

Shaken, not stirred: Test particles in binary-black-hole mergers

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Abstract

In 2015 the advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO) detected the first ever gravitational event, gravitational wave event GW150914, with multiple new gravitational wave events, originating from both binary neutron stars and binary black hole (BBH) mergers, detected in subsequent years. In light of these detections, we simulate the dynamics of ambient test particles in the gravitational potential well of a BBH system close to its inspiral phase with the goal of simulating the associated electromagnetic radiation and resulting spectral energy density distribution of such a BBH system. This could shed light on possible detection ranges of electromagnetic counterparts to BBH mergers. The potentials are numerically calculated using finite difference methods, under the assumption of non-rotating black holes with the post-Newtonian Paczynski-Wiita potential approximation in tandem with retarded time concepts analogous to electrodynamics. We find that the frequencies of potential electromagnetic radiation produced by these systems (possibly reaching earth), range between a few kHz to a few $100kHz$. The bulk of radiation is distributed at frequencies below $100kHz$.

Keywords: Binary black hole merger, binary black hole, binary black hole merger simulation, particle acceleration, gravity.

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Nomenclature

Acronyms / Abbreviations

aLIGO advanced Laser Interferometer Gravitational-Wave Observatory

BBH Binary Black Hole

BH Black Hole

CoM Center of mass

DFT discrete Fourier transform

FFT Fast Fourier Transform

HDF5 Hierarchical Data Format version 5

ISCO inner most stable circular orbit

LOFAR Low-Frequency Array

ODEs ordinary differential equations

OMP Open Multi-Processing

SED Spectral Energy Distribution

ULX Ultraluminous X-ray

Chapter 1

Introduction

In 2015 the advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO) detected the first ever gravitational wave event — gravitational wave event GW150914 (Abbott et al., 2016). This was not only the first ever direct detection of gravitational waves, but also demonstrated the existence of black holes (BHs) in the $20M_{\odot} - 50M_{\odot}$ mass range. The main objective of this dissertation will be to inspect binary black hole (BBH) environments as possible particle accelerators and possible sources of electromagnetic (EM) radiation. This chapter gives a brief summary of both classical mechanics and special relativity as well as the physical descriptions of gravity from both a Newtonian and general relativistic perspective. This includes a brief description of BHs and BBH systems, as well as the detection of gravitational waves.

1.1 Gravity

1.1.1 Newtonian gravity and classical mechanics

Newton formulated his law of universal gravitation (Cohen et al., 2016) that is defined in terms of the gravitational force F_g acting between two objects with masses m_1 and m_2 . Taking r as the distance separating the centers of mass of the two objects, then:

$$\vec{F}_g = -G \frac{m_1 m_2}{r^2} \hat{r}, \quad (1.1)$$

with G the universal gravitational constant. The Gravitational constant is experimentally determined to be $G = 6.674^{-11} m^3 \cdot kg^{-1} \cdot s^{-2}$ (Cohen et al., 2016). From the form given in

equation 1.1, it is worth noting that the universal gravitation law can be written in the form

$$\vec{F}_{g12} = m_1 \nabla \left(G \frac{m_2}{r} \right) = m_1 \nabla \Phi(r), \quad (1.2)$$

where $\vec{F}_{g12}$ is the force acting on an object of mass m_1 , due to the gravitational potential well $\Phi(r)$ produced by an object with mass m_2 (Hartle, 2003). The universal gravitation law is derived from empirical observation in contrast to Newton's three laws, which Newton refers to as axioms in his *Principia Mathematica* (Newburgh, 2001; Newton et al., 1850).

1.1.2 Special Relativity

A four-vector is defined as a directed line segment in four dimensional flat space. In other words, a four-vector $\mathbf{b} = (b^0, b^1, b^2, b^3)$ is a vector with three spatial components (b^1, b^2, b^3) and one time component b^0 . The scalar product of two four-vectors $\mathbf{b}$ and $\mathbf{d}$ is invariant under Lorentz transformations, and is given by the relation¹:

$$\mathbf{b} \cdot \mathbf{d} = \eta_{\alpha\beta} b^\alpha d^\beta. \quad (1.3)$$

Here b^α and d^β are the four components of the relevant four-vectors, while $\eta_{\alpha\beta}$ is the metric; a second order tensor. In the case of special relativity, the metric used is the metric of flat *spacetime*, which leads to the relation:

$$\mathbf{b} \cdot \mathbf{d} = -b^0 d^0 + b^1 d^1 + b^2 d^2 + b^3 d^3. \quad (1.4)$$

A quantity that we must now introduce is *proper time* τ , which is the time component of the frame of reference in which the moving object in question is at rest (Griffiths, 2012).² In this regard, Newton's first two laws can now be extended to special relativity. Taking $\mathbf{x} = (ct, x, y, z)$ to be the position four-vector, we can then define the four-velocity $\mathbf{u}$ and four-acceleration $\mathbf{a}$ as

$$\mathbf{u} = \frac{d\mathbf{x}}{d\tau}, \quad \mathbf{a} = \frac{d\mathbf{u}}{d\tau}. \quad (1.5)$$

¹Here we have introduced Einstein's summation convention wherein repeating indices on one side of the equation implies summation over all components.

²We also have that $\frac{d\tau}{dt} = \gamma^{-1}$.

It can also be shown that the four-velocity can be related to the regular three velocity vector $\vec{v}$ by the relation $\mathbf{u} = (c\gamma, \gamma\vec{v})$. As a consequence of this, in special relativity³, it holds that

$$\mathbf{u} \cdot \mathbf{u} = -c^2 \quad (1.6)$$

(Hartle, 2003). This naturally leads to the definition of the four-momentum $\mathbf{p} = m\mathbf{u}$. Additionally, from the relation between the four- and three velocities, it follows that:

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{u}}{d\tau} = \frac{d\mathbf{u}}{dt} \frac{dt}{d\tau}, \\ &= \left[c \frac{d\gamma}{dt}, \frac{d\gamma\vec{v}}{dt} \right] \gamma, \\ &= \left[c\gamma^3 \left(\frac{\vec{v} \cdot \vec{a}}{c^2} \right), \vec{v} \frac{d\gamma}{dt} + \gamma \frac{d\vec{v}}{dt} \right] \gamma, \\ &= \left[c\gamma^4 \left(\frac{\vec{v} \cdot \vec{a}}{c^2} \right), \gamma^4 \left(\frac{\vec{v} \cdot \vec{a}}{c^2} \right) \vec{v} + \gamma^2 \vec{a} \right], \end{aligned} \quad (1.7)$$

where we have used the fact that $\frac{dt}{d\tau} = \gamma$.

Finally, Newton's second law in special relativity is then given by the relation

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau} = m\mathbf{a}, \quad (1.8)$$

where $\mathbf{f}$ is the four-force and m is the rest mass. It can, however, be shown that only three of the four equations in Equation 1.8 are independent, with the fourth a linear combination of the other three (Hartle, 2003). The three-force $\vec{F}$ in special relativity is defined as $\vec{F} = \frac{d\vec{p}}{dt}$ with $\vec{p} = \gamma m \vec{v}$ the three-momentum. From this, Newton's second law takes the form:

$$\begin{aligned} \vec{F} &= \frac{d\vec{p}}{dt}, \\ &= \frac{d}{dt} (\gamma m \vec{v}), \\ &= m \left[\vec{v} \frac{d\gamma}{dt} + \gamma \vec{a} \right], \\ &= m \left[\frac{\gamma^3}{c^2} (\vec{v} \cdot \vec{a}) \vec{v} + \gamma \vec{a} \right]. \end{aligned} \quad (1.9)$$

³This also holds in general relativity.

1.1.3 General Relativity

Studying the laws of physics from the perspective of observers in different inertial frames, Einstein made the connection between the geometry of space and the distribution of matter (Roos, 2015) in his *General theory of Relativity* in 1915 (Einstein, 1915). The theory is a generalization of Einstein's theory of special relativity and Newton's law of gravitation to non-inertial reference frames.

There are conflicts between Newton's law of gravitation and special relativity. Firstly, Newton's law of gravitation cannot be written in terms of invariants. Newton's law of gravitation also does not explicitly account for time dependence, implying that the effects of gravity are instantaneous throughout the universe, which violates the principles of special relativity. Finally, Newton's gravitation law does not account for the weak equivalence principle, that states that gravitational mass and inertial mass (Newton's second law) is equivalent, which has been experimentally found to hold to 1 part in 10^{13} (Wagner et al., 2012).

Einstein set out to resolve the above mentioned conflicts and inaccuracies by generalizing the weak equivalence principle to the strong equivalence principle and combining it with the principle of general covariance (Roos, 2015).

The *strong* equivalence principle states that, to a free falling observer, the result of any locally performed experiment is independent of the magnitude of the gravitational field (Debono and Smoot, 2016; Roos, 2015).

Noting that space is only flat locally and only by approximation, Einstein combined the principle of equivalence and the generally covariant form of Newton's second law and came to the realization that an inhomogeneous gravitational field is equivalent to a curved spacetime. He found the simplest formulation describing this to be:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1.10)$$

which forms the ten independent Einstein field equations, with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (1.11)$$

where $T_{\mu\nu}$ is the stress-energy tensor, $R_{\mu\nu}$ the Ricci tensor and R the Ricci scalar (Roos, 2015). These equations reduce to Newton's equations in the weak field limit (i.e. in the limit $\frac{GM}{Rc^2} \ll 1$)

(Hartle, 2003).

Though a fully general relativistic approach is beyond the scope of this dissertation, the remainder of this section will be used to demonstrate some general relativistic concepts that will prove useful in further sections.

In order to describe a general geometry, a line element ds^2 is used to relate the infinitesimal distance between any two points in the geometry. An example of such a line element used to describe flat spacetime geometry is

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2. \quad (1.12)$$

A more general line element in a four dimensional geometry is given by the equation:

$$ds^2 = g_{\alpha\beta}(\mathbf{x}) dx^\alpha dx^\beta, \quad (1.13)$$

where $g_{\alpha\beta}(\mathbf{x})$ is a symmetric second-order tensor that is position dependent. The coordinates and metric used for any given line element can vary and still produce the same geometry. An example of this is the use of Cartesian and polar coordinates in euclidean geometry (Hartle, 2003).

Other concepts that will prove useful are that of world lines, light cones, and *geodesics*. The world line of an object is the path traced out by the object in four dimensional spacetime, and is separated into *timelike* ($ds^2 < 0$), *lightlike* ($ds^2 = 0$), and *spacelike* ($ds^2 > 0$) separations ds depending on the value of the line element at the given position of the object. Light cones, as illustrated in Figure 1.1, make up a conical surface of possible lightlike world lines of particles, such as photons, that travel at the speed of light. Light cones, therefore, define the causal limits of the possible world lines of an object.

A free *test particle* is defined as an object, with mass sufficiently small so as to not produce any significant curvature of spacetime, that is free from the influence of all forces save the curvature of spacetime. Geodesics are the extremal proper time world lines traced out by a free *test particle*. Geodesics are determined by the *geodesic equations*:

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma \quad (1.14)$$

which are, in essence, equations of motion (Hartle, 2003). In the context of GR, the affine connection $\Gamma^\alpha_{\beta\gamma}$ is called the Christoffel symbol.

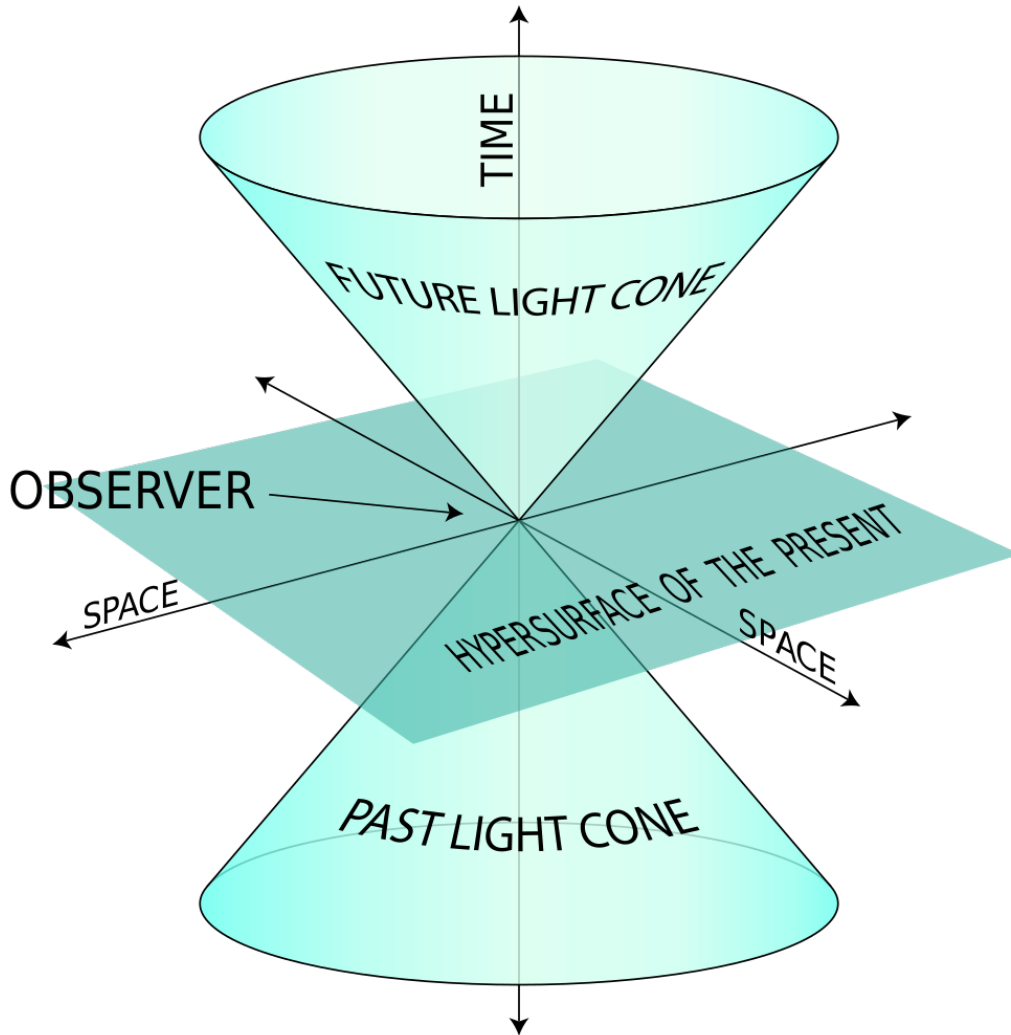


Fig. 1.1 Illustration of a light cone, that shows the canonical surface of possible light-like world lines of particles, such as photons, that travel at the speed of light.

1.2 Black Holes

BHs, in qualitative terms, are regions of spacetime for which the effect of gravity is large enough that nothing can escape from it, including light. The boundary of such a region is defined as an *event horizon*. The horizon causally separates the interior and exterior of a black hole, a notion that was first understood by David Finkelstein (Bambi, 2018; Finkelstein, 1958).

