LIE GROUP ANALYSIS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN MULTIPLE SPACE DIMENSIONS

by

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Contents

Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MISS KHADIJO RASHID ADEM

Date: ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed:..

PROF C.M. KHALIQUE

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

K. R. Adem, C. M. Khalique, Group classification, exact solutions and conservation laws of the Krichever-Novikov equation. Submitted for publication to Mathematical Methods in the Applied Sciences.

Chapter 3

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Chapter 4

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(ii) K. R. Adem, C. M. Khalique, Conservation Laws and Traveling Wave Solutions of a Generalized Nonlinear ZK-BBM Equation. Abstract and Applied Analysis, Volume 2014 (2014), Article ID 139513, 5 pages.

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Chapter 8

K. R. Adem, C. M. Khalique, A study of the Camassa-Holm-KP equation with power law nonlinearity. To be submitted for publication.

Dedication

To my parents, brothers and sisters.

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Praise, Glory, and thanks be to Allah, the most Gracious, the most Merciful. He who enabled me to successfully complete this research project, and peace be upon his Prophet Mohammad (Sallallah-Alihe-Wasallam).

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Abstract

In this thesis we study the applications of Lie group analysis to certain nonlinear partial differential equations in multiple space dimensions of mathematical physics. Exact solutions and conservation laws are obtained for such equations. The equations which are considered in this thesis are the Krichever-Novikov equation, the (2+1)-dimensional nonlinear Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony equation, the generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation, the generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation, the coupled Drinfeld-Sokolov-Satsuma-Hirota system, the (2+1)-dimensional nonlinear Kadomtsev-Petviashvili-Joseph-Egri equation with power law nonlinearity and the Camassa-Holm Kadomtsev-Petviashvili equation with power law nonlinearity.

Lie group classification is performed on the Krichever-Novikov equation. We show that the equation admits a six-dimensional equivalence Lie algebra. It is also shown that the principal Lie algebra is two-dimensional. Several possible extensions of the principal Lie algebra are computed and their associated symmetry reductions and exact solutions are obtained using the simplest equation method. Also, conservation laws are derived using the new conservation theorem due to Ibragimov.

The (2+1)-dimensional nonlinear Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony equation is studied using Lie group analysis and simplest equation method. Also conservation laws for this equation are constructed by using the multiplier method.

The generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation is analysed using Lie group analysis in conjunction with the simplest equation method and the (G'/G)-expansion method to obtain exact solutions. Conserved quantities of the underlying equation are constructed by using two distinct approches, viz., the new conservation theorem due to Ibragimov and

the multiplier method.

A generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation is investigated. The symmetries and adjoint representations for this equation are given and an optimal system of one-dimensional subalgebras is derived. The similarity reductions and exact solutions with the aid of (G'/G)-expansion method are obtained based on the optimal systems of one-dimensional subalgebras and conservation laws are constructed by using the multiplier method.

The coupled Drinfeld-Sokolov-Satsuma-Hirota system is analyzed using Lie symmetry analysis. The similarity reductions and exact solutions with the aid of simplest equation method and Jacobi elliptic function method are obtained for the coupled Drinfeld-Sokolov-Satsuma-Hirota system. In addition to this, the conservation laws are derived using the new conservation theorem due to Ibragimov and the multiplier method.

Lie group analysis and the Exp-function method are used to carry out the integration of the two-dimensional nonlinear Kadomtsov-Petviashivilli-Joseph-Egri equation with power law nonlinearity. Conservation laws are constructed by using the multiplier method.

Finally, we investigate the Camassa-Holm Kadomtsev-Petviashvili equation with power law nonlinearity. Lie group analysis, simplest equation method and the Exp-function method are used to carry out the integration of the underlying equation. Also conservation laws are constructed by using the multiplier method.

List of Acronyms

NLEE:	Nonlinear evolution equation
PDE:	Partial differential equation
ODE:	Ordinary differential equation
KdV:	Korteweg-de Vries
KP:	Kadomtsev-Petviashvili
ZK:	Zakharov-Kuznetsov
BBM:	Benjamin-Bona-Mahony
JE:	Joseph-Egri
CH:	Camassa-Holm
MEW:	Modified equal width
KP-BBM:	Kadomtsev-Petviashvili-Benjamin-Bona-Mahony
ZK-BBM:	Zakharov-Kuznetsov-Benjamin-Bona-Mahony
KP-MEW:	Kadomtsev-Petviashvil-modified equal width
DSSH:	Drinfeld-Sokolov-Satsuma-Hirota
KP-JE:	Kadomtsev-Petviashvili-Joseph-Egri
CH-KP:	Camassa-Holm Kadomtsev-Petviashvili

Introduction

The study of nonlinear evolution equations (NLEEs) has made a lot of progress in the past few decades. There has also been a growing interest in the study of NLEEs in multi-dimensions, especially for (1+2) and (1+3) dimensions. Several kinds of solutions have been obtained for NLEEs in multi-dimensions. Some of them are topological solitons, soliton manifolds, dromions and Q-balls. These equations have various applications in various areas such as the physical sciences, applied mathematics and engineering sciences. One of the most challenging aspects is the issue of integrability in solving NLEEs. During the past few decades various integration techniques have been developed by the researchers to solve these NLEEs. Some of the well-known techniques used in the literature are the inverse scattering transform method [1], the homogeneous balance method [2], the Bäcklund transformation [3], the Weierstrass elliptic function expansion method [4], the Darboux transformation [5], the ansatz method [6,7], Hirota's bilinear method [8], the (G'/G)-expansion method [9], the Jacobi elliptic function expansion method [10], the variable separation approach [11], the sine-cosine method [12], the F-expansion method [13], the Exp-function method [14], the multiple exp-function method [15] and Lie group analysis [16–19].

In the nineteenth century a great advance arose when the Norwegian mathematician Marius Sophus Lie (1842-1899) began to investigate the continuous groups of transformations leaving differential equations invariant, creating what is now called the symmetry analysis of differential equations. This is in fact a subject which permeates many branches of modern mathematics and mathematical physics. Lie symmetry analysis of differential equations provides a powerful and fundamental framework to the exploitation of systematic procedures leading to the integration by quadrature (or at least to lowering the order) of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems, to the derivation of conservation laws, to the construction of links between different differential equations that turn out to be equivalent. During the last few decades, there has been a revival of interest in Lie's theory and significant progress has been made from either a theoretical or an applied point of view, as can be seen by the number of research papers [20–30], books [16–19] and new symbolic softwares [31–37] devoted to this particular field.

Group classification of differential equations is one of the corner stones of group analysis. Such classification specifies the origin of possible applications of powerful group-theoretical tools such as constructing of exact solutions, group generation of solution families starting with known ones. One of the most impressive results in group classification belongs to Sophus Lie who was the first to intiate the problem of group classification [38]. He presented the group classification of linear second-order partial differential equations with two independent variables [19, 38]. The problem of group classification of a differential equation involving arbitrary functions is a systematic procedure that enables the specification of possible forms of arbitrary elements which appear in the equation [19, 22–25, 38–43].

Conservation laws play a very important role in the solution process of differential equations. Finding the conservation laws of a system of differential equations is often the first stage towards finding the solution. In fact, the existence of a large quantity of conservation laws of a system of partial differential equations is a possible indicator of its integrability [17]. Conservation laws aid in the numerical integration of partial differential equations [44], theory of non-classical transformations [45, 46], normal forms and asymptotic integrability [47]. In recent years, conservation laws have been used to construct exact solutions of certain partial differential equations [48, 49].

In this thesis certain nonlinear evolution equations in mulitple space dimensions will be studied. These are the Krichever-Novikov equation, the (2+1)-dimensional nonlinear Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony equation, the generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation, the generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation, the coupled Drinfeld-Sokolov-Satsuma-Hirota system, the (2+1)-dimensional nonlinear Kadomtsev-Petviashvili-Joseph-Egri equation with power law nonlinearity and the Camassa-Holm Kadomtsev-Petviashvili equation with power law nonlinearity.

The first equation that is analysed from the point of view of Lie group classification in this thesis is Krichever-Novikov equation

$$u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{H(u)}{u_x} = 0, \tag{1}$$

where $H(u) = u^3 + c_1 u + c_0$, $c_1, c_0 \in \mathbb{R}$, has first appeared in [50] in connection with the study of finite-gap solutions of the Kadomtsev-Petviashvili equation which have plenty of physical applications from plasma physics to fluid dynamics.

One of the well-known generalisations of the Korteweg-de Vries (KdV) equations is the Kadomtsov-Petviashivilli (KP) equation [51] and the Zakharov-Kuznetsov (ZK) equation [52]. The KP equation given by

$$(u_t + \alpha u u_x + u_{xxx})_x + u_{yy} = 0,$$

is a model for shallow long waves in the x-direction with some mild dispersion in the y-direction. The Benjamin-Bona-Mahony (BBM) equation [53]

$$u_t + u_x - \alpha(u^2)_x - \beta u_{xxt} = 0,$$

represents the motion of nonlinear dispersive waves, which are often encountered in a number of important physical phenomena such as shallow water and ion acoustic plasmas. The second equation that is studied in this thesis is the modified form of BBM equation formulated in the KP sense, namely the nonlinear $(2+1)$ dimensional Kadomtsev-Petviashivilli-Benjamin-Bona-Mahony equation [54]

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x + \gamma u_{yy} = 0, \tag{2}$$

where α, β and γ are real valued constants and $u(x, y, t)$ is the wave profile.

The ZK equation is given by

$$u_t + auu_x + (u_{xx} + u_{yy} + u_{zz})_x = 0,$$

governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The third equation that is studied in this thesis is the BBM equation and its modified form formulated in the ZK sense, namely generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation [55]

$$u_t + u_x + a(u^n)_x + b(u_{xt} + u_{yy})_x = 0, \tag{3}$$

where a, b and $n > 1$ are real valued constants and $u(x, y, t)$ is the wave profile.

The modified equal width (MEW) equation [56, 57]

$$u_t + 3u^2u_x - \alpha u_{xxt} = 0,$$

is a nonlinear wave equation with cubic nonlinearity with a pulse-like solitary wave solution. We next consider the fourth equation that is studied in this thesis, the MEW equation formulated in the KP sense, namely the generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation [58]

$$(u_t + \alpha(u^n)_x + \beta u_{xxt})_x + \gamma u_{yy} = 0, \tag{4}$$

where α, β, γ and $n > 1$ are real valued constants and $u(x, y, t)$ denotes the wave profile.

The Drinfeld-Sokolov-Satsuma-Hirota system [59] will be the fifth equation that will be studied in this thesis

$$\begin{aligned} u_t - 6uu_x + u_{xxx} - 6v_x &= 0, \\ v_t - 2v_{xxx} + 6uv_x &= 0. \end{aligned} \tag{5}$$

Here $u(x, t)$ and $v(x, t)$ denote the wave profile. This system was introduced independently by Drinfeld and Sokolov [60], and by Satsuma and Hirota [61]. The coupled Drinfeld-Sokolov-Satsuma-Hirota system [60] was given as one of numerous examples of nonlinear equations possessing Lax pairs of a special form. The system (5) was found as a special case of the four-reduction of the KP hierarchy, and its explicit one-soliton solution was constructed [61]. Gürses and Karasu [62] found a recursion operator and a bi-Hamiltonian structure.

The Joseph-Egri (JE) equation [63]

$$u_t + u_x + \alpha uu_x + u_{xtt} = 0.$$

shares many of the properties of both the KdV and BBM equations. We next consider JE equation formulated in the KP sense, namely the $(2+1)$-dimensional nonlinear Kadomtsev-Petviashvili-Joseph-Egri equation with power law nonlinearity [64]

$$(u_t + u_x + \alpha u^n u_x + u_{xtt})_x + \beta u_{yy} = 0, \tag{6}$$

where α, β are real valued constants with $n > 0$ and $u(x, y, t)$ denotes the wave profile.

The Camassa-Holm (CH) equation

$$u_t + 2\alpha u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{7}$$

was proposed by Camassa and Holm [65] as a model equation for unidirectional nonlinear dispersive waves in shallow water. For $\alpha = 0$, Camassa and Holm [65] showed that equation (7) has peaked solitary wave solutions.

We will consider the Camassa-Holm Kadomtsev-Petviashvili (CH-KP) equation with power law nonlinearity [66]

$$(u_t + 2\alpha u_x - u_{xxt} - \beta u^n u_x)_x + u_{yy} = 0. \tag{8}$$

This equation (8) is the KP equation formulated in the CH sense, where n is called the strength of the nonlinearity, and $\beta > 0$, $\beta, \alpha \in \mathbb{R}$.

The outline of this thesis is as follows.

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations and conservation laws are presented.

Chapter two deals with the group classification, exact solutions and conservation laws of the Krichever-Novikov equation (1).

Chapter three discusses the exact solutions and conservation laws of a (2+1)-dimensional nonlinear Kadomtsev-Petviashivilli-Benjamin-Bona-Mahony equation (2).

Chapters four deals with the exact explicit solutions and conservation laws of a generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation (3).

Chapter five discusses the symmetry analysis and conservation laws of a generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation (4).

Chapter six discusses the solutions and conservation laws of the coupled Drinfeld-Sokolov-Satsuma-Hirota system (5).

Chapter seven deals with explicit solutions and conservation laws of a (2+1)-

dimensional Kadomtsov-Petviashivilli-Joseph-Egri equation with power law nonlinearity (6).

Chapter eight discusses the study of the Camassa-Holm-KP equation with power law nonlinearity (8).

Finally, in Chapter nine, a summary of the results of the thesis is presented and future work is elaborated.

Bibliography is given at the end.

Chapter 1

Lie symmetry methods and conservation laws for differential equations

In this chapter we give some basic methods of Lie symmetry analysis and conservation laws of partial differential equations (PDEs).

1.1 Preliminaries

Sophus Lie (1842-1899) was one of the most important mathematicians of the nineteenth century. He realised that many of the methods for solving differential equations could be unifed using group theory and further developed a symmetry-based approach to obtaining exact solutions of differential equations. Symmetry methods have great power and generality. In fact, nearly all well-known techniques for solving differential equations are special cases of Lie's methods. Recently, many good books have appeared in the literature in this field. We mention a few here, Bluman and Kumei [67], Ovsiannikov [16], Olver [18], Stephani [68], Ibragimov [19,69,70], Cantwell [71] and Mahomed [72]. Definitions and results given in this Chapter are taken from these books.

Conservation laws for PDEs are constructed using two different approaches; the multiplier method [73] and the new conservation theorem due to Ibragimov [74]. Some of the preliminaries that will be needed later in the thesis are presented here while their details are in references [19,73,74].

1.2 Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a), \;\; \bar{u^\alpha} = \phi^\alpha(x, u, a). \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$
\begin{aligned}
\bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\
\bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b))
\end{aligned}
\tag{1.2}
$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$
\tilde{a} = \int_0^a \frac{ds}{w(s)}, \quad \text{where } w(s) = \left.\frac{\partial\, \phi(s, b)}{\partial b}\right|_{b=0}.
$$

1.3 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$
u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}
$$

where

$$
D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}
$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$
u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.
$$

Similarly

$$
u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n
$$

and $u_{(3)} = \{u^\alpha_{ijk}\}$ and likewise $u_{(4)}$ etc. Since $u^\alpha_{ij} = u^\alpha_{ji}$, $u_{(2)}$ contains only u^α_{ij} for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u^α_s are called differential variables [19].

We now consider a pth-order PDE(s), namely

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i.$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x, u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\, X)x, \quad \bar{u^\alpha} \approx (1 + a\, X)u,$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}\,. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots .$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha. \end{aligned} \tag{1.11}$$

This

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, a for small a, ie., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j).
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha \;=\; D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^\alpha \;=\; D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$
\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha), \\
&\ \ \vdots \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1,
\end{aligned}
$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}.$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.18}$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E_\alpha) = 0 \tag{1.19}$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]} E_\alpha \big|_{E_\alpha=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n - 1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \cdots, J_{n-1}(x, u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^n}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X. Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Conservation laws

1.7.1 Fundamental operators and their relationship

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, namely

$$E_\alpha(x. u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m. \tag{1.25}$$

The *Euler-Lagrangeoperator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m, \tag{1.26}$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \ \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.27}$$

where $\mathcal{A}$ is the space of *differential functions* [19]. The operator (1.27) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.28}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}
\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha \\
\zeta_{i_1\ldots i_s}^\alpha &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u_{ji_1\ldots i_s}^\alpha, \quad s > 1,
\end{aligned} \tag{1.29}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{1.30}$$

One can write the Lie-Bäcklund operator (1.28) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\partial}{\partial u_{i_1 i_2\ldots i_s}^\alpha}. \tag{1.31}$$

The *Noether operators* associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\frac{\delta}{\delta u_{ii_1 i_2\ldots i_s}^\alpha}, \quad i = 1,\ldots,n, \tag{1.32}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.26) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\ldots D_{j_s}\frac{\partial}{\partial u_{ij_1 j_2\ldots j_s}^\alpha}, \quad i = 1,\ldots,n, \quad \alpha = 1,\ldots,m, \tag{1.33}$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity [74]

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.34}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1,\ldots,n$, is a *conserved vector* of (1.25) if T^i satisfies

$$D_i T^i|_{(1.25)} = 0. \tag{1.35}$$

The equation (1.35) defines a *local conservation law* of system (1.25).

1.7.2 Multiplier Method

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that [73]

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.36}$$

holds identically. The right hand side of (1.36) is a divergence expression. The determining equations for the multiplier Λ_α is [73]

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.37}$$

1.7.3 Variational method for a system and its adjoint

The system of *adjoint equations* for the system of kth-order differential equations (1.25) is defined by

$$E^*_\alpha(x, u, v, \ldots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.38}$$

where $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables [74].

Assume that the system of equations (1.25) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{1.39}$$

Then the system of adjoint equations (1.38) admits the operator [74]

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha}, \quad \eta^\alpha_* = -[\lambda^\alpha_\beta v^\beta + v^\alpha D_i(\xi^i)], \tag{1.40}$$

where the operator (1.40) is an extension of (1.39) to the variable v^α and the λ^α_β are obtainable from

$$X(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{1.41}$$

Theorem 1 [74] Every Lie point, Lie-Bäcklund and non local symmetry (1.39) admitted by the system of equations (1.25) give rise to a conservation law for the system consisting of the equation (1.25) and the adjoint equation (1.38), where the components T^i of the conserved vector $T = (T^1, \ldots, T^n)$ are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta L}{\delta u^\alpha_{ii_1 i_2 \ldots i_s}}, \quad i = 1, \ldots, n, \tag{1.42}$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x, u, \ldots, u_{(k)}). \tag{1.43}$$

1.8 Conclusion

In this chapter we presented a brief introduction to the Lie group analysis and conservation laws of PDEs and gave some results which will be used throughout this thesis. We also gave the algorithm to determine the Lie point symmetries and conservation laws of PDEs.

Chapter 2

Group classification, exact solutions and conservation laws of the Krichever-Novikov equation

2.1 Introduction

In this chapter we carry out the Lie group classification of the Krichever-Novikov equation

$$u_t - u_{xxx} + \frac{3u_{xx}^2}{2u_x} - \frac{H(u)}{u_x} = 0, \tag{2.1}$$

which contains one arbitrary function $H(u)$ and then find symmetry reductions, exact solutions and conservation laws of (2.1).

This work has been submitted for publication. See [75].

2.2 Equivalence transformations

An equivalence transformation (see for example [19]) of (2.1) is an invertible transformation involving the variables t, x and u that map (2.1) into itself. The vector field

$$Y = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u + \mu(t,x,u,H)\partial_H \tag{2.2}$$

is the generator of the equivalence group for (2.1) provided it is admitted by the extended system

$$u_t - u_{xxx} + \frac{3u_{xx}^2}{2u_x} - \frac{H(u)}{u_x} = 0, \tag{2.3a}$$

$$H_t = 0, \qquad H_x = 0. \tag{2.3b}$$

The prolonged operator for the extended system (2.3) is given by

$$\widetilde{Y} = Y^{[3]} + \omega_1\partial_{H_t} + \omega_2\partial_{H_x} + \omega_3\partial_{H_u}, \tag{2.4}$$

where $Y^{[3]}$ is the third-prolongation of (2.2) given by

$$\begin{aligned} Y^{[3]} &= \tau\partial_t + \xi\partial_x + \eta\partial_u + \mu\partial_H + \zeta_1\partial_{u_t} + \zeta_2\partial_{u_x} \\ &\quad + \zeta_{22}\partial_{u_{xx}} + \zeta_{222}\partial_{u_{xxx}}. \end{aligned}$$

The variables ζ's and ω's are defined by

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_x D_t(\tau) - u_t D_t(\xi), \\ \zeta_2 &= D_x(\eta) - u_x D_x(\tau) - u_t D_x(\xi), \\ \zeta_{22} &= D_x(\zeta_2) - u_{xx} D_x(\tau) - u_{xt} D_x(\xi), \\ \zeta_{222} &= D_x(\zeta_{22}) - u_{xxx} D_x(\tau) - u_{xxt} D_x(\xi) \end{aligned}$$

and

$$\omega_1 = \widetilde{D}_t(\mu) - H_t\widetilde{D}_t(\tau) - H_x\widetilde{D}_t(\xi) - H_u\widetilde{D}_t(\eta),$$

$$\omega_2 = \widetilde{D}_x(\mu) - H_t\widetilde{D}_x(\tau) - H_x\widetilde{D}_x(\xi) - H_u\widetilde{D}_x(\eta),$$

$$\omega_3 = \widetilde{D}_u(\mu) - H_t\widetilde{D}_u(\tau) - H_x\widetilde{D}_u(\xi) - H_u\widetilde{D}_u(\eta),$$

respectively, where

$$D_t = \partial_t + u_t\partial_u + \cdots, \quad D_x = \partial_x + u_x\partial_u + \cdots$$

are the total derivative operators and

$$\begin{aligned}
\widetilde{D}_x &= \partial_x + H_x\partial_H + \cdots, \\
\widetilde{D}_t &= \partial_t + H_t\partial_H + \cdots, \\
\widetilde{D}_u &= \partial_u + H_u\partial_H + \cdots
\end{aligned}$$

are the total derivative operators for the extended system. The application of the operator (2.4) and the invariance conditions of system (2.3) leads to following equivalent generators

$$\begin{aligned}
Y_1 &= \frac{1}{2}u^2\frac{\partial}{\partial u} + 2uH\frac{\partial}{\partial H}, \\
Y_2 &= u\frac{\partial}{\partial u} + 2H\frac{\partial}{\partial H}, \\
Y_3 &= \frac{\partial}{\partial u}, \\
Y_4 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 4H\frac{\partial}{\partial H}, \\
Y_5 &= \frac{\partial}{\partial x}, \\
Y_6 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Thus the six-parameter equivalence group is given by

$$\begin{aligned}
Y_1 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{u} = \frac{2u}{2 - a_1u},\ \bar{H} = \frac{16H}{(2 - a_1u)^4}, \\
Y_2 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{u} = e^{a_2}u,\ \bar{H} = e^{2a_2}H, \\
Y_3 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{u} = a_3 + u,\ \bar{H} = H, \\
Y_4 &: \ \bar{t} = e^{3a_4}t,\ \bar{x} = e^{a_4}x,\ \bar{u} = u,\ \bar{H} = e^{-4a_4}H,
\end{aligned}$$

$$Y_5 \;:\; \bar{t} = t,\ \bar{x} = a_5 + x,\ \bar{u} = u,\ \bar{H} = H,$$

$$Y_6 \;:\; \bar{t} = a_6 + t,\ \bar{x} = x,\ \bar{u} = u,\ \bar{H} = H$$

and their composition gives

$$\begin{aligned}
\bar{t} &= e^{3a_4}(a_6 + t), \\
\bar{x} &= e^{a_4}(a_5 + x), \\
\bar{u} &= \frac{2e^{a_2}(a_3 + u)}{2 - a_1 e^{a_2}(a_3 + u)}, \\
\bar{H} &= \frac{16e^{2a_3 - 4a_4} H}{(2 - a_1 u)^4}.
\end{aligned} \tag{2.5}$$

2.3 Principal Lie algebra

The vector field

$$\Gamma = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial u} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.6}$$

is a Lie point symmetry of (2.1) if

$$\mathrm{pr}^{(3)}\Gamma\left[u_t - u_{xxx} + \frac{3u_{xx}^2}{2u_x} - \frac{H(u)}{u_x}\right] = 0 \tag{2.7}$$

whenever $u_t - u_{xxx} + 3u_{xx}^2/(2u_x) - H(u)/u_x = 0$. Here $\mathrm{pr}^{(3)}\Gamma$ denotes the third prolongation of Γ. Equation (2.7) yields the following overdetermined system of linear PDEs:

$$\begin{aligned}
&\tau_u = 0,\ \xi_u = 0,\ \eta_{uu} = 0,\ \eta_x = 0,\ \tau_x = 0,\ \eta_t = 0, \\
&-\xi_t + \xi_{xxx} = 0,\ -\tau_t + 3\xi_x = 0, \\
&H_u\eta + H\tau_t + H\xi_x - 2H\eta_u = 0.
\end{aligned} \tag{2.8}$$

For arbitrary H, the above system yields

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial t},$$

which gives us the principal Lie algebra.

2.4 Lie group classification

Solving the system (2.8), we obtain the following classifying relation:

$$\left(\alpha u^2 + \beta u + \gamma\right) H'(u) + (\lambda u + \delta) H(u) = 0,$$

where $\alpha, \beta, \gamma, \lambda$ and δ are constants. Using the equivalence transformations obtained in Section 2, this classifying relation is invariant under the equivalence transformations (2.5) if

$$a_1 = 0,\ \bar{\alpha} = e^{-2a_2-2a_3+4a_4}\alpha,\ \bar{\beta} = e^{-a_2-2a_3}\left(e^{4a_4}\beta - 2e^{4a_4}\alpha a_3\right),$$
$$\bar{\gamma} = e^{4a_4-2a_3}\left(\alpha a_3^2 - a_3\beta + \gamma\right),\ \bar{\lambda} = e^{-a_2-2a_3+4a_4}\lambda,\ \bar{\delta} = e^{4a_4-2a_3}\left(\delta - a_3\lambda\right).$$

This classifying relation leads to the following two cases for the function H and for each case we also provide the associated extended symmetries.

Case 1: $H(u) = (\alpha u + \beta)^n$, $n \neq 0, 4$.

$$\Gamma_3 = (2\alpha x - \alpha n x)\frac{\partial}{\partial x} + (6\alpha t - 3\alpha n t)\frac{\partial}{\partial t} + (4\alpha u + 4\beta)\frac{\partial}{\partial u}.$$

Case 1.1: $n = 0$

$$\Gamma_3 = \frac{\partial}{\partial u},\ \Gamma_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

Case 1.2: $n = 4$

$$\Gamma_3 = (\alpha^2 u^2 + 2\alpha\beta u + \beta^2)\frac{\partial}{\partial u},\ \Gamma_4 = -\alpha x\frac{\partial}{\partial x} - 3\alpha t\frac{\partial}{\partial t} + (2\alpha u + 2\beta)\frac{\partial}{\partial u}.$$

Case 2: $H(u) = e^{nu}$, $n \neq 0$.

$$\Gamma_3 = nx\frac{\partial}{\partial x} + 3nt\frac{\partial}{\partial t} - 4\frac{\partial}{\partial u}.$$

Case 2.1: $n = 0$

$$\Gamma_3 = \frac{\partial}{\partial u}, \ \Gamma_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

2.5 Travelling wave solutions of (2.1) for $H(u) = (\alpha u + \beta)^4$

In this case the equation (2.1) becomes

$$u_t - u_{xxx} + \frac{3u_{xx}^2}{2u_x} - \frac{(\alpha u + \beta)^4}{u_x} = 0. \tag{2.9}$$

Consider the linear combination $\nu\Gamma_1 + \Gamma_2$ of the two translation symmetries $\partial/\partial x$ and $\partial/\partial t$. The associated Lagrange's system for $\nu\Gamma_1 + \Gamma_2$ is

$$\frac{dx}{\nu} = \frac{dt}{1} = \frac{du}{0},$$

which gives the two invariants $z = x - \nu t$ and $F = u$. Thus, the group-invariant solution of the equation (2.9) is given by $u = F(z)$, where $F(z)$ satisfies the nonlinear third-order ODE

$$3F''^2 - 2(\beta + \alpha F)^4 - 2\nu F'^2 - 2F'F''' = 0. \tag{2.10}$$

We now use the simplest equation method, which was introduced by Kudryashov [76, 77] and modified by Vitanov [78], to solve the third-order ordinary differential equation (ODE) (2.10). The simplest equations that we use are the Bernoulli and Ricatti equations. Their solutions can be written in terms of elementary functions.

