

Criteria for effective mathematics teacher education with regard to mathematical content knowledge for teaching

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Dedicated to my husband Wallie, who fills my life with many happy and memorable moments.

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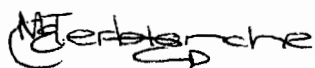
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ABSTRACT

Criteria for effective mathematics teacher education with regard to mathematical content knowledge for teaching

South African learners underachieve in mathematics. The many different factors that influence this underachievement include mathematics teachers' role in teaching mathematics with understanding. The question arises as to how teachers' mathematical content knowledge states can be transformed to positively impact learners' achievement in mathematics.

In this study, different kinds of teachers' knowledge needed for teaching mathematics were discussed against the background of research in this area, which included the work of Shulman, Ma and Ball. From this study an important kind of knowledge, namely mathematical content knowledge for teaching (MCKfT), was identified and a teacher's ability to unpack mathematical knowledge and understanding was highlighted as a vital characteristic of MCKfT.

To determine further characteristics of MCKfT, the study focussed on the nature of mathematics, different kinds of mathematical content knowledge (procedural and conceptual), cognitive processes (problem solving, reasoning, communication, connections and representations) involved in doing mathematics and the development of mathematical understanding (instrumental vs. relational understanding). The influence of understanding different problem contexts and teachers' ability to develop reflective practices in teaching and learning mathematics were discussed and connected to a teacher's ability to unpack mathematical knowledge and understanding. In this regard, the role of teachers' prior knowledge or current mathematical content knowledge states was discussed extensively. These theoretical investigations led to identifying the characteristics of MCKfT, which in turn resulted in theoretical criteria for the development of MCKfT.

The theoretical study provided criteria with which teachers' current mathematical content knowledge states could be analyzed. This prompted the development of a diagnostic instrument consisting of questions on proportional reasoning and functions. A qualitative study was undertaken in the form of a diagnostic content analysis on teachers' current mathematical content knowledge states. A group of secondary school mathematics teachers (N=128) involved in the Sediba Project formed the study population. The Sediba Project is an in-service teacher training program for mathematics teachers over a period of two years. These teachers were divided into

three sub-groups according to the number of years they had been involved in the Sediba Project at that stage.

The teachers' current mathematical content knowledge states were analyzed with respect to the theoretically determined characteristics of and criteria for the development of MCKfT. These criteria led to a theoretical framework for assessing teachers' current mathematical content knowledge states. The first four attributes consisted of the steps involved in mathematical problem solving skills, namely conceptual knowledge (which implies a deep understanding of the problem), procedural knowledge (which is reflected in the correct choice of a procedure), the ability to correctly execute the procedure and the insight to give a valid interpretation of the answer. Attribute five constituted the completion of these four attributes. The final six attributes were an understanding of different representations, communication of understanding in writing, reasoning skills, recognition of connections among different mathematical ideas, the ability to unpack mathematical understanding and understanding the context a problem is set in. Quantitative analyses were done on the obtained results for the diagnostic content analysis to determine the reliability of the constructed diagnostic instrument and to search for statistically significant differences among the responses of the different sub-groups.

Results seemed to indicate that those teachers involved in the Sediba Project for one or two years had benefited from the in-service teacher training program. However, the impact of this teachers' training program was clearly influenced by the teachers' prior knowledge of mathematics. It became clear that conceptual understanding of foundation, intermediate and senior phase school mathematics that should form a sound mathematical knowledge base for more advanced topics in the school curriculum, is for the most part procedurally based with little or no conceptual understanding. The conclusion was that these teachers' current mathematical content knowledge states did not correspond to the characteristics of MCKfT and therefore displayed a need for the development of teachers' current mathematical content knowledge states according to the proposed criteria and model for the development of MCKfT.

The recommendations were based on the fact that the training that these teachers had been receiving with respect to the development of MCKfT is inadequate to prepare them to teach mathematics with understanding. Teachers' prior knowledge should be exposed so that training can focus on the transformation of current mathematical content knowledge states according to the characteristics of MCKfT. A model for the development of MCKfT was proposed. The innermost idea behind this model is that a habit of reflective practices should be developed with respect to the

characteristics of MCKfT to enable a mathematics teacher to communicate and unpack mathematical knowledge and understanding and consequently solve mathematical problems and teach mathematics with understanding.

Key words for indexing: school mathematics, teacher knowledge, mathematical content knowledge, mathematical content knowledge for teaching, mathematical knowledge acquisition, mathematics teacher education

OPSOMMING

Kriteria vir effektiewe wiskunde-onderwyseropleiding met betrekking tot wiskundige vakkennis vir onderrig

Suid-Afrikaanse leerders onderpresteer in wiskunde. Die verskillende faktore wat hierdie onderprestasie beïnvloed sluit die rol van wiskunde-onderwysers in ten opsigte van die onderrig van wiskunde met begrip. Die vraag ontstaan hoe die stand van die onderwysers se wiskundige vakkennis verander kan word om die leerders se prestasie in wiskunde positief te beïnvloed.

Hierdie studie het verskillende soorte kennis waarvoor onderwysers moet beskik om wiskunde te onderrig bespreek teen die agtergrond van navorsing op hierdie gebied, insluitende die werk van Shulman, Ma en Ball. 'n Belangrike soort kennis, naamlik wiskundige vakkennis vir onderrig ("mathematical content knowledge for teaching") of MCKfT is geïdentifiseer, en dit beklemtoon die onderwyser se vermoë om wiskundige vakkennis en begrip te ontbondel as 'n hoogs belangrike eienskap van MCKfT.

Verdere eienskappe van MCKfT is bepaal deur te fokus op die aard van wiskunde, verskillende soorte wiskundige vakkennis (prosedureel en konseptueel), kognitiewe prosesse (probleemoplossing, beredenering, kommunikasie, verbandleggings en voorstellings) wat betrokke is wanneer mens wiskunde doen en die ontwikkeling van wiskundige begrip (instrumenteel vs. relasioneel). Die beheersing van verskillende probleemkontekste en onderwysers se vermoë om verskillende refleksiewe praktyke in die onderrig en leer van wiskunde te inkorporeer is bespreek, en dit is gekoppel aan die onderwyser se vermoë om wiskundige vakkennis en begrip te ontbondel. In hierdie opsig het die studie die rol van die onderwyser se voorafkennis of huidige wiskundige vakkennis breedvoerig bespreek. Hierdie teoretiese ondersoeke het gelei tot die identifisering van die eienskappe van MCKfT, wat weer gelei het tot teoretiese kriteria vir die ontwikkeling van MCKfT.

Die teoretiese studie het kriteria gebied waarvolgens die stand van die onderwysers se huidige wiskundige vakkennis bepaal kon word. Dit het aanleiding gegee tot die ontwikkeling van 'n diagnostiese instrument wat bestaan uit vrae oor proporsionele beredenering en funksies.

'n Kwalitatiewe studie is onderneem in die vorm van 'n diagnostiese analise van die stand van onderwysers se huidige wiskundige vakkennis. 'n Groep sekondêre wiskunde-onderwysers (N=128) wat by die Sediba Projek betrokke is, het die studiepopulasie gevorm. Die Sediba Projek is 'n in-diensopleidingsprogram van wiskunde-onderwysers wat oor 'n periode van twee jaar strek. Hierdie onderwysers is in drie sub-groepe ingedeel ooreenkomstig die tydperk wat hulle in daardie stadium by die projek betrokke was.

Die stand van die onderwysers se huidige wiskundige vakkennis is geanaliseer op grond van die teoreties-bepaalde eienskappe en kriteria vir die ontwikkeling van MCKfT. Hierdie kriteria het gelei tot 'n teoretiese raamwerk waarmee die stand van onderwysers se huidige wiskundige vakkennis gemeet is. Die kriteria is vervolgens omskep in meetbare attribute, en dit het die kriteria vir die diagnostiese meting gevorm. Die eerste vier attribute bestaan uit die stappe betrokke by wiskundige probleemoplossing, naamlik konseptuele kennis (wat begrip van die probleem impliseer), prosedurele kennis (wat gereflekteer word in die keuse van 'n prosedure), die vermoë om 'n prosedure korrek uit te voer en die insig benodig vir die korrekte interpretasie van 'n antwoord. Attribueet vyf omvat die suksesvolle uitvoering van hierdie vier attribute. Die finale ses attribute was 'n begrip van verskillende voorstellings, die kommunikasie van begrip in geskrewe vorm, beredeneringsvaardighede, die herkenning van verbande tussen verskillende wiskundige idees, die vermoë om wiskundige begrip te ontbondel en begrip van die konteks waarin 'n probleem gestel word. 'n Kwantitatiewe analise van die resultate wat met die diagnostiese inhoudsanalise verkry is, is onderneem om die betroubaarheid van die gekonstrueerde diagnostiese instrument te bepaal en om statisties-beduidende verskille tussen die antwoorde van die verskillende sub-groepe op te spoor.

Die resultate het getoon dat die onderwysers wat een of twee jaar lank by die Sediba Projek betrokke was, baat gevind het by die in-diensopleidingsprogram. Die impak van hierdie program is egter beïnvloed deur die onderwysers se bestaande vakkennis van wiskunde. Dit het duidelik geblyk dat konseptuele begrip van grondslag, intermediêre en senior fase wiskunde, wat 'n gesonde wiskunde kennisbasis moet vorm met die oog op gevorderde onderwerpe later in die skoolkurrikulum, in die meeste gevalle prosedureel van aard was met min konseptuele begrip. Die slotsom was dat die stand van hierdie onderwysers se huidige wiskundige vakkennis nie ooreenkom met die eienskappe van MCKfT nie, derhalwe dui dit op 'n behoefte aan die ontwikkeling van onderwysers se wiskundige vakkennis volgens die voorgestelde kriteria en model vir die ontwikkeling van MCKfT.

Die aanbevelings word gebaseer op die feit dat die opleiding wat hierdie onderwysers ontvang met die oog op die ontwikkeling van MCKfT, hulle nie voldoende voorberei om wiskunde met begrip te onderrig nie. Onderwysers se bestaande vakkennis moet bepaal word sodat opleiding op die transformasie van hulle huidige wiskundige vakkennis volgens die eienskappe van MCKfT kan fokus. 'n Model vir die ontwikkeling van MCKfT is voorgestel. Die wese van hierdie model is dat 'n ingesteldheid van reflektiewe praktyke ontwikkel moet word om die eienskappe van MCKfT te bevorder om 'n wiskunde-onderwyser gevolglik te bemagtig om wiskundige vakkennis en begrip te kommunikeer en te ontbondel en gevolglik met begrip wiskunde probleme op te los en wiskunde te onderrig.

Sleutelwoorde met die oog op indeksering: skoolwiskunde, onderwyserskennis, wiskundige vakkennis, wiskundige vakkennis met die oog op onderrig, wiskundige kennisverwerwing, wiskunde-onderwysersopleiding

NOTES ON TERMINOLOGY

The researcher has used the terms "learners" for children in school, "students" for people involved in studying mathematics, "teachers" for people standing in the act of teaching in classrooms, "educators" for people involved in teaching mathematics and "teacher educators" for people involved in the training of teachers.

LIST OF ABBREVIATIONS

SMK – Subject matter knowledge

PCK – Pedagogical content knowledge

MCK – Mathematical content knowledge

PUFM – Profound understanding of fundamental mathematics

MfT – Mathematics for teaching

MCKfT – Mathematical content knowledge for teaching

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CHAPTER 1

BACKGROUND AND OVERVIEW OF THE STUDY

1.1 INTRODUCTION AND PROBLEM STATEMENT

When considering the results of the Third International Mathematics and Science Study (TIMSS), it becomes clear that South African learners have performed poorly in both mathematics and science. According to Strauss and Fourie (1998:44), South African learners do not have an in-depth understanding of mathematics that allow them to apply it in practice, nor do they possess a repertoire of effective strategies for processing information, solving problems or detecting mistakes. They attribute this state of affairs to the poor quality of mathematics teaching. South African teachers have failed dismally in providing mathematics education of a high quality (Strauss & Fourie, 1998:44). Teachers form an important link in the success of any curriculum, as it is teachers who filter the curriculum through to the learners (Du Plooy, 1998:15). It is often said that what a teacher knows is one of the most important variables that impacts on what is done in the classroom (Fennema & Franke, 1992:147).

The President's Educational Initiative (PEI) project (managed by the Joint Education Trust under the leadership of N Taylor and P Vinjevold) reports that not only did South African learners perform extremely poorly when tested on mathematical concepts, but the teachers responsible for teaching these learners also did not do much better when completing exactly the same test (Taylor & Vinjevold, 1999:141). They note that the most common point of convergence across the PEI studies is that teachers' poor conceptual knowledge of the subjects they are teaching is a fundamental constraint on the quality of their teaching and learning activities, and consequently on the quality of learning outcomes. Howie (2003:228) confirms that the quality of teachers and teaching is partly to blame for the crisis in mathematics education in South Africa. She states that most mathematics teachers are not qualified to teach this subject.

It is consequently necessary to ask what the kinds of knowledge are that a mathematics teacher needs to be an effective mathematics teacher. Some researchers and policy-makers argue that South African teachers need to develop a better conceptual understanding of the subjects they

teach (Taylor & Vinjevold, 1999:141). Others argue that teachers need to develop better pedagogical content knowledge (Kahn, 2000). According to Shulman (1986:8) the major limitation on research in the field of teacher training is the lack of a coherent theoretical framework that describes how teachers transform their expertise in the subject matter into forms that learners can comprehend. There is a need for a more comprehensive look at the kinds of knowledge that should be included in teacher education programmes.

South Africa's inability to attract and keep good teachers in the country is growing worse. The task of educating teachers to be competent mathematics teachers is enormous and needs immediate and in-depth attention. The problems regarding teacher education are so extensive that institutions just carry on blindly, trying to train or re-train as many mathematics teachers as possible as quickly as possible, hoping that it will have a positive effect on matric results. However, neither sufficient time nor energy is spent to ensure that the training is effective. Typically, mathematics teaching at the tertiary level seems to be dominated by the assumption that individual concepts and skills can be presented to learners, and that they will be able to combine these skills and concepts appropriately to solve future problems (Stoker, 1993:7).

The current model of mathematics teaching in South Africa is still one of knowledge transmission: "[Teachers'] major experience of a classroom ... is that of a transmission line where the teacher is central, the source of knowledge which is to be transmitted to the students, and the students are passive receivers." (Weinzweig, 1999:26). This traditional application-of-theory model appears to be rather ineffective.

In order to upgrade mathematics teaching it is firstly necessary to improve the quality of pre-service teacher preparation, and secondly, to make quality in-service training available to all mathematics teachers. Teacher education is multi-dimensional, and this research focuses on a specific domain of mathematics teacher education, namely mathematical content knowledge and how it should be taught and learned.

So far, the changes in the preparation of mathematics teachers seem to be scattered. The problem is that mathematics teacher educators do not have a standardised set of criteria according to which the effectiveness of mathematics teacher education programs can be evaluated. Teacher education does not seem to have any action on their agenda. Intensive research is needed in order to identify what abilities, knowledge and skills a competent mathematics teacher should have, as well as what components should be included in the education of such a teacher. Criteria are

needed for preparing teachers with regard to mathematical content knowledge that will best fit the South African circumstances.

This research is an attempt to look into the above-stated dilemmas and to answer the following question: What mathematics content should students and in-service teachers be exposed to so that it impacts on teachers' mathematical content knowledge in ways that enable and improve their mathematics teaching?

1.2 RESEARCH AIMS

This research focuses on understanding the complexity of mathematics teacher education in general, and mathematics teacher education specifically within the complex South African situation. Criteria for effective mathematics teacher education with regard to mathematical content knowledge for teaching (MCKfT) are determined through a literature study and by investigating a group of teachers' current mathematical content knowledge states.

In particular, the research aims to:

- identify the different types of knowledge needed to be able to teach mathematics with understanding;
- propose criteria for the development of MCKfT;
- identify and describe teachers' current mathematical content knowledge states in terms of attribute that implies valuable MCKfT;
- propose a model for the development of MCKfT.

1.3 RESEARCH DESIGN

The research consisted of two parts. Initially an extensive literature study was conducted to investigate world-wide research on the role of teachers' mathematical content knowledge in teaching mathematics. Then an empirical study was carried out to determine teachers' current mathematical content knowledge states.

1.3.1 Literature study

A thorough literature survey was conducted by means of Nexus and Dialog searches. EBSCOhost and internet search engines were also used. The following keywords are of importance: "mathematics teacher", "mathematics teacher education", "mathematical knowledge and understanding", "diagnostic content analysis". The aim with this literature study was to identify relevant literature and research projects that have been conducted in similar areas of mathematics teacher education all over the world. The literature study consisted of three phases (see Chapter 2 and Chapter 3):

- **First phase:** An extensive study was done to (be able to) arrive at the characteristics of MCKfT.
- **Second phase:** Out of the characteristics, implications for effective mathematics teacher education with regard to MCKfT were identified. This led to a set of theoretical criteria for the development of MCKfT.
- **Third phase:** Information from the literature study was used to arrive at a theoretical framework for assessing teachers' current mathematical content knowledge states.

1.3.2 Empirical study

1.3.2.1 Design

A combination of qualitative and quantitative research methods was used in this study (see Chapter 4, Chapter 5 and Chapter 6). The goal was to do a diagnostic content analysis on a group of mathematics teachers' mathematical content knowledge states. The empirical study consisted of various phases:

- **First phase:** Information from the literature study (the theoretical criteria for the development of MCKfT and the theoretical framework for assessing MCKfT) was used to develop a diagnostic research instrument consisting of mathematical content questions.
- **Second phase:** Qualitative research in the form of a diagnostic content analysis was conducted on 128 mathematics teachers involved in the Sediba Project during 2004 and

2005 and the beginning of 2006. The goal was to diagnose and describe the teachers' current mathematical content knowledge states.

- **Third phase:** Quantitative research was carried out to investigate the internal reliability of the diagnostic instrument. Various quantitative analyses were also conducted to look for statistically significant differences in the performances of the teachers during different intervals of their training. The influence the Sediba Project had on these selected teachers, with regard to their MCKfT became evident.
- **Fourth phase:** The criteria, framework and the results from the diagnostic content analysis were used to propose a model for the development of MCKfT during mathematics teacher education.

1.3.2.2 Study population and sample

The study population for the empirical study consisted of participating mathematics teachers involved in the Sediba Project (see § 4.4.2). Sediba mathematics teachers for 2004 (Group 2, N=45), 2005 (Group 1, N=37) and the beginning of 2006 (Group 3, N=46) formed the study population. The teachers were divided into three groups according to the number of years they have been exposed to the Sediba Project.

1.3.2.3 Measuring instruments

After the literature study, a diagnostic instrument was designed with the help of the Statistical Consultation Services at the North-West University (Potchefstroom Campus). It was consequently employed in the evaluation of the teachers in the Sediba Project with regards to MCKfT (see § 4.4.4 and Appendix C).

1.3.2.4 Statistical Techniques

Qualitative as well as quantitative analyses were performed on the data to measure the internal reliability of the developed research instrument and to look for statistically significant differences in teachers' responses to the research instrument (see § 4.6).

1.4 CHAPTER OUTLINE

Chapter 1. Background and overview of the study: The purpose of this chapter is to discuss the problems that give rise to this study. The reader forms an overview of what to expect in this research.

Chapter 2. Teachers' mathematical content knowledge for teaching: This chapter pays attention to the existence of a specific kind of mathematical content knowledge, namely MCKfT. It includes a discussion of the place of MCKfT within different kinds of knowledge for teaching. This gives rise to an outline of MCKfT and criteria for the development of MCKfT.

Chapter 3. Profiling mathematical content knowledge for teaching: The chapter investigates the development of mathematical content knowledge and understanding in order to provide a basis for the theoretical development of MCKfT. The research continues to establish further criteria for the development of MCKfT and proposes a framework for assessing teachers' current mathematical content knowledge states.

Chapter 4. Empirical study: This chapter provides an outline and description of the empirical study, which includes a diagnostic content analysis.

Chapter 5. The results of the empirical study: The results and conclusions of the diagnostic content analysis on teachers' current mathematical content knowledge states become evident in this chapter.

Chapter 6. Summary, conclusions and recommendations: Finally, the study proposes criteria and a model for effective mathematics teacher education with regard to teachers' MCKfT.

1.5 VALUE OF THE RESEARCH

This research offers a formal model for the development of MCKfT. The use of the model allows for the formulation of criteria for the development of MCKfT for South African circumstances. A diagnostic test was constructed to test teachers' MCKfT for proportional reasoning and functions. This provides a first step in developing standardised diagnostic tests for other mathematical topics. The investigation of the Sediba Project reveals the extent to which the North-West University delivers well-qualified and prepared mathematics teachers.

CHAPTER 2

TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE FOR TEACHING

2.1 INTRODUCTION

For decades researchers have tried to identify what aspects are important to be a good mathematics teacher. In this study, the focus is on one specific factor, namely the mathematical content knowledge that a mathematics teacher needs to know and to use to be a competent mathematics teacher. Because there is little agreement about what such mathematical content knowledge entails, research needs to focus on what constitutes an adequate understanding of this domain and how to determine if someone has the kind of understanding that is needed. According to Fennema and Franke (1992:148), the area of teachers' knowledge is a complex area, ill-defined and often poorly studied. This research attempts to contribute to this growing body of knowledge.

The two critical questions regarding this field of study are:

- To what extent do secondary school mathematics teachers understand the mathematics that they teach?
- What mathematics do they know and how is this knowledge structured?

This chapter attempts to determine to what extent teachers' knowledge of the specific mathematics they teach is the key to developing mathematics teaching in South Africa.

2.2 DIFFERENT KINDS OF TEACHERS' KNOWLEDGE

Research on teaching suggests that one of the contributing classes of variables of the effect teachers have on learners' achievement is teaching ability, defined in terms of teachers' knowledge of subject matter and teaching strategies (Rowan et al., 1997:256). This section focuses on different kinds of teachers' knowledge, as it has been established in research conducted around the world. In order to understand the complexity of teachers' knowledge systems, there is a need to unpack the components of teachers' knowledge in order to see what underlies expertise for mathematics teaching.

As early as 1904 John Dewey noticed a tension in the preparation of teachers, namely that of the proper relationship between subject matter and teaching method. He wrote:

"Scholastic knowledge is sometimes regarded as if it was something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject matter." (Dewey, 1904/1964:327-328).

Dewey argues that this separation of substance from method fundamentally distorts knowledge. How an idea is represented is part of the idea, not merely its conveyance. Still, the problem remains today that the prevailing curriculum of teacher education is separated into different domains of knowledge, complemented by experiences such as supervised practicals, student teaching, micro lessons and practice itself. Indications are that knowledge acquired in the classroom does not transfer well to the profession (Hiebert et al. 1996:14). Hence the gap between subject matter and pedagogy fragments teacher education by fragmenting teaching (Ball & Bass, 2000:85).

Researchers, trying to bridge the gap between knowledge of a subject and pedagogical method, began to differentiate between different kinds of teachers' knowledge (see Leinhart & Smith, 1985:247; Shulman, 1986:9). In their study of expert-novice differences in specifically mathematics teaching, Leinhardt and Smith (1985:247) identified two aspects in the expertise involved in the cognitive aspect of teaching. These emerge from two core areas of knowledge, namely lesson structure knowledge and subject matter knowledge for teaching mathematics. They define lesson structure knowledge as the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly. Subject matter knowledge includes concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system drawn upon, the understanding of classes of student errors, and curriculum presentation. The recognition of the importance of subject matter knowledge for teaching mathematics is evident.

Lee Shulman made a valuable contribution to the field of teacher education in general. Shulman (1986:9) distinguishes among three categories of teacher knowledge namely subject matter content knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge. SMK refers to the amount and organization of knowledge in the mind of the teacher. Shulman (1986:9) states that to think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain to understanding the structures of the subject matter.

The introduction of PCK called attention to the fact that there exists a special kind of teacher knowledge that links subject matter and pedagogy. Shulman (1986:9) describes PCK as a

particular form of content knowledge, which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching. It is the most regularly taught topics in a subject area, the most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations and demonstration, including an understanding of what makes the learning of specific concepts easy or difficult and the conceptions and preconceptions that learners of different ages and backgrounds bring with them to the learning (Shulman, 1986:9). It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of learners and presented for instruction (Shulman, 1987:8).

Shulman (1986:10) also defines curriculum knowledge, a third kind of knowledge. It refers to the subject matter knowledge as it is represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of features that serve as both the indications and contra-indications for the use of particular curricula or program materials in particular circumstances.

In her research on English teachers, Grossman (1990:5) re-organized Shulmans' categories into four different ones, namely subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge (which includes knowledge of student understanding, curriculum knowledge, knowledge of instructional strategies) and knowledge of context.

According to Fennema and Franke (1992:148), the following components of teachers' knowledge have received the most attention from researchers: content knowledge, knowledge about learners, knowledge of mathematical representations and pedagogical knowledge. Content knowledge includes knowledge of the concepts, procedures and problem-solving processes. For mathematics education, the organization of mathematical content knowledge, which implies the connections among mathematical ideas, is an important component. Pedagogical knowledge includes components of effective strategies and planning of teaching and motivational techniques. Learners' cognition includes knowledge of how learners think and learn specific mathematical content. Knowledge of how learners think and learn is vital for teachers.

The context (teaching in a mathematics classroom) is the structure that defines the components of knowledge and beliefs that come into play. Within the given context, teachers' knowledge of content interacts with knowledge of pedagogy and learners' cognition and combines with beliefs to create a unique knowledge system that drives classroom behaviour (Fennema & Franke, 1992:162). For them knowledge of representations plays an important part in teaching

mathematics and they emphasise that some components of teachers' knowledge evolve through teaching. Hiebert et al. (1996:16) argue that teaching for understanding in mathematics requires two forms of knowledge namely knowledge of the subject in order to select tasks that encourage learners to wrestle with key ideas, and knowledge of learners' thinking, in order to select tasks that link with learners' experience so that learners can see the relevance of the ideas and skills they already possess.

It follows that SMK, PCK, curriculum knowledge and knowledge of learners form the important parts for the focus on teachers' mathematical knowledge. In mathematics education, SMK is called mathematical content knowledge (MCK).

It is clear that all these researchers acknowledge the important role that MCK plays in being a good teacher. It is also clear that MCK cannot be divorced from PCK, because PCK provides the lens through which a teacher looks when teaching mathematics. Further on, the teacher needs to know how learners think about mathematics and also needs to know how the mathematics school curriculum is structured as a whole. The figure illustrates the kinds of teacher knowledge that are important for the scope of this study.

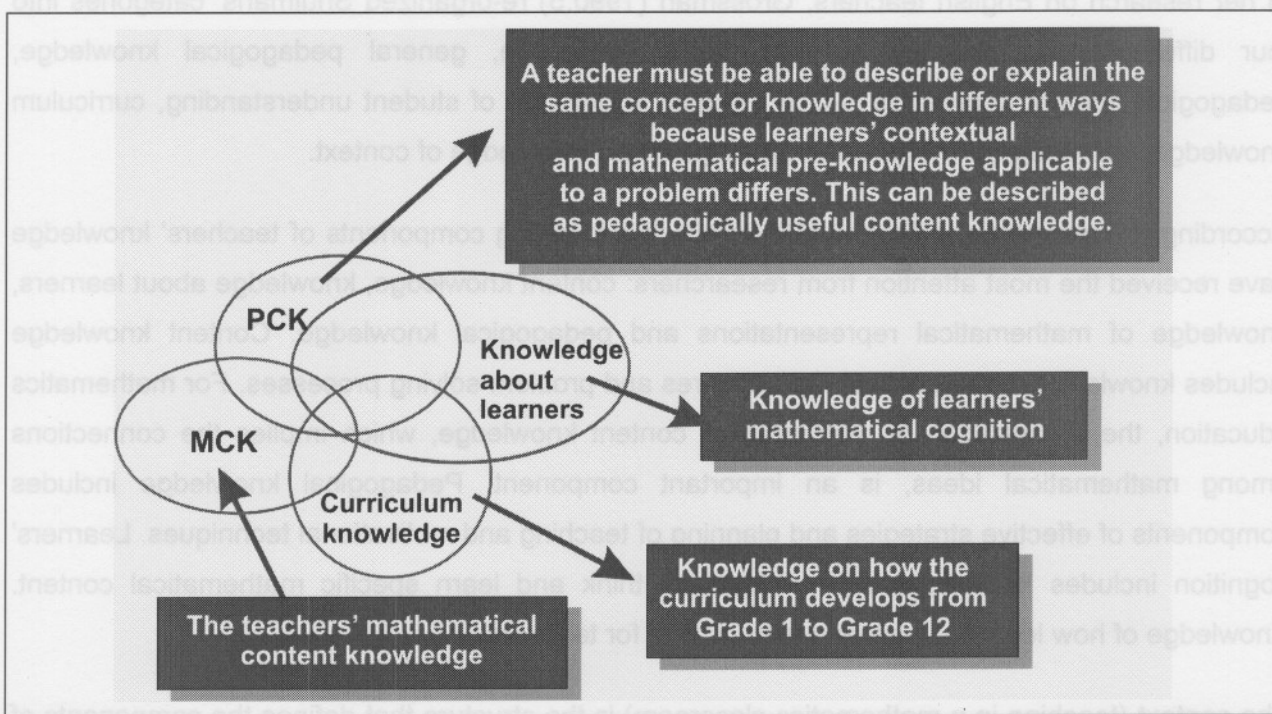


FIGURE 1: Important kinds of mathematics teachers' knowledge.

These different kinds of knowledge are interwoven. Teachers' knowledge is not monolithic, but a large, integrated, functional system with each part difficult to isolate (Fennema & Franke,

1992:148). This makes it difficult to agree on what should be classified as MCK and what as PCK. The persisting tension in the relationship between teachers' MCK and PCK leads to the following two questions (Ball and Bass, 2000:85):

- To what extent does teaching and learning to teach depend on the development of MCK? and;
- To what extent does teaching and learning to teach rely on the development of PCK?

The following section will focus on these two kinds of teachers' knowledge by discussing their respective influences.

2.3 THE INFLUENCE OF MCK AND PCK ON THE TEACHING OF MATHEMATICS

Beginning in the late 1980s, research on teaching increasingly emphasized the importance of teachers' subject matter knowledge for teaching performance. Research on teachers' education has turned from measuring teachers' education in terms of degree levels towards more detailed analyses of the subjects teachers studied in their educational programs. Researchers demonstrated that teachers who have taken more courses in the subject matter they are teaching (SMK) tend to have learners with higher levels of achievement, an indication that teachers' specific SMK is related to teaching performance (Monk, 1994:125). Rowan et al. (1997:256) for instance, identified teachers' MCK as a predictor of student achievement in the tenth grade.

Kahan et al. (2003:225-228) made an analysis of the role of teachers' MCK in their teaching. They developed a framework for research applied to a study of student teachers and identified six elements of teaching in which MCK will matter the most. These elements of teaching are: setting goals and objectives for lessons, the selection of tasks and representations, motivation of content, development (connectivity and sequencing), allocation of time, marks and class discourse. The processes of teaching that are also influenced by a teachers' MCK are planning, instruction, assessment and reflection.

Other researchers, in trying to unravel the complex interplay between teachers' MCK and teaching, also provide evidence for the findings of Kahan et al. (2003:227). According to Rowan et al. (1997:258), a deep knowledge of the subject being taught can support teachers in both the planning and interactive phase of teaching. Content knowledge is also crucial to being inventive in creating worthwhile opportunities for learning that take learners' experience, interests, and needs into account (Ball & Bass, 2000:86). Indications are that mathematically strong teachers tend to

stress more important mathematical ideas and to be less didactic in their instruction (Prawat et al., 1992:146).

Fennema and Franke (1992:149) also found that MCK influences the decisions teachers make about classroom instruction. These decisions influence important features of a lesson such as the tasks on which learners work in and out of class, the representations used by the teacher to guide investigation and explain or clarify ideas (Kahan et al., 2003:229). MCK supports lesson structure and acts as a resource in the selection of examples, formulation of explanations, and demonstrations (Leinhardt & Smith, 1985:247). It furthermore supports the organization and sequence of instruction around important concepts and related operations (Leinhardt & Smith, 1985:247). The unfolding of the mathematics in a lesson or unit is important for the effective teaching of mathematics. The content should not appear to be a collection of disjointed, isolated topics, and it should be sequenced so that topics are studied in a sensible order with prerequisite content being taught or reinforced as needed (Kahan et al., 2003:230).

A teacher's MCK will impact on the allocation of time in a lesson plan and reallocation during instruction if learners have difficulty understanding the mathematics that must be focused on (Kahan et al., 2003:230). Moreover, subject matter knowledge supports the interactive phases of instruction, acting as a resource as teachers formulate explanations and examples, both in lectures and in response to learners' questions (Leinhardt & Smith, 1985:247). According to Ball and Bass (2000:86), subject matter understanding is essential in listening flexibly to others and hearing what they are saying or where they might be heading.

The discourse in a classroom – the way of representing, thinking, talking, agreeing and disagreeing – is central to what learners learn about mathematics as a domain of human inquiry with characteristic ways of knowing (Kahan et al., 2003:230). Discourse cuts across all processes of teaching and Kahan et al. (2003:245) hypothesise that MCK is a factor in recognizing and seizing teachable moments in the mathematics classroom. Ball (as quoted by Kahan et al., 2003:231) suggests that a mathematically strong teacher is flexible enough to ask impromptu questions and to address unexpected statements or conjectures that arise in the classroom. Breen (1999:117) poses the question of whether teacher educators are paying attention to the improvement of those mathematical skills that are essential for recognizing the differing mathematical potential of responses from the class.

The richness of the material being taught appears to be directly related to the MCK of the teacher (Fennema & Franke, 1992:151). MCK may also enhance teachers' sense of the structure, power

and beauty of mathematics and help to strengthen learners' intrinsic motivation for mathematics (Kahan et al., 2003:229).

The mathematical understanding that many prospective teachers bring to classrooms is inadequate for teaching mathematics for understanding (Ball, 1990:464). Leinhardt and Smith (1985:269) asserts that teachers and textbooks often provide incomplete descriptions of the concepts and relationships in a domain and notes that the less complete a learner's knowledge base, the greater the likelihood that the learner will generate incorrect inferences, develop misconceptions, and produce inaccurate problem solutions.

Although it is clear that teachers' subject matter content knowledge plays a very important role in their effective teaching of subject matter, other studies also reveal that it alone does not ensure effective teaching performance. Grossman (1990:16) compared novice English teachers who had deep content knowledge (e.g. being well versed in Shakespeare), but different extents of education as teachers. She found that teachers with preparation as teachers tended to think of how to relate the material (e.g. Hamlet) to student experiences whereas those without preparation as teachers tried to teach secondary learners as the teachers themselves had learned in college seminars, introducing them early on to jargon-intense literary theory. The latter approach was predictably less successful at engaging student's interest, which suggests that content knowledge is not the only part of the knowledge base for teaching. Content knowledge in the subject area does not suffice for good teaching (Kahan et al., 2003:226).

Monk (1994:130) found that courses in undergraduate mathematics pedagogy contribute more to pupil performance gains than do courses in undergraduate mathematics. Ball and Wilson (1990) for instance found that mathematics majors performed no better than non-majors when asked to explain why one cannot divide 7 by 0 and worse than non-majors when asked to develop a story to model $1\frac{3}{4} \div \frac{1}{2}$. Begle (1979:51) reviewed 17 studies and found that "once a teacher reaches a certain level of understanding of the subject matter, further understanding contributes nothing to student achievement."

Counting the number of mathematics courses provides too blunt a measure of a teachers' ability to teach mathematics with understanding (Prawat et al., 1992:145). Ma (1999:xxii, 154) gives evidence that Chinese teachers with the equivalent of a 9th grade mathematics education and 2 or 3 more years of teacher education outperform college-trained United-States teachers when asked to respond to four mathematical teaching scenarios. Direct assessment of teachers' mathematical content knowledge may be a better indication of the relationship between mathematical knowledge

and teaching effectiveness. As evidence from past studies shows, MCK alone do not ensure effective teaching performance, and may not be the best investment of teacher development time.

Shulman's introduction of PCK manifests a first attempt to place greater emphasis on the special kind of knowledge teachers need as part of their subject matter preparation. Ball and Bass (2000:88) define pedagogical content knowledge as a special form of knowledge that bundles mathematical knowledge with knowledge of learners, learning, and pedagogy. Brodie (2001:18) notes that pedagogical content knowledge cannot be divorced from either teaching methods or content knowledge and in fact represents a deeper version of both. It is clear that these bundles of mathematical knowledge or deeper versions of content knowledge are part of both MCK and PCK.

Kahan et al. (2003:226) emphasises that PCK is content-specific and at the same time goes beyond simple knowledge of mathematics, because a mathematician may not possess it. Teachers must for instance not only be capable of defining for learners the accepted truths in a particular domain, they must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and beyond, both in theory and in practice. The category of PCK includes the ways of representing and formulating the subject to make it comprehensible to others (Shulman, 1986:9).

Much of the research on teachers' PCK is strongly prescriptive since it holds a strong image of what teaching should be like. However, according to Ball and Bass (2000:89), no amount of pedagogical content knowledge can prepare a teacher fully for practice, as a significant proportion of teaching is uncertain. What learners bring to the classroom in terms of social background, motivation and mathematical prior knowledge creates a unique situation and influences classroom discourse.

Ma (1999:70) sketches a scenario of how specific teachers' deficiency in understanding the meaning of division by fractions determined their inability to generate an appropriate representation. She notes that their pedagogical knowledge could not make up for their ignorance of the concept and states that to be able to generate a representation, one should first know what to represent. This statement by Ma (1999:70) raises the following questions: What comprises this mathematical content knowledge that a teacher needs in order to be able to represent mathematical concepts to learners? How can teachers' knowledge of mathematics be developed to make a positive impact on learners' understanding of mathematics?

It seems as if Shulmans' description of SMK implies strong MCK. In mathematics, different people may know mathematics in different ways according to their previous experiences with it. Therefore MCK does not imply appropriate MCK for teaching. According to Shulmans' definition of PCK, one

should think that a focus on the development of PCK for mathematics teaching, which has become known as Subject Didactics Mathematics, should make for a transformation of MCK into useable and appropriate PCK in the context of the mathematics classroom. However, this connection does not seem to occur naturally. The researcher believes that the reason for this is the way in which teachers know mathematics. There is a difference between the mathematics that someone knows (MCK) and how that mathematics is structured in the person's mind. A very different kind of mathematical knowledge exists, and this will be the focus of the next section.

2.4 MATHEMATICAL CONTENT KNOWLEDGE FOR TEACHING (MCKfT)

There appears to be an emergent attitude among researchers that there is a distinct body of knowledge associated with mathematics teaching (Davis & Simmt, 2006:294). A new discourse is emerging, attempting to distinguish and define a distinctive form of mathematical knowledge used in the practice of teaching – recognized by the way a mathematics teachers' mathematical knowledge needs to be constructed and used (Adler and Davis, 2006:271-272). The focus in this section will be on trying to isolate and call attention to a distinct kind of mathematical knowledge that is of great importance in teaching mathematics, namely mathematical content knowledge for teaching (MCKfT).

2.4.1 A description of MCKfT

Shulman (1987:8) uses the term SMK or MCK as the first element of the knowledge base for teaching, and this deserves special attention. Research in this field of study should start from teachers' current mathematical content knowledge states as this is what they have to work with when entering any kind of teacher education. It is clear that there should be a strong focus on MCK in teacher education. However, it is necessary to define and focus on a distinct kind of mathematical content knowledge. The reason is that a teacher's mathematical content knowledge needs to adhere to certain characteristics. The following perspective of Michael de Villiers (2006) highlights aspects of this:

"In teaching mathematics, no amount of pedagogical, psychological or philosophical knowledge can compensate for a lack of insight into mathematical concepts and procedures."

These characteristics make teachers' mathematical knowledge useable in teaching mathematics, and it is necessary to determine what they entail. The goal is not to argue or advocate for a "new"

kind of mathematical content knowledge for teachers, but to merely draw attention to an existing special kind of mathematical knowledge that forms part of MCK as well as PCK. Focus should shift from what teachers know mathematically to how their mathematical knowledge is structured. In the next figure MCKfT is shown to be a part of an interwoven knowledge structure.

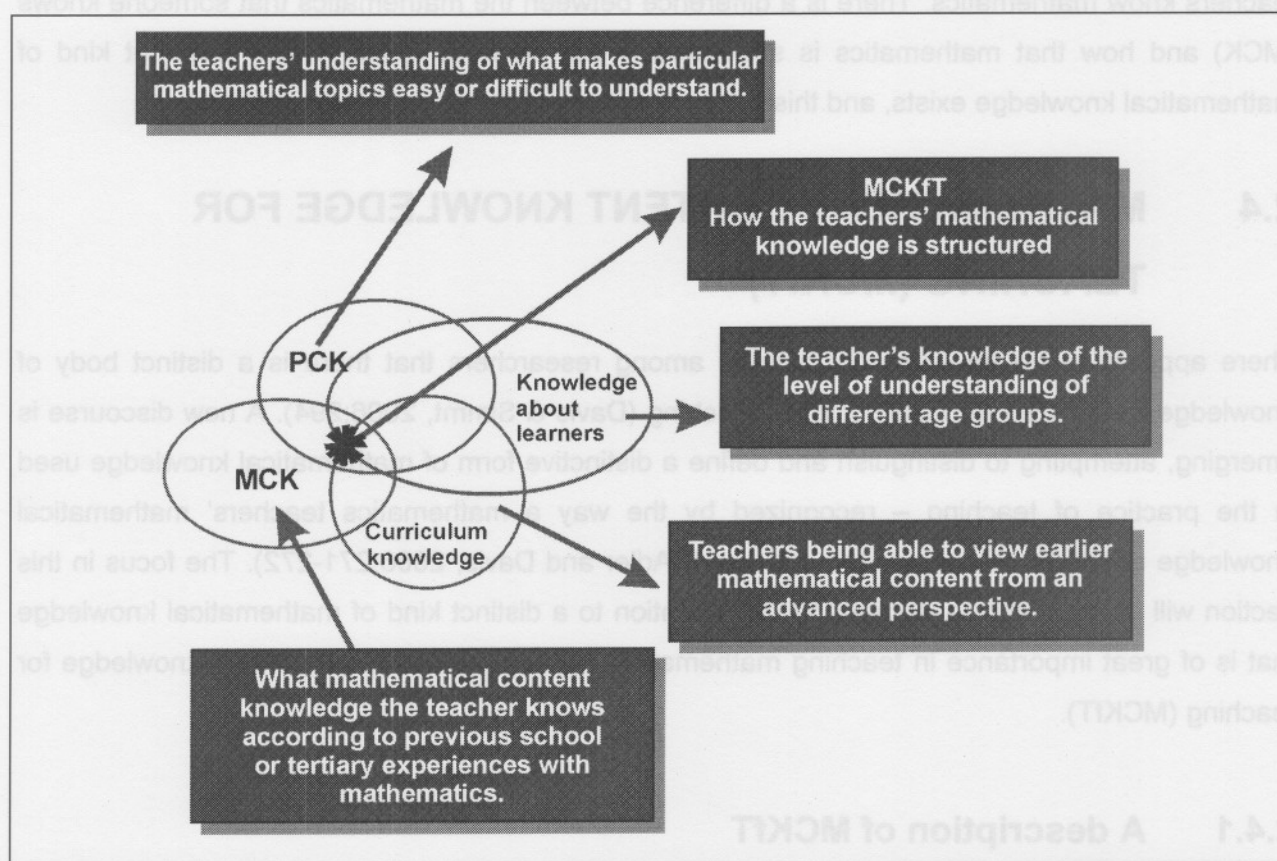


FIGURE 2: A visual description of where MCKfT fits into the big picture of different kinds of knowledge for teaching mathematics.

The importance of MCK cannot be denied, because one cannot teach what one does not know (Fennema & Franke, 1992:147). A general belief is that teachers should learn more mathematics: that the higher the level of mathematics a teacher knows the better teacher he or she becomes (Vistro-Yu, 2005). This, however, may not necessarily be true. In this sense, MCKfT is different from the mathematical content knowledge that a mathematician might have. While the mathematician knows very high levels of mathematics, MCKfT involves taking complex subject matter relevant to school mathematics and translating it into representations that learners can understand (Fennema & Franke, 1992:153). This translation of mathematics into understandable representations is what distinguishes a mathematics teacher from a career mathematician.

However, before a mathematics teacher can do this, a restructuring of his own mathematical knowledge should take place to accommodate these representations.

According to Van de Walle (2004:vii), the fundamental core of effective mathematics teaching combines an understanding of how children learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning on a daily basis. In the context of this study, the above statement entails PCK. MCKfT is less than PCK. It is the way in which the mathematics teacher's mathematical content knowledge is structured. MCKfT is the mathematical content part of PCK as it is used in teaching. Shulman (1986:9) describes it as the particular form of content knowledge that embodies the aspects of content most germane to its teachability. It is likely that Shulman naturally assumed MCKfT within PCK. However, in distinguishing MCKfT from PCK, the distinct character that a teachers' MCK must have is acknowledged. The reason for the distinct recognition of this aspect of MCK and PCK is the need for a focus on developing MCKfT. Two assumptions are made: Firstly, MCKfT is the force behind PCK because it will influence the way you are going to teach and secondly, underdeveloped MCKfT might make it difficult to attain appropriate PCK.

MCKfT also involves knowledge of learners' thinking. This includes a teacher's capability to reflect on what a learner needs to know to get to the next level of understanding and then having the necessary and available mathematical content knowledge to accomplish this goal. Studies provide evidence that teachers can understand individual learners' thinking when they have appropriate and well-organized knowledge (Fennema & Franke, 1992:156). What is not known is to which extent teachers' knowledge of learners' thinking is essential in order to have an impact on teaching.

Furthermore, knowledge of the conceptual progression of the whole school mathematics curriculum also forms a part of MCKfT. Teachers must have in-depth knowledge not only of the specific mathematics they teach, but also of the mathematics that their learners have been taught and will learn (Fennema & Franke, 1992:147). A mathematics teacher needs to be able to reflect on where a learner comes from and where learning mathematics must lead to in future grades.

Appropriate MCKfT should enable a teacher to respond to learners' problems involving the how's and the why's of understanding mathematics while teaching it. Asking and answering questions like "How do you know that?", "Why do you think that?" could be activators of MCKfT. Being able to ask questions like these motivates a teacher or student teacher to constantly reflect on what is being done while doing mathematics. The following section investigates these reflective practices.

2.4.2 The Role of reflective practices

According to Ma (1999:108) Chinese teachers adopt an old Chinese expression: "Know how and know why", which encourages one to discover a reason behind any mathematical action. This reflective practice helps a person to know how to carry out an algorithm and to know why it makes sense mathematically. Reflective practices seem to be an important part of the development and activation of MCKfT. Peterson (1988:7) argues for cognitional knowledge for classroom teaching and learning. This includes teachers' self-awareness of their own cognitive processes and being able to reflect on their own thoughts and actions as they attempt to act on their knowledge in their classroom teaching, knowledge on how learners reason in specific content areas and how to facilitate growth in learners' thinking. These are all elements of reflective teaching practices. For Peterson (1988:7), MCK isolated from learners' cognition of mathematics and from teachers' meta-cognition does not appear to be valuable in teaching mathematics. Unless a teacher can understand his/her own thinking in mathematics, MCK will not be useful in structuring the classroom so that learners can learn mathematics with understanding. It is argued here that MCKfT should be developed during teacher education so that it can be ready as a part of PCK in the practice of teaching mathematics.

The next figure provides a linear structure which indicates where MCKfT comes from and where reflective practices should ultimately lead to. A more comprehensive description of the characteristics of MCKfT will follow in Chapter 3.

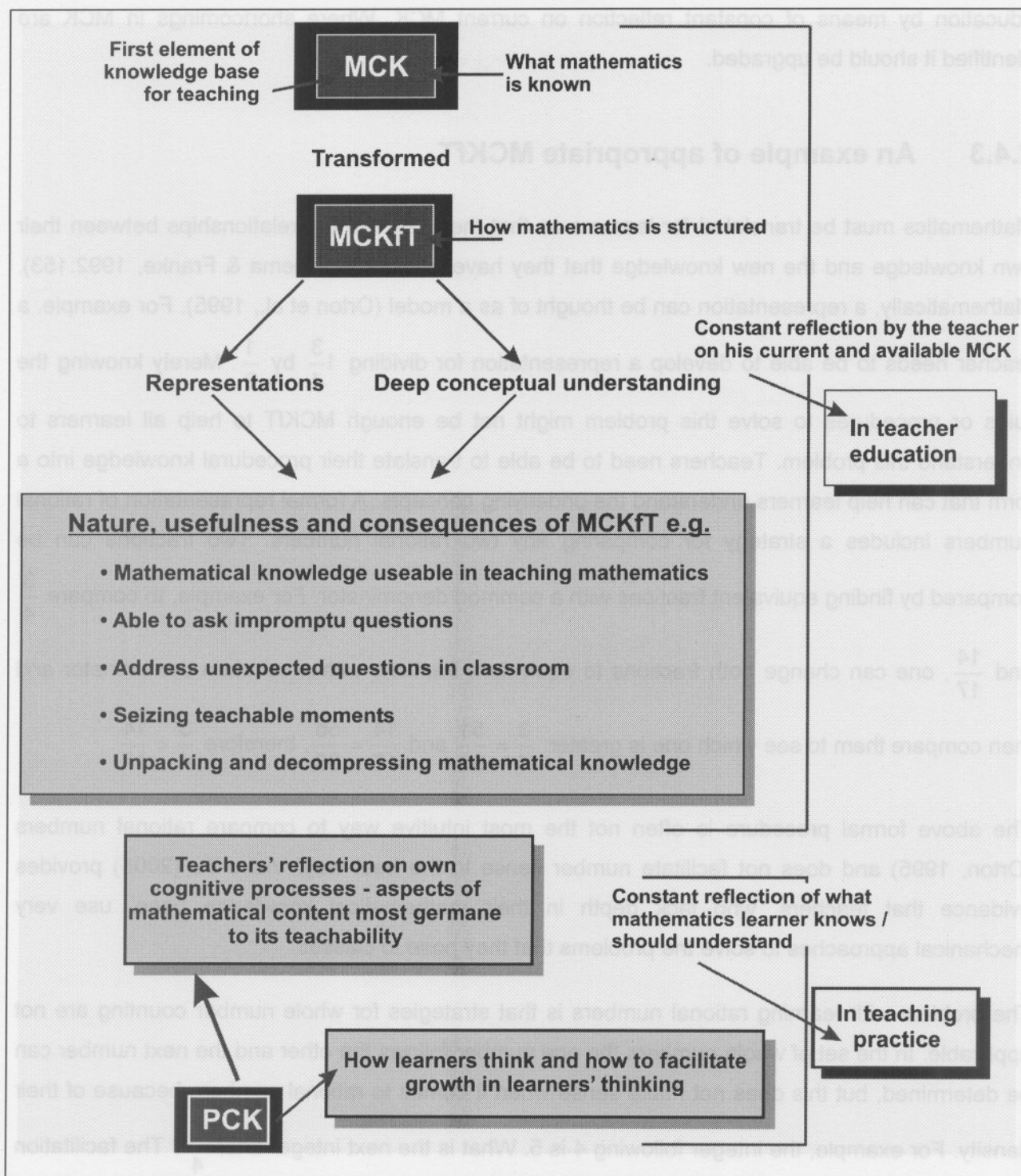


FIGURE 3: A model for the place and development of MCKfT.

Reflective practices in various aspects of teaching and learning mathematics seems to be a very important but neglected part of this field of study. MCKfT should be developed and transformed out of a teachers' current and available mathematical content knowledge states during the teachers'

education by means of constant reflection on current MCK. Where shortcomings in MCK are identified it should be upgraded.

2.4.3 An example of appropriate MCKfT

Mathematics must be translated for learners so that they can see the relationships between their own knowledge and the new knowledge that they have to learn (Fennema & Franke, 1992:153). Mathematically, a representation can be thought of as a model (Orton et al., 1995). For example, a teacher needs to be able to develop a representation for dividing $1\frac{3}{4}$ by $\frac{1}{2}$. Merely knowing the rules or procedures to solve this problem might not be enough MCKfT to help all learners to understand this problem. Teachers need to be able to translate their procedural knowledge into a form that can help learners understand the underlying concepts. A formal representation of rational numbers includes a strategy for comparing any two rational numbers. Two fractions can be compared by finding equivalent fractions with a common denominator. For example, to compare $\frac{3}{4}$ and $\frac{14}{17}$, one can change both fractions to equivalent fractions with a common denominator and then compare them to see which one is greater: $\frac{3}{4} = \frac{51}{68}$ and $\frac{14}{17} = \frac{56}{68}$, therefore $\frac{3}{4} < \frac{14}{17}$.

The above formal procedure is often not the most intuitive way to compare rational numbers (Orton, 1995) and does not facilitate number sense in the best way. Vistro-Yu (2005) provides evidence that teachers, who lack depth in their mathematical knowledge base, use very mechanical approaches to solve the problems that they pose to classes.

The problem with learning rational numbers is that strategies for whole number counting are not applicable. In the set of whole numbers the one number follows the other and the next number can be determined, but this does not make sense when it comes to rational numbers because of their density. For example, the integer following 4 is 5. What is the next integer after $\frac{3}{4}$? The facilitation of the deep concept of rational number order requires a substantial amount of pedagogical skill (Orton, 1995).

"There is a strong temptation to assume that presenting subject matter in its perfected form provides a royal road to learning. What (is) more natural than to suppose that the immature

can be saved time and energy, and be protected from needless error by commencing where competent inquirers have left off?"

(Dewey, 1916:257)

Orton (1995) argues that the "royal road to learning" is a stumbling block to the immature and that the "perfected form" of a subject matter's logic can get in the way of learning.

Research done by Behr and Post (as quoted by Orton, 1995) focuses on the strategies that learners invent for comparing rational numbers. These invented strategies might be viewed as "psychological counterparts" to the formal, common denominator procedure. Some of the strategies mentioned are:

- When comparing $\frac{5}{8}$ and $\frac{2}{5}$, some learners would argue that $\frac{5}{8}$ is greater than $\frac{1}{2}$, and $\frac{2}{5}$ is less than $\frac{1}{2}$. Thus, $\frac{5}{8}$ is greater than $\frac{2}{5}$. This is referred to as the "transitive strategy".
- When comparing $\frac{7}{8}$ and $\frac{12}{13}$, some learners would argue that both fractions are close to 1, but $\frac{7}{8}$ is $\frac{1}{8}$ away from 1 and $\frac{12}{13}$ is $\frac{1}{13}$ away from 1. Since $\frac{12}{13}$ is closer to one, it is larger. This is referred to as a "reference point strategy".
- When comparing $\frac{4}{8}$ and $\frac{7}{8}$, some learners would use a physical model. They would cut a rectangle into 8 parts and show that 4 of these parts is less area than 7 of these parts.

In the last example, a concrete representation is used to facilitate the mathematical solution to the problem.

In his research Orton (1995) used learners' solutions to these kinds of problems as a base for the assessment of teachers' pedagogical reasoning by looking at the degree to which teachers understand how rational numbers are learned. Teachers were asked to reason like a hypothetical learner or to explain a problem to a hypothetical learner in comparing $\frac{3}{4}$ and $\frac{3}{5}$. Teachers were also confronted with erroneous responses of learners to the comparison problem and then asked how they as teachers would respond.

A model of evaluation of knowledge in which teachers' descriptions of how a hypothetical learner would understand a mathematical concept can be used to explore teachers' MCKfT and can

facilitate the growth of knowledge (Orton, 1995). The results of teachers' responses to these kinds of questions provide direction as to where instruction should ultimately go.

Lesh (as quoted by Orton, 1995) describes a "translation model" according to which learners' understanding of a mathematical concept is viewed as the ability to translate within and among different representations of the concept. For example:

- The ability to translate between different expressions of the rational number concept involves being able to express the rational number $\frac{3}{4}$ symbolically as the symbol set $\frac{3}{4}$, or as a circular region cut into 4 parts with 3 of these parts shaded.
- The ability to translate among the same form of representation involves being able to use chips to represent fractions and recognizing that $\frac{4}{6}$ is equivalent to $\frac{2}{3}$ by restructuring an array of four black chips and 2 white chips into an array of two equal sets of black chips and one equal set of white chips.
- The ability to move away from concrete representations of rational numbers to more symbolic modes where in a learner can devise a strategy to find solutions situated in a more formal understanding of the ordering concept.

Two aspects in the research described above become important factors for MCKfT (see § 6.4):

- Teachers' MCKfT can be transformed and expanded by focussing on learners solutions to problems.
- Being able to represent a mathematical concept in various ways and being able to translate between different representations become important learning instruments and a vital part of MCKfT.

Mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, learners will not learn with understanding.

2.4.4 Research on MCKfT

Rowan et al. (1997:284) provides confirmation that understanding the use of mathematics in teaching is a critical area ripe for further examination. The work of Debra Ball and Liping Ma makes an enormous contribution to the ability to identify the mathematical knowledge here defined as

MCKfT, however; much more work in this field is needed. This study intends to contribute to this aspect of research on teachers' MCKfT.

In general people talk about "deep knowledge" of the subject matter (Rowan et al., 1997:258; Kahan et al., 2003:223). This shows recognition of the fact that it is not just what mathematics teachers know, but how they know it and what they are able to mobilize mathematically during the course of teaching.

The research of Ball and Bass (2000) is strongly influenced by the practice of teaching mathematics. Therefore they describe this kind of knowledge as pedagogically functional mathematical knowledge that seems to be central to effective teaching. Ball and Bass (2000:89) and Brodie (2001:19) try to identify the teacher's pedagogically functional mathematical knowledge on the basis of their practices. Rather than identifying the mathematical knowledge needed for teaching by examining the curriculum, or by interviewing teachers, Ball and Bass (2000:89) began instead with an examination of practice itself. Their reason for this is that examining the curriculum, although useful, is incomplete, for it fails to anticipate the mathematical demands of its enactment in the classroom. Interviewing teachers is incomplete because it infers the mathematical demands in teaching from teachers' account of what they think or would do. Ball and Bass (2000:94) rather look for "core activities" in teaching and they seek to identify the mathematical resources needed for these teacher activities. The core activities include such things as figuring out what learners know; choosing and managing representations of mathematical ideas; appraising, selecting and modifying textbooks; deciding between alternative courses of action and steering a productive discussion.

According to Ball and Bass (2000:97), looking at practice to find out what mathematics is used in teaching, reveals subject matter needed in teachers' work that are not seen when one begins with lists of content to be taught derived from the school curriculum. For example, they have uncovered salient issues involving mathematical language. Another question that could for example be heard is what kind of functional knowledge of proof or mathematical justification is germane to elementary instruction. They indicate that the list of what teachers should know that are produced by analyzing the school curriculum is long and largely arbitrary and little is known about how "knowing" the topics on these lists affects teachers' capabilities (Ball & Bass, 2000:95).

