

Inference for the ratio of two exponential parameters using a Bayesian approach

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Declaration

I, the undersigned, declare that the work contained in this dissertation is my own work, except for references specifically indicated in the text, and that I have not previously submitted it elsewhere for degree purposes.



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21 February 2020

Date

Abstract

In this dissertation the maximal data information prior and the probability matching prior for the ratio of two exponential parameters will be derived. The method by Datta and Ghosh (1995) will be used to derive the probability matching prior and the method proposed by Zellner (1971) will be used to derive the maximal data information prior. Simulation studies will be done to compare and evaluate the performance of the following five priors: the Jeffreys, uniform, probability matching, maximal data information priors and a prior suggested by Ghosh et al. (2011). We will investigate the performance of the credibility intervals for the ratio of two exponential parameters. These intervals will be compared with each other in terms of coverage rates and average interval lengths. It seems that if inference is made on the ratio of two exponential parameters, the Jeffreys prior performs better in terms of coverage rates, but the maximal data information prior performs better in terms of average interval lengths. Loss functions will also be used to derive Bayes estimates. The squared error loss and all-or-nothing loss functions will be compared with each other through a simulation study. The performance of each loss function will be compared by looking at the MSE and bias values of the Bayes estimates. It seems that the Jeffreys prior with the absolute error loss performs better than the other considered priors and loss functions, when Bayesian point estimates of the ratio of two exponential parameters is computed. An application is also considered where the different credibility intervals and Bayes estimates are calculated and compared.

Keywords: All-or-nothing loss, Bayesian intervals, Coverage rates, Jeffreys prior, Maximal data information prior, Probability matching prior, Ratio of two exponential parameters, Squared error loss, Uniform prior, Loss functions.

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List of Abbreviations and Notation

Abbreviations

MDI	Maximal data information
PM	Probability matching
GML	Ghosh, Mergel and Liu
MSE	Mean square error

Notation

$f(x \lambda)$	Density function
$F(x \lambda)$	Distribution function
$L(\lambda \underline{x})$	Likelihood function
$\pi(\lambda)$	Prior distribution
$I(\lambda)$	Fisher information
$\ell(\lambda, \hat{\lambda})$	Loss function
$\pi(\lambda \underline{x})$	Posterior distribution

Chapter 1

Introduction

1.1 Overview

The well-known Bayes' theorem was formulated by Reverend Thomas Bayes, and the work was published after his death in Bayes (1763). In this paper by Bayes, it shows how to make statistical inference which builds upon earlier knowledge, and how to combine it with data to update the degree of belief. The Bayes rule can thus be expressed as:

$$\textit{posterior distribution} \propto \textit{prior} \times \textit{likelihood function}.$$

Two main types of priors are found, namely subjective and objective priors. Objective priors are used when little or no prior information is available. Objective priors are also referred to as vague, flat, non-subjective or non-informative priors. In Irony and Singpurwalla (1997), José Bernardo said the following: “Non-subjective Bayesian analysis is just a part, - an important part, I believe -, of a healthy sensitivity analysis to the prior choice: it provides an answer to a very important question in scientific communication, namely, what could one conclude from the data if prior beliefs were such that the posterior distribution of the quantity of interest were dominated by the data”. The focus of this dissertation will be to compare a number of objective priors for the ratio of two exponential parameters.

1.2 Objectives

The main objectives of this dissertation can be summarised as follows:

- to derive the posterior distribution for the ratio of two exponential parameters when using the Jeffreys prior;
- to derive the Ghosh, Mergel and Lui prior for the exponential distribution, and to derive the posterior distribution for the ratio of two exponential parameters when using this prior;
- to derive the posterior distribution for the ratio of two exponential parameters when using the uniform prior;
- to derive the maximal data information prior for the exponential distribution, and to derive the posterior distribution for the ratio of two exponential parameters when using this prior;
- to obtain Bayes estimates when different loss functions are used;
- computing coverage rates and interval lengths when different objective priors are used.

1.3 Contributions

Given the objectives, the contribution of this dissertation can be summarised as follows:

- the derivation of the posterior distribution for the ratio of two exponential parameters when using the Jeffreys prior, using the method by Jeffreys (1961) to derive the prior;
- the derivation of the posterior distribution for the ratio of two exponential parameters when using the Ghosh, Mergel and Lui prior, using the method by Ghosh et al. (2011) to derive the prior;
- the derivation of the posterior distribution for the ratio of two exponential parameters, when using the maximal data information prior, using the method by Zellner (1971) to derive the prior.

As far as we know, the above mentioned objective priors have not been used to derive the posterior distribution for the ratio of two exponential parameters.

Kang et al. (2013) developed objective priors for the ratio of the scale parameters in the inverted exponential distributions. They considered first and second order matching priors, the reference prior and the Jeffreys prior. The work in this dissertation will be an extension of their work.

1.4 Dissertation outline

The literature review is done in Chapter 2. Where an overview of the exponential distribution, the Bayesian paradigm, a brief discussion of the various objective priors and loss functions are given in this chapter. The derivation of the objective priors and resulting posteriors are given in Chapter 3. An extensive simulation study is done in Chapter 4 and an application is also considered in this chapter. Concluding remarks and possibilities for future research will be discussed in Chapter 5.

Appendix A contains additional mathematical derivations. Appendix B contains additional simulation results. Appendix C contains R (R Core Team, 2019) code for the simulation studies done in Chapter 4, and Appendix D contains the MATLAB (MATLAB, 2017) code for the application in Chapter 4.

Chapter 2

Literature Study

2.1 Introduction

Bayesian methods are increasingly used in recent years in the theory and practice of statistics. Any Bayesian inference depends on a likelihood and a prior. The selection of prior has always been much debated in the Bayesian community. The Bayes rule can be expressed as: posterior distribution \propto likelihood function \times prior distribution. When we specify Bayesian models, we have to decide on prior distributions for the unknown parameters. As mentioned by Robert (2001), the most critical and most criticized point of Bayesian analysis deals with the choice of the prior distribution. For Bayesian estimates the performance depends on the prior distribution and the loss function that is used. Bayes estimators for the exponential distribution will be compared when using different vague priors.

2.2 The exponential distribution

The exponential distribution is a well-known distribution, it is a continuous distribution, with one parameter. It is often used in applications such as failure rates and reliability. Bain and Engelhardt (1991) discuss in detail the importance of the exponential distribution in reliability theory and life testing, see also Balakrishnan and Basu (1995) for further discussions on the importance and applications of the exponential distribution. The density function of an exponential random variable, X , with parameter $\lambda > 0$ is given by

$$f(x|\lambda) = \lambda \exp(-\lambda x), \quad x \geq 0. \quad (2.1)$$

The distribution function is given by

$$F(x|\lambda) = 1 - \exp(-\lambda x), \quad x \geq 0.$$

Figure 2.1 shows various plots of the density function for different values of λ .

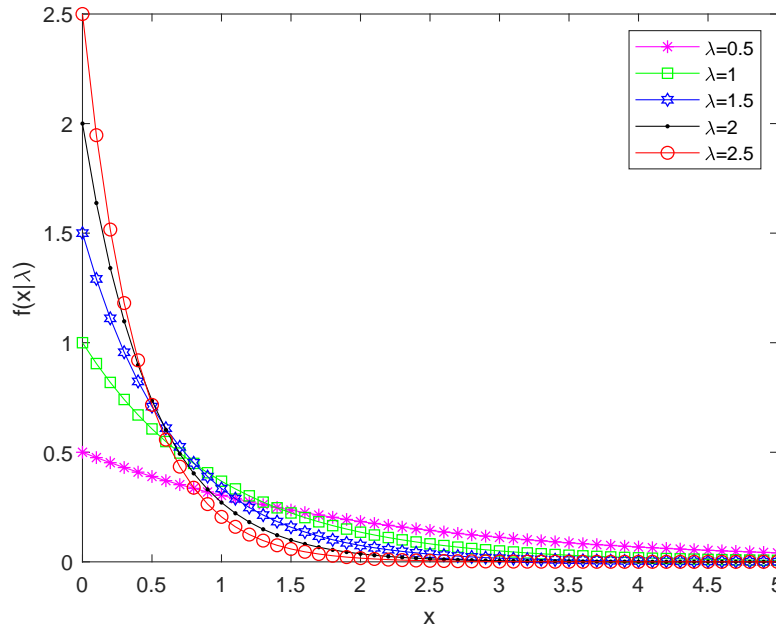


Figure 2.1: Exponential density function.

Suppose that X_1, X_2, \dots, X_n is a random sample from an exponential distribution, with unknown parameter λ , the likelihood function will then be given by

$$\begin{aligned} L(\lambda|\underline{x}) &= \prod_{i=1}^n \lambda \exp(-\lambda x_i) \\ &= \lambda^n \exp(-\lambda \sum_{i=1}^n x_i). \end{aligned}$$

From a Bayesian point of view, we will estimate this unknown parameter, λ , by giving it a prior distribution. Various objective priors will be discussed in the following section, and derived in Chapter 3.

2.3 Prior distributions

When we specify Bayesian models, we have to decide on prior distributions for unknown parameters. As mentioned by Robert (2001), the most critical and most criticised point of Bayesian analysis deals with the choice of the prior distribution. In Bayesian statistics, priors can be grouped in two main classes, namely subjective priors and objective priors. Subjective priors are used when one has actual prior knowledge about the problem and want to include this knowledge when choosing the prior. Objective priors, also known as vague or non-informative priors, is when one has little or no prior knowledge about the problem. In this dissertation the focus will be on objective priors.

In Syversveen (1998), Kass and Wasserman stated two different interpretations of non-informative priors: “Firstly there are non-informative priors which are formal representations of ignorance.” Secondly, “since there is no objective unique prior that represents ignorance, non-informative priors are selected by public agreement.” Robert (2001) disagrees, however and mentions that non-informative priors should rather be taken as default priors which the statistician may use when there is little or no prior information. It is important to note however, that, should there be no prior information the statistician must select a prior that will have the least amount of influence on the inference. The non-informative prior is used when little or no information is available to an experimenter prior to an experiment. In some cases the likelihood rules over the prior significantly, this may be the case for two reasons. The first reason is that researchers may have two different prior beliefs about an experiment, which may result in different outcomes for the same experiment. Then it is reasonable to use a reference prior which is an account of both researches prior beliefs and is ruled by the likelihood. The other reason is that the results of any experiment are meant to increase the knowledge of a reader, otherwise the experiment is a failure, if this is the case then the likelihood will dominate the prior as explained by Lee (2006).

The following five non-informative priors will be investigated in this dissertation:

- the Jeffreys prior;
- the Ghosh, Mergel and Liu prior;
- the uniform prior (also known as the Laplace prior);
- the maximal data information prior;
- the probability matching prior.

2.3.1 The Jeffreys prior

The Jeffreys prior, from Jeffreys (1961), is proportional to the square root of the determinant of the Fisher information matrix and is given by

$$\pi_A(\lambda) \propto \sqrt{|I(\lambda)|}, \quad (2.2)$$

where the Fisher information is given by

$$I(\lambda) = -E \left[\left(\frac{\partial^2 \log L(\lambda | \underline{x})}{\partial \lambda^2} \right) \right].$$

The Jeffreys prior satisfies the invariant parameterisation requirement.

According to Robert (2001), “the choice of a prior depending on Fisher information is justified by the fact that $I(\pi)$ is widely accepted as an indicator of the amount of information brought by the model (or the observation) about π .” In order to ensure that the prior distribution is as non-informative as possible, one should favour the values of π for which $I(\pi)$ is large. This is because if these values are favoured, the influence these have on the prior distribution is kept to a minimum, Gill (2008).

2.3.2 The Ghosh, Mergel and Liu prior

Ghosh et al. (2011) developed a prior where the distance between the prior and posterior is maximized by making use of the chi-square divergence. When other distances are used the Jeffreys prior is the result with adequate first order approximations but with the chi-square distance the second order approximations give this prior. Second order approximations is used since chi-square divergence approximations of the first order does not give priors. In other cases where other divergence measures are used, first order approximations gives priors that are adequate.

The prior that is suggested by Ghosh et al. (2011) can be calculated by taking the the fourth root of the Fisher information,

$$\pi_B(\lambda) \propto |I(\lambda)|^{\frac{1}{4}}.$$

2.3.3 The uniform prior

The uniform prior is used when you don't know how you would like to assign weights to the values such that some are favoured or if you don't want to let your beliefs influence inference. This is because of the fact that the weight that the uniform prior give to the values is equal, Bolstad (2007).

As mentioned by Robert and Rousseau (2010) a disadvantage of the uniform prior is that it does not have the property of invariance under reparametrisation.

We will denote the uniform prior by

$$\pi_C(\lambda) \propto \text{constant}. \quad (2.3)$$

2.3.4 The maximal data information (MDI) prior

This prior was first suggested by Zellner (1971). The maximal data information prior is chosen in such a way that the average information in the data density is maximized relative to that in the prior, Zellner (1996). Thus the use of the maximal data information prior leads to an emphasis on the information in the data density or likelihood function. The maximal data information prior is given by

$$\pi_D(\lambda) \propto \exp \{E(\log L(\lambda|x))\}. \quad (2.4)$$

2.3.5 The probability matching prior

A probability matching prior is a prior distribution under which the posterior probabilities match their frequentist coverage probabilities. The fact that the resulting Bayesian posterior intervals of level $1 - \alpha$ are also good frequentist confidence intervals at the same level is a very desirable situation. The method by Datta and Ghosh (1995) will be used to derive the probability matching prior in these cases. Datta and Ghosh (1995) derived the differential equation which a prior must satisfy if the posterior probability of a one sided credibility interval for a parametric function and its frequentist probability agree up to $O(n^{-1})$ where n is the sample size. The probability matching prior will be denoted by $\pi_E(\lambda)$.

The method to derive the probability matching prior is as follows:

1. Determine the likelihood function: $L(\underline{\lambda} | data)$, for a vector of unknown parameters $\underline{\lambda} = [\lambda_1 \lambda_2 \cdots \lambda_k]$.
2. Determine the inverse of the Fisher information matrix: $I^{-1}(\underline{\lambda})$.
3. Say for example one is interested in a probability matching prior for $t(\underline{\lambda})$, then determine:

$$\nabla'_t(\underline{\lambda}) = \left[\frac{\partial t(\underline{\lambda})}{\partial \lambda_1} \quad \frac{\partial t(\underline{\lambda})}{\partial \lambda_2} \quad \cdots \quad \frac{\partial t(\underline{\lambda})}{\partial \lambda_k} \right]$$

and

$$\nabla_t(\underline{\lambda}) = \left[\frac{\partial t(\underline{\lambda})}{\partial \lambda_1} \quad \frac{\partial t(\underline{\lambda})}{\partial \lambda_2} \quad \cdots \quad \frac{\partial t(\underline{\lambda})}{\partial \lambda_k} \right]'$$

4. Define

$$\eta'(\underline{\lambda}) = \frac{\nabla'_t(\underline{\lambda}) I^{-1}(\underline{\lambda})}{\sqrt{\nabla'_t(\underline{\lambda}) I^{-1}(\underline{\lambda}) \nabla_t(\underline{\lambda})}}$$

5. The prior $\pi(\underline{\lambda})$ is a probability matching prior if and only if the differential equation $\sum_{i=1}^k \frac{\partial}{\partial \lambda_i} \{ \eta_i(\underline{\lambda}) \pi(\underline{\lambda}) \} = 0$ is satisfied.

2.4 The posterior distribution

The posterior distribution is proportional to the prior distribution multiplied by the likelihood. It can be said that the posterior distribution is kind of a summary about what we believe regarding the parameter of interest after we have seen the data, Bolstad (2007). Interpretation of the posterior distribution is not always easy and therefore we will rather make use of measures of location or measures of spread which characterise the data, Bolstad (2007). By making use of percentiles, Bayesian credible intervals can be obtained and with a certain amount of certainty one can say that the parameter lies somewhere in that interval, Bolstad (2007). Bayesian credible intervals are related to confidence intervals but lastly mentioned can't be used to directly interpret the probability where credible intervals have the ability of direct probability interpretation, Bolstad (2007).

2.5 Loss functions

The parameter of interest is λ and we would like to estimate this parameter. There will be losses incurred when this is done. Let $\hat{\lambda}$ be the estimate of λ then a loss function $\ell(\hat{\lambda}, \lambda)$ represents the losses incurred when λ is estimated by $\hat{\lambda}$. If $\hat{\lambda} = \lambda$ there is zero loss. The most important thing about loss functions is their contribution and ability to define what a “good” estimate means, Poirier (1995). Various loss functions will give different Bayes estimates for λ . Two different loss functions will be considered in this dissertation, the squared error loss and the all-or-nothing loss.

2.5.1 Squared error loss

By far the most commonly used loss function in estimation problems is the squared error loss function. This loss function is defined as (Guure et al., 2012)

$$\ell(\lambda, \hat{\lambda}) = (\lambda - \hat{\lambda})^2.$$

As can be seen above, the squared error loss function measures the squared difference between the estimate and the true parameter value. Thus, how badly the punishment is for using the estimate and not the true value. Due to the square, we can say that for example if the absolute difference is equal to 2, the loss function says the estimate performs four times worse than the case where the absolute difference is equal to 1. Thus, the higher the absolute difference, the greater the punishment when using the squared error loss function. The reason for being the most commonly used function is due to the fact that it is symmetric, Guure et al. (2012). It does not matter if it is an over or under estimate of the true value, the squared error loss function will give equal weights to both sides.

When the squared error loss function is used, the Bayes estimate is the posterior mean.

2.5.2 All-or-nothing loss

Another commonly used loss function in estimation problems is the all-or-nothing loss function. According to Poirier (1995) this function can be defined as

$$\ell(\lambda, \hat{\lambda}) = 0 \text{ if } \hat{\lambda} = \lambda$$

$$\ell(\lambda, \hat{\lambda}) = 1 \text{ if } \hat{\lambda} \neq \lambda.$$

This is the same as to say that when the estimate is equal to the true parameter the all-or-nothing loss is zero and when the estimate differs from the true parameter, does not matter how far, the all-or-nothing loss is equal to a constant. We can say that if the absolute difference between the estimate and the true parameter is equal to 2, the loss function is also equal to two but in fact it does not matter how far the estimate is from the true value because the loss function will always be equal to the same constant. Thus, meaning that the punishment for the wrong estimate stays the same, regardless of how big the absolute difference is. To enhance your chances of getting the estimate equal to the true value, the estimate should be the value that are most likely to occur. Thus the Bayes estimate for the all-or-nothing loss is the posterior mode.

In the graph below is the illustration of how the two loss functions discussed above, differ in form from each other.

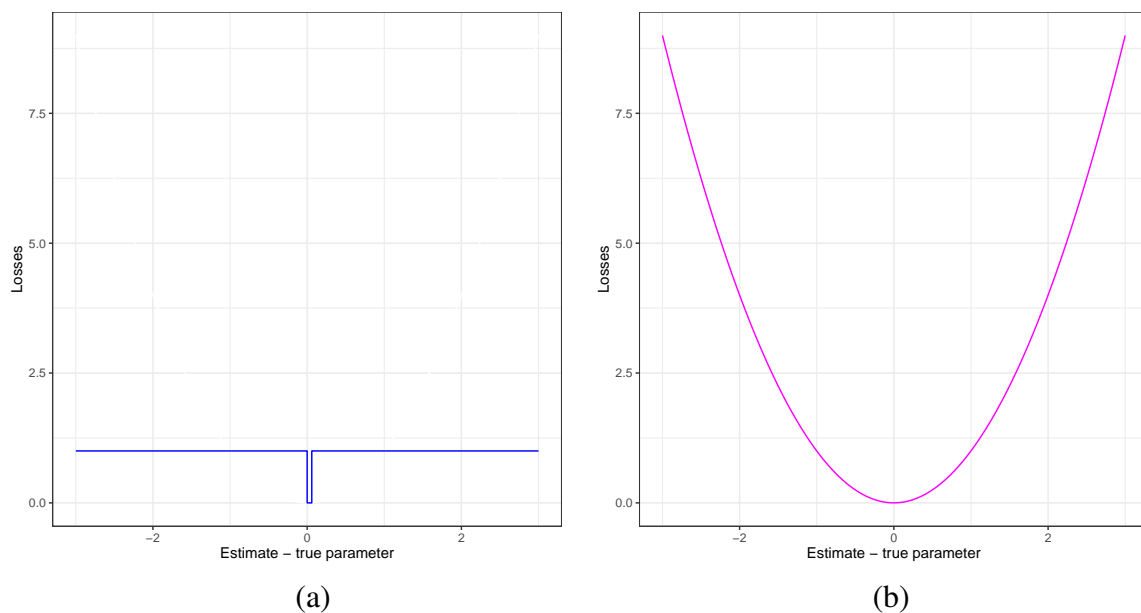


Figure 2.2: The form of two different loss functions, namely (a) the all-or-nothing loss function, and (b) the squared error loss function.

2.6 The ratio of two exponential parameters

The ratio of the parameters of two exponential distributions is important, if the ratio is equal to one, then the hazard rate at a certain time of these two distributions is identical. Kang et al. (2013) focused on the development of non-informative priors for the ratio of two scale parameters for the inverted exponential distribution. They looked at the first and second order matching priors, the reference prior and Jeffreys prior. Where they also showed that the reference prior and Jeffreys' prior are the second order matching prior. Lee (2006) investigated the reliability $R = P(X < Y)$, where X and Y are two independent exponential random variables, a classical method was used here. Polcer (2009) developed confidence intervals for the ratio of two exponential parameters, applying it to a quality control application, monitoring two processes over time. Van Zyl and Van der Merwe (2016) applied a Bayesian procedure to obtain control limits for the location and scale parameters, as well as for a one-sided upper tolerance limit in the case of the two-parameter exponential distribution where they only considered the Jeffreys prior. Madi and Tsui (1990) considered the estimation of the ratio of the scale parameters of two independent two-parameter exponential distributions with unknown location parameters. Handa and Kambo (2005) considered frequentists based tests for the equality of parameters of two exponential distributions with common known coefficient of variation.

Krishnamoorthy and Xia (2017) looked at confidence intervals for a two-parameter exponential distribution. Where they used the distribution function of a pivotal quantity to construct confidence limits, they also only looked at the difference between two exponential means using frequentist methods.

Thiagarajah and Paul (1997) investigated interval estimation for the scale parameter of the two-parameter exponential distribution, by looking at the adjusted sign root likelihood ratio, conditional likelihood, and skewness corrected score. They only considered frequentist methods, and only looked at procedures for constructing confidence intervals for the scale parameter of the two-parameter exponential distribution.

2.7 Summary

In this chapter the background knowledge that is needed for this study was discussed. The different prior distributions and how each one of them can be derived, is explained. We also

briefly discussed the posterior distribution, as well as loss functions and the exponential distribution, where the samples are drawn from. Lastly we looked at work done on the ratio of two exponential parameters. The techniques explained in this chapter will now be used in Chapter 3 to derive each of the five prior distributions together with their posterior distributions for λ_1 and λ_2 and the marginal posterior of the ratio of two exponential parameters, $\theta = \frac{\lambda_1}{\lambda_2}$.

Chapter 3

Bayesian Methods

3.1 Likelihood

Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_m denote independent random samples from exponential distributions, $X_i \sim \text{Exp}(\lambda_1)$ $i = 1, 2, \dots, n$ and $Y_j \sim \text{Exp}(\lambda_2)$ $j = 1, 2, \dots, m$.

The density functions are given by

$$\begin{aligned} f(x_i|\lambda_1) &= \lambda_1 \exp(-\lambda_1 x_i), & x_i \geq 0, & \lambda_1 > 0 \\ f(y_j|\lambda_2) &= \lambda_2 \exp(-\lambda_2 y_j), & y_j \geq 0, & \lambda_2 > 0. \end{aligned}$$

The likelihood function will then be

$$\begin{aligned} L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &= \prod_{i=1}^n \lambda_1 \exp(-\lambda_1 x_i) \prod_{j=1}^m \lambda_2 \exp(-\lambda_2 y_j) \\ &= \lambda_1^n \exp\left(-\lambda_1 \sum_{i=1}^n x_i\right) \lambda_2^m \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right) \\ &= \lambda_1^n \lambda_2^m \exp\left[-\left(\lambda_1 \sum_{i=1}^n x_i + \lambda_2 \sum_{j=1}^m y_j\right)\right]. \end{aligned} \tag{3.1}$$

Let $\theta = \frac{\lambda_1}{\lambda_2} = \prod_{i=1}^2 \lambda_i^{a_i}$, $a_1 = 1$ and $a_2 = -1$, be the parameter of interest. We will first look at the priors and posteriors for (λ_1, λ_2) and then derive the posteriors for θ . The Fisher information matrix for (λ_1, λ_2) will now be considered, since the Fisher information will be needed to obtain most of the vague priors considered in this dissertation.

The Fisher Information matrix is given by

$$I(\lambda_1, \lambda_2) = \begin{bmatrix} \frac{n}{\lambda_1^2} & 0 \\ 0 & \frac{m}{\lambda_2^2} \end{bmatrix}. \quad (3.2)$$

This proof is well-known and given in Appendix A.

3.2 Priors and posteriors

3.2.1 The Jeffreys prior and resulting posterior

The Jeffreys prior is given by

$$\begin{aligned} \pi_A(\lambda_1, \lambda_2) &\propto \sqrt{|I(\lambda_1, \lambda_2)|} = \sqrt{\frac{nm}{\lambda_1^2 \lambda_2^2}} \\ \therefore \pi_A(\lambda_1, \lambda_2) &\propto \lambda_1^{-1} \lambda_2^{-1}. \end{aligned} \quad (3.3)$$

The resulting posterior will then be

$$\begin{aligned} \pi_A(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &\propto \lambda_1^{-1} \lambda_2^{-1} \lambda_1^n \lambda_2^m \exp \left[- \left(\lambda_1 \sum_{i=1}^n x_i + \lambda_2 \sum_{j=1}^m y_j \right) \right] \\ \therefore \pi_A(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &\propto \lambda_1^{n-1} \exp \left(-\lambda_1 \sum_{i=1}^n x_i \right) \lambda_2^{m-1} \exp \left(-\lambda_2 \sum_{j=1}^m y_j \right). \end{aligned} \quad (3.4)$$

The posterior is the product of two independent gamma distributions, $\text{Gamma} \left(n, \sum_{i=1}^n x_i \right)$ and $\text{Gamma} \left(m, \sum_{j=1}^m y_j \right)$. The joint posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ and λ_2 will now be obtained, followed by the marginal posterior for $\theta = \frac{\lambda_1}{\lambda_2}$.

Theorem 3.1. *The posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ when using the Jeffreys prior, is*

$$\pi_A(\theta | \underline{x}, \underline{y}) \propto \theta^{n-1} \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j} \right) \right]^{-n-m}, \theta > 0, \quad (3.5)$$

which is a generalized beta prime distribution with shape parameters n and m , and a further shape parameter of 1 and scale parameter $\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}$.

Proof. Using (3.4), the joint posterior distribution for (θ, λ_2) will be derived. Let $\theta = \frac{\lambda_1}{\lambda_2}$, then $\lambda_1 = \theta \lambda_2$ and the resulting Jacobian will be

$$J = \begin{vmatrix} \frac{\partial \lambda_1}{\partial \theta} & \frac{\partial \lambda_1}{\partial \lambda_2} \\ \frac{\partial \lambda_2}{\partial \theta} & \frac{\partial \lambda_2}{\partial \lambda_2} \end{vmatrix} = \begin{vmatrix} \lambda_2 & \theta \\ 0 & 1 \end{vmatrix} = \lambda_2.$$

The joint posterior distribution for (θ, λ_2) will then be

$$\begin{aligned} \pi_A(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto (\theta \lambda_2)^{n-1} \exp\left(-\theta \lambda_2 \sum_{i=1}^n x_i\right) \lambda_2^{m-1} \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right) \lambda_2 \\ &= \theta^{n-1} \lambda_2^{n-1} \lambda_2^{m-1} \lambda_2 \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) \\ \therefore \pi_A(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto \theta^{n-1} \lambda_2^{n+m-1} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right). \end{aligned}$$

The marginal posterior of θ will then be

$$\begin{aligned} \pi_A(\theta | \underline{x}, \underline{y}) &= \int_0^\infty \pi_A(\theta, \lambda_2 | \underline{x}, \underline{y}) d\lambda_2 \\ &\propto \theta^{n-1} \int_0^\infty \lambda_2^{n+m-1} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) d\lambda_2 \\ &= \theta^{n-1} \frac{\Gamma(n+m)}{\left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{n+m}} \\ &= \theta^{n-1} \Gamma(n+m) \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{-n-m} \\ &= \theta^{n-1} \Gamma(n+m) \left[\sum_{j=1}^m y_j \left(1 + \theta \frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-n-m} \\ \therefore \pi_A(\theta | \underline{x}, \underline{y}) &\propto \theta^{n-1} \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-n-m}, \theta > 0, \end{aligned}$$

which is a generalized beta prime distribution. \square

3.2.2 The Ghosh, Mergel and Lui prior and resulting posterior

The GML prior is given by

$$\begin{aligned}\pi_B(\lambda_1, \lambda_2) &\propto |I(\lambda_1, \lambda_2)|^{\frac{1}{4}} = \left(\frac{nm}{\lambda_1^2 \lambda_2^2}\right)^{\frac{1}{4}} \\ \therefore \pi_B(\lambda_1, \lambda_2) &\propto \lambda_1^{-\frac{1}{2}} \lambda_2^{-\frac{1}{2}}.\end{aligned}\quad (3.6)$$

The resulting posterior will then be

$$\begin{aligned}\pi_B(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &\propto \lambda_1^{-\frac{1}{2}} \lambda_2^{-\frac{1}{2}} \lambda_1^n \lambda_2^m \exp\left[-\left(\lambda_1 \sum_{i=1}^n x_i + \lambda_2 \sum_{j=1}^m y_j\right)\right] \\ \therefore \pi_B(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &\propto \lambda_1^{n-\frac{1}{2}} \exp\left(-\lambda_1 \sum_{i=1}^n x_i\right) \lambda_2^{m-\frac{1}{2}} \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right).\end{aligned}\quad (3.7)$$

The posterior is the product of two independent gamma distributions, $Gamma\left(n + \frac{1}{2}, \sum_{i=1}^n x_i\right)$ and $Gamma\left(m + \frac{1}{2}, \sum_{j=1}^m y_j\right)$. The joint posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ and λ_2 will now be obtained, followed by the marginal posterior for $\theta = \frac{\lambda_1}{\lambda_2}$.

Theorem 3.2. *The posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ when using the Ghosh, Mergel and Lui prior, is*

$$\pi_B(\theta | \underline{x}, \underline{y}) \propto \theta^{n-\frac{1}{2}} \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-(n+m+1)}, \theta > 0, \quad (3.8)$$

which is a generalized beta prime distribution with shape parameters $n + \frac{1}{2}$ and $m + \frac{1}{2}$, and a further shape parameter of 1 and scale parameter $\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}$.

Proof. Using (3.7), the joint posterior distribution for (θ, λ_2) will be derived. Let $\theta = \frac{\lambda_1}{\lambda_2}$, then $\lambda_1 = \theta \lambda_2$ and the resulting Jacobian will be

$$J = \begin{vmatrix} \frac{\partial \lambda_1}{\partial \theta} & \frac{\partial \lambda_1}{\partial \lambda_2} \\ \frac{\partial \lambda_2}{\partial \theta} & \frac{\partial \lambda_2}{\partial \lambda_2} \end{vmatrix} = \begin{vmatrix} \lambda_2 & \theta \\ 0 & 1 \end{vmatrix} = \lambda_2.$$

Now

$$\begin{aligned}
\pi_B(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto (\theta \lambda_2)^{n-\frac{1}{2}} \exp\left(-\theta \lambda_2 \sum_{i=1}^n x_i\right) \lambda_2^{m-\frac{1}{2}} \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right) \lambda_2 \\
&= \theta^{n-\frac{1}{2}} \lambda_2^{n-\frac{1}{2}} \lambda_2^{m-\frac{1}{2}} \lambda_2 \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) \\
\therefore \pi_B(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto \theta^{n-\frac{1}{2}} \lambda_2^{n+m} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right).
\end{aligned}$$

The marginal posterior of θ will then be

$$\begin{aligned}
\pi_B(\theta | \underline{x}, \underline{y}) &= \int_0^\infty \pi_B(\theta, \lambda_2 | \underline{x}, \underline{y}) d\lambda_2 \\
&\propto \theta^{n-\frac{1}{2}} \int_0^\infty \lambda_2^{n+m} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) d\lambda_2 \\
&= \theta^{n-\frac{1}{2}} \frac{\Gamma(n+m+1)}{\left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{n+m+1}} \\
&= \theta^{n-\frac{1}{2}} \Gamma(n+m+1) \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{-n-m-1} \\
&= \theta^{n-\frac{1}{2}} \Gamma(n+m+1) \left[\sum_{j=1}^m y_j \left(1 + \theta \frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-n-m-1} \\
\therefore \pi_B(\theta | \underline{x}, \underline{y}) &\propto \theta^{n-\frac{1}{2}} \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-(n+m+1)}, \theta > 0,
\end{aligned}$$

which is a generalized beta prime distribution. □

3.2.3 The uniform prior and resulting posterior

The uniform prior is given by

$$\pi_C(\lambda_1, \lambda_2) \propto \text{constant}. \quad (3.9)$$

The resulting posterior will then be

$$\pi_C(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) \propto \lambda_1^n \exp\left(-\lambda_1 \sum_{i=1}^n x_i\right) \lambda_2^m \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right). \quad (3.10)$$

The posterior is the product of two independent gamma distributions, $\text{Gamma}\left(n+1, \sum_{i=1}^n x_i\right)$ and $\text{Gamma}\left(m+1, \sum_{j=1}^m y_j\right)$.

The joint posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ and λ_2 will now be obtained, followed by the marginal posterior for $\theta = \frac{\lambda_1}{\lambda_2}$.

Theorem 3.3. *The posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ when using the uniform prior, is*

$$\pi_C(\theta | \underline{x}, \underline{y}) \propto \theta^n \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-(n+m+2)}, \theta > 0, \quad (3.11)$$

which is a generalized beta prime distribution with shape parameters $n+1$ and $m+1$, and a further shape parameter of 1 and scale parameter $\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}$.

