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A new method for solving a coagulation-fragmentation equation

R. Guiem and S.C. Oukouomi Noutchie*

Abstract. In this paper, we study the well-posedness of a nonlinear non-autonomous integro-differential equation modeling coagulation-fragmentation processes. We construct two monotone sequences of upper and lower solutions, and establish their convergence to a unique solution thanks to the comparison principle.

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1. Introduction

Coagulation-fragmentation processes arise in applied sciences. In recent years, the solvability of such problems have been widely investigated in the literature (see [5-14]). The method using the theory semigroup of linear operator is extensively used in the literature. In [11] the author investigated the result of a Coagulation-fragmentation equation by using the substochastic semigroup theory and semilinear abstract Cauchy problem. He analyzed the space of solution of the problem

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= -a(x)u(t,x)+\int_{x+x_0}^{\infty}a(y)b(x|y)u(t,y)dy\\ &\quad+\frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0}k(x-y,y)u(t,x-y)u(t,y)dy, \qquad (1)\\ &\quad-u(t,x)\int_{x_0}^{\infty}k(x,y)u(t,y)dy,\end{aligned}$$

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*Corresponding author

and extended it to the space $X_1 = L_1([x_0, \infty), \, xdx)$. This has been done in the past only in the space $X_{0,1} = L_1([x_0, \infty), \, (1+x)dx)$ since the fragmentation equation is known to behave well in the space X_1 and the coagulation operator in the space $X_0 = L_1([x_0, \infty), \, dx)$.

In this paper, we consider a similar equation and additionally, we suppose that the fragmentation rate a is a function of mass and time. This assumption is realistic in the sense that, in certain situations observed in phytoplankton population with aggregates, the fragmentation rate evolved over time. Moreover, we present a constructive method used by Ackleh in ([1-4]). This technique allowed us to get a result similar to the resolution obtained in [11].

The model considered here is as follows:

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= -a(t,x)u(t,x) + \int_{x+x_0}^{\infty} a(t,y)b(x|y)u(t,y)dy \\ &\quad + \frac{\chi_U(x)}{2} \int_{x_0}^{x-x_0} k(x-y,y)u(t,x-y)u(t,y)dy, \qquad (2) \\ &\quad - u(t,x) \int_{x_0}^{\infty} k(x,y)u(t,y)dy, \quad (t,x) \in (0,T) \times (x_0, \infty) \\ u(0,x) &= u_0(x), \quad x \in [x_0, \infty) \end{aligned}$$

where χ_U is the characteristic function of the interval $U = [2x_0, \infty)$. In this equation u is the particle mass distribution function, a is the fragmentation rate and $b(x|y)$ is the distribution of particle masses x spawned by the fragmentation of a particle of mass y. The coagulation kernel $k(x,y)$ is the rate at which particles of mass x coalesce with particles of mass y. The characteristic function χ_U ensures no particle of mass $x < 2x_0$ can emerge as a result of coagulation.

The terms on the right side of (2) describe, from left to right, the reduction in the number of particles in the mass range $(x; x + dx)$ due to the fragmentation of particles in the same range, the increase in the number of particles in the range due to fragmentation of larger particles, the increase in the number of particles of mass $x \geq 2x_0$ as the result of particles of mass $x - y$ and mass $y \quad (x_0 \leq y \leq x - x_0)$ merging to form a particle of mass x and the last term accounts for the loss of particles of mass x because they have coalesced with particles of mass $y, \quad y \geq x_0$. Note that the factor $1/2$ takes into account that either a particle of mass $x - y$ coalesces with one of mass y or vice versa. To the best of our knowledge, comparison principle and approximation method based on monotone sequences have not been developed for the coagulation- fragmentation model (2). Therefore, in this paper we undertake such a task to establish the existence and uniqueness of the solution to (2).

For simplicity, let $D_T = (0, \infty) \times (x_0, \infty)$ and $C^1_{0,r}(D_T) = \{\psi \in C^1(D_T) : \exists x_\psi \in (x_0, \infty)$ such that $\psi \equiv 0$ for $x \geq x_\psi\}$. We begin by defining what we mean by a solution to (2).

Definition 1.1. $u(t, x)$ is called a solution of (2) on D_T if all the following hold:

(i) $u \in X_1 = L^1(D_T, xdx)$.

(ii) $u(x, 0) = u_0(x)$ a.e. in (x_0, ∞).