Let us consider the solution of (2.10) in the form

$$F(z) = \sum_{i=0}^{M} A_i (H(r))^i, \tag{2.11}$$

where $H(r)$ satisfies the Bernoulli or Ricatti equation, M is a positive integer that can be determined by balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

We note that the Bernoulli and Ricatti equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

We consider the Bernoulli equation

$$H'(r) = aH(r) + bH^2(r), \tag{2.12}$$

whose solution is

$$H(r) = a\left\{\frac{\cosh[a(r+C)] + \sinh[a(r+C)]}{1 - b\cosh[a(r+C)] - b\sinh[a(r+C)]}\right\}.$$

For the Riccati equation

$$H'(r) = aH^2(r) + bH(r) + c \tag{2.13}$$

we shall use the solutions

$$H(r) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(r+C)\right]$$

and

$$H(r) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta r\right) + \frac{\operatorname{sech}\left(\frac{\theta r}{2}\right)}{C\cosh\left(\frac{\theta r}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta r}{2}\right)},$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

Solutions of (2.10) using Bernoulli equation as the simplest equation

In this case the balancing procedure yields $M = 2$, so (2.11) takes the form

$$F(z) = A_0 + A_1 H + A_2 H^2. \tag{2.14}$$

Inserting (2.14) into (2.10) and using the Bernoulli equation and then equating all coefficients of the functions H^i to zero, we obtain the following algebraic system in terms of A_0, A_1 and A_2:

$$12\, {A_2}^2 - 2\, \alpha^4 {A_2}^4 = 0,$$

$$48\,A_2{}^2 - 8\,\alpha^4 A_1 A_2{}^3 = 0,$$

$$8\,\alpha^4 A_0{}^3 A_1 + 24\,\alpha^3\beta\,A_0{}^2 A_1 + 24\,\alpha^2\beta^2 A_0 A_1 + 8\,\alpha\,\beta^3 A_1 = 0,$$

$$2\,\alpha^4 A_0{}^4 + 8\,\alpha^3\beta\,A_0{}^3 + 12\,\alpha^2\beta^2 A_0{}^2 + 8\,\alpha\,\beta^3 A_0 + 2\,\beta^4 = 0,$$

$$76\,A_2{}^2 - 8\,\alpha^4 A_0 A_2{}^3 - 12\,\alpha^4 A_1{}^2 A_2{}^2 - 8\,\alpha^3\beta\,A_2{}^3 - 8\,\nu\,A_2{}^2 = 0,$$

$$24\,\alpha^4 A_0 A_1 A_2{}^2 + 8\,\alpha^4 A_1{}^3 A_2 + 24\,\alpha^3\beta\,A_1 A_2{}^2 + 8\,\nu\,A_1 A_2 + 16\,\nu\,A_2{}^2$$

$$-4\,A_1 A_2 - 56\,A_2{}^2 = 0,$$

$$24\,\alpha^4 A_0{}^2 A_1 A_2 + 8\,\alpha^4 A_0 A_1{}^3 + 48\,\alpha^3\beta\,A_0 A_1 A_2 + 8\,\alpha^3\beta\,A_1{}^3 - 24\,\alpha^2\beta^2 A_1 A_2$$

$$-4\,\nu\,A_1{}^2 - 8\,\nu\,A_1 A_2 + 2\,A_1{}^2 + 4\,A_1 A_2 = 0,$$

$$8\,\alpha^4 A_0{}^3 A_2 + 12\,\alpha^4 A_0{}^2 A_1{}^2 + 24\,\alpha^3\beta\,A_0{}^2 A_2 + 24\,\alpha^3\beta\,A_0 A_1{}^2 + 24\,\alpha^2\beta^2 A_0 A_2$$

$$+12\,\alpha^2\beta^2 A_1{}^2 + 8\,\alpha\,\beta^3 A_2 + 2\,\nu\,A_1{}^2 + A_1{}^2 = 0,$$

$$12\,\alpha^4 A_0{}^2 A_2{}^2 + 24\,\alpha^4 A_0 A_1{}^2 A_2 + 2\,\alpha^4 A_1{}^4 + 24\,\alpha^3\beta\,A_0 A_2{}^2 + 24\,\alpha^3\beta\,A_1{}^2 A_2$$

$$+12\,\alpha^2\beta^2 A_2{}^2 + 2\,\nu\,A_1{}^2 + 16\nu\,A_1 A_2 + 8\,\nu\,A_2{}^2 - A_1{}^2 - 8\,A_1 A_2 - 16\,A_2{}^2 = 0.$$

Solving the system of algebraic equations with the aid of Maple, we obtain

$$a = 1, \quad b = 1, \quad \nu = \frac{1}{2}, \quad A_0 = -\frac{\beta}{\alpha}, \quad A_1 = \frac{\sqrt{6}}{\alpha^2}, \quad A_2 = \frac{\sqrt{6}}{\alpha^2}.$$

Therefore the solution of (2.9), is given by

$$\begin{aligned} u(t,x) &= A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \\ &\quad + A_2 a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \end{aligned} \tag{2.15}$$

where $z = x - \nu t$ and C is a constant of integration. The profile of the solution (2.15) is given in Figure 2.1.

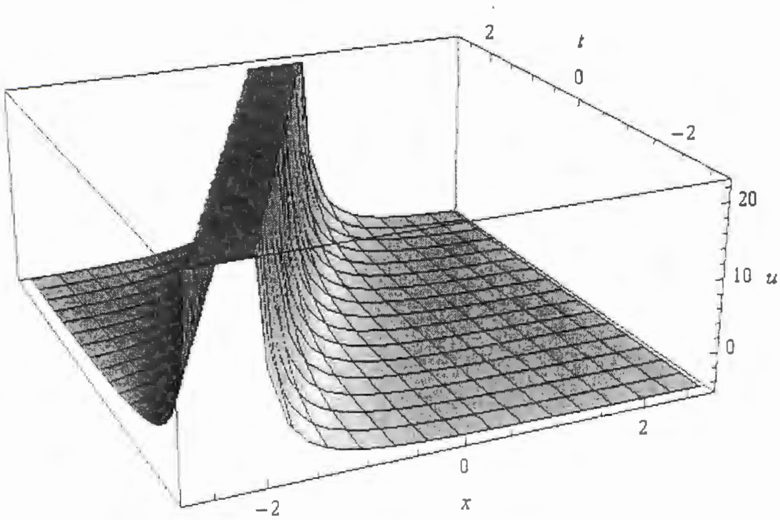

Figure 2.1: Profile of solution (2.15)

Solutions of (2.10) using Riccati equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (2.10) are of the form

$$H(z) = A_0 + A_1 H + A_2 H^2.$$

Proceeding as before and making use of the Riccati equation , we obtain

$$12\, {A_2}^2 - 2\,\alpha^4 {A_2}^4 = 0.$$

$$144\, {A_2}^2 - 8\,\alpha^4 A_1 {A_2}^3 = 0,$$

$$716\, {A_2}^2 - 12\,\alpha^4 {A_1}^2 {A_2}^2 - 8\,\alpha^4 {A_2}^3 - 8\,\alpha^3 \beta\, {A_2}^3 - 8\,\nu\, {A_2}^2 = 0,$$

$$8\,\alpha^4 {A_1}^3 A_2 + 24\,\alpha^4 A_1 {A_2}^2 + 24\,\alpha^3 \beta\, A_1 {A_2}^2 + 8\,\nu\, A_1 A_2$$

$$+48\,\nu\, {A_2}^2 - 20\, A_1 A_2 - 1848\, {A_2}^2 = 0,$$

$$2\,\alpha^4 + 8\,\alpha^3 \beta + 12\,\alpha^2 \beta^2 + 8\,\alpha\, \beta^3 + 2\,\beta^4 + 2\,\nu\, {A_1}^2 - 5\, {A_1}^2 - 12\, {A_2}^2 = 0,$$

$$8\,\alpha^4 A_1 + 24\,\alpha^3 \beta\, A_1 + 24\,\alpha^2 \beta^2 A_1 + 8\,\alpha\, \beta^3 A_1 + 12\,\nu\, {A_1}^2 + 8\,\nu\, A_1 A_2$$

$$-30\, {A_1}^2 - 20\, A_1 A_2 - 144\, {A_2}^2 = 0,$$

$$8\,\alpha^4 {A_1}^3 + 8\,\alpha^3 \beta\, {A_1}^3 + 24\,\alpha^4 A_1 A_2 + 48\,\alpha^3 \beta\, A_1 A_2 + 24\,\alpha^2 \beta^2 A_1 A_2$$

$$+12\,\nu\,{A_1}^2+88\,\nu\,A_1A_2+48\,\nu\,{A_2}^2+30\,{A_1}^2+220\,A_1A_2+1848\,{A_2}^2=0,$$

$$2\,\alpha^4{A_1}^4+24\,\alpha^4{A_1}^2A_2+24\,\alpha^3\beta\,{A_1}^2A_2+12\,\alpha^4{A_2}^2+24\,\alpha^3\beta\,{A_2}^2$$

$$+12\,\alpha^2\beta^2{A_2}^2+2\,\nu\,{A_1}^2+48\,\nu\,A_1A_2+88\,\nu\,{A_2}^2-5\,{A_1}^2-120\,A_1A_2-2560\,{A_2}^2=0,$$

$$12\,\alpha^4{A_1}^2+24\,\alpha^3\beta\,{A_1}^2+12\,\alpha^2\beta^2{A_1}^2+8\,\alpha^4A_2+24\,\alpha^3\beta\,A_2+24\,\alpha^2\beta^2A_2$$

$$+8\,\alpha\,\beta^3A_2+22\,\nu\,{A_1}^2+48\,\nu\,A_1A_2+8\,\nu\,{A_2}^2-55\,{A_1}^2-120\,A_1A_2-716\,{A_2}^2=0.$$

Solving the algebraic equations one obtains

$$a=1,\quad b=3,\quad c=1,\quad \nu=\frac{5}{2},\quad \beta=\frac{-\alpha^2+\sqrt{6}}{\alpha},\quad A_0=1,\quad A_1=\frac{3\sqrt{6}}{\alpha^2},\quad A_2=\frac{\sqrt{6}}{\alpha^2}$$

and hence the solutions of (2.9) are

$$\begin{aligned} u(t,x) &= A_0+A_1\left(-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right)\\ &\quad +A_2\left(-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right)^2 \end{aligned} \tag{2.16}$$

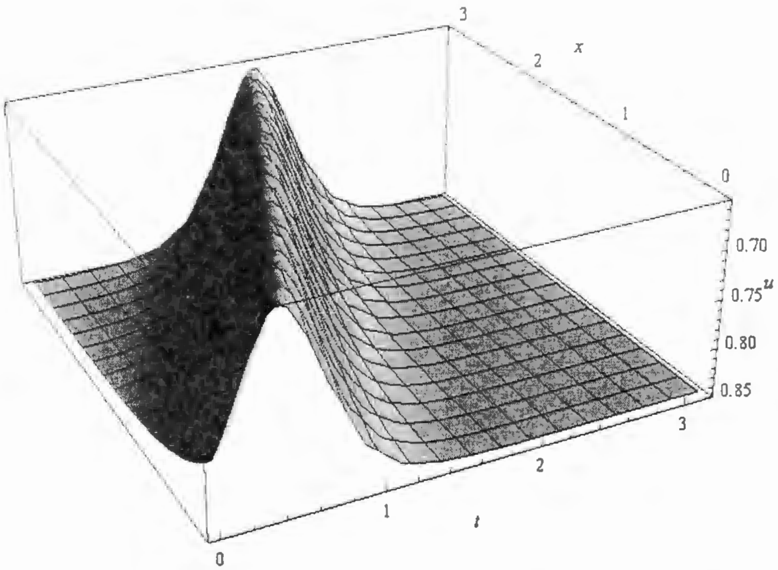

Figure 2.2: Profile of solution (2.16)

and

$$
\begin{aligned}
u(t,x) &= A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta z}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \right\} \\
&\quad + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta z}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \right\}^2, \qquad (2.17)
\end{aligned}
$$

where $z = x - \nu t$ and C is a constant of integration.

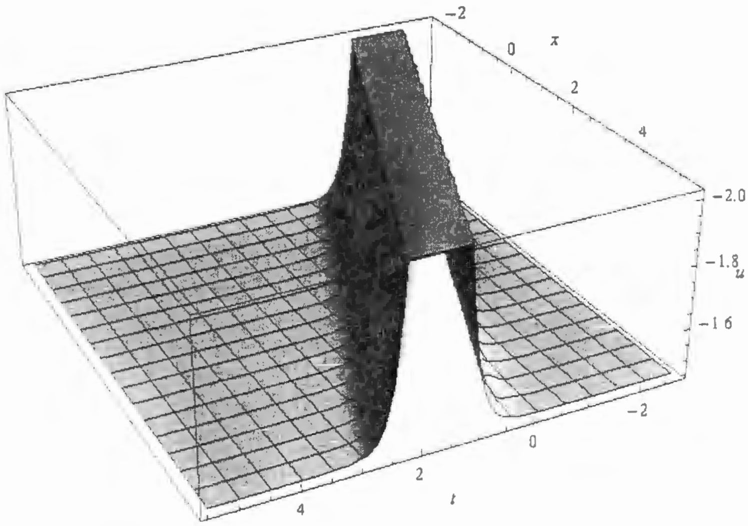

Figure 2.3: Profile of solution (2.17)

2.6 Conservation laws

In this section we construct conservation laws of (2.1) for two special values of $H(u)$. The new conservation theorem due to Ibragimov [74] will be used.

(I) $H(u) = (\alpha u + \beta)^4$

In this case the Krichever-Novikov equation together with its adjoint equation are given by

$$
E_\alpha \equiv u_t - u_{xxx} + \frac{3u_{xx}^2}{2u_x} - \frac{(\alpha u + \beta)^4}{u_x} = 0, \qquad (2.18a)
$$

$$
\begin{aligned}
E_\alpha^* \equiv\ & v_t + \frac{8\alpha v(\alpha u+\beta)^3}{u_x} + \frac{(\alpha u+\beta)^4 v_x}{u_x^2} - \frac{2(\alpha u+\beta)^4 v u_{xx}}{u_x^3} \\
& + \frac{9v_x u_{xx}^2}{2u_x^2} - \frac{3vu_{xx}^3}{u_x^3} - \frac{3u_{xx}v_{xx}}{u_x} - \frac{6v_x u_{xxx}}{u_x} + \frac{6vu_{xx}u_{xxx}}{u_x^2} \\
& - \frac{3vu_{xxxx}}{u_x} - v_{xxx} = 0.
\end{aligned} \tag{2.18b}
$$

The third-order Lagrangian for the system of equations (2.18a) and (2.18b) is given by

$$
L = v\left(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{(\alpha u+\beta)^4}{u_x}\right). \tag{2.19}
$$

We need to consider the following four cases, because the equation has four Lie point symmetries:

(i) We first consider the Lie point symmetry $\Gamma_1 = \partial_x$ of (2.18a). Corresponding to this symmetry the Lie characteristic function is $W = -u_x$. Then by using the new conservation theorem of [74], the components of the conserved vector associated with the symmetry $\Gamma_1 = \partial_x$ are given by

$$
\begin{aligned}
T_1^t &= -vu_x, \\
T_1^x &= vu_t + 3vu_{xxx} + 2v_x u_{xx} + u_x v_{xx} - \frac{1}{u_x}\Big\{2\alpha^4 vu^4 + 8\alpha^3\beta vu^3 \\
&\quad +12\alpha^2\beta^2u^2v + 8\alpha\beta^3 uv + 2\beta^4 v + 3vu_{xx}{}^2\Big\}.
\end{aligned}
$$

(ii) Likewise, the Lie point symmetry $\Gamma_2 = \partial_t$ has the Lie characteristic function $W = -u_t$. Invoking Ibragimov theorem, we obtain the conserved vector whose components are

$$
\begin{aligned}
T_2^t &= -vu_{xxx} + \frac{1}{u_x}\left\{\frac{3u_{xx}{}^2 v}{2} - \alpha^4u^4v - 4\alpha^3\beta u^3 v - 6\alpha^2\beta^2u^2v - 4\alpha\beta^3 uv - \beta^4 v\right\}, \\
T_2^x &= u_t v_{xx} - v_x u_{tx} + vu_{txx} - \frac{1}{u_x{}^2}\left\{\frac{3vu_tu_{xx}{}^2}{2} - \alpha^4u^4vu_t - 4\alpha^3\beta u^3vu_t - \beta^4 vu_t\right. \\
&\quad \left. -6\alpha^2\beta^2u^2vu_t - 4\alpha\beta^3uvu_t\right\} + \frac{1}{u_x}\left\{3u_tvu_{xxx} - 3vu_{xx}u_{tx} + 3u_tv_xu_{xx}\right\}.
\end{aligned}
$$

(iii) The Lie point symmetry $\Gamma_3 = (\alpha^2u^2 + 2\alpha\beta u + \beta^2)\partial_u$ has the Lie characteristic function $W = \alpha^2u^2 + 2\alpha\beta u + \beta^2$. Invoking Ibragimov theorem we obtain the conserved vector whose components are

$$\begin{aligned}
T_3^t &= \alpha^2u^2v + 2\alpha\beta uv + \beta^2 v,\\
T_3^x &= 2\alpha^2 u_x v_x u - 2\alpha^2 {u_x}^2 v + 4\alpha^2 uvu_{xx} - \alpha^2u^2v_{xx} - 2\alpha\beta uv_{xx} + 4\alpha\beta vu_{xx}\\
&\quad +2\alpha\beta u_x v_x - \beta^2 v_{xx} + \frac{1}{{u_x}^2}\Big\{\alpha^6u^6v + 6\alpha^5\beta u^5 v + 15\alpha^4\beta^2u^4v + \beta^6 v + 20\alpha^3\beta^3u^3v\\
&\quad +15\alpha^2\beta^4u^2v + 6\alpha\beta^5uv + 3\alpha\beta uv{u_{xx}}^2 + \frac{3\beta^2 v{u_{xx}}^2}{2} + \frac{3\alpha^2{u_{xx}}^2u^2v}{2}\Big\}\\
&\quad -\frac{1}{u_x}\Big\{6\alpha\beta uv_xu_{xx} + 6\alpha\beta uvu_{xxx} + 3\beta^2vu_{xxx} + 3\beta^2v_xu_{xx} + 3\alpha^2u^2v_xu_{xx}\\
&\quad +3\alpha^2u^2vu_{xxx}\Big\}.
\end{aligned}$$

(iv) Finally, we consider the symmetry generator $\Gamma_4 = -\alpha x\partial_x - 3\alpha t\partial_t + (2\alpha u + 2\beta)\partial_u$, which has the Lie characteristic function $W = 2\alpha u + 2\beta + \alpha x u_x + 3\alpha t u_t$. Using Ibragimov theorem [74], the components of conserved vector are given by

$$\begin{aligned}
T_4^t &= 2\beta v + 2\alpha uv + \alpha xvu_x + 3\alpha tvu_{xxx} - \frac{1}{u_x}\Big\{\frac{9\alpha tv{u_{xx}}^2}{2} - 3\alpha^5tu^4v - 12t\alpha^4\beta u^3v\\
&\quad -18\alpha^3\beta^2tu^2v - 12\alpha^2t\beta^3uv - 3\alpha\beta^4tv\Big\},\\
T_4^x &= 3\alpha tv_xu_{tx} - 2\alpha uv_{xx} - \alpha xu_xv_{xx} - 3\alpha xvu_{xxx} - \alpha xvu_t - 3\alpha tv_{xx}u_t\\
&\quad -3\alpha tvu_{txx} - 2\beta v_{xx} + \frac{1}{{u_x}^2}\Big\{\frac{9\alpha tv{u_{xx}}^2u_t}{2} + 2\alpha^5u^5v + 12t\alpha^4\beta u^3vu_t + 10\alpha^4\beta u^4v\\
&\quad +18t\alpha^3\beta^2u^2vu_t + 20\alpha^3\beta^2u^3v + 12t\alpha^2\beta^3uvu_t + 20\alpha^2\beta^3u^2v + 3\alpha uv{u_{xx}}^2\\
&\quad +3\alpha^5tu^4vu_t + 3t\alpha\beta^4vu_t + 10\alpha\beta^4uv + 3\beta v{u_{xx}}^2 + 2\beta^5v\Big\} + \frac{1}{u_x}\Big\{2\alpha^5xu^4v\\
&\quad +8x\alpha^4\beta u^3v + 12\alpha^3\beta^2xu^2v + 8\alpha^2\beta^3xuv - 6\alpha uv_xu_{xx} + 3\alpha xv{u_{xx}}^2 - 6\alpha uvu_{xxx}\\
&\quad -9\alpha tv_xu_{xx}u_t - 9\alpha tvu_{xxx}u_t + 9\alpha tvu_{xx}u_{tx} + 2\alpha\beta^4xv - 6\beta v_xu_{xx} - 6\beta vu_{xxx}\Big\}.
\end{aligned}$$

(II) $H(u) = e^{nu}$

In this case the Krichever-Novikov equation together with its adjoint equation are given by

$$E_\alpha \equiv u_t - u_{xxx} + \frac{3u_{xx}^2}{2u_x} - \frac{e^{nu}}{u_x} = 0, \tag{2.20a}$$

$$\begin{aligned} E_\alpha^* \equiv\ & \frac{v_t}{u_x} - \frac{2e^{nu}vu_{xx}}{u_x^3} - \frac{3vu_{xx}^3}{u_x^3} + \frac{6vu_{xx}u_{xxx}}{u_x^3} + \frac{2ne^{nu}v}{u_x} - \frac{3vu_{xxx}}{u_x} \\ & + \frac{e^{nu}v_x}{u_x^2} + \frac{9v_xu_{xx}^2}{2u_x^2} - \frac{6vv_xu_{xx}}{u_x} - \frac{u_xxv_{xx}}{2u_x} - \frac{v_{xxx}}{u_x} = 0. \end{aligned} \tag{2.20b}$$

The third-order Lagrangian for the system of equations (2.20a) and (2.20b) is

$$L = v\left(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{e^{nu}}{u_x}\right). \tag{2.21}$$

We have the following three cases:

(i) We first consider the Lie point symmetry $\Gamma_1 = \partial_x$. Corresponding to this symmetry the Lie characteristic function are $W = -u_x$. Thus, by using the Ibragimov theorem [74], the components of the conserved vector associated with the symmetry $\Gamma_1 = \partial_x$ are given by

$$\begin{aligned} T_1^t &= -vu_x, \\ T_1^x &= 3u_{xxx}v + u_tv + 2u_{xx}v_x + u_xv_{xx} - \frac{1}{u_x}\left\{2ve^{nu} + 3vu_{xx}^2\right\}. \end{aligned}$$

(ii) Likewise, the Lie point symmetry $\Gamma_2 = \partial_t$ has the Lie characteristic function $W = -u_t$. Invoking Ibragimov theorem we obtain the conserved vector whose components are

$$\begin{aligned} T_2^t &= -vu_{xxx} + \frac{1}{u_x}\left\{\frac{3vu_{xx}^2}{2} - e^{nu}v\right\}, \\ T_2^x &= u_tv_{xx} - v_xu_{tx} + vu_{txx} - \frac{1}{u_x^2}\left\{e^{nu}vu_t + \frac{3u_tu_{xx}^2v}{2}\right\} \\ &\quad -\frac{1}{u_x}\left\{3vu_{xx}u_{tx} - 3vu_tu_{xxx} - 3u_tu_{xx}v_x\right\}. \end{aligned}$$

(iii) The symmetry generator $\Gamma_3 = nx\partial_x + 3nt\partial_t - 4\partial_u$ has the Lie characteristic function $W = -4 - 3ntu_t - nxu_x$ and so the use of Ibragimov theorem gives the conserved vector whose components are

$$\begin{aligned}
T_3^t &= -4v - nxvu_x - 3ntvu_{xxx} + \frac{1}{u_x}\left\{\frac{9ntvu_{xx}^2}{2} - 3ntve^{nu}\right\}, \\
T_3^x &= 4v_{xx} + 3ntu_tv_{xx} - 3ntv_xu_{tx} + 2nxu_{xx}v_x - nu_xv_x + nxu_xv_{xx} - nvu_{xx} \\
&\quad +3nxvu_{xxx} + nxvu_t + 3ntvu_{txx} - \frac{1}{u_x^2}\left\{\frac{9ntvu_tu_{xx}^2}{2} + 3ntve^{nu}u_t + 4e^{nu}v\right. \\
&\quad \left. +6vu_{xx}^2\right\} - \frac{1}{u_x}\left\{3nxvu_{xx}^2 + 9ntvu_{xx}u_{tx} - 9ntvu_tu_{xxx} + 2nxe^{nu}v\right. \\
&\quad \left. -12vu_{xxx} - 9ntu_tu_{xx}v_x - 12u_{xx}v_x\right\}.
\end{aligned}$$

2.7 Conclusion

In this chapter we studied the Krichever-Novikov equation. Lie group classification was carried out on the Krichever-Novikov equation and it was shown that this equation admitted a six-dimensional equivalence generators. The principal Lie algebra consisted of two symmetries. It was also shown that the principal Lie algebra had several possible extensions. Two new cases were found and these corresponded to Cases 1-2 of Section 2.4. Exact solutions for a special and interesting case of the Krichever-Novikov equation were obtained using the two translation symmetries along with the simplest equation method. Finally, conservation laws were obtained for two cases of the Krichever-Novikov equation using the new conservation theorem due to Ibragimov.

Chapter 3

Exact solutions and conservation laws of a (2+1)-dimensional nonlinear KP-BBM equation

In this chapter, we study the (2+1)-dimensional nonlinear Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony (KP-BBM) equation, namely,

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x + \gamma u_{yy} = 0. \tag{3.1}$$

Here, in (3.1) α, β and γ are real valued constants.

The solutions of (3.1) have been studied in various aspects. Wazwaz [54, 79] used the sine-cosine method, the tanh method and the extended tanh method for finding solitonary solutions of this equation. Abdou [80] used the extended mapping method with symbolic computation to obtain some periodic solutions, solitary wave solution and triangular wave solution.

Here we use Lie group analysis in conjunction with the simplest equation method to obtain some exact solutions of (3.1). In addition to exact solutions, conservation laws will be derived for (3.1) using the multiplier method [73].

This work has been published in [28].

3.1 Symmetry analysis

In this section we first calculate the Lie point symmetries of (3.1) and later use them to construct exact solutions.

3.1.1 Lie point symmetries of (3.1)

The symmetry group of KP-BBM equation (3.1) will be generated by the vector field of the form

$$V = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u},$$

where ξ^i, $i = 1, 2, 3$ and η depend on x, y, t and u, is a Lie point symmetry of (3.1) if

$$\mathrm{pr}^{(4)}V[(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x + \gamma u_{yy}] = 0 \tag{3.2}$$

whenever $(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x + \gamma u_{yy} = 0$. Here $\mathrm{pr}^{(4)}V$ denotes the fourth prolongation of V. Expanding (3.2) and splitting on the derivatives of u, we obtain an overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^1_y = 0,\ \xi^2_t = 0,\ \xi^3_y = 0,\ \xi^2_x = 0,\ \xi^2_u = 0,\ \xi^1_t = 0,\ \xi^1_u = 0,\ \xi^3_x = 0,\ \xi^3_u = 0,\\
&\eta_{tu} = 0,\ \eta_{uu}, = 0, -\beta\,\eta_{xxxu} + \eta_{xu} = 0,\ \xi^2_{yy} - 2\,\eta_{yu} = 0,\ \eta_{xu} - \xi^1_{xx} = 0,\\
&2\,\xi^1_x - 2\,\xi^2_y - \eta_u = 0,\ 2\,\xi^2_y - \xi^3_t - 3\,\xi^1_x = 0,\ -2\,\alpha\, u\eta_{xx} - \beta\,\eta_{xxxt} + \gamma\eta_{yy} + \eta_{xt} + \eta_{xx} = 0,\\
&-4\,\alpha\, u\eta_{xu} + 2\,\alpha\, u\xi^1_{xx} - 4\,\alpha\,\eta_x + 2\,\eta_{xu} - \xi^1_{xx} = 0,\ -3\,\beta\,\eta_{xxu} + \beta\,\xi^1_{xxx} - \xi^1_x + 2\,\xi^2_y - \xi^3_t = 0,\\
&2\,\alpha\, u\xi^1_x - 2\,\alpha\, u\xi^2_y - \alpha\,\eta_1 - \xi^1_x + \xi^2_y = 0.
\end{aligned}$$

Solving resultant system of linear overdetermined partial differential equations one obtains the following four Lie point symmetries:

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial x},\\
V_2 &= \frac{\partial}{\partial y},\\
V_3 &= \frac{\partial}{\partial t},\\
\text{and} \quad V_4 &= -\alpha y\frac{\partial}{\partial y} - 2\alpha t\frac{\partial}{\partial t} + (2\alpha u - 1)\frac{\partial}{\partial u}.
\end{aligned}$$

3.1.2 Exact solutions

We utilize the symmetry $V = V_1 + V_2 + V_3$ and reduce the KP-BBM equation (3.1) to a PDE in two independent variables. It can be seen that the symmetry V yields the following three invariants:

$$f = t - y, \quad g = x - y, \quad \theta = u. \tag{3.3}$$

Now treating θ as the new dependent variable, f and g as new independent variables, the KP-BBM equation (3.1) transforms to

$$\theta_{fg} + \theta_{gg} - 2\alpha\theta_g^2 - 2\alpha\theta\theta_{gg} - \beta\theta_{fggg} + \gamma\theta_{ff} + 2\gamma\theta_{fg} + \gamma\theta_{gg} = 0, \tag{3.4}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (3.4) and transform it to an ordinary differential equation (ODE). The equation (3.4) has the following two translation symmetries:

$$\Gamma_1 = \frac{\partial}{\partial g}, \quad \Gamma_2 = \frac{\partial}{\partial f}.$$

The combination $\rho\Gamma_1 + \Gamma_2$, where $\rho \neq 0$ is a constant, of the two symmetries Γ_1 and Γ_2 yields the following two invariants:

$$r = \frac{\rho f - g}{\rho}, \quad \psi = \theta,$$

which gives rise to a group invariant solution $\psi = \psi(r)$ and consequently using these invariants, (3.4) is transformed into the fourth-order nonlinear ODE

$$-2\alpha\rho\psi'^2 - 2\alpha\rho\psi\psi'' + (\rho + \gamma\rho - \rho^2 - 2\gamma\rho^2 + \gamma\rho^3)\psi'' + \beta\psi'''' = 0. \tag{3.5}$$

Integrating the above equation twice and taking the constants of integration to be zero we obtain a second-order ODE

$$\rho(-1 - \gamma + \gamma\rho)(1 - \rho)\psi - \alpha\rho\psi^2 + \beta\psi'' = 0. \tag{3.6}$$

Multiplying equation (3.6) by ψ', integrating once and taking the constant of integration to be zero, we obtain the first-order ODE

$$\frac{1}{2}\rho(-1 - \gamma + \gamma\rho)(1 - \rho)\psi^2 - \frac{1}{3}\alpha\rho\psi^3 + \frac{1}{2}\beta\psi'^2 = 0. \tag{3.7}$$

One can integrate the above equation by separating the variables. After integrating and reverting back to the original variables, we obtain the following group-invariant solutions of the KP-BBM equation (3.1):

$$u(x, y. t) = A_1 \mathrm{sech}^2 \left[\frac{1}{2}\left(\pm A_2 r - A_3 C\right)\right],$$

where

$$\begin{aligned} A_1 &= \frac{1}{2\alpha^3}(-1 + \rho)(-1 - \gamma + \gamma\rho), \\ A_2 &= \frac{\sqrt{\rho(1 - \rho)(-1 - \gamma + \gamma\rho)}}{\sqrt{\beta}}, \\ A_3 &= \sqrt{3(-1 + \rho)(-1 - \gamma + \gamma\rho)}, \\ r &= \frac{t\rho - x + (1 - \rho)y}{\rho}. \end{aligned}$$

To find another group-invariant solution of the KP-BBM equation (3.1), we now make use of the symmetry $\Gamma = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$ ($a_i, i = 1, \cdots, 4$ are constants). The symmetry Γ yields the following three invariants:

$$f = \frac{2a_3\alpha t - a_4}{2a_3\alpha(a_2 - a_3\alpha y)^2}, \quad g = x + \frac{a_1 \ln(a_2 - a_3\alpha y)}{a_3\alpha},$$

$$\theta = a_2^2 u - 2a_2 a_3 \alpha y u + a_3^2 \alpha^2 y^2 u + a_2 a_3 y - \frac{1}{2} a_3^2 \alpha y^2. \tag{3.8}$$

Treating θ as the new dependent variable, f and g as new independent variables, the KP-BBM equation (3.1) transforms to a nonlinear fourth-order PDE in two independent variables, namely

$$\begin{aligned} & a_2^2 \theta_{gg} - 3a_2^2 a_3^2 \alpha\gamma + 14 a_3^2 \alpha^2 \gamma f \theta_f + a_1^2 \gamma \theta_{gg} - 4a_1 a_3 \alpha \gamma f \theta_{fg} - 2\alpha\theta\theta_{gg} \\ & \theta_{fg} - 2\alpha\theta_g^2 + 6a_3^2 \alpha^2 \gamma \theta + 4a_3^2 \alpha^2 \gamma f^2 \theta_{ff} - 5a_1 a_3 \alpha \gamma \theta_g - \beta \theta_{fggg} = 0. \end{aligned} \tag{3.9}$$

The equation (3.9) has a single Lie point symmetry, viz.,

$$\Gamma_1 = \frac{\partial}{\partial g}$$

and this symmetry yields the two invariants

$$r = f, \quad \psi = \theta,$$

which gives group-invariant solution $\psi = \psi(r)$ and consequently equation (3.9) is then transformed to

$$4\alpha r^2 \psi'' + 14\alpha r \psi' + 6\alpha\psi - 3a_2^2 = 0, \tag{3.10}$$

which is a second-order linear ODE. Now solving this equation and reverting back to the original variables, we obtain the following solution of the KP-BBM equation (3.1):

$$u(x, y, t) = A_1 \left(A_2 + C_1 r^{-3/2} + C_2 r^{-1} + A_3 \right),$$

where C_1 and C_2 are constants of integration and

$$\begin{aligned} A_1 &= \frac{1}{a_2^2 - 2a_2 a_3 \alpha y + a_3^2 \alpha^2 y^2}, \\ A_2 &= \frac{1}{2} a_3^2 \alpha y^2 - a_2 a_3 y, \\ A_3 &= \frac{a_2^2}{2\alpha}, \\ r &= \frac{2a_3 \alpha t - a_4}{2a_3 \alpha (a_2 - a_3 \alpha y)^2}. \end{aligned}$$

3.2 Simplest equation method

In this section we use the simplest equation method, which was used in Chapter 2, to solve the fourth-order ODE (3.5) and as a result we obtain exact solutions of our KP-BBM equation (3.1).