Proof. Using (3.10), the joint posterior distribution for (θ, λ_2) will be derived. Let $\theta = \frac{\lambda_1}{\lambda_2}$, then $\lambda_1 = \theta \lambda_2$ and the resulting Jacobian will be

$$J = \begin{vmatrix} \frac{\partial \lambda_1}{\partial \theta} & \frac{\partial \lambda_1}{\partial \lambda_2} \\ \frac{\partial \lambda_2}{\partial \theta} & \frac{\partial \lambda_2}{\partial \lambda_2} \end{vmatrix} = \begin{vmatrix} \lambda_2 & \theta \\ 0 & 1 \end{vmatrix} = \lambda_2.$$

Now

$$\begin{aligned} \pi_C(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto (\theta \lambda_2)^n \exp\left(-\theta \lambda_2 \sum_{i=1}^n x_i\right) \lambda_2^m \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right) \lambda_2 \\ &= \theta^n \lambda_2^n \lambda_2^m \lambda_2 \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) \\ \therefore \pi_C(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto \theta^n \lambda_2^{n+m+1} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right). \end{aligned}$$

The marginal posterior of θ will then be

$$\begin{aligned}
\pi_C(\theta | \underline{x}, \underline{y}) &= \int_0^\infty \pi_C(\theta, \lambda_2 | \underline{x}, \underline{y}) d\lambda_2 \\
&\propto \theta^n \int_0^\infty \lambda_2^{n+m+1} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) d\lambda_2 \\
&= \theta^n \frac{\Gamma(n+m+2)}{\left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{n+m+2}} \\
&= \theta^n \Gamma(n+m+2) \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{-n-m-2} \\
&= \theta^n \Gamma(n+m+2) \left[\sum_{j=1}^m y_j \left(1 + \theta \frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-n-m-2} \\
\therefore \pi_C(\theta | \underline{x}, \underline{y}) &\propto \theta^n \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-(n+m+2)}, \theta > 0,
\end{aligned}$$

which is a generalized beta prime distribution. \square

3.2.4 The maximal data information (MDI) prior and resulting posterior

Theorem 3.4. *The MDI prior for (λ_1, λ_2) is given by*

$$\pi_D(\lambda_1, \lambda_2) \propto \lambda_1^n \lambda_2^m. \quad (3.12)$$

Proof. Derivation of the MDI prior

$$\begin{aligned}
E [\log (L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}))] &= E \left[n \log \lambda_1 + m \log \lambda_2 - \lambda_1 \sum_{i=1}^n x_i - \lambda_2 \sum_{j=1}^m y_j \right] \\
&= n \log \lambda_1 + m \log \lambda_2 - \lambda_1 \sum_{i=1}^n \frac{1}{\lambda_1} - \lambda_2 \sum_{j=1}^m \frac{1}{\lambda_2} \\
&= n \log \lambda_1 + m \log \lambda_2 - n - m \\
&= \log \lambda_1^n + \log \lambda_2^m - n - m \\
&= \log \lambda_1^n \lambda_2^m - n - m,
\end{aligned}$$

now

$$\begin{aligned}
\exp \{E [\log (L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}))]\} &= \exp (\log \lambda_1^n \lambda_2^m - n - m) \\
&= \lambda_1^n \lambda_2^m \exp (-n) \exp (-m) \\
\therefore \pi_D(\lambda_1, \lambda_2) &\propto \lambda_1^n \lambda_2^m.
\end{aligned}$$

□

The resulting posterior will then be

$$\begin{aligned}
\pi_D(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &\propto \lambda_1^n \lambda_2^m \lambda_1^n \lambda_2^m \exp \left[- \left(\lambda_1 \sum_{i=1}^n x_i + \lambda_2 \sum_{j=1}^m y_j \right) \right] \\
\therefore \pi_D(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) &\propto \lambda_1^{2n} \exp \left(-\lambda_1 \sum_{i=1}^n x_i \right) \lambda_2^{2m} \exp \left(-\lambda_2 \sum_{j=1}^m y_j \right). \quad (3.13)
\end{aligned}$$

The posterior is the product of two independent gamma distributions, $\text{Gamma} \left(2n + 1, \sum_{i=1}^n x_i \right)$ and $\text{Gamma} \left(2m + 1, \sum_{j=1}^m y_j \right)$.

The joint posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ and λ_2 will now be obtained, followed by the marginal posterior for $\theta = \frac{\lambda_1}{\lambda_2}$.

Theorem 3.5. *The posterior for $\theta = \frac{\lambda_1}{\lambda_2}$ when using the MDI prior, is*

$$\pi_D(\theta | \underline{x}, \underline{y}) \propto \theta^{2n} \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j} \right) \right]^{-2(n+m+1)}, \theta > 0, \quad (3.14)$$

which is a generalized beta prime distribution with shape parameters $2n + 1$ and $2m + 1$, and a further shape parameter of 1 and scale parameter $\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}$.

Proof. Using (3.13), the joint posterior distribution for (θ, λ_2) will be derived. Let $\theta = \frac{\lambda_1}{\lambda_2}$, then $\lambda_1 = \theta \lambda_2$ and the resulting Jacobian will be

$$J = \begin{vmatrix} \frac{\partial \lambda_1}{\partial \theta} & \frac{\partial \lambda_1}{\partial \lambda_2} \\ \frac{\partial \lambda_2}{\partial \theta} & \frac{\partial \lambda_2}{\partial \lambda_2} \end{vmatrix} = \begin{vmatrix} \lambda_2 & \theta \\ 0 & 1 \end{vmatrix} = \lambda_2.$$

Now

$$\begin{aligned} \pi_D(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto (\theta \lambda_2)^{2n} \exp\left(\theta \lambda_2 \sum_{i=1}^n x_i\right) \lambda_2^{2m} \exp\left(-\lambda_2 \sum_{j=1}^m y_j\right) \lambda_2 \\ &= \theta^{2n} \lambda_2^{2n} \lambda_2^{2m} \lambda_2 \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) \\ \therefore \pi_D(\theta, \lambda_2 | \underline{x}, \underline{y}) &\propto \theta^{2n} \lambda_2^{2n+2m+1} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right). \end{aligned}$$

The marginal posterior of θ will then be

$$\begin{aligned} \pi_D(\theta | \underline{x}, \underline{y}) &= \int_0^\infty \pi_D(\theta, \lambda_2 | \underline{x}, \underline{y}) d\lambda_2 \\ &\propto \theta^{2n} \int_0^\infty \lambda_2^{2n+2m+1} \exp\left(-\lambda_2 \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) d\lambda_2 \\ &= \theta^{2n} \frac{\Gamma(2n+2m+2)}{\left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{2n+2m+2}} \\ &= \theta^{2n} \Gamma(2n+2m+2) \left(\theta \sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)^{-2n-2m-2} \\ &= \theta^{2n} \Gamma(2n+2m+2) \left[\sum_{j=1}^m y_j \left(1 + \theta \frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right) \right]^{-2n-2m-2} \\ \therefore \pi_D(\theta | \underline{x}, \underline{y}) &\propto \theta^{2n} \left[1 + \theta \left(\frac{\sum_{i=1}^n x_i}{\sum_{j=1}^m y_j}\right)\right]^{-2(n+m+1)}, \theta > 0, \end{aligned}$$

which is a generalized beta prime distribution. \square

3.2.5 The probability matching prior and resulting posterior

We want to find the probability matching prior for $\theta = \frac{\lambda_1}{\lambda_2}$.

Theorem 3.6. *The probability matching prior for (λ_1, λ_2) where the parameters of interest $\theta = \frac{\lambda_1}{\lambda_2}$, is*

$$\pi_E(\lambda_1, \lambda_2) \propto \lambda_1^{-1} \lambda_2^{-1}. \quad (3.15)$$

Proof. Using the result from (A.1), the inverse of the Fisher information matrix is given by

$$I^{-1}(\lambda_1, \lambda_2) = \begin{bmatrix} \frac{\lambda_1^2}{n} & 0 \\ 0 & \frac{\lambda_2^2}{m} \end{bmatrix}.$$

We are interested in the probability matching prior for $\theta = t(\lambda_1, \lambda_2) = \frac{\lambda_1}{\lambda_2}$. Now

$$\begin{aligned} \nabla'_t(\lambda_1, \lambda_2) &= \begin{bmatrix} \frac{\partial t(\lambda_1, \lambda_2)}{\partial \lambda_1} & \frac{\partial t(\lambda_1, \lambda_2)}{\partial \lambda_2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\lambda_2} & -\frac{\lambda_1}{\lambda_2^2} \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \nabla'_t(\lambda_1, \lambda_2) I^{-1}(\lambda_1, \lambda_2) &= \begin{bmatrix} \frac{1}{\lambda_2} & -\frac{\lambda_1}{\lambda_2^2} \end{bmatrix} \begin{bmatrix} \frac{\lambda_1^2}{n} & 0 \\ 0 & \frac{\lambda_2^2}{m} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\lambda_1^2}{n\lambda_2} & -\frac{\lambda_1}{m} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} \nabla'_t(\lambda_1, \lambda_2) I^{-1}(\lambda_1, \lambda_2) \nabla_t(\lambda_1, \lambda_2) &= \begin{bmatrix} \frac{\lambda_1^2}{n\lambda_2} & -\frac{\lambda_1}{m} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_2} \\ -\frac{\lambda_1}{\lambda_2^2} \end{bmatrix} \\ &= \frac{\lambda_1^2}{n\lambda_2^2} + \frac{\lambda_1^2}{m\lambda_2^2} \\ &= \frac{\lambda_1^2}{\lambda_2^2} \left(\frac{1}{n} + \frac{1}{m} \right) \\ &= \frac{\lambda_1^2}{\lambda_2^2} \left(\frac{n+m}{nm} \right). \end{aligned}$$

Define

$$\begin{aligned}\eta'(\lambda_1, \lambda_2) &= \frac{\nabla'_t(\lambda_1, \lambda_2) I^{-1}(\lambda_1, \lambda_2)}{\sqrt{\nabla'_t(\lambda_1, \lambda_2) I^{-1}(\lambda_1, \lambda_2) \nabla_t(\lambda_1, \lambda_2)}} \\ &= \left[\frac{\lambda_1^2}{n\lambda_2} \cdot \frac{\lambda_2}{\lambda_1} \left(\frac{nm}{n+m}\right)^{\frac{1}{2}} \quad \frac{-\lambda_1}{m} \cdot \frac{\lambda_2}{\lambda_1} \left(\frac{nm}{n+m}\right)^{\frac{1}{2}} \right] \\ &= \left[\frac{\lambda_1}{\sqrt{n}} \cdot \left(\frac{m}{n+m}\right)^{\frac{1}{2}} \quad \frac{-\lambda_2}{\sqrt{m}} \left(\frac{n}{n+m}\right)^{\frac{1}{2}} \right].\end{aligned}$$

The prior $\pi_E(\lambda_1, \lambda_2)$ is a probability matching prior if and only if the differential equation

$$\sum_{i=1}^2 \frac{\partial}{\partial \lambda_i} \{ \eta_i(\lambda_1, \lambda_2) \pi_E(\lambda_1, \lambda_2) \} = 0$$

is satisfied.

Let $\pi_E(\lambda_1, \lambda_2) = \lambda_1^{-1} \lambda_2^{-1}$, then

$$\eta_1(\lambda_1, \lambda_2) \pi_E(\lambda_1, \lambda_2) = \lambda_2^{-1} \left(\frac{m}{n(n+m)} \right)^{\frac{1}{2}}$$

and

$$\frac{\partial \eta_1(\lambda_1, \lambda_2) \pi_E(\lambda_1, \lambda_2)}{\partial \lambda_1} = 0$$

and

$$\eta_2(\lambda_1, \lambda_2) \pi_E(\lambda_1, \lambda_2) = -\lambda_1^{-1} \left(\frac{n}{m(n+m)} \right)^{\frac{1}{2}}$$

and

$$\frac{\partial \eta_2(\lambda_1, \lambda_2) \pi_E(\lambda_1, \lambda_2)}{\partial \lambda_2} = 0.$$

Thus

$$\pi_E(\lambda_1, \lambda_2) = \lambda_1^{-1} \lambda_2^{-1}$$

is the probability matching prior. □

From (3.15) it is clear that the probability matching prior is the same as the Jeffreys prior in (3.3).

3.3 Bayes estimates

All the posterior distributions of θ follows a generalized beta prime distribution with different shape and scale parameters. The formulas in Table 3.1 is well known and are used to calculate Bayes estimates of $\theta = \lambda_1/\lambda_2$ under the different loss functions λ considered in Chapter 2.

Table 3.1: Generalized beta prime posterior distribution of $\theta = \lambda_1/\lambda_2$.

$f(\theta; \alpha, \beta, p, q) = \frac{p \left(\frac{\theta}{q}\right)^{\alpha p - 1} \left(1 + \left(\frac{\theta}{q}\right)^p\right)^{-\alpha - \beta}}{qB(\alpha, \beta)}$	
Mean	Mode
$\frac{q\Gamma\left(\alpha + \frac{1}{p}\right)\Gamma\left(\beta - \frac{1}{p}\right)}{\Gamma(\alpha)\Gamma(\beta)} \quad \text{if } \beta p > 1$	$q\left(\frac{\alpha p - 1}{\beta p + 1}\right)^{\frac{1}{p}} \quad \text{if } \alpha p \geq 1$

The Bayes estimates of the various priors and loss functions are given in Table 3.2.

Table 3.2: Summary of Bayes estimates of $\theta = \lambda_1/\lambda_2$ under the different loss functions.

Prior	Squared error (mean)	All-or-nothing (mode)
$\pi_A(\theta)$	$\frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(n+1) \Gamma(m-1)}{\Gamma(n) \Gamma(m)}, m > 1$	$\left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{n-1}{m+1}\right), n \geq 1$
$\pi_B(\theta)$	$\frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(n + \frac{3}{2}) \Gamma(m - \frac{1}{2})}{\Gamma(n + \frac{1}{2}) \Gamma(m + \frac{1}{2})}, m > \frac{1}{2}$	$\left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{n - \frac{1}{2}}{m + \frac{3}{2}}\right), n \geq \frac{1}{2}$
$\pi_C(\theta)$	$\frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(n+2) \Gamma(m)}{\Gamma(n+1) \Gamma(m+1)}, m > 0$	$\left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{n}{m+2}\right), n \geq 0$
$\pi_D(\theta)$	$\frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(2n+2) \Gamma(2m)}{\Gamma(2n+1) \Gamma(2m+1)}, m > 0$	$\left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{2n}{2m+2}\right), n \geq 0$

3.4 Summary

The different prior distributions considered in this dissertation were derived in this chapter. Each prior distributions' posterior distribution for (λ_1, λ_2) , as well as the marginal posterior of θ , were derived. We also showed how Bayes estimates for the two different loss function can be derived. The Bayes estimates are also given for each prior used. These prior and posterior distributions will now be used in the simulation studies that follow, with an application at the end of the next chapter.

Chapter 4

Simulation Study and Application

4.1 Simulation method

In this chapter an extensive simulation study will be done. The vague priors discussed in Chapter 2 and derived in Chapter 3, will be compared. Their performances will be evaluated by looking at the resulting coverage rates and average interval lengths. This will be done in simulation study I, in Section 4.2. The preferred credibility interval, will be the interval which is the shortest and which has a coverage rate the closest to the nominal level.

The coverage of a credibility interval is the proportion of times that the credibility interval contains the true specified parameter value, Agresti and Coull (2012). Over-coverage suggests that the results are too conservative as more simulations will not find a significant result when there is a true effect, Newcombe (1998). Under-coverage indicates over confidence in the estimates since more simulations will detect a significant result. Therefore, the coverage should be approximately equal to the nominal coverage rate.

Different values for λ_1 and λ_2 , with different sample sizes, n and m , will be considered in both simulation studies. Some of the results gave the same findings and are included just for noting.

The first simulation procedure is given in Algorithm 4.1. Additional results are given in Appendix B and R code provided in Appendix C.

Algorithm 4.1 Simulation procedure I.

1. For a given λ_1 and n , simulate the data, \underline{x} , from an exponential distribution. For a given λ_2 and m , simulate the data, \underline{y} , from an exponential distribution.
2. For n and the simulated data, \underline{x} , simulate the posterior λ_1 from a gamma distribution:

$$\begin{aligned} \text{Jeffreys} - \pi_A(\theta) &\rightarrow \text{Gamma} \left(n, \sum_{i=1}^n x_i \right) \\ \text{GML} - \pi_B(\theta) &\rightarrow \text{Gamma} \left(n + \frac{1}{2}, \sum_{i=1}^n x_i \right) \\ \text{Uniform} - \pi_C(\theta) &\rightarrow \text{Gamma} \left(n + 1, \sum_{i=1}^n x_i \right) \\ \text{MDI} - \pi_D(\theta) &\rightarrow \text{Gamma} \left(2n + 1, \sum_{i=1}^n x_i \right). \end{aligned}$$

For m and the simulated data, \underline{y} , simulate the posterior λ_2 from a gamma distribution:

$$\begin{aligned} \text{Jeffreys} - \pi_A(\theta) &\rightarrow \text{Gamma} \left(m, \sum_{j=1}^m y_j \right) \\ \text{GML} - \pi_B(\theta) &\rightarrow \text{Gamma} \left(m + \frac{1}{2}, \sum_{j=1}^m y_j \right) \\ \text{Uniform} - \pi_C(\theta) &\rightarrow \text{Gamma} \left(m + 1, \sum_{j=1}^m y_j \right) \\ \text{MDI} - \pi_D(\theta) &\rightarrow \text{Gamma} \left(2m + 1, \sum_{j=1}^m y_j \right). \end{aligned}$$

Simulate a 1000 values.

3. Using the posterior λ_1 and the posterior λ_2 , determine $\theta = \frac{\lambda_1}{\lambda_2}$.
4. Sort the 1000 values of θ from smallest to largest, i.e.: $\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(1000)}$. The 95% credibility interval will then be: $(\theta_{(25)}, \theta_{(975)})$.
5. Determine the interval length, $\theta_{(975)} - \theta_{(25)}$.
6. Repeat the above steps a 1000 times. Determine the average interval lengths. To obtain the coverage rate, determine how many of these credibility intervals contained the initial parameter value, $\theta = \frac{\lambda_1}{\lambda_2}$.

In simulation study II, in Section 4.3, the various Bayes estimates will be compared. Two different loss functions are considered for each of the vague priors that were discussed in Chapter 3, namely the squared error loss and the all-or-nothing loss.

Since there are a number of ways to estimate θ it is important to choose the best estimator. Thus we want to choose an estimator that have a sampling distribution that lies, and is, concentrated around θ , the true value (Rice, 2007). The mean square error is widely used as a measurement of this concentration. The estimator with the smallest MSE value will then be chosen.

The MSE can be calculated as follows, Rice (2007),

$$\begin{aligned}MSE(\hat{\theta}) &= E(\hat{\theta} - \theta_0)^2 \\ &= Var(\hat{\theta}) + [E(\hat{\theta}) - \theta_0]^2,\end{aligned}$$

with θ_0 the true parameter, $\hat{\theta}$ the estimate of θ , $Var(\hat{\theta})$ the variance of the estimate and $E(\hat{\theta}) - \theta_0$ the bias of the estimate. When the bias is equal to zero it means that $\hat{\theta}$ is an unbiased estimate. Thus the MSE will be equal to the variance, since the MSE can also be referred to as the variance of $\hat{\theta}$ plus the squared bias.

The simulation procedure can be found in Algorithm 4.2 with the R code provided in Appendix C.

Algorithm 4.2 Simulation procedure II.

1. Simulate data, \underline{x} , from an exponential distribution for a given n and parameter λ_1 . Simulate data, \underline{y} , from an exponential distribution for a given m and parameter λ_2 .
2. Calculate the Bayes estimate under the squared error loss function, given below

$$\text{Jeffreys} - \pi_A(\theta) \rightarrow \frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(n+1) \Gamma(m-1)}{\Gamma(n) \Gamma(m)}, \quad m > 1$$

$$\text{GML} - \pi_B(\theta) \rightarrow \frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma\left(n + \frac{3}{2}\right) \Gamma\left(m - \frac{1}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(m + \frac{1}{2}\right)}, \quad m > \frac{1}{2}$$

$$\text{Uniform} - \pi_C(\theta) \rightarrow \frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(n+2) \Gamma(m)}{\Gamma(n+1) \Gamma(m+1)}, \quad m > 0$$

$$\text{MDI} - \pi_D(\theta) \rightarrow \frac{\left(\sum_{j=1}^m y_j / \sum_{i=1}^n x_i\right) \Gamma(2n+2) \Gamma(2m)}{\Gamma(2n+1) \Gamma(2m+1)}, \quad m > 0.$$

Calculate the Bayes estimate under the all-or-nothing loss function, given below

$$\text{Jeffreys} - \pi_A(\theta) \rightarrow \left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{n-1}{m+1}\right), \quad n \geq 1$$

$$\text{GML} - \pi_B(\theta) \rightarrow \left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{n - \frac{1}{2}}{m + \frac{3}{2}}\right), \quad n \geq \frac{1}{2}$$

$$\text{Uniform} - \pi_C(\theta) \rightarrow \left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{n}{m+2}\right), \quad n \geq 0$$

$$\text{MDI} - \pi_D(\theta) \rightarrow \left(\frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i}\right) \left(\frac{2n}{2m+2}\right), \quad n \geq 0.$$

3. Repeat steps 1 – 2 a 1000 times, thus the number of replications is $R = 1000$.
4. Calculate the MSE and bias of the Bayes estimates.
5. Compare the MSE and bias values to choose the best estimate.

4.2 Simulation study I

In this section the coverage rate and average interval lengths will be determined. The values for the two exponential parameters are $\lambda_1 = 0.10, 1.00, 2.50, 5.00, 7.50, 10.00$ and $\lambda_2 = 0.10, 1.00, 2.50, 5.00, 7.50, 10.00$, where we will look at the ratio of these values, $\theta = \frac{\lambda_1}{\lambda_2}$.

The following values and combinations of n and m are used, namely $n = m = 5$, $n = 5$ and $m = 10$, $n = m = 10$, $n = 10$ and $m = 20$, $n = 15$ and $m = 30$, $n = 20$ and $m = 20$. The number of simulations are equal to a 1000 and a nominal level of 95% is considered. Due to time constraints we considered only a 1000 simulations in our study.

Table 4.1: Coverage rates (CR) and average interval lengths (AL) for $n = 5$ and $m = 5$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.10	0.10	0.930	8.109	0.939	3.474	0.814	2.354	0.927	8.242
	1.00	0.937	0.238	0.955	0.363	0.790	0.148	0.920	1.067
	2.50	0.939	0.132	0.953	0.086	0.816	0.065	0.929	0.229
	5.00	0.938	0.073	0.948	0.200	0.811	0.052	0.918	0.125
	7.50	0.929	0.081	0.951	0.077	0.797	0.070	0.931	0.044
	10.00	0.939	0.011	0.945	0.045	0.826	0.023	0.924	0.032
1.00	0.10	0.938	74.464	0.950	28.809	0.811	6.437	0.924	13.985
	1.00	0.929	1.328	0.949	1.914	0.800	3.464	0.911	1.689
	2.50	0.941	3.106	0.939	1.890	0.823	0.578	0.935	1.050
	5.00	0.939	0.225	0.949	0.636	0.797	0.149	0.926	0.710
	7.50	0.927	0.291	0.956	0.161	0.803	0.187	0.921	0.524
	10.00	0.951	0.241	0.958	0.744	0.811	0.186	0.915	1.852
2.50	0.10	0.946	45.820	0.941	52.917	0.794	55.925	0.934	42.928
	1.00	0.940	13.219	0.942	13.546	0.809	2.396	0.923	12.542
	2.50	0.940	2.616	0.942	2.554	0.805	2.735	0.934	5.191
	5.00	0.930	1.441	0.943	4.361	0.791	0.572	0.931	1.791
	7.50	0.942	0.341	0.953	0.775	0.818	0.800	0.929	1.049
	10.00	0.931	0.554	0.948	1.167	0.819	0.183	0.936	0.464
5.00	0.10	0.940	193.127	0.951	105.699	0.798	59.639	0.918	179.284
	1.00	0.944	24.179	0.952	8.833	0.817	16.222	0.926	7.648
	2.50	0.927	31.710	0.933	3.810	0.780	4.375	0.929	3.216
	5.00	0.938	5.458	0.957	1.266	0.807	1.262	0.915	5.334
	7.50	0.934	1.805	0.939	4.044	0.804	0.881	0.927	2.419
	10.00	0.940	1.765	0.949	4.587	0.814	0.748	0.934	0.675
7.50	0.10	0.921	801.829	0.948	327.911	0.797	78.359	0.916	64.856
	1.00	0.949	18.633	0.953	25.299	0.809	25.555	0.933	25.015
	2.50	0.921	8.019	0.944	12.310	0.808	13.330	0.929	5.360
	5.00	0.930	9.615	0.955	13.138	0.826	1.808	0.940	5.009
	7.50	0.924	2.378	0.946	4.464	0.794	4.895	0.922	3.302
	10.00	0.934	2.974	0.950	16.270	0.804	0.364	0.926	1.987
10.00	0.10	0.943	118.206	0.951	524.132	0.816	84.402	0.927	153.018
	1.00	0.951	54.393	0.952	32.899	0.810	12.325	0.933	24.583
	2.50	0.942	18.535	0.959	7.223	0.817	6.490	0.923	10.847
	5.00	0.939	0.645	0.953	4.839	0.799	4.944	0.922	4.174
	7.50	0.936	5.255	0.953	4.248	0.805	4.643	0.924	3.919
	10.00	0.931	5.554	0.948	2.991	0.822	0.972	0.910	2.158

Table 4.2: Coverage rates (CR) and average interval lengths (AL) for $n = 10$ and $m = 20$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.10	0.10	0.956	2.001	0.953	1.453	0.814	1.151	0.933	1.452
	1.00	0.930	0.097	0.950	0.208	0.841	0.250	0.939	0.180
	2.50	0.935	0.033	0.955	0.054	0.833	0.039	0.937	0.097
	5.00	0.950	0.024	0.948	0.024	0.843	0.025	0.932	0.025
	7.50	0.955	0.016	0.946	0.030	0.837	0.032	0.953	0.026
	10.00	0.944	0.023	0.954	0.014	0.820	0.014	0.947	0.020
1.00	0.10	0.930	21.027	0.947	24.059	0.835	9.811	0.948	15.941
	1.00	0.934	1.109	0.958	1.076	0.823	0.822	0.942	0.833
	2.50	0.935	0.568	0.956	0.970	0.824	0.489	0.948	0.758
	5.00	0.948	0.279	0.951	0.383	0.828	0.288	0.932	0.420
	7.50	0.943	0.178	0.949	0.157	0.826	0.087	0.932	0.152
	10.00	0.954	0.268	0.939	0.205	0.790	0.022	0.940	0.166
2.50	0.10	0.943	61.693	0.945	39.790	0.827	30.146	0.937	75.466
	1.00	0.933	8.053	0.959	4.650	0.825	2.959	0.936	3.907
	2.50	0.952	1.720	0.947	2.116	0.837	0.767	0.934	2.313
	5.00	0.951	1.356	0.947	0.749	0.810	0.674	0.948	0.581
	7.50	0.944	0.342	0.952	0.458	0.825	0.267	0.948	0.693
	10.00	0.954	0.401	0.961	0.337	0.820	0.310	0.947	0.158
5.00	0.10	0.950	159.037	0.958	96.434	0.823	75.143	0.949	90.684
	1.00	0.950	9.358	0.952	14.613	0.829	9.386	0.919	4.097
	2.50	0.930	2.799	0.944	3.432	0.807	2.011	0.922	3.400
	5.00	0.945	3.944	0.960	2.025	0.824	0.868	0.933	1.117
	7.50	0.949	1.786	0.957	1.652	0.817	0.707	0.932	0.725
	10.00	0.956	0.892	0.951	0.584	0.816	0.989	0.938	1.363
7.50	0.10	0.957	219.510	0.961	193.026	0.826	69.466	0.919	118.868
	1.00	0.948	8.279	0.942	16.987	0.816	14.424	0.931	9.485
	2.50	0.946	3.732	0.950	3.369	0.834	5.942	0.928	2.730
	5.00	0.937	1.565	0.955	3.684	0.813	2.580	0.940	1.634
	7.50	0.947	3.955	0.949	1.547	0.811	0.986	0.939	1.658
	10.00	0.934	0.765	0.947	1.092	0.826	1.297	0.935	0.695
10.00	0.10	0.952	236.218	0.941	141.525	0.807	140.591	0.927	250.243
	1.00	0.954	27.351	0.949	27.596	0.804	13.809	0.944	9.871
	2.50	0.951	13.980	0.935	5.757	0.826	3.494	0.949	7.884
	5.00	0.948	4.710	0.936	3.892	0.830	2.190	0.941	6.284
	7.50	0.952	2.296	0.943	2.546	0.840	1.401	0.925	2.826
	10.00	0.940	1.538	0.949	1.962	0.824	1.178	0.933	1.774

Table 4.3: Coverage rates (CR) and average interval lengths (AL) for $n = 20$ and $m = 20$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.10	0.10	0.961	2.383	0.951	1.070	0.846	1.346	0.933	1.522
	1.00	0.958	0.166	0.951	0.082	0.811	0.107	0.950	0.134
	2.50	0.957	0.042	0.947	0.077	0.834	0.023	0.943	0.067
	5.00	0.939	0.023	0.937	0.051	0.839	0.015	0.938	0.023
	7.50	0.941	0.008	0.942	0.019	0.821	0.014	0.941	0.026
	10.00	0.954	0.017	0.942	0.012	0.830	0.010	0.930	0.017
1.00	0.10	0.936	10.706	0.946	12.356	0.831	5.641	0.946	10.833
	1.00	0.950	0.959	0.951	2.135	0.838	1.288	0.950	1.029
	2.50	0.934	0.771	0.941	0.682	0.838	0.429	0.946	0.498
	5.00	0.959	0.217	0.961	0.212	0.832	0.165	0.943	0.316
	7.50	0.948	0.191	0.954	0.112	0.810	0.052	0.920	0.178
	10.00	0.952	0.134	0.941	0.169	0.855	0.067	0.944	0.112
2.50	0.10	0.944	51.493	0.949	24.708	0.832	25.846	0.948	20.615
	1.00	0.952	2.450	0.934	1.617	0.849	1.318	0.941	5.218
	2.50	0.958	1.675	0.953	1.156	0.846	0.804	0.922	0.948
	5.00	0.940	1.278	0.945	0.654	0.834	0.696	0.951	0.509
	7.50	0.946	0.342	0.937	0.419	0.835	0.266	0.941	0.520
	10.00	0.962	0.334	0.941	0.405	0.827	0.143	0.931	0.343
5.00	0.10	0.945	64.558	0.947	71.108	0.829	36.472	0.943	94.010
	1.00	0.938	8.579	0.951	10.131	0.810	3.631	0.937	6.065
	2.50	0.954	1.678	0.941	2.056	0.839	1.776	0.954	2.307
	5.00	0.949	0.805	0.944	1.535	0.819	0.669	0.938	1.610
	7.50	0.949	1.065	0.952	0.828	0.849	0.949	0.935	0.719
	10.00	0.948	1.000	0.958	0.693	0.822	0.265	0.935	0.703
7.50	0.10	0.934	151.860	0.941	108.384	0.806	125.913	0.942	123.684
	1.00	0.938	11.574	0.954	10.334	0.827	7.226	0.951	8.323
	2.50	0.950	2.559	0.951	4.171	0.809	2.087	0.945	5.393
	5.00	0.951	2.080	0.943	1.534	0.849	1.716	0.937	3.274
	7.50	0.953	2.108	0.942	1.158	0.823	0.631	0.937	2.735
	10.00	0.951	0.703	0.943	1.316	0.817	0.573	0.946	1.227
10.00	0.10	0.947	130.450	0.954	149.457	0.809	109.541	0.957	89.795
	1.00	0.952	23.308	0.928	6.321	0.807	6.634	0.947	8.619
	2.50	0.940	6.485	0.951	5.703	0.826	2.250	0.943	4.862
	5.00	0.951	2.223	0.949	3.133	0.845	1.254	0.952	2.312
	7.50	0.947	1.430	0.941	4.577	0.823	1.640	0.962	1.676
	10.00	0.945	1.247	0.953	1.551	0.841	0.826	0.935	1.454

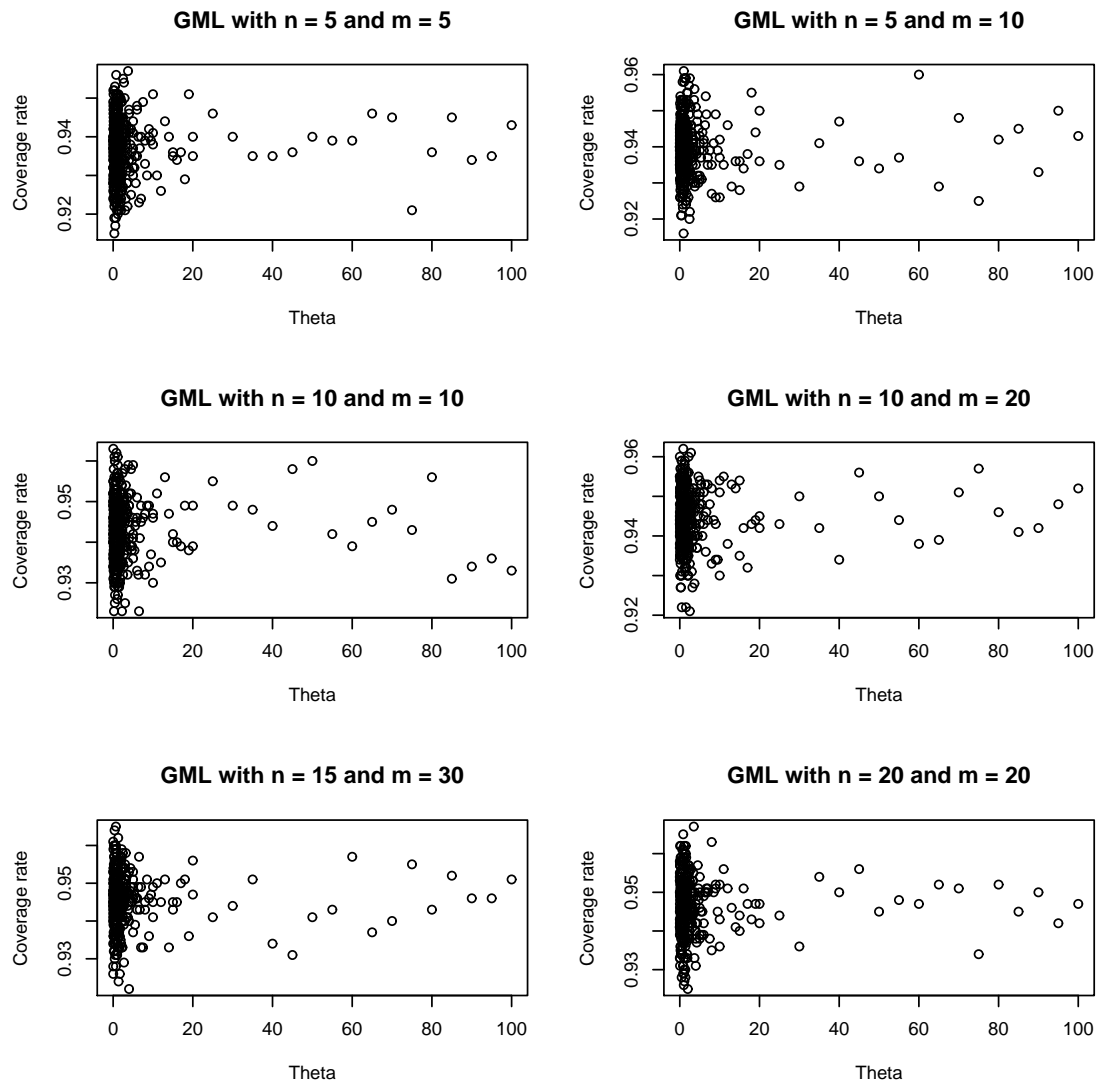


Figure 4.1: GML coverage rates for different theta values.