(iii) For every $t \in (0, T)$ and every non-negative $\xi(t, x) \in C^1_{0,r}(D_T)$,

$$\begin{aligned}
\int_{x_0}^{\infty} xu(t,x)\xi(t,x)\,dx &= \int_{x_0}^{\infty} xu(0,x)\xi(0,x)dx \\
&+ \int_{x_0}^{\infty}\int_0^t xu(s,x)\xi_s(s,x)\,ds\,dx \\
&+ \int_0^t \int_{2x_0}^{\infty} a(s,y)u(s,y)\int_{x_0}^{y-x_0} x\xi(s,x)b(x|y)\,dx\,dy\,ds \qquad (3) \\
&+ \int_{x_0}^{\infty}\int_0^t x\xi(s,x)(\mathcal{F}u)(s,x)\,ds\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{x_0}^{\infty}\int_0^t x\xi(s,x)u(s,x)\int_{x_0}^{\infty} k(x,y)u(s,y)\,dy\,ds\,dx\\
&-\int_{x_0}^{\infty}\int_0^t x\xi(s,x)a(s,x)u(s,x)\,ds\,dx
\end{aligned}
$$

where

$$
(\mathcal{F}u)(t,x)=\frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)u(t,x-y)u(t,y)\,dy. \tag{4}
$$

We organize this paper as follows. In Section 2, we give the definition of upper and lower solutions and establish a comparison principle. In Section 3, we construct two monotone sequences of upper and lower solutions and show their convergence to the unique local solution of (2).

2. Comparison principle

In order to carry out our analysis, the following assumptions will be imposed throughout the paper:

(H1) $a(t,x)\in L^{\infty}([0,T]\times(x_0,\infty))$. Particles of sizes less than $2x_0$ do not fragment since the minimum size of a particle is x_0. Therefore we assume that $a(t,x)=0$ for $x<2x_0$.

(H2) $b(x|y)$ is a continuous, non-negative function on $(x_0,\infty)\times(x_0,\infty)$ with $\|b\|_{\infty}<\infty$; $b(x|y)=0$ for $y<x+x_0$ and $\int_{x_0}^{y-x_0} xb(x|y)dx=y$ for each $y>2x_0$.

(H3) $k(x,y)\geq 0$ in $L^{\infty}((x_0,\infty)\times(x_0,\infty))$.

(H4) $u_0(x)\geq 0$ on $[x_0,\infty)$ and $u_0(x)\in L^1((x_0,\infty))\cap L^{\infty}((x_0,\infty))$.

We then introduce the definition of coupled upper and lower solutions of problem (2) as follow.

Definition 2.1. A pair of functions $\overline{u}(t,x)$ and $\underline{u}(t,x)$ are called an upper and a lower solution of (2) on D_T, respectively, if all of the following hold.

(i) $\overline{u}, \underline{u} \in X_1 = L^1(D_T, xdx)$.

(ii) $\overline{u}(0,x) \geq u_0(x) \geq \underline{u}(0,x)$ a.e. in (x_0, ∞).

(iii) For every $t \in (0,T)$ and every non-negative $\xi(t,x) \in C^1_{0,r}(D_T)$,

$$\begin{aligned}\int_{x_0}^{\infty} x\overline{u}(t,x)\xi(t,x)\,dx \geq &\int_{x_0}^{\infty} x\overline{u}(0,x)\xi(0,x)dx \\ &+\int_{x_0}^{\infty}\int_0^t x\overline{u}(s,x)\xi_s(s,x)\,ds\,dx \\ &+\int_0^t\int_{2x_0}^{\infty} a(s,y)\overline{u}(s,y)\int_{x_0}^{y-x_0} x\xi(s,x)b(x|y)\,dx\,dy\,ds \\ &+\int_{x_0}^{\infty}\int_0^t x\xi(s,x)(\mathcal{F}\overline{u})(s,x)\,ds\,dx \\ &-\int_{x_0}^{\infty}\int_0^t x\xi(s,x)\overline{u}(s,x)\int_{x_0}^{\infty} k(x,y)\underline{u}(s,y)\,dy\,ds\,dx \\ &-\int_{x_0}^{\infty}\int_0^t x\xi(s,x)a(s,x)\overline{u}(s,x)\,ds\,dx,\end{aligned} \tag{5}$$

$$\begin{aligned}\int_{x_0}^{\infty} x\underline{u}(t,x)\xi(t,x)\,dx \leq &\int_{x_0}^{\infty} x\underline{u}(0,x)\xi(0,x)dx \\ &+\int_{x_0}^{\infty}\int_0^t x\underline{u}(s,x)\xi_s(s,x)\,ds\,dx \\ &+\int_0^t\int_{2x_0}^{\infty} a(s,y)\underline{u}(s,y)\int_{x_0}^{y-x_0} x\xi(s,x)b(x|y)\,dx\,dy\,ds \\ &+\int_{x_0}^{\infty}\int_0^t x\xi(s,x)(\mathcal{F}\underline{u})(s,x)\,ds\,dx \\ &-\int_{x_0}^{\infty}\int_0^t x\xi(s,x)\underline{u}(s,x)\int_{x_0}^{\infty} k(x,y)\overline{u}(s,y)\,dy\,ds\,dx \\ &-\int_{x_0}^{\infty}\int_0^t x\xi(s,x)a(s,x)\underline{u}(s,x)\,ds\,dx.\end{aligned} \tag{6}$$

Based on such a definition, we now establish the following comparison principle.