3.2.1 Solutions of (3.5) using the equation of Bernoulli as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (3.5) are of the form

$$\psi(r) = A_0 + A_1 H + A_2 H^2. \tag{3.11}$$

Substituting (3.11) into (3.5) and making use of the Bernoulli equation and then equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 and A_2. These algebraic equations are

$$a^4 A_1\beta - 2\alpha a^2 A_0 A_1 \rho + a^2 A_1 \gamma\rho^3 - 2a^2 A_1\gamma\rho^2 + a^2 A_1\gamma\rho - a^2 A_1\rho^2 + a^2 A_1 \rho = 0,$$

$$16a^4 A_2\beta + 15a^3 A_1 b\beta - 4\alpha a^2 A_1^2\rho - 8\alpha a^2 A_0 A_2\rho + 4a^2 A_2\gamma\rho^3 - 8a^2 A_2\gamma\rho^2$$
$$+4a^2 A_2\gamma\rho - 4a^2 A_2\rho^2 + 4a^2 A_2\rho - 6\alpha a A_0 A_1 b\rho + 3aA_1 b\gamma\rho^3 - 6aA_1 b\gamma\rho^2$$
$$+3aA_1 b\gamma\rho - 3aA_1 b\rho^2 + 3aA_1 b\rho = 0,$$

$$130a^3 A_2 b\beta - 18\alpha a^2 A_1 A_2\rho + 50a^2 A_1 b^2\beta - 10\alpha a A_1^2 b\rho - 20\alpha a A_0 A_2 b\rho + 10aA_2 b\gamma\rho^3$$
$$-20aA_2 b\gamma\rho^2 + 10aA_2 b\gamma\rho - 10aA_2 b\rho^2 + 10aA_2 b\rho - 4\alpha A_0 A_1 b^2\rho$$
$$+2A_1 b^2\gamma\rho^3 - 4A_1 b^2\gamma\rho^2 + 2A_1 b^2\gamma\rho - 2A_1 b^2\rho^2 + 2A_1 b^2\rho = 0,$$

$$60aA_1\beta b^3 - 16a^2\alpha A_2^2\rho + 330a^2 A_2\beta b^2 - 42a\alpha A_1 A_2 b\rho - 6\alpha A_1^2 b^2\rho - 12\alpha A_0 A_2 b^2\rho$$
$$+6A_2 b^2\gamma\rho^3 - 12A_2 b^2\gamma\rho^2 + 6A_2 b^2\gamma\rho - 6A_2 b^2\rho^2 + 6A_2 b^2\rho = 0,$$

$$336aA_2\beta b^3 - 36a\alpha A_2^2 b\rho + 24A_1\beta b^4 - 24\alpha A_1 A_2 b^2\rho = 0,$$

$$120A_2 b^4\beta - 20\alpha A_2^2 b^2\rho = 0.$$

Solving the above system of algebraic equations with the aid of Mathematica, we obtain the following values of A_0, A_1 and A_2:

$$A_0 = \frac{a^2\beta + \rho + \gamma\rho - \rho^2 - 2\gamma\rho^2 + \gamma\rho^3}{2\alpha\rho},$$
$$A_1 = \frac{6ab\beta}{\alpha},$$
$$A_2 = \frac{6b^2\beta}{\alpha\rho}.$$

Therefore the solution of (3.1) is given by

$$\begin{aligned} u(x,y,t) &= A_0 + A_1\left(\frac{\cosh[a(r+C)] + \sinh[a(r+C)]}{1 - b\cosh[a(r+C)] - b\sinh[a(r+C)]}\right) \\ &\quad + A_2\left(\frac{\cosh[a(r+C)] + \sinh[a(r+C)]}{1 - b\cosh[a(r+C)] - b\sinh[a(r+C)]}\right)^2 \end{aligned} \tag{3.12}$$

where $r = t\rho - x + (1-\rho)y/\rho$ and C is a constant of integration.

3.2.2 Solutions of (3.5) using Riccati equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (3.5) are of the form

$$\psi(r) = A_0 + A_1 H + A_2 H^2. \tag{3.13}$$

Substituting (3.13) into (3.5) and making use of the Riccati equation, we obtain algebraic equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions H^i to zero. The corresponding algebraic equations are

$$aA_1\beta b^3 - 2a\alpha A_0 A_1 b\rho + aA_1 b\gamma\rho^3 - 2aA_1 b\gamma\rho^2 + aA_1 b\gamma\rho - aA_1 b\rho^2 + aA_1 b\rho = 0,$$
$$8a^3A_1\beta b - 2a^2\alpha A_1^2\rho - 4a^2\alpha A_0A_1\rho - 4a^2\alpha A_0A_2\rho + 2a^2A_1\gamma\rho^3 + 2a^2A_2\gamma\rho^3$$
$$-4a^2A_1\gamma\rho^2 - 4a^2A_2\gamma\rho^2 + 2a^2A_1\gamma\rho + 2a^2A_2\gamma\rho - 2a^2A_1\rho^2 - 2a^2A_2\rho^2 + 2a^2A_1\rho$$
$$+2a^2A_2\rho + 6a^2A_1\beta b^2 + 14a^2A_2\beta b^2 + 2aA_2\beta b^3 - 2a\alpha A_1^2 b\rho - 4a\alpha A_0A_2 b\rho$$

$$+2aA_2b\gamma\rho^3 - 4aA_2b\gamma\rho^2 + 2aA_2b\gamma\rho - 2aA_2b\rho^2 + 2aA_2b\rho$$

$$+A_1\beta b^4 - 2\alpha A_0A_1b^2\rho + A_1b^2\gamma\rho^3 - 2A_1b^2\gamma\rho^2 + A_1b^2\gamma\rho - A_1b^2\rho^2 + A_1b^2\rho = 0,$$

$$16a^4A_1\beta + 16a^4A_2\beta + 12a^3A_1b\beta + 72a^3A_2b\beta - 4\alpha a^2A_1^2\rho - 8\alpha a^2A_0A_2\rho$$

$$-12\alpha a^2A_1A_2\rho + 4a^2A_2\gamma\rho^3 - 8a^2A_2\gamma\rho^2 + 4a^2A_2\gamma\rho - 4a^2A_2\rho^2 + 4a^2A_2\rho$$

$$+16a^2A_1b^2\beta + 12a^2A_2b^2\beta + 6aA_1b^3\beta + 28aA_2b^3\beta - 4\alpha aA_1^2b\rho - 4\alpha aA_0A_1b\rho$$

$$-8\alpha aA_0A_2b\rho - 6\alpha aA_1A_2b\rho + 2aA_1b\gamma\rho^3 + 4aA_2b\gamma\rho^3 - 4aA_1b\gamma\rho^2 - 8aA_2b\gamma\rho^2$$

$$+2aA_1b\gamma\rho + 4aA_2b\gamma\rho - 2aA_1b\rho^2 - 4aA_2b\rho^2 + 2aA_1b\rho + 4aA_2b\rho + 2A_2b^4\beta$$

$$-2\alpha A_1^2b^2\rho - 4\alpha A_0A_2b^2\rho + 2A_2b^2\gamma\rho^3 - 4A_2b^2\gamma\rho^2 + 2A_2b^2\gamma\rho - 2A_2b^2\rho^2 + 2A_2b^2\rho = 0,$$

$$8a^4A_1\beta + 88a^4A_2\beta + 32a^3A_1b\beta + 72a^3A_2b\beta - 12\alpha a^2A_2^2\rho - 12\alpha a^2A_1A_2\rho$$

$$+12a^2A_1b^2\beta + 144a^2A_2b^2\beta + 8aA_1b^3\beta + 12aA_2b^3\beta - 4\alpha aA_1^2b\rho - 4\alpha aA_2^2b\rho$$

$$-8\alpha aA_0A_2b\rho - 24\alpha aA_1A_2b\rho + 4aA_2b\gamma\rho^3 - 8aA_2b\gamma\rho^2 + 4aA_2b\gamma\rho - 4aA_2b\rho^2$$

$$+4aA_2b\rho + 14A_2b^4\beta - 2\alpha A_1^2b^2\rho - 4\alpha A_0A_2b^2\rho - 6\alpha A_1A_2b^2\rho + 2A_2b^2\gamma\rho^3$$

$$-4A_2b^2\gamma\rho^2 + 2A_2b^2\gamma\rho - 2A_2b^2\rho^2 + 2A_2b^2\rho = 0,$$

$$16a^4A_2\beta + 8a^3A_1b\beta + 176a^3A_2b\beta - 8\alpha a^2A_2^2\rho + 16a^2A_1b^2\beta + 72a^2A_2b^2\beta + 72aA_2b^3\beta$$

$$-24\alpha aA_2^2b\rho - 12\alpha aA_1A_2b\rho - 4\alpha A_2^2b^2\rho - 12\alpha A_1A_2b^2\rho = 0,$$

$$16a^3A_2b\beta + 88a^2A_2b^2\beta + 16aA_2b^3\beta - 8\alpha aA_2^2b\rho - 12\alpha A_2^2b^2\rho = 0.$$

Solving the algebraic equations one obtains

$$A_0 = \frac{b^2\beta + \rho + \gamma\rho - \rho^2 - 2\gamma\rho^2 + \gamma\rho^3}{2\alpha\rho},$$
$$A_1 = 0,$$
$$A_2 = \frac{2(2a^3\beta + 11a^2b\beta + 2ab^2\beta)}{\alpha\rho(2a+3b)},$$

and hence the solutions of (3.1) are

$$u(x,y,t) = A_0 + A_2\left(-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(r+C)\right]\right)^2 \tag{3.14}$$

and

$$u(x,y,t) = A_0 + A_2\left(-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta r\right) + \frac{\operatorname{sech}\left(\frac{\theta r}{2}\right)}{C\cosh\left(\frac{\theta r}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta r}{2}\right)}\right)^2 \quad (3.15)$$

where $r = t\rho - x + (1-\rho)y/\rho$ and C is a constant of integration.

3.3 Conservation laws of (3.1)

We now construct conservation laws for the (2+1) dimensional nonlinear KP-BBM equation (3.1) using the multiplier approach [73]. The zeroth-order multiplier for (3.1) is, $\Lambda(t, x, y, u)$ that is given by

$$\Lambda = C_1 x + C_1 t x - \frac{C_1 y^2}{2\gamma} + yF_2(t) + F_3(t), \quad (3.16)$$

where C_1 is a constant, and $F_2(t)$ and $F_3(t)$ are arbitrary functions of t. Corresponding to the above multiplier we have the following three conserved vectors of (3.16):

$$\begin{aligned}
T_1^t &= \frac{1}{8\gamma}\Big\{-4\gamma tu - 4\gamma u + 2\beta\gamma t u_{xx} - 2\beta\gamma t x u_{xxx} + 4\gamma t x u_x \\
&\quad + 2\beta\gamma u_{xx} - 2\beta\gamma x u_{xxx} + 4\gamma x u_x + \beta y^2 u_{xxx} - 2y^2 u_x\Big\},\\
T_1^x &= \frac{1}{8\gamma}\Big\{8\alpha y^2 u_x u - 16\alpha\gamma t x u_x u - 16\alpha\gamma x u_x u + 8\alpha\gamma t u^2 + 8\alpha\gamma u^2 \\
&\quad - 8\gamma t u - 4\gamma x u - 8\gamma u + 4\beta\gamma t u_{tx} + 4\beta\gamma u_{tx} - 6\beta\gamma t x u_{txx} \\
&\quad - 6\beta\gamma x u_{txx} + 8\gamma t x u_x + 4\gamma t x u_t + 4\gamma x u_t + 3\beta y^2 u_{txx} \\
&\quad - 2y^2 u_t - 4\beta\gamma u_x + 2\beta\gamma x u_{xx} + 8\gamma x u_x - 4y^2 u_x\Big\},\\
T_1^y &= \frac{1}{2}\Big\{2yu + 2\gamma t x u_y + 2\gamma x u_y - y^2 u_y\Big\};
\end{aligned}$$

$$T_2^t = \frac{1}{2}\Big\{2yF_2(t)u_x - \beta y F_2(t) u_{xxx}\Big\},$$

$$
\begin{aligned}
T_2^x &= \frac{1}{4}\Big\{ -8\alpha y F_2(t) u_x u - 2y F_2' u - 3\beta y F_2(t) u_{txx} + 4y F_2(t) u_x \\
&\quad +2y F_2(t) u_t + \beta y F_2' u_{xx} \Big\}, \\
T_2^y &= \gamma y F_2(t) u_y - \gamma F_2(t) u;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t &= \frac{1}{4}\Big\{ 2F_3(t) u_x - \beta F_3(t) u_{xxx} \Big\}, \\
T_3^x &= \frac{1}{4}\Big\{ -8\alpha F_3(t) u_x u - 2F_3' u - 3\beta F_3(t) u_{txx} + 4F_3(t) u_x \\
&\quad +2F_3(t) u_t + \beta F_3' u_{xx} \Big\}, \\
T_3^y &= \gamma F_3(t) u_y.
\end{aligned}
$$

Remark. Due to the presence of the arbitrary function in the multiplier, one can obtain an infinitely many conservation laws for the (2+1)-dimensional nonlinear KP-BBM equation.

3.4 Conclusion

In this chapter we obtained the solutions of the two-dimensional nonlinear Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony equation by employing Lie group analysis and the simplest equation method. The solutions obtained were solitary waves and non-topological soliton. The conservation laws for the underlying equation were derived by using the multiplier method.

Chapter 4

Exact explicit solutions and conservation laws of a generalized (2+1)-dimensional nonlinear ZK-BBM equation

4.1 Introduction

In this chapter we study a generalized two-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation

$$u_t + u_x + a(u^n)_x + b(u_{xt} + u_{yy})_x = 0, \tag{4.1}$$

where a, b and $n > 1$ are real valued constants.

Several authors (see for example the papers [55, 79–83]) have studied this equation (4.1) . Wazwaz [55, 79] used the sine-cosine method, the tanh method and the extended tanh method for finding solitary solutions of this equation. Abdou [80,81] used the extended F-expansion method and the extended mapping method with symbolic computation to obtain some exact solutions. Mahmoudi [82] used the Exp-Function method to obtain some solitary solutions and periodic solutions. Song [83] used bifurcation method to obtain exact solitary wave solutions and kink wave solutions.

Here Lie group analysis in conjunction with the simplest equation method is employed to obtain some exact solutions of (4.1). Conservation laws will be derived for (4.1) using the new conservation theorem due to Ibragimov [74] and the multiplier method [73]. Moreover, the (G'/G)-expansion method [9] is used to obtain the traveling wave solutions for (4.1).

This work has been published in [29, 30].

4.2 Symmetry analysis

In this section we first calculate the Lie point symmetries of (4.1) and later use them to construct exact solutions.

4.2.1 Lie point symmetries of (4.1)

The symmetry group of ZK-BBM equation (4.1) will be generated by

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u},$$

where ξ^i, $i = 1, 2, 3$ and η depend on x, y, t and u. Applying the third prolongation $\mathrm{pr}^{(3)}X$ to (4.1) we obtain the following overdetermined system of linear partial differential equations:

$$\xi^3_y = 0,\ \xi^1_y = 0,\ \xi^1_x = 0,\ \xi^2_t = 0,\ \xi^2_x = 0,\ \xi^2_u = 0,\ \xi^1_t = 0,\ \xi^1_u = 0,\ \xi^3_x = 0,$$

$$\xi^3_u = 0,\ \eta_{tu} = 0,\ \eta_{xu} = 0,\ \eta_{uu} = 0,\ \xi^3_t - 2\xi^2_y = 0,\ \xi^2_{yy} - 2\eta_{yu} = 0,$$

$$anu^n\eta_x + u\eta_x + u\eta_t + bu\eta_{xxt} + bu\eta_{xyy} = 0,$$

$$an^2u^n\eta - anu^n\eta + u^2\xi^3_t + anu^{n+1}\xi^3_t + bu^2\eta_{yyu} = 0.$$

Solving the above system one obtains the following three Lie point symmetries:

$$X_1 = \partial/\partial x, \quad X_2 = \partial/\partial y, \quad X_3 = \partial/\partial t.$$

4.2.2 Exact solutions

First of all we utilize the symmetry $X = X_1 + \nu X_2 + X_3$ and reduce the ZK-BBM equation (4.1) to a PDE in two independent variables. It can be seen that the symmetry X yields the following three invariants:

$$f = y - \nu t, \quad g = t - x, \quad \theta = u. \tag{4.2}$$

Now treating θ as the new dependent variable and f and g as new independent variables, the KP-BBM equation (4.1) transforms to

$$-\nu\theta_f - an\theta^{n-1}\theta_g - \nu b\theta_{fgg} + b\theta_{ggg} - b\theta_{ffg} = 0, \tag{4.3}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (4.3) and transform it to an ordinary differential equation (ODE). The equation (4.3) has the two translational symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}.$$

The combination $\Gamma_1 + \Gamma_2$, of the two symmetries Γ_1 and Γ_2 yields the two invariants

$$r = f - g, \quad \psi = \theta,$$

which gives rise to a group invariant solution $\psi = \psi(r)$. Consequently using these invariants, (4.3) is transformed into the third-order nonlinear ODE

$$-\nu\psi' + an\psi^{n-1}\psi' - \nu b\psi''' = 0. \tag{4.4}$$

Integrating the above equation once and taking the constants of integration to be zero we obtain a second-order ODE

$$-\nu\psi + a\psi^{n} - \nu b\psi'' = 0. \tag{4.5}$$

Multiplying equation (4.5) by ψ', integrating once and taking the constant of integration to be zero, we obtain the first-order ODE

$$-\frac{1}{2}\nu\psi^2 + \frac{a}{n+1}\psi^{n+1} - \frac{1}{2}\nu b\psi'^2 = 0. \tag{4.6}$$

One can integrate the above equation by separating the variables. After integrating and reverting back to the original variables, we obtain the following group-invariant solutions of the ZK-BBM equation (4.1) for arbitrary values of n in the form:

$$u(x,y,t) = \left(\frac{\nu(n+1)}{2a}\right)^{\frac{1}{n-1}} \operatorname{sech}^{\frac{2}{n-1}}[Q], \tag{4.7}$$

$$Q = \gamma\left(\sqrt{\frac{2}{b\nu}}r - C\right),$$
$$\gamma = \sqrt{-\frac{\nu}{8}}(1-n),$$
$$r = (x + y - (\nu+1)t).$$

For $n = 1$,

$$u(x,y,t) = \exp\left[\mp\frac{C_1\sqrt{-\nu+a}}{\sqrt{2}} \pm \frac{\sqrt{\nu-a}}{\sqrt{-\nu b}}r\right],$$

$r = (x + y - (\nu+1)t)$,

for $n = 2$,

$$u(x,y,t) = -\frac{3}{\nu}\operatorname{sech}^2\left[\frac{1}{2}\left(\pm C_2\sqrt{-\frac{\nu}{2}} \mp \sqrt{\frac{-1}{b}}r\right)\right].$$

$r = (x + y - (\nu + 1)t)$.

By taking $n = 2$, $a = -1$, $b = 1$, $\nu = 6$, $t = 0$ and $C = 1$ in (4.7), the profile of the solution is given in Figure 4.1.

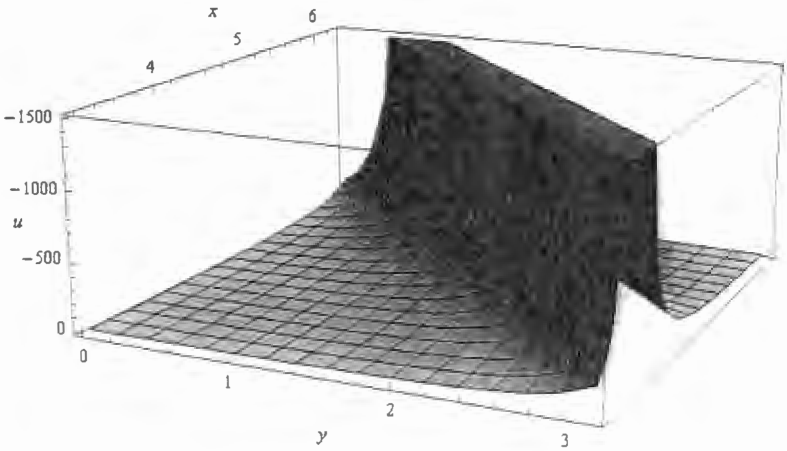

Figure 4.1: Profile of solution (4.7)

4.3 Simplest equation method

In this section we use the simplest equation method, which was used in Chapter 2, to solve the third-order ODE (4.4) for $n = 2, 3$ and as a result we obtain the exact solutions of our ZK-BBM equation (4.4).

4.3.1 Solutions of (4.4) using the equation of Bernoulli as the simplest equation

$n = 2$

The balancing procedure yields $M = 2$, so the solutions of (4.4) are of the form

$$\psi(r) = A_0 + A_1 H + A_2 H^2. \tag{4.8}$$

Substituting (4.8) into (4.4) and making use of the Bernoulli equation and then equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 and A_2. These algebraic equations are

$$
\begin{aligned}
&4aA_2^2d - 24bA_2d^3\nu = 0,\\
&2aA_0A_1c - bA_1c^3\nu - A_1c\nu = 0,\\
&6aA_1A_2d - 54bA_2cd^2\nu - 6bA_1d^3\nu + 4aA_2^2c = 0,\\
&12bA_1cd^2\nu + 38bA_2c^2d\nu - 6aA_1A_2c\\
&-2aA_1^2d + 2A_2d\nu - 4aA_0A_2d = 0,\\
&7bA_1c^2d\nu - 4aA_0A_2c + 2A_2c\nu - 2aA_1^2c\\
&-2aA_0A_1d + A_1d\nu + 8bA_2c^3\nu = 0.
\end{aligned}
$$

Solving the above system of algebraic equations with the aid of Mathematica, we obtain the following values of A_0, A_1 and A_2:

$$A_0 = \frac{\nu(1+bc^2)}{2a}, \quad A_1 = \frac{6b\nu cd}{a}, \quad A_2 = \frac{6b\nu d^2}{a}.$$

Therefore the solution of (4.1), for $n = 2$ is given by

$$u(x,y,t) = A_0 + A_1a\left(\frac{\cosh[P]+\sinh[P]}{1-d\cosh[P]-d\sinh[P]}\right) + A_2a^2\left(\frac{\cosh[P]+\sinh[P]}{1-d\cosh[P]-d\sinh[P]}\right)^2,$$

where $P = c(r+C)$ and $r = x + y - (\nu+1)t$ and C is a constant of integration.

$n = 3$

The balancing procedure yields $M = 1$, so the solutions of (4.4) are of the form

$$\psi(r) = A_0 + A_1H. \tag{4.9}$$

As before, substituting (4.9) into (4.4), we obtain the algebraic system of equations

$$3aA_1^3d - 6bA_1d^3\nu = 0,$$

$$A_1 c\nu + bA_1c^3\nu - 3aA_1A_0^2c = 0,$$

$$6aA_1^2A_0d + 3aA_1^3c - 12bA_1cd^2\nu = 0,$$

$$3aA_1A_0^2d - A_1d\nu + 6aA_1^2A_0c - 7bA_1c^2d\nu = 0,$$

which on solving yields

$$A_0 = \sqrt{\frac{2bd^2\nu}{a}}, \quad A_1 = c\sqrt{\frac{b\nu}{2a}}.$$

Therefore the solution of (4.1), for $n = 3$ is given by

$$u(x, y, t) = A_0 + A_1a\left(\frac{\cosh[P] + \sinh[P]}{1 - d\cosh[P] - d\sinh[P]}\right), \tag{4.10}$$

where $r = x + y - (\nu + 1)t$ and C is a constant of integration.

4.3.2 Solutions of (4.4) using Riccati equation as the simplest equation

$n = 2$

The balancing procedure yields $M = 2$, so the solutions of (4.4) are of the form

$$\psi(r) = A_0 + A_1H + A_2H^2. \tag{4.11}$$

Substituting (4.11) into (4.4) and making use of the Riccati equation, we obtain algebraic equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions H^i to zero. The corresponding algebraic equations are

$$4aA_2^2c - 24bA_2c^3\nu = 0,$$

$$6aA_1A_2c - 6bA_1c^3\nu + 4aA_2^2d - 54bA_2c^2d\nu = 0,$$

$$2bA_1ce^2\nu - 2aA_0A_1e + 6bA_2de^2\nu + bA_1d^2e\nu$$

$$+A_1e\nu = 0,$$

$$
\begin{aligned}
&4aA_0A_2c - 38bA_2cd^2\nu + 6aA_1A_2d - 2A_2c\nu \\
&-12bA_1c^2d\nu + 2aA_1^2c + 4aA_2^2e - 40bA_2c^2e\nu = 0, \\
&14bA_2d^2e\nu - 4aA_0A_2e + A_1d\nu + bA_1d^3\nu + 2A_2e\nu \\
&-2aA_1^2e - 2aA_0A_1d + 8bA_1cde\nu + 16bA_2ce^2\nu = 0, \\
&7bA_1cd^2\nu - 6aA_1A_2e + A_1c\nu - 2aA_0A_1c \\
&+52bA_2cde\nu - 4aA_0A_2d + 8bA_1c^2e\nu + 8bA_2d^3\nu \\
&+2A_2d\nu - 2aA_1^2d = 0.
\end{aligned}
$$

Solving the above equations yield

$$
A_0 = \frac{1}{2a}(\nu bd^2 + 8\nu bce + \nu), \quad A_1 = \frac{6\nu bcd}{a}, \quad A_2 = \frac{6\nu bc^2}{a}
$$

and hence the solutions of (4.1) for $n = 2$ are

$$
\begin{aligned}
u(x,y,t) &= A_0 + A_1\left(-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(r+C)\right]\right) \\
&\quad + A_2\left(-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(r+C)\right]\right)^2 \qquad (4.12)
\end{aligned}
$$

and

$$
\begin{aligned}
u(x,y,t) &= A_0 + A_1\left(-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta r\right) + \frac{\operatorname{sech}\left(\frac{\theta r}{2}\right)}{C\cosh\left(\frac{\theta r}{2}\right) - \frac{2c}{\theta}\sinh\left(\frac{\theta r}{2}\right)}\right) \\
&\quad + A_2\left(-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta r\right) + \frac{\operatorname{sech}\left(\frac{\theta r}{2}\right)}{C\cosh\left(\frac{\theta r}{2}\right) - \frac{2c}{\theta}\sinh\left(\frac{\theta r}{2}\right)}\right)^2,
\end{aligned}
$$

where $r = x + y - (\nu+1)t$ and C is a constant of integration.

