

**A study of Dunford-Pettis-like  
properties with applications to  
polynomials and analytic functions on  
normed spaces**

**E. D. Zeekoei**

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Supervisor: Prof. J.H. Fourie

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*I dedicate this dissertation to my mother. Rest in peace.  
I miss and will always love you.*

# Abstract

Recall that a Banach space  $X$  has the *Dunford-Pettis property* if every weakly compact operator defined on  $X$  takes weakly compact sets into norm compact sets. Some valuable characterisations of Banach spaces with the Dunford-Pettis property are:  $X$  has the *DPP* if and only if for all Banach spaces  $Y$ , every weakly compact operator from  $X$  to  $Y$  sends weakly convergent sequences onto norm convergent sequences (i.e. it requires that weakly compact operators on  $X$  are completely continuous) and this is equivalent to “if  $(x_n)$  and  $(x_n^*)$  are sequences in  $X$  and  $X^*$  respectively and  $\lim_n x_n = 0$  weakly and  $\lim_n x_n^* = 0$  weakly then  $\lim_n x_n^* x_n = 0$ ”.

A striking application of the Dunford-Pettis property (as was observed by Grothendieck) is to prove that if  $X$  is a linear subspace of  $L_\infty(\mu)$  for some finite measure  $\mu$  and  $X$  is closed in some  $L_p(\mu)$  for  $1 \leq p < \infty$ , then  $X$  is finite dimensional. The fact that the well known spaces  $L_1(\mu)$  and  $C(\Omega)$  have this property (as was proved by Dunford and Pettis) was a remarkable achievement in the early history of Banach spaces and was motivated by the study of integral equations and the hope to develop an understanding of linear operators on  $L_p(\mu)$  for  $p \geq 1$ . In fact, it played an important role in proving that for each weakly compact operator  $T : L_1(\mu) \rightarrow L_1(\mu)$  or  $T : C(\Omega) \rightarrow C(\Omega)$ , the operator  $T^2$  is compact, a fact which is important from the point of view that there is a nice spectral theory for compact operators and operators whose squares are compact.

There is an extensive literature involving the Dunford-Pettis property. Almost all the articles and books in our list of references contain some information about this property, but there are plenty more that could have been listed. The reader is for instance referred to [4], [5], [7], [8],[10], [17] and [24] for information on the role of the *DPP* in different areas of Banach space theory.

In this dissertation, however, we are motivated by the two papers [7] and

[8] to study alternative Dunford-Pettis properties, to introduce a scale of (new) alternative Dunford-Pettis properties, which we call  $DP^*$ -properties of order  $p$  (briefly denoted by  $DP^*P$ ), and to consider characterisations of Banach spaces with these properties as well as applications thereof to polynomials and holomorphic functions on Banach spaces.

In the paper [8] the class  $C_p(X, Y)$  of  $p$ -convergent operators from a Banach space  $X$  to a Banach space  $Y$  is introduced. Replacing the requirement that weakly compact operators on  $X$  should be completely continuous in the case of the  $DPP$  for  $X$  (as is mentioned above) by “weakly compact operators on  $X$  should be  $p$ -convergent”, an alternative Dunford-Pettis property (called the *Dunford-Pettis property of order  $p$* ) is introduced. More precisely, if  $1 \leq p \leq \infty$ , a Banach space  $X$  is said to have  $DPP_p$  if the inclusion  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_p(X, Y)$  holds for all Banach spaces  $Y$ . Here  $\mathcal{W}(X, Y)$  denotes the family of all weakly compact operators from  $X$  to  $Y$ . We now have a scale of “Dunford-Pettis like properties” in the sense that all Banach spaces have the  $DPP_1$ , if  $p < q$ , then each Banach space with the  $DPP_q$  also has the  $DPP_p$  and the strongest property, namely the  $DPP_\infty$  coincides with the  $DPP$ .

In the paper [7] the authors study a property on Banach spaces (called the  *$DP^*$ -property, or briefly the  $DP^*P$* ) which is stronger than the  $DPP$ , in the sense that if a Banach space has this property then it also has  $DPP$ . We say  $X$  has the  $DP^*P$ , when all weakly compact sets in  $X$  are limited, i.e. each sequence  $(x_n^*) \subset X^*$  in the dual space of  $X$  which converges weak\* to 0, also converges uniformly (to 0) on all weakly compact sets in  $X$ . It turns out that this property is equivalent to another property on Banach spaces which is introduced in [17] (and which is called the *\*-Dunford-Pettis property*) as follows: We say a Banach space  $X$  has the *\*-Dunford-Pettis property* if for all weakly null sequences  $(x_n)$  in  $X$  and all *weak\** null sequences  $(x_n^*)$  in  $X^*$ , we have  $x_n^*(x_n) \xrightarrow{\infty} 0$ . After a thorough study of the  $DP^*P$ , including characterisations and examples of Banach spaces with the  $DP^*P$ , the authors in [7] consider some applications to polynomials and analytic functions on Banach spaces.

Following an extensive literature study and in depth research into the techniques of proof relevant to this research field, we are able to present a thorough discussion of the results in [7] and [8] as well as some selected (relevant) results from other papers (for instance, [2] and [17]). This we do in Chapter 2 of the dissertation. The starting point (in Section 2.1 of Chap-

ter 2) is the introduction of the so called  $p$ -convergent operators, being those bounded linear operators  $T : X \rightarrow Y$  which transform weakly  $p$ -summable sequences into norm-null sequences, as well as the so called weakly  $p$ -convergent sequences in Banach spaces, being those sequences  $(x_n)$  in a Banach space  $X$  for which there exists an  $x \in X$  such that the sequence  $(x_n - x)$  is weakly  $p$ -summable. Using these concepts, we state and prove an important characterisation (from the paper [8]) of Banach spaces with  $DPP_p$ . In Section 2.2 (of Chapter 2) we continue to report on the results of the paper [7], where the  $DP^*P$  on Banach spaces is introduced. We focus on the characterisation of Banach spaces with  $DP^*P$ , obtaining among others that a Banach space  $X$  has  $DP^*P$  if and only if for all weakly null sequences  $(x_n)$  in  $X$  and all weak\* null sequences  $(x_n^*)$  in  $X^*$ , we have  $x_n^*(x_n) \xrightarrow{\infty} 0$ . An important characterisation of the  $DP^*P$  considered in this section is the fact that  $X$  has  $DP^*P$  if and only if every  $T \in \mathcal{L}(X, c_0)$  is completely continuous. This result proves to be of fundamental importance in the study of the  $DP^*P$  and its application to results on polynomials and holomorphic functions on Banach spaces. To be able to report on the applications of the  $DP^*P$  in the context of homogeneous polynomials and analytic functions on Banach spaces, we embark on a study of “Complex Analysis in Banach spaces” (mostly with the focus on homogeneous polynomials and analytic functions on Banach spaces). This we do in Chapter 3; the content of the chapter is mostly based on work in the books [23] and [14], but also on the work in some articles such as [15]. After we have discussed the relevant theory of complex analysis in Banach spaces in Chapter 3, we devote Chapter 4 to considering properties of polynomials and analytic functions on Banach spaces with  $DP^*P$ . The discussion in Chapter 4 is based on the applications of  $DP^*P$  in the paper [7].

Finally, in Chapter 5 of the dissertation, we contribute to the study of “Dunford-Pettis like properties” by introducing the Banach space property “ $DP^*P$  of order  $p$ ”, or briefly the  $DP^*P_p$  for Banach spaces. Using the concept “weakly  $p$ -convergent sequence in Banach spaces” as is defined in [8], we define *weakly- $p$ -compact sets* in Banach spaces. Then a Banach space  $X$  is said to have the  $DP^*$ -property of order  $p$  (for  $1 \leq p \leq \infty$ ) if all weakly- $p$ -compact sets in  $X$  are limited. In short, we say  $X$  has  $DP^*P_p$ . As in [8] (where the  $DPP_p$  is introduced), we now have a scale of  $DP^*P$ -like properties, in the sense that all Banach spaces have  $DP^*P_1$  and if  $p < q$  and  $X$  has  $DP^*P_q$  then it has  $DP^*P_p$ . The strongest property  $DP^*P_\infty$  coincides with

$DP^*P$ . We prove characterisations of Banach spaces with  $DP^*P_p$ , discuss some examples and then consider applications to polynomials and analytic functions on Banach spaces. Our results and techniques in this chapter depend very much on the results obtained in the previous three chapters, but now we have to find our own correct definitions and formulations of results within this new context. We do this with some success in Sections 5.1 and 5.2 of Chapter 5.

Chapter 1 of this dissertation provides a wide range of concepts and results in Banach spaces and the theory of vector sequence spaces (some of them very deep results from books listed in the bibliography). These results are mostly well known, but they are scattered in the literature - they are discussed in Chapter 1 (some with proof, others without proof, depending on the importance of the arguments in the proofs for later use and depending on the detail with which the results are discussed elsewhere in the literature) with the intention to provide an exposition which is mostly self contained and which will be comfortably accessible for graduate students.

The dissertation reflects the outcome of our investigation in which we set ourselves the following goals:

1. Obtain a thorough understanding of the Dunford-Pettis property and some related (both weaker and stronger) properties that have been studied in the literature.
2. Focusing on the work in the paper [8], understand the role played in the study of different classes of operators by a scale of properties on Banach spaces, called the  $DPP_p$ , which are weaker than the  $DP$ -property and which are introduced in [8] by using the weakly  $p$ -summable sequences in  $X$  and weakly null sequences in  $X^*$ .
3. Focusing on the work in the paper [7], investigate the  $DP^*P$  for Banach spaces, which is the exact property to answer a question of Pelczyński's regarding when every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous.
4. Based on the ideas intertwined in the work of the paper [8] in the study of a scale of  $DP$ -properties and the work in the paper [7], introduce the  $DP^*P_p$  on Banach spaces and investigate their applications to spaces of operators and in the theory of polynomials and analytic mappings on Banach spaces. Thereby, not only extending the results in [7] to

a larger family of Banach spaces, but also to find an answer to the question: “When will every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  be  $p$ -convergent?”

**Keyterms:** Banach space, weakly  $p$ -summable sequence, weakly  $p$ -convergent sequence,  $p$ -convergent operator, completely continuous operator, completely continuous function,  $p$ -convergent function, weakly compact set, weakly  $p$ -compact set, limited set, Dunford-Pettis property, Dunford-Pettis property of order  $p$ ,  $DP^*$ -property,  $DP^*$ -property of order  $p$ , homogeneous polynomial, analytic function on a Banach space.

# Opsomming

Die sogenaamde *Dunford-Pettis eienskap* is 'n welbekende begrip in die teorie van Banachruimtes. 'n Banachruimte  $X$  bevredig naamlik die *Dunford-Pettis eienskap* as elke swak kompakte operator op  $X$ , swak kompakte deelversamelings van  $X$  op norm kompakte versamelings afbeeld. Vervolgens noem ons enkele nuttige karakteriserings van Banachruimtes met die Dunford-Pettis eienskap:  $X$  bevredig die *DPP* as en slegs as vir alle Banachruimtes  $Y$ , elke swak kompakte operator van  $X$  in  $Y$  swak konvergente rye op norm konvergente rye afbeeld (d.i. die verlange eienskap is dat swak kompakte operatore op  $X$  volledig kontinu moet wees) en dit is weer ekwivalent met die eienskap “as  $(x_n)$  en  $(x_n^*)$  rye in  $X$  en  $X^*$  onderskeidelik is en  $\lim_n x_n = 0$  in die swak topologie van  $X$  en  $\lim_n x_n^* = 0$  in die swak topologie van  $X^*$ , dan volg  $\lim_n x_n^* x_n = 0$ ”.

'n Verbasende toepassing van die Dunford-Pettis eienskap (soos deur Grothendieck voorgestel is) is om te bewys dat as  $X$  beide 'n lineêre deelruimte van  $L_\infty(\mu)$  (vir 'n eindige maat  $\mu$ ) is en geslote is in 'n  $L_p(\mu)$ -ruimte vir een of ander  $1 \leq p < \infty$ , dan is  $X$  noodwendig eindig-dimensioneel. Die ontdekking van die feit dat die welbekende ruimtes  $L_1(\mu)$  en  $C(\Omega)$  die *DPP* bevredig (soos deur Dunford en Pettis bewys is) was 'n uitsonderlike mylpaal in die vroeëre geskiedenis van Banachruimtes; die bestudering van die *DPP* het voortgespruit uit die studie van integraalvergelykings en was gemotiveerd deur die strewe daarna om 'n beter begrip van lineêre operatore op  $L_p(\mu)$  vir  $p \geq 1$  te ontwikkel. Inderwaarheid het die feit dat  $L_1(\mu)$  en  $C(\Omega)$  die *DPP* bevredig, 'n belangrike rol gespeel in die bewys dat vir elke swak kompakte  $T : L_1(\mu) \rightarrow L_1(\mu)$  of  $T : C(\Omega) \rightarrow C(\Omega)$ , die operator  $T^2$  kompak is; dit is 'n belangrike ontdekking as in ag geneem word dat daar 'n goed ontwikkelde spektraalteorie vir sowel kompakte operatore as operatore waarvan die kwadraat kompak is, bestaan.

Daar bestaan 'n uitgebreide literatuur wat op die Dunford-Pettis eienskap

betrekking het. Byna al die artikels en boeke in ons bibliografielys bevat inligting oor hierdie eienskap, maar daar is baie meer wat ons sou kon lys. Die leser word byvoorbeeld na [4], [5], [7], [8],[10], [17] en [24] vir inligting oor die rol van die *DPP* in verskillende gebiede van Banachruimte-teorie verwys.

In hierdie verhandeling is ons egter deur die twee artikels [7] en [8] gelei om alternatiewe Dunford-Pettis eienskappe te bestudeer, om 'n skaal van (nuwe) alternatiewe Dunford-Pettis eienskappe (genoem *DP\**-eienskappe van orde  $p$  en kortliks met  $DP^*P_p$  aangedui) in te voer, Banachruimtes met hierdie eienskappe te karakteriseer en toepassings daarvan op polinome en holomorfe funksies op Banachruimtes te ondersoek.

In die artikel [8] word die familie  $C_p(X, Y)$  van  $p$ -konvergente operatore van 'n Banachruimte  $X$  na 'n Banachruimte  $Y$  ingevoer. Deur die vereiste dat swak kompakte operatore op  $X$  volledig kontinu moet wees in die geval van die *DPP* op  $X$  (soos hierbo genoem) met die vereiste “swak kompakte operatore op  $X$  moet  $p$ -konvergent wees” te vervang, word 'n alternatiewe Dunford-Pettis eienskap (genoem die *Dunford-Pettis eienskap van orde  $p$* ) ingevoer. Besonderlik, as  $1 \leq p \leq \infty$ , sê ons 'n Banachruimte  $X$  het  $DPP_p$  as die insluiting  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_p(X, Y)$  geld vir alle Banachruimtes  $Y$ . Hier dui  $\mathcal{W}(X, Y)$  die familie van alle swak kompakte operatore van  $X$  in  $Y$  aan. Sodoende het ons nou 'n skaal van “Dunford-Pettis soort eienskappe” ingevoer, in die sin dat alle Banachruimtes  $DPP_1$  het, dat as  $p < q$ , dan het elke Banachruimte met die  $DPP_q$  ook die  $DPP_p$  en die sterkste eienskap, naamlik die  $DPP_\infty$ , is dieselfde as *DPP*.

In die artikel [7] bestudeer die outeurs die sogenaamde *DP\**-eienskap (kortliks die  $DP^*P$ ) op Banachruimtes, 'n eienskap wat sterker is as die *DPP* in die sin dat as 'n Banachruimte hierdie eienskap het, dan het dit ook *DPP*. Ons sê dat  $X$  die  $DP^*P$  het, as alle swak kompakte versamelings in  $X$  limietversamelings is, d.i. as elke ry  $(x_n^*) \subset X^*$  (dus in die duaalruimte van  $X$ ) wat swak\* na 0 konvergeer, ook gelykmatig (na 0) op alle swak kompakte versamelings in  $X$  konvergeer. Dit blyk dat hierdie eienskap ekwivalent is aan 'n ander eienskap op Banachruimtes, genoem die \*-Dunford-Pettis eienskap, wat in die artikel [17] soos volg gedefinieer word: Ons sê 'n Banachruimte  $X$  het die \*-Dunford-Pettis eienskap as vir alle swak nulrye  $(x_n)$  in  $X$  en alle swak\* nulrye  $(x_n^*)$  in  $X^*$ , volg dat  $x_n^*(x_n) \xrightarrow{\infty} 0$ . Na 'n deeglike studie van die  $DP^*P$ , insluitend die karakterisering van Banachruimtes met die  $DP^*P$  en bespreking van voorbeelde daarvan, ondersoek die outeurs in [7] enkele toepassings op polinome en analitiese funksies op Banachruimtes.

Na 'n uitgebreide literatuurstudie en in diepte navorsing oor die relevante bewystegnieke in hierdie navorsingsgebied, is dit vir ons moontlik om 'n deeglike bespreking van die resultate in [7] en [8], sowel as enkele uitgesoekte (relevante) resultate in ander artikels (soos byvoorbeeld, [2] en [17]), te doen. Ons doen dit naamlik in Hoofstuk 2 van hierdie verhandeling. Die bespreking (in paragraaf 2.1 van Hoofstuk 2) begin met die invoering van die sogenaamde  $p$ -konvergente operatore, d.i. die begrensde lineêre operatore  $T : X \rightarrow Y$  wat swak  $p$ -sommeerbare rye op norm nulrye afbeeld, sowel as die invoering van die sogenaamde swak  $p$ -konvergente rye, d.i. die rye  $(x_n)$  in 'n Banachruimte  $X$  is waarvoor daar 'n  $x \in X$  bestaan só dat die ry  $(x_n - x)$  swak  $p$ -sommeerbaar is. Deur van hierdie konsepte gebruik te maak, stel en bewys ons 'n belangrike karakterisering (uit die artikel [8]) van Banachruimtes met  $DPP_p$ . In paragraaf 2.2 (van Hoofstuk 2) gaan ons voort met die bespreking van resultate in die artikel [7] waarin die  $DP^*P$  op Banachruimtes bestudeer is. Ons fokus op die karakterisering van Banachruimtes met  $DP^*P$  en bewys onder andere dat 'n Banachruimte  $X$  die eienskap  $DP^*P$  het as en slegs as dit vir alle swak nulrye  $(x_n)$  in  $X$  en alle swak\* nulrye  $(x_n^*)$  in  $X^*$  volg dat  $x_n^*(x_n) \xrightarrow{\infty} 0$ . Nog 'n belangrike karakterisering van die  $DP^*P$  wat in hierdie paragraaf bespreek word, is die feit dat  $X$  die  $DP^*P$  het as en slegs as elke operator  $T \in \mathcal{L}(X, c_0)$  volledig kontinu is. Hierdie resultaat blyk van groot belang te wees in die studie van die  $DP^*P$  en sy toepassings op resultate in die teorie van polinome en holomorfe funksies op Banachruimtes. Om in staat te wees om in hierdie verhandeling te kan rapporteer oor die toepassings van die  $DP^*P$  in die konteks van homogene polinome en analitiese funksies op Banachruimtes, is ons genoodsaak om 'n studie van “Komplekse Analise in Banachruimtes” (meesal met die fokus op homogene polinome en analitiese funksies op Banachruimtes) te loots. Ons doen dit in Hoofstuk 3; die inhoud van hierdie hoofstuk is meesal gegrond op werk in die boeke [23] en [14], maar ook op werk in enkele artikels soos byvoorbeeld die artikel [15]. Nadat ons die relevante teorie van komplekse analise op Banachruimtes in Hoofstuk 3 bespreek het, wy ons Hoofstuk 4 aan die studie van eienskappe van polinome en analitiese funksies op Banachruimtes met  $DP^*P$ . Die bespreking in Hoofstuk 4 is gebaseer op toepassings van die  $DP^*P$  soos in die artikel [7].

Ten slotte, maak ons 'n nuwe bydrae tot die studie van “Dunford-Pettis soort” eienskappe in Hoofstuk 5 van die verhandeling, deurdat ons die Banachruimte eienskap “ $DP^*P$  van orde  $p$ ”, of kortliks die  $DP^*P_p$  (vir Ba-

nachruimtes) invoer. Deur van die begrip “swak  $p$ -konvergente ry in Banachruimtes” (soos in [8] gedefinieer) gebruik te maak, definieer ons *swak- $p$ -kompakte versamelings* in Banachruimtes. Dan sê ons ’n Banachruimte  $X$  het die  $DP^*$ -eienskap van orde  $p$  (vir  $1 \leq p \leq \infty$ ) as alle swak- $p$ -kompakte versamelings in  $X$  limietversamelings is. In kort sê ons dat  $X$  die  $DP^*P_p$  het. Soos in [8] (waar die  $DPP_p$  gedefinieer is), het ons gevolglik ’n skaal van  $DP^*P$ -soort eienskappe, in die sin dat alle Banachruimtes  $DP^*P_1$  het en as  $p < q$  en  $X$  het  $DP^*P_q$ , dan het  $X$  ook  $DP^*P_p$ . Die sterkste eienskap  $DP^*P_\infty$  is dieselfde as die  $DP^*P$ . Ons bewys vervolgens in Hoofstuk 5 karakteriserings van Banachruimtes met  $DP^*P_p$ , bespreek enkele voorbeelde van sodanige ruimtes en dan ondersoek ons toepassings hiervan op polinome en analitiese funksies op Banachruimtes. Ons resultate en tegnieke in hierdie hoofstuk is grootliks geskoei op resultate wat in die voorafgaande drie hoofstukke bespreek is, maar in hierdie geval is dit ons taak om te sorg dat ons ons eie korrekte definisies en formulerings van resultate in die nuwe konteks weergee. Hierin is ons suksesvol en rapporteer graag daaroor in paragrawe 5.1 en 5.2 van Hoofstuk 5.

Hoofstuk 1 van hierdie verhandeling verskaf aan die leser ’n breë spektrum van begrippe en resultate in Banachruimtes en die teorie van vektorruimtes (sommige hiervan baie diep resultate uit boeke in die bibliografie). Die resultate is meesal welbekend, maar is oor die literatuur heen versprei - ons bespreek dit in Hoofstuk 1 (sommige met bewys en andere sonder bewys, afhange van die belang wat die bewystegnieke vir latere gebruik mag hê en ook afhange van tot welke mate van volledigheid die bewyse in ander bronne bespreek is) met die idee om die verhandeling so ver moontlik self-inhoudelik en gemaklik toeganklik vir nagraadse studente te maak.

Die verhandeling reflekteer die uitkoms van ons ondersoek waarin ons vir onself die volgende doelstellings geformuleer het:

1. Verkry ’n deeglike begrip van die Dunford-Pettis eienskap en enkele verwante (beide swakker en sterker) eienskappe op Banachruimtes wat in die literatuur bespreek is.
2. Deur op die werk in die artikel [8] te fokus, verstaan die rol wat ’n skaal van  $DPP_p$  eienskappe (wat swakker as die  $DPP$  is en wat in [8] ingevoer is deur middel van die swak  $p$ -sommeerbare rye in  $X$  en die swak nulrye in  $X^*$ ) speel in die bestudering van verskillende klasse van operatore.

3. Deur op die werk in die artikel [7] te fokus, bestudeer die  $DP^*P$  vir Banachruimtes, 'n eienskap wat blyk presies dit is wat nodig is om 'n vraag van Pelczyński met betrekking tot wanneer 'n simmetriese bilineêre afsonderlik kompakte afbeelding  $X \times X \rightarrow c_0$  volledig kontinue is, te kan beantwoord.
4. Gebaseer op die idees wat in die werk van die artikel [8] in die bestudering van 'n skaal van  $DP$ -eienskappe en die werk van die artikel [7] i.v.m. die  $DP^*P$  vervleg is, voer die konsep  $DP^*P_p$  op Banachruimtes in en ondersoek die toepassings hiervan in die studie van ruimtes van operatore en die teorie van polinome en analitiese afbeeldings op Banachruimtes. Daardeur word dan hopelik nie alleen die resultate in [7] na 'n groter familie van Banachruimtes uitgebrei nie, maar ook die presiese eienskap ingevoer wat tot 'n antwoord sal lei op die volgende oop vraag: “Wanneer sal elke simmetriese bilineêre afsonderlik kompakte afbeelding  $X \times X \rightarrow c_0$   $p$ -konvergent wees?”

**Sleutel terme:** Banachruimte, swak  $p$ -sommeerbare ry, swak  $p$ -konvergente ry,  $p$ -konvergente operator, volledig kontinue operator, volledig kontinue funksie,  $p$ -konvergente funksie, swak kompakte versameling, swak  $p$ -kompakte versameling, limietversameling, Dunford-Pettis eienskap, Dunford-Pettis eienskap van orde  $p$ ,  $DP^*$ -eienskap,  $DP^*$ -eienskap van orde  $p$ , homogene polinoom, analitiese funksie op 'n Banachruimte.

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# Introduction

In this dissertation we embark on a study of Dunford-Pettis like properties. Our main focus is on the papers [7] and [8], where in the case of [7] a stronger version of the Dunford-Pettis property, called the  $DP^*$ -property, is introduced on Banach spaces. The authors in the paper [7] then discuss several important examples of Banach spaces with this property and prove some characterisations of Banach spaces with the  $DP^*$ -property. One of the easiest ways to describe the difference (or relation) between the Dunford-Pettis property (briefly called the  $DP$ -property) and the  $DP^*$ -property is to use the characterisations “ $X$  has the  $DP$ -property if and only if for each sequence  $(x_n) \subset X$  which converges weakly to 0 and each sequence  $(x_n^*) \subset X^*$  which converges weakly to 0, we have  $x_n^*(x_n) \rightarrow 0$  if  $n \rightarrow \infty$ ” and “ $X$  has the  $DP^*$ -property if and only if for each sequence  $(x_n) \subset X$  which converges weakly to 0 and each sequence  $(x_n^*) \subset X^*$  which converges weak\* to 0, we have  $x_n^*(x_n) \rightarrow 0$  if  $n \rightarrow \infty$ ”. It is then immediately obvious that for reflexive Banach spaces  $X$  (where the weak and weak\* topologies on  $X^*$  coincide) and more general, for Grothendieck spaces (where weak sequential convergence and weak\* sequential convergence on  $X^*$  coincide) the  $DP$ -property and  $DP^*$ -property coincide. In Chapter 2 (section 2.2) we discuss an important characterisation of the  $DP^*$ -property from the paper [7] (cf. [7], Proposition 2.1)) where it is proved that  $X$  has the  $DP^*$ -property if and only if every bounded linear operator  $T \in \mathcal{L}(X, c_0)$  is completely continuous. In particular, it follows that if  $X$  has the  $DP^*$ -property, then it does not contain complemented copies of  $c_0$ . It is also proved in the same section that  $X$  has the  $DP^*$ -property, if and only if,  $X$  has the  $DP$ -property and every quotient mapping  $q : X \rightarrow c_0$  is completely continuous, identifying what exactly the “difference” between the two properties is. This result is also from the paper [7](Theorem 2.2). The authors in [7] introduce and study the  $DP^*$ -property with the application thereof in the study of completely continuous operators,

completely continuous functions and polynomials and analytic functions on Banach spaces in mind. This is for instance illustrated by the following results (which were proved in [7] and are in full detail discussed in Chapter 4 of this dissertation):

- Let  $X$  and  $Y$  be Banach spaces, where  $Y$  contains an isomorphic copy of  $c_0$ . If every  $T \in \mathcal{L}(X, Y)$  is completely continuous, then  $X$  has the  $DP^*$ -property. In this case every continuous  $n$ -homogeneous polynomial  $P \in \mathcal{P}(^n X, Y)$  is a completely continuous function.
- If  $X$  has the  $DP^*$ -property and  $Y$  is a Gelfand-Phillips space (for instance, all separable spaces Banach spaces are Gelfand-Phillips spaces), then every homogeneous polynomial  $P : X \rightarrow Y$  is completely continuous. Further, any analytic function  $f \in \mathcal{H}(X, Y)$  which is bounded on all weakly compact (resp. limited) sets is weakly continuous on them.

It is worth noting that another remarkable achievement of the authors in [7] is to apply their results to answer an open question of Pelczyński's regarding when every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous. In the paper [6] (cf. [6], page 2) it is proved that a separable Banach space is a Schur space if, and only if, every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous. This characterisation of Schur spaces is not in general true for non-separable Banach spaces. In Chapter 4 of this dissertation we discuss the following result from [7]:

- A Banach space  $X$  has the  $DP^*$ -property if, and only if, every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous.

In the paper [8] the class  $C_p(X, Y)$  of  $p$ -convergent operators from a Banach space  $X$  to a Banach space  $Y$  is introduced. We call  $T \in \mathcal{L}(X, Y)$   $p$ -convergent (for  $1 \leq p < \infty$ ) if it transforms weakly  $p$ -summable sequences into norm-null sequences. We may add  $p = \infty$ , if we agree in this case to ask that  $Tx_n \rightarrow 0$  in norm for all  $(x_n)$  which converge weakly to 0. In this case, the  $\infty$ -convergent operators are precisely the completely continuous operators, i.e.  $\mathcal{C}_\infty(X, Y) = \mathcal{V}(X, Y)$ , where  $\mathcal{V}(X, Y)$  denotes the family of all completely continuous operators from  $X$  to  $Y$ . A sequence  $(x_n)$  in a Banach space  $X$  is said to be weakly  $p$ -convergent to an  $x \in X$  if the sequence  $(x_n - x)$  is weakly  $p$ -summable. The weakly  $\infty$ -convergent sequences are simply the weakly convergent sequences, since then we agree that  $(x_n - x)_n$  is a weakly null sequence in  $X$ . Then in [8] a bounded set  $K$  in a Banach space is said to be relatively weakly  $p$ -compact, for  $1 \leq p \leq \infty$ , if every sequence in  $K$  has a weakly- $p$ -convergent subsequence. A scale of "Dunford-Pettis like

properties” is introduced in the paper [8] as follows: A Banach space  $X$  is said to have the  $DP$ -property of order  $p$  (or briefly,  $X$  has the  $DPP_p$ ) for  $1 \leq p \leq \infty$ , if the inclusion  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_p(X, Y)$  holds for all Banach spaces  $Y$ . Here  $\mathcal{W}(X, Y)$  denotes the family of weakly compact operators from  $X$  to  $Y$ . In Chapter 2 (Section 2.1) we discuss the results of the paper [8]. It is for instance proved there that:

- An operator  $T \in \mathcal{L}(X, Y)$  is  $p$ -convergent if and only if it maps relatively weakly  $p$ -compact sets of  $X$  onto relatively norm compact sets in  $Y$  (and this is the same as to say that an operator  $T \in \mathcal{L}(X, Y)$  is  $p$ -convergent if and only if it maps weakly  $p$ -compact sets onto norm compact sets in  $Y$ ).

Using this characterisation of  $p$ -convergent operators we discuss the following characterisation of spaces having  $DPP_p$  (cf. [8], Proposition 3.2):

- $X$  has  $DPP_p$  if and only if for each weakly  $p$ -summable sequence  $(x_n)$  in  $X$  and each weakly null sequence  $(x_n^*)$  in  $X^*$ , we have  $\langle x_n, x_n^* \rangle \xrightarrow{\infty} 0$ , which again is equivalent to “for any Banach space  $Y$ , every weakly compact operator  $T : X \rightarrow Y$  transforms weakly- $p$ -compact sets of  $X$  into norm-compact sets of  $Y$ ”.

Motivated by the discussion in Chapter 2 and Chapter 4 on the work in the two papers [7] and [8], we proceed in Chapter 5 to introduce a scale of (new) alternative Dunford-Pettis properties, which we call  $DP^*$ -properties of order  $p$  and to consider characterisations of Banach spaces with these properties as well as applications thereof to polynomials and holomorphic functions on Banach spaces. A Banach space  $X$  is said to have the  $DP^*$ -property of order  $p$  (for  $1 \leq p \leq \infty$ ) if all weakly- $p$ -compact sets in  $X$  are limited. In short, we say  $X$  has  $DP^*P_p$ . We prove that:

- Let  $1 \leq p < \infty$ . The Banach space  $X$  has  $DP^*P_p$  if and only if  $\langle x_n, x_n^* \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for all weak\* null sequences  $(x_n^*)$  in  $X^*$  and all weakly  $p$ -summable sequences  $(x_n)$  in  $X$ .

