Subtleties in arbitrage pricing under real market conditions for derivative instruments and counterparty credit risk

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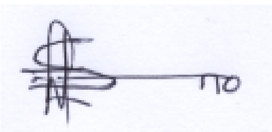

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Masimba Energy Sonono Date

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Dedication

To my wife Daphne and family for their endless love, support and encouragement. To my late mother Dorothy Maziti, thank you very much and l will cherish you forever.

"... Tomorrow, I will stand at the top of the hill, holding the staff of God in my hand."

Exodus 17 vs 9.

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Above all, I thank the Almighty God for the love, strength, wisdom and guidance. I have learnt to have faith, patience and hope in order to achieve all goals.

Authorship

All the work presented in this thesis is my own. I was involved in all aspects of this thesis which include implementation of all algorithms, data analysis and manuscript writing (lead author), though l benefited from the comments of my supervisor Professor Phillip Mashele.

The contents of Chapter 2 and Chapter 4 are published in the Journal of Mathematical Finance. The contents of Chapter 4 and Chapter 5 were submitted in peer-reviewed journals for consideration. In all the articles, Professor Phillip Mashele is the co-author.

Executive Summary

The thesis consist of four articles presented in seperate chapters, addressing selected subtle issues arising from the arbitrage pricing theory in the real market, with particular focus on derivative instruments and counterparty credit risk. The subtles include modeling of bid-ask spreads, choosing the appropriate interest rate to approximate the risk-free interest rate and modeling of counterparty credit risk. These subtles remain contentious issues in arbitrage pricing theory, hence the major interest to this work.

Chapter 2 presents an article which explores arbitrage pricing in the presence of bid-ask spreads modeled using conic finance theory. Conic finance is a brand new quantitative finance theory which models bid-ask prices of cash flows by applying the theory of acceptability to cash flows. The theory of acceptability indices combines elements of arbitrage pricing theory and expected utility theory, which captures the preferences of market participants. In the article, the theory is used to assess the risks of equity derivatives trading strategies.

Chapter 3 is an article, which is an extension of the theory introduced in the previous chapter. This article explores the theory of conic finance, with particular focus to options on LIBOR based derivative instruments. In the same article, we also explore the dynamics of the options on LIBOR based derivatives. Using the theory, an approach to estimate bid-ask prices for options on LIBOR based instruments is proposed. In particular, the proposed approach is assessed in the determination of premiums for caps and floors.

In Chapter 4, an article on simulation techniques that are useful to this work is presented. At the core of the article is comparison of various Monte Carlo methods. In this work the comparison is done in a prediction of stock price movement setup. However, the findings of this work have great bearing on the suitable simulation

techniques to be used in subsequent work.

Chapter 5 is an article on modeling credit exposures and pricing counterparty credit for OTC interest rate derivatives - caps, floors and swaptions, using the LIBOR Market Model (LMM). In the presence of counterparty credit risk, trades are no longer riskless as impact of counterparty credit risk on arbitrage pricing and hedging strategies can be significant. In the article, the Basel III standardized approach for OTC interest rate derivatives is analyzed with the objective of understanding how the CVA levels evolve under this approach. The reasons for focusing on the standardized approach being that it is widely used in financial institutions and the data required for the standardized approach is easily available. CVA stress tests towards netting agreements, ratings and maturities were implemented using a test portfolio consisting of OTC interest rate derivatives transactions with different counterparties.

Keywords: arbitrage pricing; conic finance; bid-ask prices; coherent risk measures; incomplete markets; Wang transform; Monte Carlo; Maximum Likelihood Estimation; interest rate; LIBOR; counterparty credit risk; LIBOR; credit risk; LIBOR Market Model; Basel III; credit value adjustment

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1. Introduction

Let $\mathbb{P}$ be the "real world" probability measure. A probability measure $\mathbb{Q}$ is an equivalent martingale measure if $\mathbb{Q} \sim \mathbb{P}$, that is, they define the same set of possible or impossible events (null set). Harrison & Pliska (1981) formulated the fundamental theorem of asset pricing, building on the discrete time work of Harrison & Kreps (1979), which states that: i) a market is arbitrage free if and only if there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, and ii) a market is complete if and only if there exists a unique martingale measure $\mathbb{Q} \sim \mathbb{P}$.

Intuitively, an arbitrage opportunity is an opportunity to earn a riskless profit without making a net investment. In an idealized frictionless market, with assumptions such as no bid-ask spreads, no liquidity risk, no short selling restrictions, a replicating portfolio can perfectly replicate all cash flows of an instrument in a portfolio. By replicating all the cash flows of the instrument, then the arbitrage price of the instrument in the portfolio is the market price of the replicating portfolio regardless of the risk preferences of the market participants. In other words, arbitrage opportunities arise if the instrument is not traded at its arbitrage price, and the converse.

An in-depth understanding of arbitrage opportunities requires the self-financing trading strategy concept. A self-financing trading strategy is a strategy such that no money is added or taken out of the portfolio, once the portfolio is established. Now, with the self-financing strategy an arbitrage opportunity refers to the following scenario: A portfolio with zero initial value ($V_0 = 0$ at $t = 0$) becomes greater than or equal to zero in value at a future time T (i.e. $V_T \geq 0$) with a positive expected value at time T (i.e. $E[V_T] > 0$)(Y.Tang & Li 2007).

So in general, an approach to obtain the arbitrage price of an instrument is to replicate the instrument with other instruments through self-financing trading strategy. If the replication is perfect, the present value of the instrument and the replicating portfolio should be the same. Otherwise, arbitrage opportunities arise between the instrument and the replicating portfolio by taking a long position in the less expensive one and taking a short position in the more expensive one to generate riskless portfolio. If the instrument can be replicated by more than one replicating portfolio, the present values of the instrument and all the replicating portfolios should be the same to avoid arbitrage opportunities. Theoretically, such portfolio replication strategies can be used to create perfect hedges, which are desirable as one can profit from bid-ask spread instead of the favorable market movements.

However, in the real market that has various imperfections, arbitrage pricing is more complicated and perfect replications are very difficult and expensive to achieve. Next, we discuss some of the complications which are of interest to this work.

1.1 Bid-Ask Spread

In the presence of bid-ask spread, no-arbitrage implies that the price of an instrument lies in between the corresponding bid-ask spread. In a complete market, the typical price of an arbitrage pricing model is the mid-market price approximately. The bid-ask spread is then added on for a market dealer/participant to make profit. The bid-ask spread is typically determined by market consensus.

The bid-ask spread essentially permits the market dealer to cover various costs such as replication cost and the premium of the unexpected risk. In reality, the market is incomplete and the bid-ask spread may not be enough to cover all the replication costs of an instrument.

Since the market is incomplete, an instrument cannot be hedged or replicated perfectly with other liquid instruments. The hedges on the instrument do not eliminate all risk, and instead the remaining risk is required to be an acceptable opportunity.

In such a scenario, the arbitrage pricing cannot be simply used. Instead, other factors such as consensus or risk preferences of the market participants as well as supply and demand, then become handy in determining the price and hedging of the instrument.

To address this issue, we turn to conic finance, also called the law of two prices. Conic finance is a brand new quantitative finance theory which models bid-ask prices of cash flows by applying the theory of acceptability indices to cash flows proposed by Cherny & Madan (2009). The theory of acceptability indices combines elements of arbitrage pricing theory and expected utility theory. Combining the two theories, set of arbitrage opportunities are extended to the set of all opportunities that a wide range of market participants are prepared to accept. The preferences of the market participants are captured by utility functions.

So if a market participant starts from a position with zero cost, any positions that will increase expected utility are acceptable to the market participants. These positions form a convex set that contains non-negative terminal cash flows. Every market participant has an acceptable set depending on his/her preferences. The preferences of the market participants are modeled using a set of probability measures. When a wider range of market participants are willing to accept a certain position, it is awarded a higher level of acceptability. The set accepted by all market participants is the intersection of all sets, which is a convex set. It is called the acceptance set. For further explanation on the theory of acceptance indices, one is referred to Sonono & Mashele (2013) and references therein.

1.2 Risk-free Interest Rate

The self-financing trading strategies or arbitrage strategies require borrowing and lending money. The "risk-free interest rate" is typically used to calculate the cost or the gain for borrowing or lending money.

For most market participants, the US Treasury interest rate is the risk-free benchmark against which other assets can be measured. Although the Treasury interest rate is used to discount Treasury instruments, however, it is not that precisely the "risk-free interest rate". This is because the income from US Treasury debt instruments are subject to federal tax despite being exempt from local and state taxes. As a result, the US Treasury interest rate should actually be lower than the "risk-free interest rate", which is free of default. Furthermore, many sovereign countries, especially those in emerging markets, are not default-free. Resultantly, the Treasury interest rates of these sovereign countries should actually be higher than the "risk-free interest rate" (Y.Tang & Li 2007). Now, the question is: What is a suitable "risk-free interest rate" to use in the market and in particular the derivatives market?

For derivatives pricing, it turns out several interest rates are involved. For instance, in a self-financing trading strategy for derivatives pricing, one needs to borrow and lend money for buying and selling the underlying assets, as well as for funding the derivatives instrument itself (Y.Tang & Li 2007). Thus, the "risk-free interest rate" is not required specifically; instead effective financing rates of the underlying assets or derivatives instruments are required.

If a liquid repurchase agreement(repo) market exists for an asset, then the repo rate is used as the financing rate. This rate is approximately the "risk-free interest rate", provided the underlying asset is approximately free of default and has no tax and other benefits. Derivatives instruments usually, have no liquid repo markets. The derivatives instruments are funded through unsecured borrowing, where the funding rates highly depend on the credit quality of the borrower (Y.Tang & Li 2007).

In practice, using a reference interest rate as the financing rate in arbitrage pricing models is the common approach used. In this work, we use the LIBOR (London Interbank Offered Rate) as the reference interest rate. The LIBOR is a widely used benchmark interest rate at which banks borrow large amounts of money from each other. The LIBOR is not a "risk-free interest rate", but it is an approximation to the risk-free interest rate since the participatory banks have high credit ratings.

The markets for LIBOR based instruments are among the world's largest financial markets. Most of the LIBOR based instruments are traded over-the-counter (OTC), consituting over 60% of the trades. In addition, interest rates are a cornerstone for measuring the counterparty credit risk of any other class of the OTC derivatives. Thus, an in-depth focus on the LIBOR based instruments is required. The significance of the LIBOR based instruments is that: i) They allow market participants/dealers to effectively hedge their interest rate exposures ii) They allow market participants/dealers to express their views on future levels of interest rates.

As it is commonly believed that future levels of interest rates are at least somewhat random, then there is need to model them consistently. This has led to evolution of different models to estimate interest rates. There are classic models by Vasicek (1977), Cox et al. (1985) and Ho & Lee (1986). These are all one factor models with a single factor describing the evolution of the whole yield curve. Some of these models have been extended to multifactor models which then permit several factors to describe the evolution of future rates. In the recent years, market models for modeling interest rates have been developed. The market models take the whole term structure into account and gives a complete set of forward rates for the periods ahead. They are based on the HJM framework by Heath et al. (1992) which gives forward rates in continuous time. The models were later modified into discrete time models and we then started talking of the LIBOR Market Model (LMM) developed by Brace et al. (1997). The LMM is then used to model discrete forward LIBOR rates with discrete tenors, which have the advantage of being directly observable in the market.

In a more general market model, each forward LIBOR rate is a general martingale process under its own corresponding arbitrage free measure. For the LMM, each forward LIBOR rate is a lognormal martingale diffusion process under its own forward arbitrage-free measure. Consequently, the LMM produces market consistent closed-form solutions or approximations for LIBOR based instruments such as caps, floors and European swaptions. For caps and floors, the formulae from the lognormal LMM are the same as the Black's formula. For European swaptions, the Black's formula can be used as good approximations to the formula from the lognormal LMM. This justifies the use of the Black & Cox (1976) formula by market practitioners.

1.3 Counterparty Credit Risk

Counterparty credit risk is the risk that a counterparty defaults before honoring its engagement. It comes from the notion that, when one enters into an OTC derivatives transaction, one grants one's counterparty an option to default and, at the same time, one also receives an option to default oneself (Y.Tang & Li 2007). In the presence of the counterparty credit risk, the trades are no longer riskless as impact of counterparty credit risk on arbitrage pricing and hedging strategies can be very significant.

In a rationale and efficient market, the counterparties should be compensated for the credit risks they undertake. However, the derivatives models focus on pricing of specific trades and often do not price the counterparty credit risk. In fact, they often price market risks. Now, what the derivatives market participants needed was a way to pricing of derivatives including counterparty credit risk which was not included in specific trade models. This counterparty credit risk was then priced by portfolio based models as credit value adjustment (CVA).

CVA is the difference between the risk-free portfolio and the true value of that portfolio, accounting for possible default of a counterparty. The CVA idea has been in existence for many years but was neglected by most financial institutions because transactions were made with large financial institutions which were considered as "too-big-to-fail" (Gregory 2010). The "too-big-to-fail" myth was shattered with the bankruptcy of large institutions such as AIG and Lehman brothers in 2008. The events increased the market's concern over counterparty credit risk and CVA losses could no longer be neglected. As a result, the Basel Committee for Banking Supervision (BCBS) introduced a concept for capturing CVA losses in Basel III. As specified within Basel III, CVA's purpose was to capture and measure losses on OTC derivatives due to credit spread volatility. The BCBS consequently introduced a new capital charge, which would absorb CVA losses.

The new capital charge can be calculated using either the advanced approach or standardized approach, with most banks favoring the former approach. In advanced CVA risk capital charge approach, financial institutions are authorized to use Internal Model Method (IMM) approach for calculating market risk capital and have specific interest rate risk VaR model approval. The CVA capital charge is determined in accordance with regulator approved formula based on the counterparty's CDS spread and the size of the exposure. In the standardized approach, financial institutions are not authorized to use Internal Model Method (IMM) for calculating market risk capital. Instead, they calculate their CVA charges using the standardized approach, which is based on external credit ratings supplied by ratings agencies.

However, the CVA pricing is inherently complex for two main reasons. Foremost, CVA for each transaction should reflect the consideration of collateral and netting agreements across all transactions with a counterparty. Secondly, the CVA pricing models should incorporate all risk factors of the underlying instrument including correlations between exposure and default probability, that is, right-way or wrong-way risk. It is imperative to have models that produce reasonable CVA prices so as to not subject financial institutions to massive losses.

At the core of calculating the CVA capital charge is the computation of expected exposure. Typically the industry practice is to use a Monte Carlo simulation framework to compute expected exposure and is implemented in three main steps: (i) scenario generation (ii) instrument valuation (iii) aggregation, as suggested in Giovanni et al. (2009). All instruments belonging to a counterparty are priced under a large number of market scenarios for each point in time, and the exposures for each scenario are averaged to obtain an estimate for expected exposure.

The Monte Carlo framework implementation faces some challenges which need to be taken into consideration. There can be potentially many various risk factors driving the dynamics of products in the portfolio, and so the generation of correlated scenarios can not be trivial. Furthermore, there is need to use the same family of models for all types of products in the portfolio, that is, the scenarios have to be consistent for all products in the portfolio. Consistency can however, be difficult as financial institutions often have different models for different products. To add onto the challenges is that not all of the products can be computed analytically. Diverse products have scenarios generated using either PDEs or Monte Carlo approaches. This becomes computationally unfeasible as the generated scenarios quickly use large amounts of memory. In addition, calibration will become challenging as it has to be performed at each scenario.

Canabarro & Duffie (2003) give an introduction to methods used to measure, mitigate and price counterparty risk. They use Monte Carlo simulation methods to measure counterparty risk and discuss practical calculation of CVA in currency and interest rate swaps. De Prisco & Rosen (2005) discuss counterparty risk and credit mitigation techniques at portfolio level. The paper provides a discussion of how Monte Carlo simulations, approximation methods as well as some analytical approximations that can be used to compute various statistics crucial to the measurement of counterparty credit risk. In addition, the paper also provides calculation of expected exposure in credit derivatives portfolio with wrong-way risk.

Several issues pertaining to the simulation of CVA under margin agreements are studied in Pykhtin (2009) and Pykhtin & Zhu (2007), to just name a few. Furthermore, Gregory (2010) provides thorough treatments of the methods and applications used in practice regarding counterparty credit risk.

1.4 Research Aims and Objectives

At the time of writing this thesis, the arbitrage pricing theory was a well-developed theory, with attempts of applications on the theory in practice. However, the above discussed subtles remain contentious issues, hence the major interest to this work.

Aim of the Study

The thesis aims to address the selected subtle issues arising from the arbitrage pricing theory in the real market, with particular emphasis to derivative instruments and counterparty credit risk. The subtles include modeling of bid-ask spreads, choosing the appropriate interest rate to approximate the risk-free interest rate and modeling of counterparty credit risk.

Objectives of the Study

The objectives (main contributions) of the thesis are:

1. To explore arbitrage pricing in the presence of bid-ask spreads from conic finance which factors in risk preferences of market participants.
2. To explore the modeling and methods of simulating interest rate derivatives using the LIBOR Market Model(LMM).
3. To model credit exposures and price counterparty credit risk for OTC interest rate derivatives using the LIBOR Market Model(LMM).

1.5 Outline of the thesis

This thesis is presented in six chapters that include four original articles that came out of this work. Each chapter is written as a peer-reviewed article and can be read independent of the entire thesis. The chapters are presented in the original form specified by journals in which the articles were published or are going to be published. The rest of the thesis is organized as follows:

- **Chapter 2:** Presents an original article, which explores the brand new theory of conic finance in modeling of bid-ask prices taking into consideration the risk preferences of market participants to derivatives trading strategies.
- **Chapter 3:** Presents an original article, which explores the theory of conic finance, with particular focus to LIBOR based derivative instruments. In the same article, we seek to have an in-depth understanding of the interest rate derivatives instruments.
- **Chapter 4:** Presents an original article on the simulation techniques that can be useful for this work.
- **Chapter 5:** Presents an original article on modeling credit exposures and pricing counterparty credit risk for OTC interest rate derivatives.
- **Chapter 6:** Contributions of the thesis and suggestions for future work are summarized.

  References are provided at the end of each chapter. The references used in Chapter 1 are listed according to the requirements stipulated in the manual for post-graduate studies of the North-West University. The references used in the other chapters are provided as specified by journals in which the articles were published or are going to be published.

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Paper 1

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Assessing the Risks of Trading Strategies Using Acceptability Indices

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ABSTRACT

The paper looks at the quantification of risks of trading strategies in incomplete markets. We realized that the no-arbitrage price intervals are unacceptably large. From a risk management point of view, we are concerned with finding prices that are acceptable to the market. The acceptability of the prices is assessed by risk measures. Plausible risk measures give price bounds that are suitable for use as bid-ask prices. Furthermore, the risk measures should be able to compensate for the unhedgeable risk to an extent. Conic finance provides plausible bid-ask prices that are determined by the probability distribution of the cash flows only. We apply the theory to obtain bid-ask prices in the assessment of the risks of trading strategies. We analyze two popular trading strategies—bull call the spread strategy and bear call spread strategy. Comparison of risk profiles for the strategies is done between the Variance Gamma Scalable Self Decomposable model and the Black-Scholes model. The findings indicate that using bid-ask prices compensates for the unhedgeable risk and reduces the spread between bid-ask prices.

Keywords: Conic Finance; Coherent Risk Measure; Acceptability Indices; Incomplete Markets; Bid-Ask Prices; Continuous Time Models

1. Introduction

The paper focuses on the quantification of risks of trading strategies, particularly when the market is incomplete. The incompleteness of the market gives rise to many martingales, each of which produces a no-arbitrage price. Thus there is no exact replication so as to obtain a unique price. Furthermore, the no-arbitrage price intervals are unacceptably large. From a risk management point of view, we are concerned with finding the prices which are acceptable. The acceptability of these prices is assessed by risk measures.

In the financial literature, two major classes of risk measures have gained ground in assessing the risks of financial positions. Foremost, we have coherent measures introduced by [1]. Since then, the theory of coherent risk measures has been applied to several problems in finance. Secondly, there is the grounding work of [2], in which they proposed a new class of performance measures known as acceptability indices. The acceptability indices can be considered as an extension of coherent risk measures. Under the acceptability framework, a financial position is acceptable if its distribution function withstands high levels of stress, or in other words, a stressed sampling of the financial position has a positive expectation. In this paper, our contribution is assessing the risk profiles of trading strategies using the acceptability framework.

The rest of the paper is organized as follows: Section 2 looks at the problem of pricing in incomplete markets. Section 3 gives an overview of risk measures and presents new acceptability indices based on the family of distortion functions. Section 4 presents a brief detail on conic finance and provides closed form expressions for the bid-ask prices. Section 5 presents the models that are used in this work. Section 6 presents numerical tests on assessing the risks of two trading strategies. Section 7 is the conclusion.

2. Problem of Pricing in Incomplete Markets

We start by motivating the problem through explaining the mathematical structure of good deal bounds by [3], also found in [4]. The good deal bounds determine the range of values of a risky position payoff. Let R be the set of replicable payoffs, $\Pi(Y)$ be the market price to replicate a payoff $Y \in R$, and A be an acceptance set of payoffs that are acceptable to the situation. The lower good deal bound for a payoff X is:

$$b(X) = \sup_{Y \in R} \left\{ -\Pi(Y) \middle| Y + X \in A \right\}. \quad (1)$$

This payoff might be interpreted as a bid price. Equation (1) tells us that if X can be bought for less than $b(X)$, then there is a Y that can be bought for $\Pi(Y)$ such that a cost $b(X) + \Pi(Y) < 0$. The upper good deal bound, which might be interpreted as the ask price for X, is given by:

$$a(X) = -b(-x) = \inf_{Y \in R} \left\{ \Pi(Y) \middle| Y - X \in A \right\}. \quad (2)$$

Equation (2) tells us that selling X or buying $-X$ yields the same effect. The interpretation of $-b(X)$ is the cost that renders X to be acceptable. As [5] propose: any valuation principle that gives price bounds induces a risk measure and vice versa. The acceptance set A must include the set of riskless payoffs, $\{Z | Z \geq 0\}$, which is the acceptance set that generates no-arbitrage bounds. The set A does not intersect with the set $\{Z | Z < 0\}$ of pure losses. The acceptance set A must be consistent with market prices, Π, or arbitrage occurs.

Now, an incomplete market is one in which there are many martingale measures Q. The price bounds in Equations (1) and (2) form an interval of arbitrage-free prices for X:

$$I = \left(\inf_{Q \in \mathcal{M}} E^Q[X], \sup_{Q \in \mathcal{M}} E^Q[X] \right) \quad (3)$$

where $\mathcal{M}$ is a set of equivalent martingale measures. The problem with the interval of the arbitrage-free prices for X is that it is usually too wide for the no-arbitrage bounds to serve as useful bid-ask prices.

In practice, derivatives traders are aware of the incompleteness of the markets and after making trades on certain positions, they are not able to hedge away all the risk. Instead, they must bear the risk associated with the trade. To cover their business expenses and to earn compensation for bearing the risk they are not able to hedge, traders establish bid-ask intervals around the expected discounted payoff.

Now, in constructing the bid-ask prices, the difficulty posed by incomplete markets is very significant because of adverse selection. For instance, if the ask price is too high, few potential investors will be willing to pay so much and the result is foregone profits. If the ask price is too low, the resulting trade is bad for a trader and entails likely losses. So, to ensure that trades made at bid and ask prices are beneficial, it helps to use methods that produce bounds for the prices that are suitable for use as bid-ask prices and are adequate to minimize unhedgeable risk to an extent. In the process, we will be able to quantify risk since any valuation method that yields price bounds also induces a risk measure [5].

3. Risk Performance Measures

In this section, we give a brief overview of the risk measures. In general, a risk measure, $\rho : X \to \mathbb{R}$, is a functional that assigns a numerical value to a random variable representing an uncertain payoff.

3.1. Coherent Risk Measure

Definition Coherent Risk Measure

A risk measure ρ is coherent if it satisfies the following axioms:

- Translation Invariance:

$$\rho(X + \alpha r) = \rho(X) - \alpha,$$

for all $X \in \mathcal{G}, \alpha \in \mathbb{R}$.
- Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \geq Y$ a.s.
- Positive Homogeneity:

$$\rho(\lambda X) = \lambda \rho(X),$$

for $\lambda \geq 0$.
- Subadditivity:

$$\rho(X + Y) \leq \rho(X) + \rho(Y),$$

for all $X, Y \in \mathcal{G}$
- Relevance: $\rho(X) > 0$ if $X \leq 0$ and $X \neq 0$.

The last property is included although it is not a determinant of coherency. Translation invariance axiom implies that by adding a fixed amount α to the initial position and investing it in a reference instrument, the risk $\rho(X)$ decreases by α. The monotonicity axiom postulates that if $X(\omega) > Y(\omega)$ for every state of nature ω, Y is more risk because it has higher risk potential.

The positive homogeneity axiom implies that risk linearly increases with size of the position, that is to say that the size of the risk of a position should scale with the size of the position. This is just a natural requirement, though this condition may not be satisfied in the real world since markets may be illiquid. The subadditivity axiom implies that the risk of a portfolio is always less than the sum of the risks of its subparts. This axiom ensures that diversification decreases the risk.

According to the basic representation theorem proved by [1] for a finite Ω, any coherent risk measure admits a representation of the form:

$$\rho(X) = -\inf_{Q \in \mathcal{D}} E^Q[X], \quad (4)$$

with a certain set $\mathcal{D}$ of probability measures with respect to P. A cash flow X is acceptable if it has negative risk, that is $\rho(X) < 0$.

3.2. Acceptability Indices

Cherny and Madan, defined a subclass of risk measures called acceptability indices, defined formally as:

3.3. Definition. Index of Acceptability

The acceptability index is as a mapping α from the set of bounded random variables to the extended half-line $[0,\infty]$. The index satisfies the following four properties:

- **Monotonicity**
  If Y dominates X, that is $X \le Y$, then
  $$\alpha(X) \le \alpha(Y).$$
- **Scale invariance**
  $\alpha(X)$ stays the same when X is scaled by a positive number, that is $\alpha(cX) = \alpha(X)$ for $c > 0$.
- **Quasi-concavity**
  If $\alpha(X) \ge Y$ and $\alpha(Y) \ge Y$, then
  $$\alpha(\lambda X + (1-\lambda)Y) \ge Y,$$
  for any $\lambda \in [0,1]$.
- **Fatou Property (Convergence)**
  Let $\{X_n\}$ be a sequence of random variable. $|X_n| \le 1$ and X_n converges in probability to a random variable X. If $\alpha(X_n) \ge x$, then $\alpha(X) \ge x$.

The acceptability indices are constructed by replacing the cumulative distribution function of X, $F_X(x)$, by a risk adjusted distribution, $\Psi_X(F_X(x))$. The corresponding risk measure is the negative expectation of the zero cost cash flow under the distorted distribution function:

$$\rho_\gamma(X) = -\int_{\mathbb{R}} x\mathrm{d}(\Psi_\gamma(F_X(x))), \gamma \in \mathbb{R}_+ \tag{5}$$

where Ψ_γ is a family of concave distortion functions on [0,1] increasing pointwise in the stress level parameter γ. A higher value of γ results in severe distortion of the distribution function of X. Then, the acceptability index, $\alpha(X)$, is the largest stress level γ such that the expectation of X remains positive under the distortion or in other words the distorted cash flow remains acceptable:

$$\alpha(X) = \sup\{\gamma \in \mathbb{R}_+ : \rho_\gamma(X) \le 0\}. \tag{6}$$

Cherny and Madan introduced four acceptability indices based on the family of distortion functions which are namely: AIMIN, AIMAX, AIMINMAX, AIMAXMIN.

- AIMIN is the largest number x such that the expectation of the minimum of $x+1$ draws from cash flow distribution is still positive. Let
  $$Y \overset{law}{=} \min X_1, \cdots, X_{x+1},$$
  where $X_1, \cdots, X_{x+1}$ are independent draws from X. The concave distortion function is given by:
  $$\Psi_x(y) = 1-(1-y)^{x+1}, x \in \mathbb{R}^+, y \in [0,1] \tag{7}$$
- AIMAX constructs a distribution from which one draws numerous times and takes the maximum to get the cash flow distribution being evaluated. Let,
  $$\max\{Y_1, \cdots, Y_{x+1}\} \overset{law}{=} X,$$
  where $Y_1, \cdots, Y_{x+1}$ are independent draws of Y. The concave distortion function is given by:
  $$\Psi_x(y) = y^{\frac{1}{x+1}}, x \in \mathbb{R}^+, y \in [0,1] \tag{8}$$
- AIMAXMIN is constructed by first using the MINVAR and then followed by the MAXVAR to create worst case scenarios.
  Let
  $$\max\{Y_1, \cdots, Y_{x+1}\} \overset{law}{=} \min\{X_1, \cdots, X_{x+1}\},$$
  where $X_1, \cdots, X_{x+1}$ are independent draws of X and $Y_1, \cdots, Y_{x+1}$ are independent draws of Y. Combining the MINVAR and MAXVAR, we have the distortion function:
  $$\Psi_x(y) = \left(1-(1-y)^{x+1}\right)^{\frac{1}{x+1}}, x \in \mathbb{R}^+, y \in [0,1] \tag{9}$$
- AIMAXMIN is constructed by first using the MAXVAR and then followed by the MINVAR to create worst case scenarios. Let
  $$Y \overset{law}{=} \min Z_1, \cdots, Z_{x+1},$$
  $$\max Z_1, \cdots, Z_{x+1} \overset{law}{=} X,$$
  where $Z_1, \cdots, Z_{x+1}$ are independent draws of Z. Combining the MINVAR and MAXVAR, we have the distortion function:
  $$\Psi_x(y) = 1-\left(1-y^{\frac{1}{x+1}}\right)^{x+1}, x \in \mathbb{R}^+, y \in [0,1] \tag{10}$$

The acceptability indices are more plausible in assessing the risks of financial positions. The acceptability indices have been used heavily in the theory of conic finance, which we review next.