$n = 3$

The balancing procedure yields $M = 1$, so the solutions of (4.4) are of the form

$$
\psi(r) = A_0 + A_1 H. \qquad (4.13)
$$

Substituting (4.13) into (4.4), we obtain the following algebraic system of equations:

$$6bA_1c^3\nu - 3aA_1^3c = 0,$$

$$6aA_1^2A_0c + 3aA_1^3d - 12bA_1c^2d\nu = 0.$$

$$3aA_1A_0^2e - 2bA_1ce^2\nu - A_1e\nu - bA_1d^2e\nu = 0,$$

$$8bA_1cde\nu + A_1d\nu - 6aA_1^2A_0e + bA_1d^3\nu$$

$$-3aA_1A_0^2d = 0,$$

$$A_1c\nu - 3aA_1^3e + 7bA_1cd^2\nu - 3aA_1A_0^2c$$

$$-6aA_1^2A_0d + 8bA_1c^2e\nu = 0.$$

Solving the algebraic equations one obtain

$$c = \frac{-2+bd^2}{4be}, \quad A_0 = 2bd\sqrt{\frac{\nu}{8abe^2}}, \quad A_1 = \sqrt{\frac{\nu(-2+bd^2)^2}{8abe^2}}.$$

Hence we have the following solutions of (4.1) for $n = 3$:

$$u(x,y,t) = A_0 + A_1\left(-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(r+C)\right]\right) \tag{4.14}$$

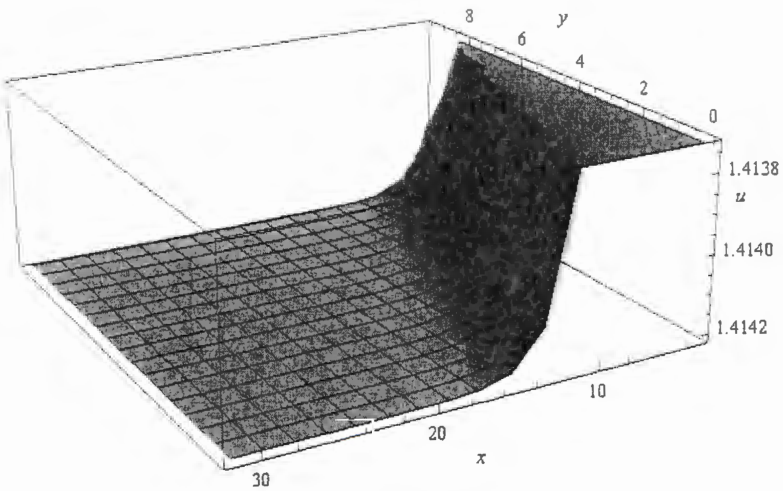

Figure 4.2: Profile of solution (4.14)

and

$$u(x,y,t) = A_0 + A_1\Bigg(-\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta r\right) + \frac{\operatorname{sech}(\frac{\theta r}{2})}{C\cosh(\frac{\theta r}{2}) - \frac{2c}{\theta}\sinh(\frac{\theta r}{2})}\Bigg), \quad (4.15)$$

where $r = x + y - (\nu + 1)t$ and C is a constant of integration.

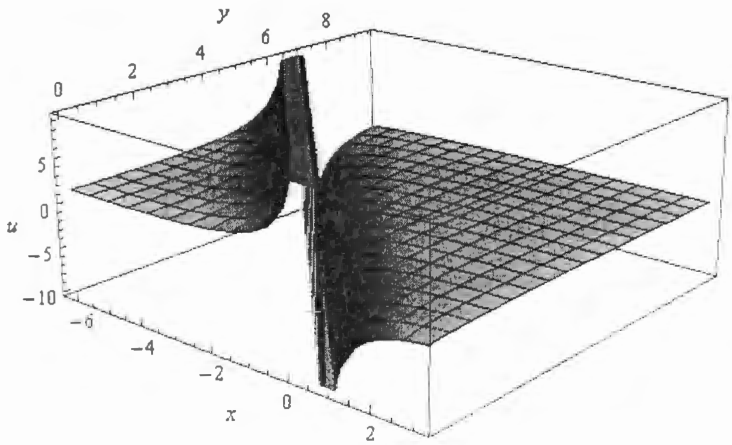

Figure 4.3: Profile of solution (4.15)

4.4 Exact solutions using (G'/G)-expansion method

In this section we use the $(G'/G)-$expansion method [9], to obtain exact solutions of the ZK-BBM equation (4.1) for $n = 2$ and $n = 3$.

Making use of the wave variable

$$z = k_1x + k_2y + k_3t + k_4,$$

where k_i, $i = 1, \cdots, 4$ are constants, the ZK-BBM equation (4.1) for $n = 2$ and $n = 3$, transforms to the third-order nonlinear ordinary differential equations (ODEs),

$$k_3\psi'(z) + k_1\psi'(z) + 2ak_1\psi(z)\psi'(z) + bk_1^2k_3\psi'''(z) + bk_2^2k_1\psi'''(z) = 0, \quad (4.16)$$

and

$$k_3\psi'(z) + k_1\psi'(z) + 3ak_1\psi^2(z)\psi'(z) + bk_1^2k_3\psi'''(z) + bk_2^2k_1\psi'''(z) = 0, \quad (4.17)$$

respectively.

We look for solutions of (4.16) and (4.17) in the form

$$\psi(z) = \sum_{i=0}^{M} A_i \left(\frac{G'(z)}{G(z)} \right)^i, \tag{4.18}$$

where $G(z)$ satisfies the second-order ODE

$$G'' + \lambda G' + \mu G = 0, \tag{4.19}$$

with λ and μ as constants. Here the constant M will be determined by the homogeneous balance procedure between the highest order derivative and highest order nonlinear term appearing in (4.16)and (4.17). $A_0, \cdots, A_M$, are parameters to be determined.

$n = 2$

Application of the balancing procedure yields $M = 2$, and so the solution of (4.16) are of the form

$$\psi(z) = A_0 + A_1 \left(\frac{G'(z)}{G(z)} \right) + A_2 \left(\frac{G'(z)}{G(z)} \right)^2. \tag{4.20}$$

Substituting (4.19) into (4.16) and using (4.20) leads to an overdetermined system of algebraic equations:

$$24\, bA_2{k_1}^2k_3 + 24\, bA_2k_1{k_2}^2 + 4\, a{A_2}^2k_1 = 0,$$

$$54\, b\lambda\, A_2{k_1}^2k_3 + 54\, b\lambda\, A_2k_1{k_2}^2 + 4\, a\lambda\, {A_2}^2k_1 + 6\, bA_1{k_1}^2k_3 + 6\, bA_1k_1{k_2}^2$$
$$+6\, aA_1A_2k_1 = 0,$$

$$b\lambda^2\mu\, A_1{k_1}^2k_3 + b\lambda^2\mu\, A_1k_1{k_2}^2 + 6\, b\lambda\, \mu^2A_2{k_1}^2k_3 + 6\, b\lambda\, \mu^2A_2k_1{k_2}^2 + 2\, b\mu^2A_1{k_1}^2k_3$$
$$+2\, b\mu^2A_1k_1{k_2}^2 + 2\, a\mu\, A_0A_1k_1 + \mu\, A_1k_1 + \mu\, A_1k_3 = 0,$$

$$38\, b\lambda^2A_2{k_1}^2k_3 + 38\, b\lambda^2A_2k_1{k_2}^2 + 12\, b\lambda\, A_1{k_1}^2k_3 + 12\, b\lambda\, A_1k_1{k_2}^2 + 40\, b\mu\, A_2{k_1}^2k_3$$
$$+40\, b\mu\, A_2k_1{k_2}^2 + 6\, a\lambda\, A_1A_2k_1 + 4\, a\mu\, {A_2}^2k_1 + 4\, aA_0A_2k_1 + 2\, a{A_1}^2k_1 + 2\, A_2k_1$$

$$+2\,A_2k_3 = 0,$$

$$b\lambda^3A_1{k_1}^2k_3 + b\lambda^3A_1k_1{k_2}^2 + 14\,b\lambda^2\mu\,A_2{k_1}^2k_3 + 14\,b\lambda^2\mu\,A_2k_1{k_2}^2 + 8\,b\lambda\,\mu\,A_1{k_1}^2k_3$$

$$+8\,b\lambda\,\mu\,A_1k_1{k_2}^2 + 16\,b\mu^2A_2{k_1}^2k_3 + 16\,b\mu^2A_2k_1{k_2}^2 + 2\,a\lambda\,A_0A_1k_1 + 4\,a\mu\,A_0A_2k_1$$

$$+2\,a\mu\,{A_1}^2k_1 + \lambda\,A_1k_1 + \lambda\,A_1k_3 + 2\,\mu\,A_2k_1 + 2\,\mu\,A_2k_3 = 0,$$

$$8\,b\lambda^3A_2{k_1}^2k_3 + 8\,b\lambda^3A_2k_1{k_2}^2 + 7\,b\lambda^2A_1{k_1}^2k_3 + 7\,b\lambda^2A_1k_1{k_2}^2 + 52\,b\lambda\,\mu\,A_2{k_1}^2k_3$$

$$+52\,b\lambda\,\mu\,A_2k_1{k_2}^2 + 8\,b\mu\,A_1{k_1}^2k_3 + 8\,b\mu\,A_1k_1{k_2}^2 + 4\,a\lambda\,A_0A_2k_1 + 2\,a\lambda\,{A_1}^2k_1$$

$$+6\,a\mu\,A_1A_2k_1 + 2\,aA_0A_1k_1 + 2\,\lambda\,A_2k_1 + 2\,\lambda\,A_2k_3 + A_1k_1 + A_1k_3 = 0.$$

Solving the system of algebraic equations, with the aid of Mathematica, we obtain

$$\begin{aligned}
A_0 &= \frac{8ab\mu A_2^2k_1^2 + abA_1^2k_1^2 - 6bA_2k_1^2 + 6bA_2k_2^2 + aA_2^2}{12abA_2k_1^2},\\
A_1 &= -\frac{\lambda(6bk_1k_3 + 6bk_2^2)}{a},\\
A_2 &= -\frac{6bk_1k_3 + 6bk_2^2}{a}.
\end{aligned}$$

Now using the general solution of (4.19) in (4.20), we have the following three types of traveling wave solutions of the ZK-BBM equation (4.1):

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$\begin{aligned}
u(x,y,t) &= A_0 + A_1\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right)\\
&\quad + A_2\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right)^2,
\end{aligned}\tag{4.21}$$

where $z = k_1x + k_2y + k_3t + k_4$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants. The profile of the solution (4.21) is given in Figure 4.4.

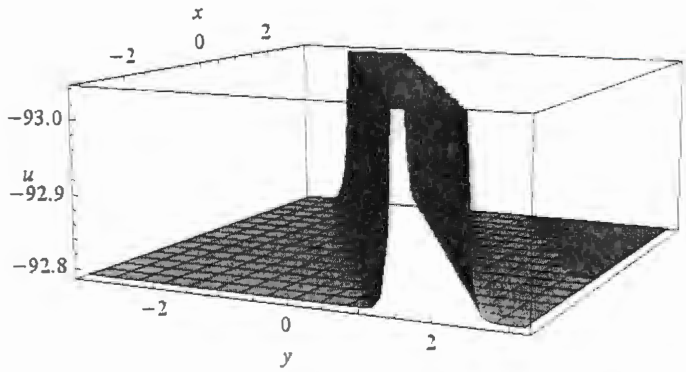

Figure 4.4: Profile of solution (4.21)

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$\begin{aligned} u(x,y,t) &= A_0 + A_1\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big) \\ &\quad + A_2\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big)^2, \end{aligned} \tag{4.22}$$

where $z = k_1x + k_2y + k_3t + k_4$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

The profile of the solution (4.22) is given in Figure 4.5.

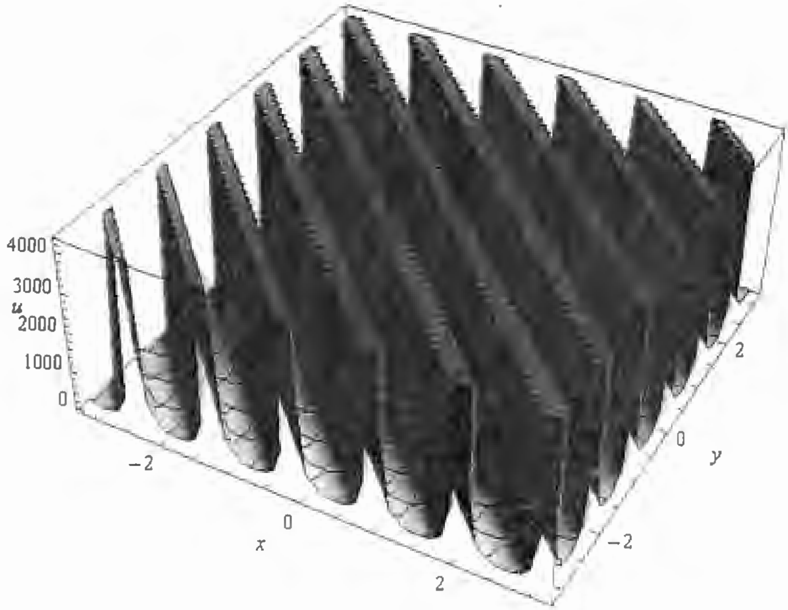

Figure 4.5: Profile of solution (4.22)

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$u(x, y, t) \;=\; A_0 + A_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + A_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2 . \quad (4.23)$$

where $z = k_1x + k_2y + k_3t + k_4$, C_1 and C_2 are arbitrary constants.

The profile of the solution (4.23) is given in Figure 4.6.

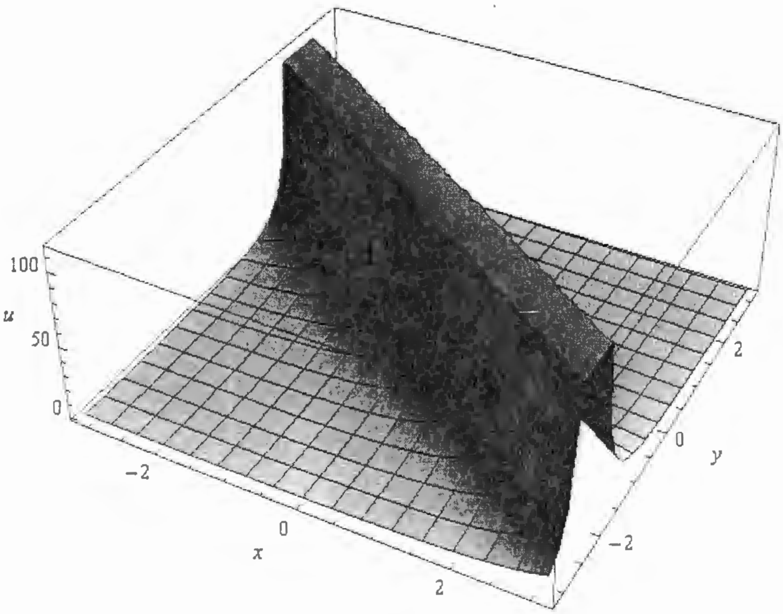

Figure 4.6: Profile of solution (4.23)

$n = 3$

Application of the balancing procedure, in this case, we obtain $M = 1$, so the solution of (4.17) are of the form

$$\psi(z) = A_0 + A_1\left(\frac{G'(z)}{G(z)}\right). \tag{4.24}$$

Substituting (4.19) into (4.17) and making use of (4.24) leads to an overdetermined system of algebraic equations:

$$3\,ak_1A_1{}^3 + 6\,bA_1k_3k_1{}^2 + 6\,bA_1k_1k_2{}^2 = 0,$$

$$3\,a\lambda\,k_1A_1{}^3 + 12\,b\lambda\,A_1k_3k_1{}^2 + 12\,b\lambda\,A_1k_1k_2{}^2 + 6\,aA_0k_1A_1{}^2 = 0,$$

$$bk_1{}^2k_3A_1\lambda^2\mu + bk_2{}^2k_1A_1\lambda^2\mu + 2\,bk_1{}^2k_3A_1\mu^2 + 2\,bk_2{}^2k_1A_1\mu^2 + 3\,ak_1A_0{}^2A_1\mu$$
$$+k_1A_1\mu + k_3A_1\mu = 0,$$

$$bA_1k_3\lambda^3k_1{}^2 + bA_1k_1\lambda^3k_2{}^2 + 8\,b\mu\,\lambda\,A_1k_3k_1{}^2 + 8\,b\mu\,\lambda\,A_1k_1k_2{}^2 + 3\,a\lambda\,A_1k_1A_0{}^2$$
$$+6\,a\mu\,A_0k_1A_1{}^2 + \lambda\,A_1k_1 + \lambda\,A_1k_3 = 0,$$

$$7\,bA_1k_3\lambda^2{k_1}^2+7\,bA_1k_1\lambda^2{k_2}^2+6\,a\lambda\,A_0k_1{A_1}^2+3\,a\mu\,k_1{A_1}^3+8\,b\mu\,A_1k_3{k_1}^2$$
$$+8\,b\mu\,A_1k_1{k_2}^2+3\,ak_1{A_0}^2A_1+k_1A_1+k_3A_1=0.$$

On solving the above algebraic equations by using Maple, we obtain the following solutions:

$$\mu=\frac{-2aA_1^2-4bk_2^2+bk_1^2\lambda^2aA_1^2+4bk_1^2}{4abk_1^2A_1^2},$$
$$A_0=\frac{\lambda\sqrt{-(2bk_1k_3+2bk_2^2)}}{2\sqrt{a}},$$
$$A_1=\sqrt{\frac{-(2bk_1k_3+2bk_2^2)}{a}}.$$

Consequently, as before, when $\lambda^2-4\mu>0$, we obtain the hyperbolic function solutions

$$u(x,y,t)\;=\;A_0+A_1\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg),\qquad(4.25)$$

where $z=k_1x+k_2y+k_3t+k_4$, $\delta_1=\frac{1}{2}\sqrt{\lambda^2-4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2-4\mu<0$, we obtain the trigonometric function solutions

$$u(x,y,t)\;=\;A_0+A_1\bigg(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\bigg),\qquad(4.26)$$

where $z=k_1x+k_2y+k_3t+k_4$, $\delta_2=\frac{1}{2}\sqrt{4\mu-\lambda^2}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2-4\mu=0$, we obtain the rational function solutions

$$u(x,y,t)\;=\;A_0+A_1\bigg(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2z}\bigg),\qquad(4.27)$$

where $z=k_1x+k_2y+k_3t+k_4$, C_1 and C_2 are arbitrary constants.

4.5 Conservation laws of (4.1)

In this section we construct conservation laws for (4.1) by using two different methods. The methods are the new conservation theorem due to Ibragimov [74] and the multiplier method [73].

4.5.1 Application of the conservation theorem

The generalized two-dimensional nonlinear ZK-BBM equation together with its adjoint equation are given by

$$E_\alpha \equiv u_t + u_x + anu^{n-1}u_x + b(u_{xt} + u_{yy})_x = 0, \tag{4.28a}$$

$$E^*_\alpha \equiv v_t + v_x + anu^{n-1}v_x + b(v_{xt} + v_{yy})_x = 0. \tag{4.28b}$$

The third-order Lagrangian for the system of equations (4.28a) and (4.28b) is given by

$$L = v(u_t + u_x + anu^{n-1}u_x + b(u_{xt} + u_{yy})_x). \tag{4.29}$$

which can be reduced to the second-order Lagrangian

$$L = v(u_t + u_x + anu^{n-1}u_x) - bv_t u_{xx} - bv_x u_{yy}. \tag{4.30}$$

We have the following three cases:

(i) We first consider the Lie point symmetry $X_1 = \partial_x$ of (4.1). Corresponding to this symmetry the Lie characteristic functions are $W^1 = -u_x$ and $W^2 = -v_x$. Thus, by using the Ibragimov theorem [74], the components of the conserved vector associated with the symmetry $X_1 = \partial_x$ are given by

$$\begin{aligned}
T_1^t &= bv_x u_{xx} - vu_x, \\
T_1^x &= vu_t - bu_x v_{tx}, \\
T_1^y &= bv_x u_{xy} - bu_x v_{xy}.
\end{aligned}$$

(ii) Likewise, the Lie point symmetry $X_2 = \partial_y$ has the Lie characteristic functions $W^1 = -u_y$ and $W^2 = -v_y$. Invoking Ibragimov theorem we obtain the conserved vector whose components are

$$T_2^t = bv_y u_{xx} - vu_y,$$

$$T_2^x = -anvu^{n-1}u_y - vu_y + bv_t u_{xy} - bu_y v_{tx} + bv_y u_{yy},$$
$$T_2^y = anvu^{n-1}u_x + vu_x + vu_t - bv_t u_{xx} - bu_y v_{xy}.$$

(iii) Finally, the Lie point symmetry $X_3 = \partial_t$ gives $W^1 = -u_t$ and $W^2 = -v_t$ and so the associated conserved vector has components

$$T_1^t = anvu_x u^{n-1} + vu_x - bv_x u_{yy},$$
$$T_1^x = -anvu^{n-1}u_t - vu_t + bv_t u_{tx} - bu_t v_{tx} + bv_t u_{yy},$$
$$T_1^y = bv_x u_{ty} - bu_t v_{xy}.$$

4.5.2 Application of the multiplier method

The zeroth-order multiplier for (4.1), viz., $\Lambda(t, x, y, u)$, is given by

$$\Lambda = Cu + F(y), \tag{4.31}$$

where C is a constant, and $F(y)$ is arbitrary function of y. Corresponding to the above multiplier we have the following two conserved vectors of (4.1):

$$T_1^t = \frac{1}{6}\left\{2buu_{xx} + 3u^2 - bu_x^2\right\},$$
$$T_1^x = \frac{1}{6(n+1)}\left\{2bnuu_{yy} + 4bnuu_{tx} + 2buu_{yy} + 4buu_{tx} + 6anu^{n+1}\right.$$
$$\left.+3nu^2 + 3u^2 - 2bnu_t u_x - bnu_y^2 - 2bu_t u_x - bu_y^2\right\},$$
$$T_1^y = \frac{1}{3}\left\{2buu_{xy} - bu_x u_y\right\}$$

and

$$T_2^t = \frac{1}{3}\left\{3uF(y) + bF(y)u_{xx}\right\},$$
$$T_2^x = \frac{1}{3}\left\{3au^n F(y) + buF''(y) + 3uF(y) - bF'(y)u_y + 2bF(y)u_{tx} + bF(y)u_{yy}\right\},$$
$$T_2^y = \frac{1}{3}\left\{2bF(y)u_{xy} - bF'(y)u_x\right\}.$$

Remark. Since $F(y)$ is an arbitrary function in the multiplier, we obtain an infinitely many conservation laws for (4.1).

4.6 Conclusion

In this chapter we obtained the solutions of the generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation by employing the Lie group analysis, the simplest equation method and (G'/G)-expansion method. The solutions obtained were solitary waves and non-topological soliton. Moreover, conservation laws were derived by using two different methods, the new conservation theorem and the multiplier method.

Chapter 5

Symmetry analysis and conservation laws of a generalized two-dimensional nonlinear KP-MEW equation

5.1 Introduction

The purpose of this chapter is to study the generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width (KP-MEW) equation [58] that is given by

$$(u_t + \alpha(u^n)_x + \beta u_{xxt})_x + \gamma u_{yy} = 0. \tag{5.1}$$

Here, in (5.1) α, β, γ and $n > 1$ are real valued constants. Equation (5.1) was studied in [58, 84, 85]. In [58], the tanh method and the sine-cosine method, for finding solitary waves and periodic solutions, were used. Asit Saha [84] used the theory of bifurcations of planar dynamical systems to prove the existence of smooth and non-smooth travelling wave solutions. Minzhi Wei *et.al* [85] used the qualitative theory of differential equations and obtained peakon, compacton, cuspons, loop soliton solutions and smooth soliton solutions.

However in this chapter we obtain symmetry reductions of (5.1) based on the optimal systems of one-dimensional subalgebras. Furthermore, the (G'/G)-expansion method is employed to obtain some exact solutions of (5.1). In addition to this conservation laws will be derived for (5.1) using the multiplier method [73].

This work has been accepted and to appear in [86].

5.2 Symmetry reductions and exact solutions of (5.1)

The vector field of the form

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u},$$

where ξ^i, $i = 1, 2, 3$ and η depend on x, y, t and u, is a Lie point symmetry of (5.1) if

$$\mathrm{pr}^{(4)}X\left[(u_t + \alpha(u^n)_x + \beta u_{xxt})_x + \gamma u_{yy}\right] = 0 \tag{5.2}$$

whenever $(u_t + \alpha(u^n)_x + \beta u_{xxt})_x + \gamma u_{yy} = 0$. Here $\mathrm{pr}^{(4)}X$ denotes the fourth prolongation of X. Expanding (5.2) and splitting on the derivatives of u, we obtain an overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^1_y, = 0,\ \xi^3_y = 0,\ \xi^2_x = 0,\ \xi^2_t = 0,\ \xi^2_u = 0,\ \xi^1_t = 0,\ \xi^1_u = 0,\ \xi^3_x = 0,\ \xi^3_u = 0,\\
&\eta_{tu} = 0,\ \eta_{uu} = 0,\ \xi^2_{yy} - 2\,\eta_{yu} = 0,\ \beta\eta_{xxxu} + \eta_{xu} = 0,\ \eta_{xu} - \xi^1_{xx} = 0,\\
&\xi^3_t - 2\,\xi^2_y + 3\,\xi^1_x = 0,\ \alpha\, n u^n \eta_{xx} + \gamma\, u\eta_{yy} + \beta\, u\eta_{xxxt} + u\eta_{xt} = 0,\\
&n\eta - 2\,u\xi^1_x + 2\,u\xi^2_y - \eta = 0,\ 2\,n\eta_x + 2\,u\eta_{xu} - u\xi^1_{xx} - 2\,\eta_x = 0,
\end{aligned}$$

$$3\,\beta\eta_{xxu} - \beta\xi^1_{xxx} - \xi^1_x + 2\,\xi^2_y - \xi^3_t = 0, \;\; n\eta + u\eta_u - 2\,u\xi^1_x + 2\,u\xi^2_y - 2\,\eta = 0.$$

Solving the above system one obtains the following four Lie point symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial y}, \\
X_3 &= \frac{\partial}{\partial t}, \\
\text{and} \quad X_4 &= y(1-n)\frac{\partial}{\partial y} + 2t(1-n)\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.
\end{aligned}$$

5.2.1 One-dimensional optimal system of subalgebras

We now calculate the optimal system of one-dimensional subalgebras for equation (5.1) and use it to find the optimal system of group-invariant solutions for equation (5.1). We follow the method given in [18]. Recall that the adjoint transformations are given by

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2}\epsilon^2[X_i,[X_i,X_j]] - \cdots,$$

where $[X_i, X_j]$ is the commutator defined by

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

We present the commutator table of the Lie symmetries and the adjoint representations of the symmetry group of (5.1) on its Lie algebra in Table 1 and Table 2, respectively. These two tables are then used to construct the optimal system of one-dimensional subalgebras for equation (5.1). As a result, after some calculations, one can obtain an optimal system of one-dimensional subalgebras given by $\{aX_1 + bX_2 + cX_3, dX_1 + X_4\}$, where $a, d \in \mathbb{R}$, $b, c = 0, \pm 1$.

Table 1. Commutator table of the Lie algebra of equation (5.1)

	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	$(1-n)X_2$
X_3	0	0	0	$(2-2n)X_3$
X_4	0	$-(1-n)X_2$	$-(2-2n)X_3$	0

Table 2. Adjoint table of the Lie algebra of equation (5.1)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	$X_4 - \varepsilon(1-n)X_2$
X_3	X_1	X_2	X_3	$X_4 - \varepsilon(2-2n)X_3$
X_4	X_1	$e^{\varepsilon(1-n)}X_2$	$e^{\varepsilon(2-2n)}X_3$	X_4

5.2.2 Symmetry reductions and exact solutions of (5.1)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions and exact solutions of the KP-MEW equation.