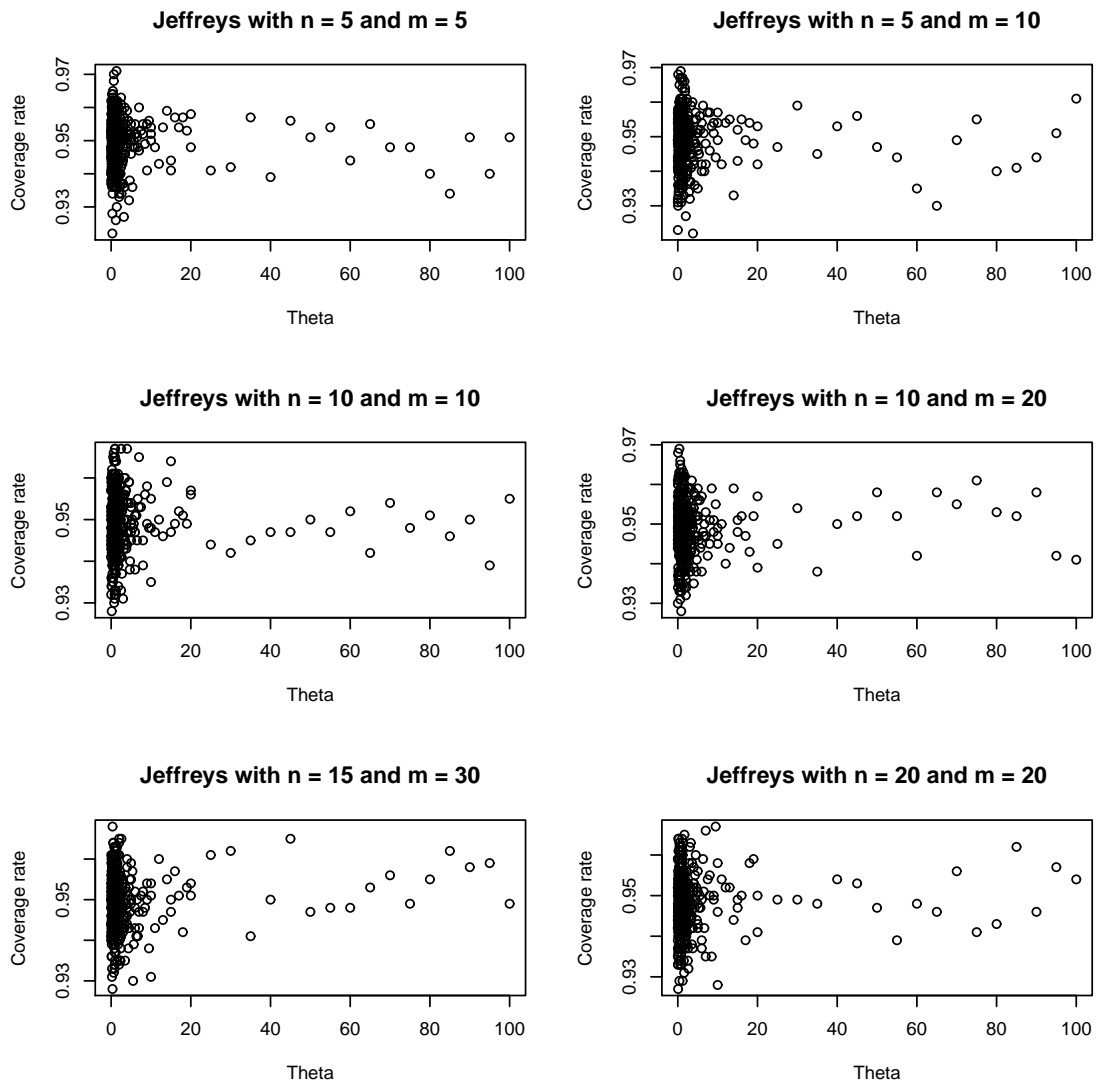


Figure 4.2: Jeffreys coverage rates for different theta values.

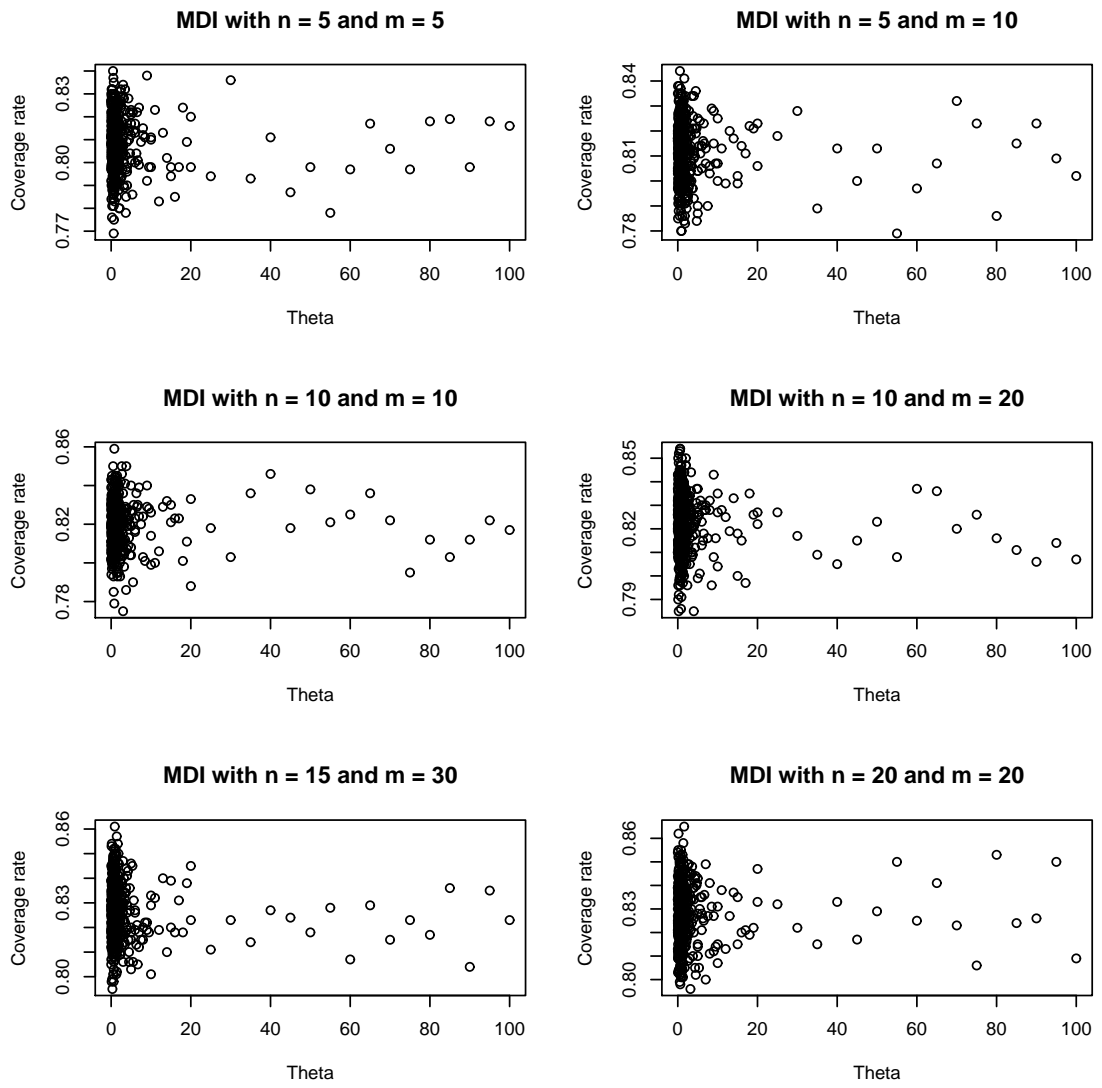


Figure 4.3: MDI coverage rates for different theta values.

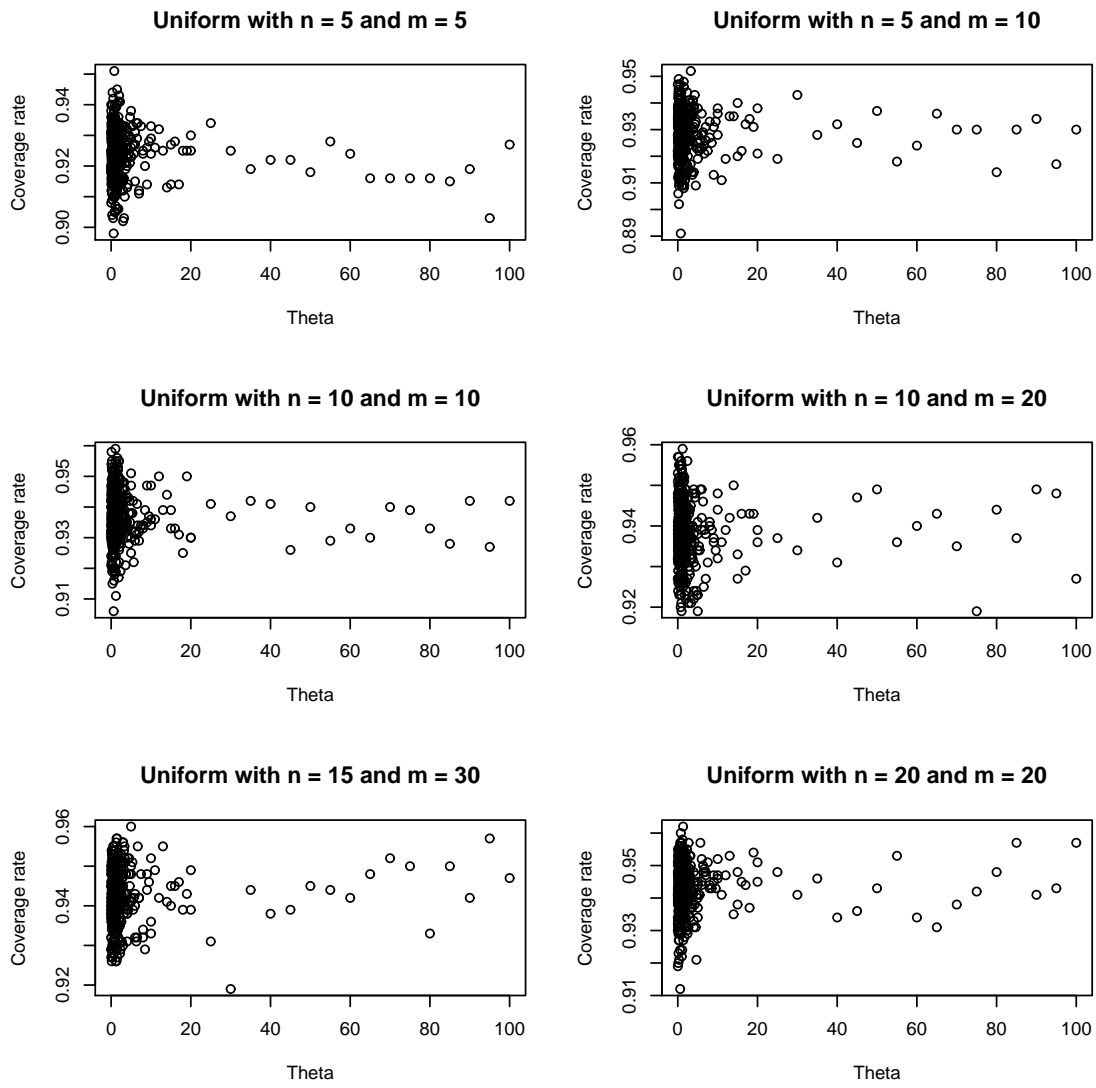


Figure 4.4: Uniform coverage rates for different theta values.

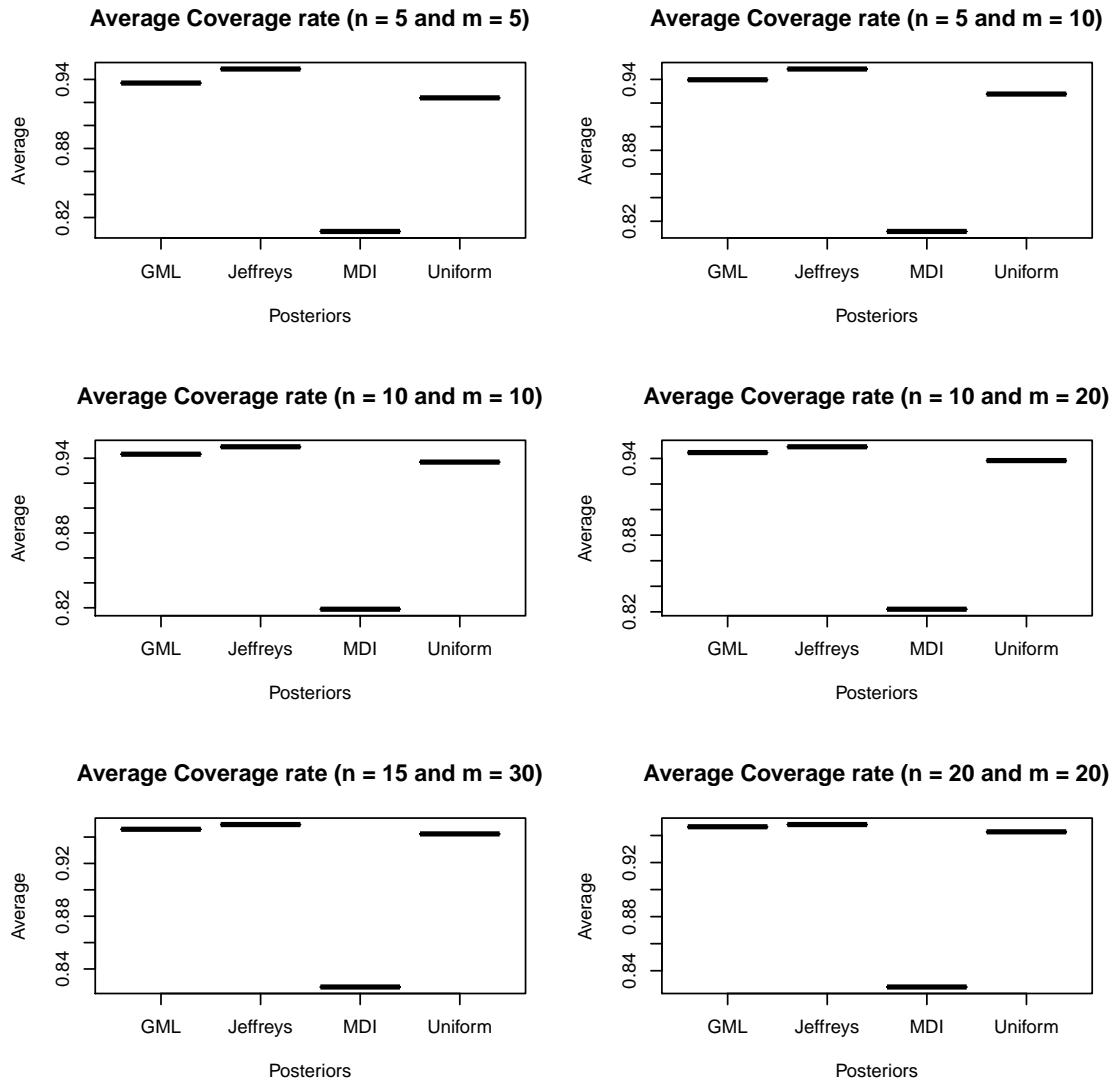
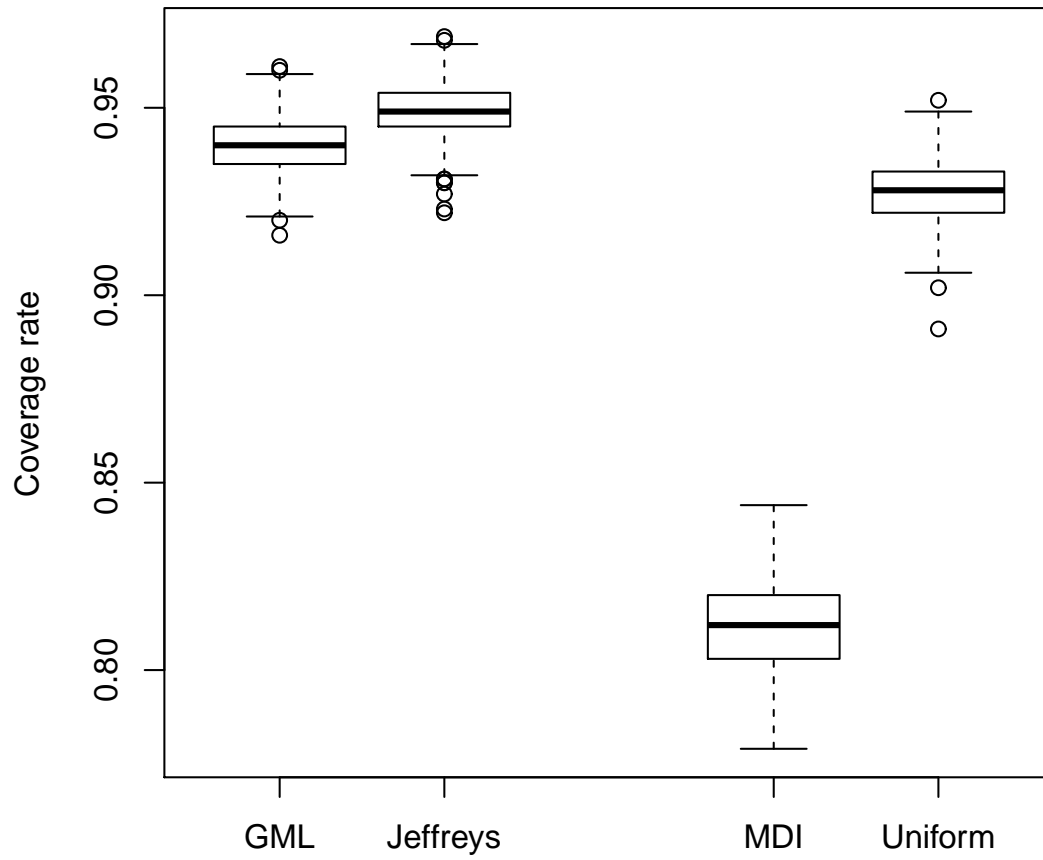
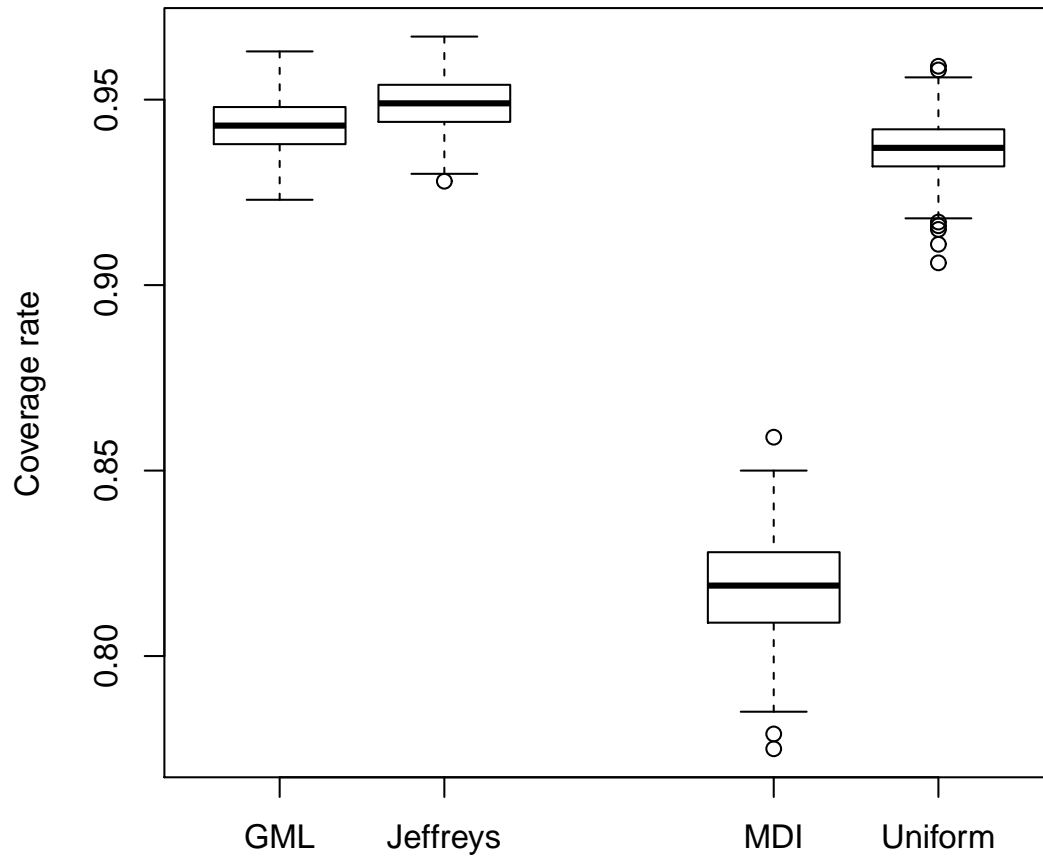
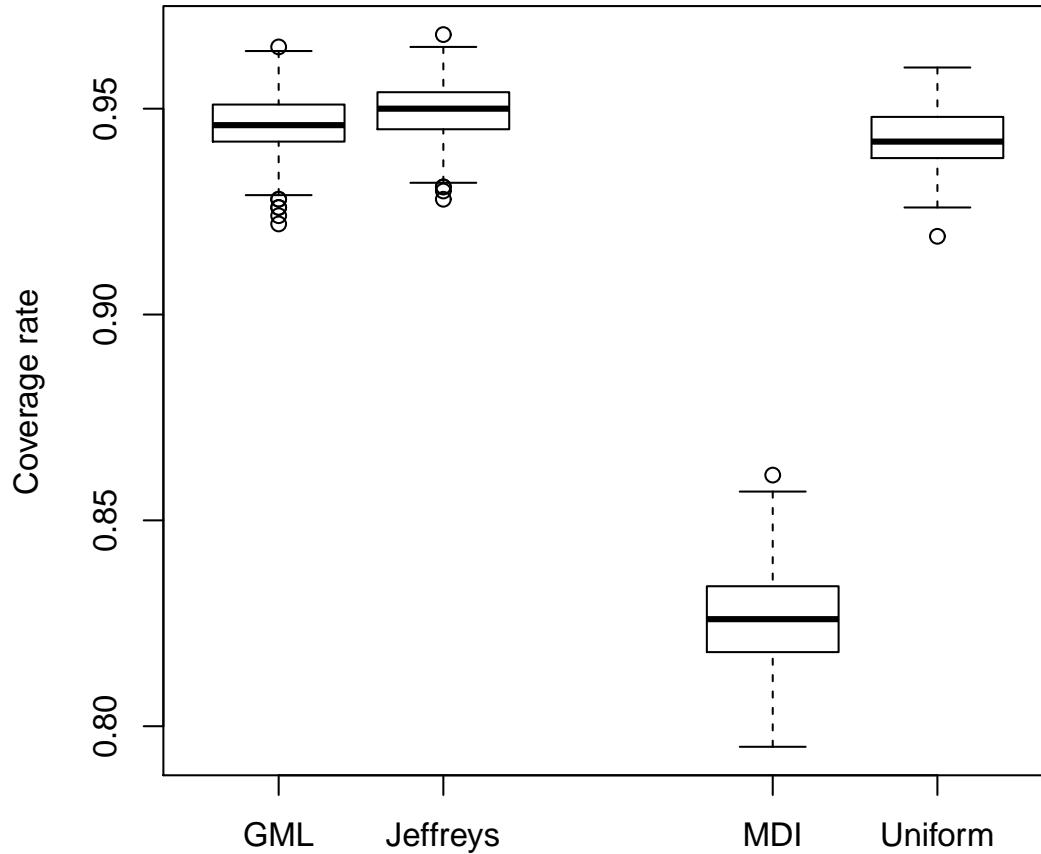


Figure 4.5: Average coverage rates for different posteriors.

Boxplot of coverage rates for $n = 5$ and $m = 10$ **Figure 4.6:** Boxplot of coverage rates for $n = 5$ and $m = 10$.

Boxplot of coverage rates for $n = 10$ and $m = 10$ **Figure 4.7:** Boxplot of coverage rates for $n = 10$ and $m = 10$.

Boxplot of coverage rates for $n = 15$ and $m = 30$ **Figure 4.8:** Boxplot of coverage rates for $n = 15$ and $m = 30$.

In Table 4.1, where $n = 5$ and $m = 5$, we can see that when using the Jeffreys prior the best coverage rates are obtained, with one or two exceptions. In Figure 4.5 the average coverage rates for different values of n and m are used to see which prior has the average coverage rate the closest to the nominal level, and also we see for $n = 5$ and $m = 5$ the Jeffreys prior has the highest rate. Also when the GML prior is used, it performs well for this combination of n and m with an average coverage rate that is just below the Jeffrey's prior average coverage rate, as can be seen in Figure 4.5. The shortest average interval length in Table 4.1 was obtained most of the time when the MDI prior was used for different parameter values but it is clear that

the coverage rates when the MDI prior is used, is far below the rest and yield unsatisfactory results. The average interval lengths of the GML and Jeffreys priors are close to each other but when the Jeffreys prior is used better coverage rates are obtained than the GML prior's coverage rates.

We can see in Table 4.2 that the highest coverage rates when $n = 10$ and $m = 20$ are obtained when the Jeffreys or GML priors are used. For some values of λ_1 and λ_2 the GML prior gives better results and for some some values of λ_1 and λ_2 , the Jeffreys prior gives better results but overall according to Table 4.2 we can say that when using the Jeffreys prior, the best coverage rates are obtained. This statement is also supported by the average coverage rates in Figure 4.5 where we see that the Jeffreys prior has average coverage rates closest to the nominal level, when $n = 10$ and $m = 20$. We can see that when the MDI prior is used, the shortest average interval length is obtained for most of the time but the MDI prior gives very bad coverage rates, in fact it gives the worst coverage rates of all the used priors for any of the n and m combinations. When the uniform prior was used, the average interval lengths were also short but the coverage rates does not give satisfactory results. As was the case with $n = 5$ and $m = 5$, the GML prior and Jeffreys prior average interval lengths are very close to each other but better coverage rates are obtained when the Jeffreys prior is used.

In Figure 4.5 when $n = 20$ and $m = 20$ we can see that we get the highest average coverage rate when the Jeffreys prior is used but the GML prior also does well and gives an average coverage rate that is just below Jeffreys. When looking at Table 4.3 we see again that the Jeffreys and GML priors coverage rates are close to each other and gives good results. The uniform prior also gives coverage rates for some θ values, that is closer to the nominal level and the other used priors, than previously but still the average coverage rate when the uniform prior is used is below the nominal level. This can also be seen in Figure 4.5. Still when the MDI prior is used, the shortest average interval lengths were obtained but with coverage rates that is very low. When we make use of either the GML prior or the Jeffreys prior very similar average coverage rates are obtained but the average coverage rate when we use the Jeffreys prior is still higher than the average coverage rates obtained with the GML prior.

In Figures 4.1, 4.2, 4.3 and 4.4 the different priors and their corresponding combination of n and m were used to see what happens with the coverage rates as the θ values increase. We can see in Figure 4.1 that as θ increases when the GML prior is used, the coverage rates vary around 0.94 for $n = m = 5$ and $n = 5$ and $m = 10$. As n and m increase, the coverage rates point where it varies around also increase and gets close to 0.95. When we look at Figure 4.2 where the Jeffreys prior is used, we see that as θ increases, the coverage rates vary around

0.95 for all the different combinations of n and m . From the MDI prior coverage rates in Figure 4.3 it is clear that the coverage rates is quite lower than the other prior distributions. For larger θ values the MDI prior's coverage rates point where it varies around, increases with an increase in sample sizes. It vary around 0.8 for $n = m = 5$ and increase to 0.83 for $n = m = 20$. In Figure 4.4 we can see that as θ increases when the uniform prior is used, the coverage rates vary around 0.92 for $n = m = 5$ and increase to 0.95 as n and m increase.

In Figure 4.5 we can see that the Jeffreys prior has the highest average coverage rate for $n = m = 5$ with the GML prior close to the average coverage rate of the Jeffreys prior. As n and m increase, the GML prior's average coverage rates also increase and when $n = m = 20$ the average coverage rate of GML is almost the same as Jeffrey's average coverage rate. It is clear that the MDI prior performed badly as it's average coverage rates is in all cases the lowest of all the used priors.

We can also see in Table 4.1 that as the θ values increase, the average interval lengths, for all the priors, increase drastically and become very large. When looking at Figure 4.5 it is clear that as n and m increase, the different prior distributions, except the MDI prior, gets more equal in terms of average coverage rates.

According to the boxplot in Figure 4.6 for $n = 5$ and $m = 10$ we can see that when the Jeffreys prior is used, the median is equal to 0.95. Also the first quartile of the Jeffreys prior is almost the same as the third quartile of the GML prior which make it also clear that it is better to make use of the Jeffreys prior due to more coverage rates obtained near the nominal level. The uniform prior's third quartile is equal to the first quartile of the GML prior which makes the GML prior better to use than the uniform prior. There is an outlier above the nominal level of 0.95 when the uniform prior is used and also an outlier below 0.90. When the MDI prior is used the lowest coverage rates are obtained and it vary from nearly 0.80 to 0.85 which is far below the expected coverage rate of 0.95.

We can see from Figure 4.7 that, as was the case with $n = 5$ and $m = 10$, when the Jeffreys prior is used the best results are obtained with a median equal to 0.95. In this case the GML prior performs better than in Figure 4.6, with the third quartile equal to the median of the Jeffrey's prior. Better results are obtained when the uniform prior is used, than was the case with $n = 5$ and $m = 10$, although there is a few outliers between the coverage rate of 0.90 and 0.92. The uniform's prior third quartile is equal to the Jeffreys prior's first quartile. Still the MDI prior gives unsatisfactory results with most of the coverage rates under 0.835.

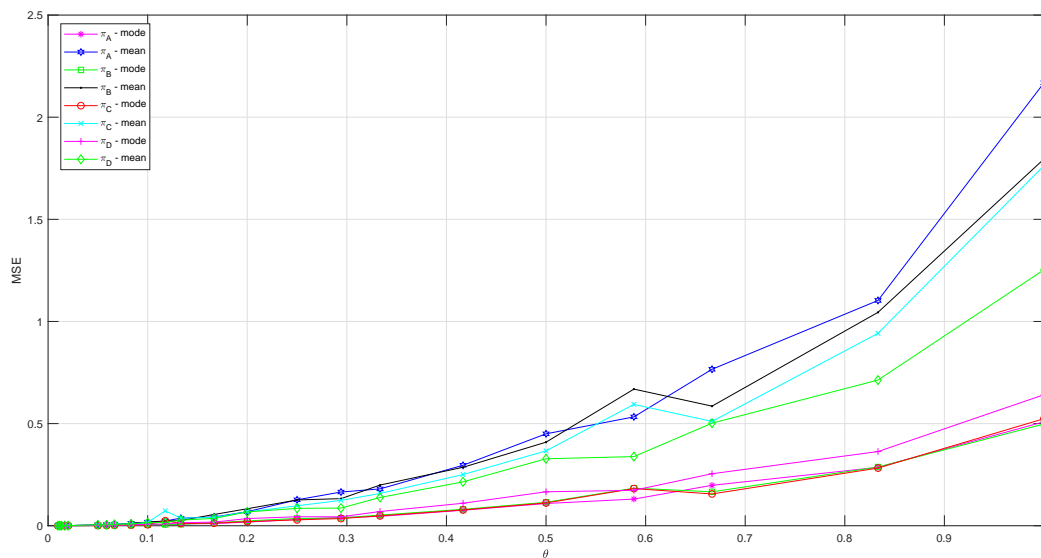
When looking at Figure 4.8 we can see that as n and m inceased, the GML prior gives coverage

rates that are closer to the nominal level. The median when the GML prior is used, is equal to the Jeffreys prior's first quartile but still when the Jeffreys prior is used there are fewer outliers and their position is closer to the rest of the coverage rates than when the GML prior is used. The uniform prior also performs better and gives coverage rates closer to the nominal level. We can see that as n and m increase, the uniform prior distribution's median gets lower and the uniform prior becomes a bit skewed to the right. With no exception the MDI prior's coverage rates are the lowest of all but also increase and get closer to the nominal level as n and m increase.

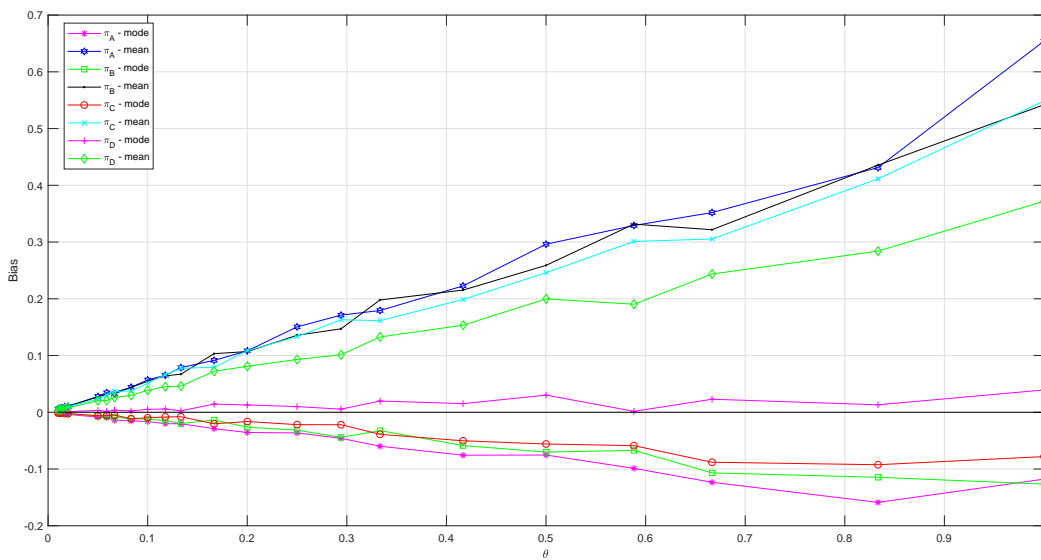
4.3 Simulation study II

The mean square error and bias will be determined in this section. The values for the ratio of two exponential parameters considered are $\lambda_1 = 0.10, 0.50, 1.00, 2.50, 5.00$ and $\lambda_2 = 5.00, 6.00, 7.50, 8.50, 10.00$, where we will look at the ratio of these values, $\theta = \frac{\lambda_1}{\lambda_2}$.