From Newtonian mechanics, John Michell and Pierre-Simon Laplace found that for highly dense objects with mass M , one could find a critical radius $R_{crit} = \frac{2GM}{c^2}$ for which the velocity needed to escape from the surface would be equal to the speed of light (Laplace, 1798; Michell, 1784; Montgomery et al., 2009). Thus, considering the finite speed of light c , such objects would not emit radiation directly. The modern concept of a BH, however, was introduced in 1916 after Karl Schwarzschild published a solution to Einstein's field equations under the assumption of vacuum space outside of a spherically symmetric source of curvature M centred at the origin of the coordinate system (Schwarzschild, 1999). It can be shown that the mass of a BH is compressed into the singularity at the center, shielded by the event horizon. This forms the *cosmic censorship conjecture*, that states that all singularities that form, are shielded by the horizon of a BH (Hartle, 2003).

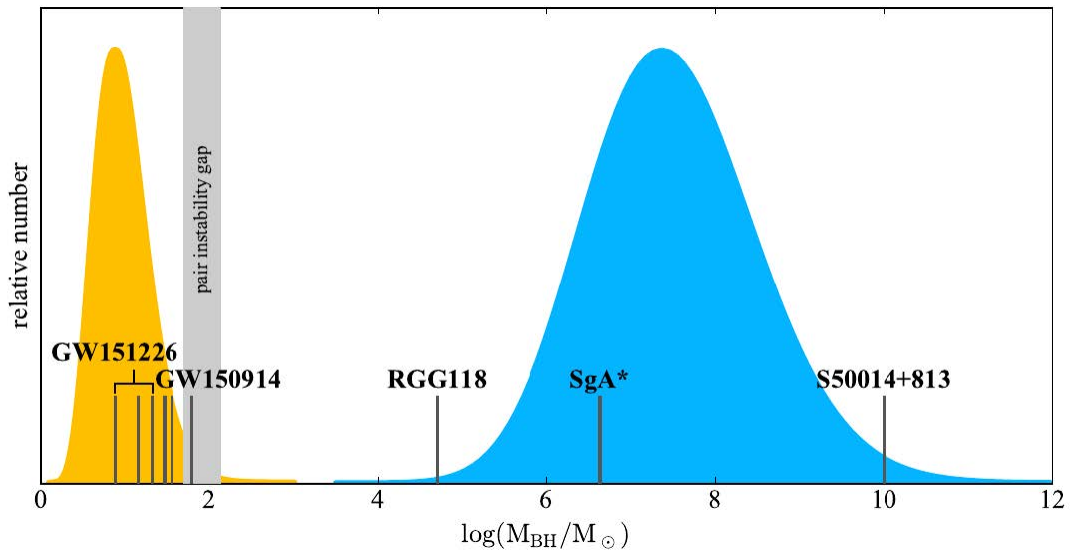


Fig. 1.2 Illustration of the BH mass function. Stellar mass BHs are depicted in orange and supermassive BHs in blue. The grey strip shows a region for which supernovae leave no remnant directly (Celoria et al., 2018).

BHs can have a vast range of masses, but are roughly classified into *stellar-mass*, *intermediate mass* and *supermassive* BHs as illustrated in Figure 1.2. Stellar-mass BHs have masses that typically range from about one to a hundred solar masses⁴ and form either as relics from stellar collapse, or from the merging of compact objects. Stellar-mass BHs have both been observed

⁴A solar mass, $M_{\odot} = 1.989 \times 10^{33}$ g, is a unit of measure for mass where one solar mass is equal to the mass of the sun.

in X-ray binary systems, and from compact object mergers (Celoria et al., 2018).

Intermediate mass BHs are believed to exist in mass ranges of $10^2 M_\odot - 10^4 M_\odot$. At the time of writing the aLIGO and Virgo collaboration detected a BBH merger (gravitational wave event GW190521) with a total mass of $150 M_\odot$, resulting in a BH remnant of $142 M_\odot$, which lies within the mass range of intermediate mass BHs (Abbott et al., 2020). Prior to this, matter accreted onto intermediate mass BHs was also suggested as possible sources for Ultraluminous X-ray (ULX) sources found within nearby galaxies (Colbert and Mushotzky, 1999; Krolik, 2004; Roberts et al., 2004; Winter et al., 2006). These ULX sources are a class of extragalactic point-like sources for which the observed X-ray luminosity is above the Eddington limit of regular stellar-mass BHs, which is $\sim 10^{39} \text{ erg} \cdot \text{s}^{-1}$ (Atapin, 2020). The Eddington luminosity (also referred to as the Eddington limit) is typically defined as the luminosity for which the force produced by the outgoing radiation pressure is equal to that of gravity (Shaviv, 1998).

The masses of supermassive BHs are in the range of $10^6 M_\odot - 10^{10} M_\odot$. Supermassive BHs are observed at the center of most galaxies and power active galactic nuclei (Celoria et al., 2018). The first direct detection of the shadow of a supermassive BH was made in 2019 by the Event Horizon Telescope collaboration, which detected the shadow at the center of the M87 Galaxy (Akiyama et al., 2019).

1.2.1 Schwarzschild Black Holes

It has previously been stated that the first solution of Einstein's field equations for a BH is the Schwarzschild metric, which has the form:

$$ds^2 = - \left(1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1.15)$$

In the weak field limit, this solution also coincides with the form of a metric expected for a Newtonian gravitational potential as defined in equation 1.2. More general BH solutions exist such as the Kerr BH solution that considers a source that is also rotating (Hartle, 2003).

From the Schwarzschild metric it becomes clear that there are singularities that appear at radial components of $r = \frac{2GM}{c^2} = r_s$ and $r = 0$, where r_s is defined as the *Schwarzschild radius*. The singularity that exists at $r = 0$ is a true physical singularity, for which it can be shown that all of the mass of a black hole is concentrated at this singularity (Carroll, 2004). Introducing a

different set of coordinates that is equivalent to those used by Schwarzschild, David Finkelstein showed that the singularity at $r = r_s$ is merely a coordinate singularity that stems from the choice of coordinates, however, this singularity does have an important *physical* interpretation as it defines the event horizon.

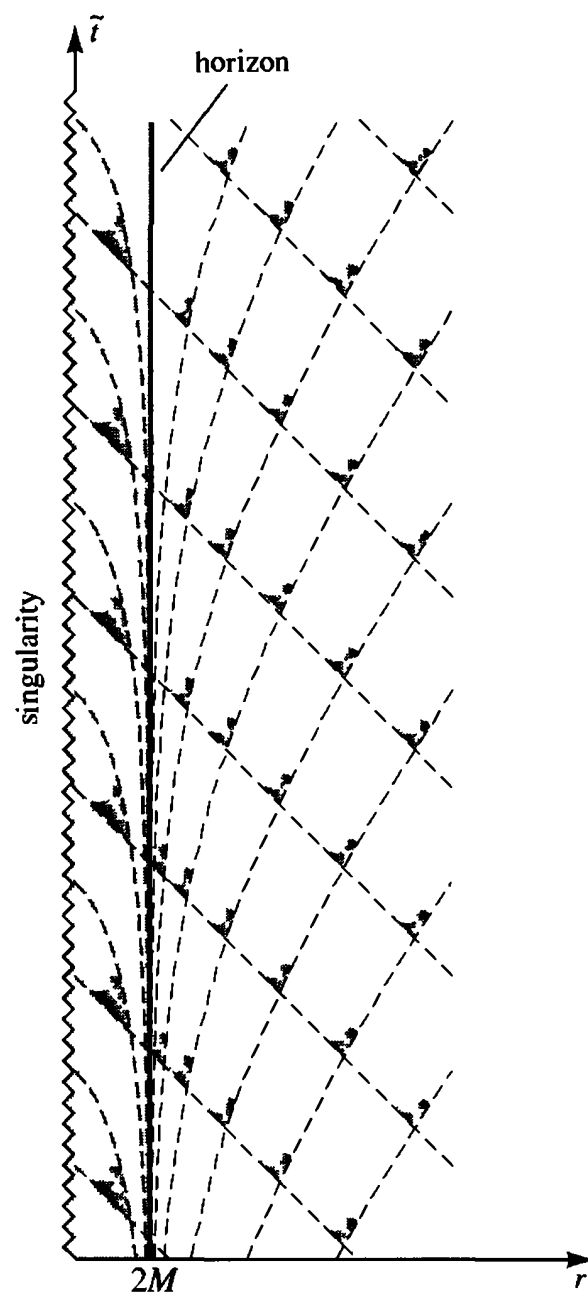


Fig. 1.3 Illustration of the world lines and light cones of free test particles close to the event horizon of a Schwarzschild BH (taken from [Hartle \(2003\)](#)).

In order to get a clearer understanding of BHs and the event horizon, it is necessary to take a brief look at world lines and light cones of test particles at regions close to the event horizon. Figure 1.3 illustrates the null geodesics⁵ in a Schwarzschild BH geometry as the dashed lines. The thick line at $r = 2M$ is the radial position of the event horizon of the BH in geometrized units, meaning that we have taken $G = c = 1$. From the figure, we can see that the null geodesics and light cones for regions inside the event horizon of the BH are directed inward and as such any matter inside this region cannot causally escape from the interior of a BH and is doomed to plummet into the singularity at $r = 0$.

In addition to the position of the event horizon of a BH, the radius of the innermost stable circular orbit (ISCO) (or radius of the marginally stable circular orbit) and the radius of the marginally bound orbit (r_{mb}) are two more important radial positions in the geometry of a BH. The ISCO radius of a BH, is the smallest radius within the geometry of a BH for which stable circular orbits around the BH are possible and is found at $r_{ISCO} = 3r_s$ for a Schwarzschild BH (Zhang and Liu, 2019). The radius of the marginally bound orbit is the smallest radius for which an unstable circular orbit around the BH is possible, and is at a radius $r_{mb} = 2r_s$ (Hartle, 2003).

1.2.2 Kerr Black Holes

Consider a rotating BH (known as a Kerr BH) with mass M and angular momentum J , described (in Boyer-Lindquist coordinates) by the metric

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mr a^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2, \quad (1.16)$$

where $a = \frac{J}{M}$ is the *Kerr parameter*, $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$ and geometrized units are used again (Abramowicz and Fragile, 2013; Hartle, 2003). The position of the event horizon is found at a radius of

$$r = r_+ = M + \sqrt{M^2 - a^2}. \quad (1.17)$$

⁵Geodesics of massless particles such as photons.

Substituting equation 1.17 and $t = \text{const}$ into the Kerr metric (equation 1.16), the two dimensional surface line element

$$d\Sigma^2 = \rho_+^2(\theta)d\theta^2 + \left(\frac{2Mr_+}{\rho_+(\theta)}\right)^2 \sin^2 \theta d\phi^2, \quad (1.18)$$

is found, with $\rho_+(\theta) = \rho(r_+, \theta)$. This defines the non-spherical event horizon of a rotating BH.

The ergosphere of a rotating BH is a region around the BH that lies between the event horizon of the BH and a surface with radius

$$r_e(\theta) = M + \sqrt{M^2 - a^2 \cos^2 \theta}, \quad (1.19)$$

where there exists no time-like solution for an observer with a four-velocity with a fixed r , θ , and ϕ component (Abramowicz and Fragile, 2013). Within the ergosphere, negative energy states are possible through which the rotational energy of the BH can be tapped through the *Penrose process* (Abramowicz and Fragile, 2013; Hartle, 2003). A complete representation of the Penrose process is given by Wagh and Dadhich (1989). Both observation (Rawlings and Saunders, 1991), and simulations (Tchekhovskoy et al., 2010, 2011) suggest that the most likely realization of Penrose processes for BHs is the Blandford-Znajek mechanism (Blandford and Znajek, 1977) for the production of jets from quasars and microquasars, however, a discussion on this mechanism is outside the scope of this dissertation.

In the case of a Kerr BH, the effective potential of a particle orbiting the BH is given by

$$U_{eff} = -\frac{1}{2} \ln \left| g^{tt} - 2lg^{t\phi} + l^2 g^{\phi\phi} \right|, \quad (1.20)$$

with l the angular momentum of the particle,

$$g^{tt} = -\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Delta}, \quad (1.21)$$

$$g^{t\phi} = -2\frac{Mar}{\rho^2 \Delta}, \quad (1.22)$$

and

$$g^{\phi\phi} = \frac{\Delta - a^2 \sin^2 \theta}{\Delta \rho^2 \sin^2 \theta}. \quad (1.23)$$

Considering slightly perturbed circular orbits around a Kerr BH, it can be shown that

$$r_{ISCO} = M \left(3 + Z_2 - [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{\frac{1}{2}} \right), \quad (1.24)$$

with $Z_1 = 1 + \left(1 - \frac{a^2}{M^2}\right)^{\frac{1}{3}} \left[\left(1 + \frac{a}{M}\right)^{\frac{1}{3}} + \left(1 - \frac{a^2}{M^2}\right)^{\frac{1}{3}} \right]$ and $Z_2 = \left(3 \frac{a^2}{M^2} + (Z_1)^2\right)^{\frac{1}{2}}$ (Abramowicz and Fragile, 2013). This reduces to $r_{ISCO} = 6M$ when $a = 0$, which corresponds to what is expected for the Schwarzschild case. Furthermore, for the case of an extremal BH ($a = M$), $r_{ISCO} = M$ for the co-rotating case and $r_{ISCO} = 9M$ for the counter-rotating case (Zhang and Liu, 2019). When considering charged BHs (Kerr-Newman BH), the position of the ISCO radius of charged particles orbiting the BH can be altered even more (Zajaček and Tursunov, 2019).

1.2.3 Particle dynamics in the vicinity of BHs and accretion disks

Though an inclusion of accretion disk dynamics in BBH merger environments is well outside the scope of this dissertation, the following section gives a brief summary of accretion disk dynamics surrounding a single BH based on the article by Abramowicz and Fragile (2013). Since accretion disks in BBH environments are beyond the scope of this dissertation, a complete discussion of the various accretion disk models is not included in this summary and the reader is instead referred to the article by Abramowicz and Fragile (2013).

In principle, a description of accretion disks depends on several factors of both the central BH as well as the surrounding matter. Three factors with respect to BHs that are particularly important in descriptions of accretion disks are the event horizon, the ergosphere⁶ and r_{ISCO} of the central BH. All three these properties depend primarily on the properties of the central BH and have been discussed briefly in the previous section.

In the context of accretion disks, only free circular orbits for radii larger than r_{ISCO} are stable orbits, this means that accretion flows, with stresses negligible compared to gravity and centrifugal effects, will resemble almost circular orbits for $r > r_{ISCO}$ and almost radial free-fall for $r < r_{ISCO}$ (Abramowicz and Fragile, 2013; Zhang and Liu, 2019).

In addition to the above discussed properties of BHs, the dynamics of the surrounding matter

⁶A region within the geometry of a rotating BH (Kerr BH) where no stationary solutions exist (Abramowicz and Fragile, 2013).

accreting onto the BH must also be considered in a description of accretion disks. The angular momentum transport and energy dissipation within an accretion disk is such that a particle of mass m must give up an approximate amount of energy equal to $\sim 0.1mc^2$, as it is quasi-steadily accreting through a Keplerian disk from a large outer radius r_{out} to an inner radius r_{in} (Abramowicz and Fragile, 2013). In order for this to occur, the particle must also lose an amount of angular momentum $\sim (GMr_{out})^{\frac{1}{2}}$, facilitated through viscous stresses in the fluid. In astrophysical accretion disks, ordinary molecular viscosity is much too weak to account for this. The stresses involved most likely come from turbulence that acts as an effective viscosity (Abramowicz and Fragile, 2013; Shakura and Sunyaev, 1973).