4. Conic Finance Theory

We look at the principles of conic finance as set out in [6]. The market serves a passive counterparty accepting the opposite side of zero cost trades proposed by market participants. The departure of conic finance from the traditional one price economy is that trade now depends on the direction of trade, with the market buying at bid price and selling at ask price. Cash flows to trade are modeled as bounded random variables on a fixed probability space $(\Omega, \mathcal{F}, P)$ for a base probability measure selected by the economy.

Now, for a risk with a cash flow outcome denoted by

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the random variable X with a distribution $F(x)$ at a fixed period, we develop bid-ask prices at which the cash flow is bought and sold such that the net cash flow is an acceptable risk. The set of acceptable risks is defined by a convex cone of random variables that contains the non-negative cash flows. [1] showed that any acceptable set (cone) $\mathcal{A}$ of acceptable risks, there exists a convex set $\mathcal{M}$ of probability measures $Q \in \mathcal{M}$, Q equivalent to P, with the property that $X \in \mathcal{A}$ if and only if:

$$E^Q[X] \geq 0, \text{ all } Q \in \mathcal{M}. \quad (11)$$

The acceptability of a cash flow can then be completely determined by its distribution function. Acceptability of cash flows is linked to positive expectation via concave distortion. So for some concave distribution function $\Psi(u)$, $0 \leq u \leq 1$ the cash flow distribution function $F(x) = P(X \leq x)$ is acceptable if:

$$\int_{-\infty}^{\infty} x \mathrm{d}\Psi(F(x)) \geq 0. \quad (12)$$

[6] show that the bid price, $b(x)$, for the cash flow X is given by:

$$b(x) = \int_{-\infty}^{\infty} x \mathrm{d}\Psi(F(x)) \geq 0 = \inf_{Q \in \mathcal{M}} E^Q[X], \quad (13)$$

and the ask price is given by:

$$a(x) = -\int_{-\infty}^{\infty} x \mathrm{d}\Psi(1 - F(-x)) \geq 0 = \sup_{Q \in \mathcal{M}} E^Q[X]. \quad (14)$$

The bid and ask prices for call and put options can be obtained by using closed formulas which are obtained on integration by parts. Let S be the random variable at time T of an underlying asset. The call option $C = (S - K)^+$ and put option $P = (K - S)^+$, where K is the strike price. The following are the closed bid and ask prices expressions:

$$a_\gamma(C) = \int_K^\infty \Psi^\gamma(1 - F_S(x)) \mathrm{d}x, \quad (14)$$

$$b_\gamma(C) = \int_K^\infty (1 - \Psi^\gamma(F_S(x))) \mathrm{d}x, \quad (15)$$

$$a_\gamma(P) = \int_0^K \Psi^\gamma(F_S(x)) \mathrm{d}x, \quad (16)$$

$$b_\gamma(P) = \int_0^K (1 - \Psi^\gamma(1 - F_S(x))) \mathrm{d}x. \quad (17)$$

F_s is the distribution function of S and is important because the bid and ask prices are determined completely by this distribution.

5. Continuous Time Models for Option Pricing

This section looks at the models that are used for option pricing. It is acknowledged that the relatively most liquid traded assets with market information are quoted vanilla options. In practice, trades mark to market their models to quoted vanilla options before they can price non-quoted options. As a result, this has led to demands for models that are capable of synthesizing the surface of vanilla options. It is well known that the geometric Brownian model is not capable of synthesizing the surface of vanilla options, although it remains a standard quoting model in the markets. Improvements on this model are offered by Lévy processes, which were found to be successful in synthesizing across strikes for a given maturity. The following is a brief overview of the models.

5.1. Black-Scholes Model

The log-normal process models continuously compounded returns using the general Brownian motion so that:

$$X(t) = \nu t + \sigma W(t), \quad (18)$$

where $W(t)$ is a standard Weiner process, ν is the instantaneous drift and σ is the instantaneous volatility of returns. The stochastic differential equation of the stock price is:

$$\mathrm{d}S(t) = S(t)(\mu \mathrm{d}t + \sigma \mathrm{d}W(t)), \quad (19)$$

where μ is the growth rate of the stock and is related to ν as follows $\nu = \mu - 1/2\sigma^2$. The stochastic differential equation can be solved to give the following dynamics of the stock price:

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}. \quad (20)$$

The characteristic function for the logarithm of the stock price is:

$$E\left[\mathrm{e}^{iu\ln(S(t))}\right] = \exp\left\{iu\left[\ln S(0) + \left(\mu - \frac{1}{2}\sigma^2\right)\right]t - \frac{1}{2}\sigma^2 u^2 t\right\} \quad (21)$$

5.2. Variance Gamma Model

[7] define a Variance Gamma process, $X(t,\nu,\sigma,\theta)$, as a time changed Brownian motion as follows:

$$X(t,\nu,\sigma,\theta) = \theta\gamma(t) + \sigma W(\gamma(t)), \quad (22)$$

where $\gamma(t)$ is a Gamma process with parameters a and b, that is, $\gamma(t) \sim Gamma(at,b)$ where the gamma probability density function $\gamma(a,b)$ is given by:

$$f_\gamma(x,a,b) = \frac{b^a x^{a-1}}{\Gamma(a)}\mathrm{e}^{-bx}, x > 0. \quad (23)$$

θ and σ are respectively the instantaneous drift and volatility and $W(t)$ is a standard Brownian motion. The Variance Gamma process uses a gamma process to time change a Brownian motion. The density function of

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a Variance Gamma process is known in closed form and requires the computation of the modified Bessel function of the second kind which can be time consuming. Thus we resort to using the characteristic function, which is found by the conditioning on the jump $\gamma(t)$ as in many Lévy processes and is given by:

$$\Phi_{X(t)}(u) = \left(1 - iu\nu\theta + \frac{1}{2}u^2\nu\sigma^2\right)^{-t/\nu}. \quad (24)$$

The dynamics of the stock price are given by:

$$S(t) = S(0)\exp\left\{(\mu+\omega)t + X(t,\nu,\sigma,\theta)\right\}, \quad (25)$$

where μ is the instantaneous expected return of the stock evaluated at calendar time and ω is a compensator term The characteristic function for the logarithm of stock price is:

$$E\left[\mathrm{e}^{iu\ln(S(t))}\right] = \exp\left\{iu\left[\ln S(0) + (\mu+\omega)t\right]\right\}\Phi_{X(t)}(u). \quad (26)$$

The compensator term can be found from the characteristic function and is given by:

$$\omega = \frac{-1}{t}\ln\left(\Phi_{X(t)}(-i)\right).$$

5.3. Variance Gamma Scalable Self Decomposable (VGSSD) Model

Sato process model was first introduced by [8]. The Sato process was shown to be effective in synthesizing many options on numerous underliers at the same time. The idea behind the model was to construct stochastic processes that had inhomogeneous independent increments from Lévy processes with homogeneous independent increments such that the higher moments are constant over the time horizon.

The starting point for the construction of the Sato model is the self-decomposable law. Loosely speaking, the self-decomposable law describes random variables that decompose into the sum of a scaled down version of themselves and an independent residual term. The scaling property means the distribution of increments of various time scales can be obtained from that of other time scale by rescaling the random variable at that time scale. Thus the distribution at larger time scales are derived from those at smaller time scales, which are easier to estimate as the data are sufficient. [9] proposed that the self-decomposable law is associated with the unit time distribution of self-similar additive process whose increments are independent, but not necessarily stationary.

It is known that stock prices are moved by many pieces of information. If the pieces of information are considered as a sequence of independent random variables $(Z_i : i = 1,2,\cdots)$, then the price changes are consequences of the impacts from all Z_i. Now, let $S_n = \sum_{i=1}^{n} Z_i$ denote their sum. Suppose that there exist centering constants c_n and scaling constants b_n such that the distribution of $b_n S_n + c_n$ converges to the distribution of the random variable X, which belongs to a family law "class *L*". Then the random variable X is said to have the class L property. So, the price change over the time horizon is the outcome of many independent random variables which can be approximated as a random variable X that has the law of "class *L*". [10] define the self-decomposable law as follows.

5.4. Definition Self Decomposable Law

A random variable X is self-decomposable if for all $c \in (0,1)$,

$$X \overset{law}{=} cX + X^c, \quad (27)$$

where X^c is a random variable independent of X.

The self-decomposable random variable X can be decomposed into a partial of itself and another independent random variable. [10] also shows that one may associate with such a self-decomposable law at unit time a process with independent but inhomogeneous increments by defining the marginal law of the process at time points t upon scaling the law at unit time. Therefore we have that:

$$X(t) = t^\gamma X, t > 0. \quad (28)$$

Thus we can study the price changes easily using self-decomposable laws, which are easier to handle than class L.

Self-decomposable laws are an important sub-class of the class of infinitely divisible laws [11]. The characteristic function of the self-decomposable laws has the form (see [10])

$$E\left[\mathrm{e}^{iux}\right] = \exp\left\{iru - \frac{1}{2}\sigma^2u^2 + \int_R \left(\mathrm{e}^{iux} - 1 - iux1_{|x|\le 1}\frac{g(x)}{|x|}\mathrm{d}x\right)\right\}, \quad (29)$$

where r, σ are constants, $\sigma^2 \ge 0$,

$$\int_R \left(|x|^2 \wedge 1\right)\frac{g(x)}{|x|}\mathrm{d}x < \infty,$$

and $g(x)$ is an increasing function when $x < 0$ and decreasing function when $x > 0$. An infinitely divisible law is self-decomposable if the corresponding Lévy density has the form

$$\frac{g(x)}{|x|},$$

where $g(x)$ is increasing for negative x and decreasing for positive x.

The dynamics of the stock price is defined as:

$$S(t) = S(0)\exp\{r(t) + X(t) + \omega(t)\}, \quad (30)$$

where $\omega(t)$ is a compensator term. The Sato process used in this work is the one constructed from the variance gamma process and is known as the Variance Gamma Scalable Self Decomposable (VGSSD) process. The variance gamma process is defined by time changing an arithmetic Brownian motion with drift θ and volatility σ by an independent gamma process with unit mean rate and variance rate ν. Let $G(t;\nu)$ be the gamma process, then the variance gamma process is written as:

$$X_{VG}(t,\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu)), \quad (31)$$

where $W(t)$ is an independent standard Brownian motion.

The gamma process is an increasing pure jump Lévy process with independent identically distributed increments over regular non overlapping intervals of length h that are gamma distributed with density $f_h(g)$ where:

$$f_h(g) = \frac{g^{\frac{h}{\nu}-1} e^{-\frac{g}{\nu}}}{\nu^{\frac{h}{\nu}} \Gamma\left(\frac{h}{\nu}\right)}, g > 0. \quad (32)$$

The VGSSD is constructed from the variance gamma process by defining the scaled stochastic process $X(t)$ such that it is equal in law to $t^{\gamma} X_{VG}(1)$ where $X_{VG}(1)$ is a variance gamma random variable at unit time. It follows that the characteristic function of $X(t)$ is given by [7]

$$\Phi_{X(t)}(u) = \Phi_{X_{VG}(1)} = \left(1 - iut^{\gamma}\nu\theta + \frac{1}{2}u^2 t^{2\gamma}\nu\sigma\right). \quad (33)$$

Since the VGSSD is a scaled stochastic process, its higher moments remain constant over time.

6. Numerical Tests

Next, focus is shifted to analyzing the risk profiles of option investing strategies. We examine two option strategies which are namely bull call spread strategy and bear call spread strategy. We determine the maximum risk, maximum reward and breakeven price for each of the strategies. Comparison of risk profiles is done between the VGSSD model and the Black-Scholes model. The Black-Scholes model is considered here since it is the one that is mostly used by industrial practioners. So, the Black-Scholes is a proxy for market prices. The theory of conic finance provides bid-ask prices, which depend on the risk appetite of investors. For evaluation of bid-ask prices, we use acceptability indices based on the MAXMINVAR. The options used in the strategies are of European type and are applied to Single Stocks Futures (SSF) options offered in the South African financial markets.

A bull call spread is a bullish strategy formed by buying an "in-the-money call option" (lower strike) and selling "out-of-the-money" (higher strike). Both call options must be on the same underlying and expiration date. The strategy's net effect is to bring down the cost and breakeven (long call strike price + net debt) on a buy call (long call) strategy.

A bear call spread is a bearish strategy formed by buying an "out-of-the-money" call option (higher strike) and selling an "in-the-money" call option (lower strike). Both call options must be on the same underlying security and expiration date. The strategy's concept is to protect the downside of the sold call option by buying a call option of higher strike price. Then, the investor receives a net credit since the call option which has been bought has a higher strike price than the sold option. The breakeven will be the sum of the strike price of the short call option plus the premium received.

For numerical illustration purposes, we used names of two large South African banks—ABSA and Standard Bank. Note that, the illustrations do not pertain to any real positions on the banks. The bid-ask prices were computed at various theoretical prices of the underlying on the expiration date. The 3-month JIBAR is used as a proxy for the risk-free interest rate. To realize model calibration, we need market prices. Simulated data set of bid-ask options at different strikes maturing on the same date were generated using the models introduced in the previous Section.

The illustrations that follow merely suggest what an investor can do given the different risk appetites on an investor. The illustrations are implemented at stress (risk) level of 0.01, 0.05 and 0.10.

6.1. Bull Call Spread Risk Profile

6.1.1. Scenario

An investor owns 100 shares in ABSA Bank (ASAQ), which in early July are trading at a Single Stock Future (SSF) fair value of R140. The investor believes the market will be bullish in the coming 6 months and decided to create a bull call spread. So the investor buys a DEC ASAQ 140 call option and sells a DEC ASAQ call option with a higher strike price, so as to create the bull call spread strategy. The concern for the investor is on the appropriate higher strike which can create an attractive strategy.

1) At different stress (risk) levels, the investor determines the bid-ask prices for the range of strike prices.

2) The investor analyzes the risk profiles at each strike price choice so as to create an appropriate trade.

3) The investor finally assesses the performance of the

strategy, given a range of possible values of the underlying at expiration for the appropriate strike price from step 2 at a stress level of 0.01.

In addition, the investor gathers the following information:

$$\text{3month JIBAR rate } = 5.01\%$$

$$\text{Time to expiration } = 6/12 \text{ yr}$$

$$\text{Dividend yield } = 0\% \text{ (assumption).}$$

In order to create the strategy appropriately, the investor implemented the following steps.

1) Bid-Ask Prices at Different Stress (γ) Levels

The calibrated parameters used for this strategy are $\sigma = 0.240$ in the Black-Scholes model and

$$\sigma = 0.226, \theta = -0.131, \nu = 0.08, \gamma = 0.480$$

in the VGSSD model. An attractive bull call spread is created when an investor buys a lower strike call and sells a higher strike call. In the scenario presented above, the investor has the choices shown in **Table 1**. **Table 2** shows the bid-ask prices for the options using both the Black-Scholes model and VGSSD model.

2) Risk Profile Analysis

Next, we look at the risk profiles for each of the choices using bid-ask prices provided in step 1 at a stress level of 0.01. Under the Black-Scholes model, an attractive strategy can be created by choice 4) as shown in **Table 3**. The reason is that the risk and breakeven point is lower whilst maximum reward and maximum Return on Investment (ROI) are high enough to be attractive.

Also under the VGSSD model, an attractive strategy can be created using choice 4) as shown in **Table 4**. The reason being again that the risk and breakeven is lower whilst the maximum reward and maximum Return on Investment (ROI) are high enough to be attractive.

3) Scenario Analysis at the Expiration Date

After choosing an attractive choice from step 2, we now look at the profit/loss of the strategy at expiration for a range of prices for the underlying. We compare the profit/loss under the two models—Black-Scholes model and VGSSD model. **Table 5** shows the profit/loss of the strategy under the two models. **Figure 1** shows the plot of the profit/loss of the strategy for a range of prices of the underlying at expiration. From **Figure 1**, it can be observed that the breakeven point is lower using the VGSSD model than the Black-Scholes model. A lower breakeven point is ideal for a strategy which intends to reduce risk.

6.1.2. Comment on the Strategy

The spread was observed to be lower in the VGSSD model than in the Black-Scholes model. Reduced spread can minimize the unhedgeable risk, which can be a major boost for option trading strategies. As a result, the cost of trade is lowered as the sold options can offset the cost of the bought option. In conclusion, the strategy becomes less risky in terms of lower risk and lower breakeven point but offers limited potential reward, which can still be highly attractive.

Table 1. Bull call spread investor choices.

Step 1	Long Call	Buy R140 Strike Call
Step 2	**1) Short Call**	Sell R141 Strike Call
Or	**2) Short Call**	Sell R142 Strike Call
Or	**3) Short Call**	Sell R143 Strike Call
Or	**4) Short Call**	Sell R144 Strike Call
Or	**5) Short Call**	Sell R145 Strike Call

6.2. Bear Call Spread Risk Profile

6.2.1. Scenario

In early July an investor believes the SSF fair price of Standard Bank (SBKQ) is going to fall from the current levels of R120 to around R117.50. The investor wants to create an attractive bear call spread. So the investor writes a SEP SBKQ 119 call option and buys a higher SEP SBKQ strike call, so as to create a bear call strategy. A little bit of concern to the investor is on the appropriate higher strike to choose so as to create an attractive strategy.

1) Now at different stress (risk) levels, the investor determines the bid-ask prices for the range of higher strike prices.

2) The investor analyzes the risk profiles at each of strike price choices so as to create an appropriate trade.

3) Finally, the investor accesses the performance of the strategy for the appropriate strike price in step 2 at a stress level of 0.01 given a range of possible values of the underlying at expiration.

1) Bid-Ask Prices at Different Stress (γ) Levels

The calibrated parameters used for this strategy are $\sigma = 0.306$ in the Black-Scholes model and

$$\sigma = 0.285, \theta = -0.070, \nu = 0.060, \gamma = 0.510$$

in the VGSSD model. An attractive bear call spread is created when an investor sells a lower strike call and buys a higher strike call. In the scenario presented here, the investor has the choices shown in **Table 6**. **Table 7** shows the bid-ask prices for the options using both the Black-Scholes model and VGSSD model.

2) Risk Profile Analysis

We now look at the risk profiles for each of the choices using bid-ask prices provided in step 1 at a stress level of 0.01. In **Table 8** a potential strategy can be created using a strike which provides reduced risk and a lower breakeven point. In addition, the gain on this strategy is the net credit received upon entering the trade. As

Table 2. Bull call spread bid-ask prices at different stress levels.

Stress Level	Black-Scholes Model					VGSSD Model				
	S	K	Bid	Ask	Spread	S	K	Bid	Ask	Spread
0.01	140	140	11.05	11.83	0.78	140	140	11.04	11.77	0.73
		141	10.56	11.32	0.76		141	10.54	11.25	0.71
		142	10.07	10.81	0.74		142	10.04	10.73	0.69
		143	9.62	10.33	0.71		143	9.59	10.26	0.67
		144	9.16	9.85	0.69		144	9.15	9.79	0.64
		145	8.71	9.37	0.66		145	8.70	9.32	0.62
0.05	140	140	10.50	14.59	4.09	140	140	10.49	14.28	3.79
		141	10.03	14.00	3.79		141	10.01	13.68	3.67
		142	9.56	13.40	3.84		142	9.53	13.09	3.56
		143	9.09	12.80	3.71		143	9.07	12.52	3.45
		144	8.67	12.28	3.61		144	8.66	12.01	3.35
		145	8.24	11.72	3.48		145	8.23	11.47	3.24
0.10	140	140	9.85	18.45	8.60	140	140	9.81	17.67	7.86
		141	9.38	17.72	8.34		141	9.35	16.99	7.64
		142	8.93	17.03	8.10		142	8.92	16.35	7.43
		143	8.49	16.33	7.84		143	8.46	15.67	7.21
		144	8.07	15.67	7.60		144	8.05	15.05	7.00
		145	7.68	15.05	7.37		145	7.67	14.46	6.79

Table 3. Bull call spread risk profile using Black-Scholes model.

Step 1	Long Call				
	Buy R140 Strike Call@R11.77				
Step 2	**Short Call**	**Risk**	**Reward**	**Breakeven**	**Max ROI**
1)	Sell R141 Strike Call@R10.54	R1.23	-R0.23	R141.23	−18.70%
2)	Sell R142 Strike Call@R10.04	R1.73	R0.27	R141.73	15.61%
3)	Sell R143 Strike Call@R9.59	R2.18	R0.82	R142.18	37.61%
4)	Sell R144 Strike Call@R9.15	R2.62	R1.38	R142.62	52.67%
5)	Sell R145 Strike Call@8.70	R3.07	R1.93	R143.07	62.87%

Table 4. Bull call spread risk profile using VGSSD model.

Step 1	Long Call				
	Buy R140 Strike Call@R11.83				
Step 2	**Short Call**	**Risk**	**Reward**	**Breakeven**	**Max ROI**
1)	Sell R141 Strike Call@R10.56	R1.27	−R0.27	R141.27	−21.26%
2)	Sell R142 Strike Call@R10.07	R1.76	R0.24	R141.76	13.64%
3)	Sell R143 Strike Call@R9.62	R2.21	R0.79	R142.21	35.75%
4)	Sell R144 Strike Call@R9.16	R2.67	R1.33	R142.67	49.81%
5)	Sell R145 Strike Call@R8.71	R3.12	R1.88	R143.12	60.26%

a result choice 4) is attractive to create the strategy since the net credit is fairly high, and the breakeven point as well as the risk are reduced.

In **Table 9** a potential strategy again can be created using a strike which provides reduced risk and lower breakeven. Also, the gain on this strategy is the net credit received upon entering the trade. As a result choice 4) is attractive to create the strategy since the net credit is fairly high, and the breakeven point is reduced and the risk is fairly low.

3) Scenario Analysis at the Expiration Date

Next, we look at the profit/loss of the strategy at expiration for a range of prices for the underlying using the choice selected in Step 2. We compare the profit/loss under the two models—Black-Scholes model and VGSSD model. **Table 10** shows the profit/loss of the strategy under the

Table 5. Bull call spread profit/loss under Black-Scholes and VGSSD models.

Black-Scholes Model		VGSSD Model	
ASAQ@expiry	**Profit/Loss**	**ASAQ@expiry**	**Profit/Loss**
135	−2.67	135	−2.62
136	−2.67	136	−2.62
137	−2.67	137	−2.62
138	−2.67	138	−2.62
139	−2.67	139	−2.62
140	−2.67	140	−2.62
141	−1.67	141	−1.62
142	−0.67	142	−0.62
142.67	0.00	142.62	0.00
143	0.33	143	0.38
144	1.33	144	1.38
145	1.33	145	1.38
146	1.33	146	1.38
147	1.33	147	1.38
148	1.33	148	1.38
149	1.33	149	1.38
150	1.33	150	1.38

Table 6. Bear call spread investor choices.

Step 1	**Short Call**	Sell R119 Strike Call
Step 2	**1) Long Call**	Buy R120 Strike Call
Or	**2) Long Call**	Buy R121 Strike Call
Or	**3) Long Call**	Buy R122 Strike Call
Or	**4) Long Call**	Buy R123 Strike Call
Or	**5) Long Call**	Buy R124 Strike Call

Table 7. Bear call spread bid-ask prices at different stress levels.

Stress Level	Black-Scholes Model					VGSSD Model				
	S	**K**	**Bid**	**Ask**	**Spread**	**S**	**K**	**Bid**	**Ask**	**Spread**
0.01	120	119	6.91	7.45	0.54	120	119	6.87	7.40	0.73
		120	6.40	6.90	0.50		120	6.37	6.86	0.71
		121	5.92	6.40	0.48		121	5.88	6.35	0.69
		122	5.47	5.93	0.46		122	5.43	5.89	0.67
		123	5.02	5.45	0.43		123	5.01	5.43	0.64
		124	4.62	5.03	0.41		124	4.61	5.02	0.41
0.05	120	119	6.57	9.42	2.85	120	119	6.55	9.34	2.79
		120	6.06	8.76	2.70		120	6.04	8.67	2.63
		121	5.61	8.16	2.55		121	5.59	8.09	2.50
		122	5.18	7.61	2.43		122	5.17	7.53	2.36
		123	4.75	7.03	2.28		123	4.73	6.95	2.22
		124	4.35	6.50	2.15		124	4.33	6.42	2.09
0.10	120	119	6.16	12.25	6.09	120	119	6.14	12.05	5.91
		120	5.67	11.44	5.77		120	5.66	11.28	5.62
		121	5.25	10.74	5.49		121	5.23	10.56	5.33
		122	4.81	10.00	5.19		122	4.80	9.85	5.05
		123	4.42	9.33	4.91		123	4.41	9.19	4.78
		124	3.71	8.08	4.37		124	3.69	7.94	4.25

Table 8. Bear call spread risk profile using Black-Scholes model.

Step 1	Short Call				
	Sell R119 Strike Call@R6.91				
Step 2	**Long Call**	**Risk**	**Reward**	**Breakeven**	**Max ROI**
1)	Buy R120 Strike Call@R6.90	R0.99	R0.01	R119.01	1.01%
2)	Buy R121 Strike Call@R6.40	R1.49	R0.51	R119.51	34.23%
3)	Buy R122 Strike Call@R5.93	R2.02	R0.98	R119.98	48.51%
4)	Buy R123 Strike Call@R5.45	R2.54	R1.46	R120.46	57.48%
5)	Buy R124 Strike Call@5.03	R3.12	R1.88	R120.88	60.26%

Table 9. Bear call spread risk profile using VGSSD model.

Step 1	Short Call				
	Sell R119 Strike Call@R6.87				
Step 2	**Long Call**	**Risk**	**Reward**	**Breakeven**	**Max ROI**
1)	Buy R120 Strike Call@R6.86	R0.99	R0.01	R119.01	1.01%
2)	Buy R121 Strike Call@R6.35	R1.48	R0.52	R119.52	35.14%
3)	Buy R122 Strike Call@R5.89	R2.02	R0.98	R119.98	48.51%
4)	Buy R123 Strike Call@R5.43	R2.56	R1.44	R120.44	56.25%
5)	Buy R124 Strike Call@5.02	R3.15	R1.85	R120.85	58.73%

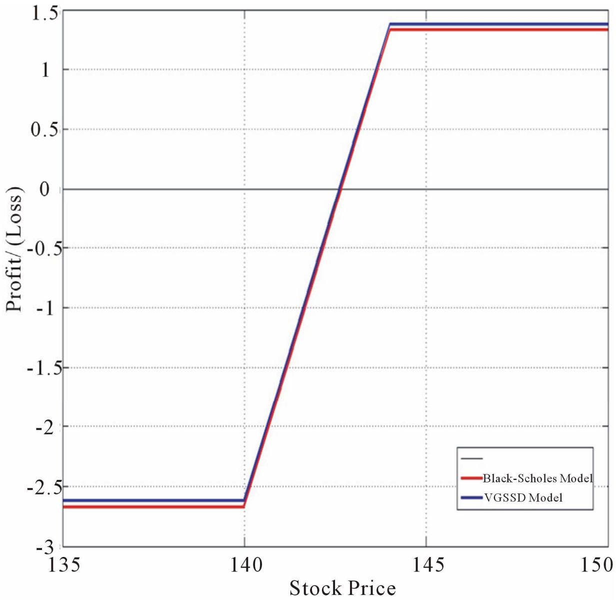

Figure 1. Plot of bull call spread profit/loss.

Table 10. Bull call spread profit/loss under the Black-Scholes and VGSSD models.