The combinations of n and m that are used, is $n = 5$ and $m = 5$, $n = 10$ and $m = 20$, $n = 20$ and $m = 20$. In the simulation study, a 1000 replications were used to calculate the MSE to be able to determine which estimator is the best to use.



(a)



(b)

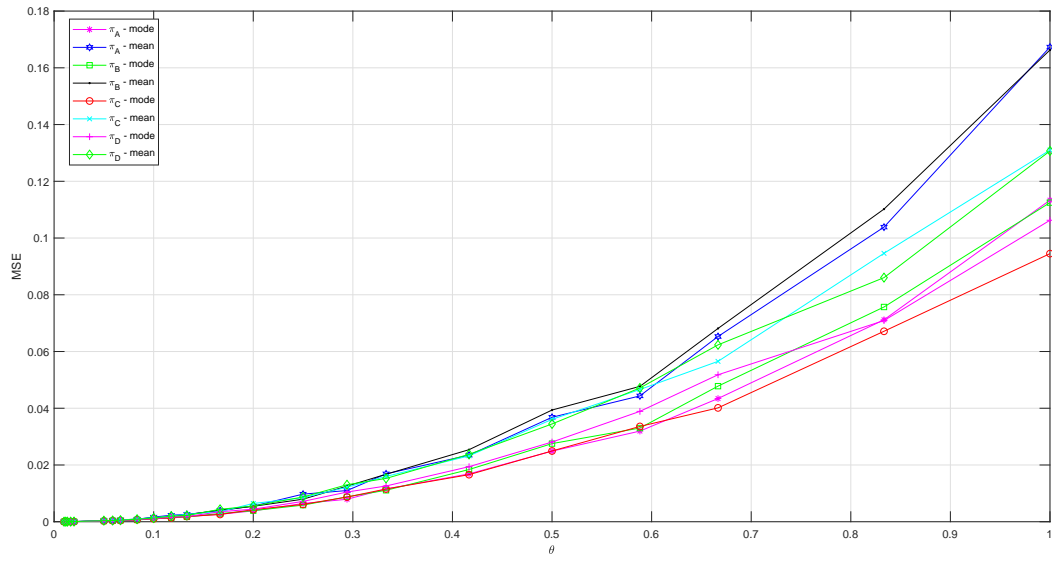
Figure 4.9: (a) the MSE and (b) the bias of the Bayes estimates when $n = 5$ and $m = 5$.

Table 4.4: MSE and Bias of the Bayes estimate for the squared error loss when $n = 10$ and $m = 20$.

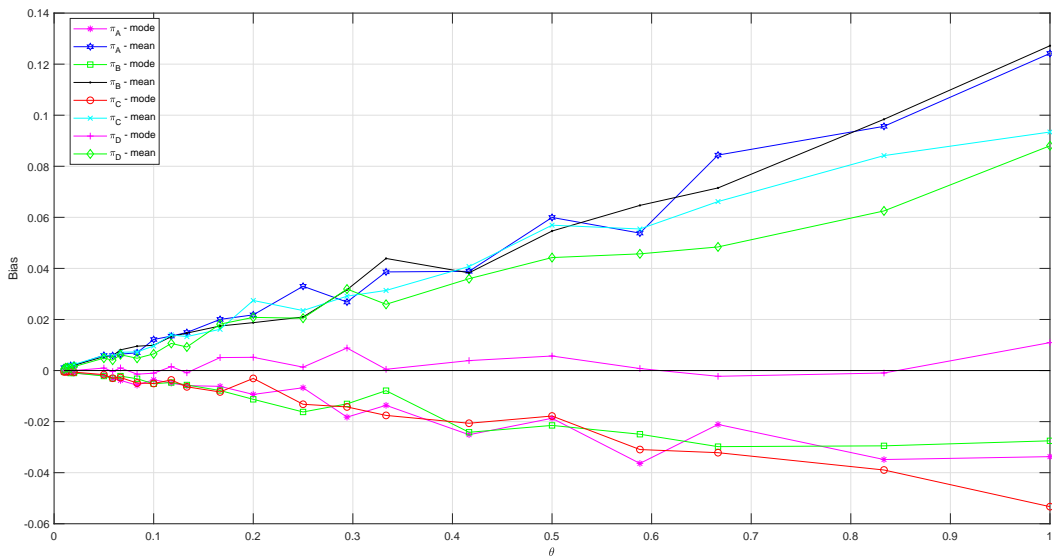
λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
0.1	5	0.00012	0.00402	0.00010	0.00295	0.00010	0.00286	0.00013	0.00439
	6	0.00009	0.00351	0.00007	0.00269	0.00007	0.00253	0.00010	0.00417
	7.5	0.00005	0.00285	0.00005	0.00239	0.00005	0.00246	0.00006	0.00286
	8.5	0.00004	0.00197	0.00004	0.00184	0.00004	0.00180	0.00005	0.00282
	10	0.00003	0.00180	0.00003	0.00178	0.00003	0.00183	0.00004	0.00252
0.5	5	0.00321	0.02043	0.00274	0.01619	0.00288	0.01726	0.00311	0.02162
	6	0.00196	0.01589	0.00190	0.01189	0.00196	0.01382	0.00212	0.01737
	7.5	0.00132	0.01183	0.00116	0.01128	0.00135	0.01183	0.00145	0.01486
	8.5	0.00110	0.01193	0.00090	0.01011	0.00098	0.01053	0.00124	0.01356
	10	0.00072	0.01052	0.00069	0.00791	0.00069	0.00876	0.00079	0.01074
1	5	0.01114	0.03587	0.01003	0.02938	0.01029	0.03541	0.01280	0.04241
	6	0.00954	0.03742	0.00788	0.03070	0.00826	0.02461	0.00903	0.03886
	7.5	0.00588	0.02777	0.00432	0.02226	0.00497	0.02302	0.00523	0.02969
	8.5	0.00405	0.02401	0.00331	0.01749	0.00381	0.01711	0.00457	0.02754
	10	0.00312	0.01981	0.00274	0.01741	0.00276	0.01690	0.00340	0.02414
2.5	5	0.07327	0.09971	0.07215	0.08295	0.07028	0.08391	0.08635	0.10883
	6	0.05025	0.07891	0.05184	0.07493	0.05685	0.08520	0.05681	0.09105
	7.5	0.03007	0.06387	0.03146	0.05364	0.03138	0.05993	0.03627	0.07338
	8.5	0.02622	0.06292	0.02395	0.05139	0.02282	0.04701	0.02575	0.06507
	10	0.01656	0.04373	0.01633	0.04411	0.01822	0.04435	0.01684	0.05102
5	5	0.29652	0.19174	0.27383	0.17443	0.25116	0.15366	0.34211	0.23657
	6	0.18780	0.15095	0.17457	0.12285	0.17954	0.12791	0.21048	0.17856
	7.5	0.11928	0.12969	0.11734	0.11138	0.10204	0.09812	0.15673	0.16247
	8.5	0.11239	0.12650	0.08603	0.08599	0.09899	0.09560	0.12185	0.14357
	10	0.06581	0.08909	0.06985	0.08205	0.07612	0.08475	0.08581	0.11235

Table 4.5: MSE and Bias of the Bayes estimate for the all-or-nothing loss when $n = 10$ and $m = 20$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
0.1	5	0.00007	-0.00029	0.00006	-0.00131	0.00008	0.00073	0.00007	0.00016
	6	0.00005	-0.00011	0.00004	-0.00091	0.00005	0.00074	0.00006	0.00055
	7.5	0.00003	-0.00006	0.00003	-0.00053	0.00004	0.00100	0.00003	0.00005
	8.5	0.00002	-0.00049	0.00002	-0.00069	0.00003	0.00054	0.00003	0.00029
	10	0.00002	-0.00032	0.00002	-0.00041	0.00002	0.00073	0.00002	0.00035
0.5	5	0.00188	-0.00118	0.00167	-0.00539	0.00216	0.00636	0.00181	0.00051
	6	0.00115	-0.00191	0.00120	-0.00579	0.00148	0.00479	0.00124	-0.00011
	7.5	0.00080	-0.00225	0.00069	-0.00319	0.00102	0.00454	0.00084	0.00071
	8.5	0.00065	-0.00077	0.00054	-0.00269	0.00073	0.00408	0.00072	0.00099
	10	0.00041	-0.00034	0.00042	-0.00285	0.00052	0.00330	0.00046	0.00019
1	5	0.00668	-0.00644	0.00625	-0.01322	0.00761	0.01353	0.00751	0.00034
	6	0.00548	0.00081	0.00463	-0.00595	0.00635	0.00682	0.00514	0.00319
	7.5	0.00344	-0.00113	0.00258	-0.00664	0.00372	0.00849	0.00298	0.00139
	8.5	0.00234	-0.00141	0.00205	-0.00760	0.00292	0.00458	0.00261	0.00234
	10	0.00184	-0.00169	0.00164	-0.00440	0.00207	0.00603	0.00193	0.00259
2.5	5	0.04271	-0.00788	0.04392	-0.02531	0.05290	0.02963	0.05090	0.00316
	6	0.02975	-0.00999	0.03092	-0.01637	0.04228	0.03854	0.03315	0.00293
	7.5	0.01756	-0.00739	0.01928	-0.01822	0.02341	0.02337	0.02111	0.00279
	8.5	0.01499	-0.00113	0.01429	-0.01277	0.01719	0.01529	0.01471	0.00274
	10	0.00994	-0.00896	0.00965	-0.01051	0.01366	0.01699	0.00973	-0.00122
5	5	0.17540	-0.02206	0.16330	-0.04368	0.18936	0.04641	0.19593	0.02196
	6	0.11177	-0.02563	0.10874	-0.05473	0.13574	0.03855	0.12199	0.00294
	7.5	0.06917	-0.01318	0.07068	-0.03312	0.07675	0.02702	0.08936	0.01857
	8.5	0.06491	-0.00173	0.05368	-0.03922	0.07494	0.03202	0.06942	0.01656
	10	0.03925	-0.01659	0.04253	-0.02604	0.05764	0.03038	0.05002	0.00607



(a)



(b)

Figure 4.10: The Squared error (mean) and All-or-nothing (mode) for $n = 20$ and $m = 20$, with (a) the MSE and (b) the bias.

When looking at Figure 4.9, we see that the mode when the GML prior is used, gives the lowest MSE value when $n = 5$ and $m = 5$. Similar MSE values are obtained for the mode when the uniform prior is used. The uniform prior does not have the bias value closest to zero, but still has a bias value that is one of the closest to zero. The mode when the Jeffreys prior is used also gives MSE values that are considerably small and close to the MSE values of the GML and uniform priors. The MDI prior's mode also gives a MSE value that are close to the rest but a bit higher. It is clear that the means of the different prior distributions gives higher MSE values than the modes. When the Jeffreys and GML priors are used, their means give the highest MSE values among all the prior distributions. The bias of the mode when the MDI prior is used, is the closest to zero, with the uniform prior's bias just a little bit higher.

As can be seen in Table 4.4 for $n = 10$ and $m = 20$, when the MSE is computed for the mean, it seems that when the Jeffreys prior is used, the smallest MSE values are obtained most of the time with a few exceptions where the use of the MDI prior gives smaller MSE values. Also the closest to zero bias values are obtained when the Jeffreys prior is used to compute the posterior mean. Again the MDI prior, as was the case with the MSE, gives bias values that are often smaller than when the Jeffreys prior is used. Looking at Table 4.5 we see that for some θ values the use of the GML prior gives lower MSE values under the all-or-nothing loss function than the Jeffreys prior, but overall the use of the Jeffreys prior gives the lowest MSE values. The bias values that are the smallest for the mode estimate, seem to be obtained when making use of the uniform prior, with a few exceptions where the GML prior gives the smallest bias values.

In Figure 4.10 we can see that when the uniform prior is used the lowest MSE values are obtained for the mode but it have a slightly bigger bias value than the rest of the modes. The use of the Jeffreys prior also give good results in terms of low MSE values for the mode which are close to those of the uniform prior but have bias values that are a little bit smaller than the uniform prior's bias. The highest MSE and bias values are obtained for the posterior mean when the Jeffreys and GML priors are used. The bias of the MDI prior's posterior mode is the smallest for this combination of n and m but it has the highest MSE values in terms of the rest of the estimations for the mode when using one of the other three priors.

From Simulation study I we could see that the MDI prior resulted in the shortest interval most of the time, and thus coincide with results in Simulation study II where the MDI prior gives smaller bias and MSE values for a number of the cases.

4.4 Application

In this section an example from Bain and Engelhardt (1991) will be considered, this study was originally investigated in Proschan (1963). None of these authors considered Bayesian methods. Proschan (1963) explains the data set as follows: “Records were kept for the time of successive failures of the air conditioning system, of each member of a fleet of Boeing 720 jet airplanes. The intervals between successive failures are listed.” Bain and Engelhardt (1991) only considered the hours of flying time between failures for two of the planes, Plane 7911 and Plane 7912. See Table 4.6 for the data.

Table 4.6: The hours of flying time between failures for Plane 7911 and Plane 7912.

Plane 7911	55	320	56	104	220	239	47	246	176	182	33
Plane 7912	23	261	87	7	120	14	62	47	225	71	246
	21	42	20	5	12	120	11	3	14	71	11
	14	11	16	90	1	16	52	95			

The posterior distributions of λ_1 and λ_2 are given in Figure 4.11. Let λ_1 be the rate parameter for the hours of flying time between failures for Plane 7911, and let λ_2 be the rate parameter for the hours of flying time between failures for Plane 7912. We first considered the posteriors of λ_1 and λ_2 separately, where these are independent gamma distributions. This was shown in Chapter 3.

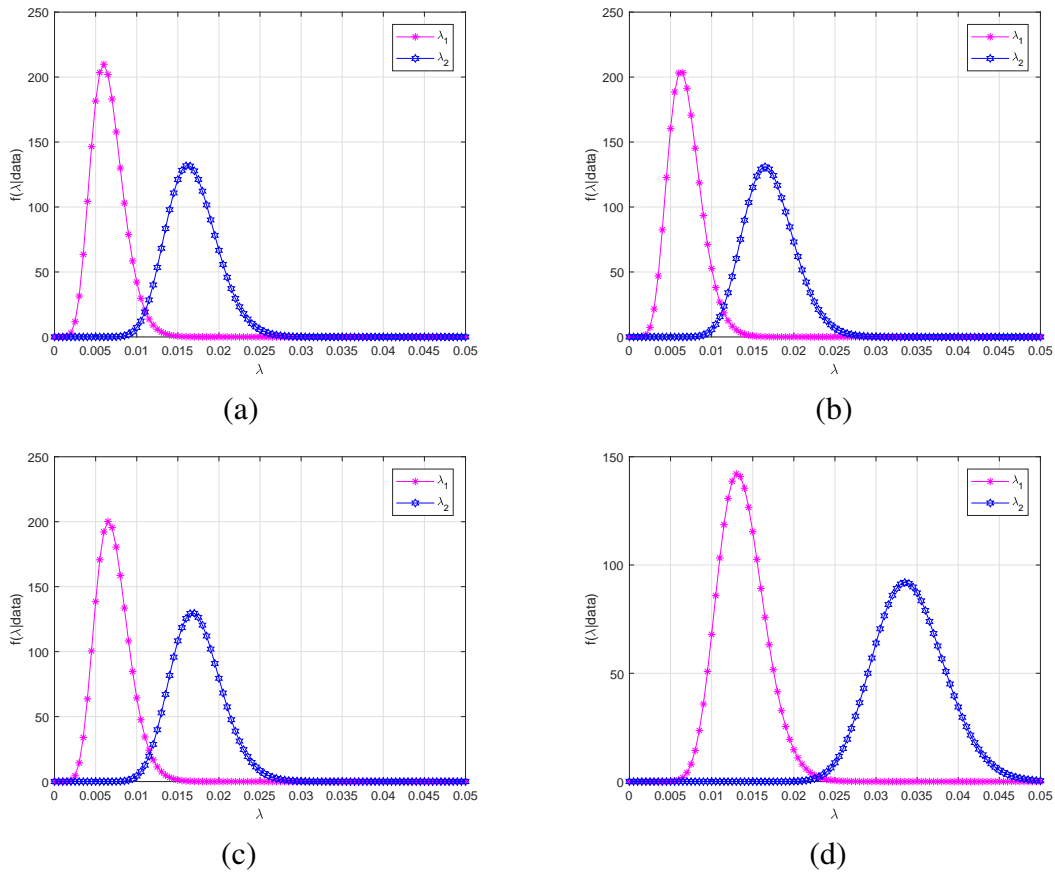


Figure 4.11: Posterior distributions for λ_1 and λ_2 , when using (a) the Jeffreys prior, (b) the GML prior, (c) the Uniform prior, and (d) the MDI prior.

The 95% credibility intervals and Bayes estimates for λ_1 and λ_2 are given in Table 4.7. From the results it can be seen that the Bayes estimates for λ_1 are much smaller than that of λ_2 for all four priors. Thus meaning that the time between failures of the airconditioning on plane 7911 is longer than on the other plane. The MDI prior gives greater values than the other three priors. We can thus say that the probability that λ_1 lies between 0.0033 and 0.0110 is 95% when using the Jeffreys prior, and that the probability that λ_2 lies between 0.0113 and 0.0233 is 95% when using the Jeffreys prior.

Table 4.7: 95% Credibility intervals, posterior means and posterior modes for λ_1 and λ_2 , when using the Jeffreys, GML, Uniform and MDI priors.

	95% CI		Posterior mean	Posterior mode
Jeffreys				
λ_1	0.0033	0.0110	0.0066	0.0072
λ_2	0.0113	0.0233	0.0168	0.0173
GML				
λ_1	0.0035	0.0113	0.0069	0.0074
λ_2	0.0116	0.0236	0.0171	0.0176
Uniform				
λ_1	0.0037	0.0117	0.0072	0.0077
λ_2	0.0118	0.0240	0.0173	0.0179
MDI				
λ_1	0.0087	0.0198	0.0137	0.0143
λ_2	0.0261	0.0432	0.0341	0.0347

The credibility intervals and Bayes estimates for the ratio of these two parameters, $\theta = \lambda_1/\lambda_2$, will now be considered. In Chapter 3 it was shown that the posterior distribution of θ is a generalized beta prime distribution. The generalized beta prime distribution is discussed in detail in McDonald (1984). In Chapter 3 the posterior distribution of θ was derived when using the Jeffreys, GML, uniform and MDI priors. Table 4.8 gives a summary of these posteriors including the normalizing constant and the actual values of the posterior parameters for this application.

Table 4.8: Posterior distributions of θ .

If $\theta _{x,y} \sim Beta'(\alpha, \beta, p, q)$ then $\pi(\theta _{x,y}) = \frac{p \left(\frac{\theta}{q}\right)^{\alpha p - 1} \left[1 + \left(\frac{\theta}{q}\right)^p\right]^{-\alpha - \beta}}{qB(\alpha, \beta)}, \theta > 0$	
Prior used	Parameters
Jeffreys	$\alpha = n = 11, \beta = m = 30, p = 1, q = \frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i} = \frac{1788}{1678}$
GML	$\alpha = n + 0.5 = 11.5, \beta = m + 0.5 = 30 + 0.5, p = 1, q = \frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i} = \frac{1788}{1678}$
Uniform	$\alpha = n + 1 = 12, \beta = m + 1 = 31, p = 1, q = \frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i} = \frac{1788}{1678}$
MDI	$\alpha = 2n + 1 = 23, \beta = 2m + 1 = 61, p = 1, q = \frac{\sum_{j=1}^m y_j}{\sum_{i=1}^n x_i} = \frac{1788}{1678}$

The four posterior distributions are displayed graphically in Figure 4.12. From this figure it can be seen that the posterior distributions when using the Jeffreys, GML and Uniform

priors, all have similar shapes, whereas the posterior distribution when using the MDI prior is slightly different, having a much higher density.

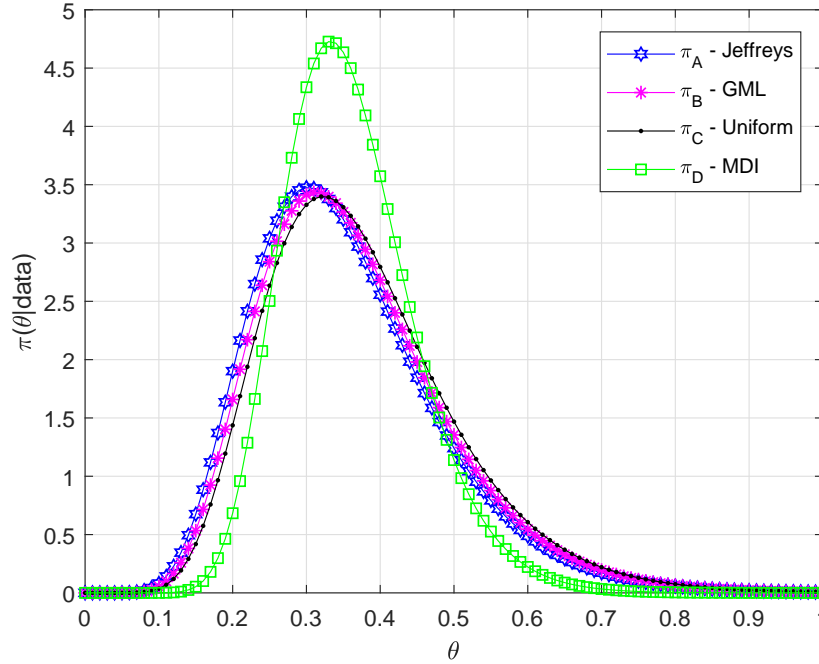


Figure 4.12: Posterior distributions of $\theta = \lambda_1/\lambda_2$ when using the Jeffreys, GML, Uniform and MDI priors.

The credibility intervals in Table 4.9 were obtained by using numerical integration and a grid, in MATLAB, to find the upper and lower limits. The 95% equal tail credibility intervals were obtained by solving the following

$$\int_0^{\text{lower limit}} \pi(\theta|\underline{x}, \underline{y}) d\theta = 0.025,$$

and

$$\int_0^{\text{upper limit}} \pi(\theta|\underline{x}, \underline{y}) d\theta = 0.975,$$

thus

$$\int_{\text{lower limit}}^{\text{upper limit}} \pi(\theta|\underline{x}, \underline{y}) d\theta = 0.95.$$

The first equation above is used to determine the lower limit of the credibility interval. As

we know, when working with a 95% credibility interval, 2.5% of the observations will lie to the left of the lower limit and 2.5% of the observations will lie to the right of the upper limit of the credibility interval. The area under the density curve between the lower and upper limits are equal to 0.95 and can be calculated by integrating the density function between the lower limit and upper limit, as illustrated above with the last equation. The density function is integrated over a grid of values and stops when 0.025 is reached, which is then the lower limit of the credibility interval. The second equation above is used to determine the upper limit of the credibility interval. The upper limit will be calculated based on the same idea explained above to calculate the lower limit, but now the density function is integrated and stops when the value 0.975 is reached. The integrated value obtained at this point is the upper limit of the credibility interval.

By using these methods as explained above, the 95% credibility intervals and Bayes estimates for θ are calculated and given in Table 4.9. The MDI prior gives a much smaller credibility interval, this is expected after seeing the graph of this posterior. The other three priors give similar results with wider credibility intervals than the intervals obtained when the MDI prior is used. None of the credibility intervals in Table 4.9 contains the value 1, which means that with 95% certainty we can say that these failure times of the airconditioning on the two different planes will never be the same. When the MDI prior is used the shortest interval lengths are obtained, with the Bayes estimate under all-or-nothing loss when using the Jeffreys prior, i.e. the posterior mode, the smallest of all the priors.

Table 4.9: 95% Credibility intervals, posterior means and posterior modes for $\theta = \lambda_1/\lambda_2$, when using the Jeffreys, GML, Uniform and MDI priors.

95% CI		Posterior mean	Posterior mode
<i>Jeffreys</i>			
0.1604	0.6573	0.3560	0.3027
<i>GML</i>			
0.1678	0.6692	0.3658	0.3128
<i>Uniform</i>			
0.1750	0.6806	0.3754	0.3226
<i>MDI</i>			
0.2123	0.5591	0.3598	0.3330

4.5 Summary

In this chapter we discussed two simulation studies and their corresponding results. In the first simulation study we obtained coverage rates and average interval lengths for different n , m and λ_1, λ_2 combinations. The second simulation study is used to evaluate Bayes estimates for different n , m and λ_1, λ_2 combinations. The coverage rates and average interval lengths for $n = 5$ and $m = 5$, $n = 10$ and $m = 20$, $n = 20$ and $m = 20$ can be found in the provided tables and figures. The results of the simulation studies is also discussed in this chapter. Lastly an application of hours of flying between successive failure times of airconditioners in planes, was considered.

Chapter 5

Conclusion

5.1 Concluding remarks

The Jeffreys, GML, MDI and uniform priors were discussed in Chapter 2 and derived in Chapter 3 together with their posterior distributions and the marginal posterior distribution for $\theta = \frac{\lambda_1}{\lambda_2}$. Bayes estimates under the squared error loss and all-or-nothing loss functions were also discussed in Chapter 2 and derived in Chapter 3. Two simulation studies were considered. In the first simulation study the coverage rates and average interval lengths were computed for different combinations of λ_1 and λ_2 and for different combinations of n and m , namely $n = 5, m = 5$ and $n = 10, m = 20$ and $n = 20, m = 20$. When the Jeffreys prior was used, coverage rates closest to the nominal level were obtained, where the MDI prior gave the shortest interval lengths. Due to the fact that the MDI prior that yielded unsatisfactory coverage rates, we can conclude that the Jeffreys prior give the best results in terms of coverage rates and average interval lengths. This was true for most of the combinations of n and m .

By looking at the average coverage rates for the different posterior distributions, we can see that as n and m increase, the average coverage rates given by the different posterior distributions get close to the nominal level, except the MDI prior's average coverage rate.

In the second simulation study the MSE and bias were computed for various Bayes estimates. We can conclude that for the combination of $n = 5$ and $m = 5$, the Bayes estimate under all-or-nothing loss, when the uniform prior is used, is the better estimate. When we look at $n = 10$ and $m = 20$, the Bayes estimate under the all-or-nothing loss, when the Jeffreys prior is used, is the best estimate. The same result holds when $n = 20$ and $m = 20$. We can thus conclude

that as n and m increase, the Bayes estimate under the all-or-nothing loss, when using the Jeffreys prior gave smaller MSE and bias values than the rest and therefore it is the better estimate to use. The posterior mean does not give estimates close to the true parameter value, and therefore will lead to high losses obtained when used to estimate the true parameter.

In conclusion we would recommend that the Jeffreys prior is used to estimate the ratio of two exponential parameters, $\theta = \frac{\lambda_1}{\lambda_2}$, with the all-or-nothing loss if Bayes point estimates have to be determined.

5.2 Future research

Future research could include further investigation of more vague priors, a sensitivity analysis which could include the comparison of the deviance information criterion. Instead of just looking at the ratio of two parameters, $\theta = \frac{\lambda_1}{\lambda_2}$, one could extend the work and look at $\theta = \prod_{i=1}^n \lambda_i^{a_i}$, where a_i could be any value, for example $\theta = \lambda_1 \lambda_2^2$ or $\theta = \lambda_1 \lambda_2 \lambda_3$, etc.

More loss functions could be investigated, for example the absolute error loss, the linear exponential loss, etc. One could also look at the highest posterior density intervals, as well as different tolerance intervals.

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Appendix A

Additional Mathematical Derivations

Theorem A.1. *The Fisher Information matrix is given by*

$$I(\lambda_1, \lambda_2) = \begin{bmatrix} \frac{n}{\lambda_1^2} & 0 \\ 0 & \frac{m}{\lambda_2^2} \end{bmatrix}. \quad (\text{A.1})$$

Proof. The Fisher Information matrix is defined by

$$I(\lambda_1, \lambda_2) = \begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial \lambda_1^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_2}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial \lambda_2 \partial \lambda_1}\right) & -E\left(\frac{\partial^2 \log L}{\partial \lambda_2^2}\right) \end{bmatrix}$$

$$\log L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) = n \log \lambda_1 + m \log \lambda_2 - \lambda_1 \sum_{i=1}^n x_i - \lambda_2 \sum_{j=1}^m y_j$$

$$\frac{\partial \log L}{\partial \lambda_1} = \frac{n}{\lambda_1} - \sum_{i=1}^n x_i$$

$$\frac{\partial \log L}{\partial \lambda_2} = \frac{m}{\lambda_2} - \sum_{j=1}^m y_j$$

$$\frac{\partial^2 \log L}{\partial \lambda_1^2} = -\frac{n}{\lambda_1^2}$$

$$\frac{\partial^2 \log L}{\partial \lambda_2^2} = -\frac{m}{\lambda_2^2}$$

$$\frac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_2} = 0 = \frac{\partial^2 \log L}{\partial \lambda_2 \partial \lambda_1}$$

$$\therefore I(\lambda_1, \lambda_2) = \begin{bmatrix} \frac{n}{\lambda_1^2} & 0 \\ 0 & \frac{m}{\lambda_2^2} \end{bmatrix}.$$

□

Appendix B

Additional Simulation Results

B.1 Complete table for $n = 5$ and $m = 5$

Table B.1: Coverage rates (CR) and average interval lengths (AL) for $n = 5$ and $m = 5$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.10	0.10	0.930	8.109	0.939	3.474	0.814	2.354	0.927	8.242
	0.50	0.940	0.560	0.945	2.276	0.822	0.189	0.911	0.312
	1.00	0.937	0.238	0.955	0.363	0.790	0.148	0.920	1.067
	1.50	0.942	0.286	0.952	0.386	0.825	0.051	0.914	0.209
	2.00	0.940	0.163	0.951	0.062	0.808	0.043	0.928	0.401
	2.50	0.939	0.132	0.953	0.086	0.816	0.065	0.929	0.229
	3.00	0.928	0.043	0.946	0.220	0.830	0.079	0.925	0.034
	3.50	0.939	0.099	0.947	0.098	0.811	0.020	0.931	0.209
	4.00	0.926	0.147	0.953	0.103	0.827	0.030	0.930	0.169
	4.50	0.937	0.129	0.962	0.046	0.812	0.041	0.918	0.034
	5.00	0.938	0.073	0.948	0.200	0.811	0.052	0.918	0.125
	5.50	0.937	0.147	0.940	0.115	0.792	0.047	0.936	0.077
	6.00	0.949	0.059	0.941	0.022	0.827	0.025	0.919	0.072
	6.50	0.940	0.123	0.938	0.094	0.822	0.113	0.908	0.061
7.00	0.936	0.099	0.952	0.052	0.811	0.022	0.925	0.039	
7.50	0.929	0.081	0.951	0.077	0.797	0.070	0.931	0.044	

	8.00	0.935	0.080	0.953	0.088	0.818	0.024	0.920	0.033
	8.50	0.930	0.014	0.945	0.101	0.798	0.025	0.938	0.156
	9.00	0.924	0.051	0.954	0.024	0.809	0.004	0.919	0.051
	9.50	0.932	0.112	0.940	0.065	0.814	0.039	0.940	0.092
	10.00	0.939	0.011	0.945	0.045	0.826	0.023	0.924	0.032
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	0.10	0.930	10.022	0.951	4.832	0.801	13.547	0.938	16.081
	0.50	0.928	2.992	0.939	4.505	0.796	2.559	0.932	12.259
	1.00	0.933	0.632	0.963	1.453	0.830	1.249	0.928	4.089
	1.50	0.940	0.960	0.938	0.971	0.820	0.704	0.940	0.632
	2.00	0.946	0.255	0.948	1.209	0.805	0.273	0.922	0.425
	2.50	0.935	1.795	0.964	1.076	0.800	0.307	0.924	0.269
	3.00	0.928	0.369	0.952	0.406	0.806	0.291	0.917	0.525
	3.50	0.934	0.587	0.948	0.327	0.804	0.155	0.929	0.294
	4.00	0.947	0.178	0.943	0.778	0.821	0.455	0.919	0.543
	4.50	0.947	1.026	0.942	0.807	0.832	0.362	0.925	0.401
0.50	5.00	0.946	0.483	0.948	0.461	0.819	0.280	0.920	0.210
	5.50	0.932	0.388	0.953	0.137	0.803	0.178	0.934	0.239
	6.00	0.932	0.507	0.937	1.488	0.812	0.117	0.919	0.098
	6.50	0.928	0.180	0.948	0.658	0.807	0.238	0.923	0.395
	7.00	0.948	0.227	0.958	0.379	0.799	0.198	0.912	0.051
	7.50	0.934	0.417	0.937	0.064	0.820	0.108	0.918	0.531
	8.00	0.932	0.089	0.947	0.129	0.784	0.094	0.933	0.446
	8.50	0.934	0.323	0.948	0.844	0.808	0.083	0.920	0.203
	9.00	0.939	0.195	0.955	0.105	0.808	0.098	0.918	0.031
	9.50	0.952	0.106	0.944	0.454	0.805	0.035	0.919	0.096
	10.00	0.928	0.177	0.956	0.087	0.802	0.167	0.916	0.193
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	0.10	0.938	74.464	0.950	28.809	0.811	6.437	0.924	13.985
	0.50	0.928	5.945	0.940	3.165	0.801	2.703	0.924	8.065
	1.00	0.929	1.328	0.949	1.914	0.800	3.464	0.911	1.689
	1.50	0.951	3.056	0.946	2.373	0.835	1.248	0.929	1.634
	2.00	0.935	5.830	0.955	2.752	0.796	1.363	0.903	2.543
	2.50	0.941	3.106	0.939	1.890	0.823	0.578	0.935	1.050
	3.00	0.952	1.602	0.943	0.960	0.832	0.499	0.929	0.441
	3.50	0.944	1.215	0.956	0.355	0.801	0.619	0.924	0.479