The term mathematics-for-teaching has also come into use. This hyphenated phrase highlights the distinct character of teachers' subject matter knowledge (Davis and Simmt, 2006:294). For these researchers the emphasis on more mathematics may be inappropriate. Davis and Simmt (2006:315) state that courses in mathematics for teaching tend to be framed in terms of either

“beyond” or “more of” the concepts that are included in the grade school curricula. They argue that what the mathematics teachers need to know is qualitatively different from the mathematics their learners are expected to master, and they state that the research community has far to go in identifying what these varieties of mathematics might be. Formal mathematics is seen as highly stable and often as pre-given and fixed. In a relatively short span of time, a transformation can be observed in an individual’s understanding of a particular piece of mathematics. Transformations to the body of mathematics take considerably longer.

For Davis and Simmt (2006:297), the key to understanding the growth of knowledge for teaching is between relatively stable aspects of mathematical knowledge and the somewhat more volatile qualities that underpin that stability. They believe that a strong sense of these dynamics is critical to effective pedagogy and a core aspect of teachers’ mathematical knowledge. For them, knowledge of established mathematics is inseparable from knowledge of how mathematics is established. Insight into the historical emergence of core concepts, interconnections among ideas and the analogies and images that have come to be associated with different principles, are of significance.

Other research strongly connected to the current study is that of Ma (1999). In her research on elementary mathematics teachers in China and the United States, Ma (1999:83) found that Chinese teachers have the ability to generate representations that use a rich variety of subjects and different models of division by fractions for example. This seemed to be based on these teachers’ solid knowledge of the topic. On the other hand, the United States teachers, who were unable to represent the operation, did not explain its meaning correctly. This suggests that in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it. She defines this special kind of mathematical knowledge as “profound understanding of fundamental mathematics (PUFM)”, and states that it means an understanding of the terrain of fundamental mathematics that is deep, broad and thorough. For her, breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. The closer an idea is to the structure of the discipline, the more powerful it will be and the more topics it will be able to support (Ma, 1999:121). Thoroughness is the capacity to connect all topics. Depth and breadth, however, depend on thoroughness – the capability to “pass through” all parts of the field – to weave them together. Indeed, it is this thoroughness which “glues” knowledge of mathematics into a coherent whole (Ma, 1999:124). Ma (1999:146) also states that teachers’ knowledge and student learning should be addressed at the same time. She does not regard improvement of teachers’ knowledge as necessarily preceding improvement of

learners' learning, because they are interdependent processes. One cannot expect to improve teachers' mathematical knowledge first, and in so doing automatically improve learners' mathematics education. For her, a teacher's subject matter knowledge of school mathematics is a product of the interaction between mathematical competence and concern about teaching and learning mathematics. Ma (1999:129) notes that PUFM, as a kind of teachers' subject matter knowledge, does not always have clear boundaries and indicates that it is hard to say that a teacher has or does not have PUFM. There is a lack of an adequate understanding of what and how mathematical knowledge is used in practice.

The research by Ball and Bass (2000), Ma (1999) and Davis and Simmt (2006) were strong driving forces behind the current focus on MCKfT. However, one deficiency in these studies is a lack of focus on secondary school mathematics. This research is an attempt to further our understanding of the characteristics of MCKfT that signals a focus on specific mathematical content knowledge needed by secondary school mathematics teachers.

Although the described research focuses on different aspects of MCKfT, the characteristics of MCKfT have not been adequately isolated and described. Research done in this field tried to identify the nature of what is here called MCKfT from practice. It is necessary that this special kind of mathematical content knowledge should be developed and mobilized in the course of teaching through the transformation of teachers' current MCK states. However, this study focuses on teacher education. Therefore an attempt will be made to isolate the characteristics of MCKfT so that criteria can be put forward for the development of MCKfT. These criteria may then possibly provide a directive to the further development of curricula for teacher education, which will include MCKfT.

2.4.5 Evidence of the usefulness of MCKfT for teaching

Teachers' mathematical thinking depends on their capacity to call into play different kinds of knowledge from different domains (Ball & Bass, 2000:88). The question arises: Can a teacher's MCKfT be developed so that a teacher can recognize and seize teachable moments?

Kahan et al. (2003:247) describes "The case of Ms Lehava", and states that her lack of MCK narrowed the scope of what was possible in her mathematics classroom. Cohen (1990:339) describes "The case of Mrs Oublier" and indicates how her ignorance kept her from imagining other ways in which she might teach mathematics.

In their research Ball and Bass (2000:98) have been increasingly intrigued by the many moments of mathematical insight, knowledge and sensibility matters in teaching and the variety of ways in which mathematics is used in practice.

Another important aspect is the creativity necessary in designing instruction in ways that are attentive to difference and which requires substantial proficiency with the material (Ball & Bass, 2000:86). "Learners get stuck: What does one do to get them to remobilize? This task of teaching is impossible without making use of mathematical understanding and insight. Herein lies a fundamental difficulty in learning to teach, for despite its centrality, usable mathematical knowledge is not something teacher education, in the main, provides effectively. Although some teachers have important understanding of the content, they often do not know it in ways that help them hear learners, select good tasks, and help all their learners learn" (Ball & Bass, 2000:94).

If research has shown that teachers' MCK is inadequate for teaching mathematics for understanding (Kahan et al., 2003:223; Monk, 1994:125), how then can this MCKfT be developed so that a teacher can manage the deeply content-related issues that can arise in the classroom? The developing MCKfT for teacher education will be discussed next.

2.5 DEVELOPMENT OF TEACHERS' MCKfT

The development of MCKfT has certain implications for mathematics teacher education. Research on teacher knowledge contains numerous examples of a mismatch between the aims of teacher education programs and prospective teachers' knowledge and beliefs (Kinach, 2002:51). This mismatch points to a need within practice of mathematics teacher education to not only assess, but to transform and deepen prospective mathematics teachers' understanding of mathematics, and to redirect their habitual ways of thinking about mathematics. The question is how and where to conduct the knowledge transformation process.

Ma (1999) provides examples of how and where teacher preparation can most effectively contribute to a teacher's MCKfT. According to Ma (1999:144), teachers' subject matter knowledge develops in a cyclic process, as illustrated in the following figure.

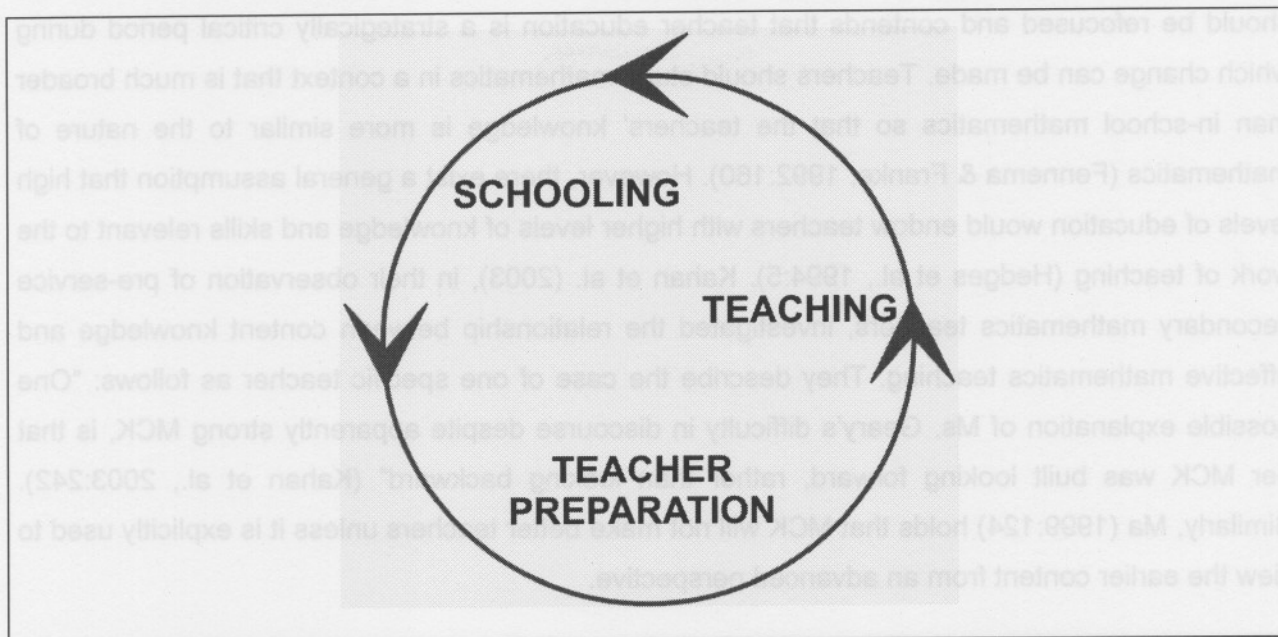


FIGURE 4: Three periods during which teachers' mathematical knowledge develops (Ma, 1999:145).

The figure illustrates three periods during which teachers' subject matter knowledge of school mathematics may be fostered. Ma's research on Chinese teachers indicates that when teachers are still learners, they attain mathematical competence. During teacher education programs, their mathematical competence starts to connect to a primary concern about teaching and learning school mathematics. Finally, during their teaching careers, as they empower learners with mathematical competence, they develop a teacher's subject matter knowledge, which Ma (1999:145) calls PUFM in its highest form.

In China, the cycle spirals upwards, but this was not the case in the United States (Ma, 1999:147). Given that the American teachers' own schooling did not yet provide future teachers with sound mathematical competence, their base for developing solid teaching knowledge was weakened (Ma, 1999:146). This suggests that although Chinese teachers develop PUFM during their teaching careers, their schooling contributes a sound basis for it (Ma, 1999:147). The conclusion thus is that low-quality school mathematics education and low-quality teacher knowledge of school mathematics reinforce each other. Teachers who do not acquire mathematical competence during schooling are unlikely to have another opportunity to acquire it.

It appears that most teachers' knowledge of mathematics may be similar to what can be described as in-school acquired knowledge (Fennema & Franke, 1992:160). Teachers' MCK is thus mostly learned by studying mathematics in schools. Ma (1999:149) believes that teacher preparation

should be refocused and contends that teacher education is a strategically critical period during which change can be made. Teachers should study mathematics in a context that is much broader than in-school mathematics so that the teachers' knowledge is more similar to the nature of mathematics (Fennema & Franke, 1992:160). However, there exist a general assumption that high levels of education would endow teachers with higher levels of knowledge and skills relevant to the work of teaching (Hedges et al., 1994:5). Kahan et al. (2003), in their observation of pre-service secondary mathematics teachers, investigated the relationship between content knowledge and effective mathematics teaching. They describe the case of one specific teacher as follows: "One possible explanation of Ms. Geary's difficulty in discourse despite apparently strong MCK, is that her MCK was built looking forward, rather than looking backward" (Kahan et al., 2003:242). Similarly, Ma (1999:124) holds that MCK will not make better teachers unless it is explicitly used to view the earlier content from an advanced perspective.

Educational mathematics courses need to (but usually do not) give teachers opportunities to unpack their understanding (Ball & Wilson, 1990). Teachers need to be able to "decompress" the mathematical knowledge that they have learned previously that may have become automated (Ball & Bass, 2000:98). The unpacking or decompressing of mathematical ideas is an important part of MCKfT. According to Ball (1990:453), the elementary as well as the secondary learners (who were majoring in mathematics) in her study, had significant difficulty "unpacking" the meaning of division with fractions.

Results from an investigation into current teacher education practices in South Africa revealed that mathematics courses designed specifically for teachers tend to be dominated by compression or abbreviation of mathematical ideas (Adler & Davis, 2006:270). Adler (2005:6) provides an example of the difference in focus of assessment in compressed settings and decompressed (unpacking of knowledge and understanding) settings:

Example 1: Solve for x : $x^2 - 2x = -1$.

Example 2: Here are three solutions to the equation $x^2 - 2x = -1$ presented by Grade 10 learners to their class. Explain clearly which of these solutions is correct/incorrect and why.

(a) Explain how you would communicate the strengths, limitations or errors in each of these solutions to the learners.

(b) What questions could you ask Learner 3 to assist her to understand and be able to formulate a more general response?

Learner 1

$x = 1$ because if $x^2 - 2x = -1$ then $x^2 - 2x + 1 = 0$ and $x = \sqrt{2x - 1}$.

x can't be 0 because we get $0 = \sqrt{-1}$.

x can't be negative because we get the square root of a negative.

$x = 1$ works because we get $1 = 1$ and no other number bigger than 1 works.

Learner 2

$x = 1$ because if $x^2 - 2x = -1$ then $x^2 = 2x - 1$ and this factorises to get $(x - 1)(x - 1) = 0$, so $x = 1$.

Learner 3

$x = 1$. I substitute a range of values for x in the equation. And 1 is the only one that works.

In Example 1 the teacher would need only the right procedural knowledge to solve the problem. In Example 2 the teacher needs to unpack his/her own knowledge and understanding to be able to answer the posed questions.

Adler (2005:10) rightfully states that assessment reflects what is valued and provides evidence that in some of the mathematics teacher education programs, teachers are not assessed on the kind of mathematical knowledge that is implied by unpacking.

Flexibility and adaptiveness are clear requirements of teaching (Ball & Bass, 2000:98). According to Ma (1999:120), teachers must be able to reorganize what they know in response to a particular context. To accomplish this, Ball and Bass (2000:98) indicate that one needs to be able to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible. They refer to this as decompression:

"Most personal knowledge of subject matter, which is desirably and usefully compressed, can be inadequate for teaching. Mathematics is a discipline in which compression is central. Its polished, compressed form can obscure one's ability to discern how learners are thinking at the roots of that knowledge. Because teachers must be able to work with content for learners in its growing, not finished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements." (Ball & Bass, 2000:98).

Ball et al. (as quoted by Adler, 2005:4) suggest that unpacking may be one of the essential and distinctive features of "knowing mathematics for teaching". The reason for this is that teachers work with mathematics as it is being learned; therefore decompression or unpacking of mathematical ideas is necessary.

Students in teacher education programs must also have the opportunity to unlearn acquired knowledge of mathematics where necessary. Mathematics teachers' knowledge needs to be cleared of misconceptions (Ma, 1999:126). One important caution is mentioned by Ma in her work with US teachers. These teachers had some misconceptions about certain topics they were teaching. Some remedial work was done. Ma (1999:126) noticed that their knowledge of the topic seemed to be "cleaned-up" – cleared of misconceptions. However, she noticed that this process may have narrowed their perspectives. Because of their caution about what is correct and incorrect, they tended not to try alternative ways where necessary.

Studies indicate that in the areas in which teachers were more knowledgeable, teachers dealt with learners' misconceptions, while in the less knowledgeable areas, they either did not recognise the misconception, agreed with the learners, or chose not to deal with the misconceptions (Fennema & Franke, 1992:151). Though it is too easy to blame teachers for learners' misconceptions, unfortunately many elementary teachers are deficient in the mathematical knowledge that they teach (Post et al. as quoted by Orton, 1995).

Thus, mathematics teachers need to be able to view earlier mathematical content from an advanced perspective; to decompress and unpack their understanding of the mathematical topics

they need to teach; to clear their mathematical knowledge of misconceptions and to reflect on their practices.

If this is the case, then we can relate to Kinach (2002:55) who contests that the transformation in a teachers' mathematical knowledge appears to be more of a dialectical conversation between prospective teachers' conceptions of mathematics and appropriate pedagogy. In the process, changes in mathematical pedagogical content knowledge occur as a result of changes in teachers' knowledge of mathematical content (MCKfT).

Further criteria for the development of MCKfT will be the focus of Chapter 3. In the following section, the South African context will be analysed.

2.6 A SITUATIONAL ANALYSIS OF THE SOUTH AFRICAN CONTEXT REGARDING MCKfT

Since 1994, South African teachers had to cope with the many different educational aspects advocated by Outcomes Based Education and Curriculum 2005. This education system lays strong emphasis on the "...quality of learning opportunities" (Department of Education, 1995:21). In short, Darling-Hammond (as quoted by Taylor and Vinjevold, 1999:110) states that learners in an Outcomes Based Education program should acquire higher order thinking skills that go beyond recall, recognition and reproduction of information to the evaluation, analysis, synthesis, production and application of ideas. Tirosh and Graeber (2003:643) state that those seeking to change mathematics teaching classroom practices and mathematics teacher education practices, need to remember that it is unwise to generalize conclusions reached in one culture to another. For this reason, a few important aspects that interact with MCKfT will be discussed.

The focus of this discussion will be on the question: Will the development of MCKfT help teachers to make the changes in classroom practices and learning materials that Curriculum 2005 and the Revised National Curriculum Statements demand from them? It is important for mathematics teachers and educators to ascertain what is possible and important given the South African context.

2.6.1 Teachers' knowledge of the subjects they are teaching

Teachers are the most important agents of instructional policy (Cohen, 1990:342). The question then remains: How much can teaching practice improve if the chief agents of change are also problematic?

In the late 90's, extensive research was commissioned by the Teacher Development Centre on behalf of the Department of Education, under the auspices of the President's Education Initiative (Taylor and Vinjevold, 1999:iii). The purpose was to provide a scientific basis for the future planning and delivery of educator development and support programmes for teachers in South Africa. The PEI research reports identified teachers' knowledge as one of the main issues in teaching and learning that were investigated. Results from this research report, fitting the context of this study, are the following:

- Widespread agreement exists that South African teachers need improved content knowledge for teaching mathematics (Taylor & Vinjevold, 1999:139).
- The PEI research studies strongly suggest that teachers' poor grasp of the knowledge structure of mathematics, science and geography acts as a major inhibitor to teaching and learning these subjects, and that this is a general problem in South African schools (Taylor & Vinjevold, 1999:142).
- One of the most consistent findings of a number of PEI projects pointed to teachers' low levels of conceptual knowledge, a poor grasp of their subjects and the range of errors made in the content and concepts presented in their lessons (Taylor & Vinjevold, 1999:139).
- "An analysis of errors indicates fundamental weaknesses in pupil understanding. Another frequently observed practice was that incorrect answers were not corrected. Nor did teachers use correct responses to questions to further develop conceptual understanding" (Taylor & Vinjevold, 1999:145).
- Teachers do not develop their own materials because of time and conceptual knowledge constraints (Taylor & Vinjevold, 1999:233).
- A positive attitude towards Curriculum 2005 exists, but even where these attitudes are supported by materials, it has not yet been accompanied by the development of the skills required to foster active learning, promote meaningful engagement with concepts, or integrate the various learning areas with each other or with everyday knowledge (Taylor & Vinjevold, 1999:157).

Apart from this evidence that there are serious constraints in South African teachers' current mathematical content knowledge states, the demand of Curriculum 2005 on teachers' current mathematical content knowledge states is even more alarming.

According to Taylor and Vinjevold (1999:128), Curriculum 2005 seems to be designed to promote superficiality at the expense of systematic and grounded conceptual development. They contest

that in terms of SAQA's critical learning outcomes, the new curriculum has overbalanced in the direction of context and attitude at the expense of knowledge and skills. For them, only the most dedicated, knowledgeable and skilled teachers are likely to achieve SAQA's learning goals using this curriculum. Even more alarming is a statement made by Taylor (2001:7) that a radical constructivist curriculum like C2005, despite a strong equity agenda, leads to a widening of social inequality, because only highly skilled teachers are able to use it effectively, while those teachers whose own knowledge resources are not strong are left to flounder. There is strong emphasis on the development of conceptual knowledge in Curriculum 2005. A lack of conceptual knowledge in the teacher's mathematical knowledge base can thus have unintended consequences in trying to implement Curriculum 2005. These statements yield overwhelming evidence that teachers in South Africa need MCKfT.

2.6.2 The real world context

An evaluation report commissioned by the Department of Education on pilot materials developed to support the introduction of C2005 in four new learning areas yields evidence that in the area of mathematics teaching and learning, the radical integration of school and everyday knowledge demanded by C2005 leads to practices in which "the body of knowledge that defines mathematics is obscured or dominated by non-mathematical considerations" (Vinjevold & Roberts, 1999:27).

According to Ma (1999:82) one should be cautious because the "real world" cannot produce mathematical content by itself:

"Without a solid knowledge of what to represent, no matter how rich one's knowledge of learners' lives, no matter how much one is motivated to connect mathematics with learners' lives, one still cannot produce a conceptually correct representation."

The topics that Chinese teachers use are broader than and less connected with learners' out-of-school lives and Ma (1999:82) contests that that might help them make more sense of mathematics.

2.6.3 Classroom practices

Research by Brodie (2001:24) indicates that South African teachers can change considerably regarding mathematical knowledge and knowledge of pedagogy, but that they have difficulty in changing their teaching practice towards methods of engaging learners in a learner-centred

approach. For her, the most glaring aspect of the practices of the mathematics teachers in her study was the lack of attention to individual learners:

"What has not shifted at all in her practice is the way in which she engages, or rather does not engage, with particular learners. She does not attempt to elicit their ideas in ways which show her how they are thinking mathematically, and upon which she can build. This is a key element of learner-centred teaching, and her difficulty in achieving this has important implications for research and teacher education in South Africa. We need to understand why this aspect of learner-centred teaching is so difficult. What is it about how this teacher thinks, how her pupils think, how her classroom and school are structured that makes this the most difficult area of change? And to what extent are this teacher's difficulties more widespread? Do many South African teachers in general struggle with engaging with individual learners? If so, what opportunities for learning should we provide for teachers to begin to make this shift?"

(Brodie, 2001:24).

Is it possible that the answers to the above questions are keys to improving teacher education in South Africa? Might it be possible that teachers cannot engage in learner-centred approaches for fear of not knowing the answers to learners' questions? To what extent can the development of MCKfT enable a mathematics teacher to have confidence in teaching mathematics?

The PEI researchers for instance concluded that:

"The most definite point of convergence across the PEI studies is the conclusion that teachers' poor conceptual knowledge of the subjects they are teaching is a fundamental constraint on the quality of teaching and learning activities, and consequently on the quality of learning outcomes. Implementing classroom practices which result in learning which is more effective and intelligent is not a question of ideology or will on the part of the teacher. Teachers support the intentions of the new curriculum, but lack the knowledge resources to give effect to these in the classroom. No amount of exhortation by politicians or pedagogical guidance by curriculum planners, university and college academics or NGOs is likely to change this situation unless the knowledge base of teachers is simultaneously strengthened" (Taylor & Vinjevold, 1999:230).

It is notable that research results indicate that teachers with little knowledge discourage learners' questions and participation in mathematical topics that can be described as low knowledge areas, whereas teachers with high knowledge actively encouraged learners' questions (Fennema &

Frake, 1992:153). Can one conclude then that South African teachers do not feel safe to work in learner-centred environments because of constraints in their mathematical knowledge base which makes it impossible to respond to learners' impromptu questions and unexpected mathematical statements in the classroom? This is a field that needs further investigation, and if this is true, it might be remedied if teachers' MCKfT can be developed.

The PEI researchers also found that:

- the majority of questions posed by teachers involved simple data recall, or were merely used to test whether the pupils were listening (Taylor and Vinjevoold, 1999:145);
- in pre-lesson interviews, teachers quoted discovery, building on prior knowledge, working in groups etc. as the way children learn. However, the methods these teachers pursued in the classroom were quite the opposite of this: pupils were never given the opportunity to discover, there was no evidence of building on prior knowledge, and exclusive whole class teaching occurred. All indications are that these teachers have accepted the desirability of learner-centred pedagogy, but are unable to practise it (Taylor & Vinjevoold, 1999:142).

Mathematics teachers became used to behaving in certain ways, and their learners became used to accepting how the teachers behaved. But there is now overwhelming evidence that in most classrooms, the normal modes of teaching and learning are not efficient for developing learners' mathematical knowledge.

Cohen (1990:330) asserts that a teacher can share the Outcomes Based Education framework's view of learner-centred teaching, but it is one thing to embrace a doctrine of instruction, and quite another to weave it into one's practice. This is also true for teachers in South Africa, as shown by the PEI researchers' research results. Outcomes Based Education clearly advocates learner-centred classroom practices. The question is, is the focus on learner-centred classrooms the best investment in the development of learners' mathematical knowledge in South Africa? Taylor (2001:2) argues that the stronger the learner-centred element of a curriculum, and the lower the socio-economic status of its recipients, the less likely it is to achieve its goal of social equity.

For Ma (1999:151) the key to reform must clearly be understood: whatever the form of classroom interactions might be, the focus must be on substantive mathematics. Mathematics teaching in Chinese classrooms, even by the teacher with PUFM, seems very "traditional"; that is, contrary to that advocated by reform. Mathematics teaching in China is clearly textbook based. In Chinese classrooms, learners sit in rows facing the teacher, who is obviously the leader and maker of the agenda and director in classroom learning. On the other hand, one can see in Chinese classrooms, particularly in those of teachers with PUFM, features advocated by reform – teaching

for conceptual understanding, learners' enthusiasm and opportunities to express their ideas, and their participation and contribution to their own learning processes. How can these seemingly contradictory features – some protested against and some advocated by reform – occur at the same time?

Cobb et al. (1992:598) argues that “meaningful learning” and “learning with understanding” may be mere speech-making in mathematics education. Their argument that the activity of following procedural instruction can be meaningful for learners seems to be at odds with mathematics educators' frequent characterization of this activity as meaningless. In a case study of two classrooms, one with “a traditional school mathematics” where knowledge is transmitted” from the teacher to “passive learners” and one with “a tradition of inquiry mathematics” in which “mathematical learning was viewed as an interactive, constructive, problem-centred process,” the researchers found that in both cases the teachers and the learners actively contributed to the development of their classroom mathematics tradition, while in both classrooms the teachers expressed their “institutionalized authority” during the process. For them, the traditional and reform instruction differ in “the quality of the taken-to-shared or normative meanings and practices of mathematics” rather than in “rhetorical characterizations” (Cobb et al., 1992:598).

Ma (1999:152) notes that although the mathematics teaching in Chinese teachers' classrooms does not meet some “rhetorical characterizations” of the reform, it is actually in the classroom mathematics tradition advocated by the current reform: “...even though the classroom of a Chinese teacher with PUFM may look very ‘traditional’ in its form, it transcends the form in many aspects. It is textbook based, but not confined to textbooks. The teacher is the leader, but learners' ideas and initiatives are highly encouraged and valued.” What Ma (1999:153) points out is that even though Chinese classrooms look very different from what has been advocated by reform, the difference is superficial. Having learners in groups facing each other and using manipulatives may mean that classrooms look more reformed-based. Ma (1999:153) states a critical point: “The real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics.” Therefore she advocates that the change of a classroom mathematics tradition may not be a revolution that simply throws out the old and adopts the new. Rather, it may be a process in which some new features develop out of the old tradition.

2.6.4 Textbooks

An important factor in the South African situation is the role well-developed textbooks can play. According to the results for TIMMS-R, more than 70% of teachers admitted that they consult

school textbooks when deciding what to teach. Vistro-Yu (2005) states that textbooks, as one's main recourse, provide inadequate ideas to teachers, who should be armed with more knowledge on how to teach a concept.

Ma (1999:147-149) believes that good textbooks can be an enormous help to teachers: "If teachers have to find out what to teach by themselves in their very limited time outside the classroom and decide how to teach it, then where is the time for them to study carefully what they are to teach?" Textbooks are the material on which Chinese teachers spend most of their time and devote most of their efforts to study intensively (Ma, 1999:131). According to Ma (1999:133), teacher's manuals provide background for the mathematics in the corresponding textbook and suggestions of how to teach it. The main body of the manual is a section-by-section discussion of each topic and subtopic of the textbook. The discussion of each topic focuses on these questions:

1. What is the concept connected with the topic?
2. What are the difficult points of teaching the concept?
3. What are the important points of teaching the concept?
4. What are the errors and confusions that learners tend to have when learning this topic?

Textbook developers' intentions may be thwarted by teachers' limited understanding of mathematics (Prawat et al., 1992:151). Only when teachers and materials developers have specific details of the concepts, content and skills to be covered in each of the learning areas in each of the school phases can learning materials be developed which address the learning goals of C2005 systematically and rigorously (Vinjevoold & Roberts, 1999:27).

2.6.5 Time allocated for teacher education

Results found by Garet et al. (2001:935) suggest several ways for improving professional development. Their results firstly indicate that sustained and intensive professional development is more likely to have an impact than professional development of shorter duration. The length of professional development was a significant predictor of an institute's effectiveness; teachers with more contact hours tended to learn more - longer sessions can enable more learning (Hill & Ball, 2005:17). Whereas much popular belief stresses the importance of duration, results by Hill and Ball (2005:22) suggest that curricular variables may play an equally important role in the quality and impact of professional development: "Our results also indicate that professional development that focuses on academic subject matter (content), gives teachers opportunities for "hands-on" work

(active learning), and is integrated into the daily life of the school (coherence), is more likely to produce enhanced knowledge and skills.”

Garet et al. (2001:936) collected data that provides empirical support that the collective participation of groups of teachers from the same school, subject, or grade is related both to coherence and active learning opportunities, which in turn are related to improvement in teacher knowledge and skill and changes in classroom practice.

2.6.6 Learning experiences for teachers

Research suggests that developing teachers' MCKfT and improving learners' mathematical education are interwoven and interdependent processes that must occur simultaneously. What is needed then is a teaching context in which it is possible for teachers to improve their knowledge of school mathematics as they work to improve their teaching of mathematics (Ma, 1999:147). In a recent study of professional development, Cohen and Hill (2001:121) also found that professional development was most likely to affect teachers' practice when it was focused on particular content as represented by curriculum, as well as on mathematical ideas, learners' thinking about that same mathematics and teaching that content.

The traditional model for teacher education is sporadically being replaced by other, more reflective approaches. It has been said that these alternative approaches will prepare teachers better for the realities of the classroom. There is a move towards linking theory and practice and looking for ways to achieve this goal. One such an approach is the realistic approach to teacher education. This approach is based on Hans Freudenthal's educational credo, 'mathematics as a human activity', based on the reality of the world around us. This is why Freudenthal called the approach 'realistic' (Korthagen & Kessels, 1999:7).

In realistic mathematics education, the emphasis shifts towards inquiry-oriented activities, interaction amongst learners, and the development of reflective skills (Korthagen & Kessels, 1999:7). The realistic approach can be applied to mathematics teacher education. Korthagen and Kessels (1999:10) define four levels in the process of learning with regard to a certain domain, namely:

1. experiences with concrete examples;
2. gestalt (the dynamic and holistic unity of needs, feelings, values, meanings and behavioural inclinations triggered by an immediate situation);

3. schema (network of elements and relations - a conscious mental framework of concepts and relationships, which gradually becomes more interrelated);
4. theory (a logical ordering of the relations in the schema).

According to the realistic approach, student teachers have as their first field-based experience a one-to-one arrangement whereby they work with one learner only. Through this experience the student teacher comes to terms with practical problems that might be experienced while teaching mathematics. A student teacher therefore experiences the problems in practice, builds a gestalt and through schematisation gets to a schema. At this point the student teacher is ready to be introduced to theories or theoretical information to be applied to solve the problem. Reflective thinking and reflection on the part of the student teacher play major roles in the realistic approach to learning.

Wubbels et al. (1997:87) note that the realistic approach to teacher education asks for very skilful teachers and teacher educators. In the South African context a lack of a strong mathematical knowledge base (especially MCKfT) might be a stumbling block in applying a realistic approach to teacher education.

2.7 CRITERIA FOR THE DEVELOPMENT OF MCKfT IN MATHEMATICS TEACHER EDUCATION

Due to the complexity of MCKfT it is necessary to establish criteria for its development. A focus on specific mathematical content and teachers' learning opportunities may have an important impact on the development of teachers' MCKfT. Teacher educators need to evaluate their own proficiency in their MCKfT base as they must illustrate this kind of mathematical thinking and understanding in mathematics teacher education classrooms.

Hill and Ball (2005:5) advocates that at least in mathematics, how teachers' knowledge is constructed may matter more than how comprehensive their knowledge base is. In other words, teaching quality might not relate so much to performance on standard tests of mathematics achievement as it does to whether teachers' knowledge is procedural or conceptual, connected to big ideas or isolated into small bits, compressed or conceptually unpacked (Ball, 1990 and Ma, 1999).

Research conducted by Leinhardt and colleagues (as quoted by Fenneman and Franke, 1992:161) shows that "good" mathematics teachers' content knowledge tends to be organized into hierarchical structures. They used rich systems of representations; teachers with more knowledge

had richer mental plans and used more representations and richer explanations than their less competent counter parts. They had a tendency to present detailed conceptual and procedural knowledge. The teachers also responded and were open for learners' comments and questions during instruction.

The focus of mathematics teacher education programs for the development of MCKfT should be on the following criteria. Education programs should:

- give mathematics teachers the opportunity to become able to view the earlier mathematical content from an advanced perspective;
- enable teachers to take complex subject matter and translate it into representations that can be understood by learners;
- facilitate connected knowledge of the different mathematical fields;
- further decompressing and unpacking teachers' understanding of mathematical topics;
- eradicate misconceptions in teachers' mathematical content knowledge;
- develop reflective practices.

It is important to understand that knowledge is never static. Teachers' knowledge are constantly changing and developing. The development of MCKfT is also a continuous process in different contexts – in the classroom, with learners and through professional experiences. The focus of this study is the development of MCKfT through professional experiences. In order to change knowledge, teachers have to restructure their current mathematical content knowledge states. Expectancies are that reflection on and establishment of the characteristics of MCKfT will provide a basis for restructured mathematical knowledge to be developed during a teachers' professional development.

According to Fennema and Franke (1992:152), many studies of teacher knowledge of content has not adequately considered the nature of mathematics. The construction of understanding in mathematics also needs to be applied to the study of teacher content knowledge. Various studies conclude that when teacher knowledge of content has been defined in a way that is congruent with the nature of mathematics and/or when a conceptual organization of knowledge was considered, a positive relationship was found between content knowledge of teachers and their instruction (Fennema & Franke, 1992:152). The teachers that were studied seem to know procedural rules but also understood the interrelationships of the procedures. Their teaching practices seem to be more conceptual and less rule-based.

There is a need to unpack how mathematical knowledge is acquired, structured and retrieved (see Chapter 3). There is also a need to re-conceptualize the MCK and PCK courses for teachers. The quality of the MCKfT in the courses should be examined.

2.8 CONCLUSION

The existence and some characteristics of MCKfT have been established in this chapter.

The following questions concerning MCKfT remain unaccounted for:

1. How do mathematics teachers understand the mathematics they plan to teach?
2. What mathematics do they know and how is their mathematical knowledge structured?
3. How must teachers' mathematical knowledge be understood and constructed so that the teacher will be capable to facilitate learners' mathematical knowledge?

There is little agreement about what would count as adequate mathematical understanding of mathematical content to be able to teach it and how to know if someone has that kind of understanding. In the next chapter, the focus will be on aspects that lead to understanding mathematics. Indications are that this search will lead to an understanding of the distinct characteristics of MCKfT and what teacher education should entail to facilitate the development of MCKfT.

Research on South African secondary school mathematics teachers should reveal whether teachers' mathematical knowledge are compressed/decompressed, connected/not connected with different mathematical fields, ready to view the earlier mathematical content from an advanced perspective, cleared/not cleared of misconceptions and if teachers can reflect on their practices by knowing how and knowing why. This aspect will be addressed in the empirical part of this research.

CHAPTER 3

PROFILING MATHEMATICAL CONTENT KNOWLEDGE FOR TEACHING

3.1 INTRODUCTION

The importance of mathematical content knowledge for teaching (MCKfT) as part of a teacher's knowledge and skills repertoire has been discussed in Chapter 2, and it has been established that it is a specific kind of mathematical knowledge. In this chapter, the development of teachers' MCKfT will be critically analysed.

To understand what MCKfT comprises, the nature of mathematics will firstly be unpacked. Secondly, different views on and beliefs about mathematics will be discussed and evaluated since views have a critical effect on what one perceives mathematical teaching and learning to be. Furthermore, mathematical knowledge, cognitive processes involved in doing mathematics and mathematical understanding will be under investigation. The different kinds of mathematical knowledge, the acquisition of mathematical knowledge and the relationships between different kinds of mathematical knowledge will be critically analysed.

The focus will then narrow down to a theoretical investigation of the characteristics of MCKfT. Theoretical criteria for the development of MCKfT will be proposed. In the last part of the chapter, a theoretical framework for assessing teacher's MCKfT is introduced.

3.2 THE NATURE AND ESSENCE OF MATHEMATICS

The nature and essence of mathematics needs to be recognised in order to determine how the teaching and learning of mathematics should take place. Individuals differ considerably in their views on what the teaching and learning of mathematics entails. Since the late 80's, efforts have been made world-wide to rethink what it means to know and do mathematics.

Ernest (1989:250) described mathematics as a process of inquiry and coming to know, not a finished product. Schoenfeld (1992:339) states that mathematics is an act of sense-making, which is socially constructed and socially transmitted. For Cangelosi (2003:v) mathematics is a human endeavour in which people construct concepts, discover relationships, invent methods, execute algorithms, communicate and solve problems posed by their own real worlds.

Where defining mathematics is concerned, there was a shift in focus from knowing mathematics to doing mathematics. The main notion is to describe mathematics as a science of systematically regulated patterns and logical order (Schoenfeld, 1992:343; Goldin, 2002:197, 213; Van de Walle, 2004:13). Doing mathematics then consists of being able to find and explore regularity and order in mathematics and then making sense of it. Classroom mathematics must mirror this view of mathematics as a sense-making activity if learners have to come to understand and use mathematics in meaningful ways (Schoenfeld, 1992:339).

It remains very important to recognize that mathematical knowledge indeed has a hierarchical network and logical structure of inter-related ideas, relations and procedures. But the focus has shifted. Nieuwoudt (1998:77) states that mathematics should not only be characterized by its hierarchical or logical structure, but rather by the actions taken to build up this logical structure of mathematics in an individual's mind. Mathematics is thus a subject that can make sense. However, most learners at school level acquire a different perception of mathematics.

3.3 MATHEMATICS THROUGH THE LENS OF THE SCHOOL CURRICULUM

Students of mathematics develop their sense of mathematics – and thus how they view and use mathematics – from their experiences with mathematics in the classroom. In the traditional mathematics classroom, mathematics is portrayed as a list of fragmented topics with seemingly no relationship. It is a rather passive activity, consisting of listening, copying, memorizing and drilling a boring sequence of terms, symbols, facts and algorithms. Learners emerge from these experiences with a view that mathematics is a series of arbitrary rules, handed down by the teacher, who in turn got them from some very smart source (Van de Walle, 2004:12).

Cangelosi (2003:113) states that there are at least four factors that contribute to the perception that mathematics is a mystifying subject that most people perceive as too difficult to learn:

- a failure to link mathematics to its historical origins;
- a failure to comprehend the language of mathematics;
- the fragmentation of mathematical topics into seemingly disconnected subtopics – especially by textbook-driven curricula;
- the failure of mathematics students to construct concepts and discover relationships for themselves.

A follow-the-rule, computation-dominated, answer-oriented perception of mathematics is a distortion of what mathematics really is and is in direct contrast to a view that involves making sense of mathematics. Views on and beliefs about mathematics and what the teaching and learning of mathematics should entail are influenced by different perceptions on the essence and nature of mathematics.

3.4 VIEWS ON AND BELIEFS ABOUT MATHEMATICS

Over the years, many definitions of what mathematics actually is have seen the light. Throughout the history of mathematics there has always been a tension between the development of beautiful abstract ideas and relationships on the one hand, and the use of mathematics to solve real world problems on the other hand (Dossey et al., 2002:6). With these two very different notions of mathematics in mind, different views on the nature of mathematics and what teaching and learning mathematics according to each of these views entail can be identified.

3.4.1 Three distinct views of mathematics

Ernest (1988) distinguished three conceptions of mathematics, namely the instrumentalist, the Platonist and the problem-solving views of mathematics.

An instrumentalist view of mathematics consists of the idea that mathematics is a “toolbox”, consisting of a set of unrelated but utilitarian rules, facts and skills. The implication of such a perspective of mathematics leads to portraying the mathematics curriculum as pragmatically reduced elements that have to be drilled in through repeated practice and application, until it can be imitated or applied in an automatic way, rather than with thinking or understanding (Nieuwoudt, 1998:75). The implication of such a view of mathematics is that a teacher who holds this belief might see mathematical knowledge as a set of unrelated rules and facts. The teacher’s mathematical knowledge will quite likely also be constructed in such a way.

A second distinguishable view of mathematics is the dynamic, problem-driven view of mathematics, also called the constructivist or relativist-dynamic view. Mathematics is seen as a dynamic, continually expanding field of human creation, in which patterns are generated and then distilled into knowledge (Thompson, 1992:132). This view portrays mathematics as a process of enquiry and coming to know. Mathematics is thus not a finished product with its origin outside the individual. Nieuwoudt (1998:71) states that it remains in the making in the individual’s mind. Consequently, mathematics is viewed as an internal act of investigation, of searching to know what to do and why and of meaningful extension of knowledge.

In the Platonist view, mathematics is viewed as a fixed and static body of certain knowledge consisting of a logical and meaningful network of inter-related truths (facts, rules and algorithms), bound together by filaments of logic and meaning (Thompson, 1992:132). Nieuwoudt (1998:69) labels this the [traditional] formalist-static view of mathematics. Dossey et al. (1992:40) indicates that this [rigid] view stems from Plato's learning, according to which the origin of mathematics is outside the individual in the "external world" of ideas.

The implication of such a view of mathematics is that a teacher who holds this belief might see mathematical knowledge as neat chunks of mathematics that can be transferred to learners. Mathematics is therefore something that can be learned through teaching, not created in the mind of the learner. The important element is the perception that the teacher is the giver of knowledge and the learners are the passive receivers of this knowledge.

However, Nieuwoudt (1998:77) contends that a person cannot receive such a logical structure of mathematics from someone else. The misconception that this is possible arises from a belief that mathematics must be taught sequentially. According to Van de Walle (2004:12), it is not acceptable to hold outdated beliefs about mathematics and expect to be a teacher of quality. Views on and beliefs about mathematics should therefore be analyzed and reformed as it greatly influences mathematical teaching and learning.

3.4.2 Beliefs and classroom practices

Teachers' views of the nature of mathematics may have a great deal to do with the way in which mathematics is characterized in classroom teaching (Prawat et al., 1992:145; Dossey, 1992:42). Research suggests a discrepancy between what people acknowledge mathematics to be and how it is portrayed in classrooms. Results from a study on South African mathematics teachers suggested that the participating teachers' beliefs on the nature of mathematics tended to be innovative and correlated with innovative views of teaching and learning. However, these views were often not reflected in their practices (Webb & Webb, 2004:13). Despite teachers' professing of beliefs in a constructivist paradigm, they used traditional approaches that led learners to see mathematics as a subject to be memorised.

Establishing someone's views on mathematics is not an easy task, as research by Cohen (1990:327-344) indicates. Through observing Mrs Oublier's classroom (see also § 2.4.5), Cohen found that although Mrs Oublier believed that she has revolutionized her mathematics teaching, her practices were filtered through a very traditional approach to instruction. A new mathematics curriculum was used, but it was portrayed in a way that conveyed a sense of mathematics as a

fixed body of right answers, rather than as a field of inquiry in which people figure out quantitative relations. Though the teacher believed that it was important to unpack answers, not one of the answers was unpacked in the classroom. Indications are that it is one thing to embrace a doctrine of instruction, and quite another to weave it into one's practice.

3.4.3 Beliefs and knowledge

There seems to be a close connection between teachers' knowledge of mathematics and their beliefs about the nature of mathematics. Thompson (1992:129) states that it is difficult to distinguish knowledge from beliefs, as teachers frequently treat their beliefs as knowledge. This means that teachers' beliefs about the nature of mathematics will influence what they believe mathematical knowledge to be.

Teachers' views on the nature of mathematics inform their philosophy of teaching and learning mathematics. Views on and beliefs about the teaching and learning of mathematics is therefore closely related to how one has learned mathematics and how one's mathematical knowledge is structured. A need arises to know how teachers' mathematical knowledge is structured.

3.4.4 Implications for MCKfT

Research shows that the general attitude towards mathematics contributes significantly to the coherence and consistency of a teacher's mathematical knowledge (Ma, 1999:120). A basic attitude such as to justify a claim with a mathematical argument, to know how as well as to know why, to keep the consistency of an idea in various contexts and to approach a topic in multiple ways, are relevant to all topics in mathematics.

Nussbaum and Novick (as quoted by Shunk, 1996:280) propose a three step model for changing students' beliefs that can be applied to beliefs about mathematics:

1. reveal and understand student beliefs about mathematics;
2. create conceptual conflict with those beliefs;
3. facilitate the development of new or revised schema and beliefs about teaching and learning mathematics.

The existence of such a model indicates the possibility of changing beliefs to influence practice. Indications are that teachers' current views on the nature of mathematics should be exposed and challenged as it might have an effect on the development of MCKfT.

3.5 MATHEMATICAL KNOWLEDGE

According to Hiebert and Lefevre (1986:1), there exists a direct link between how a teacher knows mathematics and how that teacher then teaches mathematics to learners. To be able to investigate how teachers know their mathematics, it is necessary to establish what kinds of mathematical knowledge exist. A more complete description of the nature of mathematical knowledge is a first step on the road to a better description of the acquisition of mathematical knowledge.

The aspects that will be the focus of this section includes the following: a search through literature for research on different types of mathematical knowledge, appropriate balances between different types of mathematical knowledge, some important aspects of mathematical knowledge acquisition and the relationships between different kinds of mathematical knowledge.

3.5.1 Objects of mathematical knowledge

Various attempts have been made to define the mathematical content that students of mathematics should learn. One such a description of mathematical knowledge is Bell's (1991:108), who distinguishes between direct and indirect objects of mathematical knowledge. Direct objects of mathematical learning are defined as facts, skills, concepts and principles. Bell (1991:108-109) names this the four categories into which mathematical content can be separated. Some of the indirect objects are: transfer of learning, inquiry ability, problem-solving ability, self-discipline and appreciation for the structure of mathematics.

Mathematical facts are those arbitrary conventions in mathematics, such as the symbols. For example: It is a fact that 2 is a symbol for the word two, + is a symbol for the operation of addition. Facts are learned through rote learning such as memorization, drill and practice. People are considered to have learned a fact when they can state the fact and make appropriate use of it in a number of different situations.

Also worth noticing is the term "simple knowledge" that Cangelosi (2003:167) uses to describe specified responses (but not multi-step processes) that students of mathematics remember after a specified stimulus. For example: A student states that $4 \times 4 = 16$ or that the ratio of the circumference of any circle to its diameter is π .

Mathematical skills are operations and procedures that should be carried out with speed and accuracy. Skills are specified by sets of rules and instructions or by ordered sequences of specific procedures, called algorithms. Cangelosi (2003:216) defines an algorithm as a multi-step procedure for obtaining a result. Students of mathematics apply algorithmic skills by remembering and executing a sequence of steps in a specific procedure (Cangelosi, 2003:167). For example: Fractions can be added together. Skills are learned through demonstration, drill and practice. Students have mastered a skill when they can correctly demonstrate the skill by solving different types of problems requiring the skill or by applying the skill in various situations.

Mathematical concepts are defined as abstract ideas that enable people to classify objects or events and to specify whether the object or events are examples or non-examples of the abstract idea. For example: Sets, subsets, equality, inequality, triangle, cube, radius and exponent are all examples of concepts. A concept is learned by hearing, seeing, handling, discussing, or thinking about a variety of examples and non-examples of the concept and by contrasting the examples and non-examples. A student has learned a concept when he or she is able to separate examples of the concept from non-examples.

Mathematical principles are sequences of concepts, together with relationships among these concepts.

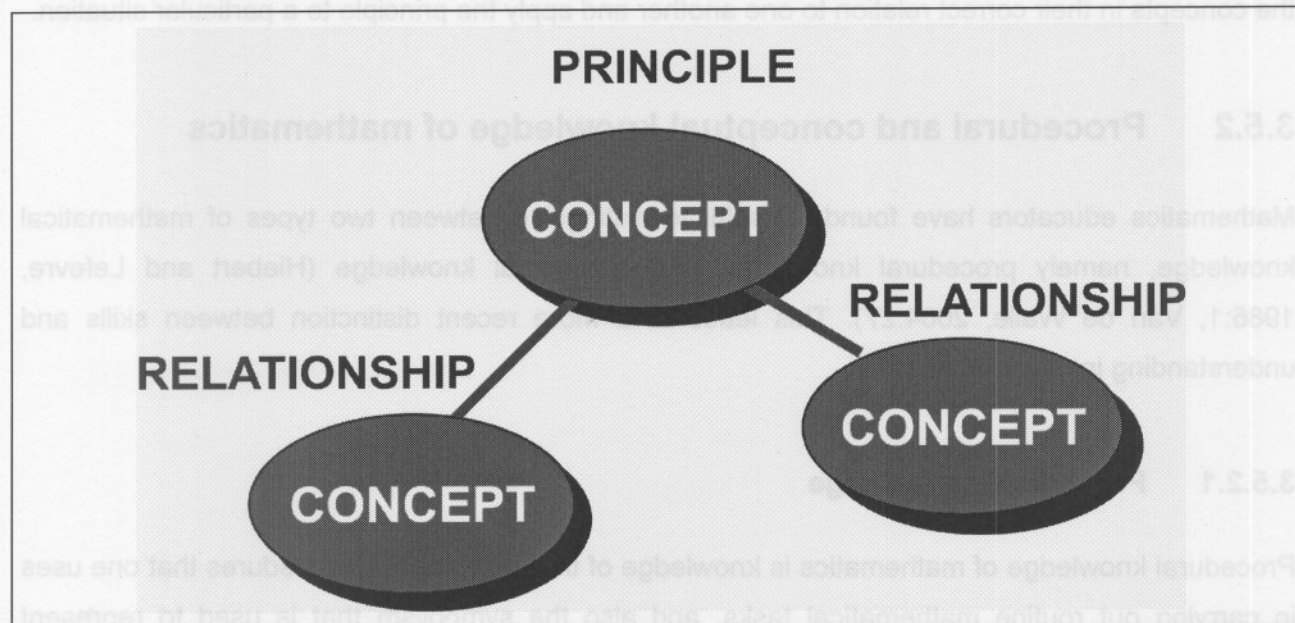


FIGURE 5 A representation of a principle

According to Bell (1991:109-110), mathematical ideas progress in order of complexity from simple facts, to skills and concepts, to complex principles.

TABLE 1: An example of the progress in order of complexity of simple facts, skills, concepts and complex principles.

Facts	A student who merely memorizes the quadratic expression $ax^2 + bx + c$ knows a fact.
Skills	A student who can put numbers into the quadratic formula and comes up with the answers, has learned a skill.
Concepts	A student who can classify 5, 3 and 4 as constants and x as a variable for the quadratic equation $5x^2 + 3x + 4 = 0$ understands some key concepts.
Principles	A student who can draw from the quadratic formula to find the roots of the equation and explain the origin to someone else has mastered a principle.

Principles can be learned through inquiry, guided discovery lessons and group discussions. A student has learned a principle when he/she can identify the concepts included in the principle, put the concepts in their correct relation to one another and apply the principle to a particular situation.

3.5.2 Procedural and conceptual knowledge of mathematics

Mathematics educators have found it useful to distinguish between two types of mathematical knowledge, namely procedural knowledge and conceptual knowledge (Hiebert and Lefevre, 1986:1, Van de Walle, 2004:27). This leads to a more recent distinction between skills and understanding in mathematics.

3.5.2.1 Procedural knowledge

Procedural knowledge of mathematics is knowledge of the rules and the procedures that one uses in carrying out routine mathematical tasks, and also the symbolism that is used to represent mathematics (Van de Walle, 2004:27). Hiebert and Carpenter (1992:78) define procedural knowledge as a sequence of actions. Procedures are thus the step-by-step routines learned to accomplish a specific task. Step-by-step procedures exist to perform tasks such as multiplying

47 × 68. Hiebert and Lefevre (1986:6) describe procedural knowledge as consisting of two distinct parts:

One part is the formal language or symbol representation system of mathematics; also called the “form” of mathematics. It includes a familiarity with the symbols used to represent mathematical ideas and an awareness of the syntactic rules for writing symbols in an acceptable form. For example: A student of mathematics should be able to know that the expression $3 \div x = 2$ is syntactically acceptable and that $6+ = x2$ is not acceptable. Knowledge of the symbols and syntax of mathematics implies only an awareness of surface features, not knowledge of meaning. This description of procedural knowledge correlates with Bell’s description of mathematical facts.

The other part of procedural knowledge consists of rules, algorithms or procedures used to complete mathematical tasks. These kinds of procedures are problem-solving strategies or actions that operate on concrete objects, visual diagrams, mental images, or other objects that are not standard symbols of our mathematical system. This part of procedural knowledge is in line with mathematical skills as indicated by Bell (see § 3.5.1).

To distinguish between the two kinds of procedures, the objects upon which they operate can be investigated (Hiebert & Lefevre, 1986:6). For example: Firstly, standard written symbols like 3 and $\sqrt{\quad}$; and secondly, objects that are non-symbolic e.g. concrete objects or mental images like a construction in geometry. After learners have been in school for a few years, the objects often are symbols. Learners are presented with problems in the form of expressions consisting of symbols.

A key feature of procedures is that they are executed in a predetermined linear sequence. It is the clearly sequential nature of procedures that sets them apart from other forms of knowledge (Hiebert & Lefevre, 1986:6). Procedures are hierarchically arranged so that some procedures are embedded in others as sub-procedures. An entire sequence of step-by-step prescriptions or sub-procedures can be characterized as a super-procedure. The advantage of creating super-procedures is that all sub-procedures in a sequence can be accessed by retrieving a single super-procedure. For example: To apply the super-procedure “multiply two decimal numbers” (e.g. 3.82×0.43) one usually applies three sub-procedures: one to write the problem in appropriate vertical form, a second to calculate the numerical part of the answer, and a third to place the decimal point in the answer. The second of these is itself made up of lower level sub-procedures for (whole numbers) multiplication. The sub-procedure is accessed as a sequential string once the super-procedure is identified.

Many of the procedures that students master probably are just chains of prescriptions for manipulating symbols (Hiebert and Lefevre, 1986:7). For example: A student can decide on using the cross-product algorithm in solving a problem. The minimal connections needed to create internal representations of a procedure are connections between succeeding actions in the procedure (Hiebert and Carpenter, 1992:78). The student thus might not have an understanding of the sequential steps in using the cross-product algorithm or why it might be used to solve the specific problem (see § 3.8.2 on using the cross-product algorithm in the wrong problem context).

Understanding at each level of the sequence then becomes a problem - students of mathematics do not always automatically understand exactly what and why they are executing sub-procedures in a super-procedure to solve a mathematical problem. The assumption is that all students construct understanding and relationships between mathematical knowledge automatically. This assumption might have an impact on using wrong procedural knowledge in a specific problem context (see § 3.8.1 for an explanation of why the cross-product algorithm always works).

It seems that as time passed, educators placed greater emphasis on understanding than on skills development. Conceptual knowledge, a key factor in mathematical understanding, will be the focus of the next section.

3.5.2.2 Conceptual knowledge

Cangelosi (2003:173-174) provides a useful explanation of what a concept consists of:

"Our world consists of specifics - unique entities, things that are not abstract. Specifics are far too many for us to think about each one as a unique entity. Therefore we categorize specifics according to certain commonalities. These categories in return provide a mental filing system for storing, thinking and retrieving information. This process by which a person groups specifics to construct a mental category is referred to as conceptualizing. The category itself is a concept."

A concept is thus an abstract category that people mentally construct from a set of two or more elements by creating a class of specifics possessing a common set of features. For example: Rational numbers, rate, function and the right triangle are all examples of concepts. A concept attribute is a characteristic common to all examples of a particular concept; a concept attribute is a necessary requirement for a specific to be subsumed within a concept (Cangelosi, 2003:417).

3.5.2.3 Conceptual knowledge and relationships

For Hiebert and Lefevre (1986:3), conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. Van de Walle (2004:27) also states that conceptual knowledge of mathematics consists of logical relationships constructed internally and existing in the mind as a part of a network of ideas.

A relationship is described as a particular association between either concepts (e.g. $\{\text{irrationals}\} \subseteq \{\text{reals}\}$), a concept and a specific (e.g. $x^2 > -4 \forall x \in \{\text{reals}\}$), a specific and a concept (e.g. 5,981 is prime), or between specifics (e.g. $\sqrt{13} \geq 1,1$) (Cangelosi, 2003:139, 421). For example: More than 1000 years before the birth of Pythagoras, Babylonians discovered that $a^2 + b^2 = c^2$, where c is the length of a right triangle and a and b are the lengths of the triangle's legs. Ideas such as seven, rectangle, ones/tens/hundreds (as in place value), sum, product, equivalent, ratio, and negative are also all examples of mathematical relationships or concepts. Cangelosi (2003:421) states that a relationship is discoverable if one can use reasoning or experimentation to find out that the relationship exists. Other relationships are relationships of convention that exists because it has been established through tradition or agreement. For example: the use of $\sqrt{3}$ to mean the positive number that when squared equals 3.

Connected webs of knowledge are networks in which the linking relationships are as prominent as the discrete pieces of information. A unit of conceptual knowledge cannot become an isolated piece of information. By definition, it is part of conceptual knowledge only if a student recognizes its relationship to other pieces of information (Hiebert & Lefevre, 1986:4; Hiebert & Carpenter, 1992:78). Concepts, together with the relationships among certain concepts, form the principles that Bell defined (see § 3.5.1).

The important point is that concepts, like procedures, are not all of the same kind. It is useful to distinguish between two levels at which relationships between pieces of conceptual mathematical knowledge can be established (Hiebert & Lefevre, 1986:4-5). The two levels are primary level relationships and reflective level relationships.

In primary level relationships, the relationship connecting the information is constructed at the same level of abstractness than that at which the information itself is represented. The relationship is no more abstract than the information it is connecting. The term abstract is used here to refer to the degree to which a unit of knowledge (or a relationship) is tied to specific context. For example: Students learn various things about decimal numbers. They learn that the position values to the right of the decimal point are tenths, hundredths etc. When they then add decimal numbers they

line up the decimal points. It is assumed that students will relate these two pieces of information and recognize that when you line up decimal points in addition you end up adding tenths with tenths, hundreds with hundreds, etc. If students do make the connection, they certainly have advanced their understanding of addition. A characteristic of this primary relationship is that it connects two pieces of information about decimal numbers and nothing more. It is tied to the decimal context.

Cangelosi (2003:194) mentions other examples of discoverable relationships typically included in the mathematics curriculum, namely the graph of a linear function on {reals} is a line; the ratio of the circumference of any circle to its diameter is π .

It is also true that abstractness increases as knowledge becomes freed from specific problem context (Hiebert and Lefevre, 1986:5). This fact leads to a second kind of relationship, namely reflective level relationships. These relationships are constructed at a higher, more abstract level than the pieces of information they connect, therefore, less tied to specific context. These relationships are created by recognizing similar core features in pieces of information that are seemingly different. The relationships transcend the level at which the knowledge is currently represented, take out the common features of different-looking pieces of knowledge, and tie them together.

