

Research Article

Exact Solutions and Conservation Laws of the Drinfel'd-Sokolov-Wilson System

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We study the Drinfel'd-Sokolov-Wilson system, which was introduced as a model of water waves. Firstly we obtain exact solutions of this system using the (G'/G) -expansion method. In addition to exact solutions we also construct conservation laws for the underlying system using Noether's approach.

1. Introduction

The classical Drinfel'd-Sokolov-Wilson (DSW) system given by

$$\begin{aligned}u_t + pvv_x &= 0, \\v_t + qv_{xxx} + svu_x + ruv_x &= 0,\end{aligned}\tag{1}$$

where p, q, r , and s are nonzero constants, has been studied by [1]. The authors obtained various types of explicit solutions for (1) by using the bifurcation method and qualitative theory of dynamical systems. Also, Yao and Li [2] and C. Liu and X. Liu [3] obtained some exact solutions for the DSW system (1) by using a direct algebra method. A special case of the classical DSW system, namely,

$$\begin{aligned}u_t + 3vv_x &= 0, \\v_t + 2v_{xxx} + vu_x + 2uv_x &= 0,\end{aligned}\tag{2}$$

was studied by several authors [4–8]. Hirota et al. [4] investigated the soliton structure of (2) and by employing an algebraic method, Fan [5] constructed some exact solutions. By using the improved generalized Jacobi elliptic function method, Yao [6] obtained some traveling wave solutions of (2), whereas by applying the Adomian decomposition method, Inc [7] obtained approximate doubly periodic wave

solutions of (2). Zhao and Zhi [8] constructed exact doubly periodic solutions of (2) by using an improved F -expansion method.

In this paper, we study the Drinfel'd-Sokolov-Wilson (DSW) system given by

$$u_t + \beta vv_x = 0,\tag{3a}$$

$$v_t + \alpha v_{xxx} + \beta vu_x + \beta uv_x = 0,\tag{3b}$$

which can be derived from (1) by taking $p = r = s = \beta$ and $q = \alpha$. We obtain exact solutions and construct conservation laws of the DSW systems (3a) and (3b).

Nonlinear partial differential equations (PDEs) model diverse nonlinear phenomena in natural and applied sciences such as mechanics, fluid dynamics, biology, plasma physics, and mathematical finance. Therefore finding exact solutions of nonlinear PDEs is very important. Unfortunately, this is a very difficult task and there are no systematic methods that can be used to find exact solutions of the nonlinear PDEs. However, in the past few decades a number of new methods have been developed to obtain exact solutions to nonlinear PDEs. Some of these methods include the exp-function method, the homogeneous balance method, the sine-cosine method, the hyperbolic tangent function expansion method, and the (G'/G) -expansion function method [9–15].

We recall that conservation laws are mathematical expressions of the physical laws, such as conservation of energy,

mass, and momentum. They are of great significance in the solution process and reduction of PDEs. In the literature, one finds that the conservation laws have been widely used in studying the existence, uniqueness, and stability of solutions of nonlinear partial differential equations [16–18], as well as in the development and use of numerical methods [19, 20]. Also, conserved vectors associated with Lie point symmetries have been employed to find exact solutions of partial differential equations [21–23]. There are various methods of constructing conservation laws. One of the methods for variational problems is by means of Noether's theorem [24]. In order to apply Noether's theorem, the knowledge of a suitable Lagrangian is necessary. For nonlinear differential equations that do not have a Lagrangian, several methods have been developed (see, e.g., [25–30]).

This paper is structured as follows. In Section 2, exact solutions of (3a) and (3b) are obtained using the (G'/G) -expansion function method. In Section 3, we construct Noether's symmetries and the conserved vectors for the DSW system (3a) and (3b). Concluding remarks are presented in Section 4.