	4.00	0.935	1.465	0.936	1.258	0.781	0.817	0.904	0.536
	4.50	0.946	0.297	0.962	0.528	0.832	0.266	0.923	0.650
1.00	5.00	0.939	0.225	0.949	0.636	0.797	0.149	0.926	0.710
	5.50	0.945	0.387	0.960	0.708	0.818	0.929	0.921	0.765
	6.00	0.927	0.435	0.960	0.574	0.821	0.171	0.922	0.440
	6.50	0.943	1.097	0.941	0.869	0.816	0.159	0.919	0.102
	7.00	0.945	0.347	0.946	0.204	0.813	0.122	0.922	0.269
	7.50	0.927	0.291	0.956	0.161	0.803	0.187	0.921	0.524
	8.00	0.944	0.314	0.955	0.237	0.807	0.147	0.919	1.043
	8.50	0.938	0.386	0.949	0.631	0.829	0.090	0.932	0.333
	9.00	0.942	0.318	0.950	0.407	0.797	0.338	0.931	0.161
	9.50	0.939	0.324	0.956	0.258	0.804	0.176	0.931	0.284
	10.00	0.951	0.241	0.958	0.744	0.811	0.186	0.915	1.852
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	0.10	0.936	76.338	0.944	50.887	0.794	6.399	0.927	78.069
	0.50	0.940	11.984	0.955	19.160	0.834	14.047	0.902	11.205
	1.00	0.939	16.384	0.953	8.173	0.813	3.870	0.925	11.713
	1.50	0.942	4.069	0.957	1.651	0.813	4.048	0.929	4.340
	2.00	0.945	3.508	0.950	0.998	0.815	0.991	0.916	1.766
	2.50	0.934	2.156	0.953	1.643	0.819	1.425	0.935	3.501
	3.00	0.936	1.466	0.943	1.421	0.804	0.403	0.910	0.629
	3.50	0.943	1.675	0.953	2.356	0.796	1.537	0.924	0.434
	4.00	0.930	0.920	0.950	0.756	0.805	2.224	0.922	0.342
	4.50	0.937	0.764	0.944	1.296	0.796	0.627	0.925	0.908
1.50	5.00	0.915	1.102	0.922	1.722	0.802	1.274	0.917	0.675
	5.50	0.929	0.868	0.951	1.691	0.788	0.716	0.916	1.448
	6.00	0.932	1.318	0.960	0.357	0.812	0.591	0.915	1.583
	6.50	0.928	1.818	0.961	0.336	0.809	0.204	0.929	2.585
	7.00	0.939	0.168	0.952	0.141	0.800	0.277	0.912	0.452
	7.50	0.930	0.904	0.950	0.639	0.806	0.150	0.934	0.714
	8.00	0.942	0.540	0.942	0.614	0.792	0.894	0.933	0.248
	8.50	0.945	0.598	0.938	0.406	0.812	0.318	0.925	0.415
	9.00	0.943	1.344	0.943	1.059	0.808	0.165	0.919	0.193
	9.50	0.946	0.494	0.954	0.147	0.808	0.139	0.922	0.263
	10.00	0.940	2.148	0.940	1.225	0.806	0.323	0.934	0.113

	0.10	0.935	106.712	0.958	303.134	0.798	26.197	0.930	47.750
	0.50	0.946	38.035	0.950	19.617	0.790	24.503	0.932	16.622
	1.00	0.937	14.605	0.955	3.669	0.800	9.934	0.919	8.897
	1.50	0.943	7.002	0.971	4.673	0.796	1.725	0.930	4.563
	2.00	0.951	2.844	0.944	3.162	0.798	0.319	0.914	6.896
	2.50	0.929	1.630	0.955	2.025	0.810	2.768	0.925	1.843
	3.00	0.938	1.752	0.968	8.552	0.815	1.123	0.935	2.251
	3.50	0.938	4.475	0.946	5.045	0.796	1.521	0.922	2.294
	4.00	0.939	1.613	0.947	0.667	0.805	0.816	0.927	0.967
	4.50	0.933	1.460	0.965	3.714	0.808	0.513	0.934	6.723
2.00	5.00	0.934	1.235	0.954	0.679	0.807	1.880	0.921	0.816
	5.50	0.923	1.564	0.934	2.057	0.790	1.105	0.930	2.151
	6.00	0.941	0.912	0.952	3.085	0.819	0.514	0.933	0.963
	6.50	0.922	0.632	0.958	1.039	0.820	0.326	0.917	2.907
	7.00	0.925	1.981	0.947	0.576	0.827	1.685	0.922	1.244
	7.50	0.940	1.747	0.955	1.319	0.814	1.233	0.924	0.233
	8.00	0.930	0.802	0.947	0.805	0.802	0.594	0.924	1.209
	8.50	0.939	0.467	0.959	0.647	0.815	0.604	0.932	1.525
	9.00	0.937	0.569	0.957	2.793	0.827	0.500	0.934	0.836
	9.50	0.934	0.621	0.952	0.926	0.812	0.448	0.924	0.128
	10.00	0.941	0.237	0.948	3.094	0.776	0.520	0.934	0.328
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	0.10	0.946	45.820	0.941	52.917	0.794	55.925	0.934	42.928
	0.50	0.932	16.370	0.950	17.267	0.823	18.704	0.923	25.368
	1.00	0.940	13.219	0.942	13.546	0.809	2.396	0.923	12.542
	1.50	0.939	2.399	0.939	4.095	0.827	8.318	0.930	2.691
	2.00	0.945	3.256	0.949	2.260	0.807	1.662	0.924	6.379
	2.50	0.940	2.616	0.942	2.554	0.805	2.735	0.934	5.191
	3.00	0.937	4.824	0.951	2.345	0.796	1.207	0.932	1.707
	3.50	0.926	9.228	0.958	7.275	0.795	0.991	0.916	1.663
	4.00	0.942	1.849	0.952	3.306	0.818	1.215	0.930	1.551
	4.50	0.942	1.395	0.948	3.082	0.803	0.671	0.920	1.259
2.50	5.00	0.930	1.441	0.943	4.361	0.791	0.572	0.931	1.791
	5.50	0.939	2.088	0.945	1.911	0.824	1.726	0.922	0.987
	6.00	0.941	1.760	0.956	2.714	0.811	0.232	0.925	0.501

6.50	0.931	1.064	0.941	0.318	0.808	1.847	0.916	1.058
7.00	0.937	1.149	0.941	1.832	0.827	0.838	0.918	0.802
7.50	0.942	0.341	0.953	0.775	0.818	0.800	0.929	1.049
8.00	0.929	0.892	0.943	2.194	0.790	1.076	0.909	0.783
8.50	0.919	0.779	0.955	0.270	0.810	0.874	0.924	3.458
9.00	0.926	1.671	0.954	0.581	0.806	0.196	0.927	0.751
9.50	0.942	0.975	0.953	1.507	0.804	1.196	0.917	0.715
10.00	0.931	0.554	0.948	1.167	0.819	0.183	0.936	0.464
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0.10	0.940	112.483	0.942	201.556	0.836	22.103	0.925	56.744
0.50	0.947	36.182	0.952	20.069	0.800	7.537	0.915	8.124
1.00	0.941	9.111	0.950	10.829	0.807	1.915	0.912	5.566
1.50	0.937	6.190	0.952	4.299	0.805	3.829	0.943	2.911
2.00	0.945	2.497	0.938	2.413	0.815	3.254	0.945	2.175
2.50	0.934	4.876	0.956	4.804	0.792	1.836	0.929	6.688
3.00	0.942	4.616	0.953	3.669	0.813	1.894	0.924	2.529
3.50	0.935	1.386	0.948	11.676	0.809	0.747	0.923	9.297
4.00	0.946	1.401	0.947	1.614	0.791	3.436	0.919	7.218
4.50	0.925	8.605	0.954	2.805	0.808	1.232	0.922	4.070
5.00	0.933	3.267	0.949	1.435	0.796	0.650	0.937	0.909
5.50	0.937	0.844	0.939	1.620	0.814	0.667	0.920	2.496
6.00	0.930	1.396	0.942	3.100	0.788	0.248	0.914	4.678
6.50	0.953	3.691	0.953	3.633	0.794	2.402	0.933	0.413
7.00	0.934	1.088	0.964	0.512	0.812	0.843	0.916	0.475
7.50	0.944	1.376	0.964	0.983	0.828	0.529	0.933	0.716
8.00	0.933	0.918	0.951	2.056	0.803	1.113	0.909	0.656
8.50	0.929	4.562	0.947	1.409	0.809	0.828	0.919	0.503
9.00	0.931	1.887	0.946	1.255	0.796	0.371	0.934	1.470
9.50	0.933	0.411	0.949	1.190	0.798	0.353	0.924	1.433
10.00	0.930	0.793	0.928	0.510	0.822	1.484	0.932	1.035
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0.10	0.935	57.366	0.957	39.766	0.793	34.626	0.919	52.219
0.50	0.924	28.413	0.947	26.996	0.824	19.277	0.911	30.999
1.00	0.928	6.806	0.946	29.104	0.823	3.850	0.921	6.760
1.50	0.941	3.770	0.940	25.972	0.827	6.025	0.928	7.480
2.00	0.931	3.641	0.950	16.058	0.818	3.690	0.914	9.805

	2.50	0.936	3.316	0.930	4.472	0.796	5.971	0.928	3.258
	3.00	0.941	3.087	0.926	8.608	0.822	1.454	0.917	6.428
	3.50	0.928	2.217	0.955	1.754	0.819	0.485	0.929	2.822
	4.00	0.940	2.229	0.955	3.813	0.792	4.154	0.927	1.766
	4.50	0.941	2.278	0.950	3.787	0.820	0.921	0.924	1.860
3.50	5.00	0.932	1.199	0.944	2.085	0.816	0.542	0.921	3.181
	5.50	0.943	3.140	0.945	1.939	0.769	0.758	0.918	0.999
	6.00	0.937	2.380	0.957	1.403	0.800	0.966	0.910	2.278
	6.50	0.917	1.957	0.951	1.423	0.818	0.647	0.942	0.550
	7.00	0.931	0.954	0.939	1.674	0.816	1.757	0.929	1.324
	7.50	0.941	1.893	0.947	0.808	0.815	0.751	0.923	1.883
	8.00	0.937	2.225	0.948	4.392	0.809	1.553	0.932	1.213
	8.50	0.944	1.211	0.943	0.475	0.812	0.899	0.918	0.703
	9.00	0.924	1.175	0.940	8.294	0.807	1.235	0.932	1.972
	9.50	0.936	2.216	0.944	2.189	0.806	1.067	0.927	1.526
	10.00	0.929	1.429	0.957	1.524	0.794	0.472	0.944	0.608
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	0.10	0.935	62.732	0.939	228.396	0.811	35.164	0.922	114.065
	0.50	0.939	6.138	0.955	60.863	0.815	12.317	0.924	24.897
	1.00	0.941	5.877	0.952	21.196	0.798	8.151	0.928	14.469
	1.50	0.941	3.103	0.951	9.477	0.794	3.188	0.926	11.510
	2.00	0.945	4.897	0.949	13.456	0.794	12.953	0.916	3.046
	2.50	0.930	2.759	0.940	7.860	0.803	4.117	0.916	2.638
	3.00	0.941	2.347	0.948	6.733	0.821	6.461	0.922	7.034
	3.50	0.931	1.916	0.941	7.962	0.798	1.582	0.928	11.426
	4.00	0.944	0.857	0.945	3.743	0.792	1.717	0.927	1.721
	4.50	0.934	1.394	0.946	2.530	0.803	3.958	0.924	2.923
4.00	5.00	0.938	1.092	0.945	8.321	0.808	3.290	0.922	0.906
	5.50	0.946	2.911	0.951	1.576	0.808	0.507	0.928	2.653
	6.00	0.941	2.273	0.953	1.283	0.795	1.539	0.926	0.687
	6.50	0.934	2.556	0.947	1.164	0.781	0.414	0.926	3.980
	7.00	0.938	1.897	0.954	3.525	0.784	2.238	0.929	1.905
	7.50	0.938	1.489	0.947	2.069	0.784	0.481	0.922	1.276
	8.00	0.948	1.444	0.949	1.222	0.800	1.364	0.923	0.478
	8.50	0.941	2.743	0.940	0.856	0.786	1.312	0.916	0.494

	9.00	0.943	0.448	0.940	1.453	0.820	0.433	0.928	1.551
	9.50	0.925	0.650	0.945	1.795	0.807	0.393	0.915	0.966
	10.00	0.928	1.030	0.952	0.911	0.796	0.833	0.920	2.243
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	0.10	0.936	256.991	0.956	72.313	0.787	151.414	0.922	64.813
	0.50	0.942	20.353	0.955	28.748	0.792	18.770	0.914	7.161
	1.00	0.925	16.186	0.932	11.014	0.814	5.988	0.924	6.178
	1.50	0.933	18.707	0.954	4.198	0.829	5.465	0.929	3.378
	2.00	0.939	2.325	0.940	3.176	0.826	6.333	0.925	5.700
	2.50	0.943	5.290	0.954	1.995	0.810	2.991	0.924	16.338
	3.00	0.939	5.020	0.944	6.611	0.818	1.548	0.917	17.728
	3.50	0.925	6.973	0.941	2.473	0.802	2.551	0.914	5.597
	4.00	0.930	6.080	0.955	3.395	0.800	2.047	0.929	2.674
	4.50	0.931	3.968	0.947	6.711	0.795	0.658	0.927	7.420
4.50	5.00	0.937	2.820	0.961	2.999	0.791	2.782	0.925	6.263
	5.50	0.938	1.107	0.952	1.357	0.810	1.945	0.933	9.161
	6.00	0.940	2.114	0.950	1.024	0.802	1.089	0.923	1.453
	6.50	0.947	3.437	0.952	3.051	0.808	0.835	0.931	4.945
	7.00	0.949	6.213	0.941	0.672	0.799	3.306	0.935	0.964
	7.50	0.938	2.160	0.961	3.704	0.789	2.107	0.924	0.767
	8.00	0.938	1.742	0.951	1.677	0.804	0.565	0.924	1.293
	8.50	0.930	0.528	0.952	2.252	0.829	0.709	0.935	2.154
	9.00	0.943	1.478	0.947	1.735	0.814	0.814	0.926	1.125
	9.50	0.943	1.385	0.944	6.293	0.840	1.565	0.932	1.263
	10.00	0.945	0.934	0.944	1.994	0.823	1.150	0.924	3.760
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	0.10	0.940	193.127	0.951	105.699	0.798	59.639	0.918	179.284
	0.50	0.941	26.176	0.954	10.824	0.798	41.541	0.929	18.789
	1.00	0.944	24.179	0.952	8.833	0.817	16.222	0.926	7.648
	1.50	0.935	4.367	0.951	5.878	0.789	5.553	0.928	7.739
	2.00	0.955	5.062	0.954	16.187	0.825	8.096	0.924	13.292
	2.50	0.927	31.710	0.933	3.810	0.780	4.375	0.929	3.216
	3.00	0.933	3.699	0.937	6.228	0.808	3.419	0.928	3.499
	3.50	0.921	7.459	0.955	4.245	0.803	3.440	0.923	0.600
	4.00	0.936	6.712	0.948	6.903	0.804	2.109	0.911	16.409
	4.50	0.946	1.254	0.945	2.383	0.829	2.590	0.911	3.184

5.00	5.00	0.938	5.458	0.957	1.266	0.807	1.262	0.915	5.334
	5.50	0.921	8.795	0.959	2.062	0.793	1.509	0.929	4.434
	6.00	0.939	1.224	0.946	1.946	0.829	0.572	0.936	1.979
	6.50	0.938	2.735	0.936	3.323	0.822	0.652	0.930	2.333
	7.00	0.949	1.969	0.937	2.290	0.824	1.607	0.926	4.974
	7.50	0.934	1.805	0.939	4.044	0.804	0.881	0.927	2.419
	8.00	0.943	0.686	0.948	2.944	0.790	1.455	0.914	0.373
	8.50	0.932	1.676	0.960	1.802	0.794	0.331	0.917	0.764
	9.00	0.938	3.654	0.944	3.587	0.837	1.918	0.927	2.692
	9.50	0.949	1.528	0.944	2.055	0.826	0.562	0.921	1.190
	10.00	0.940	1.765	0.949	4.587	0.814	0.748	0.934	0.675
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	0.10	0.939	478.891	0.954	97.791	0.778	136.707	0.928	497.390
	0.50	0.930	21.903	0.948	22.458	0.823	18.001	0.926	15.602
	1.00	0.935	8.009	0.948	50.677	0.822	6.758	0.927	18.872
	1.50	0.937	6.951	0.951	3.067	0.778	8.304	0.930	7.686
	2.00	0.942	6.443	0.959	21.996	0.803	3.049	0.933	6.437
	2.50	0.949	8.489	0.957	36.875	0.805	16.507	0.921	1.328
	3.00	0.950	1.799	0.960	5.216	0.811	1.719	0.936	4.559
	3.50	0.922	2.936	0.944	2.768	0.808	1.781	0.925	1.580
	4.00	0.937	3.708	0.951	7.941	0.815	3.250	0.906	0.834
	4.50	0.940	3.566	0.936	3.058	0.820	2.158	0.917	14.024
5.50	5.00	0.948	2.770	0.945	1.937	0.796	1.270	0.911	3.622
	5.50	0.945	5.510	0.955	2.012	0.821	1.081	0.928	4.908
	6.00	0.937	1.927	0.944	2.376	0.790	1.578	0.918	3.673
	6.50	0.936	2.014	0.953	3.335	0.783	0.850	0.930	2.044
	7.00	0.944	1.946	0.951	2.617	0.800	2.396	0.951	4.030
	7.50	0.938	2.902	0.941	4.846	0.812	1.886	0.925	1.445
	8.00	0.938	0.607	0.947	4.201	0.786	0.738	0.918	1.940
	8.50	0.933	4.337	0.947	1.037	0.797	0.728	0.925	1.421
	9.00	0.936	0.624	0.946	6.551	0.804	2.270	0.925	3.066
	9.50	0.939	1.064	0.961	1.491	0.807	1.297	0.914	1.044
	10.00	0.930	1.084	0.941	1.138	0.798	1.241	0.929	3.399
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	0.10	0.939	159.708	0.944	773.847	0.797	216.664	0.924	462.100
	0.50	0.926	152.209	0.943	56.080	0.783	77.878	0.932	21.489

	1.00	0.935	35.070	0.948	56.761	0.813	14.811	0.931	8.335
	1.50	0.934	13.006	0.948	8.413	0.809	5.986	0.920	16.629
	2.00	0.942	27.194	0.948	5.758	0.815	10.953	0.915	3.837
	2.50	0.934	5.318	0.945	6.928	0.804	0.695	0.932	5.273
	3.00	0.921	8.348	0.940	4.097	0.815	4.742	0.923	3.194
	3.50	0.930	5.622	0.945	2.752	0.827	2.396	0.906	4.877
	4.00	0.931	7.968	0.956	16.573	0.807	3.178	0.912	2.632
	4.50	0.942	8.725	0.958	4.196	0.790	3.654	0.926	5.486
6.00	5.00	0.929	7.273	0.949	3.309	0.818	2.282	0.922	2.560
	5.50	0.929	1.862	0.943	2.685	0.814	3.132	0.925	3.787
	6.00	0.944	2.318	0.961	7.080	0.798	2.789	0.936	5.514
	6.50	0.948	0.973	0.945	2.366	0.815	2.194	0.929	1.481
	7.00	0.940	2.010	0.956	2.099	0.794	1.466	0.922	1.734
	7.50	0.941	5.077	0.950	1.039	0.811	0.856	0.924	2.031
	8.00	0.947	1.942	0.953	2.924	0.813	2.584	0.923	1.061
	8.50	0.940	2.868	0.953	4.810	0.775	0.607	0.934	1.271
	9.00	0.951	1.128	0.962	23.659	0.793	4.320	0.930	2.727
	9.50	0.929	2.259	0.950	3.186	0.806	1.175	0.898	1.968
	10.00	0.934	3.439	0.960	5.713	0.789	0.865	0.915	1.493
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	0.10	0.946	130.433	0.955	231.971	0.817	176.068	0.916	292.714
	0.50	0.944	104.399	0.954	32.371	0.813	23.242	0.925	109.877
	1.00	0.923	15.974	0.951	14.569	0.822	14.239	0.929	10.036
	1.50	0.938	14.413	0.956	13.489	0.828	2.646	0.930	11.860
	2.00	0.939	4.391	0.957	8.776	0.800	7.339	0.903	14.987
	2.50	0.933	2.572	0.934	8.511	0.803	8.330	0.927	11.998
	3.00	0.938	5.639	0.940	4.088	0.815	5.895	0.933	18.736
	3.50	0.923	20.090	0.951	9.674	0.806	2.731	0.916	6.128
	4.00	0.950	6.667	0.950	9.528	0.824	7.137	0.923	4.986
	4.50	0.924	0.857	0.947	5.244	0.785	2.368	0.917	7.152
6.50	5.00	0.948	0.752	0.944	16.812	0.818	2.022	0.930	2.331
	5.50	0.933	2.403	0.947	2.287	0.811	1.713	0.917	12.770
	6.00	0.939	7.446	0.957	1.910	0.810	2.113	0.905	4.047
	6.50	0.922	9.893	0.962	0.933	0.824	2.053	0.928	0.801
	7.00	0.935	7.439	0.948	0.901	0.805	2.137	0.935	1.119

7.50	0.936	3.869	0.943	11.514	0.830	2.444	0.935	2.245	
8.00	0.928	1.848	0.944	1.023	0.809	1.315	0.905	2.521	
8.50	0.956	4.469	0.952	1.658	0.795	2.131	0.920	2.119	
9.00	0.940	1.450	0.954	3.973	0.807	0.382	0.932	3.781	
9.50	0.930	1.968	0.970	0.682	0.814	2.709	0.938	1.891	
10.00	0.923	7.593	0.947	1.262	0.823	2.812	0.918	1.856	
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0.10	0.945	390.246	0.948	359.869	0.806	73.666	0.916	170.133	
0.50	0.940	24.333	0.959	118.340	0.802	34.245	0.913	7.466	
1.00	0.940	66.421	0.960	72.445	0.799	9.318	0.912	11.391	
1.50	0.936	26.738	0.938	10.002	0.822	16.181	0.921	21.209	
2.00	0.928	9.105	0.946	18.950	0.832	15.066	0.933	2.116	
2.50	0.934	12.768	0.943	20.866	0.805	8.039	0.924	23.115	
3.00	0.940	12.451	0.951	3.451	0.829	5.506	0.926	4.212	
3.50	0.929	5.558	0.934	4.568	0.806	2.179	0.920	5.995	
4.00	0.930	1.171	0.945	3.358	0.805	1.159	0.922	6.817	
4.50	0.929	3.116	0.949	3.268	0.792	1.933	0.929	3.264	
7.00	5.00	0.931	2.408	0.943	1.881	0.802	11.310	0.934	6.163
5.50	0.933	2.510	0.952	8.566	0.797	1.444	0.925	4.228	
6.00	0.948	2.256	0.952	3.979	0.826	0.936	0.920	2.799	
6.50	0.934	6.294	0.946	3.489	0.819	0.977	0.907	14.667	
7.00	0.924	1.572	0.951	2.192	0.804	1.359	0.925	0.600	
7.50	0.943	3.864	0.957	14.947	0.816	0.782	0.927	5.775	
8.00	0.948	3.613	0.943	4.959	0.814	0.700	0.929	1.220	
8.50	0.934	2.289	0.953	2.142	0.818	1.173	0.925	2.250	
9.00	0.930	2.702	0.949	2.545	0.798	2.700	0.915	2.663	
9.50	0.943	1.855	0.961	7.312	0.812	3.496	0.930	4.801	
10.00	0.919	3.515	0.944	3.508	0.796	1.181	0.923	1.712	
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0.10	0.921	801.829	0.948	327.911	0.797	78.359	0.916	64.856	
0.50	0.935	22.328	0.941	56.315	0.798	34.563	0.914	37.966	
1.00	0.949	18.633	0.953	25.299	0.809	25.555	0.933	25.015	
1.50	0.941	50.699	0.946	7.586	0.812	6.874	0.929	12.345	
2.00	0.957	8.119	0.951	12.181	0.785	6.690	0.921	13.806	
2.50	0.921	8.019	0.944	12.310	0.808	13.330	0.929	5.360	
3.00	0.927	8.312	0.961	11.543	0.832	13.454	0.921	10.908	

	3.50	0.930	8.302	0.940	5.007	0.816	3.096	0.915	5.535
	4.00	0.944	6.973	0.954	3.937	0.802	3.218	0.932	13.068
	4.50	0.950	1.482	0.958	7.123	0.825	1.646	0.916	6.702
7.50	5.00	0.930	9.615	0.955	13.138	0.826	1.808	0.940	5.009
	5.50	0.921	1.918	0.953	12.816	0.817	3.083	0.932	1.039
	6.00	0.945	4.895	0.936	6.658	0.792	2.892	0.913	2.311
	6.50	0.931	0.971	0.956	8.046	0.805	2.664	0.939	8.589
	7.00	0.940	3.074	0.950	3.639	0.797	1.530	0.925	1.949
	7.50	0.924	2.378	0.946	4.464	0.794	4.895	0.922	3.302
	8.00	0.941	1.446	0.947	3.940	0.794	0.603	0.922	3.121
	8.50	0.939	4.637	0.943	8.303	0.808	3.252	0.929	7.928
	9.00	0.939	17.467	0.943	2.746	0.802	5.225	0.930	11.330
	9.50	0.943	1.833	0.955	1.432	0.798	1.751	0.931	1.134
	10.00	0.934	2.974	0.950	16.270	0.804	0.364	0.926	1.987
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	0.10	0.936	227.293	0.940	372.785	0.818	39.099	0.916	272.465
	0.50	0.934	17.691	0.957	13.904	0.785	100.185	0.928	49.299
	1.00	0.933	27.850	0.952	18.698	0.812	9.935	0.925	56.693
	1.50	0.932	13.857	0.936	3.802	0.786	9.372	0.933	10.664
	2.00	0.947	8.080	0.949	13.403	0.796	3.358	0.913	10.388
	2.50	0.938	33.110	0.927	1.151	0.828	2.488	0.916	13.808
	3.00	0.943	2.931	0.951	11.038	0.818	8.931	0.916	7.787
	3.50	0.938	4.077	0.944	7.838	0.807	1.629	0.920	5.846
	4.00	0.924	10.355	0.949	4.645	0.798	4.081	0.934	15.044
	4.50	0.944	5.120	0.944	4.412	0.814	3.163	0.941	1.813
8.00	5.00	0.938	6.101	0.957	14.219	0.803	1.557	0.929	3.099
	5.50	0.935	8.198	0.947	6.437	0.798	4.194	0.922	2.512
	6.00	0.939	5.136	0.945	7.790	0.799	2.009	0.918	3.409
	6.50	0.921	2.544	0.944	3.465	0.798	1.041	0.915	4.246
	7.00	0.932	1.903	0.962	3.076	0.804	2.943	0.913	2.162
	7.50	0.935	5.311	0.956	4.563	0.798	2.511	0.915	3.088
	8.00	0.929	1.829	0.951	2.555	0.809	2.079	0.920	2.207
	8.50	0.936	5.273	0.952	7.742	0.822	3.604	0.940	3.544
	9.00	0.929	3.432	0.946	4.298	0.814	0.822	0.922	1.362
	9.50	0.932	4.017	0.945	3.859	0.823	1.940	0.918	1.956

	10.00	0.938	2.551	0.953	4.313	0.805	0.428	0.929	2.254
	0.10	0.945	354.163	0.934	88.541	0.819	110.940	0.915	161.828
	0.50	0.936	229.359	0.954	25.389	0.798	39.312	0.914	38.933
	1.00	0.930	23.906	0.949	16.105	0.811	13.198	0.920	24.762
	1.50	0.928	13.868	0.950	36.127	0.814	20.391	0.924	6.325
	2.00	0.931	12.501	0.956	9.224	0.815	5.225	0.914	21.268
	2.50	0.943	12.645	0.960	11.389	0.819	7.004	0.910	8.150
	3.00	0.933	9.706	0.950	9.050	0.797	8.107	0.931	10.951
	3.50	0.944	6.106	0.937	7.427	0.793	2.655	0.914	7.102
	4.00	0.936	3.531	0.948	5.467	0.780	3.019	0.932	1.749
	4.50	0.933	4.419	0.949	5.279	0.795	5.202	0.916	4.358
8.50	5.00	0.937	4.497	0.947	2.747	0.818	1.647	0.916	7.313
	5.50	0.927	22.761	0.962	2.803	0.808	2.868	0.924	5.562
	6.00	0.939	2.581	0.946	1.319	0.819	6.015	0.930	3.580
	6.50	0.938	4.695	0.960	2.417	0.823	2.886	0.925	3.342
	7.00	0.930	1.428	0.945	1.967	0.800	1.131	0.914	2.878
	7.50	0.942	3.037	0.947	8.399	0.821	1.448	0.939	3.903
	8.00	0.933	1.737	0.945	3.236	0.819	0.785	0.911	4.451
	8.50	0.929	2.566	0.942	2.820	0.826	4.119	0.924	3.392
	9.00	0.927	3.414	0.940	6.598	0.809	3.941	0.935	2.739
	9.50	0.942	2.079	0.956	6.751	0.813	1.705	0.921	1.982
	10.00	0.941	5.624	0.940	1.296	0.788	1.437	0.925	3.459
	0.10	0.934	886.061	0.951	91.965	0.798	52.968	0.919	100.296
	0.50	0.929	57.492	0.957	51.166	0.824	27.360	0.925	40.228
	1.00	0.940	43.083	0.941	73.339	0.838	15.278	0.926	12.849
	1.50	0.948	11.404	0.955	41.834	0.817	7.691	0.927	13.458
	2.00	0.942	16.690	0.952	19.481	0.810	10.293	0.930	9.534
	2.50	0.922	7.222	0.955	13.516	0.803	5.764	0.919	6.634
	3.00	0.943	9.127	0.937	6.164	0.804	12.078	0.933	11.563
	3.50	0.936	9.696	0.946	6.306	0.814	1.786	0.931	2.274
	4.00	0.926	13.674	0.950	12.750	0.831	7.730	0.916	13.921
	4.50	0.949	3.880	0.943	3.327	0.808	2.355	0.929	3.676
9.00	5.00	0.937	5.995	0.939	4.394	0.819	0.730	0.906	3.872
	5.50	0.950	3.310	0.946	5.684	0.797	3.814	0.927	20.710

6.00	0.944	1.005	0.961	5.527	0.806	1.072	0.930	9.369
6.50	0.949	1.994	0.950	6.369	0.823	2.301	0.933	2.493
7.00	0.945	3.072	0.959	3.137	0.805	1.496	0.928	7.473
7.50	0.931	1.469	0.952	6.956	0.818	2.359	0.917	8.312
8.00	0.927	7.550	0.960	11.841	0.821	1.561	0.931	2.095
8.50	0.949	2.204	0.954	3.467	0.796	0.982	0.920	0.830
9.00	0.947	1.830	0.957	3.024	0.820	1.915	0.931	2.418
9.50	0.933	5.269	0.953	5.825	0.806	0.939	0.916	3.092
10.00	0.933	4.813	0.950	4.261	0.790	1.088	0.921	2.512
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0.10	0.935	141.184	0.940	273.279	0.818	356.339	0.903	581.889
0.50	0.951	111.127	0.953	73.102	0.809	26.860	0.925	57.772
1.00	0.939	28.623	0.956	46.440	0.798	23.084	0.928	42.017
1.50	0.940	21.790	0.952	27.480	0.804	10.061	0.934	29.276
2.00	0.934	34.620	0.952	23.310	0.804	8.507	0.936	21.985
2.50	0.932	18.184	0.948	8.454	0.797	7.509	0.926	3.812
3.00	0.924	6.847	0.950	6.193	0.790	10.467	0.925	4.398
3.50	0.954	11.883	0.945	9.297	0.807	1.208	0.923	16.739
4.00	0.938	7.163	0.950	7.373	0.816	3.055	0.912	3.435
4.50	0.936	7.461	0.936	5.460	0.821	1.641	0.934	8.099
9.50 5.00	0.926	4.561	0.955	21.161	0.788	4.342	0.915	3.443
5.50	0.943	6.623	0.944	7.746	0.805	4.608	0.923	14.345
6.00	0.945	9.207	0.940	5.967	0.791	2.579	0.924	4.543
6.50	0.925	1.656	0.956	6.197	0.810	1.113	0.927	1.456
7.00	0.951	7.963	0.947	3.883	0.800	3.627	0.925	2.176
7.50	0.947	9.065	0.956	2.131	0.800	7.425	0.925	1.679
8.00	0.936	1.756	0.947	3.757	0.802	4.956	0.932	2.091
8.50	0.941	2.063	0.950	2.824	0.791	1.334	0.935	2.651
9.00	0.937	4.565	0.948	1.705	0.803	0.930	0.916	0.508
9.50	0.942	1.794	0.943	9.991	0.815	1.084	0.921	2.731
10.00	0.938	2.282	0.960	4.029	0.808	2.800	0.913	1.721
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0.10	0.943	118.206	0.951	524.132	0.816	84.402	0.927	153.018
0.50	0.940	115.877	0.948	18.731	0.820	41.018	0.925	58.428
1.00	0.951	54.393	0.952	32.899	0.810	12.325	0.933	24.583
1.50	0.937	18.756	0.948	19.119	0.801	3.319	0.934	37.676

	2.00	0.937	10.006	0.955	18.431	0.821	32.203	0.923	20.749
	2.50	0.942	18.535	0.959	7.223	0.817	6.490	0.923	10.847
	3.00	0.941	22.760	0.936	27.796	0.814	2.126	0.925	11.395
	3.50	0.950	11.523	0.953	6.166	0.810	6.741	0.916	9.014
	4.00	0.945	13.635	0.963	3.584	0.824	5.249	0.926	8.147
	4.50	0.939	6.891	0.951	2.262	0.808	10.102	0.941	0.921
10.00	5.00	0.939	0.645	0.953	4.839	0.799	4.944	0.922	4.174
	5.50	0.935	6.921	0.956	1.079	0.821	1.902	0.914	6.938
	6.00	0.941	12.871	0.951	6.065	0.810	1.019	0.921	10.860
	6.50	0.939	4.021	0.952	12.549	0.816	0.889	0.920	5.647
	7.00	0.931	4.354	0.942	10.803	0.812	2.981	0.924	9.115
	7.50	0.936	5.255	0.953	4.248	0.805	4.643	0.924	3.919
	8.00	0.931	5.856	0.945	4.144	0.814	1.010	0.923	3.404
	8.50	0.920	1.424	0.938	5.073	0.813	1.839	0.932	5.124
	9.00	0.931	4.183	0.954	2.351	0.812	0.380	0.920	3.361
	9.50	0.942	1.689	0.945	5.672	0.808	0.793	0.934	0.583
	10.00	0.931	5.554	0.948	2.991	0.822	0.972	0.910	2.158