Another factor that is important to a description of accretion disks, is the radiative processes within the accretion disk. This depends on the thermodynamic state of matter, the magnetic field, and motion within the disk as well as the optical depth τ of the accretion disk (Abramowicz and Fragile, 2013; Davis and Hubeny, 2006; Shimura et al., 1995).

As the characteristic time-scale of dynamical processes within the accretion disk is much smaller than that of thermal and viscous time-scales, one can typically, as a first approximation, consider only the dynamical structure of the disk (Abramowicz and Fragile, 2013).

Models describing accretion disks for various dynamical situations are typically characterised by the relative thickness $h = \frac{H}{R}$; dimensionless accretion rate $\dot{m} = \frac{0.1\dot{M}c^2}{L_{Edd}}$ (with L_{Edd} the Eddington luminosity); optical depth τ ; importance of advection $q = \frac{Q_{adv}}{Q_{rad}}$ (with Q an energy flux); importance of radiation pressure $\beta = \frac{P_{gas}}{P_{gas}+P_{rad}}$; location of the inner edge of the disk r_{in} and the accretion efficiency⁷ η of the accretion disk (Abramowicz and Fragile, 2013; Bian and Zhao, 2003; Krolik, 2004). The most prominent such models are *Thick disk*⁸ models, *Thin disk*⁹ models, *Slim disk*¹⁰ models, and *Advection-dominated accretion flow* (ADAF)¹¹ models (Abramowicz and Fragile, 2013).

⁷The accretion efficiency is typically defined as $\eta = \frac{b_{in}}{c^2}$, where b_{in} is the specific binding energy at the inner edge of the accretion disk (Beloborodov, 1999).

⁸ $h > 1, \dot{m} \gg 1, \tau \gg 1, q \sim 1, \beta \ll 1, r_{mb} \sim r_{in}, \eta \ll 0.1$

⁹ $h \ll 1, \dot{m} < 1, \tau \gg 1, q = 0, \beta \sim 1, r_{in} = r_{ISCO}, \eta \sim 0.1$

¹⁰ $h \sim 1, \dot{m} \gtrsim 1, \tau \gg 1, q \sim 1, \beta < 1, r_{mb} < r_{in} < r_{ISCO}, \eta < 0.1$

¹¹ $h < 1, \dot{m} \ll 1, \tau \ll 1, q \sim 1, \beta = 1, r_{mb} < r_{in} < r_{ISCO}, \eta \ll 0.1$

All four the aforementioned models are built upon the conservation of rest mass

$$\nabla_\mu (\rho_{Den} u^\mu) = 0, \quad (1.25)$$

and the conservation of energy-momentum

$$\nabla_\mu T_V^\mu = 0, \quad (1.26)$$

with ρ_{Den} the rest mass density and u^μ the four-velocity of the matter in the accretion disk. ∇_μ denotes the covariant derivative. Typically, T_V^μ is written as the sum of the contributions from its fluid $(T_V^\mu)_{fluid}$, viscous $(T_V^\mu)_{visc}$, Maxwell $(T_V^\mu)_{Max}$ and radiation $(T_V^\mu)_{rad}$ components. With

$$\begin{aligned} (T_V^\mu)_{fluid} &= (\rho_{Den} u^\mu) (W u_\nu) + \delta_V^\mu P, \\ (T_V^\mu)_{visc} &= v_* \sigma_V^\mu, \\ (T_V^\mu)_{Max} &= F^{\mu\alpha} F_{\alpha\nu} - \frac{1}{4} \delta_V^\mu F_{\alpha\beta} F^{\alpha\beta}, \\ (T_V^\mu)_{rad} &= \frac{4}{3} E u^\mu u_\nu + u^\mu F_\nu + u_\nu F^\mu. \end{aligned} \quad (1.27)$$

In equation 1.27 W is the entropy, δ_V^μ the Kronecker delta tensor, P the pressure, v_* the kinematic viscosity, σ_V^μ shear, $F^{\mu\nu}$ the Faraday electromagnetic field tensor, E the radiation energy density, and F^μ the radiation flux (Abramowicz and Fragile, 2013).

In addition to the models discussed above, the stability of the accretion disk, oscillations within the disk and the formation of jets should also be considered, however, the reader is referred to the article by Abramowicz and Fragile (2013) for a more thorough discussion of these factors and the individual models. Additionally, accretion disks in BBH environments is discussed in section 1.2.7 later in this chapter.

1.2.4 The Paczynski-Wiita Potential

The Paczynski-Wiita potential is an important pseudo-Newtonian approximation to the general relativistic Schwarzschild geometry developed by B. Paczynski and P. Wiita while studying

thick accretion disks and super-Eddington luminosities (Paczynsky and Wiita, 1980):

$$\Phi_{PW}(r) = -\frac{GM}{r-r_s}, \quad r_s = \frac{2GM}{c^2}. \quad (1.28)$$

The Paczynski-Wiita potential is a Newton-like potential that (for the case of non-rotating BHs) exactly reproduces the marginally bound ($r_{mb} = 2r_s$) and marginally stable circular ($r_{ms} = 3r_s$) orbits of the Schwarzschild geometry, as well as the form of the Keplerian angular momentum

$$L(r) = \frac{GM r^3}{(r-r_s)^2}, \quad (1.29)$$

but not that of the Keplerian angular velocity (Abramowicz, 2009; Torkelsson, 1998). In fact, comparing the effective potentials, defined as

$$V(r, l) = \Phi(r) + \frac{l^2}{2r^2}, \quad (1.30)$$

with l the angular momentum of the orbiting particle (Abramowicz, 2009), for the Paczynski-Wiita potential and the Schwarzschild geometry in figure 1.4, it is clear that though the location of r_{ms} is the same for both cases, the effective potentials do differ. This difference in the effective potentials results in different orbits when calculating the orbit of a particle around a BH when using the Paczynski-Wiita potential in comparison with a fully relativistic approach, particularly when considering regions close to the BH. Witzany et al. (2015), however, has shown that orbits resulting from a Paczynski-Wiita potential exhibit satisfactory correspondence between the orbits, with the caveat that these orbits are likely to be more chaotic than those expected from a fully relativistic approach. Hence, any predictions made in this dissertation regarding particle acceleration and electromagnetic radiation may be considered as upper limits to the radiative output in full GR.

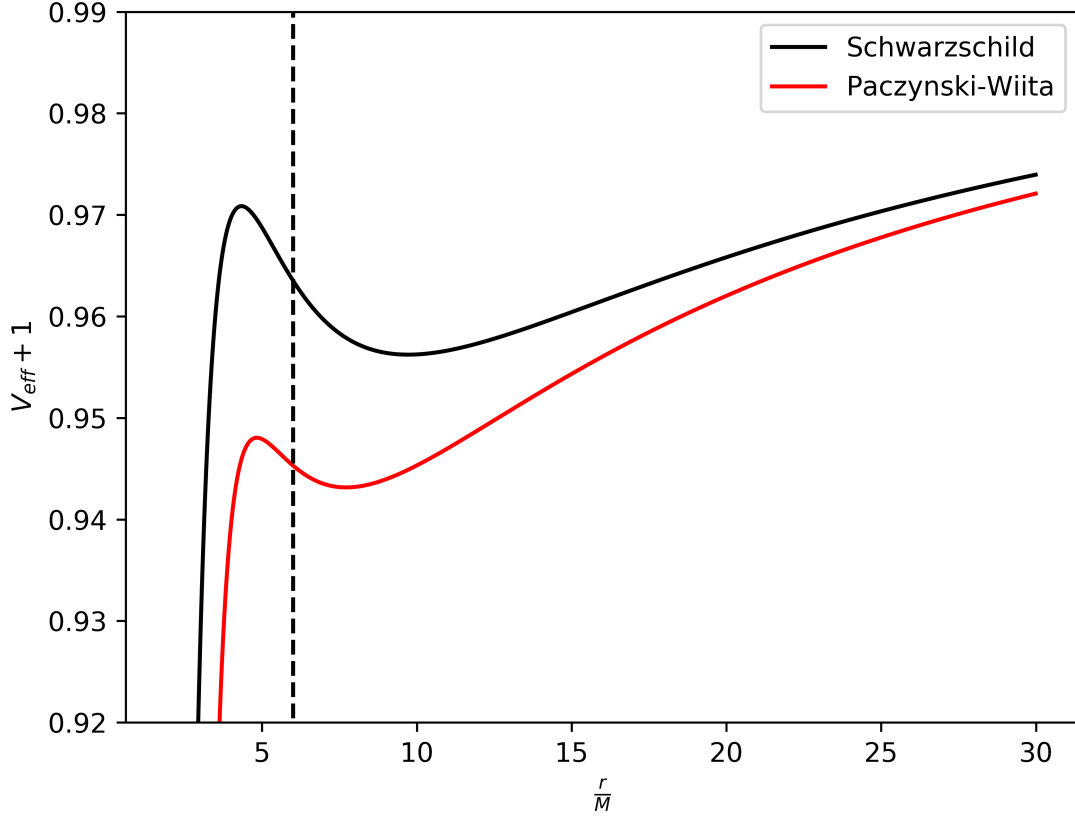


Fig. 1.4 Comparison of the effective potentials $V(r) + 1$ resulting from the Paczynski-Wiita potential and from the Schwarzschild geometry for an angular momentum of $l = 3.75M$ in geometrized units. The effective potential resulting from the Schwarzschild geometry is $V_{\text{eff},\text{Swarz}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}$, and from the Paczynski-Wiita potential is $V_{\text{eff},\text{PW}} = -\frac{M}{(r-r_s)} + \frac{l^2}{2r^2}$.

Precession due to the Paczynski-Wiita potential

An important result of GR is the orbital precession of the periastron of test particles orbiting a Schwarzschild BH, which can be calculated by

$$\Delta\Theta_{GR} = 2 \int_{r_-}^{r_+} \left[\frac{(E+1)^2}{h} - \left(\frac{1}{h^2} + \frac{1}{r^2} \right) \left(1 - \frac{2M}{r} \right) \right]^{-\frac{1}{2}} \frac{dr}{r^2}, \quad (1.31)$$

with $E = -\frac{p_0}{m} - 1$ the specific energy without the rest mass energy of the orbiting test particle, $h = \frac{p_\phi}{m}$ the specific angular momentum of the test particle, r_+ the apoapsis and r_- the periapsis

of the test particle orbit (Misner et al., 1973; Wegg, 2012). The apoapsis and periapsis of the orbiting particle can be calculated from the largest roots of

$$(E + 1)^2 - \left(1 - \frac{2M}{r_{\pm}}\right) \left(1 + \frac{h^2}{r_{\pm}^2}\right) = 0. \quad (1.32)$$

Though the Paczynski-Wiita potential does not exactly replicate the precession of the apoapsides expected from GR (Wegg, 2012), it does also cause a precession of the apoapsides which can be determined.

From classical mechanics, the change in angle between two consecutive apoapsides for a given central potential $\Phi(r)$ can be calculated by

$$\Delta\Theta = 2 \int_{r_-}^{r_+} \left[\frac{h}{r^2 \left(1 + \frac{h^2}{r^2}\right)^{\frac{1}{2}}} \left(\left[\left(1 - (e - \Phi)^{-2}\right) \left(1 + \frac{h^2}{r^2}\right) - \frac{h^2}{r^2} \right]^{-1} - 1 \right)^{\frac{1}{2}} \right] dr, \quad (1.33)$$

where $r_{\pm}$ can be found from the roots of

$$\left[\left(1 - (e - \Phi)^{-2}\right) \left(1 + \frac{h^2}{r^2}\right) - \frac{h^2}{r^2} \right]^{-1} - 1 = 0, \quad (1.34)$$

and $e = \frac{E}{m}$ is the specific energy of the particle (Landau and Lifshitz, 1976; Wegg, 2012). Equations 1.33 and 1.34 are derived from the Lagrangian,

$$L = -\gamma^{-1}m - \Phi, \quad (1.35)$$

and energy,

$$E = \gamma m + \Phi, \quad (1.36)$$

for a particle in a central potential consistent with special relativity (Hand and Finch, 1998; Landau and Lifshitz, 1976).

Substituting the Paczynski-Wiita potential into equations 1.33 and 1.34, we find

$$\Delta\Theta = 2 \int_{r_-}^{r_+} \left[\frac{h}{r^2 \left(1 + \frac{h^2}{r^2}\right)^{\frac{1}{2}}} \left(\left[\left(1 - \left(e + \frac{M}{r - r_s}\right)^{-2}\right) \left(1 + \frac{h^2}{r^2}\right) - \frac{h^2}{r^2} \right]^{-1} - 1 \right)^{\frac{1}{2}} \right] dr, \quad (1.37)$$

and

$$\left(1 - \left(e + \frac{M}{r_{\pm} - r_s}\right)^{-2}\right) \left(1 + \frac{h^2}{r_{\pm}^2}\right) - \frac{h^2}{r_{\pm}^2} = 0. \quad (1.38)$$

The roots of equation 1.38 can be found numerically and used to numerically integrate equation 1.37, as done in Section 3.4.2. This serves as a test case with which the code developed in Chapter 3 will be benchmarked.

1.2.5 Gravitational waves

An important consequence of general relativity is the existence of ripples of curved spacetime that propagate away at the speed of light, caused by the non-spherical, non-uniform motion of mass. These ripples are known as gravitational waves (Hartle, 2003). Due to the weakness of gravitational interactions only the most extreme and energetic processes in the universe, such as asymmetric supernovae and the merger of compact objects (such as Neutron stars and BHs), are capable of producing detectable gravitational waves. Even in these cases extremely precise measurements are required to detect propagating gravitational waves.

The detection of gravitational waves is, in qualitative terms, done by comparing the relative motion of at least two test bodies. The motion of a single test body in a curved spacetime is locally indistinguishable from that of a test body in a flat spacetime consistent with what is expected from the equivalence principle (Hartle, 2003). For this reason, gravitational wave detectors such as aLIGO and Virgo are based on the Michelson Interferometer.