Case 1. $aX_1 + bX_2 + cX_3; \quad a \in \mathbb{R}, \;\; b, c = \pm 1$

The symmetry $aX_1 + bX_2 + cX_3$ gives rise to the following three invariants:

$$f = \frac{bt - cy}{b}, \quad g = \frac{bx + ay}{b}, \quad \phi = u. \tag{5.3}$$

Now treating ϕ as the new dependent variable and f and g as new independent variables, the KP-MEW equation (5.1) transforms to

$$(2ac\gamma + b^2)\phi_{fg} + a^2\gamma\phi_{gg} + \alpha n(n - b^2)\phi^{n-2}\phi_g^2 + b^2\alpha n\phi^{n-1}\phi_{gg} + b^2\beta\phi_{fggg}$$

$$+c^2\gamma\phi_{ff} = 0, \tag{5.4}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (5.4) and transform it to an ordinary differential equation (ODE). The equation (5.4) has the two translational symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}.$$

The linear combination $\Gamma_1 + \Gamma_2$ of the two symmetries Γ_1 and Γ_2 yields the two invariants

$$z = f - g, \quad F = \phi,$$

which gives rise to a group invariant solution $F = F(z)$. Consequently using these invariants, (5.4) is transformed into the fourth-order nonlinear ODE

$$\gamma(a+c)^2F'' + \alpha nb^2F^{n-1}F'' - b^2F'' + \alpha nb^2(n-1)F^{n-2}F'^2 - \beta b^2F'''' = 0. \tag{5.5}$$

Integrating the above equation twice and taking the constants of integration to be zero we obtain a second-order ODE

$$(\gamma(a+c)^2 - c^2)F + \alpha b^2F^n - \beta b^2F'' = 0. \tag{5.6}$$

Multiplying equation (5.6) by F', integrating once and taking the constant of integration to be zero, we obtain the first-order ODE

$$\frac{1}{2}(\gamma(a+c)^2 - b^2)F^2 + \frac{\alpha b^2}{n+1}F^{n+1} - \frac{1}{2}\beta b^2F'^2 = 0. \tag{5.7}$$

One can integrate the above equation by separating the variables. After integrating and reverting back to the original variables, we obtain the following group-invariant solutions of the KP-MEW equation (5.1) for arbitrary values of n in the form:

$$u(x,y,t) = \left(\frac{(n+1)(b^2 - \gamma(a+c)^2)}{2\beta b^2}\right)^{\frac{1}{n-1}} \operatorname{sech}^{\frac{2}{n-1}}[Q], \tag{5.8}$$

where

$$Q = \frac{\sqrt{\gamma(a+c)^2+b^2}}{2\sqrt{2}}\left(\frac{\sqrt{2}(1-n)}{c\sqrt{\beta}}z+(n-1)C\right),$$

$$z = t - x - \frac{(a+c)}{b}y$$

and C is a constant of integration. By taking $n=2$, $\alpha=\frac{1}{2}$, $\beta=1$, $\gamma=1$, $a=1$, $b=1$, $c=1$, $t=0$ and $C=1$ in (5.8), the profile of the solution is given in Figure 5.1.

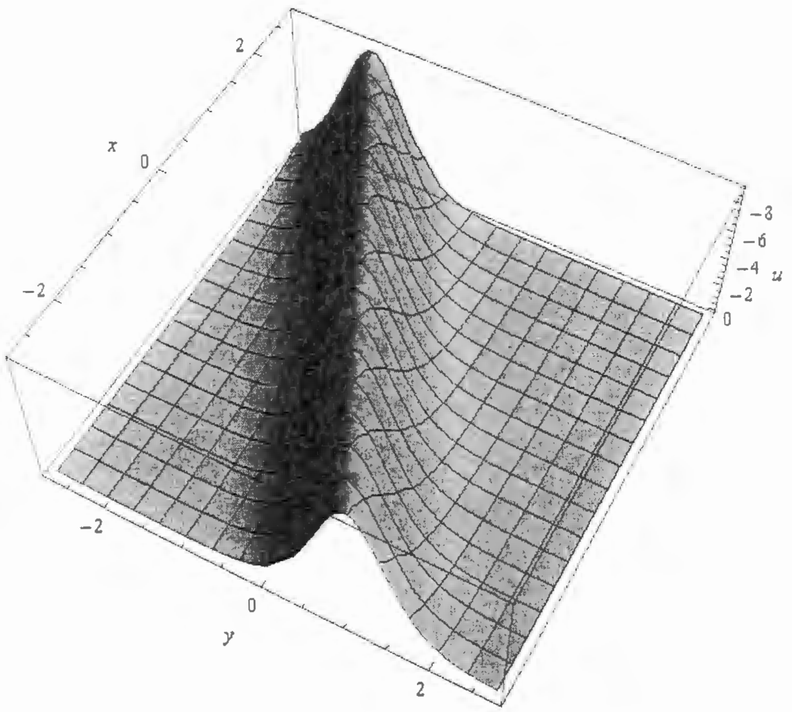

Figure 5.1: Profile of solution (5.8)

Case 2. $dX_1 + X_4$

The symmetry $dX_1 + X_4$ gives rise to the three invariants

$$f = \frac{t}{y^2}, \quad g = \frac{d\ln y + nx - x}{n-1}, \quad \phi = uy^{\frac{2}{n-1}}. \tag{5.9}$$

By treating ϕ as the new dependent variable and f and g as new independent variables, the KP-MEW equation (5.1) transforms to

$$\beta n^2\phi_{fggg} - 2\beta n\phi_{fggg} + 2d\gamma\phi_{gg} + 2n\gamma\phi + 4f^2\gamma\phi_{ff} - 2f\gamma\phi_f - 3a_1\gamma\phi_g$$
$$+\phi_{fg} - \alpha n\phi^{n-2}\phi_g^2 + \alpha n^4\phi^{n-2}\phi_g^2 + 3\alpha n^2\phi^{n-2}\phi_g^2 + 6fn^2\gamma\phi_f - 4fn\gamma\phi_f$$
$$-3\alpha n^3\phi^{n-2}\phi_g^2 + \alpha n\phi^{n-1}\phi_{gg} + \alpha n^3\phi^{n-1}\phi_{gg} - 8f^2n\gamma\phi_{ff} + 4f^2n^2\gamma\phi_{ff}$$
$$+4df\gamma\phi_{fg} - 2\alpha n^2\phi^{n-1}\phi_{gg} + 2\gamma\phi + \beta\phi_{fggg} + fn^2\phi_g - 2fn\phi_g - 4dfn\gamma\phi_{fg}$$
$$-dfn\gamma\phi_g = 0. \tag{5.10}$$

The equation (5.10) has a single Lie point symmetry, viz.,

$$\Gamma_1 = \frac{\partial}{\partial g}$$

and this symmetry yields the two invariants

$$r = f, \quad F = \phi,$$

which gives rise a group invariant solution $F = F(r)$ and consequently using these invariants, (5.10) is then transformed to a second-order Cauchy-Euler ODE

$$(2n^2r^2 - 4nr^2 + 2r^2)F'' + (3n^2r - 2nr - r)F' + (1+n)F = 0. \tag{5.11}$$

Now solving this equation and reverting back to the original variables, we obtain the following solution of the KP-MEW equation (5.1):

$$u(x,y,t) = \frac{1}{y^{2/(n-1)}}\left(C_1 r^{-\frac{n+1}{2(n-1)}} + C_2 r^{-\frac{1}{n-1}}\right), \tag{5.12}$$

where $r = t/y^2$ and C_1 and C_2 are constants of integration.

5.3 (G'/G)-expansion method

In this section we use the $(G'/G)-$expansion method [9], which was used in Chapter 4, to obtain a few exact solutions of the KP-MEW equation (5.1) for $n = 2$ and $n = 3$.

n=2

Application of the balancing procedure to fourth-order ODE (5.5) yields $M = 2$, so the solution of (5.5) are of the form

$$F(z) = A_0 + A_1\left(\frac{G'(z)}{G(z)}\right) + A_2\left(\frac{G'(z)}{G(z)}\right)^2. \tag{5.13}$$

Substituting (5.13) into (5.5) leads to an overdetermined system of algebraic equations:

$$42\,\alpha\,\lambda\,b^2A_1A_2 - 330\,\beta\,b^2A_2\lambda^2 - 240\,\mu\,\beta\,b^2A_2 + 6\,\gamma\,A_2c^2 + 12\,ac\gamma\,A_2 - 60\,\beta\,\lambda\,b^2A_1$$
$$+16\,\alpha\,\lambda^2b^2{A_2}^2 + 6\,\alpha\,{A_1}^2 - 6\,b^2A_2 + 12\,\alpha\,b^2A_0A_2 + 32\,\mu\,\alpha\,b^2{A_2}^2 + 6\,\gamma\,A_2a^2 = 0,$$
$$2\,\gamma\,A_1ac\lambda\,\mu + 2\,\alpha\,b^2{A_1}^2\mu^2 - 16\,\beta\,b^2A_2\mu^3 - b^2A_1\lambda\,\mu + 2\,\gamma\,A_2a^2\mu^2 + 2\,\gamma\,A_2c^2$$
$$-14\,\beta\,b^2A_2\lambda^2\mu^2 - 8\,\beta\,b^2A_1\lambda\,\mu^2 - 2\,b^2A_2\mu^2 - \beta\,b^2A_1\lambda^3\mu + \gamma\,A_1a^2\lambda\,\mu + \gamma\,A_1c^2\lambda\,\mu$$
$$+4\,\gamma\,A_2ac\mu^2 + 4\,\alpha\,b^2A_0A_2\mu^2 + 2\,\alpha\,b^2A_0A_1\lambda\,\mu = 0.$$
$$4\,\alpha\,b^2A_0A_1 - 10\,\lambda\,b^2A_2 + 36\,\mu\,\alpha\,b^2A_1A_2 + 28\,\mu\,\alpha\,\lambda\,b^2{A_2}^2 + 20\,ac\gamma\,\lambda\,A_2 + 10\,\gamma\,\lambda\,A_2a^2$$
$$+18\,\alpha\,b^2A_1A_2\lambda^2 + 2\,\gamma\,A_1a^2 - 130\,\beta\,b^2A_2\lambda^3 + 2\,\gamma\,A_1c^2 + 10\,\alpha\,\lambda\,b^2{A_1}^2 + 10\,\gamma\,\lambda\,A_2c^2$$
$$+20\,\alpha\,\lambda\,b^2A_0A_2 - 440\,\mu\,\beta\,\lambda\,b^2A_2 - 2\,b^2A_1 - 50\,\beta\,b^2A_1\lambda^2 - 40\,\mu\,\beta\,b^2A_1 + 4\,\gamma\,A_1ac = 0,$$
$$22\,\mu\,\beta\,b^2A_1\lambda^2 + b^2A_1\lambda^2 - 4\,ac\mu\,\gamma\,A_1 - 2\,\alpha\,b^2A_0A_1\lambda^2 + 120\,\beta\,\lambda\,b^2A_2\mu^2 - 2\,\mu\,\gamma\,A_1c^2$$
$$+30\,\mu\,\beta\,b^2A_2\lambda^3 - 6\,\mu\,\alpha\,\lambda\,b^2{A_1}^2 - 12\,\alpha\,b^2A_1A_2\mu^2 - \gamma\,A_1c^2\lambda^2 - 6\,\mu\,\gamma\,\lambda\,A_2a^2 + 2\,\mu\,b^2A_1$$
$$-12\,\mu\,\alpha\,\lambda\,A_0A_2 + 16\,\beta\,b^2A_1\mu^2 + 6\,\mu\,\lambda\,b^2A_2 - 12\,ac\mu\,\gamma\,\lambda\,A_2 - 2\,\mu\,\gamma\,A_1a^2 - 6\,\mu\,\gamma\,\lambda\,A_2c^2$$
$$-\gamma\,A_1a^2\lambda^2 - 2\,ac\gamma\,A_1\lambda^2 + \beta\,b^2A_1\lambda^4 - 4\,\mu\,\alpha\,b^2A_0A_1 = 0,$$
$$15\,\beta\,b^2A_1\lambda^3 - 12\,\alpha\,b^2\mu^2{A_2}^2 - 8\,\alpha\,b^2A_0A_2\lambda^2 + 4\,b^2A_2\lambda^2 + 16\,\beta\,b^2A_2\lambda^4 - 3\,\gamma\,\lambda\,A_1c^2$$
$$-8\,\mu\,\gamma\,A_2a^2 + 60\,\mu\,\beta\,\lambda\,b^2A_1 - 8\,\mu\,\gamma\,A_2c^2 - 4\,\alpha\,\lambda^2b^2{A_1}^2 - 8\,\mu\,\alpha\,b^2{A_1}^2 - 16\,\mu\,\alpha\,b^2A_0A_2$$
$$-30\,\mu\,\alpha\,\lambda\,b^2A_1A_2 - 8\,ac\gamma\,A_2\lambda^2 + 232\,\mu\,\beta\,b^2A_2\lambda^2 - 4\,\gamma\,A_2a^2\lambda^2 - 4\,\gamma\,A_2c^2\lambda^2$$
$$-16\,ac\mu\,\gamma\,A_2 - 6\,ac\gamma\,\lambda\,A_1 + 3\,\lambda\,b^2A_1 + 136\,\beta\,b^2A_2\mu^2 + 8\,\mu\,b^2A_2 - 3\,\gamma\,\lambda\,A_1a^2$$
$$-6\,\alpha\,\lambda\,b^2A_0A_1 = 0,$$
$$20\,\alpha\,{A_2}^2 - 120\,\beta\,A_2 = 0,$$

$$36\,\alpha\,\lambda\,{A_2}^2+24\,\alpha\,A_1A_2-336\,\beta\,\lambda\,A_2-24\,\beta\,A_1=0.$$

Solving this system of algebraic equations with the aid of Maple, we obtain

$$A_0=\frac{b^2\alpha\,\lambda^2A_2+8\,b^2\mu\,\alpha\,A_2-6\,\gamma\,a^2-12\,ac\gamma-6\,\gamma\,c^2+6\,b^2}{12\alpha\,b^2},$$
$$A_1=\frac{6\beta\lambda}{\alpha},$$
$$A_2=\frac{6\beta}{\alpha}.$$

Therefore we have the following three types of travelling wave solutions of the KP-MEW equation (5.1):

When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solution

$$\begin{aligned} u(x,y,t) &= A_0+A_1\left(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\right)\\ &\quad +A_2\left(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\right)^2, \end{aligned} \qquad (5.14)$$

where $z=t-x-\frac{(a+c)}{b}y$, $\delta_1=\frac{1}{2}\sqrt{\lambda^2-4\mu}$, C_1 and C_2 are arbitrary constants.

The profile of the solution (5.14) is given in Figure 5.2.

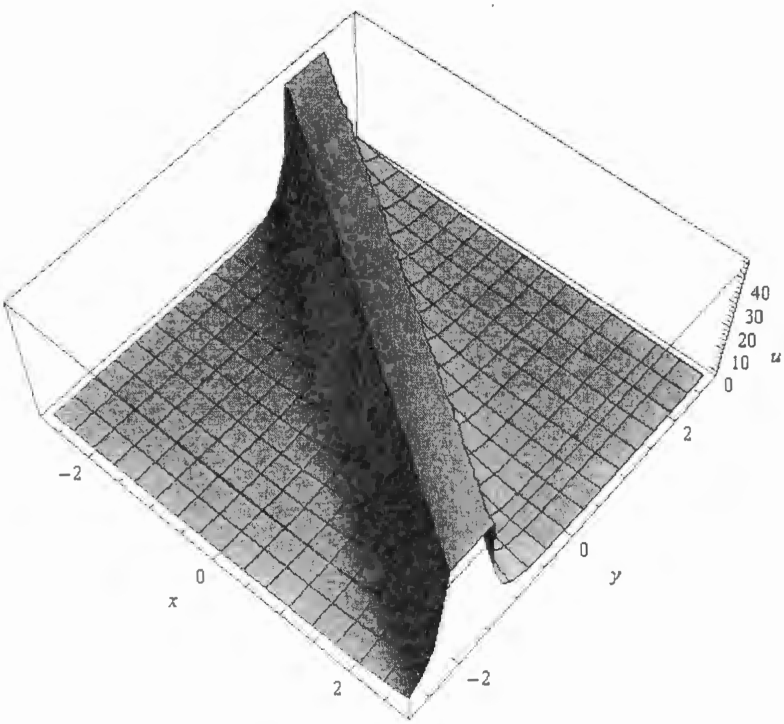

Figure 5.2: Profile of solution (5.14)

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$\begin{aligned} u(x,y,t) &= A_0 + A_1\Big(-\frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 z) + C_2 \cos(\delta_2 z)}{C_1 \cos(\delta_2 z) + C_2 \sin(\delta_2 z)} \Big) \\ &\quad + A_2 \Big(-\frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 z) + C_2 \cos(\delta_2 z)}{C_1 \cos(\delta_2 z) + C_2 \sin(\delta_2 z)} \Big)^2, \end{aligned} \tag{5.15}$$

where $z = t - x - \frac{(a+c)}{b} y$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

The profile of the solution (5.15) is given in Figure 5.3.

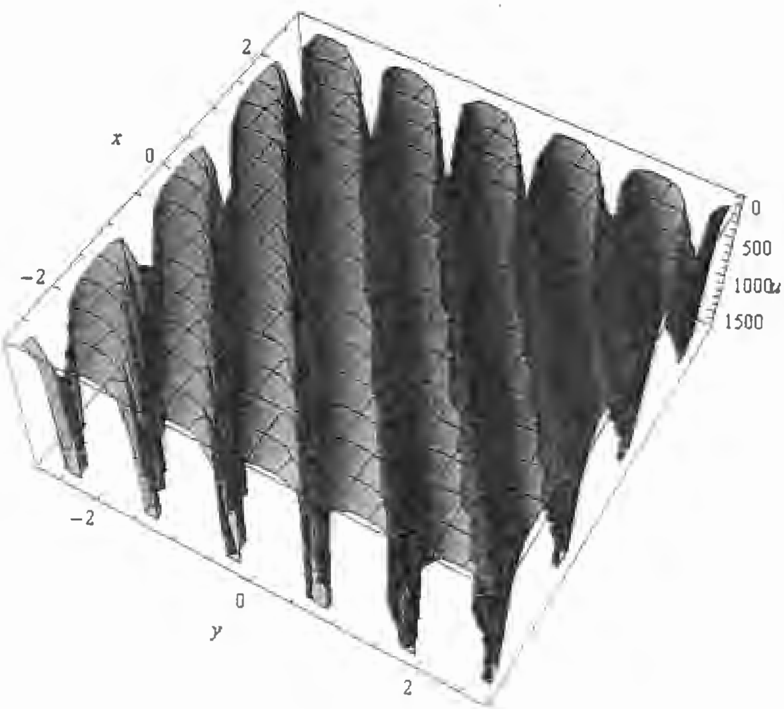

Figure 5.3: Profile of solution (5.15)

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u(x,y,t) = A_0 + A_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + A_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \quad (5.16)$$

where $z = t - x - \frac{(a+c)}{b}y$, C_1 and C_2 are arbitrary constants.

The profile of the solution (5.16) is given in Figure 5.4.

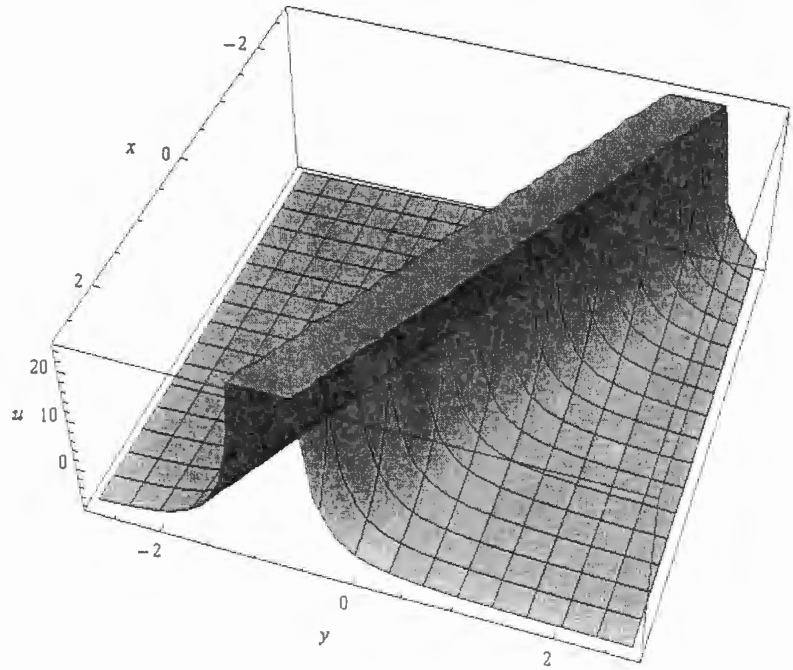

Figure 5.4: Profile of solution (5.16)

n=3

Again the application of the balancing procedure to fourth-order ODE yields $M = 1$, so the solution of (5.5) are of the form

$$F(z) = A_0 + A_1\left(\frac{G'(z)}{G(z)}\right). \tag{5.17}$$

Substituting (5.17) into (5.5) leads to an overdetermined system of algebraic equations:

$$3\,\alpha\, b^2 {A_0}^2 A_1 \lambda\,\mu + 6\,\alpha\, b^2 A_0 {A_1}^2 \mu^2 - \beta\, b^2 A_1 \lambda^3 \mu - 8\,\beta\, b^2 A_1 \lambda\,\mu^2 + \gamma\, A_1 a^2 \lambda\,\mu$$
$$+2\,\gamma\, A_1 ac\lambda\,\mu + \gamma\, A_1 c^2 \lambda\,\mu - b^2 A_1 \lambda\,\mu = 0,$$
$$12\,\alpha\, b^2 A_0 \lambda^2 {A_1}^2 + 15\,\mu\,\alpha\,\lambda\, b^2 {A_1}^3 + 9\,\alpha\,\lambda\, b^2 A_1 {A_0}^2 + 24\,\mu\,\alpha\, b^2 A_0 {A_1}^2 - 15\,\beta\, b^2 A_1 \lambda^3$$
$$-60\,\mu\,\beta\,\lambda\, b^2 A_1 + 3\,\gamma\,\lambda\, A_1 a^2 + 6\, ac\gamma\,\lambda\, A_1 + 3\,\gamma\,\lambda\, A_1 c^2 - 3\,\lambda\, b^2 A_1 = 0,$$
$$9\,\alpha\,\lambda^2 b^2 {A_1}^3 + 30\,\alpha\,\lambda\, b^2 A_0 {A_1}^2 + 18\,\mu\,\alpha\, b^2 {A_1}^3 + 6\,\alpha\, b^2 A_1 {A_0}^2 - 50\,\beta\, b^2 A_1 \lambda^2$$

$$-40\,\mu\,\beta\,b^2A_1 + 2\,\gamma\,A_1a^2 + 4\,ac\gamma\,A_1 + 2\,\gamma\,A_1c^2 - 2\,b^2A_1 = 0,$$

$$3\,\alpha\,b^2A_1\lambda^2{A_0}^2 + 18\,\mu\,\alpha\,\lambda\,b^2A_0{A_1}^2 + 6\,\alpha\,\mu^2b^2{A_1}^3 - \beta\,b^2A_1\lambda^4 + 6\,\mu\,\alpha\,b^2A_1{A_0}^2$$

$$-22\,\mu\,\beta\,b^2A_1\lambda^2 + \gamma\,A_1a^2\lambda^2 + 2\,ac\gamma\,A_1\lambda^2 - 16\,\beta\,b^2A_1\mu^2 + \gamma\,A_1c^2\lambda^2 + 2\,\mu\,\gamma\,A_1a^2$$

$$+4\,ac\mu\,\gamma\,A_1 - A_1b^2\lambda^2 + 2\,\mu\,\gamma\,A_1c^2 - 2\,\mu\,b^2A_1 = 0,$$

$$12\,\alpha\,{A_1}^3 - 24\,\beta\,A_1 = 0,$$

$$21\,\alpha\,\lambda\,{A_1}^3 + 18\,\alpha\,A_0{A_1}^2 - 60\,\beta\,\lambda\,A_1 = 0.$$

Solving the above system of algebraic equations with the aid of Maple, we obtain

$$\beta = \frac{\alpha}{2}A_1^2,$$

$$A_0 = \frac{\lambda\sqrt{-a^2\gamma - 2ac\gamma + b^2 - c^2\gamma}}{\sqrt{\alpha b^2\lambda^2 - 4\alpha b^2\mu}},$$

$$A_1 = \frac{2\sqrt{-a^2\gamma - 2ac\gamma + b^2 - c^2\gamma}}{\sqrt{\alpha b^2\lambda^2 - 4\alpha b^2\mu}},$$

and hence, we have the following two types of travelling wave solutions of the KP-MEW equation (5.1):

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u(x,y,t) \;=\; A_0 + A_1\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right), \qquad (5.18)$$

where $z = t - x - \frac{(a+c)}{b}y$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$u(x,y,t) \;=\; A_0 + A_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right), \qquad (5.19)$$

where $z = t - x - \frac{(a+c)}{b}y$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

5.4 Conservation laws of (5.1)

In this section we construct conservation laws for (5.1). The multiplier method will be used.

The zeroth-order multiplier $\Lambda(t,x,y,u)$ for (5.1) is given by

$$\Lambda = -\frac{y^3}{6\gamma}f_1'(t) + xyf_1(t) - \frac{y^2}{2\gamma}f_2'(t) + xf_2(t) + yf_3(t) + f_4(t), \tag{5.20}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ are arbitrary functions of t. Corresponding to the above multiplier we have the following four conserved vectors of (5.1):

$$\begin{aligned}
T_1^t &= \frac{1}{24\gamma}\Big\{ -12\gamma yf_1(t)u - 6\beta\gamma yf_1(t)u_{xx} + 6\beta\gamma xyf_1(t)u_{xxx} + 12\gamma xyf_1(t)u_x \\
&\quad -\beta y^3 f_1'(t)u_{xxx} - 2y^3 f_1'(t)u_x\Big\},\\
T_1^x &= -\frac{y}{24\gamma u}\Big\{4\alpha ny^2 f_1'(t)u_xu^n - 24\alpha\gamma nxf_1(t)u_xu^n - \beta y^2 f_1''(t)u_{xx}u \\
&\quad +3\beta y^2 f_1'(t)u_{txx}u + 2y^2 f_1'(t)u_tu - 12\beta\gamma f_1'(t)u_xu + 6\beta\gamma xf_1'(t)u_{xx}u \\
&\quad +12\beta\gamma f_1(t)u_{tx}u - 18\beta\gamma xf_1(t)u_{txx}u - 12\gamma xf_1(t)u_tu + 24\alpha\gamma f_1(t)u^{n+1} \\
&\quad -2y^2 f_1''(t)u^2 + 12\gamma xf_1'(t)u^2\Big\},\\
T_1^y &= \frac{1}{6}\Big\{3y^2 f_1'(t)u - 6\gamma xf_1(t)u + 6\gamma xyf_1(t)u_y - y^3 f_1'(t)u_y\Big\};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{8\gamma}\Big\{ -4\gamma f_2(t)u - 2\beta\gamma f_2(t)u_{xx} + 2\beta\gamma xf_2(t)u_{xxx} + 4\gamma xf_2(t)u_x - \beta y^2 f_2'(t)u_{xxx} \\
&\quad -2y^2 f_2'(t)u_x\Big\},\\
T_2^x &= -\frac{1}{8\gamma u}\Big\{4\alpha ny^2 f_2'(t)u_xu^n - 8\alpha\gamma nxf_2(t)u_xu^n - \beta y^2 f_2''u_{xx}u + 3\beta y^2 f_2'u_{txx}u \\
&\quad +2y^2 f_2'(t)u_tu - 4\beta\gamma f_2'(t)u_xu + 2\beta\gamma xf_2'(t)u_{xx}u + 4\beta\gamma f_2(t)u_{tx}u - 6\beta\gamma xf_2(t)u_{txx}u \\
&\quad -4\gamma xf_2(t)u_tu + 8\alpha\gamma f_2(t)u^{n+1} - 2y^2 f_2''(t)u^2 + 4\gamma xf_2'(t)u^2\Big\},\\
T_2^y &= \frac{1}{2}\Big\{2yf_2'(t)u + 2\gamma xf_2(t)u_y - y^2 f_2'(t)u_y\Big\};
\end{aligned}$$

$$\begin{aligned}
T_3^t &= \frac{1}{4}\Big\{\beta yf_3(t)u_{xxx} + 2yf_3(t)u_x\Big\},\\
T_3^x &= -\frac{y}{4u}\Big\{ -4\alpha nf_3(t)u_xu^n + \beta f_3'(t)u_{xx}u - 3\beta f_3(t)u_{txx}u - 2f_3(t)u_tu + 2f_3'(t)u^2\Big\},
\end{aligned}$$

$$
\begin{aligned}
T_3^y &= \gamma y f_3(t) u_y - \gamma f_3(t) u; \\
\\
T_4^t &= \frac{1}{4}\Big\{\beta f_4(t) u_{xxx} + 2 f_4(t) u_x\Big\}, \\
T_4^x &= -\frac{1}{4u}\Big\{-4\alpha n f_4(t) u_x u^n + \beta f_4'(t) u_{xx} u - 3\beta f_4(t) u_{txx} u - 2 f_4(t) u_t u + 2 f_4'(t) u^2\Big\}, \\
T_4^y &= \gamma f_4(t) u_y.
\end{aligned}
$$

Remark. The presence of the arbitrary functions in the multiplier, leads to a family of infinitely many conservation laws for (5.1).

5.5 Concluding remarks

In this chapter we obtained the solutions of a generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation by employing the Lie group analysis and the (G'/G)-expansion method. The solutions obtained were solitary waves and non-topological soliton. The conservation laws for the underlying equation were derived by using the multiplier method.

Chapter 6

On the solutions and conservation laws of the coupled DSSH system

6.1 Introduction

In this chapter we study the coupled Drinfeld-Sokolov-Satsuma-Hirota (DSSH) system

$$\begin{aligned} u_t - 6uu_x + u_{xxx} - 6v_x &= 0, \\ v_t - 2v_{xxx} + 6uv_x &= 0. \end{aligned} \tag{6.1}$$

This system was introduced independently by Drinfeld and Sokolov [60], and by Satsuma and Hirota [61]. The coupled DSSH system [60] was given as one of numerous examples of nonlinear equations possessing Lax pairs of a special form. Also, the coupled DSSH system [61] was found as a special case of the four-reduction of the KP hierarchy, and its explicit one-soliton solution was constructed. Gürses and Karasu [67] found a recursion operator and a bi-Hamiltonian structure for (6.1). Wazwaz [59] used three distinct methods, namely, the Cole-Hopf transformation, Hirota's bilinear and the Exp-function methods and obtained solitons, multiple soliton solutions, multiple singular soliton solutions, and plane periodic solutions. Zheng [87] used the $(G'/G)-$expansion method and obtained travelling wave solutions of (6.1).