B.2 Complete table for $n = 10$ and $m = 20$

Table B.2: Coverage rates (CR) and average interval lengths (AL) for $n = 10$ and $m = 20$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.10	0.10	0.956	2.001	0.953	1.453	0.814	1.151	0.933	1.452
	0.50	0.938	0.698	0.947	0.700	0.812	0.217	0.942	0.407
	1.00	0.930	0.097	0.950	0.208	0.841	0.250	0.939	0.180
	1.50	0.936	0.121	0.960	0.092	0.827	0.078	0.933	0.140
	2.00	0.943	0.048	0.954	0.069	0.811	0.077	0.946	0.107
	2.50	0.935	0.033	0.955	0.054	0.833	0.039	0.937	0.097
	3.00	0.943	0.080	0.960	0.086	0.825	0.055	0.927	0.049
	3.50	0.941	0.037	0.944	0.036	0.815	0.019	0.936	0.045
	4.00	0.955	0.081	0.946	0.080	0.810	0.033	0.951	0.030
	4.50	0.946	0.057	0.952	0.029	0.810	0.039	0.957	0.030
	5.00	0.950	0.024	0.948	0.024	0.843	0.025	0.932	0.025
	5.50	0.938	0.026	0.949	0.034	0.828	0.022	0.940	0.020
	6.00	0.937	0.019	0.961	0.027	0.810	0.011	0.941	0.025
	6.50	0.940	0.015	0.957	0.027	0.819	0.016	0.934	0.055
	7.00	0.948	0.043	0.955	0.015	0.826	0.015	0.938	0.013
	7.50	0.955	0.016	0.946	0.030	0.837	0.032	0.953	0.026
	8.00	0.934	0.020	0.945	0.016	0.826	0.014	0.948	0.014
	8.50	0.945	0.018	0.930	0.017	0.832	0.017	0.939	0.015
	9.00	0.943	0.016	0.944	0.023	0.822	0.019	0.927	0.016
	9.50	0.960	0.025	0.949	0.007	0.830	0.010	0.932	0.018
10.00	0.944	0.023	0.954	0.014	0.820	0.014	0.947	0.020	
	0.10	0.948	6.596	0.959	4.525	0.799	8.270	0.923	8.721
	0.50	0.937	2.527	0.952	1.700	0.830	0.785	0.948	1.324
	1.00	0.938	0.716	0.956	0.857	0.844	0.834	0.940	0.574
	1.50	0.949	0.376	0.949	0.790	0.822	0.468	0.945	1.093
	2.00	0.940	0.328	0.956	0.524	0.831	0.198	0.938	0.562
	2.50	0.947	0.666	0.949	0.337	0.826	0.210	0.945	0.348
	3.00	0.950	0.438	0.953	0.288	0.808	0.435	0.938	0.200

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.50	3.50	0.957	0.106	0.954	0.351	0.814	0.156	0.946	0.512
	4.00	0.951	0.232	0.958	0.202	0.828	0.317	0.937	0.166
	4.50	0.951	0.127	0.947	0.096	0.796	0.172	0.950	0.112
	5.00	0.938	0.124	0.952	0.122	0.836	0.104	0.940	0.137
	5.50	0.942	0.198	0.961	0.090	0.821	0.074	0.937	0.115
	6.00	0.940	0.142	0.950	0.197	0.831	0.083	0.924	0.148
	6.50	0.941	0.109	0.953	0.188	0.843	0.071	0.941	0.142
	7.00	0.935	0.161	0.937	0.076	0.808	0.088	0.935	0.183
	7.50	0.952	0.193	0.968	0.174	0.839	0.060	0.948	0.102
	8.00	0.943	0.068	0.954	0.060	0.814	0.059	0.931	0.080
	8.50	0.944	0.140	0.947	0.110	0.813	0.079	0.935	0.120
	9.00	0.952	0.075	0.961	0.058	0.814	0.061	0.936	0.069
	9.50	0.942	0.042	0.934	0.141	0.836	0.054	0.941	0.200
	10.00	0.951	0.208	0.951	0.062	0.832	0.035	0.939	0.065
1.00	0.10	0.930	21.027	0.947	24.059	0.835	9.811	0.948	15.941
	0.50	0.937	3.372	0.942	3.606	0.821	2.240	0.943	3.990
	1.00	0.934	1.109	0.958	1.076	0.823	0.822	0.942	0.833
	1.50	0.949	1.520	0.937	2.007	0.808	0.628	0.936	0.873
	2.00	0.942	1.475	0.952	1.244	0.806	1.309	0.940	0.792
	2.50	0.935	0.568	0.956	0.970	0.824	0.489	0.948	0.758
	3.00	0.942	0.474	0.948	0.696	0.817	0.316	0.947	0.365
	3.50	0.957	0.327	0.942	0.672	0.814	0.382	0.934	0.624
	4.00	0.946	0.622	0.950	0.376	0.835	0.110	0.935	0.480
	4.50	0.951	0.492	0.956	0.260	0.811	0.190	0.937	0.371
	5.00	0.948	0.279	0.951	0.383	0.828	0.288	0.932	0.420
	5.50	0.954	0.298	0.938	0.375	0.814	0.105	0.946	0.455
	6.00	0.939	0.188	0.949	0.477	0.807	0.134	0.945	0.234
	6.50	0.940	0.186	0.946	0.135	0.815	0.149	0.942	0.184
7.00	0.952	0.279	0.937	0.104	0.834	0.257	0.944	0.318	
7.50	0.943	0.178	0.949	0.157	0.826	0.087	0.932	0.152	
8.00	0.947	0.289	0.952	0.335	0.815	0.120	0.926	0.219	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	8.50	0.935	0.189	0.955	0.225	0.823	0.117	0.941	0.571
	9.00	0.948	0.158	0.951	0.329	0.821	0.143	0.941	0.222
	9.50	0.946	0.141	0.952	0.080	0.850	0.095	0.929	0.149
	10.00	0.954	0.268	0.939	0.205	0.790	0.022	0.940	0.166
	0.10	0.954	29.328	0.951	33.138	0.800	10.683	0.927	14.777
	0.50	0.931	6.692	0.946	4.961	0.822	2.599	0.944	5.209
	1.00	0.953	4.135	0.959	3.162	0.839	0.860	0.932	1.807
	1.50	0.942	1.779	0.938	1.633	0.818	1.021	0.954	1.090
	2.00	0.937	2.092	0.954	1.317	0.830	0.784	0.934	1.001
	2.50	0.940	1.384	0.936	1.475	0.832	0.738	0.937	0.730
	3.00	0.930	1.125	0.950	0.715	0.829	0.493	0.932	0.864
	3.50	0.955	0.662	0.945	1.207	0.819	0.327	0.955	0.610
	4.00	0.950	0.543	0.958	0.929	0.822	0.295	0.955	0.721
	4.50	0.941	1.031	0.948	0.537	0.822	0.417	0.952	0.476
1.50	5.00	0.953	0.383	0.961	0.441	0.828	0.450	0.937	0.367
	5.50	0.947	0.420	0.947	0.368	0.852	0.220	0.940	0.501
	6.00	0.941	0.320	0.956	0.433	0.792	0.213	0.950	0.398
	6.50	0.945	0.418	0.950	0.197	0.848	0.181	0.933	0.457
	7.00	0.946	0.395	0.957	0.269	0.803	0.327	0.939	0.529
	7.50	0.933	0.521	0.958	0.293	0.814	0.331	0.947	0.323
	8.00	0.951	0.315	0.951	0.218	0.812	0.141	0.945	0.234
	8.50	0.938	0.201	0.962	0.259	0.818	0.412	0.934	0.295
	9.00	0.944	0.217	0.948	0.205	0.826	0.235	0.929	0.265
	9.50	0.943	0.226	0.936	0.202	0.833	0.156	0.931	0.791
	10.00	0.944	0.285	0.950	0.190	0.811	0.106	0.944	0.394
	0.10	0.942	20.088	0.939	20.257	0.827	28.141	0.939	25.149
	0.50	0.949	3.263	0.958	6.653	0.821	5.098	0.924	4.800
	1.00	0.948	3.308	0.939	5.598	0.814	2.514	0.936	3.351
	1.50	0.936	2.117	0.953	1.386	0.818	1.615	0.927	2.620
	2.00	0.942	3.561	0.949	2.268	0.832	0.836	0.935	1.335
	2.50	0.952	1.469	0.945	0.970	0.798	0.709	0.945	1.590

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
2.00	3.00	0.954	0.781	0.944	0.668	0.838	0.909	0.949	1.424
	3.50	0.940	1.749	0.936	1.881	0.823	0.686	0.943	0.683
	4.00	0.947	1.020	0.937	1.295	0.822	0.291	0.948	1.381
	4.50	0.959	1.347	0.953	0.565	0.799	0.559	0.932	1.363
	5.00	0.939	1.718	0.948	0.545	0.842	0.635	0.950	0.623
	5.50	0.942	0.938	0.961	0.758	0.829	0.428	0.941	0.805
	6.00	0.945	0.497	0.946	0.840	0.821	0.389	0.949	0.650
	6.50	0.944	0.821	0.955	0.735	0.842	0.451	0.936	0.445
	7.00	0.927	0.332	0.945	0.383	0.809	0.159	0.946	0.857
	7.50	0.937	0.237	0.956	0.402	0.823	0.207	0.939	0.533
	8.00	0.951	0.407	0.945	0.315	0.808	0.181	0.939	0.723
	8.50	0.935	0.335	0.948	0.512	0.839	0.382	0.941	0.325
	9.00	0.930	0.470	0.949	0.368	0.785	0.407	0.935	0.520
	9.50	0.945	0.506	0.950	0.183	0.814	0.179	0.947	0.296
10.00	0.927	0.394	0.946	0.304	0.821	0.152	0.943	0.749	
2.50	0.10	0.943	61.693	0.945	39.790	0.827	30.146	0.937	75.466
	0.50	0.943	6.717	0.956	11.599	0.837	6.895	0.934	7.292
	1.00	0.933	8.053	0.959	4.650	0.825	2.959	0.936	3.907
	1.50	0.949	1.453	0.955	1.729	0.834	1.823	0.944	1.602
	2.00	0.942	1.334	0.950	2.612	0.823	1.475	0.948	2.604
	2.50	0.952	1.720	0.947	2.116	0.837	0.767	0.934	2.313
	3.00	0.956	0.989	0.946	0.767	0.845	2.121	0.945	1.272
	3.50	0.952	1.019	0.935	0.777	0.822	0.949	0.928	0.794
	4.00	0.939	0.912	0.952	1.310	0.827	0.975	0.927	0.479
	4.50	0.949	0.738	0.952	1.298	0.817	0.646	0.936	1.215
	5.00	0.951	1.356	0.947	0.749	0.810	0.674	0.948	0.581
	5.50	0.959	0.716	0.952	1.388	0.798	1.115	0.948	0.485
	6.00	0.944	0.607	0.933	0.594	0.808	1.011	0.941	0.423
	6.50	0.942	0.719	0.945	0.868	0.813	0.513	0.942	0.549
7.00	0.939	1.268	0.937	0.503	0.826	0.509	0.940	0.547	
7.50	0.944	0.342	0.952	0.458	0.825	0.267	0.948	0.693	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	8.00	0.944	0.883	0.950	0.774	0.827	0.286	0.940	0.602
	8.50	0.944	0.866	0.956	0.356	0.814	0.271	0.938	0.241
	9.00	0.942	0.394	0.951	0.266	0.829	0.228	0.957	0.706
	9.50	0.930	0.267	0.958	0.297	0.817	0.386	0.946	0.230
	10.00	0.954	0.401	0.961	0.337	0.820	0.310	0.947	0.158
	0.10	0.950	74.192	0.954	64.181	0.817	33.988	0.934	41.070
	0.50	0.945	10.482	0.957	19.716	0.813	3.802	0.942	15.631
	1.00	0.940	4.131	0.953	4.595	0.827	6.187	0.940	5.337
	1.50	0.930	4.386	0.949	4.740	0.837	1.370	0.943	2.871
	2.00	0.934	1.506	0.962	1.785	0.840	0.975	0.947	1.951
	2.50	0.942	1.963	0.956	2.205	0.838	2.032	0.934	5.810
	3.00	0.938	2.623	0.954	2.538	0.829	0.858	0.935	1.574
	3.50	0.948	2.007	0.944	1.363	0.813	0.958	0.956	1.099
	4.00	0.941	5.607	0.951	1.532	0.844	0.985	0.934	1.746
	4.50	0.947	0.880	0.957	2.025	0.816	0.837	0.933	1.412
3.00	5.00	0.946	1.644	0.965	0.907	0.800	0.474	0.938	1.636
	5.50	0.942	1.882	0.949	0.678	0.817	0.796	0.943	1.800
	6.00	0.939	1.252	0.954	0.429	0.853	0.451	0.927	0.518
	6.50	0.953	0.601	0.942	0.453	0.839	0.375	0.942	1.148
	7.00	0.938	0.485	0.943	1.451	0.815	0.711	0.941	0.728
	7.50	0.948	1.207	0.945	0.594	0.803	0.858	0.946	1.191
	8.00	0.939	0.534	0.951	0.438	0.815	0.374	0.937	0.458
	8.50	0.947	0.634	0.949	0.559	0.820	0.197	0.939	0.551
	9.00	0.935	0.605	0.948	0.429	0.820	0.404	0.935	0.405
	9.50	0.941	0.665	0.935	0.349	0.810	0.416	0.949	0.603
	10.00	0.955	0.248	0.956	0.501	0.817	0.519	0.955	0.421
	0.10	0.942	56.602	0.938	43.729	0.809	28.311	0.942	49.164
	0.50	0.944	6.933	0.947	11.442	0.828	5.193	0.927	8.390
	1.00	0.952	11.253	0.944	5.721	0.821	3.858	0.933	4.085
	1.50	0.937	3.411	0.950	5.987	0.817	2.214	0.939	6.075
	2.00	0.944	3.185	0.947	6.677	0.835	1.257	0.923	3.547

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
3.50	2.50	0.947	3.241	0.959	2.113	0.840	1.548	0.946	1.755
	3.00	0.956	1.815	0.959	2.000	0.822	0.699	0.939	1.585
	3.50	0.944	1.886	0.953	1.158	0.828	1.949	0.931	2.471
	4.00	0.951	1.671	0.955	2.412	0.831	0.639	0.936	1.405
	4.50	0.946	0.811	0.948	1.624	0.805	0.855	0.939	3.106
	5.00	0.947	0.920	0.956	1.636	0.808	0.899	0.936	0.995
	5.50	0.943	0.873	0.952	0.824	0.843	1.033	0.932	1.169
	6.00	0.940	1.010	0.933	0.960	0.831	0.328	0.937	1.118
	6.50	0.937	1.151	0.943	0.724	0.815	0.382	0.931	0.530
	7.00	0.948	0.725	0.954	0.683	0.830	0.634	0.927	0.506
	7.50	0.953	1.849	0.950	0.793	0.820	0.324	0.931	1.301
	8.00	0.953	0.758	0.942	1.214	0.845	1.165	0.940	1.071
	8.50	0.936	0.743	0.955	0.669	0.828	0.451	0.931	0.559
	9.00	0.950	0.647	0.969	0.493	0.798	0.305	0.938	1.238
	9.50	0.942	0.664	0.947	0.788	0.823	0.363	0.923	0.653
10.00	0.944	0.807	0.963	1.560	0.813	0.602	0.931	0.429	
4.00	0.10	0.934	29.674	0.950	48.031	0.805	31.880	0.931	102.519
	0.50	0.933	11.587	0.951	20.600	0.828	6.732	0.940	13.843
	1.00	0.947	6.359	0.954	4.213	0.826	4.088	0.922	5.190
	1.50	0.949	3.875	0.946	12.192	0.833	3.984	0.940	5.883
	2.00	0.945	5.294	0.948	3.179	0.817	1.223	0.926	3.591
	2.50	0.946	3.582	0.950	5.785	0.806	2.298	0.933	4.094
	3.00	0.948	1.939	0.954	1.803	0.819	1.231	0.945	2.438
	3.50	0.955	3.354	0.946	1.616	0.829	1.587	0.936	2.069
	4.00	0.941	1.659	0.956	1.704	0.842	0.639	0.940	2.031
	4.50	0.942	1.016	0.950	2.373	0.843	0.868	0.949	1.472
	5.00	0.942	1.763	0.948	0.888	0.828	0.910	0.939	1.125
	5.50	0.945	0.487	0.946	0.759	0.827	1.416	0.955	1.048
6.00	0.954	0.894	0.951	1.134	0.847	1.278	0.940	0.892	
6.50	0.949	0.700	0.952	0.933	0.810	0.410	0.928	1.467	
7.00	0.940	0.932	0.942	0.900	0.854	0.446	0.942	0.815	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	7.50	0.950	0.341	0.947	1.551	0.823	0.429	0.943	0.855
	8.00	0.945	0.913	0.943	0.968	0.826	0.495	0.925	1.340
	8.50	0.940	0.743	0.949	1.521	0.823	0.315	0.943	0.767
	9.00	0.947	0.774	0.943	0.647	0.804	0.434	0.942	0.626
	9.50	0.952	0.611	0.946	0.777	0.808	0.522	0.941	0.333
	10.00	0.944	0.530	0.949	0.741	0.838	0.407	0.943	0.303
	0.10	0.956	48.148	0.952	28.367	0.815	45.849	0.947	65.063
	0.50	0.943	15.616	0.948	15.873	0.808	12.453	0.936	17.051
	1.00	0.945	9.111	0.957	5.949	0.818	7.792	0.930	22.394
	1.50	0.944	5.859	0.950	11.622	0.828	3.814	0.944	3.272
	2.00	0.950	2.031	0.940	2.199	0.821	2.689	0.945	2.029
	2.50	0.934	1.986	0.962	2.264	0.811	1.461	0.931	6.989
	3.00	0.932	1.712	0.948	2.105	0.818	2.009	0.933	1.621
	3.50	0.954	1.979	0.951	1.769	0.821	1.370	0.941	2.179
	4.00	0.941	6.014	0.949	1.163	0.828	1.673	0.928	4.014
	4.50	0.941	1.340	0.956	1.506	0.825	0.881	0.944	1.471
4.50	5.00	0.942	1.393	0.954	2.543	0.812	0.750	0.936	1.031
	5.50	0.941	1.689	0.942	0.991	0.817	0.683	0.939	1.193
	6.00	0.952	0.826	0.950	0.998	0.816	1.318	0.942	2.683
	6.50	0.949	0.935	0.959	0.699	0.808	0.601	0.925	0.880
	7.00	0.948	2.206	0.946	1.977	0.826	1.000	0.936	0.964
	7.50	0.941	1.442	0.938	0.529	0.854	0.737	0.926	2.179
	8.00	0.947	1.430	0.946	1.162	0.796	0.804	0.951	0.839
	8.50	0.938	1.030	0.948	1.764	0.835	0.513	0.950	0.750
	9.00	0.922	0.787	0.951	0.510	0.807	0.538	0.931	0.728
	9.50	0.935	1.038	0.944	0.881	0.802	0.834	0.930	0.570
	10.00	0.950	0.659	0.949	0.853	0.834	0.611	0.934	0.704
	0.10	0.950	159.037	0.958	96.434	0.823	75.143	0.949	90.684
	0.50	0.951	19.164	0.945	30.457	0.827	11.127	0.932	8.628
	1.00	0.950	9.358	0.952	14.613	0.829	9.386	0.919	4.097
	1.50	0.939	3.190	0.943	5.770	0.836	8.440	0.939	5.606

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
5.00	2.00	0.921	6.739	0.954	4.443	0.819	3.969	0.937	4.912
	2.50	0.930	2.799	0.944	3.432	0.807	2.011	0.922	3.400
	3.00	0.939	4.947	0.938	1.326	0.835	2.071	0.934	3.324
	3.50	0.944	1.482	0.948	2.585	0.824	2.319	0.939	6.133
	4.00	0.943	1.485	0.961	3.249	0.840	1.387	0.943	1.729
	4.50	0.941	3.567	0.962	1.262	0.806	1.750	0.947	3.260
	5.00	0.945	3.944	0.960	2.025	0.824	0.868	0.933	1.117
	5.50	0.951	1.232	0.933	0.722	0.812	1.145	0.941	2.185
	6.00	0.939	0.785	0.933	1.278	0.827	1.088	0.954	1.871
	6.50	0.938	2.324	0.951	0.899	0.825	0.609	0.940	1.327
	7.00	0.943	1.435	0.956	0.985	0.805	0.884	0.944	1.469
	7.50	0.949	1.786	0.957	1.652	0.817	0.707	0.932	0.725
	8.00	0.935	0.792	0.954	0.949	0.831	1.022	0.937	0.527
	8.50	0.949	0.773	0.946	1.697	0.812	0.700	0.928	1.005
	9.00	0.953	0.893	0.936	0.768	0.844	0.531	0.934	1.476
9.50	0.945	1.386	0.948	0.993	0.825	0.922	0.936	0.864	
10.00	0.956	0.892	0.951	0.584	0.816	0.989	0.938	1.363	
5.50	0.10	0.944	54.001	0.952	61.505	0.808	67.076	0.936	74.571
	0.50	0.955	12.442	0.950	17.629	0.828	24.837	0.936	15.427
	1.00	0.948	5.917	0.956	8.524	0.801	4.433	0.934	14.989
	1.50	0.928	7.495	0.950	8.576	0.820	10.493	0.932	3.360
	2.00	0.944	3.810	0.956	9.239	0.832	3.740	0.930	5.602
	2.50	0.960	4.997	0.944	4.990	0.831	2.145	0.933	6.093
	3.00	0.935	5.019	0.939	4.088	0.836	2.507	0.938	3.196
	3.50	0.954	2.338	0.953	1.192	0.830	2.414	0.942	3.174
	4.00	0.957	3.922	0.960	2.395	0.828	1.675	0.940	1.497
	4.50	0.932	2.307	0.954	2.528	0.825	1.438	0.944	2.487
5.00	0.943	3.082	0.952	1.850	0.821	1.209	0.938	1.535	
5.50	0.947	2.341	0.947	1.570	0.817	0.991	0.938	1.711	
6.00	0.949	0.648	0.948	1.296	0.827	0.890	0.927	1.406	
6.50	0.942	2.845	0.948	1.572	0.824	0.800	0.949	1.774	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	7.00	0.941	0.964	0.957	0.667	0.820	0.954	0.949	0.983
	7.50	0.937	1.503	0.935	0.986	0.828	0.535	0.920	1.102
	8.00	0.944	1.588	0.951	1.193	0.837	1.149	0.934	0.640
	8.50	0.946	0.918	0.951	0.741	0.840	1.000	0.934	1.237
	9.00	0.944	0.981	0.966	0.846	0.821	1.089	0.938	1.347
	9.50	0.955	1.244	0.938	1.033	0.816	0.788	0.938	0.538
	10.00	0.931	1.442	0.962	0.594	0.817	1.138	0.938	0.861
	0.10	0.938	170.485	0.942	138.291	0.837	62.903	0.940	98.773
	0.50	0.938	8.497	0.940	12.861	0.825	7.224	0.939	46.194
	1.00	0.943	9.569	0.945	8.924	0.823	7.852	0.949	9.486
	1.50	0.942	18.399	0.939	5.632	0.785	2.504	0.931	2.639
	2.00	0.937	8.087	0.947	6.981	0.816	2.717	0.929	4.090
	2.50	0.952	4.593	0.951	4.478	0.796	1.984	0.956	7.438
	3.00	0.938	4.923	0.950	4.340	0.850	2.714	0.936	4.221
	3.50	0.948	2.438	0.959	3.119	0.813	2.667	0.936	4.036
	4.00	0.941	1.572	0.958	4.520	0.814	0.606	0.944	1.744
	4.50	0.950	3.650	0.939	2.500	0.809	3.348	0.943	1.963
6.00	5.00	0.940	2.660	0.940	2.773	0.830	1.843	0.940	1.635
	5.50	0.942	2.308	0.952	1.481	0.816	0.693	0.944	3.815
	6.00	0.954	1.115	0.950	2.359	0.830	1.370	0.955	1.504
	6.50	0.945	0.628	0.954	1.092	0.823	0.832	0.935	1.622
	7.00	0.946	1.861	0.928	1.245	0.835	0.769	0.950	1.250
	7.50	0.937	1.414	0.958	1.098	0.826	0.837	0.947	0.962
	8.00	0.954	0.765	0.956	1.667	0.808	0.382	0.946	2.119
	8.50	0.949	1.010	0.949	1.077	0.818	0.882	0.925	0.991
	9.00	0.943	1.842	0.950	1.435	0.815	0.708	0.932	0.516
	9.50	0.941	0.689	0.931	1.366	0.829	0.719	0.938	1.340
	10.00	0.936	0.972	0.938	1.580	0.825	0.992	0.937	1.565
	0.10	0.939	76.698	0.958	128.791	0.836	63.724	0.943	92.859
	0.50	0.953	28.728	0.944	17.840	0.819	13.303	0.942	30.076
	1.00	0.936	12.283	0.947	13.164	0.818	4.534	0.925	8.228

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
6.50	1.50	0.937	9.798	0.943	7.838	0.812	12.831	0.947	5.459
	2.00	0.939	4.075	0.945	6.696	0.826	3.813	0.945	7.002
	2.50	0.948	5.756	0.950	4.846	0.819	1.792	0.947	4.739
	3.00	0.935	2.947	0.953	3.898	0.817	2.517	0.944	2.747
	3.50	0.954	3.047	0.952	4.880	0.827	3.246	0.938	14.691
	4.00	0.947	4.625	0.934	4.229	0.805	2.905	0.936	2.064
	4.50	0.942	3.200	0.941	2.097	0.835	1.431	0.939	2.995
	5.00	0.943	2.748	0.953	4.197	0.821	1.098	0.933	1.868
	5.50	0.953	2.168	0.940	2.214	0.811	0.900	0.929	1.071
	6.00	0.942	1.856	0.957	3.497	0.838	3.444	0.941	1.577
	6.50	0.950	1.060	0.945	1.757	0.839	1.259	0.937	2.347
	7.00	0.936	1.718	0.953	2.517	0.804	1.128	0.930	2.094
	7.50	0.954	1.489	0.948	1.385	0.815	0.910	0.921	1.063
	8.00	0.947	1.894	0.951	0.560	0.786	1.118	0.923	1.685
	8.50	0.945	1.214	0.946	1.237	0.845	1.732	0.945	1.649
	9.00	0.957	1.474	0.941	1.722	0.804	0.764	0.943	1.875
9.50	0.954	1.447	0.963	0.997	0.822	0.617	0.941	1.020	
10.00	0.952	1.603	0.949	1.809	0.819	1.058	0.942	1.285	
7.00	0.10	0.951	136.035	0.955	122.205	0.820	177.957	0.935	136.447
	0.50	0.952	30.254	0.959	15.059	0.833	21.023	0.950	14.855
	1.00	0.953	18.207	0.953	9.992	0.830	6.902	0.938	19.424
	1.50	0.955	6.306	0.938	7.544	0.831	5.146	0.924	17.557
	2.00	0.941	2.450	0.961	4.177	0.808	3.009	0.935	6.434
	2.50	0.961	2.712	0.942	3.648	0.823	2.004	0.934	2.666
	3.00	0.946	4.983	0.942	3.779	0.819	1.020	0.942	3.195
	3.50	0.944	4.163	0.948	3.285	0.817	1.991	0.949	3.735
	4.00	0.947	2.855	0.947	4.138	0.827	2.704	0.948	5.622
	4.50	0.922	1.640	0.952	2.243	0.827	1.878	0.927	1.166
5.00	0.954	2.355	0.963	2.389	0.832	1.518	0.936	1.041	
5.50	0.942	3.747	0.956	3.205	0.804	1.486	0.948	2.844	
6.00	0.939	2.157	0.947	3.379	0.808	1.882	0.953	1.810	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	6.50	0.937	2.697	0.957	3.315	0.803	0.760	0.931	2.063
	7.00	0.949	3.530	0.953	2.565	0.826	1.079	0.926	2.671
	7.50	0.942	1.291	0.952	2.010	0.832	1.656	0.919	1.517
	8.00	0.962	1.941	0.954	0.886	0.811	1.217	0.945	1.112
	8.50	0.953	1.808	0.940	1.491	0.822	0.856	0.923	1.802
	9.00	0.951	0.720	0.955	1.364	0.818	1.053	0.927	1.384
	9.50	0.955	1.735	0.942	1.209	0.812	1.869	0.928	0.665
	10.00	0.941	1.326	0.942	0.859	0.832	0.345	0.934	1.237
	0.10	0.957	219.510	0.961	193.026	0.826	69.466	0.919	118.868
	0.50	0.935	20.944	0.948	40.894	0.818	13.521	0.933	20.649
	1.00	0.948	8.279	0.942	16.987	0.816	14.424	0.931	9.485
	1.50	0.954	8.485	0.951	10.000	0.825	4.922	0.938	9.386
	2.00	0.950	4.405	0.953	7.969	0.809	2.901	0.940	7.535
	2.50	0.946	3.732	0.950	3.369	0.834	5.942	0.928	2.730
	3.00	0.943	1.714	0.952	5.400	0.819	1.822	0.939	2.631
	3.50	0.939	3.389	0.944	3.526	0.805	2.158	0.941	2.157
	4.00	0.953	2.117	0.952	4.045	0.830	2.525	0.932	2.423
	4.50	0.932	2.691	0.949	8.295	0.811	1.301	0.945	2.647
7.50	5.00	0.937	1.565	0.955	3.684	0.813	2.580	0.940	1.634
	5.50	0.931	1.646	0.949	3.095	0.830	1.886	0.940	2.265
	6.00	0.947	2.145	0.940	2.601	0.837	1.603	0.935	1.098
	6.50	0.943	1.060	0.940	1.117	0.835	1.158	0.951	2.149
	7.00	0.958	2.551	0.945	1.961	0.814	1.196	0.949	2.221
	7.50	0.947	3.955	0.949	1.547	0.811	0.986	0.939	1.658
	8.00	0.937	1.395	0.939	1.673	0.832	2.089	0.931	1.088
	8.50	0.943	1.504	0.943	1.945	0.840	0.963	0.931	1.431
	9.00	0.954	1.206	0.952	1.366	0.823	0.667	0.940	2.237
	9.50	0.954	1.395	0.954	1.036	0.824	0.999	0.943	1.539
	10.00	0.934	0.765	0.947	1.092	0.826	1.297	0.935	0.695
	0.10	0.946	114.144	0.953	59.756	0.816	306.390	0.944	171.041
	0.50	0.942	16.125	0.952	33.444	0.815	25.411	0.943	27.754

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
8.00	1.00	0.952	6.393	0.945	16.897	0.832	8.107	0.941	7.174
	1.50	0.953	7.210	0.946	13.527	0.826	7.990	0.935	5.701
	2.00	0.940	5.484	0.946	11.150	0.821	6.568	0.938	7.675
	2.50	0.943	7.186	0.949	4.265	0.822	2.779	0.934	3.613
	3.00	0.949	9.515	0.941	4.703	0.813	2.510	0.939	12.111
	3.50	0.940	3.974	0.943	2.707	0.835	3.509	0.938	5.669
	4.00	0.939	1.363	0.944	3.626	0.847	1.447	0.931	3.787
	4.50	0.940	4.510	0.951	2.271	0.814	1.077	0.933	2.839
	5.00	0.934	2.970	0.938	10.082	0.819	1.625	0.945	1.591
	5.50	0.941	1.508	0.951	2.334	0.830	2.446	0.938	1.912
	6.00	0.950	2.426	0.943	1.387	0.827	1.772	0.936	1.860
	6.50	0.941	2.147	0.958	2.885	0.823	1.020	0.928	0.673
	7.00	0.938	4.981	0.946	2.213	0.827	0.915	0.928	1.570
	7.50	0.950	2.459	0.939	1.312	0.828	0.970	0.930	2.106
	8.00	0.953	1.545	0.962	2.509	0.831	1.129	0.930	3.185
	8.50	0.953	0.978	0.964	2.115	0.809	0.493	0.941	3.196
	9.00	0.945	2.590	0.957	1.226	0.847	0.895	0.944	1.350
9.50	0.948	0.776	0.940	0.654	0.817	1.325	0.934	1.110	
10.00	0.937	1.869	0.954	0.930	0.817	0.428	0.936	0.842	
8.50	0.10	0.941	178.243	0.952	83.013	0.811	100.050	0.937	178.852
	0.50	0.932	19.908	0.947	23.356	0.797	16.452	0.929	36.838
	1.00	0.944	8.023	0.959	15.453	0.796	7.486	0.939	20.877
	1.50	0.940	14.063	0.944	7.240	0.832	6.576	0.949	6.888
	2.00	0.937	7.119	0.947	10.692	0.822	4.510	0.936	3.971
	2.50	0.952	9.219	0.954	10.190	0.810	9.203	0.936	8.943
	3.00	0.948	6.710	0.953	4.925	0.830	2.282	0.939	11.984
	3.50	0.944	2.321	0.945	2.783	0.824	2.700	0.927	3.876
	4.00	0.930	7.197	0.934	3.077	0.814	3.341	0.924	10.386
	4.50	0.946	2.339	0.956	2.890	0.811	1.649	0.941	5.504
5.00	0.936	1.574	0.949	3.224	0.822	1.072	0.939	1.175	
5.50	0.950	4.643	0.948	2.676	0.817	1.085	0.926	2.357	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	6.00	0.940	2.220	0.952	1.755	0.838	3.061	0.944	3.566
	6.50	0.955	2.466	0.935	1.801	0.825	1.424	0.939	3.389
	7.00	0.942	1.688	0.948	3.405	0.834	0.916	0.934	1.696
	7.50	0.940	2.180	0.948	1.180	0.812	1.124	0.943	1.805
	8.00	0.948	1.884	0.950	1.053	0.820	1.759	0.937	2.876
	8.50	0.946	1.657	0.957	1.784	0.804	1.307	0.933	1.551
	9.00	0.947	1.547	0.951	1.461	0.825	1.453	0.935	1.861
	9.50	0.952	3.475	0.946	1.124	0.821	1.094	0.952	1.350
	10.00	0.949	0.993	0.952	1.106	0.820	0.960	0.944	0.701
	0.10	0.942	99.442	0.958	90.780	0.806	118.838	0.949	152.312
	0.50	0.943	37.446	0.943	30.020	0.835	34.629	0.943	33.565
	1.00	0.934	17.623	0.951	14.926	0.843	12.295	0.937	8.090
	1.50	0.951	9.025	0.938	7.073	0.830	7.855	0.939	6.052
	2.00	0.940	3.110	0.952	6.738	0.823	5.099	0.948	13.159
	2.50	0.946	8.899	0.951	4.499	0.810	5.852	0.937	7.839
	3.00	0.951	4.983	0.942	4.427	0.805	1.410	0.937	2.579
	3.50	0.951	3.990	0.947	5.376	0.810	2.463	0.937	3.339
	4.00	0.938	1.987	0.955	3.853	0.821	1.742	0.924	5.195
	4.50	0.950	3.212	0.932	3.504	0.847	2.201	0.934	3.518
9.00	5.00	0.944	2.282	0.942	4.042	0.812	2.085	0.946	1.972
	5.50	0.945	1.813	0.941	2.785	0.844	1.180	0.952	1.916
	6.00	0.945	2.310	0.945	1.187	0.827	1.844	0.938	1.349
	6.50	0.953	2.020	0.948	2.435	0.832	1.796	0.939	2.113
	7.00	0.953	2.492	0.957	1.960	0.801	2.638	0.938	2.739
	7.50	0.953	3.466	0.942	2.852	0.802	1.171	0.959	1.985
	8.00	0.953	2.382	0.949	1.662	0.839	1.804	0.931	1.003
	8.50	0.938	1.175	0.940	0.804	0.818	1.593	0.938	1.229
	9.00	0.953	2.057	0.952	2.579	0.835	1.042	0.948	0.835
	9.50	0.942	1.210	0.953	1.570	0.800	2.121	0.937	0.627
	10.00	0.934	1.550	0.951	0.921	0.791	1.244	0.933	2.566
	0.10	0.948	60.079	0.942	169.615	0.814	140.282	0.948	222.372

