law nonlinearity

In this chapter we obtain both topological as well as non-topological soliton solutions of the ZK-MEW equation (3.1) with power law nonlinearity using the solitary wave ansatz method [1]. The constraint relation between the parameters is also determined.

4.1 Topological solitons

In order to obtain the topological 1-soliton solution to (3.1), the starting hypothesis is given by

$$u(x, y, t) = A \tanh^{p} \tau, \tag{4.1}$$

where

$$\tau = B_1 x + B_2 y - vt. \tag{4.2}$$

Here in (4.1) and (4.2), A, B_1 and B_2 are free parameters while v is the velocity of the soliton. Thus from (4.1),

$$u_t = pAv\left(\tanh^{p+1}\tau - \tanh^{p-1}\tau\right), \tag{4.3}$$

$$u_x = pAB_1\left(\tanh^{p-1}\tau - \tanh^{p+1}\tau\right), \tag{4.4}$$

$$\begin{aligned} u_{xxt} &= -pvAB_1^2\left[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau\right. \\ &+ \left.\left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\right] \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} u_{xyy} &= pAB_1B_2^2\left[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau\right. \\ &+ \left.\left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\right]. \end{aligned} \tag{4.6}$$

Substituting (4.3)-(4.6) into (3.1) gives

$$\begin{aligned} & pAv\left(\tanh^{p+1}\tau - \tanh^{p-1}\tau\right) \\ + \; & apA^{n+1}B_1\left(\tanh^{np+p-1}\tau - \tanh^{np+p+1}\tau\right) \\ + \; & bpAB_1\left(B_2^2 - vB_1\right)\left[(p-1)(p-2)\tanh^{p-3}\tau - \left\{2p^2+(p-1)(p-2)\right\}\tanh^{p-1}\tau\right. \\ + \; & \left.\left\{2p^2+(p+1)(p+2)\right\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\right] = 0. \end{aligned} \tag{4.7}$$

Now from (4.7), setting the exponents $np+p-1$ and $p+1$ equal to one another gives

$$np+p-1 = p+1, \tag{4.8}$$

which gives

$$p = \frac{2}{n}. \tag{4.9}$$

The same result is yielded when the exponents $np+p+1$ and $p+3$ are equated. Now in (4.7) the linearly independent functions are $\tanh^{p+j}\tau$, where $j=-3,-1,1$ and 3. However $\tanh^{p-3}\tau$ is a stand alone linearly independent function. So setting its coefficient to zero yields

$$p = 1 \tag{4.10}$$

or

$$p = 2. \tag{4.11}$$

Now, if $p = 1$, then

$$n = 2 \tag{4.12}$$

while if $p = 2$, then

$$n = 1. \tag{4.13}$$

This shows that topological solitons of the ZK-MEW equation with power law non-linearity will only exist if $n = 1$ or $n = 2$ and for no other values of n. This is a very important observation that is being made here for the first time. Now, setting the coefficients of the other linearly independent functions to zero gives

$$v = \frac{-aA^n B_1 - 2p^2 bB_1 B_2^2 - bB_1 B_2^2 (p+1)(p+2)}{1 - 2p^2 bB_1^2 - bB_1^2 (p+1)(p+2)}, \tag{4.14}$$

$$v = \frac{-2p^2 bB_1 B_2^2 - bB_1 B_2^2 (p-1)(p-2)}{1 - 2p^2 bB_1^2 - bB_1^2 (p-1)(p-2)} \tag{4.15}$$

and

$$A = \left[\frac{b}{a}(p+1)(p+2)\left(vB_1 - B_2^2\right)\right]^{\frac{1}{n}}. \tag{4.16}$$

Now equating the two values of the velocity, v, from (4.14) and (4.15) also gives (4.16) and thus this closes the system of topological soliton parameters. In the following two subsections the special cases where $n = 1$ and $n = 2$ will be studied.

4.1.1 p=1 OR n=2

In this case the power law ZK-MEW equation is

$$u_t + au^2 u_x + b\left(u_{xxt} + u_{xyy}\right) = 0. \tag{4.17}$$

Thus the velocity of the soliton is given by

$$v = \frac{-aA^2 B_1 - 8bB_1 B_2^2}{1 - 8bB_1^2} \tag{4.18}$$

or

$$v = \frac{-2bB_1B_2^2}{1-2bB_1^2}, \tag{4.19}$$

which leads to

$$A = \left[\frac{6b}{a}\left(vB_1 - B_2^2\right)\right]^{\frac{1}{2}}. \tag{4.20}$$

For this case the topological 1-soliton solution to (4.17) is

$$u(x,y,t) = A\tanh\left(B_1x + B_2y - vt\right) \tag{4.21}$$

where the free parameters A, B_1 and B_2 are in (4.20) and the velocity of the soliton is given by (4.18) or (4.19).