Theorem 2.2. *Suppose that* $(H1)-(H4)$ *hold. Let* $\overline{u}$ *and* $\underline{u}$ *be an upper solution and a lower solution of (2), respectively. Then* $\overline{u} \geq \underline{u}$ *a.e. in* D_T.

Proof. Let $v = \underline{u} - \overline{u}$ and choose $\xi \in C^1_{0,r}((0,T)\times(x_0,n))$, where $C^1_{0,r}((0,T)\times(x_0,n)) = \{\psi \in C^1((0,T)\times(x_0,n)) : \exists x_\psi \in (x_0,n)$ such that $\psi \equiv 0$ for $x \geq x_\psi\}$. Then w satisfies

$$v(0,x) = \underline{u}(0,x) - \overline{u}(0,x) \leq 0 \quad \text{a.e. in } [x_0,\infty) \tag{7}$$

and

$$\begin{aligned}
\int_{x_0}^{\infty} xv(t,x)\xi(t,x)\,dx &\leq \int_{x_0}^{\infty} xv(0,x)\xi(0,x)dx \\
&+ \int_{x_0}^{\infty}\int_0^t xv(s,x)\xi_s(s,x)\,ds\,dx \\
&+ \int_0^t \int_{2x_0}^{\infty} a(s,y)v(s,y)\int_{x_0}^{y-x_0} x\xi(s,x)b(x|y)\,dx\,dy\,ds \\
&+ \int_{x_0}^{\infty}\int_0^t x\xi(s,x)\left[(\mathcal{F}\underline{u})(s,x) - (\mathcal{F}\overline{u})(s,x)\right]\,ds\,dx \\
&+ \int_{x_0}^{\infty}\int_0^t x\xi(s,x)\overline{u}(s,x)\int_{x_0}^{\infty} k(x,y)v(s,y)\,dy\,ds\,dx \\
&- \int_{x_0}^{\infty}\int_0^t x\xi(s,x)v(s,x)\int_{x_0}^{\infty} k(x,y)\overline{u}(s,y)\,dy\,ds\,dx \\
&- \int_{x_0}^{\infty}\int_0^t x\xi(s,x)a(s,x)v(s,x)\,ds\,dx.
\end{aligned} \tag{8}$$

Upon manipulation, we have

$$\begin{aligned}
& \int_{x_0}^{\infty}\int_0^t x\xi(s,x)\left[(\mathcal{F}\underline{u})(s,x) - (\mathcal{F}\overline{u})(s,x)\right]\,ds\,dx \\
= \; & \frac{1}{2}\int_{x_0}^{\infty}\int_0^t x\xi(s,x)\chi_{U}(x)\int_{x_0}^{x-x_0} k(x-y,y)\left[\underline{u}(s,x-y)v(s,y)\right. \\
& \left. + v(s,x-y)\overline{u}(s,y)\right]\,dy\,ds\,dx \\
= \; & \frac{1}{2}\int_0^t\int_{x_0}^{\infty} v(s,y)\int_{y+x_0}^{\infty} x\xi(s,x)\chi_{U}(x)k(x-y,y) \\
& \underline{u}(s,x-y)\,dx\,dy\,ds + \frac{1}{2}\int_0^t\int_{x_0}^{\infty}\overline{u}(s,y)\int_{y+x_0}^{\infty} x\xi(s,x)\chi_{U}(x) \\
& k(x-y,y)v(s,x-y)\,dx\,dy\,ds \\
= \; & \frac{1}{2}\int_0^t\int_{x_0}^{\infty} v(s,y)\int_{x_0}^{\infty}(y+z)\xi(s,y+z)\chi_{U}(y+z)
\end{aligned} \tag{9}$$

$$k(z,y)\underline{u}(s,z)\,dz\,dy\,ds + \frac{1}{2}\int_0^t \int_{x_0}^\infty \overline{u}(s,y)\int_{x_0}^\infty (y+z)\xi(s,y+z)$$
$$\chi_{_U}(y+z)k(z,y)v(s,z)\,dz\,dy\,ds.$$

Let $\xi(t,x) = e^{\lambda t}\zeta(t,x)$, where $\zeta \in C^1_{0,r}((0,T)\times(x_0,n))$ and $\lambda(>0)$ is chosen so that $\lambda - a(t,x) - \displaystyle\int_{x_0}^\infty k(x,y)\overline{u}(s,y)\,dy \geq 0$ on D_T. Then we find