For example: Students might notice that you line up numbers on the right to add whole numbers and end up adding units with units. When adding common fractions, you look for common denominators and end up adding same size pieces together. Now the connection between the position value and lining up decimal points to add decimal numbers is recognized as a special case of the general idea that you always add things that are alike in some crucial way, things that have been measured with a common unit, that are the "same size".

This kind of connection is at a reflective level because building it requires a process of stepping back and reflecting on the information being connected. It is at a higher level than the primary level, because at this level, connections in different aspects of the mathematical terrain can be made. According to Cangelosi (as quoted by Cangelosi, 2003:138), from a constructivist perspective, how well students of mathematics are able to apply mathematics to new, meaningful situations depends on their having constructed certain key concepts and discovered key relationships in their own minds.

3.5.2.4 Linking procedural and conceptual knowledge

Linking conceptual knowledge and procedural knowledge has many advantages (Hiebert & Lefevre, 1986:16). Procedural knowledge that is supported by conceptual knowledge results in symbols that have meaning and procedures that can be remembered better and used more effectively. Procedural knowledge also provides a formal language and action sequences that raise the level and applicability of conceptual knowledge. Unpacking a procedure could lead to understanding a concept. Procedures can also condense the concepts because a procedure defines a concept very clearly. Concepts in return, are connected by procedures, as is evident in the description of what a principle is (see § 3.5.1). An example of an algorithm that is based on a relationship is: The computation to find the distance between the two coordinate points A(8;7) and B(3;-5) is based on the Pythagorean Theorem:

$$AB = \sqrt{(8-3)^2 + (7-(-5))^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

According to Van de Walle (2004:24), even rote learning is a construction. The question then is: What ideas are used for construction in rote learning? To what is rotely learned knowledge connected? Cangelosi's (2003:138) view that multiple expressions for the same concept or relationship and multiple algorithms for reaching the same result are ubiquitous in mathematics, might bring us closer to make connections even between rotely learned mathematical knowledge. For example: Almost everything in calculus revolves around the limit concept. Students' algorithms skills are developed in determining limits, reduction takes place and limits are linked to only procedures knowledge. Students can for instance use limit laws to determine the following limit:

$\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x - 2}$. An understanding of the limit concept is not needed in solving problems concerning limits because all one needs is the laws of limits. However, students understand mathematics best when they understand concepts.

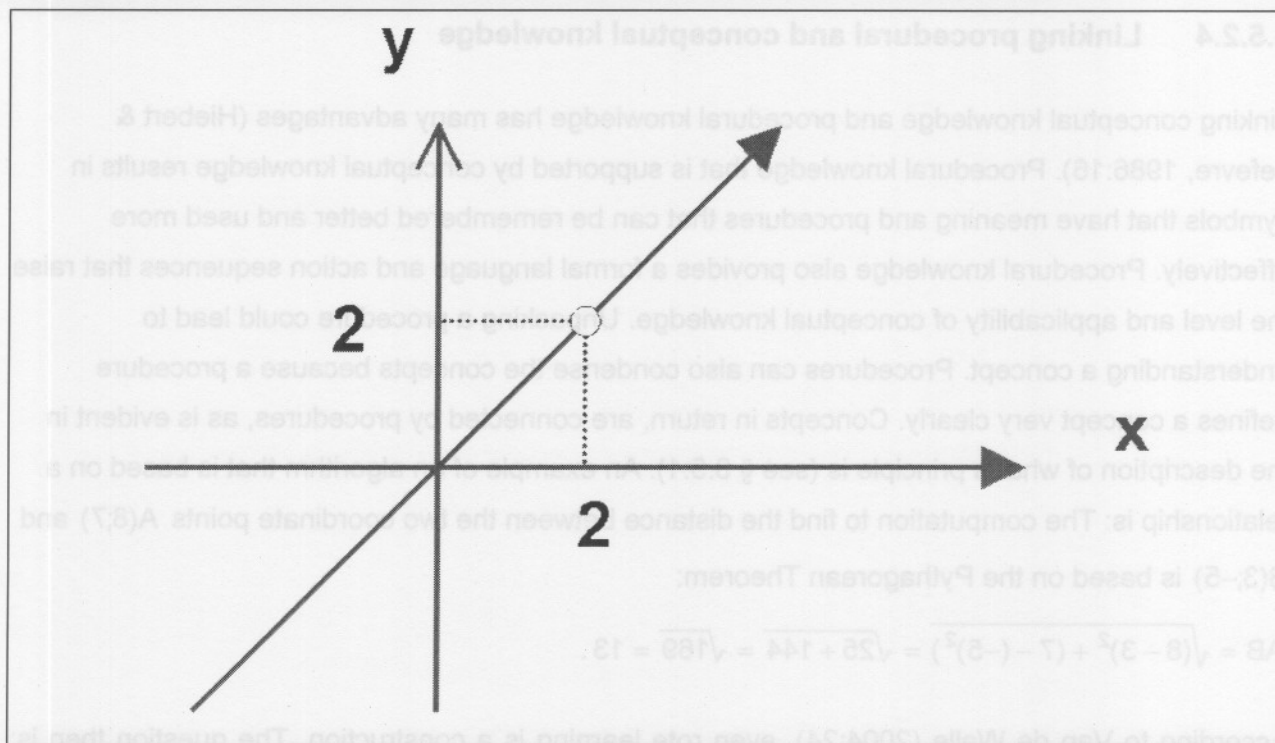


FIGURE 6: A sketch of the function $f(x) = \frac{x^2 - 2x}{x - 2}$.

The value of the $\lim_{x \rightarrow 2}$ is equal to 2. The value of the function at $x = 2$ does not exist, which is clear from the sketch of the function. If students do not understand the limit concept and blindly use limit laws to determine limits, an understanding of the fact that the limit in a specific function value can exist although the function might not be defined in that specific point, might not be developed. This has an effect on students' ability to understand the concept of a continuous function.

Students of mathematics learn arithmetic procedures by rote rather than by constructing them on the basis of their conceptual understanding of arithmetic principles (Ohlsson & Rees, 1991:103). The consequences of rote learning are a lack of flexibility in using procedures and making non-sensical errors. Indications are that the transfer of knowledge to new unfamiliar problem contexts are altered if for example, the limit concept is not connected to the procedures involved in determining a first derivative (see § 3.7.3 and § 5.5.8.1 for the results on teachers' response to Question 10).

It should be noted that it is not very useful to classify all the objects of secondary school mathematics into categories of mathematical knowledge – even experts in mathematics would disagree about the proper category for many mathematical objects (Bell, 1991:109, Hiebert and

Lefevre, 1986:9). Not all knowledge fits into one class or the other. According to Hiebert and Lefevre (1986:3), the relationship between procedural and conceptual knowledge are not yet well understood. This is true because of the fact that the types of knowledge themselves are difficult to define. The core of each is easy to describe, but it is harder to make a distinction concerning the outside edges. However, their distinction between the two types of knowledge provides a way of interpreting the learning process that helps us better understand students' failures and successes in the acquisition of mathematical knowledge.

3.5.3 Implications for MCKfT

According to Kinach (2002:51), the procedural understanding of mathematics that pre-service mathematics teachers typically exhibit in university mathematics courses, mathematics methods courses, and other teacher education coursework is not adequate to teach mathematics with understanding. Research also indicates that with isolated and underdeveloped knowledge packages, the mathematical understanding of a teacher with a procedural perspective is fragmentary (Ma, 1999:118-119). The question then is: Is it possible for a teacher with unconnected fragmented procedural knowledge to unpack mathematical knowledge that leads to understanding?

Anderson (as quoted by Hiebert and Carpenter, 1992:79) describes that as problems are solved, some pieces of knowledge are connected as steps in a procedure. As the procedure is practiced repeatedly, the individual pieces of knowledge lose their identity and become part of a single procedure, making it difficult to reflect on individual steps. This point is very important for MCKfT, as teachers must be able to unpack a merged specific rule to find the meaning of any given step.

Another aspect that adds to the complexity of the situation is the already mentioned fact that classifying mathematical content into simple knowledge, concepts, procedures or principles or any of the other categories described, is not such a clear-cut process. Previously π has been given as an example of simple knowledge (see § 3.5.1). However, an understanding of π 's origin cannot be described as simple knowledge for everybody studying mathematics. Mathematical content that have been developed as conceptual knowledge can become simple knowledge later on.

Where MCKfT is concerned, it is for instance important for a mathematics teacher to understand where π comes from. A simple investigation of the ratio of the circumference of any circle to its diameter will reveal that it will always be π .

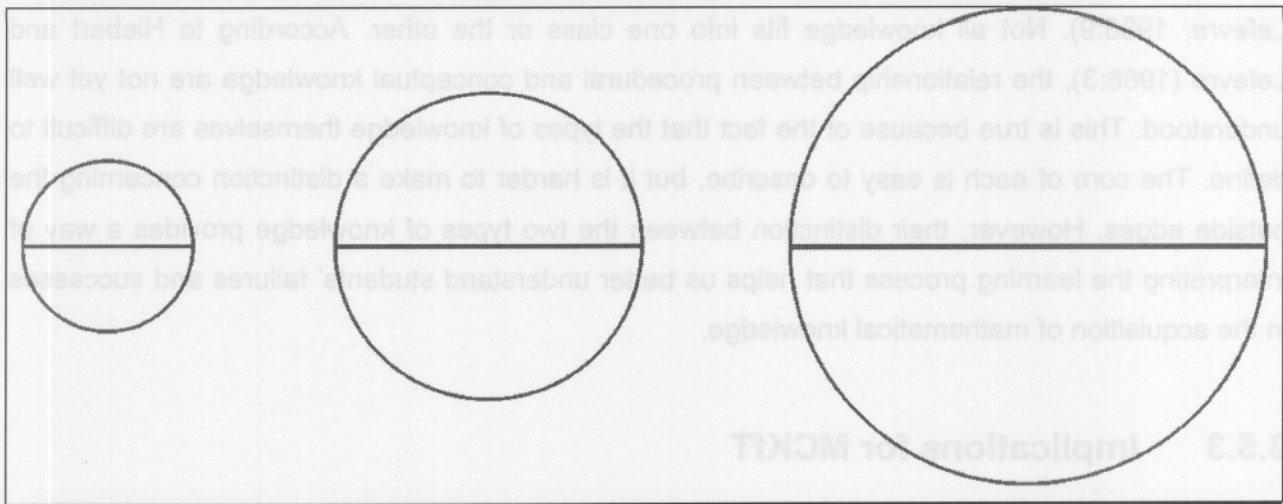


FIGURE 7: A representation of the $\frac{\text{circumference}}{\text{diameter}} = \pi$ relationship.

There is a difference between simple knowledge of knowing that the ratio of the circumference of any circle to its diameter is π and being able to unpack an understanding of the ratio π to explain it to learners in a conceptual way.

Ma (1999) argues that Chinese primary mathematics teachers understand mathematics for teaching in deeply structural ways. They have knowledge packages (p. 113) which represent webs of mathematical concepts and their relationships in ways in which they are best learned. A key feature of these knowledge packages is conceptual knots (p. 115) which tie together meanings of concepts, representations, algorithms and procedural and conceptual knowledge.

It seems as if the development of conceptual and procedural knowledge and relationships (see § 3.5.2.3; § 3.5.2.4) can have a positive influence on the development of MCKfT. In the next section, cognitive processes involved in the development of mathematical understanding will come under the spotlight in an attempt to find out in what way these cognitive processes can help in the development of MCKfT.

3.6 COGNITIVE PROCESSES THROUGH WHICH MATHEMATICAL UNDERSTANDING DEVELOPS

A distinction between the processes of mathematical thinking and the products of mathematical thought has become more and more evident. According to Mamona-Downs and Downs (2002:167), traditional methods of teaching tertiary mathematics stress content of mathematical

theory rather than the motivations and thoughts that underlie this content. This brings the focus back to rote learning versus meaningful learning by thinking about the mental processes involved in becoming mathematically competent and the mental processes that inhibit such competency from developing (see § 3.7.1).

There are thus two focuses: Firstly, a focus on content, which implies the mathematical content that should be taught at school level. Secondly, a focus on the cognitive processes, which implies the various ways in which the mind works with mathematical topics. Processes through which mathematical understanding can be developed are problem solving, reasoning and proof, communication, connections and representations.

These are the cognitive processes through which students of mathematics should acquire and use mathematical knowledge. They direct the methods of doing all mathematics and, therefore, should be seen as integral components of all mathematics learning and teaching (Van de Walle, 2004:5). There is an urgent need to focus on the cognitive processes we use to do and learn mathematics as it might have an effect on the method of knowing and thus teaching mathematics. In the following section the value of these cognitive processes in the development of mathematical knowledge and understanding will become evident.

3.6.1 Problem Solving

Problem solving is an important process in doing mathematics. It is essential to understand that problem solving is much more than finding answers to word problems and exercises. It entails engaging in a task for which the solution method is not known in advance (NCTM, 2000:52).

The process of problem solving involves using prior knowledge in new or different ways, formulating a plan or strategy to reach the desired goal, and possibly acquiring new knowledge about the given situation (Dossey et al., 2002:72; Van de Walle, 2004:4). Strategies and knowledge should be developed for real-life problem solving. Word problems from mathematics textbooks provide convenient exercises for students to experience some, but not all, aspects of real-life problem solving (Cangelosi, 2003:159).

A distinction is thus made between two different kinds of problems that can be solved: mathematics textbook problems that have been altered for conventions' sake and genuine real-life problems as they occur in everyday practices. Mathematical textbook problems try to imitate a real-life situation. These problems can usually also be sub-divided into two groups: Those from a familiar problem contexts and those from an unfamiliar problem contexts (see § 3.7.3).

The type of problem-solving strategy used depends on the skills of the student solving the problem (Dossey et al. et al., 2002:71). A difficult problem for one student may be a routine and quick computation for another. Most students do not confidently address problems, nor do they systematically formulate solutions (Cangelosi, 2003:158). It is often difficult for students to translate a problem from a linguistic representation to a mental representation (Shunk, 1996:270).

Cangelosi (2003:158-159) describes valuable prerequisites for being able to do problem solving as it is argued above:

- confidence and willingness to pursue solutions to problems;
- conceptual-level understanding of the mathematical concepts, relationships and algorithms from which the problem solutions are built;
- skills in recalling or retrieving formulas and executing algorithms;
- comprehension of necessary language and structural conventions for organizing, retaining and relating mathematics to problem solving;
- ability to discriminate between appropriate and inappropriate mathematical concepts, relationships and algorithms, depending on the problem situation.

3.6.2 Reasoning

Mathematical reasoning is a key process in making sense of the world around us. Being able to reason is essential to understanding mathematics (NCTM, 2000:56). Ultimately, reasoning is the logical thinking that helps us decide if and why our answers make sense (Van de Walle, 2004:4). The habit of providing reasons for answers and giving logical arguments and realising that the reasons for an answer is as important as the answer itself, should form an integral part of doing mathematics.

Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many problem contexts (NCTM, 2000:56). According to Dossey et al. (2002:76), reasoning occurs when students are given opportunities to explore mathematics on their own and are expected to verify the results of their explorations. Students who reason and think analytically tend to note patterns (NCTM, 2000:56).

A distinction is made between deductive reasoning and inductive reasoning. According to Cangelosi (2003:161), deductive reasoning is used to formulate proofs, whereas concept

construction and relationships discovery resulting in conjectures require the use of inductive reasoning.

Deductive reasoning is deciding that a specific or particular problem is subsumed by a generality (Cangelosi, 2003:255, 418). It is the cognitive process through which people determine whether what they know about a concept or abstract relationship is applicable to some specific situation. Problem solving, (also real-life problem solving) depends on deductive reasoning. When confronted with questions about a specific situation, one reasons deductively by deciding how, if at all, a previously learned generality, such as a concept, relationship or algorithm, is relevant to that situation (Cangelosi, 2003:158, 255). Deductive reasoning is used to decide how to utilize particular mathematical content to solve problems.

Also inherent in deductive reasoning, is the use of syllogisms (Cangelosi, 2003:255). A syllogism is a scheme for inferring problem solutions – a scheme in which a conclusion is drawn from a major premise and a minor premise. The major premise is a general rule or abstraction. The minor premise is the relationship of a specific to the general rule or abstraction. The conclusion is a logical consequence of the combined premises. An example is: Major premise: If the discriminant $b^2 - 4ac$ of a quadratic equation $ax^2 + bx + c = 0$ is positive and not a perfect square, the equation has two irrational roots. Minor premise: The discriminant of $x^2 - x - 18 = 0$ (i.e. 73) is positive and not a perfect square. Conclusion: $x^2 - x - 18 = 0$ has two irrational roots.

To construct a concept, inductive reasoning is used to distinguish examples from non-examples of the concept (see § 3.6.2). Inductive reasoning is therefore generalizing from encounters with specifics. It is the cognitive process through which people discover commonalities among specific examples, thus leading them to formulate abstract categories (i.e., concepts) or discover abstract relationships (Cangelosi, 2003:167, 177, 419). As with constructing concepts, students need to reason inductively to discover relationships or to figure out why the relationship exists. In these cases, students formed hypotheses or formulated a proposition from their experiments with specifics (Cangelosi, 2003:167, 192).

Dossey et al. (2002:77) note that inductive reasoning, or generalizing from a series of examples, is often mistaken for proving a conjecture to be true. Inductive reasoning relies on collecting examples or discerning patterns. This process does not verify or prove the validity of the statement. Although many examples can fit the conjecture, one example that does not fit the conjecture can prove that the conjecture cannot be generalized. However, it can be the pointer as to something that may possibly be always true and it can be investigated whether a deductive

proof can be found that will prove that the conjecture can be generalized. Deductive reasoning is used to formulate proofs.

If students have usable mathematical knowledge and strive to make sense of mathematics, they need to be able to communicate their knowledge to others. Reasoning and proof are methods of communicating mathematical ideas to others.

3.6.3 Communication

Being able to talk about, write about, describe and explain mathematical ideas is a way of sharing ideas and clarifying understanding. When students are challenged to think and reason about mathematics and to communicate the results of their thinking process to others orally or in writing, they learn to be clear and convincing. According to Van de Walle (2004:5) there exists no better way for wrestling with an idea than to attempt to articulate it to others. In addition to that, listening to other students' explanations creates opportunities to develop understanding. Students gain insight into their thinking when they present their methods for solving problems, when they justify their reasoning to a classmate, or when they formulate a question about something that is puzzling to them (NCTM, 2000:60-61).

Communication can support students' learning of new mathematical concepts as they act out a situation, draw, use objects, give verbal accounts and explanations, use diagrams, write, and use mathematical symbols (NCTM, 2000:60-61). Misconceptions can also be identified and addressed during the communication of knowledge and understanding.

Mathematical symbols are powerful tools for communicating mathematics. Symbols for instance, describe relationships or actions in mathematics. Mathematics is thus a language spoken in symbols. A person who understands the language of mathematics has gone a long way toward understanding mathematics (Dossey et al., 2002:79). However, communicating in mathematics is not simply learning the meaning of mathematical symbols and terminology in order to respond to questions. Because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognized as an important part of mathematics education (NCTM, 2000:60-61). Communication skills in mathematics involve reading, writing, speaking, listening, modelling, as well as pictorial, symbolic, and possibly tabular representations (Dossey et al., 2002:80). Communication is a two-way process, so listening and comprehending is just as important as speaking.

It is important to work with mathematical tasks that are worthwhile topics of discussion. Procedural tasks for which students are expected to have well developed algorithmic approaches are usually not good candidates for discourse (NCTM, 2000:60-61). One should not underestimate the importance of learning to communicate mathematically, as well as the role that communication plays in building mathematical understanding. Conversations in which mathematical ideas are explored from multiple perspectives help the participants to make connections (NCTM, 2000:60-61). When a problem is solved in multiple ways, it serves as a tie connecting several pieces of mathematical knowledge.

3.6.4 Connections (Relationships)

If mathematical knowledge comprises of well-developed, connected knowledge packages, it forms a network of mathematical understanding, solidly supported by the structure of the subject. Establishing well-developed, connected knowledge packages is a matter of making connections (Ma, 1999:121).

3.6.4.1 Different kinds of connections

Connections can be made on different levels. Firstly, connections within and among mathematical concepts and ideas should be encouraged. Students who can connect various mathematical concepts are more likely to develop a deeper mathematical understanding. Such connections give students greater flexibility in problem solving and in constructing mathematical arguments (Dossey et al., 2002:82). For example: An important connection is that points on a graph, representing equivalent fractions, ratios, or rate pairs will define a straight line through the origin - a linear function of the form $y = mx$.

Secondly, connections within and among various areas that demonstrate the overall coherence of the subject mathematics should also be encouraged. We often divide mathematics into subcategories like Algebra and Geometry, and frequently view them as unrelated subjects (Dossey et al., 2002:82). Earlier content lay the foundation for understanding more advanced mathematics (see § 2.5), but the separation of mathematical content can discourage students of mathematics from making connections that promote understanding. Van de Walle (2004:436) states that algebraic reasoning involves a search for regularity in all of mathematics and that functions are one of the most powerful tools in this endeavour. Furthermore, functions allow us to see, generalize and represent relationships between variables in every area of mathematics in which there are variables to be related.

The following figure provides a graphical representation of the mathematical content connections that a function has with other parts of mathematics.

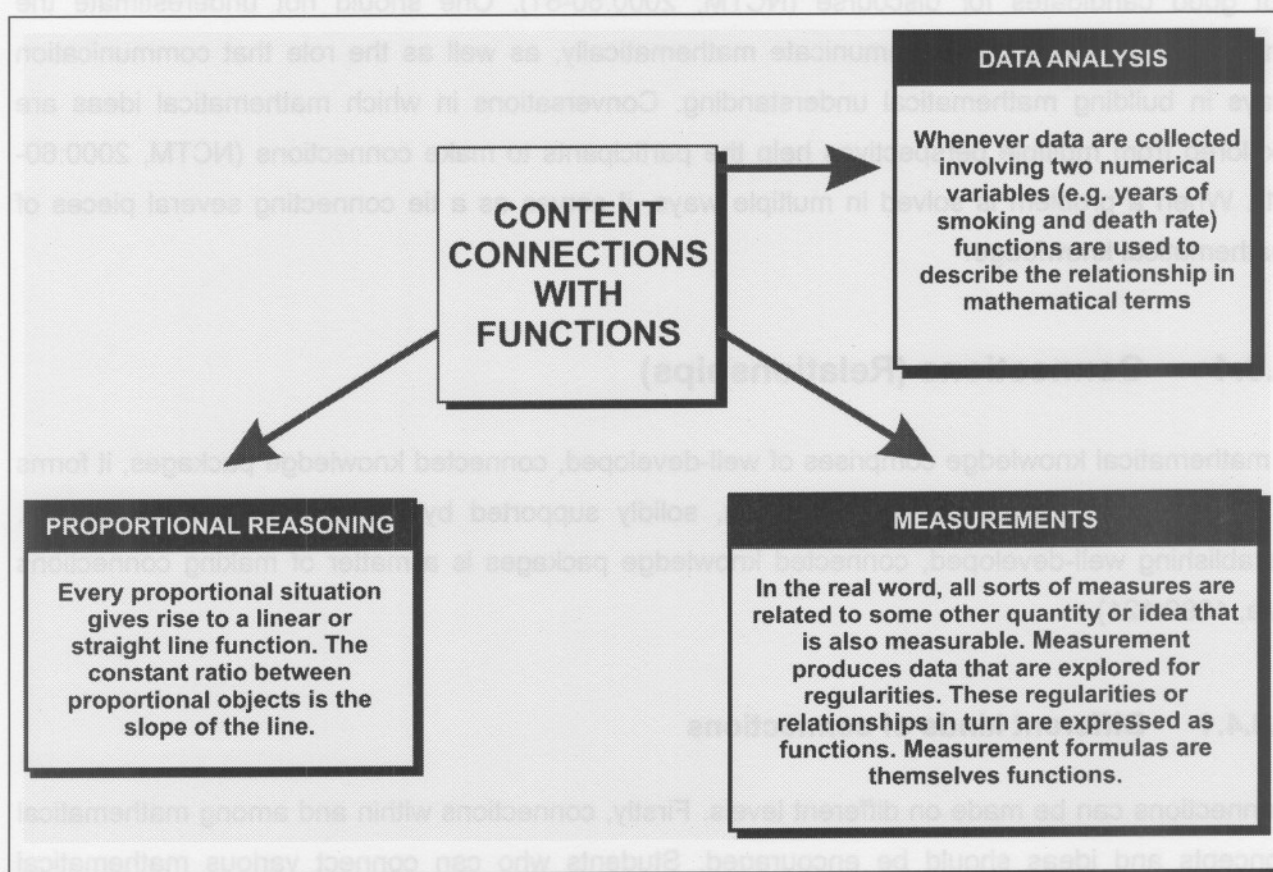


FIGURE 8: Mathematical content connections with functions (Van de Walle, 2004:436).

Ma (1999:112) argues for a practice of solving problems in more than one way. Being able to and trying to solve a problem in more than one way reveal the ability and the predilection to make connections between and among mathematical areas and topics. Mathematics does not consist of isolated rules, but connected ideas, therefore problems can be solved in multiple ways.

Further on, connections between conceptual and procedural knowledge contribute to the development of a sound mathematical knowledge base (see § 3.5.2.4). Students of mathematics are not fully competent in mathematics if either kind of knowledge (conceptual and procedural) is deficient or if they both have been acquired but remain separate entities (Hiebert and Lefevre, 1986:9). When concepts and procedures are not connected, students may have an intuitive feel for mathematics but not be able to solve the problem, or they may generate answers but not understand what they are doing (see § 5.5.2.1 for teachers' interpretation of answers).

Connections between conceptual and procedural knowledge appear to increase the usefulness of procedural knowledge (Hiebert and Lefevre, 1986:10). Meaning for symbols is developed and retrieval of the right procedures for the specific mathematical problem is fostered. This benefit is very important for solving real-life problems. Connecting conceptual knowledge with rules, algorithms, or procedures, reduces the number of procedures that must be learned and increases the likelihood that an appropriate procedure will be recalled and used effectively (Hiebert and Lefevre, 1986:14).

Thirdly, connections between mathematics and other contexts, for instance the real world and to other disciplines should also be explored (Van de Walle, 2004:5). If students are unable to make all these kinds of connections, their mathematical understanding will be severely jeopardized. However, Taylor and Vinjevoold (1999:115) note that mathematics is a collection of deductive systems and leading students of mathematics to believe that all knowledge is derivable from real world, or that all formal knowledge has practical application, is not only bad pedagogy, but wrong epistemology.

3.6.4.2 Making connections

It is argued that it is not wise to assume that the connections taught explicitly are internalized by students (Hiebert & Carpenter, 1992:86). Students of mathematics do not naturally and routinely connect their conceptual and procedural knowledge. Factors that inhibit the creation and recognition of relationships between conceptual and procedural knowledge are (Hiebert & Lefevre, 1986:17-18):

- Deficits in the knowledge base. Connections between items of knowledge cannot be constructed if the knowledge does not exist.
- Difficulties of encoding connections. Connections between units of mathematical knowledge, although taught by educators using seemingly appropriate methods, may not be picked up and internalized by students.
- Tendency to compartmentalize knowledge. Things learned in a particular context are initially tied to surface features of that problem context (see § 3.7.3). This prevents one from noticing similarities between the newly acquired knowledge and previously acquired knowledge held in memory. Specific compartments are accessed only when surface features of the problem context are recognized as similar to those in which the knowledge was acquired.

It is also worth noticing that there exists a potential danger in teaching connections explicitly. The information required to make the connections explicitly can be internalized as one more piece of

isolated knowledge rather than supporting the construction of useful connections (Hiebert and Carpenter, 1992:86).

According to Dossey et al. (2002:82), students of mathematics can be guided in the connection process through the learning experiences they are exposed to. Carefully designed learning experiences can demonstrate connections with previously learned concepts. The learning experiences should provide a good chance for students to connect new ideas to concepts and ideas that are already a part of their mathematical and non-mathematical repertoire.

Connections could be structured to start with an analysis of how knowledge should be connected once it is acquired and work backwards to decide how those connections might be formed through learning experiences. The goal is to attempt to teach students to make the same kinds of connections observed in experts (Hiebert and Carpenter, 1992:81).

3.6.4.3 Connections and misconceptions

Hiebert and Carpenter (1992:88) states that students' errors and misconceptions can also be understood in terms of connections that have been formed. Wrong relationships are sometimes formed, which might lead to incorrect connections between mathematical knowledge. For example:

Our intuition might tell us that an equation such as $\frac{6}{2} = 4 - 1$ should not be called a proportion,

although the two sides are equal, because the two sides are not structurally similar or they do not involve the same pattern or the components are not multiplicatively related (Lesh et al. 1988). However, if learners are taught that "everything is OK as long as you do the same thing to both sides of the equation", it might lead to misconceptions in learners' proportional reasoning.

Resnick et al. (as quoted by Hiebert and Carpenter, 1992:88) argue that errors are a natural consequence of attempting to integrate new procedures into prior knowledge. Thus, the problem is not that incorrect procedures are isolated from other knowledge, but critical connections that would make clear the nature of the errors are absent.

Certain instructional approaches are required to correct these errors (Hiebert & Carpenter, 1992:89). Unless students are forced to confront explicitly the conflict between their misconceptions and certain mathematical principles, the connections may never be made and the misconception and mathematical knowledge may exist as separate islands of knowledge.

3.6.4.4 Assessing connections

It should be understood that it might not be easy to assess the different kinds of connections (Hiebert & Carpenter, 1992:90). A difficulty in assessing connections between symbols and referents is that it is not always easy to tell whether meaningful connections exist or whether students are simply applying a procedure they have learned mechanically. Connecting conceptual knowledge with symbols creates a meaningful representation system, which is an essential prerequisite for intelligent mathematical learning and performance (Hiebert and Levere, 1986:14).

3.6.5 Representations

A representation is an object that describes or models a situation (Dossey et al., 2002:83). The abstract notion of representations involves a relation between two (or more) configurations, with one representing another (Goldin, 2002:207). In paragraph 3.2 it was argued that mathematics is a study of pattern and order. According to Goldin (2002:212), the word pattern, describing the fundamental object(s) of study in mathematics, is already strongly suggestive of some sort of representation. Because we need to communicate mathematical ideas, we need to represent them in some or the other way in the mind or on paper. Representations are very important because the way in which mathematical ideas are represented is fundamental to how students understand and use those ideas. Part of problem solving is to relate different types of representations. The more different types of representations a student of mathematics has, the more expanded his/her conceptual knowledge is.

3.6.5.1 Internal and external representations

A distinction is made between external representations and internal representations. Mathematical ideas need to be communicated, and to accomplish this, external representations are used. Goldin (2002:207) describes external representations as configurations external to the individual, generally observable in the immediate environment (such as real-life objects or events, spoken or written words, formulas and equations, geometric figures or graphs). External representations are thus represented in the form of spoken language, written symbols, pictures or physical objects.

Mathematical ideas also need to be represented internally because we need to think about mathematical ideas in a way that allows the mind to operate on them (Hiebert and Carpenter, 1992:66). This is called internal representations. It is configurations internal to the individual, presumed to be encoded in the brain but mainly to be described at more holistic levels such as verbal and syntactic configurations (natural language and personal symbolization constructs),

visual and spatial imagery, internalized mathematical symbols, rules and algorithms and problem solving strategies (Goldin, 2002:207, 210).

It is of course not possible to observe the internal or mental representations of someone else directly. One way to explore what is involved in a student's understanding of a mathematical concept is to consider the variety of distinct, appropriate (or inappropriate) internal representations formed and to try to describe and analyze the representing relations that were developed (Goldin, 2002:211).

3.6.5.2 Processes in and products of representations

The graph of $f(x) = x^3$ is a representation of a specific function. The term "representations" refers both to process and to product. According to Dossey et al. (2002:83), the process of representing is just as important as the product or object that is represented. This process is the act of capturing a mathematical concept or relationship in some form that conveys an idea, a picture, or a mathematical connection to the viewer.

3.6.5.3 The usefulness of representations

Mathematical representations have not always been recognized as an important element in understanding mathematics. What makes mathematics so powerful is the multiple ways in which mathematical ideas can be expressed with symbols, charts, graphs, diagrams and physical models. Moving from one representation to another is an important means to add understanding to an idea (Van de Walle, 2004:5).

Dossey et al. (2002:83) provide a few reasons why representations are so important in mathematical understanding. Representations are useful tools for learning and doing mathematics, as well as for communicating and making connections. Representations extend a person's understanding of a concept, and shed light on an idea not fully understood in another form.

Most if not all important mathematics concepts and procedures can best be taught through problem solving (Van de Walle, 2004:36). According to Hiebert and Lefevre (1986:11-12), problems are solved by building mental representations of the problems and then dealing with the representation to select appropriate procedures. Relevant conceptual knowledge can be supportive of solving the problem through enrichment of the problem context. Data supporting the importance of problem representations that are heavily conceptual come from Silver (1986:191). Comparisons between experts and novices suggest that expert problem solvers represent

problems by using underlying structures and conceptual features of the problem context, whereas novices focus more on superficial features and specific symbols.

A student's ability to develop and interpret various representations increases the ability to do and understand mathematics. When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically (NCTM, 2000:67). Because of these reasons, representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understanding to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through a variety of modelling. Representations are thus a useful tool in developing mathematical understanding.

3.6.5.4 Models of representations

A representation is a configuration that can represent something else in some manner: the representing configuration might, for instance, act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, encode, evoke, label, link with, mean, produce, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, suggest or symbolize the represented one (Goldin, 2002:208). Dossey et al. (2002:84) states that students often need to see multiple representations before their sense of the concept begins to take shape. Some students develop stronger understanding when they see drawings or graphs. Others prefer algebraic or symbolic representations. Exposing the student to a variety of representations is also useful after students have developed a sense of the concept because representations strengthen students' understanding. The concept of a half can for example be represented in the following different

ways: $\frac{1}{2}$, half, , a half glass of water, 30 min = $\frac{1}{2}$ hour etc.

In their research, Lesh, Post and Behr (1987:34) work with five representations for mathematical concepts. Two of these representations are manipulative models and pictures. They also consider written symbolism, oral language and real world situations to be representations or models of mathematical concepts. Translations between and within each can help develop new understanding.

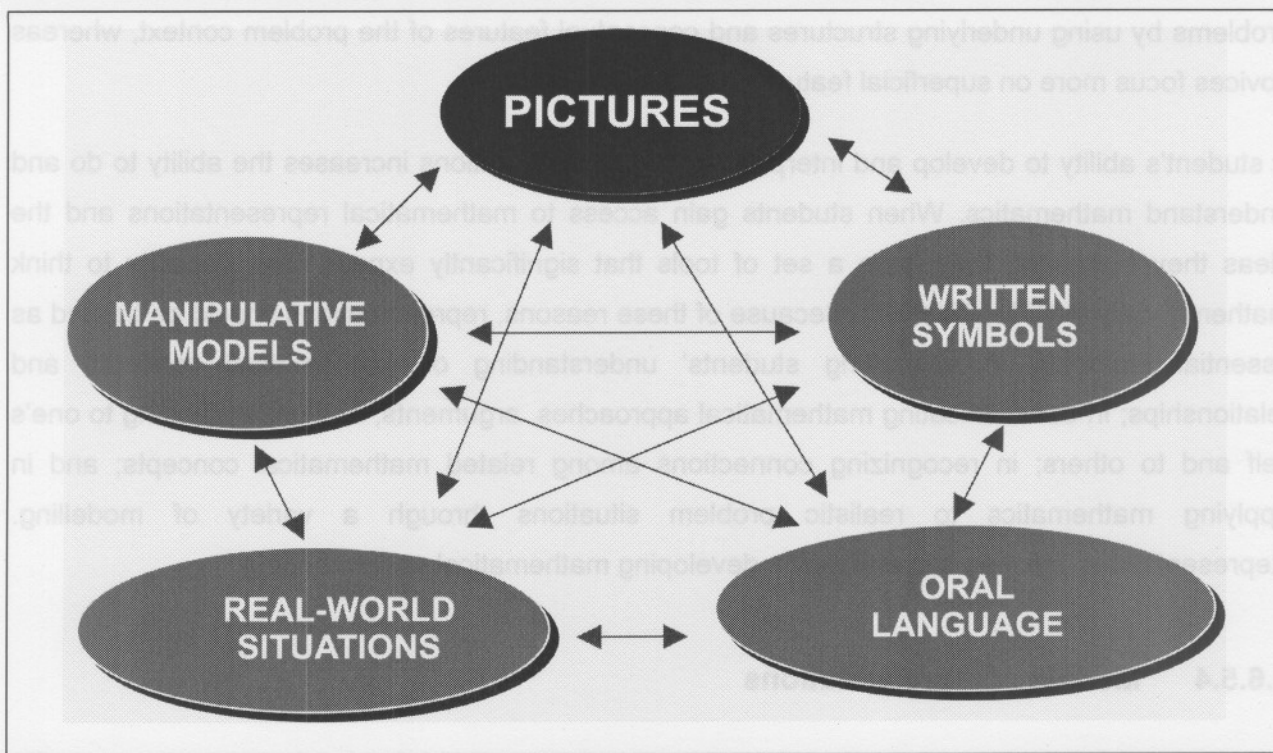


FIGURE 9: Five different representations of mathematical ideas
(Lesh, Post and Behr, 1987:34)

Learners who have difficulty translating a concept from one representation to another are the same learners who have difficulty solving problems and understanding computations (Lesh, Post and Behr, 1987:34). Strengthening the ability to move between and among these different representations improve the growth of students' concepts (Van de Walle, 2004:30). Chances increase for concepts being formed correctly and integrated into a rich web of ideas and relational understanding if students can think about a mathematical idea in more than one way (see § 3.7.1).

Dossey et al. (2002:84) describes another model for mathematical representations. The figure implies that different representations emphasize various facets of the mathematics. The edges of the tetrahedron symbolize the transformation from one representation to another. Students need to learn to shift from one representation to another (one vertex to another) in order to view the different facets of mathematics, and to make the appropriate connections. As students work with and create different representations of a concept or situation, they move to the different vertices and, along the way, discover new aspects of the concept. When students analyze representations they can better decide which representations provide useful information, and which do not.

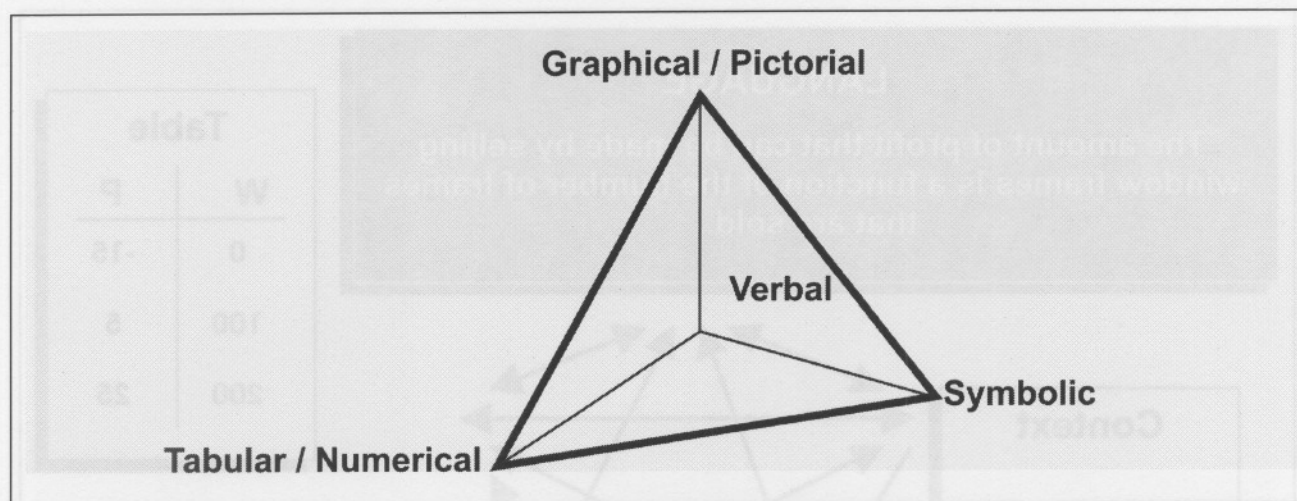


FIGURE 10: Representations in mathematics (Dossey et al., 2002:84)

3.6.5.5 Representations and connections

According to Hiebert and Carpenter (1992:66), connections between representational forms and within representational forms (external and internal), play a role in learning mathematics with understanding. When relationships among internal representations of ideas are constructed, they produce networks of interconnected knowledge. For example: The study of functions should focus on transforming connections in different problem contexts (Van de Walle, 2004:436). Algebraically a function can be represented by $y = x^2$, one quantity y , depends on another quantity x . Students experience the function concept whenever they consider how change in one variable can cause or have a corresponding effect on another (Van de Walle, 2004:436). Van Dyke (2002:vii) states that students will benefit from exploring the relationships between a graph and a verbal statement describing a function before looking at a table or an algebraic representation. The five different representations of functions is provided in the following figure.

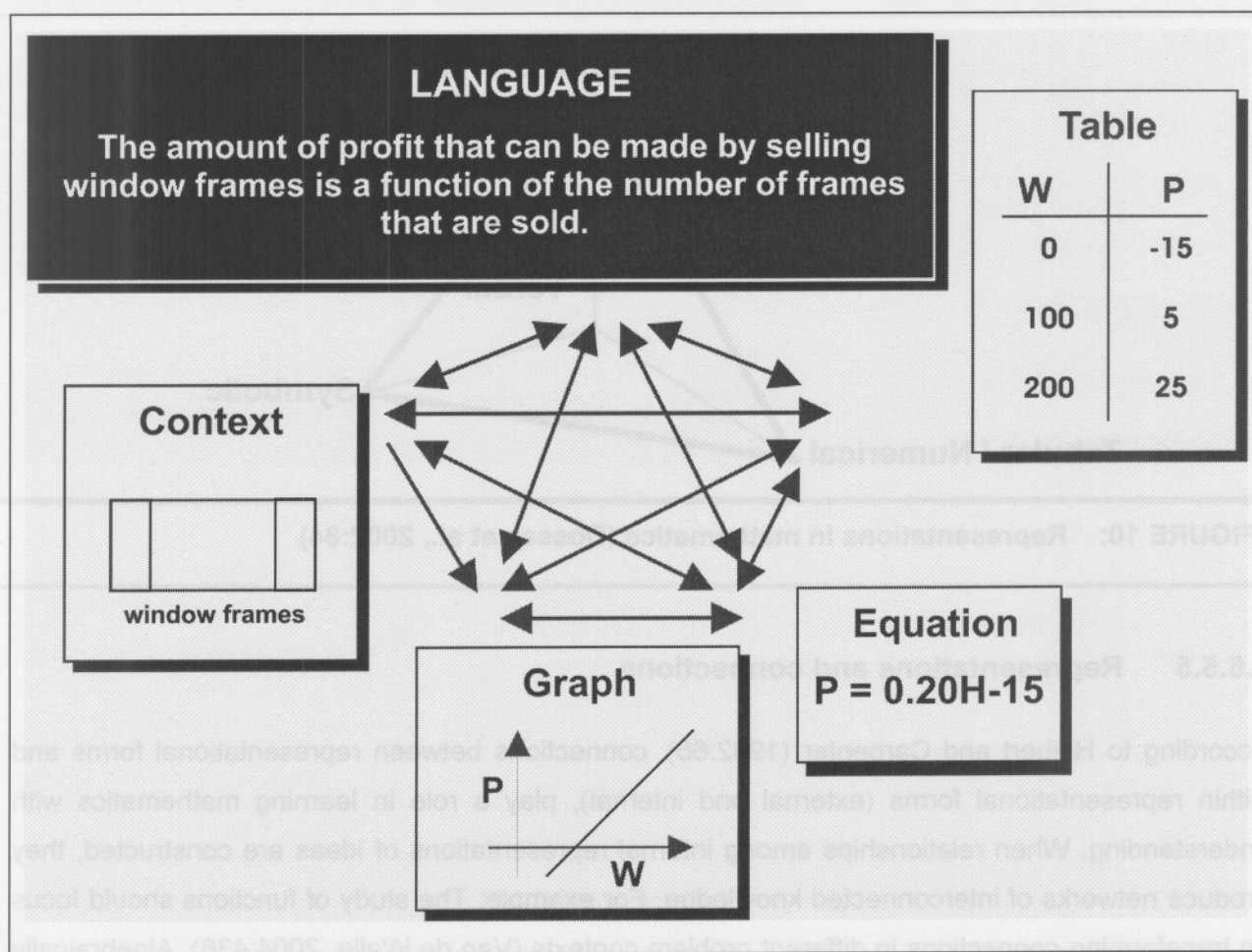


FIGURE 11: Five different representations of a function.

For any given function, the different representations are all connected and illustrate the same relationship. It is important to notice that each representation provides a different perspective on the function. The deliberate construction of connections between different forms of external and internal representations should be encouraged to enhance understanding of mathematics.

3.6.6 Implications of the cognitive processes on the development of MCKfT

The cognitive processes involved in doing and understanding mathematics seems to be closely linked to the nature of MCKfT and is therefore important for the development of a teacher's MCKfT. The following aspects are important for MCKfT:

- Teachers should be able to solve problems in a familiar/unfamiliar context, therefore the teacher's knowledge should not be context bound; it should be transferable to new situations. It should also be conceptually understood. A teacher should be able to translate a problem from a linguistic representation to a mental representation. The teacher should have the ability to discriminate between appropriate and inappropriate mathematical concepts, relationships and algorithms depending on the problem situation.
- Teachers should be able to reason logically to help them decide if and why answers make sense. Reasoning mathematically should be evident in many different problem contexts. The habit of providing reasons for answers and giving logical arguments and learning that the reasons for an answer is as important as the answer itself, should be developed. Reasoning should help the teacher to recognise patterns.
- A teacher communicates mathematical knowledge and understanding to a learner in the way in which the teacher understands the mathematics. Being able to communicate mathematical understanding in various ways is thus an important part of MCKfT. Conversations in which mathematical ideas are explored from multiple perspectives should be encouraged.
- Different connections between and within mathematical knowledge should be developed. Compartmentalized knowledge should be reorganized. Surface features of problems that tie it to a specific concept should be investigated. The habit to solve problems in more than one way should be encouraged as it reveals the ability to make connections between and among mathematical areas and topics.
- Multiple ways to represent problems should be encouraged. Teachers with appropriate mathematical knowledge will be able to decide on the best representation to model a situation that will also appeal the most to learners. Connections among various representations should be encouraged and might help to enforce a habit of providing reasons for answers (Dossey et al., 2002:73).

The main focus is on the knowledge transformation process and on the cognitive processes used to shift prospective teachers' instructional explanations from an instrumental to a relational understanding of mathematics (Kinach, 2002:51).

3.7 UNDERSTANDING MATHEMATICAL CONTENT

It is an undeniable fact that mathematics has structured and clearly defined content, however, distinction should be made between knowing mathematics as isolated packages of knowledge and understanding mathematics. It is possible to say that one knows something or one does not know something. This means that knowledge is something that one either has or does not have.

Understanding entails something else. Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas or that new knowledge has with prior knowledge (Van de Walle, 2004:24). Hiebert and Carpenter (1992:74) state that students who understand mathematics will be able to retain what they have learned and they will also be able to transfer it to novel situations.

This section will investigate how understanding mathematics might have positive repercussions for the development of MCKfT. Important elements in the development of different kinds of mathematical knowledge will be discussed.

3.7.1 Instrumental and relational understanding

Skemp (1976/2005:2) made a groundbreaking theoretical formulation of two competing views or conceptions of mathematical understanding – mathematics with understanding or mathematics without understanding. Skemp called these two conceptions instrumental understanding and relational understanding. These two different conceptions of what constitutes mathematics account for sharp differences in instructional practices, goals and emphases in the mathematics classroom.

Instrumental understanding refers to the “what and how” of mathematics. Instrumental knowledge of mathematics is knowledge of a set of “fixed plans” for performing mathematical tasks. A feature of these plans is that they prescribe a step-by-step procedure to be followed in performing a given task, with each step determining the next. The kind of learning that leads to instrumental [knowledge of] mathematics (see § 3.4.1) consists of the learning of an increased number of fixed plans, by which learners can find the answer to the question (Skemp, 1976/2005:14). Within its own context, instrumental understanding is easier to understand and provides quick and easy ways to get answers to problems. It has been described as “rules without reasons”. For some learners and teachers, the ability to use procedures and get answers is what they mean by understanding mathematics.

On the other hand, relational understanding includes insight into the why of mathematics, or the reasons for the “what and how”. Relational understanding thus refers to knowing both what to do and why as far as mathematics is concerned (Skemp, 1976/2005:14). Relational knowledge of mathematics is characterized by the possession of conceptual structures that enable the possessor to construct several plans for performing a given task. Relational understanding is situated at a rich interconnected end of the continuum. At the other end of the continuum, ideas are completely isolated or nearly so. Understanding at this end is called instrumental understanding.

Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The more connections are made to a network of ideas, the better the understanding (Fennema & Franke, 1992:152, Van de Walle, 2004:25). Changes in networks can be described as reorganizations (Hiebert and Carpenter, 1992:69). Representations are rearranged; new connections are formed, and old connections may be modified or abandoned. Reorganization is manifested both as new insight, local or global, and as temporary confusion (Hiebert and Carpenter, 1992:69). Understanding increases as the reorganizations yield more richly connected, cohesive networks.

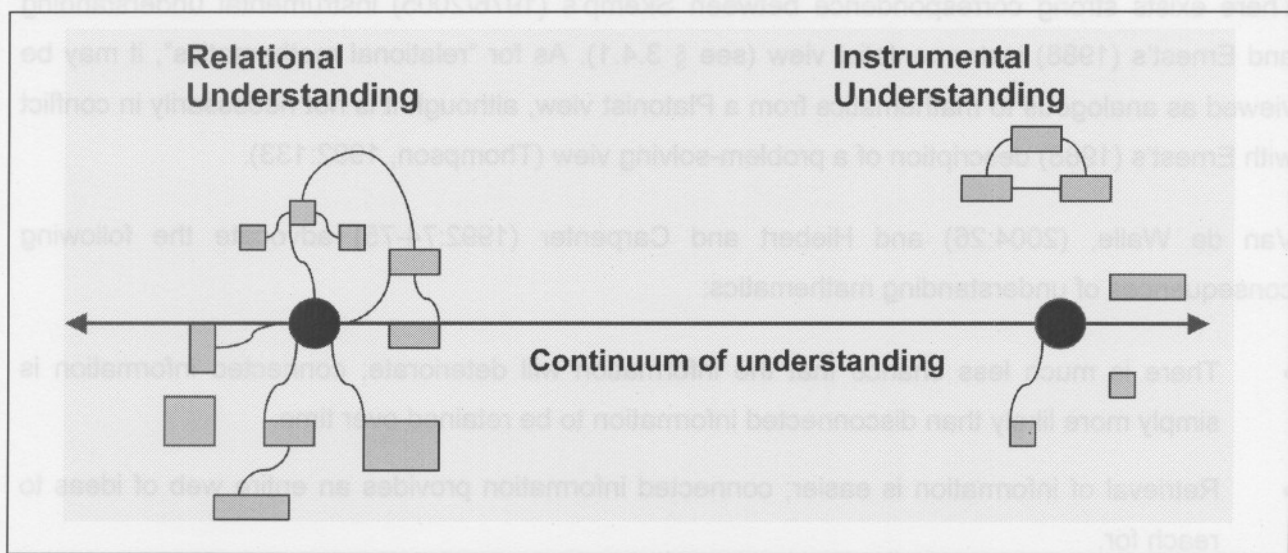


FIGURE 12: A representation of relational and instrumental understanding (Van de Walle, 2004:25).

According to Hiebert and Carpenter (1992:69), understanding increases as networks grow and as connections become strengthened with reinforcing experiences and tighter network structuring. Growth of networks can be characterized as changes in networks, as well as additions to networks. Growth of networks can be seen as temporary regressions as well as progressions, which appear to be intermittent and somewhat unpredictable. Understanding is built sporadically rather than through smooth, monotonic increases (Hiebert and Carpenter, 1992:69)

What counts as learning, understanding or student achievement on these two views is fundamentally different (Kinach, 2002:54). An instrumentalist would perceive remembering rules, facts and procedures as clear evidence of student achievement. On the other hand, for a teacher who embraces a relational view of mathematics, a student's achievement is much broader than

remembering. Problem posing, critical and contextual thinking, the ability to justify and represent one's thinking mathematically, are part of what mathematics achievement means.

For Thompson (1992:133), distinction resides in the type of knowledge each reflects. Teaching integer subtraction according to the instrumentalist view would mean teaching the subtraction algorithm without attention to its underlying concepts and principles. By contrast, teachers espousing a relational view would likely expand the instrumentalist's presentation of integer subtraction by justifying why the subtraction algorithm works.

There exists strong correspondence between Skemp's (1976/2005) instrumental understanding and Ernest's (1988) instrumentalist view (see § 3.4.1). As for "relational mathematics", it may be viewed as analogous to mathematics from a Platonist view, although it is not necessarily in conflict with Ernest's (1988) description of a problem-solving view (Thompson, 1992:133).

Van de Walle, (2004:26) and Hiebert and Carpenter (1992:74-75) advocate the following consequences of understanding mathematics:

- There is much less chance that the information will deteriorate; connected information is simply more likely than disconnected information to be retained over time.
- Retrieval of information is easier; connected information provides an entire web of ideas to reach for.
- When learning relationally, the student tends to develop a positive self-concept about his or her ability to learn and understand mathematics. There is a definite sense of "I can do this! I understand!"
- Understanding is generative. Implication of generative understanding is that students can be placed in settings in which they can construct useful connections (Mayer, 1989:47). Understanding is more accurately viewed as generated by individual students than as provided by the teacher.
- Understanding reduces the amount that has to be remembered.
- Understanding enhances transfer. Transfer is usually quite specific and contextually bounded. Transfer is most apparent if there are specific external and internal elements in common and if the tasks are situated in a common context.

One can have an opinion on what teaching and learning procedural mathematical knowledge entails. The important point is to notice that one can have an instrumental or relational under-

standing of a procedure. Hiebert and Carpenter (1992:67) refer to “webs” of interrelated ideas. These webs or networks of knowledge will be the focus of the next section.

3.7.2 Networks of knowledge and big ideas

The network theory of information processing makes it possible to explain the growth of mathematical knowledge and understanding. Bits of knowledge and the relationships or connections between them all form part of networks of knowledge and understanding.

Hiebert and Lefevre (1986:xii) state that connections between procedural and conceptual knowledge is the key to understanding how students of mathematics do mathematics and how they think about what they do. An understood idea is associated with many other existing ideas in a meaningful network of concepts and procedures (Van de Walle, 2004:24). Networks of knowledge and understanding can be structured in two different ways, namely vertical hierarchies or like webs. Webs are structures of networks of knowledge where nodes can be thought of as the pieces of represented information, and the threads between them as the connections of relationships (Hiebert and Carpenter, 1992:67). It is also noted that the nodes in a web of knowledge and understanding are ultimately connected, making it possible to travel between them by following established connections.

Vertical hierarchies are structures of networks of knowledge where some representations subsume other representations, representations fit as details underneath or within more general representations (Hiebert and Carpenter, 1992:67). Insofar as connections are hierarchical, it may be possible to identify key constructs that provide structure to a number of related concepts (Hiebert and Carpenter, 1992:86). These key constructs refer to the notion of big ideas.

Big ideas are large networks of interrelated concepts (Van de Walle, 2004:26). Ideas are learned relationally when they are integrated into a large web of information, a big idea. According to Wiebe (2006), learning the big ideas in mathematics is the most effective way to empower students mathematically. For example:

- Proportional reasoning, with its single overarching but utterly simple pattern $\frac{a}{b} = \frac{c}{d}$, is possibly the biggest idea in elementary and middle school mathematics.
- A function is a relationship or rule that uniquely associates members of one set with members of another set.

The following figure illustrates the concept of big ideas and networks of connected knowledge.

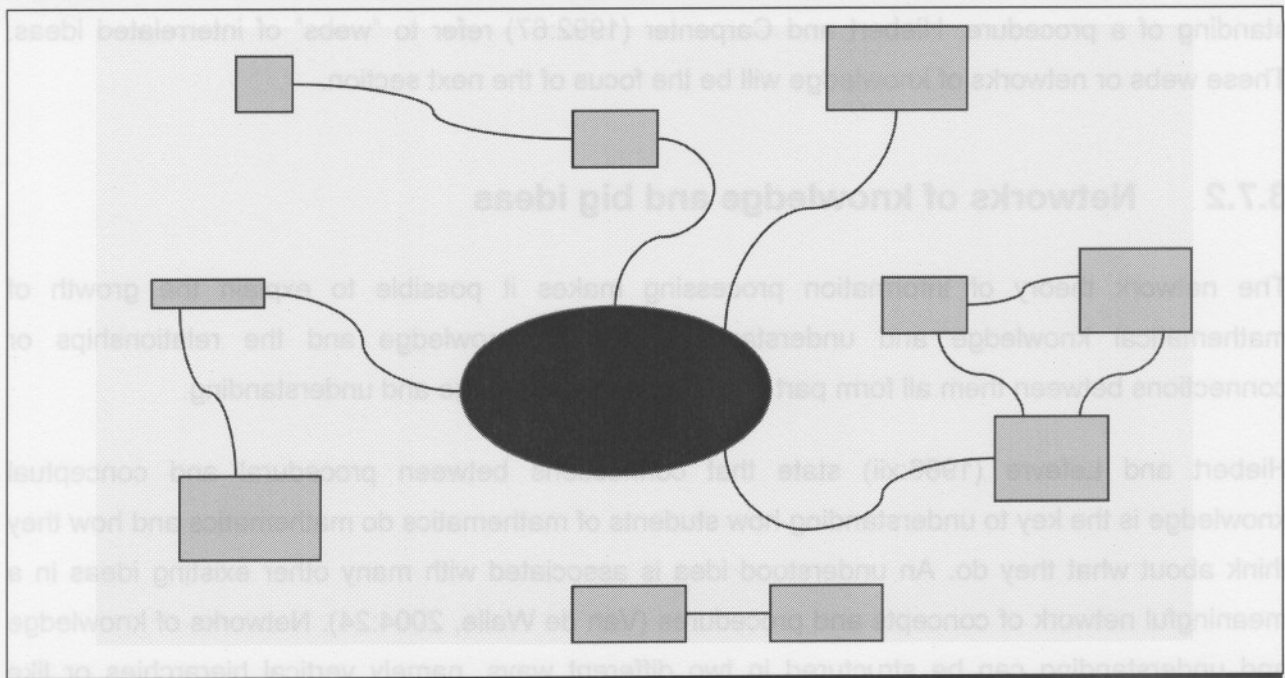


FIGURE 13: A metaphor for the construction of ideas (Van de Walle, 2004:23).