Here we perform symmetry reductions of (6.1), which are based on the optimal systems of one-dimensional subalgebras. The simplest equation method and Jacobi elliptic function method [88] is later employed to obtain some exact solutions of (6.1). In addition to this conservation laws are derived for (6.1) using the new conservation theorem [74] and the multiplier method [73].

This work has been accepted and to appear in [89].

6.2 Symmetry analysis of (6.1)

The symmetry group of the coupled DSSH system (6.1) will be generated by the vector field of the form

$$X=\xi^1(t,x,u,v)\frac{\partial}{\partial t}+\xi^2(t,x,u,v)\frac{\partial}{\partial x}+\eta^1(t,x,u,v)\frac{\partial}{\partial u}+\eta^2(t,x,u,v)\frac{\partial}{\partial v}.$$

The application of the third prolongation $\mathrm{pr}^{(3)}X$ to (6.1) results in an overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\eta^1_v=0,\ \xi^1_x=0,\ \xi^2_v=0,\ \xi^1_v=0,\ \xi^1_u=0,\ \xi^2_u=0,\ \eta^2_u=0,\ \eta^2_{vv}=0,\ \eta^1_{uu}=0,\\
&\eta^1_{xu}-\xi^2_{xx}=0,\ -\eta^2_{xv}+\xi^2_{xx}=0,\ -\xi^1_t+3\,\xi^2_x=0,\ -3\,\xi^2_x+\xi^1_t=0,\\
&6\,u\eta^2_x+\eta^2_t-2\,\eta^2_{xxx}=0,\ -6\,u\eta^1_x+\eta^1_t+\eta^1_{xxx}-6\,\eta^2_x=0,\ -\xi^1_t+\xi^2_x+\eta^1_u-\eta^2_v=0,\\
&-6\,\xi^1_tu+6\,u\xi^2_x-6\,\eta^1+3\,\eta^1_{xxu}-\xi^2_t-\xi^2_{xxx}=0,\ 6\,\xi^1_tu-6\,u\xi^2_x+6\,\eta^1-6\,\eta^2_{xxv}-\xi^2_t\\
&+2\,\xi^2_{xxx}=0.
\end{aligned}$$

The general solution of these equations with the aid of Maple is given by

$$\xi^1(t,x,u,v)=C_1+3tC_4,$$

$$\xi^2(t, x, u, v) = C_2 + xC_4,$$
$$\eta^1(t, x, u, v) = -2C_4u,$$
$$\eta^2(t, x, u, v) = C_3 - 4vC_4.$$

where C_i, $i = 1, \cdots, 4$ are arbitrary constants. The above general solution contains four arbitrary constants. Hence the infinitesimal symmetries of (6.1) form the four-dimensional Lie algebra spanned by the following linearly independent operators:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial v},\ X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}.$$

6.2.1 One-dimensional optimal system of subalgebras

The commutator table of the Lie point symmetries of the equation (6.1) and the adjoint representations of the symmetry group of (6.1) on its Lie algebra are given in Table 1 and Table 2, respectively. The Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for equation (6.1).

Table 1. Commutator table of the Lie algebra of equation (6.1)

	X_1	X_2	X_3	X_4
X_1	0	0	0	$3X_2$
X_2	0	0	0	X_2
X_3	0	0	0	$-2X_3$
X_4	$-3X_2$	$-X_2$	$2X_3$	0

Table 2. Adjoint table of the Lie algebra of equation (6.1)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$-3\varepsilon X_1 + X_4$
X_2	X_1	X_2	X_3	$-\varepsilon X_2 + X_4$
X_3	X_1	X_2	X_3	$4\varepsilon X_3 + X_4$
X_4	$3e^{\varepsilon} X_1$	$e^{\varepsilon} X_2$	$e^{-4\varepsilon} X_3$	X_4

From Tables 1 and 2 and following [18] one can obtain an optimal system of one-dimensional subalgebras given by $\{X_4, bX_1 + cX_2 + dX_3\}$, where $b, c, d = 0, \pm 1$.

6.2.2 Symmetry reductions of (6.1)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions

Case 1. X_4

The operator X_4 gives rise to the group-invariant solution

$$u = t^{-\frac{2}{3}} F(z), \quad v = t^{-\frac{4}{3}} G(z), \tag{6.2}$$

where $z = xt^{-1/3}$ is an invariant of the symmetry X_4. Substitution of (6.2) into (6.1) results in the system of ordinary differential equations (ODEs) where F and G satisfy

$$4G(z) + 6G'''(z) - 18F(z)G'(z) + zG'(z) = 0,$$
$$2F(z) - 3F'''(z) + zF(z) + 18F(z)F'(z) + 18G(z) = 0.$$

Case 2. $bX_1 + cX_2 + dX_3; \quad b, c, d = 0, \pm 1$

The symmetry $bX_1 + cX_2 + dX_3$ gives rise to the group-invariant solution

$$u = F(z), \quad v = G(z) + td/b, \tag{6.3}$$

where $z = (bx - ct)/b$ is an invariant of the symmetry $bX_1 + cX_2 + dX_3$. The insertion of (6.3) into (6.1) results in the system of ODEs

$$6bF(z)F'(z) + cF'(z) - bF'''(z) + 6bG'(z) = 0,$$
$$cG'(z) - 6bF(z)G'(z) + 2bG'''(z) - d = 0.$$

6.3 Exact solutions of (6.1) using simplest equation method

Taking the linear combination of the translation symmetries, viz., $X_1 + \lambda X_2$ and solving the corresponding Lagrange system for the symmetry $X_1+\lambda X_2$, one obtains an invariant $z = x - \lambda t$, and the group-invariant solution of the form

$$u = F(z), \quad v = G(z), \tag{6.4}$$

where the functions F and G satisfy

$$F'''(z) - \lambda F'(z) - 6F(z)F'(z) - 6G'(z) = 0, \tag{6.5a}$$
$$6F(z)G'(z) - 2G'''(z) - \lambda G'(z) = 0. \tag{6.5b}$$

Now we use the simplest equation method, which was used in Chapter 2, to solve the system (6.5) and as a result we obtain the exact solutions of our coupled DSSH system (6.1).

Let us consider the solutions of (6.5) in the form

$$F(z) = \sum_{i=0}^{M} \mathcal{A}_i (H(z))^i, \quad G(z) = \sum_{i=0}^{N} \mathcal{B}_i (H(z))^i, \tag{6.6}$$

where $H(z)$ satisfies the Bernoulli or Riccati equation, M and N are positive integers that can be determined by a balancing procedure and $\mathcal{A}_i$'s and $\mathcal{B}_i$'s are parameters to be determined.

6.3.1 Solutions of (6.1) using the Bernoulli equation as the simplest equation

The balancing procedure gives $M = 2$ and $N = 4$ and hence the solutions of (6.5) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2, \tag{6.7a}$$

$$G(z) = B_0 + B_1 H + B_2 H^2 + B_3 H^3 + B_4 H^4. \tag{6.7b}$$

Substituting (6.7) into (6.5) and making use of the Bernoulli equation and then equating the coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_i $(i = 0, 1, 2)$ and B_i $(i = 0, 1, 2, 3, 4)$:

$$24\, bA_2B_4 - 240\, b^3B_4 = 0,$$

$$24\, b^3A_2 - 12\, bA_2{}^2 - 24\, bB_4 = 0,$$

$$6\, aA_0B_1 - 2\, a^3B_1 - a\nu\, B_1 = 0,$$

$$a^3A_1 - a\nu\, A_1 - 6\, aA_0A_1 - 6\, aB_1 = 0,$$

$$24\, aA_2B_4 - 600\, ab^2B_4 - 120\, b^3B_3 + 24\, bA_1B_4 + 18\, bA_2B_3 = 0,$$

$$54\, ab^2A_2 + 6\, b^3A_1 - 12\, aA_2{}^2 - 18\, bA_1A_2 - 24\, aB_4 - 18\, bB_3 = 0,$$

$$6\, aA_1B_1 - 16\, a^3B_2 - 14\, a^2bB_1 - 2\, a\nu\, B_2 + 12\, aA_0B_2 - b\nu\, B_1 + 6\, bA_0B_1 = 0,$$

$$38\, a^2bA_2 + 12\, ab^2A_1 - 18\, aA_1A_2 - 2\, b\nu\, A_2 - 12\, bA_0A_2 - 6\, bA_1{}^2 - 18\, aB_3$$
$$-12\, bB_2 = 0,$$

$$8\, a^3A_2 + 7\, a^2bA_1 - 2\, a\nu\, A_2 - 12\, aA_0A_2 - 6\, aA_1{}^2 - b\nu\, A_1 - 6\, bA_0A_1 - 12\, aB_2$$
$$-6\, bB_1 = 0,$$

$$24\, bA_0B_4 - 488\, a^2bB_4 - 288\, ab^2B_3 - 48\, b^3B_2 + 24\, aA_1B_4 + 18\, aA_2B_3 - 4\, b\nu\, B_4$$
$$+18\, bA_1B_3 + 12\, bA_2B_2 = 0,$$

$$12\, bA_0B_2 - 54\, a^3B_3 - 76\, a^2bB_2 - 24\, ab^2B_1 - 3\, a\nu\, B_3 + 18\, aA_0B_3 + 12\, aA_1B_2$$
$$+6\, aA_2B_1 - 2\, b\nu\, B_2 + 6\, bA_1B_1 = 0,$$

$$12\,aA_2B_2 - 128\,a^3B_4 - 222\,a^2bB_3 - 108\,ab^2B_2 - 12\,b^3B_1 - 4\,a\nu\,B_4 + 24\,aA_0B_4$$
$$+18\,aA_1B_3 - 3\,b\nu\,B_3 + 18\,bA_0B_3 + 12\,bA_1B_2 + 6\,bA_2B_1 = 0.$$

Solving the resultant system of algebraic equations with the aid of Maple, one possible set of values of A_i and B_i are

$$\lambda = \frac{7}{2}a^2,\ A_0 = \frac{11}{12}a^2,\ A_1 = 10ab,\ A_2 = 10b^2,$$
$$B_1 = -\frac{40}{3}a^3b,\ B_2 = -\frac{160}{3}a^2b^2,\ B_3 = -80b^3a,\ B_4 = -40b^4.$$

As a result, a solution of (6.1) is

$$u(t,x) = A_0 + A_1a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} + A_2a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \tag{6.8a}$$

$$v(t,x) = B_0 + B_1a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} + B_2a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2 + B_3a^3\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^3 + B_4a^4\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^4, \tag{6.8b}$$

where $z = x - \lambda t$. The profile of the solution (6.8) is given in Figure 6.1.

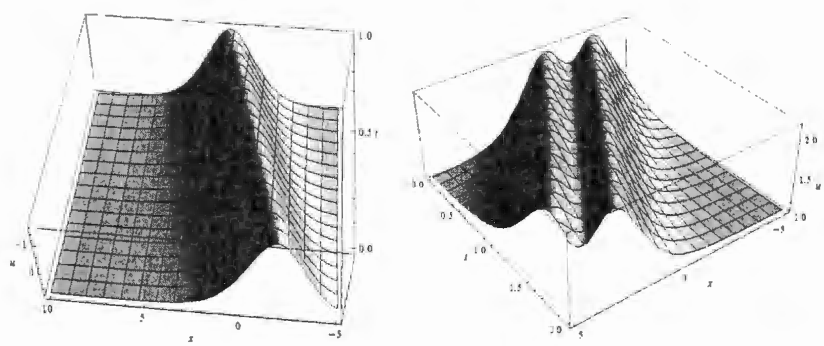

Figure 6.1: Profile of (6.8)

6.3.2 Solutions of (6.1) using Riccati equation as the simplest equation

In this case the balancing procedure also gives same values of M and N, i.e., $M = 2$ and $N = 4$. Thus the solutions of (6.5) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2, \tag{6.9a}$$

$$G(z) = B_0 + B_1 H + B_2 H^2 + B_3 H^3 + B_4 H^4. \tag{6.9b}$$

Substituting (6.9) into (6.5) and making use of the Riccati equation, we obtain an algebraic system of equations in terms of A_i and B_i:

$$24\,aA_2B_4 - 240\,a^3B_4 = 0,$$

$$24\,a^3A_2 - 12\,aA_2{}^2 - 24\,aB_4 = 0,$$

$$24\,bA_2B_4 - 120\,a^3B_3 - 600\,a^2bB_4 + 24\,aA_1B_4 + 18\,aA_2B_3 = 0,$$

$$6\,a^3A_1 + 54\,a^2bA_2 - 18\,aA_1A_2 - 12\,bA_2{}^2 - 18\,aB_3 - 24\,bB_4 = 0,$$

$$2\,ac^2A_1 + b^2cA_1 + 6\,bc^2A_2 - c\nu\,A_1 - 6\,cA_0A_1 - 6\,cB_1 = 0,$$

$$6\,cA_0B_1 - 4\,ac^2B_1 - 2\,b^2cB_1 - 12\,bc^2B_2 - 12\,c^3B_3 - c\nu\,B_1 = 0,$$

$$8\,abcA_1 + 16\,ac^2A_2 + b^3A_1 + 14\,b^2cA_2 - b\nu\,A_1 - 6\,bA_0A_1 - 2\,c\nu\,A_2 - 12\,cA_0A_2$$
$$-6\,cA_1{}^2 - 6\,bB_1 - 12\,cB_2 = 0,$$

$$12\,a^2bA_1 + 40\,a^2cA_2 + 38\,ab^2A_2 - 2\,a\nu\,A_2 - 12\,aA_0A_2 - 6\,aA_1{}^2 - 12\,aB_2$$
$$-18\,bA_1A_2 - 12\,cA_2{}^2 - 18\,bB_3 - 24\,cB_4 = 0,$$

$$18\,bA_2B_3 - 48\,a^3B_2 - 288\,a^2bB_3 - 496\,a^2cB_4 - 488\,ab^2B_4 - 4\,a\nu\,B_4 + 24\,aA_0B_4$$
$$+18\,aA_1B_3 + 12\,aA_2B_2 + 24\,bA_1B_4 + 24\,cA_2B_4 = 0,$$

$$6\,cA_1B_1 - 16\,abcB_1 - 32\,ac^2B_2 - 2\,b^3B_1 - 28\,b^2cB_2 - 72\,bc^2B_3 - 48\,c^3B_4 - b\nu\,B_1$$
$$+6\,bA_0B_1 - 2\,c\nu\,B_2 + 12\,cA_0B_2 = 0,$$

$$8\,a^2cA_1 + 7\,ab^2A_1 + 52\,abcA_2 + 8\,b^3A_2 - a\nu\,A_1 - 6\,aA_0A_1 - 2\,b\nu\,A_2 - 12\,bA_0A_2$$
$$-6\,bA_1{}^2 - 18\,cA_1A_2 - 6\,aB_1 - 12\,bB_2 - 18\,cB_3 = 0,$$

$$12\,bA_0B_2 - 16\,a^2cB_1 - 14\,ab^2B_1 - 104\,abcB_2 - 120\,ac^2B_3 - 16\,b^3B_2 - 114\,b^2cB_3$$
$$-216\,bc^2B_4 - a\nu\,B_1 + 6\,aA_0B_1 - 2\,b\nu\,B_2 + 6\,bA_1B_1 - 3\,c\nu\,B_3 + 18\,cA_0B_3$$
$$+12\,cA_1B_2 + 6\,cA_2B_1 = 0,$$
$$6\,aA_2B_1 - 12\,a^3B_1 - 108\,a^2bB_2 - 228\,a^2cB_3 - 222\,ab^2B_3 - 784\,abcB_4 - 128\,b^3B_4$$
$$-3\,a\nu\,B_3 + 18\,aA_0B_3 + 12\,aA_1B_2 - 4\,b\nu\,B_4 + 24\,bA_0B_4 + 18\,bA_1B_3 + 12\,bA_2B_2$$
$$+24\,cA_1B_4 + 18\,cA_2B_3 = 0,$$
$$12\,cA_2B_2 - 24\,a^2bB_1 - 80\,a^2cB_2 - 76\,ab^2B_2 - 336\,abcB_3 - 304\,ac^2B_4 - 54\,b^3B_3$$
$$-296\,b^2cB_4 - 2\,a\nu\,B_2 + 12\,aA_0B_2 + 6\,aA_1B_1 - 3\,b\nu\,B_3 + 18\,bA_0B_3 + 12\,bA_1B_2$$
$$+6\,bA_2B_1 - 4\,c\nu\,B_4 + 24\,cA_0B_4 + 18\,cA_1B_3 = 0.$$

Solving the resultant system, one possible set of values are

$$\lambda = -\frac{7}{200}\left(\frac{400a^3c - A_1^2}{a^2}\right),\ A_0 = \frac{7600a^3c + 11A_1^2}{1200a^2},\ A_1 = 10ab,\ A_2 = 10a^2,$$
$$B_1 = -\frac{A_1\,(200a^3c + A_1^2)}{75a^2},\ B_2 = -\frac{8}{15}A_1^2 - \frac{80}{3}a^3c,\ B_3 = -8a^2A_1,\ B_4 = -40a^4.$$

Hence solutions of (6.1) are

$$u(t,x) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} + A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \tag{6.10a}$$

$$\begin{aligned} v(t,x) = B_0 &+ B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \\ &+ B_3\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^3 \\ &+ B_4\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^4 \end{aligned} \tag{6.10b}$$

and

$$u(t,x) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$$

$$+A_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \quad (6.11a)$$

$$v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$$

$$+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2$$

$$+B_3\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^3$$

$$+B_4\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^4, \quad (6.11b)$$

where $z = x - \lambda t$.

6.3.3 Solutions of (6.1) in terms of Jacobi elliptic functions

We now present exact solutions of the coupled DSSH system (6.1) that are expressed in Jacobi elliptic functions. The cosine-amplitude function, $\operatorname{cn}(z|\omega)$ and the sine-amplitude function, $\operatorname{sn}(z|\omega)$ satisfy the first-order differential equations

$$H'(z) = -\left\{\left(1-H^2(z)\right)\left(1-\omega+\omega H^2(z)\right)\right\}^{\frac{1}{2}} \quad (6.12)$$

and

$$H'(z) = \left\{\left(1-H^2(z)\right)\left(1-\omega H^2(z)\right)\right\}^{\frac{1}{2}}, \quad (6.13)$$

respectively [88].

Treating the above first-order ODEs as our simplest equations and then proceeding as before, we obtain the cnoidal and snoidal wave solutions that given by

$$\begin{aligned} u(t,x) &= A_0 + A_1\operatorname{cn}(z|\omega) + A_2\operatorname{cn}^2(z|\omega), && (6.14a)\\ v(t,x) &= B_0 + B_1\operatorname{cn}(z|\omega) + B_2\operatorname{cn}^2(z|\omega) \\ &\quad + B_3\operatorname{cn}^3(z|\omega) + B_4\operatorname{cn}^4(z|\omega), && (6.14b)\end{aligned}$$

where

$$\lambda = 42k - 280\,\omega + 140,\ A_0 = k,\ A_1 = -10\,\omega,\ A_2 = 0,$$
$$B_1 = 0,\ B_2 = 80k\omega - 480\,\omega^2 + 240\,\omega,\ B_3 = 0,\ B_4 = -40\,\omega^2,$$
$$k \quad \text{is any root of} \quad 3k^2 + (-40\omega + 20)k + 133\omega^2 - 133\omega + 33\omega = 0$$

and

$$\begin{aligned} u(t,x) &= A_0 + A_1\mathrm{sn}(z|\omega) + A_2\mathrm{sn}^2(z|\omega), \\ v(t,x) &= B_0 + B_1\mathrm{sn}(z|\omega) + B_2\mathrm{sn}^2(z|\omega) \\ &\quad + B_3\mathrm{sn}^3(z|\omega) + B_4\mathrm{sn}^4(z|\omega), \end{aligned}$$

where

$$\lambda = 42k + 140\,\omega + 140,\ A_0 = k(\nu + 4\omega + 4),\ A_1 = 0,\ A_2 = 10\omega,$$
$$B_1 = 0,\ B_2 = -80k\omega - 240\,\omega^2 - 240\,\omega,\ B_3 = 0,\ B_4 = -40\,\omega^2,$$
$$k \quad \text{is any root of} \quad 3k^2 + (20\omega + 20)k + 33\omega^2 + 67\omega + 33 = 0$$

and $z = x - \nu t$. The profile of the solution (6.14) is given in Figure 6.2.

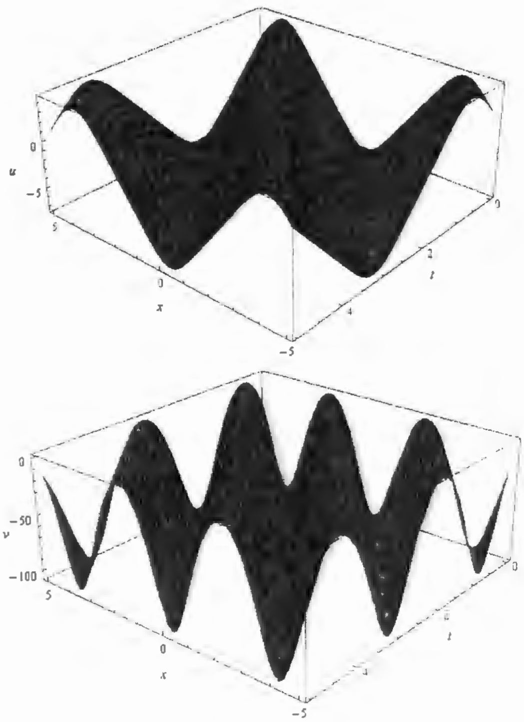

Figure 6.2: Profile of cniodal waves (6.14)

6.4 Conservation laws of (6.1)

In this section we construct conservation laws for the coupled Drinfeld-Sokolov-Satsuma-Hirota system (6.1). The new conservation theorem due to Ibragimov [74] and the multiplier method [73] will be used.

6.4.1 Construction of conservation laws using the new conservation theorem

In this subsection we construct conservation laws for (6.1), by applying the new conservation theorem.

The coupled DSSH system together with its adjoint equation are given by

$$
\begin{aligned}
E_{\alpha_1} &\equiv u_t - 6uu_x + u_{xxx} - 6v_x = 0,\\
E_{\alpha_2} &\equiv v_t - 2v_{xxx} + 6uv_x = 0,
\end{aligned}
\tag{6.15}
$$

$$
\begin{aligned}
E^*_{\alpha_1} &\equiv -p_t + 6up_x - p_{xxx} + 6qv_x = 0,\\
E^*_{\alpha_2} &\equiv -q_t - 6uq_x + 2q_{xxx} + 6p_x - 6qu_x = 0.
\end{aligned}
\tag{6.16}
$$

One can easily verify that the third-order Lagrangian for the system of equations (6.15) and (6.16) is given by

$$L = p(u_t - 6uu_x + u_{xxx} - 6v_x) + q(v_t - 2v_{xxx} + 6uv_x). \tag{6.17}$$

We recall that the coupled DSSH system admits the following four Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial v},\ X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}.$$

Thus we have the following four cases:

(i) For the Lie point symmetry $X_1 = \partial_t$ the corresponding Lie characteristic functions are $W^1 = -u_t$ and $W^2 = -v_t$. Thus, by using the Ibragimov theorem [74], the components of the conserved vector are given by

$$
\begin{aligned}
T_1^t &= -6pu_xu + pu_{xxx} - 6pv_x + 6quv_x - 2qv_{xxx},\\
T_1^x &= 6puu_t - pu_{txx} + 6pv_t - 6quv_t + 2qv_{txx} - u_tp_{xx} + p_xu_{tx} + 2v_tq_{xx} - 2q_xv_{tx}.
\end{aligned}
$$

(ii) The Lie point symmetry $X_2 = \partial_x$ has the Lie characteristic functions that are given by $W^1 = -u_x$ and $W^2 = -v_x$. Hence by the application of Ibragimov theorem [74], the conserved vector (T_2^t, T_2^x) is given by

$$T_2^t = -pu_x - qv_x,$$

$$T_2^x = pu_t + qv_t - u_x p_{xx} + p_x u_{xx} + 2v_x q_{xx} - 2q_x v_{xx}.$$

(iii) The symmetry generator $X_3 = \partial_v$ has the Lie characteristic functions given by $W^1 = 0$ and $W^2 = 1$ and hence in this case one can obtain the conserved vector whose components are

$$\begin{aligned} T_3^t &= q, \\ T_3^x &= -6p + 6uq - 2q_{xx}. \end{aligned}$$

(iv) Finally, we consider the symmetry generator $X_4 = -3t\partial_t - x\partial_x + 2u\partial_u + 4v\partial_v$, which has the Lie characteristic functions $W^1 = 2u + 3tu_t + xu_x$ and $W^2 = 4v + 3tv_t + xv_x$. By invoking Ibragimov theorem [74], the components of conserved vector are given by

$$\begin{aligned} T_4^t &= 18tpuu_x + xpu_x - 3tpu_{xxx} + 18tpv_x - 18tquv_x + xqv_x + 6tqv_{xxx} + 2pu + 4qv, \\ T_4^x &= 2up_{xx} - 18tpuu_t + 4pu_{xx} - xpu_t + 3tpu_{txx} - 18tpv_t + 18tquv_t - 8vq_{xx} - 12qv_{xx} \\ &\quad -xqv_t - 6tqv_{txx} - 12u^2p - 24pv + 24quv + 3tu_tp_{xx} - 3tp_xu_{tx} - 3p_xu_x + xu_xp_{xx} \\ &\quad -xp_xu_{xx} - 6tv_tq_{xx} + 6tq_xv_{tx} + 10q_xv_x - 2xv_xq_{xx} + 2xq_xv_{xx}. \end{aligned}$$

6.4.2 Construction of conservation laws using the multiplier method

Here we use the multiplier method [73] to construct conservation laws for the coupled DSSH system (6.1). The second-order multipliers, $\Lambda_1 = \Lambda_1(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ and $\Lambda_2 = \Lambda_2(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ are given by

$$\begin{aligned} \Lambda_1 &= \frac{1}{2}\left(3u^2 - u_{xx} + 2v\right)C_2 + C_1tu + C_3u + \frac{1}{6}C_1x, \\ \Lambda_2 &= C_2u + C_1t + C_3, \end{aligned}$$

where C_i, $i = 1, \cdots, 3$ are arbitrary constants. Corresponding to the above multipliers we obtain the following three local conserved vectors of (6.1):

$$\begin{aligned} T_1^t &= \frac{1}{6}\Big\{3tu^2 + xu + 6tv\Big\}, \\ T_1^x &= \frac{1}{6}\Big\{6tu_{xx}u - 12tu^3 - 3xu^2 - 6xv - 3tu_x^2 - 12tv_{xx} - u_x + xu_{xx}\Big\}; \end{aligned}$$

$$\begin{aligned} T_2^t &= \frac{1}{4}\Big\{-u_{xx}u + 4uv + 2u^3\Big\}, \\ T_2^x &= \frac{1}{4}\Big\{-8v_{xx}u + 4u_{xx}v + 6u_{xx}u^2 + uu_{tx} - 12u^2v - 9u^4 - 12v^2 \\ &\quad -u_tu_x + 8u_xv_x - u_{xx}^2\Big\}; \end{aligned}$$

$$\begin{aligned} T_3^t &= \frac{1}{2}\Big\{u^2 + 2v\Big\}, \\ T_3^x &= \frac{1}{2}\Big\{2u_{xx}u - 4u^3 - u_x^2 - 4v_{xx}\Big\}. \end{aligned}$$

Remark 1. It should be noted that higher order conservation laws of (6.1) can be computed by increasing the order of the multipliers.

6.5 Concluding remarks

In this paper chapter we obtained the solutions of the Drinfeld-Sokolov-Satsuma-Hirota equation by employing Lie group analysis together with the simplest and Jacobi elliptic equation methods. Also symmetry reductions were obtained based on the optimal systems of one-dimensional subalgebras. The exact solutions obtained were travelling wave solutions, cnoidal and snoidal wave solutions. Furthermore, the conservation laws for the underlying equation were derived by using two different approaches; namely the new conservation theorem and the multiplier method.

Chapter 7

Explicit solutions and conservation laws of a (2+1)-dimensional KP-Joseph-Egri equation with power law nonlinearity

7.1 Introduction

The purpose of this chapter is to study the (2+1)-dimensional Kadomtsev-Petviashvili-Joseph-Egri (KP-JE) equation with power law nonlinearity that is given by

$$(u_t + u_x + \alpha u^n u_x + u_{xtt})_x + \beta u_{yy} = 0, \tag{7.1}$$

where α and β are real-valued constants with $n > 0$. Taghizadeh [64] used the cosine-function method and the (G'/G)-expansion method for finding solitary waves and periodic solutions of (7.1).

Here Lie group analysis in conjunction with the Exp-function method [14] is employed to obtain some exact solutions of (7.1). In addition to this conservation laws will be derived for (7.1) using the multiplier method [73].

7.2 Symmetry analysis

In this section we first calculate the Lie point symmetries of (7.1) and latter use them to construct exact solutions.