Black-Scholes Model		VGSSD Model	
SBKQ@expiry	**Profit/Loss**	**SBKQ@expiry**	**Profit/Loss**
125	−2.54	125	-2.56
124	−2.54	124	-2.56
123	−2.54	123	-2.56
122	−1.54	122	-1.56
121	−0.54	121	-0.56
120.46	0.00	120.44	0.00
120	0.46	120	0.44
119	1.46	119	1.44
118	1.46	118	1.44
117	1.46	117	1.44
116	1.46	116	1.44
115	1.46	115	1.44

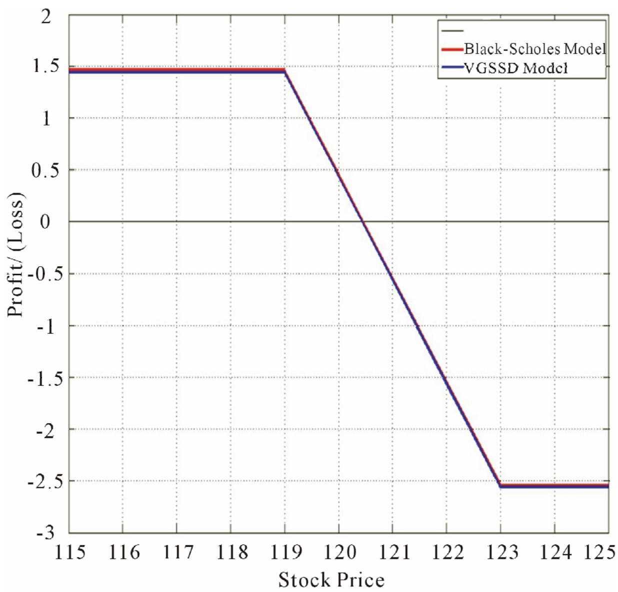

Figure 2. Plot of bear call spread profit/loss.

two models. **Figure 2** shows the plot of the profit/loss of the strategy for a range of prices of the underlying at expiration. The breakeven point is lower in the VGSSD model than the Black-Scholes model, which is ideal in creating a strategy with reduced risk.

6.2.2. Comment on the Strategy

Under this strategy, the spread is reduced under the VGSSD model than the Black-Scholes model. The lower spread implies reduced cost of risk and lower breakeven point. However, in this strategy reducing the risk impacts on the potential reward under the VGSSD model as compared to the Black-Scholes model.

7. Conclusions

In this paper, we have looked at the quantification of risks of trading strategies in incomplete markets. We established that the no-arbitrage price intervals are unacceptably large. We need intervals with prices which are acceptable to the market. The acceptability of the prices is assessed by risk measures. Ideal risk measures are those that produce price bounds that are suitable for use as bid-ask prices and are able to compensate for unhedgeable risk. Plausible risk measures we look at are coherent risk measure and acceptability indices. Acceptability indices are heavily used in the theory of conic finance, which we used to assess the risk of trading strategies. Conic finance provides plausible bid-ask prices which are determined only by the probability distribution of the cash flow.

We assess the risks of financial positions using two strategies-bull call spread and bear call spread. Comparison of the risk profiles for the trading strategies is done between the VGSSD model and the Black-Scholes model. The findings showed that the spread was reduced, especially using the VGSSD model as compared to the Black-Scholes model. In addition, the findings showed that the bid-ask price intervals are able to compensate for the unhedgeable risk. Ultimately, reward from the strategies had a potential of increasing.

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Paper 2

Estimation of bid-ask prices for options on LIBOR based instruments

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Abstract

Interest rate options are the most liquid traded derivatives in the markets. We observe the following from the markets: (i) Market dealers usually quote the mid-price. The mid-price is a subjective and hypothetical price. (ii) OTC interest rate options market are incomplete, and options cannot always be costlessly replicated. (iii) The bid-ask prices are not widely available for the market as a whole. With these observations in mind, we propose an approach to estimate the bid-ask prices for options on LIBOR based instruments. In particular, we assess the proposed approach in the determination of premiums for caps and floors.

Keywords:
Interest rate, LIBOR, Caps, Floors, Bid-ask prices, Wang transform

1. Introduction

Over-the-counter (OTC) interest rate options such as caps and floors are amongst the most liquid options that are trade in the global financial markets. In the OTC markets, where there is absence of centralized trading platform, prices are negotiated bilaterally between buyers and sellers. The buyers of caps and floors in the OTC market are typically corporations trying to hedge their interest rate risk. On the other hand, the sellers are usually derivatives desks at large commercial and investment banks. The sellers are concerned

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with hedging the risks of cap and floor contracts they sell. The size of the individual trades are relatively large, with the contracts being long dated portfolios of options. The long dated nature of the contracts create enormous transactions costs if a seller hedges dynamically using the underlying interest rate markets. Also the dealers in these markets cannot hedge perfectly the risks since OTC interest rate options market are incomplete (due to maturity, transaction costs, basis difference, contract size considerations, liquidity, bid-ask spread etc). In particular, the dealers are interested in holding little positions as much as possible after hedging their trades.

In the idealized world of the hedging paradigm of the Black-Scholes, both the buyer and seller can hedge continuously, perfectly and costlessly in the underlying market. However in the real world, options cannot always be exactly and costlessly replicated due to frictions in the market such as discrete rebalancing or transaction costs. With frictions in the market, option pricing models may not satisfy the martingale restriction, that is, the price of the underlying asset implied by the option pricing model must equal its actual value. In such a scenario, the no-arbitrage framework can only place bounds on option prices. Constantides (1997) argues that with transaction costs, the concept of no-arbitrage price of a derivative is replaced by a range of prices, which is likely to be wider for OTC derivatives (which includes most interest rate options), as opposed to plain-vanilla exchange-traded contracts, since the seller has to incur the high costs to cover for the unhedgeable risk. In a similar vein, Longstaff (1995) shows that in the presence of frictions, option pricing models may not satisfy the martingale restriction. These factors result in some part of the risk in options becoming unhedgeable.

To recoup their business expenses and to earn compensation for bearing the risk that cannot be completely hedged, the dealers establish a bid-ask price intervals around the option prices. Now in constructing the bid-ask prices, the difficulty posed by incomplete markets becomes apparent because of adverse selection. For instance, if the ask price is too high few potential investors will be willing to pay so much and the result is foregone profits. If the ask price is too low, the resulting trade is bad for the dealer and entails likely losses (Sonono and Mashele (2013)).

However, the dealers usually quote the mid-prices (the average of the bid-ask prices) to the market participants. The mid-price is subjective and hypothetical price which may differ from other dealers or market participants. It does not include funding, hedging or any other costs or adjustments, and may not necessarily reflect the price at which one can enter into a deal with a dealer or another market participant. Deuskar et al. (2004) find out that using the mid-price of caps and floors may significantly distort the estimation of pricing models, since the shape of the volatility smile and the information therein are significantly different on the ask-side from those on the bid-side. Taking the mid-prices to calibrate models may lead to losing important information about the dynamic evolution of market prices in response to different demands by market participants.

One of the major challenges in the area of OTC securities including caps and floors is to establish sound basis for pricing decisions based on quantitative models rather than leaving it to intuition. The bid-ask prices set by the market dealers depend on informal considerations of several factors such as the dealer's outlook on likely market events, trading inquiries and execution prices for similar transactions and macroeconomic events affecting the market. In addition, the data on bid-ask prices are not widely available for the markets as a whole. In the absence of bid-ask prices for caps and floors, the major aim in this paper is to propose an approach to estimate the bid-ask prices for caps and floors, which factors into consideration the risk sentiments prevailing in the market. There are variety of theoretical approaches that try to model bid-ask prices. The approach relating to this work is the one studied in Cochrane and Saá-Requejo (2000), Bernardo and Ledoit (2000), Jaschke and Küchler (2001) and Carr et al. (2001), cited in Cherny and Madan (2010). We believe the risk sentiment captures the expected hedging costs and the risks over the lives of the caps and floors. The approach can be seen as a first line of thought which can be possibly further conditioned by further plausible considerations.

Although there has been some work in the existing literature that deals with estimating bid-ask prices, the work has been in the context of equity options. In this regard we cite works by Vijh (1990), George and F.A.Longstaff (1993), Pena et al. (2001) and Mayhew (2002). The conclusions from equity option markets cannot be extended to interest rate option markets including caps and floors, since these markets differ significantly from each other. To the best of our knowledge, there are few studies which have looked at the estimation of bid-ask prices for interest rate options.

The outline of the rest of the paper is as follows: In Section 2 we discuss the options on LIBOR based instruments and the market practice for pricing these instruments. Section 3 is devoted to the approach used to come up with bid-ask prices for the options on LIBOR based instruments. We test the proposed approach to determine premiums for caps and floors in Section

4 and Section 5 is the conclusion.

2. Options on LIBOR based instruments

In this section we discuss the options on LIBOR based instruments and the market practice for pricing these instruments. In particular, we focus on caps and floors. The reader is referred to Björk (2004) for the exposition in this section.

Definition 2.1. LIBOR Rate

The LIBOR rate $L(t, t+\delta)$ contracted at time t is the solution to the equation:

$$1 + \delta L(t, t+\delta) \quad = \quad \frac{1}{p(t, t+\delta)}, \tag{1}$$

where $\delta > 0$ is the time length covered by the LIBOR interest rate.

Consider a fixed set of increasing maturities also known as the tenor structure for the LIBOR given by $T_1 < \ldots T_i < T_{i+1} < \ldots T_N$. Define $\delta_i = T_{i+1} - T_i$, $i = 1, 2, \ldots, N-1$, where δ_i is the tenor and $1/\delta_i$ is the day-count factor. Let $p_i(t) \equiv p(t, T_i)$ denote the time t price of a default-free discount bond maturing at time T_i. Let $L_i(t) \equiv L_i(t, T_i, T_{i+1})$ denote the forward LIBOR rate contracted at time t (where $t \leq T_i$), for the period $[T_i, T_{i+1}]$, that is $L_i(t)$ is reset at dates $T_i, i = 1, \ldots, N-1$ known as the reset dates and is valid for the period $\delta_i = T_{i+1} - T_i$. Equation 1 can be written as:

$$1 + \delta L(t, t+\delta) \quad = \quad \frac{p_i(t)}{p_{i+1}(t,)}, \quad i = 1, 2, \ldots, N-1 \tag{2}$$

which implies that the T_i forward LIBOR rate at time t is given by:

$$L_i(t) \quad = \quad \frac{1}{\delta_i}\left(\frac{p_i(t)}{p_{i+1}(t)} - 1\right), \; i = 1, 2, \ldots, N-1, \;\; t \geq 0 \tag{3}$$

with initial term structure $L_i(0)$.

Definition 2.2. Cap

A cap with cap rate K and reset dates $T_1, T_2, \ldots, T_{N-1}$ is a contract which at time T_{i+1} gives the holder of the cap the amount:

$$Cap = \delta_i\{L_i(T_i) - K\}^+ = \delta_i \max\{L_i(T_i) - K, 0\}. \quad (4)$$

The forward LIBOR rate $L_i(T_i)$ is the reference floating rate and the cap rate, K, is the fixed rate for the forward agreement. From the definition above, a cap can be viewed as call option written on the forward LIBOR rate with cap rate K.

A cap consists in a portfolio a number of more basic contracts named caplets. Thus a cap is a sum of caplets. Let $Cpl_i(T_{i+1})$ denote the payoff of the i^{th} caplet received at time T_{i+1}, then $Cpl_i(T_{i+1}) = \delta_i\{L_i(T_i) - K\}^+$. The caplet payoff is known at time T_i but received at time T_{i+1}. Thus for forward LIBOR rates $L_i(.), i = 1, 2, \ldots, N-1$, we have the following vector of corresponding caplet payoffs for $i = 1, 2, \ldots, N-1$:

$$\begin{aligned} Cpl_1(T_2) &= \delta_1\{L_1(T_1) - K\}^+ \\ Cpl_2(T_3) &= \delta_2\{L_2(T_2) - K\}^+ \\ &\vdots \\ Cpl_{N-1}(T_N) &= \delta_{N-1}\{L_{N-1}(T_{N-1}) - K\}^+, \end{aligned} \quad (5)$$

where $Cpl_{N-1}(T_N)$ denotes the payoff at maturity T_N for a caplet contracted at time T_{N-1}.

Next, we determine the arbitrage-free price of a caplet under the forward-neutral probability measure. Let $Cpl_i(t)$ denote the time t price of the i^{th} caplet. Let $h(Cpl_i(T_{i+1})) = \delta_i\{L_i(T_i) - K\}^+$ denote the payoff function of the caplet at time T_{i+1}, $p_{i+1}(t)$ denote the time t price of a default-free discount bond maturing at time T_{i+1} and $\mathbf{E}_t^{T_{i+1}}$ denote the conditional expectation under the forward-neutral probability measure $P^{T_{i+1}}$, given filtration $\mathcal{F}_t$. Then the time t price of the i^{th} caplet is given by:

$$\begin{aligned} Cpl_i(t) &= p_{i+1}\mathbf{E}_t^{T_{i+1}}[h(Cpl_i(T_{i+1}))] \\ &= \delta_i p_{i+1}\mathbf{E}_t^{T_{i+1}}\left[\{L_i(T_i) - K\}^+\right], \quad 0 \le t \le T_i. \end{aligned} \quad (6)$$

Since $Cpl_i(t)$ is the payoff from a call option on the underlying $h_i(T_i)$, the market practice is to use Black (1976) for a call option to price the caplet. The following definition states the Black's formula for pricing an i^{th} caplet.

Definition 2.3. Black's formula for Caplets

Let $Cpl_i^{Black}(t)$ denote the time t price of a caplet. Then the Black's price of a caplet written on the forward LIBOR rate $L_i(t)$ with maturity T_i and cap rate K is given by:

$$Cpl_i^{Black}(t) = \delta_i p_{i+1}(t)\{L_i(t)N(d_1) - KN(d_2)\}, \tag{7}$$

where

$$d_1 = \frac{1}{\sigma_i\sqrt{T_i - t}}\left\{\ln\left(\frac{L_i(t)}{K}\right) + \frac{1}{2}\sigma_i^2(T_i - t)\right\}, \tag{8}$$

$$\begin{aligned} d_2 &= \frac{1}{\sigma_i\sqrt{T_i - t}}\left\{\ln\left(\frac{L_i(t)}{K}\right) - \frac{1}{2}\sigma_i^2(T_i - t)\right\} \\ &= d_1 - \sigma_i\sqrt{T_i - t}, \end{aligned} \tag{9}$$

σ_i denotes the volatility of the forward LIBOR rate $L_i(t)$ and $N(d_i)$ is the distribution function of the standard normal distribution.

In the market, cap prices are quoted in terms of implied Black volatilities and these volatilities can be quoted as either flat volatilities or forward volatilities (see Björk (2004)). For a sequence of implied flat volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$, the implied flat volatilities are defined as solutions of the equations:

$$Cap_i^m(t) = \sum_{k=1}^{i} Cpl_k^{Black}(t, \bar{\sigma}_i),\ i = 1, \ldots, N, \tag{10}$$

and the implied spot or forward volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as solutions of equations:

$$Cap_i^m(t) = Cpl_k^{Black}(t, \bar{\sigma}_i),\ i = 1, \ldots, N. \tag{11}$$

Definition 2.4. Floor

A floor with floor rate K and reset dates $T_1, T_2, \ldots, T_{N-1}$ is a contract which at time T_{i+1} gives the holder of the floor the amount:

$$Floor = \delta_i\{K - L_i(T_i)\}^+ = \delta_i \max\{K - L_i(T_i), 0\}. \tag{12}$$

The forward LIBOR rate $L_i(T_i)$ is the reference floating rate and the floor rate, K, is the fixed rate for the forward agreement. From the definition, a floor can be viewed as a put option written on the forward LIBOR rate with floor rate K.

A floor consists in a portfolio a number of more basic contracts called floorlets. A floor is thus a sum of floorlets. Let $Flr(T_{i+1})$ denote the payoff of the i^{th} floorlet received at time T_{i+1}, then $Flr(T_{i+1}) = \delta_i\{K - Flr(T_i)\}^+$. For the forward LIBOR rates $L_i(.), i = 1, 2, \ldots, N-1$ we have the following vector of corresponding floorlet payoff for $i = 1, 2, \ldots, N-1$:

$$\begin{aligned} Flr_1(T_2) &= \delta_1\{K - L_1(T_1)\}^+ \\ Flr_2(T_3) &= \delta_2\{K - L_2(T_2)\}^+ \\ &\vdots \\ Flr_{N-1}(T_N) &= \delta_{N-1}\{K - L_{N-1}(T_{N-1})\}^+, \end{aligned} \tag{13}$$

Analogous to caps, we can determine the arbitrage free price of a floorlet under the forward-neutral probability measure. If we let $h(Flr_i(T_{i+1})) = [\delta_i\{K - Flr_i(T_i)\}^+]$, denote the payoff function of the floorlet at time T_{i+1}, $p_{i+1}(t)$ denote the time t price of a default-free discount bond maturing at time T_{i+1} and $\mathbf{E}_t^{T_{i+1}}$ denote the conditional expectation under the forward-neutral probability measure $P^{T_{i+1}}$ given filtration $\mathcal{F}_t$. Then the time t price of the i^{th} floorlet is given by:

$$\begin{aligned} Flr_i(t) &= p_{i+1}\mathbf{E}_t^{T_{i+1}}[h(Flr_i(T_{i+1}))] \\ &= \delta_i p_{i+1}\mathbf{E}_t^{T_{i+1}}\left[\{K - L_i(T_i)\}^+\right], \quad 0 \le t \le T_i. \end{aligned} \tag{14}$$

From the equation above, we can subsequently derive the Black's formula for a floorlet. The next definition states the Black's formula for pricing an i^{th} floorlet.

Definition 2.5. Black's formula for Floorlets
Let $Flr_i^{Black}(t)$ denote the time t price of a floorlet. Then the Black's price of a floorlet written on the forward LIBOR rate $L_i(t)$ with maturity T_i and cap rate K is given by:

$$Flr_i^{Black}(t) = \delta_i p_{i+1}(t)\{KN(-d_2 - L_i(t)N(-d_1))\}, \tag{15}$$

where

$$d_1 = \frac{1}{\sigma_i\sqrt{T_i - t}}\left\{\ln\left(\frac{L_i(t)}{K}\right) + \frac{1}{2}\sigma_i^2(T_i - t)\right\}, \tag{16}$$

$$\begin{aligned} d_2 &= \frac{1}{\sigma_i\sqrt{T_i - t}}\left\{\ln\left(\frac{L_i(t)}{K}\right) - \frac{1}{2}\sigma_i^2(T_i - t)\right\} \\ &= d_1 - \sigma_i\sqrt{T_i - t}, \end{aligned} \tag{17}$$

σ_i denotes the volatility of the forward LIBOR rate $L_i(t)$ and $N(d_i)$ is the distribution function of the standard normal distribution.

Analogous to caps, if we denote the corresponding floor market price by $Floor_i^m(t)$, then the implied flat volatilities are defined as solutions to the equations:

$$Floor_i^m(t) = \sum_{k=1}^{i} Flr_k^{Black}(t, \bar{\sigma}_i),\ i = 1, \ldots, N, \tag{18}$$

and the implied forward volatilities are defined as solutions of equations:

$$Flr_i^m(t) = Flr_k^{Black}(t, \bar{\sigma}_i),\ i = 1, \ldots, N. \tag{19}$$

3. Wang Transform Approach and the Bid-Ask Formulas

In this section we introduce our approach which we used to come up with the formulas for bid-ask prices for cap and floor contracts discussed in the previous section. The approach is borrowed from the actuarial literature and is consistent with Bühlmann (1980). Bühlmann (1980)'s model was developed for the pricing and hedging of insurance risk. Insurance market is incomplete, in the sense that risks in the market cannot be completely replicated by other assets in the market, and so is the market for cap and floor contracts.

In the actuarial literature, many probability transforms have been developed for pricing financial and insurance risks. Wang (2000)and Wang (2002) proposed a pricing method based on the following transformation from $G(x)$ to $G^Q(x)$:

$$G^Q(x) = \Phi[\Phi^{-1}(G(x)) + \gamma], \quad \gamma \in [0, \infty) \tag{20}$$

where Φ denotes the standard normal cumulative distribution function. The transform is called the Wang transform and $G^Q(x)$ is considered to be a risk-adjusted cumulative distribution function under the risk-neutral probability measure Q. We shall employ the Wang transform approach in constructing the bid-ask formulas for the cap and floor contracts. The Wang transform not only possess the desirable properties as a pricing method, but also has sound economic interpretation. Wang (2003) observed that the transform in (20) is consistent with Bühlmann's economic pricing model, hence the Wang transform is appropriate for our approach.

More precisely in the seminal paper, Bühlmann (1980) defines the following economic premium principle:

$$\rho^\gamma[X, Y] = \frac{E[X \exp(\gamma Y)]}{E[\exp(\gamma Y)]}, \tag{21}$$

where X and Y are random variables on the same probability space that describe the stochastic cash flow and a market index, respectively. Bühlmann (1980) also shows that in the case X and $Y - X$ are independent, Y drops out of the equation and the expression reduces to the following premium known as the Esscher principle:

$$\varrho^\gamma[X, Y] = \frac{E[X \exp(\gamma X)]}{E[\exp(\gamma X)]}. \tag{22}$$

Wang (2003) establishes a formal connection between Bühlmann's economic premium principle and the distorted expectations approach. Under the assumption of co-monotonicity between X and Y, Wang shows that the Bühlmann's premium can be written as a distorted expectation using the following general distortion function of X wrt Y:

$$\Psi^{\gamma}(u) = \frac{\int_0^u \exp(\gamma F_Y^{-1}(v))dv}{E(\exp(\gamma Y))} \tag{23}$$

where F_Y is the cumulative distribution function of the random variable Y. The derivative of the above function yields the corresponding Radon-Nikodym derivative:

$$\theta^{\gamma}(u) = \frac{\exp(\gamma F_Y^{-1}(u))}{E(\exp(\gamma Y))}. \tag{24}$$

This function distorts the distribution function of X using the distortion function of Y. When Y has the normal distribution, the general distortion function reduces to the Wang transform in (20).

3.1. Theoretical Bid-Ask Prices

Before we move to the derivation of the bid-ask formulas in the Black's model framework using the Wang transform approach, first we present an overview on expressions for theoretical bid-ask prices. The expressions are key to steps in the derivation of the bid-ask formulas. The concepts of acceptability indices and distortion functions are used to derive the expressions. One is referred to Cherny and Madan (2010) for more on the concepts of acceptability and distortion functions.

The market is seen as a counterparty willing to accept all stochastic cash flows X that have an acceptability level γ. Bid price $b_{\gamma}(X)$ is the price that an investor gets for selling an asset to the market. The ask price $a_{\gamma}(X)$ is the price an investor pays to buy an asset from the market. The market maker due to competition, will set the minimal ask price such that the net cash flow resulting from a sell is acceptable at level γ, or $a_{\gamma}(X) - X \in \mathcal{A}_{\gamma}$. Similarly, the market maker will set the maximal bid price such that the net cash flow resulting from a buy is acceptable at level γ, or $X - b_{\gamma}(X) \in \mathcal{A}_{\gamma}$. Cherny and Madan (2010) show that:

$$\alpha(a - X) \geq \gamma \iff \int_{-\infty}^{\infty} x \, d\Psi^{\gamma}(F_{a-X}(x)) \geq 0 \tag{25}$$

$$\iff a + \int_{-\infty}^{\infty} x \, d\Psi^{\gamma}(F_{-X}(x)) \geq 0, \tag{26}$$

so that the minimum value of a leads to the ask price:

$$a_{\gamma}(X) = -\int_{-\infty}^{\infty} x \, d\Psi^{\gamma}(F_{-X}(x)). \tag{27}$$

Analogously, for the bid price Cherny and Madan (2010) show that:

$$\alpha(X - b) \geq \gamma \iff \int_{-\infty}^{\infty} x \, d\Psi^{\gamma}(F_{X-b}(x)) \geq 0 \tag{28}$$

$$\iff -b + \int_{-\infty}^{\infty} x \, d\Psi^{\gamma}(F_{-X}(x)) \geq 0, \tag{29}$$

so that the maximum of b leads to the bid price:

$$b_{\gamma}(X) = \int_{-\infty}^{\infty} x \, d\Psi^{\gamma}(F_{X}(x)). \tag{30}$$

So due to competition, the bid price is raised and the ask price is lowered so as for the position to remain acceptable. Consequently, the spread is narrowed and the risk of a position is minimized.

3.2. Explicit Bid-Ask Formulas for Caplets and Floorlets

The market standard for quoting prices on caplets/floorlets is in terms of Black's model. We assume that a LIBOR forward rate $F(t)$ follows a driftless lognormal process:

$$dF_t = \sigma F_t dW_t, \; 0 \leq t \leq T, \tag{31}$$

where W_t is a Weiner process and σ is the lognormal volatility under a risk measure Q. The solution to this stochastic differential equation between times t and T is:

$$F_T = F_t \exp\left\{\frac{1}{2}\sigma^2(T-t) + \sigma[W(T) - W(t)]\right\}. \tag{32}$$

The difference $\ln F$ between times t and T follows a normal distribution:

$$\ln F_T - \ln F_t \sim N\left(-\frac{1}{2}\sigma^2(T-t), \sigma^2(T-t)\right). \tag{33}$$

Then

$$\ln F_T \sim N\left(-\frac{1}{2}\sigma^2(T-t) + \ln F_t, \sigma^2(T-t)\right). \tag{34}$$

$\ln F_T$ follows a normal distribution and this means that the distribution of F_T is:

$$F_T \sim \mathit{lognormal}\left(-\frac{1}{2}\sigma^2(T-t) + \ln F_t, \sigma^2(T-t)\right). \tag{35}$$

The cumulative distribution of F_T can be written as:

$$\begin{aligned} F_{F_T}(X) &= \Phi\left[\frac{\ln X - \ln F_t + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right] \\ &= \Phi\left[\frac{\ln\left(\frac{X}{F_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right] \end{aligned} \tag{36}$$

The distorted cumulative distribution function from applying the Wang transform follows. If F has a $\mathit{lognormal}(\mu, \sigma^2)$ distribution, $\Psi^\gamma(F_X)$ is another lognormal distribution with $\mu^* = \mu + \gamma\sigma$ and $\sigma^* = \sigma$. Thus:

$$\Psi^\gamma(F_{F_T}(X)) = \Phi\left[\frac{\ln\left(\frac{X}{F_t}\right) + \frac{1}{2}\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right] \tag{37}$$

Next, we present the bid-ask formulas for the caplets and floorlets based on the general expressions for bid-ask prices found in Cherny and Madan (2010). In the following expressions, N is the notional principal amount, $P(0,T_i)$ is the price of a zero-coupon bond maturing at time T_i, δ_i is the tenor and γ is the stress level (risk sentiment in the market). The derivations for the prices are found in Appendix A.

$$CapletBid = N\delta_i P(0,T_i)[F_t e^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1) - K\Phi(d_2)], \quad (38)$$

$$\text{where } d_1 = \frac{\ln(F_t/K) + 1/2\sigma^2(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \; and \; d_2 = d_1 - \sigma\sqrt{T-t}.$$

$$CapletAsk = N\delta_i P(0,T_i)[F_t e^{\gamma\sigma\sqrt{T-t}}\Phi(d_1) - K\Phi(d_2)], \quad (39)$$

$$where \; d_1 = \frac{\ln(F_t/K) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \; and \; d_2 = d_1 - \sigma\sqrt{T-t}.$$

$$FloorletBid = N\delta_i P(0,T_i)[K\Phi(d_2) - F_t e^{\gamma\sigma\sqrt{T-t}}\Phi(d_1)], \quad (40)$$

$$where \; d_1 = \frac{\ln(K/F_t) - 1/2\sigma^2(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \; and \; d_2 = d_1 + \sigma\sqrt{T-t}.$$

$$FloorletAsk = N\delta_i P(0,T_i)[K\Phi(d_2) - F_t e^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1)], \quad (41)$$

$$where \; d_1 = \frac{\ln(K/F_t) - 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \; and \; d_2 = d_1 + \sigma\sqrt{T-t}.$$

4. Numerical Tests

In this section we apply the concepts discussed in the previous sections to determine the premiums for cap and floor contracts. We want to obtain an idea on the approximate numerical magnitudes using the suggested approach. The cap and floor contracts are structured on the 3 month LIBOR reference rate at predetermined strike levels. Also, the cap and floor contracts are defined on pre-specified notional principal amount of US$ 1 million each with quarterly reset dates. All calculations are made on *Actual*/360 day-count convention.

4.1. Market Data

We present the actual market data inputs to be used in the numerical implementation. All the market data are taken from 4 September 2014. The data obtained from Bloomberg consists of benchmark rates and also ATM cap volatilities.

Benchmark Rates

The benchmark rates are the benchmark instruments used as inputs in the construction of the interest rate yield curve. We take into consideration the following rates for the construction of the yield curve: 3M LIBOR rates, 3 monthly FRAs and IRS exchanging fixed cash flows for floating cash flows linked to the 3M LIBOR. The data for 4 September are presented in Table 1. The London Interbank Offer Rate (LIBOR) is a daily reference rate based on interest rates at which banks borrow unsecured funds from other banks. The FRAs are considered as forward rates for specific 3 month time periods. The IRS are fixed rates that will ensure swaps price back to zero.

Table 1: Benchmark Instruments as of 4 September 2014

Benchmark Instrument	**Rate (%)**
LIBOR 3M	0.23
FRA 3x6	0.23
FRA 6x9	0.33
FRA 9x12	0.52
FRA 12x15	0.76
FRA 15x18	0.99
FRA 18x21	1.23
FRA 21x24	1.50
IRS 3Y	1.17
IRS 5Y	1.81
IRS 10Y	2.53
IRS 15Y	2.87
IRS 20Y	3.03
IRS 25Y	3.10
IRS 30Y	3.14
IRS 40Y	3.15

Cap Volatilities

The market quoted volatilities for ATM caps as of 4 September 2014 are given in Table 2. Market quoted cap volatilities are typically termed flat volatilities. The flat volatility has the property that when applied to each caplet yields the option premium. Note that the cap volatility is the same as the floor volatility when at-the-money (ATM).