$$\begin{aligned}
&e^{\lambda t}\int_{x_0}^\infty xv(t,x)\zeta(t,x)\,dx\\
&\leq \int_{x_0}^\infty xv(0,x)\zeta(0,x)dx + \int_{x_0}^\infty \int_0^t xv(s,x)e^{\lambda s}\zeta_s(s,x)\,ds\,dx\\
&+\int_0^t \int_{2x_0}^\infty a(s,y)v(s,y)\int_{x_0}^{y-x_0} xe^{\lambda s}\zeta(s,x)b(x|y)\,dx\,dy\,ds\\
&+\int_{x_0}^\infty \int_0^t xv(s,x)e^{\lambda s}\zeta(s,x)\bigg(\lambda - a(s,x) - \int_{x_0}^\infty k(x,y)\\
&\overline{u}(s,y)\,dy\bigg)dsdx \qquad\qquad (10)\\
&+\frac{1}{2}\int_0^t \int_{x_0}^\infty v(s,y)\int_{x_0}^\infty (y+z)e^{\lambda s}\zeta(s,y+z)\chi_{_U}(y+z)\\
&k(z,y)\underline{u}(s,z)\,dz\,dy\,ds\\
&+\frac{1}{2}\int_0^t \int_{x_0}^\infty \overline{u}(s,y)\int_{x_0}^\infty (y+z)e^{\lambda s}\zeta(s,y+z)\chi_{_U}(y+z)\\
&k(z,y)v(s,z)\,dz\,dy\,ds\\
&+\int_{x_0}^\infty \int_0^t xe^{\lambda s}\zeta(s,x)\overline{u}(s,x)\int_{x_0}^\infty k(x,y)v(s,y)\,dy\,ds\,dx.
\end{aligned}$$

We now set up a backward problem as follows:

$$\begin{aligned}
\zeta_s(s,x) &= 0, \qquad 0<s<t, \quad x_0<x<n\\
\zeta(s,n) &= 0, \qquad 0<s<t \qquad\qquad (11)\\
\zeta(t,x) &= \chi^1(x), \quad x_0 \leq x \leq n.
\end{aligned}$$

Here $\chi^1 \in C_0^\infty((x_0,n))$, $0 \leq \chi^1 \leq 1$.

The existence of $\zeta \in C^1_{0,r}((0,T)\times(x_0,n))$ follows from the fact that by the variable change $\tau = t - s$, the above problem (11) can be written into

$$\begin{array}{rcl}\zeta_\tau(\tau,x) & = & 0, \qquad 0<s<t, \quad x_0<x<n \\ \zeta(\tau,n) & = & 0, \qquad 0<s<t \\ \zeta(0,x) & = & \chi^1(x), \quad x_0 \le x \le n.\end{array} \tag{12}$$

Note that the initial and boundary values for ζ imply that $0 \le \zeta \le 1$ on $(0,T)\times(x_0,n)$.

Substituting such a ζ in (10) yields

$$\begin{array}{rcl}\int_0^n xv(t,x)\chi^1(x)dx & \le & \int_{x_0}^\infty xv(0,x)^+\,dx \\ & & +\int_0^t\int_{2x_0}^\infty xa(s,x)v(s,x)^+\,dx\,ds \\ & & +\nu\int_0^t\int_{x_0}^\infty xv(s,x)^+\,dx\,ds\end{array} \tag{13}$$

where

$$\begin{array}{rcl}\nu & = & \sup_{\overline{D_T}}\Bigg[\left(\lambda-a(s,x)-\int_{x_0}^\infty k(x,y)\overline{u}(t,y)\,dy\right) \\ & & +\frac{1}{2}\int_{x_0}^\infty \chi_{U}(x+z)k(z,x)\underline{u}(t,z)\,dz \\ & & +\frac{3}{2}k_0\int_{x_0}^\infty(1+\chi_{U}(x+y))\overline{u}(t,y)\,dy\Bigg]\end{array}$$

and

$$v(s,x)^+ = \sup_{\overline{D_T}}\{v(s,x),0\}.$$

From the condition on initial data in (7), we have

$$\begin{array}{rcl}\int_{x_0}^n xv(t,x)\chi^1(x)dx & \le & \int_0^t\int_{2x_0}^\infty xa(s,x)v(s,x)^+\,dx\,ds \\ & & +\nu\int_0^t\int_{x_0}^\infty xv(s,x)^+\,dx\,ds.\end{array}$$

From the assumption $(H1)$ we have $a(t,x)=0$ for $x<2x_0$, and then

$$\int_{x_0}^{n} xv(t,x)\chi^1(x)dx \leq \int_0^t \int_{x_0}^{\infty} (a(s,x)+\nu)xv(s,x)^+\,dx\,ds.$$

Since this inequality holds for every χ^1, we can choose a sequence $\{\chi_k^1\}$ on $(0,n)$ converging to

$$\chi^1 = \begin{cases} 1 & \text{if} \quad v(t,x)>0 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we find

$$\int_{x_0}^{n} xv(t,x)^+dx \leq \int_0^t \int_{x_0}^{\infty} (a(s,x)+\nu)xv(s,x)^+\,dx\,ds.$$

Note that ν is independent of n, letting $n \longrightarrow \infty$ we further have

$$\int_{x_0}^{\infty} xv(t,x)^+dx \leq \int_0^t \int_{x_0}^{\infty} (a(s,x)+\nu)xv(s,x)^+\,dx\,ds.$$