2. Exact Solutions of the DSW System (3a) and (3b)

In this section, we obtain exact solutions of the DSW system (3a) and (3b). We first transform the system (3a) and (3b) into a system of ordinary differential equations by using the substitutions

$$u(x, t) = U(z), \quad v(x, t) = V(z), \quad (4)$$

where $z = x - ct$. Substituting (4) into the system (3a) and (3b) and integrating with respect to z , we obtain the following ordinary differential equations (ODEs):

$$\begin{aligned} -cU + \frac{\beta}{2}V^2 &= 0, \\ -cV + \alpha V'' + \beta VU &= 0, \end{aligned} \quad (5)$$

where integration constants are taken to be zero. From the first equation, we obtain $U = \beta V^2/(2c)$. Substituting this value of U in the second equation of the system, we obtain

$$-cV + \alpha V'' + \frac{\beta^2}{2c}V^3 = 0. \quad (6)$$

Now multiplying the above equation by V' and integrating while taking the constant of integration to be zero, we arrive at a first-order variables separable equation. Integrating this equation and reverting back to our original variables, we obtain

$$v(x, t) = \frac{4c^2 \exp \left[\sqrt{c/\alpha} (x - ct + A) \right]}{1 + \beta c^2 \exp \left[2\sqrt{c/\alpha} (x - ct + A) \right]}, \quad (7)$$

where A is an arbitrary constant of integration. Since $U = \beta V^2/(2c)$, we have

$$u(x, t) = \frac{\beta}{2c} \left\{ \frac{4c^2 \exp \left[\sqrt{c/\alpha} (x - ct + A) \right]}{1 + \beta c^2 \exp \left[2\sqrt{c/\alpha} (x - ct + A) \right]} \right\}^2. \quad (8)$$

Thus, we have obtained one exact solution of the DSW system (3a) and (3b).

To obtain more exact solutions of the DSW system (3a) and (3b), we employ the (G'/G) -expansion function method [15]. We assume that the solutions of the ODE (6) can be expressed as a polynomial in (G'/G) by

$$V(z) = \sum_{i=0}^n \phi_i \left(\frac{G'}{G} \right)^i, \quad (9)$$

where n is the balancing number to be determined and the function $G(z)$ satisfies the second-order linear ODE given by

$$G''(z) + \lambda G'(z) + \mu G(z) = 0 \quad (10)$$

with λ and μ being arbitrary constants. In our case, the balancing procedure gives $n = 1$. Thus

$$V(z) = \phi_0 + \phi_1 \left(\frac{G'}{G} \right). \quad (11)$$

Substituting (11) into (6) and making use of (10) and then equating all the terms with the same powers of (G'/G) to zero yield the following system of algebraic equations:

$$\begin{aligned} \alpha \phi_1 \lambda^2 + 2\alpha \phi_1 \mu + \frac{3\beta^2 \phi_0^2 \phi_1}{2c} - c\phi_1 &= 0, \\ 3\alpha \phi_1 \lambda + \frac{3\beta^2 \phi_0 \phi_1^2}{2c} &= 0, \\ 2\alpha \phi_1 + \frac{\beta^2 \phi_1^3}{2c} &= 0, \\ \alpha \phi_1 \lambda \mu + \frac{\beta^2 \phi_0^3}{2c} - c\phi_0 &= 0. \end{aligned} \quad (12)$$

Solving the above equations, with the aid of Mathematica, we obtain

$$\begin{aligned} \alpha &= -\frac{2c}{(\lambda^2 - 4\mu)}, \\ \phi_0 &= -\frac{\sqrt{2}\sqrt{c^2 - 2\alpha c\mu}}{\beta}, \\ \phi_0 &= \frac{\sqrt{2}\sqrt{c^2 - 2\alpha c\mu}}{\beta}, \\ \phi_1 &= \frac{2\phi_0}{\lambda}. \end{aligned} \quad (13)$$

Consequently, we obtain the following two types of travelling wave solutions of the DSW system.

For $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic functions travelling wave solutions

$$\begin{aligned}
 v_1(t, x) &= \pm \frac{\sqrt{2(c^2 - 2\alpha c\mu)}}{\beta} \pm \frac{2\sqrt{2(c^2 - 2\alpha c\mu)}}{\lambda\beta} \\
 &\times \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \right. \\
 &\times \left(\left(c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct)\right) \right. \right. \\
 &\quad \left. \left. + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct)\right) \right) \right. \\
 &\times \left(c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct) + c_2 \right. \right. \\
 &\quad \left. \left. \times \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct)\right)^{-1} \right) \right) \left. \right],
 \end{aligned}$$

$$\begin{aligned}
 u_1(t, x) &= \frac{\beta}{2c} \left\{ \pm \frac{\sqrt{2(c^2 - 2\alpha c\mu)}}{\beta} \pm \frac{2\sqrt{2(c^2 - 2\alpha c\mu)}}{\lambda\beta} \right. \\
 &\times \left[-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \right. \\
 &\times \left(\left(c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct) + c_2 \right) \right. \right. \\
 &\quad \left. \left. \times \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct)\right) \right) \right. \\
 &\times \left(c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct) + c_2 \right) \right. \\
 &\quad \left. \left. \times \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(x - ct)\right)^{-1} \right) \right) \left. \right\}^2.
 \end{aligned}
 \tag{14}$$