Table 2: ATM cap volatilities for 4 September 2014

Tenor	**Cap Vol (%)**
1Y	66.68
2Y	64.42
3Y	56.63
4Y	50.42
5Y	46.06
6Y	42.64
7Y	40.19
8Y	38.28
9Y	36.58
10Y	35.05
12Y	32.50
15Y	29.79
20Y	26.53
25Y	26.41
30Y	26.45

4.2. Curve Bootstrapping

Now, given that our interest rate cap and floor contracts are indexed on the 3M LIBOR rate, our implementation requires forward rates for intermediate dates which lie in between the standard market quoted maturity dates. The data in the market are not quoted with 3 month intervals and therefore in order to obtain the rates on the missing maturity dates, we use an interpolation scheme to bootstrap the input benchmark rates. Figure 1 shows the bootstrapped forward rates on 4 September over a period of 10 years. The bootstrapped curve is upsloping such that the consensus is that the LIBOR will rise and so the cap contracts are expected to be more expensive than the floor contracts

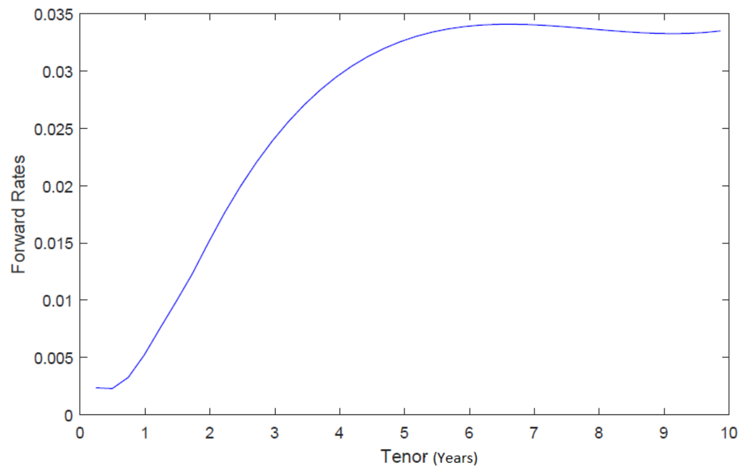

Figure 1: Bootstrapped forward rates on 4 September 2014 using Cubic Spline Interpolation

4.3. Cap and Floor Premiums

Having extracted the yield curve from the data and found the implied volatilities for the cap and floor contracts, next we find the premiums for the cap and floor contracts. Using our suggested approach, we provide the bid-and-ask premiums for the various cap and floor contracts at different strikes, maturities and risk levels (γ). Table 3 and Table 4 provide summary information for the premiums on various cap and floor contracts on 3M LIBOR as of the 4th of September 2014. The premiums are quoted in basis points based on the notional principle only. For the cap contracts, the strike rates are 2.0%, 2.5% and 3%. For the floor contracts, the strike rates are 0.5%, 1.0% and 1.5%. The bid premium presents what the contract seller receives while the ask premium represents what the contract buyer pays.

The premiums are provided for risk levels (γ) of 0, 0.01, 0.05 and 0.1 . For $\gamma = 0$, the bid and ask premiums are equivalent and they reduce to the regular Black's formula prices. The premiums at $\gamma = 0$ were benchmarks which suggested that the derivations presented in the previous section are in fact accurate. At all the other risk levels (γ), it can be observed that the bid premiums are always less than the ask premiums. This is in line with what is observed in the financial markets.

Second, the premiums increase with maturity. This reflects that a contract seller must be compensated more for committing to a cap or floor

Table 3: Premiums for Interest Rate Caps on 3M LIBOR (4 September 2014)

Risk Level (γ)	**Term Cap**	**Bid**	**Ask**	**Bid**	**Ask**	**Bid**	**Ask**
		2.0%		**2.5%**		**3.0%**	
0	1year	0.055	0.055	0.019	0.019	0.007	0.007
	2years	15	15	10	10	7	7
	3years	78	78	58	58	44	44
	5years	316	316	254	254	206	206
	7years	600	600	492	492	408	408
	10years	1007	1007	838	838	703	703
0.01	1year	0.053	0.057	0.018	0.020	0.007	0.008
	2years	14	15	9	10	7	7
	3years	77	80	57	60	44	45
	5years	312	321	250	258	202	210
	7years	591	608	484	500	401	415
	10years	993	1022	825	851	691	715
0.05	1year	0.047	0.064	0.016	0.023	0.006	0.009
	2years	13	16	9	11	6	8
	3years	72	85	53	64	40	49
	5years	293	341	233	276	188	225
	7years	556	645	454	533	374	444
	10years	936	1083	774	907	646	765
0.1	1year	0.040	0.075	0.013	0.027	0.005	0.011
	2years	12	18	8	12	5	8
	3years	66	93	48	70	36	54
	5years	271	368	214	299	172	245
	7years	516	694	418	577	343	483
	10years	868	1163	713	980	592	831

Table 4: Premiums for Interest Rate Floors on 3M LIBOR (4 September 2014)

Risk Level (γ)	**Term Floor**	**Bid 0.5%**	**Ask 0.5%**	**Bid 1.0%**	**Ask 1.0%**	**Bid 1.5%**	**Ask 1.5%**
0	1year	21	21	68	68	117	117
	2years	25	25	94	94	178	178
	3years	26	26	104	104	207	207
	5years	27	27	115	115	244	244
	7years	28	28	124	124	274	274
	10years	29	29	137	137	316	316
0.01	1year	21	21	68	68	117	117
	2years	25	25	93	94	177	178
	3years	26	26	103	104	206	208
	5years	27	27	114	116	242	245
	7years	27	28	123	125	272	275
	10years	29	29	136	138	314	319
0.05	1year	21	21	67	68	116	118
	2years	24	26	92	95	175	180
	3years	25	27	101	106	203	212
	5years	26	28	112	119	237	251
	7years	27	29	120	128	265	283
	10years	28	30	132	143	304	328
0.1	1year	20	22	67	69	116	118
	2years	24	26	90	97	172	183
	3years	24	27	99	109	199	216
	5years	25	29	108	122	231	258
	7years	26	30	116	133	256	292
	10years	27	31	127	149	293	341

contract for a longer period of time. However, the bid premium is raised and the ask premium is lowered, as suggested in the literature, for the premiums to remain acceptable to the market.

5. Conclusion

One of the major challenges in the area of OTC derivatives including caps and floors is to establish sound basis for pricing decisions based on quantitative models rather than leaving it to intuition and subjectivity. The market dealers usually quote the mid-prices (the average of the bid-ask prices) to the market. The mid-price is subjective and hypothetical price which does not include funding, hedging or any other costs or adjustments, and may not necessarily reflect the price at which one can enter into a deal.

Furthermore, the OTC interest rate options market are incomplete, and options cannot always be exactly and costlessly replicated. As a result, the option pricing models may not satisfy the martingale restriction. In such a scenario, the no-arbitrage framework can only place bounds on option prices. However, the range of prices is likely to be wider for OTC derivatives since the seller has to incur the high costs to cover the unhedgeable risk.

In light of the above arguments, we put across an approach to estimate the bid and ask prices for options on LIBOR based instruments, which factors into consideration the risk sentiments prevailing in the market. In particular, we assess the approach in the determination of premiums for cap and floor contracts.

From the numerical work, we conclude that indeed as the risk level increases, market participants cannot be completely hedged. There is always some unhedgeable risk. However, we believe with the approach suggested in this work, the spread is minimized. We also conclude that the premiums increase as the maturity increases. Although, the premiums do not decrease as the maturity increases, we believe the premiums obtained from this approach can make market participants reach consensus and also reduces transactions costs than what literature suggests.

This approach can be seen as a first line of thought in determining bid and ask premiums, which can be possibly further conditioned by further plausible considerations.

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Appendix A. Wang Transform Theoretical Bid-Ask Prices

Appendix A.1. Caplet Bid Formula

$$CapletBid = N\delta_i P(0,T_i) b_\gamma(C), \tag{A.1}$$

Now,

$$\begin{aligned} b_\gamma(C) &= \int_0^\infty x d\Psi^\gamma(F_C(x)) \\ &= \int_K^\infty (x-K) d\Psi^\gamma(F_F(x)) \\ &= \int_K^\infty (x-K) d\Phi\left(\frac{\ln(x/F_t) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\ &= \underbrace{\int_K^\infty x d\Phi\left(\frac{\ln(x/F_t) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T}}\right)}_{(i)} \\ &\quad \underbrace{-\int_K^\infty K d\Phi\left(\frac{\ln(x/F_t) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(ii)} \end{aligned} \tag{A.2}$$

Now, if $X \sim Lognormal(\mu, \sigma^2)$ with CDF $F_X(x)$, then:

$$\int_K^\infty x dF(x) = e^{\mu + 1/2\sigma^2}\Phi\left(\frac{\mu + \sigma^2 - \ln K}{\sigma}\right) \tag{A.3}$$

Integral (i) in equation (A.2) becomes:

$$\begin{aligned} & \int_K^\infty x d\Phi\left(\frac{\ln(x/S_0) - (r - 1/2\sigma^2)T + \gamma\rho\sigma\sqrt{T}}{\sigma\sqrt{T}}\right) \\ = \; & e^{\ln F_t - 1/2\sigma^2(T-t) - \gamma\sigma\sqrt{T-t} + 1/2\sigma^2(T-t)} \\ & \Phi\left(\frac{\ln F_t - 1/2\sigma^2(T-t) - \gamma\sigma\sqrt{T-t} + \sigma^2(T-t) - \ln K}{\sigma\sqrt{T-t}}\right) \\ = \; & F_t e^{-\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(F_t/K) + 1/2\sigma^2(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{A.4}$$

Integral (ii) in equation (A.2) is the partial integral of the lognormal density and is given by:

$$\begin{aligned}
&\int_K^\infty K d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\\
=\ & K\left(1-\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\right). \qquad \text{(A.5)}
\end{aligned}$$

Substituting the results of the two integrals in (A.2) we get:

$$\begin{aligned}
b_\gamma(C) &= F_t e^{-\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(F_t/K)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\\
&\quad -K\left(1-\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\right) \qquad \text{(A.6)}
\end{aligned}$$

This can be written in a more convenient way as:

$$b_\gamma(C) = F_t e^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1) - K\Phi(d_2), \qquad \text{(A.7)}$$

$$\textit{where } d_1 = \frac{\ln(F_t/K)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \textit{ and } d_2 = d_1 - \sigma\sqrt{T-1}.$$

From (A.1), the caplet bid formula is:

$$CapletBid = N\delta_i P(0,T_i)[F_t e^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1) - K\Phi(d_2)]. \qquad \text{(A.8)}$$

Appendix A.2. Caplet Ask Formula

$$CapletAsk = N\delta_i P(0,T_i) a_\gamma(C), \qquad \text{(A.9)}$$

Now,

$$a_\gamma(C) = -\int_0^\infty x d\Psi^\gamma(F_{-C_T}(x))$$

$$
\begin{aligned}
&= -\int_K^\infty (x-K)d\Psi^\gamma(1-F_{F_T}(x)) \\
&= -\int_K^\infty (x-K)d\Psi^\gamma\left(1-\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= -\int_K^\infty (x-K)d\Psi^\gamma\Phi\left(\frac{\ln(F_t/x)-1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= -\int_K^\infty (x-K)d\Phi\left(\frac{\ln(F_t/x)-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= \int_K^\infty (x-K)d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T}}{\sigma\sqrt{T-t}}\right) \\
&= \underbrace{\int_K^\infty xd\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(i)} \\
&= -\underbrace{\int_K^\infty Kd\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(ii)} \qquad \text{(A.10)}
\end{aligned}
$$

Integral (i) in (A.10) becomes:

$$
\begin{aligned}
&\int_K^\infty xd\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= e^{\ln F_t-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}+1/2\sigma^2(T-t)} \\
&\quad \Phi\left(\frac{\ln F_t-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}+\sigma^2(T-t)-\ln K}{\sigma\sqrt{T-t}}\right) \\
&= F_t e^{\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(F_t/K)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \qquad \text{(A.11)}
\end{aligned}
$$

Integral (ii) in (A.10) is the partial integral of the lognormal density and is given by:

$$
\begin{aligned}
&\int_K^\infty Kd\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= K\left(1-\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\right). \qquad \text{(A.12)}
\end{aligned}
$$

Substituting the results of the two integrals in (A.10), we get:

$$\begin{aligned} a_\gamma(C) &= F_t e^{\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(F_t/K)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\ & -K\left(1-\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\right). \end{aligned} \tag{A.13}$$

This can be written in a more convenient way as:

$$a_\gamma(C) = F_t e^{\gamma\sigma\sqrt{T-t}}\Phi(d_1) - K\Phi(d_2), \tag{A.14}$$

$$where\ d_1 = \frac{\ln(F_t/K)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\ and\ d_2 = d_1 - \sigma\sqrt{T-1}.$$

From (A.9), the caplet ask formula is:

$$CapletAsk = N\delta_i P(0,T_i)[F_t e^{\gamma\sigma\sqrt{T-t}}\Phi(d_1) - K\Phi(d_2)]. \tag{A.15}$$

Appendix A.3. Floorlet Bid Formula

$$FloorletBid = N\delta_i P(0,T_i) b_\gamma(P), \tag{A.16}$$

$$\begin{aligned} b_\gamma(P) &= \int_0^\infty x d\Psi^\gamma(F_{P_T}(x)) \\ &= -\int_0^K (K-x) d\Psi^\gamma(1-F_{F_T}(x)) \\ &= -\int_0^K (K-x) d\Psi^\gamma\left(1-\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\ &= -\int_0^K (K-x) d\Psi^\gamma\left(\Phi\left(\frac{\ln(F_t/x)-1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\ &= -\int_0^K (K-x) d\Phi\left(\frac{\ln(F_t/x)-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^K (K-x)\,d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= \underbrace{\int_0^K K\,d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(i)} \\
&\quad \underbrace{-\int_0^K x\,d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(ii)} \qquad \text{(A.17)}
\end{aligned}
$$

Integral (i) in equation (A.17) gives:

$$
\begin{aligned}
&\int_0^K K\,d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= K\left[\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\right]_0^K \\
&= K\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right). \qquad \text{(A.18)}
\end{aligned}
$$

Integral (ii) in equation (A.17) gives:

$$
\begin{aligned}
&\int_0^K x\,d\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= e^{\ln F_t-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T}+1/2\sigma^2(T-t)} \\
&\quad \left(1-\Phi\left(\frac{\ln F_t-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}+\sigma^2(T-t)-\ln K}{\sigma\sqrt{T-t}}\right)\right) \\
&= F_t e^{\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(K/F_t)-1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T}}\right) \qquad \text{(A.19)}
\end{aligned}
$$

Substituting the results of the two integrals in (A.17) gives:

$$
\begin{aligned}
b_\gamma(P) &= K\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\
&\quad -F_t e^{\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(K/F_t)-1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T}}\right) \qquad \text{(A.20)}
\end{aligned}
$$

This can be written more conveniently as:

$$\begin{aligned} b_\gamma(P) &= K\Phi(d_2) - F_t e^{\gamma\sigma\sqrt{T-t}}\Phi(d_1), \quad \text{(A.21)} \\ \textit{where } d_1 &= \frac{\ln(K/F_t) - 1/2\sigma^2(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} \textit{ and } d_2 = d_1 + \sigma\sqrt{T-t}. \end{aligned}$$

From (A.16), the floorlet bid formula is:

$$FloorletBid = N\delta_i P(0,T_i)[K\Phi(d_2) - F_t e^{\gamma\sigma\sqrt{T-t}}\Phi(d_1)]. \quad \text{(A.22)}$$

Appendix A.4. Floorlet Ask Formula

$$FloorletAsk = N\delta_i P(0,T_i) a_\gamma(P), \quad \text{(A.23)}$$

$$\begin{aligned} a_\gamma(P) &= -\int_{-\infty}^{0} x d\Psi^\gamma(F_{-P_T}(x)) \\ &= \int_0^K (K-x) d\Psi^\gamma(F_{F_T}(x)) \\ &= \int_0^K (K-x) d\Psi^\gamma\left(\Phi\left(\frac{\ln(x/F_t) + 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\ &= \int_0^K (K-x) d\Phi\left(\frac{\ln(x/S_0) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right) \\ &= \underbrace{\int_0^K K d\Phi\left(\frac{\ln(x/F_t) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(i)} \\ &\quad - \underbrace{\int_0^K x d\Phi\left(\frac{\ln(x/F_t) + 1/2\sigma^2(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)}_{(ii)} \quad \text{(A.24)} \end{aligned}$$

Integral (i) in equation (A.24) gives:

$$\begin{aligned}
&\int_0^K Kd\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\\
&= K\left[\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\right]_0^K\\
&= K\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right). \qquad \text{(A.25)}
\end{aligned}$$

Integral (ii) in equation (A.24) gives:

$$\begin{aligned}
&\int_0^K xd\Phi\left(\frac{\ln(x/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\\
&= e^{\ln F_t-1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}+1/2\sigma^2(T-t)}\\
&\quad\left(1-\Phi\left(\frac{\ln F_t+1/2\sigma^2(T-t)-\gamma\sigma\sqrt{T-t}+\sigma^2(T-t)-\ln K}{\sigma\sqrt{T-t}}\right)\right)\\
&= F_te^{-\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(K/F_t)-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right). \qquad \text{(A.26)}
\end{aligned}$$

Substituting the results of the two integrals in A.24 gives:

$$\begin{aligned}
a_\gamma(P) &= K\Phi\left(\frac{\ln(K/F_t)+1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right)\\
&\quad -F_te^{-\gamma\sigma\sqrt{T-t}}\Phi\left(\frac{\ln(K/F_t)-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\right). \qquad \text{(A.27)}
\end{aligned}$$

This can be written more conveniently as:

$$a_\gamma(P) = K\Phi(d_2) - F_te^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1), \qquad \text{(A.28)}$$

$$where\ d_1 = \frac{\ln(K/F_t)-1/2\sigma^2(T-t)+\gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}\ and\ d_2 = d_1+\sigma\sqrt{T-t}.$$

From (A.23), the floorlet ask formula is:

$$FloorletAsk = N\delta_iP(0,T_i)[K\Phi(d_2) - F_te^{-\gamma\sigma\sqrt{T-t}}\Phi(d_1)]. \qquad \text{(A.29)}$$

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Paper 3

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Prediction of Stock Price Movement Using Continuous Time Models

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Abstract

Predicting stock price movement is generally accepted to be challenging such that until today it is continuously being attempted. This paper attempts to address the problem of stock price movement using continuous time models. Specifically, the paper provides comparative analysis of continuous time models—General Brownian Motion (GBM) and Variance Gamma (VG) in predicting the direction and accurate stock price levels using Monte Carlo methods—Quasi Monte Carlo (QMC) and Least Squares Monte Carlo (LSMC). The hit ratio and mean-absolute percentage error (MAPE) were used to evaluate the models. The empirical tests suggest that either the GBM model or VG model in any Monte Carlo method can be used to predict the direction of stock price movement. In terms of predicting the stock price values, the empirical findings suggest that the GBM model performs well in the QMC method and the VG model performs well in the LSMC method.

Keywords

Monte Carlo, Stock Price, Geometric Brownian Motion, Variance Gamma, Maximum Likelihood Estimation, MAPE, Hit Ratio

1. Introduction

In this paper we deal with the problem of prediction of stock price movement (increase or decrease) that has been there over years. Several methods have been proposed and have predicted stock price movement with va-

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riable degrees of accuracy such that until today, prediction of stock price movement is continuously being attempted. A manifold of factors such as economical, political, social and psychological factors interact in a complex way influencing stock price movement. It is no doubt that prediction of stock price movement is quite challenging. This paper is an attempt to predict stock price movement using continuous time models. We believe that continuous models are suitable to capture the unpredictable dynamics of stock prices to a certain extent.

The models for prediction of stock price movement have several uses to researchers and practitioners alike which include optimal portfolio construction and executing best informed buy/sell orders. Also, a major boost for the models of stock price movement is in derivatives models to determine fair values of derivatives and simulation models for risk management purposes.

Most studies have focused on the accurate estimation of the value of stock price. In most cases, the accuracy of the estimations is measured by the error between the estimates and observed values. However, different investors use diverse trading strategies and strategies based on minimizing the error between observed values and the estimates may not appeal to them [1]. Some recent studies have illustrated that forecasts from trading strategies based on the direction of stock price change may be more effective and can generate higher returns. Thus, trading strategies based on predictability of direction of stock need attention in the effective development of market trading strategies.

In the literature, there exist a vast number of articles addressing the accurate estimation of the value of stock price. However, to the best of our knowledge, there are few studies which look at the predictability of the direction of stock price. In this respect, we cite studies by [2], who explores the relationship between the direction of interday and intraday price changes on the S&P 500 futures. [3] investigated on the predictability of the direction of change in the future spot exchange. [4] concluded that the performance of cross-hedging improves if the direction of changes in exchange rates can be predicted. [1] provided a comparative evaluation of the forecasting performance of a group of classification models to that of a group of level estimation models. The classification models were used to forecast the direction of index returns and the level estimation models were used to estimate the value of the return. Recently, [5] attempted to predict the direction of stock price movement with focus on emerging markets using data mining techniques.

We follow the approach of [1] in this work. Instead of making a comparison between classification models, which predict direction based on probability, and level estimation models, which forecast the accurate price level, we resort to using continuous time models in Monte Carlo framework to achieve both objectives of predicting the direction of stock price and accurate stock price level. The major contributions of this work are comparative analysis of continuous time models to predict the direction and accurate price levels of stocks in the Monte Carlo framework.

[6] and [7] used the Geometric Brownian Motion (GBM) model to forecast share prices for short-term investments. Though the GBM model is good for forecasting share price movements, there is ongoing active research to improve upon it substantially. In the past decades, there have been several theoretical and empirical studies that have tried to address this issue. Amongst these studies, the most important are studies by [8] [9] and [10]. Of the proposed models from the studies, we consider the model proposed by [10] in this work. [10] proposed a stochastic process for stock price movement called the Variance Gamma (VG) model. The empirical findings of the authors claim that the VG model is a good contender for forecasting share price movements. Hence, the interest to this work is the comparative analysis of continuous time models—GBM model and VG model in stock price movement.

The assumptions on which the continuous models are based meet the rules imposed by the weak Efficient Market Hypothesis. The weak Efficient Market Hypothesis guarantees transparency in the sense that it gives everyone the same information about a stock. According to the hypothesis, the only relevant information about a stock is the current value, so as to be able to determine future stock price movement.

The rest of the paper is organized as follows: the Monte Carlo techniques used for simulating stock price processes are discussed in the next section. In Section 3 we discuss the dynamics and parameter estimation of the models used in this work. Then we discuss the design of the experiments in Section 4. In this section we also provide results of the performance of the models. Section 5 formulates our conclusions and carries summary of our findings.

2. Monte Carlo Methods

We look at the techniques for simulating the stock price processes encountered in this work. The Monte Carlo methods lend themselves naturally to this task as they are useful in estimating numerically the values of integral expressions especially in high dimensions. The simulation procedures used in here are found in [11].

2.1. Crude Monte Carlo Method

The integral of a Lebesgueintegrable function $f(x)$ can be expressed as the expectation of a function *f* evaluated at a random point. Consider an integral on the unit cube $I^d = [0,1]^d$ in *d* dimension. Let *x* be the uniformly distributed random variable on the interval. Then:

$$I = E\left[f(x)\right] = \int_{I^d} f(x)\,\mathrm{d}x. \tag{1}$$

The Monte Carlo quadrature formula is based on the probabilistic interpretation of the integral. Now, consider a sequence $\{x_N\}$ which is sampled from the uniform distribution. An empirical approximation to the expectation then is:

$$\hat{I}_N = \frac{1}{N}\sum_{k=1}^{N} f(x_k). \tag{2}$$

According to the Strong Law of Large Numbers, the approximation converges to the true value of the integral:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(x_k) = I. \tag{3}$$

This means that $\hat{I} \to I$ with probability 1 as $N \to \infty$. $\hat{I}$ is therefore an unbiased estimator for *I* (see [12]).

Now in crude form, Monte Carlo simulations are computationally inefficient. A large number of simulations are generally required so as to achieve high degree of accuracy. However, the efficiency can be improved by either using other methods such as variance reduction method, quasi-Monte Carlo method or least-squaresregression Monte Carlo method [13]. Of these methods, quasi-Monte Carlo and least-squares regression Monte Carlo methods are of interest to our work.

2.2. Quasi-Monte Carlo Method

Quasi-Monte Carlo (QMC) method, also called low-discrepancy, can be described in simple terms as the deterministic method of the crude Monte Carlo method. The random samples in the Monte Carlo method are instead replaced by well-chosen deterministic points. Quasi-Monte Carlo thus makes no attempt to mimic randomness. It rather generates sample points that are literally too evenly distributed to be random and thus selectively tries to increase accuracy [11].

Suppose for the unit cube integration domain $I^d = [0,1]^d$, we have the quasi-Monte Carlo approximation:

$$\int_{[0,1]^d} f(x)\,\mathrm{d}x = \frac{1}{N}\sum_{k=1}^{N} f(x_k), \tag{4}$$

which formally looks like the crude Monte Carlo estimate but is now used with the deterministic points $x_1, x_2, \cdots, x_N \in I^d$. These points are chosen judiciously so as to guarantee a small error.

It is intuitively clear that:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(x_k) = \int_{[0,1]^d} f(x)\,\mathrm{d}x, \tag{5}$$

and that the points x_k are chosen so as to fill the hyper-cube uniformly, and achieve a maximal degree of uniformity and a low degree of discrepancy. The discrepancy is a measure of the "level of uniformity" or more exactly the deviation from uniformity. It is defined as:

$$D_N^* = \max_{A\in\mathcal{A}} \left| \frac{\#\{x_k \in \mathcal{A}\}}{n} - vol(A) \right|, \tag{6}$$

where $\mathcal{A}$ is a collection of subsets of $[0,1)^d$, $\#\{x_k \in \mathcal{A}\}$ is the number of points x_k contained in $\mathcal{A}$ and

$vol(A)$ is volume (measure) of $\mathcal{A}$. If we choose $\mathcal{A}$ to be the collection of all rectangles in $[0,1)^d$ of the form:

$$\prod_{j=1}^{d}\left[0,u_j\right), 0\le u_j \le 1 \tag{7}$$

we define the star discrepancy $D_N^*\left(x^1,x^2,\cdots,x^N\right)$. The lower the star discrepancy, the more uniformly distributed the points are. There exist different kinds of pseudo-random sequences.

In this work, we use Halton sequences. Halton sequences are generally *d*-dimensional sequences with values in the unit hypercube $I^d=[0,1)^s$. The first dimension of the Halton sequence is the van der Corput sequence base 2, the second dimension is the van der Corput sequence using base 3, the third dimension is the van der Corput using base 5, and so on. Dimension *d* of the Halton sequence is the van der Corput sequence using the d^{th} prime numbers as the base.

2.3. Least Squares Regression Monte Carlo Method

Consider a reward function that depends on both X_t and time *t* such that:

$$g(t,x)=\mathrm{e}^{-rt}X_t,\ X_0=x>0, \tag{8}$$

where *r* is the discount factor. Suppose $X=\left(X_t\right)_{t\ge 0}$ is a stochastic process defined on a probability space $(\Omega,\mathcal{F},P)$. For any point in time *n* and a given stopping time τ with $n\le\tau\le T$, we define the value process $J(\tau)$ by:

$$J_n(\tau)=E\left[g(\tau,X_\tau)\middle|\mathcal{F}_n\right]=E\left[Z_\tau\middle|\mathcal{F}_n\right], \tag{9}$$

where we let $Z_\tau=g(\tau,X_\tau)$ and the optimal value process *V* is defined by:

$$V_n=\sup_{n\le\tau\le T}E\left[Z_\tau\middle|\mathcal{F}_n\right]=E\left[Z_{\tau^*}\middle|\mathcal{F}_n\right], \tag{10}$$

where τ^* signifies the optimal stopping time.

This is a typical optimal stopping problem whereby an investor has to decide the right time or optimal time to sell a stock in order to maximize the expected reward. In particular, we are interested in the optimal expected reward at an optimal stopping time. The findings from a study by [14] and further supported in [15] suggest that over a finite horizon $[0,T]$, a selling strategy is optimal at the terminal time *T* or at the moment when the stock price hits the maximum price. In our context, we are interested in the optimal strategy at the terminal time *T*.

To tackle this stochastic control problem, we use Dynamic Programming. The main idea originated from [16]'s principle which states that: An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Thus, the optimal value process *V* is the solution of the following backward recursion:

$$V_n=\max\left\{Z_n,E\left[V_{n+1}\middle|\mathcal{F}_n\right]\right\}. \tag{11}$$

$$V_T=Z_T. \tag{12}$$

It is optimal to stop at time *n* if and only if $V_n=Z_n$. If stopping at *n* is not optimal, then $V_n>Z_n$ and $V_n=E\left[V_{n+1}\middle|\mathcal{F}_n\right]$. The region where $V_n>Z_n$, that is, the region where it is optimal to terminate the process and receive the reward is called the "exercise region" and the compliment to this region is called the "continuation region".