Another important characterisation of  $DP^*P_p$  which we prove in Chapter 5 (Section 1) is:

- The Banach space  $X$  has  $DP^*P_p$  if and only if every operator  $T \in \mathcal{L}(X, c_0)$  is  $p$ -convergent, i.e. if and only if  $\mathcal{L}(X, c_0) = C_p(X, c_0)$ .

From the definition of the  $DP^*P_p$  it is clear that as in [8] (where the  $DPP_p$  is introduced), we now have a scale of  $DP^*P$ -like properties, in the sense that all Banach spaces have  $DP^*P_1$  and if  $p < q$  and  $X$  has  $DP^*P_q$  then it has  $DP^*P_p$ . The strongest property  $DP^*P_\infty$  coincides with  $DP^*P$ . Moreover,

if we compare our new property  $DP^*P_p$  with the  $DPP_p$ , then we can also describe exactly what the “difference” between them is. To be precise, we prove in Chapter 5 that:

- A Banach space  $X$  has the  $DP^*P_p$  if and only if it has the  $DPP_p$  and every quotient mapping  $q : X \rightarrow c_0$  is  $p$ -convergent.

In Chapter 5 (Section 2) we consider the applications of our work in the first section to polynomials and analytic functions on Banach spaces. A function  $f$  from a normed space  $X$  into a normed space  $Y$  is called  $p$ -convergent if it maps weakly  $p$ -convergent sequences onto norm convergent sequences. The first important result in Section 2 of Chapter 5 states that:

- Let  $X, Y$  be Banach spaces, where  $Y$  contains an isomorphic copy of  $c_0$ . If every  $T \in \mathcal{L}(X, Y)$  is  $p$ -convergent, then  $X$  has the  $DP^*P_p$ . In this case, every  $n$ -homogeneous polynomial  $P \in \mathcal{P}(^n X, Y)$  is a  $p$ -convergent function for all  $n \in \mathbb{N}$ .

Another application of the  $DP^*P_p$  to polynomials and holomorphic functions which we prove in Chapter 5 is:

- If  $X$  has  $DP^*P_p$  and  $Y$  is a Gelfand-Phillips space, then every  $P \in \mathcal{P}(^n X, Y)$  is  $p$ -convergent. Furthermore, each  $f \in \mathcal{H}(X, Y)$  which is bounded on weakly  $p$ -compact sets, is weakly continuous on them.

The  $DP^*P_p$  also proves to be exactly the right property to find an answer to the question: “When will every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  be  $p$ -convergent”? This we see in the following result:

- A Banach space  $X$  has the  $DP^*P_p$  if, and only if, every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is  $p$ -convergent.

Chapter 1 of this dissertation provides a wide range of concepts and results in Banach spaces and the theory of vector sequence spaces. The results listed (and some proved) in this chapter are important to understand the work in the subsequent chapters of the dissertation. These results are mostly well known, but they are scattered in the literature. Some results in this chapter we could not find in the literature (although most possibly they are known), in which case we then provide a detailed discussion.

Chapter 3 provides the background knowledge of homogeneous polynomials and analytic functions on Banach spaces. This study of “Complex Analysis in Banach spaces” is mostly based on work in the books [23] and [14], but also on the work in some articles such as [15]. Having this theory available, we are able to report on the applications of the  $DP^*P_p$  in the context of homogeneous polynomials and analytic functions on Banach spaces.

# Chapter 1

## Preliminaries

### 1.1 Notation and basic information on Banach spaces

Throughout the dissertation we use  $X, Y, Z, E, F, G$  etc. to denote Banach spaces, unless otherwise specified. The close unit ball of  $X$  is denoted by  $\mathcal{B}_X$  and we agree to use  $X^*$  for the continuous dual space of  $X$ . The weak topology on  $X$  is denoted by  $\sigma(X, X^*)$  and convergence of sequences (or nets) in this topology is called “weak convergence” or “wk-convergence”. Similarly,  $\sigma(X^*, X)$  denotes the weak\* topology on  $X^*$  and convergence in this topology is called “weak\* convergence” or “wk\*-convergence”. We also agree to denote by  $\mathcal{L}(X, Y)$  the space of bounded linear operators from  $X$  to  $Y$ . For an operator  $T$ , we will denote the kernel of  $T$  and the range of  $T$  by  $\ker T$  and  $\mathcal{R}(T)$  respectively.

The Banach spaces of  $p$ -summable scalar sequences (for  $1 \leq p < \infty$ ) is denoted  $\ell_p$  and  $\ell_\infty$  is the space of bounded scalar sequences. The closed subspace of  $\ell_\infty$  consisting of the scalar sequences which converges to 0 in the norm of  $\ell_\infty$  is denoted by  $c_0$ . The unit coordinate vector  $e_n$  in these sequence spaces, is the sequence  $e_n = (\delta_{n,j})_j$ , where  $\delta_{n,j} = 0$  if  $j \neq n$  and  $\delta_{n,n} = 1$ .

**Definition 1.1.1** (cf. [9], page 187) *Let  $X, Y$  be Banach spaces.  $T \in \mathcal{L}(X, Y)$  is called weakly compact if  $\overline{T(\mathcal{B}_X)}$  is weakly compact, i.e compact in  $(Y, \sigma(Y, Y^*))$ .*

Denote by  $\mathcal{K}(X, Y)$  and  $\mathcal{W}(X, Y)$  the linear spaces of compact linear operators and weakly compact operators from  $X$  to  $Y$  respectively. Then

$\mathcal{K}(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$  and clearly  $\mathcal{K}(X, Y) \subseteq \mathcal{W}(X, Y)$ .

**Definition 1.1.2** (cf. [9], Definition V.3.2) A linear operator  $T : X \rightarrow Y$  is said to be completely continuous if for all sequences  $(x_n) \subset X$  such that  $x_n \rightarrow 0$  weakly, the sequence  $(Tx_n)$  in  $Y$  satisfies  $Tx_n \xrightarrow[n \rightarrow \infty]{} 0$  (in norm).

Completely continuous operators are also called Dunford-Pettis operators (DP-operators). Of course,  $T$  is completely continuous if and only if it takes weakly convergent sequences to norm convergent sequences.

Denote by  $\mathcal{V}(X, Y)$  the linear space of completely continuous operators from  $X$  to  $Y$ . Then  $\mathcal{V}(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$ . It was proved in [9] (Proposition 3.3; Chapter VI) that  $\mathcal{K}(X, Y) \subseteq \mathcal{V}(X, Y)$  and that  $\mathcal{K}(X, Y) = \mathcal{V}(X, Y)$  if  $X$  is reflexive.

We recall the well known Dunford-Pettis property (abbreviated by *DPP* in the sequel) in the following definition:

**Definition 1.1.3** (cf. [8], page 50) A Banach space  $X$  is said to have *DPP* if any weakly compact operator  $T : X \rightarrow Y$  transforms weakly compact sets of  $X$  into norm compact sets of  $Y$ .

This is equivalent to say that: A Banach space  $X$  has *DPP* if  $\mathcal{W}(X, Y) \subseteq \mathcal{V}(X, Y)$ .

Recall that a Banach space  $X$  has *DPP* if, and only if, for every weak null sequence in  $X$  and every weak\* null sequence  $(x_n^*)$  in  $X^*$ , the scalar sequence  $(x_n^*(x_n))_{n=1}^{\infty}$  converges to 0 (cf. [1], Theorem 5.4.4 (p.115) for instance).

For example  $c_0$  and  $\ell_1$  has *DPP*, but no infinite-dimensional reflexive Banach space  $X$  has *DPP* (cf. [1], page 115). The well known spaces  $L_1(\mu)$  and  $C(\Omega)$  have the *DPP* (as was proved by Dunford and Pettis).

We recall the concept “limited set” from the literature (cf. for example [7], [11], [20] and [15]):

**Definition 1.1.4** A subset  $L$  of a Banach space  $X$  is called limited if weak\* null sequences in  $X^*$  converge uniformly on  $L$ .

Properties of limited sets and results concerning limited sets are scattered through the literature. We recall some information from [11]:

- (i) Limited sets are bounded.
- (ii) Relatively compact sets are limited.
- (iii) In separable Banach spaces, limited sets are relatively compact (cf. [11], page 116).
- (iv) The set  $\{e_n : n \geq 1\}$  of unit coordinate vectors is limited in  $\ell_\infty$ , but not in  $c_0$  (cf. [11], page 116). The set  $\{e_n : n \in \mathbb{N}\}$  is however a weakly relatively compact set in  $c_0$ , since for each  $x^* = (\xi_i) \in c_0^* = \ell_1$  we have

$$|\langle e_n, x^* \rangle| = |\xi_n| \xrightarrow{\infty} 0,$$

i.e.  $e_n \rightarrow 0$  weakly and so does each subsequence of  $(e_n)$ . The set  $\{e_n : n \in \mathbb{N}\} \cup \{\underline{0}\}$  (where  $\underline{0}$  now denotes the sequence whose terms are all equal to 0) is thus an example of a weakly compact subset of  $c_0$  which is not limited.

- (v) If  $\mathcal{B}_{X^*}$  is weak\* sequentially compact, then limited sets in  $X$  are relatively compact (cf. [11], page 238).

The following are also well known concepts.

**Definition 1.1.5** (1)  $X$  is called a Gelfand-Phillips space if each limited set in  $X$  is relatively compact.

- (2)  $X$  has the Grothendieck property if weak\* convergent sequences in  $X^*$  are weakly convergent.

**Example 1.1.6** From (iii) in the above list of properties of limited sets it is clear that all separable Banach spaces are Gelfand-Phillips spaces. This of course includes the family of all  $\ell_p$  spaces (for  $1 \leq p < \infty$ ) and  $c_0$ . The fact that  $c_0$  is a Gelfand-Phillips space will later in the dissertation be of great importance in the characterisation of different Dunford-Pettis-like properties on Banach spaces.

**Definition 1.1.7** (cf. [7], page 206) A Banach space  $X$  has the Schur property (or  $X$  is a Schur space) if weak and norm sequential convergence coincide in  $X$ , i.e. a sequence  $(x_n)_{n=1}^\infty$  in  $X$  converges to 0 weakly if and only if  $(x_n)_{n=1}^\infty$  converges to 0 in norm.

Recall that  $\ell_1$  is an example of a Schur space (cf. [9], Proposition V.5.2 on page 135). As a matter of fact, any infinite dimensional Banach space with the Schur property contains a subspace isomorphic to  $\ell_1$ . It is remarked in [11] (cf. [11], page 212) that the dual space  $X^*$  of a Banach space  $X$  is a Schur space if and only if  $X$  has the Dunford-Pettis property and does not contain a copy of  $\ell_1$ . If both  $X$  and  $X^*$  have the Schur property, then  $X$  has to be finite dimensional.

Recall that  $\{x_n : n \in \mathbb{N}\}$  is a (Schauder) basis in a Banach space  $X$  if for each  $x \in X$  there is a unique sequence of scalars  $(\alpha_i) \subset \mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$  such that

$$x = \sum_{i=1}^{\infty} \alpha_i x_i,$$

i.e.

$$\|x - \sum_{j=1}^n \alpha_j x_j\| \xrightarrow[n]{\infty} 0.$$

It is well known that  $\{e_n : n \in \mathbb{N}\}$  (where  $e_n$  denotes the  $n^{\text{th}}$  unit coordinate vector in  $\ell_p$ ) is a Schauder basis for  $\ell_p$  (when  $1 \leq p < \infty$ ) and  $c_0$ , sometimes called the “standard basis”. All Banach spaces which have Schauder bases are separable (and hence have the Gelfand-Phillips property).

Two bases  $(x_n)_{n \in \mathbb{N}}$  of  $X$  and  $(y_n)_{n \in \mathbb{N}}$  of  $Y$  are called equivalent provided a series  $\sum_{n=1}^{\infty} a_n x_n$  converges (in  $X$ ) if, and only if,  $\sum_{n=1}^{\infty} a_n y_n$  converges (in  $Y$ ). It follows immediately from the Closed Graph Theorem that  $(x_n)_n$  is equivalent to  $(y_n)_n$  if, and only if, there is an isomorphism  $T$  from  $X$  to  $Y$  for which  $Tx_n = y_n$  for all  $n \in \mathbb{N}$ . For a proof of this fact, the reader is referred to Theorem 1.3.2 on page 10 of [1]. We agree to use  $(x_n)_n \sim (y_n)_n$  to indicate that the two bases are equivalent. We also refer to [1], Corollary 1.3.3 for the following result:

**Theorem 1.1.8** *Let  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  be two bases for the Banach spaces  $X$  and  $Y$  respectively. Then  $(x_n)_n \sim (y_n)_n$  if, and only if, there exists a constant  $C > 0$  such that for all finitely nonzero sequence of scalars  $(a_i)_{i=1}^{\infty}$  we have*

$$C^{-1} \left\| \sum_{i=1}^{\infty} a_i y_i \right\|_Y \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_X \leq C \left\| \sum_{i=1}^{\infty} a_i y_i \right\|_Y.$$

If in this case  $C = 1$  then the bases  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  are said to be isometrically equivalent.

## 1.2 Vector-valued sequence spaces

**Definition 1.2.1** (cf. [12], page 32) We call a sequence  $(x_n)$  in  $X$  a strongly  $p$ -summable sequence (or  $(x_n)$  is a strong  $\ell_p$  sequence) if  $(\|x_n\|) \in \ell_p$ .

We let

$$\ell_p^{strong}(X) = \{(x_n) \in X^\mathbb{N} : (\|x_n\|) \in \ell_p\},$$

normed by

$$\|(x_n)\|_p^{strong} = \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} = \|(\|x_n\|)\|_p.$$

The space  $(\ell_p^{strong}(X), \|\cdot\|_p^{strong})$  is a Banach space (cf. [12]).

For any  $(x_i) \in \ell_p^{strong}(X)$ , let  $(x_i)(\leq N) = (x_1, x_2, \dots, x_N, 0, 0, \dots)$ . Then clearly  $(x_i)(\leq N) \in \ell_p^{strong}(X)$  for all  $N \in \mathbb{N}$  and if  $N \rightarrow \infty$ ,

$$\begin{aligned} \|(x_i) - (x_i)(\leq N)\|_p^{strong} &= \|(0, 0, \dots, x_{N+1}, x_{N+2}, \dots)\|_p^{strong} \\ &= \left( \sum_{n=N+1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} \\ &\rightarrow 0. \end{aligned}$$

Thus  $(x_i) = \lim_{N \rightarrow \infty} (x_i)(\leq N)$  in  $\ell_p^{strong}(X)$ .

If we let  $\Phi(X) = \{(x_i) \in X^\mathbb{N} : \exists \text{ finite index set } \Lambda \text{ such that } x_i = 0 \forall i \notin \Lambda\}$ , then  $\Phi(X) \subseteq \ell_p^{strong}(X)$ . Also, since for each  $(x_i) \in \ell_p^{strong}(X)$  it follows that  $(x_i) = \overline{\lim_{N \rightarrow \infty} (x_i)(\leq N)}$ , where each  $(x_i)(\leq N) \in \Phi(X)$ , it follows that  $\ell_p^{strong}(X) = \overline{\Phi(X)}$  in the norm of  $\ell_p^{strong}(X)$ .

The natural analogue for the weak topology on  $X$  is:

**Definition 1.2.2** (cf. [12], page 32) A sequence  $(x_n)$  in  $X$  is called weakly  $p$ -summable (or it is said to be a weak  $\ell_p$  sequence) if  $(\langle x_n, x^* \rangle) \in \ell_p$  for each  $x^* \in X^*$ .

We let

$$\ell_p^{weak}(X) = \{(x_n) \in X^{\mathbb{N}} : (\langle x_n, x^* \rangle) \in \ell_p, \forall x^* \in X^*\}.$$

This is a normed space with norm:

$$\|(x_n)\|_p^{weak} = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p\right)^{\frac{1}{p}} : x^* \in \mathcal{B}_{X^*}\right\}.$$

The fact that  $\|\cdot\|_p^{weak}$  is well defined, follows by application of the Closed Graph Theorem. To demonstrate the application of the Closed Graph Theorem, we discuss the proof of this fact:

**Proposition 1.2.3** (cf. [12], page 32)

$$\|(x_n)\|_p^{weak} = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p\right)^{\frac{1}{p}} : x^* \in \mathcal{B}_{X^*}\right\}$$

is well-defined, for all  $(x_n) \in \ell_p^{weak}(X)$ .

**Proof** Let  $(x_i) \in \ell_p^{weak}(X)$  and define  $U : X^* \rightarrow \ell_p$  by  $U(x^*) = (\langle x_i, x^* \rangle)_i$ .

Clearly,  $U$  is a linear operator. Denote by  $Gr(U)$  the graph of  $U$ , with

$$Gr(U) = \{(x^*, Ux^*) : x^* \in X^*\} \subseteq X^* \times \ell_p.$$

Let  $(x_0^*, (\xi_i)) \in \overline{Gr(U)}$ . There exists a sequence  $(x_n^*, Ux_n^*) \subset Gr(U)$  such that  $x_n^* \xrightarrow[\infty]{n} x_0^*$  and  $Ux_n^* \xrightarrow[\infty]{n} (\xi_i)$ .

Denote the  $j^{th}$  term of  $Ux_n^*$  by  $Ux_n^*(j)$ . Then we have

$$|Ux_n^*(j) - \xi_j| \leq \left(\sum_{i=1}^{\infty} |Ux_n^*(i) - \xi_i|^p\right)^{\frac{1}{p}} \xrightarrow[\infty]{n} 0.$$

Therefore,  $Ux_n^*(j) \xrightarrow[\infty]{n} \xi_j, \forall j \in \mathbb{N}$ .

However,

$$Ux_n^*(j) = \langle x_j, x_n^* \rangle \xrightarrow[\infty]{n} \langle x_j, x_0^* \rangle.$$

Hence,  $\xi_j = \langle x_j, x_0 \rangle \forall j$ . This implies that  $(\xi_j) = (\langle x_j, x_0^* \rangle) = Ux_0^*$ . Thus  $(x_0^*, (\xi_j)) = (x_0^*, Ux_0^*) \in Gr(U)$ , showing that  $Gr(U) = \overline{Gr(U)}$  and hence that  $U$  has a closed graph.

The Closed Graph Theorem says that  $U$  is bounded, i.e.  $\|U\| < \infty$  and since

$$\|U\| = \sup_{\|x^*\| \leq 1} \|Ux^*\| = \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}},$$

the result follows.  $\square$

If  $X$  is a Banach space, then  $(\ell_p^{weak}(X), \|\cdot\|_p^{weak})$  is a Banach space (cf. [12], page 32). This also follows from the following well known result, which is also discussed in [12].

**Theorem 1.2.4** (cf. [12], page 36) *Let  $X$  be a Banach space and  $1 < p < \infty$ . Then  $\ell_p^{weak}(X)$  is isometrically isomorphic to  $\mathcal{L}(\ell_q, X)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $p = 1$ , then  $\mathcal{L}(c_0, X) \cong \ell_1^{weak}(X)$ .*

**Proof** Let  $(x_i) \in \ell_p^{weak}(X)$ . For each  $(\alpha_i) \in \ell_q$  we have

$$\begin{aligned} \left\| \sum_{i=m+1}^n \alpha_i x_i \right\| &= \sup_{\|x^*\| \leq 1} \left| \left\langle \sum_{i=m+1}^n \alpha_i x_i, x^* \right\rangle \right| \\ &= \sup_{\|x^*\| \leq 1} \left| \sum_{i=m+1}^n \alpha_i \langle x_i, x^* \rangle \right| \\ &\leq \left( \sum_{i=m+1}^n |\alpha_i|^q \right)^{\frac{1}{q}} \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p \right)^{\frac{1}{p}} \\ &\xrightarrow[m, n \rightarrow \infty]{} 0. \end{aligned}$$

Thus  $(\sum_{i=1}^n \alpha_i x_i)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ ; therefore the series  $\sum_{i=1}^{\infty} \alpha_i x_i$  converges in  $X$  for all  $(\alpha_i) \in \ell_q$ .

If we define  $T_{(x_i)} : \ell_q \rightarrow X$  by

$$T_{(x_i)}((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i x_i,$$

then  $T_{(x_i)}$  is linear and  $\|T_{(x_i)}((\alpha_i))\| \leq \|(\alpha_i)\|_q \| (x_i) \|_p^{weak}$ . Therefore,  $T_{(x_i)}$  is bounded and  $\|T_{(x_i)}\| \leq \| (x_i) \|_p^{weak}$ .

Conversely, suppose  $T \in \mathcal{L}(\ell_q, X)$  and put  $x_n = Te_n$ , with  $e_n = (0, \dots, 1, 0, \dots) \in \ell_q$ . Let  $(\alpha_i) \in \ell_q$ . Then for each  $x^* \in X^*$  we have

$$\begin{aligned}
\left| \sum_{i=m+1}^n \alpha_i \langle x_i, x^* \rangle \right| &= \left| \langle T \left( \sum_{i=m+1}^n \alpha_i e_i \right), x^* \rangle \right| \\
&\leq \|T \left( \sum_{i=m+1}^n \alpha_i e_i \right)\| \|x^*\| \\
&\leq \|T\| \left\| \sum_{i=m+1}^n \alpha_i e_i \right\|_q \|x^*\| \\
&\xrightarrow[\infty]{m,n} 0.
\end{aligned}$$

Thus  $\sum_{i=1}^{\infty} \alpha_i \langle x_i, x^* \rangle$  converges for all  $x^* \in X^*$  and for all  $(\alpha_i) \in \ell_q$ .

Since for each  $(\alpha_i) \in \ell_q$ , also the sequence  $(\beta_i)$  with  $\beta_i = |\alpha_i| \operatorname{sgn}(\langle x_i, x^* \rangle)$  belongs to  $\ell_q$ , we therefore have that  $\sum_{i=1}^{\infty} |\alpha_i| |\langle x_i, x^* \rangle| = \sum_{i=1}^{\infty} \beta_i \langle x_i, x^* \rangle < \infty$  for all  $(\alpha_i) \in \ell_q$ . This shows that  $(\langle x_i, x^* \rangle) \in \ell_p$  for all  $x^* \in X^*$ , i.e.  $(x_i) \in \ell_p^{\text{weak}}(X)$  and

$$\begin{aligned}
\|(\langle x_i, x^* \rangle)\|_p &= \sup_{\|(\alpha_i)\|_q \leq 1} \left| \sum_{i=1}^{\infty} \alpha_i \langle x_i, x^* \rangle \right| \\
&\leq \sup_{\|(\alpha_i)\|_q \leq 1} \|T \left( \sum_{i=1}^{\infty} \alpha_i e_i \right)\| \|x^*\|.
\end{aligned}$$

Therefore,  $\|(x_i)\|_p^{\text{weak}} \leq \|T\|$ . Note that:

$$T((\alpha_i)) = T\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{\infty} \alpha_i x_i.$$

Therefore,  $T = T_{(x_i)}$ .

In summary:  $\|T_{(x_i)}\| = \|(x_i)\|_p^{\text{weak}}$ .

We have thus proved that  $(x_i) \mapsto T_{(x_i)}$  defines an isometric isomorphism between  $\ell_p^{\text{weak}}(X)$  and  $\mathcal{L}(\ell_q, X)$ .  $\square$

If  $p = \infty$ , then  $\ell_{\infty}^{\text{weak}}(X) = \ell_{\infty}^{\text{strong}}(X)$  and  $T_{(x_i)} : \ell_1 \rightarrow X : (\xi_i) \mapsto \sum_i \xi_i x_i$  is a bounded linear operator for all  $(x_i) \in \ell_{\infty}^{\text{weak}}(X)$ . Clearly for all  $n \in \mathbb{N}$ ,

$$\begin{aligned}
\|T_{(x_i)}\| &= \sup_{\|(\xi_i)\|_1 \leq 1} \left\| \sum_i \xi_i x_i \right\| \\
&\leq \sup_i \|x_i\| \\
&= \|(x_i)\|_{\ell_\infty^{strong}(X)} \\
&= \|(x_i)\|_{\ell_\infty^{weak}(X)}.
\end{aligned}$$

Also,  $\|T_{(x_i)}\| \geq \|T_{(x_i)}e_n\| = \|x_n\|$  for all  $n \in \mathbb{N}$  and thus  $\|T_{(x_i)}\| \geq \sup_i \|x_i\|$ . For any  $T \in \mathcal{L}(\ell_1, X)$ , the sequence  $(x_n)$ , with  $x_n = Te_n$  for all  $n \in \mathbb{N}$ , is in  $\ell_\infty^{strong}(X)$ . Clearly,  $T_{(x_i)} = T$ . Therefore,  $\mathcal{L}(\ell_1, X) \cong \ell_\infty^{strong}(X) (= \ell_\infty^{weak}(X))$ .

Now  $c_0^{weak}(X)$  is a (strict) closed subspace of  $\ell_\infty^{weak}(X) = \mathcal{L}(\ell_1, X)$ . It is therefore natural to ask which operators are defined by elements of  $c_0^{weak}(X)$ . Since we do not know the answer to this question from the literature, we'll now attempt to provide an answer in the following discussion:

Let  $(x_n) \in c_0^{weak}(X)$ . Then as before  $T_{(x_n)} \in \mathcal{L}(\ell_1, X)$ , with

$$\|(x_n)\|_{c_0^{weak}(X)} = \|(x_n)\|_{\ell_\infty^{strong}(X)} = \|T_{(x_n)}\|.$$

Consider the set

$$A := T_{(x_n)}(\mathcal{B}_{\ell_1}) = \left\{ \sum_{i=1}^{\infty} \lambda_i x_i : \sum_{i=1}^{\infty} |\lambda_i| \leq 1 \right\}.$$

We show that  $A$  is weakly compact in  $X$ :

Take any sequence  $(\sum_{i=1}^{\infty} \lambda_{i,n} x_i)_n \subset A$ . Then  $((\lambda_{i,n})_i)_n \subset \mathcal{B}_{\ell_1}$ . The set  $\mathcal{B}_{\ell_1}$  is  $\sigma(\ell_1, c_0)$  compact. Since  $c_0$  is separable, it follows from [9], Theorem V.5.1 that  $(\mathcal{B}_{\ell_1}, \sigma(\ell_1, c_0))$  is metrizable. Thus it is sequentially  $\sigma(\ell_1, c_0)$ -compact. Therefore, there is a subsequence  $((\lambda_{i,n_k})_i)_k \subseteq \mathcal{B}_{\ell_1}$  such that  $((\lambda_{i,n_k})_i)_k \xrightarrow{k \rightarrow \infty} (\lambda_i) \in \mathcal{B}_{\ell_1}$  in the  $\sigma(\ell_1, c_0)$ -topology. Consider the subsequence  $(\sum_{i=1}^{\infty} \lambda_{i,n_k} x_i)_k$  of  $(\sum_{i=1}^{\infty} \lambda_{i,n} x_i)_n$ . For each  $x^* \in X^*$  we have  $(\langle x_i, x^* \rangle) \in c_0$  and therefore

$$\sum_{i=1}^{\infty} \lambda_{i,n_k} \langle x_i, x^* \rangle \xrightarrow[k \rightarrow \infty]{k} \sum_{i=1}^{\infty} \lambda_i \langle x_i, x^* \rangle,$$

that is

$$\left\langle \sum_{i=1}^{\infty} \lambda_{i,n_k} x_i, x^* \right\rangle \xrightarrow{\infty} \left\langle \sum_{i=1}^{\infty} \lambda_i x_i, x^* \right\rangle$$

for all  $x^* \in X^*$ . This shows that  $A$  is weakly sequentially compact, i.e. weakly compact, by the Eberlein-Smulion Theorem (cf. [9], Theorem V.13.1 or Theorem 1.3.4 below).

The operator  $T_{(x_i)}$  is therefore a weakly compact operator. So we have in conclusion that

**Proposition 1.2.5** *With each  $(x_i) \in c_0^{weak}(X)$  we associate a weakly compact operator  $T_{(x_i)} : \ell_1 \rightarrow X$ , where  $\|T_{(x_i)}\| = \|(x_n)\|$  and  $T_{(x_n)}((\xi_i)) = \sum_{i=1}^{\infty} \xi_i x_i$ .*

**Remark 1.2.6** (a) *If  $X$  is a Banach space, then  $\mathcal{L}(\ell_q, X)$  is a Banach space, thus  $\ell_p^{weak}(X)$  is a Banach space.*

(b)  $\ell_p^{strong}(X) \subsetneq \ell_p^{weak}(X)$  if  $\dim X = \infty$ .

The following definition is well known. See for instance [18], Section 19.4.

**Definition 1.2.7** *A sequence  $(x_n^*)$  in  $X^*$  is called weak\*- $p$ -summable (or it is said to be a weak\*  $\ell_p$  sequence) if  $(\langle x, x_n^* \rangle) \in \ell_p$  for each  $x \in X$ .*

We let

$$\ell_p^{weak^*}(X^*) = \{(x_n^*) \in (X^*)^{\mathbb{N}} : (\langle x, x_n^* \rangle) \in \ell_p, \forall x \in X\}.$$

This is a normed space with norm:

$$\|(x_n^*)\|_p^{weak^*} = \sup \left\{ \left( \sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{\frac{1}{p}} : x \in \mathcal{B}_X \right\}.$$

It is moreover also complete, as is clear from the following result:

**Theorem 1.2.8** (cf. [18], Proposition 19.4.3) *Let  $X$  be a Banach space and  $1 \leq p \leq \infty$ . Then :*

(a)  $\ell_p^{weak^*}(X^*)$  is isometrically isomorphic to  $\mathcal{L}(X, \ell_p)$ .

(b)  $c_0^{weak^*}(X^*)$  is isometrically isomorphic to  $\mathcal{L}(X, c_0)$ .

**Proof** (a) Let  $(x_i^*) \in \ell_p^{weak^*}(X^*)$  and define  $T_{(x_i^*)} : X \rightarrow \ell_p$  by

$$T_{(x_i^*)}(x) = (\langle x, x_i^* \rangle).$$

Then  $T_{(x_i^*)}$  is a linear operator and for all  $x \in X$ ,

$$\|T_{(x_i^*)}(x)\|_p = \|(\langle x, x_i^* \rangle)\|_p \leq \|(x_i^*)\|_p^{weak^*} \|x\|,$$

with  $\|(x_i^*)\|_p^{weak^*} = \sup_{\|x\| \leq 1} (\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p)^{\frac{1}{p}}$ .

Therefore,  $T_{(x_i^*)}$  is bounded and  $\|T_{(x_i^*)}\| \leq \|(x_i^*)\|_p^{weak^*}$ .

Let  $J : \ell_p^{weak^*}(X^*) \rightarrow \mathcal{L}(X, \ell_p)$  such that  $J((x_i^*)) = T_{(x_i^*)}$ . It follows that  $J$  is linear and furthermore that

$$\|J((x_i^*))\| = \|T_{(x_i^*)}\| \leq \|(x_i^*)\|_p^{weak^*},$$

showing that  $J$  is bounded and  $\|J\| \leq 1$ .