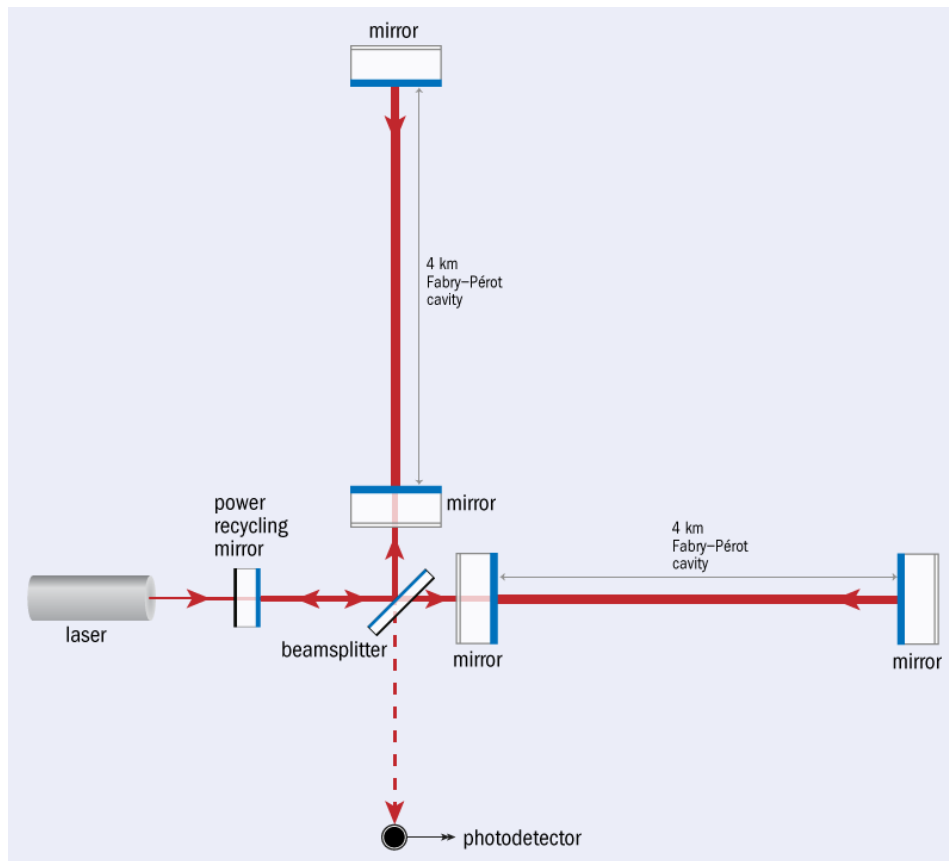


Fig. 1.5 Depiction of the aLIGO Fabry-Pérot interferometer that builds upon the Michelson interferometer through the addition of Fabry-Pérot cavities that artificially extend the length of the interferometer arms by reflecting the incident laser light within each arm multiple times, before recombination at the beam splitter. The image is taken from [aLIGO's website](#)

The interferometer detectors, an example of which is illustrated in Figure 1.5, are constructed with three test masses free to move in horizontal directions. One of the test masses supports a beam splitter that splits a beam of incident Laser light into two beams along the two arms of the interferometer at a right angle (Hartle, 2003). The other two test masses support mirrors that reflect the beams back to the beam splitter, where they are recombined and directed into a photodetector. If the lengths of the two arms differ by an integral number of wavelengths of the incident light, the two beams interfere constructively when they are recombined. If the lengths of the two arms differ by an odd number of half-wavelengths of the incident light, the beams interfere destructively when recombined. If the difference in length between the two arms of the interferometer varies, the intensity of the recombined beam will vary as the two arms move between the conditions for constructive and destructive interference. An incident gravitational wave passing through the detector will change the relative distance between each of the test

masses supporting the mirrors and the beam splitter resulting in a variation in the intensity of the detected recombined beam (Hartle, 2003).

Though current detectors are based on the principle of the Michelson interferometer described above, the design of the detectors is necessarily much more complex in order to achieve the necessary precision needed to detect gravitational waves (Hartle, 2003). One improvement present in the aLIGO interferometer design that increases the sensitivity of the interferometer, is the addition of Fabry-Pérot cavities within each of the interferometer arms (as illustrated in Figure 1.5), that artificially extend the length of the arms.

Since the first detection of a gravitational wave by the aLIGO-Virgo collaboration in 2015 (Abbott et al., 2016), multiple new detections of gravitational waves originating from both binary neutron star mergers and BBH mergers have been made and are summarized in Figure 1.6.

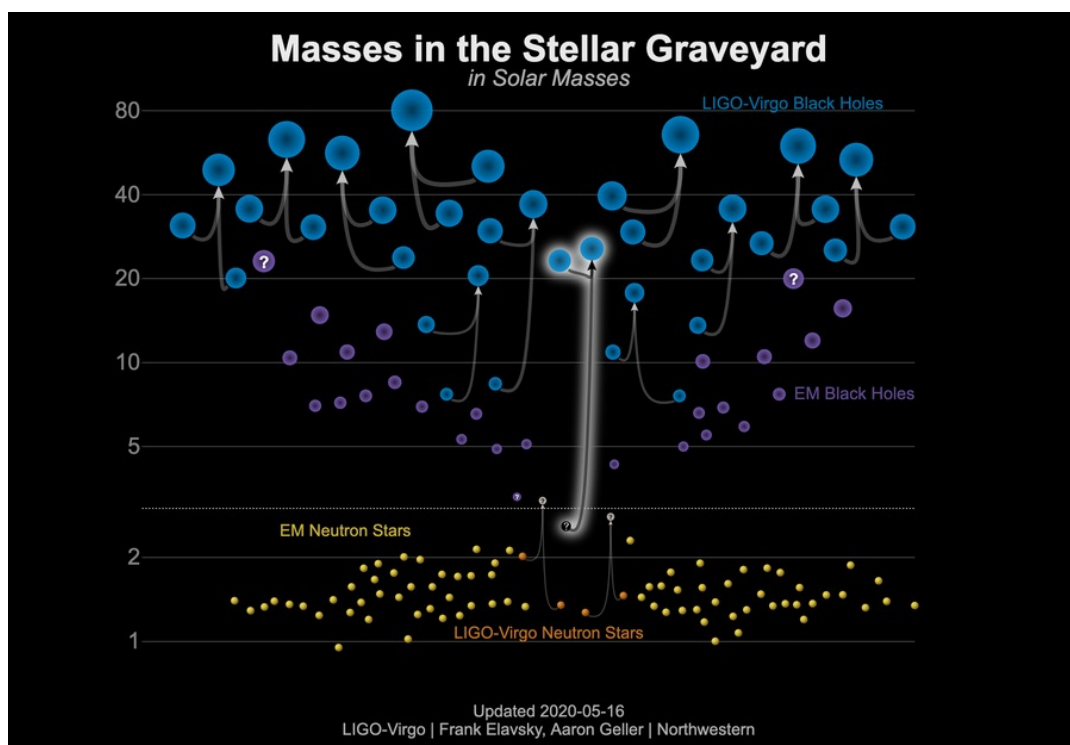


Fig. 1.6 Summary of merging compact binaries by aLIGO-Virgo collaboration, with neutron stars depicted in yellow and BHs in blue. The image is taken from aLIGO's website

1.2.6 Binary Black Hole mergers

From the observations of aLIGO and Virgo it has become apparent that BBHs are fairly common (Celoria et al., 2018). BBH mergers can be characterized by three different phases. The first and longest phase of a BBH merger is the *Inspiral phase* during which the BBHs are in a gradually shrinking orbit around their common center. It is believed that gravitational radiation is not the dominant reason for the inspiral of a BBH, but rather that BBH systems lose angular momentum due to interactions between the BHs and surrounding matter. Once the BBHs are sufficiently close to each other, the gravitational radiation emitted by the system causes the orbiting BHs to rapidly lose angular momentum and enter a plunging orbit. This is known as the *merging phase* which ends once the two black holes coalesce. The final phase of a BBH merger is the *ringdown phase*, during which the merging BHs have become a single BH undergoing perturbations of the horizon of the BH. These perturbations of the BH are quickly damped out.

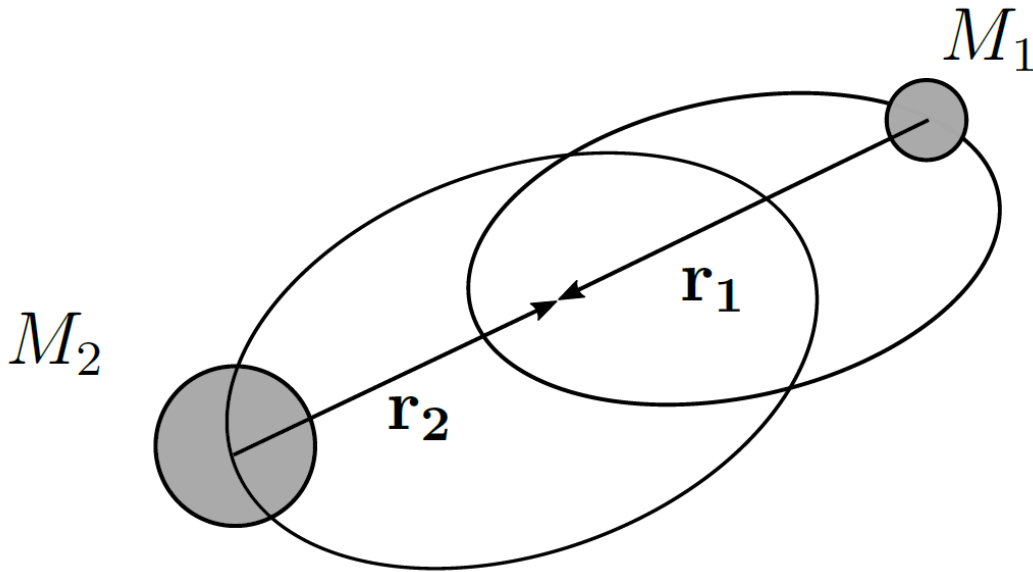


Fig. 1.7 Illustration of the two-body problem as taken from Celoria et al. (2018).

From a Newtonian perspective, the orbit of a binary system can be described by classical Keplerian motion. The setup, as illustrated in Figure 1.7, of two point masses M_1 and M_2 with a separation vector of $\vec{r}$ moving around a common center of mass (CoM) in elliptical orbits, is the classic *two-body problem* (Celoria et al., 2018). The two-body problem can be classically

solved by reducing it to an effective one-body problem. This is done by introducing the reduced mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$, with $\vec{r}_1$ and $\vec{r}_2$ the vectors from the respective masses to the CoM. The effective one-body problem is then to determine the motion of an object with mass μ in a gravitational potential $V(r) = -\frac{GM}{r}$, with $M = M_1 + M_2$ the total mass. From this we have that

$$\vec{r}_1 = \frac{M_2}{M_1 + M_2} \vec{r}, \quad \vec{r}_2 = -\frac{M_1}{M_1 + M_2} \vec{r}. \quad (1.39)$$

Using these relations, and some initial position and velocities for the point masses, we can calculate the evolution of the orbits (Celoria et al., 2018). This can also be solved analytically in terms of the angular momentum l and eccentricity e of the system using Lagrangian mechanics.

The solutions discussed above, however, are only based upon the assumption of a purely Newtonian gravitational potential. To calculate more accurate orbits for BBH systems in the late inspiral phase, one must consider a full general relativistic approach which involves solving Einstein's equations which contain non-linear higher order differential equations that do not have known analytical solutions.

Taking into consideration the energy radiated away by a BBH system in the form of gravitational waves, it is possible to get an estimate of how the separation distance between the BHs decrease. From this an estimate of the coalescence time scale of the BBH merger due to the emission of gravitational waves can be calculated as

$$t_{coal} = \frac{5c^5}{256G^3} \frac{r_0^4}{M_1 M_2 (M_1 + M_2)} \frac{1}{F(e)}, \quad (1.40)$$

where r_0 is the initial separation of the BBHs, and

$$F(e) = (1 - e^2)^{-\frac{7}{2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right), \quad (1.41)$$

is a function of eccentricity e (Celoria et al., 2018).

1.2.7 Accretion disks in BBH systems

It is expected that supermassive BBHs form in gas rich environments from galaxy mergers (Rodriguez et al., 2009), with circumbinary disks forming around the BBH system that provide a possible source for EM counterparts to possible GW signals. Gold et al. (2014) proposed that

it is possible for minidisks, fed from the circumbinary disk, to form in such systems when there are stable circular orbits within the Hill sphere¹² around each BH. This proposition has found some numerical support (Paschalidis et al., 2021). In contrast to this, it is believed that GW events resulting from stellar-mass BBH mergers are unlikely to have EM counterparts (De Mink and King, 2017), however, the detection of a potential transient signal to GW150914 by the Fermi gamma-ray burst monitor (Connaughton et al., 2016) encouraged theoretical speculation on possible scenarios for which EM counterparts to GW signals are possible. Some of this work has investigated the possibility of circumbinary accretion disks and mindisks as the origin, however, the large amount of uncertainties in such scenarios makes any conclusions impossible (De Mink and King, 2017; Khan et al., 2018).

The aim of this thesis is to simulate the trajectories of test particles in the gravitational field of merging BBHs, evaluate the resulting time-dependent particle acceleration and the emanating radiation signatures. To our knowledge, no such simulations can be found in the literature.

1.3 Dissertation Structure

In chapter two the relativistic Larmor formula is derived. Additionally, a brief summary of Fourier transforms and the calculation of the Fourier transform of a discrete dataset is given in chapter two, as this is used to determine the spectral energy distribution (SED) resulting from the Larmor formula. The model is constructed in chapter three. Finally, an example of results obtained from the model for a given BBH system is given in chapter four and chapter five is dedicated to conclusions and possible future work.

¹²The Hill sphere of a body is defined as the region in which the body dominates the gravitational attraction of satellites.

Chapter 2

Radiation from accelerated test particles

The majority of radiation detected from astrophysical sources is produced by charged particles that are accelerated by mechanisms specific to the source. Regardless of the mechanism, the total radiative output of an accelerated charged particle, as a function of time, is given by the *Larmor formula*. Taking the Fourier transform (which is a transformation that projects a function in the time domain onto the frequency domain) of the Larmor formula gives the spectral distribution of energy resulting from the accelerated particle. In this chapter a brief derivation of the Larmor formula, and an introduction of the Fourier transform is provided.

2.1 The Larmor formula

Classical electrodynamics yields the electric fields of an accelerated point charge as

$$\vec{E} = -\frac{q}{4\pi\epsilon_0} \frac{\wp}{(\vec{\wp} \cdot \vec{u})^3} \left[\left(c^2 - |\dot{\vec{r}}|^2 \right) \vec{u} + \vec{\wp} \times (\vec{u} \times \ddot{\vec{r}}) \right], \quad (2.1)$$

with $\vec{u} = (c\hat{\wp} - \dot{\vec{r}})$ and q the charge of the particle, where the separation vector, $\vec{\wp}$, describes the separation between the source and the observer, as defined by $\wp = |\vec{\wp}| \equiv |\vec{r} - \vec{r}'|$. Only the second term of Equation 2.1 contributes to EM radiation due to the dependencies of each term on distance. As we are only interested in the component of the electric field that is radiated away from the source, the first term in Equation 2.1 can be dropped, so that

$$\vec{E}_{rad} = -\frac{q}{4\pi\epsilon_0} \frac{\wp}{(\vec{\wp} \cdot \vec{u})^3} [\vec{\wp} \times (\vec{u} \times \ddot{\vec{r}})], \quad (2.2)$$

where $\vec{E}_{rad}$ is the component of the electric field that is radiated away.

The directional *energy flux* (i.e. the energy transfer per unit area per unit time) of an EM field is represented by the *Poynting vector*:

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}), \quad (2.3)$$

which can be rewritten as

$$\vec{S} = \frac{1}{\mu_0 c} E^2 \hat{\phi} \quad (2.4)$$

for radiation, since

$$\vec{B}_{rad} = c^{-1} (\hat{\phi} \times \vec{E}_{rad}), \quad (2.5)$$

with $\vec{B}_{rad}$ the radiated magnetic field.

Substituting Equation 2.2 into Equation 2.4, then for ($\dot{r} = 0$), the result is

$$\vec{S} = \frac{\mu_0 q^2 |\ddot{\vec{r}}|^2 \sin^2 \theta}{16\pi^2 c \rho^2} \hat{\phi}, \quad (2.6)$$

where θ is the angle between $\hat{\phi}$ and $\ddot{\vec{r}}$.

By integrating over a spherical surface around the particle, at radius ρ , the total power, P , radiated by the particle can be determined as follows:

$$\begin{aligned} P &= \oint \vec{S} \cdot d\vec{a}, \\ &= \frac{\mu_0 q^2 |\ddot{\vec{r}}|^2}{16\pi^2 c} \int \frac{\sin^2 \theta}{\rho^2} \rho^2 \sin \theta d\theta d\phi, \\ &= \frac{\mu_0 q^2 |\ddot{\vec{r}}|^2}{6\pi c}. \end{aligned} \quad (2.7)$$

The result of Equation 2.7 is known as the Larmor formula.