4.1.2 p=2 OR n=1

In this case the power law ZK-MEW equation is

$$u_t + auu_x + b\left(u_{xxt} + u_{xyy}\right) = 0. \tag{4.22}$$

Thus the velocity of the soliton is given by

$$v = \frac{-aAB_1 - 20bB_1B_2^2}{1-20bB_1^2} \tag{4.23}$$

or

$$v = \frac{-8bB_1B_2^2}{1-8bB_1^2}, \tag{4.24}$$

which leads to

$$A = \frac{12b}{a}\left(vB_1 - B_2^2\right). \tag{4.25}$$

For this case the topological 1-soliton solution to (4.22) is

$$u(x,y,t) = A\tanh^2\left(B_1x + B_2y - vt\right), \tag{4.26}$$

where the free parameters A, B_1 and B_2 are in (4.25) and the velocity of the soliton is given by (4.23) or (4.24).

4.2 Non-topological solitons

For non-topological solitons the starting hypothesis is given by

$$u(x,y,t) = \frac{A}{\cosh^p \tau}. \tag{4.27}$$

In this case A represents the amplitude of the soliton while B_1 and B_2 are the inverse widths in the x and y directions, respectively, while v is the velocity of the soliton. Here, from (4.27),

$$u_t = pvA\frac{\tanh\tau}{\cosh^p \tau}, \tag{4.28}$$

$$u_x = -pAB_1\frac{\tanh\tau}{\cosh^p \tau}, \tag{4.29}$$

$$u_{xxt} = p^3vAB_1^2\frac{\tanh^3\tau}{\cosh^p \tau} - p(3p+2)vAB_1^2\frac{\tanh\tau}{\cosh^{p+2} \tau} \tag{4.30}$$

and

$$u_{xyy} = -p^3AB_1B_2^2\frac{\tanh^3\tau}{\cosh^p \tau} + p(3p+2)AB_1B_2^2\frac{\tanh\tau}{\cosh^{p+2} \tau}. \tag{4.31}$$

Substituting (4.28)-(4.31) into (3.1) gives

$$\begin{aligned} & pvA\frac{\tanh\tau}{\cosh^p\tau} - apA^{n+1}B_1\frac{\tanh\tau}{\cosh^{np+p}\tau} \\ + \ & bp^3AB_1(vB_1 - B_2^2)\frac{\tanh^3\tau}{\cosh^p\tau} - bp(3p+2)AB_1(vB_1 - B_2^2)\frac{\tanh\tau}{\cosh^{p+2}\tau} = 0. \end{aligned} \tag{4.32}$$

From (4.32), equating the exponents $np+p$ and $p+2$ gives

$$np + p = p + 2, \tag{4.33}$$

which leads to the same value of p as in (4.9). Now from (4.32) setting the coefficients of the linearly independent functions $1/\cosh^{p+j}\tau$ for $j = 0, 2$ to zero gives

$$v = \frac{4bB_1B_2^2}{n^2 + 4bB_1^2} \tag{4.34}$$

and

$$A = \left[\frac{2b}{an}(n+3)(B_2^2 - vB_1)\right]^{\frac{1}{n}}. \tag{4.35}$$

Thus the non-topological 1-soliton solution to (3.1) is given by

$$u(x, y, t) = \frac{A}{\cosh^{\frac{2}{n}}(B_1 x + B_2 y - vt)}, \tag{4.36}$$

where the amplitude A is related to the other parameters as in (4.35) while the velocity of the soliton is given by (4.34).

Solution (4.36) has no singularities. By taking $a = 1$, $b = 1$, $n = 2$, $B_1 = 1$, $B_2 = 1$ and $y = 0$ in (4.36) we have the following profile.