Let  $T \in \mathcal{L}(X, \ell_p)$  and  $\pi_i : \ell_p \rightarrow \mathbb{K} : (\xi_i) \mapsto \xi_i$  be the (continuous) linear  $i^{th}$  coordinate projection of  $\ell_p$ . For each  $i$ , set  $x_i^* = \pi_i \circ T$  and thus  $x_i^* \in X^*$ . For each  $x \in X$ , we have that  $\langle x, x_i^* \rangle = \pi_i(Tx)$  and thus

$$(\langle x, x_i^* \rangle)_i = (\pi_i(Tx))_i \in \ell_p.$$

For all  $x \in X$ ,

$$\begin{aligned} \left(\sum_{i=1}^{\infty} |\langle x, x_i^* \rangle|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{\infty} |\pi_i(Tx)|^p\right)^{\frac{1}{p}} \\ &= \|Tx\|_p \\ &\leq \|T\| \|x\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|(x_i^*)\|_p^{weak^*} &= \sup_{\|x\| \leq 1} \left(\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p\right)^{\frac{1}{p}} \\ &\leq \|T\|. \end{aligned}$$

For all  $x \in X$ , we have that

$$Tx = (\langle x, x_i^* \rangle) = T_{(x_i^*)}(x).$$

We obtain  $J((x_i^*)) = T$  and

$$\begin{aligned} \|(x_i^*)\|_p^{weak^*} &\leq \|T\| \\ &= \|J((x_i^*))\|. \end{aligned}$$

Therefore,  $J : \ell_p^{weak^*}(X^*) \rightarrow \mathcal{L}(X, \ell_p)$  is an isometry of  $\ell_p^{weak^*}(X^*)$  onto  $\mathcal{L}(X, \ell_p)$ .

(b) Use the arguments in the proof of (a) to prove the second part by replacing  $\ell_p$  with  $c_0$ .  $\square$

It is clear that  $\ell_p^{weak}(X^*) \subseteq \ell_p^{weak^*}(X^*)$ . In [17] (page 427) it is mentioned that the converse inclusion is also true (isometrically) for  $1 \leq p \leq \infty$ . For the sake of completeness we now show how this follows from Theorem 1.2.4 and Theorem 1.2.8:

**Lemma 1.2.9** *Let  $1 \leq p \leq \infty$ . Then  $\ell_p^{weak^*}(X^*) = \ell_p^{weak}(X^*)$ .*

**Proof** We only need to show that  $\ell_p^{weak^*}(X^*) \subseteq \ell_p^{weak}(X^*)$ . Recall from Theorem 1.2.4 (for  $1 < p < \infty$ ) and the discussion following Theorem 1.2.4 (for  $p = \infty$ ) that

$$\ell_p^{weak}(X^*) = \mathcal{L}(\ell_q, X^*) : (x_i^*) \mapsto T_{(x_i^*)}, \text{ with } T_{(x_i^*)}((\xi_i)) = \sum_{i=1}^{\infty} \xi_i x_i^*,$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Recall from Theorem 1.2.8 that

$$\ell_p^{weak^*}(X^*) = \mathcal{L}(X, \ell_p) : (x_i^*) \mapsto S_{(x_i^*)}, \text{ with } S_{(x_i^*)}(x) = (\langle x_i^*, x \rangle)$$

for  $1 \leq p \leq \infty$ . From Theorem 1.2.4 we have

$$\ell_1^{weak}(X^*) = \mathcal{L}(c_0, X^*)$$

and from Theorem 1.2.8 we also have

$$c_0^{weak^*}(X^*) = \mathcal{L}(X, c_0).$$

Now, let  $(x_i^*) \in \ell_p^{weak^*}(X^*)$ . For each  $(\xi_i) \in \ell_q$  and each  $x \in X$  we have  $\sum_{i=1}^{\infty} |\xi_i| |\langle x_i^*, x \rangle| < \infty$ , i.e. the sequence  $\left( \sum_{i=1}^n \xi_i x_i^* \right)_n$  converges in the strong

operator topology to  $\sum_{i=1}^{\infty} \xi_i x_i^*$  in  $X^*$ , i.e the operator

$$T : \ell_q \rightarrow X^* : (\xi_i) \mapsto \sum_{i=1}^{\infty} \xi_i x_i^*$$

is well defined. Also,

$$\|T((\xi_i))\| \leq \sup_{\|x\| \leq 1} \sum_{i=1}^{\infty} |\xi_i| |\langle x_i^*, x \rangle| \leq \|(\xi_i)\|_{\ell_q} \| (x_i^*) \|_p^{weak*}.$$

We thus have  $T \in L(\ell_q, X^*)$ . Therefore, for  $1 < p \leq \infty$ , there exists a unique  $(y_i^*) \in \ell_p^{weak}(X^*)$  such that  $T = T_{(y_i^*)}$  as above. Clearly, since  $T(e_i) = x_i^*$  and  $T_{(y_i^*)}(e_i) = y_i^*$  for all  $i$ , it follows that  $(x_i^*) \in \ell_p^{weak}(X^*)$ . In case of  $p = 1$ , it follows from  $(x_i^*) \in \ell_1^{weak*}(X^*)$ , that  $S_{(x_i^*)} \in \mathcal{L}(X, \ell_1)$ . Thus,  $S_{(x_i^*)}^*$  restricted to  $c_0$  is in  $\mathcal{L}(c_0, X^*)$  and maps  $(\xi_i) \in c_0$  onto  $\sum_{i=1}^{\infty} \xi_i x_i^*$ . As before, since the identification of each operator in  $\mathcal{L}(c_0, X^*)$  with a sequence in  $\ell_1^{weak}(X^*)$  in this way is unique, it follows that  $(x_i^*) \in \ell_1^{weak}(X^*)$ . We have thus proved the equality  $\ell_p^{weak*}(X^*) = \ell_p^{weak}(X^*)$  for all  $1 \leq p \leq \infty$ .  $\square$

### 1.3 Results on weakly compact operators

We refer to Definition 1.1.1 for the notion of “weakly compact operator”.

**Proposition 1.3.1** (cf. [9], Proposition VI.5.1)

- (a) *If either  $X$  or  $Y$  is reflexive, then every  $T \in \mathcal{L}(X, Y)$  is weakly compact.*
- (b) *If  $T : X \rightarrow Y$  is weakly compact and  $A \in \mathcal{L}(Y, Z)$ , then  $A \circ T$  is weakly compact.*
- (c) *If  $T : X \rightarrow Y$  is weakly compact and  $B \in \mathcal{L}(Z, X)$ , then  $T \circ B$  is weakly compact.*

**Proof** (a) Suppose  $X$  is reflexive. Then  $\mathcal{B}_X$  is weakly compact (cf. [9], Theorem V.4.2). If  $T \in \mathcal{L}(X, Y)$ , then  $T$  is weakly continuous. Thus  $T(\mathcal{B}_X)$  is weakly compact in  $Y$ . If, on the other hand,  $Y$  is reflexive, then  $\mathcal{B}_Y$  is weakly compact. Also, since  $T(\mathcal{B}_X) \subset \lambda \mathcal{B}_Y$  for some  $\lambda > 0$  and since

$T(\mathcal{B}_X)$  is convex, we have (by [9], Theorem V.1.4)  $\overline{T(\mathcal{B}_X)}^{wk} = \overline{T(\mathcal{B}_X)} \subseteq \lambda\mathcal{B}_Y$ , showing that  $T(\mathcal{B}_X)$  is weakly relatively compact in  $Y$ .

(b) Let  $T : X \rightarrow Y$  be weakly compact and  $A \in \mathcal{L}(Y, Z)$ . Then  $\overline{T(\mathcal{B}_X)}$  is weakly compact. Since  $A$  is weakly continuous,  $A(\overline{T(\mathcal{B}_X)})$  is weakly compact. Being convex, it is also norm closed. Therefore  $\overline{(A \circ T)(\mathcal{B}_X)} \subseteq A(\overline{T(\mathcal{B}_X)})$  is clear. Conversely, let  $Ay \in A(\overline{T(\mathcal{B}_X)})$ , i.e. there exists  $(x_n) \subset \mathcal{B}_X$  such that  $y = \lim_n Tx_n$ . Then,

$$Ay = \lim_{n \rightarrow \infty} A(Tx_n) \in \overline{A(\overline{T(\mathcal{B}_X)})}.$$

This proves that  $\overline{(A \circ T)(\mathcal{B}_X)} = A(\overline{T(\mathcal{B}_X)})$ . Hence,  $A \circ T$  is weakly compact.

(c) Let  $B \in \mathcal{L}(Z, X)$  and  $T : X \rightarrow Y$  weakly compact. Then  $B(\mathcal{B}_Z) \subseteq \lambda\mathcal{B}_X$  for some  $\lambda > 0$ . Thus  $(TB)(\mathcal{B}_Z) \subseteq \lambda T(\mathcal{B}_X)$  and  $\overline{\lambda T(\mathcal{B}_X)}$  is weakly compact. Therefore,

$$\overline{(TB)(\mathcal{B}_Z)}^{wk} = \overline{(TB)(\mathcal{B}_Z)} \subseteq \overline{\lambda T(\mathcal{B}_X)}.$$

Thus  $\overline{TB(\mathcal{B}_Z)}$  is weakly compact. □

We refer the reader to [9] (Theorem VI.5.4) for a proof of the following characterization of weakly compact bounded linear operators:

**Theorem 1.3.2** *Let  $T \in \mathcal{L}(X, Y)$ . Then  $T$  is weakly compact if and only if there exists a reflexive Banach space  $R$  and operators  $A \in \mathcal{L}(R, Y)$ ,  $B \in \mathcal{L}(X, R)$  such that  $T = AB$ .*

One of the immediate corollaries of this theorem is the following:

**Theorem 1.3.3** *(cf. [9], Theorem VI.5.5) If  $X, Y$  are Banach spaces and  $T \in \mathcal{L}(X, Y)$ , then the following statements are equivalent:*

- (a)  $T$  is weakly compact.
- (b)  $T^{**}(X^{**}) \subseteq Y$ .
- (c)  $T^*$  is weakly compact.

Several proofs of the following important result exist in the literature. We refrain from considering a proof, but refer the reader to [9] for the following formulation of the Eberlein-Smulion Theorem:

**Theorem 1.3.4** (*Eberlein-Smulion Theorem*)(cf. [9], Theorem V.13.1) *If  $X$  is a Banach space and  $A \subseteq X$ , then the following statements are equivalent:*

- (a) *Each sequence in  $A$  has a weakly convergent subsequence.*
- (b) *Each sequence in  $A$  has a weak cluster point.*
- (c) *The weak closure of  $A$  is weakly compact.*

**Corollary 1.3.5** (cf. [9], page 163) *If  $X$  is a Banach space and  $A \subseteq X$ , then  $A$  is weakly compact if and only if  $A \cap M$  is weakly compact for every separable subspace  $M$  of  $X$ .*

We state without proof another important property of weakly compact sets.

**Theorem 1.3.6** (*Krein-Smulian Theorem*)(cf. [9], Theorem V.13.4) *In any Banach space, the closed convex hulls of weakly compact sets are weakly compact.*

The Eberlein-Smulion Theorem (Theorem 1.3.4) which characterises weak relative compactness as weak relative sequential compactness, plays an important role in our work. We discuss the proof of the fact that  $\mathcal{W}(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$ , thereby illustrating how Theorem 1.3.4 comes into play.

**Proposition 1.3.7**  *$\mathcal{W}(X, Y)$  is a closed linear subspace of  $\mathcal{L}(X, Y)$ .*

**Proof** Let  $T \in \overline{\mathcal{W}(X, Y)}$ . There exists a sequence  $(T_n) \subset \mathcal{W}(X, Y)$  such that  $T_n \rightarrow T$  in the norm of  $\mathcal{L}(X, Y)$  (also known as the uniform norm). We show that  $T \in \mathcal{W}(X, Y)$ .

Consider any sequence  $(Tx_n) \subset T(\mathcal{B}_X)$ . For  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\|T - T_n\| < \frac{\varepsilon}{2}, \quad \forall n \geq N, \text{ i.e.}$$

$$\|Tx - T_n x\| < \frac{\varepsilon}{2}, \quad \forall n \geq N; \quad \forall x \in \mathcal{B}_X.$$

In particular,  $\|Tx - T_N x\| < \frac{\varepsilon}{2}, \quad \forall x \in \mathcal{B}_X$ .

Since  $T_N$  is weakly compact, then according to Theorem 1.3.4, the sequence  $(T_N x_n)_n$  has a weakly convergent subsequence in  $Y$ , i.e. there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $T_N x_{n_k} \xrightarrow[k]{\infty} y \in Y$  weakly.

Let  $x^* \in \mathcal{B}_{Y^*}$ . There exists  $k_0 \in \mathbb{N}$  such that

$$|\langle y - T_N x_{n_k}, x^* \rangle| < \frac{\varepsilon}{2}, \quad \forall k \geq k_0 \quad (k_0 = k_0(x^*)).$$

Therefore,  $\forall k \geq k_0$

$$\begin{aligned} |\langle y - T x_{n_k}, x^* \rangle| &\leq |\langle y - T_N x_{n_k}, x^* \rangle| + |\langle T_N x_{n_k} - T x_{n_k}, x^* \rangle| \\ &\leq |\langle y - T_N x_{n_k}, x^* \rangle| + \|T_N x_{n_k} - T x_{n_k}\| \|x^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This shows that  $\langle y - T x_{n_k}, x^* \rangle \xrightarrow[k]{\infty} 0, \quad \forall x^* \in Y^*$ ; i.e.  $T x_{n_k} \xrightarrow[k]{\infty} y$  weakly.

Thus  $T$  is weakly compact, and  $\overline{\mathcal{W}(X, Y)}^{\|\cdot\|} = \mathcal{W}(X, Y)$ .  $\square$

**Remark 1.3.8** In Theorem 1.2.8 we discussed the isometry of the Banach spaces  $c_0^{weak^*}(X^*)$  and  $\mathcal{L}(X, c_0)$ , where with each  $(x_i^*) \in c_0^{weak^*}(X^*)$  we associate uniquely the operator  $T_{(x_i^*)} : X \rightarrow c_0 : x \mapsto (\langle x, x_i^* \rangle)$ , such that  $\|(x_i^*)\|_{weak^*} = \|T_{(x_i^*)}\|$ . Now  $c_0^{weak}(X^*)$  is a subspace of  $c_0^{weak^*}(X^*)$  with the same norm. In the following proposition we discuss the subspace of  $\mathcal{L}(X, c_0)$  generated by the sequences in  $c_0^{weak}(X^*)$ .

Before discussing the identification of  $c_0^{weak}(X^*)$  as a subspace of  $\mathcal{L}(X, c_0)$ , we recall the well known Goldstine's Theorem:

**Theorem 1.3.9** (Goldstine's Theorem)(cf. [1], page 345) Let  $X$  be a normed space. Then  $\mathcal{B}_X$  is weak\* (i.e.  $\sigma(X^{**}, X^*)$ ) dense in  $\mathcal{B}_{X^{**}}$ .

Using Goldstine's Theorem, we obtain

**Lemma 1.3.10** For  $(x_i^*) \in c_0^{weak}(X^*) \subseteq c^{weak^*}(X^*)$  we have

$$\|(x_i^*)\|_{weak^*} = \|(x_i^*)\|_{weak}.$$

**Proof** Let  $(x_i^*) \in c_0^{weak}(X^*)$ . For  $\varepsilon > 0$ , let  $x_\varepsilon^{**} \in \mathcal{B}_{X^{**}}$  such that

$$\begin{aligned} \|(x_i^*)\|_{weak} &< \|(\langle x_i^*, x_\varepsilon^{**} \rangle)_i\|_{c_0} + \frac{\varepsilon}{4} \\ &= \sup_i |\langle x_i^*, x_\varepsilon^{**} \rangle| + \frac{\varepsilon}{4}. \end{aligned}$$

Then we let  $j \in \mathbb{N}$  such that  $\sup_i |\langle x_i^*, x_\varepsilon^{**} \rangle| < |\langle x_j^*, x_\varepsilon^{**} \rangle| + \frac{\varepsilon}{4}$ . Using Goldstine's Theorem we then let  $x_\varepsilon \in \mathcal{B}_X$  such that  $|\langle x_j^*, x_\varepsilon^{**} \rangle| < |\langle x_\varepsilon, x_j^* \rangle| + \frac{\varepsilon}{4}$ . Then we have

$$\begin{aligned} \|(x_i^*)\|_{weak} &< |\langle x_\varepsilon, x_j^* \rangle| + \varepsilon \\ &\leq \sup_i |\langle x_\varepsilon, x_i^* \rangle| + \varepsilon \\ &\leq \|(x_i^*)\|_{weak^*} + \varepsilon. \end{aligned}$$

We conclude that  $\|(x_i^*)\|_{weak} \leq \|(x_i^*)\|_{weak^*}$ .

On the other hand,

$$\begin{aligned} \|(x_i^*)\|_{weak^*} &= \sup_{x \in \mathcal{B}_X} \|(\langle x, x_i^* \rangle)_i\| \\ &\leq \sup_{x^{**} \in \mathcal{B}_{X^{**}}} \|(\langle x_i^*, x^{**} \rangle)_i\|_{c_0} \\ &= \|(x_i^*)\|_{weak}. \end{aligned}$$

This completes our proof.  $\square$

**Proposition 1.3.11** *Let  $X$  be a Banach space. Then  $c_0^{weak}(X^*) = \mathcal{W}(X, c_0)$ , where the isometry is defined by  $(x_i^*) \leftrightarrow T_{(x_i^*)}$  as in Remark 1.3.8.*

**Proof** Let  $(x_i^*) \in c_0^{weak}(X^*)$ . Then  $T_{(x_i^*)} : X \rightarrow c_0 : x \mapsto (\langle x, x_i^* \rangle)$  is a bounded linear operator with  $\|T_{(x_i^*)}\| = \|(x_i^*)\|_{weak^*} = \|(x_i^*)\|_{weak}$ , by Lemma 1.3.10. We prove that  $T_{(x_i^*)}^{**}(X^{**}) \subseteq c_0$ :

Let  $x^{**} \in X^{**}$ . For  $(\xi_i) \in \ell_1$ , we observe that

$$\begin{aligned} \langle x, T_{(x_i^*)}^{**}((\xi_i)) \rangle &= \langle T_{(x_i^*)}x, (\xi_i) \rangle \\ &= \langle (\langle x, x_i^* \rangle)_i, (\xi_i)_i \rangle \\ &= \sum_{i=1}^{\infty} \xi_i \langle x, x_i^* \rangle \\ &= \left( \sum_{i=1}^{\infty} \xi_i x_i^* \right) (x) \text{ for all } x \in X. \end{aligned}$$

Therefore,  $T_{(x_i^*)}^*((\xi_i)) = \sum_{i=1}^{\infty} \xi_i x_i^*$ .

Now let  $(\gamma_i) = T_{(x_i^*)}^{**}(x^{**})$  for some  $x^{**} \in X^{**}$ . Then we have  $(\gamma_i) \in \ell_{\infty}$  and  $\gamma_i = T_{(x_i^*)}^{**}(x^{**})(e_i)$  for all  $i \in \mathbb{N}$  (cf. [21], page 121). Thus

$$\begin{aligned} \gamma_i &= \langle e_i, T_{(x_i^*)}^{**}x^{**} \rangle \\ &= \langle T_{(x_i^*)}^*e_i, x^{**} \rangle \\ &= \langle x_i^*, x^{**} \rangle. \end{aligned}$$

Since  $(\langle x_i^*, x^{**} \rangle) \in c_0$  by assumption, we have  $(\gamma_i) \in c_0$ . This shows that  $T_{(x_i^*)}^{**}(X^{**}) \subseteq c_0$ . By Theorem 1.3.3,  $T_{(x_i^*)}$  is weakly compact.

Conversely, suppose that  $T : X \rightarrow c_0$  is weakly compact. Since  $T$  is bounded, there exists a unique  $(x_i^*) \in c_0^{weak^*}(X^*)$  such that  $T = T_{(x_i^*)}$ . We show that  $(x_i^*) \in c_0^{weak}(X^*)$ :

To see this, recall that since  $T_{(x_i^*)} (= T)$  is weakly compact, we have

$$T_{(x_i^*)}^{**}(X^{**}) \subseteq c_0.$$

Also, if  $x^{**} \in X^{**}$  is given, then for all  $(\xi_i) \in \ell_1$  we have

$$\begin{aligned} \langle T_{(x_i^*)}^{**}x^{**}, (\xi_i) \rangle &= \langle x^{**}, T_{(x_i^*)}^*((\xi_i)) \rangle \\ &= \langle x^{**}, \sum_{i=1}^{\infty} \xi_i x_i^* \rangle \\ &= \sum_{i=1}^{\infty} \xi_i \langle x_i^*, x^{**} \rangle. \end{aligned}$$

Since  $T_{(x_i^*)}^{**}x^{**} \in c_0$ , it follows that both  $(\langle x_i^*, x^{**} \rangle) \in \ell_{\infty}$  and  $T_{(x_i^*)}^{**}x^{**} = (\langle x_i^*, x^{**} \rangle)_i$ , i.e.  $(\langle x_i^*, x^{**} \rangle)_i \in c_0$ . Since this holds for all  $x^{**} \in X^{**}$ , it follows that  $(x_i^*) \in c_0^{weak}(X^*)$ .  $\square$

## 1.4 Results on completely continuous operators

Refer to Definition 1.1.2 for the concept of ‘‘completely continuous operator’’. The following fact is well known (cf. [18], page 115).

**Theorem 1.4.1**  $T \in \mathcal{L}(X, Y)$  is completely continuous if and only if  $T$  maps weakly compact sets into norm compact sets.

**Proof** Suppose  $T : X \rightarrow Y$  is completely continuous and let  $A \subset X$  be weakly compact. Consider any sequence  $(Tx_n) \subseteq T(A)$ . By Theorem 1.3.4, there exists a subsequence  $(x_{n_k}) \subset A$  such that  $x_{n_k} \xrightarrow[k]{\infty} x \in A$  weakly. Thus,  $T(x_{n_k}) \xrightarrow[k]{\infty} Tx$  in norm. This shows that  $T(A)$  is (norm) compact.

Conversely, if  $T \in \mathcal{L}(X, Y)$  maps weakly compact sets into norm compact sets, then if  $(x_n) \subset X$  such that  $x_n \rightarrow x$  weakly, we have a weakly compact set

$$A := \{x_n : n \in \mathbb{N}\} \cup \{x\}$$

– so by assumption, the set  $T(A)$  is contained in a norm compact set.

Thus, if  $(Tx_{n_k}) \subset T(A)$  is any subsequence of  $T(x_n)$ , then it has a subsequence  $(Tx_{n_{k_l}})_l$  which converges in norm, i.e. there exists  $y \in Y$  such that  $Tx_{n_{k_l}} \xrightarrow[l]{\infty} y$  in norm. But  $x_n \xrightarrow{wk}{x}$  and  $T : X \rightarrow Y$  is weakly continuous – Thus  $Tx_n \xrightarrow[n]{\infty} Tx$  weakly. Therefore,  $Tx_{n_{k_l}} \xrightarrow[l]{\infty} Tx$  weakly, i.e.  $Tx = y$ , i.e.  $y \in T(A)$ . So we have that  $Tx_{n_{k_l}} \xrightarrow[l]{\infty} Tx$  in norm.

Suppose  $Tx_n \not\xrightarrow{\infty} Tx$  in norm. Then there exists  $\varepsilon > 0$  such that for all  $k \in \mathbb{N}$  there is  $n_k \in \mathbb{N}$  with  $n_k < n_{k+1} < \dots$  and  $\|Tx_{n_k} - Tx\| \geq \varepsilon$ .

As before, there exists a subsequence  $(Tx_{n_{k_l}})_{l \in \mathbb{N}}$  such that  $Tx_{n_{k_l}} \xrightarrow[l]{\infty} Tx$  in norm – this contradicts that  $\|Tx_{n_{k_l}} - Tx\| \geq \varepsilon, \forall l$ . Hence we have  $Tx_n \xrightarrow{\infty} Tx$  in norm.  $\square$

**Lemma 1.4.2** (cf. [11], page 49) A sequence  $(x_n)$  in  $X$  is a weakly (respectively, norm) Cauchy sequence if and only if given strictly increasing sequences  $(j_n)$  and  $(k_n)$  of positive integers, then  $(x_{k_n} - x_{j_n})$  is weakly null (respectively, norm null).

**Proof** Suppose  $(x_n)$  is a weakly Cauchy sequence. Let  $(j_n)$  and  $(k_n)$  be strictly increasing sequences of positive numbers. Let  $x^* \in X^*$ . For  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $|\langle x_j - x_k, x^* \rangle| < \varepsilon \forall j, k \geq N$ .

Since  $j_n \geq n$  and  $k_n \geq n$  for all  $n \in \mathbb{N}$ , it follows that

$$|\langle x_{j_n} - x_{k_n}, x^* \rangle| < \varepsilon, \quad \forall n \geq N,$$

i.e.  $\langle x_{j_n} - x_{k_n}, x^* \rangle \xrightarrow[\infty]{n} 0$ . This is true for all  $x^* \in X^*$ , i.e.  $x_{j_n} - x_{k_n} \xrightarrow[\infty]{n} 0$  weakly.

Conversely, suppose that  $(x_{k_n} - x_{j_n})$  is weakly null for all strictly increasing sequences  $(j_n)$  and  $(k_n)$  of positive integers. Assume that  $(x_n)$  is not a weakly Cauchy sequence.

Then there exists  $x^* \in X^*$  such that  $(\langle x_n, x^* \rangle)_{n \in \mathbb{N}}$  is not a Cauchy sequence, i.e. there exists  $\varepsilon > 0$  such that for each  $k \in \mathbb{N}$ , we have  $m_k \geq k$  and  $n_k \geq k$  with

$$|\langle x_{n_k} - x_{m_k}, x^* \rangle| \geq \varepsilon$$

– this contradicts that  $\lim_{k \rightarrow \infty} |\langle x_{n_k} - x_{m_k}, x^* \rangle| = 0$ , as was assumed.  $\square$

**Proposition 1.4.3** (cf. [12], page 49) *If  $T : X \rightarrow Y$  is completely continuous, then for each weakly Cauchy sequence  $(x_n)$ , the sequence  $(Tx_n)$  is a norm Cauchy sequence.*

**Proof** Suppose  $(x_n)$  is weakly Cauchy in  $X$ . Then if  $(j_n)$  and  $(k_n)$  are strictly increasing sequences of positive integers, we by the preceding lemma, have that  $(x_{j_n} - x_{k_n})$  is a weakly null sequence in  $X$ . Therefore,  $(Tx_{j_n} - Tx_{k_n})$  is norm null in  $Y$ .

Since  $(j_n)$  and  $(k_n)$  were arbitrarily chosen, again by the preceding lemma (for sequences in normed spaces), we have that  $(Tx_n)$  is a norm Cauchy sequence.  $\square$

# Chapter 2

## Dunford-Pettis-like and stronger Dunford-Pettis properties on normed spaces

### 2.1 Dunford-Pettis-like properties

In the paper [8] the family of “ $p$ -convergent” operators was introduced. Since these operators will play an important role in our discussion later on, we recall the definition as given in [8]:

**Definition 2.1.1** (cf. [8], Definition 1.2) *Let  $1 \leq p < \infty$ . We call  $T \in \mathcal{L}(X, Y)$   $p$ -convergent if it transforms weakly  $p$ -summable sequences into norm-null sequences, i.e.  $(Tx_n) \in c_0^{\text{strong}}(Y)$  (or  $\|Tx_n\| \xrightarrow{\infty} 0$ ) for all  $(x_n) \in \ell_p^{\text{weak}}(X)$ .*

Let us denote the class of  $p$ -convergent operators from  $X$  into  $Y$ , by  $\mathcal{C}_p(X, Y)$ .

**Remark 2.1.2** (a) *In Definition 2.1.1 we may add  $p = \infty$ , if we agree in this case to ask that  $(Tx_n) \in c_0^{\text{strong}}(Y)$  for all  $(x_n) \in c_0^{\text{weak}}(X)$ . In this case, the  $\infty$ -convergent operators are precisely the completely continuous operators, i.e.  $\mathcal{C}_\infty(X, Y) = \mathcal{V}(X, Y)$ .*

(b) *If  $p < q$ , then  $\ell_p^{\text{weak}}(X) \subseteq \ell_q^{\text{weak}}(X)$ , thus  $\mathcal{C}_q(X, Y) \subseteq \mathcal{C}_p(X, Y)$ .*

- (c) The scale of  $\mathcal{C}_p(X, Y)$  ideals (for  $1 \leq p \leq \infty$ ) are intermediate between the ideals of completely continuous operators and unconditionally summing operators.
- (d) The class  $\mathcal{C}_p$  of  $p$ -convergent operators on the family of all Banach spaces is an injective operator ideal (for  $1 \leq p \leq \infty$ ).  $\mathcal{C}_p$  is however not a surjective (operator) ideal (cf. [8], page 44).
- (e)  $\mathcal{C}_p(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$ : Let  $T \in \mathcal{L}(X, Y)$  and  $(T_n) \subset \mathcal{C}_p(X, Y)$  so that  $\|T_n - T\| \xrightarrow{\infty} 0$ . For  $\varepsilon > 0$ , choose  $n_1 \in \mathbb{N}$  such that  $\|T_n - T\| < \frac{\varepsilon}{2}$  for all  $n \geq n_1$ . Let  $(x_i) \in \ell_p^{weak}(X)$ ; we may assume that  $\|x_j\| \leq 1$  for all  $j \in \mathbb{N}$ . Now let  $n_0 \in \mathbb{N}$  such that  $\|T_{n_1}x_j\| < \frac{\varepsilon}{2}$  for all  $j \geq n_0$ . We have

$$\begin{aligned} \|Tx_j\| &\leq \|Tx_j - T_{n_1}x_j\| + \|T_{n_1}x_j\| \\ &< \varepsilon, \text{ for all } j \geq n_0. \end{aligned}$$

**Definition 2.1.3** (cf. [8], page 44) A sequence  $(x_n)$  in a Banach space  $X$  is said to be weakly  $p$ -convergent to an  $x \in X$  if the sequence  $(x_n - x)$  is weakly  $p$ -summable, i.e.  $(x_n - x) \in \ell_p^{weak}(X)$ .

The weakly  $\infty$ -convergent sequences are simply the weakly convergent sequences, since then we agree to have  $(x_n - x) \in c_0^{weak}(X)$ .

Recall that in a Banach space  $X$  a subset  $M$  is relatively compact if and only if for each sequence  $(x_n)$  in  $M$  there is a convergent subsequence  $(x_{n_k})$ , i.e.  $x_{n_k} \xrightarrow{\infty} x \in X$ . According to Theorem 1.3.4, a similar characterization holds for relatively weakly compact sets. It is therefore natural to introduce the following:

**Definition 2.1.4** (cf [8], Definition 1.3) A bounded set  $K$  in a Banach space is said to be relatively weakly  $p$ -compact,  $1 \leq p \leq \infty$ , if every sequence in  $K$  has a weakly- $p$ -convergent subsequence.

In other words, the bounded set  $K$  is relatively weakly  $p$ -compact if for every sequence  $(x_n) \subset K$  there is a subsequence  $(x_{n_k})$  and  $x \in X$  such that  $(x_{n_k} - x)_k \in \ell_p^{weak}(X)$ . If the ‘‘limit point’’ of each weakly  $p$ -convergent subsequence is in  $K$ , then we call  $K$  a weakly- $p$ -compact set.

**Lemma 2.1.5** *If  $A \subset X$  is weakly- $p$ -compact, then  $T(A)$  is weakly- $p$ -compact in  $Y$  for each  $T \in \mathcal{L}(X, Y)$ .*

**Proof** Consider any sequence  $(Tx_n) \subset T(A)$ .

Since  $A$  is weakly- $p$ -compact, there is a subsequence  $(x_{n_k})$  and  $x \in A$  such that  $(x_{n_k} - x)_k \in \ell_p^{weak}(X)$ . Let  $y^* \in Y^*$ . Then  $y^* \circ T \in X^*$ , thus  $(\langle Tx_{n_k} - Tx, y^* \rangle)_k = (\langle x_{n_k} - x, y^* \circ T \rangle)_k \in \ell_p$ .  $\square$

**Definition 2.1.6** *An operator  $T \in \mathcal{L}(X, Y)$  is called weakly  $p$ -compact,  $1 \leq p \leq \infty$ , if  $T(\mathcal{B}_X)$  is relatively weakly  $p$ -compact.*

We denote the vector space of all weakly  $p$ -compact operators by  $\mathcal{W}_p(X, Y)$  and the identity operator on  $X$  by  $id_X$ .