The grey blocks refer to existing ideas (the ideas we already have). These existing ideas are used to construct a new idea (the black circle). The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. Through this process, a network of connections between ideas develops. Understanding increases as more and more ideas are used and more and more connections are made.

A web of associations that could contribute to the understanding of ratio is provided next.

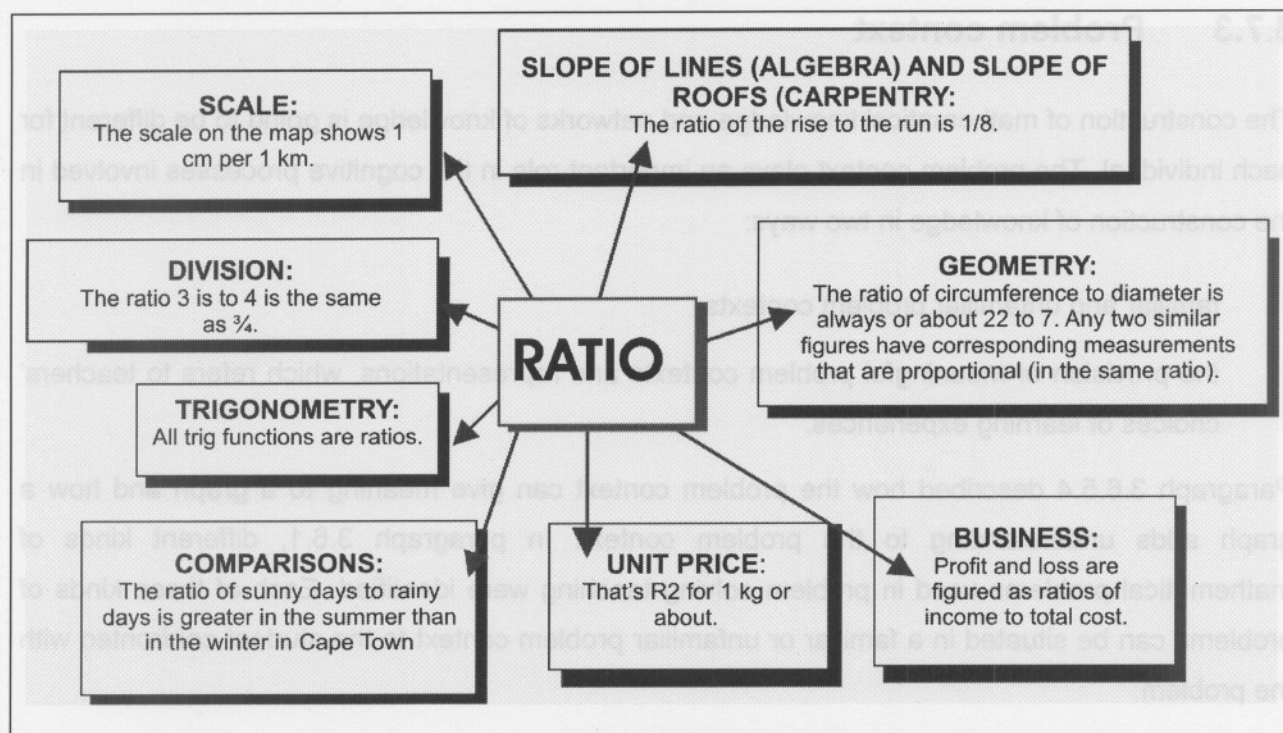


FIGURE 14: A potential web of associations that could contribute to the understanding of ratio (Van de Walle, 2004:25).

According to Wiebe (2006), big ideas help reduce student memory load, broaden understanding of appreciation for mathematical concepts, provide connections across the discipline and reveal the beauty of mathematics. A mathematical idea, procedure or fact is understood if it is part of an internal network of connected knowledge. Mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections (Hiebert and Carpenter, 1992:67).

Because mathematics is not a list of isolated rules and formulas (Van de Walle, 2004:5), students should be helped to see how mathematical knowledge is constructed into a useful network of interconnected knowledge. Cangelosi (2003:162) argues for methods for connecting and integrating topics so that students have fewer concepts and relationships with which to contend. Indications are that if knowledge is not conceptually connected, familiar and unfamiliar problem contexts play a role in solving mathematical problems. Problem context will be the focus of the next section.

3.7.3 Problem context

The construction of mathematical knowledge and networks of knowledge is going to be different for each individual. The problem context plays an important role in the cognitive processes involved in the construction of knowledge in two ways:

- familiar and unfamiliar problem contexts;
- the provision of meaningful problem contexts and representations, which refers to teachers' choices of learning experiences.

Paragraph 3.6.5.4 described how the problem context can give meaning to a graph and how a graph adds understanding to the problem context. In paragraph 3.6.1, different kinds of mathematical problems used in problem solving teaching were identified. Each of these kinds of problems can be situated in a familiar or unfamiliar problem context to the student confronted with the problem.

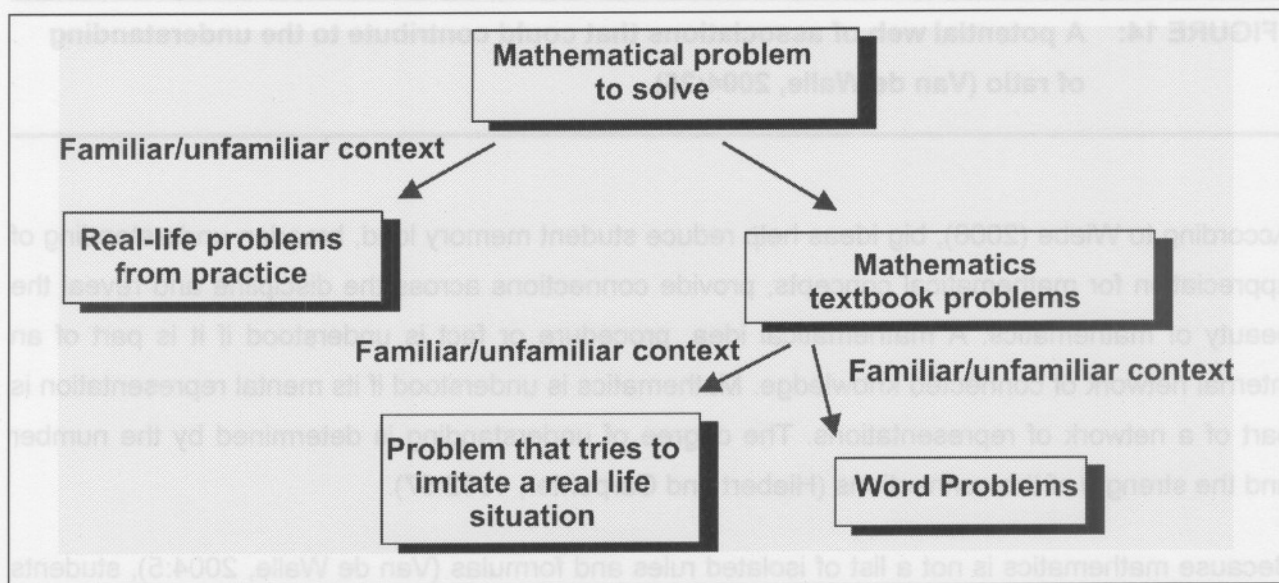


FIGURE 15: Different kinds of mathematical problems.

According to Hiebert and Carpenter (1992:76), mathematical learning experiences are filled with overly restricted activities. The potential for students to build restricted representations is exacerbated by the fact that, in school, problem situations often are printed symbols on textbook pages or worksheets. These problems encourage students to think primarily about symbols. Internal representations are formed for written symbols and manipulations on symbols. If the

problems do not promote connections to other representations, the internal networks of symbols that are formed remain severely bounded and unconnected to other knowledge.

Knowledge acquired through activities set in a problem context, which enable a learner to connect the knowledge to his or her broader culture, is generalizable and useful (Fennema & Franke, 1992:160). Hiebert and Carpenter (1992:79) distinguish between street mathematics and school mathematics. According to them, a frequent criticism of school mathematics is that procedures for solving problems are learned in an overly mechanical way. The result is then that school-learned procedures often cannot be used flexibly to solve problems other than those on which they were practiced and, thus, does not “transfer” well to other problem contexts.

Taylor (2001:3) states that not all everyday problem contexts provide suitable entry points into school knowledge. For instance teachers cannot allow learners free reign in discussing everyday examples at random and hope that this will lead inexorably to the acquisition of sophisticated mathematical concepts. Teachers with a conceptually connected mathematical knowledge base will be able to distinguish between appropriate and inappropriate everyday examples.

Hiebert and Carpenter (1992:70) also indicate that concrete materials are sometimes ineffective in the teaching and learning of mathematics because of the following reasons: “Students of mathematics do not always bring with them the kind of knowledge of quantities that the instructor of mathematics expects. It might not necessarily be easy for the students of mathematics to relate their interactions with the materials to existing networks. They do not interpret the materials in the way that the instructor expects, and the use of concrete materials is then likely to generate only haphazard connections.”

Negative results with concrete materials may be traced to two features of the activities in which students of mathematics engage (Hiebert & Carpenter, 1992:70-71):

- The distance between the concrete material and the mathematical relationships that we intend the students of mathematics to represent. This is called the contextual distance.
- A potential interaction between the materials and the social situation in which the materials are used.

These factors are especially important in the South African context because of the multiple cultural groups and the different contexts in which each of them grew up (see § 5.5.2.2).

There has always been a discussion about whether instruction should focus on connections derived from a problem context that gives particular mathematical procedures and concepts meaning or whether instruction should be based on analysis of mathematical structure (Hiebert &

Carpenter, 1992:81). The notion of situated cognition and conceptual fields shed light on this point of conflict.

Situated cognition refers to the process in which abstract mathematical concepts and procedures are imbedded and therefore taught in the problem context (problem situation) that initially gave them meaning. By learning concepts and procedures in problem solving contexts, it is presumed that the knowledge is connected so that it is accessible for problem solving (Hiebert & Carpenter, 1992:81).

The idea of conceptual fields (Vergnaud, 1983, 1988 as quoted by Hiebert and Carpenter, 1992:81-82) is also concerned with the problem context that underlies mathematical abstractions. Conceptual fields are defined through an analysis of the properties of problems or situations represented by an interconnected set of mathematical abstractions. Conceptual fields are defined in terms of problems, but the emphasis is on fundamental semantic properties that define similarities and differences among problems rather than on the particular context of the problem.

Situated cognition and conceptual fields are perspectives of how knowledge is organized or how it should be structured. An analysis of how problem situations fit in a conceptual field may help to identify appropriate problem contexts in which to situate particular concepts and procedures.

It should be acknowledged that the notion of problem context should be handled with caution. Cobb (1988), Erlwanger (1975) and Lawler (1981) as quoted by Hiebert and Carpenter (1992:90), have documented how students fail to acknowledge the connections of different solutions to identical problems presented in different problem contexts. There are boundaries that the problem context seems to place on the transfer of knowledge because of the surface features of the problem (see § 3.6.4.2). Hiebert and Carpenter (1992:76) notes that the way internal representations are connected helps to explain the potential for transfer. The problem context influences the amount of transfer of existing knowledge that actually occurs. It seems that often students fail to make connections because of unfamiliar problem contexts. Students represent information as isolated pieces, compartmentalize their knowledge, and fail to recognize or create connections in mathematical situations that appear obvious to adults (Hiebert & Carpenter, 1992:76).

Contextualized mathematics should not be romanticized and the abstract devalued, because the problem context can sometimes result in a cognitive obstacle to the more abstract mathematical understanding (Goldin, 2002:215). The pattern where the contextualized representations first assist and then constrain the subsequent cognitive development is quite common in mathematics. Goldin (2002:214) states that familiar problem contexts are encoded internally as representational

configurations in common words, images, formal notations, strategies and operations, and (ideally) comfortable affects. These internal structures in return (with their familiar or common-sense nature – expectations, contingencies, beliefs, competencies) serve as the templates for the construction of in-context mathematical representations, which may reasonably be said to encode contextualized understandings. Familiar, concrete objects might be used to serve as an external representational system for connections. Goldin (2002:216) argues against insisting that all mathematics be in context, especially when the contexts are those that will pose natural obstacles to later abstraction. Because most initial contexts eventually create some cognitive obstacles, the process requires the progressive detachment of representations from their initial contexts as structure is built.

It has already been noted that abstractness increases as knowledge becomes freed from specific context (see § 3.5.2.3). Reinforcing procedural knowledge with the relevant conceptual knowledge can help to generalize procedural knowledge as it is freed from a specific context. Procedural knowledge can become more useful when it is recognized as being applicable to other contexts. The ability to generalize a concept or procedure to new situations is the ability which distinguishes concept learning from other forms of learning – the learner can generalize the concept or procedure in an unfamiliar situation (Bell, 1991:117). Thus, procedural rules should never be learned in the absence of a concept (Van de Walle, 2004:28).

It is important to realize that the problem situations with which students of mathematics interact may set boundaries on their internal representations of the problems to be solved (Hiebert and Carpenter, 1992:79). If the instructional program presents problems only as written symbols and does not support connections with other external representation forms and problem situations, the knowledge acquired has limited transfer potential because the internal representations is severely constrained.

Heller et al. (1989:206) states that factors such as problem format, the particular numbers used in problems, problem context, and even the problem(s) preceding a task, influence students' ability to solve proportional problems. According to Shunk (1996:272), experts in problem solving develop sophisticated procedural knowledge for classifying mathematical problems according to type. The development of these kinds of knowledge for solving different kinds of problems in different problem contexts should receive more attention as it is important for the development of mathematical understanding. The surface features of a problem should not stand in the way of doing mathematics. The importance of reflective practices in the development of mathematical understanding and consequently, MCKfT, will be the focus of the next section.

3.7.4 Reflective practices

As early as 1904, Dewey (as quoted by Mewborn, 1999:316 and Hiebert et al. 1996:14) argued that teachers who are proficient in the skills of teaching, but who lack an inquiring mind, will have their professional growth curtailed because reflective practices is the key to move beyond the distinction between knowing and doing. The preparation of reflective student teachers is a necessary first step for those who work in university programs of teacher education (Zeichner & Listin, 1987:45). This viewpoint can also be applied to the teaching and learning of mathematics in various aspects.

Three aspects of a reflective frame of mind are important:

- Firstly, being able to reflect on one's current knowledge and understanding in the construction of new mathematical knowledge and understanding.
- Secondly, being able to look back and evaluate one's answers for the validity of the answer. Without a sound conceptually connected knowledge base, it might be impossible for a person to evaluate answers. Teachers must learn to validate their own answers (Ball, 1990:457).
- Thirdly, a teacher's ability to reflect on his/her own mathematical knowledge and understanding to be able to teach mathematics, keeping learners' level of understanding in mind (see § 2.4.1).

When confronted with a new problem, one has available and appropriate knowledge to solve the problem. Firstly then, reflective thought means sifting through existing ideas to find those that seem to be the most useful in giving meaning to a new idea (Van de Walle, 2004:23). To construct and understand a new mathematical idea or concept requires of the individual to actively think about it. The tools we use to build understanding are our existing ideas, and the effort that must be supplied is active and reflective thought (Van de Walle, 2004:22).

Integrated networks of connected knowledge are both the product of constructing knowledge and the tools with which additional new knowledge can be constructed (Van de Walle, 2004:23). As learning occurs and understanding is built, the networks are rearranged, added to, or modified. When there is active, reflective thought, schemas are constantly being modified or changed so that ideas fit better with what is known (Van de Walle, 2004:23).

Hiebert and Carpenter (1992:73) state that understanding written mathematical symbols requires students to consciously reflect on the symbols as elements of a system rather than simply moving them around on paper according to memorized rules. It is true that symbols can be manipulated

without reflection, but manipulation without reflection is unlikely to stimulate construction of the relationships that lead to understanding. Making connections thus requires a reflective frame of mind.

The third application of a reflective frame of mind deals with a mathematics teacher's teaching practices. Kinach (2002:69) proposes a meta-cognitive approach to teachers' development of MCKfT. She refers to this as cognitively self-guided professional development. This model has been altered to fit our definition of MCKfT. The model offers three types of meta-cognitive prompts:

1. Active focused prompts, which require the reflective practitioner to explain and represent the topic in different instructional or problem contexts.
2. Cognitive focused prompts, which require the reflective practitioner to assess the adequacy of their MCKfT.
3. Self-monitoring prompts that provide the reflective practitioner with a five-element meta-cognitive scaffolding routine to self-monitor the development of an individual's own MCKfT.

Constructing knowledge, doing problems and teaching mathematics requires a reflective frame of mind, which entails actively thinking about what you are doing. Van de Walle (2004:32) goes as far as to say that reflective thinking is the single most important ingredient for effective learning, and moreover, teaching. However, one can only reflect on what you know, therefore information on teachers' current knowledge states (also called prior knowledge) is a prerequisite for teacher education.

3.7.5 Prior knowledge

Prior knowledge has long been acknowledged as a very important factor in teaching and learning. "If I could reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows." (D. Ausubel as quoted by Hiebert and Carpenter, 1992:80). A large body of research on student learning demonstrates how prior knowledge can influence subsequent action and performance (Prawat et al., 1992:146). Educational researchers claim that mathematical understanding builds on current understanding or prior knowledge (Dossey et al., 2002:81). In other words, we learn mathematics by making connections between new ideas and those ideas that are already part of what we know. The amount of prior knowledge (what the learner already knows) and the way that knowledge is structured influence current learning (Hiebert and Carpenter, 1992:80). Prior knowledge that is well understood influences learning differently than prior knowledge that is less understood.

Parrott (2005) proposes the following framework for understanding the relationship between meta-cognition, prior knowledge and conceptual change.

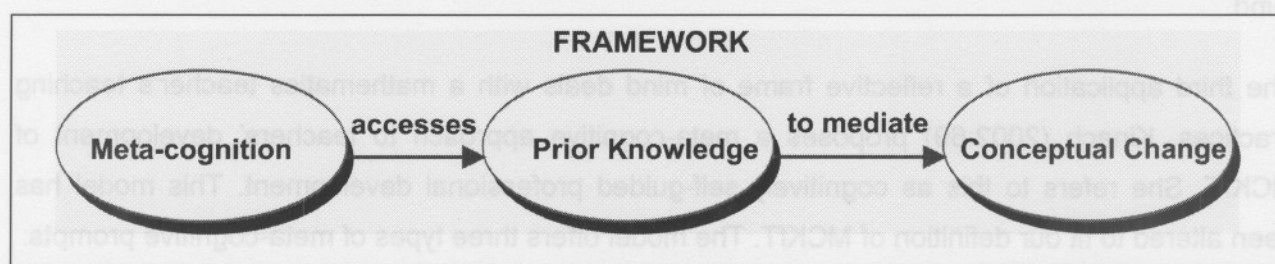


FIGURE 16: A theoretical framework for accessing prior knowledge through meta-cognition and mediating conceptual change in knowledge.

Meta-cognition refers to one's knowledge concerning one's own cognitive processes or anything related to them, e.g. the learning-relevant properties of information or data (Flavell as quoted by Schoenfeld, 1992:347). It can thus be described as a process of reflection on current mathematical knowledge and understanding. Prior knowledge differs for each person. For a teacher of mathematics, prior knowledge must include various aspects. Except for the MCKfT, the teachers must have knowledge of the learner's level of mathematical knowledge and understanding and what can reasonably be expected from the learner in the teaching and learning of mathematics.

If one wants to work on a teacher's MCKfT, one first has to know something about a teacher's current mathematical content knowledge. Therefore a diagnostic test on prior knowledge would be a necessary prerequisite for working on MCKfT (see framework in § 3.10). The construction of connections could then for instance be a process of starting with the MCKfT students already have and focus on how that knowledge can be extended.

3.7.6 Understanding mathematics and developing MCKfT

There exist no absolute levels of the development of knowledge and understanding (Bell, 1991:109). It seems to be more of a cyclic process of developing and building mathematical knowledge. It is a constant cycle of procedural knowledge reinforcing conceptual knowledge and conceptual knowledge reinforcing procedural knowledge.

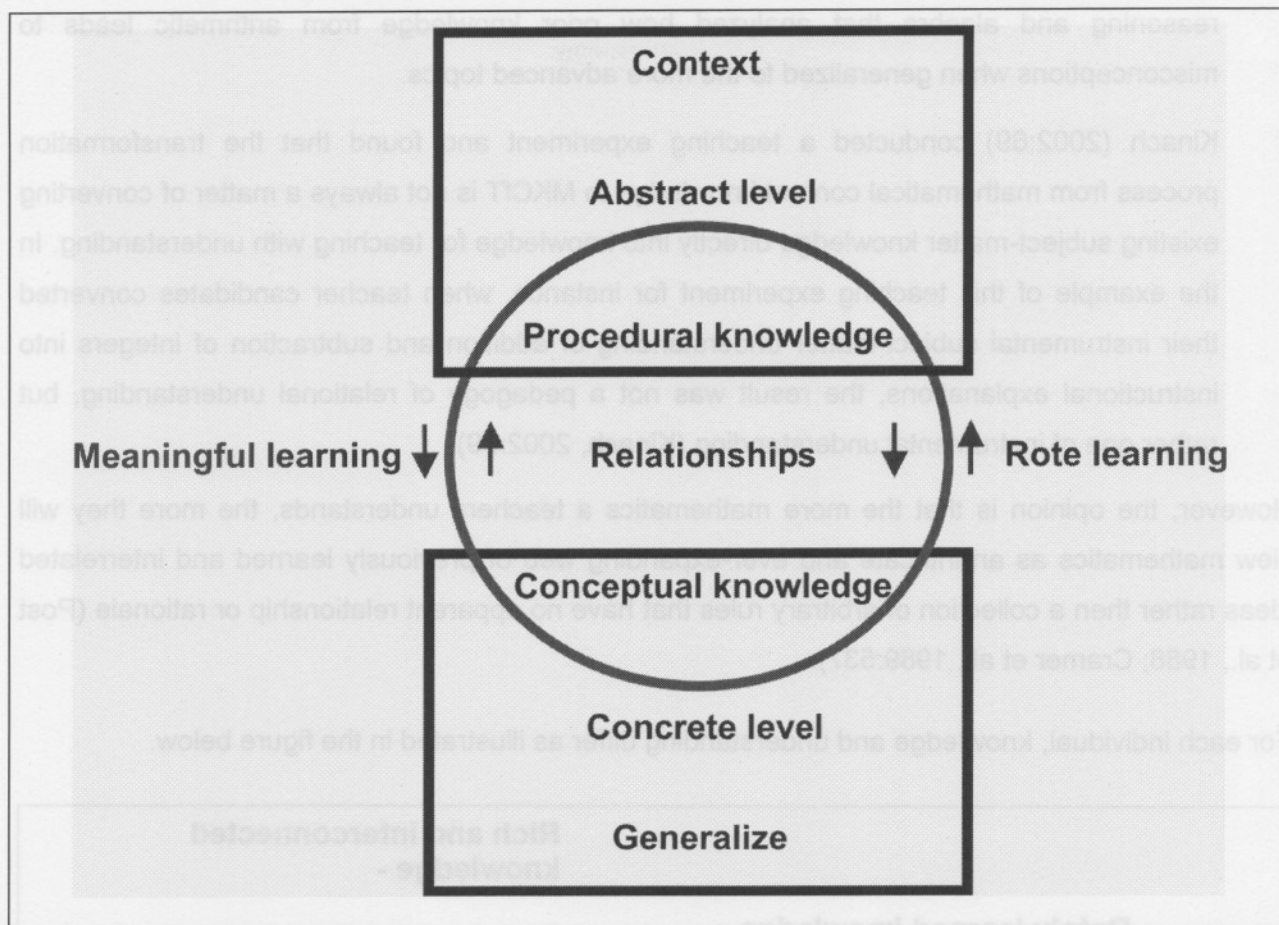


FIGURE 17: A cyclic process of the development of mathematical knowledge and understanding.

It is useful to think of a student's knowledge of mathematics as internal networks of representations. Meaningful learning occurs as these representations get connected into increasingly structured and cohesive networks.

Alarming research results should however be acknowledged:

- Hiebert and Carpenter (1992:78) provide evidence that suggests that learners who possess well-practiced, automatized rules for manipulating symbols are reluctant to connect the rules with other representations that might give them meaning. This can be an important aspect in the development of MCKfT as one will have to work on and transform teachers' current or prior mathematical content knowledge states that are already established and held in some way.
- Hart (1988) and Matz (1980) as quoted by Hiebert and Carpenter (1992: 85) provides evidence through research on learning advanced mathematical topics like proportional

reasoning and algebra that analyzed how prior knowledge from arithmetic leads to misconceptions when generalized to the more advanced topics.

- Kinach (2002:69) conducted a teaching experiment and found that the transformation process from mathematical content knowledge to MKCfT is not always a matter of converting existing subject-matter knowledge directly into knowledge for teaching with understanding. In the example of this teaching experiment for instance, when teacher candidates converted their instrumental subject-matter understanding of addition and subtraction of integers into instructional explanations, the result was not a pedagogy of relational understanding, but rather one of instrumental understanding (Kinach, 2002:69).

However, the opinion is that the more mathematics a teachers understands, the more they will view mathematics as an intricate and ever-expanding web of previously learned and interrelated ideas rather than a collection of arbitrary rules that have no apparent relationship or rationale (Post et al., 1988; Cramer et al., 1989:537).

For each individual, knowledge and understanding differ as illustrated in the figure below.

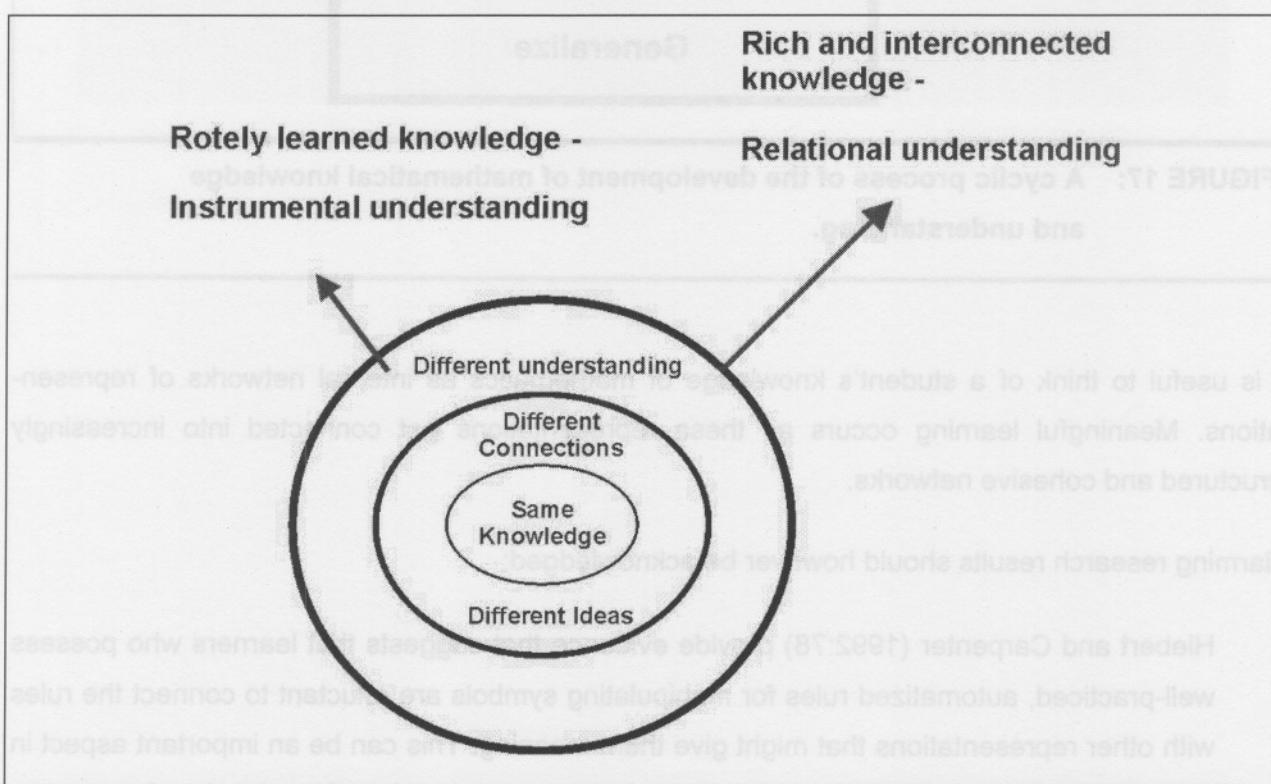


FIGURE 18: Different people can have different understanding of the same knowledge.

The preference to ask “Why does it make sense?” is the first stepping stone to conceptual understanding of mathematics (Ma, 1999:109). Exploring the mathematical reasons underlying algorithms might lead students to more important ideas of the discipline. It might be necessary for mathematics teachers to unlearn acquired knowledge of maths (Cohen, 1990:342). There is a need for assessment to change. The development of MCKfT can be done through interrogating and challenging prospective teachers’ instrumental knowledge with relational mathematical knowledge questions. The focus of assessment should change from what a mathematics teacher knows to how the mathematical knowledge is structured. A question like “Does he/she know it?” should be replaced with “How does he/she understand it?” and “What knowledge does he/she connect with it?” These questions entail unpacking of mathematical knowledge and understanding.

3.8 DEVELOPING MCKfT THROUGH UNPACKING MATHEMATICAL CONTENT

In this section, examples will be given to illustrate the meaning of being able to unpack mathematical content knowledge. It will be shown that unpacking mathematical knowledge entails understanding mathematics. Understanding mathematics will be shown to enable a mathematics teacher to unpack their mathematical knowledge. The belief is that a focus on the right questions, namely “Why does something make sense?” and developing reflective practices can be used in the education of mathematics teachers to strengthen MCKfT. To fully understand what MCKfT entails, one has to look at it through the lens of specific mathematical content. The chosen mathematical content includes aspects of proportional reasoning and functions.

3.8.1 Unpacking the cross-product algorithm

Traditionally, proportional situations have been embedded in missing-value problems. The standard algorithm or cross-product algorithm, for example ($\frac{a}{b} = \frac{c}{x}$), entails the recognition of a problem context in which three values are given and one value is missing: a , b and c given, find x . The standard solution procedure is to cross-multiply and solve for x : $ax = cb \rightarrow x = \frac{cb}{a}$. This algorithm entails a mechanical process stripped of meaning in a real world context.

A teacher should know the cross-product algorithm and be able to identify appropriate proportional situations to use the cross-product algorithm. It seems that a much more valuable question to the one “Solve the problem using the cross-product algorithm”, would be “In which problem context

and why does the cross-product algorithm work?" The second question will not be so straight forward for teachers to answer. Will teachers for instance be able to provide the following explanations:

- A For $y = mx$, $y = 5x$, different rate pairs will be $\frac{5}{1}$, $\frac{10}{2}$ and $\frac{15}{3}$.
- B All the above fractions have a value of 5.
- The presence of the constant factor 5 in A and B is evident. The reciprocal rate pairs are $\frac{1}{5}$, $\frac{2}{10}$ and $\frac{3}{15}$. All the above fractions have a value of $\frac{1}{5}$.

This special characteristic of the equivalent rate pairs that exist in all proportional situations enables one to use the cross-product algorithm (Post et al. 1988).

A teachers' ability to unpack understanding of proportional situations can be strengthened with knowledge of mathematical content connections that ratio and proportion have with other parts of mathematics. The following figure illustrates some of these connections:

3.8.1 Unpacking the cross-product algorithm

Traditionally, proportional situations have been embedded in missing-value problems. The standard algorithm or cross-product algorithm, for example ($\frac{a}{b} = \frac{c}{x}$), entails the recognition of a problem context in which three values are given and one value is missing: a , b and c given, find x . The standard solution procedure is to cross-multiply and solve for x : $ax = cb \rightarrow x = \frac{cb}{a}$. This algorithm entails a mechanical process stripped of meaning in a real word context. A teacher should know the cross-product algorithm and be able to identify appropriate proportional situations to use the cross-product algorithm. It seems that a much more valuable question to the one "Solve the problem using the cross-product algorithm," would be "In which problem context

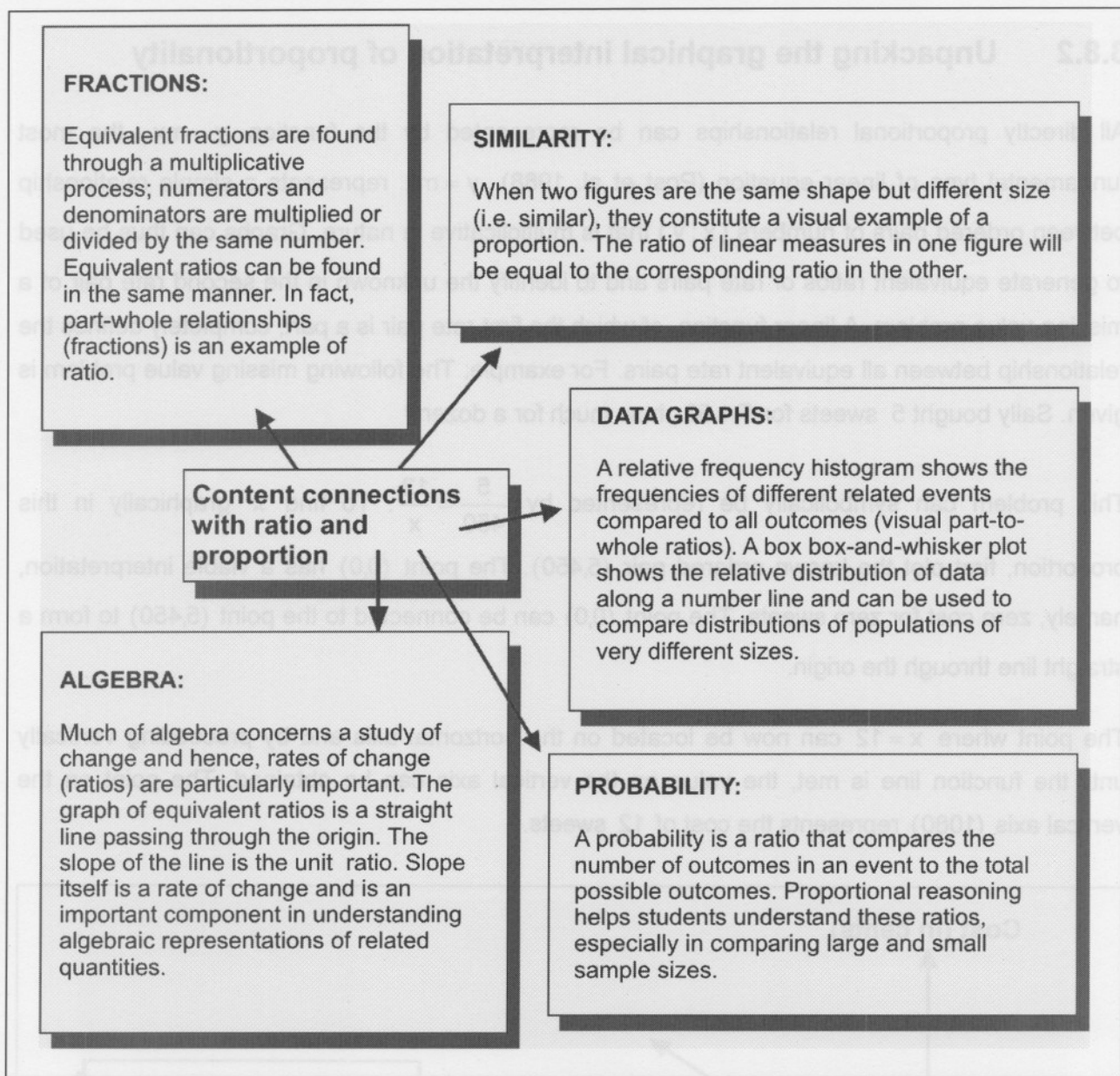


FIGURE 19: Mathematical content connections with ratio and proportion
(Van de Walle, 2004:298).

Example of content connections between proportional situations and functions will be provided in the next section.

3.8.2 Unpacking the graphical interpretation of proportionality

All directly proportional relationships can be represented by the function $y = mx$, the most fundamental type of linear equation (Post et al. 1988). $y = mx$ represents a simple relationship between ordered pairs of numbers $(x; y)$ that is multiplicative in nature. Graphs can thus be used to generate equivalent ratios or rate pairs and to identify the unknown in the second rate pair of a missing-value problem. A linear function, of which the first rate pair is a part, completely defines the relationship between all equivalent rate pairs. For example: The following missing value problem is given. Sally bought 5 sweets for R4.50, how much for a dozen?

This problem can symbolically be represented by $\frac{5}{450} = \frac{12}{x}$. To find x graphically in this proportion, first plot the known ordered pair $(5,450)$. The point $(0,0)$ has a viable interpretation, namely, zero cost for zero sweets. The point $(0,0)$ can be connected to the point $(5,450)$ to form a straight line through the origin.

The point where $x = 12$ can now be located on the horizontal axis and by proceeding vertically until the function line is met, the value on the vertical axis can be obtained. The point on the vertical axis (1080) represents the cost of 12 sweets.

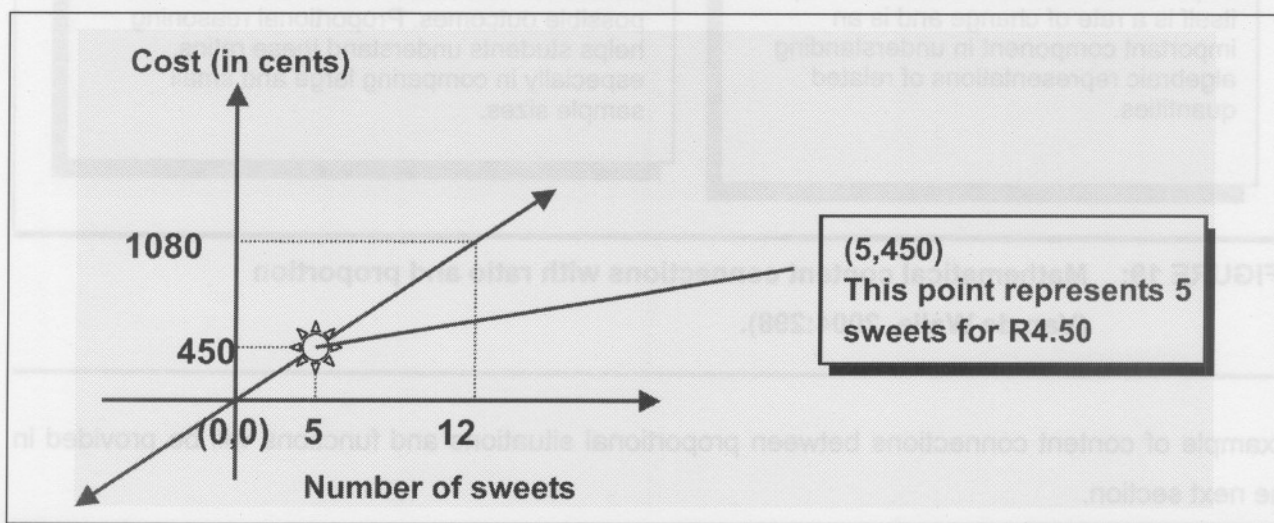


FIGURE 20: A graphical representation of $y = 90x$.

The equation of the straight line through the coordinates $(0,0)$ and $(5,450)$ is the line $y = 90x$.

The following connections can now be made:

- The slope of the line is 90.
- 90 is also the cost of a single sweet, therefore the unit rate. The unit rate can also be located by getting 1 on the horizontal axis and the corresponding y -value on the y -axis. The unit rate is always the slope of the line if the slope is expressed with a denominator of 1.
- The slope of the line represents the multiplicative nature of the relationship between the variables. The slope (m) of a line of the form $y = mx$ (straight line through origin) is $\frac{y}{x}$

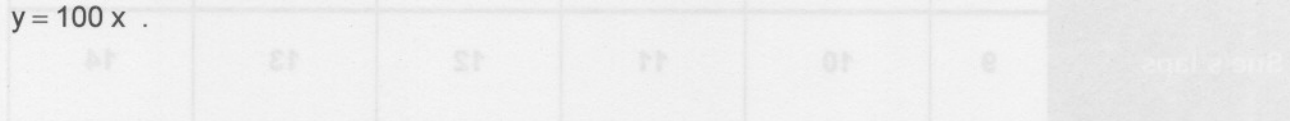
(solving for m). But $\frac{y}{x}$ is the same division used to determine the rate pair and to determine

the unit rate. It follows then that, the unit rate and the slope of any line expressed with a denominator of 1 is the same thing.

- Most proportional situations are restricted to the first quadrant because the values generated by “real-life” are normally positive.

There exists a very important link between a proportion and the concept of what a function is. The central concept of proportional reasoning is the representation of one quantity as a certain proportion of another (Wiebe, 2006). This central concept of proportional reasoning can also be described as the multiplicative relationship that exists among specific components in a specific situation. Because of this relationship, all proportional situations can be expressed through an algebraic rule of the form $y = mx$ (Cramer and Post, 1993a; Van de Walle, 2004:298, 445). This means that the graphs of all proportional situations (equivalent ratios) are straight lines that pass through the origin, as is indicated by $y = mx$. The slope of these lines will always be the unit ratio or rate between the two variables indicated by x and y . Slope itself is a rate of change and is an important component in understanding algebraic representations of related quantities (Van de Walle, 2004:298).

We know that 1 meter is equal to 100 centimeter. Therefore 2 meters must be equal to 200 centimeter and so on. A graphical representation of the above relationship follows in the next figure. y Centimeters is equal to 100 times x centimeters, and one writes it algebraically as $y = 100x$.



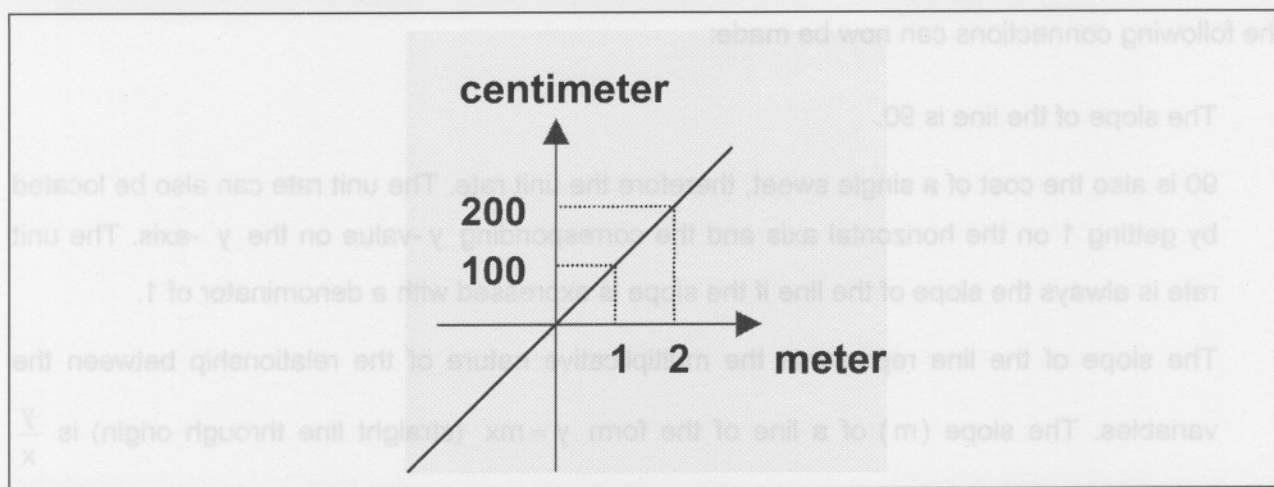


FIGURE 21: A graphical representation of $y = 100x$

One can say that every proportional situation gives rise to a linear or straight-line function where the constant ratio (100 in this case) between proportional objects is the slope of the line. Many relationships involving rates or proportions offer a valuable opportunity for examining functions (Van de Walle, 2004:444).

A graph is one of the most powerful representations of a function and can also be useful in determining if a relationship is a proportional situation. An example of a non-proportional situation that could mistakably be interpreted as a proportional situation is as follows: Sue and Julie were running equally fast around a track. Sue started first. When she had run nine laps, Julie had run three laps. When Julie had completed fifteen laps, how many laps had Sue run?

The context in which the problem is set might look like a missing value problem because of the fact that three values are given and the fourth one must be determined (Post et al. 1988). Students might use the cross-product algorithm, but constructing a table helps to identify the numerical relationship between the two quantities.

TABLE 2: A table representation of the function $y = x + 6$.

Julie's laps	3	4	5	6	7	8
Sue's laps	9	10	11	12	13	14

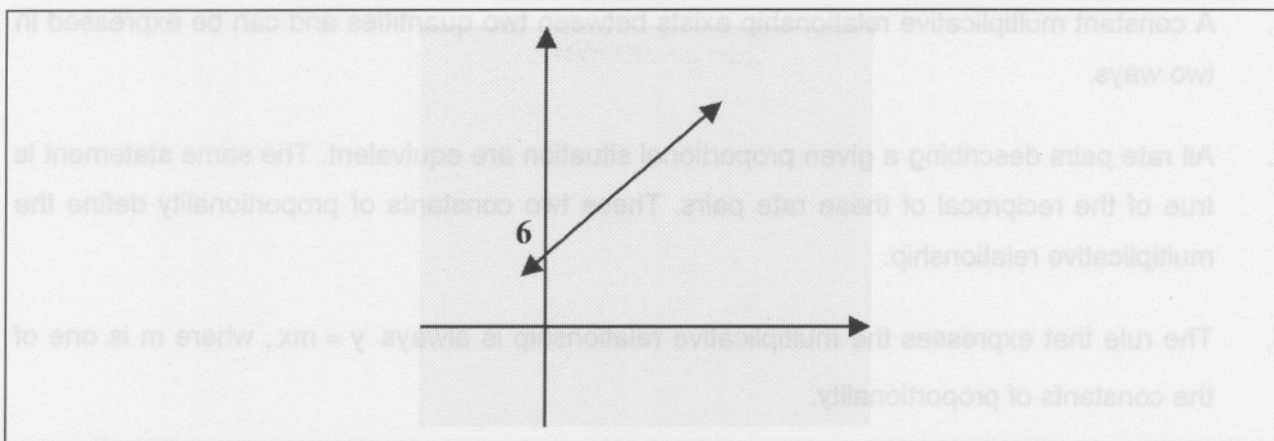


FIGURE 22: A graphical representation of $y = x + 6$.

The graph of $y = mx + c$ is not a proportional solution. The numerical relationship is a constant sum, not a constant factor as indicated by $y = mx + 6$. Sue will always be running 6 laps more than Julie – reflected by addition.

Other connections are:

- If the situation were proportional, then the graph would have intersected the origin.
- If the rate pairs were to be expressed as fractions – e.g. $\frac{9}{3}$, $\frac{10}{4}$, $\frac{11}{5}$, $\frac{12}{6}$ – it can be shown that they do not have the same value.

Silver (1986:191) (see § 3.6.5.3) and Shunk (1996:271) state that experts overlook surface features of a problem and analyze it in terms of the operations required for solution, whereas novices are swayed more by surface features (see § 3.6.4.2 and § 3.7.3). The fundamental assumption is that skilled performance in a complex domain like mathematics teaching and learning requires being able to unpack mathematical understanding, which entails developing MCKfT.

A deeper understanding of the mathematical features of proportional situations will enable a teacher to solve problems in multiple ways. The following list summarizes the mathematical features of proportional situations (Post et al. 1988):

1. A constant multiplicative relationship exists between two quantities and can be expressed in two ways.
2. All rate pairs describing a given proportional situation are equivalent. The same statement is true of the reciprocal of these rate pairs. These two constants of proportionality define the multiplicative relationship.
3. The rule that expresses the multiplicative relationship is always $y = mx$, where m is one of the constants of proportionality.
4. Graphically, all points for a proportional situation fall on a straight line passing through the origin. In real world settings these lines always have positive slopes. All rate pairs for the particular situation fall on the line.
5. The slope of the line is the m in the equation $y = mx$ and is one of the constant multiplicative factors relating the two quantities y and x .

Previously in this section an example was given where the cross-product algorithm might be used if a student recognises wrongly that three values are available and one is missing. An understanding of why procedures work and under what conditions specific procedures can be applied, are objectives that are often lacking in mathematics instruction (Post et al. 1988). Fostering the habit to reflect on why a specific procedure is suitable in solving a specific mathematical problem is a key aspect of building MCKfT. A superficial understanding of a concept makes it easy to apply memorized rules in the wrong situation. Being able to look beyond the surface features of proportional situations will enable students to make appropriate decisions as to when and if a procedure can be applied (Post et al. 1988).

Building MCKfT is also a question of being able to unpack across topics in the school curriculum. A teacher of mathematics needs to have a big picture of the structure of mathematics and how the mathematics curriculum develops from Grade 1 through to Grade 12. Post et al. (1988) cites the following example of connecting proportions to other topics: "Looking back, proportional situations in the form of missing value problems can be rethought from familiar multiplication and division perspectives. Looking ahead, unit rate is also interpretable as the slope (m) of a linear function of the form $y = mx$."

Important characteristics of MCKfT became dominant in the theoretical study. It is above all a case of being able to unpack knowledge and understanding or struggling to teach compressed

compartmentalized knowledge of mathematics. Criteria for the development of MCKfT will be the focus of the next section.

3.9 CRITERIA FOR DEVELOPING MATHEMATICAL CONTENT KNOWLEDGE FOR TEACHING

The following figure summarizes the important characteristics of MCKfT and also criteria for the development of MCKfT.

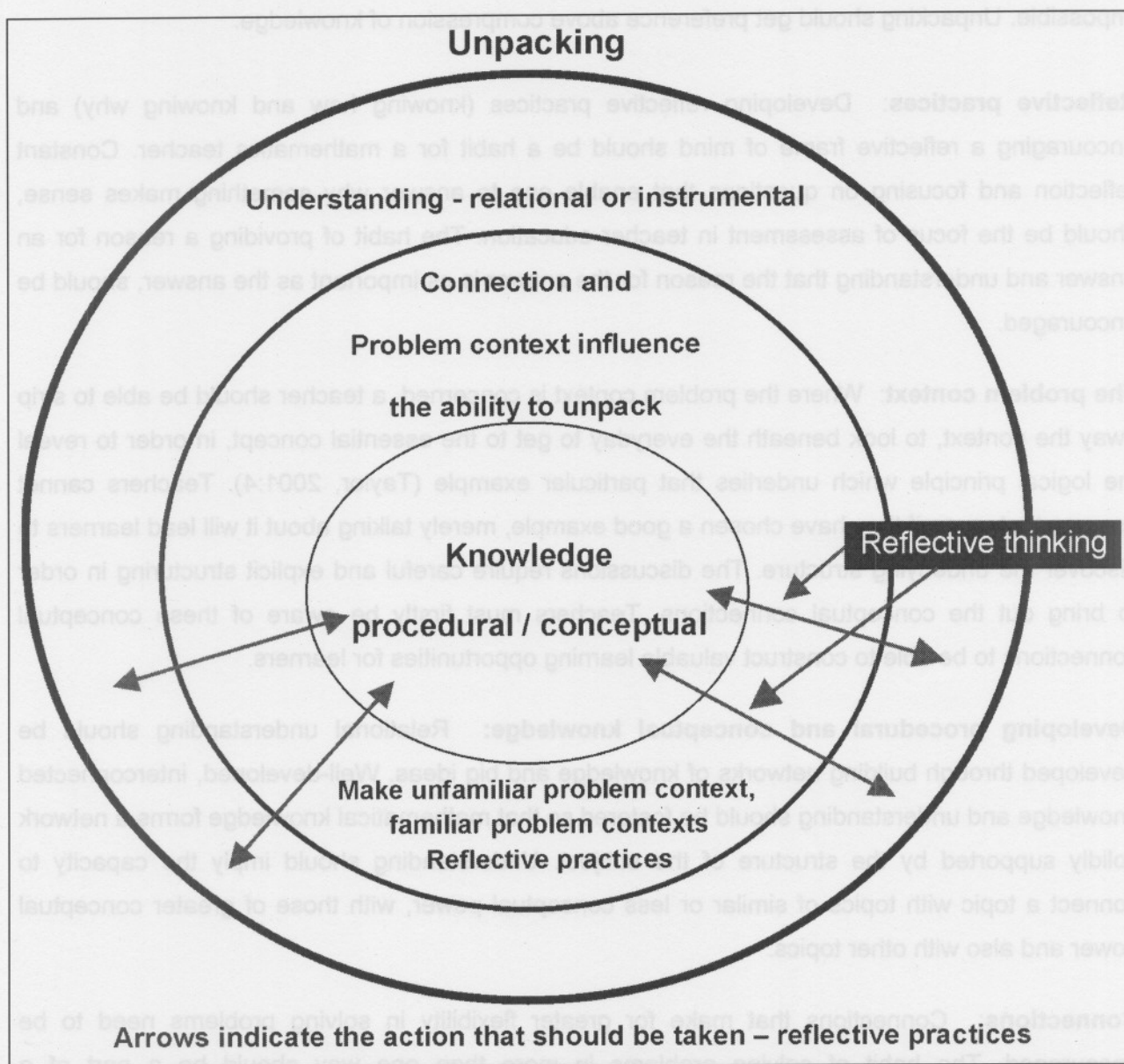


FIGURE 23: Unpacking knowledge and developing MCKfT.

Criteria for the development of mathematical content knowledge and understanding and fostering MCKfT are:

Unpacking: Unpacking of mathematical content knowledge and understanding should be the focus of mathematics teacher training programs. As one moves deeper into the mathematics field, knowledge becomes more compressed. However, the focus should be on a sound and connected knowledge and understanding basis on which more advanced mathematics can be built. A teacher's MCKfT should be decompressed and stripped of contextually bound stumbling stones that make the transfer of knowledge to new situations, and therefore problem solving, almost impossible. Unpacking should get preference above compression of knowledge.

Reflective practices: Developing reflective practices (knowing how and knowing why) and encouraging a reflective frame of mind should be a habit for a mathematics teacher. Constant reflection and focusing on questions that enable one to answer why something makes sense, should be the focus of assessment in teacher education. The habit of providing a reason for an answer and understanding that the reason for the answer is as important as the answer, should be encouraged.

The problem context: Where the problem context is concerned, a teacher should be able to strip away the context, to look beneath the everyday to get to the essential concept, in order to reveal the logical principle which underlies that particular example (Taylor, 2001:4). Teachers cannot assume that, even if they have chosen a good example, merely talking about it will lead learners to discover the underlying structure. The discussions require careful and explicit structuring in order to bring out the conceptual connections. Teachers must firstly be aware of these conceptual connections to be able to construct valuable learning opportunities for learners.

Developing procedural and conceptual knowledge: Relational understanding should be developed through building networks of knowledge and big ideas. Well-developed, interconnected knowledge and understanding should be fostered so that mathematical knowledge forms a network solidly supported by the structure of the subject. Understanding should imply the capacity to connect a topic with topics of similar or less conceptual power, with those of greater conceptual power and also with other topics.

Connections: Connections that make for greater flexibility in solving problems need to be encouraged. The habit of solving problems in more than one way should be a part of a mathematics teacher's repertoire and thus be encouraged in teacher education. Solving a problem in different problem contexts makes for the development of connected knowledge and understanding. Connections between conceptual and procedural knowledge should be fostered for

developing a sound mathematical knowledge base. Concepts and procedures should be connected for teachers to have both an intuitive feel for mathematics and being able to solve the mathematical problems. Generating answers without understanding what they are doing should be discouraged. New knowledge should be connected to teachers' prior knowledge.

Communication: The ability to communicate mathematics in different ways (orally and in writing) should be encouraged, as this is what a teacher needs to do to teach mathematics.

Representations: Knowledge of different mathematical representations and the ability to use different representations of mathematical concepts to reveal and represent connections among mathematical concepts and procedures to learners, should be encouraged. Representations that further mathematical understanding should be part of teacher education.

Reasoning: Students of mathematics should be exposed to opportunities where they can reason inductively to foster the discovery of connections. A way to build understanding might be to use and encourage more intuitive and qualitative approaches in solving problems during teacher education.

Developing contextual as well as abstract understanding: Investigating and understanding the logical structure of mathematics should be encouraged. The structure of mathematics, as well as problems in different contexts, is important. Understanding concerning mathematical ways of knowing as well as mathematical substance should be focused on in teacher education. A fundamental understanding of the whole elementary and secondary mathematics curriculum should be fostered. Teachers must be ready to use an opportunity to review concepts that learners have previously studied or to lay groundwork for a concept to be studied later. Teachers should also be made aware that not all mathematical knowledge can be connected to a real world situation and that not all formal knowledge has practical applications.

Since it has been established what teachers' mathematical content knowledge for teaching should entail, an empirical investigation has been undertaken to determine a group of teachers' current mathematical content knowledge states. The following section focuses on a framework for assessment of teachers' mathematical content knowledge states against important aspects of MCKfT. The experimental design and results of this empirical study are reported in the next two chapters.

3.10 A THEORETICAL FRAMEWORK FOR ASSESSING MATHEMATICAL CONTENT KNOWLEDGE STATES FOR TEACHING

According to Goldin (2002:198), the language that is used to describe mathematical learning and teaching entails assumptions that are increasingly treated as ideological rather than scientific. There is a need for a shared, scientific, non-ideological framework for empirical and theoretical research in mathematical learning.

Teacher education traditionally focuses on what teachers need to know, rather than on what they actually know. Prior knowledge has been established as an integrally important aspect of the development of understanding in mathematics, which can in turn have an influence on the development of MCKfT.

A useful framework is proposed here. The idea is that the elements in the framework must be used as indicators of aspects that should form part of the assessment of teachers' current mathematical content knowledge states.

Assessing teachers' prior MCKfT will entail:

- determining what their views on and beliefs about teaching and learning mathematics are as this influence their teaching and learning of mathematics;
- determining the state of their prior procedural and conceptual knowledge by determining
 - if they know the how and why of every step in a procedure so that they can be able to decompress / unpack their understanding of steps in a procedure
 - to what extent their procedural knowledge is connected to use in different problem contexts;
- assess their problem solving skills;
- assess their reasoning skills;
- assess if they can communicate orally and in writing and comprehend the language of mathematics;
- assess different types of connections
 - between procedural knowledge and conceptual knowledge
 - among mathematical topics;

- assess how their knowledge and understanding is represented (externally and internally), movement between representations and if they can give multiple representations for the same mathematical idea;
- determining if teachers have instrumental understanding (rules without reasons) or relational understanding (knowing how and knowing why) and connected networks of knowledge;
- assessing understanding in different problem contexts;
- assess abstract understanding (the logical mathematical structure);
- assessing for a reflective frame of mind and reflective practices in doing mathematical problems.

The rationale behind this study is to evaluate what teachers actually know by mapping it on what they should know to have sufficient MCKfT.