7.2.1 Lie point symmetries of (7.1)

The symmetries of KP-JE equation with power law nonlinearity (7.1) will be generated by the vector field of the form

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u},$$

where ξ^i, $i = 1,2,3$ and η depend on x, y, t and u. Applying the fourth prolongation $\mathrm{pr}^{(4)}X$ to (7.1) we obtain an overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^2_u = 0,\ \xi^2_t = 0,\ \xi^2_x = 0,\ \xi^1_u = 0,\ \xi^1_t = 0,\ \xi^1_y = 0,\ \xi^3_u = 0,\ \xi^3_x = 0,\ \xi^3_y = 0,\\
&\xi^1_{xx} = 0,\ \eta_{uu} = 0,\ \eta_{xu} = 0,\ 2\,\eta_{tu} - \xi^3_{tt} = 0,\ \xi^2_{yy} - 2\,\eta_{yu} = 0,\\
&2\,\alpha\, n u^n \eta_x + u\eta_{tu} = 0,\ \xi^2_y + \xi^1_x + \xi^3_t = 0,\ 2\,\xi^2_y - \xi^1_x - \xi^3_t = 0,\\
&\alpha u^n \eta_{xx} + \beta\eta_{yy} + \eta_{xx} + \eta_{xxtt} + \eta_{xt} = 0,\ n\eta + u\eta_u - 2\,u\xi^1_x + 2\,u\xi^2_y - \eta = 0,\\
&\alpha\, n u^n \eta - 2\,\alpha\, u^{n+1}\xi^1_x - 2\,u\xi^1_x + 2\,\alpha\, u^{n+1}\xi^2_y + 2\,u\xi^2_y + u\eta_{ttu} = 0.
\end{aligned}$$

Solving this system one obtains the following three symmetries.

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_t.$$

7.2.2 Exact solutions

First of all we utilize the point symmetry $X = X_1+X_2+X_3$ and reduce the KP-JE equation with power law nonlinearity (7.1) to a PDE in two independent variables. Solving the corresponding Lagrange's system of X, one obtains the following three invariants:

$$f = t - y, \quad g = x - y, \quad \theta = u. \tag{7.2}$$

The variable θ is now considered as the new dependent variable whereas f and g are treated as new independent variables. The KP-JE equation (7.1) then transforms to

$$\alpha n\theta^{n-1}\theta_g^2 + \alpha\theta^n\theta_{gg} + \beta\theta_{ff} + (2\beta+1)\theta_{fg} + (\beta+1)\theta_{gg} + \theta_{ffgg} = 0. \tag{7.3}$$

We now use the Lie point symmetries of (7.3) and transform it to an ordinary differential equation (ODE). The equation (7.3) has the two translation symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}.$$

The combination $\nu\Gamma_1+\Gamma_2$. of the two symmetries Γ_1 and Γ_2 yields the two invariants

$$z = f - \nu g, \quad \psi = \theta$$

and consequently the group invariant solution $F = F(z)$. Hence by using these invariants, (7.3) transforms into the fourth-order nonlinear ODE

$$\alpha\nu^2 nF^{n-1}F'^2 + \alpha\nu^2F^nF'' + \beta(\nu-1)^2F'' + \nu^2F'' - \nu F'' + \nu^2F'''' = 0. \tag{7.4}$$

Integrating the above equation twice and taking the constants of integration to be zero, we obtain a second-order ODE

$$(\nu-1)(\beta\nu-\beta+\nu)F + \frac{\alpha\nu^2}{n+1}F^{n+1} + \nu^2F'' = 0. \tag{7.5}$$

Multiplying equation (7.5) by F', integrating once and taking the constant of integration to be zero, we obtain the first-order variable separable ODE

$$\frac{1}{2}(\nu-1)(\beta\nu-\beta+\nu)F^2+\frac{\alpha\nu^2}{(n+1)(n+2)}F^{n+2}+\frac{1}{2}\nu^2F'^2=0. \tag{7.6}$$

After integrating (7.6) and then reverting back to the original variables, we obtain the group-invariant solution of the KP-JE equation (7.1), which is a non-topological soliton solution, given by

$$u(x,y,t)=\left(-\frac{A_2}{A_1}(n+1)(n+2)\right)^{\frac{1}{n}}\operatorname{sech}^{\frac{2}{n}}\left[\frac{1}{2}nC\sqrt{A_2}\,(z-C)\right], \tag{7.7}$$

where

$$\begin{aligned} A_1 &= -2\alpha, \\ A_2 &= -\frac{(\nu-1)(\beta\nu-\beta+\nu)}{\nu^2}, \\ z &= t-\nu x+(\nu-1)y \end{aligned}$$

and C is a constant of integration.

By taking $n=1$, $\alpha=2$, $\beta=-1$, $\nu=-3$, $t=0$ and $C=1$ in (7.7), the profile of the solution is given in Figure 7.1.

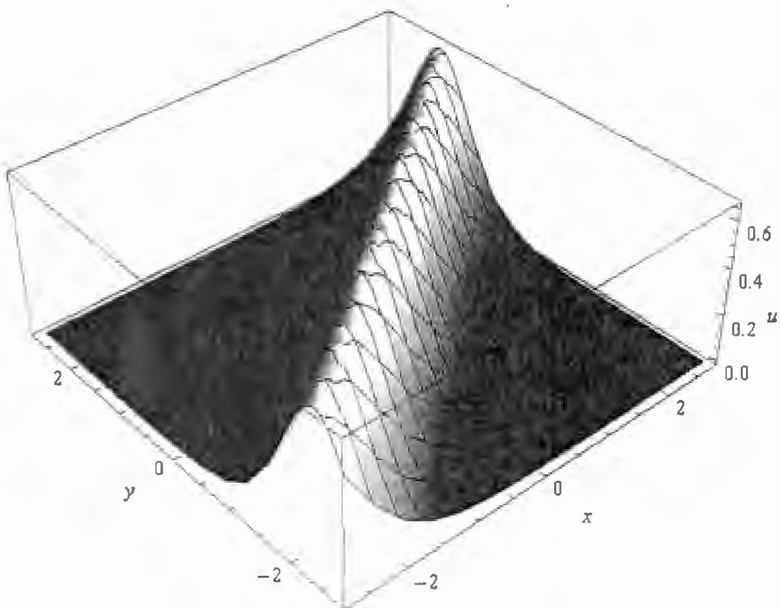

Figure 7.1: Profile of solution (7.7)

7.3 Exp-function method

In this section we invoke the Exp-function method to solve (7.4) for $n = 1$. The Exp-function method results in the traveling wave solutions based on the assumption that the solution can be expressed in the form

$$F(z) = \frac{\sum_{n=-c}^{d} a_n exp(nz)}{\sum_{m=-p}^{q} b_m exp(mz)}, \tag{7.8}$$

where c, d, p and q are positive integers that can be determined, a_n and b_m are unknown constants.

For simplicity, we set $c = p = 1$ and $d = q = 1$, then (7.8) reduced to

$$F(z) = \frac{a_1 exp(z) + a_0 + a_{-1} exp(-z)}{b_1 exp(z) + b_0 + b_{-1} exp(-z)}. \tag{7.9}$$

Substituting (7.9) into the fourth-order nonlinear ODE (7.4) and setting the numerator to zero leads to a system of algebraic equations. Solving the resultant system we find the following cases and corresponding solutions:

Case 1.

We have the following:

$$\beta = \frac{\nu(1+10\nu)}{(\nu-1)^2},\ a_{-1} = -\frac{12b_{-1}}{\alpha},\ a_0 = -\frac{6b_0}{\alpha},\ a_1 = -\frac{3b_0^2}{\alpha b_{-1}},\ b_1 = \frac{b_0^2}{4b_{-1}}.$$

As a result one of the solutions of (7.1) is given by

$$u_1(x,y,t) = \frac{a_1 exp(z) + a_0 + a_{-1} exp(-z)}{b_1 exp(z) + b_0 + b_{-1} exp(-z)}, \tag{7.10}$$

where $z = t - \nu x + (\nu - 1)y$.

As a special case, if we choose $b_0 = 2$, $b_{-1} = 1$ in (8.20), we obtain

$$u(x,y,t) = -\frac{3}{\alpha}\left\{1 + 2\cosh(t - \nu x + (\nu-1)y)\right\}\text{sech}^2\left[\frac{t - \nu x + (\nu-1)y}{2}\right], \tag{7.11}$$

The profile of the solution (7.11) is given in Figure 7.2.

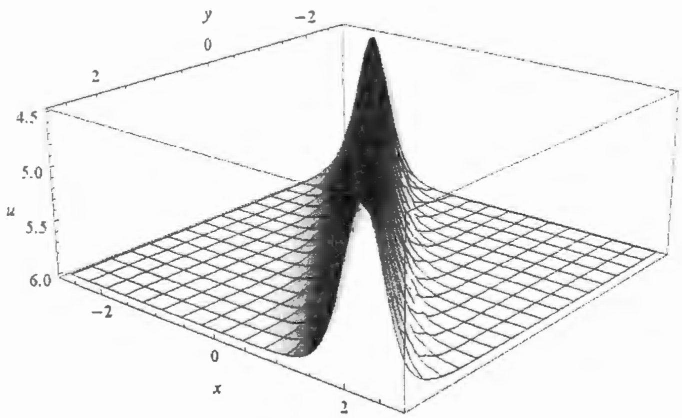

Figure 7.2: Profile of solution (7.11)

Case 2.

For this case we have the following:

$$\alpha = -\frac{b_0\left(\nu^2\beta - 4\nu^2 - 2\nu\beta - \nu + \beta\right)}{\nu^2 a_0},\ a_{-1} = \frac{a_0 b_{-1}\left(\nu^2\beta + 2\nu^2 - 2\nu\beta - \nu + \beta\right)}{b_0\left(\nu^2\beta - 4\nu^2 - 2\nu\beta - \nu + \beta\right)},$$

$$a_1 = \frac{(\nu^2\beta + 2\,\nu^2 - 2\,\nu\,\beta - \nu + \beta)\,a_0 b_0}{4b_{-1}\,(\nu^2\beta - 4\,\nu^2 - 2\,\nu\,\beta - \nu + \beta)},\ b_1 = \frac{b_0^2}{4b_{-1}}.$$

Hence one of the solutions of (7.1) is given by

$$u_2(x,y,t) = \frac{a_1 exp(z) + a_0 + a_{-1} exp(-z)}{b_1 exp(z) + b_0 + b_{-1} exp(-z)}, \tag{7.12}$$

where $z = t - \nu x + (\nu - 1)y$.

The profile of the solution (7.12) is given in Figure 7.3.

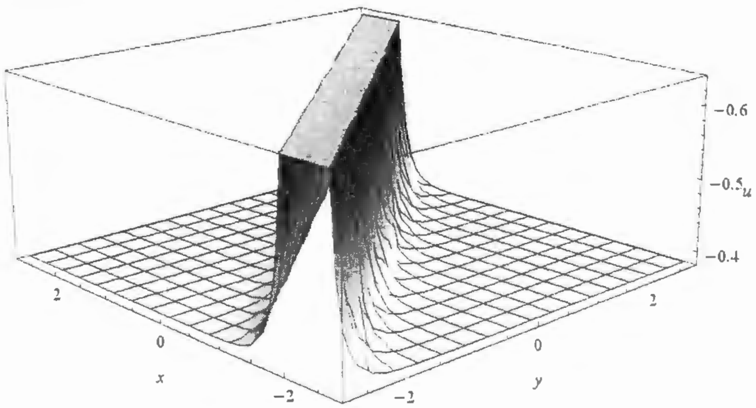

Figure 7.3: Profile of solution (7.12)

Case 3.

In this case we have

$$\alpha = \frac{6b_0}{a_0},\ \beta = -\frac{\nu(2\nu - 1)}{(\nu - 1)^2},\ b_{-1} = \frac{b_0^2}{4b_1},\ a_{-1} = 0,\ a_1 = 0.$$

As a result one of the solutions of (7.1) is given by

$$u_3(x,y,t) = \frac{a_1 exp(z) + a_0 + a_{-1} exp(-z)}{b_1 exp(z) + b_0 + b_{-1} exp(-z)}, \tag{7.13}$$

where $z = t - \nu x + (\nu - 1)y$.

The profile of the solution (7.13) is given in Figure 7.4.

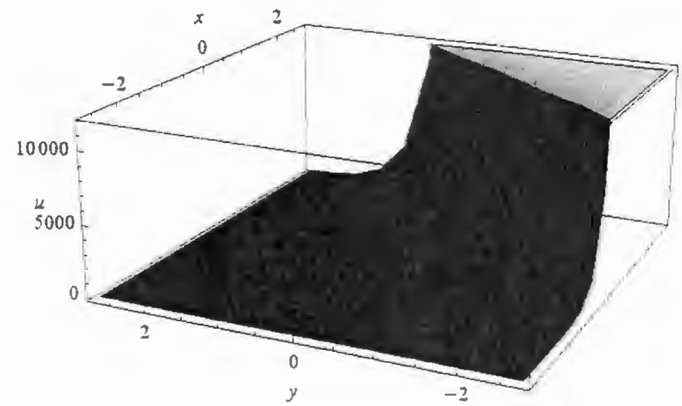

Figure 7.4: Profile of solution (7.13)

Case 4.

We have

$$\alpha = -\frac{6b_{-1}}{a_{-1}},\ \beta = \frac{\nu(4\nu+1)}{(\nu-1)^2},\ a_1 = \frac{a_{-1}b_0^2}{4b_{-1}^2},\ b_1 = \frac{b_0^2}{4b_{-1}},\ a_0 = 0.$$

Thus one of the solutions of (7.1) is given by

$$u_4(x,y,t) = \frac{a_1 exp(z) + a_0 + a_{-1}exp(-z)}{b_1 exp(z) + b_0 + b_{-1}exp(-z)}, \tag{7.14}$$

where $z = t - \nu x + (\nu - 1)y$.

The profile of the solution (7.14) is given in Figure 7.5.

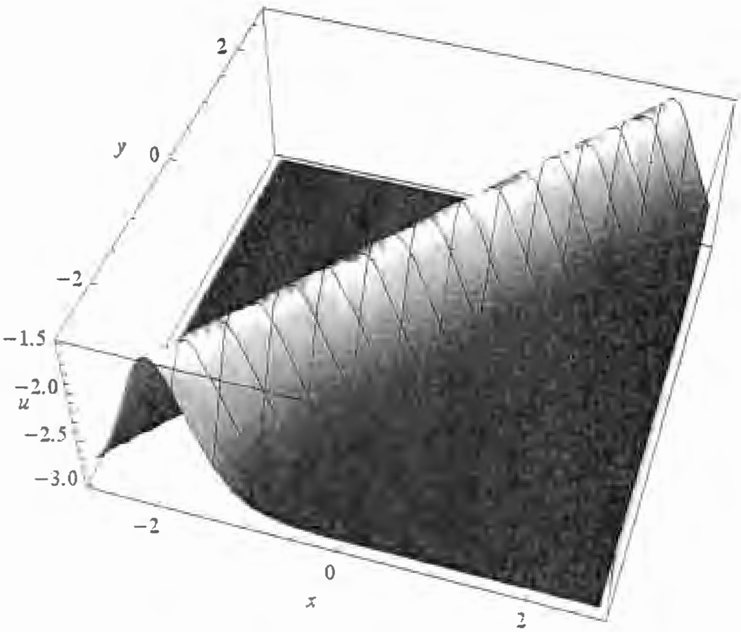

Figure 7.5: Profile of solution (7.14)

7.4 Conservation laws of (7.1)

In this section we construct conservation laws for (7.1). The multiplier method [73] will be used.

The zeroth-order multiplier $\Lambda(t, x, y, u)$ for (7.1) is given by

$$\begin{aligned}\Lambda &= (-y\beta f_1(t) - \beta f_2(t) + C_1 y + C_2)x + \frac{1}{6}y^3 f_1'(t) + \frac{1}{2}y^2 f_2'(t) \\ &\quad + y f_3(t) + f_4(t), \qquad (7.15)\end{aligned}$$

where C_1 and C_2 are constants, and $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ are arbitrary functions of t. Corresponding to the above multiplier we have the following six conserved vectors of (7.1):

$$T_1^t = \frac{1}{36}\Big\{3f_1'(t)u_x y^3 - f_1''(t)u_{xx}y^3 + 3f_1' u_{txx}y^3 + 18\beta u f_1(t)y - 18x\beta f_1(t)u_x y$$

$$-12\beta f_1'(t)u_x y + 6x\beta f_1'(t)u_{xx}y + 12\beta f_1(t)u_{tx}y - 18x\beta f_1(t)u_{txx}y\Big\},$$

$$\begin{aligned}
T_1^x &= \frac{1}{36(n+1)}\Big\{-36nxy\alpha\beta f_1(t)u_x u^n - 36xy\alpha\beta f_1(t)u_x u^n + 6ny^3\alpha f_1'(t)u_x u^n \\
&\quad +6y^3\alpha f_1'(t)u_x u^n + 36y\alpha\beta f_1(t)u^{n+1} + 36ny\beta f_1(t)u + 36y\beta f_1(t)u \\
&\quad +18nxy\beta f_1'(t)u + 18xy\beta f_1'(t)u - 3ny^3 f_1''(t)u - 3y^3 f_1''(t)u + 18ny\beta f_1''(t)u \\
&\quad +18y\beta f_1''(t)u - 36nxy\beta f_1(t)u_x - 36xy\beta f_1(t)u_x + 6ny^3 f_1'(t)u_x + 6y^3 f_1'(t)u_x \\
&\quad -6nxy\beta f_1''(t)u_x - 6xy\beta f_1''(t)u_x + ny^3 f_1'''(t)u_x + y^3 f_1'''(t)u_x - 18nxy\beta f_1(t)u_t \\
&\quad -18xy\beta f_1(t)u_t + 3ny^3 f_1'(t)u_t + 3y^3 f_1'(t)u_t - 12ny\beta f_1'(t)u_t - 12y\beta f_1'(t)u_t \\
&\quad +12nxy\beta f_1'(t)u_{tx} + 12xy\beta f_1'(t)u_{tx} - 2ny^3 f_1''(t)u_{tx} - 2y^3 f_1''(t)u_{tx} + 6ny\beta f_1(t)u_{tt} \\
&\quad +6y\beta f_1(t)u_{tt} - 18nxy\beta f_1(t)u_{ttx} - 18xy\beta f_1(t)u_{ttx} + 3ny^3 f_1'(t)u_{ttx} + 3y^3 f_1'(t)u_{ttx}\Big\},\\
T_1^y &= \frac{1}{6}\Big\{\beta f_1'(t)u_y y^3 - 3\beta u f_1'(t)y^2 - 6x\beta^2 f_1(t)u_y y + 6x\beta^2 u f_1(t)\Big\};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{12}\Big\{3f_2'(t)u_x y^2 - f_2''(t)u_{xx}y^2 + 3f_2'(t)u_{xx}y^2 + 6\beta u f_2(t) - 6x\beta f_2(t)u_x \\
&\quad -4\beta f_2'(t)u_x + 2x\beta f_2'(t)u_{xx} + 4\beta f_2(t)u_{tx} - 6x\beta f_2(t)u_{txx}\Big\},\\
T_2^x &= \frac{1}{12(n+1)}\Big\{-12nx\alpha\beta f_2(t)u_x u^n - 12x\alpha\beta f_2(t)u_x u^n + 6ny^2\alpha f_2'(t)u_x u^n \\
&\quad +6y^2\alpha f_2'(t)u_x u^n + 12\alpha\beta f_2(t)u^{n+1} + 12n\beta f_2(t)u + 12\beta f_2(t)u + 6nx\beta f_2'(t)u \\
&\quad +6x\beta f_2'(t)u - 3ny^2 f_2''(t)u - 3y^2 f_2''(t)u + 6n\beta f_2''(t)u + 6\beta f_2''(t)u - 12nx\beta f_2(t)u_x \\
&\quad -12x\beta f_2(t)u_x + 6ny^2 f_2'(t)u_x + 6y^2 f_2'(t)u_x - 2nx\beta f_2''(t)u_x - 2x\beta f_2''(t)u_x \\
&\quad +ny^2 f_2'''(t)u_x + y^2 f_2'''(t)u_x - 6nx\beta f_2(t)u_t - 6x\beta f_2(t)u_t + 3ny^2 f_2'(t)u_t \\
&\quad +3y^2 f_2'(t)u_t - 4n\beta f_2'(t)u_t - 4\beta f_2'(t)u_t + 4nx\beta f_2'(t)u_{tx} + 4x\beta f_2'(t)u_{tx} \\
&\quad -2ny^2 f_2''(t)u_{tx} - 2y^2 f_2''(t)u_{tx} + 2n\beta f_2(t)u_{tt} + 2\beta f_2(t)u_{tt} - 6nx\beta f_2(t)u_{ttx} \\
&\quad -6x\beta f_2(t)u_{ttx} + 3ny^2 f_2'(t)u_{ttx} + 3y^2 f_2'(t)u_{ttx}\Big\},\\
T_2^y &= \frac{1}{2}\Big\{\beta f_2'(t)u_y y^2 - 2\beta u f_2'(t)y - 2x\beta^2 f_2(t)u_y\Big\};
\end{aligned}$$

$$
\begin{aligned}
T_3^t &= \frac{1}{6}\Big\{3yf_3(t)u_x + 3yf_3(t)u_{txx} - yf_3'u_{xx}\Big\},\\
T_3^x &= \frac{1}{6}y\Big\{6\alpha f_3(t)u_xu^n - 3f_3'u - 2f_3'u_{tx} + 6f_3(t)u_x + 3f_3(t)u_{ttx} + 3f_3(t)u_t + f_3''(t)u_x\Big\},\\
T_3^y &= \beta yf_3(t)u_y - \beta f_3(t)u;
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= \frac{1}{6}\Big\{3f_4(t)u_x + 3f_4(t)u_{txx} - f_4'(t)u_{xx}\Big\},\\
T_4^x &= \frac{1}{6}\Big\{6\alpha f_4(t)u_xu^n - 3f_4'(t)u - 2f_4'(t)u_{tx} + 6f_4(t)u_x + 3f_4(t)u_{ttx} + 3f_4(t)u_t + f_4''(t)u_x\Big\},\\
T_4^y &= \beta f_4(t)u_y;
\end{aligned}
$$

$$
\begin{aligned}
T_5^t &= \frac{1}{6}\Big\{-3yu - 2yu_{tx} + 3xyu_{txx} + 3xyu_x\Big\},\\
T_5^x &= \frac{1}{6(n+1)}\Big\{6y\alpha nxu_xu^n + 6\alpha xu_xu^n - 6\alpha u^{n+1} - 6nu - 6u + 3nxu_t + 3nxu_{ttx}\\
&\quad -nu_{tt} + 6nxu_x + 3xu_t + 3xu_{ttx} - u_{tt} + 6xu_x\Big\},\\
T_5^y &= \beta xyu_y - \beta xu;
\end{aligned}
$$

$$
\begin{aligned}
T_6^t &= \frac{1}{6}\Big\{-3u - 2u_{tx} + 3xu_{txx} + 3xu_x\Big\},\\
T_6^x &= \frac{1}{6(n+1)}\Big\{6\alpha nxu_xu^n + 6\alpha xu_xu^n - 6\alpha u^{n+1} - 6nu - 6u + 3nxu_t + 3nxu_{ttx}\\
&\quad -nu_{tt} + 6nxu_x + 3xu_t + 3xu_{ttx} - u_{tt} + 6xu_x\Big\},\\
T_6^y &= \beta xu_y;
\end{aligned}
$$

Remark. Due to the presence of the arbitrary functions in the multiplier, one can obtain an infinitely many conservation laws for the (2+1)-dimensional nonlinear KP-JE equation.

7.5 Concluding remarks

In this chapter we obtained the solutions of the (2+1)-dimensional nonlinear Kadomtsev-Petviashvili-Joseph-Egri equation by employing the Lie group analysis together with the Exp-function method. The solutions obtained were solitary waves and non-topological soliton. The conservation laws for the underlying equation were also derived by using the multiplier method.

Chapter 8

A study of the Camassa-Holm-KP equation with power law nonlinearity

8.1 Introduction

The aim of this chapter is to study the Camassa-Holm Kadomtsev-Petviashvili (CH-KP) equation with power law nonlinearity [66] that is given by

$$(u_t + 2\alpha u_x - u_{xxt} - \beta u^n u_x)_x + u_{yy} = 0, \tag{8.1}$$

where α and β are real valued constants with $\beta > 0$ and $n > 0$.

The solutions of (8.1) have been studied in various aspects. Wazwaz [66] studied two variants of CH-KP equations, by employing the sine-cosine method and the tanh method and obtained the solitons, compactons, solitary patterns and periodic solutions and their analytic expressions. Lai and Xu [90] studied the generalized forms of CH-KP, and derived the families of exact travelling wave solutions. Xie *et.al* [91] used the dynamical system theory and found periodic waves, periodic cusp waves, solitary waves, peakons, loops and kink waves for (8.1) .

However in this chapter Lie group analysis in conjunction with the Exp-function method and the simplest method are employed to obtain some exact solutions of (8.1). Finally, conservation laws will be derived for (8.1) using the multiplier method [73].

8.2 Lie point symmetries of (8.1)

In this section we first calculate the Lie point symmetries of (8.1) and use them to construct exact solutions. The vector field of the form

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u},$$

where ξ^i, $i = 1,2,3$ and η depend on x, y, t and u, is a Lie point symmetry of (8.1) if

$$\mathrm{pr}^{(4)}X[(u_t + 2\alpha u_x - u_{xxt} - \beta u^n u_x)_x + u_{yy}] = 0 \tag{8.2}$$

whenever $(u_t + 2\alpha u_x - u_{xxt} - \beta u^n u_x)_x + u_{yy} = 0$. Here $\mathrm{pr}^{(4)}X$ denotes the fourth prolongation of X. Expanding (8.2) and splitting on the derivatives of u, we obtain an overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^3_u = 0,\ \xi^3_x = 0,\ \xi^3_y = 0,\ \xi^2_u = 0,\ \xi^2_t = 0,\ \xi^2_x = 0,\ \xi^1_u = 0,\ \xi^1_t = 0,\ \xi^1_y = 0,\\
&\eta_{tu} = 0,\ \eta_{uu} = 0,\ \eta_{xu} - \xi^1_{xx} = 0,\ 2\,\eta_{yu} - \xi^2_{yy} = 0,\ \eta_{xxxu} - \eta_{xu} = 0,\\
&2\,\xi^2_y - \xi^3_t - 3\,\xi^1_x = 0,\ \beta u^n \eta_{xx} - 2\,\alpha\eta_{xx} + \eta_{xxxt} - \eta_{yy} - \eta_{xt} = 0,\\
&\beta\, n u^n \eta - 2\,\beta\, u^{n+1}\xi^1_x + 4\,\alpha\, u\xi^1_x + 2\,\beta\, u^{n+1}\xi^2_y - 4\,\alpha\, u\xi^2_y = 0,\\
&n\eta + u\eta_u - 2\,u\xi^1_x + 2\,u\xi^2_y - \eta = 0,\ 2\,\xi_y - 3\,\eta_{xxu} - \xi^3_t - \xi^1_x + \xi^1_{xxx} = 0,\\
&2\,\beta\, n u^n \eta_x - \beta\, u^{n+1}\xi^1_{xx} + 2\,\alpha\, u\xi^1_{xx} + 2\,\beta\, u^{n+1}\eta_{xu} - 4\,\alpha\, u\eta_{xu} = 0.
\end{aligned}$$

After some lengthy calculations, one can solve the above system and obtain the values of the coefficient functions ξ^1, ξ^2 ξ^3 and η. Consequently we obtain the three translation symmetries of (8.1):

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_t.$$

8.3 Solutions of (8.1) using the three translation symmetries

By using the symmetry $X = X_1 + X_2 + X_3$ and reduce the CH-KP equation with power law nonlinearity (8.1) to a PDE in two independent variables. This is done by computing the invariant of X. It can be seen that the symmetry X yields the following three invariants:

$$f = t - y, \quad g = x - y, \quad \theta = u. \tag{8.3}$$

Now treating θ as the new dependent variable and f and g as new independent variables, the CH-KP equation (8.1) transforms to

$$3\alpha\theta_{gg} + 3\theta_{fg} + \theta_{gg} + \theta_{ff} - \beta n\theta^{n-1}\theta_g^2 - \beta\theta^n\theta_{gg} - \theta_{fggg} = 0, \tag{8.4}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (8.4) and transform it to an ordinary differential equation (ODE). The equation (8.4) has the two translational symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}.$$

The combination $\nu\Gamma_1 + \Gamma_2$, of the two symmetries Γ_1 and Γ_2 yields the two invariants

$$z = f - \nu g, \quad F = \theta,$$

which gives rise to a group invariant solution $F = F(z)$. Consequently using these invariants, (8.4) is transformed into the fourth-order nonlinear ODE

$$2\alpha\nu^2 F'' + (\nu-1)^2 F'' - \beta\nu^2 F^n F'' - \nu F'' - \beta\nu^2 n F^{n-1} F'^2 + \nu^3 F'''' = 0. \quad (8.5)$$

Integrating the above equation twice and taking the constants of integration to be zero we obtain a second-order ODE

$$\left(2\alpha\nu^2 + \nu^2 - 3\nu + 1\right) F - \frac{\beta\nu^2}{n+1} F^{n+1} + \nu^3 F'' = 0. \quad (8.6)$$

Multiplying equation (8.6) by F', integrating once and taking the constant of integration to be zero, we obtain the first-order ODE

$$\frac{1}{2}\left(2\alpha\nu^2 + \nu^2 - 3\nu + 1\right) F(z)^2 - \frac{\beta\nu^2}{(n+1)(n+2)} F^{n+2} + \frac{\nu^3}{2} F'(z)^2 = 0. \quad (8.7)$$

One can integrate the above equation by separating the variables. After integrating and reverting back to the original variables, we obtain the following group-invariant solutions of the CH-KP equation with power law nonlinearity (8.1)

$$u(x, y, t) = \left(-\frac{A_2}{A_1}(n+1)(n+2)\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}\left[\frac{1}{2} n C \sqrt{A_2}\,(z - C)\right], \quad (8.8)$$

where

$$\begin{aligned} A_1 &= -2\beta/\nu, \\ A_2 &= (2\alpha\nu^2 + \nu^2 - 3\nu + 1)/\nu^3, \\ z &= t - y - \nu(x - y) \end{aligned}$$

and C is a constant of integration. We note that the solution (8.8) represents a non-toplogical soliton. By taking $n = 1$, $a = -1$, $b = 1$, $\nu = 6$, $t = 0$ and $C = 1$ in (8.8), the profile of the solution is given in Figure 8.1.