For $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling solutions

$$\begin{aligned}
 v_2(t, x) &= \pm \frac{\sqrt{2(c^2 - 2\alpha c\mu)}}{\beta} \pm \frac{2\sqrt{2(c^2 - 2\alpha c\mu)}}{\lambda\beta} \\
 &\times \left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \right. \\
 &\times \left(\left(-c_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct)\right) \right. \right. \\
 &\quad \left. \left. + c_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct)\right) \right) \right. \\
 &\times \left(c_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct) + c_2 \right) \right. \\
 &\quad \left. \left. \times \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct)\right)^{-1} \right) \right) \left. \right],
 \end{aligned}$$

$$\begin{aligned}
 u_2(t, x) &= \frac{\beta}{2c} \left\{ \pm \frac{\sqrt{2(c^2 - 2\alpha c\mu)}}{\beta} \pm \frac{2\sqrt{2(c^2 - 2\alpha c\mu)}}{\lambda\beta} \right. \\
 &\times \left[-\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \right. \\
 &\times \left(\left(-c_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct) + c_2 \right) \right. \right. \\
 &\quad \left. \left. \times \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct)\right) \right) \right. \\
 &\times \left(c_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct) + c_2 \right) \right. \\
 &\quad \left. \left. \times \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(x - ct)\right)^{-1} \right) \right) \left. \right\}^2.
 \end{aligned}
 \tag{15}$$

3. Conservation Laws of the DSW Equations (3a) and (3b)

In this section, we construct the conservation laws of the DSW system (3a) and (3b). Since the third-order DSW system (3a) and (3b) does not have a Lagrangian, we cannot apply the Noether theorem. However, if we transform the third-order DSW system (3a) and (3b) to a fourth-order with the aid of the transformation $u = U_x$, $v = V_x$ [31], we obtain

$$U_{tx} + \beta V_x V_{xx} = 0, \quad (16a)$$

$$V_{tx} + \alpha V_{xxxx} + \beta V_x U_{xx} + \beta U_x V_{xx} = 0. \quad (16b)$$

This system has a Lagrangian given by

$$L = \frac{1}{2} (\alpha V_{xx}^2 - \beta U_x V_x^2 - U_x U_t - V_x V_t) \quad (17)$$

and it satisfies the Euler-Lagrange equations

$$\frac{\delta L}{\delta U} = 0, \quad \frac{\delta L}{\delta V} = 0, \quad (18)$$

where $\delta/\delta U$ and $\delta/\delta V$ are defined by

$$\begin{aligned} \frac{\delta}{\delta U} &= \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} \\ &\quad + D_t^2 \frac{\partial}{\partial U_{tt}} + D_x^2 \frac{\partial}{\partial U_{xx}} + D_x D_t \frac{\partial}{\partial U_{tx}} - \dots, \\ \frac{\delta}{\delta V} &= \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} \\ &\quad + D_x^2 \frac{\partial}{\partial V_{xx}} + D_x D_t \frac{\partial}{\partial V_{tx}} - \dots, \end{aligned} \quad (19)$$

respectively.

Let us now consider the vector field

$$\begin{aligned} X &= \xi^1(t, x, U, V) \frac{\partial}{\partial t} + \xi^2(t, x, U, V) \frac{\partial}{\partial x} \\ &\quad + \eta^1(t, x, U, V) \frac{\partial}{\partial U} + \eta^2(t, x, U, V) \frac{\partial}{\partial V}. \end{aligned} \quad (20)$$