The form of the Larmor formula, given in Equation 2.7, only holds in non-relativistic cases and must be adapted to for relativistic cases. From Equation 1.7 in the previous chapter, the

four-acceleration of the particle, $\mathbf{a}$, can be formulated as

$$\mathbf{a} = \left[\gamma^4 c \left(\frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{c^2} \right), \gamma^2 \ddot{\vec{r}} + \gamma^4 \dot{\vec{r}} \left(\frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{c^2} \right) \right]. \quad (2.8)$$

In Equation 2.8, $\vec{r}$ is measured in the frame of the observer. The four-acceleration $\mathbf{a}_0$, as measured in the instantaneous rest frame of the particle, is given by $\mathbf{a}_0 = [0, \ddot{\vec{r}}_0]$. By making use of the invariance of the scalar product of the four-acceleration with itself, discussed in the previous chapter, the following relation is found:

$$|\ddot{\vec{r}}_0|^2 = \gamma^4 \left[\ddot{\vec{r}}^2 + \gamma^2 \left(\frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{c} \right)^2 \right]. \quad (2.9)$$

If this is substituted into the Larmor formula, it becomes

$$P = \frac{\mu_0 q^2 \gamma^4}{6\pi c} \left[\ddot{\vec{r}}^2 + \gamma^2 \left(\frac{\dot{\vec{r}} \cdot \ddot{\vec{r}}}{c} \right)^2 \right]. \quad (2.10)$$

Additionally, the acceleration can be broken up into components that are perpendicular and parallel to the velocity using the definitions

$$\vec{a}_{\parallel} = \left(\frac{\vec{v}}{|\vec{v}|} \cdot \vec{a} \right) \hat{v}, \quad (2.11)$$

and

$$a^2 = a_{\perp}^2 + a_{\parallel}^2. \quad (2.12)$$

This means that Equation 2.10 can be rewritten as

$$P = \frac{\mu_0 q^2}{6\pi c} \left[|\gamma^2 \ddot{\vec{r}}_{\parallel}|^2 + |\gamma^3 \ddot{\vec{r}}_{\perp}|^2 \right], \quad (2.13)$$

which will be more convenient than the form of the equation given in Equation 2.10 when applying the Fourier transform.

2.2 Fourier Transforms

The Fourier transform is an extension of the Fourier series found by Joseph Fourier (Bracewell, 1986). It is a mathematical transform that decomposes a function (often a function of time) into its consistent frequencies (Giordano and Nakanishi, 2006). Another way to state this is that it projects a function in the time domain onto the frequency domain. The Fourier transform $\hat{f}(\nu)$, of a function $f(t)$, is defined by

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} dt. \quad (2.14)$$

An important theorem from Fourier analysis, that will be used in subsequent chapters, is Parseval's theorem, which states that

$$\int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 d\nu = \int_{-\infty}^{\infty} |f(t)|^2 dt, \quad (2.15)$$

or in the case where $f(t)$ is a real function,

$$\int_0^{\infty} |\hat{f}(\nu)|^2 d\nu = \int_{-\infty}^0 |f(t)|^2 dt. \quad (2.16)$$

Applying Parseval's theorem to Equation 2.13,

$$P = \int_0^{\infty} I(\nu) d\nu = \int_0^{\infty} \frac{\mu_0 q^2}{3\pi c} \left[|\gamma^2 \ddot{r}_{\parallel}(\nu)|^2 + |\gamma^3 \ddot{r}_{\perp}(\nu)|^2 \right] d\nu, \quad (2.17)$$

where $\ddot{r}_{\parallel}(\nu)$, $\ddot{r}_{\perp}(\nu)$ and γ are the Fourier transforms of the corresponding terms (Longair, 2011).

From Equation 2.17, it follows that the spectral energy distribution, $I(\nu)$, of radiation from an accelerated particle is given by

$$I(\nu) = \frac{\mu_0 q^2}{3\pi c} \left[|\gamma^2 \ddot{r}_{\parallel}(\nu)|^2 + |\gamma^3 \ddot{r}_{\perp}(\nu)|^2 \right]. \quad (2.18)$$

2.2.1 Discrete Fourier transform

The discrete Fourier transform (DFT), like the regular Fourier transform, converts a data set from the time domain to the frequency domain. However, the DFT is not applied on a continuous function, but rather on a finite set of equally spaced samples (Giordano and Nakanishi, 2006). If $\{x_n\} = x_0, x_1, \dots, x_{N-1}$ is a set of N complex numbers, and $\{y_k\} = y_0, y_1, \dots, y_{N-1}$ is the resulting set of complex numbers in the frequency space, then the DFT can be defined as

$$y_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn}. \quad (2.19)$$

Furthermore, if $\{x_n\}$ is a set of samples in the time domain — which is equally spaced on the time domain, with a spacing of dt — then the resulting transformed dataset $\{y_k\}$ will have a frequency spacing of $df = \frac{1}{Ndt}$. There also exists an important limit known as the *Nyquist Frequency* ($f_{Nyquist} = \frac{1}{2dt}$) which is the largest spectral component that can be recovered using a DFT for the given dataset (Giordano and Nakanishi, 2006).

2.2.2 Fast Fourier transform and the FFTW package

Though the DFT is a useful and fairly robust tool with which to numerically calculate Fourier transforms of a given data set, calculating the summation given in Equation 2.19 is very computationally heavy for large datasets and, as such, requires N^2 calculations. If N is sufficiently large, the computational time needed to calculate the Fourier transform becomes unfeasible (Giordano and Nakanishi, 2006). For this reason, different computational algorithms, grouped under fast Fourier transforms (FFTs), have been developed that rely on grouping the various terms and factors in the DFT. This is done to minimise the amount of calculations necessary to determine the Fourier transform of a given dataset. These algorithms are typically dependent on the number of samples in the given dataset N (Giordano and Nakanishi, 2006), and can typically reduce the number of calculations needed from N^2 calculations to $N \log(N)$.

For particularly large datasets, or in the case where many FFTs are necessary, further acceleration of the calculations can be achieved by calibrating the FFT algorithm to the hardware used (Frigo and Johnson, 2005). It is in this area that the usefulness of the FFTW package arises, as it relies on a *planner* routine that runs pre-written *codelets* based on various FFT algorithms (such as the Cooley-Tukey and prime factor algorithms). This is to establish a FFT algorithm that is optimized for both the properties of the dataset that is transformed, and the

hardware in use ([Frigo and Johnson, 2005](#)).

As the model and code implementation requires taking multiple Fourier transforms of large datasets, I make use of the FFTW package in order to minimise the computational time required.

Chapter 3

The model

This chapter describes the model used to determine the gravitational acceleration of a charged particle in a near inspiral-phase BBH system. The gravitational acceleration of the particle will be determined using the Paczynski-Wiita gravitational potential, as described in Section 1.2.4. Additionally, we introduce the concept of retarded time (analogous to how it is used in electrodynamics) in order to determine the acceleration of the particle. For this purpose, the time dependent motion of the BHs in the BBH system, as well as that of the particle, has to be determined.

3.1 Solving the effective one-body problem

Taking the definition of the effective one-body problem (as discussed in section 1.2.6), the analytical solution of the circularized, near-inspiral BBH orbit can now be derived by using the classical Newtonian gravitational potential. Firstly, from the classical definition of angular momentum $\vec{l} = \vec{r} \times m\vec{r}$, it is evident that the motion of the effective one-body problem is restricted to one plane of motion;

$$\vec{r} \cdot \vec{l} = \vec{r} \cdot (\vec{r} \times m\vec{r}) = 0. \quad (3.1)$$

Taking this into consideration, the Lagrangian L can be constructed for the one-body system using the classical Newtonian gravitational potential $\Phi = -\frac{GM}{r}$ in the $r - \phi$ -plane as follows:

$$\begin{aligned} L &= K - V \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{GM}{r}. \end{aligned} \quad (3.2)$$

where, as described in Section 1.2.6, μ is the reduced mass of the binary system, and M is the total mass of the binary system. Substituting the constructed Lagrangian into Lagrange's equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \quad (3.3)$$

with q the relevant coordinate, for both the r and ϕ coordinates, it follows that

$$-\mu \ddot{r} + \mu r \dot{\phi}^2 - \frac{GM}{r^2} = 0 \quad (3.4)$$

$$-\frac{d}{dt} (\mu r^2 \dot{\phi}) = 0. \quad (3.5)$$

From Equation 3.5, it is evident that

$$\mu r^2 \dot{\phi} = P_\phi, \quad (3.6)$$

with P_ϕ a constant in time. Solving Equation 3.6 for $\dot{\phi}$ and substituting it back into Equation 3.4, it becomes

$$-\mu \ddot{r} + \frac{P_\phi^2}{\mu r^3} - \frac{GM}{r^2} = 0. \quad (3.7)$$

Considering that a typical BBH orbit is expected to be circular when it is close to the inspiral-phase, r can be taken as $r = r_0$, with r_0 a constant. From this it follows that Equation 3.7 becomes

$$\begin{aligned} \frac{P_\phi^2}{\mu r_0^3} - \frac{GM}{r_0^2} &= 0 \\ \Rightarrow r_0 &= \frac{P_\phi^2}{\mu GM}. \end{aligned} \quad (3.8)$$

Substituting this back into Equation 3.6, it becomes

$$\begin{aligned} \dot{\phi} &= \frac{\mu G^2 M^2}{P_\phi^3} \\ \Rightarrow \phi &= \frac{\mu G^2 M^2}{P_\phi^3} t, \end{aligned} \quad (3.9)$$

3.2 Calculating the acceleration of a charged particle in a near-inspiral phase BBH merger 32

where $t_0 = 0$ and $\phi_0 = 0$. Transforming this result from polar coordinates to Cartesian coordinates, the position vector $\vec{r}$ of the reduced mass in the CoM frame is given by

$$\vec{r} = \begin{pmatrix} \frac{P_\phi^2}{\mu GM} \cos\left(\frac{\mu G^2 M^2}{P_\phi^3} t\right) \\ \frac{P_\phi^2}{\mu GM} \sin\left(\frac{\mu G^2 M^2}{P_\phi^3} t\right) \end{pmatrix}. \quad (3.10)$$

Furthermore, the position vectors of the BHs, with masses M_1 and M_2 , as a function of time in the CoM frame is given by

$$\begin{aligned} \vec{r}_1 &= \begin{pmatrix} \frac{M_2 P_\phi^2}{\mu G M^2} \cos\left(\frac{\mu G^2 M^2}{P_\phi^3} t\right) \\ \frac{M_2 P_\phi^2}{\mu G M^2} \sin\left(\frac{\mu G^2 M^2}{P_\phi^3} t\right) \end{pmatrix}, \quad \text{and} \\ \vec{r}_2 &= - \begin{pmatrix} \frac{M_1 P_\phi^2}{\mu G M^2} \cos\left(\frac{\mu G^2 M^2}{P_\phi^3} t\right) \\ \frac{M_1 P_\phi^2}{\mu G M^2} \sin\left(\frac{\mu G^2 M^2}{P_\phi^3} t\right) \end{pmatrix}. \end{aligned} \quad (3.11)$$

If $\vec{r}_{CoM}$ is the position vector of the CoM in the lab frame, the position vectors of the BHs in the lab frame are

$$\begin{aligned} \vec{r}'_1 &= \vec{r}_{CoM} - \vec{r}_1 \\ \vec{r}'_2 &= \vec{r}_{CoM} - \vec{r}_2. \end{aligned} \quad (3.12)$$

3.2 Calculating the acceleration of a charged particle in a near-inspiral phase BBH merger

Considering a charged particle, with mass m_e , position $\vec{r}_{m_e}$, and velocity $\vec{v}_{m_e}$ in the lab frame, if the particle is under the influence of a force $\vec{F}$, then

$$\frac{d\vec{p}}{dt} = \vec{F}, \quad (3.13)$$

3.2.1 Determining $\vec{F}$

The Newton-Raphson method

The Newton-Raphson method, also known as Newton's method, is a rather simple method to numerically find the root of a given function by approximating the function around some initial estimate for the root by the tangent of the function (Atkinson, 2008). Suppose we have a differentiable function $f(x)$, with the derivative of the function $f'(x)$. Taking x_r as one of the roots of the function, it then follows that $f(x_r) = 0$. Approximating $f(x)$ at some point x_n close to the root x_r by the tangential curve,

$$y = f'(x_n)(x - x_n) + f(x_n), \quad (3.14)$$

some point x_{n+1} , that is closer to the root x_r , can be found from the fact that $y = 0$ by iteratively calculating

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad (3.15)$$

where after each iteration, x_{n+1} is taken to be the new estimate. This is done until $|x_{n+1} - x_n| < \epsilon$, where ϵ is a chosen acceptable threshold.

Determining the retarded position of the BHs

To determine $\vec{F}$, the retarded positions of the BHs, with respect to the position of the charged particle, has to be determined first. For this purpose, the definition of retarded time t_r is

$$t_r = t - \frac{1}{c} |\vec{r} - \vec{r}_k|, \quad (3.16)$$

where $\vec{r}$ is the position of the particle and $\vec{r}_k$ is the position of the BH at time t_r . Substituting the position of the particle at time t and the position of the BH, described by Equations 3.11 and 3.12 as evaluated at time t_r , gives

$$t_r = t - \frac{1}{c} \left[(x - A_k \cos(\omega t_r))^2 + (y - A_k \sin(\omega t_r))^2 + z^2 \right]^{\frac{1}{2}}, \quad (3.17)$$

where $x = x_{m_e} - x_{CoM}$, $y = y_{m_e} - y_{CoM}$, $z = z_{m_e} - z_{CoM}$, and $\omega = \frac{\mu G^2 M^2}{P_\phi^3}$. Additionally, $k = 1, 2$ with $A_1 = \frac{M_2 P_\phi^2}{\mu G M^2}$ and $A_2 = -\frac{M_1 P_\phi^2}{\mu G M^2}$. Applying the Newton-Raphson method to this, as described

in the previous section, t_r can numerically be found with

$$f(t_r) = t - \frac{1}{c} \left[(x - A_k \cos(\omega t_r))^2 + (y - A_k \sin(\omega t_r))^2 + z^2 \right]^{\frac{1}{2}} - t_r, \quad (3.18)$$

$$f'(t_r) = -\frac{1}{2c} \frac{(2[x - A_k \cos(\omega t_r)] A_k \omega \sin(\omega t_r) - 2[y - A_k \sin(\omega t_r)] A_k \omega \cos(\omega t_r))}{\left[(x - A_k \cos(\omega t_r))^2 + (y - A_k \sin(\omega t_r))^2 + z^2 \right]^{\frac{1}{2}}} - 1, \quad (3.19)$$

and

$$t_{r,n+1} = t_{r,n} - \frac{f(t_{r,n})}{f'(t_{r,n})}, \quad (3.20)$$

where $t_{r,0} = t - \frac{|\vec{r}|}{c}$. Doing this for both BHs with Masses M_1 and M_2 , gives the respective retarded times t_{r,M_1} and t_{r,M_2} with which to determine the retarded positions of the individual BH at time t .