To continue with calculations, we need to estimate the conditional expectation term, that is, the continuation value $E\left[V_{n+1}\middle|\mathcal{F}_n\right]$ at each time step. The most popular approach to estimate the continuation values is the Least Squares regression-based Monte Carlo (LSMC) method suggested by [17]. The main idea of the Least Squares Regression is to approximate the conditional expectation with linear combination of a set of *R* basis functions $\Phi_0,\Phi_1,\cdots,\Phi_R$, at each time step. Denote $C_n=E\left[V_{n+1}\middle|\mathcal{F}_n\right]=E\left[V_{n+1}\middle|X_n\right]$ in this setup. Now, for each time step, we assume the following multi-linear model:

$$E\left[V_{n+1}\middle|X_n\right]\approx\sum_{r=0}^{R}\beta_{n,r}\Phi_r\left(X_n\right). \tag{13}$$

We generate N independent sample paths of the process X, denoted by X^k, for $k=1,\cdots,N$. The squared residual along path k is:

$$r_k^2=\left[E\left[V_{n+1}\left|X_n^k\right.\right]-\sum_{r=0}^{R}\beta_{n,r}\Phi_r\left(X_n^k\right)\right]^2. \tag{14}$$

Ordinary Least Squares (OLS) regression is performed to find the parameter vector $\beta_n=\left(\beta_{n,0},\cdots,\beta_{n,R}\right)'$ that minimizes the sum of all squared residuals:

$$\min_{\beta_n}\sum_{j=1}^{N}r_{k_j}^2=\min_{\beta_n}\sum_{j=1}^{N}\left[E\left[V_{n+1}\left|X_n^{k_j}\right.\right]-\sum_{r=0}^{R}\beta_{n,r}\Phi_r\left(X_n^{k_j}\right)\right]^2. \tag{15}$$

Once we have determined the vector $\hat{\beta}_n$, the continuation value at time t_n along path k_j is estimated by the fitted value of regression:

$$\hat{C}_n^{k_j}=\sum_{r=0}^{R}\hat{\beta}_{n,r}\Phi_r\left(X_n^{k_j}\right). \tag{16}$$

For the basis regression functions, we choose the weighted Laguerre polynomials suggested by [11], which are defined as:

$$\Phi_j\left(x\right)=\mathrm{e}^{-x/2}L_k\left(x\right),\ k=j-1 \tag{17}$$

where $L_k\left(x\right)$ is the Laguerre polynomial defined as:

$$L_k\left(x\right)=\frac{\mathrm{e}^x}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\mathrm{e}^{-x}x^k\right)=\begin{cases}1, & \text{for } k=0\\ 1-x, & \text{for } k=1\\ \frac{1}{k}\left(\left(2k-1-x\right)L_{k-1}\left(x\right)\right)-\left(k-1\right)L_{k-2}\left(x\right), & \text{for } k\geq 1.\end{cases} \tag{18}$$

3. Simulation of Stock Price Processes

In this section, we discuss the dynamics of the models used in this work. The price of a particular stock at a future time t is usually unknown at the present time. Thus, we think of the stock price as a random variable. The stock price process is denoted by $S=\left\{S_t,t\geq 0\right\}$, where S_t is the price at time $t\geq 0$. The following is a brief overview of the dynamics used in simulating the stock price process.

3.1. Dynamics of Geometric Brownian Motion Process

The Geometric Brownian Motion (GBM) stock price dynamics under the risk neutral measure are given by:

$$\mathrm{d}S_t=rS_t\mathrm{d}t+\sigma S_t\mathrm{d}W_t,\ \ 0\leq t\leq T \tag{19}$$

where r is the risk-free rate and σ is the volatility parameter. Applying the Itóformula to the log-returns $\ln\left(S_t\right)$ yields:

$$\begin{aligned}\mathrm{d}\left(\ln\left(S_t\right)\right)&=\frac{1}{S_t}\mathrm{d}S_t+\frac{1}{2}\left(-\frac{1}{S_t^2}\right)\mathrm{d}S_t\mathrm{d}S_t\\ &=\frac{1}{S_t}\left(rS_t\mathrm{d}t+\sigma S_t\mathrm{d}W_t\right)+\frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2S_t^2\mathrm{d}t\\ &=\left(r-\frac{1}{2}\sigma^2\right)\mathrm{d}t+\sigma\mathrm{d}W_t.\end{aligned} \tag{20}$$

Integrating the above stochastic differential equation over the time interval $\left[t_i,t_{i+1}\right]$, where $0\leq t_i<t_{i+1}\leq T$, gives:

$$\ln\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right) = \ln S_{t_{i+1}} - \ln S_{t_i} = \int_{t_i}^{t_{i+1}}\left(r-\frac{1}{2}\sigma^2\right)\mathrm{d}t + \int_{t_i}^{t_{i+1}}\sigma \mathrm{d}W_t$$
$$= \left(r-\frac{1}{2}\sigma^2\right)\left(t_{i+1}-t_i\right) + \sigma\left(W_{t_{i+1}} - W_{t_i}\right). \tag{21}$$

This gives us the recursive expression for $S_{t_{i+1}}$:

$$S_{t_{i+1}} = S_{t_i}\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)\left(t_{i+1}-t_i\right)+\sigma\left(W_{t_{i+1}}-W_{t_i}\right)\right\} \tag{22}$$

Assuming an equidistant grid, let $t_{i+1}-t_i=\Delta t$, for all *i*. Let $S_{i+1}=S_{t_{i+1}}$, $\Delta W_{i+1}=W_{t_{i+1}}-W_{t_i}$, for $i=0,\cdots,N-1$. Notice that $\Delta W_{i+1}\sim N\left(0,\Delta t\right)$, so we can replace ΔW_{i+1} by $\sqrt{\Delta t}Z_{i+1}$ where Z_{i+1} are independent standard normal random variables for all $i=0,1,\cdots,N-1$. The discretized form becomes:

$$S_{i+1} = S_i\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}Z_{i+1}\right\} \tag{23}$$

3.2. Variance Gamma Process

Variance Gamma (VG) process falls under the class of infinite activity pure jump models. In the pure jump models, one does not have to introduce a diffusion component since the dynamics of jumps already are rich enough to generate non-trivial small time behavior. However, models of this type can be constructed by Brownian subordination. Subordinating Brownian motion with drift μ by the (subordinator) process *S*, we obtain a new Lévy process $L_t=\mu S_t+\sigma W_{S_t}$.

The subordinating process under the Variance Gamma model is the Gamma process. The Gamma process, like the Poisson process, is a pure jump process with no diffusion component. Jumps of negligible size arrive infinitely often in the Variance Gamma model and the infinite activity allows the model to behave like a diffusion process for small jumps. Jumps of non-negligible size occur with a finite frequency and the arrival rate of these jumps decrease monotonically with the jump size.

So, the Variance Gamma process uses a Gamma process to time change a Brownian motion. Instead of evaluating a Brownian motion at time *t*, rather it is evaluated at time γ_t, where γ_t follows a gamma process with $E\left[\gamma_t\right]=t$ and $Var\left[\gamma_t\right]=\nu t$. [18] define a Variance Gamma process, $X_t\left(\sigma,\nu,\theta\right)$, as a time changed Brownian motion as follows:

$$X_t^{VG}\left(\sigma,\nu,\theta\right)=\theta\gamma_t+\sigma W_{\gamma_t}. \tag{24}$$

The dynamics of the stock price under the risk-neutral measure are:

$$S\left(t\right)=S\left(0\right)\exp\left\{\left(r+\omega\right)t+X_t^{VG}\left(\sigma,\nu,\theta\right)\right\}, \tag{25}$$

where S_t, S_0 are stock prices at times *t* and 0, respectively, *r* is the risk-free rate of return of the stock and ω is a compensator term chosen to make sure that the Variance Gamma process is a martingale.

We can write the dynamics in Equation (25) as:

$$S_t=S_0\exp\left\{\left(r+\omega\right)t+\theta\gamma_t+\sigma W_{\gamma_t}\right\} \tag{26}$$

Consider the dynamics over the time interval $\left[t_i,t_{i+1}\right]$, where $0\le t_i<t_{i+1}\le T$. As above, assume an equidistant grid so that $t_{i+1}-t_i=\Delta t$, for all $i=0,1,\cdots,N-1$. Also, let $S_{i+1}=S_{t_{i+1}}$ and $\Delta W_{i+1}=W_{t_{i+1}}-W_{t_i}$. Since the Gamma process is used to subordinate the Weiner process, the discretized Variance Gamma process becomes:

$$S_{i+1}=S_i\exp\left\{\left(r+\omega\right)\Delta t+\theta\gamma_{i+1}+\sigma\sqrt{\gamma_{i+1}}Z_{i+1}\right\} \tag{27}$$

3.3. Parameter Estimation

The parameters of the General Brownian model (with parameter σ) and Variance-Gamma model (with parameters σ, θ, ν) were estimated using the maximum likelihood estimation (MLE) method. The maximum likelihood method postulates that the most sensible values of the parameters are those that maximize the likelihood of the observations. The likelihood function of the sample data is given by:

$$\mathcal{L}(x;\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i;\boldsymbol{\theta}). \tag{28}$$

where $\boldsymbol{\theta} = \{\theta_1, \theta_2, \cdots, \theta_k\}$ is the parameter set and $f(x_i;\boldsymbol{\theta})$ is the probability distribution function (pdf) of the proposed distribution. A value of θ that maximizes $\mathcal{L}$ is known as the maximum likelihood estimate (MLE), that is, $\hat{\theta}$ is a MLE of $\boldsymbol{\theta}$ if:

$$\mathcal{L}(x;\hat{\theta}) = \arg\max_{\boldsymbol{\theta}} \prod_{i=1}^{n} f(x_i;\boldsymbol{\theta}). \tag{29}$$

It is often simpler to maximize the log-likelihood function:

$$l(x;\boldsymbol{\theta}) = \sum_{i=1}^{n} \log(f(x_i;\boldsymbol{\theta})). \tag{30}$$

[19] suggested a method, which is used in this work, to approximate more efficiently the likelihood function using the fast Fourier Transform. For a given parameter vector, the density function is calculated at N points $y_1, y_2, \cdots, y_N$ (where $N = 2^{14}$) over a finite range by inverting the characteristic function with the use of fast Fourier Transform. Given m observed data points, [13] arranged the observed data x_i for $i = 1, \cdots, m$ into their corresponding intervals $x_i \in [y_j, y_{j+1}]$ for $j = 1, \cdots, N-1$ and counted the number of observed data points that fell into each interval. The intervals for the bins are formed using the density evaluated at some pre-specified points. The likelihood of observing this binned data is then maximized by appropriate choice of the parameter vector.

However, solving the likelihood equations is often non-trivial. We had to rely on global optimization algorithms to obtain the maximum likelihood estimates. It should be hinted that the solutions yielded by these algorithms were sensitive to the choice of initial values because the log-likelihood function may have several local minimas.

4. Data and Experimental Results

4.1. Data Description

We selected 19 stocks from the ALSI40 (JSE Top 40 Index) for the purpose of this work, which are representative of the different industry sector categories. The share code, share name and industry sector categories are given in **Table 1**. The data employed in this paper comprise the returns, open prices, close prices and midprices. The data were sampled every 5 minutes from 0900 hrs until 1700 hrs CAT. One of the indisputable stylized features of financial time series is that they exhibit periodicities, or recurring patterns. We examined returns for the ALSI40 (JSE Top 40 Index) stocks at 5 minute intervals and found out that significant returns were on average earned during the first 30 minutes of trading.

Continuous time models provide better predictions when used to model stock prices over longer periods of time rather than short periods of time. As a result, we try to predict the stock price movement on a weekly basis.

The predictions were made at the start of the first business day of the week.

The models were tested for their prediction capabilities when the market was generally trending downwards as well as when the market was generally trending upwards. For the former, the sample data set runs from May 2008 to April 2009, and for the latter, the sample data set runs from September 2010 to August 2011. Both sample data sets span for a period of approximately a year each. About two thirds of the observations were used for In-Sample predictions and the remainder of the observations were used for one-step ahead forecast, which we consider as our out-of-sample prediction forecast. The statistical tests were performed at 5% significance level $(\alpha = 0.05)$. The results reported here are for 2-tailed tests.

Table 1. Stocks selected from the JSE Top 40 (ALSI).

Share Code	Share Name	Trading Sector
AGL	Anglo American Plc	General Mining
AMS	Anglo American Platinum Plc	Platinum and Precious Minerals
ANG	Anglo Gold Ashanti Ltd	Gold Mining
APN	Aspen Pharmacare Holding Ltd	Pharmaceuticals
BGA	Barclays Africa Group Ltd	Banking
BIL	BHP Billiton Plc	General Mining
BVT	Bidvest Group Ltd	Business Support Services
GRT	Growthpoint Property Ltd	Real Estate Investment and Holdings
IMP	Impala Platinum Holdings Ltd	Platinum and Precious Minerals
INL	Investec Ltd	Investment Services
IPL	Imperial Holdings Ltd	Transportation Services
MTN	MTN Group Ltd	Mobile Telecommunications
NPN	Naspers Ltd	Media
OML	Old Mutual Plc	Life Insurance
SAB	SabmillerPlc	Beverages
SBK	Standard Bank Group Ltd	Banking
SHP	Shoprite Holdings Ltd	Food Retailers and Wholesalers
SOL	Sasol Ltd	Oil and Gas Producers
WHL	Woolworths Holdings Ltd	Clothes Retailers

4.2. Performance Evaluation of the Models

Now, we evaluate the performance of the models discussed in the previous sections. From the numerous choices of performance evaluation metrics, we choose to use the mean-absolute percentage error (MAPE) and hit ratio to evaluate the performance of the models. The mean-absolute percentage error (MAPE) measures the magnitude of error from the observed prices in percentage terms. The formula for calculating MAPE is:

$$\text{MAPE} = \frac{100}{N}\sum_{i=1}^{N}\left|\frac{\hat{y}_i - y_i}{y_i}\right|, \tag{31}$$

where y_i is the i^{th} actual observation (price) and $\hat{y}_i$ is the predicted observation (price).

The hit ratio is simply the accuracy of the predicting model measured in percentage over number of predictions. It is determined by the number of correct signs (correct price direction) divided by the total number of predictions. The hit ratio is given as:

$$\text{Hit Ratio} = \frac{\text{Number of correct signs}}{\text{Total number of predictions}} \times 100\%. \tag{32}$$

For each performance evaluation metric, we make a comparative analysis of the models—GBM or VG, under each simulation method—QMC or LSMC. Furthermore for each performance evaluation metric, we make a comparative analysis of the simulation models—QMC and LSMC, for each type model—GBM or VG. The following are the findings of the evaluations for the two different time periods selected.

4.2.1. Performance Evaluation—Downward Trend Analysis

Table 2 shows the average hit ratios of the stocks from the sample obtained using the quasi-Monte Carlo method. The In-Sample average hit ratio for the GBM model is 55.41%, whereas for the VG model is 55.71%. For the Out-Sample, the average hit ratio for the GBM model is 51.64% and the VG model is 51.97%. The All Periods average hit ratios for the GBM model and VG model are 54.25% and 54.55%, respectively. The statistical t-tests showed no significant differences in the hit ratios from using either of the models for all the different testing periods (In-Sample, Out-Sample, All Periods).

Table 3 shows the average hit ratios for the stocks from the sample acquired using the Least Squares Regression Monte Carlo Method. The average hit ratios for the different testing periods (In-Sample, Out-Sample, All Periods) are within a range of 49% - 52%. The statistical t-tests showed no significant difference from using either the GBM model or VG model for all the different testing periods.

In addition, to assess which simulation method is better for hit ratios we made a comparative analysis of each type of model (GBM model or VG model) under each type of simulation method. Table 4 gives a comparison of the GBM model hit ratios under the different simulation methods.

Table 5 shows the performance of the VG model under different simulation procedures. The t-tests for results in Table 4 and Table 5 show no significant differences from using either simulation method when using either the GBM model or VG model in predicting the hit ratios.

Table 2. Average hit ratios of the models using quasi-Monte Carlo method—downward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	55.70%	52.05%	1.45	0.16
Out-Sample	51.97%	49.01%	0.80	0.43
All Periods	54.25%	50.81%	1.90	0.06

Table 3. Average hit ratios of the models using least squares regression Monte Carlo method—downward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	55.41%	55.70%	−0.13	0.90
Out-Sample	51.64%	51.97%	−0.11	0.92
All Periods	54.25%	54.55%	−0.17	0.87

Table 4. Average GBM model hit ratios using different simulation methods—downward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	51.02%	52.05%	−0.43	0.67
Out-Sample	50.33%	49.01%	0.33	0.74
All Periods	50.81%	51.11%	−0.15	0.88

Table 5. Average VG model hit ratios using different simulation methods—downward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	55.41%	51.02%	2.00	0.05
Out-Sample	51.64%	50.33%	0.38	0.71
All Periods	54.25%	50.81%	1.90	0.06

Next, we give a comparative analysis for the downward trend using the MAPE as the performance measure. **Table 6** reports the average MAPEs of the models obtained using the quasi-Monte Carlo method.

Under the quasi-Monte Carlo method, the MAPEs for the VG model is greater than those of the GBM model for all the different testing periods. The statistical tests confirm no significant differences between the two models in predicting stock price values.

Table 7 reports the MAPEs of the models using the Least Squares Regression Monte Carlo method. From using the Least Squares Regression Monte Carlo method, the MAPEs are lower for VG model as compared to the GBM model for all the different time periods. The t-tests showed that there are significant differences when using either the GBM model or VG model in predicting the stock price values.

Again, we test to assess which simulation method is better at predicting stock price values. **Table 8** gives a comparison of the GBM model under the different simulation methods. The mean MAPEs for the GBM model under quasi-Monte Carlo method are lower than the mean MAPEs under the Least Squares Regression Monte Carlo method. The statistical tests showed significant differences between the simulation methods in predicting stock price values when using the GBM model.

Table 9 gives the performance of the VG model under the different simulation methods. The mean MAPEs are lower under the Least Squares Regression Monte Carlo method as compared to the quasi-Monte Carlo Method for all different time periods. The statistical tests confirm the significant differences between the simulations methods in predicting stock price values when using the VG model.

Table 6. Average MAPEs of the models using quasi-Monte Carlo method—downward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	13.70%	26.41%	−14.03	3.87E−12
Out-Sample	14.61%	25.72%	−7.40	3.78E−07
All Periods	13.98%	26.29%	−16.73	3.17E−13

Table 7. Average MAPEs of the models using least squares regression Monte Carlo method—downward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	13.70%	20.59%	−16.59	7.23E−18
Out-Sample	14.61%	21.15%	−8.89	8.79E−10
All Periods	13.98%	20.75%	−18.64	6.05E−17

Table 8. Average GBM model MAPEs using different simulation methods—downward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	26.41%	7.55%	18.49	5.98E−18
Out-Sample	25.72%	8.00%	10.84	2.43E−11
All Periods	26.29%	7.68%	20.74	7.13E−21

Table 9. Average VG model MAPEs using different simulation methods—downward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	20.59%	7.55%	20.81	2.21E−19
Out-Sample	21.15%	8.00%	13.43	2.27E−15
All Periods	20.75%	7.68%	20.77	5.87E−19

4.2.2. Performance Evaluation—Upward Trend Analysis

Table 10 shows the average hit ratios of the stocks from the sample for different time periods obtained using the quasi-Monte Carlo method. The average mean hit ratios were within a range of 46% - 49%. The results from the statistical tests indicate no significant differences in the hit-ratios from either using any of the models for the different test periods.

Table 11 shows the average hit ratios of the stocks obtained using the Least Squares Regression Monte Carlo method. The average hit ratios for the GBM model are greater than those of the VG model in the In-Sample period as well as for All Periods combined. There are statistical differences from using the models in predicting the hit ratios for the In-Sample period. However, the average hit ratio for the VG model exceeds that of the GBM model in an Out-Sample. We fail to find any significant difference between the models in the Out-Sample and All Periods combined.

Table 12 gives a comparison of the GBM model average hit ratios under the different simulation methods. The results in **Table 12** show that the GBM model has high hit ratios under the Least Squares Regression Monte Carlo method as compared to the quasi-Monte Carlo method. We find significant differences between the simulation methods in the In-Sample and for All Periods combined. However, there is no significant difference for the Out-Sample.

Table 13 shows a comparison of the VG model average hit ratios under the different simulation methods. The average hit ratios increase as we move from the quasi-Monte Carlo method to the Least Squares Regression

Table 10. Average hit ratios of the models using quasi-Monte Carlo method—upward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	46.35%	47.66%	−0.61	0.55
Out-Sample	48.30%	46.44%	0.57	0.57
All Periods	46.97%	47.27%	−0.17	0.86

Table 11. Average hit ratios of the models using least squares regression Monte Carlo method—upward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	52.63%	46.78%	2.87	0.01
Out-Sample	51.70%	53.25%	−0.42	0.68
All Periods	52.33%	48.86%	2.00	0.05

Table 12. Average GBM model hit ratios using different simulation methods—upward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	46.35%	52.63%	−3.26	0.00
Out-Sample	48.30%	51.70%	−0.92	0.37
All Periods	46.97%	52.33%	−3.35	0.00

Table 13. Average VG model hit ratios using different simulation methods—upward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	47.66%	46.78%	0.39	0.70
Out-Sample	46.44%	53.25%	−2.11	0.04
All Periods	47.27%	48.86%	−0.86	0.40

Monte Carlo method for the Out-Sample and All Periods combined, except for the In-Sample. We find statistical difference between the simulation methods only for the Out-Sample period.

We move on to give analysis of the models and simulation methods using MAPE. In **Table 14** we report the average MAPEs obtained using the quasi-Monte Carlo method. The average MAPEs for the GBM model are less than those of the VG model for all different time periods. We find significant statistical differences from the t-tests in predicting the hit ratios for all the different time periods.

In **Table 15** we report the average MAPEs for the models obtained using the Least Squares Regression Monte Carlo method. There are significant reductions in the average MAPEs as we move from using the GBM model to using the VG model under the Least Squares Regression Monte Carlo method for all different time periods. This is further supported by the statistical t-tests which showed significant differences between the models in predicting the stock prices.

Next, we give a comparative analysis of the simulation methods in estimating the stock prices. **Table 16** gives a comparison of the GBM model average MAPEs under the different simulation methods. The quasi-Monte Carlo method has lower MAPEs in comparison to the Least Squares Regression Monte Carlo method for all the different time periods. The t-tests showed significant differences between the simulation methods.

Table 17 gives the results of the performance of the VG model under the different simulation methods. The margin of error (MAPE) in predicting stock prices using the VG model reduces drastically when moving from the quasi-Monte Carlo method to the Least Squares Regression Monte Carlo method. The statistical t-tests show that there are significant differences between the simulation methods.

Table 14. Average MAPEs of the models using quasi-Monte Carlo method—upward trend.

Period	Mean GBM	Mean VG	t-value	p-value
In-Sample	16.16%	22.75%	−4.12	5.90E−04
Out-Sample	14.99%	21.82%	−4.16	3.30E−04
All Periods	15.78%	22.45%	−4.25	4.30E−04

Table 15. Average MAPEs of the models using least squares regression Monte Carlo method—upward trend.

Period	Mean GBM	Mean VG	t-Value	p-Value
In-Sample	33.79%	3.32%	11.07	1.83E−09
Out-Sample	30.94%	3.57%	9.65	1.53E−08
All Periods	32.90%	3.40%	10.89	2.37E−09

Table 16. Average GBM model MAPEs using different simulation methods—upward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	16.16%	33.79%	−5.57	5.27E−06
Out-Sample	14.99%	30.94%	−4.99	3.16E−05
All Periods	15.78%	32.90%	−5.50	6.37E−06

Table 17. Average VG model MAPEs using different simulation methods—upward trend.

Period	Mean QMC	Mean LSMC	t-Value	p-Value
In-Sample	22.75%	3.32%	60.96	2.57E−31
Out-Sample	21.82%	3.57%	25.82	2.15E−17
All Periods	22.45%	3.40%	60.31	1.80E−29

5. Conclusions

This paper addressed the problem of stock price movement using continuous time models. Specifically, the paper provides comparative analysis of continuous time models—GBM and VG in predicting the direction and accurate price levels of stocks using Monte Carlo methods—QMC and LSMC. The performance evaluation metrics used in this paper were hit ratio and MAPE. The t-test was used to show significance. For each performance evaluation metric, we made a comparative analysis of the models—GBM and VG under each simulation method—QMC or LSMC. Furthermore, we made a comparative analysis of the simulation models—QMC or LSMC for each model—GBM or VG. The models were tested for their prediction capabilities when the market was generally trending downwards as well as when the market was generally trending upwards.

For the downtrend analysis, we found no significant difference between the GBM model and VG model in terms of the hit ratio (number of times the predicted direction is correct). We also found no significant difference between the Monte Carlo methods—QMC and LSMC in terms of hit ratios for the downward trend period.

In terms of the MAPEs for the downward trend, there were no significant differences between GBM model and VG model under the QMC method. However, there were significant differences between the GBM model and VG model under the LSMC method. The VG model performs better than the GBM model under the LSMC method in predicting stock price values.

The Monte Carlo methods assessment for the downtrend showed significant differences either using the GBM model or VG model as shown by the MAPEs. The findings hint that the GBM model works well when used in the QMC method whereas the VG model works well when used in the LSMC method.

For the uptrend analysis, we found no significant difference between the GBM model and VG model under the QMC method in terms of the hit ratios. Under the LSMC method there were no significant differences between the GBM model and VG model except for the In-Sample period. In this case the GBM model predicted the direction correctly most of the times in comparison to the VG model.

In the comparison of the Monte Carlo methods, there were significant differences for the GBM model In-Sample and All Periods when used in the Monte Carlo methods. The GBM model predicted the hit ratios most of the times in the LSMC method in comparison to the QMC method. Comparison of the VG model in the Monte Carlo methods showed significant difference for the Out-Sample only with the VG model fairing well in the LSMC method as compared to the QMC method.

In terms of the MAPEs for the uptrend, we found significant differences between the GBM model and VG model. The GBM model fairs better under the QMC method whereas the VG model fairs well under the LSMC method in predicting the stock price values. The findings also show significant differences between the Monte Carlo methods in predicting the stock price values. Again the GBM model performs better in the QMC method and the VG model performs better in the LSMC method.

We summarize the findings as follows: for predicting the direction of stock price (as indicated by the hit ratios), the GBM model or VG model can be used in any Monte Carlo method as most of the times we found no significant differences as evidenced from the t-tests. The hit ratios we obtained are “near” random walk behavior. This hints on how challenging it is in predicting stock price movement. Since the results of the hit ratios are at the same level where even a random predictor can produce them, then our results are justifiable.

For predicting the stock price values (as indicated by the MAPEs), the GBM model performs well under the QMC method and the VG model performs well under the LSMC method. The finding has important implications in risk management simulations.

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Counterparty credit risk for interest rate derivatives using Basel III standardized approach

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Abstract In this article, we analyze the Basel III standardized approach for OTC interest rate derivatives - caps, floors and swaptions, with the objective of understanding how the CVA levels evolve under this approach. We implement a market model approach to model and simulate the interest rate derivative instruments. For this, a LIBOR Market Model (LMM) is used. The Monte Carlo method was used to simulate the CVA exposures. The Internal Model Method (IMM) was used to determine the Exposure at Default (EAD). CVA stress tests towards netting agreements, ratings and maturities were implemented using a test portfolio consisting of OTC interest rate derivatives transactions with different counterparties. Based on the observations from the stress tests, we conclude that the more the portfolio is diversified, the less the CVA cost. Also, we conclude that long-dated instruments pose great uncertainty with adverse shift in market conditions, hence such type of instruments should be carefully monitored.

Keywords counterparty credit risk · interest rate derivatives · LIBOR Market Model · Basel III · standardized approach · credit value adjustment

1 Introduction

Counterparty credit risk (CCR) has become one of the highest profile risks facing financial markets participants in recent times, largely due to credit crisis that started in 2007. Concerns about counterparty risk were significantly heightened in early 2008 by the collapse of Bear Sterns, but intensified later in the same year when Lehman Brothers defaulted on its debt and swap obligations and declared Chapter 11 bankruptcy. In the aftermath of the Lehman bankruptcy, the frequent threat of bankruptcy among high profile institutions

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demonstrated the necessity of having accurate exposure figures to the counterparties. In addition, the misconception that the counterparty risk on certain entities (assumed to be risk-free) could be overlooked was set aside by the filing of bankruptcy by Lehman Brothers [1]. These events brought counterparty risk to the fore such that currently it is a pertinent area of focus.

Under the Basel II framework, financial institutions held capital against the variability in the market value of OTC transactions, but there was no requirement to hold capital against variability in counterparty credit risk. As counterparty credit risk caused turmoil and caused huge losses, the Basel Committee for Banking Supervision (BCBS) introduced a risk capital charge called the credit valuation adjustment (CVA) capital charge in Basel III. A financial institution is now required to hold additional capital when entering an OTC transaction. The capital charge is designed to cover losses arising from the deterioration of the credit worthiness of trading counterparties.