**Remark 2.1.7** (1) *Weakly  $\infty$ -compact operators are precisely those  $T \in \mathcal{L}(X, Y)$  for which  $T(\mathcal{B}_X)$  is relatively weakly  $\infty$ -compact, i.e. for which  $T(\mathcal{B}_X)$  is relatively weakly compact; thus  $\mathcal{W}_\infty(X, Y) = \mathcal{W}(X, Y)$ .*

(2)  *$id_X \in \mathcal{W}_1(X, X) \Leftrightarrow id_X(\mathcal{B}_X) = \mathcal{B}_X$  is relatively weakly 1-compact  $\Leftrightarrow$  for all sequences  $(x_n) \subset \mathcal{B}_X$ , there is a subsequence  $(x_{n_k})$  which is weakly 1-convergent, i.e. there exists  $x \in X$  such that  $(x_{n_k} - x) \in \ell_1^{weak}(X)$ .*

In the paper [8] the concept ‘‘Dunford-Pettis property of order  $p$ ’’ (which we will abbreviate by  $DPP_p$  in the sequel) is introduced as follows:

**Definition 2.1.8** (cf. [8], Definition 3.1) *Let  $1 \leq p \leq \infty$ . A Banach space  $X$  has the  $DPP_p$  if the inclusion  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_p(X, Y)$  holds for all Banach spaces  $Y$ .*

Recall that if  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_\infty(X, Y) = \mathcal{V}(X, Y)$  for all Banach spaces  $Y$ , then  $X$  is said to have the  $DPP$ . Clearly  $DPP$  is a stronger property than  $DPP_p$  for all  $1 \leq p < \infty$  and moreover we have:

**Remark 2.1.9** (1) *If  $X$  has  $DPP_p$  and  $q \leq p$ , then  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_p(X, Y) \subseteq \mathcal{C}_q(X, Y)$ . Thus  $X$  has  $DPP_q$ .*

(2) *All Banach spaces have  $DPP_1$ : The fact that  $\mathcal{W}(X, Y) \subseteq \mathcal{C}_1(X, Y)$  for all Banach spaces  $X$  and  $Y$ , follows from some deep results in Banach space theory. We only give the outline:*

- (i) A series  $\sum_n x_n$  in a Banach space  $X$  is called *weakly unconditionally Cauchy* (or *weakly unconditionally convergent* – a *wuc-series*) if for each  $x^* \in X^*$ ,  $\sum_n |\langle x_n, x^* \rangle| < \infty$ , i.e.  $(x_n) \in \ell_1^{weak}(X)$ .
- (ii) A series  $\sum_n x_n$  in a Banach space  $X$  is called *unconditionally convergent* if  $\sum_n t_n x_n$  converges in  $X$  for all  $(t_n) \in \ell_\infty$ . This implies in particular that  $\sum_n x_n$  converges in  $X$ .
- (iii) If  $X$  is weakly sequentially complete, then every wuc-series in  $X$  is unconditionally convergent, e.g. all reflexive Banach spaces are weakly sequentially complete.
- (iv) Suppose  $T \in \mathcal{W}(X, Y)$  and let  $(x_n) \in \ell_1^{weak}(X)$ . There exists a reflexive space  $R$  and  $Q \in \mathcal{L}(X, R)$ ,  $P \in \mathcal{L}(R, Y)$  such that  $T = PQ$  (cf. Theorem 1.3.2). Now  $(Qx_n) \in \ell_1^{weak}(R)$ . By (ii) and (iii) the series  $\sum_n Qx_n$  converges in  $R$ . Therefore  $\sum_n Tx_n = \sum_n PQx_n$  converges in the norm of  $Y$ . This implies that  $\|Tx_n\| \xrightarrow{\infty} 0$ , i.e.  $(Tx_n) \in c_0^{strong}(Y)$ . Thus  $T \in \mathcal{C}_1(X, Y)$ . It follows that all Banach spaces  $X$  have  $DPP_1$ .
- (3) Suppose  $id_X \in \mathcal{C}_p(X, X)$ . For any Banach space  $Y$  and  $T \in \mathcal{W}(X, Y)$  it follows that  $T = T \circ id_X$ . If  $(x_n) \in \ell_p^{weak}(X)$ , then  $(id_X x_n) \in c_0^{strong}(X)$ . Therefore,  $(Tx_n) = ((T \circ id_X)x_n) \in c_0^{strong}(Y)$ . This shows that  $T \in \mathcal{C}_p(X, Y)$ . Since this holds for any  $T \in \mathcal{W}(X, Y)$ , it follows that  $X$  has  $DPP_p$ .

To avoid having to repeat the same compactness argument in several proofs in the sequel, we discuss a “general compactness argument” in the following lemma.

**Lemma 2.1.10** (A compactness argument) *Let  $(X, d)$  be a metric space and denote the metric topology on  $X$  by  $\tau_d$ . Let  $A \subset X$  be a  $\tau_d$ -compact subset of  $X$ . For any weaker topology  $\tau$  on  $X$  (i.e.  $\tau \leq \tau_d$ ) such that  $(X, \tau)$  is a Hausdorff topological space and any  $\tau$ -convergent sequence  $(x_n)$  in  $A$  (i.e.  $x_n \xrightarrow{\tau} x \in A$ ) we have  $x_n \xrightarrow{\tau_d} x$ .*

**Proof** Suppose  $(x_n) \subset A$  and  $x_n \xrightarrow{\tau} x \in A$ . Since  $A$  is compact in the metric space  $(X, \tau_d)$ , a subsequence  $(x_{n_k})$  converges in the metric topology to some  $y \in A$ . Certainly,  $x_{n_k} \xrightarrow{\infty} y$  in the weaker topology  $\tau$ ; this topology

being Hausdorff, we conclude  $y = x$ , i.e.  $\tau_d - \lim_k x_{n_k} = x$ .

If we assume that the original sequence  $(x_n)$  does not converge to  $x$  in the metric topology, then there exist an  $\varepsilon > 0$  and a subsequence  $(x_{n_m})$  such that  $d(x_{n_m}, x) \geq \varepsilon$  for all  $m \in \mathbb{N}$ . Being a sequence in the compact set  $A$ , there exists a subsequence  $(x_{n_{m_j}})_j$  which converges in  $A$ . As before, we must have  $\lim_{j \rightarrow \infty} x_{n_{m_j}} = x$ ; this contradicts the fact that  $d(x_{n_{m_j}}, x) \geq \varepsilon$  for all  $j \in \mathbb{N}$ .

We conclude that  $x_n \xrightarrow{\tau} x \in A$ , implies  $x_n \xrightarrow{\tau_d} x \in A$ .  $\square$

**Lemma 2.1.11** *An operator  $T \in \mathcal{L}(X, Y)$  is  $p$ -convergent (i.e.  $T \in \mathcal{C}_p(X, Y)$ ) if and only if  $T$  maps relatively weakly  $p$ -compact sets of  $X$  onto relatively norm compact sets in  $Y$ .*

**Proof** Let  $T \in \mathcal{C}_p(X, Y)$  and assume that  $A \subset X$  is relatively weakly  $p$ -compact.

Let  $(Tx_n) \subset T(A)$ , i.e.  $(x_n) \subset A$ . There is a subsequence  $(x_{n_k})$  and  $x \in X$  such that  $(x_{n_k} - x)_k \in \ell_p^{weak}(X)$ . By assumption,  $(Tx_{n_k} - Tx)_k \in c_0(Y)$ , i.e.  $Tx_{n_k} \xrightarrow[k]{k} Tx$  in norm. Therefore, we conclude that  $T(A)$  is relatively norm compact in  $Y$ .

Conversely, suppose  $T$  transforms relatively weakly  $p$ -compact sets in  $X$  into relatively norm compact sets in  $Y$ . Let  $(x_n) \in \ell_p^{weak}(X)$ . Then  $\{x_n : n \in \mathbb{N}\}$  is relatively weakly  $p$ -compact (in fact,  $\{x_n : n \in \mathbb{N}\} \cup \{0\}$  is weakly  $p$ -compact). Let  $B = \{Tx_n : n \in \mathbb{N}\}$ . By assumption,  $B$  is relatively norm compact. Thus there exists a subsequence  $(Tx_{n_k})$  which converges in  $Y$ . However, since  $x_n \xrightarrow[n]{n} 0$  weakly in  $X$  and  $T \in \mathcal{L}(X, Y)$  we also have  $Tx_n \xrightarrow[n]{n} 0$  weakly in  $Y$ . Therefore,  $Tx_{n_k} \xrightarrow[k]{k} 0$  in norm, i.e.  $0 \in \overline{B}$ . Since  $Tx_n \xrightarrow[k]{k} 0$  weakly, it follows from Lemma 2.1.10 that  $\|Tx_n\| \xrightarrow[n]{n} 0$ . Hence we conclude that  $T \in \mathcal{C}_p(X, Y)$ .  $\square$

**Lemma 2.1.12** *Let  $T \in \mathcal{L}(X, Y)$ . Then  $T$  maps relatively weakly  $p$ -compact sets in  $X$  onto relatively norm compact sets in  $Y$ , if and only if,  $T$  maps weakly  $p$ -compact sets onto norm compact sets in  $Y$ .*

**Proof** Suppose  $T$  maps relatively weakly  $p$ -compact sets in  $X$  onto relatively norm compact sets in  $Y$ . Let  $A \subset X$  be weakly  $p$ -compact. Then

$T(A)$  is relatively norm compact. We show that  $T(A)$  is norm compact: Let  $(Tx_n) \subset T(A)$ . There is a subsequence  $(Tx_{n_k})$  such that  $Tx_{n_k} \xrightarrow[\infty]{k} y \in Y$ . Now  $(x_{n_k})$  has a subsequence  $(x_{n_{k_l}})$  such that  $(x_{n_{k_l}} - x)_l \in \ell_p^{weak}(X)$ ,  $x \in A$ . In particular,  $x_{n_{k_l}} \rightarrow x$  weakly. Therefore,  $Tx_{n_{k_l}} \rightarrow Tx$  weakly. But  $Tx_{n_{k_l}} \rightarrow y$  in norm, thus  $y = Tx \in T(A)$ .

Conversely, let  $T$  map weakly  $p$ -compact sets onto norm compact sets. Let  $A \subset X$  be relatively weakly  $p$ -compact. Consider  $(Tx_n) \subset T(A)$ . There is a subsequence  $(x_{n_k}) \subset A$  and  $x \in X$  such that  $(x_{n_k} - x) \in \ell_p^{weak}(X)$ . Now the set  $B = \{x_{n_k} : k \in \mathbb{N}\} \cup \{x\}$  is weakly  $p$ -compact. Therefore,  $T(B) = \{Tx_{n_k} : k \in \mathbb{N}\} \cup \{Tx\}$  is norm compact. Thus there exists  $(Tx_{n_{k_l}})_l$  such that  $Tx_{n_{k_l}} \xrightarrow[\infty]{l} y \in T(B) \subseteq Y$ .  $\square$

**Remark 2.1.13** *From Lemmas 2.1.11 and 2.1.12 it follows that  $T \in \mathcal{C}_p(X, Y)$  if, and only if,  $T$  maps weakly  $p$ -compact sets onto norm compact sets.*

Using the characterisation of  $p$ -convergent operators in Remark 2.1.13 we may now discuss the following characterisation of spaces having  $DPP_p$ :

**Proposition 2.1.14** *(Characterisation of  $DPP_p$ )(cf. [8], Proposition 3.2) For a given Banach space,  $X$ , the following are equivalent:*

- (a)  $X$  has  $DPP_p$  ( $1 \leq p \leq \infty$ ).
- (b) If  $(x_n) \in \ell_p^{weak}(X)$  and  $(x_n^*) \in c_0^{weak}(X^*)$ , then  $\langle x_n, x_n^* \rangle \xrightarrow[\infty]{n} 0$ .
- (c) For any Banach space  $Y$ , every weakly compact operator  $T : X \rightarrow Y$  transforms weakly- $p$ -compact sets of  $X$  into norm-compact sets of  $Y$ .

**Proof** (a)  $\Rightarrow$  (b): Suppose  $X$  has  $DPP_p$ . Then  $\mathcal{W}(X, c_0) \subseteq \mathcal{C}_p(X, c_0)$ . Let  $(x_n) \in \ell_p^{weak}(X)$  and  $(x_n^*) \in c_0^{weak}(X^*)$ . By Proposition 1.3.11 the operator  $T_{(x_n^*)} : X \rightarrow c_0$  is weakly compact, hence also  $p$ -convergent. Therefore,  $T_{(x_n^*)}x_n \xrightarrow[\infty]{n} 0$  in norm. However,

$$\begin{aligned} |\langle x_n, x_n^* \rangle| &\leq \sup_i |\langle x_n, x_i^* \rangle| \\ &= \|(\langle x_n, x_i^* \rangle)_i\|_{c_0} \\ &= \|T_{(x_i^*)}x_n\|_{c_0} \xrightarrow[\infty]{n} 0. \end{aligned}$$

(b)  $\Rightarrow$  (a): Suppose (b) holds, but there exists a Banach space  $Y$  and  $T \in \mathcal{W}(X, Y)$  such that  $T \notin \mathcal{C}_p(X, Y)$ . Thus for all  $n \in \mathbb{N}$  and  $\varepsilon > 0$ , there exists  $(x_n) \in \ell_p^{weak}(X)$  such that  $\|Tx_n\| \geq \varepsilon$ . Therefore there exists  $y_n^* \in Y^*$  such that  $\|y_n^*\| = 1$  and  $|\langle Tx_n, y_n^* \rangle| > \frac{\varepsilon}{2}$ . Thus  $|\langle x_n, T^*y_n^* \rangle| > \frac{\varepsilon}{2}$  for all  $n \in \mathbb{N}$ . Since  $T^*$  is weakly compact,  $T^*(\mathcal{B}_{Y^*})$  is relatively weakly compact in  $X^*$ . Therefore there exists a subsequence  $(T^*y_{n_k}^*)$  such that  $T^*y_{n_k}^* \xrightarrow[k]{\infty} x^*$  weakly, by Theorem 1.3.4. Put  $x_k^* = T^*y_{n_k}^* - x^*$ . Then  $x_k^* \xrightarrow[k]{\infty} 0$  weakly in  $X^*$ . By (b),  $\langle x_{n_k}, x_k^* \rangle \xrightarrow[k]{\infty} 0$  and since  $(x_{n_k}) \in \ell_p^{weak}(X) \subseteq c_0^{weak}(X)$ ,

$$\begin{aligned} |\langle x_{n_k}, T^*y_{n_k}^* \rangle| &= |\langle x_{n_k}, x_k^* \rangle + \langle x_{n_k}, x^* \rangle| \\ &\xrightarrow[k]{\infty} 0, \end{aligned}$$

contradicting the fact that  $|\langle x_n, T^*y_n^* \rangle| > \frac{\varepsilon}{2}$  for all  $n \in \mathbb{N}$ .

(a)  $\Leftrightarrow$  (c) follows from Definition 2.1.8 and Remark 2.1.13.  $\square$

In the paper [8] the authors introduce the concept of  $DPP_p$  and discuss the examples of classical Banach spaces with  $DPP_p$ . Their results are summarised (but not proved) in the following remark.

**Remark 2.1.15** *In the following,  $r^*$  denotes the conjugate number of  $r$  (i.e.  $\frac{1}{r} + \frac{1}{r^*} = 1$ ). The reader is referred to examples 3.3 and 1.9 in [8] for the following facts.*

- (a)  $C(K)$  and  $L_1$  have the  $DPP$ , and therefore the  $DPP_p$  for all  $p$ .
- (b) If  $1 < r < \infty$ ,  $\ell_r$  has the  $DPP_p$  for  $p < r^*$ .
- (c) If  $1 < r < \infty$ ,  $L_r(\mu)$  has the  $DPP_p$  for  $p < \min(2, r^*)$ .
- (d) We refer the reader to the book [1] (pages 247 to 260) for an interesting discussion about Tsirelson's space  $T$  and its role in the study of the geometry of Banach spaces. During the 1960s the study of the geometry of Banach spaces very much figured around the question whether every Banach space must contain a copy of either  $\ell_p$  space or  $c_0$ . In 1974, Tsirelson published a paper in which he was able to construct a reflexive Banach space (nowadays called Tsirelson's space  $T$ ) which does not contain an isomorphic copy of  $\ell_p$  or  $c_0$ . In [1] (cf. [1], Theorem

10.3.2) the authors state (and prove) Tsirelson's result as a theorem which says "There is a reflexive Banach space  $T$  which contains no copy of  $\ell_p$  for  $1 \leq p \leq \infty$ , or  $c_0$ ". It is beyond the scope of our discussion to try to go into the details of Tsirelson's space.

It is however interesting to learn from the paper [8] that the space  $T$  has  $DPP_p$  for all  $p < \infty$ . However, since  $T$  is reflexive, it does not have  $DPP$ .

(e) Tsirelson's dual space  $T^*$  does not have  $DPP_p$  for any  $p > 1$ .

## 2.2 A stronger Dunford-Pettis property

In the paper [7] the authors define a stronger version of the  $DPP$  on Banach spaces, which they call the  $(DP^*)$  property. We agree to abbreviate " $(DP^*)$  property" by  $DP^*P$  and recall the definition from [7] (cf. [7], Definition 1.1) as follows:

**Definition 2.2.1**  $X$  has the  $DP^*P$  when all weakly compact sets in  $X$  are limited, i.e. each  $(x_n^*) \in c_0^{weak^*}(X^*)$  converges uniformly to 0 on all weakly compact sets in  $X$ .

**Remark 2.2.2** It is clear that for Grothendieck spaces the concepts of  $DPP$  and  $DP^*P$  are equivalent. Thus, for instance  $\ell_\infty$  has  $DP^*P$ .

In the paper [3] there is also a so called " $DP^*$  property" defined. This is introduced by considering when Mackey and  $weak^*$  convergence coincide sequentially in  $X^*$ . Recall that the Mackey topology  $\tau(X^*, X)$  is the finest (strongest) locally convex topology on  $X^*$  which is consistent with the duality  $(X, X^*)$ . i.e.  $(X^*, \tau(X^*, X))^* = X$ . A net  $(x_\delta^*) \subset X^*$  converges to  $x^* \in X^*$  in the Mackey topology (or  $(x_\delta^*)$  is Mackey convergent for  $x^* \in X^*$ ) if for each absolutely convex (i.e. balanced and convex) weakly compact set  $A \subset X$  and each  $\varepsilon > 0$  there exists an index  $\delta_0 = \delta_0(\varepsilon; A)$  such that

$$|\langle x, x_\delta^* - x^* \rangle| < \varepsilon$$

for all  $\delta \geq \delta_0$  and for all  $x \in A$ .

Thus, Mackey convergence in  $X^*$  is uniform convergence on absolutely convex weakly compact sets in  $X$ . The  $DP^*$  property in [3], page 60 is defined as follows:

**Definition 2.2.3** A Banach space  $X$  is said to have the  $DP^*$  property if  $weak^*$  and Mackey convergence coincide sequentially in  $X^*$ , i.e. if a sequence  $(x_n^*) \subset X^*$  converges  $weak^*$  if and only if it converges in the Mackey topology.

From our definition (or rather the definition in the paper [7]) as is stated in Definition 2.2.3, it is clear that if  $X$  has  $DP^*P$ , then each  $(x_n^*) \in c_0^{weak^*}(X^*)$  converges uniformly on all absolutely convex weakly compact sets in  $X$ . Obviously, Definition 2.2.1 is thus stronger than Definition 2.2.3. Since in general the closed absolutely convex hull of a weakly compact set needs not be weakly compact, it is not clear (at first sight) that the  $DP^*$  property in Definition 2.2.3 implies the  $DP^*P$  in Definition 2.2.1. However, if we choose  $(x_n) \in c_0^{weak}(X)$ , then the set

$$A := \left\{ \sum_{i=1}^{\infty} \lambda_i x_i : (\lambda_i) \in \mathcal{B}_{\ell_1} \right\}$$

is weakly compact by our discussion of the proof of Proposition 1.2.5. It is also absolutely convex. Therefore, if we assume that Mackey and  $weak^*$  null sequences in  $X^*$  are the same, it follows for each  $(x_n^*) \in c_0^{weak^*}(X^*)$  that  $x_n^* \rightarrow 0$  uniformly on  $A$ . In particular, we have

$$\begin{aligned} |\langle x_n, x_n^* \rangle| &\leq \sup_j |\langle x_j, x_n^* \rangle| \\ &\leq \sup_{x \in A} |\langle x, x_n^* \rangle| \\ &\xrightarrow[\infty]{n} 0. \end{aligned}$$

From the characterization of  $DP^*P$  in Theorem 2.2.8 below, we thus see that the  $DP^*$  property introduced in [3] and the  $DP^*P$  considered in [7] are the same.

It is interesting to note that the authors in [3] had some application with respect to differentiability of Lipschitz convex functions in mind when they introduced the  $DP^*$  property. Recall that a function  $f : X \rightarrow \mathbb{R}$  (for a real Banach space  $X$ ) is called Gateaux differentiable (briefly,  $G$ -differentiable) at  $x \in X$  if there exists an  $x^* \in X^*$  such that

$$\lim_{t \rightarrow 0} \frac{[f(x+h) - f(x) - x^*(th)]}{t} = 0,$$

for each  $h \in X$  with  $\|h\| = 1$ . If the above limit is uniform on weakly compact subsets of  $X$ , then  $f$  is called weak Hadamard differentiable (briefly,  $WH$ -differentiable) at  $x$ . By its very definition it is clear that  $WH$ -differentiable implies  $G$ -differentiable. From Theorem 3.1. in [3] we have

**Theorem 2.2.4** (c.f. [3], Theorem 3.1) *Let  $X$  be a (real) Banach space. The following are equivalent:*

- (a)  $X$  has the  $DP^*$  property.
- (b)  $WH$ -differentiability and  $G$ -differentiability coincide for all Lipschitz convex functions on  $X$ .
- (c) Every continuous convex function is bounded on weakly compact subsets of  $X$ .
- (d) A sequence of Lipschitz convex functions  $(f_n)$  converges uniformly to a continuous affine function  $\phi$  on weakly compact sets, provided that  $(f_n)$  converges pointwise to  $\phi$  on  $X$ .

For the proofs of Theorem 2.2.4 as well as the following characterization of the  $DP^*$  property, the reader is referred to the paper [3].

**Theorem 2.2.5** (c.f. [3], Theorem 3.3) *For a Banach space  $X$ , the following are equivalent:*

- (a)  $X$  has the  $DP^*$  property.
- (b) For each Banach space  $Y$  such that  $\mathcal{B}_{Y^*}$  is weak\*-sequentially compact, each bounded linear  $T : X \rightarrow Y$  is completely continuous.

Also in the paper [17] on page 471 there is a property, called the  $*$ -Dunford-Pettis property, introduced. The definition in this case states the following:

**Definition 2.2.6** (cf. [17], page 471) *We say a Banach space  $X$  has the  $*$ -Dunford-Pettis property if for all weakly null sequences  $(x_n)$  in  $X$  and all weak\* null sequences  $(x_n^*)$  in  $X^*$ ,  $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ .*

Again, by Theorem 2.2.8 (below) we see that the  $*$ -Dunford-Pettis property is the same as our  $DP^*P$ .

In addition to Theorem 2.2.5, if we assume that  $X$  has  $DP^*P$  and  $\mathcal{B}_{X^*}$  is  $weak^*$  sequentially compact, then  $X$  is a Schur space. This is proved in the paper [17]. Having our machinery available, the result now follows as an easy consequence of Theorem 2.2.5.

**Corollary 2.2.7** (cf. Proposition 5 in [17]) *Suppose  $X$  has  $DP^*P$  and  $\mathcal{B}_{X^*}$  is  $weak^*$  sequentially compact. Then  $X$  is a Schur space.*

**Proof** In Theorem 2.2.5, let  $Y = X$ . By Theorem 2.2.5, we have that each bounded linear operator  $T : X \rightarrow X$  is completely continuous. In particular,  $id_X : X \rightarrow X$  is completely continuous. Thus, if  $x_n \rightarrow x$  weakly in  $X$ , then  $x_n \rightarrow x$  in norm; i.e.  $X$  is a Schur space.  $\square$

**Theorem 2.2.8**  *$X$  has  $DP^*P$  if and only if  $\langle x_n, x_n^* \rangle \xrightarrow{\infty} 0$  for all  $(x_n) \in c_0^{weak}(X)$  and  $(x_n^*) \in c_0^{weak^*}(X^*)$ .*

**Proof** Suppose  $X$  has  $DP^*P$ . Let  $(x_n) \in c_0^{weak}(X)$ ,  $(x_n^*) \in c_0^{weak^*}(X^*)$ . The set

$$A = \{x_n : n \in \mathbb{N}\} \cup \{0\}$$

is weakly compact.

Thus  $x_n^* \rightarrow 0$  uniformly on  $A$ ;  $\sup_{x \in A} |\langle x, x_n^* \rangle| \xrightarrow{\infty} 0$ .

In particular,

$$|\langle x_n, x_n^* \rangle| \leq \sup_{x \in A} |\langle x, x_n^* \rangle| \xrightarrow{\infty} 0.$$

Conversely, suppose  $\langle x_n, x_n^* \rangle \xrightarrow{\infty} 0$  for all  $(x_n) \in c_0^{weak}(X)$  and  $(x_n^*) \in c_0^{weak^*}(X^*)$ .

Suppose there exists a weakly compact set  $A \subset X$  which is not limited. Thus there exists  $(x_n^*) \subset X^*$  such that  $x_n^* \xrightarrow{wk^*} 0$ , but not uniformly on  $A$ . Then there exists  $\varepsilon > 0$  such that for each  $k \in \mathbb{N}$ , there are  $x_k \in A$  and  $n_k \geq k$  such that  $|\langle x_k, x_{n_k}^* \rangle| \geq \varepsilon$ . The sequence  $(x_k)$  has a weakly convergent subsequence  $(x_{k_m})_m$ , i.e. there exists  $x \in A$  such that  $(x_{k_m} - x)_{m \in \mathbb{N}} \in c_0^{weak}(X)$ .

Put  $y_m = x_{k_m} - x$  and  $\tilde{x}_m^* = x_{n_{k_m}}^*$ . Then  $|\langle x_{k_m}, \tilde{x}_m^* \rangle| \geq \varepsilon$ ,  $\forall m \in \mathbb{N}$ . Also,  $(y_m) \in c_0^{weak}(X)$  and  $(\tilde{x}_m^*) \in c_0^{weak^*}(X^*)$ . We have

$$\begin{aligned} |\langle y_m, \tilde{x}_m^* \rangle| &= |\langle x_{k_m} - x, \tilde{x}_m^* \rangle| \\ &\geq |\langle x_{k_m}, \tilde{x}_m^* \rangle| - |\langle x, \tilde{x}_m^* \rangle| \\ &\geq \varepsilon - |\langle x, \tilde{x}_m^* \rangle|, \forall m \in \mathbb{N}. \end{aligned}$$

Let  $m_0 \in \mathbb{N}$  such that  $|\langle x, \tilde{x}_m^* \rangle| < \frac{\varepsilon}{2}$ ,  $\forall m \geq m_0$ .

Then  $|\langle y_m, \tilde{x}_m^* \rangle| \geq \frac{\varepsilon}{2}$ ,  $\forall m \geq m_0$  – contradictory to our assumption.

So  $X$  has the  $DP^*P$ . □

Although the  $DP^*P$  is a strictly stronger property than the  $DPP$ , it often happens that “ $DP^*P = \text{something else} + DPP$ ”. This is illustrated by examples in the paper [7], some of which we summarise in the Remark 2.2.9 below. Some of the results in Remark 2.2.9 are easy (almost trivial) and some follow from other “deep” results in Banach space theory. We will see later on (in Theorem 2.2.15) what exactly the “something else” together with the  $DPP$  will give us the  $DP^*P$ .

**Remark 2.2.9** (cf. [7], page 207)

- (a) If  $X$  is a Schur space, then  $X$  has the  $DP^*P$ .
- (b) Every Grothendieck Banach space with the  $DPP$  enjoys the  $DP^*P$ .
- (c) Every Banach space with the Pelczynski ( $P$ ) property, the  $DPP$  and without complemented copies of  $c_0$  enjoys the  $DP^*P$ .
- (d) As a consequence,  $H^\infty$  has the  $DP^*P$  since Bourgain showed that it has the  $DPP$  and it is a Grothendieck space.

**Proposition 2.2.10** (cf. [7], Proposition 1.5) If  $X$  and  $Y$  are Banach spaces with the  $DP^*P$ , then so is  $X \times Y$  as well.

**Proof** Recall that  $\|(x, y)\|_1 = \|x\| + \|y\|$  defines the norm on  $X \times Y$ . Let  $(\phi_n^*) \in c_0^{weak^*}((X \times Y)^*)$  and  $(x_n, y_n) \in \ell_p^{weak}(X \times Y)$ . We show that  $(x_n) \in \ell_p^{weak}(X)$  and  $(y_n) \in \ell_p^{weak}(Y)$ :

Let  $x^* \in X^*$ . Define

$$\phi_{x^*} : X \times Y \rightarrow \mathbb{K}$$

by

$$\phi_{x^*} = x^* \circ P_1,$$

where

$$P_1 : X \times Y \rightarrow X : (x, y) \mapsto x$$

is the coordinate projection onto  $X$ . Then  $x^* \circ P_1 \in (X \times Y)^*$  and

$$\langle x_n, x^* \rangle = \langle P_1(x_n, y_n), x^* \rangle = \langle (x_n, y_n), x^* \circ P_1 \rangle$$

for all  $n \in \mathbb{N}$ , i.e.

$$(\langle x_n, x^* \rangle)_n = (\langle (x_n, y_n), x^* \circ P_1 \rangle)_n \in \ell_p.$$

Similarly, for each  $y^* \in Y^*$  we have

$$(\langle y_n, y^* \rangle)_n = (\langle (x_n, y_n), y^* \circ P_2 \rangle)_n \in \ell_p,$$

where

$$P_2 : X \times Y \rightarrow Y : (x, y) \mapsto y$$

is the coordinate projection onto  $Y$ .

Now let  $(\phi_n^*) \subset (X \times Y)^*$  be a weak\* null sequence. Then, if  $x_n^* = \phi_n^* \circ j_1$ , where  $j_1 : X \rightarrow X \times Y$  is the (continuous) embedding  $j_1 x = (x, 0)$ , then  $(x_n^*) \subset X^*$  and for each  $x \in X$  we have

$$\begin{aligned} \langle x, x_n^* \rangle &= \langle x, \phi_n^* \circ j_1 \rangle \\ &= \langle (x, 0), \phi_n^* \rangle \\ &\longrightarrow 0 \end{aligned}$$

if  $n \longrightarrow \infty$ , i.e.  $x_n^* \xrightarrow[\infty]{n} 0$  weak\*.

Similarly, if  $y_n^* = \phi_n^* \circ j_2$ , where  $j_2 : Y \rightarrow X \times Y$  is the (continuous) embedding  $j_2 y = (0, y)$ , then  $(y_n^*) \subset c_0^{weak^*}(Y^*)$ .

By assumption we then have:

$$\begin{aligned} \langle (x_n, y_n), \phi_n^* \rangle &= \langle x_n, x_n^* \rangle + \langle y_n, y_n^* \rangle \\ &\xrightarrow[\infty]{n} 0. \end{aligned}$$

By Theorem 2.2.8 the space  $X \times Y$  has  $DP^*P$ . □

**Remark 2.2.11** (cf. [7], page 207-208)

- (a) In general, a quotient space of a space with the  $DP^*P$  does not share this property.
- (b) The  $DP^*P$  is neither inherited from the dual, as in the case of  $c_0$  (refer to Theorem 2.2.13 below), nor inherited by the dual, as in the case of  $\ell_1(\ell_2^n)$  which is a Schur space whose dual  $\ell_\infty(\ell_2^n)$  lacks the DPP.
- (c) The preceding proposition enlarges the class of spaces enjoying the  $DP^*P$  and provides examples, like  $\ell_1 \times \ell_\infty$ , of spaces with the  $DP^*P$  which are neither Schur nor Grothendieck.
- (d) The tensor product  $\ell_\infty \hat{\otimes}_\pi \ell_\infty$  lacks the DPP, hence also the  $DP^*P$ .
- (e) In a weakly sequentially complete Banach space  $X$  (i.e. weak Cauchy sequences in  $X$  are weakly convergent) the notions of conditionally weakly compact (i.e. every sequence in a subset  $B$  of  $X$  contains a weak Cauchy subsequence) and relatively weakly compact set coincide. Therefore if  $X$  has the  $DP^*P$ , then the limited sets in  $X$  are relatively weakly compact if, and only if,  $X$  is weakly sequentially complete.