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
9.50	0.50	0.944	38.687	0.952	28.842	0.826	25.190	0.943	18.525
	1.00	0.934	26.596	0.955	15.963	0.816	10.735	0.934	27.325
	1.50	0.953	16.303	0.948	3.986	0.815	5.976	0.946	17.084
	2.00	0.947	13.938	0.941	6.741	0.837	3.402	0.923	7.340
	2.50	0.943	3.837	0.949	7.765	0.817	5.982	0.948	8.477
	3.00	0.927	14.169	0.938	4.852	0.828	4.362	0.932	5.848
	3.50	0.946	4.976	0.939	4.419	0.818	4.121	0.921	5.429
	4.00	0.940	3.750	0.957	4.649	0.833	3.097	0.944	2.319
	4.50	0.945	2.383	0.941	4.315	0.814	1.720	0.939	3.041
	5.00	0.945	3.947	0.937	2.018	0.834	1.721	0.949	3.186
	5.50	0.953	5.865	0.942	2.447	0.815	2.035	0.942	1.305
	6.00	0.938	3.469	0.951	2.647	0.800	1.325	0.951	1.390
	6.50	0.934	1.105	0.951	3.170	0.818	1.742	0.938	2.553
	7.00	0.940	2.127	0.954	1.676	0.813	0.939	0.942	2.651
	7.50	0.959	2.689	0.955	1.337	0.797	0.973	0.940	1.000
	8.00	0.958	2.810	0.948	1.443	0.833	0.919	0.952	2.716
	8.50	0.951	1.149	0.943	1.222	0.832	2.128	0.930	1.605
	9.00	0.945	1.084	0.945	1.997	0.810	1.222	0.941	1.967
9.50	0.938	1.534	0.954	1.684	0.850	0.570	0.944	2.308	
10.00	0.936	1.952	0.934	1.964	0.817	1.403	0.939	1.228	
10.00	0.10	0.952	236.218	0.941	141.525	0.807	140.591	0.927	250.243
	0.50	0.945	23.888	0.957	81.982	0.822	17.177	0.936	21.391
	1.00	0.954	27.351	0.949	27.596	0.804	13.809	0.944	9.871
	1.50	0.946	7.449	0.950	9.375	0.828	14.470	0.939	11.079
	2.00	0.951	6.730	0.947	7.970	0.827	7.146	0.941	4.961
	2.50	0.951	13.980	0.935	5.757	0.826	3.494	0.949	7.884
	3.00	0.948	3.947	0.959	5.241	0.844	5.362	0.921	4.022
	3.50	0.942	11.132	0.959	6.968	0.826	1.771	0.943	6.577
	4.00	0.946	4.920	0.948	2.486	0.830	5.242	0.938	3.214
	4.50	0.956	1.900	0.948	2.573	0.826	1.003	0.934	3.394
5.00	0.948	4.710	0.936	3.892	0.830	2.190	0.941	6.284	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	5.50	0.956	2.068	0.946	4.562	0.830	2.224	0.945	1.810
	6.00	0.953	2.592	0.941	4.467	0.818	1.320	0.931	2.734
	6.50	0.942	2.500	0.955	5.374	0.814	1.590	0.937	5.054
	7.00	0.946	1.330	0.945	2.494	0.831	3.940	0.942	7.200
	7.50	0.952	2.296	0.943	2.546	0.840	1.401	0.925	2.826
	8.00	0.947	3.008	0.948	1.563	0.830	1.301	0.928	1.429
	8.50	0.952	1.460	0.950	1.788	0.835	6.261	0.941	2.599
	9.00	0.942	1.849	0.961	0.729	0.835	1.781	0.929	1.616
	9.50	0.939	2.304	0.952	1.162	0.804	1.267	0.942	0.760
	10.00	0.940	1.538	0.949	1.962	0.824	1.178	0.933	1.774

B.3 Complete table for $n = 20$ and $m = 20$

Table B.3: Coverage rates (CR) and average interval lengths (AL) for $n = 20$ and $m = 20$.

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.10	0.10	0.961	2.383	0.951	1.070	0.846	1.346	0.933	1.522
	0.50	0.949	0.326	0.947	0.251	0.822	0.211	0.947	0.168
	1.00	0.958	0.166	0.951	0.082	0.811	0.107	0.950	0.134
	1.50	0.962	0.090	0.944	0.140	0.820	0.074	0.948	0.087
	2.00	0.933	0.056	0.946	0.087	0.819	0.047	0.946	0.064
	2.50	0.957	0.042	0.947	0.077	0.834	0.023	0.943	0.067
	3.00	0.943	0.038	0.959	0.038	0.812	0.039	0.933	0.068
	3.50	0.947	0.034	0.947	0.040	0.839	0.019	0.942	0.032
	4.00	0.953	0.033	0.945	0.016	0.824	0.033	0.940	0.027
	4.50	0.942	0.035	0.944	0.034	0.820	0.028	0.939	0.032
	5.00	0.939	0.023	0.937	0.051	0.839	0.015	0.938	0.023
	5.50	0.943	0.015	0.959	0.016	0.827	0.020	0.943	0.023
	6.00	0.945	0.017	0.952	0.047	0.813	0.011	0.941	0.019
	6.50	0.955	0.023	0.939	0.021	0.844	0.009	0.940	0.022
	7.00	0.953	0.016	0.935	0.019	0.825	0.010	0.951	0.022
	7.50	0.941	0.008	0.942	0.019	0.821	0.014	0.941	0.026
	8.00	0.958	0.011	0.952	0.016	0.839	0.010	0.948	0.013
	8.50	0.950	0.021	0.944	0.017	0.854	0.011	0.934	0.015
	9.00	0.933	0.014	0.950	0.027	0.834	0.006	0.919	0.008
	9.50	0.941	0.009	0.948	0.020	0.834	0.008	0.932	0.017
10.00	0.954	0.017	0.942	0.012	0.830	0.010	0.930	0.017	
	0.10	0.946	9.890	0.942	5.860	0.815	3.734	0.940	3.118
	0.50	0.943	1.311	0.959	1.442	0.818	1.927	0.941	1.228
	1.00	0.940	1.154	0.950	0.630	0.832	0.676	0.953	0.540
	1.50	0.938	0.254	0.943	0.482	0.823	0.235	0.948	0.312
	2.00	0.954	0.276	0.950	0.488	0.809	0.105	0.941	0.303
	2.50	0.946	0.162	0.959	0.339	0.821	0.070	0.938	0.388
	3.00	0.939	0.293	0.946	0.286	0.846	0.188	0.944	0.166

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
0.50	3.50	0.951	0.197	0.945	0.211	0.852	0.136	0.952	0.322
	4.00	0.950	0.164	0.943	0.126	0.836	0.102	0.940	0.120
	4.50	0.939	0.192	0.950	0.104	0.826	0.105	0.932	0.112
	5.00	0.955	0.127	0.943	0.163	0.822	0.119	0.940	0.152
	5.50	0.943	0.161	0.944	0.104	0.837	0.069	0.947	0.084
	6.00	0.950	0.098	0.927	0.166	0.815	0.062	0.931	0.190
	6.50	0.941	0.070	0.946	0.085	0.834	0.103	0.945	0.079
	7.00	0.946	0.148	0.953	0.090	0.832	0.085	0.951	0.111
	7.50	0.941	0.073	0.935	0.059	0.819	0.068	0.947	0.037
	8.00	0.931	0.041	0.956	0.094	0.809	0.027	0.947	0.061
	8.50	0.949	0.056	0.933	0.090	0.803	0.060	0.953	0.132
	9.00	0.957	0.066	0.954	0.067	0.837	0.078	0.932	0.102
	9.50	0.957	0.102	0.933	0.093	0.823	0.032	0.939	0.050
	10.00	0.945	0.095	0.946	0.105	0.839	0.035	0.955	0.060
1.00	0.10	0.936	10.706	0.946	12.356	0.831	5.641	0.946	10.833
	0.50	0.952	2.575	0.938	2.269	0.838	0.973	0.942	2.674
	1.00	0.950	0.959	0.951	2.135	0.838	1.288	0.950	1.029
	1.50	0.936	0.938	0.940	1.298	0.819	0.722	0.950	0.983
	2.00	0.942	0.723	0.941	0.753	0.817	0.401	0.938	0.422
	2.50	0.934	0.771	0.941	0.682	0.838	0.429	0.946	0.498
	3.00	0.942	0.399	0.954	0.391	0.828	0.238	0.921	0.450
	3.50	0.952	0.424	0.964	0.212	0.825	0.183	0.953	0.488
	4.00	0.942	0.297	0.944	0.394	0.838	0.172	0.945	0.277
	4.50	0.944	0.294	0.951	0.260	0.833	0.122	0.935	0.243
	5.00	0.959	0.217	0.961	0.212	0.832	0.165	0.943	0.316
	5.50	0.945	0.174	0.955	0.166	0.821	0.133	0.939	0.115
	6.00	0.955	0.152	0.949	0.182	0.803	0.110	0.948	0.260
	6.50	0.953	0.320	0.951	0.180	0.841	0.115	0.929	0.147
7.00	0.948	0.249	0.949	0.349	0.823	0.083	0.948	0.230	
7.50	0.948	0.191	0.954	0.112	0.810	0.052	0.920	0.178	
8.00	0.942	0.114	0.942	0.228	0.841	0.145	0.931	0.166	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	8.50	0.950	0.179	0.937	0.179	0.835	0.138	0.936	0.366
	9.00	0.940	0.130	0.936	0.127	0.809	0.086	0.950	0.115
	9.50	0.949	0.205	0.964	0.107	0.839	0.146	0.938	0.181
	10.00	0.952	0.134	0.941	0.169	0.855	0.067	0.944	0.112
	0.10	0.940	18.879	0.949	21.612	0.815	10.291	0.948	20.873
	0.50	0.942	3.883	0.947	4.930	0.822	2.369	0.949	4.162
	1.00	0.944	1.962	0.953	1.525	0.817	1.463	0.954	2.577
	1.50	0.947	1.286	0.942	1.312	0.823	1.039	0.954	1.857
	2.00	0.948	0.966	0.934	0.914	0.827	0.473	0.938	2.129
	2.50	0.953	0.738	0.955	1.029	0.835	0.470	0.957	0.709
	3.00	0.945	0.599	0.945	1.132	0.799	0.434	0.955	0.842
	3.50	0.946	0.653	0.962	0.631	0.825	0.279	0.938	0.551
	4.00	0.953	0.396	0.961	0.452	0.822	0.316	0.955	0.584
	4.50	0.950	0.396	0.957	0.328	0.840	0.297	0.938	0.370
1.50	5.00	0.959	0.607	0.947	0.335	0.834	0.413	0.948	0.421
	5.50	0.934	0.590	0.941	0.521	0.809	0.313	0.942	0.347
	6.00	0.946	0.496	0.952	0.511	0.847	0.209	0.944	0.247
	6.50	0.936	0.197	0.953	0.182	0.804	0.190	0.927	0.408
	7.00	0.948	0.215	0.961	0.189	0.819	0.406	0.939	0.325
	7.50	0.954	0.304	0.954	0.240	0.830	0.098	0.955	0.295
	8.00	0.935	0.114	0.952	0.300	0.810	0.236	0.923	0.369
	8.50	0.945	0.207	0.951	0.196	0.862	0.171	0.940	0.203
	9.00	0.940	0.223	0.940	0.267	0.843	0.124	0.951	0.148
	9.50	0.936	0.199	0.951	0.237	0.818	0.107	0.948	0.163
	10.00	0.946	0.284	0.940	0.147	0.813	0.056	0.954	0.259
	0.10	0.947	27.466	0.941	22.141	0.847	17.744	0.951	34.567
	0.50	0.947	6.358	0.953	3.989	0.839	4.434	0.931	6.139
	1.00	0.949	2.119	0.942	1.944	0.814	1.285	0.934	4.576
	1.50	0.950	3.425	0.949	3.118	0.844	1.059	0.938	1.399
	2.00	0.948	1.089	0.953	1.098	0.813	0.740	0.941	1.803
	2.50	0.949	1.141	0.945	0.865	0.801	1.282	0.935	1.669

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
2.00	3.00	0.949	0.819	0.958	0.771	0.816	0.648	0.949	1.031
	3.50	0.955	0.776	0.956	0.665	0.824	0.418	0.924	0.877
	4.00	0.955	0.864	0.942	0.780	0.829	0.608	0.934	0.486
	4.50	0.945	0.432	0.947	0.364	0.836	0.479	0.949	0.960
	5.00	0.940	0.230	0.947	0.748	0.826	0.687	0.939	0.657
	5.50	0.946	0.538	0.963	0.434	0.833	0.288	0.945	0.704
	6.00	0.934	0.452	0.953	0.424	0.838	0.430	0.934	0.540
	6.50	0.943	0.383	0.945	0.373	0.852	0.153	0.934	0.644
	7.00	0.950	0.429	0.941	0.338	0.836	0.291	0.942	0.233
	7.50	0.948	0.342	0.947	0.552	0.845	0.262	0.936	0.200
	8.00	0.942	0.174	0.938	0.196	0.823	0.215	0.949	0.330
	8.50	0.945	0.328	0.952	0.395	0.837	0.177	0.947	0.332
	9.00	0.939	0.211	0.951	0.287	0.824	0.102	0.943	0.484
	9.50	0.941	0.303	0.953	0.449	0.839	0.239	0.941	0.377
10.00	0.946	0.324	0.955	0.244	0.828	0.155	0.939	0.271	
2.50	0.10	0.944	51.493	0.949	24.708	0.832	25.846	0.948	20.615
	0.50	0.939	9.158	0.956	9.192	0.814	6.009	0.945	9.169
	1.00	0.952	2.450	0.934	1.617	0.849	1.318	0.941	5.218
	1.50	0.944	2.208	0.942	1.812	0.818	2.139	0.928	2.791
	2.00	0.941	1.304	0.949	3.750	0.839	1.330	0.938	1.825
	2.50	0.958	1.675	0.953	1.156	0.846	0.804	0.922	0.948
	3.00	0.949	0.625	0.942	1.256	0.825	0.632	0.939	0.864
	3.50	0.939	0.631	0.934	1.494	0.835	0.986	0.947	0.910
	4.00	0.952	1.954	0.934	0.891	0.802	0.506	0.942	1.381
	4.50	0.947	0.739	0.945	1.005	0.839	0.359	0.942	0.829
	5.00	0.940	1.278	0.945	0.654	0.834	0.696	0.951	0.509
	5.50	0.956	0.786	0.944	0.230	0.827	0.453	0.935	0.305
	6.00	0.945	0.425	0.941	0.294	0.824	0.433	0.944	0.370
	6.50	0.948	0.762	0.948	0.410	0.823	0.290	0.952	0.624
7.00	0.944	0.371	0.952	0.315	0.814	0.220	0.948	0.465	
7.50	0.946	0.342	0.937	0.419	0.835	0.266	0.941	0.520	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	8.00	0.945	0.673	0.934	0.753	0.845	0.243	0.941	0.258
	8.50	0.953	0.508	0.947	0.531	0.820	0.161	0.946	0.543
	9.00	0.952	0.342	0.950	0.346	0.828	0.291	0.936	0.301
	9.50	0.943	0.452	0.952	0.497	0.825	0.219	0.948	0.379
	10.00	0.962	0.334	0.941	0.405	0.827	0.143	0.931	0.343
	0.10	0.936	47.854	0.949	27.578	0.822	25.649	0.941	24.353
	0.50	0.939	5.189	0.937	8.874	0.828	3.598	0.945	6.226
	1.00	0.946	4.202	0.950	3.539	0.845	2.938	0.946	2.976
	1.50	0.945	2.262	0.948	2.452	0.819	1.373	0.940	2.480
	2.00	0.962	3.389	0.947	1.723	0.832	0.661	0.932	2.154
	2.50	0.950	2.449	0.940	1.214	0.816	0.984	0.942	1.893
	3.00	0.929	0.883	0.949	0.558	0.823	0.432	0.942	1.742
	3.50	0.965	0.741	0.945	0.605	0.842	0.404	0.955	0.646
	4.00	0.949	0.888	0.954	0.669	0.842	1.030	0.943	1.394
	4.50	0.956	1.036	0.940	1.000	0.822	0.516	0.929	1.101
3.00	5.00	0.934	0.378	0.943	0.561	0.841	0.445	0.937	1.005
	5.50	0.946	0.536	0.953	0.506	0.813	0.522	0.935	0.595
	6.00	0.940	0.474	0.942	0.949	0.845	0.429	0.947	0.786
	6.50	0.948	0.285	0.953	0.618	0.832	0.434	0.937	0.613
	7.00	0.947	0.660	0.948	0.602	0.838	0.210	0.944	0.671
	7.50	0.935	0.519	0.952	0.409	0.839	0.203	0.942	0.481
	8.00	0.954	0.538	0.952	0.461	0.833	0.281	0.941	0.320
	8.50	0.952	0.736	0.957	0.427	0.845	0.425	0.951	0.395
	9.00	0.940	0.427	0.944	0.322	0.804	0.356	0.931	0.500
	9.50	0.945	0.328	0.959	0.391	0.816	0.279	0.951	0.409
	10.00	0.958	0.586	0.944	0.410	0.842	0.475	0.945	0.545
	0.10	0.954	45.739	0.948	39.973	0.815	25.419	0.946	37.116
	0.50	0.950	16.796	0.935	10.431	0.849	7.975	0.947	6.321
	1.00	0.967	3.032	0.938	3.991	0.845	2.868	0.942	5.339
	1.50	0.947	3.983	0.941	3.319	0.844	2.269	0.955	3.747
	2.00	0.946	1.162	0.955	2.721	0.828	1.160	0.945	2.019

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
3.50	2.50	0.936	2.781	0.949	1.853	0.823	1.263	0.954	1.683
	3.00	0.939	1.170	0.937	1.370	0.822	1.301	0.945	2.958
	3.50	0.948	1.230	0.942	1.248	0.842	0.636	0.943	1.157
	4.00	0.958	1.285	0.956	0.916	0.850	0.973	0.943	1.366
	4.50	0.953	0.731	0.955	1.495	0.837	0.894	0.956	0.756
	5.00	0.934	0.978	0.933	0.617	0.820	0.581	0.935	1.280
	5.50	0.947	0.816	0.946	1.205	0.842	0.382	0.946	0.854
	6.00	0.950	0.565	0.937	0.677	0.806	0.669	0.942	0.878
	6.50	0.942	0.478	0.942	0.619	0.821	0.574	0.934	0.623
	7.00	0.953	0.611	0.941	0.525	0.805	0.439	0.943	0.765
	7.50	0.955	0.682	0.945	0.665	0.824	0.247	0.947	0.495
	8.00	0.944	0.557	0.952	0.753	0.850	0.339	0.940	0.574
	8.50	0.941	0.493	0.949	0.274	0.830	0.337	0.943	0.749
	9.00	0.940	0.602	0.943	0.558	0.811	0.432	0.948	0.583
	9.50	0.948	0.468	0.949	0.713	0.829	0.289	0.940	0.420
10.00	0.941	0.583	0.951	0.606	0.839	0.177	0.927	0.778	
4.00	0.10	0.950	54.440	0.954	38.556	0.833	41.371	0.934	99.448
	0.50	0.963	10.126	0.955	8.060	0.841	10.693	0.944	12.869
	1.00	0.931	8.808	0.953	2.948	0.834	3.421	0.943	2.902
	1.50	0.950	2.856	0.943	3.842	0.819	2.528	0.931	4.696
	2.00	0.954	2.852	0.940	2.149	0.819	1.972	0.951	3.216
	2.50	0.945	1.375	0.938	2.890	0.865	1.473	0.941	0.822
	3.00	0.952	2.216	0.950	2.394	0.836	0.861	0.938	2.917
	3.50	0.950	1.700	0.943	2.237	0.828	0.630	0.945	1.329
	4.00	0.949	0.871	0.942	1.024	0.805	1.225	0.936	1.149
	4.50	0.941	1.306	0.957	1.316	0.816	0.669	0.942	1.597
	5.00	0.941	1.073	0.945	0.813	0.820	0.502	0.947	1.267
	5.50	0.950	0.849	0.949	0.929	0.834	0.672	0.932	1.145
	6.00	0.941	0.792	0.946	0.945	0.817	0.503	0.934	1.149
	6.50	0.948	0.597	0.938	1.043	0.837	0.387	0.944	0.631
	7.00	0.958	0.903	0.950	0.684	0.845	0.464	0.948	0.968

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	7.50	0.953	0.492	0.947	0.512	0.838	0.614	0.940	0.818
	8.00	0.942	0.745	0.941	0.764	0.826	0.361	0.956	1.148
	8.50	0.928	0.445	0.956	0.446	0.816	0.316	0.948	0.996
	9.00	0.948	0.421	0.938	0.671	0.834	0.459	0.950	0.628
	9.50	0.950	0.612	0.946	0.535	0.818	0.359	0.947	0.654
	10.00	0.945	0.710	0.929	0.441	0.836	0.708	0.944	0.503
	0.10	0.956	80.306	0.953	47.399	0.817	48.950	0.936	64.198
	0.50	0.951	14.815	0.949	19.970	0.812	5.438	0.947	12.685
	1.00	0.957	5.269	0.949	5.841	0.802	3.766	0.947	6.682
	1.50	0.945	4.648	0.962	5.518	0.827	2.381	0.934	3.889
	2.00	0.944	2.923	0.946	2.372	0.829	1.833	0.941	2.790
	2.50	0.948	2.016	0.941	3.507	0.832	1.307	0.931	2.688
	3.00	0.945	2.507	0.950	1.767	0.847	1.492	0.944	1.286
	3.50	0.944	2.387	0.943	1.640	0.801	1.500	0.941	1.133
	4.00	0.950	1.842	0.958	1.533	0.801	1.407	0.933	1.411
	4.50	0.940	1.233	0.949	0.820	0.823	0.828	0.938	1.180
4.50	5.00	0.950	1.804	0.949	0.909	0.812	0.596	0.948	1.078
	5.50	0.959	0.841	0.961	0.787	0.840	0.441	0.946	1.036
	6.00	0.944	0.638	0.952	1.301	0.817	0.899	0.951	1.306
	6.50	0.948	1.231	0.952	0.755	0.837	0.548	0.935	1.160
	7.00	0.951	0.489	0.944	0.332	0.808	0.502	0.939	0.577
	7.50	0.947	0.714	0.939	1.517	0.843	0.710	0.912	0.959
	8.00	0.958	1.215	0.952	0.514	0.802	0.438	0.938	0.847
	8.50	0.948	0.717	0.959	0.500	0.835	0.546	0.936	0.674
	9.00	0.944	0.651	0.955	0.894	0.837	0.465	0.942	0.975
	9.50	0.950	0.523	0.958	0.403	0.849	0.290	0.944	0.536
	10.00	0.948	0.609	0.946	0.350	0.842	0.332	0.944	0.717
	0.10	0.945	64.558	0.947	71.108	0.829	36.472	0.943	94.010
	0.50	0.943	16.564	0.958	11.413	0.815	11.445	0.952	14.716
	1.00	0.938	8.579	0.951	10.131	0.810	3.631	0.937	6.065
	1.50	0.950	4.984	0.942	5.071	0.834	5.289	0.937	4.603

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
5.00	2.00	0.953	3.155	0.947	3.265	0.835	2.330	0.948	2.763
	2.50	0.954	1.678	0.941	2.056	0.839	1.776	0.954	2.307
	3.00	0.951	2.648	0.949	1.423	0.829	1.963	0.944	1.281
	3.50	0.951	2.771	0.949	2.042	0.840	1.190	0.953	1.544
	4.00	0.945	1.149	0.951	4.125	0.816	1.544	0.950	1.101
	4.50	0.954	1.757	0.948	1.901	0.834	1.045	0.924	1.115
	5.00	0.949	0.805	0.944	1.535	0.819	0.669	0.938	1.610
	5.50	0.951	1.732	0.950	1.404	0.821	0.526	0.953	1.051
	6.00	0.942	1.170	0.954	1.011	0.836	0.940	0.942	0.877
	6.50	0.952	0.552	0.950	1.057	0.835	0.690	0.935	1.156
	7.00	0.940	1.040	0.956	0.766	0.837	0.668	0.932	2.371
	7.50	0.949	1.065	0.952	0.828	0.849	0.949	0.935	0.719
	8.00	0.945	0.877	0.940	0.894	0.798	0.460	0.948	0.506
	8.50	0.950	0.787	0.947	1.117	0.820	0.530	0.934	0.482
	9.00	0.943	1.262	0.961	0.962	0.817	0.637	0.929	0.514
9.50	0.947	0.734	0.947	0.606	0.813	0.874	0.949	0.475	
10.00	0.948	1.000	0.958	0.693	0.822	0.265	0.935	0.703	
5.50	0.10	0.948	97.649	0.939	90.350	0.850	50.407	0.953	86.712
	0.50	0.956	12.377	0.954	20.983	0.838	12.768	0.941	11.936
	1.00	0.939	21.956	0.950	6.382	0.830	5.658	0.941	11.542
	1.50	0.933	6.869	0.948	2.991	0.827	2.108	0.938	4.781
	2.00	0.944	3.045	0.946	4.452	0.849	2.171	0.935	3.853
	2.50	0.939	3.235	0.943	2.638	0.817	1.856	0.949	1.614
	3.00	0.934	2.966	0.943	1.811	0.825	1.174	0.941	1.723
	3.50	0.951	1.468	0.948	2.028	0.834	1.671	0.943	1.890
	4.00	0.930	2.563	0.954	2.452	0.838	1.334	0.955	3.000
	4.50	0.940	1.914	0.960	1.874	0.828	1.247	0.938	1.490
5.00	0.947	0.987	0.951	1.086	0.839	0.842	0.947	1.870	
5.50	0.939	1.219	0.949	1.232	0.821	0.671	0.943	2.263	
6.00	0.953	0.985	0.961	1.427	0.836	1.118	0.946	1.076	
6.50	0.951	1.292	0.951	1.392	0.822	0.566	0.941	1.347	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	7.00	0.953	0.961	0.948	0.901	0.826	0.665	0.956	0.477
	7.50	0.945	0.860	0.946	0.845	0.827	0.652	0.939	0.730
	8.00	0.951	0.454	0.933	0.874	0.829	0.291	0.947	1.923
	8.50	0.948	0.647	0.951	0.729	0.832	1.089	0.937	0.579
	9.00	0.948	0.930	0.944	0.648	0.833	0.441	0.941	0.549
	9.50	0.945	0.451	0.950	0.683	0.841	0.363	0.930	0.952
	10.00	0.951	0.854	0.957	0.734	0.812	0.484	0.943	0.556
	0.10	0.947	81.585	0.948	84.562	0.825	77.561	0.934	91.590
	0.50	0.951	11.272	0.952	8.015	0.813	17.818	0.947	15.596
	1.00	0.949	7.119	0.949	8.336	0.821	9.535	0.952	13.322
	1.50	0.938	4.355	0.957	4.752	0.820	2.541	0.948	7.691
	2.00	0.944	4.547	0.945	3.041	0.836	2.734	0.947	2.719
	2.50	0.956	2.247	0.953	4.159	0.838	1.675	0.952	2.736
	3.00	0.943	2.850	0.944	2.994	0.823	2.052	0.939	2.384
	3.50	0.946	1.772	0.945	5.404	0.805	1.141	0.940	2.240
	4.00	0.960	1.470	0.942	2.177	0.835	0.778	0.952	1.805
	4.50	0.946	1.595	0.946	1.075	0.816	0.984	0.940	0.989
6.00	5.00	0.951	1.115	0.929	1.764	0.837	0.557	0.941	2.359
	5.50	0.958	1.349	0.948	1.212	0.836	0.953	0.939	2.365
	6.00	0.942	1.021	0.949	1.754	0.835	0.718	0.952	1.440
	6.50	0.948	0.863	0.953	1.150	0.816	0.751	0.948	0.879
	7.00	0.940	0.785	0.933	1.430	0.828	0.892	0.937	0.694
	7.50	0.952	1.547	0.950	1.565	0.855	0.649	0.936	1.936
	8.00	0.937	0.621	0.943	0.952	0.813	0.487	0.960	0.665
	8.50	0.951	1.593	0.951	0.751	0.817	0.700	0.943	0.989
	9.00	0.952	0.662	0.945	0.525	0.853	0.576	0.946	1.465
	9.50	0.961	0.506	0.947	0.683	0.820	0.408	0.947	0.569
	10.00	0.950	0.678	0.957	1.104	0.828	0.549	0.949	0.815
	0.10	0.952	107.078	0.946	82.041	0.841	56.577	0.931	90.068
	0.50	0.946	23.463	0.952	19.467	0.827	22.538	0.953	21.298
	1.00	0.950	11.002	0.951	7.697	0.825	4.340	0.948	13.799

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
6.50	1.50	0.942	4.337	0.955	6.622	0.805	2.169	0.945	4.359
	2.00	0.944	3.153	0.963	3.306	0.821	4.386	0.950	4.626
	2.50	0.952	4.482	0.937	1.556	0.826	1.858	0.942	3.405
	3.00	0.942	2.369	0.950	3.303	0.841	3.090	0.931	3.966
	3.50	0.950	2.890	0.952	1.928	0.814	1.663	0.947	2.284
	4.00	0.949	4.226	0.942	2.113	0.838	1.759	0.949	1.483
	4.50	0.952	2.089	0.942	1.481	0.827	1.164	0.941	1.881
	5.00	0.945	1.323	0.945	2.446	0.817	1.102	0.950	1.889
	5.50	0.944	2.272	0.955	0.901	0.813	1.249	0.958	0.946
	6.00	0.962	0.897	0.946	2.098	0.839	0.978	0.943	1.339
	6.50	0.933	0.754	0.953	0.703	0.811	0.717	0.937	1.372
	7.00	0.940	1.782	0.936	0.988	0.823	1.084	0.944	1.351
	7.50	0.929	1.005	0.957	1.426	0.829	0.581	0.941	0.556
	8.00	0.947	1.166	0.946	1.039	0.821	0.985	0.957	1.357
	8.50	0.937	1.579	0.954	1.167	0.851	0.489	0.942	0.869
9.00	0.943	0.883	0.954	0.772	0.825	0.708	0.948	1.517	
9.50	0.948	1.002	0.957	1.169	0.844	0.804	0.947	0.767	
10.00	0.949	0.902	0.938	0.598	0.826	0.353	0.944	0.661	
7.00	0.10	0.951	109.112	0.956	156.708	0.823	43.099	0.938	134.931
	0.50	0.941	23.782	0.944	17.975	0.837	12.057	0.935	13.754
	1.00	0.951	9.873	0.966	6.957	0.800	6.593	0.948	9.040
	1.50	0.951	11.566	0.952	6.638	0.840	6.349	0.921	4.767
	2.00	0.949	4.609	0.945	3.816	0.824	3.240	0.939	3.692
	2.50	0.946	4.999	0.949	3.865	0.824	2.642	0.946	4.150
	3.00	0.955	2.308	0.938	2.927	0.820	1.617	0.943	3.946
	3.50	0.954	3.976	0.953	3.970	0.825	1.647	0.947	1.763
	4.00	0.945	2.351	0.947	2.932	0.825	1.152	0.934	1.946
	4.50	0.952	1.814	0.956	0.919	0.836	1.111	0.936	2.268
	5.00	0.928	2.317	0.954	1.956	0.851	0.983	0.940	1.808
5.50	0.944	0.914	0.947	2.342	0.810	1.114	0.939	1.595	
6.00	0.950	3.639	0.955	3.081	0.839	0.843	0.936	1.723	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	6.50	0.930	1.183	0.964	2.183	0.810	0.819	0.953	1.189
	7.00	0.946	1.200	0.952	1.371	0.819	0.784	0.957	0.908
	7.50	0.937	0.856	0.961	2.219	0.824	1.166	0.935	1.674
	8.00	0.953	0.548	0.953	1.776	0.828	0.682	0.946	0.638
	8.50	0.952	0.722	0.944	0.758	0.824	0.789	0.933	1.140
	9.00	0.944	1.406	0.935	1.244	0.832	0.501	0.947	1.201
	9.50	0.946	0.802	0.953	0.415	0.824	1.021	0.940	0.708
	10.00	0.939	1.308	0.944	1.072	0.830	0.588	0.942	1.012
	0.10	0.934	151.860	0.941	108.384	0.806	125.913	0.942	123.684
	0.50	0.944	16.924	0.947	19.572	0.835	23.088	0.938	25.543
	1.00	0.938	11.574	0.954	10.334	0.827	7.226	0.951	8.323
	1.50	0.950	6.854	0.948	6.939	0.843	3.069	0.934	6.378
	2.00	0.940	8.027	0.937	4.304	0.839	1.928	0.941	8.394
	2.50	0.950	2.559	0.951	4.171	0.809	2.087	0.945	5.393
	3.00	0.940	2.841	0.943	3.745	0.832	1.559	0.944	3.129
	3.50	0.944	3.710	0.956	3.779	0.816	1.906	0.942	3.649
	4.00	0.949	2.318	0.950	2.598	0.822	1.282	0.943	2.093
	4.50	0.937	1.610	0.945	1.417	0.831	1.758	0.947	2.015
7.50	5.00	0.951	2.080	0.943	1.534	0.849	1.716	0.937	3.274
	5.50	0.951	1.623	0.953	1.789	0.827	1.048	0.927	1.876
	6.00	0.956	1.758	0.952	1.077	0.833	1.096	0.945	1.273
	6.50	0.947	1.550	0.946	1.700	0.812	1.013	0.946	2.153
	7.00	0.945	1.738	0.951	1.571	0.848	0.780	0.934	1.491
	7.50	0.953	2.108	0.942	1.158	0.823	0.631	0.937	2.735
	8.00	0.940	1.082	0.944	1.052	0.803	0.497	0.944	0.906
	8.50	0.941	0.905	0.962	1.111	0.823	1.065	0.944	0.884
	9.00	0.950	1.083	0.949	1.254	0.825	0.466	0.938	1.065
	9.50	0.941	0.656	0.942	0.917	0.826	0.942	0.949	0.814
	10.00	0.951	0.703	0.943	1.316	0.817	0.573	0.946	1.227
	0.10	0.952	60.036	0.943	66.884	0.853	93.695	0.948	131.396
	0.50	0.951	15.448	0.950	19.664	0.820	15.489	0.945	29.932