The CVA is part of the risk-weighted assets (RWA) under Basel III. Risk-weighted assets (RWA) was introduced under the first Basel accord as a measure of minimum capital requirements for financial institutions . Now, the first step in computing capital requirements for counterparty credit risk is to calculate the risk weights of the transactions against every counterparty. Basel III framework proposes two ways for measuring the risk weights which are namely the standardized approach and advanced approach. In this work we will concentrate on the standardized approach to compute the risk weights. The reasons for focusing on the standardized approach being that the standardized approach is widely used in financial institutions and the data required for the standardized approach is easily available as compared to the data for the advanced approach.

Issues related to CVA such as netting, collateral management, wrong-way risk, volatility of CVA, backtesting and stress testing also received significant attention in Basel III. Some of these issues still pose some challenges to financial institutions and cannot be disregarded. It is with this notion, that we will focus on the measuring of counterparty risk (or CVA pricing) including some of the issues raised in Basel III using the standardized approach. We put strong focus on the practicality of the above mentioned issues in the modeling of credit exposures and pricing the counterparty credit risk.

Canabarro and Duffie (2003) give an introduction to methods used to measure, mitigate and price counterparty risk. They use Monte Carlo simulation methods to measure counterparty risk and discuss practical calculation of CVA in currency and interest rate swaps. De Prisco and Rosen (2005) discuss counterparty risk and credit mitigation techniques at portfolio level. The paper provides a discussion of how Monte Carlo simulations, approximation methods as well as some analytical approximations that can be used to compute various statistics crucial to the measurement of counterparty credit risk. In addition, the paper also provides calculation of expected exposure in credit derivatives portfolio with wrong-way risk. Several issues pertaining to the simulation of

[1] Lehman Brothers held an investment grade credit rating at the time of default.

CVA under margin agreements are studied in Pykhtin (2009) and Pykhtin and Zhu (2007), to just name a few. Furthermore, Gregory (2010) provides thorough treatments of the methods and applications used in practice regarding counterparty credit risk.

Interest rate derivatives, which will be the main focus of this work, are the largest class of derivatives traded on the OTC derivatives market. Thus, a deep understanding and adequate analysis of counterparty credit risk of interest rate derivatives is very important. In addition, interest rates are the cornerstone for measuring the counterparty credit risk of other classes of OTC derivatives, hence our interest. In particular, we implement a market model approach to model and simulate the interest rate risk exposures. For this, a LIBOR Market Model (LMM) calibrated to the evolution of the historical yield curve is used. Based on the simulated interest rates, we evaluate interest rate derivatives and the exposures to a counterparty and hence measure the counterparty credit risk. We cite few papers on counterparty credit risk related to interest rate derivatives such as Sorensen and Bollier (1994), Brigo and Masetti (2006), Brigo and Pallavicini (2007) and Segoviano and Manmohan (2008).

The paper is organized as follows: The construction of the LIBOR Market Model (LMM) and the calibration of the model to data is discussed in the next section. In Section 3 we focus on the modeling of credit exposures and pricing of counterparty credit risk. In Section 4 we present numerical results from CVA calculations of OTC interest rate derivatives using the standardized approach for several counterparties. Section 5 formulates our conclusions and carries summary of our findings.

2 LIBOR Market Model

In this section we discuss the construction of the LIBOR Market Model (LMM). The discussion is along the lines of construction done by Musiela and Rutkowski (1997) and Jamshidian (1997). The LIBOR Market Model describes the arbitrage-free dynamics of the term structure of interest rates through the evolution of forward rates. We will work in terms of simple forward rates, that is, rates that are really quoted in the market. The forward LIBOR rate $L(0,T)$ is the rate set at time 0 for the interval $[T, T+\delta]$. The forward rates cannot be traded on the market thus the underlying asset for pricing interest rate derivatives is the time t price of a family of zero-coupon (default-free) bonds.

The idea behind the LMM is to model the forward LIBOR process under the terminal measure where the choice of numeraire is the bond with longest maturity.

Definition 2.1 ***LIBOR Rate***

The LIBOR rate $L(t, t+\delta)$ contracted at time t is the solution to the equation:

$$1 + \delta L(t, t+\delta) = \frac{1}{p(t, t+\delta)}, \tag{1}$$

where $\delta > 0$ is the time length covered by the LIBOR interest rate.

Consider a fixed set of increasing maturities also known as the tenor structure for the LIBOR given by $T_1 < \ldots T_i < T_{i+1} < \ldots T_N$. Define $\delta_i = T_{i+1} - T_i$, $i = 1, 2, \ldots, N-1$, where δ_i is the tenor and $1/\delta_i$ is the day-count factor. Let $p_i(t) \equiv p(t, T_i)$ denote the time t price of a default-free dsicount bond maturing at time T_i. Let $L_i(t) \equiv L_i(t, T_i, T_{i+1})$ denote the forward LIBOR rate contracted at time t (where $t \leq T_i$), for the period $[T_i, T_{i+1}]$, that is $L_i(t)$ is reset at dates $T_i, i = 1, \ldots, N-1$ known as the reset dates and is valid for the period $\delta_i = T_{i+1} - T_i$. Equation 1 can be written as:

$$1 + \delta L(t, t+\delta) = \frac{p_i(t)}{p_{i+1}(t,)}, \quad i = 1, 2, \ldots, N-1 \tag{2}$$

which implies that the T_i forward LIBOR rate at time t is given by:

$$L_i(t) = \frac{1}{\delta_i}\left(\frac{p_i(t)}{p_{i+1}(t)} - 1\right), \ i = 1, 2, \ldots, N-1, \ \ t \geq 0 \tag{3}$$

with initial term structure $L_i(0)$.

As shown in equation 3, associated with the forward LIBOR rate is the time t price of bond prices with corresponding tenor dates $T_i, \ i = 1, 2, \ldots, N$.

The dynamics of the forward LIBOR $L_i(t)$ are summarized in the following proposition.

Proposition 2.2 *Suppose the process $\{L_i(t)\}$ is a solution to the stochastic differential equation of forward LIBOR rates $L_i(t), \ i = 1, \ldots, N-1$, defined by:*

$$dL_i(t) = \mu_i^L(t)dt + \sigma_i(t)dz, \quad i = 1, \ldots, N-1 \tag{4}$$

where $\mu_i^L(t)$ is the drift, $\sigma_i(t)$ is the volatility function and $z = \{z(t)\}$ is a standard Brownian motion under empirical probability measure P. Then the log-normal LMM exists if and only if:

a) The forward LIBOR rate $L_i(t)$ is given by equation 3 and the initial term structure $L_i(0)$ is known.

b) $L_i(t)$ is a martingale under forward probability measure $P^{T_{i+1}}$ with the SDE defined as:

$$dL_i(t) = L_i(t)\sigma_i(t)dz^{i+1}, \quad i = 1, 2, \ldots, N-1 \tag{5}$$

for some volatility $\sigma_i(t)$ where $z^{i+1} = \{z^{i+1}(t)\}$ is a standard Brownian motion under $P^{T_{i+1}}$.

c) The volatility function $\sigma_i(t), t \leq T_i$ is a deterministic function of time $t \leq T_i$ for each settlement date, that is, the forward LIBOR rates $L_i(t), \ i = 1, \ldots, N-1$, are log-normally distributed with mean, $m_i(t) = -\int_t^{T_i} 1/2 \ \sigma_i^2(s)ds$ and variance, $\nu_i^2(t) = \int_t^{T_i} \sigma_i^2 ds$.

Proof See Proposition 3.1 (Mutengwa, 2011) □

2.3 Dynamics of the forward LIBOR rates under the terminal measure

The notion behind the LMM is to model the forward LIBOR processes under the terminal measure, where the choice of numeriare is the bond with the longest maturity. The dynamics of each forward LIBOR rate is defined under its own forward measure depending on the effective period of the forward rate. Here we outline the construction of the stochastic differential equation for forward LIBOR rates under the terminal measure.

Using equation 2, the Radon-Nikodym derivative for the change of measure from P^{T_i} to $P^{T_{i+1}}$ for the forward LIBOR rate is:

$$\frac{\mathrm{d}P^{T_i}}{\mathrm{d}P^{T_{i+1}}} = \frac{v_i(t)/v_i(0)}{v_{i+1}(t)/v_{i+1}(0)} = \frac{v_i(t)}{v_{i+1}(t)}\frac{v_{i+1}(0)}{v_i(0)}. \tag{6}$$

Let $\eta_i(t)$ denote $\frac{\mathrm{d}P^{T_i}}{\mathrm{d}P^{T_{i+1}}}$, then:

$$\eta_i(t) = c\frac{v_i(t)}{v_{i+1}(t)} = c(1+\delta_i L_i(t)), \tag{7}$$

where $c = \frac{v_{i+1}(0)}{v_i(0)}$ is a normalizing constant. Using Girsanov's Theorem it follows that:

$$\mathrm{d}z^i = \mathrm{d}z^{i+1} + \beta_i(t)\mathrm{d}t, \tag{8}$$

where $\beta_i(t)$ satisfies the condition:

$$\eta_i(t) = \exp\left(\int_0^t -\beta_i(s)\mathrm{d}z^{i+1} - \frac{1}{2}\int_0^t \beta^2(s)\mathrm{d}t\right) \tag{9}$$

From equations 7 and 9, it follows that:

$$\eta_i(t) = c(1+\delta_i L_i(t)) = \exp\left(\int_0^t -\beta_i(s)\mathrm{d}z^{i+1} - \frac{1}{2}\int_0^t \beta^2(s)\mathrm{d}t\right) \tag{10}$$

The above equation in differential form is written as:

$$\mathrm{d}\eta_i(t) = c\delta_i \mathrm{d}L_i(t) = -\eta_i\beta_i(t)\mathrm{d}z^{i+1}. \tag{11}$$

Substituting equations 5 and 7 into equation 9 implies that:

$$c\delta_i L_i(t)\sigma_i(t)\mathrm{d}z^{i+1} = -c(1+\delta_i L_i(t))\beta_i(t)\mathrm{d}z^{i+1}. \tag{12}$$

Thus we obtain:

$$\beta_i(t) = -\frac{\delta_i L_i(t)\sigma_i(t)}{1+\delta_i L_i(t)}. \tag{13}$$

Substituting $\beta_i(t)$ in equation 8 we get that:

$$\mathrm{d}z^i = \mathrm{d}z^{i+1} - \frac{\delta_i L_i(t)\sigma_i(t)}{1+\delta_i L_i(t)}\mathrm{d}t. \tag{14}$$

Under the forward LIBOR rate, $L_{N-1}(t)$ is a martingale under terminal measure P^{T_N}. Using equation 14, it follows that:

$$\begin{aligned}
\mathrm{d}L_{N-2}(t) &= L_{N-2}(t)\sigma_{N-2}(t)\mathrm{d}z^{N-1} \\
&= L_{N-2}(t)\sigma_{N-2}(t)\left[\mathrm{d}z^N - \frac{\delta_{N-1}L_{N-1}(t)\sigma_{N-1}(t)}{1+\delta_{N-1}L_{N-1}(t)}\right] \\
&= L_{N-2}(t)\left[-\frac{\delta_{N-1}L_{N-1}(t)\sigma_{N-2}(t)}{1+\delta_{N-1}L_{N-1}(t)}\right] + L_{N-2}(t)\sigma_{N-2}(t)\mathrm{d}z^N
\end{aligned} \tag{15}$$

Thus $L_{N-2}(t)$ is a martingale under $P^{T_{N-1}}$ but not a martingale under P^{T_N}. As suggested in Jamshidian (1997), we can deductively obtain the forward rates $L_{N-2}, \ldots, L_1$. If the solution $\{L_i(t)\}$ exists, then the i^{th} component of the forward LIBOR rate follows the SDE:

$$\mathrm{d}L_t = L_i(t)\mu_i^*(t)\mathrm{d}t + L_i(t)\sigma_i(t)\mathrm{d}z^N,\ i = 1, \ldots, N-1 \tag{16}$$

where

$$\mu_i^*(t) = \begin{cases} \sum_{j=i+1}^{N-1} -\frac{\delta_j L_j(t)\sigma_j^{'}(t)\sigma_i(t)}{1+\delta_j L_j(t)} & \text{for } i < N-1 \\ 0 & \text{for } i = N-1 \end{cases}$$

and $z^N \equiv z^N(t)$ is a standard Brownian motion under the terminal measure P^{T_N}.

In the LMM model, the payoff of interest rate derivatives depend on several forward rates at the same time. So correlations between the different forward rates affects the payoffs and must therefore be taken into consideration. Under the terminal measure, we now have forward rate dynamics with correlated Brownian motion:

$$\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{ij}(t)\mathrm{d}t. \tag{17}$$

Introducing the correlation changes equation 16 to:

$$\mathrm{d}L_t = L_i(t)\mu_i^*(t)\rho_{ij}(t)\mathrm{d}t + L_i(t)\sigma_i(t)\mathrm{d}z^N,\ i = 1, \ldots, N-1 \tag{18}$$

where

$$\mu_i^*(t) = \begin{cases} \sum_{j=i+1}^{N-1} -\frac{\delta_j L_j(t)\sigma_j'(t)\sigma_i(t)}{1+\delta_j L_j(t)} & \text{for } i < N-1 \\ 0 & \text{for } i = N-1 \end{cases}$$

From above it is evident that the forward LIBOR rates are specified in terms of volatilities and correlations. Next, we need to consider the different possible specifications for volatilities and correlations which we will use for this work.

The two main representatives of the family of market models are the LMM, which models the forward LIBOR rates, and the Swap Market Model (SMM), which models the dynamics of the forward swap rate. The state variables in the LMM are market observable rates and are assumed to be lognormally distributed. This allows the pricing of caplets (floorlets) and hence caps(floors) that are consistent with Black (1976) formula. The state variables in the SMM are also market observables and are assumed to be lognormally distributed. This allows the pricing of swaptions in a manner that is consistent with the Black (1976) formula.

However, LIBOR and swap market prices are not compatible (see for example Brigo and Mercurio (2001)) because the dynamics that would be obtained for the swap rate, by starting from the dynamics of the LIBOR market model are not log-normal as in the swap market model. This then implies that one has to decide which of the models to work with. We will focus our attention on the LIBOR market model. This implies simple calibration to cap(floor) prices and use the approximations from the calibration to swaption prices.

2.4 Calibration of Instantaneous Volatilities

The instantaneous volatility is commonly represented using the approach of piecewise-constant volatilities or by using an approach in which the volatilities are represented by a parametric form. Piecewise-constant approach volatilities imply that for short intervals of time the volatility of a forward rate is constant. Using a parameter form imply that the volatility is represented by a function of time, maturity and a set of parameters. The aim is to obtain flexible specifications that are able to reflect observable features of the market behaviour of volatilities which are financially meaningful. In this work we follow the latter approach.

Now, the relationship between Black caplet volatilities and instantaneous forward rates in the LMM framework for a general caplet is given by (see Rebonato (2002)):

$$\sigma_{T_i,Black} = \sqrt{\frac{1}{T_i}\int_0^{T_i} \sigma_i^2 \mathrm{d}s}. \tag{19}$$

Thus the Black implied volatility is the root-mean square instantaneous volatility and that the term structure of volatilities can be represented using instantaneous volatilities. Given a quote for $\sigma_{i,Black}$, it is not possible to uniquely determine the instantaneous volatility $\sigma_i(t)$ as there exist plenty functions $\sigma_i(t)$ that would integrate to $\sigma_{i,Black}$. To calibrate a LMM to a caplet market is then a matter of choosing a well-behaved function for the instantaneous volatility. An ideal function is one that is time homogeneous, that is a function should be able to reproduce the current shape of the volatility curve in the future.

The idea of imposing time-homogeneity originates from observing term structures of forward rate volatilities of caplets in the market. As time passes one after another the various forward rates will have the same remaining time to expiry. Assuming time homogeneous volatility implies that we expect the forward rates to react similarly in terms of volatility as time passes. In order to make the term structure of volatilities remain the same as time passes, the volatility can be expressed as a function of time until the forward rate resets, that is:

$$\sigma_i(t) = g(T_i - t), \tag{20}$$

for some function $g(.)$ so that the volatility is only a function of time to maturity. Several specifications of the instantaneous volatility of forward rates have been proposed throughout the literature. A widely used specification $g(.)$ due to Rebonato is:

$$\sigma_i(t) = [a + b(T_i - t)] \exp\{-c(T_i - t)\} + d. \tag{21}$$

The specification is often called the *abcd*-formula. The Rebonato specification is popular due to its ability to fit humped shaped volatility structures often observed in the market.

Let $\mathcal{A} = \{a, b, c, d\}$. We choose the *abcd*-parametric form for the instantaneous forward rate via a minimization problem. Calibrating to market caplets is a minimization problem of the form:

$$\min_{\mathcal{A}} \sum_{i=1}^{M} \left(\sigma_{i,Black} - \sqrt{\frac{1}{T_i - t} \int_t^{T_i} [(a + b(T_i - t)) \exp\{-c(T_i - t)\} + d]^2 \mathrm{d}s} \right), \tag{22}$$

where M is the number of caplets under consideration and optimization is subject to the constraints $a + d > 0, c > 0, d > 0$. Fitting all caplet volatilities with the *abcd*-formula will generally not suffice and so we introduce the factor k_i, that measure the extent to which time-homogeneity is lost. Including k_i term in equation 21 changes it to:

$$\sigma_i(t) = k_i[a + b(T_i - t)]\exp\{-c(T_i - t)\} + d, \tag{23}$$

and changes equation 19 to:

$$\sigma_{T_i,Black} = k_i\sqrt{\frac{1}{T_i}\int_0^{T_i}\sigma_i^2 \mathrm{d}s}. \tag{24}$$

If the time-homogeneous specification is able to recover market prices well, then most of the times k_i values are approximately unit.

As for swaption volatilities, we will use the Rebonato approximation formula for swaptions where the squared swaption volatility can be approximated by (see Gatarek et al (2006)):

$$(v_{n,N})^2 = \sum_{i,j=n+1} \frac{\omega_i(0)\omega_j(0)L_i(0)L_j(0)\rho_{i,j}}{S_{n,N}(0)^2}\int_0^{T_n}\sigma_i(t)\sigma_j(t)\mathrm{d}t \tag{25}$$

and the swap rates are expressed as linear combination of forward rates:

$$S_{n,N}(0) = \sum_{i=n+1}^{N}\omega_i(t)L_i(t) \overset{assumption}{=} \sum_{i=n+1}^{N}\omega_i(0)L_i(0). \tag{26}$$

For the purpose of the calibration all $\omega_i(t)$ and $L_i(t)$ are frozen to the value at time 0. So we have the swap rate volatilities expressed as a linear combination of forward rates. Also note that the swaption volatilities will be dependent on the shape of the instantaneous volatility functions of forward rates.

2.5 Calibration of Instantaneous Correlations

Forward rate correlations are important inputs to the LMM framework. This model feature is included into the LMM because the value of a cap(floor) or swaption at maturity is influenced by the joint distribution of forward rates as thus by the correlation amongst the forward rates. Instantaneous correlations are defined as:

$$\rho_{ij} = \frac{< \mathrm{d}L_i(t), \mathrm{d}L_j(t) >}{\sqrt{< \mathrm{d}L_i(t) >}\sqrt{< \mathrm{d}L_j(t) >}}. \tag{27}$$

In general, a correlation matrix $\{\rho_{ij}\}$ must satisfy the following properties:

- $\rho_{ii} = 1$ for all i
- $-1 \leq \rho_{ij} \leq 1$ for all ij combinations

- The correlation matrix should be symmetric
- The correlation matrix should be positive definite

There are two main approaches to estimate the correlation matrix using historical data to estimate the future correlations empirically, or assume that the correlation matrix follows a certain parametric form. A natural choice to estimate the correlations would be to use historical data. However, correlation matrices produced by this approach do not always satisfy all of the conditions above. On the otherhand, the parametric form approach gives us a lot of flexibility and we can choose to work with parametric forms that satisfy some of the conditions above.

In this work, we will consider the estimation of correlation matrices using historical data. As pointed out above, the correlation matrices might display some unwanted properties due to issues with statistical estimation. However, we will fit a parametric form onto the historical matrix in order to smooth out some of the noise.

The approach assumes that the log-returns of the forward rates are normally distributed. In order to calculate the correlations, we calculate the sample means and sample covariances first. Hence we have that:

$$\hat{\mu}_i = \frac{1}{n}\sum_{k=1}^{n-1} \ln\left[\frac{L_i(t_{k+1})}{L_i(t_k)}\right], \tag{28}$$

and

$$\hat{V}_{ij} = \frac{1}{n}\sum_{k=1}^{n-1}\left[\left(\ln\left(\frac{L_i(t_{k+1})}{L_i(t_k)}\right) - \mu_i\right)\left(\ln\left(\frac{L_j(t_{k+1})}{L_j(t_k)}\right) - \mu_j\right)\right], \tag{29}$$

where n is the number of observed log-returns for each rate. The correlation matrix is then obtained as:

$$\hat{\rho}_{ij} = \frac{\hat{V}_{ij}}{\sqrt{\hat{V}_{ii}}\sqrt{\hat{V}_{jj}}}. \tag{30}$$

We can then use this matrix either directly in the calibration process or fit it to a plausible parametric form.

One of the simplest parametric functional form of correlation function is:

$$\rho_{ij} = \exp(-\beta|T_i - T_j|), \tag{31}$$

where T_i and T_j are the expiries of the i^{th} and j^{th} forward rates and β is a positive constant. Although the exponential function is convenient to work with, it contains only small amount of correlation structures. Moreover, the correlation among forward rates goes asymptotically to zero as their distance increases, which is not consistent with what is often observed in the market. Based on these setbacks, Rebonato suggested a generalization of the exponential form as follows:

$$\rho_{ij} = \rho_\infty + (1 - \rho_\infty) \exp(-\beta_{ij}|T_i - T_j|), \tag{32}$$

where

$$\rho_\infty = \lim_{|T_i - T_j| \longrightarrow \infty} \rho_{ij}.$$

β is no longer a constant but is now a function of the forward rates. However, the parameters ρ_∞ and β_{ij} must be carefully chosen to produce reasonable results.

2.6 Definitions of Common Interest Rate Derivatives

Next, we present the payoffs of two main interest rate derivatives, which are caps/floors and swaptions.

Definition 2.7 ***Caps/Floors***

The discounted payoff at time t of a cap contract associated to the tenor structure $\mathcal{T} = \{T_\alpha, \ldots, T_\beta,\}$, with corresponding set of year fractions $\tau = \{\tau_{\alpha+1}, \ldots, \tau\beta\}$ and principal N is:

$$\sum_{i=\alpha+1}^{\beta} p(t, T_i) N \tau_i (L(T_{i-1}, T_i) - K)^+ \tag{33}$$

Analogously, the discounted payoff at time t of a floor contract is:

$$\sum_{i=\alpha+1}^{\beta} p(t, T_i) N \tau_i (K - L(T_{i-1}, T_i))^+ \tag{34}$$

Definition 2.8 ***Swaption***

A $T_\alpha x(T_\alpha - T_\beta)$ payer swaption with strike K is a contract that gives the right (but not the obligation) to enter a payer IRS with tenor $T_\beta - T_\alpha$ and fixed rate K at a future time T_α, that is, the swaption maturity, and has the following payoff:

$$\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_\beta) N \tau_i (L(T_{i-1}, T_i) - K) \right)^+ \tag{35}$$

3 Counterparty Credit Risk (CCR)

In this section we will focus on the modeling of credit exposure and pricing counterparty risk along the lines of Pykhtin and Zhu (2007) with a strong consideration for the practical issues in the implementation of CCR. We will define counterparty risk measures, introduce netting and margin agreements as risk management tools for reducing counterparty-level exposure and model credit exposure. For the pricing of counterparty risk, we will define credit value adjustment (CVA) as the price of counterparty credit risk and discuss methods to its calculation.

3.1 Counterparty Risk Measures

The two main reasons for measuring CCR are the need to limit the risk to the counterparties and the need to determine the proper amount of reserve capital to cushion from potential defaults from counterparties. Here we discuss typical measures used to gain insight into risk exposures of counterparties. Assume that $V(t)$ is the total value of the portfolio to a particular counterparty at time t and $V_i(t)$ where $i = 1, \dots, N$, is the sum of the individual instruments in the portfolio.

Current Exposure

The Basel Committee on Banking Supervision (BCBS) defines it as:

– the larger of zero, or the market value of a transaction or portfolio of transactions within a netting set with a counterparty that would be lost upon the default of the counterparty, assuming no recovery on the value of those transactions in bankruptcy. Current exposure is also called Replacement Cost.

  Without netting, current exposure is defined as:

$$E(t) = \sum_i E_i(t) = \sum_i \max[V_i(t), 0]. \tag{36}$$

  With netting, current exposure is defined as:

$$E(t) = \max[\sum_i V_i(t), 0]. \tag{37}$$

A netting agreement is a contract that permits aggregation of transactions between the counterparties, that is, transactions with negative values can be used to offset ones with positive values. Then only the net positive value represents credit exposure at the time of default. Netting plays a crucial role in reducing counterparty exposure.

In general, there can be several netting agreements with one counterparty. However, there may be partial netting agreement covering only K trades. The current exposure is then defined as:

$$E(t) = \max[\sum_{i=1}^{K} V_i(t), 0] + \sum_{i=K+1}^{N} \max[V_i(t), 0]. \tag{38}$$

Potential Future Exposure (PFE)

The Basel Committee on Banking Supervision (BCBS) defines it as:

- a high percentile (typically 95% or 99%) of the distribution of exposures at any particular future date before the maturity date of the longest transaction in the netting set.

  The metric is given by:

$$PFE^{\alpha}(t) = \inf\{x|\mathbb{P}(V_t \leq x) \geq \alpha\}, \tag{39}$$

  where α is the given confidence level.

Expected Exposure (EE)

The Basel Committee on Banking Supervision (BCBS) defines it as:

- the mean of the distribution of exposures at any particular future date before the longest maturity transaction in the netting set matures.

  The metric is given by the formula:

$$EE(t) = \mathbb{E}[\max\{\sum_{i=1}^{N} V_i(t)\}] = \mathbb{E}[V^{+}(t)]. \tag{40}$$

  In the presence of partial netting agreement, we have that:

$$EE(t) = \mathbb{E}[\max\{\sum_{i=1}^{K} V_i(t)\} + \sum_{i=K+1}^{N} \max\{V_i(t)\}]. \tag{41}$$

  Thus EE is the average exposure after considering different scenarios.

Expected Positive Exposure (EPE)

The Basel Committee on Banking Supervision (BCBS) defines it as:

– the weighted average over time of expected exposures where the weights are the proportion that an individual expected exposure represents of the entire time interval.

The EPE is computed as follows:

$$EPE = \frac{1}{t_k}\int_t^{t_k} EE(s)\mathrm{d}s \approx \frac{1}{t_k}\sum_{i=1}^{N} EE(t_i)(t_i - t_{i-1}), \tag{42}$$

where t_k is the exposure horizon and $t < t_0 < t_1 < \ldots < t_N = t_k$.

Effective EPE

The Basel Committee on Banking Supervision (BCBS) defines it as:

– the weighted average over time of effective expected exposure over the first year, or over the time period of the longest maturity contract in the netting set where the weights are the proportion that an individual exposure represents of the entire time interval.

As the number of trades with a counterparty and the number of unrealized cash flows in a portfolio decrease over time, the portfolio exposure decreases as a function of time. In particular, as short-term trades expire, the EE profile decreases. These short-term trades are likely to be replaced by new ones, but however, the EPE does not take this into account and thus underestimating the risk.

To account for this, the EPE definition is modified as follows. We now find the Effective EE profile over the first year. This is found from the EE profile by adding the non-decreasing constraint represented by the following recursive relation:

$$\text{Effective } EE(k) = \max[\text{Effective } EE(k-1), EE(k)], \tag{43}$$

where the initial condition Effective EE(0) is equal to current exposure.

The Effective EPE is computed from Effective EE profile in exactly the same way as EPE is computed from EE:

$$\text{Effective } EPE = \sum_{i=1}^{\min(1yr,\text{maturity})} (\text{Effective } EE)(t_i - t_{i-1}). \tag{44}$$

The Effective EPE is calculated from the Effective EE profile up to one year. If all contracts in the netting set mature before one year, the Effective EPE is the average of Effective EE until all contracts in the netting set mature.

Peak Exposure (PE)

PE is a high percentile of the distribution of exposures at any particular future date before the maturity of the longest transaction in the netting set.

3.2 Mitigating Counterparty Credit Risk

Firms that are engaged in OTC derivatives markets employ various techniques that help to reduce exposure to a counterparty risk. The most commonly used techniques are netting and margining (or collateralization).

3.2.1 Netting Agreements

If an investor has more than one trade with a particular counterparty, and the credit risk is not mitigated, the investor's exposure to the counterparty is the sum of the exposures on each of the individual contracts with the counterparty.