**Remark 2.2.12** Another way of looking at the different properties thus far discussed, is to introduce “multiplier spaces”. The advantage of doing this, is that one can then easily compare the different properties. We will not go into any details, but only briefly give the definitions in terms of multiplier spaces.

Let  $\Lambda$  be a scalar sequence space. Denote by  $\Lambda^{weak}(X)$  the space of all sequences  $(x_n)$  in  $X$  such that the scalar sequence  $(\langle x_n, x^* \rangle) \in \Lambda$ ,  $\forall x^* \in X^*$ . We denote by  $Mult(\Lambda^{weak}(X), c_0)$  the space of all sequences  $(x_n^*) \subset X^*$  such that  $(\langle x_n, x_n^* \rangle) \in c_0$ ,  $\forall (x_n) \in \Lambda^{weak}(X)$ . In this notation we have:

- (i)  $X$  has DPP  $\Leftrightarrow c_0^{weak}(X^*) \subseteq Mult(c_0^{weak}(X), c_0)$ .
- (ii)  $X$  has  $DP^*P$   $\Leftrightarrow c_0^{weak^*}(X^*) \subseteq Mult(c_0^{weak}(X), c_0)$ .
- (iii)  $X$  has  $DPP_p$   $\Leftrightarrow c_0^{weak}(X^*) \subseteq Mult(\ell_p^{weak}(X), c_0)$ .  
Later on in the dissertation we will introduce and discuss the “missing link”
- (iv)  $X$  has  $DP^*P_p$   $\Leftrightarrow c_0^{weak^*}(X^*) \subseteq Mult(\ell_p^{weak}(X), c_0)$ .

**Theorem 2.2.13** (cf. [7], Proposition 2.1) *X* has  $DP^*P$  if and only if every  $T \in \mathcal{L}(X, c_0)$  is completely continuous. In particular, if *X* has  $DP^*P$ , then it does not contain complemented copies of  $c_0$ .

**Proof** Assume each  $T \in \mathcal{L}(X, c_0)$  is completely continuous. Let  $(x_n) \in c_0^{weak}(X)$ ,  $(x_n^*) \in c_0^{weak^*}(X^*)$ . Define  $T : X \rightarrow c_0$  by  $Tx = (\langle x, x_n^* \rangle)_{n \in \mathbb{N}}$ . Then  $T$  is bounded and

$$\|T\| = \|(x_n^*)\|_{c_0^{weak^*}(X^*)} = \sup_{\|x\| \leq 1} \|(\langle x, x_n^* \rangle)\|_{c_0}.$$

By assumption,  $\|Tx_n\| \xrightarrow{\infty} 0$ , i.e.  $\|(\langle x_n, x_j^* \rangle)_j\|_{c_0} \xrightarrow{\infty} 0$ . Therefore,

$$\begin{aligned} |\langle x_n, x_n^* \rangle| &\leq \sup_j |\langle x_n, x_j^* \rangle| \\ &= \|(\langle x_n, x_j^* \rangle)\|_{c_0} \xrightarrow{\infty} 0. \end{aligned}$$

Thus *X* has  $DP^*P$  by Theorem 2.2.8.

Conversely, assume *X* has  $DP^*P$ . Also assume  $T \in \mathcal{L}(X, c_0)$  is not completely continuous. Then there exists  $(x_n) \in c_0^{weak}(X)$  such that  $\|Tx_n\|_{c_0} \not\rightarrow 0$  if  $n \rightarrow \infty$ . We may assume there exists  $\delta > 0$  such that  $\|Tx_m\| \geq \delta$  for all  $m \in \mathbb{N}$  (or, otherwise we may choose a subsequence with this property). Now,  $\|Tx_m\|_{c_0} = \sup_j |\pi_j(Tx_m)|$ , where  $\pi_j : c_0 \rightarrow \mathbb{K} : (\xi_j) \mapsto \xi_j$  is the (continuous) linear  $j^{th}$  coordinate projection of  $c_0$ . This implies that there exists  $k_m \in \mathbb{N}$  such that

$$\frac{\delta}{2} < |\pi_{k_m}(Tx_m)| \leq \|Tx_m\|_{c_0}.$$

Let  $(x_i^*) \in c_0^{weak^*}(X^*)$ , such that  $T = T_{(x_n^*)}$  where  $T_{(x_n^*)}(x) = (\langle x, x_n^* \rangle)$ ,  $\forall x \in X$  (by Theorem 1.2.8). It follows that

$$\frac{\delta}{2} < |\pi_{k_m}((\langle x_m, x_i^* \rangle)_i)| = |\langle x_m, x_{k_m}^* \rangle| \quad \forall m \in \mathbb{N}. \quad (*)$$

Suppose there are only a finite number of  $k_m$  where  $m$  runs through  $\mathbb{N}$  such that  $(*)$  holds. Then there exists  $N \in \mathbb{N}$  such that  $k = k_m = k_{m+1} = \dots$  for all  $m \geq N$ . Thus  $\frac{\delta}{2} < |\langle x_m, x_k^* \rangle| \quad \forall m \geq N$  – contradicting that  $\langle x_m, x_k^* \rangle \xrightarrow{\infty} 0$ .

So, the set  $\{k_m : m \in \mathbb{N}\}$  is infinite. Taking a subsequence if necessary,

we may assume that  $k_1 < k_2 < \dots$

The subsequence  $(x_{k_m}^*)$  of  $(x_n^*)$  is in  $c_0^{weak^*}(X^*)$ .

Since  $A := \{x_m : m \in \mathbb{N}\} \cup \{0\}$  is weakly compact, we should therefore have by assumption that

$$\sup_{x \in A} |\langle x, x_{k_m}^* \rangle| \xrightarrow{\infty} 0; \text{ i.e.}$$

$$|\langle x_m, x_{k_m}^* \rangle| \leq \sup_{x \in A} |\langle x, x_{k_m}^* \rangle| \xrightarrow{\infty} 0$$

– contradicting (\*).

We conclude that  $T$  is completely continuous.  $\square$

The proof of Theorem 2.2.5 as is discussed in [3] relies on an earlier result about the coincidence of weak Hadamard and Gateaux differentiability, which was proved in the paper [2]. We are now ready to discuss the proof without referring to these kind of differentiabilitys and by relying on the characterizations Theorem 2.2.8 and Theorem 2.2.13 only.

**Proof** Proof of Theorem 2.2.5

(a)  $\Rightarrow$  (b): Assume  $X$  has  $DP^*P$ . Let  $Y$  be a Banach space for which  $\mathcal{B}_{Y^*}$  is weak\* sequentially compact. Assume there exists a  $T \in \mathcal{L}(X, Y)$  which is not completely continuous, i.e. there is a  $(x_n) \in c_0^{weak}(X)$  such that  $\|Tx_n\| \not\rightarrow 0$  if  $n \rightarrow \infty$ . Using a subsequence (which for simplicity we denote by  $(Tx_n)$  again) we may assume there exists  $\varepsilon > 0$  such that  $\|Tx_n\| \geq \varepsilon$  for all  $n \in \mathbb{N}$ . By Hahn-Banach we may find for each  $n \in \mathbb{N}$  a  $y_n^* \in Y^*$  such that  $\|y_n^*\| = 1$  and  $y_n^*(Tx_n) = \|Tx_n\|$ . The unit ball  $\mathcal{B}_{Y^*}$  being weak\* sequentially compact, there exists a subsequence  $(y_{n_k}^*)$  such that  $y_{n_k}^* \xrightarrow{\infty} y^*$  weak\* in  $\mathcal{B}_{Y^*}$ . Since  $T^*$  is weak\* continuous and  $X$  has  $DP^*P$ , it follows that

$$\langle x_{n_k}, T^*(y_{n_k}^* - y^*) \rangle \xrightarrow{\infty} 0.$$

Moreover, since  $x_{n_k} \xrightarrow{\infty} 0$  weakly, we also have  $\langle x_{n_k}, T^*y^* \rangle \xrightarrow{\infty} 0$ . Together this gives us  $\langle x_{n_k}, T^*y_{n_k}^* \rangle \xrightarrow{\infty} 0$ , contradicting that  $\langle x_{n_k}, T^*y_{n_k}^* \rangle = \|Tx_{n_k}\| \geq \varepsilon$  for all  $k \in \mathbb{N}$ . Thus  $T$  should be completely continuous.

(b)  $\Rightarrow$  (a): If we let  $Y = c_0$ , then  $\mathcal{B}_{Y^*} = \mathcal{B}_{\ell_1}$  is weak\* sequentially compact. By our assumption it follows that each  $T \in \mathcal{L}(X, c_0)$  is completely continuous. By Theorem 2.2.13 this implies that  $X$  has  $DP^*P$ .  $\square$

Recall that a sequence  $(e_n)$  in a Banach space  $X$  is called a “basic sequence” if  $(e_n)$  is a Schauder basis for  $\text{span}\{e_n : n \in \mathbb{N}\} \subseteq X$ . We refer the reader to [1], Theorem 10.2.1 for a proof of the following fundamental theorem in Banach space theory:

**Theorem 2.2.14** (*Rosenthal’s  $\ell_1$  Theorem*) *Let  $(x_n)$  be a bounded sequence in an infinite dimensional Banach space  $X$ . Then:*

- (a) *Either  $(x_n)$  has a subsequence which is weakly Cauchy,*
- (b) *or  $(x_n)$  has a subsequence which is basic and equivalent to the canonical basis of  $\ell_1$ .*

**Theorem 2.2.15** (*cf. [7], Theorem 2.2*)  *$X$  has the  $DP^*P$ , if and only if,  $X$  has the  $DPP$  and every quotient mapping  $q : X \rightarrow c_0$  is completely continuous.*

**Proof** Clearly we only need to show that under these conditions,  $X$  has  $DP^*P$ . Suppose that  $\phi_n \xrightarrow{wk^*} 0$  in  $X^*$ ,  $x_n \xrightarrow{wk} 0$  in  $X$  and that  $\phi_n(x_n) \not\rightarrow 0$  as  $n \rightarrow \infty$ .

There exists  $\delta > 0$  such that for all  $n \in \mathbb{N}$  there exists  $k_n \geq n$ , with  $|\phi_{k_n}(x_{k_n})| \geq \delta$ .

Since the subsequences  $(\phi_{k_n})$  and  $(x_{k_n})$  also satisfy  $\phi_{k_n} \rightarrow 0$  weak\*,  $x_{k_n} \rightarrow 0$  weak and  $\phi_{k_n}(x_{k_n}) \not\rightarrow 0$  as  $n \rightarrow \infty$ , we may assume for simplicity that

$$|\phi_n(x_n)| \geq \delta \text{ for all } n \in \mathbb{N}.$$

If  $x_m$  is given, then  $\phi_n(x_m) \xrightarrow[n \rightarrow \infty]{} 0$ . Therefore, there exists  $n_m \in \mathbb{N}$  such that  $n_m \geq m$  and  $|\phi_n(x_m)| < \frac{\delta}{2} \forall n \geq n_m$ . Thus

$$\begin{aligned} |(\phi_{n_m} - \phi_m)(x_m)| &\geq ||\phi_{n_m}(x_m)| - |\phi_m(x_m)|| \\ &\geq |\phi_m(x_m)| - |\phi_{n_m}(x_m)| \\ &\geq \delta - \frac{\delta}{2} \\ &= \frac{\delta}{2}. \quad (*) \end{aligned}$$

Because of Theorem 2.2.14 there are two possibilities (either (i) or (ii) below holds):

- (i)  $(\phi_n)$  is weakly Cauchy.
- (ii)  $(\phi_n)$  is equivalent to the  $\ell_1$  basis.

Suppose (i):

Then for all  $x^{**} \in X^{**}$ , we have that

$$\langle \phi_j - \phi_k, x^{**} \rangle \longrightarrow 0 \text{ if } j, k \longrightarrow \infty.$$

In particular,

$$\langle \phi_{n_m} - \phi_m, x^{**} \rangle \xrightarrow[\infty]{m} 0.$$

Therefore,  $(\phi_{n_m} - \phi_m)$  is weakly null in  $X^*$ . Since  $X$  has the *DPP*, it follows that

$$\lim_m (\phi_{n_m} - \phi_m)(x_m) = 0,$$

contradictory with (\*).

Suppose (ii):

Consider the mapping  $q : X \rightarrow c_0$ , defined by  $q(x) = (\phi_n(x))$ .

Then  $q^* : \ell_1 \rightarrow X^*$  is given by

$$\begin{aligned} \langle q^*((\alpha_i)), x \rangle &= \langle (\alpha_i), q(x) \rangle \\ &= \langle (\alpha_i), (\phi_i(x)) \rangle \\ &= \sum_{i=1}^{\infty} \alpha_i \phi_i(x) \\ &= \left( \sum_{i=1}^{\infty} \alpha_i \phi_i \right)(x), \quad \forall x \in X. \end{aligned}$$

Therefore,  $q^*((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i \phi_i$ , and

$$\begin{aligned}
\|q^*((\alpha_i))\| &= \left\| \sum_{i=1}^{\infty} \alpha_i \phi_i \right\| \\
&= \sup_{\|x\| \leq 1} \left| \left( \sum_{i=1}^{\infty} \alpha_i \phi_i \right)(x) \right| \\
&= \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{\infty} \alpha_i \phi_i(x) \right| \\
&\leq \sup_{\|x\| \leq 1} \sup_i |\phi_i(x)| \sum_{i=1}^{\infty} |\alpha_i| \\
&= \left( \sup_i \|\phi_i\| \right) \|(\alpha_i)\|_1.
\end{aligned}$$

Thus  $q^*$  is bounded, with  $\|q^*\| \leq \sup_i \|\phi_i\|$ .

Now, since  $(\phi_n) \sim (e_n)$ , by Theorem 1.1.8, there exists a  $C > 0$  such that

$$C^{-1} \left\| \sum_{i=1}^{\infty} \alpha_i \phi_i \right\|_{X^*} \leq \left\| \sum_{i=1}^{\infty} \alpha_i e_i \right\|_{\ell_1} \leq C \left\| \sum_{i=1}^{\infty} \alpha_i \phi_i \right\|_{X^*},$$

implies that

$$C^{-1} \|q^*((\alpha_i))\| \leq \|(\alpha_i)\|_{\ell_1} \leq C \|q^*((\alpha_i))\|. \quad (**)$$

From (\*\*) we have:

- (i)  $q^*$  is continuous (bounded) and  $\|q^*((\alpha_i))\| \leq C \|(\alpha_i)\|_{\ell_1}$ .  
Therefore,  $\|q^*\| \leq C$ .
- (ii)  $q^*$  is 1-1-mapping:

$$\begin{aligned}
q^*((\alpha_i)) = 0 &\Rightarrow \|(\alpha_i)\|_{\ell_1} \leq C \|q^*((\alpha_i))\| = 0 \\
&\Rightarrow (\alpha_i) = 0.
\end{aligned}$$

Thus  $\ker q^* = \{0\}$ .

Consider  $(q^*)^{-1} : \mathcal{R}(q^*) \rightarrow \ell_1$ .

Then  $(q^*)^{-1}(q^*((\alpha_i))) = (\alpha_i)$ , and

$$\|(q^*)^{-1}(q^*((\alpha_i)))\|_{\ell_1} = \|(\alpha_i)\|_{\ell_1} \leq C\|q^*((\alpha_i))\|.$$

Therefore  $(q^*)^{-1}$  is continuous and  $\|(q^*)^{-1}\| \leq C$ .

Thus  $q^*$  is an isomorphic embedding.

This shows that  $q$  is a quotient mapping onto  $c_0$  by Proposition VI.1.8 in [9], since  $\mathcal{R}(q)^\perp = \ker q^* = \{0\}$ . By our assumption,  $q$  is completely continuous. Therefore,  $q(x_n) \xrightarrow[\infty]{n} 0$  in norm.

This implies that

$$\|(\phi_j(x_n))_j\|_{c_0} = \|q(x_n)\| \xrightarrow[\infty]{n} 0,$$

i.e.  $\sup_j |\phi_j(x_n)| \xrightarrow[\infty]{n} 0$ .

However,

$$\sup_j |\phi_j(x_n)| \geq |\phi_n(x_n)| \geq \delta$$

for all  $n$ .

This contradiction leaves us with no other choice than to conclude that  $X$  has  $DP^*P$ .  $\square$

**Remark 2.2.16** *Suppose  $X$  has the property that  $X^*$  does not contain a copy of  $\ell_1$ . If we assume that  $X$  has DPP, then in the proof of (2.2.15), we are left with the option that  $(\phi_n)$  is a weakly Cauchy sequence (as given by Rosenthal's  $\ell_1$ -theorem).*

*We then do not need that every quotient mapping  $q : X \rightarrow c_0$  is completely continuous when we prove that  $X$  has  $DP^*P$ ; thus in this case DPP and  $DP^*P$  are equivalent.*

We end this chapter with a discussion of operators  $T \in \mathcal{L}(X, Y)$  which “fix a copy of  $\ell_1$ ”, meaning that  $T(X)$  contains a subspace isomorphic to  $\ell_1$ . First we introduce a class of operators, called “limited operators” which do not satisfy this property and then show how the  $DP^*P$  on  $X$  and  $Y$  assures that non-limited operators do fix copies of  $\ell_1$ .

**Definition 2.2.17** (i)  $T : X \rightarrow Y$  is called limited if  $T(B_X)$  is contained in a limited subset of  $Y$ .

(ii)  $T : X \rightarrow Y$  is called conditionally weakly compact if  $T(B_X)$  is contained in a conditionally weakly compact subset of  $Y$ .

**Theorem 2.2.18** Assume  $X$  and  $Y$  have the  $DP^*P$ . If  $T : X \rightarrow Y$  is a non-limited operator, then  $T$  fixes a copy of  $\ell_1$ .

**Proof** If  $T$  is non-limited, then  $T(B_X)$  is not a limited set in  $Y$ ; thus there exists a sequence  $(\phi_n) \subset Y^*$  such that  $\phi_n \xrightarrow{wk^*} 0$ , but the convergence is not uniform on  $T(B_X)$ . Therefore, there exists an  $\varepsilon > 0$  such that for all  $k \in \mathbb{N}$  we can find  $x_k \in B_X$  and  $n_k \in \mathbb{N}$  so that

$$(2.1) \quad |\phi_{n_k}(Tx_k)| \geq \varepsilon.$$

The subsequence  $(\phi_{n_k})$  of  $(\phi_n)$  converges weak\* to 0 and satisfies (2.1). For simplicity we assume that the sequence  $(\phi_k) \subset Y^*$  satisfies  $\phi_k \xrightarrow{wk^*} 0$  and

$$|\phi_k(Tx_k)| \geq \varepsilon, \quad \forall k \in \mathbb{N}.$$

**Claim 2.2.19** The sequence  $(x_k)$  has no weakly Cauchy subsequences.

We first prove the claim: If (2.2.19) is false, there exists a subsequence of  $(x_k)$  which is weakly Cauchy. Again, for simplicity, we assume that  $(x_k)$  itself is weakly Cauchy. For each  $m \in \mathbb{N}$ ,  $\phi_n(Tx_m) \xrightarrow[n \rightarrow \infty]{} 0$ ; i.e. there is a  $n_m \in \mathbb{N}$  such that

$$|(\phi_{n_m} \circ T)(x_m)| < \frac{\varepsilon}{2}.$$

We also have,

$$\begin{aligned} |\phi_{n_m}(Tx_{n_m} - Tx_m)| &\geq |\phi_{n_m}(Tx_{n_m})| - |\phi_{n_m}(Tx_m)| \\ &\geq \varepsilon - \frac{\varepsilon}{2} \\ &= \frac{\varepsilon}{2}. (*) \end{aligned}$$

However,  $(x_k)$  being weakly Cauchy, the sequence  $(x_{n_m} - x_m)_{m \in \mathbb{N}}$  is weakly null in  $X$  and  $\phi_{n_m} \circ T \xrightarrow{wk^*} 0$  if  $m \rightarrow \infty$  in  $X^*$ . It follows from the  $DP^*P$  (on  $X$ ) that

$$(\phi_{n_m} \circ T)(x_{n_m} - x_m) \xrightarrow[m \rightarrow \infty]{} 0.$$

This contradicts (\*) above. Thus the claim holds. The sequence  $(x_k)$  being bounded in  $X$ , the Rosenthal's  $\ell_1$ -theorem therefore implies the existence of a subsequence of  $(x_n)$  which is equivalent to the  $\ell_1$  bases. Denote this subsequence by  $(x_n)$  again.

**Claim 2.2.20**  $(Tx_n)$  does not have a weakly Cauchy subsequence.

Again, if (2.2.20) is false, we may assume that  $(Tx_n)$  is weakly Cauchy. As before, we find a subsequence  $(\phi_{n_m})$  of  $(\phi_n)$  such that

$$|\phi_{n_m}(Tx_m)| < \frac{\varepsilon}{2}$$

and

$$(2.2) \quad |\phi_{n_m}(Tx_{n_m} - Tx_m)| \geq \frac{\varepsilon}{2}, \quad \forall m \in \mathbb{N}.$$

Now,  $Tx_{n_m} - Tx_m \xrightarrow{wk} 0$  if  $m \rightarrow \infty$ . Since  $Y$  has  $DP^*P$ , it follows that  $\phi_{n_m}(Tx_{n_m} - Tx_m) \xrightarrow{m} 0$ , contradicting (2.2) above. Again by Rosenthal's  $\ell_1$ -theorem there is a subsequence of  $(Tx_n)$  which is equivalent to the  $\ell_1$  basis. Therefore a copy of  $\ell_1$  in  $Y$  is fixed by  $T$ ; i.e. there is a subspace of  $T(X) \subseteq Y$  which is isomorphic to  $\ell_1$ .  $\square$

# Chapter 3

## Polynomials and analytic functions on Banach spaces

### 3.1 Polynomials

In this chapter we will follow the conventions used by Dineen (cf. [14], pages 2-16) and use the following notation:

**Notation 3.1.1** *Let  $E$  and  $F$  be vector spaces over  $\mathbb{C}$ . Then:*

- (a) *We denote by  $x^n y^m$  the element  $(x, x, \dots, x, y, y, \dots, y)$  of  $E^n \times F^m$ , where  $x \in E^n$  and  $y \in F^m$ .*
- (b) *We denote by  $\mathcal{L}_a(^n E; F)$  the space of all  $n$ -linear mappings from  $E$  to  $F$ , i.e.  $\mathcal{L}_a(^n E; F) = \{f : E^n \rightarrow F : f \text{ is } n\text{-linear}\}$ .
  - (i) *When  $n = 1$ :  $f \in \mathcal{L}_a(E; F)$  is linear.*
  - (ii) *When  $n = 2$ :  $\mathcal{B}_a(E; F) = \mathcal{L}_a(^2 E; F)$  is the space of 2-linear (or bilinear) functions from  $E^2$  into  $F$ .**
- (c) *When  $F = \mathbb{C}$ , we let  $\mathcal{L}_a(^n E) = \mathcal{L}_a(^n E; \mathbb{C})$  and  $\mathcal{L}_a(E) = E'$ . Also, we let  $\mathcal{L}_a(^0 E, F)$  be the set of constant mappings from  $E$  into  $F$ . If  $\alpha$  is a seminorm on  $F$  and  $A \subseteq \mathcal{D}(f)$ , we let*

$$\|f\|_{\alpha, A} := \sup_{x \in A} \alpha(f(x)).$$

(d) Let  $E, F$  and  $G$  be vector spaces over  $\mathbb{C}$  and  $m, n \in \mathbb{N}$ . Then the mappings  $I_m$  and  $J_m$  are given by:

$$I_m : \mathcal{L}_a^{(m+n} E; F) \rightarrow \mathcal{L}_a^{(m} E; \mathcal{L}_a^{(n} E; F)) : A \mapsto I_m A,$$

with

$$[I_m A(x)](y) := A(x; y), \text{ where}$$

$$A \in \mathcal{L}_a^{(m+n} E; F), \quad x \in E^m \text{ and } y \in E^n;$$

$$J_m : \mathcal{L}_a^{(m} E; \mathcal{L}_a^{(n} F; G)) \rightarrow \mathcal{L}_a^{(n} F; \mathcal{L}_a^{(m} E; G)), \text{ such that}$$

$$[J_m A(y)](x) := [A(x)](y), \text{ where}$$

$$A \in \mathcal{L}_a^{(m} E; \mathcal{L}_a^{(n} F; G)), \quad y \in F^n \text{ and } x \in E^m.$$

**Remark 3.1.2** The mappings  $I_m$  and  $J_m$  can be shown to be linear isomorphisms.

A particular case, in the linear theory when  $m, n = 1$  and  $F = \mathbb{C}$ , is when

$$\mathcal{L}_a^{(2} E) \simeq \mathcal{L}_a(E, E'), \text{ by the isomorphism } I_1.$$

In this case, for  $\phi : E \times E \rightarrow \mathbb{C}$ , we define the mapping  $T_\phi : E \rightarrow \mathcal{L}(E, \mathbb{C}) = E'$  by  $(T_\phi x)(y) = \phi(x, y)$ . Then  $I_1 \phi = T_\phi$  defines the isomorphism.

**Definition 3.1.3** The diagonal mapping  $\Delta_n : E \rightarrow E^n$  is defined by  $\Delta_n x = (x, x, \dots, x)$ .

**Definition 3.1.4** Let  $E, F$  be vector spaces over  $\mathbb{C}$ . A mapping  $P : E \rightarrow F$  is called an  $n$ -homogeneous polynomial from  $E$  to  $F$  if there exists an element  $L \in \mathcal{L}_a^{(n} E; F)$  such that  $P = L \circ \Delta_n$ ; that is we have  $P(x) = L(x, x, \dots, x)$  for all  $x \in E$ .

**Definition 3.1.5** An  $n$ -linear mapping  $L$  from  $E^n$  into  $F$  is said to be symmetric if

$$L(x_1, x_2, \dots, x_n) = L(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}),$$

for any  $(x_1, x_2, \dots, x_n) \in E^n$  and any permutation  $\sigma$  of  $\{1, 2, \dots, n\}$ .

**Notation 3.1.6** (a) We let  $\mathcal{P}_a^{(n} E; F)$  be the vector space of all  $n$ -homogeneous polynomials from  $E$  to  $F$ . A polynomial from  $E$  into  $F$  is any finite sum of homogeneous polynomials from  $E$  into  $F$ . The vector space of all polynomials from  $E$  into  $F$  is denoted by  $\mathcal{P}_a(E; F)$ .

(b) We let  $\mathcal{L}_a^s({}^n E; F)$  be the vector space of all symmetric  $n$ -linear mappings from  $E$  into  $F$ .

**Remark 3.1.7** Given a  $L \in \mathcal{L}_a({}^n E; F)$ , we associate with  $L$  a symmetric  $n$ -linear mapping by

$$s(L)(x_1, x_2, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} L(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$$

where  $S_n$  is the set of all permutations of the first  $n$  natural numbers.

The following properties are easily verified (cf. [14], page 6):

- (a)  $s(L) \in \mathcal{L}_a^s({}^n E; F)$  for all  $L \in \mathcal{L}_a({}^n E; F)$ .
- (b)  $s(L) = L$  if and only if  $L \in \mathcal{L}_a^s({}^n E; F)$ .
- (c)  $s : \mathcal{L}_a({}^n E; F) \rightarrow \mathcal{L}_a^s({}^n E; F) : L \mapsto s(L)$  is an idempotent mapping, i.e.  $s^2 = s$ .
- (d)  $s$  is a linear operator.
- (e) If  $L \in \mathcal{L}_a({}^n E; F)$  and  $x \in E$ , then  $L(x^n) = s(L)(x^n)$ .

It follows from (a), (b), (c) and (d) that  $s : \mathcal{L}_a({}^n E; F) \rightarrow \mathcal{L}_a^s({}^n E; F)$  is a surjective linear projection. Also, if we let  $\hat{L} = L \circ \Delta_n$ , then it is clear from (e) that

$$\begin{aligned} \widehat{s(L)}(x) &= (s(L) \circ \Delta_n)(x) \\ &= s(L)(x^n) \\ &\stackrel{(e)}{=} L(x^n) \\ &= (L \circ \Delta_n)(x) \\ &= \hat{L}(x) \end{aligned}$$

for all  $x \in E$ .

We call  $s(L)$  the symmetrization of  $L$ . Clearly, since  $\widehat{s(L)} = \hat{L}$  for all  $L \in \mathcal{L}_a({}^n E; F)$ , we have  $P \in \mathcal{P}_a({}^n E; F)$  if and only if there exists a symmetric  $n$ -linear mapping  $L \in \mathcal{L}_a^s({}^n E; F)$  such that  $P = L \circ \Delta_n$ , i.e. such that

$$P = \hat{L}.$$

Given  $L \in \mathcal{L}_a^s({}^n E; F)$  we obtain a  $P \in \mathcal{P}_a({}^n E; F)$  by  $P = \hat{L}$ . Suppose  $P \in \mathcal{P}_a({}^n E; F)$  is given. How do we obtain  $L \in \mathcal{L}_a^s({}^n E; F)$  such that  $P = \hat{L}$ ? The answer is given by the following classical result:

**Corollary 3.1.8** (cf. [14], Corollary 1.6) (Polarization formula) *Let  $P \in \mathcal{P}_a({}^n E; F)$ , where  $E$  and  $F$  are vector spaces over  $\mathbb{C}$ , and let  $L \in \mathcal{L}_a^s({}^n E; F)$ . If  $\hat{L} = P$ , then*

$$L(x_1, x_2, \dots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1}^n \epsilon_i x_i\right).$$

Since we can obtain the result (3.1.8) (refer to [14], p.9) it follows that the mapping  $L \mapsto \hat{L}$  from  $\mathcal{L}_a^s({}^n E; F)$  to  $\mathcal{P}_a({}^n E; F)$  is a linear isomorphism, i.e. the spaces  $\mathcal{L}_a^s({}^n E; F)$  and  $\mathcal{P}_a({}^n E; F)$  are linearly isomorphic.

So far we have considered algebraic results in the theory of polynomials. We now embark on a brief discussion of topological results which in the book of Dineen (cf. [14], page 11) pave the way for the study of continuous holomorphic mappings between locally convex spaces.

We suppose that  $E, F$  are locally convex spaces over  $\mathbb{C}$ . For a seminorm  $\alpha$  on  $F$  and  $x \in F$ , we let

$$B_\alpha(x; r) := \{y \in F : \alpha(x - y) < r\} \text{ and} \\ \bar{B}_\alpha(x; r) := \{y \in F : \alpha(x - y) \leq r\}.$$

If  $x = 0$ , we write  $B_\alpha(r)$  and  $\bar{B}_\alpha(r)$  respectively. Now suppose  $\alpha$  is a continuous seminorm on  $F$  and let  $F_\alpha := (F, \alpha)/\alpha^{-1}(0)$  be the quotient space, where  $(F, \alpha)$  is the seminormed space defined by the vector space  $F$  and seminorm  $\alpha$  and where on  $F_\alpha$  we consider the quotient topology. Let  $\pi_\alpha$  denote the quotient mapping. Then  $\alpha$  defines a quotient norm on  $F_\alpha$  by

$$\|x + \alpha^{-1}(0)\| = \inf\{\alpha(y) : x - y \in \alpha^{-1}(0)\},$$

since

$$\begin{aligned} \|x + \alpha^{-1}(0)\| = 0 &\Leftrightarrow \inf\{\alpha(y) : x - y \in \alpha^{-1}(0)\} = 0 \\ &\Leftrightarrow x \in \alpha^{-1}(0) \\ &\Leftrightarrow x + \alpha^{-1}(0) = \alpha^{-1}(0). \end{aligned}$$

Recall that  $f : E \rightarrow F$  is continuous if and only if  $\pi_\alpha \circ f$  is continuous for all continuous seminorms  $\alpha$  on  $F$ . Therefore, a study of polynomials and holomorphic functions taking their values in normed spaces is sufficient.