Determining $\vec{F}$ using the retarded times t_{r,M_1} and t_{r,M_2}

Once we have the retarded times t_{r,M_1} and t_{r,M_2} , $\vec{F}$ can be determined. Using the Paczynski-Wiita potential, as described in Section 1.2.4,

$$\vec{F} = m_e \vec{\nabla} \left[\Phi_{PW,1}(|\vec{r}_{m_e} - \vec{r}'_1(t_{r,M_1})|) + \Phi_{PW,2}(|\vec{r}_{m_e} - \vec{r}'_2(t_{r,M_2})|) \right], \quad (3.21)$$

where

$$\Phi_{PW,s}(r) = -\frac{GM_s}{r - r_{G,s}}, \quad r_{G,s} = \frac{2GM_s}{c^2}, \quad (3.22)$$

$s = 1, 2$. Since

$$\begin{aligned} \vec{\nabla} \Phi_{PW,s}(r) &= -GM_s \vec{\nabla} (r - r_{G,s})^{-1} \\ &= -\frac{GM_s}{r(r - r_{G,s})^2} (x\hat{x} + y\hat{y} + z\hat{z}), \end{aligned} \quad (3.23)$$

it follows that Equation 3.21 simplifies to

$$\vec{F} = -m_e \left[\frac{GM_1}{r_{d,1} (r_{d,1} - r_{G,1})^2} (x_{d,1}\hat{x} + y_{d,1}\hat{y} + z_{d,1}\hat{z}) + \frac{GM_2}{r_{d,2} (r_{d,2} - r_{G,2})^2} (x_{d,2}\hat{x} + y_{d,2}\hat{y} + z_{d,2}\hat{z}) \right], \quad (3.24)$$

where $r_{d,v} = |\vec{r}_{m_e} - \vec{r}'_v(t_{r,M_v})|$, $x_{d,v} = (\vec{r}_{m_e} - \vec{r}'_v(t_{r,M_v}))_x$, $y_{d,v} = (\vec{r}_{m_e} - \vec{r}'_v(t_{r,M_v}))_y$, $z_{d,v} = (\vec{r}_{m_e} - \vec{r}'_v(t_{r,M_v}))_z$, and $v = 1, 2$.

3.2.2 Calculating the position $\vec{r}_i$ and acceleration $\vec{a}_i$ of the particle

Determining $\vec{v}$ from $\vec{p}$

From the definition $\vec{p} = \gamma m_e \vec{v}$,

$$\begin{aligned} p^2 &= m_e^2 \frac{v^2}{1 - \frac{v^2}{c^2}} \\ \Rightarrow \frac{p^2}{m_e^2} \left(1 - \frac{v^2}{c^2} \right) &= v^2 \\ \Rightarrow \frac{p^2}{m_e^2} &= v^2 \left(1 + \frac{p^2}{m_e^2 c^2} \right) \\ \Rightarrow v^2 &= \frac{p^2}{m_e^2} \left(1 + \frac{p^2}{m_e^2 c^2} \right)^{-1}. \end{aligned} \quad (3.25)$$

Substituting this back into the definition of momentum for $\vec{p}$,

$$\begin{aligned} \vec{p} &= \gamma m_e \vec{v} \\ &= m_e \frac{\vec{v}}{\sqrt{1 - \frac{p^2}{m_e^2 c^2} \left(1 + \frac{p^2}{m_e^2 c^2} \right)^{-1}}} \\ \Rightarrow \vec{v} &= \frac{\vec{p}}{m_e} \sqrt{1 - \frac{p^2}{m_e^2 c^2} \left(1 + \frac{p^2}{m_e^2 c^2} \right)^{-1}}. \end{aligned} \quad (3.26)$$

Calculating $\vec{r}$ from $\vec{v}$

As stated earlier in this section, the position $\vec{r}$ of the particle at time t is required to calculate the retarded force $\vec{F}$. For a given velocity we can calculate the position from the definition of

velocity,

$$\vec{v} = \frac{d\vec{r}}{dt}. \quad (3.27)$$

Substituting equation 3.26 into this and converting to geometrized units¹³ we find a set of coupled first order ordinary differential equations (ODEs)

$$\begin{aligned} \frac{d\vec{p}}{dt} &= \vec{F}(t, \vec{r}) \\ \frac{d\vec{r}}{dt} &= \frac{\vec{p}}{m_e} \sqrt{1 - \frac{p^2}{m_e^2} \left(1 + \frac{p^2}{m_e^2}\right)^{-1}} \end{aligned} \quad (3.28)$$

that must be solved simultaneously, with $\vec{F}$ calculated as described in equation 3.24. In order to simultaneously solve this set of ODEs we use the 4th-order Runge-Kutta method, which is a very stable numerical technique for solving ODEs, that is accurate to an order of $O(dx^4)$; with dx the step size used in the calculation (Faires and Burden, 2003). The 4th-order Runge-Kutta method, in essence, iteratively calculates the values of a set of ODEs from preceding values by evaluating the ODEs at 4 points across the range $(x, x+dx)$ and taking the new value as the weighted average of the calculated values (Süli, 2010).

Let

$$\vec{Y} = \begin{bmatrix} \vec{p} \\ \vec{r} \end{bmatrix}, \quad (3.29) \quad \text{and} \quad \vec{f}(t, \vec{Y}) = \begin{bmatrix} \vec{F}(t, \vec{r}) \\ \frac{\vec{p}}{m_e} \sqrt{1 - \frac{p^2}{m_e^2} \left(1 + \frac{p^2}{m_e^2}\right)^{-1}} \end{bmatrix}. \quad (3.30)$$

Then we iteratively calculate the values of $\vec{Y}_{n+1}$ from preceding values using the weighted average

$$\vec{Y}_{n+1} = \vec{Y}_n + \frac{dt}{6} (\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4), \quad (3.31)$$

¹³We convert to geometrized units here for better numerical accuracy as terms containing G and c can suffer from numerical rounding errors.

where

$$\begin{aligned}
 \vec{k}_1 &= \vec{f}(t_n, \vec{Y}_n) \\
 \vec{k}_2 &= \vec{f}\left(t_n + \frac{1}{2}dt, \vec{Y}_n + \frac{1}{2}dt\vec{k}_1\right) \\
 \vec{k}_3 &= \vec{f}\left(t_n + \frac{1}{2}dt, \vec{Y}_n + \frac{1}{2}dt\vec{k}_2\right) \\
 \vec{k}_4 &= \vec{f}(t_n + dt, \vec{Y}_n + dt\vec{k}_3)
 \end{aligned} \tag{3.32}$$

(Faires and Burden, 2003; Süli, 2010).

Calculating the acceleration $\vec{a}_i$ from the velocity $\vec{v}_i$

In order to calculate the acceleration $\vec{a}_i$ of the particle under the influence of the retarded force $\vec{F}_i$, Equation 1.9 has to be rewritten. Taking the scalar product of Equation 1.9 with $\vec{v}$,

$$\begin{aligned}
 \vec{F} \cdot \vec{v} &= m \left[\frac{\gamma^3}{c^2} (\vec{v} \cdot \vec{a}) \vec{v} + \gamma \vec{a} \right] \cdot \vec{v}, \\
 &= m \left[\frac{\gamma^3}{c^2} (\vec{v} \cdot \vec{a}) (\vec{v} \cdot \vec{v}) + \gamma \vec{a} \cdot \vec{v} \right], \\
 &= m [\beta^2 + \gamma^{-2}] (\gamma^3 (\vec{a} \cdot \vec{v})), \\
 &= m [\beta^2 + 1 - \beta^2] (\gamma^3 (\vec{a} \cdot \vec{v})), \\
 &= m \gamma^3 (\vec{a} \cdot \vec{v}),
 \end{aligned} \tag{3.33}$$

where $\beta^2 = \frac{\vec{v} \cdot \vec{v}}{c^2}$. Substituting this back into Equation 1.9,

$$\begin{aligned}
 \vec{F} &= m \left[\frac{(\vec{v} \cdot \vec{F})}{mc^2} \vec{v} + \gamma \vec{a} \right], \\
 \Rightarrow \vec{a} &= \frac{1}{m\gamma} \left[\vec{F} - \frac{(\vec{v} \cdot \vec{F})}{c^2} \vec{v} \right],
 \end{aligned} \tag{3.34}$$

where the discretized form applied to the particle is given by

$$\vec{a}_i = \frac{1}{m_e \gamma} \left[\vec{F}_i - \frac{(\vec{v}_i \cdot \vec{F}_i)}{c^2} \vec{v}_i \right]. \tag{3.35}$$

Finally, the acceleration of the particle $\vec{a}_i$ is separated into its parallel and perpendicular components, with respect to the motion of the particle. To this end, the parallel component $\vec{a}_{\parallel,i}$ is calculated as

$$\vec{a}_{\parallel,i} = \left(\vec{a}_i \cdot \frac{\vec{v}_i}{|\vec{v}_i|} \right) \hat{v}_i, \quad (3.36)$$

with the corresponding perpendicular component

$$\vec{a}_{\perp,i} = \vec{a}_i - \vec{a}_{\parallel,i}. \quad (3.37)$$

3.2.3 Cubic spline interpolation

The cubic spline interpolation method is a special case of spline interpolation that attempts to avoid Runge's phenomenon, which is the problem of oscillation of a higher degree polynomial interpolation at the end points of equally spaced intervals (De Boor, 1978). This is done by interpolating between the points of a given data set, using a twice continuously differentiable function $S(x)$. Given a set of $n + 1$ data points (x_i, y_i) , with $x_0 = \alpha$, $x_n = \beta$ and x_i in ascending order, the spline $S(x)$ is a function that meets the following requirements:

- $S(x) \in \mathcal{C}^2[\alpha, \beta]$
- On the subintervals $[x_{i-1}, x_i]$, with $i = 1, 2, \dots, n$, $S(x)$ is a 3rd degree polynomial
- $S(x_i) = y_i$ for $i = 0, 1, \dots, n$

$S(x)$ is then defined as

$$S(x) = \sum_{i=1}^n ([u(x - x_{i-1}) - u(x - x_i)] C_i(x)), \quad (3.38)$$

where $u(x)$ is the Heaviside step function that is defined as:

$$u(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1, & \text{for } x \leq 0 \end{cases} \quad (3.39)$$

and

$$C_i(x) = a_i + b_i x + c_i x^2 + d_i x^3 \quad (3.40)$$

(De Boor, 1978). From the fact that $S(x)$ should be twice continuously differentiable, constraints on $C_i(x)$ can be determined. For the case where $i = 1, 2, \dots, n$,

$$\begin{aligned} C_i(x_i) &\stackrel{!}{=} y_i & \text{and} \\ C_i(x_{i-1}) &\stackrel{!}{=} y_{i-1}. \end{aligned} \quad (3.41)$$

Additionally, for $i = 1, 2, \dots, n-1$ it follows that

$$C'_i(x_i) \stackrel{!}{=} C'_{i+1}(x_i), \quad (3.42)$$

and

$$C''_i(x_i) \stackrel{!}{=} C''_{i+1}(x_i). \quad (3.43)$$

From the constraints given in equations 3.41 through to 3.43, there is a set of $4n-2$ equations with which to determine the $4n$ variables in $S(x)$. In order to find two additional equations with which to determine the spline $S(x)$, constraints on the endpoints of $S(x)$ have to be found by considering the first derivative of $S(x)$ on the boundaries at the endpoints of $S(x)$. Typically these constraints are formulated based on predetermined knowledge of the dataset, but in the case where nothing is known about the derivatives on the endpoints of $S(x)$, the *not-a-knot* condition should be used (De Boor, 1978). The *not-a-knot* condition adds the two additional constraints

$$C'''_{k-1}(x_k) = C'''_k(x_k), \quad \text{with } k = 2, n. \quad (3.44)$$

With the added constraint in Equation 3.44, we then have a set of $4n$ equations and $4n$ unknowns with which to determine $S(x)$.

3.3 Calculating the SED of charged particles in a near-inspiral BBH merger

From Equation 2.18, the SED of a single accelerated charged particle is given by

$$I(\nu) = \frac{\mu_0 q^2}{3\pi c} \left[|\gamma^2(\nu) \vec{a}_{\parallel}(\nu)|^2 + |\gamma^3(\nu) \vec{a}_{\perp}(\nu)|^2 \right]. \quad (3.45)$$

From this, it is evident that $I_{\parallel,k} = |\gamma^2(v_k) \vec{a}_{\parallel}(v_k)|^2$, and $I_{\perp,k} = |\gamma^3(v_k) \vec{a}_{\perp}(v_k)|^2$ needs to be determined in frequency space. This is done by simply taking the DFT¹⁴, as defined in Equation 2.19, of the relevant products as defined in Equation 2.18,

$$I_{\parallel,k} = \sum_{j=0}^{N-1} |\gamma_j^2 \vec{a}_{\parallel,j}|^2 e^{-\frac{2\pi i}{N}kj} \quad (3.46)$$

$$I_{\perp,k} = \sum_{j=0}^{N-1} |\gamma_j^3 \vec{a}_{\perp,j}|^2 e^{-\frac{2\pi i}{N}kj}, \quad (3.47)$$

with $i = \sqrt{-1}$ and N is the number of data points. From this

$$I(v_k) = \frac{\mu_0 q^2}{3\pi c} [I_{\parallel,k} + I_{\perp,k}], \quad (3.48)$$

where $v_k = kd v$.

Equation 3.48 gives the SED of a single particle that radiates due to gravitational acceleration in a near-inspiral BBH merger. Considering a set of $N_{particles}$ particles, each with an SED $I_n(v)$, the total SED of the system $I_{Total}(v)$ is given by

$$I_{Total}(v) = \sum_{n=1}^{N_{particles}} I_n(v). \quad (3.49)$$

Since it is expected that the radiation contribution of each particle is incoherent, the total SED can be calculated in this manner without the need to consider possible interference when summing over the fluxes of the particles. When considering a set of discrete SEDs $I_n(v_k)$ for each particle, however, this process becomes somewhat more complicated. When applying the DFT in the calculation of the SEDs of each of the $N_{particles}$ particles, the implicit discrete frequency steps dv_n within the DFTs are not necessarily of the same size.¹⁵ To account for this, each individual particle SED $I_n(v)$ is interpolated to comparable frequency increments, using Cubic Spline Interpolation (as described in Section 3.2.3).

¹⁴As will be discussed in the final section of this chapter, though we do take the DFT of the datasets, it is calculated using FFT algorithms by using the FFTW C++ Fourier Transform Library.

¹⁵This comes from the fact that the size N of the input data set of the DFT will not be the same size for each particle. From the definition of the DFT as given in section 2.2.1, we have that $dv = \frac{1}{Ndt}$, which leads to different frequency increment sizes for variable sizes of the data sets.

3.4 Implementing the model in code

The model described in the earlier sections of this chapter is implemented in two separate programs. The first program is written in C++, with the time evolution of each of the position, velocity and acceleration vectors, as well as that of the Lorentz factor, saved as separate output datasets. The second program is a *Python* program that takes the acceleration and Lorentz factor datasets of the previous program as input, and calculates the Fourier transform of each particle (as described in Section 3.3). This is done by using the pyFFTW package that gives a convenient way of implementing the FFTW C++ library within Python.