The second prolongation operator, $X^{[2]}$ of X , is given by

$$\begin{aligned} X^{[2]} &= \xi^1(t, x, U, V) \frac{\partial}{\partial t} \\ &\quad + \xi^2(t, x, U, V) \frac{\partial}{\partial x} + \eta^1(t, x, U, V) \frac{\partial}{\partial U} \\ &\quad + \eta^2(t, x, U, V) \frac{\partial}{\partial V} + \zeta_t^1 \frac{\partial}{\partial U_t} \\ &\quad + \zeta_t^2 \frac{\partial}{\partial V_t} + \zeta_x^1 \frac{\partial}{\partial U_x} + \zeta_x^2 \frac{\partial}{\partial V_x} + \dots, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), \\ \zeta_t^2 &= D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), \\ \zeta_x^1 &= D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2), \\ \zeta_x^2 &= D_x(\eta^2) - V_t D_x(\xi^1) - V_x D_x(\xi^2), \\ D_t &= \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + U_{tt} \frac{\partial}{\partial U_t} \\ &\quad + V_{tt} \frac{\partial}{\partial V_t} + U_{tx} \frac{\partial}{\partial U_x} + V_{tx} \frac{\partial}{\partial V_x} + \dots, \\ D_x &= \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} \\ &\quad + V_{xx} \frac{\partial}{\partial V_x} + U_{tx} \frac{\partial}{\partial U_t} + V_{tx} \frac{\partial}{\partial V_t} + \dots. \end{aligned} \quad (22)$$

We recall that X , given by (20), is a Noether symmetry of (16a) and (16b), if it satisfies

$$X(L) + L[D_t(\xi^1) + D_x(\xi^2)] = D_t(B^1) + D_x(B^2), \quad (23)$$

where $B^1(t, x, U, V)$ and $B^2(t, x, U, V)$ are the gauge functions. Expanding the above equation gives

$$\begin{aligned} &-\frac{1}{2} U_x [\eta_t^1 + U_t \eta_U^1 + V_t \eta_V^1 - U_t \xi_t^1 \\ &\quad - U_t^2 \xi_U^1 - U_t V_t \xi_V^1 \\ &\quad - U_x \xi_t^2 - U_t U_t \xi_U^2 - U_x V_t \xi_V^2] \\ &-\frac{1}{2} V_x [\eta_t^2 + U_t \eta_U^2 + V_t \eta_V^2 - V_t \xi_t^1 \\ &\quad - U_t V_t \xi_U^1 - V_t^2 \xi_V^1 - V_x \xi_t^2 \\ &\quad - U_t V_x \xi_U^2 - V_t V_x \xi_V^2] \\ &-\frac{1}{2} (\beta V_x^2 + U_t) \\ &\quad \times [\eta_x^1 + U_x \eta_U^1 + V_x \eta_V^1 \\ &\quad - U_t \xi_x^1 - U_t U_x \xi_U^1 - U_t V_x \xi_V^1 \\ &\quad - U_x \xi_x^2 - U_x^2 \xi_U^2 - U_x V_x \xi_V^2] \\ &-\left(\frac{1}{2} V_t + \beta U_x V_x\right) \\ &\quad \times [\eta_x^2 + U_x \eta_U^2 + V_x \eta_V^2 - V_t \xi_x^1 - U_x V_t \xi_U^1 \\ &\quad - V_t V_x \xi_V^1 - V_x \xi_x^2 - U_x V_x \xi_U^2 - V_x^2 \xi_V^2] \end{aligned}$$

$$\begin{aligned}
 & + \alpha V_{xx} \left[D_x^2 \eta^2 - V_t D_x^2 \xi^1 - V_x D_x^2 \xi^2 \right. \\
 & \quad - 2V_{tx} \left(\xi_x^1 + U_x \xi_U^1 + V_x \xi_V^1 \right) \\
 & \quad \left. - 2V_{xx} \left(\xi_x^2 + U_x \xi_U^2 + V_x \xi_V^2 \right) \right] \\
 & + \frac{1}{2} \left[\alpha V_{xx}^2 - \beta U_x V_x^2 - U_x U_t - V_x V_t \right] \\
 & \quad \times \left[\xi_t^1 + U_t \xi_U^1 + V_t \xi_V^1 \right. \\
 & \quad \left. + \xi_x^2 + U_x \xi_U^2 + V_x \xi_V^2 \right] \\
 & = B_t^1 + U_t B_u^1 + V_t B_v^1 + B_x^2 + U_x B_u^2 + V_x B_v^2.
 \end{aligned} \tag{24}$$