**Notation 3.1.9** (a) We let  $\mathcal{P}(^n E; F)$ ,  $\mathcal{L}(^n E; F)$  and  $\mathcal{L}^s(^n E; F)$  denote respectively the spaces of continuous  $n$ -homogeneous polynomials from  $E$  into  $F$ , the continuous  $n$ -linear mappings from  $E$  into  $F$  and the continuous symmetric  $n$ -linear mappings from  $E$  into  $F$ . If  $F = \mathbb{C}$ , these spaces are denoted by  $\mathcal{P}(^n E)$ ,  $\mathcal{L}(^n E)$  and  $\mathcal{L}^s(^n E)$  respectively. In case of  $n = 1$ , we write  $E^* = \mathcal{L}(E) = \mathcal{L}(E, \mathbb{C})$ , i.e.  $E^*$  is the (continuous) dual space of  $E$ .

(b) We denote the mapping

$$\mathcal{L}^s(^n E; F) \rightarrow \mathcal{P}(^n E; F) : L \mapsto \hat{L}$$

by  $\hat{\cdot}$ , i.e.  $\hat{L}(x) = \hat{L}(x)$ . Since this is an isomorphism by Corollary 3.1.8, we may consider the inverse  $\check{\cdot} : \mathcal{P}(^n E; F) \rightarrow \mathcal{L}^s(^n E; F)$  such that  $\check{\cdot}(P) = L$ , where  $\hat{L} = P$  (or where  $L$  is given by the Polarization formula (Corollary 3.1.8)). Hence  $\hat{L} = P$  and  $\check{P} = L$ .

Using the Binomial Theorem and the one variable Maximum Modulus Theorem (refer to the proof in [14] on page 12), the following can be derived:

**Lemma 3.1.10** (cf. [14], Lemma 1.9) Let  $E$  and  $F$  be vector spaces over  $\mathbb{C}$ ,  $P \in \mathcal{P}_a(^n E; F)$  and  $x, y \in E$ . Then

$$(a) \quad P(x + y) = \sum_{j=0}^n \binom{n}{j} \check{P}(x^j, y^{n-j})$$

$$P(x) - P(y) = \sum_{j=0}^{n-1} \binom{n}{j} \check{P}(y^j, (x - y)^{n-j}).$$

(b) If  $F = \mathbb{C}$ , then

$$\sup_{|\lambda| \leq 1} |P(x + \lambda y)|^2 \geq |P(x)|^2 + |P(y)|^2.$$

**Definition 3.1.11** (cf. [14], page 14) If  $B$  is a convex balanced subset of the vector space  $E$ , we consider the Minkowski functional  $\|\cdot\|_B$  of  $B$ , given by

$$\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\} \text{ and}$$

$\|x\|_B = \infty$  if  $x \notin \lambda B$  for all  $\lambda > 0$  where  $x \in E$ .

We have

$$\{x : \|x\|_B < 1\} \subset B \subset \{x : \|x\|_B \leq 1\}.$$

Consider the subspace  $E_B = \bigcup_{n>0} nB$  of  $E$ . Then  $\|\cdot\|_B$  is a seminorm on  $E_B$ .

It is proved in [14] (Lemma 1.10, page 14) that:

**Lemma 3.1.12** *Let  $E$  and  $F$  be vector spaces over  $\mathbb{C}$ ,  $P \in \mathcal{P}_a({}^n E; F)$ ,  $B \subset E$  and let  $\alpha$  denote a seminorm on  $F$ . Then:*

(a) *For a balanced  $A \subset E$ , let*

$$\|P\|_{\alpha, A} := \sup_{y \in A} \alpha(P(y)) \text{ and } \|\check{P}\|_{\alpha, A^n} = \sup_{(y_1, \dots, y_n) \in A^n} \alpha(\check{P}(y_1, \dots, y_n)).$$

*If  $B$  is balanced and  $x \in E$ , then  $\|P\|_{\alpha, B} \leq \|P\|_{\alpha, x+B}$ , where  $x+B = \{x+y : y \in B\}$ .*

(b) *If  $B$  is convex and balanced,  $\lambda \in \mathbb{R}$ ,  $\lambda > 0$  and  $\lambda x \in B$ , then*

$$\|P\|_{\alpha, x+B} \leq (1 + \frac{1}{\lambda})^n \|P\|_{\alpha, B}.$$

(c) *If  $B$  is convex and balanced,  $x$  and  $y$  in  $B$ ,  $\|y\|_B \leq \varepsilon$  and  $\|x-y\|_B \leq 1$ , then  $\alpha(P(x) - P(y)) \leq \frac{n^n}{n!} (1 + \varepsilon)^n \|x-y\|_B \|P\|_{\alpha, B}$ .*

The reader is referred to [14] for the proofs of (a) and (b). To gain some understanding of the arguments used in these kind of proofs and to demonstrate the role played by the Polarization formula, we prove (c):

**Proof** By Lemma 3.1.10(a) and the Polarization formula,

$$\begin{aligned} \alpha(P(x) - P(y)) &= \alpha \left( \sum_{r=0}^{n-1} \binom{n}{r} \check{P}(y^r, (x-y)^{n-r}) \right) \\ &\leq \sum_{r=0}^{n-1} \binom{n}{r} \alpha(\check{P}(y^r, (x-y)^{n-r})). \end{aligned}$$

Now, since for  $z \in E$  we have  $z^s \in B^s \Leftrightarrow z \in B$ , it follows that

$$\frac{y^r}{\|y\|_B^r} \in B^r \text{ and } \frac{(x-y)^{n-r}}{\|x-y\|_B^{n-r}} \in B^{n-r},$$

i.e.

$$\left( \frac{y^r}{\|y\|_B^r}, \frac{(x-y)^{n-r}}{\|x-y\|_B^{n-r}} \right) \in B^n.$$

Thus,

$$\begin{aligned} \|\check{P}\|_{\alpha, B^n} &\geq \alpha \left( \check{P} \left( \frac{y^r}{\|y\|_B^r}, \frac{(x-y)^{n-r}}{\|x-y\|_B^{n-r}} \right) \right) \\ &= \alpha \left( \check{P} \left( \frac{y}{\|y\|_B}, \frac{y}{\|y\|_B}, \dots, \frac{y}{\|y\|_B}, \frac{x-y}{\|x-y\|_B}, \frac{x-y}{\|x-y\|_B}, \dots, \frac{x-y}{\|x-y\|_B} \right) \right) \\ &= \frac{1}{\|y\|_B^r} \frac{1}{\|x-y\|_B^{n-r}} \alpha(\check{P}(y^r, (x-y)^{n-r})). \end{aligned}$$

Using this inequality and the Polarization formula we obtain

$$\begin{aligned} \alpha(P(x) - P(y)) &\leq \sum_{r=0}^{n-1} \binom{n}{r} \|\check{P}\|_{\alpha, B^n} \|y\|_B^r \|x-y\|_B^{n-r} \\ &\leq \sum_{r=0}^{n-1} \binom{n}{r} \frac{n^n}{n!} \|P\|_{\alpha, B} \|y\|_B^r \|x-y\|_B^{n-r} \\ &\leq \frac{n^n}{n!} \|P\|_{\alpha, B} \|x-y\|_B \sum_{r=0}^{n-1} \binom{n}{r} \|y\|_B^r \\ &\leq \frac{n^n}{n!} \|P\|_{\alpha, B} \|x-y\|_B (1+\varepsilon)^n, \text{ since} \end{aligned}$$

$$\sum_{r=0}^{n-1} \binom{n}{r} \varepsilon^r = (1+\varepsilon)^n \text{ by the Binomial Theorem.} \quad \square$$

Using Lemma 3.1.12 it is proved in ([14], page 15) that

**Proposition 3.1.13** *Let  $E$  be a locally convex space,  $F$  a normed linear space and suppose  $P \in \mathcal{P}_a(nE; F)$ . The following are equivalent:*

- (a)  $P$  is locally uniformly continuous (i.e. for each  $x \in E$  there exists a neighbourhood  $V$  of  $x$  such that  $P|_V$  is uniformly continuous).
- (b)  $P$  is everywhere continuous.
- (c)  $P$  is continuous at some point.
- (d)  $P$  is bounded on a neighbourhood of some point in  $E$ .

(e)  $P$  is a locally bounded function (i.e. every point in  $E$  contains a neighbourhood on which  $P$  is bounded).

**Corollary 3.1.14** (cf. [14], Corollary 1.12) *Let  $E$  and  $F$  be locally convex spaces over  $\mathbb{C}$  and let  $P \in \mathcal{P}_a({}^n E; F)$ . Then  $P \in \mathcal{P}({}^n E; F)$  if and only if  $P$  is continuous at one point.*

Next we consider a useful factorisation lemma. If  $E$  and  $F$  are locally convex spaces,  $\alpha$  is a continuous seminorm on  $E$  and  $P \in \mathcal{P}({}^n E_\alpha; F)$ , then

$$\pi_\alpha^*(P) := P \circ \pi_\alpha \in \mathcal{P}({}^n E; F), \text{ i.e.}$$

$$\pi_\alpha^* : \mathcal{P}({}^n E_\alpha; F) \rightarrow \mathcal{P}({}^n E; F) : P \mapsto P \circ \pi_\alpha.$$

Denote by  $cs(E)$  the set of all continuous seminorms on  $E$ . Of course the union

$$\bigcup_{\alpha \in cs(E)} \pi_\alpha^*(\mathcal{P}({}^n E_\alpha; F))$$

can (by the above identification) then be considered to be a subset of  $\mathcal{P}({}^n E; F)$ . Moreover, when  $F$  is a normed linear space we have the following ‘‘Factorization Lemma’’

**Lemma 3.1.15** (cf. [14], Lemma 1.13) *(Factorization Lemma) If  $E$  is a locally convex space and  $F$  is a normed sapce, then*

$$\mathcal{P}({}^n E; F) = \bigcup_{\alpha \in cs(E)} \pi_\alpha^*(\mathcal{P}({}^n E_\alpha; F)).$$

**Proof** Let  $P \in \mathcal{P}({}^n E; F)$ . Since  $F$  is a normed space, there exists a convex balanced neighbourhood  $V$  of zero such that by Proposition 3.1.13(e),

$$\|P\|_V = \sup_{x \in V} \|Px\| < \infty.$$

Let  $\alpha = \|\cdot\|_V$ . If  $x, y \in E$  and  $\alpha(y) = 0$ , then by Lemma 3.1.12(c) we have

$$\begin{aligned} \|P(x+y) - P(x)\| &\leq \frac{n^n}{n!} (1 + \alpha(x))^n \alpha(y) \|P\|_V \\ &= 0. \end{aligned}$$

Hence  $P(x+y) = P(x)$  for all  $y \in E$  such that  $\alpha(y) = 0$ . Let  $\hat{x} \in E_\alpha$  and suppose  $x_1, x_2 \in \hat{x}$  (i.e.  $\pi_\alpha x_1 = \pi_\alpha x_2 = \hat{x}$ ). Then  $x_1 - x_2 \in \alpha^{-1}(0)$ , i.e.

$$\alpha(x_1 - x_2) = 0.$$

Thus, by the above argument  $P(x_1) = P(x_2 + x_1 - x_2) = P(x_2)$ ; therefore  $\tilde{P} : E_\alpha \rightarrow F : \hat{x} \mapsto P(z)$  for any  $z \in \hat{x}$ , is well defined. Put

$$\tilde{L}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n \tilde{P}\left(\sum_{i=1}^n \epsilon_i \hat{x}_i\right).$$

If  $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n)$ , then  $\hat{x}_i = \hat{y}_i \forall i = 1, 2, \dots, n$  and hence  $\alpha(x_i - y_i) = 0$  for any choice of  $x_i \in \hat{x}_i$  and  $y_i \in \hat{y}_i$ . As before we have  $P(x_i) = P(y_i)$  for  $i = 1, 2, \dots, n$ . Also,

$$\begin{aligned} \tilde{L}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) &= \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1}^n \epsilon_i x_i\right) \\ &= \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1}^n \epsilon_i y_i\right), \end{aligned}$$

since  $\sum_{i=1}^n \epsilon_i y_i \in \sum_{i=1}^n \epsilon_i \hat{y}_i = \sum_{i=1}^n \epsilon_i \hat{x}_i$ .

Thus,  $\tilde{L}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = \tilde{L}(\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n)$ , showing that  $\tilde{L}$  is well defined.

Moreover,

$$\begin{aligned} &\tilde{L}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{j-1}, \hat{x}_j + \beta \hat{y}_j, \hat{x}_{j+1}, \dots, \hat{x}_n) \\ &= \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1}^n \epsilon_i x_i + \epsilon_j \beta y_j\right) \\ &= \tilde{P}(x_1, x_2, \dots, x_{j-1}, x_j + \beta y_j, x_{j+1}, \dots, x_n) \\ &= \tilde{P}(x_1, x_2, \dots, x_n) + \beta \tilde{P}(x_1, x_2, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n) \\ &= \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1}^n \epsilon_i x_i\right) + \beta \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1, i \neq j}^n \epsilon_i x_i + \epsilon_j y_j\right) \\ &= \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n \tilde{P}\left(\sum_{i=1}^n \epsilon_i \hat{x}_i\right) + \beta \frac{1}{2^{nn!}} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n \tilde{P}\left(\sum_{i=1, i \neq j}^n \epsilon_i \hat{x}_i + \epsilon_j \hat{y}_j\right) \\ &= \tilde{L}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) + \beta \tilde{L}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{j-1}, \hat{y}_j, \hat{x}_{j+1}, \dots, \hat{x}_n). \end{aligned}$$

Thus,  $\tilde{L} : E_\alpha^n \rightarrow F$  is  $n$ -linear. Also,

$$\begin{aligned}
(\tilde{L} \circ \Delta_n)(\hat{x}) &= \tilde{L}(\hat{x}, \hat{x}, \dots, \hat{x}) \\
&= \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n \tilde{P}\left(\sum_{i=1}^n \epsilon_i \hat{x}\right) \\
&= \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_n P\left(\sum_{i=1}^n \epsilon_i x\right) \\
&= \check{P}(x, x, \dots, x), \text{ with } x \in \hat{x} \\
&= P(x), \text{ with } x \in \hat{x} \\
&= \tilde{P}(\hat{x}).
\end{aligned}$$

Therefore,  $\tilde{L} \circ \Delta_n = \tilde{P}$ , showing that  $\tilde{P} \in \mathcal{P}_a({}^n E_\alpha; F)$ .

Also,

$$\begin{aligned}
\|\tilde{P}\|_{\pi_\alpha(V)} &= \sup_{\hat{x} \in \pi_\alpha(V)} \|\tilde{P}\hat{x}\| \\
&= \sup_{x \in V} \|Px\| \\
&= \|P\|_V \\
&< \infty.
\end{aligned}$$

Thus  $\tilde{P} \in \mathcal{P}({}^n E_\alpha; F)$ . By definition we have  $P = \tilde{P} \circ \pi_\alpha$ , i.e.  $P \in \pi_\alpha^*(\mathcal{P}({}^n E_\alpha; F))$  for this choice of  $\alpha \in cs(E)$ .  $\square$

Some interesting results concerning polynomials on Banach spaces with the *DPP* are considered in ([14]). It is for instance proved in ([14]) that:

**Proposition 3.1.16** *Let  $E$  be a Banach space with the Dunford-Pettis property and  $P \in \mathcal{P}({}^n E)$ , then  $P$  is weakly sequentially continuous.*

**Corollary 3.1.17** *If  $E$  is a Banach space with the Dunford-Pettis property and  $\ell_1$  is not isomorphic to a subspace of  $E$ , then continuous polynomials on  $E$  are (uniformly) weakly continuous on bounded sets.*

## 3.2 Holomorphic mappings between locally convex spaces

We now refer to [23] for the following information.  
Let  $E$  and  $F$  be Banach spaces.

**Definition 3.2.1** A power series from  $E$  into  $F$  around the point  $a \in E$  is a series of mappings of the form  $\sum_{m=0}^{\infty} P_m(x - a)$ , where  $P_m \in \mathcal{P}_a({}^m E; F)$  for every  $m \in \mathbb{N} \cup \{0\}$  (i.e. the  $m$ -th term in the series is the value of a  $m - 1$ -homogeneous polynomial at the point  $x - a \in E$  and the series converges in the norm of  $F$ ). Since we know that  $P_m = A_m \circ \Delta_m$  for some  $A_m \in \mathcal{L}_a^s({}^m E; F)$  (unique), we have  $\sum_{m=0}^{\infty} P_m(x - a) = \sum_{m=0}^{\infty} A_m((x - a)^m)$ ; hence  $\hat{A}_m = P_m$  for all  $m \in \mathbb{N} \cup \{0\}$ .

As usual, the radius of (uniform) convergence of the power series

$$\sum_{m=0}^{\infty} P_m(x - a)$$

is the supremum of all  $r \geq 0$  such that the series converges uniformly on the ball  $\bar{B}(a, r)$ . As we would expect, the radius of convergence  $R$  is also given by the Cauchy-Hadamard Formula

$$\frac{1}{R} = \limsup_{m \rightarrow \infty} \|P_m\|^{\frac{1}{m}}.$$

The reader is referred to ([23], Theorem 4.3) for the proof of this fact and that the series converges uniformly on  $\bar{B}(a, r)$  whenever  $0 \leq r < R$ . If for some  $0 < r < R$  we have  $\sum_{m=0}^{\infty} P_m(x - a) = 0$  for all  $x \in B(a, r)$ , then  $P_m = 0$  for every  $m \in \mathbb{N} \cup \{0\}$  (cf. [23], Proposition 4.4).

For the discussion on holomorphic mappings to follow, we assume  $E$  and  $F$  to be complex Banach spaces.

**Definition 3.2.2** Let  $U$  be an open subset of  $E$ . A mapping  $f : U \rightarrow F$  is said to be holomorphic (or analytic) if for each  $a \in U$  there exists a ball  $B(a, r) \subset U$  and a sequence of polynomials  $P_m \in \mathcal{P}({}^n E; F)$  such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - a),$$

uniformly on  $B(a, r)$ .

**Notation 3.2.3** The vector space of all holomorphic mappings from  $U$  into  $F$  is denoted by  $\mathcal{H}(U; F)$ . When  $F = \mathbb{C}$ , we write  $\mathcal{H}(U; \mathbb{C}) = \mathcal{H}(U)$ .

If  $f(x) = \sum_{m=0}^{\infty} P_m(x-a) = \sum_{m=0}^{\infty} Q_m(x-a)$ , for all  $x \in B(a, r)$  as in Definition 3.2.2, then

$$\sum_{m=0}^{\infty} (P_m - Q_m)(x-a) = 0, \text{ with } P_m - Q_m \in \mathcal{P}(^m E; F), \text{ i.e. } P_m = Q_m \text{ for all } m.$$

Therefore the sequence  $(P_n)$  in the power series expansion depends only on  $f$  and  $a \in U$ .

We call the unique power series expansion of  $f$  around  $a$  the Taylor series of  $f$  at  $a$ ; sometimes the notation  $P_m = P^m f(a)$  is used, i.e.

$$\sum_{m=0}^{\infty} P_m(x-a) = \sum_{m=0}^{\infty} P^m f(a)(x-a).$$

Moreover, the unique member  $\check{P}_m \in \mathcal{L}_s(^m E; F)$  such that  $P_m = \check{P}_m \circ \Delta_m$  is in this notation usually denoted by  $A^m f(a)$ , i.e.

$$f(x) = \sum_{m=0}^{\infty} A^m f(a)(x-a), \text{ with } (A^m f(a))^\wedge = P_m = P^m f(a).$$

**Remark 3.2.4**  $\mathcal{P}(E; F) \subset \mathcal{H}(E; F)$ , i.e. each polynomial (as a linear combination of  $m$ -homogeneous polynomials) is holomorphic.

**Examples 3.2.5** (cf. [23], pages 34-36)

(i) Let  $P_m \in \mathcal{P}(^m E; F)$  for all  $m \in \mathbb{N} \cup \{0\}$  and suppose that  $\sum_{m=0}^{\infty} P_m(x)$  has an infinite radius of convergence. Then the mapping

$$f(x) = \sum_{m=0}^{\infty} P_m(x) \quad (x \in E)$$

is holomorphic from  $E$  into  $F$ .

(ii) An important example, which we will use in the proof of Theorem 4.2.5 is discussed in [23] (cf. [23], Example 5.5, page 35). Suppose  $(\phi_m) \subset E^*$  converges weak\* to 0. If we define

$$f(x) = \sum_{m=0}^{\infty} (\phi_m(x))^m$$

for all  $x \in E$ , then

$$f \in \mathcal{H}(E) = \mathcal{H}(E; \mathbb{C}).$$

**Remark 3.2.6** *Many properties of holomorphic mappings in Banach spaces can be derived from corresponding properties of holomorphic functions of one complex variable with the aid of the following result (cf. [23], Lemma 5.6): Let  $U$  be an open subset of  $E$  and let  $f \in \mathcal{H}(U; F)$ . Then:*

(a)  *$f$  is continuous.*

(b)  *$f$  is locally bounded, that is,  $f$  is bounded on a suitable neighbourhood of each point of  $U$ .*

(c) *For each  $a \in U$ ,  $b \in E$  and  $\phi \in F^*$ , the function*

$$\lambda \mapsto \phi \circ f(a + \lambda b)$$

*is holomorphic on the open set  $\{\lambda \in \mathbb{C} : a + \lambda b \in U\}$ .*

## Chapter 4

# Properties of polynomials and analytic functions on Banach spaces with $DP^*P$

### 4.1 Completely continuous polynomials on Banach spaces with $DP^*P$ .

**Definition 4.1.1** A function  $f$  from a normed space  $X$  into a normed space  $Y$  is called completely continuous if it maps weakly convergent sequences into norm convergent sequences, i.e.  $x_n \xrightarrow{wk} x \Rightarrow f(x_n) \rightarrow f(x)$  in norm.

**Proposition 4.1.2** Let  $X$  and  $Y$  be Banach spaces,  $c_0 \subseteq Y$  (i.e., there exists an isomorphism  $j : c_0 \rightarrow Y$  into  $Y$ ). If every  $T \in \mathcal{L}(X, Y)$  is completely continuous, then  $X$  has the  $DP^*P$ . In this case every  $P \in \mathcal{P}(^n X, Y)$  is a completely continuous function.

**Proof** Let  $T : X \rightarrow c_0$  be bounded. Then  $j \circ T \in \mathcal{L}(X, Y)$ , hence completely continuous by assumption.

Thus, if  $x_n \xrightarrow{wk} 0$  in  $X$ , then  $\|(j \circ T)x_n\| \xrightarrow{\infty} 0$  in  $Y$ .

Since  $j^{-1}$  is continuous,  $\|Tx_n\|_{c_0} \xrightarrow{\infty} 0$ , i.e.  $T$  is completely continuous. By Theorem 2.2.13,  $X$  has the  $DP^*P$ .

The result about polynomials is proved by induction. If the degree  $n = 1$ ,

then  $P \in \mathcal{P}({}^1X, Y)$  is a bounded linear operator – thus by assumption it is completely continuous.

Assume each  $P \in \mathcal{P}({}^nX, Y)$  is completely continuous. Let  $P \in \mathcal{P}({}^{n+1}X, Y)$  and suppose  $\check{P}$  is the unique symmetric  $n+1$ -linear mapping from  $X^{n+1}$  into  $Y$  such that  $P(x) = \check{P}(x, x, \dots, x)$ .

Fix  $x \in X$  and consider the  $n$ -linear mapping  $\check{Q} : X^n \rightarrow Y$  defined by

$$\check{Q}(x_1, x_2, \dots, x_n) = \check{P}(x, x_1, x_2, \dots, x_n).$$

Corresponding to  $\check{Q}$ , there is a unique  $Q \in \mathcal{P}({}^nX, Y)$  such that

$$\begin{aligned} Q(z) = \check{Q}(z, z, \dots, z) &= \check{P}(x, z, z, \dots, z) \\ &= \check{P}(x, z^n) \quad \forall z \in X. \end{aligned}$$

By the induction assumption,  $Q$  is completely continuous.

Hence, if  $x_m \xrightarrow{wk} 0$  if  $m \rightarrow \infty$ , we have  $Q(x_m) = \check{P}(x, x_m^n) \rightarrow Q(0) = 0$  in norm if  $m \rightarrow \infty$ . We now show that  $P(x_m) \rightarrow 0$  in the norm of  $Y$ :

By the Hahn-Banach Theorem, there exists for each  $m \in \mathbb{N}$  a  $\phi_m \in Y^*$  such that  $\|\phi_m\| = 1$  and  $\|P(x_m)\| = \phi_m(P(x_m))$ . We have

$$|\phi_m(\check{P}(x, x_m^n))| \leq \|\phi_m\| \|\check{P}(x, x_m^n)\| \xrightarrow{\infty} 0.$$

This holds for all  $x \in X$ . Therefore,

$$T : X \rightarrow c_0 : x \mapsto (\phi_m(\check{P}(x, x_m^n)))_m$$

is well defined.

Clearly,  $T$  is linear and since

$$|\phi_m(\check{P}(x, x_m^n))| \leq \|\phi_m\| \|\check{P}(x, x_m^n)\| \leq \|\check{P}\| \|x\| \|x_m\|^n,$$

i.e.

$$\begin{aligned} \|Tx\| = \|(\phi_m(\check{P}(x, x_m^n)))_m\|_{c_0} &= \sup_m |\phi_m(\check{P}(x, x_m^n))| \\ &\leq (\|\check{P}\| \sup_m \|x_m\|^n) \|x\|, \end{aligned}$$

$T$  is bounded.

Since  $X$  has  $DP^*P$  (by the first part of the proof), it follows that  $T$  is completely continuous by Theorem 2.2.13. Therefore

$$\begin{aligned} 0 = \lim_m \|Tx_m\|_{c_0} &= \lim_m \|(\phi_j(\check{P}(x_m, x_j^n)))_j\|_{c_0} \\ &= \lim_{m \rightarrow \infty} \sup_j |\phi_j(\check{P}(x_m, x_j^n))|. \end{aligned}$$

Thus,

$$\begin{aligned} \|P(x_m)\| = \phi_m(P(x_m)) &= \phi_m(\check{P}(x_m, x_m^n)) \\ &\leq \sup_j |\phi_j(\check{P}(x_m, x_j^n))| \\ &\xrightarrow[\infty]{m} 0. \end{aligned}$$

Since  $P$  is not linear, to show that it is completely continuous, we have to prove that if  $x_m \xrightarrow{wk} a$ , then  $P(x_m) \rightarrow P(a)$  in norm.

We consider the identity

$$\begin{aligned} P(x_m) - P(a) &= \sum_{j=0}^n \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j) \\ &= P(x_m - a) + \sum_{j=1}^n \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j), \end{aligned}$$

where  $z \mapsto \check{P}(z^{n+1-j}, a^j)$ , ( $j = 1, 2, \dots, n$ ) is a polynomial of degree  $\leq n$ .

Since  $x_m \xrightarrow{wk} a$ , we have that  $x_m - a \xrightarrow{wk} 0$ .

Thus  $\lim_{m \rightarrow \infty} \|P(x_m - a)\| = 0$ , as was proved above.

Also,  $\check{P}((x_m - a)^{n+1-j}, a^j)$  has degree  $\leq n$  for  $j = 1, 2, \dots, n$ . Thus by the induction assumption,

$$\sum_{j=1}^n \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j) \xrightarrow[\infty]{m} 0 \text{ (in norm).}$$

Therefore,  $\lim_{m \rightarrow \infty} P(x_m) = P(a)$ . □

In Proposition 4.1.2 we have the assumption  $c_0 \subseteq Y$  in order to use Theorem 2.2.13. However, the following example indicates that one should not expect a more general space  $Y$ .

Consider the case  $X = \ell_2$  and  $Y = \ell_1$ . In this case  $\mathcal{L}(X, Y) = \mathcal{K}(X, Y)$  by Pitt's Theorem (cf. [1], Theorem 2.1.4, page 32); in particular each  $T \in \mathcal{L}(X, Y)$  is completely continuous. However,  $\ell_2$  lacks  $DP^*P$  (as reflexive space, it also lacks the (weaker)  $DPP$ ). Moreover, the 2-homogeneous polynomial  $P : \ell_2 \rightarrow \ell_1$ , defined by  $P((\xi_n)) = (\xi_n^2) = L \circ \Delta_2$ , where  $L(x, y) = (\xi_i \eta_i)_{i \in \mathbb{N}}$  with  $x = (\xi_i)$ ,  $y = (\eta_i)$ , is not completely continuous: We know that  $e_n \xrightarrow[\infty]{n} 0$  weakly in  $\ell_2$ ; however  $(P((e_n))) = (e_n)_{n \in \mathbb{N}}$  is not a norm convergent sequence in  $\ell_1$ .

For the following results, we refer the reader to [15].

**Theorem 4.1.3** (cf. [15], Theorem 3) *Every scalar valued polynomial  $P : X \rightarrow \mathbb{K}$  is weakly continuous on limited sets.*

**Theorem 4.1.4** (cf. [15], Theorem 4) *If  $L \subset X$  is a limited set and  $(P_n)$  is a sequence in  $\mathcal{P}({}^k X, \mathbb{K})$  pointwise convergent to 0, then  $(P_n)$  is uniformly convergent on  $L$  to 0.*

**Corollary 4.1.5** (cf. [15], Corollary 6) *If  $L \subset E$  is limited and  $P \in \mathcal{P}({}^k E, F)$ , then  $P(L)$  is a limited set in  $F$ .*

**Theorem 4.1.6** (cf. [15], Proposition 7) *If  $P \in \mathcal{P}({}^k E, F)$  is weakly sequentially continuous on limited sets and  $L \subset E$  is limited, then  $P$  is weakly continuous in  $L$  and  $P(L)$  is a relatively compact set.*

**Proposition 4.1.7** *Let  $X$  be a Banach space with  $DP^*P$  and let  $(P_n) \subset \mathcal{P}({}^k X)$  converge pointwise to a  $P \in \mathcal{P}({}^k X)$ , then  $P_n(x_n) \xrightarrow[\infty]{n} 0$ , for all  $(x_n) \subset X$  such that  $x_n \xrightarrow{wk} 0$ .*

**Proof** Suppose  $P_n \rightarrow P$  pointwise. By Theorem 4.1.4,  $P_n \rightarrow P$  uniformly on all limited sets. Now let  $x_n \rightarrow 0$  weakly. Then  $\{x_n : n \in \mathbb{N}\} \cup \{0\}$  is weakly compact and hence limited, since we assume that  $X$  has  $DP^*P$ . Therefore  $P_n \rightarrow P$  uniformly on  $\{x_n : n \in \mathbb{N}\} \cup \{0\}$ . By Theorem 4.1.3,  $P$

is weakly continuous on limited sets, hence also on  $\{x_n : n \in \mathbb{N}\} \cup \{0\}$ . Thus  $P(x_n) \rightarrow P(0) = 0$ . Consequently, for each  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$|P_n(x_m) - P(x_m)| < \frac{\varepsilon}{2} \text{ for all } n \geq n_0, m \in \mathbb{N}$$

and  $|P(x_n)| < \frac{\varepsilon}{2}$  for all  $n \geq n_0$ .