3.4.1 Calculating the position, velocity and acceleration of the charged particles

Due to the fact that the calculation of the position, velocity and acceleration vectors of each particle (described in Section 3.2) depends only on the positions of the BHs and the position of the particle itself, it is convenient to make use of the object-oriented class structures of the C++ language. It is also essential that the Open Multi-Processing (OMP) programming interface is used to parallelize the calculations. This is done by constructing an object class that calculates the relevant vectors described in Section 3.2. Each particle is then created as an object of this class, and given a set of initial position and velocity vectors determined from a uniform probability distribution. The evolution of these particle objects is then calculated independently of one another using multiple CPU cores through the *pragmaOMP* function of the OMP interface, taking care to terminate the calculation of the time evolution of the particles if the separation length between the particle and either of the BHs is larger than a given length or is smaller than the Swarzschild radius of the corresponding BHs. This is done to both reduce the computation time of the simulation by reducing the amount of unnecessary calculations, and in the case where the particle lies within the Swarzschild radius of one of the BHs, because it is unphysical for a particle to escape from inside the Swarzschild radius of a BH.

The use of the object-orientated class structures of C++ simplifies the parallelization process, as all of the variables for each of the particles are stored in their own memory segment of the machine in use. This helps to avoid multiple processes attempting to access the same memory segment of the machine at a given time. As the resulting datasets of the simulation of each particle are expected to be very large, the outputs are all stored in binary files to ensure both sufficient precision is maintained of each dataset, and to minimize the total storage space required by the outputs.

3.4.2 Testing the stability of the code

In the following section the numerical integrity of the developed code is tested. First, the stability of circular orbits around a single BH is tested. Secondly, the precession angle of a bound particle orbiting a Single BH is compared with what is expected from Equation 1.37.

Testing circular orbits

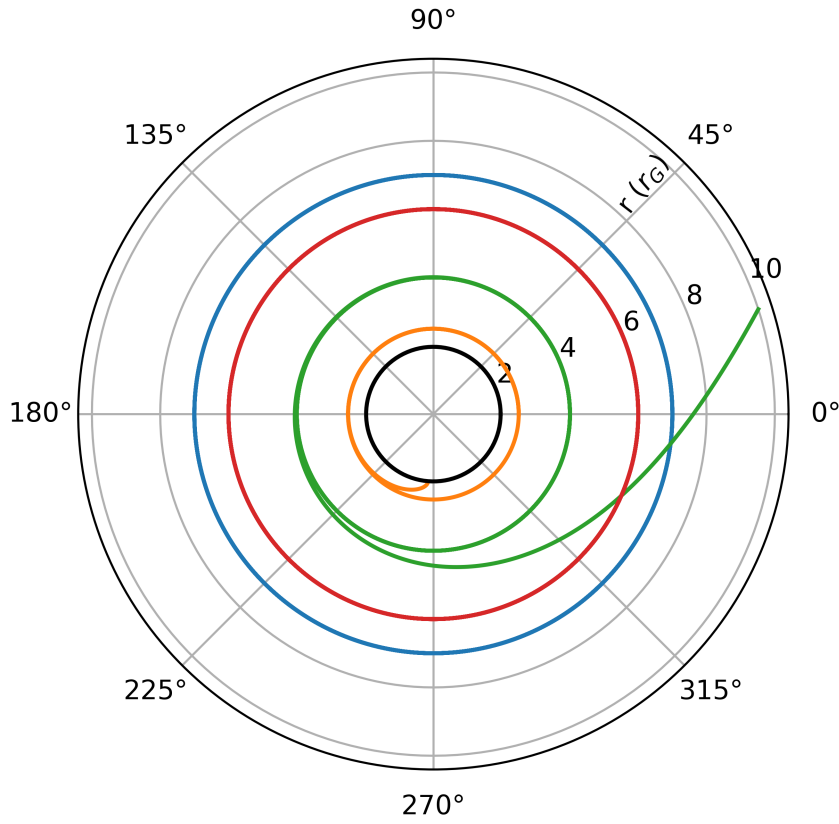


Fig. 3.1 Particle trajectories produced by the code for particles with initial radii $r_0 = 2.5r_G, 4r_G, 6r_G, 7r_G$. The results show stable circular orbits for radii larger than $6r_G$.

The angular velocity $\dot{\phi}$ of a particle on a circular orbiting in a central force field $\vec{F}(r) = F(r)\hat{r}$, is given by

$$\dot{\phi} = c\sqrt{2}\sqrt{\sqrt{g^4 + g^2} - g^2}, \quad (3.50)$$

where the dimensionless quantity g , for a given radius $r = r_0$, is defined as

$$g \equiv -\frac{r_0 F(r_0)}{2m_e c^2} = \frac{r_0 G M}{2c^2 (r_0 - r_s)^2}, \quad (3.51)$$

(Aguirregabiria et al., 2004). Taking this as initial velocities for several initial radii, the resulting orbits produced by the code are shown in Figure 3.1. The resulting orbits show stable circular orbits for circular orbits with $r > 6r_G$. For the orbit at $r = 4r_G$, 3 full circular orbits are executed before the accumulated truncation error of the 4th-order Runge-Kutta method becomes large enough to influence the circular trajectory of the particle.

Testing precessing orbits

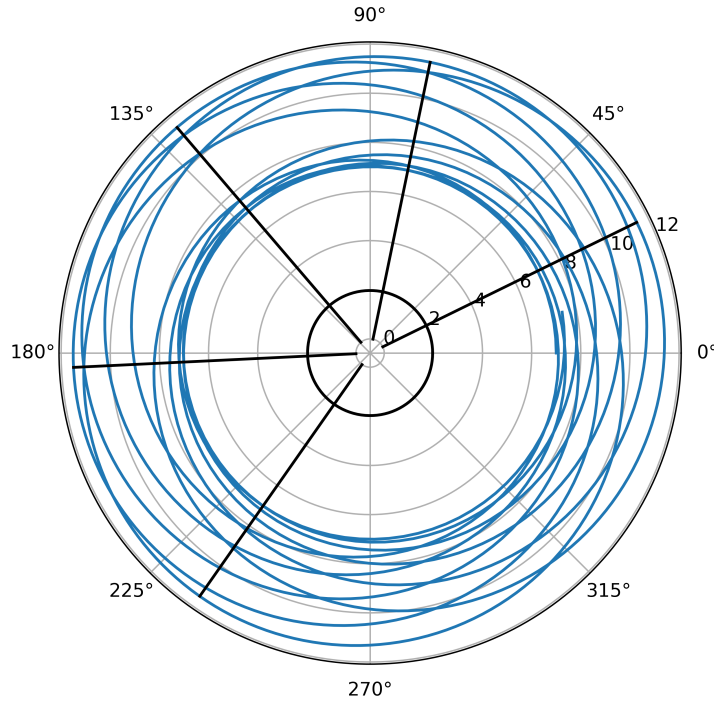


Fig. 3.2 Particle trajectory of a particle orbiting a single BH of mass $M = 30M_\odot$ with an initial velocity of $\vec{v} = [0, 0.5c]$, and position of $\vec{r} = [7r_G, 0]$. The black radial lines, show the positions of the apoapsides of the orbit at a radius of $r_+ = 11.509r_G$. The black circle at $r = 2r_G$ indicates the position of the event horizon of the BH.

In order to test the numerical stability of the developed code, the precession angle of the apoapsides of a precessing orbit produced by the code, as illustrated in Figure 3.2, is compared

to that calculated from Equation 1.37. To do this, it is necessary to numerically integrate Equation 1.37, noting that the integrand goes to infinity as $r \rightarrow r_{\pm}$, which may introduce numerical errors in the calculation. In order to account for this, Equation 1.37 is rewritten as

$$\begin{aligned} \frac{1}{2}\Delta\Theta &= I_1 + I_{mid} + I_2 \\ &= \int_{r_-}^{r_-+\epsilon} G(r) dr + \int_{r_-+\epsilon}^{r_+-\epsilon} G(r) dr + \int_{r_+-\epsilon}^{r_+} G(r) dr, \end{aligned} \quad (3.52)$$

with

$$G(r) = \left[\frac{h}{r^2 \left(1 + \frac{h^2}{r^2}\right)^{\frac{1}{2}}} \left(\left[\left(1 - \left(e + \frac{M}{r-r_s}\right)^{-2}\right) \left(1 + \frac{h^2}{r^2}\right) - \frac{h^2}{r^2} \right]^{-1} - 1 \right)^{\frac{1}{2}} \right], \quad (3.53)$$

and $\epsilon \ll r_{\pm}$.

Equation 3.53 can be rewritten as

$$G(r) = \frac{h}{r^2 \left(1 + \frac{h^2}{r^2}\right)^{\frac{1}{2}} (f(r))^{\frac{1}{2}}} (1 - f(r))^{\frac{1}{2}}, \quad (3.54)$$

where $f(r)$ is defined as

$$f(r) \equiv \left(1 - \left(e + \frac{M}{r-r_s}\right)^{-2}\right) \left(1 + \frac{h^2}{r^2}\right) - \frac{h^2}{r^2}, \quad (3.55)$$

and it follows that

$$f'(r) = -\frac{2M \left(\frac{h^2}{r^2} + 1\right)}{(r-r_s)^2 \left(\frac{M}{r-r_s} + e\right)^3} - \frac{2h^2 \left(1 - \frac{1}{\left(\frac{M}{r-r_s} + e\right)^2}\right)}{r^3} + \frac{2h^2}{r^3} \quad (3.56)$$

Performing a first order Taylor expansion of r^{-2} , $\left(1 + \frac{h^2}{r^2}\right)^{-\frac{1}{2}}$, $f(r)$, and $(1 - f(r))^{\frac{1}{2}}$ around r_- and changing the variable of integration to $y_1 = r - r_-$, it follows that

$$I_1 \approx \int_0^\epsilon h \left[\left(1 + \frac{h^2}{r_-^2}\right)^{-\frac{1}{2}} + \frac{h^2}{r_-^3} \left(1 + \frac{h^2}{r_-^2}\right)^{-\frac{3}{2}} y_1 \right] \left[\frac{1}{r_-^2} - \frac{2y_1}{r_-^3} \right] \frac{\left(1 + \frac{1}{2}f'(r_-)y_1\right)}{(f'(r_-)y_1)^{\frac{1}{2}}} dy_1. \quad (3.57)$$

Similarly, it follows that

$$I_2 \approx \int_{-\epsilon}^0 h \left[\left(1 + \frac{h^2}{r_+^2}\right)^{-\frac{1}{2}} + \frac{h^2}{r_+^3} \left(1 + \frac{h^2}{r_+^2}\right)^{-\frac{3}{2}} y_2 \right] \left[\frac{1}{r_+^2} - \frac{2y_2}{r_+^3} \right] \frac{(1 + \frac{1}{2}f'(r_+)y_2)}{(f'(r_+)y_2)^{\frac{1}{2}}} dy_2, \quad (3.58)$$

where $y_2 = r - r_+$.

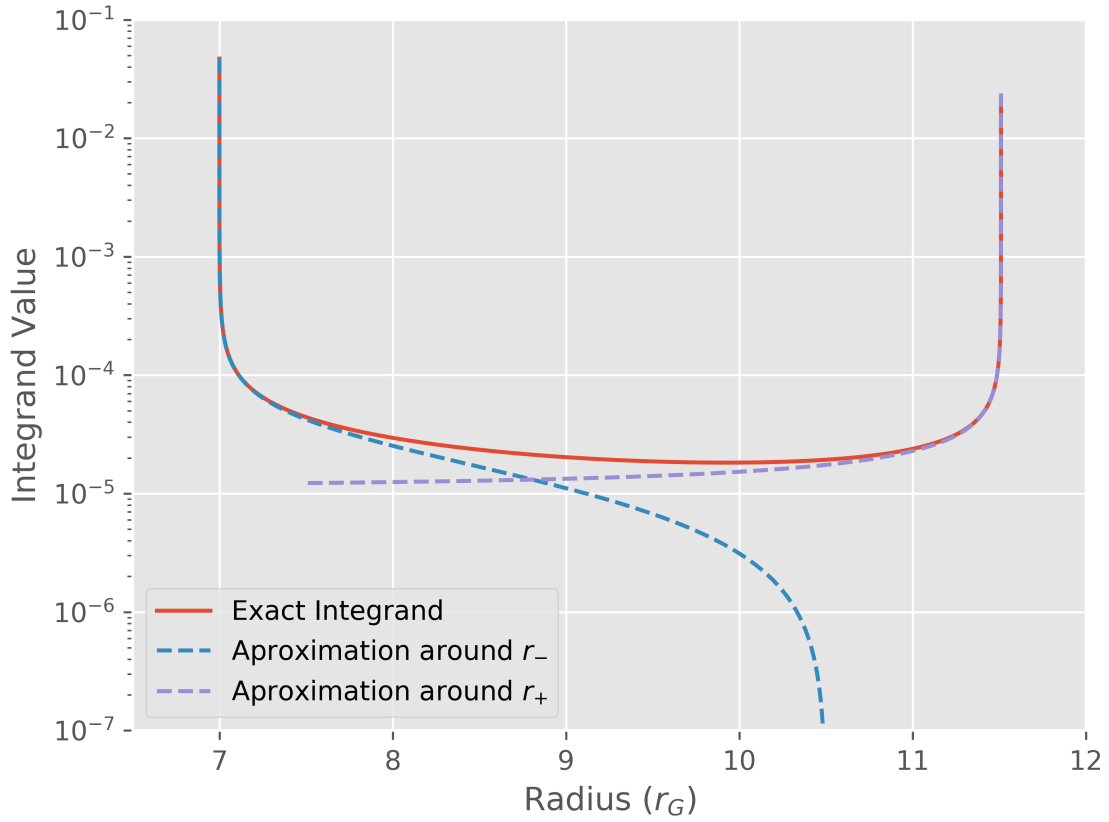


Fig. 3.3 Comparison of approximate integrands of I_1 and I_2 in Equations 3.57 and 3.58 for given values of $e = 0.955$, $h = 181865.335m$, for which $r_- = 7r_G$ and $r_+ = 11.509r_G$.

Figure 3.3 shows a comparison of the approximated integrands of I_1 and I_2 as found in Equations 3.57 and 3.58 to that of Equation 3.53. The figure confirms that the approximate integrands fit the exact integrand very well in the regions $r_- + \epsilon$ and $r_+ - \epsilon$, for I_1 and I_2 respectively.

Evaluating the integrals of Equations 3.57 and 3.58 analytically, it follows that

$$I_1 = \frac{\sqrt{f'(r_-)}}{\sqrt{\frac{h^2}{r_-^2} + 1}} \left[\frac{2}{5} \epsilon^{\frac{5}{2}} \left(\frac{h^3}{2h^2r_-^3 + 2r_-^5} - \frac{h}{r_-^3} - \frac{2h^3}{f'(r_-)(h^2r_-^4 + r_-^6)} \right) - \frac{2h^3\epsilon^{\frac{7}{2}}}{7(h^2r_-^4 + r_-^6)} \right] \\ + \frac{1}{\sqrt{f'(r_-)}\sqrt{\frac{h^2}{r_-^2} + 1}} \left[\frac{2\epsilon^{\frac{3}{2}}}{3} \left(\frac{f'(r_-)h}{2r_-^2} + \frac{h^3}{h^2r_-^3 + r_-^5} - \frac{2h}{r_-^3} \right) + \frac{2h\sqrt{\epsilon}}{r_-^2} \right], \quad (3.59)$$

and

$$I_2 = \frac{\sqrt{-f'(r_+)}}{\sqrt{\frac{h^2}{r_+^2} + 1}} \left[-\frac{2}{5} \epsilon^{\frac{5}{2}} \left(\frac{h^3}{2h^2r_+^3 + 2r_+^5} - \frac{h}{r_+^3} - \frac{2h^3}{f'(r_+)(h^2r_+^4 + r_+^6)} \right) - \frac{2h^3\epsilon^{\frac{7}{2}}}{7(h^2r_+^4 + r_+^6)} \right] \\ + \frac{1}{\sqrt{-f'(r_+)}\sqrt{\frac{h^2}{r_+^2} + 1}} \left[\frac{2\epsilon^{\frac{3}{2}}}{3} \left(\frac{f'(r_+)h}{2r_+^2} + \frac{h^3}{h^2r_+^3 + r_+^5} - \frac{2h}{r_+^3} \right) - \frac{2h\sqrt{\epsilon}}{r_+^2} \right]. \quad (3.60)$$

Finally, I_{mid} is evaluated numerically using the *Python Scipy.integrate.quad()* function.