This leads to an overdetermined system of PDEs for the functions $\xi^1, \xi^2, \eta^1, \eta^2, B^1$, and B^2 . Solving the system of PDEs gives

$$\begin{aligned}
 \xi^1 &= c_1, & \xi^2 &= c_2, & \eta^1 &= f(t), & \eta^2 &= g(t), \\
 B^1 &= A(t, x), \\
 B^2 &= \frac{-1}{2} f'(t) - \frac{1}{2} g'(t) + H(t, x), \\
 A_t + H_x &= 0.
 \end{aligned} \tag{25}$$

We may choose $H = 0$ and $A = 0$ as they contribute to the trivial part of the conserved vector. The conserved vector for the second-order Lagrangian L is given by [24, 31]

$$\begin{aligned}
 T^1 &= -B^1 + \xi^1 L \\
 &+ W^1 \left[\frac{\partial L}{\partial U_t} - D_t \frac{\partial L}{\partial U_{tt}} - D_x \frac{\partial L}{\partial U_{tx}} \dots \right] \\
 &+ W^2 \left[\frac{\partial L}{\partial V_t} - D_t \frac{\partial L}{\partial V_{xt}} - D_x \frac{\partial L}{\partial V_{tt}} \dots \right] \\
 &+ D_t (W^1) \frac{\partial L}{\partial U_{tt}} + D_t (W^2) \frac{\partial L}{\partial V_{tt}}, \\
 T^2 &= -B^2 + \xi^2 L \\
 &+ W^1 \left[\frac{\partial L}{\partial U_x} - D_t \frac{\partial L}{\partial U_{xt}} - D_x \frac{\partial L}{\partial U_{xx}} \dots \right] \\
 &+ W^2 \left[\frac{\partial L}{\partial V_x} - D_t \frac{\partial L}{\partial V_{xt}} - D_x \frac{\partial L}{\partial V_{xx}} \dots \right] \\
 &+ D_x (W^1) \frac{\partial L}{\partial U_{xx}} + D_x (W^2) \frac{\partial L}{\partial V_{xx}},
 \end{aligned} \tag{26}$$

where $W^1 = \eta^1 - U_t \xi^1 - U_x \xi^2$ and $W^2 = \eta^2 - V_t \xi^1 - V_x \xi^2$ are the characteristic functions. Now using (26) in conjunction

with (25), $u = U_x$, and $v = V_x$, we obtain the following independent conserved vectors for system (3a) and (3b):

$$\begin{aligned}
 T_1^1 &= \frac{1}{2} (\alpha v_x^2 - \beta uv^2), \\
 T_1^2 &= -\frac{1}{2} \beta \int u_t dx + \frac{1}{2} \int u_t dx \int u_t dx \\
 &\quad + \beta uv \int v_t dx + \frac{1}{2} \int v_t dx \int v_t dx \\
 &\quad + \alpha u_{xx} \int v_t dx - \alpha v_t v_x,
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 T_2^1 &= \frac{1}{2} (u^2 + v^2), \\
 T_2^2 &= -\frac{1}{2} \alpha v_x^2 + \alpha v v_{xx} + \beta uv^2.
 \end{aligned} \tag{28}$$

We note that (27) is a nonlocal conserved vector, whereas (28) is a local conserved vectors. Also, for the arbitrary functions $f(t)$ and $g(t)$, we obtain the following conserved vectors:

$$\begin{aligned}
 T_{(f,g)}^1 &= -\frac{1}{2} u f(t) - \frac{1}{2} v g(t), \\
 T_{(f,g)}^2 &= \frac{1}{2} f'(t) \int u dx + \frac{1}{2} g'(t) \int v dx \\
 &\quad + f(t) \left[-\frac{1}{2} \beta v^2 - \frac{1}{2} \int u_t dx \right] \\
 &\quad + g(t) \left[-\frac{1}{2} \beta uv - \frac{1}{2} \int v_t dx - \alpha v_{xx} \right],
 \end{aligned} \tag{29}$$

which gives us infinitely many nonlocal conservation laws.

4. Conclusion

The third-order DSW system (3a) and (3b) was studied. Exact solutions of the DSW system were obtained using direct integration and the (G'/G) -expansion function method. The solutions obtained were hyperbolic and trigonometric solutions. In addition conservation laws were also derived. This system does not have a Lagrangian. In order to invoke Noether's theorem we used the transformations $u = U_x$ and $v = V_x$ to convert the DSW system to a fourth-order system, which has a Lagrangian. The conservation laws were then obtained and consisted of a local and infinite number of nonlocal conserved vectors.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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