Thus

$$\begin{aligned} |P_n(x_n)| &\leq |P_n(x_n) - P(x_n)| + |P(x_n)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for all } n \geq n_0. \end{aligned}$$

Therefore,  $P_n(x_n) \xrightarrow[\infty]{n} 0$ . □

**Remark 4.1.8** Consider the following assertions:

- (i)  $X$  has the  $DP^*P$ .
- (ii) Every operator  $T : X \rightarrow c_0$  is completely continuous.
- (iii) For all integers  $k$ , each polynomial  $P \in \mathcal{P}(^k X, c_0)$  is completely continuous.
- (iv) For some integer  $k$ , each polynomial  $P \in \mathcal{P}(^k X, c_0)$  is completely continuous.

Then we have:

- (i) if and only if (ii) if and only if (iii) if and only if (iv).

Note that

- (i) if and only if (ii) follows from Theorem 2.2.13.

Furthermore,

- (ii) if and only if (iii) and (iii) if and only if (iv) follow from Theorem 17 in the paper [16].

However,

- (ii) implies (iii) and (i) implies (iii) also follow from Proposition 4.1.2 and its proof.

## 4.2 Completely continuous holomorphic functions on Banach spaces with $DP^*P$ .

We refer to [22] (Corollary 7.3, p. 47) for a Cauchy integral formula of holomorphic functions on Banach spaces and the following consequence of this result:

**Theorem 4.2.1** (Cauchy Inequality, [22] Corollary 7.4) *Let  $U$  be an open set in the Banach space  $X$  and let  $f \in \mathcal{H}(U, Y)$  for some Banach space  $Y$ . For  $a \in U$  and  $x \in X$ , let  $r > 0$  be such that  $a + \rho x \in U$  for all  $\rho \in \mathbb{K}$  with  $|\rho| \leq r$ . Then for each  $m \in \mathbb{N} \cup \{0\}$  we have the Cauchy Inequality*

$$\|P^m f(a)(x)\| \leq r^{-m} \sup_{|\rho|=r} \|f(a + \rho x)\|.$$

**Lemma 4.2.2** *Let  $f_n, f : X \rightarrow Y$  (with  $n = 1, 2, \dots$ ) be functions from  $X$  into  $Y$ . Suppose  $L$  is a subset of  $X$  such that each  $f_n$  is weakly continuous on  $L$  and  $f_n \xrightarrow[n \rightarrow \infty]{} f$  uniformly on  $L$ . Then  $f$  is weakly continuous on  $L$ .*

**Proof** Fix any  $x_0 \in L$ . We show that  $f$  is weakly continuous at  $x_0$ : Let  $0 \neq y^* \in Y^*$  be given. For any  $\varepsilon > 0$  there exists an  $n_0 \in \mathbb{N}$  such that  $\|f_n(x) - f(x)\| < \frac{\varepsilon}{3\|y^*\|}$  for all  $n \geq n_0$  and for all  $x \in L$ . Since  $f_{n_0}$  is weakly continuous at  $x_0$ , there exists a  $\delta > 0$  such that  $|\langle f_{n_0}(x) - f_{n_0}(x_0), y^* \rangle| < \frac{\varepsilon}{3}$  for all  $x \in L$  such that  $|\langle x - x_0, y^* \rangle| < \delta$ . We have

$$\begin{aligned} |\langle f(x) - f(x_0), y^* \rangle| &\leq |\langle f(x) - f_{n_0}(x), y^* \rangle| + |\langle f_{n_0}(x) - f_{n_0}(x_0), y^* \rangle| \\ &\quad + |\langle f_{n_0}(x_0) - f(x_0), y^* \rangle| \\ &\leq \|f(x) - f_{n_0}(x)\| \|y^*\| + |\langle f_{n_0}(x) - f_{n_0}(x_0), y^* \rangle| \\ &\quad + \|f_{n_0}(x_0) - f(x_0)\| \|y^*\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all  $x \in L$  such that  $|\langle x - x_0, y^* \rangle| < \delta$ . □

Let  $L \subset X$ ,  $r > 0$ . We agree to use the notation

$$L_r = \{\rho x : x \in L, \rho \in \mathbb{K} \text{ and } |\rho| = r\}.$$

**Lemma 4.2.3** (i) *Suppose  $L \subset X$  is a limited set. Then also  $L_r$  is a limited set.*

(ii) Suppose  $L \subset X$  is weakly compact, then  $L_r$  is weakly compact as well.

**Proof**

(i) Let  $x_n^* \xrightarrow{weak^*} 0$ . For  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that

$$|\langle x, x_n^* \rangle| < \frac{\varepsilon}{r} \text{ for all } n \geq n_0 \text{ for all } x \in L.$$

Then

$$|\langle \rho x, x_n^* \rangle| = r |\langle x, x_n^* \rangle| < \varepsilon$$

for all  $n \geq n_0$  and for all  $|\rho| = r$  and for all  $x \in L$ .

(ii) We only need to prove that  $L_r$  is weakly sequentially compact. Let  $(\alpha_n x_n) \subset L_r$  with  $|\alpha_n| = r$  for all  $n \in \mathbb{N}$  and  $(x_n) \subset L$ . Since the sphere  $S_r = \{\lambda \in \mathbb{K} : |\lambda| = r\}$  is compact, there is  $\alpha \in S_r$  and a subsequence  $(\alpha_{n_k})$  such that  $|\alpha_{n_k} - \alpha| \xrightarrow[k]{\infty} 0$ . Now  $(x_{n_k}) \subset L$  and  $L$  is weakly compact. Thus there exists a subsequence  $(x_{n_{k_l}})$  such that  $x_{n_{k_l}} \xrightarrow[l]{\infty} x \in L$  weakly. Therefore, for all  $x^* \in X^*$  we have

$$\begin{aligned} |\langle \alpha x - \alpha_{n_{k_l}} x_{n_{k_l}}, x^* \rangle| &\leq |\langle \alpha x - \alpha_{n_{k_l}} x, x^* \rangle| + |\alpha_{n_{k_l}}| |\langle x - x_{n_{k_l}}, x^* \rangle| \\ &= |\alpha - \alpha_{n_{k_l}}| |\langle x, x^* \rangle| + |\alpha_{n_{k_l}}| |\langle x - x_{n_{k_l}}, x^* \rangle| \\ &\xrightarrow[l]{\infty} 0, \end{aligned}$$

i.e. the subsequence  $(\alpha_{n_{k_l}} x_{n_{k_l}})$  of  $(\alpha_n x_n)$  converges weakly to  $(\alpha x) \in L_r$ .  $\square$

**Lemma 4.2.4** *Let  $f \in \mathcal{H}(X, Y)$  be bounded on weakly compact (resp. limited) sets in  $X$ . Then  $f$  may be approximated uniformly on weakly compact (resp. limited) sets in  $X$  by its Taylor series at 0.*

**Proof** If in Theorem 4.2.1 we let  $U = X$  and  $a = 0$ , then clearly we may fix any  $r > 1$  and have  $\rho x \in X$  for all  $\rho \in \mathbb{K}$  such that  $|\rho| \leq r$ . Therefore, by Theorem 4.2.1, we have for any  $x \in X$  that:

$$\begin{aligned} \|P_m(x)\| &= \|P^m f(0)(x)\| \\ &\leq r^{-m} \sup_{|\rho|=r} \|f(\rho x)\| \end{aligned}$$

for all  $m \in \mathbb{N} \cup \{0\}$  and where  $f(y) = \sum_{m=0}^{\infty} P_m(y)$  uniformly on a ball  $\mathcal{B}(0, R) \subseteq X$ . Thus, if  $L \subset X$  is a weakly compact set (resp. a limited set) then by Lemma 4.2.3 so is  $L_r$ . By assumption  $f$  is bounded on  $L_r$ . Thus there exists  $\beta > 0$  such that  $\|P_m(x)\| \leq \beta r^{-m}$  uniformly on  $L$  – i.e. the Taylor series  $\sum_m P_m(x)$  is uniformly convergent on  $L$  to its limit function  $f$ .  $\square$

**Theorem 4.2.5** (cf. [17], Theorem 3, page 472) *If  $X$  has DPP but not  $DP^*P$ , then there exists at least one scalar-valued holomorphic function  $f$  on  $X$  which does not map all weakly null sequences onto null sequences in  $\mathbb{C}$ .*

**Proof** Take  $(x_k)$  weakly null in  $X$  and  $(x_k^*)$  weak\* null in  $X^*$  such that  $x_k^*(x_k)$  does not converge to 0. There exists  $\delta > 0$  and subsequences  $(x_{k_n})$  and  $(x_{k_n}^*)$  such that

$$|\langle x_{k_n}, x_{k_n}^* \rangle| \geq \delta \quad \forall n.$$

Put  $\tilde{x}_{k_n} = \frac{x_{k_n}}{\langle x_{k_n}, x_{k_n}^* \rangle}$  for all  $n \in \mathbb{N}$ . Then  $\tilde{x}_{k_n} = \lambda_{k_n} x_{k_n}$ , whereby  $|\lambda_{k_n}| \leq \frac{1}{\delta}$  for all  $n$  (i.e.  $(\tilde{x}_{k_n}) \in c_0^{\text{weak}}(X)$ ) and  $\langle \tilde{x}_{k_n}, x_{k_n}^* \rangle = 1$  for all  $n \in \mathbb{N}$ .

The above argument shows that we may (for simplicity) assume that our sequences  $(x_k)$  and  $(x_k^*)$  satisfy  $x_k^*(x_k) = 1$  for all  $k \in \mathbb{N}$ . Now let  $0 < \varepsilon < 1$ . We will extract subsequences  $(x_{k_j})$  and  $(x_{k_j}^*)$  inductively. Set  $k_1 = 1$ , and having defined  $k_1, \dots, k_{r-1}$ , define  $k_r$  as follows: since  $(x_k^*)$  is weak\* null, choose  $n_r > k_{r-1}$  and such that

$$|x_k^*(x_{k_{r-1}})| < \frac{\varepsilon}{2 + \varepsilon}, \quad \text{if } k \geq n_r,$$

and since  $(x_k)$  is weakly null, take  $m_r > k_{r-1}$  and such that

$$\sum_{j < r} |x_{k_j}^*(x_k)|^j < \frac{\varepsilon}{2}, \quad \text{if } k \geq m_r.$$

Define  $k_r = \max\{n_r, m_r\}$ . Now construct  $f : X \rightarrow \mathbb{C}$  as  $f(x) = \sum_{j=1}^{\infty} (x_{k_j}^*(x))^j$ , which is well defined and holomorphic by Example 3.2.5(ii). Note that since  $(x_{k_j}^*)$  is weak\* null and  $0 < \limsup_j \|x_{k_j}^*\| < \infty$ , this is an entire function on

$X$  which is not of bounded type. Finally, for each  $r \in \mathbb{N}$  we have

$$\begin{aligned} |f(x_{k_r}) - 1| &\leq \sum_{j < r} |x_{k_j}^*(x_{k_r})|^j + \sum_{j > r} |x_{k_j}^*(x_{k_r})|^j \\ &< \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} \left(\frac{\varepsilon}{2+\varepsilon}\right)^j \\ &= \varepsilon, \end{aligned}$$

bearing in mind that  $k_r \geq m_r$  and that for  $j > r$ , we have  $k_j \geq n_{r+1}$ . Thus  $|f(x_{k_r}) - 1| < \varepsilon < 1$ , and  $f(x_{k_r})$  cannot tend to  $0 = f(0)$ .  $\square$

**Proposition 4.2.6** *If  $X$  has the  $DP^*P$  and  $Y$  is a Gelfand-Phillips space, then every homogeneous polynomial  $P : X \rightarrow Y$  is completely continuous. Further, any  $f \in \mathcal{H}(X, Y)$  which is bounded on all weakly compact (resp. limited) sets is weakly continuous on them.*

**Proof** Since  $P(L)$  is limited for each limited set  $L \subset X$  (by Corollary 4.1.5) and since each weakly compact set in  $X$  is limited (by assumption that  $X$  has  $DP^*P$ ), it is clear that  $P(A)$  is limited for each weakly compact set  $A \subset X$ . Each limited set in  $Y$  being relatively compact (because  $Y$  is assumed to be a Gelfand-Phillips space), it follows that  $\overline{P(A)}$  is norm compact in  $Y$  for each weakly compact set  $A \subseteq X$ .

To show that  $P$  is completely continuous, assume that  $x_n \rightarrow a$  weakly in  $X$ . The set  $A = \{x_n : n \in \mathbb{N}\} \cup \{a\}$  is weakly compact, so that  $\overline{P(A)}$  is norm compact in  $Y$ .

We show that  $P(x_n) \rightarrow P(a)$  weakly in  $Y$ :

For each  $\phi \in Y^*$ , the function  $\phi \circ P : X \rightarrow \mathbb{K}$  is a scalar-valued polynomial. By Theorem 4.1.3 each  $\phi \circ P$  is weakly continuous on limited sets, i.e. also on the set  $\{x_n : n \in \mathbb{N}\} \cup \{a\}$ . Therefore,  $(\phi \circ P)(x_n) \xrightarrow[n \rightarrow \infty]{} (\phi \circ P)(a)$ , (i.e.  $\phi(P(x_n)) \xrightarrow[n \rightarrow \infty]{} \phi(P(a))$  for each  $\phi \in Y^*$ ), i.e.  $P(x_n) \rightarrow P(a)$  weakly in  $Y$ .

Replacing  $\tau$  by  $\sigma(Y, Y^*)$  and  $\tau_d$  by the norm topology on  $Y$  in Lemma 2.1.10, we have that in the norm compact set  $\overline{P(A)}$ , weakly convergent sequences

are norm convergent. Thus,  $P(x_n) \longrightarrow P(a)$  in the norm of  $Y$ . This proves that  $P$  is completely continuous. Note that we have also proved that  $P$  is weakly sequentially continuous. By Theorem 4.1.6,  $P$  is therefore weakly continuous on limited sets (i.e. also on compact sets in this case).

Application of Lemma 4.2.4 and Lemma 4.2.2 therefore yields that  $f$  is weakly continuous on weakly compact and limited subsets of  $X$ .  $\square$

Recall that a Banach space  $X$  has the Schur property (or  $X$  is a Schur space) if weak and norm sequential convergence coincide in  $X$ , i.e. a sequence  $(x_n)_{n=1}^{\infty}$  in  $X$  converges to 0 weakly if, and only if,  $(x_n)_{n=1}^{\infty}$  converges to 0 in norm.

**Definition 4.2.7** *Let  $X, Y$  and  $Z$  be Banach spaces. A bilinear operator*

$$\phi : X \times Y \rightarrow Z$$

*is called separately compact if for each fixed  $y \in Y$ , the linear operator  $T_y : X \rightarrow Z : x \mapsto \phi(x, y)$  is compact and for each fixed  $x \in X$ , the linear operator  $T_x : Y \rightarrow Z : y \mapsto \phi(x, y)$  is compact.*

**Remark 4.2.8** *If  $\phi$  is compact, then  $\phi$  is separately compact.*

Recall that if  $X, Y$  are Banach spaces and  $T : X \rightarrow Y$  is a bounded linear operator, then  $T$  is weakly continuous (cf. Theorem 1.1, Chapter VI of [9]). Therefore, if  $X, Y$  are Banach spaces and  $(X \times Y, \|\cdot\|_1)$  is the Banach space with norm  $\|(x, y)\|_1 = \|x\|_1 + \|y\|_1$ , then the coordinate projections  $P_X : X \times Y \rightarrow X : (x, y) \mapsto x$  and  $P_Y : X \times Y \rightarrow Y : (x, y) \mapsto y$  are weakly continuous.

In Fact 1 of the paper [6] (cf. [6], page 2) it is proved that: A separable Banach space is a Schur space if, and only if, every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous.

This characterization of Schur spaces is not in general true for non-separable Banach spaces (also remarked in Fact 1 of [6]). Proposition 4.2.6 however, results in a more general characterization of Banach spaces with  $DP^*P$  as follows:

**Corollary 4.2.9** *A Banach space  $X$  has the  $DP^*P$  if, and only if, every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous.*

**Proof** Assume that  $X$  has  $DP^*P$  and let  $L : X \times X \rightarrow c_0$  be a symmetric bilinear separately compact map. If  $(x_{1,n}, x_{2,n}) \xrightarrow[n]{\infty} (0, 0)$  in  $X \times X$  weakly, we have  $x_{1,n} \xrightarrow[n]{\infty} 0$  and  $x_{2,n} \xrightarrow[n]{\infty} 0$  weakly in  $X$ . Consider the unique polynomial  $P = L \circ \Delta_2$ , i.e.  $P = \hat{L}$ , such that by the polarization formula

$$L(x_1, x_2) = \frac{1}{8} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 P\left(\sum_{i=1}^2 \epsilon_i x_i\right).$$

By Proposition 4.2.6, the mapping  $P$  is completely continuous (i.e.  $P : X \rightarrow c_0$ , with  $c_0$  a Gelfand-Phillips space is completely continuous). Since

$$\sum_{i=1}^2 \epsilon_i x_{i,n} \xrightarrow[n]{\infty} 0$$

weakly, the above polarization formula gives us  $L(x_{1,n}, x_{2,n}) \xrightarrow[n]{\infty} 0$  in norm. Thus we have verified that  $L$  is completely continuous.

To prove the converse, the assumption that the bilinear mappings should be separately compact comes into play:

Assume that every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is completely continuous. Let  $T \in L(X, c_0)$  be arbitrary.

Define  $T \otimes T : X \times X \rightarrow c_0$  by  $(T \otimes T)(x, y) = T(x)T(y)$  (coordinate-wise product of two sequences in  $c_0$ ).

Then  $T \otimes T$  is bilinear and satisfies the following properties:

- (i)  $(T \otimes T)(x, y) = (T \otimes T)(y, x)$  (so it is symmetric).
- (ii)  $T \otimes T$  is separately compact: For each fixed  $x \in X$ , the operator

$$T_x : X \rightarrow c_0 : y \mapsto T(x)T(y) = (T \otimes T)(x, y)$$

is linear and bounded with  $\|T_x\| \leq \|T\|^2 \|x\|$ . We may assume without loss of generality that  $\|T\| = 1$ . Since  $T(x)$  is a sequence in  $c_0$ , we agree to denote the  $m$ -th term of  $T(x)$  by  $T(x)_m$ . Then, for  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that  $|T(x)_m| \leq \varepsilon$  for all  $m \geq n$ .

Denote  $(T(x)_1, T(x)_2, \dots, T(x)_n, 0, 0, \dots)$  by  $T(x)(\leq n)$  and  $(0, 0, \dots, 0, T(x)_{n+1}, T(x)_{n+2}, \dots)$  by  $T(x)(> n)$ . Then

$$\begin{aligned} T_x(y) &= T(x)T(y) \\ &= T(x)(\leq n)T(y) + T(x)(> n)T(y). \end{aligned}$$

The set  $\{T(x)(\leq n)T(y) : y \in X\}$  is contained in a finite-dimensional subspace

$$M := \{(\xi_i)(\leq n) : (\xi_i) \in c_0\}$$

of  $c_0$ . We have

$$\{T(x)(\leq n)T(y) : y \in B_X\} \subseteq \|T(x)\|B_M,$$

which is compact.

For each  $\varepsilon > 0$  there exists  $z_1, \dots, z_p \in \|T(x)\|B_M$  such that

$$\|T(x)\|B_M \subseteq \bigcup_{i=1}^p (z_i + \varepsilon B_{c_0}).$$

Thus,

$$\begin{aligned} T_x(B_X) &= T(x)T(B_X) \\ &\subseteq \bigcup_{i=1}^p (z_i + \varepsilon B_{c_0}) + \varepsilon B_{c_0} \\ &\subseteq \bigcup_{i=1}^{p+1} (z_i + 2\varepsilon B_{c_0}) \text{ with } z_{p+1} = 0. \end{aligned}$$

This shows that  $T_x(B_X)$  is relatively compact. Hence  $T \otimes T$  is separately compact.

By assumption  $T \otimes T$  is completely continuous. Therefore if  $x_n \rightarrow 0$  weakly, then

$$\begin{aligned} \|T(x_n)\|^2 &= \sup_j (|T(x_n)_j|^2) \\ &= \|(T(x_n))^2\| \\ &= \|(T \otimes T)(x_n, x_n)\| \xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

This shows that  $T$  is completely continuous. Since  $T \in \mathcal{L}(X, c_0)$  was arbitrarily chosen it follows from Proposition 4.1.2 that  $X$  has  $DP^*P$ .  $\square$

**Remark 4.2.10** From Corollary 4.2.9 and the discussion preceding the corollary it is clear that, on the class of separable Banach spaces, the Schur property coincides with  $DP^*P$ .

**Definition 4.2.11** A subset  $L$  of  $X$  is called bounding if every  $f \in \mathcal{H}(X)$  is bounded on  $L$ .

**Lemma 4.2.12** Let  $X, Y$  be Banach spaces. If  $f \in \mathcal{H}(X, Y)$  is completely continuous, then it is bounded on all weakly compact subsets of  $X$ .

**Proof** Suppose  $f \in \mathcal{H}(X, Y)$  is unbounded on a weakly compact subset  $L$  of  $X$ . Then for each  $n \in \mathbb{N}$  there exists  $x_n \in L$  such that  $\|f(x_n)\| \geq n$ .  $L$  being weakly compact, there exists a weakly convergent subsequence  $(x_{n_k})$ , i.e.  $x_{n_k} \xrightarrow[\infty]{k} x \in L$  weakly. By assumption,  $f(x_{n_k}) \rightarrow f(x)$  in norm; this however, contradicts the fact that  $\|f(x_{n_k})\| \geq n_k$  for all  $k$ .  $\square$

Let us denote by  $\mathbb{B}(Y)$  the class of all Banach spaces  $X$  such that all  $f \in \mathcal{H}(X, Y)$  are bounded on the weakly compact subsets of  $X$ .  $\mathbb{B}(\mathbb{C})$  is denoted by  $\mathbb{B}$ . Bounding sets in a Banach space  $X$  are limited (cf. [20], page 321). Thus  $\mathbb{B}$  is the class of all Banach spaces whose weakly compact subsets are bounding. If  $Y$  is a Banach space and  $f \in \mathcal{H}(X, Y)$ , then for each weakly compact set  $A \subset X$  and each  $y^* \in Y^*$  we have

$$y^*(f(A)) = (y^* \circ f)(A) \text{ and } y^* \circ f \in \mathcal{H}(X).$$

Thus, if  $X \in \mathbb{B}$ , then  $y^*(f(A))$  is bounded for all  $y^* \in Y^*$ , i.e.  $f(A)$  is weakly bounded hence bounded in  $Y$ , i.e.  $X \in \mathbb{B}(Y)$ . We conclude that

$$X \in \mathbb{B} \Leftrightarrow X \in \mathbb{B}(Y), \text{ for all Banach spaces } Y.$$

Let  $\mathcal{C}_c(Y)$  denote the class of all Banach spaces  $X$  such that each  $f \in \mathcal{H}(X, Y)$  is completely continuous.  $\mathcal{C}_c(\mathbb{K})$  is denoted by  $\mathcal{C}_c$  and coincides with the class of all Banach spaces  $X$  such that each  $f \in \mathcal{H}(X)$  is weakly sequentially continuous. Applying Proposition 4.1.2 we may give a description of the property “ $X \in \mathbb{B}$ ” as follows:

**Proposition 4.2.13** (i)  $X \in \mathbb{B}$  (i.e.  $X \in \mathbb{B}(Y)$  for all Banach spaces  $Y$ ) if and only if  $X \in \mathcal{C}_c$ .

(ii) If  $X \in \mathbb{B}$ , then  $X$  has  $DP^*P$  and for each Gelfand-Phillips space  $Y$ , we have  $X \in \mathcal{C}_c(Y)$ .

**Proof** Proof of (i): If  $X \in \mathcal{C}_c$ , then  $X \in \mathbb{B}$  follows from Lemma 4.2.12. Let  $X \in \mathbb{B}$ . Then the weakly compact sets in  $X$  are bounding, hence they are limited (cf. [20], page 321). Therefore  $X$  has  $DP^*P$ . Since  $\mathbb{K}$  is a Gelfand-Phillips space, it follows from Proposition 4.2.6 that each  $f \in \mathcal{H}(X)$  is weakly continuous on the weakly compact subsets of  $X$ . Hence if  $x_n \xrightarrow[n \rightarrow \infty]{w} x$  weakly, then since  $L = \{x_n : n \in \mathbb{N}\} \cup \{x\}$  is weakly compact, we have  $f(x_n) \xrightarrow[n \rightarrow \infty]{w} f(x)$  in  $\mathbb{C}$  for all  $f \in \mathcal{H}(X)$ . Thus  $X \in \mathcal{C}_c$ .

Proof of (ii): If  $X \in \mathbb{B}$  then  $X$  has  $DP^*P$  (as was seen in the proof of (i) above). Let  $Y$  be a Gelfand-Phillips space. Since  $X \in \mathbb{B}(Y)$ , it follows that each  $f \in \mathcal{H}(X, Y)$  is bounded on the weakly compact sets in  $X$ . Therefore, using Lemma 4.2.4, each  $f \in \mathcal{H}(X, Y)$  can be uniformly approximated on weakly compact sets of  $X$  by its Taylor series at 0, i.e. if we let  $f_n(x) = \sum_{k=0}^n P_k(x)$ , then  $f_n \rightarrow f$  uniformly on all weakly compact subsets of  $X$ . By Proposition 4.2.6 each  $f_n$  is completely continuous.

We show that  $f$  is completely continuous: Let  $x_n \rightarrow x$  weakly in  $X$ . Then  $f_n \rightarrow f$  uniformly on  $L = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ , i.e. for given  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $\|f_n(x) - f(x)\| < \frac{\varepsilon}{3}$  for all  $n \geq n_0$  and for all  $x \in L$ . Since  $f_{n_0}$  is completely continuous, there exists  $k_0 \in \mathbb{N}$  such that  $\|f_{n_0}(x_n) - f_{n_0}(x)\| < \frac{\varepsilon}{3}$  for all  $n \geq k_0$ . It follows that for all  $n \geq k_0$ ,

$$\begin{aligned} \|f(x_n) - f(x)\| &\leq \|f(x_n) - f_{n_0}(x_n)\| + \|f_{n_0}(x_n) - f_{n_0}(x)\| \\ &\quad + \|f_{n_0}(x) - f(x)\| < \varepsilon. \end{aligned}$$

□

**Remark 4.2.14** Notice that in Proposition 4.2.13 we have to deal with all holomorphic functions  $f \in \mathcal{H}(X)$  on the Banach space  $X$  – i.e. we can not conclude from Proposition 4.2.13 that a holomorphic function  $f \in \mathcal{H}(X)$  is completely continuous if and only if it is bounded on weakly compact sets. There are however examples of infinite dimensional Banach spaces  $X$  for which it is true that  $f \in \mathcal{H}(X)$  is completely continuous if and only if  $f$  is bounded on weakly compact sets.

Proposition 4.2.13 characterizes the class  $\mathbb{B}$  in terms of holomorphic (complex valued) functions on the Banach spaces in  $\mathbb{B}$ . There is an alternative characterization of  $\mathbb{B}$  which is analogous to Proposition 4.2.6 (in [7]):

**Proposition 4.2.15**  $X \in \mathbb{B}$  if and only if  $X \in \mathcal{C}_c(c_0)$ .

**Proof** We observe that  $\mathcal{C}_c(c_0) \subseteq \mathcal{C}_c$ : Let  $X \in \mathcal{C}_c(c_0)$ . If  $f \in \mathcal{H}(X)$ , then  $j_1 \circ f \in \mathcal{H}(X, c_0)$  where  $j_1 : \mathbb{C} \rightarrow c_0$  is the isometric embedding  $\xi \mapsto (\xi, 0, 0, \dots)$ . If  $x_n \rightarrow x$  weakly in  $X$ , then  $(j_1 \circ f)(x_n) \xrightarrow[n \rightarrow \infty]{} (j_1 \circ f)(x)$  in norm. Thus

$$\begin{aligned} |f(x_n) - f(x)| &= \|j_1(f(x_n) - f(x))\|_{c_0} \\ &= \|j_1(f(x_n)) - j_1(f(x))\|_{c_0} \\ &= \|(j_1 \circ f)(x_n) - (j_1 \circ f)(x)\|_{c_0} \\ &\xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

i.e.  $f$  is completely continuous. This holds for all  $f \in \mathcal{H}(X)$ , so that we conclude that  $X \in \mathcal{C}_c$ . It follows from Proposition 4.2.13 that  $X \in \mathcal{C}_c(c_0) \Rightarrow X \in \mathbb{B}$ .

Conversely, since  $c_0$  is a Gelfand-Phillips space (see Example 1.1.6) if  $X \in \mathbb{B}$ , it follows from Proposition 4.2.13 (part (ii)) that  $X \in \mathcal{C}_c(c_0)$ .  $\square$

**Remark 4.2.16** (i) It is maintained in [7] that  $\ell_\infty \in \mathbb{B}$  follows from a result of Josefson (Theorem 1 in [19]).

(ii) It is proved in [7] (cf. [7], Proposition 3.6) that if  $X, Y \in \mathbb{B}$ , then also  $X \times Y \in \mathbb{B}$ .

(iii) Using an example from [20] it is also showed in [7] (Example 3.7) that there are spaces with  $DP^*P$  which are not in  $\mathbb{B}$ .

# Chapter 5

## On a scale of stronger Dunford-Pettis-like properties in Banach spaces.

In this chapter we introduce the (new) concept of  $DP^*$ -property of order  $p$  (for  $1 \leq p \leq \infty$ ) and discuss Banach spaces with this property.

### 5.1 On a scale of $DP^*$ -properties on Banach spaces

**Definition 5.1.1** *A Banach space is said to have the  $DP^*$ -property of order  $p$  (for  $1 \leq p \leq \infty$ ) if all weakly- $p$ -compact sets in  $X$  are limited. In short, we say  $X$  has  $DP^*P_p$ .*

**Theorem 5.1.2** *Let  $1 \leq p < \infty$ . The Banach space  $X$  has  $DP^*P_p$  if and only if  $\langle x_n, x_n^* \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for all  $(x_n^*) \in c_0^{weak^*}(X^*)$  and all  $(x_n) \in \ell_p^{weak}(X)$ .*

**Proof** Suppose  $X$  has  $DP^*P_p$  and let  $(x_n) \in \ell_p^{weak}(X)$ . The set  $A := \{x_n : n \in \mathbb{N}\} \cup \{0\}$  is weakly- $p$ -compact. Thus,  $x_n^* \rightarrow 0$  uniformly on  $A$ , i.e.  $\sup_{x \in A} |\langle x, x_n^* \rangle| \rightarrow 0$  as  $n \rightarrow \infty$ . In particular,  $\langle x_n, x_n^* \rangle \rightarrow 0$  as  $n \rightarrow \infty$ .