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
8.00	1.00	0.935	11.864	0.950	12.302	0.811	8.054	0.943	8.246
	1.50	0.945	6.901	0.947	9.053	0.805	5.249	0.941	4.583
	2.00	0.942	8.082	0.952	4.147	0.839	2.328	0.933	10.366
	2.50	0.953	7.816	0.952	3.391	0.841	1.144	0.942	2.877
	3.00	0.945	2.900	0.932	2.201	0.828	2.273	0.939	4.709
	3.50	0.944	4.487	0.950	3.137	0.833	2.545	0.930	2.336
	4.00	0.938	2.734	0.951	2.802	0.839	2.430	0.932	2.136
	4.50	0.952	1.625	0.954	2.633	0.836	1.566	0.941	2.055
	5.00	0.959	2.181	0.955	1.638	0.825	1.286	0.939	2.047
	5.50	0.951	1.779	0.941	2.135	0.824	1.563	0.946	1.936
	6.00	0.960	1.972	0.949	1.587	0.830	1.315	0.940	1.893
	6.50	0.943	1.392	0.955	1.073	0.817	1.228	0.948	2.872
	7.00	0.938	2.363	0.954	1.971	0.804	0.835	0.949	2.251
	7.50	0.948	1.024	0.943	1.417	0.846	1.097	0.946	1.591
	8.00	0.951	0.855	0.938	1.691	0.841	0.941	0.941	1.173
	8.50	0.942	1.436	0.955	1.138	0.839	1.159	0.929	0.679
	9.00	0.931	1.650	0.953	0.499	0.843	0.679	0.950	1.360
9.50	0.937	0.549	0.938	1.584	0.814	0.901	0.944	0.794	
10.00	0.931	1.046	0.957	1.014	0.831	0.838	0.940	0.872	
8.50	0.10	0.945	107.558	0.962	96.248	0.824	75.904	0.957	136.593
	0.50	0.947	32.442	0.939	21.595	0.821	22.547	0.944	13.352
	1.00	0.950	4.339	0.935	11.362	0.831	9.541	0.943	12.965
	1.50	0.945	8.437	0.947	8.755	0.835	5.917	0.957	5.365
	2.00	0.937	7.917	0.947	3.832	0.843	6.060	0.938	5.423
	2.50	0.955	4.440	0.945	7.325	0.830	2.328	0.940	5.165
	3.00	0.947	2.821	0.946	3.831	0.829	1.692	0.941	3.763
	3.50	0.948	4.787	0.948	3.998	0.836	3.138	0.929	1.640
	4.00	0.957	4.447	0.944	2.122	0.828	1.622	0.939	4.162
	4.50	0.956	4.078	0.942	1.922	0.827	1.335	0.952	1.807
5.00	0.943	2.247	0.953	1.378	0.834	1.639	0.948	2.487	
5.50	0.954	2.142	0.942	3.524	0.839	1.341	0.947	2.199	

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	6.00	0.933	1.778	0.956	4.060	0.840	1.666	0.942	1.201
	6.50	0.943	1.028	0.955	2.917	0.819	0.921	0.940	2.366
	7.00	0.941	1.793	0.951	3.608	0.814	1.345	0.947	1.192
	7.50	0.927	1.257	0.956	2.034	0.817	1.467	0.943	1.253
	8.00	0.954	2.906	0.947	1.567	0.816	1.201	0.943	3.743
	8.50	0.926	1.102	0.962	1.786	0.818	0.864	0.942	0.927
	9.00	0.943	1.035	0.960	1.488	0.824	0.748	0.942	1.370
	9.50	0.938	1.114	0.944	1.802	0.812	1.377	0.936	1.957
	10.00	0.946	0.978	0.942	0.893	0.818	1.040	0.934	1.516
	0.10	0.950	163.842	0.946	168.264	0.826	52.585	0.941	94.609
	0.50	0.943	15.257	0.958	33.663	0.819	14.345	0.937	37.600
	1.00	0.952	13.234	0.950	12.430	0.827	5.429	0.945	11.216
	1.50	0.942	7.469	0.939	7.626	0.835	3.785	0.944	9.053
	2.00	0.949	10.933	0.960	6.186	0.844	3.926	0.931	4.738
	2.50	0.945	4.085	0.958	3.750	0.848	2.878	0.937	4.220
	3.00	0.946	1.726	0.949	3.435	0.836	1.728	0.953	3.026
	3.50	0.943	2.529	0.953	3.972	0.808	1.063	0.944	3.205
	4.00	0.957	4.007	0.945	3.512	0.836	1.823	0.949	3.410
	4.50	0.925	1.737	0.947	2.272	0.827	1.656	0.949	2.263
9.00	5.00	0.945	2.533	0.939	2.394	0.821	2.716	0.946	3.256
	5.50	0.948	1.803	0.950	1.759	0.829	1.806	0.947	2.282
	6.00	0.957	1.996	0.934	3.295	0.839	1.514	0.953	2.217
	6.50	0.952	1.589	0.940	3.098	0.858	1.112	0.955	1.598
	7.00	0.949	0.902	0.945	2.818	0.823	0.876	0.946	2.141
	7.50	0.951	1.491	0.955	1.743	0.812	1.005	0.948	2.940
	8.00	0.946	1.708	0.953	1.752	0.815	0.712	0.954	2.239
	8.50	0.944	1.543	0.949	1.339	0.822	1.766	0.939	1.544
	9.00	0.938	1.088	0.950	1.365	0.844	0.663	0.947	1.387
	9.50	0.942	1.321	0.938	1.790	0.832	0.940	0.933	1.261
	10.00	0.957	1.980	0.957	1.073	0.836	0.551	0.939	0.852
	0.10	0.942	163.618	0.957	146.395	0.850	86.236	0.943	141.797

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
9.50	0.50	0.947	21.713	0.959	31.183	0.822	13.932	0.954	31.125
	1.00	0.945	8.927	0.967	16.160	0.814	14.240	0.943	10.495
	1.50	0.950	5.720	0.959	7.822	0.826	6.155	0.950	7.610
	2.00	0.948	5.660	0.944	7.293	0.841	4.465	0.941	9.151
	2.50	0.950	5.140	0.942	6.232	0.810	3.133	0.942	5.140
	3.00	0.941	7.196	0.948	4.233	0.796	1.915	0.953	5.956
	3.50	0.942	5.798	0.946	3.770	0.833	1.267	0.957	3.363
	4.00	0.944	3.273	0.948	2.095	0.837	2.338	0.946	2.443
	4.50	0.946	2.229	0.940	3.835	0.819	1.487	0.945	2.364
	5.00	0.933	3.402	0.947	2.666	0.834	1.272	0.937	1.865
	5.50	0.944	1.996	0.947	2.245	0.834	2.044	0.948	4.368
	6.00	0.948	2.623	0.931	2.287	0.835	0.885	0.934	1.980
	6.50	0.945	2.788	0.955	2.707	0.811	3.222	0.946	1.231
	7.00	0.953	2.387	0.943	1.062	0.831	1.217	0.948	1.669
	7.50	0.947	1.199	0.953	1.139	0.845	0.949	0.943	1.314
	8.00	0.940	1.453	0.961	1.727	0.834	1.868	0.953	1.372
	8.50	0.941	1.050	0.953	2.498	0.828	0.851	0.943	0.973
	9.00	0.937	1.619	0.948	1.102	0.833	0.984	0.940	2.796
9.50	0.939	1.215	0.956	1.629	0.816	0.648	0.942	0.685	
10.00	0.959	1.262	0.955	0.737	0.807	0.658	0.946	0.518	
10.00	0.10	0.947	130.450	0.954	149.457	0.809	109.541	0.957	89.795
	0.50	0.942	17.893	0.950	19.470	0.833	15.980	0.945	18.715
	1.00	0.952	23.308	0.928	6.321	0.807	6.634	0.947	8.619
	1.50	0.951	9.776	0.942	4.573	0.836	6.383	0.949	9.228
	2.00	0.954	8.934	0.943	9.561	0.828	3.831	0.944	5.827
	2.50	0.940	6.485	0.951	5.703	0.826	2.250	0.943	4.862
	3.00	0.953	5.727	0.944	4.107	0.834	4.591	0.937	6.046
	3.50	0.956	4.164	0.958	3.185	0.823	4.366	0.949	4.020
	4.00	0.938	4.088	0.948	3.229	0.846	1.743	0.947	2.379
	4.50	0.952	3.608	0.948	1.853	0.829	1.682	0.951	3.355
	5.00	0.951	2.223	0.949	3.133	0.845	1.254	0.952	2.312

λ_1	λ_2	GML		Jeffreys		MDI		Uniform	
		CR	AL	CR	AL	CR	AL	CR	AL
	5.50	0.946	2.696	0.937	1.951	0.814	2.274	0.939	1.910
	6.00	0.946	2.007	0.965	1.328	0.817	1.547	0.938	1.496
	6.50	0.938	1.829	0.943	2.350	0.828	1.050	0.945	1.351
	7.00	0.950	1.850	0.943	2.238	0.841	1.081	0.952	1.130
	7.50	0.947	1.430	0.941	4.577	0.823	1.640	0.962	1.676
	8.00	0.945	1.080	0.963	2.069	0.807	1.437	0.947	1.111
	8.50	0.939	1.845	0.949	1.368	0.853	1.119	0.953	1.955
	9.00	0.959	1.638	0.945	2.177	0.836	0.699	0.953	1.347
	9.50	0.946	1.421	0.952	0.541	0.834	0.985	0.946	2.241
	10.00	0.945	1.247	0.953	1.551	0.841	0.826	0.935	1.454

Appendix C

Code Simulation Studies

C.1 Code simulation study I

```
#-----#  
#-----Define n and m-----#  
  
Packages = c("csv")  
  
for(a in Packages){  
    if(!require(a, character.only = TRUE)){  
        install.packages(a)  
    }  
    library(a, character.only = TRUE)  
}  
  
seq1 = seq(0, 10, 0.5)  
lambda1 = c(0.1, seq1[-1])  
lambda2 = c(0.1, seq1[-1])
```

```
n = 20
m = 20
num = 1000
iter = 1000
```

```
#-----#
```

```
#-----Jeffreys-----#
```

```
theta = rep(0, num)
ave_int_len = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
Cov_rate = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
thet = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb1 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb2 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
theta0 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
count = 0

for(i in 1:21){

  for(j in 1:21){

    theta0 = lambda1[i]/lambda2[j]
    int_len = rep(0, iter)
    count = 0

    for(k in 1:iter){

      x = rexp(n, lambda1[i])
```

```

y = rexp(m, lambda2[j])

lambda_1 = rgamma(num, shape = n,
                  rate = sum(x))
lambda_2 = rgamma(num, shape = m,
                  rate = sum(y))

theta = lambda_1/lambda_2
values = sort(theta)
low_lim = values[0.025*num]
upp_lim = values[0.975*num]
int_len[k] = upp_lim - low_lim

if((low_lim < theta0) && (upp_lim >
theta0)){

    count = count + 1
} else {
    count = count
}
}

ave_int_len[j,i] = mean(int_len[k])
Cov_rate[j,i] = count/iter
thet[j,i] = theta0
lamb1[,i] = lambda1[i]
lamb2[j,] = lambda2[j]
}
}

ave_int_len = round(as.vector(ave_int_len), 3)
Cov_rate = as.vector(Cov_rate)
thet = round(as.vector(thet), 3)
lamb1 = as.vector(lamb1)

```

```

lamb2 = as.vector(lamb2)

results = cbind(ave_int_len , Cov_rate , thet , lamb1 , lamb2)
colnames(results) = c("Average interval length", "Coverage rate",
"Theta", "Lambda 1", "Lambda 2")
write.csv(results , paste("C:/Users/user/Desktop/Codes/Results /
Results_Jeffreys", "n =", n, "m =", m, ".csv"))

#-----#
#-----GML-----#

theta = rep(0, num)
ave_int_len = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
Cov_rate = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
thet = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb1 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb2 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
theta0 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
count = 0

for(i in 1:21){

    for(j in 1:21){

        theta0 = lambda1[i]/lambda2[j]
        int_len = rep(0, iter)
        count = 0

        for(k in 1:iter){

```

```

x = rexp(n, lambda1[i])
y = rexp(m, lambda2[j])

lambda_1 = rgamma(num, shape = (n + 1/2),
rate = sum(x))
lambda_2 = rgamma(num, shape = (m + 1/2),
rate = sum(y))
theta = lambda_1/lambda_2
values = sort(theta)
low_lim = values[0.025*num]
upp_lim = values[0.975*num]
int_len[k] = upp_lim - low_lim

if((low_lim < theta0) && (upp_lim >
theta0)){

        count = count + 1
    }else{
        count = count
    }
}

ave_int_len[j,i] = mean(int_len[k])
Cov_rate[j,i] = count/iter
thet[j,i] = theta0
lamb1[,i] = lambda1[i]
lamb2[j,] = lambda2[j]
}
}

ave_int_len = round(as.vector(ave_int_len), 3)
Cov_rate = as.vector(Cov_rate)
thet = round(as.vector(thet), 3)
lamb1 = as.vector(lamb1)

```

```

lamb2 = as.vector(lamb2)

results = cbind(ave_int_len , Cov_rate , thet , lamb1 , lamb2)
colnames(results) = c("Average interval length", "Coverage rate",
"Theta", "Lambda 1", "Lambda 2")
write.csv(results , paste("C:/Users/user/Desktop/Codes/Results/
Results_GML", "n =", n, "m =", m, ".csv"))

#-----#
#-----Uniform-----#

theta = rep(0, num)
ave_int_len = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
Cov_rate = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
thet = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb1 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb2 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
theta0 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
count = 0

for(i in 1:21){

    for(j in 1:21){

        theta0 = lambda1[i]/lambda2[j]
        int_len = rep(0, iter)
        count = 0

        for(k in 1:iter){

```

```

x = rexp(n, lambda1[i])
y = rexp(m, lambda2[j])

lambda_1 = rgamma(num, shape = (n + 1),
rate = sum(x))
lambda_2 = rgamma(num, shape = (m + 1),
rate = sum(y))
theta = lambda_1/lambda_2
values = sort(theta)
low_lim = values[0.025*num]
upp_lim = values[0.975*num]
int_len[k] = upp_lim - low_lim

if((low_lim < theta0) && (upp_lim >
theta0)){

        count = count + 1
} else {
        count = count
}
}

ave_int_len[j,i] = mean(int_len[k])
Cov_rate[j,i] = count/iter
thet[j,i] = theta0
lamb1[,i] = lambda1[i]
lamb2[j,] = lambda2[j]
}
}

ave_int_len = round(as.vector(ave_int_len), 3)
Cov_rate = as.vector(Cov_rate)
thet = round(as.vector(thet), 3)
lamb1 = as.vector(lamb1)

```

```

lamb2 = as.vector(lamb2)

results = cbind(ave_int_len , Cov_rate , thet , lamb1 , lamb2)
colnames(results) = c("Average interval length", "Coverage rate",
"Theta", "Lambda 1", "Lambda 2")
write.csv(results , paste("C:/Users/user/Desktop/Codes/Results/
Results_Uniform", "n =", n, "m =", m, ".csv"))

#-----#
#-----MDI-----#

theta = rep(0, num)
ave_int_len = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
Cov_rate = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
thet = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb1 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
lamb2 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
theta0 = matrix(c(rep(0, 21*21)), ncol=21, nrow=21)
count = 0

for(i in 1:21){

    for(j in 1:21){

        theta0 = lambda1[i]/lambda2[j]
        int_len = rep(0, iter)
        count = 0

        for(k in 1:iter){

```

```

x = rexp(n, lambda1[i])
y = rexp(m, lambda2[j])

lambda_1 = rgamma(num, shape = (2*n + 1),
rate = sum(x))
lambda_2 = rgamma(num, shape = (2*m + 1),
rate = sum(y))
theta = lambda_1/lambda_2
values = sort(theta)
low_lim = values[0.025*num]
upp_lim = values[0.975*num]
int_len[k] = upp_lim - low_lim

if((low_lim < theta0) && (upp_lim >
theta0)){

        count = count + 1
    }else{
        count = count
    }
}

ave_int_len[j,i] = mean(int_len[k])
Cov_rate[j,i] = count/iter
thet[j,i] = theta0
lamb1[,i] = lambda1[i]
lamb2[j,] = lambda2[j]
}
}

ave_int_len = round(as.vector(ave_int_len), 3)
Cov_rate = as.vector(Cov_rate)
thet = round(as.vector(thet), 3)
lamb1 = as.vector(lamb1)

```

```
lamb2 = as.vector(lamb2)
```

```
results = cbind(ave_int_len, Cov_rate, thet, lamb1, lamb2)
colnames(results) = c("Average interval length", "Coverage rate",
"Theta", "Lambda 1", "Lambda 2")
write.csv(results, paste("C:/Users/user/Desktop/Codes/Results/
Results_MDI", "n =", n, "m =", m, ".csv"))
```

```
#-----#
#-----#
```

C.2 Code simulation study II

```
#-----#
#-----Define n and m-----#

lambda1 = c(0.1, 0.5, 1, 2.5, 5)
lambda2 = c(5, 6, 7.5, 8.5, 10)

iter = 1000
n = 15
m = 30

#-----#
#-----GML-----#

lamb1 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
lamb2 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE1GML = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE2GML = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b1GML = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b2GML = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
theta0 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
thet = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)

for(k in 1:5){

  for(j in 1:5){

    Res1GML = rep(0, iter)
    Res2GML = rep(0, iter)
```

```

theta0 = lambda1[k]/lambda2[j]
theta1GML = rep(0, iter)
theta2GML = rep(0, iter)

  for(i in 1:iter){

      x = rexp(n, lambda1[k])
      y = rexp(m, lambda2[j])

      theta1GML[i] = (((sum(y))/(sum(x)))*
                      (gamma(n+1.5))*(gamma
                      (m-0.5)))/((gamma(n+0.5))
                      *(gamma(m+0.5)))
      theta2GML[i] = ((sum(y))/(sum(x)))*
                      ((n-0.5)/(m+1.5))
      diff1_2GML = (theta1GML[i] - theta0)^2
      diff2_2GML = (theta2GML[i] - theta0)^2
      Res1GML[i] = diff1_2GML
      Res2GML[i] = diff2_2GML
  }

  MSE1GML[j,k] = sum(Res1GML)/iter
  MSE2GML[j,k] = sum(Res2GML)/iter
  b1GML[j,k] = mean(theta1GML) - theta0
  b2GML[j,k] = mean(theta2GML) - theta0
  lamb1[,k] = lambda1[k]
  lamb2[j,] = lambda2[j]
  thet[j,k] = theta0
}

MSE1GML = as.vector(MSE1GML)
MSE2GML = as.vector(MSE2GML)

```

```

b1GML = as.vector(b1GML)
b2GML = as.vector(b2GML)
lamb1 = as.vector(lamb1)
lamb2 = as.vector(lamb2)
thet = as.vector(thet)

```

```

Results_GML = cbind(lamb1, lamb2, b1GML, MSE1GML, b2GML, MSE2GML)
colnames(Results_GML) = c("Lambda 1", "Lambda 2", "Bias (mean)",
"Squared error (mean)", "Bias (mode)", "All-or-nothing(mode)")
write.csv(Results_GML, paste("H:/M/ Skripsie_Ratio of Exp
Parameters/Codes/Results/Simulation Study II/Results_GML",
"n =", n, "m =", m, ".csv"))

```

```

mse1 = data.frame(thet, MSE1GML)
mse2 = data.frame(thet, MSE2GML)
bias1 = data.frame(thet, b1GML)
bias2 = data.frame(thet, b2GML)

```

```

mse1_sort = mse1[order(mse1$thet),]
mse2_sort = mse2[order(mse2$thet),]
bias1_sort = bias1[order(bias1$thet),]
bias2_sort = bias2[order(bias2$thet),]

```

```

#-----#
#-----Jeffreys-----#

```

```

lamb1 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
lamb2 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE1Jef = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE2Jef = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b1Jef = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)

```

```

b2Jef = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
theta0 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
thet = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)

for(k in 1:5){

  for(j in 1:5){

    Res1Jef = rep(0, iter)
    Res2Jef = rep(0, iter)

    theta0 = lambda1[k]/lambda2[j]
    theta1Jef = rep(0, iter)
    theta2Jef = rep(0, iter)

    for(i in 1:iter){

      x = rexp(n, lambda1[k])
      y = rexp(m, lambda2[j])

      theta1Jef[i] = (((sum(y))/(sum(x)))*
                      (gamma(n+1))*(gamma
                      (m-1)))/((gamma(n))*
                      (gamma(m)))
      theta2Jef[i] = ((sum(y))/(sum(x)))*
                      ((n-1)/(m+1))
      diff1_2Jef = (theta1Jef[i] - theta0)^2
      diff2_2Jef = (theta2Jef[i] - theta0)^2
      Res1Jef[i] = diff1_2Jef
      Res2Jef[i] = diff2_2Jef
    }

    MSE1Jef[j,k] = sum(Res1Jef)/iter
  }
}

```

```

        MSE2Jef[j,k] = sum(Res2Jef)/iter
        b1Jef[j,k] = mean(theta1Jef) - theta0
        b2Jef[j,k] = mean(theta2Jef) - theta0
        lamb1[,k] = lambda1[k]
        lamb2[j,] = lambda2[j]
        thet[j,k] = theta0
    }
}

MSE1Jef = as.vector(MSE1Jef)
MSE2Jef = as.vector(MSE2Jef)
b1Jef = as.vector(b1Jef)
b2Jef = as.vector(b2Jef)
lamb1 = as.vector(lamb1)
lamb2 = as.vector(lamb2)
thet = as.vector(thet)

Results_Jef = cbind(lamb1, lamb2, b1Jef, MSE1Jef, b2Jef, MSE2Jef)
colnames(Results_Jef) = c("Lambda 1", "Lambda 2", "Bias (mean)",
"Squared error (mean)", "Bias (mode)", "All-or-nothing(mode)")
write.csv(Results_Jef, paste("H:/M/ Skripsie_Ratio of Exp
Parameters/Codes/Results/Simulation Study II/Results_Jeffreys",
"n =", n, "m =", m, ".csv"))

mse1 = data.frame(thet, MSE1Jef)
mse2 = data.frame(thet, MSE2Jef)
bias1 = data.frame(thet, b1Jef)
bias2 = data.frame(thet, b2Jef)

mse1_sort = mse1[order(mse1$thet),]
mse2_sort = mse2[order(mse2$thet),]
bias1_sort = bias1[order(bias1$thet),]
bias2_sort = bias2[order(bias2$thet),]

```

```

#-----#
#-----MDI-----#

lamb1 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
lamb2 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE1MDI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE2MDI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b1MDI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b2MDI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
theta0 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
thet = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)

for(k in 1:5){

  for(j in 1:5){

    Res1MDI = rep(0, iter)
    Res2MDI = rep(0, iter)

    theta0 = lambda1[k]/lambda2[j]
    theta1MDI = rep(0, iter)
    theta2MDI = rep(0, iter)

    for(i in 1:iter){

      x = rexp(n, lambda1[k])
      y = rexp(m, lambda2[j])

      theta1MDI[i] = (((sum(y))/(sum(x)))*
                      (gamma((2*n)+2))*(gamma
                      ((2*m) - 0)))/((gamma((2*n)

```

```

+ 1))*(gamma((2*m)+1)))
theta2MDI[i] = ((sum(y))/(sum(x)))*
               (((2*n)-0)/((2*m)+2))
diff1_2MDI = (theta1MDI[i] - theta0)^2
diff2_2MDI = (theta2MDI[i] - theta0)^2
Res1MDI[i] = diff1_2MDI
Res2MDI[i] = diff2_2MDI
}

MSE1MDI[j,k] = sum(Res1MDI)/iter
MSE2MDI[j,k] = sum(Res2MDI)/iter
b1MDI[j,k] = mean(theta1MDI) - theta0
b2MDI[j,k] = mean(theta2MDI) - theta0
lamb1[,k] = lambda1[k]
lamb2[j,] = lambda2[j]
thet[j,k] = theta0
}
}

MSE1MDI = as.vector(MSE1MDI)
MSE2MDI = as.vector(MSE2MDI)
b1MDI = as.vector(b1MDI)
b2MDI = as.vector(b2MDI)
lamb1 = as.vector(lamb1)
lamb2 = as.vector(lamb2)
thet = as.vector(thet)

Results_MDI = cbind(lamb1, lamb2, b1MDI, MSE1MDI, b2MDI, MSE2MDI)
colnames(Results_MDI) = c("Lambda 1", "Lambda 2", "Bias (mean)",
"Squared error (mean)", "Bias (mode)", "All-or-nothing(mode)")
write.csv(Results_MDI, paste("H:/M/Skripsie_Ratio of Exp
Parameters/Codes/Results/Simulation Study II/Results_MDI",
"n =", n, "m =", m, ".csv"))

```

```

mse1 = data.frame(thet , MSE1MDI)
mse2 = data.frame(thet , MSE2MDI)
bias1 = data.frame(thet , b1MDI)
bias2 = data.frame(thet , b2MDI)

mse1_sort = mse1[order(mse1$thet),]
mse2_sort = mse2[order(mse2$thet),]
bias1_sort = bias1[order(bias1$thet),]
bias2_sort = bias2[order(bias2$thet),]

#-----#
#-----Uniform-----#

lamb1 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
lamb2 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE1UNI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
MSE2UNI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b1UNI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
b2UNI = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
theta0 = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)
thet = matrix(c(rep(0, 5*5)), ncol = 5, nrow = 5)

for(k in 1:5){

  for(j in 1:5){

    Res1UNI = rep(0, iter)
    Res2UNI = rep(0, iter)

    theta0 = lambda1[k]/lambda2[j]
    theta1UNI = rep(0, iter)

```

```

theta2UNI = rep(0, iter)

for(i in 1:iter){

    x = rexp(n, lambda1[k])
    y = rexp(m, lambda2[j])

    theta1UNI[i] = (((sum(y))/(sum(x)))*
                    (gamma(n+2))*(gamma
                    (m-0)))/((gamma(n+1))*
                    (gamma(m+1)))
    theta2UNI[i] = ((sum(y))/(sum(x)))*
                    ((n-0)/(m+2))
    diff1_2UNI = (theta1UNI[i] - theta0)^2
    diff2_2UNI = (theta2UNI[i] - theta0)^2
    Res1UNI[i] = diff1_2UNI
    Res2UNI[i] = diff2_2UNI
}

MSE1UNI[j,k] = sum(Res1UNI)/iter
MSE2UNI[j,k] = sum(Res2UNI)/iter
b1UNI[j,k] = mean(theta1UNI) - theta0
b2UNI[j,k] = mean(theta2UNI) - theta0
lamb1[,k] = lambda1[k]
lamb2[j,] = lambda2[j]
thet[j,k] = theta0
}

MSE1UNI = as.vector(MSE1UNI)
MSE2UNI = as.vector(MSE2UNI)
b1UNI = as.vector(b1UNI)
b2UNI = as.vector(b2UNI)
lamb1 = as.vector(lamb1)

```

```
lamb2 = as.vector(lamb2)
```

```
thet = as.vector(thet)
```

```
Results_UNI = cbind(lamb1, lamb2, b1UNI, MSE1UNI, b2UNI, MSE2UNI)
colnames(Results_UNI) = c("Lambda 1", "Lambda 2", "Bias (mean)",
"Squared error (mean)", "Bias (mode)", "All-or-nothing(mode)")
write.csv(Results_UNI, paste("H:/M/Skripsie_Ratio of Exp
Parameters/Codes/Results/Simulation Study II/Results_Uniform",
"n =", n, "m =", m, ".csv"))
```

```
mse1 = data.frame(thet, MSE1UNI)
```

```
mse2 = data.frame(thet, MSE2UNI)
```

```
bias1 = data.frame(thet, b1UNI)
```

```
bias2 = data.frame(thet, b2UNI)
```

```
mse1_sort = mse1[order(mse1$thet),]
```

```
mse2_sort = mse2[order(mse2$thet),]
```

```
bias1_sort = bias1[order(bias1$thet),]
```

```
bias2_sort = bias2[order(bias2$thet),]
```

Appendix D

Code Application

D.1 Code for intervals and estimates for λ_1 and λ_2

```
clear
clc
xdata = [55, 320, 56, 104, 220, 239, 47, 246, 176, 182, 33];
ydata = [23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42,
20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95];
n = length(xdata);
m = length(ydata);
x = 0:0.0005:0.05;
y = 0:0.0005:0.05;

post_lambda1 = gampdf(x,2*n+1,1/(sum(xdata)));
post_lambda2 = gampdf(y,2*m+1,1/(sum(ydata)));
lower_limit_CI1 = gaminv(0.025,2*n+1,1/(sum(xdata)));
upper_limit_CI1 = gaminv(0.975,2*n+1,1/(sum(xdata)));
CI1 = [lower_limit_CI1 upper_limit_CI1]
lower_limit_CI2 = gaminv(0.025,2*m+1,1/(sum(ydata)));
upper_limit_CI2 = gaminv(0.975,2*m+1,1/(sum(ydata)));
CI2 = [lower_limit_CI2 upper_limit_CI2]

posterior_mean1 = (2*n+1)/(sum(xdata))
```

```

posterior_mean2 = (2*m+1)/(sum(ydata))

posterior_mode1 = (2*n+1+1)/(sum(xdata))
posterior_mode2 = (2*m+1+1)/(sum(ydata))

plot(x, post_lambda1, '*-m')
hold on
plot(y, post_lambda2, 'h-b')
hold off
xlabel('\lambda')
ylabel('f(\lambda | data)')
legend("\lambda_1", "\lambda_2")
grid

```

D.2 Code for intervals and estimates for θ

```

clear
clc
xdata = [55, 320, 56, 104, 220, 239, 47, 246, 176, 182, 33];
ydata = [23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42,
20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95];
n = length(xdata);
m = length(ydata);
theta = 0:0.01:1;
alphaA = n;
betA = m;
pA = 1;

qA = (sum(xdata))/(sum(ydata));
beta_prime_densityA = (pA.*((theta ./qA).^(alphaA.*pA-1)).*
((1+(theta ./qA).^pA).^(-alphaA-betA)))./(qA.*(beta(alphaA, betA)));
plot(theta, beta_prime_densityA, 'h-b')
hold on
alphaB = n+0.5;
betB = m+0.5;

```

```

pB = 1;

qB = (sum(xdata))/(sum(ydata));
beta_prime_densityB = (pB.*((theta./qB).^(alphaB.*pB-1)).*
((1+(theta./qB).^pB).^(-alphaB-betB)))/(qB.*(beta(alphaB,betB)));
plot(theta,beta_prime_densityB,'*-m')
hold on
alphaC = n+1;
betC = m+1;
pC = 1;

qC = (sum(xdata))/(sum(ydata));
beta_prime_densityC = (pC.*((theta./qC).^(alphaC.*pC-1)).*
((1+(theta./qC).^pC).^(-alphaC-betC)))/(qC.*(beta(alphaC,betC)));
plot(theta,beta_prime_densityC,'.-k')
hold on
alphaD = 2*n+1;
betD = 2*m+1;
pD = 1;
qD = (sum(xdata))/(sum(ydata));

beta_prime_densityD = (pD.*((theta./qD).^(alphaD.*pD-1)).*
((1+(theta./qD).^pD).^(-alphaD-betD)))/(qD.*(beta(alphaD,betD)));
plot(theta,beta_prime_densityD,'s-g')
hold off
xlabel('\theta')
ylabel('\pi(\theta|data)')
legend("\pi_A - Jeffreys ","\pi_B - GML"," \pi_C - Uniform ","
\pi_D - MDI")
grid

posterior_meanA = (qA*(gamma(alphaA+1))*(gamma(betA-1)))/
((gamma(alphaA))*(gamma(betA)))
posterior_modeA = qA*(alphaA-1)/(betA+1)

```

```

posterior_meanB = (qB*(gamma(alphaB + 1))*(gamma(betB - 1)))/
((gamma(alphaB))*(gamma(betB)))
posterior_modeB = qB*(alphaB - 1)/(betB + 1)
posterior_meanC = (qC*(gamma(alphaC + 1))*(gamma(betC - 1)))/
((gamma(alphaC))*(gamma(betC)))
posterior_modeC = qC*(alphaC - 1)/(betC + 1)
posterior_meanD = (qD*(gamma(alphaD + 1))*(gamma(betD - 1)))/
((gamma(alphaD))*(gamma(betD)))
posterior_modeD = qD*(alphaD - 1)/(betD + 1)

intA1 = [];
intB1 = [];
intC1 = [];
intD1 = [];

for i = 0:0.0001:1.3

funA = @(x) (pA.*((x./qA).^(alphaA.*pA - 1)).*((1+(x./qA).^pA).^
(-alphaA - betA)))./(qA.*(beta(alphaA, betA)));
integralA1 = integral(funA, 0, i);
intA1 = [intA1 integralA1];
funB = @(x) (pB.*((x./qB).^(alphaB.*pB - 1)).*((1+(x./qB).^pB).^
(-alphaB - betB)))./(qB.*(beta(alphaB, betB)));
integralB1 = integral(funB, 0, i);
intB1 = [intB1 integralB1];

funC = @(x) (pC.*((x./qC).^(alphaC.*pC - 1)).*((1+(x./qC).^pC).^
(-alphaC - betC)))./(qC.*(beta(alphaC, betC)));
integralC1 = integral(funC, 0, i);
intC1 = [intC1 integralC1];

funD = @(x) (pD.*((x./qD).^(alphaD.*pD - 1)).*((1+(x./qD).^pD).^
(-alphaD - betD)))./(qD.*(beta(alphaD, betD)));
integralD1 = integral(funD, 0, i);

```

```
intD1 = [intD1 integralD1];
```

```
end
```

```
Results = [intA1 ' intB1 ' intC1 ' intD1 '];
```