If $E_i(t)$ is the exposure on the i^{th} contract, the counterparty level exposure, $E(t)$, is:

$$E(t) = \sum_i E_i(t) = \sum_i \max[V_i(t), 0]. \tag{45}$$

The exposure can be reduced significantly by entering into a netting agreement. This is a legally binding contract between two counterparties that allows the aggregaton of transactions between two counterparties in the event of default. Thus, transactions with negative value can be used to offset those with a positive value. Under a netting agreement, the net counterparty exposure is:

$$E(t) = \sum_i \max[V_i(t), 0]. \tag{46}$$

In some cases, netting agreements only cover certain types of transaction (e.g. fixed income), while other agreements allow cross-product netting, where transactions of different product categories are included within a netting set.

3.2.2 Collateral and Margin Agreement

Although netting significantly reduces counterparty risk, it can still limit trading with certain counterparties. The use of margin agreements can further reduce counterparty exposure and permit trade with less credit-worthy counterparties. Pykhtin and Zhu (2007) define a margin agreement as a legally binding contract that requires one or both counterparties to post collateral when the uncollateralized exposure exceeds a threshold and to post additional collateral if this excess grows.

The modeling of collateralized exposure poses a challenge. We discuss typical modeling approach in Pykhtin and Zhu (2007), which is commonly used in financial institutions. We denote the collateral at time t by $C(t)$. This amount at a date t is determined by comparing the uncollateralized exposure at time $t-s$ against the threshold value H:

$$C(t) = \max[E(t-s) - H, 0], \tag{47}$$

where s is the margin period of risk, and collateral is set to zero if it is less than the MTA. Then the collateralized exposure at time t is calculated by subtracting the collateral $C(t)$ from the uncollateralized exposure:

$$E_C(t) = \max[E(t) - C(t), 0]. \tag{48}$$

To compute exposure at time $t-s$, additional dates (secondary time buckets) are placed prior to the main dates.

3.3 Credit Valuation Adjustment (CVA)

Whilst counterparty risk can be reduced using some combination of methods described above, it cannot be eradicated completely. Thus, it is crucial for a financial institution to quantify correctly the remaining counterparty risk and ensure that they are compensated for taking the risk. A more advanced measure for quantifying counterparty risk, required by Basel III is Credit Valuation Adjustment (CVA). CVA is an adjustment to the price of an OTC derivative to take into account counterparty credit risk/exposure. In otherwords, the CVA value gives an indication of the cost of compensation for the potential loss of a defaulting counterparty. Formally, CVA is defined as:

Definition 3.4 ***Credit Valuation Adjustment*** *Let the value of a portfolio of trades be V_t and let the value of a corresponding portfolio which takes into account counterparty credit risk be $\hat{V}_t$. The CVA is defined as:*

$$CVA = V_t - \hat{V}_t. \tag{49}$$

CVA is the market value of the counterparty credit risk. If the time of the default was assumed to be known, then the discounted loss would be:

$$L = \mathbf{1}_{\{\tau \leq T\}}(1 - R_c)D(\tau)EE(\tau). \tag{50}$$

We can observe that the main components in CVA calculation are:

- Loss given default - $(1 - R_c)$
  This is the percentage of the exposure expected to be lost in case of default by the counterparty.

- Discount factor - $D(\tau)$
  $D(\tau)$ is used to discount future losses back to the current time.
- Expected Exposure (EE)

CVA comes in a number of different forms. There are:

Unilateral CVA

Unilateral CVA is the adjustment in valuation where one of the two parties being considered default risky. The setback of this form of CVA is that it will different values for the same trade since the valuation method is symmetric.

Bilateral CVA

Bilateral CVA assumes that both parties are default risky. The valuation method is symmetric and both parties are able to reach an agreement on the fair value of a trade.

3.5 Right and Wrong Way Risk

The concepts of right and wrong way risk are integral to the discussion of CVA. For explanation purposes, we shall use the unilateral CVA. Unilateral CVA including right or wrong way risk is given by the risk-neutral expectation of the discounted loss:

$$\text{uCVA} = \mathbb{E}^{\mathbb{Q}} L = \mathbb{E}^{\mathbb{Q}} \int_0^T (1 - R_c) D(\tau) EE(\tau) q_c(\tau) \mathrm{d}\tau, \tag{51}$$

where $q_c(\tau)$ is the risk neutral probability of counterparty default at time τ. In general, R_c, D, EE and q_c can correlate. Now,

- if $\text{corr}(EE, q_c) > 0$, this is called wrong-way risk.
- if $\text{corr}(EE, q_c) < 0$, this is called right-way risk.

In otherwords, the risk is wrong-way if exposure tends to increase when counterparty credit quality worsens. The risk is right-way if exposure tends to decrease when counterparty credit quality worsens. Calculating CVA without handling correlation between exposure and default is common practice but it may underestimate the CVA value in the case of WWR. However, this correlation is in reality very difficult to capture.

3.6 CVA Computation

In this work we make use of the Exposure Profile Method for CVA computations. The Exposure Profile Method uses counterparty credit risk metrics in pricing the exposure. Giovanni et al (2009) suggest valuing CVA as follows:

$$\begin{aligned} CVA &= \int_0^T EPE(s)D(0,s)C_s(s)\mathrm{d}s \\ &\approx \sum_i EPE_i(T_i - T_{i-1})D(0,T_i)C_{si}, \end{aligned} \tag{52}$$

where $EPE(t)$ is the expected positive exposure assumed to be piecewise constant between T_{i-1} and T_i. C_{si} is the credit spread of a forward starting CDS beginning at T_{i-1} and maturing at T_i. The above expression is essential particularly for contracts with no closed-form valuations or where a liquid market is not available on the contracts.

A similar approach approach adopted by Pykhtin and Zhu (2007) looks at the loss L incurred at a default time τ:

$$L = \mathbf{1}_{\{\tau \le T\}}(1-R)D(0,\tau)CE, \tag{53}$$

where CE is the counterparty exposure and R is the recovery rate. The CVA is considered to be the cost of hedging the incurred loss L, that is:

$$CVA = \mathbb{E}^{\mathbb{Q}}[L] = (1-R)\int_0^T \mathbb{E}^{\mathbb{Q}}[D(0,t)CE(t)|\tau = t]\mathrm{d}\mathbb{Q}(0,t), \tag{54}$$

where $\mathbb{Q}(s,t)$ are the risk-neutral probabilities of counterparty default between times s and t, which are usually backed out from CDS spreads. If we assume independence, then we have:

$$CVA = (1-R)\int_0^T \mathbb{E}^{\mathbb{Q}}D(0,t)EE(t)\mathrm{d}\mathbb{Q}(0,t), \tag{55}$$

where $EE(t)$ is the expected exposure.

The calculation of exposure profiles is the backbone of capital requirements for regulatory compliance. The industry practice uses a Monte Carlo framework to compute exposure with three main steps: (1) scenario generation, (2) instrument valuation, and (3) aggregation.

In the first step, possible scenarios of the underlying risk factors at future dates are generated and hence generate the empirical price distributions. Relevant statistical quantities can be obtained from the price distribution at each time. Exposure of portfolios are computed by pricing different products (instrument valuation) on the same underlying scenarios and aggregating the results taking into account possible netting and collateral agreement with the counterparty Giovanni et al (2009).

In Basel III, the total counterparty credit risk capital charge is the sum of the default risk capital charge and CVA capital charge, that is:

$$Total\ CCR\ Capital = \begin{matrix} Default\ Risk \\ Capital\ Charge \end{matrix} + \begin{matrix} CVA\ Risk \\ Capital\ Charge \end{matrix} \tag{56}$$

3.6.1 Default Risk Capital Charge

The Default Risk Capital Charge covers the risk of counterparty default. It is based on counterparty exposure and more specifically the metric Exposure at Default (EAD). The Basel Committee specifies four different methods for determining the EAD which are namely:

- Original Exposure Method
- Current Exposure Method
- Standardised Method
- Internal Model Method (IMM)

The methods differ in their risk sensitivity and using a less sensitive method generates larger capital requirement. In this work we shall use the Internal Model Method (IMM) for determining EAD. The IMM makes use of the counterparty risk measures discussed earlier. Under the IMM, EAD is calculated as the product of a multiplier α and EEPE:

$$EAD = \alpha.EEPE \tag{57}$$

The α multiplier is formally defined as the ratio between economic capital and one calculation carried out with deterministic exposures set to EPE. The existence of α is meant to account for WWR, exposures' volatility, model estimation errors and numerical errors.

3.6.2 CVA Risk Capital Charge

CVA Risk Capital Charge covers changes in CVA due to changes in credit worthiness of the counterparty. There are two different methods to calculate CVA capital charge which are the standard approach and the advanced approach. The approach a bank uses depends on the method approved for the bank to use in calculating capital charge for counterparty default risk.

Standardized Approach

For banks that do not have IMM approval, they must calculate the CVA capital charge using the following formula:

$$K = 2.33.\sqrt{h}.\beta \tag{58}$$

where

$$\begin{aligned}\beta^2 &= \left[0.5\sum_{i=1}^{N} w_i(M_i EAD_i^{total} - M_i^{hedge}B_i) - \sum_{ind} w_{ind}.M_{ind}.B_{ind}\right]^2 \\ &= +\sum_{i=1}^{N} 0.75 w_i^2.(M_i EAD_i^{total} - M_i^{hedge}B_i)^2,\end{aligned}$$

where

- h is the one-year risk horizon.
- w_i is the weight applicable to counterparty i, which should be weighted according to the external rating or internal according to Basel III.
- EAD_i^{total} is the exposure at default of counterparty i who is not granted the approval for IMM. The exposure is discounted using the factor: $[1 - \exp(-0.05\ M_i)]/(0.05\ M_i)$.
- B_{ind} is the full notional of one or more index CDS used to hedge the CVA risk. The risk is discounted using the factor: $[1-\exp(-0.05\ M_{ind})]/(0.05\ M_{ind})$.
- w_{ind} is the weight applicable to index hedges.
- M_i is the effective maturity of the transactions with maturity i.
- M_i^{hedge} is the maturity of the hedge instrument with notional B_i.
- M_{ind} is the maturity of the index hedge ind, which is the notional weight average maturity in case of several hedge positions.

The weights used in the formula are obtained from Table 1

Table 1 Basel III Counterparty rating and CVA weights

Rating	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
Weight (%)	0.7	0.7	0.8	1	2	3	10

Advanced Approach

In the advanced approach, the CVA capital charge is composed of the sum of a stressed and a non-stressed VaR component, and due to its property, the calibration of the expected exposure is normally done using the credit spread calibration of the worst one-year period contained in the three-year period. The CVA capital charge for the advanced approach are made using the following formula:

$$CVA = (LGD_{MKT}) \sum_{i=1}^{T} \left(\frac{EE_{t_{i-1}} * D_{t_{i-1}} + EE_{t_i} * D_{t_i}}{2} \right) * \max \left[0; \exp\left(\frac{-s_{t_{i-1}} * t_{t_{i-1}}}{LGD_{MKT}} \right) - \exp\left(\frac{-s_{t_i} * t_{t_i}}{LGD_{MKT}} \right) \right] \quad (59)$$

where

- t_i is the i^{th} revaluation time bucket.
- T is the contractual maturity with the counterparty.
- s_i is the CDS-spread of the counterparty at time t_i.
- LGD_{MKT} is the loss given default of the counterparty, based on the spread of a market instrument of the counterparty, which must be assessed instead of using an internal estimate.
- EE_i is the expected exposure to the counterparty at time t_i.
- D_i is the default risk-free discount factor at time t_i, and $D_0 = 1$.

However, as mentioned earlier we will focus on the standardized approach in this work.

4 Numerical Tests

In this section, we present numerical results from CVA calculations of OTC interest rate derivatives using the standardized approach for several counterparties. The LIBOR Market Model is calibrated as discussed in Section 2 and the CVA calculations are implemented as discussed in Section 3. This section starts with the description of the data used for numerical tests. We also describe the construction of the test portfolio (consisting of interest rate derivatives transactions with different counterparties) used in this work. Then we present results of CVA stress tests towards netting, maturities and ratings.

4.1 Data Description

The interest rate derivative contracts are structured on the 3M LIBOR reference rate at predetermined ATM strike levels. The interest rate derivative contracts are defined on pre-specified notion principal amounts with quarterly resets. All calculations are made on 30/360 day-count convention. The calculations are based on data from the 4th of September 2014 [2].

The interest yield curve is constructed from the following benchmark rates: 3M LIBOR rate, 3M FRAs (forward rate agreements) and IRS (interest rate

[2] The data are obtained from Bloomberg

swaps) exchanging fixed cash flows for floating cash flows linked to the 3M LIBOR. The benchmark rates used to construct the interest rate yield curve are shown in Table 2. 92

Table 2 Benchmark Rates on the 4th of Septmember 2014

Benchmark Instrument	**Rate (%)**
LIBOR 3M	0.23
FRA 3x6	0.23
FRA 6x9	0.33
FRA 9x12	0.52
FRA 12x15	0.76
FRA 15x18	0.99
FRA 18x21	1.23
FRA 21x24	1.50
IRS 3Y	1.17
IRS 5Y	1.81
IRS 10Y	2.53
IRS 15Y	2.87
IRS 20Y	3.03
IRS 25Y	3.10
IRS 30Y	3.14
IRS 40Y	3.15

Fig 1 shows the bootstrapped forward rates on the 4th of September 2014 over a period of 10 years. Since the bootstrapped curve is upsloping,the consensus is that the LIBOR will rise.

The other input data needed to calibrate the LMM to market data is the swaption volatility matrix that contains the Black implied volatilities with different exercise dates and underlying swap maturities. The swaption data used in this work as of the 4th of September 2014 is depicted in Table 3, where the rows represent option expiries and the columns represent the underlying swap maturities.

Table 3 Swaption Volatilities on the 4th of September 2014

	Tenor												
Expiry	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**6Y**	**7Y**	**8Y**	**9Y**	**10Y**	**12Y**	**15Y**	**20Y**
1Y	56.59	47.2	41.17	37.13	34.2	32.19	30.68	29.89	28.6	27.33	26.39	25.12	24.48
2Y	44.45	38.83	35.67	33.32	31.53	30.31	29.27	28.70	28.09	26.85	26.11	25.17	24.69
3Y	38.30	34.71	32.79	31.19	29.91	29.00	28.15	27.91	27.27	26.37	25.74	24.91	24.45
4Y	34.58	32.32	30.97	29.86	28.76	28.01	27.33	27.14	26.68	25.83	25.30	24.56	24.15
5Y	32.29	30.46	29.47	28.59	27.74	27.07	26.58	26.62	26.32	25.58	25.05	24.28	23.97
6Y	30.70	29.18	28.26	27.46	26.49	26.19	25.75	25.94	25.72	25.04	24.55	23.84	23.60
7Y	29.21	27.93	27.03	26.21	25.61	25.28	25 .00	25.34	25.18	24.55	24.10	23.43	23.25
8Y	28.13	26.69	25.77	25.47	24.91	24.66	24.45	24.92	24.8	24.13	23.73	23.13	22.99
9Y	27.01	25.29	25.12	24.69	24.33	24.15	24.00	24.58	24.49	23.77	23.43	22.87	22.76
10Y	25.38	24.79	24.44	24.07	23.88	23.75	23.63	24.33	24.26	23.47	23.18	22.66	22.57
12Y	24.61	24.04	23.68	23.40	23.30	23.21	23.15	23.95	23.99	23.26	22.99	22.50	22.49
15Y	23.52	23.06	22.80	22.50	22.44	22.46	22.55	23.52	23.67	23.00	22.75	22.31	22.39
20Y	23.02	22.67	22.39	22.22	22.26	22.44	22.67	23.97	24.16	23.32	23.10	22.68	22.94

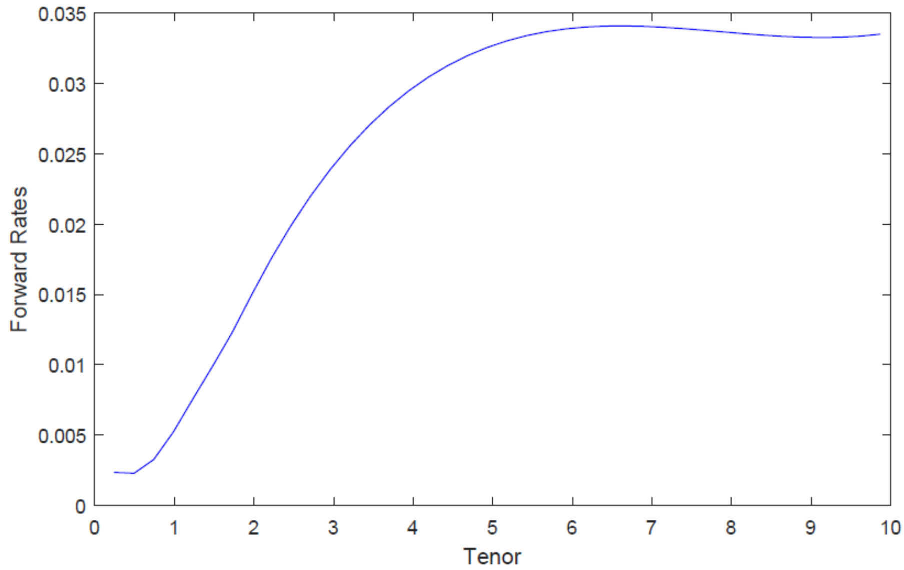

Fig. 1 Bootstrapped forward rates on 4 September 2014 using Cubic Spline Interpolation

The volatility structures are then estimated as explained in Section 2.3. The correlation structures are estimated according to the theoretical aspects explained in Section 2.4. The historical correlation matrices obtained by market data span the year before the 4th of September 2014, and are then calibrated to the Rebonato exponential form in Equation 32.

4.2 Test Portfolio

The test portfolio contains portfolios for three counterparties, denoted 1,2 and 3, with three types of interest rate derivatives transactions - caps, floors and swaptions. The test portfolio will be used to highlight some key results on CVA calculations for interest rate derivatives. To gain a better intuition, we considered single netting agreements to each counterparty with no hedge instruments.

Counterparty 1 is A rated and trades to counterparty 1 include caps and floors contracts only. Counterparty 2 is BBB rated and trades to counterparty 2 consists of swaptions only. Counterparty 3 is AA rated and trades to counterparty 3 is a mixture of caps, floors and swaptions. The notional amount for each of the contracts in the portfolios is assumed to be US$ 1 million. The portfolio is given in Appendix A. The estimations were all done under the establishment of the netting agreements and varying start and maturity dates.

4.3 Monte Carlo Exposure Results

Figure 2 summarizes results of counterparty credit exposures using the Monte Carlo approach. The exposure profile for each of the three counterparties increase from their current values and remain positive for some period during the simulation. Also, the plots in Figure 2 indicate that the exposures for the three counterparties start trending upwards after about three quarters from the settlement date. This can be attributed to the notion that the forward rates during this period are at a low level and so the instruments are mostly OTM, resulting in the observed low exposures.

After the first three quarters, the exposures for each of the three counterparties rise sharply and then decrease to zero after the peaks are reached as the different derivatives instruments mature. This is so because the uncertainty in future rates increases resulting in high exposures. After the peaks there are less instruments still to mature, hence the drop in the exposures. Of the three counterparties, Counterparty 2 has derivative instruments which mature after 10 years, hence the lingering on of exposure for some time.

From the three counterparties, the exposure to Counterparty 3 is the least. This implies that trades held against Counterparty 3 offset each other due to diversity of the instruments held in the portfolio.

4.4 CVA Stress Tests

Here, we present results of CVA stress tests towards netting agreements, maturity and ratings.

Stress Test 1 - CVA sensitivity to netting agreements

We consider exposure calculations taking into account the impact of netting. We compare the results with and without netting in Figure 3. Our simulations show marginal reduction in CVA when netting agreements are taken into consideration. This confirms with the theory which says transactions with negative values offset those with positive values. Though the reduction in CVA was marginal, we believe that for larger portfolios the reduction with netting agreements can be enormous.

Stress Test 2 - CVA sensitivity to ratings

We simulated CVAs for the three different counterparties where all parameters are fixed and only the ratings drop down by a notch. Figure 4 shows the CVA outcomes where there is drop in ratings. The CVAs for all the three counterparties increase with drop in ratings. Counterparty 2 is severely affected by drop in the ratings. This maybe due to long-dated instruments in the portfolio.

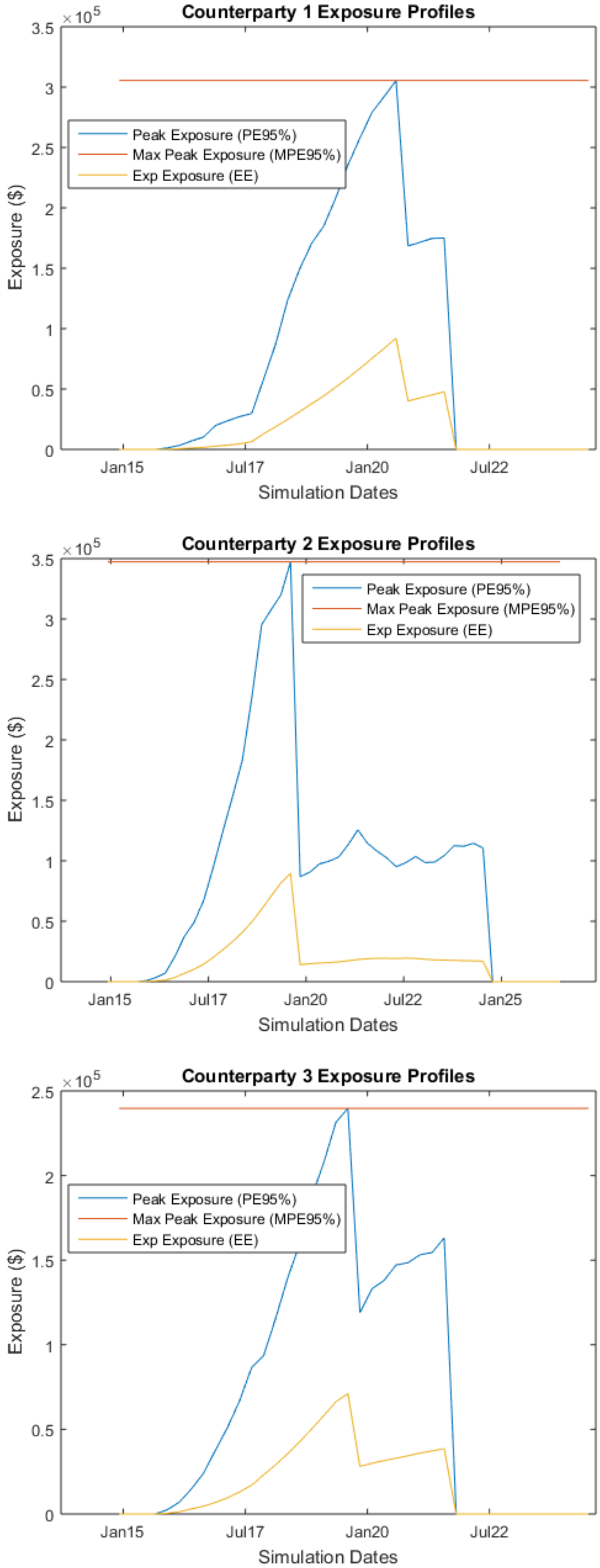

Fig. 2 Exposure profiles for Counterparty 1, 2 and 3

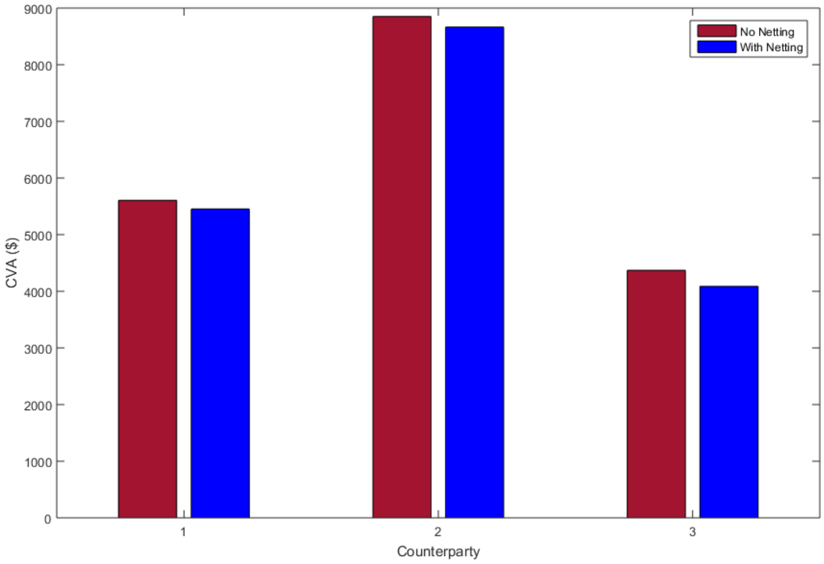

Fig. 3 Exposure Calculations With and Without Netting

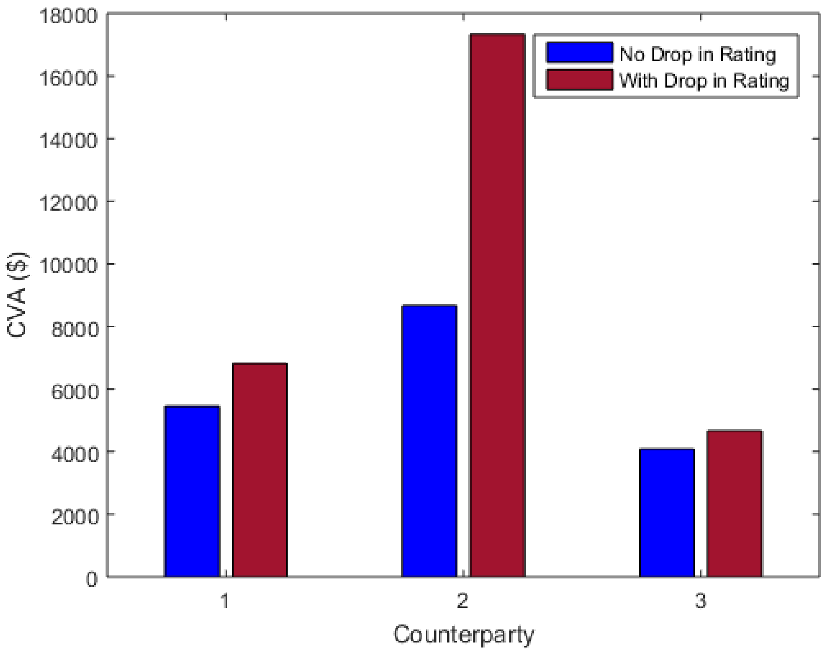

Fig. 4 CVA computed for the different counterparties with drop in ratings

Stress Test 3 - CVA sensitivity to maturity

Figure 5 shows CVA simulations for the three different counterparties with increasing time to maturity from 5 to 10 years. We can observe that the CVAs of all the three counterparties increase as time to maturity increases. Again, Counterparty 2 is the most affected with increase in time to maturity.

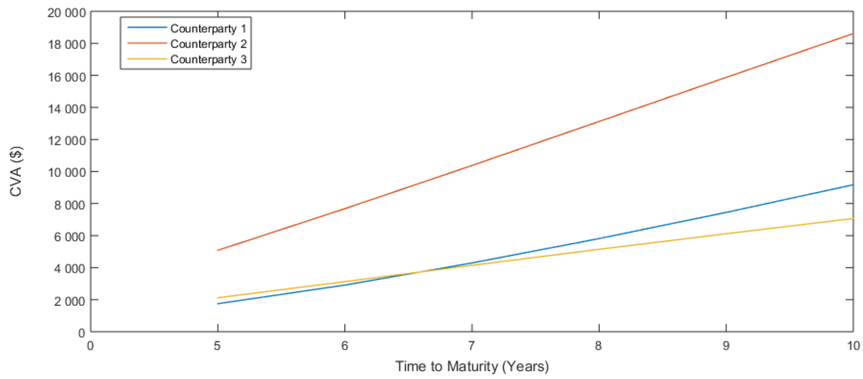

Fig. 5 CVA computed for the different counterparties with change in maturity

5 Conclusion

In this work we have looked at the dynamics of the Basel III standardized approach for OTC interest rate derivatives with the objective of understanding how the CVA levels evolve under this approach. We used the LMM, a sophisticated underlying interest rate model which gives a great flexibility with regard to interest rate derivative instruments, where we had to specify the payoff functions only. The Monte Carlo method was used to estimate the CVA exposures based on a set of market scenarios. The IMM was used to determine the exposure at default (EAD).

In the numerical work, we evaluated our CVA test portfolio to sensitivities due to netting agreements, ratings change and also change in time to maturity. We observed that the CVA reduces when netting agreements are taken into consideration. On the contrary, the CVA increases with drop in ratings and with increase in time to maturity.

From the numerical work we conclude that the more the portfolio is diversified in terms of instruments, the less the CVA cost. Also, we conclude that long-dated instruments pose great uncertainty in the portfolio such that adverse shift in market conditions will affect the magnitude of the CVA cost. Hence, the exposures due to such type of instruments should be carefully monitored.