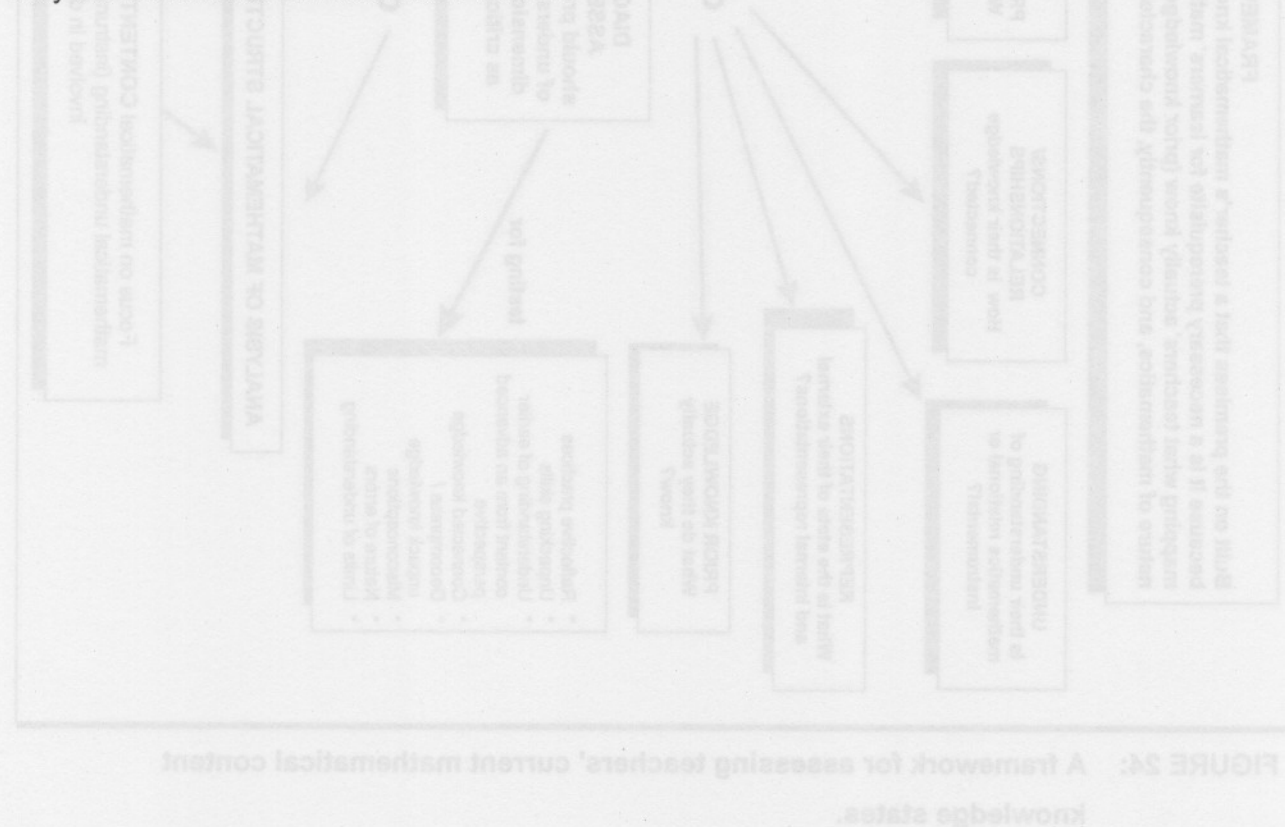
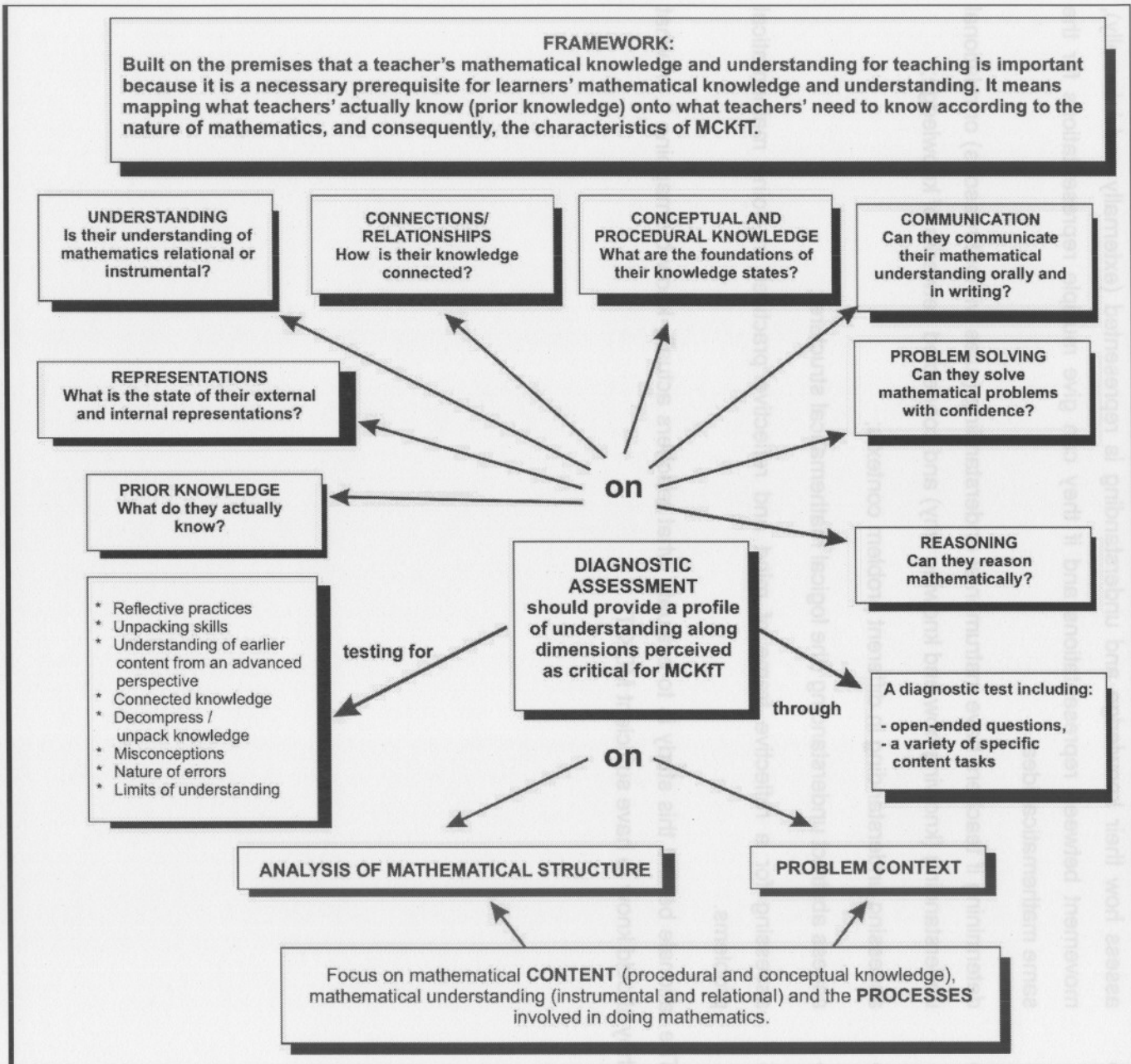


FIGURE 24: A framework for assessing teachers' current mathematical content knowledge states.



3.11 CONCLUSION

An extensive literature study has revealed some important prerequisites for teaching and learning mathematical content knowledge with understanding. The establishment of the characteristics of MCKfT has led to theoretical criteria for the development of MCKfT. A theoretical framework for diagnostic assessment of mathematics teachers MCKfT was established, built on the premises that a teacher's mathematical knowledge and understanding for teaching is important because it is a necessary prerequisite for learners' mathematical knowledge and understanding. Since it has theoretically been established what teachers' mathematical content knowledge should entail being able to teach mathematics with understanding, we should take this a step further by determining what mathematics teachers' actually know (prior knowledge or current mathematical content knowledge states) in relation to the established criteria for the development of MCKfT. The focus of the next chapter will thus be on planning and performing diagnostic assessment on a group of mathematics teachers to determine these teachers' current mathematical content knowledge states.

CHAPTER 4

EMPIRICAL STUDY

4.1 INTRODUCTION

The aim of the literature study in Chapter 2 and Chapter 3 was threefold: Firstly to argue for a distinct kind of mathematical content knowledge for teaching, namely MCKfT, secondly, to establish what mathematics teachers need to know and be able to do in terms of MCKfT (the characteristics of MCKfT) and thirdly, what the development of MCKfT entails (criteria for the development of MCKfT). A framework for assessing teachers' current mathematical content knowledge states was established at the end of Chapter 3.

The aim of this chapter is to describe a plan for investigating a group of teachers' current mathematical content knowledge states. The focus of the chapter will firstly be on the research methodology to be used. The components needed for carrying out the experiment will be discussed in the first part of this chapter. In the last part of the chapter, the focus will be on the statistical techniques that fit the empirical design of this study.

4.2 THE RESEARCH METHODOLOGY

Research methodology refers to the methods used to extract meaning from data. The format of the data dictates the research methodology. In this section the problem under investigation and the purpose of this study will be established. This will lead to an indication of the format of the data. The rationale behind the chosen research methodology will become clear in this section.

4.2.1 The problem under investigation

Indications are that past research in the field of mathematics teacher education falls short of investigating what mathematical content knowledge and understanding prospective and in-service mathematics teachers actually bring to teacher education programs (Carter, 1990:291).

Since 1990, valuable research on how MCKfT is acquired has seen the light (Ball, 1990, Ma, 1999). Collaboration with Ball had an influence on the focus of researchers in South Africa. Adler (2005) for instance focuses on what mathematical (and teaching) knowledge are assessed in mathematics teacher education programmes.

However, another aspect of research in this field is of interest, namely teacher's current mathematical content knowledge states. The first question that comes to mind is: What mathematical content knowledge does mathematics teachers actually have? The second question is: Is it a kind of content knowledge that adheres to the characteristics of MCKfT? The theoretical investigation of the characteristics of MCKfT and the criteria for the development of MCKfT led to a framework for assessing teachers' current mathematical content knowledge states.

This empirical study focused on teacher's current mathematical content knowledge states with respect to specific mathematical content (see § 4.4.3). The purpose of the research was to explore, analyse, describe, explain and interpret teachers' current mathematical content knowledge states. A large part of the research has thus been conducted in a qualitative format. A part of the research was also done in a quantitative format to investigate differences among different groups' of teachers. The qualitative and quantitative research methodologies will be discussed in the next section.

4.2.2 Qualitative and quantitative research

In this empirical study, both a qualitative and a quantitative research methodology were used in a cyclic fashion. Leedy and Ormrod (2001:103) shed some light on how qualitative and quantitative research methodologies can be used alternately during research: "After qualitative processes have revealed a theme in data through an inductive process, a move into more deductive modes enables one to verify or modify it with additional data."

Qualitative research is descriptive as well as interpretative in nature (Leedy & Ormrod, 2001:101). The descriptive feature was applied in order to reveal teachers' current mathematical content knowledge states. The interpretative feature enabled the researcher to gain insight in the nature of teachers' current mathematical content knowledge states and discover problems that exist in these knowledge states.

For the quantitative research part, certain statistical procedures were used to analyse and draw conclusions from the qualitative data analyses. The descriptive quantitative analyses of findings will be communicated in numbers, or statistics as we call it.

4.2.3 The focus on a specific qualitative research design –

A diagnostic content analysis

A specific qualitative research design, named a diagnostic content analysis formed a large part of this empirical study. The diagnostic content analysis is a detailed and systematic examination of the contents of a particular body of material for the purpose of identifying patterns, themes or biases (Leedy & Ormrod, 2001:155). The diagnostic content analysis was used to analyse teacher's written responses to a research instrument in terms of assigned attributes (see § 4.4).

The following steps typify a diagnostic content analysis (Leedy & Ormrod, 2001:156 - 157):

Step 1: The researcher identifies / describes the specific body of materials to be studied.

Step 2: The researcher defines and describes the characteristics to be examined.

Step 3: The coding or rating procedure – the coding of the material in terms of predetermined and precisely defined characteristics.

Step 4: Tabulations of the frequency or percentages of each characteristic.

Step 5: Descriptive or inferential statistical analyses as needed to answer the research question.

A content analysis is thus in itself also qualitative and quantitative in nature. After a qualitative content analysis, a statistical analysis is performed on the frequencies obtained to determine whether significant differences exist relevant to the research question. The researcher then uses such tabulations and statistical analyses to interpret the data as they reflect on the problem under investigation (Leedy & Ormrod, 2001:156).

The first three steps in a content analysis, which consist of the components needed in a diagnostic content analysis, will be discussed in the next section. The fourth and fifth components of a content analysis will be dealt with in paragraph 4.5.

4.3 THE EXECUTION OF A DIAGNOSTIC CONTENT ANALYSIS ON TEACHERS' CURRENT MATHEMATICAL CONTENT KNOWLEDGE STATE

The literature research evoked the necessity for a type of diagnostic assessment in which teachers' current mathematical content knowledge states could be viewed through the lenses of necessary knowledge states for MCKfT (see § 3.9 and § 3.10).

This study is mostly a diagnostic study of secondary school mathematics teachers' mathematical content knowledge for teaching. The rationale behind the study is that if one can diagnose prior knowledge, one knows which aspects to focus on in the future education of mathematics teachers. The diagnostic assessment used here fits the description of a qualitative content analysis.

4.4 COMPONENTS NEEDED IN THE EXECUTION OF A DIAGNOSTIC CONTENT ANALYSIS

To be able to conduct a diagnostic content analysis, the following components are needed (see § 4.2.4):

For step 1: A specific body of materials to be studied. The content analysis firstly entails something that can be diagnosed. The body of material to be diagnosed here is MCKfT, which implies specific mathematical content. Proportional reasoning and functions were chosen as the mathematical content under investigation (see § 4.4.3). Items on this content were constructed (see § 4.4.4 and Appendix C). Teachers' written responses to these items were used to analyse teachers' mathematical content knowledge states.

For step 2: A definition and description of the characteristics to be examined. The researcher needed to diagnose necessary knowledge states for MCKfT. These knowledge states were called attributes (see § 4.4.1). The chosen attributes provided a way of looking at the underlying factors measured by a specific item in terms of the characteristics of MCKfT. The attributes provided something according to which the mathematics teachers' responses to the items could be analyzed.

For step 3: The coding or rating procedure. The researcher decided to do all the evaluations in terms of 1's and 0's. A "1" was rewarded if an answer was correct or sufficient

evidence was exhibited of the existence of an attribute. A "0" was awarded if an answer was incorrect or if no evidence of the existence of an attribute could be seen. The established item-attribute matrix (see § 4.4.5) provided an instrument with which all responses could be measured.

Another important aspect was the population under investigation. For this study, it had to be mathematics teachers.

Each of these components had an influence on the choice of the other. Each of these choices for this study will be described in detail in the following section.

4.4.1 The attributes

4.4.1.1 Defining an attribute

The researcher originally came across the concept of an attribute in the work of Tatsuoka and Boodoo (2000:821) in their description of the Rule Space Model. Birenbaum and Tatsuoka (1993:256) describe an attribute of a task as an expression of an underlying dimension of the task that is required in order to complete the target task successfully. Later on, Tatsuoka and Boodoo (2000:825) describe an attribute for a specific task as a description of the content and processes that an examinee must know in order to complete the task successfully.

Broadly speaking, an attribute refers to any procedure, knowledge component, strategy, skill or any cognitive process that a specific cognitive task for a specific domain might entail.

It is clear that a specific attribute should be very closely linked to a specific task in the sense that the attributes give a description of the task's cognitive requirements for a specific domain of knowledge. With this in mind, it is clear that the task attributes should be generated by a domain expert.

4.4.1.2 Establishing the attributes for this study

The framework for assessing teachers' MCKfT was established according to the criteria perceived as critical for the development MCKfT (see § 3.9 and § 3.10). These criteria were transformed into measurable attributes. The attributes each describes an observable aspect of MCKfT. These attributes were determined in order to have a vehicle with which a teacher's current mathematical content knowledge states could be described or analysed in terms of the characteristics of MCKfT.

4.4.1.3 A description of the attributes that relate to this study

It has already been stated that the choice of attributes is closely linked to the characteristics of and criteria for the development of MCKfT, specific mathematical topics and the population under investigation. Although some of these aspects will be described later on in this chapter, a complete list and description of the attributes will follow:

- A1 **Conceptual knowledge** – This attribute refers to a teachers' deep conceptual understanding of a mathematical problem. In the context of this study, multiplicative proportional reasoning versus additive proportional reasoning (misconception) and an understanding of the ratio concept will be diagnosed.
- A2 **Procedural knowledge** – This attribute includes a teachers' choice of procedural knowledge to solve a problem. The context of procedural proportional reasoning will mostly be evaluated.
- A3 **Procedure solved** – Teachers' procedural fluency and the ability to solve a procedure correctly are diagnosed.
- A4 **Interpretation of answers** – The ability to interpret an answer correctly after doing some calculations is diagnosed.
- A5 **Problem solving** – A teachers ability to understand and unpack a mathematical problem and then solve the problem is diagnosed.
- A6 **Representations** – An ability to interpret different representations (tables, graphs, words) and move among different representations is investigated.
- A7 **Communication** – Teachers' ability to communicate mathematical knowledge and understanding and the results of their thinking and findings in words and in writing is diagnosed.
- A8 **Reasoning** – Teachers' reasoning skills in mathematical situations is investigated.
- A9 **Connections** – The quality of teachers' connections among different mathematical content and ideas (relational understanding) is diagnosed.
- A10 **Unpacking** – The ability to unpack mathematical knowledge and understanding is diagnosed.

A11 **Problem contexts** - Understanding of different / alternative / changed problem contexts is investigated.

Reflective practices were determined as being an essential part of MCKfT. A reflective frame of mind is, however, very difficult to infer from written responses to mathematical problems. Reflective practices thus were not specified as an attribute.

4.4.2 The population under investigation

The mathematical content knowledge states of a group of 128 mathematics teachers were investigated. These teachers were all part of the Sediba Project of the North-West University (Potchefstroom Campus) at the time of the research and all willingly took part in this study. The research on the teachers was conducted during September 2005 and January 2006.

Sediba is a two-year (part time) professional development program, aimed at improving the quality of mathematics teaching in schools. Through the Sediba Project, a mathematics teacher can obtain an ACE (Advanced Certificate in Education) with specialization in Mathematics. The qualifications of a Sediba mathematics teacher are thus upgraded from a [matric plus] 3 year qualification to a [matric plus] 4 year qualification. The course is offered on a part-time basis, comprising of both distance and contact teaching. The lectures take place during school holidays on the Potchefstroom campus.

The Sediba Project is aimed at retraining previously disadvantaged high school mathematics teachers mostly from the North-West Province in South Africa. These teachers have to adhere to certain criteria before they are enrolled to take part in the program. They should all be in possession of a three-year teacher's diploma in mathematics teaching and they should also be teaching Grade 12 mathematics learners. The majority of black secondary teachers trained under Apartheid only had access to a three-year College training diploma. The selection criteria for these teachers result in a reasonably homogeneous group of mathematics teachers, except for their own schooling backgrounds.

The group of 128 mathematics teachers (N=128) was divided into 3 separate groups according to the number of years they were part of the Sediba Project at the time the research was conducted:

Group 1: (Sediba 2005/2006) was a group of 37 teachers (N=37). They had been part of the Project for 1 year at the time the research was conducted in September 2005 during the last contact session of their first year of study.

Group 2: (Sediba 2004/2005) was a group of 45 teachers (N=45). They had been part of the Project for 2 years at the time the research was conducted in September 2005 during the last contact session of their second year of study.

Group 3: (Sediba 2006/2007) was a group of 46 teachers (N=46). At the time the research was conducted they were all new recruits for the course and took part in the research on the first day (January 2006) of their 2 year study period at Sediba.

It is thus clear that the 3 groups of mathematics teachers took part at different stages of their studies for the ACE. This fact definitely had an influence on the results of the research study which will be elaborated on in chapter 5.

At the end of 2004, the Sediba mathematical content courses were restructured. Group 1 and Group 2 were exposed to different mathematical content courses. The differences in the courses will be explained later on in this section.

The course content for the Sediba ACE in Mathematics consists mainly of Mathematics, Didactics of mathematics, Computer Literacy and Education. The structure of ACE in mathematics education (FET-band) is given below:

Group 2 was exposed to the following mathematical content courses in their two years of study at Sediba:

NWSK 511: FOUNDATIONS OF MATHEMATICS (16 credits) After completion of this module the teacher will be able to convey to his learners the fundamental principles of algebra, functional theory, analytical and Euclidean geometry and the application of these principles by means of facilitating in the FET-band.

NWSK 512: PROBLEM SOLVING IN MATHEMATICS (16 credits) After completion of this module the teacher should be able to facilitate the development of problem solving skills involving the following equations: the solution of polynomial, exponential and logarithmic equations; and systems of linear equations are discussed, as well as the axiomatic system for plane geometry involving circles and theorems in mathematical and real-life contexts by learners in the FET-band.

NWSK 521: CONIC GENERATED AND PERIODIC FUNCTIONS (16 credits) After completion of this module the teacher should be able to facilitate the mastering of the different facets of trigonometry and conics and their application to real-life situations by learners in the FET-band.

NWSK 522: OPTIMISATION AND STATISTICS (16 credits) After completion of this module the teacher should be able to facilitate the introduction of these powerful mathematical tools (an introduction to calculus and simple probability theory and their applications to rate of change, optimisation and statistical problem respectively) to learners in the FET-band to empower them to solve scientific, social and commercially related problems.

NDWK 521: DIDACTICS OF MATHEMATICS (16 credits) After completion of this module the teacher ought to be able to demonstrate his knowledge, skills and values with regard to teaching and learning strategies, as well as technological and learning aids in learning mathematics. The teacher also ought to convey to his learners an insight into methods of solving mathematics problems in real-life situations and inspire his learners by his methods to become enthusiastic problem solvers themselves.

Re-structuring of the mathematics courses offered through the Sediba Project took place at the end of 2004. Teachers from Group 1 were exposed to different mathematical content courses than those offered for Group 2. A change in philosophy took place in that the mathematical content courses were integrated with the didactics of mathematics teaching. A separate module on functions and graphs were introduced in the first year.

Group 1 was exposed to the following mathematical content courses in their first year of studies at Sediba:

NWSK 513: MATHEMATICS FOR ACE IA (16 credits) After completion of this module the teacher should be able to facilitate the construction of basic concepts of algebra, as well as the development of problem solving skills in this domain.

NWSK 514: MATHEMATICS FOR ACE IB (16 credits) After completion of this module the teacher should be able to facilitate the development of the function concept, as well as problem solving skills in this domain.

Group 3 were not exposed to any of these course offered at Sediba at the time the diagnostic instrument was written. However, they did comply with the same selection criteria as Group 1 and Group 2.

4.4.3 The mathematical content under investigation

Due to the centrality and importance of proportional reasoning and functions, they were used as examples for the explanations that needed mathematical knowledge to be defined or illustrated. Proportional reasoning and functions were also chosen to form the mathematical knowledge under investigation in the diagnostic content analysis. Proportional reasoning was selected as a mathematical topic in the investigation due to the demand for the application and understanding of this concept across the school curriculum and beyond. Proportional reasoning is not a topic that is explicitly taught in the Sediba Project. Functions were mainly chosen because of the special connection with proportional situations (see § 3.8.2) and also the representational aspect involved in functions (see § 3.6.5 and § 4.4.3.2). Different representations of functions were introduced as part of the modules offered at the Sediba Project from the beginning of 2005.

4.4.3.1 Proportional reasoning

Mathematics teachers should be able to facilitate a learner's development of proportional reasoning. To be able to achieve this goal, mathematics teachers must be "proportional reasoners" themselves and be able to determine whether their learners are capable of reasoning proportionally. This section will focus on what proportional reasoning entails.

4.4.3.1.1 A description of proportional reasoning

A proportion is defined as a mathematical statement that expresses the equality of two ratios (Math Vantage, 1998). A ratio is an ordered pair of numbers or measurements that expresses a comparison between the numbers or measures, and a proportion is a statement of equality between two ratios (Van de Walle, 2004:299-300). The essential feature of proportional reasoning involve reasoning about a holistic relationship between two rational expressions such as rates, ratios, quotients and fractions (Lesh, Post and Behr, 1988).

Proportional thinking involves the ability to understand and compare ratios, to predict and produce equivalent ratios, to compare quantities and the relationships between quantities (Math Vantage, 1998). Post et al. (1988) advocates a firm grasp of various rational number concepts such as order and equivalence, relationships between unit and its parts, meaning and interpretation of ratio and issues dealing with division, especially dividing smaller numbers by larger ones.

Post et al. (1988) describes proportional reasoning as a form of mathematical reasoning, involving a sense of co-variation, multiple comparisons, an ability to mentally store and process several

pieces of information concerned with inference and prediction, and involving qualitative and quantitative methods of thought.

A distinction is made between additive (absolute) and multiplicative (relative) thinking (Van de Walle, 2004:300). For example: $\frac{8}{12}$ and $\frac{12}{16}$ are not equivalent ratios, but corresponding differences are the same: $12 - 8 = 16 - 12$. A focus on this additive relationship implies not seeing the multiplicative relationship of proportionality. A feature of proportional situations is the multiplicative relationship among the quantities. Ratios and rates are used to make statements about one measurement in relation to another.

According to Van de Walle (2004:300), a learner may need as much as three years' worth of opportunities to reason in multiplicative situations in order to adequately develop proportional reasoning skills. The ability to reason proportionally should be developed over time and in many different problem contexts (Thompson & Bush, 2003:400). Only then will learners be able to recognize a proportional relationship. A proportional reasoner has the mental flexibility to approach problems from multiple perspectives, and at the same time has understanding that is stable enough not to be radically affected by large or "awkward" numbers or by the context within which a problem is posed (Post et al., 1988).

4.4.3.1.2 Development as a proportional reasoner – an intuitive understanding

Proportional reasoning is not such a simple thing to define. It is difficult to tell if a person is a proportional thinker or not. The reason for this is because proportional thinking is both a qualitative and quantitative process (Lesh, Post and Behr, 1988, Heller et al., 1989:205).

The qualitative process of proportional reasoning involves a way of thinking and asking questions like: Does this answer makes sense?, Should it be larger or smaller? Such thinking requires a comparison that is not dependent on specific values.

An understanding of proportions implies the ability to recognize a proportional relationship between two quantities. All persons who solve a problem involving proportions do not necessarily use proportional reasoning. The study of proportional reasoning is often portrayed as the solving of proportionality problems where the cross-multiple rule is used to solve problems demanding only that one knows the algorithm. However, the algorithm involves only the quantitative side of proportional reasoning. Thompson and Bush (2003:400) warns that the study of proportions should not be equated to the study of the cross-multiple rule only. It is not dependent on a skill with a mechanical or algorithmic procedure (Math Vantage, 1998). Teaching students to solve proportions

by using the cross-product method alone does not develop the student's proportional reasoning skills.

A study by Moore et al. (1991:441) led to the distinction between intuitive knowledge and formal computational knowledge of proportional reasoning. An intuitive understanding involves estimation or intuition, whereas a computational understanding only involves explicit calculations. Mapping relations from one domain onto another requires understanding of how variables function and an intuitive understanding of the computational task. If intuitive understanding is not present, then the computational scheme is based primarily on the memory available for mathematical operations.

An intuitive understanding of ratios (qualitative reasoning) should precede numerical proportional exercises, and even small increases in familiarity of the problem context and units may be helpful in teaching ratios (Heller et al. 1989:219). Premature use of rules without experiences with intuitive and conceptual methods encourages learners to apply rules without thinking, and thus the ability to reason proportionally often does not develop (Van de Walle, 2004:300). It is inadequate to describe computational proportional reasoning as automatically arising from mature intuitive proportional reasoning (Moore et al., 1991:441). Cramer and Post (1993b) suggest that teachers should begin with more intuitive strategies and focus learners' attention to multiple strategies for any given problem. This helps emphasise learning concepts over procedures. Van de Walle (2004:301) also advocates informal activities to develop an intuitive concept of ratio and proportion. These informal activities must then be followed by a more informal approach before the introduction of the cross-multiply procedure. The informal approaches involve working with "within ratios" and "between ratios" (see § 4.4.3.1.3).

Researchers found that the problem context and the nature of the numerical relationship influenced problem difficulty (Heller et al., 1989:205; Cramer and Post, 1993a). Instruction in proportional reasoning should therefore start with familiar contexts and extend to the unfamiliar. A proportional reasoner should not be affected by an awkward numerical relationship or the problem context in which the problem is situated. Tourniaire and Pulos (1985:191) and McLaughlin (2003) investigated many variables affecting a person's ability to reason proportionally. Among these is the M-capacity, which reverses to one's ability to attend to a number of schemes at the same time. There are relationships between M-capacity and ability to reason proportionally for specific types of proportional problems. Different kinds of proportional reasoning problems will be the focus of the next section.

4.4.3.1.3 Different kinds of proportional reasoning problems

It is useful to differentiate among three main categories of proportional reasoning problems namely, missing value problems, numerical comparison problems and qualitative prediction / reasoning problems.

Within the category of missing value problems, a sub-distinction can be made between within ratios and between ratios (Van de Walle, 2004:309; Post et al., 1988). These kinds of ratios can be used in an informal approach to the development of proportional reasoning – that is, before the cross-product algorithm is employed.

A ratio of two measures in the same setting is a within ratio (see Question 27.1 in Appendix C). These problems have the most intuitive appeal (Post et al., 1988). It is easy to determine the cost of one sweet (the unit rate or unit price) – this is referred to as the unit-rate method of solving proportions. Research by Cramer et al. (1989:537-539) indicates that the unit rate approach was found to be more meaningful than the traditional cross-multiplication algorithm because students have an intuitive understanding of unit rate from their shopping experiences. This makes for a connection to a familiar problem context.

A between ratio is a ratio of two corresponding measures in different situations (see Question 27.2 in Appendix C). A unit price should be used, but the division $\frac{R2}{3}$ is not easy. Because 9 is a multiple of 3, it is easier to intuitively notice that the cost of 9 will be 3 times the cost of 3, which is $3 \times R2$. This is called the factor-of-change method (“times as many”).

According to Post et al. (1988) and Cramer and Post (1993b), in the past, students who were able to answer the numerically awkward situations containing non-integer multiples within and between the rate pairs in missing value problems were thought to be at the highest level and proportional reasoners. However, this is a necessary but not sufficient condition because these kinds of problems lend themselves to purely algorithmic and possibly rote solutions. Proportional reasoning is in the first place a case of intuitively recognising a proportional situation and not of computational fluency.

In certain situations the unit price or factor of change methods cannot be used. This calls for the use of the cross-product algorithm (see Question 27.3 in Appendix C). The problems discussed previously in this section are all examples of missing value problems - that is, where three values in two rate pairs are given and the fourth is to be found.

Numerical comparison problems are a second type of proportionally-related problems, which involves the comparison of two rates (see Questions 22 and 24 in Appendix C). In these kinds of problems, the unit rate approach can be used to generate two unit rates that can then be compared. An important aspect that comes into play here is the relationships between a unit and the parts it consists of – an important aspect in proportional understanding. There are always two unit rates for a given rate pair, each being the reciprocal of the other (Post et al., 1988). One is usually more useful and more easily interpretable than the other. Being able to reflect on the answer and correctly interpreting an answer reflects understanding of the proportional relationship given.

Qualitative prediction/reasoning proportional problems refer to problems that require a comparison, not depending on specific values (see Questions 18 and 29.2 in Appendix C).

4.4.3.1.4 The importance of proportional reasoning in mathematics

Indications are that proportional reasoning is among the most well researched ideas in school mathematics due to its importance for various topics. Researchers working on proportional reasoning constantly refer back to the work of Jean Piaget as the starting point on research on proportional reasoning. Piaget's main interest was developmental psychology and knowledge growth (Smith, 2000, McLaughlin, 2003:4). Piaget identified proportional reasoning as an important aspect in the developmental stages of a child.

More recently, valuable research on proportional reasoning has been done by researchers (see over 86 research publications by Behr, Cramer, Harel, Lesh, Post and others) involved in the rational number project (Post, 2002). Lesh et al. (1988) view proportional reasoning as a pivotal concept that is the capstone of the elementary school curriculum and the cornerstone of high school mathematics and science. The following topics in the school curriculum is based on proportional reasoning: division (multiplication), fractions (equivalent fractions and ratios, part-whole relationships, percent), proportional parts, similarity and similar figures and their area and volume relationships, circle graph computations, data graphs (relative frequency histogram and the box-and-whisker plot), probability, algebra (rates of change, multiplication by unit rates), the study of linear equations, slope, graphing and scaling or scale relationships, rational numbers and expressions (Van de Walle, 2004:298; Thompson & Bush, 2003:399; Math Vantage, 1988; Horak, 2006:361, Wiebe, 2006). Proportional reasoning is also necessary for solving problems in all branches of science (Thompson & Bush, 2003:399; Horak, 2006:360; McDermott et al., 1987:503). Heller et al. (1989:205) point out that proportionality is one of the most ancient and fundamental connections between math and science. Evidence thus exists that proportional reasoning are at the heart of the school mathematics curriculum.

4.4.3.2 Functions

An understanding of different representations of mathematical knowledge was identified as an important aspect of MCKfT and therefore identified as one of the attributes (see § 3.9). Functions is an ideal topic for testing the mathematics teachers' representational knowledge states.

4.4.3.2.1 A description of what a function is

In the simplest sense, a function expresses a relationship between two different quantities. Van de Walle (2004:436) describes functions as relationships or rules that uniquely associate members of one set with members of another set. One experiences the function concept whenever you consider how change in one variable can cause or have a corresponding effect on another. If you think "The faster I drive, the quicker I will arrive at Spar", you have expressed the idea of a function. You are stating that there is a relationship between the speed at which you drive and the time it takes before you arrive at your destination.

Functional relationships can be expressed in different representations. Each different representation (language, context, table, equation, graph) is simply a different way of expressing the same idea. The five different ways to interpret or represent a function were stated and described in paragraph 3.6.5.

4.4.3.3 The importance of graphs and functions

The concept of functions is one of the big ideas in mathematics. Algebraic reasoning involves a search for regularity in mathematics. According to Van de Walle (2004:436) functions is one of the most powerful tools in this endeavour. Van Dyke (2002:vii) notes that for the past several years, many articles have been written urging us to make functions a central theme in high school algebra. It has been convincingly argued that students will learn mathematics best if it is presented in the context of a meaningful application. Science also aims to understand the relationship between different quantities and to make predictions based on this understanding. As with proportional reasoning, functions is also an important part of understanding a real-life context necessary for solving problems in all branches of science. McDermott et al. (1987:513) states that the ability to draw and interpret graphs is perhaps one of the most important among the many skills that can be developed in the study of physics. The importance of functions in science can thus not be overstated. Functions are the tools used for mathematical modelling in all types of real world change.

4.4.4 The Measurement instrument

To investigate the established attributes, the researcher focussed on a specific group of teachers, using questions based on proportional reasoning and functions. There was no standardized instrument in the literature that could give sufficient information about the pre-determined attributes that had to be investigated in the teachers' mathematical content knowledge states. A 34-item diagnostic test (instrument) was constructed by the researcher with some items adapted from sources on proportional reasoning and functions (see Appendix C).

4.4.4.1 The purpose of the diagnostic test

The entire empirical research effort depended on the constructed diagnostic test (measurement instrument). Although the procedure (the construction and use of this measurement instrument) is very subjective in nature because of the personal view of the researcher and various unknown variables that might be involved, Bart et al. (1994) nevertheless states that written answers by students can provide detailed information about student thought processes.

The purpose of using such a measuring instrument is to be able to identify and analyse at best the teachers' current mathematical content knowledge states according to specific attributes that were essential in terms of MCKfT. Its main goal was to provide information concerning students' functioning in the area of proportional reasoning and functions.

4.4.4.2 The nature of the diagnostic items / Instrument specifications

The diagnostic test items were set for a specific target group, namely secondary school mathematics teachers. There was a time limitation on completing the items. Teachers were allowed to rely only on their prior knowledge to complete the test. The format of the items was mostly free response questions and also some multi-choice questions. The purpose of the items was to measure the assigned attributes.

The items had to meet the following important requirements:

- Teachers' responses to the items had to reflect information about each attribute (in terms of MCKfT) that was assigned to each of the items.
- The items had to reflect information about a teacher's MCKfT on proportional reasoning and certain aspects of functions.

The final and complete diagnostic test instrument with information on the origin of each item is shown in Appendix C.

4.4.4.3 Mathematical diagnostic items on proportional reasoning

The purpose of these items was to evaluate teachers' understanding of proportional reasoning in different settings of proportional reasoning problems. Proportional problems were divided into two broad categories, namely numerical problems with computation encouraged, and problems using qualitative or intuitive reasoning. Each can be subdivided into two smaller sub-categories:

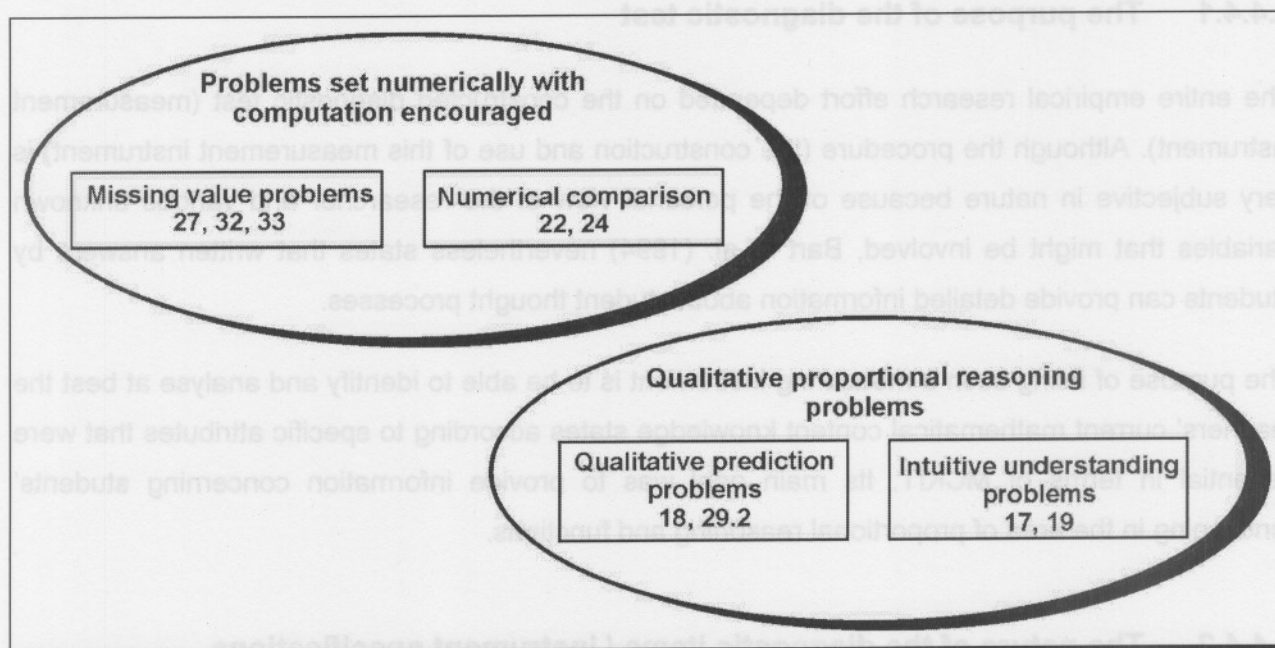


FIGURE 25: Different categories of proportional reasoning problems used in the diagnostic instrument.

Questions 27, 32 and 33 are in the category of missing value problems. The standard algorithm taught to solve missing value proportional problems involves setting up a proportion and using the cross-product algorithm. Questions 32 and 33 are classical proportional missing value problems also used by Jean Piaget (as quoted by Mayer, 2003; Singh, 2000:582). Question 27 was also chosen to be able to examine the strategies (within ratio, between ratio or cross-product algorithm) used to solve these kinds of problems.

Questions 22 and 24 are numerical comparison problems where comparing given ratios can lead to the correct answer. In these items, the relationships between the unit of measurement and its parts needed to be understood for the teacher to provide the correct interpretation of the answer.

There are always two unit rates for each rate pair, one being the reciprocal of the other (see § 3.8.1). For Question 22 (see Appendix C), the ratio can be expressed in two ways:

- $\frac{R6}{4 \text{ kg}}$ tomatoes (interpreted as R1.50 for 1kg of tomatoes, the result of dividing $\frac{600}{4} = 1.50$) or
- $\frac{4 \text{ kg}}{R6}$ tomatoes (interpreted as 0.666666.....kg per R1 of tomatoes, the result of the division $\frac{4}{600} = 0.666666.....$).

The choice of ratio influenced the interpretation that was given in each case.

Questions 18 and 29.2 fall in the qualitative prediction (reasoning) category. These items are not represented numerically and no computations are needed to solve the problem.

Questions 17 and 19 also fall into the type of question where intuition is required. These items are difficult problems which require higher levels of reasoning and overlap with the missing value category. These two items are special in the sense that an intuitive understanding is needed of proportional reasoning to be able to recognise that these problems are set in a proportional situation. Responses to Question 17 (see Appendix C) can be put into two categories, namely:

- An answer indicating that both grew the same amount which is 4 cm. This response is based on additive reasoning where a single quantity was added to each measurement to result in the two new measurements.
- A second way to look at the problem is to compare the amount of growth to the original height of the flower. The first one grew $\frac{4}{8}$ of its height and second one grew $\frac{4}{12}$ of its height.

Based on this multiplicative view the first flower grew more.

Van de Walle (2004:301) states that an ability to understand the difference between these situations is an indication of proportional reasoning.

Proportional reasoning problems where qualitative reasoning is required, calls for the ability to interpret the meaning of two ratios, store that information, and then compare these interpretations according to some predetermined criteria. This situation requires multi-level comparative thinking different from an algorithmic approach where a rule is used to solve predictable problems in

predetermined ways (Post et al. 1988). The mental capability needed to solve these kinds of problems – a process of operating on operations - might be problematic to most students of mathematics. However, mathematics teachers should be classified as proportional reasoners where MCKfT is concerned.

4.4.4.4 Mathematical diagnostic items on proportional reasoning in a percentage context

Questions 1 and 31 (see Appendix C) were used to search for specific connections and understanding a specific representation in the teachers' current mathematical content knowledge states. Points representing equivalent fractions, ratios, or rate pairs will define a straight line through the origin (see § 3.8.2). The linear function in Question 1 is represented by $y = \frac{11}{100}x$.

4.4.4.5 Mathematical diagnostic items on proportional reasoning in a function context

The purpose of these items is to evaluate teachers' proportional reasoning in the context of functions. Questions 4, 6 and 28 (see Appendix C) formed part of this evaluation. A link between conceptual understanding and the use of correct procedures in the specific problem context were investigated. The focus was also on looking for teachers' unpacking of their understanding in these situations.

4.4.4.6 Mathematical diagnostic items on different representations of functions

Questions 2, 3, 7, 8, 9, 11 and 14 (see Appendix C) formed part of this sub-group. The purpose of some of these items was to investigate understanding in making connections between a graphical representation and the real-life situation it represents.

In Question 14, teachers' ability to "read" an answer from a graphical representation of the mathematical information was investigated. According to McDermott et al. (1987:504) it is more difficult to interpret curved graphs than straight line graphs.

Questions 8 and 9 focused on the task of matching the information in a narrative passage to a graphical representation.

4.4.4.7 Mathematical diagnostic items on familiar/unfamiliar problem contexts

Questions 5 and 10 (see Appendix C) focused on teachers' ability to solve problems in familiar and unfamiliar problem contexts. Question 10 was constructed in such a format that only certain features of the problem setting were needed to activate appropriate procedural knowledge to solve the problem. To solve Question 5, the same procedural knowledge was required, but the problem setting involved a very different representation.

4.4.4.8 Mathematical diagnostic items on ratios in a probability and combinations context

The purpose of Questions 29 and 34 (see Appendix C) were to look for connections in knowledge on ratios represented in the contexts of probability and combinations.

A full explication of which attributes were investigated by means of which items, will be provided in the next section.

4.4.5 The item-attribute matrix

4.4.5.1 The purpose of the item-attribute matrix

An item-attribute matrix was established to provide a way in which the teacher's written responses on the measurement instrument could be evaluated.

4.4.5.2 Obtaining an item-attribute matrix

The rows of the item-attribute matrix consist of the items developed and the columns contain the attributes of each item. The attributes that were assigned to specific items, reflected the mathematical content, cognitive processes and problem contexts that a teacher had to know in order to complete each item successfully. The item-attribute matrix is thus a reflection of the relationship that the constructed attributes had with the constructed items.

Although the teachers completed all 34 items in the diagnostic instrument, only 23 items were chosen to be analysed as they provided the necessary information for this study (see § 4.4.4).

Some of the excluded items were familiar to the teachers in Group 2 who have completed the Trigonometry module. The 23 chosen items and the attributes linked to them are provided in the next matrix.

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TABLE 3: The item-attribute matrix

Items / Questions	ATTRIBUTES										
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11
	Conceptual knowledge	Procedural knowledge	Procedure solved	Interpretation of answers	Problem solving	Representations	Communication	Reasoning	Connections	Unpacking	Problem contexts
1		X	X			X			X		
1.1			X			X					
1.2		X	X						X		
2						X	X		X		
2.1						X					
2.2									X		
2.4							X				
3						X	X		X		
3.1						X					
3.2						X					
3.7							X		X		
4	X	X	X	X	X					X	
4.1					X						
4.2	X	X	X	X						X	
5		X	X	X	X	X				X	X
5.1						X					X
5.2		X								X	
6	X					X	X			X	X
6.1	X					X					X
6.2							X			X	
7		X	X			X					
7.1		X	X			X					
7.3		X	X								
8						X					

8.1(B)						X					
8.2(F)						X					
8.3(A)						X					
8.4(E)						X					
8.5(C)						X					
9						X	X				
9.1.1(D)						X	X				
9.1.2(H)						X	X				
9.1.3(I)						X	X				
9.2(A)						X	X				
10.1		X	X								X
11						X	X			X	X
11.1						X					
11.2							X			X	
14						X					
14.1						X					
18							X	X			
18.1								X			
18.2							X				
19	X	X	X	X	X					X	
19.1	X										
19.2		X	X	X	X					X	
22	X	X	X	X	X					X	
22.1	X										
22.2		X	X	X	X					X	
24	X	X	X	X	X					X	
24.1	X										
24.2		X	X	X	X					X	
27.1		X	X								
27.2		X	X								
27.3		X									

TABLE 3: The non-additive matrix

CHAPTER 4:

28						X	X			X	X
28.1						X					
28.2							X			X	X
29		X					X	X	X	X	
29.1		X							X		
29.2								X			
29.3							X			X	
31		X					X				
31.1		X									
31.2							X		X		
32	X	X	X	X	X					X	
32.1	X										
32.2		X	X	X	X					X	
33	X	X	X	X	X					X	
33.1	X										
33.2		X	X	X	X					X	
34		X							X		
34.1		X									
34.2									X		

Proportional reasoning in a percentages context

Different representations of functions

Proportional reasoning problems

Familiar/Unfamiliar problem contexts

Proportional reasoning in a functions context

Proportional reasoning in a probability and combinations context

4.5 THE EXECUTION OF THE EXPERIMENT

4.5.1 The diagnostic content analysis

It has already been indicated that a large part of the empirical study consists of a diagnostic assessment also referred to as a qualitative diagnostic content analysis. The components needed for conducting the first three steps of the content analysis were already discussed in § 4.4. As indicated in § 4.2.3, the last two parts of a diagnostic content analysis consists of the following:

Step 4: Tabulations of the frequency or percentages of each characteristic. This part refers to the collection of the data for this study (see § 4.5.2).

Step 5: Descriptive or inferential statistical analyses as needed too answer the research question. This part refers to the analysis of the data (see § 4.5.3).

The collection and analysis of the data as well as other issues that form part of the empirical design will be discussed in the following section. An outline of the successive parts that were followed for the empirical study will also be provided.

4.5.2 The collection of the data and the scale of measurement

The research instrument was written by Group 1 (first year group) and Group 2 (second year group) in September 2005. Group 3 (new recruits) wrote the diagnostic test in January 2006 (see information on the groups in paragraph 4.4.2). The collection of the actual data (diagnostic evaluation of the 128 teachers' written responses) was done solely by the researcher. These diagnostic evaluations provided the data that the results of this study are based on.

The scale of measurement ultimately dictates the statistical procedures that one uses in processing the data. According to Leedy and Ormrod (2001:156), a crucial step in a diagnostic content analysis is to tabulate the frequency of each characteristic (attribute in this case) found in the documents being studied.

Because a diagnostic content analysis is both qualitative and quantitative in nature, teachers' written responses to the measurement instrument had to be quantified. The scoring criteria or point scale used was as follows:

- A “1” was also awarded if an item was correct and “0” if an item had a wrong answer.
- A “1” was awarded if evidence appears of the existence of an attribute or “0” if no evidence shows the existence of an attribute.

4.5.3 Analysis of the data

4.5.3.1 The item-attribute matrices

The 128 teachers’ written responses on the diagnostic instrument (items) were quantified, using the method described above, into an item-attribute matrix for each of the 128 teachers. The item-attribute matrices then provided frequencies or percentages for each correct item and observed attribute (see Chapter 5 for these results).

The frequencies or percentages for each correct item and observed attribute were used in reporting and describing the data in terms of observed mathematical content knowledge states. These frequencies were also used to conduct the quantitative part of the research in an organized fashion. Appropriate statistical analyses were performed on the frequencies or percentages obtained. The researcher then used these tabulations and statistical analyses to interpret the data as they reflect on the mathematical content knowledge states of the group of teachers under investigation. Trends could be identified in the data through a description of the patterns that the data reflected.

4.5.3.2 The successive parts of analyzing the data

The successive parts were:

1. A qualitative diagnosis content analysis of items correct / incorrect. A teacher gets a “1” if the answer of an item was right. A teacher gets a “0” if the answer to an item was wrong.
2. A qualitative diagnosis content analysis of attributes obtained that were assigned to the items. A teacher gets a “1” if evidence exists that an assigned attribute was obtained. A teacher gets a “0” if no evidence exists that an assigned attribute was obtained. Criteria for the assignment of an attribute are given in Chapter 5 together with the results for each item.
3. A quantitative analysis of the internal reliability of the instrument. The diagnostic assessment in parts 1 and 2 yielded results in terms of 1’s and 0’s. These results were used to determine the internal reliability of the instruments, indicating how consistent each item and attribute

was measured and also how well assigned attributes measured against the total item score (see results in § 5.2).

4. A quantitative diagnostic content analysis of the different groups of teachers' performance on the attributes (see results in § 5.3).
5. A quantitative diagnostic content analysis of the different groups of teachers' performance on the different categories of items (see results in § 5.4).
6. A qualitative diagnostic content analysis of teacher's responses in each category of the mathematical knowledge tested (see results in § 5.5).

Qualitative and various quantitative methods had to be used to conduct the above parts of the analysis. The qualitative parts followed by the quantitative methods will be discussed in the next section.

4.5.3.3 The qualitative analysis

A qualitative analysis was done in two phases of the diagnostic content analysis. Firstly, a descriptive content analysis of the mathematics teachers' understanding of items in terms of the assigned attribute was conducted.

Secondly, in order to understand the results more clearly, an item analysis was conducted. The items were analysed according to the categories established previously (see § 4.4.4). The categories are:

- proportional reasoning problems with subdivisions
 - missing value problems
 - numerical comparison
 - qualitative category;
- proportional reasoning in a percentages context;
- proportional reasoning in a functions context;
- different representations of functions;
- familiar/unfamiliar problem contexts;
- proportional reasoning in a probability and combinations context.

The results are represented in the above categories (see § 5.5).

4.5.3.4 The quantitative analyses

Quantitative analyses had to be done on the data in order to determine:

- the internal reliability of the items and the attributes;
- statistical significant differences among the results obtained by the three different groups;
- the 70% percentile profiles on teachers' performance in the items and the attributes.

4.5.3.5 Reliability and validity of the research instrument

The reliability and validity of any measurement instrument influence the extent to which one can learn something about the aspects one is studying. It refers to the extent to which you can draw meaningful conclusions from your data (Leedy & Ormrod, 2001:31) if statistical significance was obtained. Therefore it was important to subject the research instruments to tests on its reliability and validity.

4.5.3.5.1 The reliability of the instrument

Reliability refers to the consistency with which a measuring instrument yields a certain result when the entity being measured has not changed (Leedy & Ormrod, 2001:31). The diagnostic items were designed for the sole purpose of this study. Therefore it was an unstandardized test only earmarked for the internal use of diagnosing a group of teachers' mathematical content knowledge for teaching. A statistical technique had to be used to determine the reliability of the items and attributes measured. The statistical technique that was used to determine the internal consistency reliability of the diagnostic instrument is Cronbach's alpha, α (see § 4.6.1), which is the most common measure of scale reliability (Field, 2005:667).

4.5.3.5.2 The validity of the instrument

Validity of a measurement is the extent to which the instrument measures what it is supposed to be measuring (Leedy & Ormrod, 2001:31). The researcher had to make sure that the diagnostic items were valid for measuring the established attributes (see § 4.4.1.3) latent in the constructed items on proportional reasoning and functions (see § 4.4.3 and Appendix C).

4.5.3.5.3 The validity of the content

Content validity is the extent to which a measurement instrument is a representative sample of the content area being measured (Leedy & Ormrod, 2001:98). Content validity is a consideration when we want to assess people's achievement in some area. The validity of the content of the instrument was evaluated by a group of experts who were considered to be knowledgeable about the content validity and wording of each item. The validity of the attributes that were assigned to each item was also judged by the group of experts. These experts consist of university faculty members who are mathematicians and mathematics educators at the North-West University (Potchefstroom Campus). There was agreement that the content of the instrument was valid for measuring the assigned attributes.

4.6 STATISTICAL TECHNIQUE

A small part of the content analysis was done in a quantitative way. The statistical techniques that were used will be discussed in the next section.

4.6.1 Statistical techniques for measuring reliability

The reliability coefficient and the standard error for the items and assigned attributes for the instrument were computed using Cronbach's alpha. For each item, the variance within the item, and the covariance between a particular item and any other item on the scale can be calculated. We can thus construct a variance-covariance matrix of all items. (Field, 2005:667). Cronbach's

alpha, α , is a measure of the reliability of a scale defined by: $\alpha = \frac{N^2 \overline{\text{Cov}}}{\sum s_{\text{item}}^2 + \sum \text{cov}_{\text{item}}}$ in which the

top half of the equation is the number of items (N) squared multiplied by the average covariance between items (the average of the off-diagonal elements in the variance – covariance matrix). The bottom half is the sum of all the items variances and item covariance (i.e. the sum of everything in the variance – covariance matrix).

A value of 0.5 is seen as an acceptable value of α for cognitive tests such as intelligence tests (Field, 2005:668).

4.6.2 Techniques for measuring statistical significance among or between the responses of different groups

There was a need to look for statistically significant differences among the results obtained in the different groups on the items and the attributes. A statistical significance test (t-test) is used to show that the results (that is, the difference between two means) are significant (Ellis & Steyn, 2003:51). Analysis of variance or ANOVA was used to analyse situations in which there are more than two groups of people under investigation (Field, 2005:309).

The p-values are criterions of this, giving the probability that the obtained value could be obtained under the assumption that the null hypothesis (e.g. no difference between the population means) is true (Ellis & Steyn, 2003:51). A p-value smaller than 0.05 is considered as sufficient evidence that the results are statistically significant. Statistical significance is strictly speaking not relevant as the study population did not consist of randomly chosen groups. Instead of only reporting descriptive statistics, effect sizes are calculated to determine if the groups differed in practice.

Cohen's d is a measure of effect size which is useful because it provides an objective measure of the importance of an effect (Field, 2005:32). Cohen's d is given by the following formula:

$$d = \frac{|\bar{x}_1 - \bar{x}_2|}{S_{\max}} \quad \text{where } |\bar{x}_1 - \bar{x}_2| \text{ is the difference between } \bar{x}_1 \text{ and } \bar{x}_2 \text{ without taking the sign into}$$

consideration. When no control group exists, the division by $S_{\max}$ gives rise to a conservative effect size in the sense that a practically significant result will not be concluded too easily.

Cohen (1988) (as quoted by Ellis & Steyn, 2003:51), gives the following guidelines for the interpretation of the effect size:

- small effect: $0.2 \leq d < 0.5$;
- medium effect and possibly practically significant: $0.5 \leq d < 0.8$;
- large effect and practically significant: $d \geq 0.8$.

4.6.3 The 70% percentile range

There was a need to have an overview of teachers' responses to the items and attributes. The 70% percentile can be defined as the value to which 70% of the respondents adhere to. A 70%

percentile range was decided on as this seems to be an appropriate level of mathematical proficiency for a mathematics teacher in mathematical topics (Maree & Crafford, 2005:90).

4.7 ETHICAL ISSUES

Participants were given a letter to ask for their consent in writing the diagnostic instrument. They all signed a letter of consent. The researcher is in possession of these letters. See Appendix A and B for the letters of consent.

4.8 CONCLUSION

The experimental design and statistical techniques were discussed in this chapter. The next chapter consist of the results obtained during each part of the diagnostic content analysis and other techniques described in this chapter.

CHAPTER 5

RESULTS OF THE EMPIRICAL STUDY

5.1 INTRODUCTION

In Chapter 4 a complete description of all the aspects of the empirical study was given. The aim of this chapter is to give the results of the diagnostic content analysis of the 128 teachers' mathematical content knowledge states. The results of the successive parts of the empirical study, which includes the qualitative and quantitative processes of data handling, will be reported in the following order:

1. The results of a quantitative analysis of the internal reliability of the designed research instrument (diagnostic test) in terms of the items and in terms of the assigned attributes (§ 5.2).
2. The results and the interpretation of the results of a quantitative diagnosis content analysis of the attributes obtained that were assigned to the items (§ 5.3). Criteria for the assignment of an attribute are provided as part of the analysis for each item.
3. The results and the interpretation of the results of the quantitative diagnosis content analysis of teachers' performance on the items (§ 5.4). The analysis and interpretation of the results will be given according to the categories described in paragraph 4.5.3.3.
4. The results and the interpretation of the results of the qualitative diagnosis content analysis of teachers' responses to the items (§ 5.5). The qualitative analysis and interpretation of results are also given according to the different categories of mathematical problems. Some examples of individual teachers' responses to items will also be provided.

5.2 THE INTERNAL RELIABILITY OF THE DIAGNOSTIC INSTRUMENT

The results of the quantitative analysis on the internal reliability consistency (measured in terms of Cronbach's alpha) of the items and attributes for the group of 128 teachers is given in the following tables. Table 4 shows Cronbach's alpha values for the items in terms of the assigned attributes.

TABLE 4: Cronbach's alpha values for the items in terms of the attributes.