From the assumption $(H1)$, there exists $M \in \mathbf{R}_+$ such that $\|a(t,x)\|_\infty \leq M$. Thus

$$\int_{x_0}^{\infty} xv(t,x)^+dx \leq (\nu+M)\int_0^t \int_{x_0}^{\infty} xv(s,x)^+\,dx\,ds. \tag{14}$$

Upon application of Gronwall's inequality, (14) leads to

$$\int_{x_0}^{\infty} xv(t,x)^+dx = 0.$$

Thus, the proof is completed. □

Remark 2.3. From the proof of Theorem 2.2, it easily follows that for any function $v \in L^1(D_T, xdx)$, if $v(0,x) \leq 0$ a.e. in (x_0,∞), and the following inequality holds for every non-negative $\xi \in C^1_{0,r}(D_T)$:

$$\begin{aligned}\int_{x_0}^{\infty} xv(t,x)\xi(t,x)\,dx &\leq \int_{x_0}^{\infty} xv(0,x)\xi(0,x)dx \\ &+ \int_{x_0}^{\infty}\int_0^t xv(s,x)\xi_s(s,x)\,ds\,dx\end{aligned}$$

$$+\int_0^t\int_{2x_0}^\infty a(s,y)v(s,y)\int_{x_0}^{y-x_0} x\xi(s,x)b(x|y)\,dx\,dy\,ds \tag{15}$$
$$+\int_{x_0}^\infty\int_0^t x\xi(s,x)A(s,x)v(s,x)\,ds\,dx$$
$$+\int_{x_0}^\infty\int_0^t xv(s,x)\int_{x_0}^\infty \xi(s,x+y)B(s,x,y)\,dy\,ds\,dx,$$

with $A\in L^\infty(D_T)$, $B\geq 0$, and $\int_{x_0}^\infty B(t,x,y)\,dy\in L^\infty(D_T)$, then $v(t,x)\leq 0$ a.e. in D_T. Such a result will be used in Section 3.

Corollary 2.4. *Suppose that $(H1)-(H4)$ hold. Let $\underline{u}$ and $\overline{u}$ be a non-negative lower solution and a non-negative upper solution of (2), respectively. If u is the solution of (2), then*

$$\underline{u}\leq u\leq\overline{u}\quad \text{a.e. in } D_T.$$

Proof. We first claim that $u\geq 0$, since if $v=-u$, v satisfies (15) with

$$A(t,x)=-a(t,x)-\int_{x_0}^\infty k(x,y)\overline{u}(t,y)\,dy$$

and

$$B(t,x,y)=\frac{1}{2}\chi_U(x+y)k(x,y)u(t,y).$$

Then let $v=u-\overline{u}$. Since $-u(t,x)\int_{x_0}^\infty k(x,y)u(t,y)\,dy\leq 0$ and $\overline{u}$ satisfies

$$\int_{x_0}^\infty x\overline{u}(t,x)\xi(t,x)\,dx\geq\int_{x_0}^\infty x\overline{u}(0,x)\xi(0,x)dx$$
$$+\int_{x_0}^\infty\int_0^t x\overline{u}(s,x)\xi_s(s,x)\,ds\,dx$$
$$+\int_0^t\int_{2x_0}^\infty a(s,y)\overline{u}(s,y)\int_{x_0}^{y-x_0} x\xi(s,x)b(x|y)\,dx\,dy\,ds$$
$$+\int_{x_0}^\infty\int_0^t x\xi(s,x)(\mathcal{F}\overline{u})(s,x)\,ds\,dx$$
$$-\int_{x_0}^\infty\int_0^t x\xi(s,x)a(s,x)\overline{u}(s,x)\,ds\,dx.$$

One can see that v satisfies (15) with $A(t,x) = -a(t,x)$, and

$$B(t,x,y) = \frac{1}{2}\chi_U(x+y)[k(x,y)u(t,y) + k(x,y)\overline{u}(t,y)],$$

which shows $u \le \overline{u}$. Now let $v = \underline{u} - u$. Since

$$\int_{x_0}^{\infty} k(x,y)u(t,x)u(t,y)dy \le \int_{x_0}^{\infty} k(x,y)u(t,x)\overline{u}(t,y)dy,$$

v satisfies (15) with

$$A(t,x) = -a(t,x) - \int_{x_0}^{\infty} k(x,y)\overline{u}(t,y)\,dy$$

and

$$B(t,x,y) = \frac{1}{2}\chi_U(x+y)[k(x,y)\underline{u}(t,y) + k(x,y)u(t,y)],$$

hence $\underline{u} \le u$.