A Test Portfolio

Table 4 Test Portfolio used to test the CVA model

Counterparty 1	**(Caps/Floors)**		**Rating: A (0.8)**					
Product	**Currency**	**Start**	**Maturity**	**Notional**	**Strike**	**Buy/Sell**	**Pay Freq**	**Basis**
Floor	USD	0.25	1	1000000	0.0038	Sell	Quarterly	30/360
Cap	USD	0.25	3	1000000	0.0127	Buy	Quarterly	30/360
Cap	USD	2	3	1000000	0.0030	Sell	Quarterly	30/360
Cap	USD	0.25	7	1000000	0.0226	Buy	Quarterly	30/360
Cap	USD	3	6	1000000	0.0172	Buy	Quarterly	30/360
Counterparty 2	**(Swaptions)**		**Rating: BBB (1.0)**					
Product	**Currency**	**Start**	**Maturity**	**Notional**	**Strike**	**Buy/Sell**	**Pay Freq**	**Basis**
Swaption	USD	1	10	1000000	0.0193	Buy	Quarterly	30/360
Swaption	USD	5	10	1000000	0.0114	Sell	Quarterly	30/360
Swaption	USD	1	5	1000000	0.0102	Buy	Quarterly	30/360
Swaption	USD	2	5	1000000	0.0103	Buy	Quarterly	30/360
Counterparty 3	**(Caps/Floors/Swaptions)**		**Rating: AA (0.7)**					
Product	**Currency**	**Start**	**Maturity**	**Notional**	**Strike**	**Buy/Sell**	**Pay Freq**	**Basis**
Floor	USD	0.25	1	1000000	0.0038	Sell	Quarterly	30/360
Cap	USD	0.25	3	1000000	0.0127	Buy	Quarterly	30/360
Swaption	USD	5	10	1000000	0.0114	Sell	Quarterly	30/360
Cap	USD	2	3	1000000	0.0030	Sell	Quarterly	30/360
Swaption	USD	1	5	1000000	0.0102	Buy	Quarterly	30/360
Cap	USD	0.25	7	1000000	0.0226	Buy	Quarterly	30/360

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6. Conclusions

The thesis has shed some light on the subtleties in arbitrage pricing under real market conditions, which make arbitrage pricing very complicated and make perfect replications very difficult and expensive to achieve. The subtleties of interest to this work were modeling of bid-ask spreads, choosing the appropriate interest rate to approximate the risk-free interest rate and modeling of counterparty credit risk. The thesis focused on derivative instruments and counterparty credit risk.

Chapter 2 explored the arbitrage pricing theory in the presence of bid-ask spreads, which were modeled using conic finance theory. The theory was used to assess the risks of equity derivatives trading strategies. It had been noted that the no-arbitrage price intervals for the equity derivative trades were unacceptably large. From a risk management point of view, prices that are acceptable to the market are required. The acceptability of the prices is assessed by risk measures. Plausible risk measures give rise to price bounds that are suitable for use as bid-ask prices. This is where conic finance theory comes in handy and provides plausible bid-ask prices. The empirical findings indicated that the bid-ask prices can compensate for the unhedgeable risk and reduce the spread between the bid-ask prices.

The theory of conic finance was extended to options on LIBOR based derivatives in Chapter 3. It is in this chapter focus on interest rate derivatives instruments - caps and floors began. Recall that arbitrage strategies require borrowing and lending money. The "risk-free interest rate" is typically used as the financing rate. In practice, using a reference interest rate as the financing rate in arbitrage pricing is the common approach. In this work, the LIBOR was used as the reference interest rate. For the OTC interest rate derivatives market, it had been realized that the market was incomplete, and the options cannot always be exactly and costlessly replicated. Also, it had been realized that the bid-ask prices were not widely available for this market. As a result, an approach using the conic finance theory was proposed to estimate the bid-ask prices for options on LIBOR based instruments. In particular, the proposed approach was assessed in the determination of premiums for caps and floors. It was concluded that the approach can be important to market dealers and market participants to make sound basis for pricing decisions based on quantitative models rather than leaving it mostly to subjectivity.

The scope of potential applications of conic finance theory in financial applications is large. In view of the above conclusions, the plan is to extend the results into asset and liability management, where we seek to assess risks of various hedging strategies in the presence of bid-ask spreads. Another direction the results of this work is intended to extend is in counterparty credit risk, where we seek to incorporate the bid-ask spreads in the counterparty credit risk calculations.

Chapter 4 looked at simulation methods that are useful to this work. In this chapter, comparison of various Monte Carlo methods were done in a prediction of stock price movements setup. The findings from the simulations work had a great bearing on the suitable simulation methods to use in subsequent work.

Chapter 5 looked at the other subtlety of modeling counterparty credit risk. In the presence of counterparty credit risk, trades are no longer riskless as impact of counterparty credit risk on arbitrage pricing and hedging strategies can be significant. In a rational and efficient market, the counterparties should be compensated for the credit risks they undertake. The compensation is embedded in the pricing of counterparty credit risk known as credit value adjustment (CVA). In the empirical analysis, the Basel III standardized approach for OTC interest rate derivatives- caps, floors and swaptions, was analyzed with the objective of understanding how the CVA levels evolve under this approach. CVA stress tests towards netting agreements, ratings and maturities were implemented using a test portfolio consisting of OTC interest rate derivatives with different counterparties.

The thesis scrapped the surface of counterparty credit risk. The next step is to analyze counterparty credit risk using the advanced approach and make comparison with results produced in this thesis. We also intend to look into aspects of collateral agreements, modeling wrong way risk (WWR), incorporating other instruments into the CVA risk engine as well as carry out further stress tests. On the methodology part, we intend to use advanced Monte Carlo methods based on the findings from one of the articles presented in this thesis.

Appendix A. Mathematical Toolbox

This appendix summarizes the mathematics that underpin the work in this thesis.

A.1 The Multivariate Normal Distribution

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ be an $n \times 1$ random vector, with mean μ and variance-covariance matrix $\mathbf{\Sigma}$. We write $\mathbf{Y} \sim \mathbf{N}(\mu, \mathbf{\Sigma})$ and say that $\mathbf{Y}$ has a multivariate normal distribution with vector μ and variance matrix $\mathbf{\Sigma}$ if:

$$f_{\mathbf{Y}} = f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2}|\mathbf{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{y} - \mu)'\mathbf{\Sigma}\mathbf{y} - \mu)\right\}, \qquad \text{(A.1.1)}$$
$$-\infty < y_i < \infty,\ i = 1, \ldots, n$$

where we assume that $\mathbf{\Sigma}$ is non-negative definite.

The result follows from an application of the Aitken integral for quadratic forms $\mathbf{y}'\mathbf{A}\mathbf{y}$ with a positive definite defining matrix $\mathbf{A}$, given below:

$$\int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\mathbf{y}'\mathbf{A}\mathbf{y}\right\} \mathrm{d}y_1 \ldots \mathrm{d}y_n = (2\pi)^{n/2}|\mathbf{A}|^{-1/2}. \qquad \text{(A.1.2)}$$

If $\mathbf{\Sigma}$ is non-singular, then there is a unique $n \times n$ matrix S which is lower triangular with $\mathbf{\Sigma} = \mathbf{SS}'$. $\mathbf{S}$ is called the Chleosky decomposition of $\mathbf{\Sigma}$. We can then write $\mathbf{Y} = \mu + \mathbf{SZ}$, where $\mathbf{Z} = (Z_1, \ldots, Z_n)'$ and the Z_i are independent and identically distributed standard normal random variables, that is, $Z_i \sim N(0, 1)$.

A.2 Brownian Motion

The Brownian motion is one of the simplest stochastic processes and is a adynamic counterpart of the Normal distribution. It was first introduced by Robert Brown to describe the movement of particles contained in the pollen grains of plants. The Brownian motion was first introduced into finance by Louis Bachelier in 1900 but was first proved mathematically by Norbert Weiner in 1923. Hence in honor of this, the Brownian motion is also known as the Weiner process.

A.2.1 Definition. A stochastic process $W = (W_t)_{t \geq 0}$ is a standard (one-dimensional) Brownian motion, W, on some probability space if:

(i) $W(0) = 0$ almost surely,

(ii) W has independent increments, that is, $W(t + \mu) - W(t)$ is independent of $\{W(s), s \leq t\}$, for $u \geq 0$,

(iii) W has stationary increments, that is, the distribution of $W(t + \mu) - W(u)$ depends only on u,

(iv) W has Gaussian increments, that is, $W(t + \mu) - W(t) \sim N(0, u)$, and

(v) W has continuous sample paths $t \longrightarrow W(t, \omega)$ for all $\omega \in \Omega$. This means that the graph of $W(t, \omega)$ as a function of t does not have any breaks in it.

For an n-dimensional process, $W(t) = (W_1(t), \ldots, W_n(t))'$, if each of the $W_i(t)$ is a standard one-dimensional Brownian motion and if the $W_i(t)$ are all independent of one another, then $W(t)$ is called a standard n-dimensional Brownian motion.

A.3 Strong Markov Property and Markov Generator

A time homogeneous diffusion process X_t subject to the stochastic differential equation:

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + b(X_t)\mathrm{d}W_t \tag{A.3.1}$$

satisfies the strong Markov property that states that:

The dynamics of X to the future of any stopping time depends on the value of X at the stopping time, but not on the values of X before the stopping time.

A.4 Theorem. ***(Strong Markov Property)***

Let f be a bounded Borel function on $\mathbb{R}^n$, τ be a stopping time with respect to $\mathcal{F}_t$, with $\tau < \infty$ a.s. Then for all $h \geq 0$:

$$\mathbb{E}_X[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[f(X_h)]. \tag{A.4.1}$$

A.4.1 Definition. The Markov generator of the process X_t is defined by:

$$\mathcal{L}f(x) = \lim_{t \longrightarrow 0^+} \frac{\mathbb{E}_X[f(X_t) - f(x)]}{t}, \tag{A.4.2}$$

for functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that the limit exists at x. For $C^{0,2}$ functions f is given by:

$$\mathcal{L}f(x) = \sum_i a_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{ij}(bb^{'})_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{A.4.3}$$

A.5 One-dimensional Itô and Diffusion Processes

Supppose that μ is adapted and locally integrable and σ is adapted and measurable so that $\int_0^t \sigma(s)\mathrm{d}W(s)$ is defined as a stochastic integral. Then:

$$X(t) = x_0 + \int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W(s), \tag{A.5.1}$$

is an Itô (stochastic) process. The above equation has a stochastic differential represenation of the form:

$$\mathrm{d}X(t) = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \quad X(0) = x_0. \tag{A.5.2}$$

Now, let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $C^{1,2}$ continuously differentiable function once in its first argument (usually time) and twice in its second argument (usually space). The following theorem summarizes the Itô's Lemma.

A.6 Theorem. ***(Itô's Lemma)***

If a stochastic process $X(t)$ has a stochastic differential of the form $dX(t) = b(t)dt + \sigma(t)dW(t)$, then $f = f(t, X(t))$ has a stochastic differential:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX(t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX(t))^2. \tag{A.6.1}$$

A.7 Multi-dimensional Diffusion Process

Consider a vector process $X = (X_1, \ldots, X_n)^T$, where the component X_i has a stochastic differential equation:

$$\mathrm{d}X_i(t) = \mu_i(t)\mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij}(t)\mathrm{d}W_j(t), \tag{A.7.1}$$

and $W_1, \ldots, W_d$ are independent Weiner processes. The drift vector μ is defined by:

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}, \tag{A.7.2}$$

the d-dimensional vector Weiner process W is defined by:

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_d \end{bmatrix}, \tag{A.7.3}$$

and the $n \times d$-dimensional diffusion matrix σ by:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma 1d \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma 2d \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nd} \end{bmatrix}. \tag{A.7.4}$$

The dynamics maybe written as:

$$\mathrm{d}X(t) = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t). \tag{A.7.5}$$

Define the process Z by:

$$Z(t) = f(t, X(t)), \tag{A.7.6}$$

where $f : \mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a $C^{1,2}$ mapping. The following theorem summarizes the Itô's Lemma in multidimensional form.

A.8 Theorem. *Let the n-dimensional process X have dynamics given by:*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t). \tag{A.8.1}$$

The process $f(t, X(t))$ has a stochastic differential equation given by:

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_i dX_j. \tag{A.8.2}$$

A.9 Feynman-Kac Theorem

The Feynman-Kac Theorem relates stochastic differential equations and partial differential equations. When the partial differential equation (usually numerically), it gives a solution of a derivative security price. Now, consider the n-dimensional stochastic differential equation:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t. \tag{A.9.1}$$

Through the Itô's formula, the process above is closely conncted to a partial differential operator $\mathcal{A}$, defined below. The operator $\mathcal{A}$ is also known as the inifinitesmal operator.

A.9.1 Definition. Given the stochastic differential equation in equation A.9.1, the partial differential operator $\mathcal{A}$, is defined for any function $h(x)$ with $h \in C^2(\mathbb{R}^n)$, by:

$$\mathcal{A}h(t, x) = \sum_{i=1}^{n} \mu_i(t, x_i) \frac{\partial h}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij}(t, x) \frac{\partial^2 h}{\partial x_i \partial x_j}(x). \tag{A.9.2}$$

This operator is also known as the Dynkin operator, the Itô generator, or the Kolmogorov backward operator. In terms of the infinitesmal generator, the Itô formula takes the form:

$$\mathrm{d}f(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \mathcal{A}f \right\} \mathrm{d}t + [\nabla_x f]\sigma \mathrm{d}W_t, \tag{A.9.3}$$

where the gradient ∇_x is defined for $h \in C'(\mathbb{R}^n)$ as:

$$\nabla_x h = \left[\frac{\partial h}{\partial x_1}, \cdots\cdots, \frac{\partial h}{\partial x_n} \right]. \tag{A.9.4}$$

Suppose we are given the following boundary value equation on $[0, T] \times \mathbb{R}$:

$$\frac{\partial F}{\partial t}(t, x) + \mu(t, x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 F}{\partial x^2}(t, x) = 0, \tag{A.9.5}$$

$$F(T, x) = \Phi(x), \tag{A.9.6}$$

with scalar functions $\mu(t, x)$, $\sigma(t, x)$ and $\Phi(x)$. The task is to find a function $F(x)$ which satisfies the above boundary value problem. So, assume that there actually exists a solution F to equations A.9.1-A.9.6. Then fix a point in time and a point in space x. Having fixed these, we define the stochastic process X on the time interval $[t, T]$ as the solution to the stochastic differential equation:

$$\mathrm{d}X_s = \mu(s, X_s)\mathrm{d}t + \sigma(s, X_s)\mathrm{d}W_s, \tag{A.9.7}$$

$$X_t = x. \tag{A.9.8}$$

The infinitesmal generator $\mathcal{A}$ for this process is given by:

$$\mathcal{A} = \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{A.9.9}$$

Thus, the boundary value problem may be written as:

$$\frac{\partial F}{\partial t}(t,x) + \mathcal{A}F(t,x) = 0, \tag{A.9.10}$$

$$F(T,x) = \Phi(x). \tag{A.9.11}$$

Applying the Itô's formula to the process $F(s, X(s))$ gives:

$$\begin{aligned} F(T, X_T) = F(t, X_t) + \int_t^T \left\{ \frac{\partial F}{\partial t}(s, X_s) + \mathcal{A}F(s, X_s) \right\} \mathrm{d}s \\ \int_t^T \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)\mathrm{d}W_s. \end{aligned} \tag{A.9.12}$$

By assumption, F satisifies A.9.10, he time integral will vanish. Furthermore, the process $\sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)$ is sufficiently integrable. Taking expected values, the stochastic integral will also vanish. The initial value $X_t = x$ and the boundary condition $F(T, x) = \Phi$ will also vanish and we are left with the formula:

$$F(t, x) = E_{t,x}[\Phi(X_T)]. \tag{A.9.13}$$

The above result is summarized in the following proposition.

A.9.2 Proposition. Feyman-Kac

Assume that F is a solution to the boundary value problem:

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) + \mu(t, x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 F}{\partial x^2}(t, x) = 0,& \\ F(T, x) = \Phi(x).& \end{aligned} \tag{A.9.14}$$

Assume that the process:

$$\sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s), \tag{A.9.15}$$

is in $\mathcal{L}^2$, where X satisfies the stochastic differential equation:

$$\mathrm{d}X_s = \mu(s, X_s)\mathrm{d}t + \sigma(s, X_s)\mathrm{d}W_s, X_t \qquad = x.$$

Then F has the representation:

$$F(t,x) = E_{t,x}[\Phi(X_T)]. \tag{A.9.16}$$

A.9.3 Proposition. Assume that F is a solution to the boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0,$$
$$F(T,x) = \Phi(x). \tag{A.9.17}$$

Assume that the process:

$$e^{-rs}\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s), \tag{A.9.18}$$

is in $\mathcal{L}^2$, where X satisfies the stochastic differential equation:

$$\mathrm{d}X_s = \mu(s,X_s)\mathrm{d}t + \sigma(s,X_s)\mathrm{d}W_s, X_t \qquad = x.$$

Then F has the representation:

$$F(t,x) = e^{-r(T-t)}E_{t,x}[\Phi(X_T)]. \tag{A.9.19}$$

A.10 Girsanov Theorem

The Girsanov Theorem allows us to change the drift of an Itô diffusion by switching to an equivalent martingale measure $\mathbb{Q}$ from the original $\mathbb{P}$.

Let $0 < T < \infty$ and suppose that $W(t,\omega) : [0,T] \times \Omega \longrightarrow \mathbb{R}^n$ is an n-dimensional Brownian motion under $\mathbb{P}$. Let $\gamma(t,\omega) = (\gamma_1(t,\omega), \ldots, \gamma_n(t,\omega))'$, where the $\gamma_i(t,\omega)$ are previsible diffusions satisfying the Novikov condition:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T |\gamma(t,\omega)|^2 \mathrm{d}t\right)\right] < \infty. \tag{A.10.1}$$

Define:

$$Z(t,\omega) = \exp\left[-\int_0^t \gamma(u,\omega)'\mathrm{d}W(u,\omega) - \frac{1}{2}\int_0^T |\gamma(u,\omega)|^2 \mathrm{d}u\right] \tag{A.10.2}$$

Given that $\gamma(t,\omega)$ satisfies the Novikov condition, $Z(t,\omega)$ is a martingale under $\mathbb{P}$ for $0 \leq t \leq T$.

A.11 Theorem. ***Girsanov Theorem***

(a) Suppose that $W(t,\omega)$ is a standard n-dimensional Brownian motion under P and $\gamma(t,\omega)$ is an n-dimensional diffusion that satisfies the Novikov condition. Then there exists a measure $\mathbb{Q}$ such that:

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$,

(ii) $\overline{W}(t,\omega) = W(t,\omega) + \int_0^t \gamma(u,\omega)du$ is a standard n-dimensional Brownian motion under $\mathbb{Q}$.

(b) If $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $[0,T]\times\Omega$, then there exists an n-dimensional previsible diffusion $\gamma(t,\omega)$ such that $\overline{W}(t,\omega) = W(t,\omega) + \int_0^t \gamma(u,\omega)du$ is a Brownian motion under $\mathbb{Q}$.

A.11.1 Definition. Radon-Nikodym Derivative

The Radon-Nikodym derivative measures the relative likelihood for a given point ω under the measures $\mathbb{P}$ and $\mathbb{Q}$ over the interval $[0,T]$ and is defined as:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}}(\omega) = Z(T,\omega) \text{ or } \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}}(\omega) = Z(T) \tag{A.11.1}$$

The Radon-Nikodym derivative is applied as follows:

Suppose X is any $\mathcal{F}_t$-measurable random variable. Then:

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\right] = \mathbb{E}^{\mathbb{P}}[Z(T)X]. \tag{A.11.2}$$

Furthermore, if X is $\mathcal{F}_t$-measurable for some $t < T$, then:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[X] &= \mathbb{E}^{\mathbb{P}}[Z(T)X] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}\{Z(T)X|\mathcal{F}_t\}] \\ &= \mathbb{E}^{\mathbb{P}}[X\mathbb{E}^{\mathbb{P}}\{Z(T)|\mathcal{F}_t\}] = \mathbb{E}^{\mathbb{P}}[XZ(t)]. \end{aligned} \tag{A.11.3}$$

A.12 Change of Numeraire

A.12.1 Definition. A numeraire N_t is any $\mathbb{P}$ a.s. strictly positive traded asset. The price of an asset normalized by the numeraire N has the form X_t/N_t. The standard example of a numeraire is the money market B_t, in terms of which normalization is discounting.

The concepts of arbitrage, replicability and completeness are all expressed in terms of self-financing trading strategies, and the invariance lemma shows that the self-financing condition is invariant with respect to a change of numeraire.

A.12.2 Lemma. Invariance Lemma

Let N_t be any $\mathbb{P}$ a.s. strictly postive process. A trading strategy is self-financing in terms of the traded assets with prices $S_t = (B_t, S_t^1, \ldots, S_t^d)$ if and only if it is self-financing in terms of the normalized asset prices S_t/N_t.

A.13 Arbitrage Pricing Theory

Consider a finite time horizon T and an economy consisting of $d+1$ non-dividend paying traded securities whosee prices are modelled by the $d+1$ dimensional adapted semi-martingales $S_t = (S_t^0 = B_t, S_t^1, \ldots, S_t^d)$.

A.13.1 Definition. Trading Strategies

(i) A trading strategy or portfolio is represented by a vector $\theta = \{\theta_t : t = 1, 2, \ldots, T\}$, which describes the number of securities off each type held. The trading strategy or portfolio should be a predictable vector-valued stochastic process, that is, for each $t < T$, θ_{t+1} should be $\mathcal{F}_t$-measurable. With each trading strategy or portfolio, we can associate a value process $V(\theta)$ and a gains process G.

(ii) The value process $V(\theta) = (V_t)_{t\in\mathbb{T}}$ is a stochastic process which gives the value of the investor's portfolio θ at the different trading times. Thus:

$$V_0(\theta) = \theta_1.S_0 = \sum_{n=0}^{N} \theta_1^n S_0^n,$$
$$V_t(\theta) = \theta_t.S_t = \sum_{n=0}^{N} \theta_t^n S_t^n \tag{A.13.1}$$

The value $V_0(\theta)$ is the investor's initial endowment. The investors selct their time t portfolio once the stock prices at time $t-1$ are konw, and they hold this portfolio during the time interval $(t-1, t]$.

(iii) A portfolio is said to be self-financing if:

$$\theta_t.S_t = \theta_{t+1}.S_t \text{ i.e. } \sum_{n=0}^{N} \theta_t^n S_t^n = \sum_{n=0}^{N} \theta_{t+1}^n S_t^n, \tag{A.13.2}$$

for all $t \in \mathbb{T}$, $1 \leq t \leq T-1$.

A portfolio is self-financing if no money is added or taken away from the portfolio between times $t = 0$ and $t = T$. Thus it has the same value just before it is restructured and just after since no value is added or taken away from the portfolio at any trading date. Since share prices will change from time $t-1$ to t,so will the value of the portfolio. Thus the change in value is:

$$V_t - V_{t-1} = \sum_{n=0}^{N} \theta_t^n (S_t^n - S_{t-1}^n)$$
$$= \sum_{n=0}^{N} \theta_t^n \Delta S_t^n, \quad \text{(A.13.3)}$$

where $\Delta S_t^n = S_t^n - S_{t-1}^n$. The total (cumulative) gain $G_t(\theta)$ from time 0 to time t is:

$$G_t = V_t - V_0 = \sum_{u=1}^{t} (V_u - V_{u-1}) \quad \text{(A.13.4)}$$
$$= \sum_{u=1}^{t} \sum_{n=0}^{N} \theta_u^n \Delta S_u^n \quad \text{(A.13.5)}$$
$$= \sum_{u=1}^{t} \theta_u . \Delta S_u. \quad \text{(A.13.6)}$$

Alternatively, we again define the gains process associated with θ by setting:

$$G_0(\theta) = 0, G_t(\theta) = \theta_1.\Delta S_1 + \theta_2.\Delta S_2 + \ldots + \theta_t.\Delta S_t. \quad \text{(A.13.7)}$$

We can realize that θ is self-financing if and only if:

$$V_t(\theta) = V_0(\theta) + G_t(\theta) \text{ for all } t \in \mathbb{T}. \quad \text{(A.13.8)}$$

(iv) Let $\beta_t = \frac{1}{S_t^0}$ be the discount factor. We define the discounted price process $\overline{S}_t^n$ by:

$$\overline{S}_t^n = \beta_t S_t^n \;\; \Delta\overline{S}_t^n = \overline{S}_t^n - \overline{S}_{t-1}^n, \tag{A.13.9}$$

and the discounted value and gains processes as follows:

$$\overline{V}_t = \beta_t V_t \; \overline{G} = \sum_{u=1}^{t} \theta_u . \Delta\overline{S}_u = \sum_{u=1}^{t} \sum_{n=0}^{N} \theta_u^n \Delta\overline{S}_u^n. \tag{A.13.10}$$

A trading strategy θ is self-financing if and only if:

$$\overline{V}_t = \overline{V}_0 + \overline{G}_t. \tag{A.13.11}$$

A.13.2 Definition. Admissible Strategies

Let Θ be the class of all self-financing strategies. If we impose the restriction $V_t(\theta) \geq 0$ for all $t \in \mathbb{T}$, then we call such self-financing strategies admissible.

(i) A self-financing trading strategy θ is called an arbitrage opportunity or arbitrage strategy if and only if:

(a) $V_0(\theta) = 0$,

(b) $V_T(\theta) = 0$, for all $t \in \mathbb{T}$, and

(c) $\mathbb{E}[V_T(\theta) > 0]$.

Thus an arbitrage strategy is a trading strategy with zero initial cost, no chance of making a loss, and some (non-zero) chance of making a profit.

A.13.3 Proposition. There is an arbitrage strategy if and only if there is trading strategy θ with:

(a) $\overline{G}(\theta) \geq 0$,

(b) $\mathbb{E}[G_T(\theta)] > 0$.

(ii) An admissible strategy is a self-financing trading strategy θ with the property that $V_t(\theta) \geq 0$ for all $t \in \mathbb{T}$.

A.13.4 Definition. Arbitrage Pricing

(i) A contingent claim is a random variable that represents a time T payoff. The payoff depends on the state of the world, first observed at time T.

(ii) A contingent claim is said to be attainable if and only if there is a self-financing trading strategy θ such that:

$$X = V_T(\theta). \tag{A.13.12}$$

The random variables X and $V_T(\theta)$ have the same value in all states of the world and θ is called a replicating portfolio for X.

A.13.5 Proposition. Suppose the market is arbitrage-free. If θ, φ are trading strategies with $V_T(\theta) = V_T(\varphi)$, then $V_t(\theta) = V_t(\varphi)$ for all trading times t.

So in the absence of arbitrage, any two portfolios must have the same value at all times.

(iii) A market $\mathcal{M}$ is said to be arbitrage-free if no admissible arbitrage portfolios exist in $\mathcal{M}$.

A.13.6 Definition. Equivalent Martingale Measure (EMM)

(i) A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure if and only if:

(a) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, that is, $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$ for all $A \in \mathcal{F}$.

(b) Each discounted price process $\overline{S}^n$ is a martingale with respect to $\mathbb{Q}$, that is:

$$\mathbb{E}[\overline{S}^n_{t+1}|\mathcal{F}_t] = \overline{S}^n_t, \quad t = \overline{0, T-1}, \quad n = \overline{0, N}. \tag{A.13.13}$$

(ii) If $\mathbb{Q}$ is an EMM, and θ is a self-financing trading strategy, then the discounted value and gains process, $\overline{V}_t(\theta)$ and $\overline{G}_t(\theta)$, are also $\mathbb{Q}$ martingales.

A.14 Theorem. ***Absence of Aribtrage***

If an equivalent martingale measure exists for the market $\mathcal{M}$, then $\mathcal{M}$ is arbitrage-free.

A.14.1 Definition. Risk-Neutral Pricing and Complete Markets

A contingent claim X is said to be attainable (or marketable, or replicable) if and only if there is a replicating portfolio θ with the property $V_T(\theta) = \theta$. We now tackle the problem of calculating the $t = 0$ value of an attainable contingent claim X without knowing the replicating portfolio.

(i) In an arbitrage-free market, the $t = 0$ price of an attainable contingent claim X is its discounted expected value under an EMM $\mathbb{Q}$:

$$X_0 = \mathbb{E}^{\mathbb{Q}}[\overline{X}], \tag{A.14.1}$$

where $\overline{X} = \frac{X}{S^0_T}$ is the discounted value of X.

A.15 Theorem. ***First Fundamental Theorem of Asset Pricing***

If there exists an equivalent martingale measure $\mathbb{Q}$ for the market $\mathcal{M}$ and if such a measure $\mathbb{Q}$ is unique, then every claim X is attainable in the market $\mathcal{M}$.

(ii) A market $\mathcal{M}$ is said to be complete if every contingent claim is attainable. The Second Fundamental Theorem of Asset Pricing connects the market completeness to the martingale measure.

A.16 Theorem. ***Second Fundamental Theorem of Asset Pricing***

A market is complete if and only if there exists a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.