Question	1	2	3	4	5	6	7	8	9	10	11	14
Cronbach's alpha	0.80	0.44	0.65	0.93	0.93	0.98	0.92	0.41	0.87	0.85	0.82	Not determined
Question	17	18	19	22	24	27	28	29	31	32	33	34
Cronbach's alpha	Not determined	0.68	0.40	0.96	0.94	0.78	0.94	0.76	0.63	0.99	0.98	0.61

Except for Questions 2, 8 and 19, the rest of the values were significantly above a Cronbach alpha value of 0.5 ($\alpha > 0.5$) (see § 4.6.1). Question 19 forms part of the qualitative analysis where it will become clear why this item didn't have an appropriate Cronbach alpha value. Questions 2 and 8 are included in the analysis on different categories of items because Cronbach's alpha value for the function category was appropriate. Cronbach's alpha values could not be determined for Questions 14 and 17. Question 14 only represented the response to one of the attributes, which makes it inappropriate to determine Cronbach's alpha. Responses to Question 17 yielded nearly no correct answers and therefore Cronbach's alpha could not be determined. Responses to these two items will, however, be discussed as part of the qualitative analysis on the items.

Table 5 shows Cronbach's alpha value for the assigned attributes in terms of the items.

TABLE 5: Cronbach's alpha values for the attributes in terms of the items.

Attributes	1	2	3	4	5	6	7	8	9	10	11
Cronbach's alpha	0.56	0.75	0.71	0.61	0.62	0.68	0.60	0.07	0.43	0.66	0.25

Table 5 is a reflection of how consistent assigned attributes was measured. Except for Attribute 8 (Reasoning), Attribute 9 (Connections) and Attribute 11 (Problem contexts), the rest of Cronbach's alpha values were all above 0.5 ($\alpha > 0.5$) (see § 4.6.1).

Cronbach's alpha provided evidence for the internal reliability of the diagnostic instrument (in terms of most of the attributes and items) for this specific group of teachers. Explanations can be given for inappropriate Cronbach alpha values for some of the attributes and items. These explanations will form part of the qualitative analysis on attributes and items.

5.3 THE QUANTITATIVE DIAGNOSTIC CONTENT ANALYSIS ON THE ATTRIBUTES

This section will consist of a broad overview of teachers' performance in the attributes. The following quantitative analysis and results were only done on Attributes 1, 2, 3, 4, 5, 6, 7 and 10 because of appropriate Cronbach alpha values.

An ANOVA test was performed to look for statistically significant differences among the responses of the three groups of teachers. Table 6 is a reflection of the means per attribute obtained by teachers in Group 1, Group 2, Group 3 and the whole groups of 128 teachers.

No significant differences exist between the results from Group 1 and Group 2. These groups consist of teachers at different stages in the Sediba Project. A decision was made to combine these two groups in doing a t-test to determine p-values and ultimately Cohen's d-values, which is a reflection of the effect size between the responses of Group 1 & 2 and Group 3.

TABLE 6: Means per attribute obtained by the teachers.

N	Attribute 1 Means	Attribute 1 Std. Dev.	Attribute 2 Means	Attribute 2 Std. Dev.	Attribute 3 Means	Attribute 3 Std. Dev.	Attribute 4 Means	Attribute 4 Std. Dev.	Attribute 5 Means	Attribute 5 Std. Dev.	Attribute 6 Means	Attribute 6 Std. Dev.	Attribute 7 Means	Attribute 7 Std. Dev.	Attribute 10 Means	Attribute 10 Std. Dev.
Group 1 (N=37)1	50.19305	26.85765	58.34658	18.62117	54.26195	24.18602	34.23423	26.91606	37.38739	27.04746	53.48506	17.99907	34.45946	20.80203	30.71253	22.30428
Group 2 (N=45)	48.57143	27.47912	53.98693	16.79254	49.91453	19.85844	28.88889	24.46395	31.11111	25.52479	49.23977	17.85372	32.77778	17.89066	25.25253	18.47785
Group 3 (N=46)	37.57764	18.65274	37.97954	16.39134	31.10368	17.90886	17.02899	18.08387	18.84058	19.11909	36.04119	12.79463	26.81159	16.75099	17.78656	13.81453
All groups (N=128)	45.08929	24.93363	49.49449	19.21444	44.41106	22.78668	26.17188	24.09187	28.51563	24.93670	45.72368	17.78090	31.11979	18.54520	24.14773	18.83016

TABLE 7: t-values, p-values, Cohen's d-values and effect sizes of teachers' performance in the attributes

	Mean Group 1 & 2	Std.Dev. Group 1 & 2	Mean Group 3	Std.Dev. Group 3	t-value	p-value	Cohen's d-value	Effect size
Attribute 1	49.30	27.04	37.58	18.65	2.61	0.0101	0.43	Small
Attribute2	55.95	17.67	37.98	16.39	5.67	0.0000	1.02	Large
Attribute 3	51.88	21.88	31.10	17.91	5.49	0.0000	0.95	Large
Attribute 4	31.30	25.58	17.03	18.08	3.34	0.0011	0.56	Medium
Attribute5	33.94	26.25	18.84	19.12	3.42	0.0008	0.58	Medium
Attribute 6	51.16	17.93	36.04	12.79	5.04	0.0000	0.84	Large
Attribute 7	33.54	19.15	26.81	16.75	1.99	0.0486	0.35	Small
Attribute 10	27.72	20.35	17.79	13.81	2.95	0.0038	0.49	Small

Table 7 represents teachers in the different groups' responses to the diagnostic items in terms of the attributes. The results in the table above show large differences between Group 1 & 2 and Group 3's responses concerning Attribute 2 (Procedural knowledge), Attribute 3 (Procedure solved) and Attribute 6 (Representations). Medium differences between Group 1 & 2 and Group 3's responses concerning Attribute 4 (Interpretation of answers) and Attribute 5 (Problem solving) are reflected by the results. For Attribute 1 (Conceptual knowledge), Attribute 7 (Communication) and Attribute 10 (Unpacking), small differences between Group 1 & 2 and Group 3's responses are shown.

A 70% percentile threshold was then determined according to the groups' performance according to the attributes. Figure 26 shows that there was no significant difference in performance on the attributes for Group 1 and Group 2. This was to be expected according to the explanation given in § 4.4.2. However, there were statistically significant differences between the performances of Group 1 & 2 and Group 3.

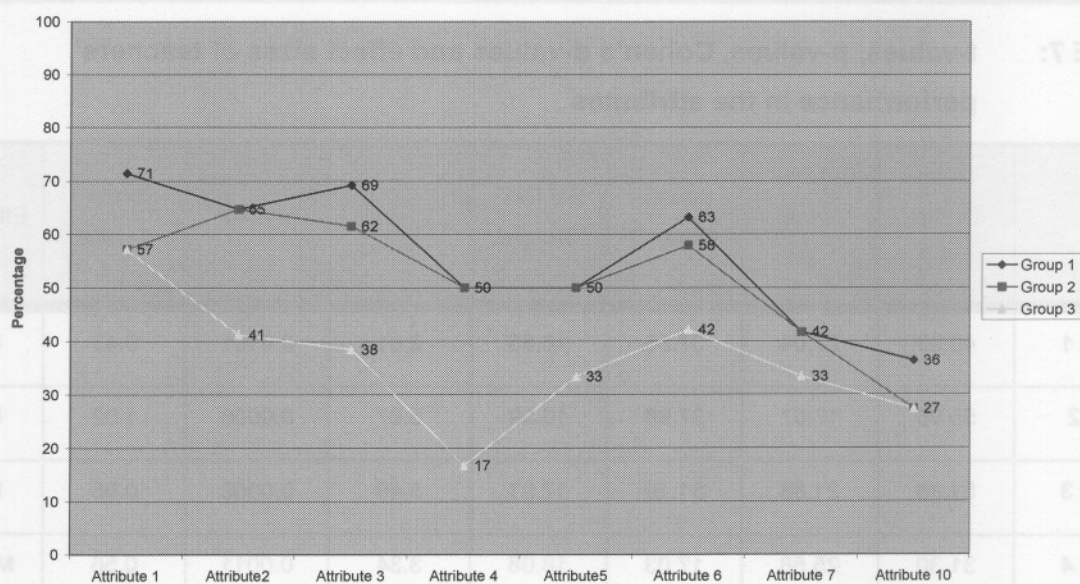


FIGURE 26: 70% Percentile profiles of the groups in terms of the attributes

In the next graph, the performances for Group 1 & 2 are illustrated together.

70% Percentile profile of the groups in terms of the attributes

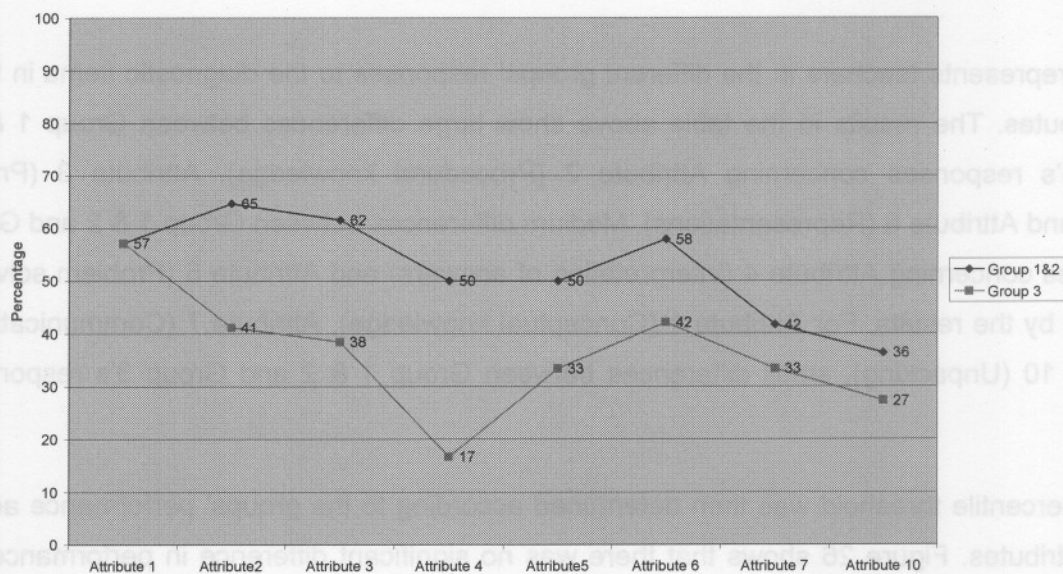


FIGURE 27: 70% Percentile profiles of the groups in terms of the attributes

The graphs in Figure 26 and Figure 27 show two very important aspects of the results of this study. A hopeful sign is the difference in the profiles of the teachers who had been part of the Sediba Project (Group 1 & 2) and those who had not been exposed to any mathematical modules offered through the Sediba Project (Group 3). Secondly, it is alarming that none of these groups of teachers obtained a value of 70% on the percentile profile on any one of the attributes, which reflects important aspects of MCKfT in proportional reasoning and functions.

Attribute 1 which is an indication of teachers' performance on four different categories of proportional reasoning problems, is portrayed as an aspect where nothing happened during exposure to an in-service training program to improve teachers' knowledge of proportional reasoning. Because the Sediba Project focuses on teachers who are currently teaching Grade 12 learners, proportional reasoning per se does not form part of their mathematical modules. Expectations are that a teacher teaching Grade 12 learners will be competent in important concepts like proportional reasoning which actually forms part of the middle school curriculum. Indications are that these teachers cannot be classified as proportional reasoners and that their proportional reasoning skills need to be addressed at this level of training, as proportional reasoning forms an important part of mathematical performance across the mathematics curriculum. A teacher who is not a proportional reasoner will not be able to assist a learner to become a proportional reasoner. The results show that 70% of the group of 128 teachers had 57% or less for items measuring proportional reasoning skills.

For Attribute 2 and Attribute 3, practically significant differences in the groups' performances are evident. Indications are that Group 1 & 2s' teachers benefited a great deal with regard to aspects of procedural proficiency. 70% of teachers in Group 1 & 2 obtained 65% or less on items measuring procedural knowledge, while 70% of teachers in Group 3 obtained only 41% on these items. 70% of teachers in Group 1 & 2 obtained 62% or less on items measuring teachers' ability to solve a procedure without making mistakes, while 70% of teachers in Group 3 obtained only 38% or less on these items.

Cohen's d-values for Attribute 4 and Attribute 5 indicate medium size differences in the responses of the groups. Although it is higher for Group 1 & 2, special attention must be paid to the fact that Attribute 4 says something of teachers' reflective behaviour. If a teacher is not consciously reflecting on obtained answers, they do not develop a tendency to interpret answers correctly or evaluate if an answer can be valid. 70% of teachers in Group 1 & 2 obtained 50% or less on items that measure a teachers' ability to interpret an answer correctly, while 70% of teachers in Group 3 obtained only 17% on these items.

An inability to interpret obtained answers negatively impact on the ability to solve a problem (Attribute 5). 70% of teachers in Group 1 & 2 obtained 50% or less on items measuring the teachers' ability to solve problems while 70% of teachers in Group 3 obtained only 33% on these items.

Indications are that teachers benefited to a large extent with regard to Attribute 6. The performance of Group 1 was better than the performance of Group 2. Results show that teachers from Group 1 who were exposed to the new revised curriculum (see § 4.4.2), which included more material on graphs (representations), benefited from these experiences with graphs. 70% of teachers in Group 1 & 2 obtained 58% or less on items measuring representational understanding in the context of functions, while 70% of teachers in Group 3 obtained only 42% on these items.

Results on Attribute 7 are alarming. Indications are that teachers are not able to communicate their understanding of mathematical concepts or ideas in writing. As teachers they should be able to communicate mathematical understanding in writing to learners. 70% of teachers in Group 1 & 2 obtained only 42% or less on items measuring the teachers' ability to communicate understanding of the mathematical problem or give an understandable explanation in writing of his/her thinking. 70% of teachers in Group 3 obtained only 33% or less on these items.

Attribute 10 indicates that teachers are not able to effectively unpack their understanding of a mathematical concept or idea. 70% of teachers in Group 1 & 2 obtained only 36% or less on items measuring the teachers' ability to unpack his/her understanding of a problem. 70% of teachers in Group 3 obtained only 27% or less on these items.

Attribute 8, Attribute 9 and Attribute 11 were not measured with sufficient reliability and were therefore not included in the above analysis. However, these three attributes will come into consideration in the qualitative analysis of the obtained results. Indications are that although reasoning, connections and an understanding of a problem context are important aspects of teachers' MCKFT, teachers' performances in these attributes were not consistent.

5.4 THE QUANTITATIVE DIAGNOSIS CONTENT ANALYSIS OF TEACHERS' PERFORMANCE ON THE ITEMS

In this paragraph a broad overview of teachers' performance on the items is given. The results of the quantitative content analyses will be reported and discussed according to the category each item fits into (see § 4.5.3.3).

5.4.1 The internal reliability consistency of the categories of items

The results of the quantitative diagnostic content analysis on the internal reliability consistency (measured in terms of Cronbach's alpha) of the different categories of items for the group of 128 teachers is given in the following table.

TABLE 8: Cronbach's alpha values of teachers' performance on the different categories of problems.

CATEGORIES		CRONBACH'S ALPHA
1	Missing value proportional reasoning problems	0.71
2	Numerical comparison proportional reasoning problems	0.58
3	Intuitive proportional reasoning problems	1.00
4	Qualitative proportional reasoning problems	0.54
5	Proportional reasoning in a percentage problem context	0.50
6	Proportional reasoning in a function context	0.65
7	Functions	0.69
8	Familiar / Unfamiliar problem contexts	0.53
9	Proportional reasoning in a probability and combinations context	0.42

Except for Category 9, the rest of the values were significantly above a Cronbach alpha value of 0.5 ($\alpha > 0.5$) (see § 4.6.1). However, Category 9 forms part of the qualitative discussion. Category 3 was not used in the further quantitative analysis. This category consisted of Questions 17 and 19. Question 17 could not form a part of the quantitative analysis as nearly no one of the 128 teachers responded correctly to this item. Responses on Question 19 were also not in such a way that it gave a workable Cronbach alpha. Questions 17 and 19 will, however, be discussed in the qualitative part of the results (see § 5.5.3).

Due to the fact that the differences between the responses of the teachers in Groups 1 & 2 were not statistically significant, a t-test was done to look for statistical significant differences between the responses on the items between Group 1 & 2 and Group 3. The results of a t-test and Cohen's

d-values, which is a reflection of the effect size between the responses of Group 1 & 2 and Group 3 are reflected in the following table.

TABLE 9: t-values, p-values, Cohen's d-values and effect sizes of teachers' performance on different categories of problems.

Category	Mean Group 1 & 2	Std. Dev. Group 1 & 2	Mean Group 3	Std. Dev. Group 3	t-value	p-value	Cohen's d-value	Effect size
1	47.90941	26.02129	27.32919	19.93779	4.64994	0.000008	0.790899	Large
2	52.43902	33.47418	45.65217	29.95568	1.14199	0.255624	0.202749	Small
4	56.09756	34.29800	64.49275	34.71377	1.32300	0.188232	0.241840	Small
5	54.57317	27.25283	33.69565	27.99672	4.11811	0.000069	0.745713	Medium
6	52.84553	31.40567	46.73913	25.72878	1.12354	0.263346	0.194436	Small
7	56.95122	15.86510	41.08696	15.94829	5.41806	0.000000	0.994732	Large
8	26.42276	29.03940	9.42029	15.17477	3.69384	0.000328	0.585497	Medium

The table represents teachers in the different groups' responses to the diagnostic items in terms of the different categories of mathematical items. The results in the table above show practically significant differences between Group 1 & 2 and Group 3's responses concerning Categories 1 and 7. Indications are that Group 1 & 2 benefited from their studies in the Sediba Project on aspects of procedural fluency and understanding different representations of functions.

Categories 5 and 8 yielded a medium size effect in responses. Group 1 & 2 seem to be better in practice with understanding and working within a specific problem context that formed part of the mathematical topics presented in the Sediba Project.

Categories 2, 3 and 6 showed small size effects in responses between Group 1 & 2 and Group 3. These were items that required understanding of a problem context that were not part of their mathematical coursework at Sediba. A 70% percentile threshold was then determined according to Group 1 & 2 and Groups 3 performances according to the different categories of items.

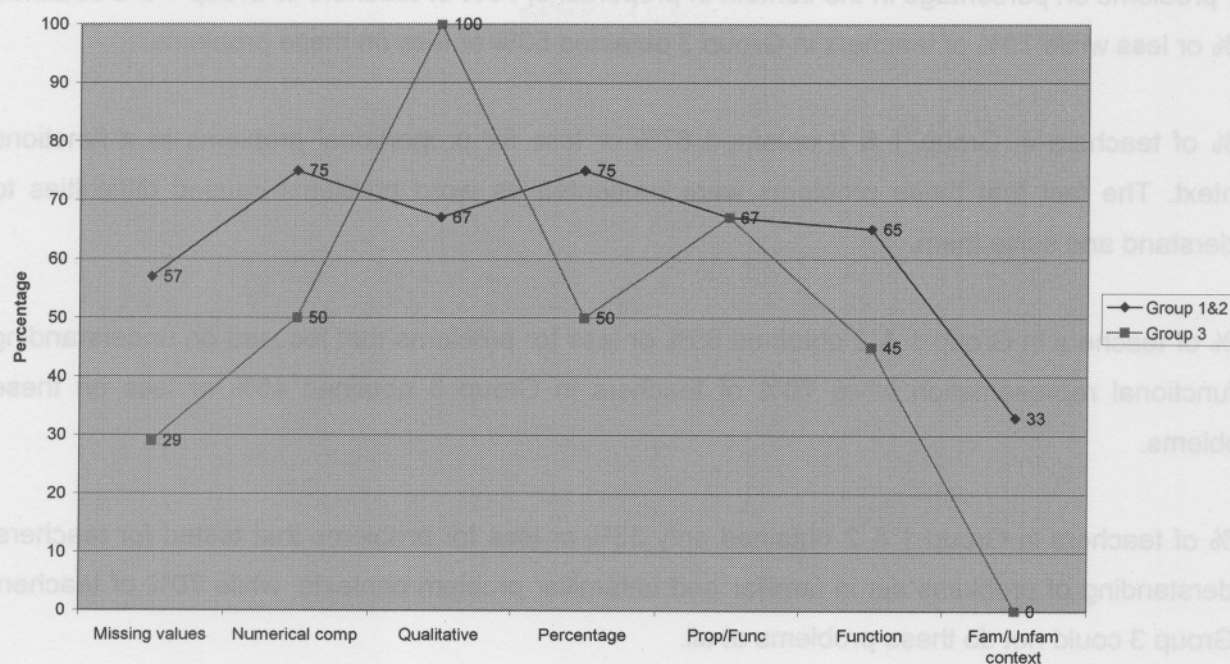


FIGURE 28: 70% Percentile profiles of teachers' performance within the different categories of problems

The results show that 70% of teachers in Group 1 & 2 had 57% or less for missing value problems while 70% of the teachers in Group 3 had only 29% or less for missing value problems.

Numerical comparison problems were one of the few kinds of problems where teachers' showed reasonably good skills in solving the problems. 70% of teachers in Group 1 & 2 obtained 75% or less for numerical comparison problems while 70% of teachers in Group 3 obtained 50% or less on numerical comparison problems.

The qualitative reasoning category yield surprising results. 70% of teachers in Group 3 obtained 100% or less for qualitative reasoning problems while 70% of teachers in Group 1 & 2 obtained only 67% or less on these problems. Teachers who were part of the Sediba Project had a tendency to state that the question could not be solved because there was supposedly not enough information. These teachers wanted numerical values to work with. Because the problem was not set in such a problem context, the teachers indicated that they could not determine an answer to the problem.

For problems on percentage in the context of proportions, 70% of teachers in Group 1 & 2 obtained 75% or less while 70% of teachers in Group 3 obtained 50% or less on these problems.

70% of teachers in Group 1 & 2 obtained 67% or less for proportional problems in a functions context. The fact that these problems were presented as word problems caused difficulties to understand and solve them.

70% of teachers in Group 1 & 2 obtained 65% or less for problems that focused on understanding a functional representation while 70% of teachers in Group 3 obtained 45% or less on these problems.

70% of teachers in Group 1 & 2 obtained only 33% or less for problems that tested for teachers' understanding of problems set in familiar and unfamiliar problem contexts, while 70% of teachers in Group 3 could not do these problems at all.

5.5 THE QUALITATIVE DIAGNOSIS CONTENT ANALYSIS OF TEACHERS' RESPONSES ON THE DIFFERENT CATEGORIES OF ITEMS

5.5.1 Missing value proportional reasoning problems

Questions 27, 32 and 33 form part of the analysis of items in the missing value category.

5.5.1.1 Question 27

The following table indicates teachers' performance on Question 27 in terms of assigned attributes to this item.

TABLE 10: Results of the qualitative diagnostic analysis on Question 27.

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
27.1	A2	Could decide on a procedure to solve the problem.	No / wrong response	87	95	91	76
27.1	A3	Made no procedural mistakes.	No / wrong response	76	89	73	67
27.2	A2	Could decide on a procedure to solve the problem.	No / wrong response	91	89	96	89
27.2	A3	Made no procedural mistakes.	No / wrong response	77	81	80	70
27.3	A2	Could decide on a procedure to solve the problem.	No / wrong response	77	92	82	61

Questions 27.1 and 27.2 seemed to be easier than Question 27.3 because of the numerical relationships, as this influences problem difficulty. Results however, do not quite reflect the notion that Questions 27.1 and 27.2 were in fact easier. The next table shows the results on an analysis of the strategies used (with-in ratio, between ratio or cross-product algorithm) by the teachers to solve Question 27.

TABLE 11: Results of teachers' solution strategies in Question 27.

		Group 1	Group 2	Group 3
27.1	With-in ratio / unit price	38	24	40
	Between ratio	0	0	2
	Cross-product algorithm	51	67	40
	Wrong or unknown strategy	11	11	18
27.2	With-in ratio / unit price	27	9	20
	Between ratio	5	16	30
	Cross-product algorithm	57	67	28
	Wrong or unknown strategy	11	11	22
27.3	With-in ratio / unit price	32	16	17
	Between ratio	0	0	2
	Cross-product algorithm	57	67	40
	Wrong or unknown strategy	8	20	41

The results of the strategies used to solve the problems indicate that the teachers who had not yet started their participation in the Sediba Project, had a tendency to use strategies with more intuitive appeal (§ 4.4.3.1.2). With-in ratios for Question 27.1 and between ratios for Question 27.2 were used to a greater extent by Group 3.

The cross-product algorithm was not used as explicitly by Group 3. Group 1 & 2 had a tendency to use the cross-product algorithm to a greater extent in solving all three problems. Blindly using that cross-product algorithm in these kinds of problems was more evident in Group 1 & 2. It seems as if there is less of a tendency for Group 1 & 2 to reflect on the situation and use more informal approaches like a with-in ratio or a between ratio, rather than more procedural approaches like the cross-product algorithm.

The response from Respondent 7 in Group 1 illustrates a lack of reflective practices in looking back and reflecting on the validity of an answer.

27. Remember, you are not allowed to use a calculator in this question.
Show all your reasoning steps in this question.

27.1 Paul bought 3 sweets for R2.40. At the same price, what would 10 sweets cost?

$$\frac{R2.40}{3} = R8 \times 10$$

$$= R80$$

$\therefore$ 10 sweets will cost R80.00

27.2 Paul bought 3 sweets for R2.00. At the same price, what would 9 sweets cost?

$$\frac{R2.00}{3} = R6.50 \times 9$$

$$= R57.00$$

27.3 Paul bought 13 sweets for R8.40. At the same price, what would 17 sweets cost?

$$\frac{R8.40}{13} = R78.60 \times 17$$

$$= R1442.20$$

- Group 1, Respondent 7, Question 27

Some of the teachers had a problem with computational accuracy when solving problems. Some of the mistakes were simply writing a "22" instead of a "24". However, especially in Group 3, 17 % of the group made mistakes like the following:

27.1 Paul bought 3 sweets for R2.40. At the same price, what would 10 sweets cost?

$$x + 10 = 2,40$$

$$\frac{2,40}{3}$$

$\frac{2,40}{3}$ = this will give the amount of 3 sweets.

$$\therefore \text{answer} \times 10 - 3$$

$$18,5 \times 10 - 3$$

$$185 - 3$$

$$182$$

$$\frac{10 \times 2,40}{3}$$

• Group 3, Respondent 92, Question 27.1

The teacher did not recognise that $\frac{24}{3} = 8$.

27.1 Paul bought 3 sweets for R2.40. At the same price, what would 10 sweets cost?

$$3 \text{ sweets} = R2,40$$

$$10 \text{ sweets} = ? \text{ let the ? be } x$$

$$3 = 2,40 \dots ①$$

$$10 = x \dots ②$$

$$\text{Sub } ② \text{ in } ① \quad 3x = 10 \times 2,40$$

$$3x = 24,00$$

$$x = \frac{24}{3}$$

$$x = 8$$

$$10 \text{ sweets} = R8$$

• Group 3, Respondent 97, Question 27.1

The teacher indicates that $\frac{24}{3} = 6$.

27.1 Paul bought 3 sweets for R2.40. At the same price, what would 10 sweets cost?

$$\begin{array}{l}
 3 \rightarrow 2,40 \\
 10 \rightarrow x \\
 3x = 2,40 \times 10 \\
 3x = R24,00 \\
 3x = 2,400 \\
 x = \frac{2,400}{3} \\
 x = 800
 \end{array}$$

- Group 3, Respondent 124, Question 27.1

The teacher indicates that $R2,40 \times 10 = R2,400$ and shows that $\frac{2,400}{3} = 800$.

27.1 Paul bought 3 sweets for R2.40. At the same price, what would 10 sweets cost?

$$\begin{array}{l}
 3 = R2,40 \\
 10 = Rx \\
 3x = 2400 \\
 x = 700 \\
 x = R7.00
 \end{array}$$

- Group 1 Respondent 9 Question 32

The teacher indicated that $\frac{2400}{3} = 700$.

Overall, teachers in Group 1 & 2 had more procedural fluency in solving these problems and made less obvious mistakes. It might be argued that simple mathematical knowledge like knowing your tables could have assisted these teachers in making fewer mistakes in solving the problems. It is clear that there is a lack in some of these teachers' prior knowledge in that they do not know their tables, could therefore not identify the unit rate easily and therefore resort to blindly using the cross-product rule to solve questions like Question 27.1 where more obvious, less procedural

methods could be used. The question is: Are teachers' simple knowledge in place so that they can recognise intuitive methods e.g. the unit rate?

5.5.1.2 Question 32

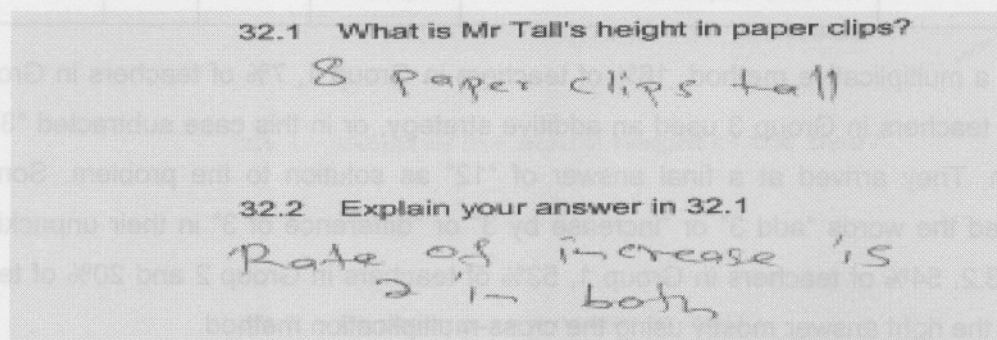
The following table is a reflection of teachers' performance on Question 32 in terms of assigned attributes to this item.

TABLE 12: Results of the qualitative diagnostic analysis on Question 32

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
32.1	A1	Teacher indicates that a multiplicative relationship exists.	No / wrong response	24	35	33	7
32.2	A2	Could choose a procedure to determine the correct answer.	No / wrong response	22	30	31	7
32.2	A3	Made no procedural mistakes in determining the correct answer.	No / wrong response	21	30	29	7
32.2	A4	Interpreted the answer correctly.	No / wrong response	23	32	31	7
32.2	A5	Could solve the problem.	No / wrong response	21	30	29	7
32.2	A10	Could unpack the problem context and solve the problem.	No / wrong response	19	27	29	2

Rather than a multiplicative method, 43% of teachers in Group 1, 22% of teachers in Group 2 and 46 % of the teachers in Group 3 used an additive strategy, to solve this problem. They arrived at a final answer of "8" as the solution to the problem. Teachers used the words "add 2" or "2 more" in their unpacking of the answer in 32.2. 30% of teachers in Group 1, 31% of teachers in Group 2 and 7% of teachers in Group 3 got the right answer. Most of them used the cross-multiplication method.

The example of a response below shows the teacher's additive proportional thinking and inability to communicate the proportional situation (use appropriate language) in using the word "ratio" while not recognizing a proportional situation.



• Group 3, Respondent 98, Question 32

The teacher's unpacking of the solution to the problem and communication of understanding of the solution shows serious misconceptions and an inability to unpack or communicate the solution to this problem.

5.5.1.3 Question 33

The following table is a reflection of teachers' performance on Question 33 in terms of assigned attributes to this item.

TABLE 13: Results of the qualitative diagnostic analysis on Question 33

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
33.1	A1	Teacher indicates that a multiplicative relationship exists.	No / wrong response	41	54	53	20
33.2	A2	Could choose a procedure to determine the correct answer.	No / wrong response	41	54	53	20
33.2	A3	Made no procedural mistakes in determining the correct answer.	No / wrong response	38	46	49	20
33.2	A4	Interpreted the answer correctly.	No / wrong response	38	49	49	20

33.2	A5	Could solve the problem.	No / wrong response	38	49	49	20
33.2	A10	Could unpack the problem context and solve the problem.	No / wrong response	25	30	36	11

Rather than a multiplicative method, 16% of teachers in Group 1, 7% of teachers in Group 2 and 13 % of the teachers in Group 3 used an additive strategy, or in this case subtracted “3” to solve this problem. They arrived at a final answer of “12” as solution to the problem. Some of the teachers used the words “add 3” or “increase by 3” or “difference of 3” in their unpacking of the answer in 33.2. 54% of teachers in Group 1, 53% of teachers in Group 2 and 20% of teachers in Group 3 got the right answer mostly using the cross-multiplication method.

One teacher from Group 1, two teachers from Group 2 and two teachers from Group 3 indicated that they used the “Pythagoras theorem” to find the shadow.

33.1 What is the actual height of the tree?

~~221m~~ $\sqrt{221m} = 14,7m$

33.2 Explain your answer in 33.1.

When using Pythagoras theorem we find that x is given as 15 and x as $2m$

$$\therefore x^2 = 2^2 + 15^2$$

$$x^2 = 2^2 + 9$$

$$\therefore x^2 = 221m^2$$

$$\therefore x = 221m$$

• Group 2, Respondent 59, Question 33

27% of teachers in Group 1, 20% of teachers in Group 2 and 9% of teachers in Group 3 solved both Question 32 and 33 correctly – thus could recognise a multiplicative relationship in both problem contexts.

The teacher in the response below could recognise the multiplicative relationship in Question 32, but the problem context in Question 33 made it impossible for the teacher to solve the problem

correctly, although the teacher had the necessary procedural knowledge to have solved Question 33 the same way as in Question 32.

32.1 What is Mr Tall's height in paper clips?

4 buttons = 6 paper clips
 6 buttons = x
 $x = \frac{6 \times 6}{4} = \frac{36}{4} = 9$ paper clips.

32.2 Explain your answer in 32.1

$\frac{4}{6} = \frac{2}{3}$
 $\frac{6}{9} = \frac{2}{3}$

They have the same ratio or they are in the same proportion.

- Group 1 Respondent 9 Question 32

33.1 What is the actual height of the tree?

$\frac{2}{5} = \frac{x}{15}$
 $x = \frac{2 \times 15}{5} = 6$ meters

33.2 Explain your answer in 33.1.

They are in the same ratio because they are parallel to each other

$\frac{2}{5} = \frac{x}{15}$
 $x = \frac{2 \times 15}{5}$
 $= 6$

- Group 1 Respondent 9 Question 32

Jean Piaget reported the use of an additive strategy in solving Question 32 and 33 more than 26 years ago. These exact problems still seem to be solved using additive strategies after all these years.

5.5.2 Numerical comparison proportional reasoning problems

Questions 22 and 24 fall in the numerical comparison category. All three groups performed better in Question 24 than in Question 22. Although the teachers could get the correct answer for Question 24.1, they had difficulty explaining and unpacking their solution to the problem.

5.5.2.1 Question 22

The following table is a reflection of teachers' performance on Question 22 in terms of assigned attributes to this item.

TABLE 14: Results of the qualitative diagnostic analysis on Question 22

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
22.1	A1	Teacher knew that a numerical comparison ratio should be used.	No / wrong response	44	54	42	37
22.2	A2	Could choose a procedure to determine the correct answer, 5 kg for R7, 4 kg for R6	No / wrong response	42	57	40	33
22.2	A3	Made no procedural mistakes in determining the correct answer.	No / wrong response	40	54	40	28
22.2	A4	Interpreted the answer correctly – knew whether they were working with a R to kg ratio or a kg to R ratio.	No / wrong response	27	43	22	20

22.2	A5	Could solve the problem.	No / wrong response	27	43	22	20
22.2	A10	Could unpack the problem and solve the problem.	No / wrong response	27	43	20	20

40% of the teachers seems to have an intuitive feeling for solving the problem, but did not know what they were doing, as is evident from their interpretation of the results (see § 3.6.4.1). From the 54% of teachers in Group 1 who could solve the problem, only 43% could correctly interpret their answers. From the 40% of teachers in Group 2 who could solve the problem, only 22% could correctly interpret their answers. From the 28% of teachers in Group 3 who could solve the problem, only 20% could correctly interpret their answers. A habit to reflect on the validity or meaning of an answer to a procedurally solved problem does not seem to be present in these teachers.

22. At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.

22.1 Whose tomatoes are cheaper?
Simphiwe's tomatoes are cheaper

22.2 Explain your answer in 22.1.

*Abel: 5kg = R7
 Cost of one tomato = $\frac{R7}{5}$
 $\frac{5kg}{R7} = \frac{5}{7}$
 $= R0,7$*

*Simphiwe: 4kg for R6
 $\frac{4kg}{R6} = \frac{4}{6}$
 $= \frac{2}{3}$
 $= R0,6$*

22.3 Solve the problem in another way if possible?

*5kg = R7
 1kg = x
 5x = 7.
 $\therefore x = \frac{7}{5}$
 1kg = 1,4*

*4kg = R6
 1kg = x
 4x = R6
 $x = \frac{6}{4} = \frac{3}{2}$
 1kg = R1,5*

- Group 1, Respondent 20, Question 22

22.	At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.			
22.1	Whose tomatoes are cheaper? Simphiwe's			
22.2	Explain your answer in 22.1. $\frac{5}{7} \times \frac{6}{6} \quad \text{and} \quad \frac{4}{6} \times \frac{7}{7}$ $= \frac{30}{42} \quad = \frac{28}{42}$ $\frac{30}{42} > \frac{28}{42} \quad \therefore 4 \text{ kg for R6 is cheaper than } 5 \text{ kg for R7}$			
22.3	Solve the problem in another way if possible?			

• Group 2, Respondent 49, Question 22

These teachers unpacked the problem correctly, chose the right procedure to solve the problem, but could not interpret their answer correctly. Some of these teachers are thus unable to interpret the unit rates and reciprocals of the unit rates.

Evidence exists that most of the teachers who did not solve the problem, used additive thinking in getting a solution to the problem.

22. At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.

22.1 Whose tomatoes are cheaper?

None.

22.2 Explain your answer in 22.1.

5kg is more 1kg ~~than~~ 4kg and also R7 is more by R1 to R6 therefore the more the kg the more the money and the lesser the kg the lesser the amount. In this case the difference is 1 for both.

22.3 Solve the problem in another way if possible?

$$5\text{kg} = \text{R}7$$

$$4\text{kg} = \text{R}6$$

$$\therefore 5\text{kg} - 4\text{kg} = \text{R}7 - \text{R}6$$

$$1\text{kg} = \text{R}1.$$

- Group 2, Respondent 42, Question 22

22. At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.

22.1 Whose tomatoes are cheaper?

Both sells at a cheaper prices ?

22.2 Explain your answer in 22.1.

Both sell a kilo at R1

They both gain R2 for every packet they sell. There is only a difference of R1 and per kilo.

22.3 Solve the problem in another way if possible?

IF They sell their tomatoes per 1kg. then Abel will be cheaper, because he will be selling his tomatoes at R1.40 and Simphiwe sell at R1.50 per kg.

- Group 1, Respondent 3, Question 22

22. At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.

22.1 Whose tomatoes are cheaper?

None.

22.2 Explain your answer in 22.1.

5kg is more 1kg ~~than~~ 4kg and also R7 is more by R1 to R6 therefore the more the kg the more the money and the lesser the kg the lesser the amount. In this case the difference is 1 for both

22.3 Solve the problem in another way if possible?

$$\begin{aligned} 5\text{kg} &= \text{R}7 \\ 4\text{kg} &= \text{R}6 \\ \therefore 5\text{kg} - 4\text{kg} &= \text{R}7 - \text{R}6 \\ 1\text{kg} &= \text{R}1. \end{aligned}$$

- Group 2, Respondent 65, Question 22

When they were looking at whether two proportions were equivalent, they looked for a relationship between the first ratio and then matched it with the second one. Because of their inability to reason multiplicatively, their reasoning on the relation between ratios were based on an additive structure.

22. At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.

22.1 Whose tomatoes are cheaper?

~~Abel~~ Abel

22.2 Explain your answer in 22.1.

There is only 1kg different and one Rand. You can't buy another kg with that one Rand.

- Group 3, Respondent 90, Question 22

5.5.2.2 Question 24

The following table is a reflection of teachers' performance on Question 24 in terms of assigned attributes to this item.

TABLE 15: Results of the qualitative diagnostic analysis on Question 24

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
24.1	A1	Teacher knew that a numerical comparison ratio should be used.	No / wrong response	68	70	71	63
24.2	A2	Could choose a procedure to determine the correct answer, 2 to 3 ratio for grade 7 3 to 5 ratio for grade 8	No / wrong response	63	65	67	57
24.2	A3	Made no procedural mistakes in determining the correct answer.	No / wrong response	53	57	58	46
24.2	A4	Interpreted the answer correctly.	No / wrong response	39	43	42	33
24.2	A5	Could solve the problem.	No / wrong response	39	46	42	30
24.2	A10	Could unpack the problem context and solve the problem.	No / wrong response	33	41	33	26

A potential interaction between Question 24 and the social situation in which Question 24 were set, interfered with the teachers' search for a solution to the problem.

24. Two classes order family size pizzas for break time. The grade 7 class orders pizzas so that every three students will share 2 pizzas. The grade 8 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person?

Grade 7

24.2 Solve this problem.

They must buy equal pizzas for every class, so that each class must share equally.

24.3 Solve the problem by using a different method.

Combine both classes and count the pizzas.
Check whether 2 students can share one pizza.
If not increase the number of pizzas so that 2 students can share one pizza.

• Group 1, Respondent 1, Question 24

24. Two classes order family size pizzas for break time. The grade 7 class orders pizzas so that every three students will share 2 pizzas. The grade 8 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person?

Grade 7

24.2 Solve this problem.

Combine two class so that 8 student can share 5 p
so that each student can get equal parts.

- Group 2, Respondent 40, Question 24

24. Two classes order family size pizzas for break time. The grade 7 class orders pizzas so that every three students will share 2 pizzas. The grade 8 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person? Grade 7

24.2 Solve this problem. they must use the same number
for dividing the grade 7 and grade 8 class
in order to get the same food.

24.3 Solve the problem by using a different method.

$$\frac{2}{3}$$

$$1,5$$

and

$$\frac{5}{3}$$

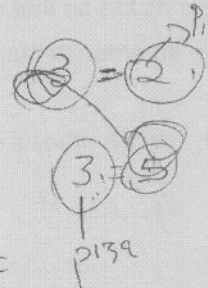
$$1,66 \text{ or } 1,7$$

- Group 3, Respondent 99, Question 24

24. Two classes order family size pizzas for break time. The grade 7 class orders pizzas so that every three students will share 2 pizzas. The grade 8 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person?

GRADE 7



24.2 Solve this problem.

I will combine the 2 pizzas of Grade 7 and 3 pizzas of Grade 8 so that they can eat the equally.

24.3 Solve the problem by using a different method.

Motivating learners by saying those performed very excellent in my class will get pizza for lunch.

• Group 3, Respondent 113, Question 24

The teacher in the preceding response stated that the solution to the problem is to buy enough so that each student can get the same size of pizza. It was thus not a mathematical solution to the problem, but rather grounded in a social context of equal pieces of food for every person involved.

Also in this question, the teachers had difficulty to interpret their answer correctly. The unit rate and reciprocals of the unit rate were not clearly understood by the teachers. From the 58% of teachers in Group 1 who could solve the problem, 42% could correctly interpret their answers. From the 57% of teachers in Group 2 who could solve the problem, 43% could correctly interpret their answers. From the 46% of teachers in Group 3 who could solve the problem, 33% could correctly interpret their answers.

24. Two classes order family size pizzas for break time. The grade 8 class orders pizzas so that every three students will share 2 pizzas. The grade 7 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person?

Grade 8

24.2 Solve this problem.

$$\text{Grd 8, } \frac{5}{2} = 1,7$$

$$\text{Grd 7, } \frac{3}{2} = 1,5$$

24.3 Solve the problem by using a different method.

$$\frac{5}{2} = \frac{3}{2}$$

$$10 = 9$$

$$10 - 9 = 1$$

- Group 1, Respondent 5, Question 24

24. Two classes order family size pizzas for break time. The grade 7 class orders pizzas so that every three students will share 2 pizzas. The grade 8 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person?

Grade 8

24.2 Solve this problem.

$$\frac{3}{2} = 1,5 \text{ and } \frac{5}{2} = 1,67$$

$$\begin{array}{r} 1,67 \\ - 1,50 \\ \hline 0,17 \end{array} \text{ is } \frac{1}{2}$$

24.3 Solve the problem by using a different method.

$$5x + 3 = 3x + 2$$

$$2x = -1$$

$$x = -\frac{1}{2}$$

- Group 1, Respondent 19, Question 24

Looking back and reflecting on the meaning of an answer does not seem to be a habit for these teachers.

5.5.3 Estimation / Intuition / Intuitive understanding of proportional reasoning problems

Questions 17 and 19 form part of the analysis on intuitive understanding in proportional situations. Appropriate performance in these two items was almost non-existent. In paragraph 4.4.3.1.2 it has been stated that proportional reasoning is in the first place a case of intuitively recognising a proportional situation, as required in Questions 17 and 19. It is thus alarming that teachers' responses to these kinds of items are very deprived of conceptual understanding in the context of proportional reasoning.

5.5.3.1 Question 17

The results for Question 17 stood out in that almost none of the teachers recognized a multiplicative relationship between the growth of the two plants. Only Respondent 20 and Respondent 114 recognised a multiplicative relationship between the growth of the two plants, but the values chosen to complete the procedural solution part of the problem were not appropriate.

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

The 12cm tall has grown the fastest

17.2 Explain your answer in 17.1.

1st plant: $\frac{8}{12} = \frac{2}{3}$

2nd plant: $\frac{12}{16} = \frac{3}{4}$. $\therefore \frac{3}{4} > \frac{2}{3}$ i.e. the 2nd plant is faster than the 1st.

$\therefore$ 2nd plant has grown faster

• Group 1, Respondent 20, Question 17

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

12 cm to 16 cm plant

17.2 Explain your answer in 17.1.

*rate of growth: $\frac{8}{12}$
 $= 0,6$
 $= 60\%$*

*$\frac{12}{16}$
 $= 0,75$
 $= 75\%$*

$\therefore$ The 12 cm to 16 cm plant grows faster.

- **Group 3, Respondent 114, Question 17**

These teachers only compared the real growth and not the tempo of growth i.e. they did not make a comparison between the growth and the original height of the tree.

Some teachers were unable to recognise that the problem context makes for being able to coordinate the two ratios simultaneously.

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

The 12 cm.

17.2 Explain your answer in 17.1.

Because it is now the tallest of the two.

- **Group 2, Respondent 45, Question 17**

This teacher simple states that the 12 cm plant has grown the most because it is now the highest.

Since the difference in growth of the two plants is the same (4 cm), most of the teachers responded that the plants have grown equally.

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

None, they were growing with the same rate.

17.2 Explain your answer in 17.1.

The difference that the plants ^{with} grow is the same. They are growing at the same rate.

- **Group 1, Respondent 1, Question 17**

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

None

17.2 Explain your answer in 17.1.

They grow at the same pace

$$(12 - 8) \text{ cm} = 4 \text{ cm}$$

$$(16 - 12) \text{ cm} = 4 \text{ cm}$$

- **Group 1, respondent 6, Question 17**

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

none of them, the grow at the same length.

17.2 Explain your answer in 17.1.

the plants were 8cm and 12cm respectively
then they are now 12cm and 16 cm respectively

$12 - 8 = 4$
 $16 - 12 = 4$ } the length is the same.

- Group 2, respondent 64, Question 17

17. Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.

17.1 Which of the 8-cm or the 12-cm plant has grown the fastest? Both

17.2 Explain your answer in 17.1.

Because they increase in same rate, which is

4

- Group 3, respondent 99, Question 17

Teachers' use of the word rate while describing an "additive relationship" is evidence from responses to this problem.

5.5.3.2 Question 19

The following table is a reflection of teachers' performance on Question 19 in terms of assigned attributes to this item.

TABLE 16: Results of the qualitative diagnostic analysis on Question 19

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
19.1	A1	A teachers' response indicates that he/she recognises that the machines work at the same time – together – each machine does a proportion of the work.	No / wrong response	30	27	31	30
19.2	A2	Could decide on a procedure to solve the problem.	No / wrong response	5	11	4	2
19.2	A3	Used an appropriate procedure or any other strategy and made no mistakes.	No / wrong response	0	0	0	0
19.2	A4	Interpret the answer correctly.	No / wrong response	0	0	0	0
19.2	A5	Could solve the problem.	No / wrong response	1	3	0	0
19.2	A10	Could unpack the problem context and solve the problem.	No / wrong response	2	0	4	0

This problem required teachers to look at the problem intuitively with no ready-made procedure available to solve the problem. The teachers are clearly not used to approaching a problem intuitively; they want a ready-made algorithm. This is evident from the next table, which reflects the drastic difference in teachers who indicate that they recognised that the two machines were working together, but then could not get a mathematical way of showing the inverse proportional relationship between more machines that take less time to do the job. Teachers needed to

recognise that they are working with an inverse proportion. If two machines work together, it takes less time to do the work.

The following table is a reflection of the teachers' responses to this question.

TABLE 17: Results of teachers' responses to Question 19

	% of teachers' responses in terms of the number of hours the two machines will take to do the work.					
	< 2 hours	2 hours	3 hours	4 hours	6 hours	8 hours
Group 1	14	11	30	11	22	3
Group 2	22	11	27	16	9	2
Group 3	22	9	20	17	15	2
Whole group	19	10	26	15	15	2

26% of the whole group stated that it will take the two machines 3 hours to complete the job. These teachers argued that $\frac{4+2}{2} = 3$. They took the "average time" (machine 1 takes 4 hours to shred a truckload and machine 2 takes 2 hours to shred a truckload) to solve the problem. This response indicates that a misconception exists here because the machines work at different speeds, therefore average hours could not be used to solve the problem. 15% of the whole group just added the amount of hours it will take each machine to do the same job. They did not take into account that the two machines were working simultaneously and thus the job will be done in less time than it takes the fastest of the two machines. Only 19% of the whole group indicated that it will take the two machines less than 2 hours to complete the job. Although these teachers could not prove their thinking mathematically correct in some way, they argued that it will at least take the two machines together less time than it will take the fastest machine. These teachers' responses included "30 min, $\frac{3}{4}$, 1h, 1h15, 1h30, 1h45".

Most of the teachers fail to construct an inverse relationship in thinking about this problem. Teachers showed no habit of reflecting on a solution to a problem to decide if an obtained answer could be valid. This is an indication of an inability to realize that an answer was unrealistic.

Results indicate that none of these groups showed natural proportional reasoning in intuitive versions of proportional reasoning problems. Van de Walle (2004:15) states that components would be understood initially at an intuitive level, as shown on an estimation task, and only later would understanding of the same components be expressed computationally. There is thus an alarming gap in these teachers' knowledge base.

5.5.4 Qualitative prediction or reasoning proportional reasoning problems

Questions 18 and 29.2 formed part of the qualitative prediction category. Only qualitative prediction or reasoning is required for solving these kinds of problems. These items are not represented numerically and no calculations are needed.

5.5.4.1 Question 18

The following table is a reflection of teachers' performance on Question 18 in terms of assigned attributes to this item.

TABLE 18: Results of the qualitative diagnostic analysis on Question 18

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
18.1	A8	Teacher used the word "slower" in the answer.	No / wrong response	72	68	64	83
18.2	A7	Teacher used the words "more time" or "fewer laps" in the answer.	No / wrong response	45	51	36	48

Here the teachers had to construct the relationship that fewer laps in more time imply slower running. Some teachers were unable to construct this inverse relationship. Notable is that while 83% of the teachers in Group 3 could understand the problem setting and see that Mary was running slower, only 48% of them could communicate their understanding of the inverse relationship. The most frequent responses to this problem are reflected in the following table:

TABLE 19: Results of teachers' responses to Question 18.

	% of teachers' responses	
	"cannot tell"	"same speed"
Group 1	19	8
Group 2	31	2
Group 3	15	0
Whole group	22	3

22% of the whole group of teachers' responses show that they do not believe there is enough information available to determine the answer to the problem. This question needed the teachers to coordinate multiple pieces of information. They were unable to coordinate two "non-mathematical" ratios simultaneously.

5.5.4.2 Question 29.2

The following table is a reflection of teachers' performance on Question 29.2 in terms of assigned attributes to this item.

TABLE 20: Results of the qualitative diagnostic analysis on Question 29.2

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
29.2	A8	Teacher used the word "increase" in the answer	No / wrong response	45	54	36	46

The table represents teachers' ability to comprehend the problem context. They were able to understand that a persons chances increase each time he/she is not voted out of the competition.

5.5.5 Proportional reasoning in a percentage context

Questions 1 and 31 were analysed for this category. Teachers overall did better in Question 31 than in Question 1.

5.5.5.1 Question 1

The following table is a reflection of teachers' performance on Question 1 in terms of assigned attributes to this item.

TABLE 21: Results of the qualitative diagnostic analysis on Question 1

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
1.1	A3	A procedure was used to complete the table.	No / wrong response	51	54	62	37
1.1	A6	Subject understood the specific representation and could complete the table.	No / wrong response	74	76	80	67
1.2	A2	The following procedure was used to determine the answer: $\frac{2.20}{20} \times \frac{100}{1} = 11\%.$	No / wrong response	39	54	40	26
1.2	A3	Subject made no procedural mistake and arrived at an answer of 11%.	No / wrong response	23	46	18	11
1.2	A9	A connection could be made between tax rate and %.	No / wrong response	27	41	24	20

Teachers seem to have a tendency to make procedural mistakes while solving a problem. Results also show unsatisfactory knowledge of connections between tax rate and percentage. Some teachers stated that "tax rate" refers to an amount of money.

- 1.1 Answer the following question by completing the table below.
If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20	4.4	0.55	4.95

- 1.2 What is the tax rate on these items?

R 0,11 =
11 cents

• **Group 1, Respondent 14 , Question 1**

- 1.1 Answer the following question by completing the table below.
If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20	4.40	55c	4.95

- 1.2 What is the tax rate on these items?

-the tax rate on these items is 11c

• **Group 1, Respondent 24 , Question 1**

1.1 Answer the following question by completing the table below.

If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20	4.40	0.60	5.00

1.2 What is the tax rate on these items?

The tax rate is R10.00

• **Group 3, Respondent 126 , Question 1**

Some of the wrong solutions to Question 1.2 include:

1.1 Answer the following question by completing the table below.

If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20	4,4	0,55	4,95

1.2 What is the tax rate on these items?

$$\frac{2,20 + 4,4 + 0,55 + 4,95}{4} = 3,025 = 3,02$$

• **Group 2, Respondent 51 , Question 1**

1.1 Answer the following question by completing the table below.

If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20	4,40	0,7	5

1.2 What is the tax rate on these items?

$$\text{Tax rate} = \frac{45-20}{5-2,20} = \frac{25}{2,8} = 8,9$$

- Group 2, Respondent 39 , Question 1**

1.1 Answer the following question by completing the table below.

If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20	4,40	0,55	4,95

1.2 What is the tax rate on these items?

$$\begin{aligned} \text{Tax Rate} &= \frac{\Delta \text{cost}}{\Delta \text{tax}} \\ &= \frac{20}{2,20} \\ &= 9,09 \end{aligned}$$

- Group 2, Respondent 67, Question 1**

Except for some teachers trying to determine something like a gradient for a line, none of the teachers gave evidence that they made the connection that a rate (ratio) can be expressed as the

slope of the line of a function - $y = \frac{11}{100}x$ could be the line drawn for this proportional situation (see § 4.4.4.4).

5.5.5.2 Question 31

The following table is a reflection of teachers' performance on Question 31 in terms of assigned attributes to this item.

TABLE 22: Results of the qualitative diagnostic analysis on Question 31

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
31.1	A2	$\frac{3}{5} = \frac{60}{100} = 60\%$.		72	84	80	54
31.2	A7	The teacher uses the necessary words to communicate in writing that percentage is a ratio.		26	19	47	11
31.2	A9	The connection was made that percentage is a ratio.		33	51	40	11

The following two respondents stated that the connection between percentage and ratio is that they are inverses of each other.

31.1 Convert the fraction to percentages.

$$\frac{3}{5} = 60\%$$

31.2 What is the relationship between proportions and percentages?

They are inverse.

- Group 1, Respondent 1, Question 31

31.1 Convert the fraction to percentages.

$$\begin{aligned}\frac{3}{5} &= \frac{3}{5} \times 100\% \\ &= 1,2 \times 100 \\ &= 120\%.\end{aligned}$$

31.2 What is the relationship between proportions and percentages?

It is the inverse of the other.

- Group 1, Respondent 3, Question 31

31.1 Convert the fraction to percentages.

$$\frac{3}{5} = 0,6$$

31.2 What is the relationship between proportions and percentages?

Proportions is leads to the triangle to be
of equal size.

Percentage leads to the amount per 100% (total)

- **Group 1, Respondent 15, Question 31**

These teachers clearly have never made the connection that a percentage implies a proportion. From the response of the teacher above, it is clear that the teacher sees percentage and proportion as two distinct entities with no relationship to each other, while some other teachers, likely out of a lack of profound understanding of the concepts, states that percentage and proportions are inverses of each other.

Teacher who could do Question 31 could not necessarily do Question 1. The different problem contexts in which the two questions were set seem to have affected some of the teachers.

5.5.6 Proportional reasoning in a function context

Questions 4, 6 and 28 were constructed to investigate teachers' understanding of proportional situations in a functions context.

5.5.6.1 Question 4

The following table is a reflection of teachers' performance on Question 4 in terms of assigned attributes to this item.

TABLE 23: Results of the qualitative diagnostic analysis on Question 4

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
4.1	A5	Could explain how to get the correct answer of 15 litre.	No / wrong response	46	54	49	37
4.2	A1	Applied proportional reasoning skills and got 60 or 45 students.	No / wrong response	63	68	60	61
4.2	A2	Use the following procedure: $\frac{2}{3} \times 90 = 60 \times 250\text{ml} = 15000\text{ml} = 15\text{l}$	No / wrong response	73	78	69	72
4.2	A3	Made no mistakes in finding an answer of 15 litre.	No / wrong response	45	54	47	35
4.2	A4	Correct interpretation of the answer been 15 litre.	No / wrong response	30	38	29	24
4.2	A10	Could unpack understanding and explain how to get to an answer of 15 litre.	No / wrong response	46	54	49	37

The reason why teachers performed better in the second part of Question 4 is because they could get a correct mark although their answer in Question 4.1 was not right. In Question 4.2 the researcher looked for any indication that the teacher could unpack his understanding of the problem.

Two consecutive procedures need to be executed. First $\frac{2}{3} \times 90 = 60$ and then $60 \times 250\text{ml} = 15000\text{ml} = 15\text{l}$. Indications are that the teachers have difficulty doing these consecutive procedures. This problem requires multi-level comparative thinking (see § 4.3.5.3.1) from these teachers. Some teachers clearly had a problem in this area.