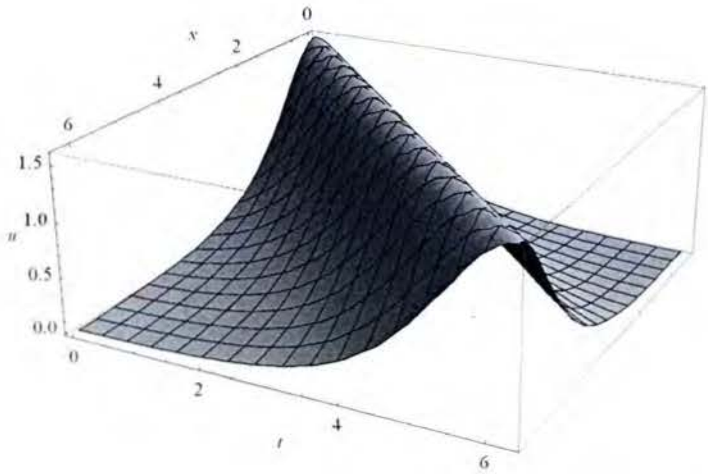

Figure 4.1: Profile of solution (4.36)

4.3 Conclusion

In this chapter the topological as well as non-topological soliton solutions are obtained for the ZK-MEW equation with power law nonlinearity by ansatz method. The solutions obtained are the solitary wave solutions. In this context it has been shown that the topological solitons will only exit for the two special cases when $n = 1$ and $n = 2$. These solutions are going to be useful in further analysis of this equation, for example when time dependent or stochastic coefficients are taken into consideration.

Chapter 5

Concluding remarks

In this research project we first recalled some important definitions and results from Lie group theory, which were later used in the dissertation.

In Chapter two we obtained the non-topological soliton solution, cnoidal waves and the traveling wave solutions of the generalized $(2+1)$-dimensional ZK-MEW equation (2) with $n = 3$. The integration of this ZK-MEW equation was performed by the aid of Lie symmetry analysis. Profiles of these exact solutions are plotted.

In Chapter three we first obtained exact solutions of the $(2+1)$-dimensional ZK-MEW equation with power law non-linearity by using the Lie symmetry method along with the simplest equation method. Furthermore, conservation laws of this equation are derived using the new conservation theorem and the multiplier method.

Finally, in chapter four topological as well as non-topological soliton solutions are obtained for the ZK-MEW equation with power law nonlinearity by solitary wave ansatz method.

In the future, exact, numerical solutions and conservation laws of certain nonlinear partial differential equations, such as coupled nonlinear Schrodinger, Gross-Pite and coupled Ramani equations will be studied.

Bibliography

[1] A. Biswas, Topological and non-topological solitons for the generalized Zakharov-Kuznetsov modified equal width equation, Int. J. Theor. Phys. 48 (2009) 2698-2703.

[2] A. Biswas, E. Zerrad, Solitary wave solution of the Zakharov-Kuznetsov equation in plasmas with power law nonlinearity, Nonlinear Anal. Real World Appl. 11 (2010) 3272-3274.

[3] A. Biswas, 1-soliton solution of the generalized Zakharov-Kuznetsov modified equal width equation, Appl. Math. Lett. 22 (2009) 1775-1777.

[4] A. Biswas, E. Zerrad, Soliton solution of the Zakharov-Kuznetsov equation with dual-power law nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 3574-3577.

[5] R. Hirota, Exact solution of the Kortewegde Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192-1194.

[6] M.R. Miura, Backlund Transformation, Springer, Berlin, 1978.

[7] J. Weiss, M. Tabor, G. Carnevale, The Painleve property for partial differential equations, J. Math. Phys. 24 (1983) 522-526.

[8] S.K. Liu, Z.T. Fu, S.D. Liu, Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.

[9] E. Yomba, The modified extended fan sub-equation method and its application to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation, Chaos Solitons Fractals 27 (2006) 187-196.

[10] J.Q. Hu, An algebraic method exactly solving two high-dimensional nonlinear evolution equations, Chaos Solitons Fractals 23 (2005) 391-398.

[11] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, in: Applied Mathematical Sciences, vol. 81, Springer-Verlag, New York, 1989.

[12] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York, 1993.

[13] L.V. Ovsiannikov, Group Analysis of Differential Equations, English translation by Chapovsky Y and Ames W F, Academic presss, New York, 1982.

[14] N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, Wiley, Chichester, 1999.

[15] N.H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, Vol 1-3, CRC Press, Boca Raton, Florida. 1994-1996.

[16] C.M. Khalique, A. Biswas, A Lie symmetry approach to nonlinear Schrdingers equation with non-Kerr law nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 4033-4040.

[17] F.M. Mahomed, Symmetry group classification of ordinary differential equations: survey of some results, Math. Methods App. Sci. 30 (2007) 1995-2012.

[18] D.J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Phil. Mag. 39 (1895) 422-443.

[19] P.G Drazin, R.S Johnson, Solitions, An introduction. Cambridge University, New York, 1989.