3. Existence and uniqueness of the solution

We begin this section by constructing monotone sequences of upper and lower solutions. Suppose that $\underline{u}^0(t,x)$ and $\overline{u}^0(t,x)$ are a lower solution and an upper solutions of (2), respectively, and they are continuously differentiable in t. Under the hypothesis $(H1)$, we can choose a positive constant M such that $M - a(t,x) - \int_{x_0}^{\infty} k(x,y)\overline{u}(t,y)\,dy \ge 0$ for $(x,y) \in \overline{D}_T$ and $\underline{u}^0(t,x) \le u(t,x) \le \overline{u}^0(t,x)$. Based on the general idea from [4], we then set up two sequences $\{\underline{u}^k\}_{k=0}^{\infty}$ and $\{\overline{u}^k\}_{k=0}^{\infty}$ by the following procedure:

For $k = 1, 2, \ldots$ let $\underline{u}^k$ and $\overline{u}^k$ satisfy the equation

$$\begin{aligned}
&\underline{u}^k_t = -a(t,x)\underline{u}^{k-1} + \int_{x+x_0}^{\infty} a(t,y)b(x|y)\underline{u}^{k-1}(t,y)dy - M(\underline{u}^k - \underline{u}^{k-1}) \\
&\qquad + \mathcal{F}\underline{u}^{k-1} - \underline{u}^{k-1}\int_{x_0}^{\infty} k(x,y)\overline{u}^{k-1}(t,y)\,dy \quad \text{ on } D_T \\
&\underline{u}(0,x) = u_0(x) \quad \text{ in } [0,\infty),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\overline{u}^k_t = -a(t,x)\overline{u}^{k-1} + \int_{x+x_0}^{\infty} a(t,y)b(x|y)\overline{u}^{k-1}(t,y)dy - M(\overline{u}^k - \overline{u}^{k-1})\\
&\qquad +\mathcal{F}\overline{u}^{k-1} - \overline{u}^{k-1}\int_{x_0}^{\infty} k(x,y)\underline{u}^{k-1}(t,y)\,dy \quad \text{ on } D_T\\
&\overline{u}(0,x) = u_0(x) \quad \text{ in } [0,\infty).
\end{aligned} \tag{17}$$

The existence of solutions to problems (16) and (17) follows from the fact that (16) and (17) are both linear problems with initial condition. We first show that $\underline{u}^0 \le \underline{u}^1 \le \overline{u}^1 \le \overline{u}^0$. Let $v(t,x) = \underline{u}^0 - \underline{u}^1$. Then v satisfies (15) with $A(t,x) = -M$, and $B(t,x,y) = 0$. Thus by Remark 2.3 $v \le 0$, which implies $\underline{u}^0 \le \underline{u}^1$. In a similar manner, it can be shown that $\overline{u}^1 \le \overline{u}^0$.

Then let $v(t,x) = \underline{u}^1 - \overline{u}^0$. Since $\underline{u}^0 \le \underline{u}^1$ and $\overline{u}^1 \le \overline{u}^0$, v satisfies (15) with $A(t,x) = -a(t,x) - \int_{x_0}^{\infty} k(x,y)\underline{u}^0(t,y)\,dy$, and $B(t,x,y) = \frac{1}{2}\chi_U(x+y)[k(x,y)\underline{u}^1(t,y) + k(x,y)\overline{u}^0(t,y)]$. Thus by comparison $\underline{u}^1 \le \overline{u}^0$. Similarly, it can be seen that $\underline{u}^0 \le \overline{u}^1$.

We now claim that $\underline{u}^1$ and $\overline{u}^1$ are a lower solution and an upper solution of (2), respectively. Since $\underline{u}^0 \le \underline{u}^1$ and $\overline{u}^1 \le \overline{u}^0$, on the one hand, the right-hand side of the equation in (16) satisfies

$$\begin{aligned}
&-a(t,x)\underline{u}^0 + \int_{x+x_0}^{\infty} a(t,y)b(x|y)\underline{u}^0(t,y)dy - M(\underline{u}^1 - \underline{u}^0)\\
&\quad +\mathcal{F}\underline{u}^0 - \underline{u}^0\int_{x_0}^{\infty} k(x,y)\overline{u}^0(t,y)\,dy\\
= &\int_{x+x_0}^{\infty} a(t,y)b(x|y)\underline{u}^0(t,y)dy - M\underline{u}^1\\
&\quad + \left(M - a(t,x) - \int_{x_0}^{\infty} k(x,y)\overline{u}^0(t,y)\,dy\right)\underline{u}^0\\
&\quad + \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\underline{u}^0(t,x-y)\underline{u}^0(t,y)dy\\
\le &\int_{x+x_0}^{\infty} a(t,y)b(x|y)\underline{u}^1(t,y)dy - M\underline{u}^1
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(M - a(t,x) - \int_{x_0}^{\infty} k(x,y)\overline{u}^1(t,y)\,dy\right)\underline{u}^1 \\
&\quad + \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\underline{u}^1(t,x-y)\underline{u}^1(t,y)dy \\
= &\quad -a(t,x)\underline{u}^1 + \int_{x+x_0}^{\infty} a(t,y)b(x|y)\underline{u}^1(t,y)dy + \mathcal{F}\underline{u}^1 - \underline{u}^1 \\
&\int_{x_0}^{\infty} k(x,y)\overline{u}^1(t,y)\,dy.
\end{aligned}
$$