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$\frac{2}{3} \times \frac{90}{1} \times 250 = \frac{500}{270} = 1,85 \text{ ml}$$

$$\frac{2}{3} \times \frac{x}{90} = 3x = 180$$

$$x = 60 \text{ ml}$$

4.2 Explain your answer in 4.1.

• **Group 1, Respondent 4, Question 4**

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$90 \times 250 \text{ ml} = 22500$$

$$22500 \times \frac{2}{3} = 3750 \text{ litres}$$

$$\frac{2}{3} \times 90 = 60$$

$$30 \text{ don't drink milk}$$

$$60 \times 250 \text{ ml} = 1500$$

$$\frac{2}{3} \times 90 = 60$$

$$\frac{60}{1000} = 0,24 \text{ l.}$$

$$90 \times 250 \text{ ml}$$

$$22500$$

$$15000 \text{ l.}$$

$$3750 \text{ l.}$$

4.2 Explain your answer in 4.1.

90 student ~~eat~~ ^{eat cafeteria} ~~drink glass of milk~~ but $\frac{2}{3}$ don't drink milk
 Multiply 90 by $\frac{2}{3}$ then the answer is
 multiply by 250ml of milk to find
 how many were consumed.

• **Group 1, Respondent 21, Question 4**

Some teachers had difficulty converting millilitres to litres.

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

150 litres of milk.

4.2 Explain your answer in 4.1.

If 90 student eat in cafeteria, only 60 students will drink a glass of milk, the you multiply 60 by 250ml. The answer you get you change it to litre by dividing that answer by 100, because 1ml is equal to 1002.

- Group 1 Respondent 1, Question 4

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

15000 ^{ml} litres of milk.
750
20 litres.

4.2 Explain your answer in 4.1.

$\frac{2}{3}$ of 90 = 60 the number of students drinking milk. The 60 x 250ml = 15000 litres of ml

- Group 1 Respondent 18, Question 4

Teachers had difficulty translating the problem from the linguistic representation to a mental representation because they needed to coordinate multiple pieces of information.

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$\frac{2}{3} \rightarrow 250 \text{ ml}$$

$$\frac{2 \times 90}{3} \rightarrow 250 \text{ ml } x$$

$$x =$$

$$\frac{2}{3} \text{ drink} \rightarrow 1(250)$$

$$\frac{2 \times 90}{3} \rightarrow x$$

$$x = 22500 \text{ ml}$$

$$1 \text{ l} = 1000 \text{ ml}$$

$$y = 22500 \text{ ml}$$

4.2 Explain your answer in 4.1.

If 2 out of 3 drinks 250 ml.
and 60 out of 90 will drink 22500 ml
and 22500 ml is converted to
litres which gives 22,5 l.

• Group 1, Respondent 31, Question 4

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$1 \text{ g} = 250 \text{ ml}$$

$$45 \text{ g} = x$$

$$x = 11250 \text{ l}$$

4.2 Explain your answer in 4.1.

It is because not all 90 students were drinking milk, therefore out of 90 students only 45 did drink milk according to the data given

• Group 1, Respondent 6, Question 4

The teachers substituted the words "a glass of milk" with 250 ml.

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

45 litres

4.2 Explain your answer in 4.1.

Because if 90 students is divided by the two students the answer is 45, and $45 \times 1 \text{ glass} = 45 \text{ litres}$

- Group 1, Respondent 7, Question 4

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

3750 mL of milk

4.2 Explain your answer in 4.1.

90 students eat in the cafeteria
of which two of them drink milk
it means 15 students drink milk
of 3750 mL

- Group 1, Respondent 24, Question 4

The most obvious other tendency for mistakes was getting an answer of 22,5 litres. 14% of the teachers in Group 1, 9% of teachers in Group 2 and 15% of teachers in Group 3 arrived at an answer of 22,5 litres.

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$\frac{2}{3} \quad 250 \times 90 = 22.500 \text{ litres}$$

4.2 Explain your answer in 4.1.

each learner drink 250.
If there are 90 learner.
we be multiplying $250 \times 90 = 22500$
then we got the correct answer

- Group 1, Respondent 22, Question 4

7% of teachers in Group 3 arrived at an answer of 11250 litres.

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$\begin{array}{r} 250 \text{ litres} \\ 45 \\ \hline 1250 \\ 1000 \\ \hline 11250 \end{array}$$

11250 litres of milk is consumed

$$\frac{90}{3} = 30$$

$$\begin{array}{r} 2 \overline{) 90} \\ \underline{30} \\ 60 \\ \underline{30} \\ 30 \\ \underline{30} \\ 0 \end{array}$$

18
5
90

4.2 Explain your answer in 4.1.

- Group 3, Respondent 84, Question 4

4.1 Two out of every three students who eat in the cafeteria drink a glass (250 ml) of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$2 = 250 \text{ ml}$$

$$90 = x$$

$$90 \text{ students} \times 250 \text{ ml} = 22500 \text{ ml}$$

$$2 \text{ students} \times x \text{ ml}$$

$$\Rightarrow \frac{22500 \text{ ml}}{2 \text{ students}} = 11250 \text{ ml}$$

$$\frac{22500 \text{ ml}}{2} = 11250 \text{ ml}$$

4.2 Explain your answer in 4.1.

11250 l were consumed.

The more the students the more milk were consumed.

• Group 3, Respondent 106, Question 4

4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

$$2 \rightarrow 250 \text{ ml}$$

$$90 \rightarrow x$$

$$90 = \# \quad \frac{90 \times 250}{2} = 11250 \text{ L.}$$

4.2 Explain your answer in 4.1.

• Group 3, Respondent 125, Question 4

The teachers could not unpack the problem and work with multiple pieces of linguistic information to transform it into a mathematical representation.

5.5.6.2 Question 6

The following table is an illustration of teachers' performance on Question 6 in terms of assigned attributes to this item. Teachers had to understand that the equation represents an inverse proportional relationship between x and y .

TABLE 24: Results of the qualitative diagnostic analysis on Question 6

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
6.1	A1	Evidence of multiplicative proportional reasoning.	No / wrong response	46	43	49	46
6.1	A6	Could work from a representation in the form of a formula to the same representation in the form of words.	No / wrong response	46	43	49	46
6.1	A11	Could understand the problem context.	No / wrong response	46	43	49	46
6.2	A7	Could communicate the solution to the problem.	No / wrong response	39	43	40	35
6.2	A10	Could unpack the explanation to the solution of the problem.	No / wrong response	37	43	38	30

In this task, 49% of the teachers in Group 1, 38% of teachers in Group 2 and 39% of teachers in Group 3 failed to see the inverse proportional relationship between the values of x and y . Those who unpacked their solution to Question 6.2, solved the equation for y : $4y = 2x, y = \frac{1}{2}x$. Teachers had to apply inverse proportional reasoning to be able to come to the correct solution to this problem. If teachers choose (b) it means they did not apply inverse proportional reasoning. Some teachers did choose (c), but did not make y the subject of the formula. A "0" was then awarded for attribute A10.

Some teachers choose (b) because of solving for x : $2y = x$.

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

6.2 Explain your answer in 6.1

If we look at our equation $4y = 2x$ we can simplify by dividing both sides by 2.

$$\text{we } \frac{4y}{2} = \frac{2x}{2}$$

$$2y = x$$

Then we have y increasing 2 times as fast as the value of x .

• Group 2, Respondent 55, Question 6

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

6.2 Explain your answer in 6.1

The ratio of the equation is
 $4:2$ so y is 2 units fast
 than x .

• Group 2, Respondent 62, Question 6

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

$$\frac{4y}{4} = \frac{2x}{4}$$

$$y = \frac{1}{2}x$$

$$\frac{4y}{2} = \frac{2x}{2}$$

$$x = 2y$$

6.2 Explain your answer in 6.1

* Because when we make y the subject of the formula y becomes $\frac{1}{2}$ more than x and also when we make x the subject of the formula y becomes 2 times x .

• Group 2, Respondent 76, Question 6

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

6.2 Explain your answer in 6.1

if $4y = 2x$ then $2y = x$ it stands to reason that each value on y is double half of x .
 $\therefore$ (b) will be my answer

• Group 2, Respondent 81, Question 6

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

6.2 Explain your answer in 6.1

If I have to make x the subject of the formula I will ~~have~~ get $4y = 2x$
 $\frac{4y}{2} = x$
 $2y = x$
 Thus the value of y increases 2 times as fast as that of x .

• Group 3, Respondent 108, Question 6

These teachers failed to construct an inverse relationship.

Some teachers who had chosen (c) correctly had a problem to unpack their understanding of the solution to the problem.

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

$$y = \frac{1}{2}x$$

6.2 Explain your answer in 6.1

The statement is true if you substitute the y -value with $\frac{1}{2}$ this will give us an equal side that is LHS = RHS

• Group 2, Respondent 74, Question 6

6.1 Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

6.2 Explain your answer in 6.1

Because we are going to multiply the value of y by 4.

• Group 3, Respondent 103, Question 6

Inverse proportional reasoning was clearly a problem for these teachers in that they could either not reason inversely, or that they could not explain the reason for their answer.

5.5.6.3 Question 28

The following table is a reflection of teachers' performance on Question 28 in terms of assigned attributes to this item.

TABLE 25: Results of the qualitative diagnostic analysis on Question 28

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
28.1	A6	(a) was chosen as the answer.		70	73	67	72
28.2	A7	Evidence that the teacher could communicate his understanding of the problem and the solution.		57	59	60	52
28.2	A10	An appropriate explanation of the solution of the problem.		54	59	53	50
28.2	A11	Evidence of understanding of the problem context in which the problem were set.		60	70	58	54

19% of teachers in Group 1, 24% of teachers in Group 2 and 13% of teachers in Group 3 choose (b). Their reason for choosing (b) was because $\frac{L}{3} = T$ could be written as $L = 3T$ which was the form they wanted the relationship between L and T in.

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

$\therefore \frac{L}{3} = T$
 $\therefore L = 3T$

• **Group 2, Respondent 61, Question 28**

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$ ✓

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

This ^(T) year he earned 3 times ^(L) last year
 $3T = L$ which is $T = \frac{L}{3}$

• **Group 3, Respondent 94, Question 28**

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

You will cross multiply and get 3 more for last year.

- Group 3, Respondent 113, Question 28

Some of the teachers communicated that T and L had an additive relationship.

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

His earnings were increased by three !!

- Group 1, Respondent 16, Question 28

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

The amount that Jose earned for last year
it is increased by 3.

- Group 2, Respondent 44, Question 28

Some teachers communicated using terms such as ratio and proportion. No evidence exists that these teachers were using these words in the right context to describing the relationship between L and T .

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

T is directly proportional.

- Group 1, Respondent 3, Question 28

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

It must be in the ratio form.

• **Group 1, Respondent 5, Question 28**

5.5.7 Functions - looking for understanding of representations

In this category, Questions 2, 3, 7, 8, 9, 11 and 14 were evaluated.

5.5.7.1 Question 2

Table 26 is a reflection of teachers' performance on Question 2 in terms of assigned attributes to this item.

TABLE 26: Results of the qualitative diagnostic analysis on Question 2

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
2.1	A6	Could understand the way the information was represented in the table.	No / wrong response	81	78	84	80
2.2	A9	Could make the connection that meter per minute is a rate.	No / wrong response	45	59	38	41
2.4	A7	Could communicate their understanding in the correct mathematical terms – 6 meters per minute.	No / wrong response	55	62	56	48

19% of the teachers in Group 1 described the rate in Question 2.2 as "6 meters per second." This group did functions during the year in which they completed their first year studies at the Sediba Project.

2.1 How far was the horse from the stable when he started walking?

6 m

2.2 At what rate is the horse walking away from the stable?

2 s

2.3 Is the horse walking at a constant rate?

Yes

2.4 Explain your answer in 2.3.

Time is constantly increasing by 1 second
distance is constantly increasing by 6 m

5

• Group 1, Respondent 6, Question 2

The following is an example of a teacher who could not understand how the information to the answer in Question 2.1 was represented in the table and who had to use a formula to determine the answer.

2.1 How far was the horse from the stable when he started walking?

$$\text{Speed} = \frac{\text{distance}}{\text{time}} \\ = \frac{12}{1} = 12 \text{ m/s}$$

2.2 At what rate is the horse walking away from the stable?

$$t = ? \quad \Delta t = \text{Final time} - \text{initial time} \\ = 3 - 1 \\ = 2 \text{ s}$$

2.3 Is the horse walking at a constant rate?

NO Yes

2.4 Explain your answer in 2.3.

$$\text{velocity} = \frac{\text{displacement}}{\text{time}} \quad \text{acceleration} = \frac{\text{change velocity}}{\text{change in time}} \\ = \frac{12}{2} \quad = \frac{12}{2} \\ = 6 \quad = 6 \text{ m/s}^2$$

• Group 2, Respondent 47, Question 2

Notice the language: "walking at a rate of 6 m."

2.1 How far was the horse from the stable when he started walking?

6 metres

2.2 At what rate is the horse walking away from the stable?

6 metres

2.3 Is the horse walking at a constant rate? ~~NO~~ YES

2.4 Explain your answer in 2.3.

Because in every after a minute it
walks at a rate of 6 metres.

- **Group 2, Respondent 53, Question 2**

In the next example of teachers' responses the teacher uses appropriate language to explain his/her understanding of rate. The teachers' understanding of rate in this problem context is evident from the response to these questions.

2.1 How far was the horse from the stable when he started walking?

6 meters.

2.2 At what rate is the horse walking away from the stable?

6 meter per minute

2.3 Is the horse walking at a constant rate?

Yes .

2.4 Explain you answer in 2.3.

There is a proportionality -
between Time & distance walked
by the horse .

5

- **Group 3, Respondent 104, Question 2**

2.1 How far was the horse from the stable when he started walking?

It was 6m far.

2.2 At what rate is the horse walking away from the stable?

On a changing rate. As the time increases the
distance also increases.

2.3 Is the horse walking at a constant rate? No .

2.4 Explain you answer in 2.3.

An increase in time tends to an increase
in distance, thus the far it moves from the
stable the more the time .

5

- **Group 3, Respondent 108, Question 2**

- 2.1 How far was the horse from the stable when he started walking? 6 meters
- 2.2 At what rate is the horse walking away from the stable? $\frac{6}{12} = 0,5$
= 50% rate
- 2.3 Is the horse walking at a constant rate? $\frac{12}{18} = 0,6$
= 60% rate
- 2.4 Explain you answer in 2.3. NO
- THE RATE IS DIFFERENT BECAUSE THE SPEED IS DIFFERENT. $\frac{12}{1} = 12 \text{ m/s}$ and $\frac{18}{2} = 9 \text{ m/s}$
and $\frac{24}{3} = 8 \text{ m/s}$

• **Group 3, Respondent 114, Question 2**

In the next responses the teachers indicated that rate is a percentage (%).

- 2.1 How far was the horse from the stable when he started walking? 6 metres.
- 2.2 At what rate is the horse walking away from the stable? 6%
- 2.3 Is the horse walking at a constant rate? YES
- 2.4 Explain you answer in 2.3. The distance from 0 - 3 minutes has a d.f of 6 metres.

• **Group 1, Respondent 25, Question 2**

2.1 How far was the horse from the stable when he started walking?

6 m

2.2 At what rate is the horse walking away from the stable?

6%

2.3 Is the horse walking at a constant rate?

YES

2.4 Explain your answer in 2.3.

The difference between the 1st term and the 2nd term and the 2nd term and the 3rd term and so on are equal which is 6.

- Group 2, Respondent 58, Question 2

5.5.7.2 Question 3

The following table is a reflection of teachers' performance on Question 3 in terms of assigned attributes to this item.

TABLE 27: Results of the qualitative diagnostic analysis on Question 3

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
3.1	A6	Could take the necessary information represented in the word sum to complete the table.	No / wrong response	59	70	71	39
3.2	A6	Could understand the representation of the information in the table to draw the graph.	No / wrong response	55	65	62	41
3.7	A7	Could communicate understanding of the connection between the constant rate at which the petrol becomes less and the gradient of the straight line.	No / wrong response	59	57	64	54
3.7	A9	Could make the connection that the constant rate at which the petrol becomes less is represented by the gradient of the line.	No / wrong response	25	30	31	15

Two of the teachers in Group 1 responded that the gradient of the line represents "the slope at which the car is travelling". This implies a serious misconception in the teachers' mathematical content knowledge base. These teachers interpret the slope literally as part of the road travelled.

3.7 What does the gradient of the line represent?

the slope the car is travelling after some time (in minutes)

- Group 1, Respondent 24, Question 3.7**

3.7 What does the gradient of the line represent?

It represent the Slope in which the car travels.

- **Group 1, Respondent 25, Question 3.7**

11% of the teachers in Group 1 state that the gradient is an indication of the speed or velocity or deceleration of the car.

3.7 What does the gradient of the line represent?
The car was moving at a constant speed.

- **Group 1, Respondent 3, Question 3.7**

3.7 What does the gradient of the line represent?

Speed/Velocity

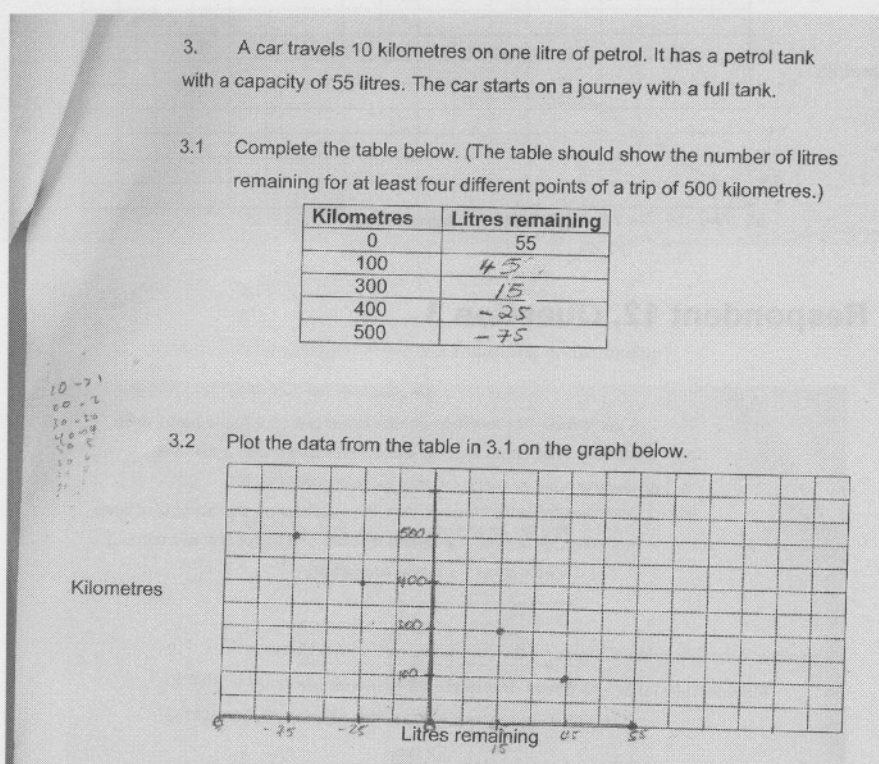
- **Group 1, Respondent 17, Question 3.7**

3.7 What does the gradient of the line represent?

*decelerate opposite / slow down
decelerate*

• **Group 1, Respondent 19, Question 3.7**

The following responses are a reflection of teachers' inability to understand different representations of information.



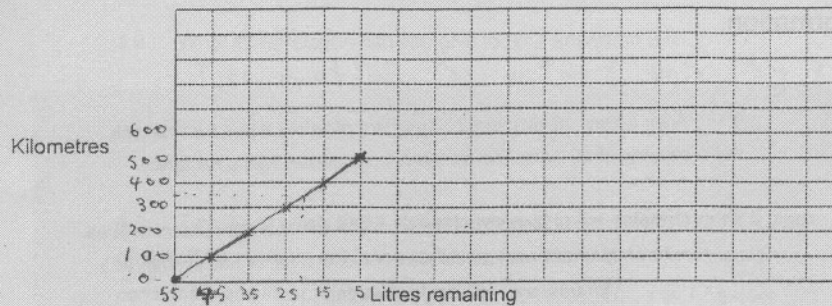
• **Group 1, Respondent 4, Question 3**

3. A car travels 10 kilometres on one litre of petrol. It has a petrol tank with a capacity of 55 litres. The car starts on a journey with a full tank.

3.1 Complete the table below. (The table should show the number of litres remaining for at least four different points of a trip of 500 kilometres.)

Kilometres	Litres remaining
0	55
100	45
300	25
400	15
500	05

3.2 Plot the data from the table in 3.1 on the graph below.



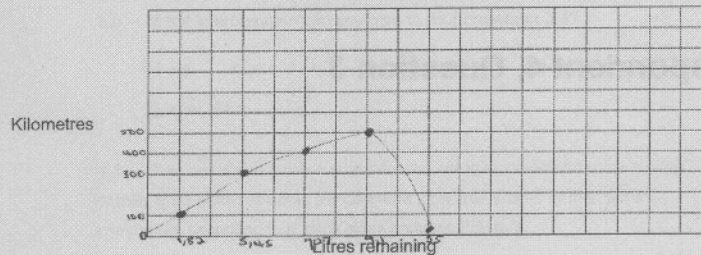
• Group 1, Respondent 12, Question 3

3. A car travels 10 kilometres on one litre of petrol. It has a petrol tank with a capacity of 55 litres. The car starts on a journey with a full tank.

3.1 Complete the table below. (The table should show the number of litres remaining for at least four different points of a trip of 500 kilometres.)

Kilometres	Litres remaining
0	55
100	1,82
300	5,45
400	7,27
500	9,1

3.2 Plot the data from the table in 3.1 on the graph below.



• Group 3, Respondent 99, Question 3

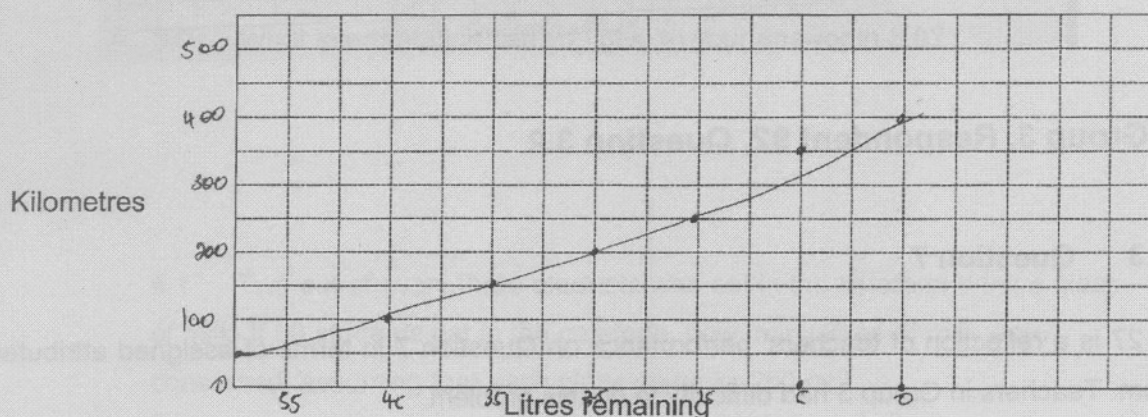
Some teachers from Group 3 used decreasing values on x-axis.

3.2 Plot the data from the table in 3.1 on the graph below.



- **Group 3, Respondent 93, Question 3.2**

3.2 Plot the data from the table in 3.1 on the graph below.



- **Group 3, Respondent 110, Question 3.2**

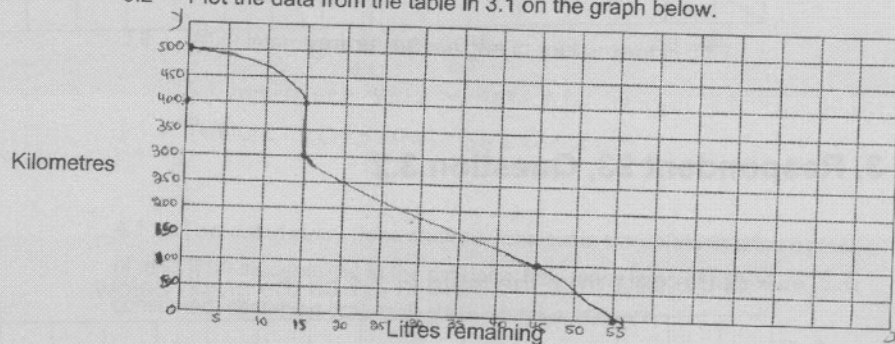
The following mistakes occurred frequently amongst Group 3 – some of the teachers did not recognise that the available information represents a graph of a straight line.

3. A car travels 10 kilometres on one litre of petrol. It has a petrol tank with a capacity of 55 litres. The car starts on a journey with a full tank.

- 3.1 Complete the table below. (The table should show the number of litres remaining for at least four different points of a trip of 500 kilometres.)

Kilometres	Litres remaining
0	55
100	45
300	15
400	0
500	0

- 3.2 Plot the data from the table in 3.1 on the graph below.



- Group 3, Respondent 92, Question 3.2

5.5.7.3 Question 7

Table 27 is a reflection of teachers' performance on Question 7 in terms of assigned attributes to this item. Teachers in Group 3 had difficulty to do this problem.

TABLE 27: Results of the qualitative diagnostic analysis on Question 7

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
7.1	A2	Knew the general formula for a straight line $y = mx + c$.	No / wrong response	76	89	89	52
7.1	A3	Could solve the problem without making any procedural mistakes.	No / wrong response	59	76	73	30
7.1	A6	Could see that the problem represented a straight line.	No / wrong response	81	89	93	63
7.3	A2	Could work with general formula for a straight line: $y = mx + c$.	No / wrong response	68	81	82	43
7.3	A3	Could solve the problem without making any procedural mistakes.	No / wrong response	53	65	69	28

It was clear that teachers from Group 3 could not remember / work with the general formula for a straight line.

7.1 Determine the equation of a function f if the point $(1 ; 3350)$ is on the graph of the function and the gradient of the graph is 2000.

$$y = mx + c$$

$$3350 = 2000x + c$$

$$c = \frac{3350}{2000} \quad \therefore 3350y = 2000x + \frac{3350}{2000}$$

7.2 How did you know which equation to use for the function?

For gradient and a function we have
 $y = mx + c$ as the eq.

7.3 Determine $f(5)$.

$$f(5) = 2000x + \frac{3350}{2000}$$

- Group 3, Respondent 89, Question 7

7.1 Determine the equation of a function f if the point $(1 ; 3350)$ is on the graph of the function and the gradient of the graph is 2000.

$$x = 1 \quad y = 3350, \quad m = 2000$$

$$y = mx + c$$

$$\frac{3350}{2000} = \frac{2000 \times 1 + c}{2000} \quad \therefore \frac{3350}{2000} = c$$

$$f(y) = \frac{1}{2}x + 1$$

7.2 How did you know which equation to use for the function?

7.3 Determine $f(5)$.

- Group 3, Respondent 98, Question 7

5.5.7.4 Question 8

The following table is a reflection of teachers' performance on Question 8 in terms of assigned attributes to this item.

TABLE 28: Results of the qualitative diagnostic analysis on Question 8

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
8.1	A6	Could match a representation of a scenario in words with a mathematical graph.	No / wrong response	52	62	69	28
8.2	A6	Could match a representation of a scenario in words with a mathematical graph.	No / wrong response	34	54	31	22
8.3	A6	Could match a representation of a scenario in words with a mathematical graph.	No / wrong response	32	38	31	28
8.4	A6	Could match a representation of a scenario in words with a mathematical graph.	No / wrong response	66	68	76	57
8.5	A6	Could match a representation of a scenario in words with a mathematical graph.	No / wrong response	33	49	31	22

5.5.7.5 Question 9

The following table is a reflection of teachers' performance on Question 9 in terms of assigned attributes to this item. The teachers were not awarded a point when they just indicated A, B, C, ... with no explanation of why they chose the particular graph.

TABLE 29: Results of the qualitative diagnostic analysis on Question 9

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
9.1.1	A6	Understood that you cannot go back in time.		28	35	29	22
9.1.1	A7	Could communicate the represented understanding.		16	22	18	9
9.1.2	A6	Understood that you cannot go back in time.		19	32	20	7
9.1.2	A7	Could communicate the represented understanding.		15	24	18	4
9.1.3	A6	Understood that you cannot travel a distance while no time passes.		13	19	13	7
9.1.3	A7	Could communicate the represented understanding.		12	16	13	7
9.2	A6	Understood that constant speed is represented by graph (A).		41	43	47	35
9.2	A7	Could communicate the represented understanding.		18	19	20	15

In Question 9.1 the teachers had to state that some of the graphs showed impossible situations (D, H – Cannot go back in time and I – cannot travel a distance in no time.)

In Question 9.2 the teachers had to indicate that they recognise the graphical representation of constant speed. A “1” was allocated for A6 if the teacher indicated that he was thinking in terms of time. Question 9.3 was not evaluated because teachers took scenarios from Question 8 to answer this question and did not use their own ideas.

Some interesting responses for the teachers include the following examples:

9.1 Identify the impossible graphs. Explain why each of the graphs you have chosen is impossible. A because before any distance can be covered already time is covered.

D. Time is covered then after some time it went back to zero which is impossible

E time is covered before distance can be covered.

H time cannot be reversed and be covered again.

I time is reverse a little which is impossible.

9.2 Which graph reflects constant speed? Explain your answer.

A as time is covered, the distance is covered and the gradient is the same through-out the line.

- Group 1, Respondent 12, Question 9

9.1 Identify the impossible graphs. Explain why each of the graphs you have chosen is impossible.

A - Because you cannot go straight upwards without any curve or steep somewhere on the road.

9.2 Which graph reflects constant speed? Explain your answer.

A - constant because shows no change.

- Group 2, Respondent 41, Question 9

9.1 Identify the impossible graphs. Explain why each of the graphs you have chosen is impossible.

A - Impossible - starting point is zero not start at 0.
 B - Impossible - not starting at 0
 C - Impossible - not starting at 0
 E - Impossible - not starting at 0

9.2 Which graph reflects constant speed? Explain your answer.

A - no disturbance. went straight to the required point.

Group 3, Respondent 94, Question 9

9.1 Identify the impossible graphs. Explain why each of the graphs you have chosen is impossible.

D
B
G
H } All these graphs curve sharply

9.2 Which graph reflects constant speed? Explain your answer.

E.

• Group 3, Respondent 118, Question 9

Group 1 did better than Group 2 because the teachers from Group 1 experienced similar problems in the Sediba coursework. Teachers experienced difficulty in connecting graphs to the real world.

5.5.7.6 Question 11

The following table is a reflection of teachers' performance on Question 11 in terms of assigned attributes to this item.

TABLE 30: Results of the qualitative diagnostic analysis on Question 11

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
11.1	A6	Teachers' understood how the mathematical representation could be understood in words, real-life situation.		23	30	20	20
11.2	A7	Teachers could communicate their understanding of the mathematical representation.		13	19	4	15
11.2	A10	Teachers could unpack their understanding of the mathematical representation.		10	19	4	9

Group 1 did patterns in one of their mathematics modules offered. Table 31 represents teachers' choices of answer and understanding of the mathematical representation:

TABLE 31: Percentages of teachers' responses to each of the possible answers.

	(a)	(b)	(c)	(d)
Group 1	27	11	35	22
Group 2	27	11	13	36
Group 3	24	24	22	22

27% of teachers in Group 1, 27% of teachers in Group 2 and 24% of teachers in Group 3 chose the correct answer. 11% of teachers in Group 1, 11% of teachers in Group 2 and 24% of teachers in Group 3 indicated that they understood the representation $R(m) = S(m + 12)$ as meaning salary after 12 months on the job. 11% of teachers in Group 1, 11% of teachers in Group 2 and 24% of teachers in Group 3 indicated that they understood the representation $R(m) = S(m + 12)$ as meaning R12 more than the salary of someone who has worked for m months. 22% of teachers in Group 1, 36% of teachers in Group 2 and 22% of teachers in Group 3 indicated that they could not solve the problem because not enough information was given. The following teachers' response indicates an inability to work with unknowns in the context of this problem.

11.1 If $S(m)$ represents the salary of an employee after m months on the job, what would the function $R(m) = S(m + 12)$ represent?

- (a) The salary of an employee after $m + 12$ months on the job.
- (b) The salary of an employee after 12 months on the job.
- (c) R12 more than the salary of someone who has worked for m months.
- (d) Not enough information.

11.2 Explain your answer in 11.1.

The question is not clear, we don't know what m represent and also R .

• Group 2, Respondent 41, Question 11

These teachers are not competent in understanding different representations of a function. Their ability to move between an equation and a linguistic representation of a function is not sufficient.

5.5.7.7 Question 14

Table 32 is a reflection of teachers' performance on Question 14 in terms of assigned attributes to this item.

TABLE 32: Results of the qualitative diagnostic analysis on Question 14

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
14.1	A6	Could read the answer from the graph.	No / wrong response	48	76	47	28

22% of teachers in Group 1, 51% of teachers in Group 2 and 52% of teachers in Group 3 chose answer (c); an indication of teachers inability to extract information from a representation.

Some teachers experienced difficulty to read the information from the curved graph. Question 14.2 could not be evaluated because there were not enough responses from the teachers.

5.5.8 Familiar / unfamiliar problem contexts

Questions 5 and 10 formed part of this category. Question 10 was taken as correct if the teacher got $8x + 6$ as the answer. The teachers did considerably better in Question 10. This question was set in a familiar problem context (see § 3.7.3). The same procedural knowledge was needed to solve Question 5, but the surface features (problem context) in which the problem was set, made it difficult for the teachers to connect their available procedural knowledge with the problem context of Question 5.

5.5.8.1 Question 5

Table 33 is a reflection of teachers' performance on Question 5 in terms of assigned attributes to this item.

TABLE 33: Results of the qualitative diagnostic analysis on Question 5

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
5.1	A6	Could work from a representation in words to a representation of mathematical information in symbols.	No / wrong response	11	16	16	2
5.1	A11	Could solve the problem in this specific problem context.	No / wrong response	11	16	16	2
5.2	A2	The following procedure was used: $y = x^2$ $y = (x+3)^2$ $= x^2 + 6x + 9$ $(x+3)^2 - x^2$ $= x^2 + 6x + 9 - x^2$ $= 6x + 9$	No / wrong response	7	14	9	0
5.2	A10	Could unpack the solution to the problem (A2).	No / wrong response	6	14	7	0

The most frequent wrong response was (b). 70% of Group 1, 71% of Group 2 and 76% of Group 3 responded that (b) was the correct answer. Some of the explanations for them choosing (b) were:

5.1 In the equation $y = x^2$, if x is increased by 3, then y is increased by

- (a) 6
- ☒ (b) 9
- (c) $6x+6$
- (d) $6x+9$

5.2 Explain your answer in 5.1.

Substituting the value of x by 3 gives you 9 and if you add 3 to x on the equation and let x to zero you get the value of y being 9.

• **Group 1, Respondent 8, Question 5**

5.1 In the equation $y = x^2$, if x is increased by 3, then y is increased by

- (a) 6
- ☒ (b) 9
- (c) $6x+6$
- (d) $6x+9$

5.2 Explain your answer in 5.1.

If we substitute 3 where x is we have 9
for y .
 $y = x^2$
 $y = (3)^2$
 $= 9$

• **Group 2, Respondent 55, Question 5**

5.1 In the equation $y = x^2$, if x is increased by 3, then y is increased by

- (a) 6
- (b) 9
- (c) $6x+6$
- (d) $6x+9$

$$y = x^2$$

$$y = 3^2 = 9$$

$\therefore 9$ is increased by 9

5.2 Explain your answer in 5.1.

If the value of x is 3, then the value of y will be 9.

• **Group 3, Respondent 84, Question 5**

A few teachers choose (d) and their explanations looked like the following two examples:

5.1 In the equation $y = x^2$, if x is increased by 3, then y is increased by

- (a) 6
- (b) 9
- (c) $6x+6$
- (d) $6x+9$

5.2 Explain your answer in 5.1.

$$y = x^2$$

$$y = (x+3)^2$$

$$y = x^2 + 6x + 9$$

~~$y = x^2 + 6x + 9$~~ y is increased by $6x+9$ by
 $y = x^2$.

• **Group 1, Respondent 9, Question 5**

5.1 In the equation $y = x^2$, if x is increased by 3, then y is increased by

- (a) 6
- (b) 9
- (c) $6x+6$
- (d) $6x+9$

5.2 Explain your answer in 5.1.

doubling 3 for x will give you 6 and squaring it for y you'll get 9.

• Group 2, Respondent 45, Question 5

5.5.8.2 Question 10

The following table is a reflection of teachers' performance on Question 10 in terms of assigned attributes to this item.

TABLE 34: Results of the qualitative diagnostic analysis on Question 10

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
10	A2	$f(x+4) - f(x) = ((x+4)^2 + 5) - (x^2 + 5)$ $= x^2 + 8x + 16 + 5 - x^2 - 5$ $= 8x + 16$	No / wrong response	57	70	62	41
10	A3	No procedural mistakes were made and an answer of $8x + 16$ was given as the solution.	No / wrong response	43	54	51	26
10	A11	The problem context was familiar to the teacher.	No / wrong response	74	73	80	70

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$f(x+4) = (x+4)^2 + 5$
 $= x^2 + 8x + 16 + 5$

$f(x) = x^2 + 5$

$\lim_{h \rightarrow 0} \frac{x^2 + 8x + 16 + 5 - (x^2 + 5)}{h}$

$\lim_{h \rightarrow 0} \frac{x^2 + 8x + 16 + 5 - x^2 - 5}{h}$

$\lim_{h \rightarrow 0} \frac{8x + 16}{h}$

$\lim_{h \rightarrow 0} \frac{8x + 16}{h} = 8$

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 5 - (x^2 + 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h) \\
 &= 2x
 \end{aligned}$$

- Group 2, Respondent 57, Question 10

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$$\lim_{h \rightarrow 0} \frac{(x+4)^2 + 5 - (x^2 + 5)}{h}$$

$$\lim_{h \rightarrow 0} \frac{x^2 + 8x + 16 + 5 - x^2 - 5}{h}$$

$$\lim_{h \rightarrow 0} \frac{8x + 16}{h}$$

- Group 2, Respondent 68, Question 10

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$$\begin{aligned} f(x) &= x^2 + 5 \\ f(x+4) &= (x+4)^2 + 5 \\ f(x+4) - f(x) &= (x+4)^2 + 5 - (x^2 + 5) \\ &= x^2 + 8x + 16 + 5 - x^2 - 5 \\ &= 8x + 16 \end{aligned}$$

$$f'(x) = 2x$$

- Group 2, Respondent 69, Question 10

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$$f(x+4) - f(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 + 5 - (x^2 + 5)}{h}$$

$$\lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h}$$

$$\lim_{h \rightarrow 0} \frac{2xh + h^2}{h}$$

$$\lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = 2x$$

- Group 2, Respondent 72, Question 10

It clearly shows that teachers memorise procedures and rules rather than reflect if the outcome makes sense.

Surprisingly, two teachers from Group 3 who has never been exposed to limits or derivatives at the Sediba Project, also tried to solve this problem using the following methods.

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$$\lim_{h \rightarrow 0} \frac{f(x+4+h) - f(x)}{h}$$

$$\lim_{h \rightarrow 0} \frac{x+4+h-x}{h}$$

$$\lim_{h \rightarrow 0} \frac{4+h}{h}$$

$$\lim_{h \rightarrow 0} \frac{4}{h}$$

- Group 3, Respondent 92, Question 10

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x+4) - f(x)$.

$$\begin{aligned} & f(x+4) - f(x) \\ & \frac{x+4 - (x^2+5)}{x+4} \\ & = \frac{x+4 - x^2 - 5}{x+4} \\ & = \frac{-x^2 + x - 1}{4} = \frac{x^2 - x + 1}{4} = \frac{1}{4} \end{aligned}$$

- Group 3, Respondent 98, Question 10

These teachers recognised that $f(x+4) - f(x)$ looks like a part of the formula for determining the derivative of a function from first principles ($\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$). It is clear that these teachers have learned arithmetic procedures for determining limits and the derivative of a function from first principles by rote rather than by constructing them on the basis of conceptual understanding of a limit or a derivative. The surface features of the problem made them think that they should determine the derivative from first principles ($f'(x)$) and their procedural knowledge connected to this to solve the problem. This is evidence again that familiar and unfamiliar problem contexts vary according to a person's previous experiences with mathematics. These teachers did not have the

conceptual knowledge needed to increase the usefulness of their procedural knowledge in the context of this problem. This is an example of how knowledge that is not conceptually connected plays a role in familiar and unfamiliar problem contexts and solving mathematical problems.

5.5.9 Ratios in a probability and combinations context

Questions 29 and 34 were analysed to describe teachers' understanding of ratio in the context of probability and combinations.

5.5.9.1 Question 29

The following table is a reflection of teachers' performance on Question 29 in terms of assigned attributes to this item.

TABLE 35: Results of the qualitative diagnostic analysis on Question 29

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
29.1	A2	$P = \frac{4}{12} = \frac{1}{3}$	No / wrong response	16	19	18	13
29.1	A9	Connection was made that you are working with a fraction.	No / wrong response	14	16	16	11
29.3	A7	Teacher could communicate his understanding of the problem in words.	No / wrong response	21	22	18	24
29.3	A10	The meaning of the fraction could be unpacked.	No / wrong response	8	8	4	11

32% of teachers from Group 1, 11% of teachers from Group 2 and 22% of teachers from Group 3 stated that the 12 participant had 3 chances to be among the top 4 contestants. An example of one of these responses is given:

29. The number of Idols participants is decreasing as they are voted out of the competition by the public. There are now just 12 participants left in the competition.

29.1 What chance does each of the 12 participants have to be among the top 4 contestants?

Only 3 chances.

29.2 Does a contestant's chance decrease or increase each time he/she is not voted out?

Increase.

29.3 Explain your answer in 29.2.

The fewer the contestants the more the chance of winning the competition.

• Group 1, Respondent 8, Question 29

An example of how these teachers thought about the problem and came to a response of "3 chances" is given below:

29. The number of Idols participants is decreasing as they are voted out of the competition by the public. There are now just 12 participants left in the competition.

29.1 What chance does each of the 12 participants have to be among the top 4 contestants?

$$\frac{12}{4} = 3$$

∴ 3 chances

29.2 Does a contestant's chance decrease or increase each time he/she is not voted out?

Increases

29.3 Explain your answer in 29.2.

If 2 contestants are voted out = 10 contestants and two chances will be left.

$$\therefore \frac{10}{2} = 5,0$$

⇒ Contestant's chance increases.

• Group 1, Respondent 34, Question 29

16% of the teachers from Group 2 tried to solve the problem by using formulas like $P(12;4)$, $C(12;4)$, $12 \times 11 \times 10 \times \dots$ and $\frac{12!}{4!}$. This group of teachers did Permutations, Combinations and the Fundamental principle of counting during the first semester of the year in which this test was conducted. The problem context led them to believe that these concepts had something to do with this kind of problem.

29. The number of Idols participants is decreasing as they are voted out of the competition by the public. There are now just 12 participants left in the competition.

29.1 What chance does each of the 12 participants have to be among the top 4 contestants?

They have 495 chances.

$$= \frac{12!}{8!4!} = \frac{12 \times 11 \times 10 \times 9 \times 8!}{8! \times 4 \times 3 \times 2 \times 1} = 11 \times 5 = 55$$

29.2 Does a contestant's chance decrease or increase each time he/she is not voted out? *The chances increase.*

29.3 Explain your answer in 29.2.

The more the number of contestant decreases the more the chances of the remaining contestants.

- Group 2, Respondent 68, Question 29

29. The number of Idols participants is decreasing as they are voted out of the competition by the public. There are now just 12 participants left in the competition.

29.1 What chance does each of the 12 participants have to be among the top 4 contestants?

$$p(n; k) = \frac{n!}{k!(n-k)!} = \frac{12!}{4!(12-4)!} = \frac{12!}{4!8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{12 \cdot 11 \cdot 5}{2} = \frac{12 \cdot 11}{2} = 66$$

29.2 Does a contestant's chance decrease or increase each time he/she is not voted out?

Decreases.

29.3 Explain your answer in 29.2.

* Because they have less chances to be selected among the top 4

• Group 2, Respondent 76, Question 29

29. The number of Idols participants is decreasing as they are voted out of competition by the public. There are now just 12 participants left in the competition.

29.1 What chance does each of the 12 participants have to be among the top 4 contestants?

$$\frac{12}{4} = 3 \text{ chances}$$

29.2 Does a contestant's chance decrease or increase each time he/she is voted out? decreases

29.3 Explain your answer in 29.2.

Once the top 4 is reached only one ~~chance~~ chance now

• Group 2, Respondent 51, Question 29

The respondent confused a chance or probability with an opportunity.

5.5.9.2 Question 34

The following table is a reflection of teachers' performance on Question 34 in terms of assigned attributes to this item.

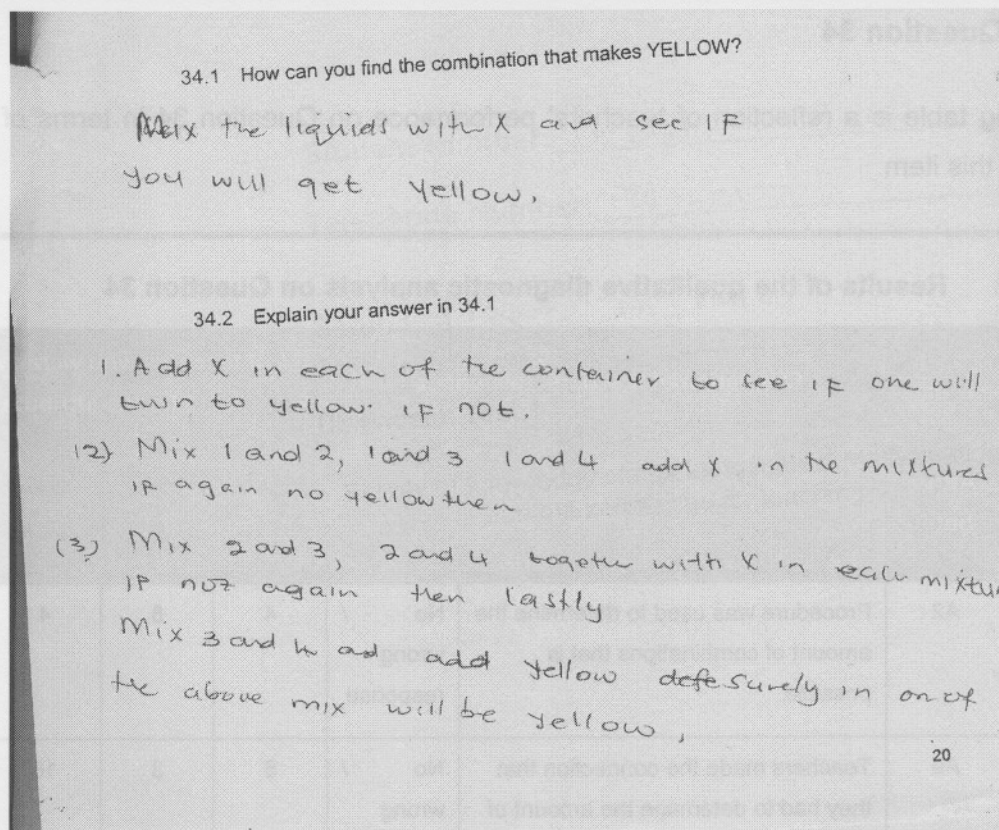
TABLE 36: Results of the qualitative diagnostic analysis on Question 34

Question	Attribute	Evaluation of teachers' responses		% of teachers who were awarded a 1 for the attribute			
		1 if	0 if	Whole group	Group 1	Group 2	Group 3
34.1	A2	Procedure was used to determine the amount of combinations that is possible.	No / wrong response	4	8	4	0
34.2	A9	Teachers made the connection that they had to determine the amount of all possible combinations.	No / wrong response	6	3	16	0

Some of the proposed solutions included: $1 + x, 2 + x, \dots, 1, 2, 3, 4$ with x and $1 + 2 + 3 + 4 + x$.

Some of the teachers in Group 2 use formulas like $C(5,1)$ and $C(4,1)$.

The following response was the only one that included a description of most of the different combinations:



- **Group 1, Respondent 37, Question 34**

5.6 AN OVERALL EVALUATION OF THE RESULTS

For this group of teachers, proportional reasoning is a case of computational fluency in a familiar problem context rather than intuitive recognition of a proportional situation. Indications are that some of these teachers do not have an intuitive understanding of ratio. They do not have a deep understanding of the purpose of a proportional problem, namely to keep the value of a ratio equivalent under transformation (like in the numerical comparison problems).

Indications are that these teachers would use additive rather than multiplicative reasoning in an unknown problem context because their first quantification in proportional reasoning items is not multiplicative in nature, but additive - an additive structure of $(a - b)$ rather than a ratio $\frac{a}{b}$.

For all three groups, Questions 27.3 and 32 were more difficult to solve. Question 27.3 because of the numerical difficulty the problem was set in, and Question 32 because of the context the problem was set in. The problem context and the nature of the numerical relationship influenced the teachers' ability to solve the proportional problems. Teachers can only solve proportional

problems from a familiar problem context in which a proportion features as a direct proportion. Evidence exist that an understanding of an inversely proportional situation is a problem for these teachers (see § 5.5.6.2). However, good proportional reasoners should not be affected by the numerical setting or the context the proportional problem is set in (see § 4.4.3.1.2).

Teachers' (mostly Groups 1 & 2) use of the cross-product algorithm is largely mechanical and stripped of meaning. Results indicate that the cross-product algorithm is not sufficiently understood by the teachers and is not generated naturally. Although it should not be understated that it is an efficient procedure, it is clearly used as a means of getting an answer rather than to understand the problem. The findings largely indicate that the teachers have an instrumental rather than a relational understanding of proportional problems because of their inability to translate their knowledge to new contextual situations. Teachers can only use the most basic properties of proportion. It seems as if using the cross-product algorithm is simply an act of symbolic manipulation. Teachers are clearly not making sense of what they are doing when they are working with the cross-product algorithm.

It is very likely that these teachers' school based exposure to proportional reasoning problems has been and are only procedural, without sense-making. It is also likely that they will teach learners in the same way. Although the cross-product algorithm is an efficient way of getting answers, it is clearly used without meaning. These teachers can be regarded as novices in solving proportional reasoning problems.

The following aspects stand out:

- Overall, Groups 1 & 2 showed more procedural fluency in solving problems (see § 5.5.1.1).
- Teachers' solutions to some of the problems show a lack in their ability to communicate or unpack understanding to problems (see § 5.5.1.2, § 5.5.1.3, § 5.5.3.1).
- An absence of a tendency to reflect on a required answer is evident among these teachers. There is inconsistent behaviour to reflect on an answer and evaluate it for its validity or to make a correct interpretation of the answer (see § 5.5.2.1, § 5.5.2.2, § 5.5.3.2).
- The social setting of a problem provides a barrier in mathematically solving problems (see § 5.5.2.2). Problems set in the context of food makes for social justifications like equal amounts of food for every person involved seemed to overshadow the mathematical solution.
- Various misconceptions in the teachers' mathematical knowledge base prevails (see § 5.5.3.2, § 5.5.6.1, § 5.5.6.2, § 5.5.8.2).

- The teachers have difficulty to work with multiple pieces of linguistic information and are unable to translate these multiple pieces of linguistic information into corresponding mathematical language (see § 5.5.7.1).
- Additive reasoning in unfamiliar proportional situations is practiced (see § 5.5.7.3).
- Teachers cannot use language of ratio and proportion accurately and flexibly to compare quantities or describe a proportional relationship.
- Teachers cannot move between different representations of a function with understanding (see §5.5.7.5).

These teachers have gaps in their mathematical knowledge base concerning primary and middle school mathematics: their tendency to use additive reasoning when dealing with unfamiliar proportional problems and also their inability to convert millilitres to litres. This has implications for the development of appropriate MCKfT as a sound knowledge base in school mathematics is important for the development of MCKfT (see § 2.5). These findings have serious implications for the development of MCKfT.

5.7 CONCLUSION

The goal is to describe the teachers' current mathematical content knowledge states in terms of important characteristics of MCKfT. It is apparent that there are several constraints in these teachers' mathematical content knowledge base. In the next chapter criteria and a model for the development of MCKfT will be proposed.

CHAPTER 6

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

6.1 INTRODUCTION

The aim of this chapter is to give a summary of the theoretical research and to state the most apparent findings and conclusions from the empirical study. Recommendations concerning the development of MCKfT during teacher education (in the form of a model and specific criteria) will then be proposed.

6.2 A CURSORY SUMMARY

The persisting problem of learners who underachieve in school mathematics in the South African context was posed in Chapter 1. This research focused on one particular factor, which influences the teaching and learning of mathematics, namely teachers' mathematical content knowledge for teaching. The following questions were raised: What kinds of mathematics do teachers know and how do they understand the mathematics they know? How can teachers' current mathematical content knowledge states be transformed so that they are able to facilitate learners' achievement in school mathematics?

The theoretical study in Chapter 2 led to the acknowledgment of a special kind of mathematical knowledge, which can enable a teacher to teach mathematics with understanding. This important kind of knowledge was named mathematical content knowledge for teaching (MCKfT). The characteristics of MCKfT were established in the rest of the theoretical study (see Chapter 3). These characteristics were posed as theoretical criteria for the development of MCKfT and a theoretical framework for assessing teachers' mathematical content knowledge states were proposed.

Through planning and conducting a qualitative experiment (see Chapter 4) in the form of a diagnostic content analysis, teachers' current mathematical content knowledge states could be investigated and described. The characteristics of and criteria for the development of MCKfT formed the diagnostic indicators and were called attributes. The attributes include aspects of conceptual knowledge, which includes understanding a problem, procedural knowledge needed to solve a problem, the ability to correctly apply a procedure, to interpret an answer correctly, to solve

mathematical problems, to understand different representations, to communicate understanding in writing, reasoning skills, the state of connections to different mathematical ideas, the ability to unpack mathematical understanding and the ability to understand the context a problem is set in.

In Chapter 5 a thorough description and examples of teachers' responses to the diagnostic content analysis were given. The most prominent findings and conclusions will be discussed in the following section. Findings from the theoretical study will be connected with results from the empirical study.

6.3 FINDINGS AND CONCLUSIONS

In the empirical study, teachers' current mathematical content knowledge states were investigated according to the characteristics of MCKFT. The most noticeable findings are the following:

Procedural knowledge and conceptual knowledge: These teachers' mathematical knowledge is overshadowed by unconnected procedural knowledge. Their procedural knowledge has become automatised. This is concluded from the various examples in which teachers use wrong procedural knowledge in a specific problem context (see § 3.6.2.1, Question 10). Because of the teachers' inability to use available mathematical knowledge in an unfamiliar problem context, it can be said that their mathematical knowledge base is not rich in interconnected connections. This can point to a lack of conceptual knowledge (see § 3.6.2.3). These teachers' response to mathematical problems indicates that they have an instrumental rather than relational understanding of mathematics as a whole. The procedural understanding of mathematics exhibited by these teachers is not adequate for teaching mathematics with understanding. Teacher current mathematical content knowledge states can be described as follows: "...islands of superficial knowledge without a canoe to get from one to the other." (Singh, 2000:598).

Teachers as proportional reasoners: These teachers do not have a proficient understanding of the biggest mathematical idea in elementary and middle school, namely proportionality with its pattern $\frac{a}{b} = \frac{c}{d}$ (see § 3.8.2).

Misconceptions: Various misconceptions have been identified in these teachers' mathematical knowledge base through their written communication in the diagnostic instrument (see § 5.5.7.2).

Communication: One of the most important tasks of a mathematics teacher is to be able to communicate mathematical understanding in writing. These teachers do not show efficient levels of being able to communicate mathematical understanding (see § 5.6).

Connections: Teachers' mathematical knowledge is fragmented into unrelated pieces of information. Teachers' responses showed an inability to solve a problem in more than one way, which also indicates an unconnected mathematical knowledge base (see § 3.7.4.1). Teachers' conceptual and procedural knowledge are not connected, which is evident from the fact that they can generate an answer to a problem but do not understand the meaning of the answer or what they actually did to solve the problem (see § 3.7.4.1).

Representations: Teachers' understanding of different representations and the ability to move among different representations of mathematical knowledge are not sufficient (see § 5.6).

Problem context: Teachers' knowledge seems to be contextually bound and not transferable to an unfamiliar problem context. Recognition of the appropriate problem context in which a specific procedure should be used was described in paragraph 3.7.3 and 5.5.8.

Unpacking: These teachers have difficulty unpacking their understanding of a mathematical idea because of their mathematical content knowledge states that can largely be described as unconnected and fragmented procedural knowledge packages (see § 3.6.3).

Problem solving: These teachers experienced difficulty in translating a problem from a linguistic representation to a mental representation (see § 5.5.6.1). Connections that lead to greater flexibility in problem solving are not evident from results and therefore teachers are unable to approach problems with alternative solving strategies.

In-service teacher training: The impact of the Sediba Project is clearly influenced by teachers' prior knowledge of mathematics. Conceptual understanding of primary and intermediate school mathematics that should form a sound mathematical basis for more advanced topics in the school curriculum (see § 2.5) is a stumbling block for the acquisition of more advanced mathematical topics and is also for the most part procedurally based with little or no conceptual understanding. Indications are that teachers involved in the Sediba Project for one or two years benefited from the in-service teacher training program in almost all the attributes measured. However, appropriate levels of mathematical understanding are not reached and these teachers are thus not able to teach mathematics with understanding. Their current mathematical content knowledge states do not correspond to an acceptable extent with the characteristics of MCKfT.

In conclusion, teacher education programs should focus on exposing teachers' prior knowledge and transforming teachers' current mathematical content knowledge states according to the theoretically proposed criteria for the development of MCKfT. In the next section recommendations for the development of MCKfT are made.

6.4 RECOMMENDATIONS

More than 15 years ago Kathy Carter stated that in teacher education attention has traditionally been paid to what teachers need to know and how they can be trained, rather than on what teachers actually know or how that knowledge is acquired (Carter, 1990:291). This is still true for the most part today.

There is a need to find ways to determine what teachers know mathematically and how far off teachers' current mathematical content knowledge states are from the identified characteristics of MCKfT. How to strip mathematical content knowledge states for misconceptions and how to go about developing and acquiring MCKfT, with everything this special kind of content knowledge implies, also need attention.

The work done in this study focused on assessing what teachers know mathematically by mapping their current mathematical content knowledge states of certain mathematical content onto what they should actually know (MCKfT) to be competent mathematics teachers. Research on how MCKfT can be acquired needs further attention as it is a rather unexplored field. There is a need to identify MCKfT according to each topic in the mathematics school curriculum and accompanying diagnostic content test so that teachers' prior knowledge states can be identified before further education is applied. Ways of identifying individual teachers' content knowledge states should be explored.

Finally, criteria for the development of MCKfT will now be proposed in the form of a model for mathematics teacher education with regard to teachers' MCKfT. The proposed model for the development of MCKfT is one that is congruent with the nature of mathematics. The components are identified as being valuable for MCKfT. Focus should shift from what teachers know mathematically to how they know mathematics. The model for the development of MCKfT includes the following:

Reflective mathematics teaching and learning: The foundation of the model is constant awareness of what you are doing while you are teaching and learning mathematics. The development of MCKfT should be strengthened through encouraging a reflective frame of

mind. The end results should be mathematical knowledge and understanding that a teacher is able to set in motion while teaching mathematics.