Conversely, suppose the condition  $\langle x_n, x_n^* \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for all  $(x_n^*) \in c_0^{weak^*}(X^*)$  and all  $(x_n) \in \ell_p^{weak}(X)$  holds. If some weakly- $p$ -compact set  $A$  in  $X$  is not limited, then there exist  $(x_n^*) \in c_0^{weak^*}(X^*)$  and an  $\epsilon > 0$  such that

for each  $k \in \mathbb{N}$  there is  $x_k \in A$  and an index  $n_k \geq k$  such that  $|\langle x_k, x_{n_k}^* \rangle| \geq \epsilon$ . The sequence  $(x_k)$  has a weakly- $p$ -convergent subsequence  $(x_{k_m})$ , i.e. there exists  $x \in A$  such that  $(x_{k_m} - x)_m \in \ell_p^{weak}(X)$ . Put  $y_m = x_{k_m} - x$ . Recall that  $(y_m)$  is a subsequence of  $(x_k - x)_k$ , whereby  $|\langle x_k, x_{n_k}^* \rangle| \geq \epsilon$  holds for all  $k$ . Therefore,  $|\langle x_{k_m}, x_{n_{k_m}}^* \rangle| \geq \epsilon$  for all  $m \in \mathbb{N}$ . Put  $y_m^* = x_{n_{k_m}}^*$  for all  $m \in \mathbb{N}$ , i.e.  $(y_m^*)$  is a subsequence of the original sequence  $(x_n^*)$ . Then  $(y_m^*) \in c_0^{weak^*}(X^*)$  and  $|\langle x_{k_m}, y_m^* \rangle| \geq \epsilon$  for all  $m \in \mathbb{N}$ . Therefore, it follows that

$$|\langle y_m, y_m^* \rangle| = |\langle x_{k_m} - x, y_m^* \rangle| \geq \epsilon - |\langle x, y_m^* \rangle|.$$

Since  $\langle x, y_m^* \rangle \rightarrow 0$  as  $m \rightarrow \infty$ , it follows that there exists an index  $m_0 \in \mathbb{N}$  such that  $|\langle y_m, y_m^* \rangle| \geq \frac{\epsilon}{2}$  for all  $m \geq m_0$ , contradicting that by assumption  $\langle y_m, y_m^* \rangle \rightarrow 0$  as  $m \rightarrow \infty$ .  $\square$

**Theorem 5.1.3** *Let  $1 \leq p < \infty$ . The Banach space  $X$  has  $DP^*P_p$  if and only if every operator  $T \in \mathcal{L}(X, c_0)$  is  $p$ -convergent, i.e. if and only if  $\mathcal{L}(X, c_0) = C_p(X, c_0)$  (See Definition 2.2.1).*

**Proof** Assume that  $\mathcal{L}(X, c_0) = C_p(X, c_0)$ . Let  $(x_n) \in \ell_p^{weak}(X)$  and  $(x_n^*) \in c_0^{weak^*}(X^*)$ . Identify  $(x_n^*)$  with the operator  $T : X \rightarrow c_0$  such that  $Tx = (\langle x, x_n^* \rangle)$ . Then,  $T \in \mathcal{L}(X, c_0)$  and

$$\|T\| = \sup_{\|x\| \leq 1} \|(\langle x, x_n^* \rangle)\|_{c_0} = \|(x_n^*)\|_{c_0^{weak^*}(X^*)}.$$

By assumption  $(Tx_n) \in c_0^{strong}(c_0)$ , i.e

$$|\langle x_n, x_n^* \rangle| \leq \sup_j |\langle x_n, x_j^* \rangle| = \|Tx_n\| \rightarrow 0$$

as  $n \rightarrow \infty$ . By Theorem 5.1.2,  $X$  has  $DP^*P_p$ .

Conversely, assume that  $X$  has  $DP^*P_p$  and let  $T \in \mathcal{L}(X, c_0)$ . If we assume that  $T \notin C_p(X, c_0)$ , then there exists  $(x_n) \in \ell_p^{weak}(X)$  such that  $\|Tx_n\| \not\rightarrow 0$ , i.e. we may assume that there exists  $\delta > 0$  such that  $\|Tx_m\| \geq \delta$  for all  $m$  (where, if necessary we can use a subsequence). Let  $\pi_j : c_0 \rightarrow \mathbb{K}$  denote the  $j^{th}$  coordinate projection. Then there exists for each  $m \in \mathbb{N}$  an index  $k_m$  such that

$$\delta/2 < |\pi_{k_m}(Tx_m)| \leq \|Tx_m\|.$$

Consider the (unique)  $(x_i^*) \in c_0^{weak^*}(X^*)$  such that  $Tx = (\langle x, x_i^* \rangle)_i$  for all  $x \in X$  (again, using Theorem 1.2.8). Then,  $\delta/2 < |\langle x_m, x_{k_m}^* \rangle|$  for all  $m \in \mathbb{N}$ .

If this inequality holds for only a finite number of different (positive integer) indexes  $k_m$  as  $m$  runs through  $\mathbb{N}$ , then this would imply that there exists a positive integer  $N$  such that  $k_m = k_{m+1} = \dots = k$  (say) for all  $m \geq N$ ; i.e.  $\delta/2 < |\langle x_m, x_k^* \rangle|$  for all  $m \geq N$ , contradicting that  $\langle x_m, x_k^* \rangle \rightarrow 0$  as  $m \rightarrow \infty$ . Therefore, there has to be infinitely many different positive integers  $k_m$  such that  $\delta/2 < |\langle x_m, x_{k_m}^* \rangle|$  as  $m$  runs through  $\mathbb{N}$ . By taking an infinite subset of  $\{k_m : m \in \mathbb{N}\}$  if necessary, we may in this case assume that  $k_1 < k_2 < \dots$ . The sequence  $(x_{k_m}^*)$  is then a subsequence of  $(x_i^*)$ , so that  $(x_{k_m}^*) \in c_0^{weak^*}(X^*)$ . Since the set  $A := \{x_m : m \in \mathbb{N}\} \cup \{0\}$  is weakly- $p$ -compact, the assumption applies that  $\sup_{x \in A} |\langle x, x_{k_m}^* \rangle| \rightarrow 0$  as  $m \rightarrow \infty$ . Since this implies in particular that  $\langle x_m, x_{k_m}^* \rangle \rightarrow 0$  as  $m \rightarrow \infty$ , we have a contradiction. Thus  $T$  has to be  $p$ -convergent.  $\square$

Clearly, the  $DP^*P_p$  property on Banach spaces is stronger than the  $DPP_p$  property. The following theorem indicates when they are the same. When  $X$  is a Grothendieck space, they coincide. More precisely, we have

**Theorem 5.1.4** *The Banach space  $X$  has the  $DP^*P_p$  if and only if it has the  $DPP_p$  and every quotient mapping  $q : X \rightarrow c_0$  is  $p$ -convergent.*

**Proof** Suppose  $X$  has the  $DPP_p$  and every quotient mapping  $q : X \rightarrow c_0$  is  $p$ -convergent. If we assume that  $X$  does not have  $DP^*P_p$ , then there are sequences  $(\phi_n)$  in  $X^*$  and  $(x_n)$  in  $X$  such that  $\phi_n \rightarrow 0$  weak\*,  $(x_n) \in \ell_p^{weak}(X)$  and  $\phi_n(x_n) \not\rightarrow 0$  if  $n \rightarrow \infty$ . We may assume that  $|\phi_n(x_n)| \geq \delta > 0$  for all  $n$ . By assumption, for each  $m \in \mathbb{N}$  there exists an index  $n_m \in \mathbb{N}$  such that  $|\phi_n(x_m)| < \delta/2$  for all  $n \geq n_m$ . Also,

$$|(\phi_{n_m} - \phi_m)(x_m)| \geq |\phi_m(x_m)| - |\phi_{n_m}(x_m)| \geq \delta/2$$

for all  $m$ . By Rosenthal's  $\ell_1$  theorem, we may assume that either  $(\phi_n)$  is a weakly Cauchy sequence or that it is equivalent to the  $\ell_1$  basis. In the first case, we then verify as in the proof of Theorem 2.2.14 that  $(\phi_{n_m} - \phi_m)$  is weakly null in  $X^*$ , so that by assumption that  $X$  has  $DPP_p$ , it follows that  $\lim_{m \rightarrow \infty} (\phi_{n_m} - \phi_m)(x_m) = 0$ . This contradiction then leaves us with the second case, i.e. that  $(\phi_n)$  is equivalent to the  $\ell_1$  basis. Consider the mapping  $q : X \rightarrow c_0 : x \mapsto (\phi_n(x))$ . We have  $q^* : \ell_1 \rightarrow X^*$ , whereby

$$\langle q^*(e_n), x \rangle = \langle e_n, (\phi_j(x))_j \rangle = \phi_n(x)$$

for all  $x \in X$ , i.e.  $q^*(e_n) = \phi_n$  for all  $n \in \mathbb{N}$ . This shows that  $q^*$  is the isomorphic embedding of  $\ell_1$  into  $X^*$ . Therefore  $q$  is a continuous surjection (hence quotient mapping). By assumption,  $q$  is  $p$ -convergent; i.e.  $(q(x_n))$  is a null sequence in  $c_0$ . This contradicts the fact that

$$\|q(x_n)\| \geq |\phi_n(x_n)| \geq \delta$$

for all  $n$ . It follows that  $X$  has  $DP^*P_p$ .

The converse statement of the theorem follows from Theorem 5.1.3.  $\square$

## 5.2 Application to properties of polynomials and analytic functions.

We begin by introducing a new class of functions on normed spaces.

**Definition 5.2.1** *A function  $f$  from a normed space  $X$  into a normed space  $Y$  is called  $p$ -convergent if it maps weakly  $p$ -convergent sequences into norm convergent sequences, i.e. if  $(x_n) \in \ell_p^{weak}(X)$ , then  $(f(x_n))$  is norm null in  $Y$ .*

**Proposition 5.2.2** *Let  $X, Y$  be Banach spaces,  $c_0 \subseteq Y$ . If every  $T \in \mathcal{L}(X, Y)$  is  $p$ -convergent, then  $X$  has the  $DP^*P_p$ . In this case, every polynomial  $P \in \mathcal{P}(^n X, Y)$  is a  $p$ -convergent function for all  $n \in \mathbb{N}$ .*

**Proof** Let  $T : X \rightarrow c_0$  be bounded and  $j : c_0 \rightarrow Y$  the isomorphic embedding. Then  $j \circ T \in \mathcal{L}(X, Y)$ , hence  $p$ -convergent by assumption. Thus, if  $(x_n) \in \ell_p^{weak}(X)$ , then  $(j(Tx_n)) \in c_0^{strong}(Y)$ . Since  $j^{-1}$  is continuous,  $\|Tx_n\|_{c_0} \xrightarrow{\infty} 0$ , i.e.  $T$  is  $p$ -convergent. By Theorem 5.1.3,  $X$  has the  $DP^*P_p$ .

The result about polynomials is proved by induction. If the degree  $n = 1$ , then  $P \in \mathcal{P}(^1 X, Y)$  is a bounded linear operator – thus by assumption it is  $p$ -convergent.

Assume each  $P \in \mathcal{P}(^n X, Y)$  is  $p$ -convergent. Let  $P \in \mathcal{P}(^{n+1} X, Y)$  and suppose  $\check{P}$  is the unique symmetric bounded  $(n + 1)$ -linear mapping from  $X^{n+1}$

into  $Y$  such that  $P(x) = \check{P}(x, x, \dots)$ .

Fix  $x \in X$  and consider the  $n$ -linear mapping  $\check{Q} : X^n \rightarrow Y$  defined by

$$\check{Q}(x_1, x_2, \dots, x_n) = \check{P}(x, x_1, x_2, \dots, x_n).$$

Corresponding to  $\check{Q}$ , there is a unique  $Q \in \mathcal{P}(^n X, Y)$  such that

$$\begin{aligned} Q(z) = \check{Q}(z, z, \dots, z) &= \check{P}(x, z, z, \dots, z) \\ &= \check{P}(x, z^n) \quad \forall z \in X. \end{aligned}$$

By the induction assumption,  $Q$  is  $p$ -convergent.

Hence, if  $(x_m) \in \ell_p^{weak}(X)$ , we have  $\check{P}(x, x_m^n) = Q(x_m) \rightarrow 0$  in norm if  $m \rightarrow \infty$ . We now show that  $P(x_m) \rightarrow 0$  in the norm of  $Y$ :

By the Hahn-Banach Theorem, there exists for each  $m \in \mathbb{N}$  a  $\phi_m \in Y^*$  such that  $\|\phi_m\| = 1$  and  $\|P(x_m)\| = \phi_m(P(x_m))$ . We have

$$|\phi_m(\check{P}(x, x_m^n))| \leq \|\phi_m\| \|\check{P}(x, x_m^n)\| \xrightarrow{\infty} 0.$$

This holds for all  $x \in X$ . Therefore,

$$T : X \rightarrow c_0 : x \mapsto (\phi_m(\check{P}(x, x_m^n)))_m$$

is well defined.

Clearly,  $T$  is linear and since

$$|\phi_m(\check{P}(x, x_m^n))| \leq \|\phi_m\| \|\check{P}(x, x_m^n)\| \leq \|\check{P}\| \|x\| \|x_m\|^n,$$

we have

$$\begin{aligned} \|Tx\| = \|(\phi_m(\check{P}(x, x_m^n)))_m\|_{c_0} &= \sup_m |\phi_m(\check{P}(x, x_m^n))| \\ &\leq (\|\check{P}\| \sup_m \|x_m\|^n) \|x\|, \end{aligned}$$

where the sequence  $(x_m)$  is norm bounded. Thus  $T : X \rightarrow c_0$  is a bounded linear operator.

Since  $X$  has  $DP^*P_p$ , it follows that  $T$  is  $p$ -convergent (by Theorem 5.1.3). Therefore

$$\begin{aligned} 0 = \lim_m \|Tx_m\|_{c_0} &= \lim_m \|(\phi_j(\check{P}(x_m, x_j^n)))_j\|_{c_0} \\ &= \lim_{m \rightarrow \infty} \sup_j |\phi_j(\check{P}(x_m, x_j^n))|. \end{aligned}$$

Thus,

$$\begin{aligned} \|P(x_m)\| = \phi_m(P(x_m)) &= \phi_m(\check{P}(x_m, x_m^n)) \\ &\leq \sup_j |\phi_j(\check{P}(x_m, x_j^n))| \\ &\xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Since  $P$  is not linear, to show that it is  $p$ -convergent, we have to prove that if  $(x_m - a) \in \ell_p^{weak}(X)$ , then  $P(x_m) \rightarrow P(a)$  in norm.

We consider the identity

$$\begin{aligned} P(x_m) - P(a) &= \sum_{j=0}^n \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j) \\ &= P(x_m - a) + \sum_{j=1}^n \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j), \end{aligned}$$

where  $z \mapsto \check{P}(z^{n+1-j}, a^j)$ , ( $j = 1, 2, \dots, n$ ) is a polynomial of degree  $\leq n$ . By the induction assumption,  $\check{P}(x_m^{n+1-j}, a^j) \xrightarrow{m \rightarrow \infty} 0$ . Since  $(x_m - a) \in \ell_p^{weak}(X)$ , we have that  $\lim_{m \rightarrow \infty} \|P(x_m - a)\| = 0$ , as was proved above.

Also,  $\check{P}((x_m - a)^{n+1-j}, a^j)$  has degree  $\leq n$  for  $j = 1, 2, \dots, n$ . Thus by the induction assumption,

$$\sum_{j=1}^n \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j) \xrightarrow{m \rightarrow \infty} 0 \text{ (in norm).}$$

Therefore,  $\lim_{m \rightarrow \infty} P(x_m) = P(a)$ . □

**Theorem 5.2.3** *Let  $X$  be a Banach space with  $DP^*P_p$  and let  $(P_n) \subset \mathcal{P}({}^k X)$  such that  $P_n \rightarrow P \in \mathcal{P}({}^k X)$  pointwise. Then  $P_n \rightarrow P$  uniformly on weakly  $p$ -compact subsets of  $X$  and for each  $(x_n) \in \ell_p^{weak}(X)$  we have  $P_n(x_n) \xrightarrow{n \rightarrow \infty} 0$ .*

**Proof** Since each weakly  $p$ -compact set in  $X$  is limited, it follows from Theorem 4.1.4 that  $P_n \rightarrow P$  uniformly on each weakly  $p$ -compact subset of  $X$ . Let  $(x_n) \in \ell_p^{weak}(X)$ . Then the set  $A := \{x_n : n \in \mathbb{N}\} \cup \{0\}$  is weakly  $p$ -compact. Therefore,  $P_n \rightarrow P$  uniformly on  $A$ . Let  $\varepsilon > 0$  be given. There exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  such that  $|P_n(y) - P(y)| < \frac{\varepsilon}{2}$  for all  $n \geq n_0$  and for all  $y \in A$ ; in particular, this says that  $\sup_j |P_n(x_j) - P(x_j)| < \frac{\varepsilon}{2}$  for all  $n > n_0$ . Thus, we have  $|P_n(x_n) - P(x_n)| < \frac{\varepsilon}{2}$  for all  $n \geq n_0$ . Also, since  $P$  is  $p$ -convergent by Proposition 5.2.2 and  $(x_n)$  is weakly  $p$ -convergent with limit 0, we may also assume that  $|P(x_n)| < \frac{\varepsilon}{2}$  for all  $n \geq n_0$ . It follows that

$$|P_n(x_n)| \leq |P_n(x_n) - P(x_n)| + |P(x_n)| < \varepsilon$$

for all  $n \geq n_0$ . □

**Lemma 5.2.4** *If  $\sup_i |\xi_{i,j}|^n \xrightarrow{j} 0$ , then  $\sup_i |\xi_{i,j}| \xrightarrow{j} 0$ .*

**Proof** Suppose  $\sup_i |\xi_{i,j}| \not\rightarrow 0$  if  $j \rightarrow \infty$ . Then there exists  $\varepsilon > 0$  such that

$$\sup_i |\xi_{i,j_k}| \geq \varepsilon,$$

for some  $j_k \in \mathbb{N}$ . Therefore, for all  $k \in \mathbb{N}$  there exists  $j_k \in \mathbb{N}$  such that

$$|\xi_{i_k, j_k}| \geq \frac{\varepsilon}{2}.$$

Thus

$$|\xi_{i_k, j_k}|^n \geq \left(\frac{\varepsilon}{2}\right)^n$$

and we conclude that

$$\sup_i |\xi_{i_k, j_k}|^n \geq |\xi_{i_k, j_k}|^n \geq \left(\frac{\varepsilon}{2}\right)^n > 0,$$

for all  $k \in \mathbb{N}$  – leaving us with a contradiction. □

Let us agree to denote the class of all  $p$ -convergent  $k$ -homogeneous polynomials from  $X$  into  $c_0$  by  $\mathbb{P}_{pc}(^k X, c_0)$ . In general,  $\mathbb{P}_{pc}(^k X, Y) = \{P \in \mathcal{P}(^k X, Y) : P \text{ is } p\text{-convergent}\}$ .

**Proposition 5.2.5** *The following assertions are equivalent:*

(i)  $X$  has the  $DP^*P_p$ .

(ii) Every operator  $T : X \rightarrow c_0$  is  $p$ -convergent.

(iii) For all integers  $k$ , each polynomial  $P \in \mathcal{P}(^kX, c_0)$  is  $p$ -convergent.

(iv) For some integer  $k$ , each polynomial  $P \in \mathcal{P}(^kX, c_0)$  is  $p$ -convergent.

**Proof** (i)  $\Leftrightarrow$  (ii) follows from Theorem 5.1.3.

(i)  $\Rightarrow$  (iii) follows from Proposition 5.2.2.

(iii)  $\Rightarrow$  (iv) follows trivially.

We prove (iv)  $\Rightarrow$  (ii) (hence (i)): Let  $T \in \mathcal{L}(X, c_0)$  and define  $\theta : X^k \rightarrow c_0$  by

$$\begin{aligned}\theta(x_1, \dots, x_k) &= (Tx_1) \cdot (Tx_2) \cdots (Tx_k) \\ &= (\xi_{i,1}\xi_{i,2} \cdots \xi_{i,k})_{k \in \mathbb{N}},\end{aligned}$$

where  $Tx_j = (\xi_{i,j})_i$  for  $j = 1, \dots, k$ .

Clearly,  $\theta \in \mathcal{L}(^kX, c_0)$ . Thus if  $\check{\theta} = \theta \circ \Delta_k$ , then  $\check{\theta} \in \mathcal{P}(^kX, c_0)$ .

By assumption,  $\check{\theta} \in \mathbb{P}_{pc}(^kX, c_0)$ . Let  $(x_n) \in \ell_p^{weak}(X)$ . Then

$$\check{\theta}(x_n) \longrightarrow \check{\theta}(0) = 0.$$

Therefore,

$$\begin{aligned}(\theta \circ \Delta_k)(x_n) \xrightarrow[\infty]{n} 0 &\Rightarrow \theta(x_n, \dots, x_n) \xrightarrow[\infty]{n} 0 \\ &\Rightarrow (Tx_n) \cdot (Tx_2) \cdots (Tx_n) \xrightarrow[\infty]{n} 0, \text{ i.e.}\end{aligned}$$

$$(Tx_n)^k \xrightarrow[\infty]{n} 0.$$

If  $Tx_n = (\xi_{i,n})_i \in c_0$ , then this says that  $(\xi_{i,n}^k) \xrightarrow[\infty]{n} 0$  in  $c_0$ . Thus,

$$\sup_i |\xi_{i,n}|^k \xrightarrow[\infty]{n} 0.$$

By Lemma 5.2.4 this implies that

$$\sup_i |\xi_{i,n}| \xrightarrow[\infty]{n} 0, \text{ i.e.}$$

$\|Tx_n\|_{c_0} \xrightarrow[\infty]{n} 0$ . Therefore  $T \in \mathcal{C}_p(X, c_0)$  for all  $T \in \mathcal{L}(X, c_0)$ . □

**Proposition 5.2.6** *If  $X$  has  $DP^*P_p$  and  $Y$  is a Gelfand-Phillips space, then every  $P \in \mathcal{P}(^nX, Y)$  is  $p$ -convergent. Furthermore, each  $f \in \mathcal{H}(X, Y)$  which is bounded on weakly  $p$ -compact sets, is weakly continuous on them.*

**Proof** Since the arguments are similar to those in the proof of Proposition 4.2.6, we will leave the details to the reader. In this case each weakly  $p$ -compact set in  $X$  is limited. Therefore,  $P(A)$  is limited in  $Y$  for each weakly  $p$ -compact subset  $A$  of  $X$ . Using that  $Y$  is a Gelfand-Phillips space, it follows that  $\overline{P(A)}$  is norm compact in  $Y$ . To show that  $P$  is  $p$ -convergent, assume that  $(x_n)$  is weakly  $p$ -convergent in  $X$ , i.e. there exists  $x \in X$  such that  $(x_n - x)_n \in \ell_p^{weak}(X)$ . Then the set  $A := \{x_n : n \in \mathbb{N}\} \cup \{x\}$  is weakly  $p$ -compact in  $X$ , hence  $\overline{P(A)}$  is norm compact in  $Y$ . Now we follow the arguments in the proof of Proposition 4.2.6 to verify that  $P(x_n) \xrightarrow{\infty} P(x)$  (in the norm of  $Y$ ). Thus it is clear that  $P$  is a  $p$ -convergent function.

Using Theorem 4.1.6 and lemmas 4.2.2 and 4.2.4 we show that all  $f \in \mathcal{H}(X, Y)$  which are bounded on weakly  $p$ -compact sets are weakly continuous on them.  $\square$

The following (general) result will be needed in our proof of yet another characterisation of Banach spaces with  $DP^*P_p$  to follow.

**Lemma 5.2.7** *Let  $X, Y$  be Banach spaces and  $(x_n) \in X^{\mathbb{N}}$ ,  $(y_n) \in Y^{\mathbb{N}}$  such that  $(x_n, y_n) \in \ell_p^{weak}(X \times Y)$ . Then  $(x_n) \in \ell_p^{weak}(X)$  and  $(y_n) \in \ell_p^{weak}(Y)$ .*

**Proof** Let  $P_X : X \times Y \rightarrow X : (x, y) \mapsto x$  and  $P_Y : X \times Y \rightarrow Y : (x, y) \mapsto y$  be the bounded linear coordinate projections. For  $x^* \in X^*$  we have  $x^* \circ P_X \in (X \times Y)^*$  and thus we have

$$\begin{aligned} (\langle x_n, x^* \rangle) &= (\langle P_X(x_n, y_n), x^* \rangle)_n \\ &= (\langle (x_n, y_n), x^* \circ P_X \rangle)_n \in \ell_p. \end{aligned}$$

This shows that  $(x_n) \in \ell_p^{weak}(X)$ . Similarly,  $(y_n) \in \ell_p^{weak}(Y)$ .  $\square$

**Proposition 5.2.8** *A Banach space  $X$  has the  $DP^*P_p$  if, and only if, every symmetric bilinear separately compact map  $X \times X \rightarrow c_0$  is  $p$ -convergent.*

**Proof** Assume  $X$  has  $DP^*P_p$  and let  $L : X \times X \rightarrow c_0$  be a symmetric bilinear separately compact map. If  $(x_{1,n}, x_{2,n})_n$  is weakly  $p$ -convergent in  $X \times X$ , then there exists  $(x_1, x_2) \in X \times X$  such that

$$(x_{1,n} - x_1, x_{2,n} - x_2)_n \in \ell_p^{weak}(X \times X).$$

By Lemma 5.2.7, we have  $(x_{1,n} - x_1)_n \in \ell_p^{weak}(X)$  and  $(x_{2,n} - x_2)_n \in \ell_p^{weak}(X)$ . Since  $X$  has  $DP^*P_p$ , the (unique) 2-homogeneous polynomial  $P = L \circ \Delta_2 : X \rightarrow c_0$  (i.e.  $P = \hat{L}$ ) defined by the polarization formula

$$L(x_1, x_2) = \frac{1}{8} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 P \left( \sum_{i=1}^2 \epsilon_i x_i \right)$$

is  $p$ -convergent. Also

$$\left( \sum_{i=1}^2 \epsilon_i (x_{i,n} - x_i) \right)_n = \left( \sum_{i=1}^2 \epsilon_i x_{i,n} - \sum_{i=1}^2 \epsilon_i x_i \right)_n \in \ell_p^{weak}(X);$$

thus

$$P \left( \sum_{i=1}^2 \epsilon_i x_{i,n} \right) \xrightarrow[\infty]{n} P \left( \sum_{i=1}^2 \epsilon_i x_i \right)$$

in norm.

We therefore conclude that

$$\begin{aligned} L(x_{1,n}, x_{2,n}) &= \frac{1}{8} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 P \left( \sum_{i=1}^2 \epsilon_i x_{i,n} \right) \\ &\xrightarrow[\infty]{n} \frac{1}{8} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 P \left( \sum_{i=1}^2 \epsilon_i x_i \right) \\ &= L(x_1, x_2) \text{ in norm.} \end{aligned}$$

Conversely, if  $T \in \mathcal{L}(X, c_0)$  is given, then we follow the arguments in the proof of Corollary 4.2.9 to show that

$$T \otimes T : X \times X \rightarrow c_0 : (x, y) \mapsto T(x)T(y)$$

is both symmetric and separately compact. By our assumption,  $T \otimes T$  is therefore also  $p$ -convergent. Let  $(x_i) \in \ell_p^{weak}(X)$ . We prove that

$$\|Tx_i\| \longrightarrow 0, \text{ if } i \longrightarrow \infty :$$

Let  $j_1 : X \rightarrow X \times X$  and  $j_2 : X \rightarrow X \times X$  be the embedding mappings, i.e.  $j_1(x) = (x, 0)$  and  $j_2(x) = (0, x)$ . They are bounded linear operators. We

show that  $(x_n, x_n) \in \ell_p^{weak}(X \times X)$ : Let  $\phi \in (X \times X)^*$ , then  $\phi \circ j_1, \phi \circ j_2 \in X^*$ . Then

$$\begin{aligned}\phi(x_n, x_n) &= \phi((x_n, 0) + (0, x_n)) \\ &= \phi(j_1(x_n) + j_2(x_n)) \\ &= (\phi \circ j_1)(x_n) + (\phi \circ j_2)(x_n) \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Thus

$$\begin{aligned}(\phi(x_n, x_n))_{n \in \mathbb{N}} &= ((\phi \circ j_1)(x_n))_n + ((\phi \circ j_2)(x_n))_n \\ &\in \ell_p.\end{aligned}$$

This shows that  $(x_n, x_n)_n \in \ell_p^{weak}(X \times X)$ . Since  $T \otimes T$  is  $p$ -convergent, it follows that

$$\begin{aligned}\|Tx_n\|_{c_0}^2 &= \|(T \otimes T)(x_n, x_n)\|_{c_0} \\ &\xrightarrow[\infty]{n} \|(T \otimes T)(0, 0)\|_{c_0} \\ &= 0.\end{aligned}$$

By Theorem 5.1.3,  $X$  has  $DP^*P_p$ . □

Using a similar argument as for the proof of Lemma 4.2.12, we easily verifies that

**Lemma 5.2.9** *Let  $X, Y$  be Banach spaces. If  $f \in \mathcal{H}(X, Y)$  is  $p$ -convergent, then it is bounded on all weakly  $p$ -convergent subsets of  $X$ .*

Denote by  $\mathbb{B}_p$  the class of all Banach spaces whose weakly  $p$ -compact subsets are bounding.

**Proposition 5.2.10**  *$X \in \mathbb{B}_p$  if and only if each  $f \in \mathcal{H}(X)$  is  $p$ -convergent.*

**Proof** Assume each  $f \in \mathcal{H}(X)$  is  $p$ -convergent. Then by Lemma 5.2.9 each  $f \in \mathcal{H}(X)$  is bounded on all weakly  $p$ -compact subsets of  $X$ , i.e. all weakly  $p$ -compact subsets of  $X$  are bounding. Thus  $X \in \mathbb{B}_p$ .

Conversely, let  $X \in \mathbb{B}_p$ . Then all the weakly  $p$ -compact subsets of  $X$  are bounding, hence they are limited. Thus  $X$  has  $DP^*P_p$ . Since  $\mathbb{K}$  is a Gelfand-Phillips space, it follows from Proposition 5.2.6 that each  $f \in \mathcal{H}(X)$  is  $p$ -convergent. □

**Proposition 5.2.11** *Let  $X \in \mathbb{B}_p$  and let  $Y$  be a Gelfand-Phillips space. Then each  $f \in \mathcal{H}(X, Y)$  is  $p$ -convergent.*

**Proof** From the proof of Proposition 5.2.10, we see that  $X$  has  $DP^*P_p$ . Also, since  $X \in \mathbb{B}_p$  it follows that each  $f \in \mathcal{H}(X, Y)$  is bounded on the weakly  $p$ -compact subsets of  $X$  – this is clear, since for each  $f \in \mathcal{H}(X, Y)$  and each  $y^* \in Y^*$  we have  $y^* \circ f \in \mathcal{H}(X)$ , showing that for each weakly  $p$ -compact subset  $A$  of  $X$ , the set  $f(A)$  is weakly bounded and hence bounded in  $Y$ . Weakly  $p$ -compact subsets of  $X$  are limited (since  $X$  has  $DP^*P_p$ ) and therefore it follows from Lemma 4.2.4 that  $f$  can be approximated uniformly on all weakly  $p$ -compact subsets of  $X$  by its Taylor series at 0, i.e.  $f_n \rightarrow f$  uniformly on all weakly  $p$ -compact subsets of  $X$ , where each  $f_n(x) = \sum_{k=0}^n P_k(x)$  is the  $n$ -th partial sum of the Taylor series expansion  $f(x) = \sum_{k=0}^{\infty} P_k(x)$  of  $f$ . We show that  $f$  is  $p$ -convergent:

Let  $(x_n) \subset X$  be a weakly  $p$ -convergent sequence, i.e. there exists  $x \in X$  such that  $(x_n - x)_n \in \ell_p^{weak}(X)$ . Now  $f_n \rightarrow f$  uniformly on the weakly  $p$ -compact set  $L = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ . So, for given  $\varepsilon > 0$  we let  $n_0 \in \mathbb{N}$  such that  $\|f_n(x) - f(x)\| < \frac{\varepsilon}{3}$  for all  $n \geq n_0$  and all  $x \in L$ . Each  $P_k$  being  $p$ -convergent by Proposition 5.2.6, it follows that each  $f_n$  (hence also  $f_{n_0}$ ) is  $p$ -convergent. Therefore,  $f_{n_0}(x_n) \rightarrow f_{n_0}(x)$  in norm if  $n \rightarrow \infty$ , i.e. there exists  $k_0 \in \mathbb{N}$  such that  $\|f_{n_0}(x_n) - f_{n_0}(x)\| < \frac{\varepsilon}{3}$  for all  $n \geq k_0$ . As in the proof of Proposition 4.2.13 (ii) we finally conclude that  $\|f(x_n) - f(x)\| < \varepsilon$  for all  $n \geq k_0$ , i.e.  $f(x_n) \rightarrow f(x)$  in norm. Thus we have verified that each  $f \in \mathcal{H}(X, Y)$  is  $p$ -convergent.  $\square$

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