On the other hand, the right-hand side of the equation in (17) satisfies

$$
\begin{aligned}
&-a(t,x)\overline{u}^0 + \int_{x+x_0}^{\infty} a(t,y)b(x|y)\overline{u}^0(t,y)dy - M(\overline{u}^1 - \overline{u}^0) \\
&\quad + \mathcal{F}\overline{u}^0 - \overline{u}^0\int_{x_0}^{\infty} k(x,y)\underline{u}^0(t,y)\,dy \\
= &\int_{x+x_0}^{\infty} a(t,y)b(x|y)\overline{u}^0(t,y)dy - M\overline{u}^1 \\
&\quad + \left(M - a(t,x) - \int_{x_0}^{\infty} k(x,y)\underline{u}^0(t,y)\,dy\right)\overline{u}^0 \\
&\quad + \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\overline{u}^0(t,x-y)\overline{u}^0(t,y)dy \\
&\geq \int_{x+x_0}^{\infty} a(t,y)b(x|y)\overline{u}^1(t,y)dy - M\overline{u}^1 \\
&\quad + \left(M - a(t,x) - \int_{x_0}^{\infty} k(x,y)\underline{u}^1(t,y)\,dy\right)\overline{u}^1 \\
&\quad + \frac{\chi_U(x)}{2}\int_{x_0}^{x-x_0} k(x-y,y)\overline{u}^1(t,x-y)\overline{u}^1(t,y)dy \\
= &\quad -a(t,x)\overline{u}^1 + \int_{x+x_0}^{\infty} a(t,y)b(x|y)\overline{u}^1(t,y)dy + \mathcal{F}\overline{u}^1 - \overline{u}^1 \\
&\int_{x_0}^{\infty} k(x,y)\underline{u}^1(t,y)\,dy.
\end{aligned}
$$

We then assume that for some $k > 1$, $\underline{u}^k$ and $\overline{u}^k$ are a lower solution and an upper solution of (2), respectively. Proceeding analogously, we can show that $\underline{u}^k \leq \underline{u}^{k+1} \leq \overline{u}^{k+1} \leq \overline{u}^k$ and that $\underline{u}^{k+1}$ and $\overline{u}^{k+1}$ are also a lower solution and an upper solution of (2), respectively. Hence by induction, we obtain two monotone sequences that satisfy

$$\underline{u}^0 \leq \underline{u}^1 \leq \cdots \leq \underline{u}^k \leq \overline{u}^k \leq \cdots \leq \overline{u}^1 \leq \overline{u}^0 \quad \text{a.e in } \overline{D}_T$$

for each $k = 0, 1, 2, \cdots$. From the monotonicity of the sequences $\{\underline{u}^k\}_{k=0}^{\infty}$ and $\{\overline{u}^k\}_{k=0}^{\infty}$, it follows that there exist functions $\underline{u}$ and $\overline{u}$ such that $\underline{u}^k \longrightarrow \underline{u}$ and $\overline{u}^k \longrightarrow \overline{u}$ pointwise in D_T. Clearly $\underline{u} \leq \overline{u}$ a.e in D_T.

We now show that $\underline{u} = \overline{u}$. To this end, let $v = \overline{u} - \underline{u}$. Since $\overline{u} \geq \underline{u}$, $v(t,x) \geq 0$ and $v(0,x) = 0$. In view of (8), by choosing $\xi(t,x) = \xi(x)$, where $\xi(x) \equiv 1$ for $x_0 \leq x \leq n$, $\xi(x) \equiv 0$ for $n+2 \leq x < \infty$, and $-1 \leq \xi' \leq 0$ for $n \leq x \leq n+2$, we have that

$$\begin{aligned}\int_{x_0}^{n} xv(t,x)\,dx &\leq \int_0^t \int_{x_0}^{\infty} x\left[(\mathcal{F}\overline{u})(s,x) - (\mathcal{F}\underline{u})(s,x)\right]\,dx\,ds \\ &\quad + \int_0^t \int_{x_0}^{\infty} x\overline{u}(s,x) \int_{x_0}^{\infty} k(x,y)v(s,y)\,dy\,dx\,ds \qquad (18)\\ &\leq \tilde{\nu} \int_0^t \int_{x_0}^{\infty} xv(s,x)\,dx\,ds,\end{aligned}$$

where

$$\begin{aligned}\tilde{\nu} &= \sup_{\overline{D_T}} \left[\frac{1}{2}\int_{x_0}^{\infty} \chi_U(x+z)k(z,x)\underline{u}(t,z)\,dz \right.\\ &\quad \left. + \frac{3}{2}k_0 \int_{x_0}^{\infty} (1+\chi_U(x+y))\overline{u}(t,y)\,dy\right].\end{aligned}$$

Since $\tilde{\nu}$ is independent of n, letting $n \longrightarrow \infty$, we have

$$\int_{x_0}^{\infty} xv(t,x)\,dx \leq \tilde{\nu} \int_0^t \int_{x_0}^{\infty} xv(s,x)\,dx\,ds.$$

Hence, it follows from Gronwall's inequality that $v(t,x) = 0$, i.e. $\underline{u} = \overline{u}$. Defining this common limit by u, we find that u is a solution of (2).