Procedural understanding: There should be constant reflection by the teacher on knowing how to carry out an algorithm and knowing why it makes sense mathematically. Teachers should reflect on limits in their understanding of procedural knowledge. What they are doing in every step of a procedure and why they can do it should be a case of continuously unpacking, decompressing and then connecting procedural knowledge to be able to use it in different appropriate problem contexts.

Conceptual understanding: Reflection on the existence and understanding of important concepts should be fostered. To construct concepts and discover relationships or connections between concepts and between concepts and procedures should be a focus in mathematics teacher education.

Representational understanding: Because understanding is stimulated by reflection on different representations of the same concept, exposure to multiple representations of a concept should be a focus of mathematics teacher education.

Reasoning: The teachers' ability to use reasoning to reflect on the validity of the answer of a mathematical problem should be fostered. Experience with inductive and deductive reasoning should be created (see § 3.7.2). The idea that the reason for an answer is as important as the answer itself should be fostered.

Connections: There must be reflection on prior knowledge before connections can be made because one can only connect what one knows. Connected knowledge of the different mathematics fields or concepts connected with a particular mathematical topic should be fostered. Fragmented mathematical topics should be connected. Teachers must be able to identify a concept and assess connections between concepts. Attention should be paid to the different types of connections that exist throughout the school curriculum. Measures of the quality and quantity of connections that an idea has with existing ideas should be reflected upon.

Problem context: The degree to which a unit of knowledge is tied to specific problem context should be challenged and this should include reflection on the transferability of knowledge to new situations. The context that the mathematics teachers are used to should be assessed. Difficult aspects of teaching a particular concept and the effect of the problem context should also be reflected on.

Communication: Teachers should constantly reflect on their ability to comprehend the language of mathematics, orally and in writing. The ability to articulate understanding to others must be fostered and teachers should learn to be clear and convincing.

Unpacking: Reflection on previously learned mathematics that may have become automated should be fostered. The ability to decompress and unpack mathematical content knowledge should become a habit for a mathematics teacher.

Problem solving: Teachers must be able to reflect on their prior knowledge in new problem situations. Problem solving should be fostered by representing a mathematical topic or concept in various ways, and by translating between different representations of concepts. This aspect is important because learners have different prior knowledge and new knowledge should be connected to prior knowledge to be able to solve problems.

The mathematical content: Teachers need to experience mathematics as a subject of regulated pattern and logical order. Focus should be on building the logical structure of mathematics. A broad reflection on mathematical content should include: working with misconceptions, viewing earlier content from an advanced perspective and researching the historical origins of mathematical knowledge. Mathematical content and cognitive processes should be important, that is, analysis of mathematical structure and working with the context that problems are set in. Teachers should be exposed to learning experiences where they can view some important mathematical topics (like proportional reasoning) across Grade 1 to Grade 12.

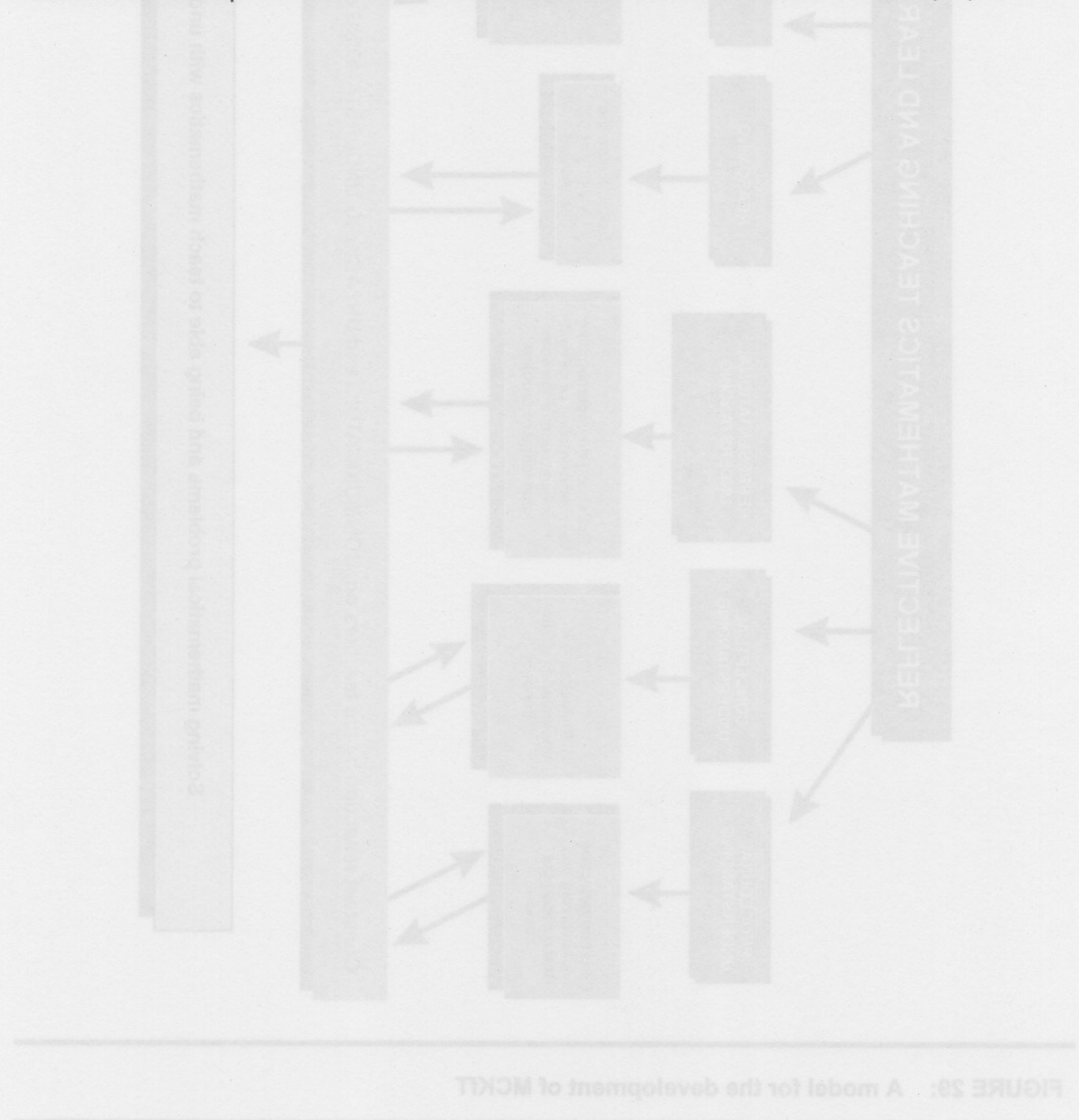
The tasks: Teachers should be exposed to meaningful tasks that include the various aspects of the development of MKCfT. Tasks should include aspects of learners' understanding in certain problem contexts and imaginary learners' solutions to problems. This is important because teachers need to reflect on what learners need to know to get to the next level of understanding.

Assessment: Assessment in a teacher education program should reflect what kind of mathematical knowledge is valued (see § 2.5). For the development of MCKfT, assessment that implies teachers' unpacking of their mathematical knowledge and understanding should be valued. Instrumental understanding of mathematics (rules without reasons) should be discouraged by not assessing just that. Relational understanding (knowing both what and why) should be fostered by assessing teachers' ability to answer questions like: "Why can I use this procedure in this problem context?" or "Why does this make sense?" or "How do you

know that?" or "Why do you think that?" A question like "Do you know it?" Should be exchanged with "How do you understand this?".

Indications are that teachers' current beliefs about the teaching and learning of mathematics could be challenged and altered through exposure to valuable learning experiences by themselves (see § 3.5.4).

Out of the previously discussed important component for the development of MCKfT, the following model for the development of MCKfT at mathematics teacher education institutions is proposed:



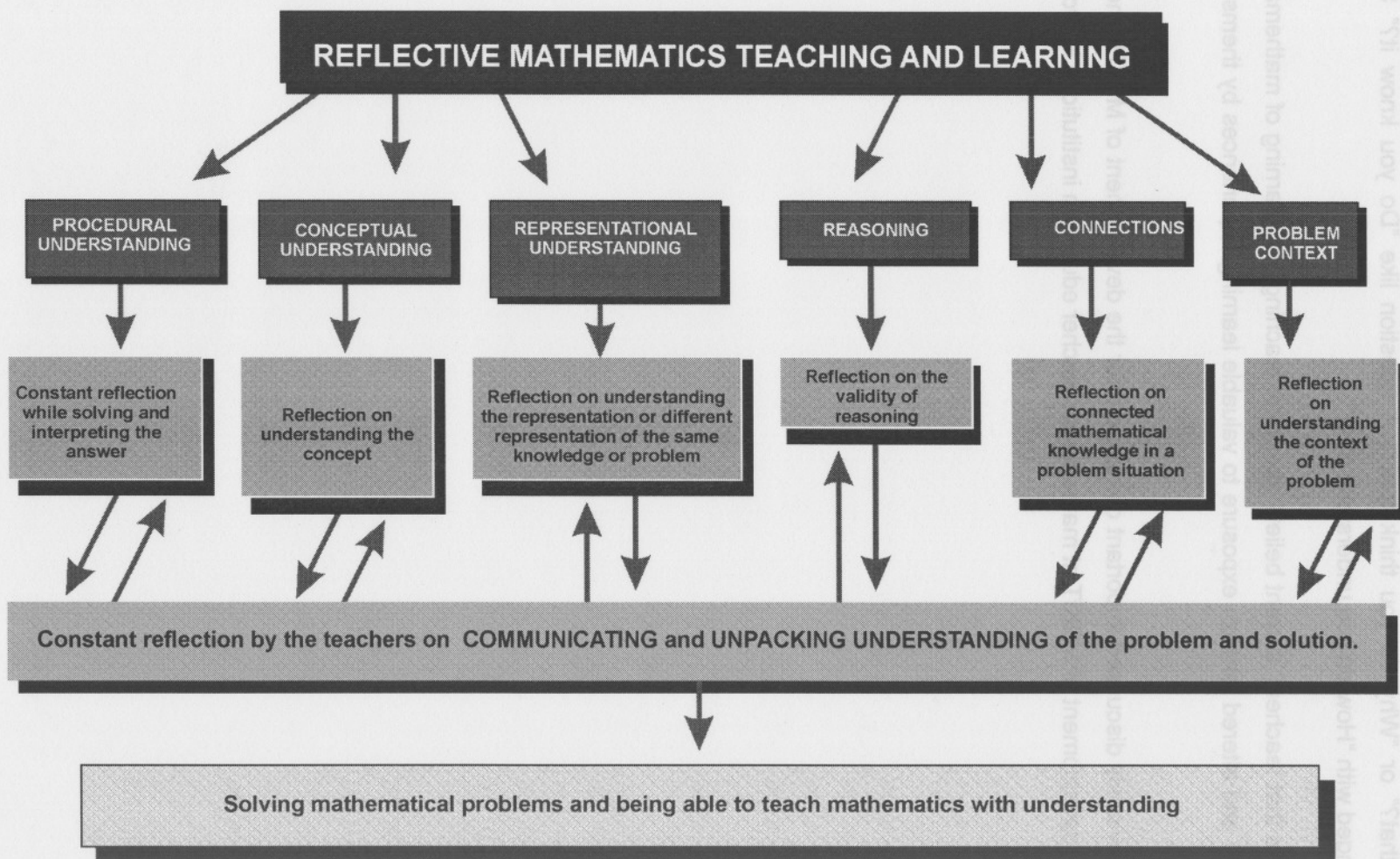


FIGURE 29: A model for the development of MCKFT

The main idea behind this model is that a habit of reflective practices in teaching and learning mathematics should be developed. The focus of reflection should be on developing the characteristics of MCKfT. This should enable a mathematics teacher to communicate mathematical understanding to learners. The habit of unpacking mathematical knowledge and understanding should be encouraged throughout mathematics teacher education. Consequently, mathematical problem solving and teaching mathematics with understanding should benefit from this model of mathematics teacher education.

6.5 A FINAL NOTE

"If it is implausible to expect learners to understand mathematics simply by being told, why is it any less implausible to expect teachers to learn a "new" mathematics simply by being told? If learners need new instruction to learn to understand mathematics, would not teachers need new instruction to learn to teach a "new" mathematics?"

(Cohen, 1990:343)

Everything that had been proposed in this chapter has to start at teacher education institutions. This is a huge task because evidence from teachers' current mathematical content knowledge states indicates huge gaps concerning mathematical knowledge from school experiences. The gaps between teachers' current mathematical content knowledge states and ideal MCKfT pose an enormous task to accomplish. Hopefully, the proposed model can assist various role players in mathematics teacher education to rethink mathematics teacher education where MCKfT is concerned.

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APPENDIX A

Research on the Sediba Project in 2005

28-09-2005

Dear Sediba Class of 2005,

You are hereby kindly requested to participate in a research activity for mathematics teachers. You might be aware that we are constantly trying to improve your learning experiences at Sediba. In order to further this improvement of the program we need your valuable input.

The focus of the research is on developing mathematics teachers' content knowledge for teaching. You are thus asked to complete two diagnostic tests on mathematical content knowledge.

You are asked to do your utmost best in answering the questions. Only then will it be possible to get a true reflection of your mathematical content knowledge states for this specific small section of mathematics after one year of study for the first year students and after two years of study for the second year students.

This is not an exam and it will not be evaluated as a pass or a fail. It is an attempt to investigate the **knowledge patterns** of mathematics teachers.

This is also a declaration that except for the researcher, nobody else will have access to an individual's knowledge states.

Thank you very much for your participation.

Mrs M Plotz

Student number

Signature of student

APPENDIX B

Research on the Sediba Project in 2006

04-01-2006

Dear Sediba Class of 2006

You are hereby kindly requested to participate in a research activity for mathematics teachers. We are constantly trying to improve your learning experiences at Sediba. In order to further this improvement of the program we need your valuable input.

The focus of the research is on developing mathematics teachers' content knowledge for teaching. You are thus asked to complete two diagnostic tests on mathematical content knowledge.

You are asked to do your utmost best in answering the questions. Only then will it be possible to get a true reflection of your mathematical content knowledge states for this specific small section of mathematics. You will be evaluated again at the end of your two years of study at Sediba.

This is not an exam and it will not be evaluated as a pass or a fail. It is an attempt to investigate the **knowledge patterns** of mathematics teachers.

This is also a declaration that except for the researcher, nobody else will have access to an individual's knowledge states.

Thank you very much for your participation.

Mrs M Plotz

Student number

Signature of student

APPENDIX C

The diagnostic instrument

- 1.1 Answer the following question by completing the table below. If the tax on a purchased item that costs R20 is R2.20, how much tax will there be on an item of R45?

Cost of item	20	40	5	45
Tax on item	2.20			

- 1.2 What is the tax rate on these items?

2. A horse walks away from its stable. His distance from the stable as time passes is shown in the table below:

Time (Minutes)	Distance (meters)
0	6
1	12
2	18
3	24

- 2.1 How far was the horse from the stable when he started walking?

- 2.2 At what rate is the horse walking away from the stable?

- 2.3 Is the horse walking at a constant rate?

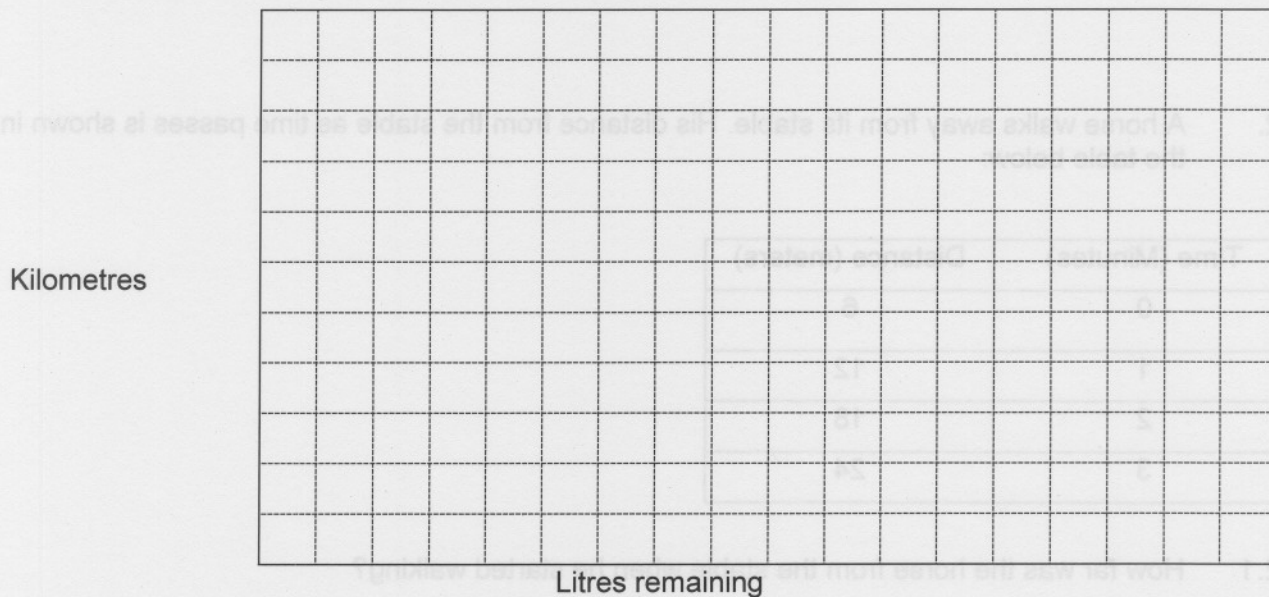
- 2.4 Explain your answer in 2.3.

3. A car travels 10 kilometres on one litre of petrol. It has a petrol tank with a capacity of 55 litres. The car starts on a journey with a full tank.

- 3.1 Complete the table below. (The table should show the number of litres remaining for at least four different points of a trip of 500 kilometres.)

Kilometres	Litres remaining
0	55
100	
300	
400	
500	

- 3.2 Plot the data from the table in 3.1 on the graph below.



- 3.3 How can you tell from the graph how much petrol will be left after 200 kilometres?
- 3.4 How many kilometres did the car travel before it had only 20 litres left?
- 3.5 Explain your answer in 3.4
- 3.6 Determine the gradient of the line you have plotted?
- 3.7 What does the gradient of the line represent?
- 3.8 Is the gradient of the line increasing or decreasing?

- 3.9 What interpretation can you give to your answer in 3.8?
- 4.1 Two out of every three students who eat in the cafeteria drink a glass of milk. If 90 students eat in the cafeteria, how many litres of milk were consumed, assuming that each glass contains 250 ml?

4.2 Explain your answer in 4.1.

5.1 (Educational testing services, 2005)

In the equation $y = x^2$, if x is increased by 3, then y is increased by

- (a) 6
- (b) 9
- (c) $6x+6$
- (d) $6x+9$

5.2 Explain your answer in 5.1.

6.1 (Educational testing services, 2005)

Which of the following statements are true if $4y = 2x$?

- (a) The value of y increases 4 times as fast as the value of x .
- (b) The value of y increases 2 times as fast as the value of x .
- (c) The value of y increases $\frac{1}{2}$ as fast as the value of x .
- (d) The value of y increases $\frac{1}{4}$ as fast as the value of x .

6.2 Explain your answer in 6.1

7.1 Determine the equation of a function f if the point (1 ; 3350) is on the graph of the function and the gradient of the graph is 2000.

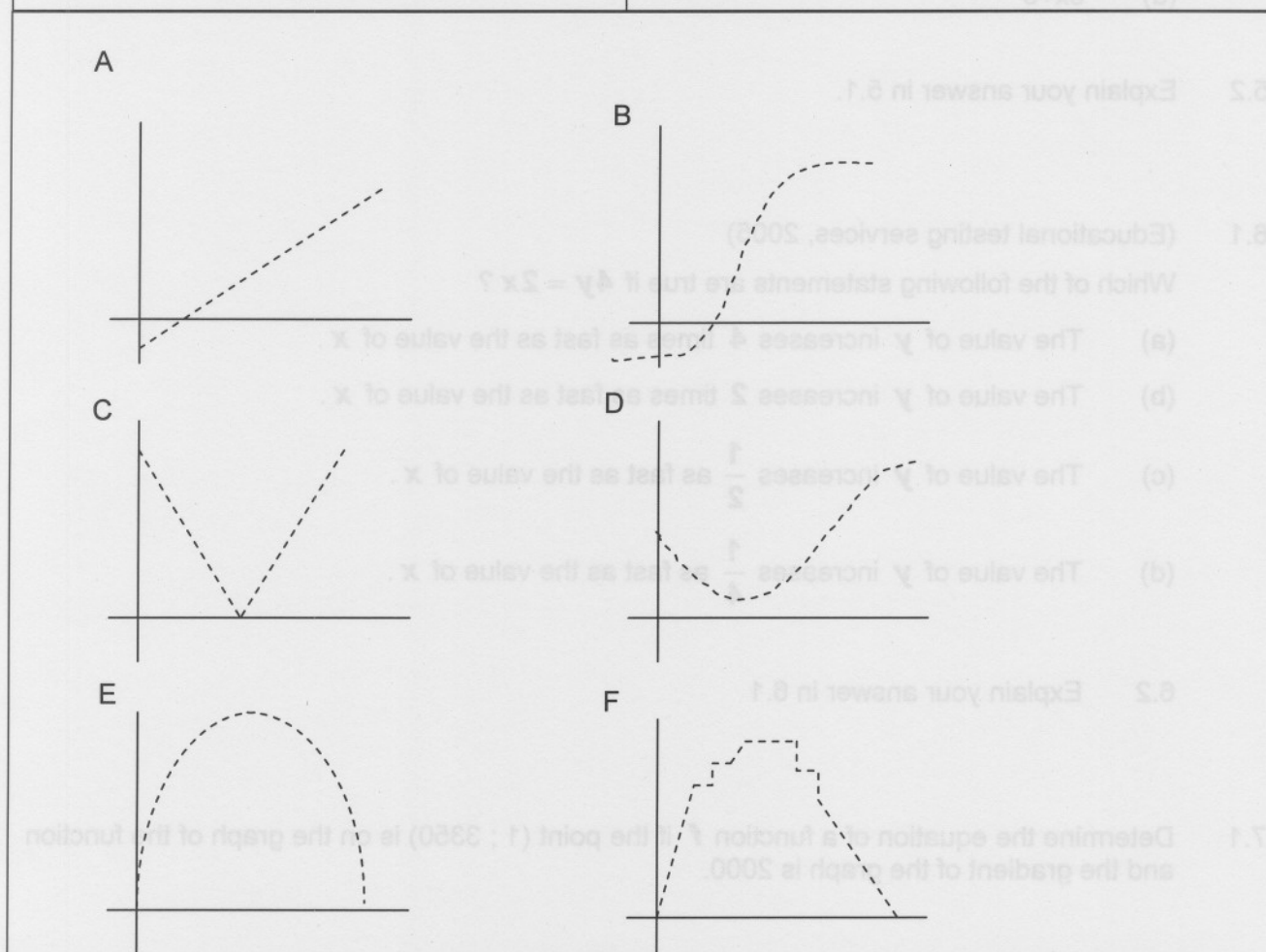
7.2 How did you know which equation to use for the function?

7.3 Determine $f(5)$.

8. (Van de Walle, 2004:448)

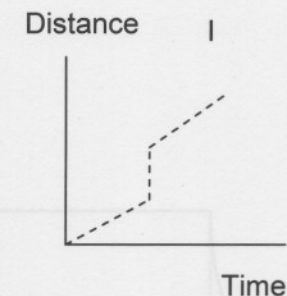
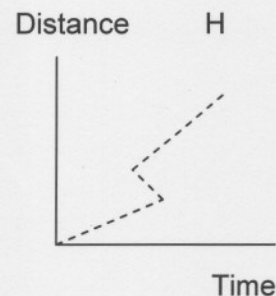
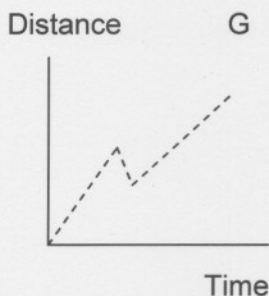
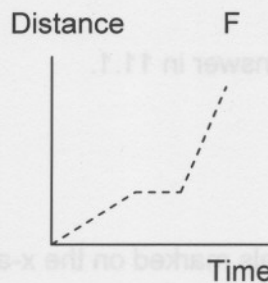
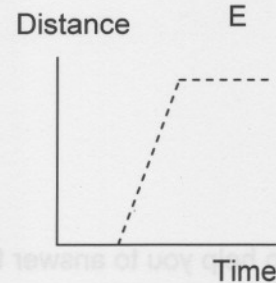
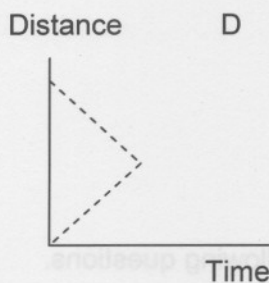
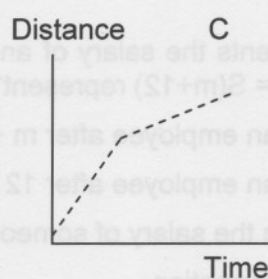
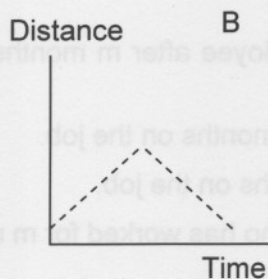
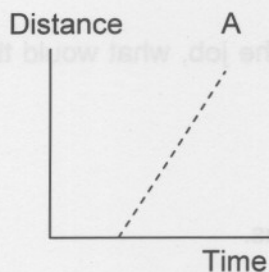
Match each situation described in the table with one of the graphs given.

1. The temperature of a frozen dinner from 30 minutes before it is removed from the freezer until it is removed from the microwave and placed on the table.	4. The height of a baseball in terms of time from when it is thrown straight up into the air to the time it hits the ground.
2. The level of water in a bathtub from the time you begin to fill it until it is completely empty.	5. The speed of the baseball in situation 4.
3. Profit in terms of number of items sold.	



9. (Van de Walle, 2004:449)

Consider the time-distance graphs plotted below. Each is supposed to represent a journey of a single vehicle or person. Some of them are impossible and could not represent any journey.



9.1 Identify the impossible graphs. Explain why each of the graphs you have chosen is impossible.

9.2 Which graph reflects constant speed? Explain your answer.

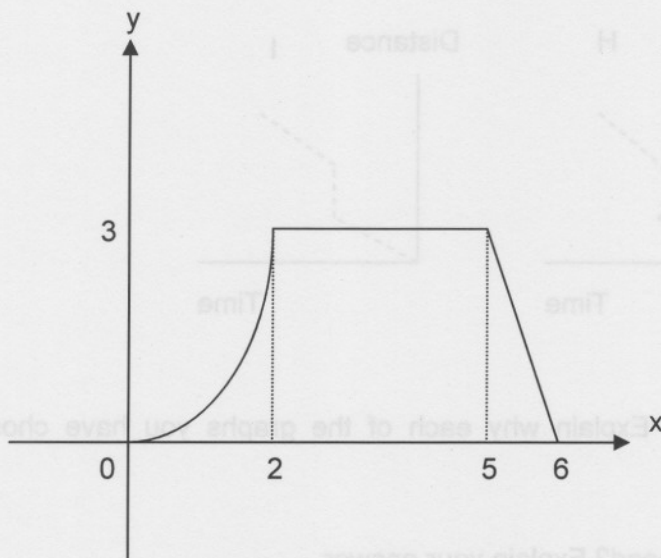
9.3 Create a journey story for each of the graphs A, E and G.

Graph A:

Graph E:

Graph G:

10. The function $f(x) = x^2 + 5$ is given. Determine $f(x + 4) - f(x)$.
- 11.1 If $S(m)$ represents the salary of an employee after m months on the job, what would the function $R(m) = S(m+12)$ represent?
- The salary of an employee after $m + 12$ months on the job.
 - The salary of an employee after 12 months on the job.
 - R12 more than the salary of someone who has worked for m months.
 - Not enough information.
- 11.2 Explain your answer in 11.1.
12. Use the intervals marked on the x-axis to help you to answer the following questions.

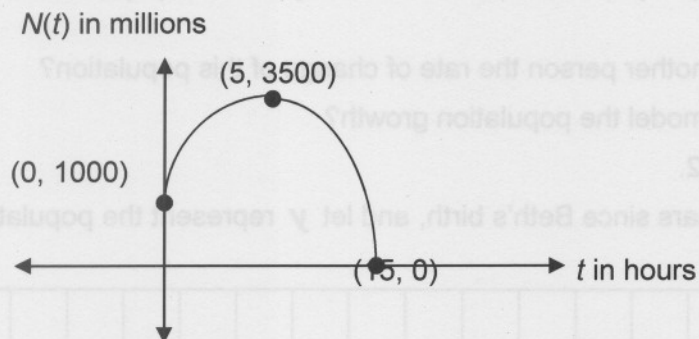


- On which interval(s) is the graph increasing?
- Explain your answer in 12.1.
- On which interval(s) is the graph decreasing?
- Explain your answer in 12.3.
- On which interval(s) is the graph a linear function?
- Explain your answer in 12.5.

12.7 On which interval(s) is the graph non-linear?

12.8 Explain your answer in 12.7.

13. A wound was treated with a medicine to reduce the bacteria present. The number of bacteria, $N(t)$, remaining t hours after the medicine has been administered is shown in the graph below.



13.1 How many bacteria are present at the moment the medicine is administered?

13.2 What is the maximum number of bacteria that were present at any stage?

13.3 After how many hours is the number of bacteria a maximum?

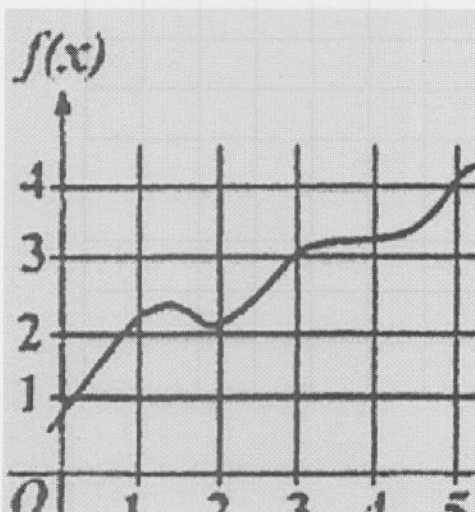
13.4 During which interval is the number of bacteria increasing?

13.5 During which interval is the number of bacteria decreasing?

13.6 How long does it take to kill all the bacteria?

14.1 (Educational testing services, 2005)

The figure below shows a section of the graph of a function f .



According to the graph, if $f(x) = 3.6$, then x is between which of the following?

- (a) 1 and 2
- (b) 2 and 3
- (c) 3 and 4

(d) 4 and 5

14.2 Explain your answer in 14.1.

15. (Murdock et al., 2002:270)

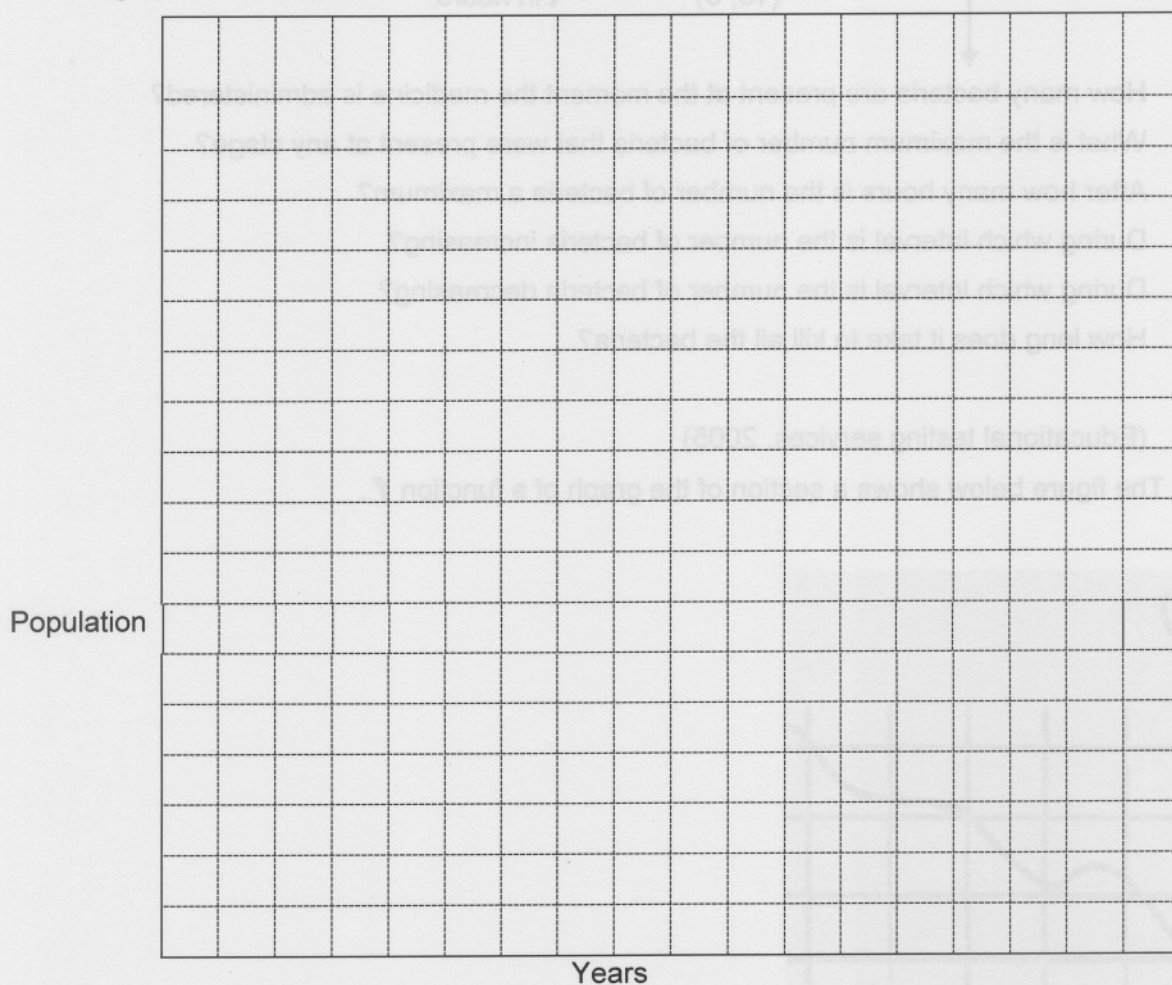
Since the time Beth was born, the population of her town has increased with approximately 850 people per year. On Beth's 9th birthday the total population was nearly 307 650.

15.1 How will you describe to another person the rate of change of this population?

15.2 What type of function can model the population growth?

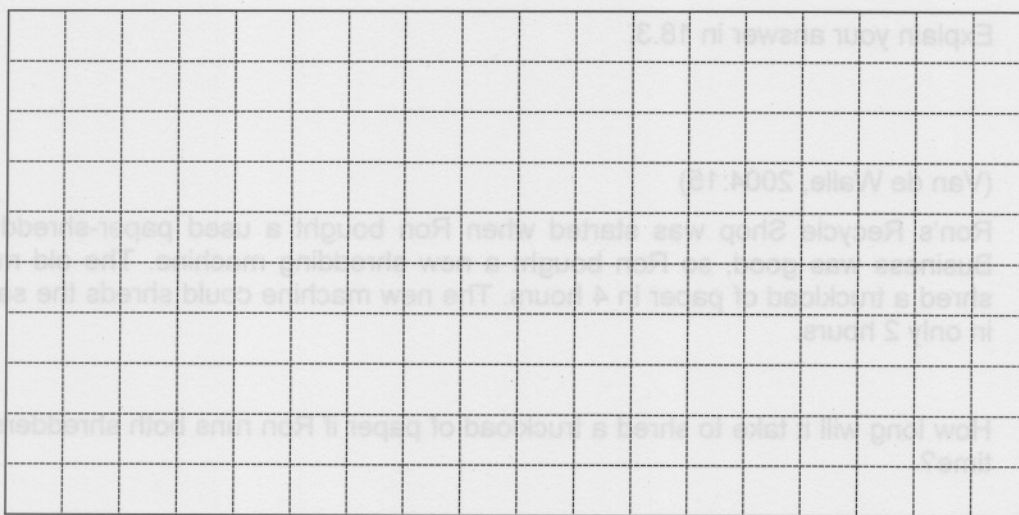
15.3 Explain your answer in 15.2.

15.4 Let x represent time in years since Beth's birth, and let y represent the population, draw a graph of the function.



15.5 If this rate of growth continues, what will the population be on Beth's 16th birthday?

16. The n th term of an arithmetic sequence is given as $a_n = 1 + 2(n - 1)$.
- 16.1 Determine the first 10 terms of the sequence.
- 16.2 Explain in words what the “common difference” of an arithmetic sequence is.
- 16.3 Plot the graph of the arithmetic sequence $a_n = 1 + 2(n - 1)$ on the set of axes. Use the first 10 terms of the sequence that you have determined in 16.1.



- 16.3 What do you notice about the points you have plotted on your graph?
- 16.4 Give an explanation of your findings in 16.3.
17. (Van de Walle, 2004:300)
Two weeks ago the height of two plants were measured. The plants were 8 cm and 12 cm tall respectively. Today they are respectively 12 cm and 16 cm tall.
- 17.1 Which of the 8-cm or the 12-cm plant has grown the fastest?

17.2 Explain your answer in 17.1.

18. Yesterday, Mary counted the number of laps she ran and recorded the amount of time it took. Today she ran fewer laps but it took more time.

18.1 Did she run faster, slower, or at about the same speed today as she did yesterday or is it not possible to tell?

18.2 Explain your answer in 18.1.

18.3 What would your answer be if she took more time but also ran more laps than yesterday?

18.4 Explain your answer in 18.3.

19. (Van de Walle, 2004:15)

Ron's Recycle Shop was started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours.

19.1 How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

19.2 Explain your answer in 19.1.

20.1 How would you describe the following statement in words?

$$3:9 = 4:12$$

20.2 Rewrite the above statement in another mathematical format?

21.1 Select the equivalent fractions in the following sequence:

$$\frac{5}{8}; \frac{9}{12}; \frac{10}{16}; \frac{5}{10}; \frac{8}{11}$$

21.2 Explain your answer in 21.1.

22. At the farmer's market Abel sells tomatoes at 5 kg for R7 whereas Simphiwe sells his tomatoes at 4 kg for R6.

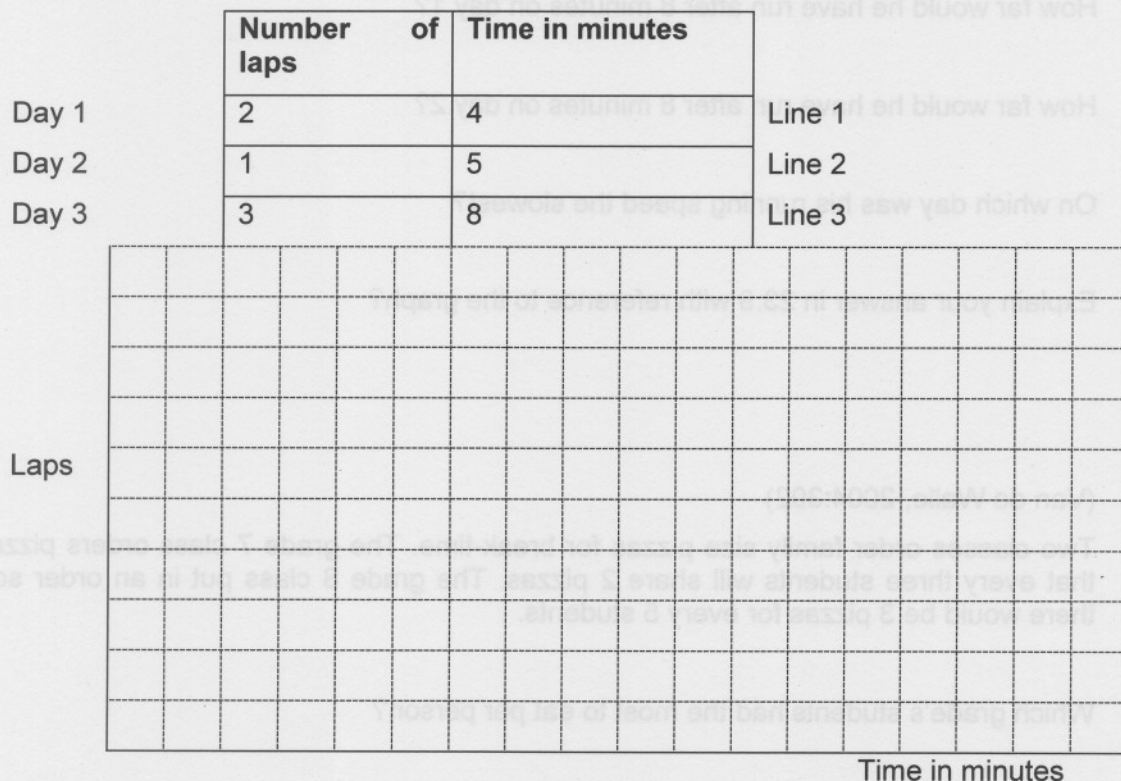
22.1 Whose tomatoes are cheaper?

22.2 Explain your answer in 22.1.

22.3 Solve the problem in another way if possible?

23. Hezekiel Sepeng counted the number of laps he ran and recorded the time it took. The data for 3 days were recorded.

Use this data to draw three lines on the graph which will reflect Hezekiel's running speed on each of the three days.



Answer the following questions:

23.1 On which day did he run the fastest?

23.2 Explain your answer in 23.1.

23.3 Determine the slope of each line.

Slope of line 1:

Slope of line 2:

Slope of line 3:

23.4 Determine the equation of each line.

Equation of line 1:

Equation of line 2:

Equation of line 3:

23.5 Is there a connection between the slope of each line and the speed with which he ran on that day?

23.6 Explain your answer in 23.5.

23.7 How far would he have run after 8 minutes on day 1?

23.8 How far would he have run after 8 minutes on day 2?

23.9 On which day was his running speed the slowest?

23.10 Explain your answer in 23.9 with reference to the graph?

24. (Van de Walle, 2004:302)

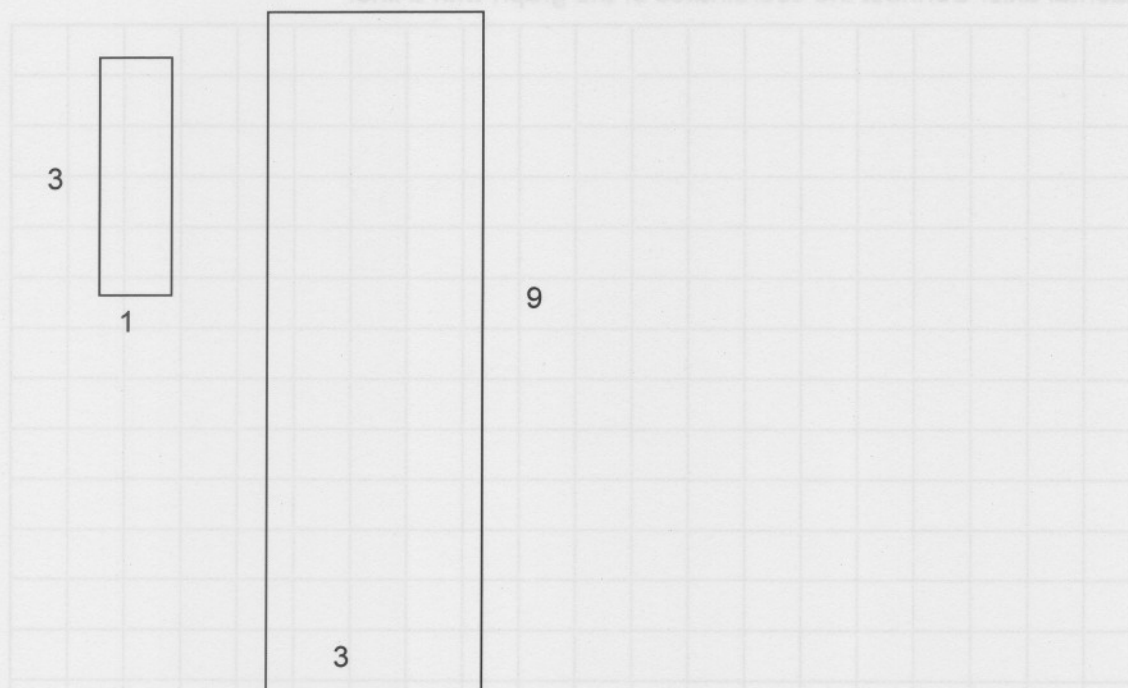
Two classes order family size pizzas for break-time. The grade 7 class orders pizzas so that every three students will share 2 pizzas. The grade 8 class put in an order so that there would be 3 pizzas for every 5 students.

24.1 Which grade's students had the most to eat per person?

24.2 Solve this problem.

24.3 Solve the problem by using a different method.

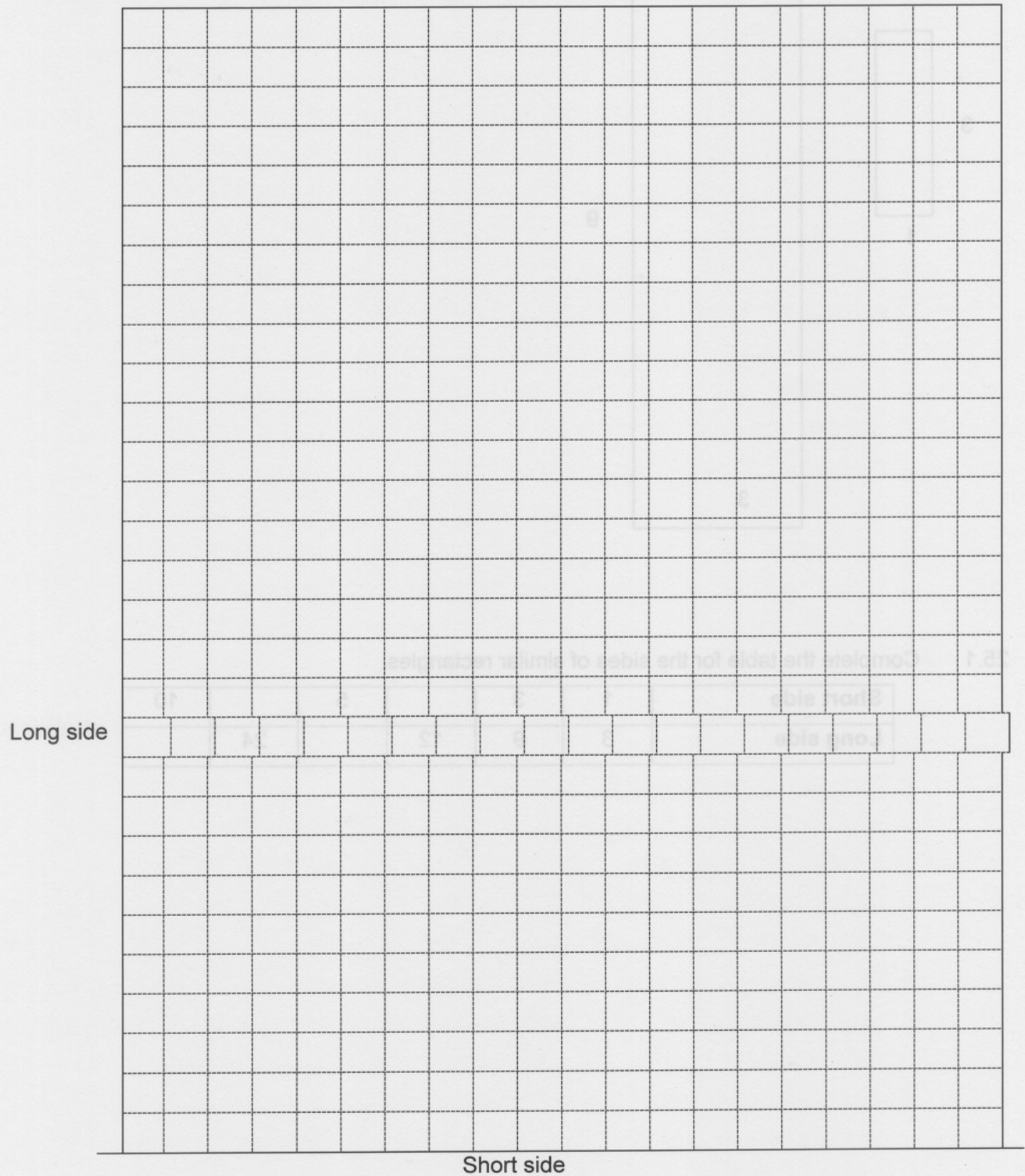
25. The proportions of two similar rectangles are given below:



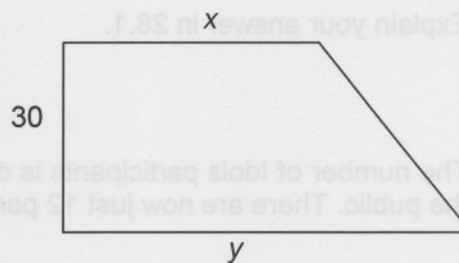
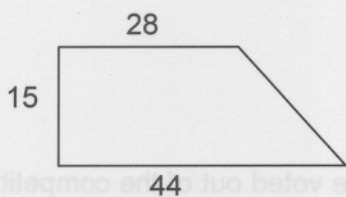
- 25.1 Complete the table for the sides of similar rectangles.

Short side	1	3		5		10
Long side	3	9	12		24	

- 25.2 Draw a graph to illustrate the ratio of the sides of the similar rectangles that you have determined above. Plot the long side on the vertical axis and the short side on the horizontal axis. Connect the coordinates of the graph with a line.



- 25.3 If the short side is 2, find the value of the long side from the graph.
- 25.4 If the long side is 6, find the value of the short side from the graph.
- 25.5 Determine the unit ratio for the graph.
- 25.6 Is there any relationship between the gradient of the line through the coordinates and the unit ratio?
- 25.7 If your answer was yes in 25.6, explain the relationship.
- 25.8 Can you determine the gradient of the line?
- 25.9 Can you determine the equation of the line?
- 25.10 What does the gradient of the line indicate?
26. A trapezium is enlarged as shown in the drawing of two similar trapeziums. Note:
Drawing not according to scale.
- 26.1 What will be the lengths of the sides marked x and y ?
- $x =$
- $y =$



- 26.2 Explain your answer in 26.1.

27. (Van de Walle, 2004:299)

Remember, you are not allowed to use a calculator in this question.

Show all your reasoning steps in this question.

27.1 Paul bought 3 sweets for R2.40. At the same price, what would 10 sweets cost?

27.2 Paul bought 3 sweets for R2.00. At the same price, what would 9 sweets cost?

27.3 Paul bought 13 sweets for R8.40. At the same price, what would 17 sweets cost?

28.1 This year Jose earned 3 times as much money as he earned last year. If Jose earned T rand this year and L rand last year, which of the following equations represent the relationship between T and L ?

(a) $3L = T$

(b) $\frac{L}{3} = T$

(c) $T \times L = 3$

(d) $\frac{L}{3} = \frac{T}{3}$

(e) $\frac{L}{3} = \frac{3}{T}$

28.2 Explain your answer in 28.1.

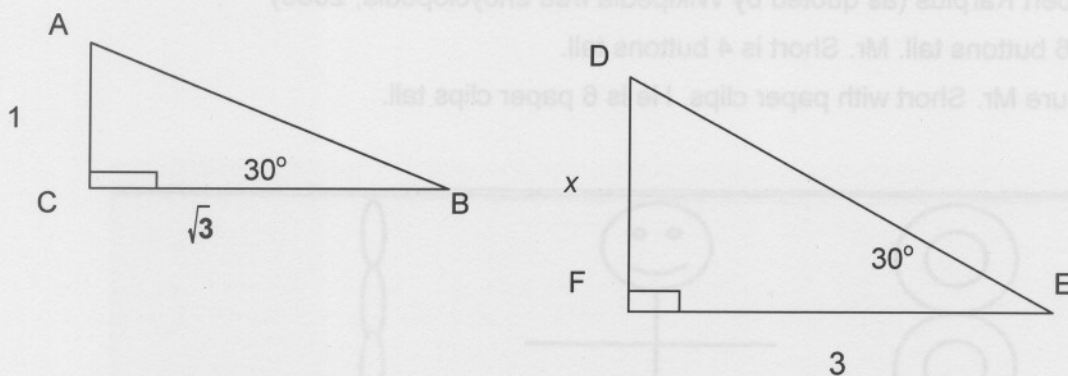
29. The number of Idols participants is decreasing as they are voted out of the competition by the public. There are now just 12 participants left in the competition.

29.1 What chance does each of the 12 participants have to be among the top 4 contestants?

29.2 Does a contestant's chance decrease or increase each time he/she is not voted out?

29.3 Explain your answer in 29.2.

30. Consider the two triangles given below. Use ratios to work out your answers.



- 30.1 What do the equal angles tell you about the two triangles?

- 30.2 Determine the value of side x .

- 30.3 Determine the ratios of the sides for each triangle by completing the table:

Triangle ABC	$\frac{AC}{AB} =$	$\frac{AC}{BC} =$	$\frac{BC}{AB} =$
Triangle DEF	$\frac{DF}{DE} =$	$\frac{DF}{EF} =$	$\frac{EF}{DE} =$

- 30.4 What can you conclude from the values in the table?

- 30.5 Will this be true for all similar right-angled triangles with base angle equal to 30° ?

- 30.6 Explain your answer in 30.5.

- 30.7 For which section of mathematics would this be a good introductory question?

- 30.8 Explain your answer in 30.7

- 31.1 Convert the fraction to percentages.

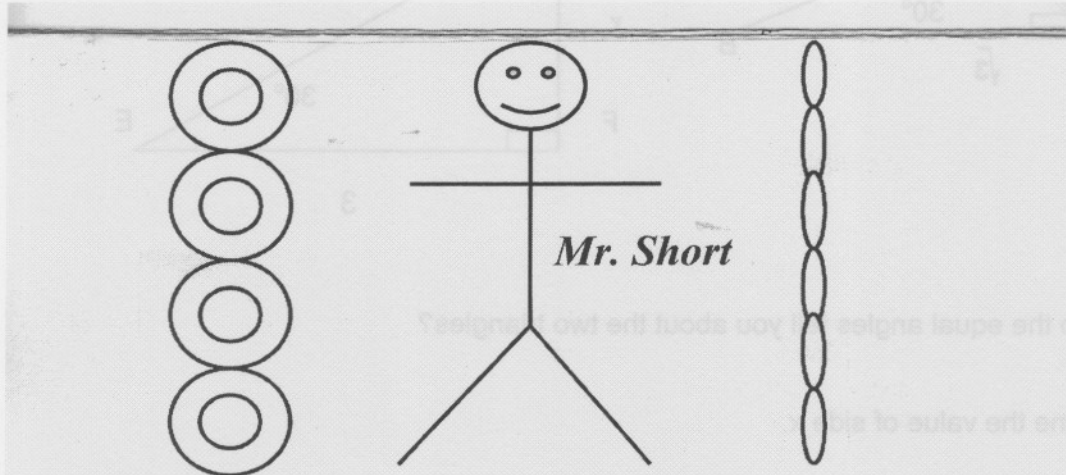
$$\frac{3}{5} =$$

31.2 What is the relationship between proportions and percentages?

32. Robert Karplus (as quoted by Wikipedia free encyclopedia, 2006)

Mr. Tall is 6 buttons tall. Mr. Short is 4 buttons tall.

Now measure Mr. Short with paper clips. He is 6 paper clips tall.

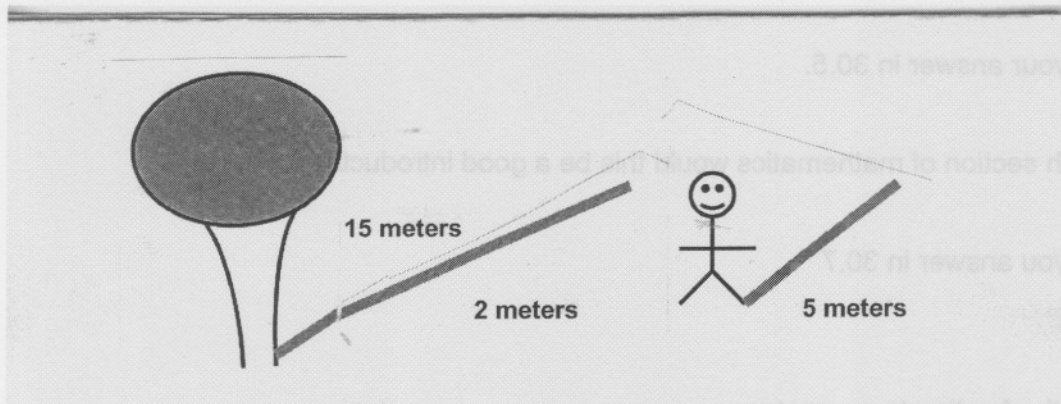


32.1 What is Mr Tall's height in paper clips?

32.2 Explain your answer in 32.1

33. (Mayer, 2003)

Walking back to my room after class yesterday afternoon, I noticed my 2 meter frame cast a shadow 5 meters long. A rather small tree next to the sidewalk cast a shadow of 15 meters long.

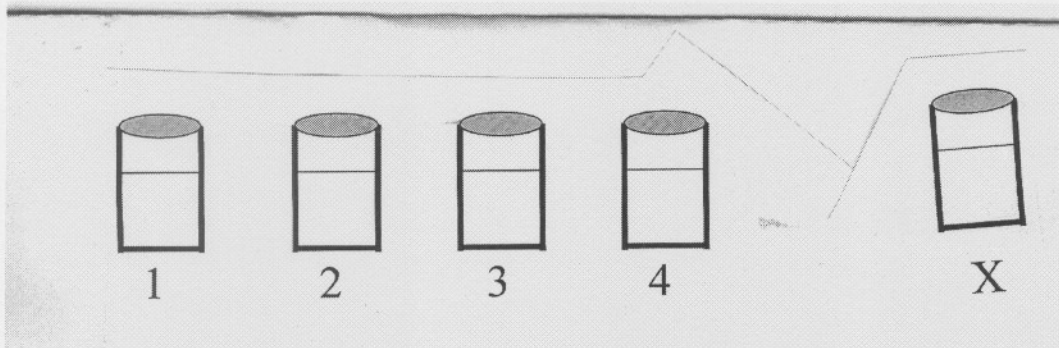


33.1 What is the actual height of the tree?

33.2 Explain your answer in 33.1.

34. (Inhelder & Piaget, 1958:108-109)

- 1, 2 3 and 4 contain colorless, odorless liquids.
- X contains an "activating solution".
- Some combination of liquids (always including X) will give a YELLOW color.



34.1 How can you find the combination that makes YELLOW?

34.2 Explain your answer in 34.1