[20] L. Debnath, Nonlinear Partial Differential Equations For Scientists and Engineers. Birkhauser, Berlin. 48 1998

[21] V.E. Zakharov, E.A. Kuznetsov, On three-dimensional Solitons, Sov. Phys. 39 (1974) 285-288.

[22] A.M.Wazwaz, A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompact solutions, Appl. Math. Comput. 165 (2005) 485-501.

[23] A.M. Wazwaz, The extended tanh method for the Zakharov-Kuznetsov (ZK) equation, the modified ZK equation, and its generalized forms, Commun. Nonlinear Sci. Numer. Simul. 13 (2008) 1039-1047.

[24] S. Monro, E.J. Parkes, The derivation of a modified Zakharov-Kuznetsov equation and the stability of its solutions, J. Plasma Phys. 62 (1999) 305-317.

[25] S. Monro, E.J. Parkes, Stability of solitary-wave solutions to a modified Zakharov-Kuznetsov equation, J. Plasma Phys. 64 (2000) 411-426.

[26] A.M.Wazwaz, New compact and noncompact solutions for two variants of a modified Camassa-Holm equation, Appl. Math. Comput. 163 (2005) 1165-1179.

[27] M. Inc, Exact solutions with solitary patterns for the Zakharov-Kuznetsov equations with fully nonlinear dispersion, Chaos Solitons Fractals 33 (2007) 1783-1790.

[28] B.K. Shivamoggi, The Painleve analysis of the Zakharov-Kuznetsov equation, Phys. Scr. 42 (1990) 641-642.

[29] S.I. Zaki, Solitary wave interactions for the modified equal width wave equation, Comput. Phys. Commun. 126 (2000) 219-231.

[30] A.M. Wazwaz, The tanh and the sine-cosine methods for a reliable treatment of the modified equal width equation and its variants, Commun. Nonlinear Sci. Numer. Simul. 11 (2006) 148-160.

[31] A. Esen, A lumped Galerkin method for the numerical solution of the modified equal-width equation using quadratic B-splines, Int. J. Comput. Math. 83 (2006) 449-459.

[32] A. Esen, S. Kutluay, Solitary wave solutions of the modified equal width wave equation, Commun. Nonlinear Sci. Numer. Simul. 13 (2008) 1538-1546.

[33] A.M. Wazwaz, Exact solutions for the ZK-MEW equation by using the tanh and sine-cosine methods, Int. J. Comput. Math. 82 (2005) 699-708.

[34] M. A. Noor, S. T. Mohyud-Din, A. Waheed, E. A. Al-Said, Exp-function method for traveling wave solutions of nonlinear evolution equations, Appl. Math. Comput. 216 (2010) 477-483.

[35] C.M. Khalique and K.R. Adem, Exact solutions of the (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation using Lie group analysis, Math. Comput. Modelling 54 (2011) 184-189.

[36] N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos Solitons Fractals 24 (2005) 1217-1231.

[37] N.A. Kudryashov, Exact solitary waves of the Fisher equation, Phys. Lett. A 342 (2005) 99-106.

[38] N.K Vitanov, Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2050-2060.

[39] N.K Vitanov, Application of the method of simplest equation for obtaining exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2836-2845.

[40] N.H. Ibragimov, A new conservation theorem, J. Math. Anal. Appl. 333 (2007) 311-328.

[41] R.W. Atherton, G.M. Homsy, On the existence and formulation of variational principles for nonlinear differential equations, Studies Appl. Math. 54 (1975) 31-60.

[42] W. Hereman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quant. Chem. 106 (2006) 278-299.

[43] S. C. Anco, G. W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications, European J. Appl. Math. 13 (2002) 545-566.

[44] M. Anthonyrajah, D.P. Mason, Conservation laws and invariant solutions in the Fanno model for turbulent compressible flow, Math. Comput. Appl. 15 (2010) 529-542.

[45] K.R. Adem, C.M. Khalique, Exact solutions and conservation laws of Zakharov-Kuznetsov modified equal width equation with power law nonlinearity, Nonlinear Anal. Real World Appl. 13 (2012) 1692-1702.

[46] H. Stephani, Differential Equations: their solutions using symmetries, Cambridge University Press, Cambridge, 1989.

[47] .J. Cantwell , Introduction to Symmetry Analysis, Cambridge University Press, Cambridge, 2002.

[48] M. Abramowitz, I.A. Stegun, Handbook of Mathematical functions with formulas, graphs, and Mathematical Tables, Dover Publications, Inc, New York, 1972.