Suppose that w is another solution of (2). Since for each k, $\underline{u}^k$ and $\overline{u}^k$ are a lower solution and an upper solution of (2), respectively, by Corollary 2.4, $\underline{u}^k \leq w \leq \overline{u}^k$, which shows $u \equiv w$.

To sum up, we have the following existence-uniqueness result.

Theorem 3.1. *Suppose that hypotheses* $(H1)-(H4)$ *hold. Furthermore, suppose that* $\underline{u}^0(t,x)$ *and* $\overline{u}^0(t,x)$ *are a non-negative lower solution and a non-negative upper solution of (2), respectively. Then there exist monotone sequences* $\left\{\underline{u}^k(t,x)\right\}$ *and* $\left\{\overline{u}^k(t,x)\right\}$ *converging to the unique solution* u *of (2).*

We now show that the solution of (2) possesses the following property.

Theorem 3.2. *Suppose that hypotheses* $(H1)-(H4)$ *hold. Then for the solution* $u(t,x)$ *of (2),* $P(t)=\displaystyle\int_{x_0}^{\infty} xu(t,x)dx$ *is continuous in the existence interval.*

Proof. In view of hypothesis $(H1)-(H4)$, to show $P(t)\in C([0,T])$, it suffices to establish the following equality:

$$\begin{aligned}\int_{x_0}^{\infty} xu(t,x)\,dx &= \int_{x_0}^{\infty} xu(0,x)dx+\int_{x_0}^{\infty}\int_0^t x(\mathcal{F}u)(s,x)\,ds\,dx\\ &\quad -\int_{x_0}^{\infty}\int_0^t xu(s,x)\int_{x_0}^{\infty} k(x,y)u(s,y)\,dy\,ds\,dx.\end{aligned} \tag{19}$$

To this end, we again choose $\xi(t,x)=\xi(x)$, where $\xi(x)\equiv 1$ for $x_0\le x\le n$, $\xi(x)\equiv 0$ for $n+2\le x<\infty$, and $-1\le\xi'\le 0$ for $n\le x\le n+2$. By the definition of the solution of (2) we find

$$\begin{aligned}&\left|\int_{x_0}^{\infty} xu(t,x)\,dx-\int_{x_0}^{\infty} xu(0,x)dx-\int_{x_0}^{\infty}\int_0^t x(\mathcal{F}u)(s,x)\,ds\,dx\right.\\ &\qquad\left.+\int_{x_0}^{\infty}\int_0^t xu(s,x)\int_{x_0}^{\infty} k(x,y)u(s,y)\,dy\,ds\,dx\right|\\ =\;&\left|\int_n^{\infty} x[u(t,x)-u(0,x)][1-\xi(x)]\,dx\right.\\ &\qquad-\int_n^{\infty}\int_0^t x(\mathcal{F}u)(s,x)[1-\xi(x)]\,ds\,dx \qquad\qquad (20)\\ &\qquad\left.+\int_n^{\infty}\int_0^t xu(s,x)[1-\xi(x)]\int_{x_0}^{\infty} k(x,y)u(s,y)\,dy\,ds\,dx\right|\end{aligned}$$

$$\leq \left(2+\frac{3}{2}\|k\|_\infty \sup_{[0,T]} \parallel u(t,\cdot) \parallel_1\right) \sup_{[0,T]} \int_n^\infty xu(t,x)\,dx.$$

Since $u \in L^1(D_T, x\,dx)$, $\sup\limits_{[0,T]} \int_n^\infty xu(t,x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$, which yields (19).

As a consequence, we obtain the following global existence result.

Theorem 3.3. *Suppose that hypotheses* $(H1)-(H4)$ *hold. Then the unique solution of (2), exists for* $0 \leq t < \infty$.

Proof. By the definition of the solution, we only need to show that $P(t)$ does not blow-up at infinity. Making use of (19) we have that

$$\begin{aligned} 0 \leq P(t) &= P(0) - \frac{1}{2}\int_{x_0}^\infty \int_0^t xu(s,x) \int_{x_0}^\infty k(x,y)u(s,y)\,dy\,ds\,dx \\ &\leq P(0). \end{aligned}$$

It follows that $P(t)$ is bounded at all time. Thus, the proof is completed. □

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MaSIM Focus Area
North-West University
Mafikeng 2735
South Africa

and

Higher Institute of the Sahel
University of Maroua
P.O. Box 46 Maroua
Cameroon
E-mail: guiemrichard@yahoo.fr

MaSIM Focus Area*
North-West University
Mafikeng 2735
South Africa
E-mail: 23238917@nwu.ac.za

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