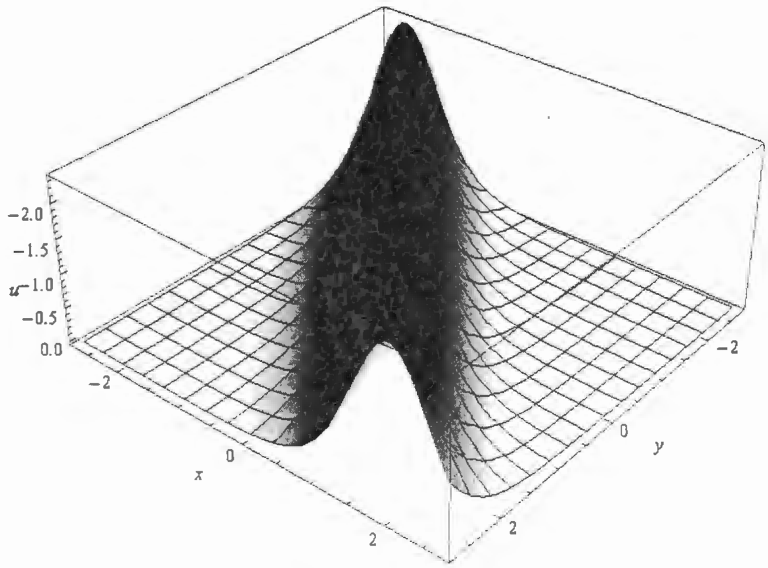

Figure 8.1: Profile of solution (8.8)

8.4 Simplest equation method

In this section we use the simplest equation method, which was used in Chapter 2, to solve the third-order ODE (8.5) for $n = 1, 2$.

8.4.1 Solutions of (8.5) using the Bernoulli equation as the simplest equation

$n = 1$

The balancing procedure yields $M = 2$, so the solutions of (8.5) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2. \tag{8.9}$$

Substituting (8.9) into (8.5) and making use of the Bernoulli equation and then equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 and A_2. These algebraic equations are

$$
\begin{aligned}
&1920\,\nu^3 A_2 - 40\,\nu^2\beta\, {A_2}^2 = 0,\\
&2688\,\nu^3 A_2 - 48\,\nu^2\beta\, A_1 A_2 - 36\,\nu^2\beta\, {A_2}^2 + 384\,\nu^3 A_1 = 0,\\
&A_1 - \nu^2\beta\, A_0 A_1 + \nu^3 A_1 + 2\,\nu^2\alpha\, A_1 + \nu^2 A_1 - 3\,\nu\, A_1 = 0,\\
&24\,\nu^2 A_2 - 24\,\nu^2\beta\, A_0 A_2 - 12\,\nu^2\beta\, {A_1}^2 - 42\,\nu^2\beta\, A_1 A_2 - 8\,\nu^2\beta\, {A_2}^2 + 480\,\nu^3 A_1\\
&+1320\,\nu^3 A_2 + 48\,\nu^2\alpha\, A_2 - 72\,\nu\, A_2 + 24\, A_2 = 0,\\
&8\,\nu^2\alpha\, A_2 - 6\,\nu^2\beta\, A_0 A_1 - 4\,\nu^2\beta\, A_0 A_2 - 2\,\nu^2\beta\, {A_1}^2 + 30\,\nu^3 A_1 + 16\,\nu^3 A_2 + 12\,\nu^2\alpha\, A_1\\
&+6\,\nu^2 A_1 + 4\,\nu^2 A_2 - 18\,\nu\, A_1 - 12\,\nu\, A_2 + 6\, A_1 + 4\, A_2 = 0.\\
&16\,\nu^2\alpha\, A_1 - 8\,\nu^2\beta\, A_0 A_1 - 20\,\nu^2\beta\, A_0 A_2 - 10\,\nu^2\beta\, {A_1}^2 - 9\,\nu^2\beta\, A_1 A_2 + 200\,\nu^3 A_1\\
&+260\,\nu^3 A_2 + 40\,\nu^2\alpha\, A_2 + 8\,\nu^2 A_1 + 20\,\nu^2 A_2 - 24\,\nu\, A_1 - 60\,\nu\, A_2 + 8\, A_1 + 20\, A_2 = 0.
\end{aligned}
$$

Solving the above system of algebraic equations with the aid of Mathematica, we obtain the following values of A_0, A_1 and A_2:

$$
a = 1, \quad b = 2, \quad A_0 = \frac{(\nu^3 + 2\alpha\nu^2 + \nu^2 - 3\nu + 1)}{\beta\nu^2}, \quad A_1 = \frac{24\nu}{\beta}, \quad A_2 = \frac{48\nu}{\beta}.
$$

Therefore the solution of (8.1), for $n = 1$ is given by

$$
\begin{aligned}
u(x,y,t) \;&=\; A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\\
&\quad + A_2 a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \qquad (8.10)
\end{aligned}
$$

where $z = t - y - \nu(x - y)$ and C is a constant of integration.

$n = 2$

The balancing procedure yields $M = 1$, so the solutions of (8.5) are of the form

$$
F(z) = A_0 + A_1 H. \qquad (8.11)
$$

As before, substituting (8.11) into (8.5), we obtain the algebraic system of equations

$$
\begin{aligned}
&24\,b^4\nu^3 A_1 - 4\,b^2\nu^2\beta\,{A_1}^3 = 0,\\
&60\,ab^3\nu^3 A_1 - 7\,ab\nu^2\beta\,{A_1}^3 - 6\,b^2\nu^2\beta\,A_0{A_1}^2 = 0,\\
&a^4\nu^3 A_1 - a^2\nu^2\beta\,{A_0}^2 A_1 + 2\,a^2\nu^2\alpha\,A_1 + a^2\nu^2 A_1 - 3\,a^2\nu\,A_1 + a^2 A_1 = 0,\\
&15\,a^3 b\nu^3 A_1 - 4\,a^2\nu^2\beta\,A_0{A_1}^2 - 3\,ab\nu^2\beta\,{A_0}^2 A_1 + 6\,ab\nu^2\alpha\,A_1 + 3\,ab\nu^2 A_1\\
&-9\,ab\nu\,A_1 + 3\,abA_1 = 0,\\
&50\,a^2b^2\nu^3 A_1 - 3\,a^2\nu^2\beta\,{A_1}^3 - 10\,ab\nu^2\beta\,A_0{A_1}^2 - 2\,b^2\nu^2\beta\,{A_0}^2 A_1 + 4\,b^2\nu^2\alpha\,A_1\\
&+2\,b^2\nu^2 A_1 - 6\,b^2\nu\,A_1 + 2\,b^2 A_1 = 0.
\end{aligned}
$$

which on solving yields

$$
\alpha = \frac{a^2b^2\nu + (2\nu^2 - 6\nu + 2)b^2}{4\nu^2 b^2}, \quad A_0 = \sqrt{\frac{3a^2\nu}{2\beta}}, \quad A_1 = \sqrt{\frac{6b^2\nu}{\beta}}.
$$

Consequently, the solution of (8.1), for $n = 2$ is given by

$$
u(x,y,t) = A_0 + A_1 a\left(\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right), \tag{8.12}
$$

where $z = t - y - \nu(x-y)$ and C is a constant of integration.

8.4.2 Solutions of (8.5) using Riccati equation as the simplest equation

$n = 1$

The balancing procedure yields $M = 2$, so the solutions of (8.5) are of the form

$$
F(z) = A_0 + A_1 H + A_2 H^2. \tag{8.13}
$$

Substituting (8.13) into (8.5) and making use of the Riccati equation, we obtain algebraic equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions

H^i to zero. The corresponding algebraic equations are

$$120\,\nu^3 A_2 - 10\,\nu^2\beta\, {A_2}^2 = 0,$$

$$24\,\nu^3 A_1 - 12\,\nu^2\beta\, A_1 A_2 - 54\,\nu^2\beta\, {A_2}^2 + 1008\,\nu^3 A_2 = 0,$$

$$6\,\nu^2 A_2 - 6\,\nu^2\beta\, A_0 A_2 - 3\,\nu^2\beta\, {A_1}^2 - 63\,\nu^2\beta\, A_1 A_2 - 88\,\nu^2\beta\, {A_2}^2 + 180\,\nu^3 A_1$$
$$+3210\,\nu^3 A_2 + 12\,\nu^2\alpha\, A_2 - 18\,\nu\, A_2 + 6\, A_2 = 0,$$

$$4\,\nu^2\alpha\, A_2 - 3\,\nu^2\beta\, A_0 A_1 - 2\,\nu^2\beta\, A_0 A_2 - \nu^2\beta\, {A_1}^2 + 51\,\nu^3 A_1 + 142\,\nu^3 A_2 + 6\,\nu^2\alpha\, A_1$$
$$+3\,\nu^2 A_1 + 2\,\nu^2 A_2 - 9\,\nu\, A_1 - 6\,\nu\, A_2 + 3\, A_1 + 2\, A_2 = 0,$$

$$11\, A_1 + 18\, A_2 - 11\,\nu^2\beta\, A_0 A_1 - 18\,\nu^2\beta\, A_0 A_2 - 9\,\nu^2\beta\, {A_1}^2 - 6\,\nu^2\beta\, A_1 A_2 + 295\,\nu^3 A_1$$
$$+1170\,\nu^3 A_2 + 22\,\nu^2\alpha\, A_1 + 36\,\nu^2\alpha\, A_2 + 11\,\nu^2 A_1 + 18\,\nu^2 A_2 - 33\,\nu\, A_1 - 54\,\nu\, A_2 = 0,$$

$$9\,\nu^2 A_1 - 9\,\nu^2\beta\, A_0 A_1 - 44\,\nu^2\beta\, A_0 A_2 - 22\,\nu^2\beta\, {A_1}^2 - 45\,\nu^2\beta\, A_1 A_2 - 6\,\nu^2\beta\, {A_2}^2$$
$$+585\,\nu^3 A_1 + 3520\,\nu^3 A_2 + 18\,\nu^2\alpha\, A_1 + 88\,\nu^2\alpha\, A_2 + 44\,\nu^2 A_2 - 27\,\nu\, A_1$$
$$-132\,\nu\, A_2 + 9\, A_1 + 44\, A_2 = 0,$$

$$2\, A_1 - 2\,\nu^2\beta\, A_0 A_1 - 30\,\nu^2\beta\, A_0 A_2 - 15\,\nu^2\beta\, {A_1}^2 - 99\,\nu^2\beta\, A_1 A_2 - 42\,\nu^2\beta\, {A_2}^2 + 490\,\nu^3 A_1$$
$$+4830\,\nu^3 A_2 + 4\,\nu^2\alpha\, A_1 + 60\,\nu^2\alpha\, A_2 + 2\,\nu^2 A_1 + 30\,\nu^2 A_2 - 6\,\nu\, A_1 - 90\,\nu\, A_2 + 30\, A_2 = 0.$$

Solving the above equations yields

$$a = 1, \quad b = 3, \quad c = 1, \quad A_0 = \frac{(17\nu^3 + 2\alpha\nu^2 + \nu^2 - 3\nu + 1)}{\beta\nu^2}, \quad A_1 = \frac{36\nu}{\beta}, \quad A_2 = \frac{12\nu}{\beta}$$

and hence the solutions of (8.1) for $n = 1$ are

$$\begin{aligned} u(x,y,t) &= A_0 + A_1\left(-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right) \\ &\quad + A_2\left(-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right)^2 \end{aligned} \tag{8.14}$$

and

$$u(x,y,t) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{\theta z}{2}\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$$

$$+A_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{\theta z}{2}\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2 \quad (8.15)$$

where $z = t - y - \nu(x-y)$ and C is a constant of integration.

$n = 2$

The balancing procedure yields $M = 1$, so the solutions of (8.5) are of the form

$$F(z) = A_0 + A_1 H. \quad (8.16)$$

Substituting (8.16) into (8.5), we obtain the following algebraic system of equations:

$$24\,\nu^3 A_1 - 4\,\nu^2\beta\, A_1{}^3 = 0,$$

$$60\,\nu^3 A_1 - 6\,\nu^2\beta\, A_0 A_1{}^2 - 7\,\nu^2\beta\, A_1{}^3 = 0,$$

$$2\,\nu^2\alpha\, A_1 - \nu^2\beta\, A_0{}^2 A_1 - 2\,\nu^2\beta\, A_0 A_1{}^2 + 9\,\nu^3 A_1 + \nu^2 A_1 - 3\,\nu\, A_1 + A_1 = 0,$$

$$3\,\nu^2 A_1 - 3\,\nu^2\beta\, A_0{}^2 A_1 - 12\,\nu^2\beta\, A_0 A_1{}^2 - 5\,\nu^2\beta\, A_1{}^3 + 75\,\nu^3 A_1 + 6\,\nu^2\alpha\, A_1$$

$$-9\,\nu\, A_1 + 3\, A_1 = 0,$$

$$3\,\nu^2 A_1 - 3\,\nu^2\beta\, A_0{}^2 A_1 - 6\,\nu^2\beta\, A_0 A_1{}^2 - 2\,\nu^2\beta\, A_1{}^3 + 39\,\nu^3 A_1 + 6\,\nu^2\alpha\, A_1$$

$$-9\,\nu\, A_1 + 3\, A_1 = 0,$$

$$2\,\nu^2 A_1 - 2\,\nu^2\beta\, A_0{}^2 A_1 - 10\,\nu^2\beta\, A_0 A_1{}^2 - 9\,\nu^2\beta\, A_1{}^3 + 90\,\nu^3 A_1$$

$$+4\,\nu^2\alpha\, A_1 - 6\,\nu\, A_1 + 2\, A_1 = 0.$$

Solving the algebraic equations one obtains

$$a = 1, \quad b = 1, \quad c = 1, \quad \alpha = \frac{432\nu\beta - 216\nu^3 - 144\nu^2\beta - 144\beta}{288\beta\nu},$$

$$A_0 = \sqrt{\frac{3\nu}{2\beta}}, \quad A_1 = \sqrt{\frac{6\nu}{\beta}}.$$

Hence we have the following solutions of (8.1) for $n = 2$:

$$u(x,y,t) = A_0 + A_1\left(-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right) \quad (8.17)$$

and

$$u(x,y,t) = A_0 + A_1\left(-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right) \quad (8.18)$$

where $z = t - y - \nu(x-y)$ and C is a constant of integration. The profile of the solutions (8.17) and (8.18) is given in Figure 8.2 and Figure 8.3, respectively

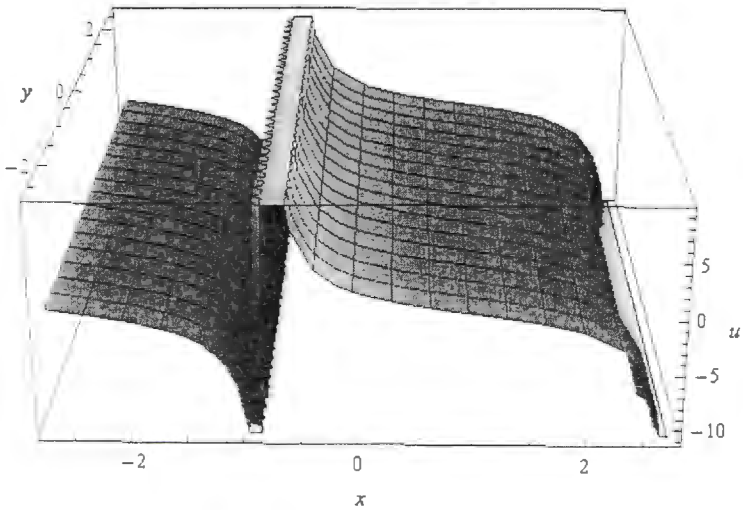

Figure 8.2: Profile of solution (8.17)

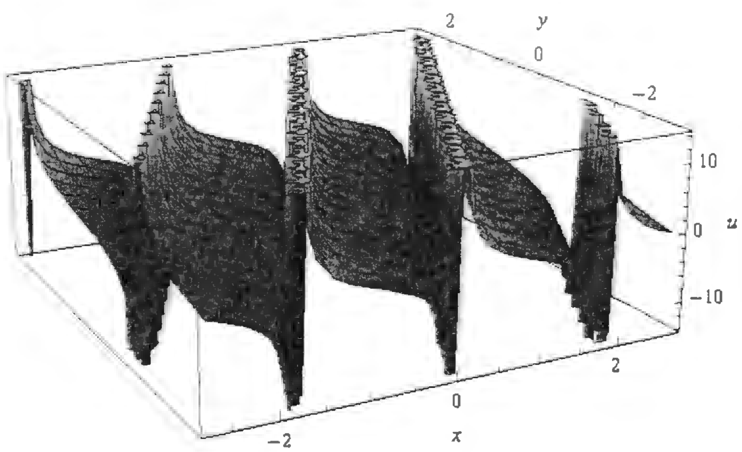

Figure 8.3: Profile of solution (8.18)

8.5 Exp-function method

In this section we invoke the Exp-function method, which was used in Chapter 7, to solve the third-order ODE (8.5) for $n = 1$. This will then give us the exact solution for Camassa-Holm Kadomtsev-Petviashvili (CH-KP) equation with power law nonlinearity (8.1).

For simplicity, we set $c = p = 1$ and $d = q = 1$,

$$F(z) = \frac{a_1 exp(z) + a_0 + a_{-1} exp(-z)}{b_1 exp(z) + b_0 + b_{-1} exp(-z)}. \tag{8.19}$$

Substituting (8.19) into (8.5) and by setting the numerator to be zero leads to system of algebraic equations. Solving the resultant system we obtain

$$a_{-1} = \frac{{b_0}^2 (\nu^3 + 2\,\alpha\,\nu^2 + \nu^2 - 3\,\nu + 1)}{4\nu^2 \beta\, b_1}, \quad a_0 = \frac{b_0 (-5\,\nu^3 + 2\,\alpha\,\nu^2 + \nu^2 - 3\,\nu + 1)}{\beta\,\nu^2},$$
$$a_1 = \frac{b_1 (\nu^3 + 2\,\alpha\,\nu^2 + \nu^2 - 3\,\nu + 1)}{\beta\,\nu^2}, \quad b_{-1} = \frac{{b_0}^2}{4b_1}.$$

As a result one of the solutions of (8.1) is given by

$$u(x, y. t) = \frac{a_1 exp(z) + a_0 + a_{-1} exp(-z)}{b_1 exp(z) + b_0 + b_{-1} exp(-z)},$$

where $z = t - y - \nu(x - y)$.

As a special case, if we choose $b_0 = 2$, $b_{-1} = 1$ in (8.20), we obtain the solution

$$u(x, y, t) = \frac{1}{2\beta\nu^2}\left\{A_1 + A_2 \cosh(z)\right\}\operatorname{sech}^2\left(\frac{z}{2}\right), \tag{8.20}$$

where

$$\begin{aligned} A_1 &= 1 - 3\,\nu + (1 + 2\alpha)\nu^2 - 5\,\nu^3, \\ A_2 &= 1 - 3\,\nu + (1 + 2\alpha)\nu^2 + \nu^3, \\ z &= t - y - \nu(x - y). \end{aligned}$$

The profile of the solution (8.20) is given in Figure 8.4.

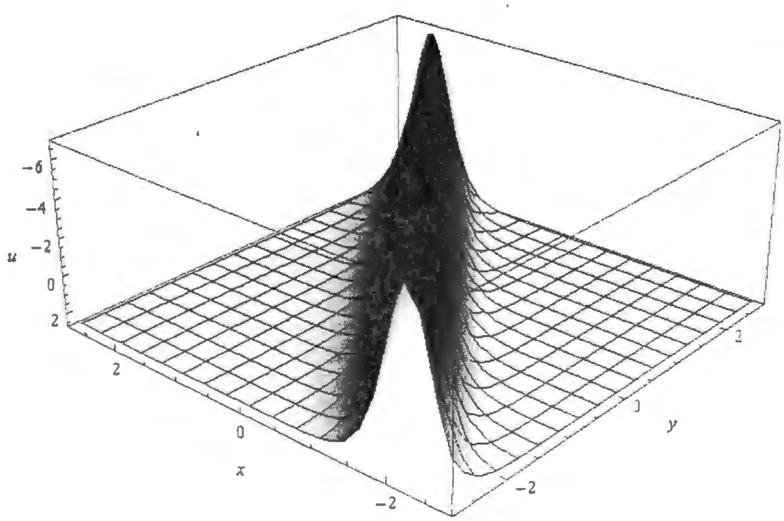

Figure 8.4: Profile of solution (8.20)

8.6 Conservation laws of (8.1)

In this section we construct conservation laws for (8.1). The multiplier method [27,73] will be used. We first obtain the zero-order multiplier $\Lambda(t,x,y,u)$ for (8.1). Following the proceedure given in [73], we see that the zeroth-order multiplier $\Lambda(t,x,y,u)$ for (8.1) is given by

$$\Lambda = -\frac{1}{6}y^3 f_1'(t) - \frac{1}{2}y^2 f_2'(t) + xyf_1(t) + xf_2(t) + yf_3(t) + f_4(t), \quad (8.21)$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$ and $f_4(t)$ are arbitrary functions of t. Consequently, corresponding to the above multiplier we have the following four conserved vectors of (8.1):

$$\begin{aligned} T_1^t &= \frac{1}{24}\Big\{ -12yf_1(t)u + 12xyf_1(t)u_x + 6yf_1(t)u_{xx} - 6xyf_1(t)u_{xxx} \\ &\quad -2y^3 f_1'(t)u_x + y^3 f_1'(t)u_{xxx}\Big\}, \\ T_1^x &= \frac{1}{24(n+1)}\Big\{4\beta ny^3 f_1'(t)u_x u^n + 4\beta y^3 f_1'(t)u_x u^n - 24\beta nxyf_1(t)u_x u^n \end{aligned}$$

$$
\begin{aligned}
& -24\beta xyf_1(t)u_xu^n + 2ny^3 f_1''(t)u - 48\alpha nyf_1(t)u + 24\beta yf_1(t)u^{n+1} \\
& -12nxyf_1'(t)u + 2y^3 f_1''(t)u - 48\alpha yf_1(t)u - 12xyf_1'(t)u + 3ny^3 f_1'(t)u_{txx} \\
& +48\alpha nxyf_1(t)u_x + 12nxyf_1(t)u_t + 12nyf_1(t)u_{tx} - 18nxyf_1(t)u_{txx} \\
& -2ny^3 f_1'(t)u_t - 8\alpha ny^3 f_1'(t)u_x - ny^3 f_1''(t)u_{xx} - 12nyf_1'(t)u_x + 6nxyf_1'(t)u_{xx} \\
& +3y^3 f_1'(t)u_{txx} + 48\alpha xyf_1(t)u_x + 12xyf_1(t)u_t + 12yf_1(t)u_{tx} - 18xyf_1(t)u_{txx} \\
& -2y^3 f_1'(t)u_t - 8\alpha y^3 f_1'(t)u_x - y^3 f_1''(t)u_{xx} - 12yf_1'(t)u_x + 6xyf_1'(t)u_{xx}\Big\}, \\
T_1^y &= \frac{1}{6}\Big\{3y^2 f_1'(t)u - 6xf_1(t)u + 6xyf_1(t)u_y - y^3 f_1'(t)u_y\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_2^t &= \frac{1}{8}\Big\{-4f_2(t)u + 4xf_2(t)u_x + 2f_2(t)u_{xx} - 2xf_2(t)u_{xxx} - 2y^2 f_2'(t)u_x + y^2 f_2'(t)u_{xxx}\Big\}, \\
T_2^x &= \frac{1}{8(n+1)}\Big\{4\beta ny^2 f_2'(t)u_xu^n + 4\beta y^2 f_2'(t)u_xu^n - 8\beta nxf_2(t)u_xu^n - 8\beta xf_2(t)u_xu^n \\
& +2ny^2 f_2''(t)u - 16\alpha nf_2(t)u + 8\beta f_2(t)u^{n+1} - 4nxf_2'(t)u + 2y^2 f_2''(t)u - 16\alpha f_2(t)u \\
& -4xf_2'(t)u + 16\alpha nxf_2(t)u_x + 3ny^2 f_2'(t)u_{txx} + 4nxf_2(t)u_t + 4nf_2(t)u_{tx} \\
& -6nxf_2(t)u_{txx} - 2ny^2 f_2'(t)u_t - 8\alpha ny^2 f_2'(t)u_x - ny^2 f_2''(t)u_{xx} - 4nf_2'(t)u_x \\
& +2nxf_2'(t)u_{xx} + 16\alpha xf_2(t)u_x + 3y^2 f_2'(t)u_{txx} + 4xf_2(t)u_t + 4f_2(t)u_{tx} - 6xf_2(t)u_{txx} \\
& -2y^2 f_2'(t)u_t - 8\alpha y^2 f_2'(t)u_x - y^2 f_2''(t)u_{xx} - 4f_2'(t)u_x + 2xf_2'(t)u_{xx}\Big\}, \\
T_2^y &= \frac{1}{2}\Big\{2yf_2'(t)u + 2xf_2(t)u_y + -y^2 f_2'(t)u_y\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_3^t &= \frac{1}{4}\Big\{2yf_3(t)u_x - yf_3(t)u_{xxx}\Big\}, \\
T_3^x &= -\frac{1}{4}y\Big\{4\beta f_3(t)u_xu^n + 2f_3'(t)u - 8\alpha f_3(t)u_x + 3f_3(t)u_{txx} - 2f_3(t)u_t - f_3'(t)u_{xx}\Big\}, \\
T_3^y &= yf_3(t)u_y - f_3(t)u;
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= \frac{1}{4}\Big\{2f_4(t)u_x - f_4(t)u_{xxx}\Big\}, \\
T_4^x &= \frac{1}{4}\Big\{-4\beta f_4(t)u_xu^n - 2f_4'(t)u + 8\alpha f_4(t)u_x - 3f_4(t)u_{txx} + 2f_4(t)u_t + f_4'(t)u_{xx}\Big\},
\end{aligned}
$$

$$T_4^y = f_4(t)u_y.$$

Remark. The presence of the arbitrary functions in the multiplier, leads to a family of infinitely many conservation laws for (8.1).

8.7 Concluding remarks

In this chapter we obtained the solutions of Camassa-Holm Kadomtsev-Petviashvili with power law nonlinearity. We first used the three translation symmetries of (8.1) and obtained a non-topological soliton solution. By employing the simplest equation and Exp-function methods, we computed travelling wave solutions for (8.1). The conservation laws for the underlying equation were also derived by using the multiplier method.

Chapter 9

Concluding remarks

In this research project we first evoked some important definitions and results from Lie group theory and conservation laws, which were later used in the thesis. In Chapter two, Lie group classification was performed on the Krichever-Novikov equation. The Krichever-Novikov equation admitted six-dimensional equivalence generators. It was also shown that the Krichever-Novikov equation consisted of two symmetries and several possible extensions of the principal Lie algebra were computed, two new cases were found and exact solutions for a special and interesting case of the Krichever-Novikov equation was obtained using the two translation symmetries along with the simplest equation method. Finally, conservation laws were obtained for two cases of the Krichever-Novikov equation using the new conservation theorem due to Ibragimov.

In Chapter three we studied the two-dimensional nonlinear Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony equation by employing Lie group analysis and the simplest equation method. The solutions obtained were solitary waves and non-topological soliton. The conservation laws for the underlying equation were derived by using the multiplier method.

In Chapter four we computed the solutions of the generalized (2+1)-dimensional nonlinear Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation by employing the Lie group analysis, the simplest equation method and (G'/G)-expansion method. The solutions obtained were solitary waves and non-topological soliton. Moreover, conservation laws were derived by using two different methods, the new conservation theorem and the multiplier method.

In Chapter five we obtained the solutions of a generalized two-dimensional nonlinear Kadomtsev-Petviashivilli-modified equal width equation by employing the Lie group analysis, in conjunction with the (G'/G)-expansion method. The solutions obtained were solitary waves and non-topological soliton. The conservation laws for the underlying equation were also derived by using the multiplier method.

In Chapter six firstly we obtained the solutions of the Drinfeld-Sokolov-Satsuma-Hirota equation by employing Lie group analysis together with the simplest and Jacobi elliptic equation methods. Also symmetry reductions were obtained based on the optimal systems of one-dimensional subalgebras. The exact solutions obtained were travelling wave solutions, cnoidal and snoidal wave solutions. Furthermore, the conservation laws for the underlying equation was derived by using two different approaches; namely the new conservation theorem and the multiplier method.

In Chapter seven we obtained the solutions of the (2+1)-dimensional Kadomtsev-Petviashvili-Joseph-Egri equation with power law nonlinearity by employing the Lie group analysis together with the Exp-function method. The solutions obtained are solitary waves and non-topological soliton. The conservation laws for the underlying equation were also derived by using the multiplier method.

In Chapter eight we computed the solutions of Camassa-Holm Kadomtsev-Petviashvili equation with power law nonlinearity. We first used the three translation symmetries and obtained a non-topological soliton solution. By employing the simplest equation and Exp-function methods, we computed travelling wave solutions. The conservation laws for the underlying equation were also derived by using the multiplier method.

In the future we intend to study the stability of the exact solutions found in this thesis and also use them as benchmarks against possible numerical schemes for the underlying equations.

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