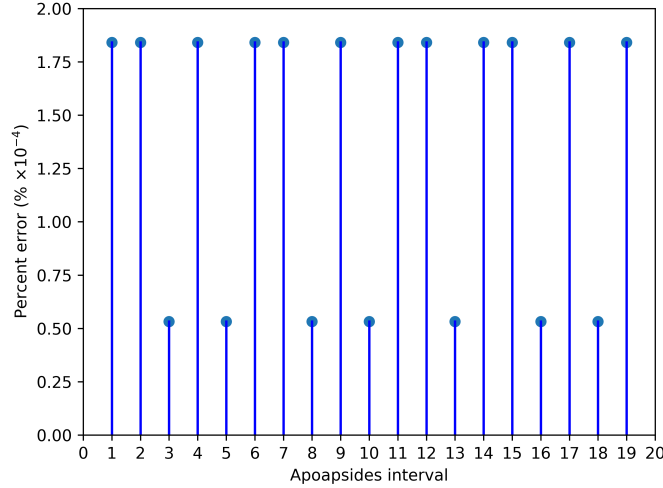


Fig. 3.4 Comparison of the precession angle determined for 19 consecutive precession intervals from the developed code for an initial velocity $\vec{v} = [0, 0.5c]$, and position of $\vec{r} = [7r_G, 0]$ to that expected from Equation 3.52.

Figure 3.4 shows the comparison of the precession angle $\Delta\Theta$ determined for 19 consecutive precession intervals from the developed code, to that calculated as described in the preceding part of this section. Here the initial velocity, and position is taken as $\vec{v} = [0, 0.5c]$, and $\vec{r} = [7r_G, 0]$ respectively and ϵ is set to $\epsilon = 0.001r_G$. The maximum deviation found for this comparison is $1.841 \times 10^{-4}\%$.

3.4.3 Calculating the SEDs of the particles

The acceleration and Lorentz factor outputs produced by the C++ program for each of the particles are loaded into the *Python* program. From here the two terms given in Equation 3.45 are calculated. The DFT of each term described in Equations 3.46 and 3.47 is calculated using the `pyfftw.interfaces.numpy_fft.fft()` function of the pyFFTW package. From here Equation 3.48 is evaluated and the result is stored into a *Hierarchical Data Format version 5 (HDF5)*¹⁶ file. The resulting SED of each particle is iteratively calculated and stored in this way.

After the SED of each particle is calculated and stored, each SED dataset is interpolated, using the `scipy.interpolate.CubicSpline()` function of the *scipy Python* package, which uses the *not-a-knot* constraint (described in Section 3.2.3) by default. This is done in order to have each dataset distributed over the same discrete frequency space. Finally, the total SED is calculated and stored into the same HDF5 file.

¹⁶The HDF5 file type is an open source file type that supports large, complex data, and has the benefit of storing the data in a *directory*-like structure that gives a more convenient method for accessing and manipulating the data than binary file types, while maintaining small storage sizes.

Chapter 4

Results

The model, as described in the previous chapter, is applied to a BBH system with two BHs with masses $M_1 = 30M_\odot$ and $M_2 = 35M_\odot$. The initial separation of the BHs are taken to be $r_0 = 1.0 \times 10^9 \text{ cm}$, with the position of the CoM of the system centred at the origin in the lab frame. A set of 40 particles are then randomly distributed in the system, with a uniform probability distribution.

An estimate of the expected coalescence time scale for the system is calculated, using equation 1.40, as $t_{\text{coal}} = 2.96 \times 10^3 \text{ s}$. Comparing this to the average time scale of the simulation that is $\sim 10^2 \text{ s}$, it is found that the time scale of the simulation is sufficiently small compared to the expected coalescence time scale. Regardless of this, the focus of this dissertation is the acceleration on sub-millisecond time scales and an initial estimate of possible accelerations achieved within the system.

Figure 4.1 shows the trajectories of three particles with initial positions

$$\vec{r}_1 = (-2.734 \times 10^{10}, -7.215 \times 10^{10}, 1.740 \times 10^{10}) \text{ cm},$$

$$\vec{r}_2 = (8.811 \times 10^9, -9.305 \times 10^9, -2.213 \times 10^{10}) \text{ cm}, \text{ and}$$

$$\vec{r}_3 = (3.318 \times 10^9, 1.356 \times 10^{10}, 6.008 \times 10^9) \text{ cm};$$

and initial velocities

$$\vec{v}_1 = (1.124, 4.636, -1.395) \times 10^{-5} c,$$

$$\vec{v}_2 = (-47.04, 32.97, 1.657) \times 10^{-6} c, \text{ and}$$

$\vec{v}_3 = (-23.09, -49.19, -6.877) \times 10^{-6} c$. The orbits of all three particles start out relatively close to the BHs, and chaotically orbit around one or both of the BHs. All of the particles distributed throughout the system follow similar trajectories, with the majority of the particles on trajectories eventually plunging into one of the BHs.

Figures 4.2, 4.3, and 4.4 show the components of the acceleration of the particles parallel and perpendicular to the direction of motion as a function of time. Figures 4.5, 4.6, and 4.7 show the speed of the orbiting particles as a function of time. Each of the figures indicates that the orbiting particles reach speeds upward of $0.1c$, justifying the use of the relativistic Larmor formula.

Figures 4.8, 4.9 and 4.10 show the SEDs of each of the three particles, as calculated from Equation 3.48. Figures 4.8, 4.9 and 4.10 indicate that the interpolated function of the SED of each particle fit the generated data quite well. Furthermore, the SEDs of each of the particles show a steep drop-off at $10^4 - 10^5 \text{ Hz}$, with an initial hard radiation spectrum at lower frequencies. Figure 4.11 shows the total SED when summing over all of the particles. The total SED shows the same trend as each of the individual particle SEDs, with a steep cut-off at 10^5 Hz , due to the end of our current resolution range.

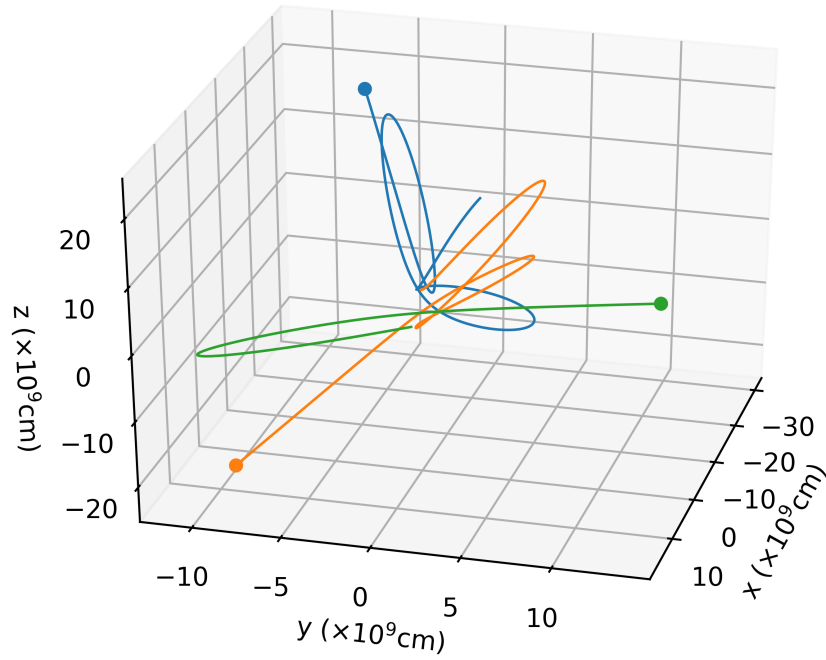


Fig. 4.1 Figure showing the trajectories for 3 particles in the BBH system with initial positions $\vec{r}_1 = (-2.734 \times 10^{10}, -7.215 \times 10^{10}, 1.740 \times 10^{10}) \text{ cm}$, $\vec{r}_2 = (8.811 \times 10^9, -9.305 \times 10^9, -2.213 \times 10^{10}) \text{ cm}$, and $\vec{r}_3 = (3.318 \times 10^9, 1.356 \times 10^{10}, 6.008 \times 10^9) \text{ cm}$; and initial velocities $\vec{v}_1 = (1.124, 4.636, -1.395) \times 10^{-5} c$, $\vec{v}_2 = (-47.04, 32.97, 1.657) \times 10^{-6} c$, and $\vec{v}_3 = (-23.09, -49.19, -6.877) \times 10^{-6} c$. The dots indicate the initial position of each of the particles.

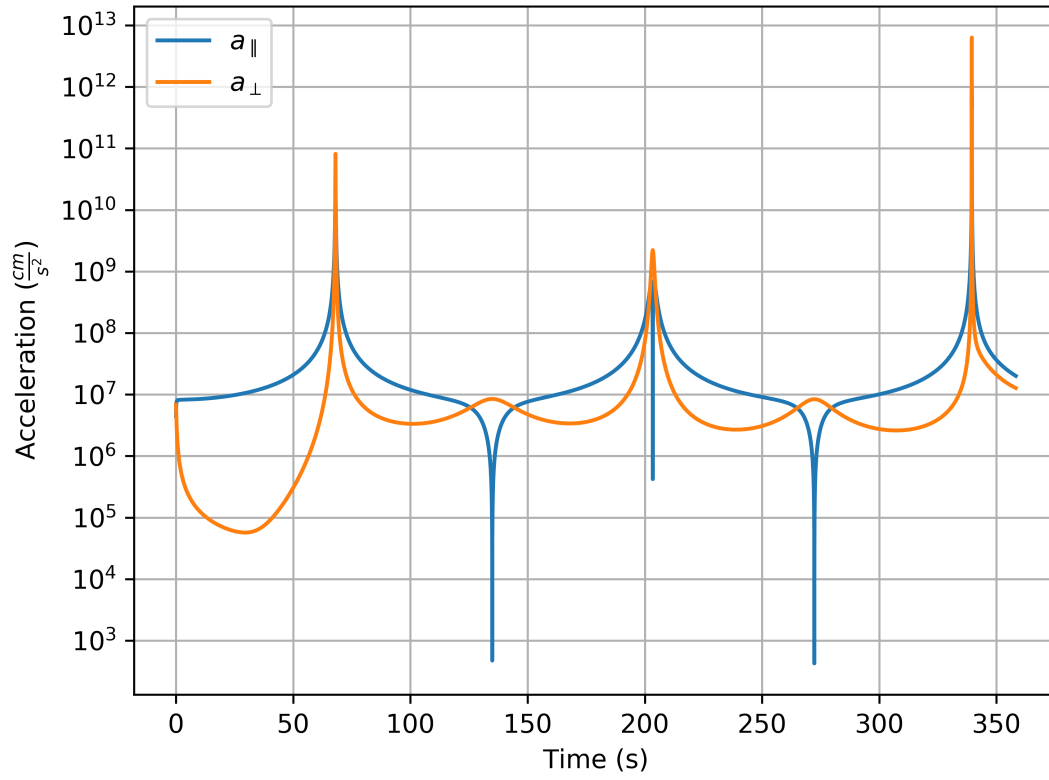


Fig. 4.2 Acceleration components as a function of time for particle 1, with an initial position of $\vec{r}_1 = (-2.734 \times 10^{10}, -7.215 \times 10^{10}, 1.740 \times 10^{10}) \text{ cm}$ and initial velocity $\vec{v}_1 = (1.124, 4.636, -1.395) \times 10^{-5} c$.

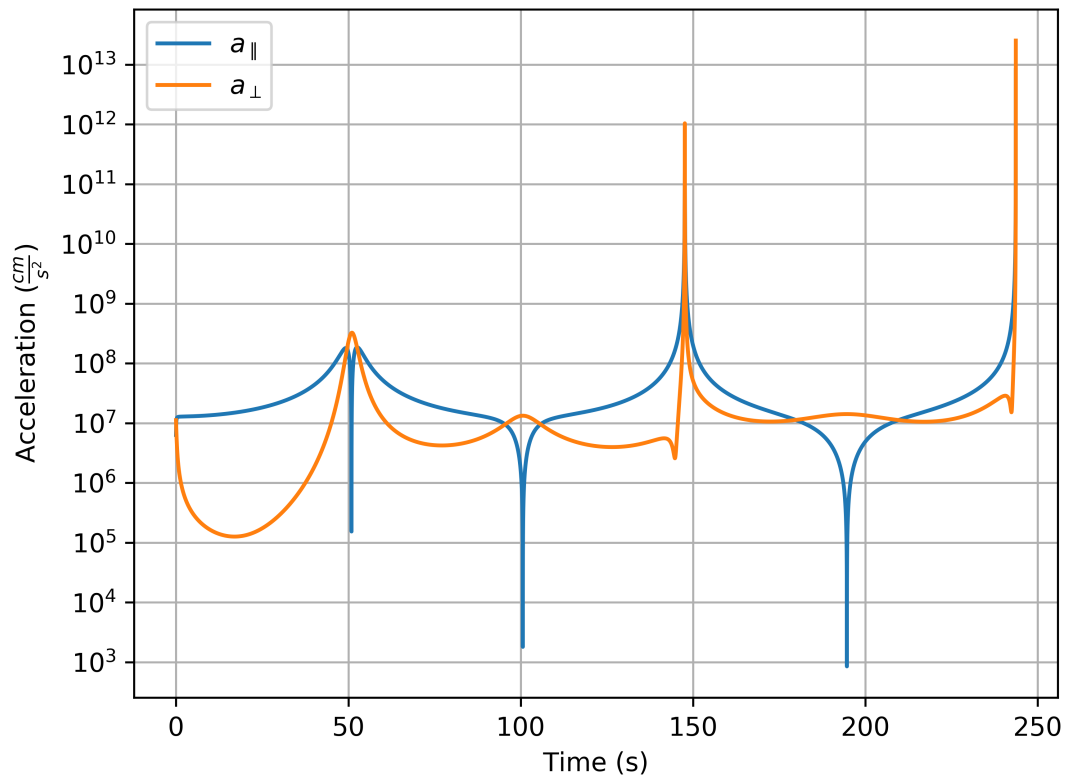


Fig. 4.3 Acceleration components as a function of time for particle 2, with an initial position of $\vec{r}_2 = (8.811 \times 10^9, -9.305 \times 10^9, -2.213 \times 10^{10}) \text{ cm}$ and initial velocity $\vec{v}_2 = (-47.04, 32.97, 1.657) \times 10^{-6} c$.

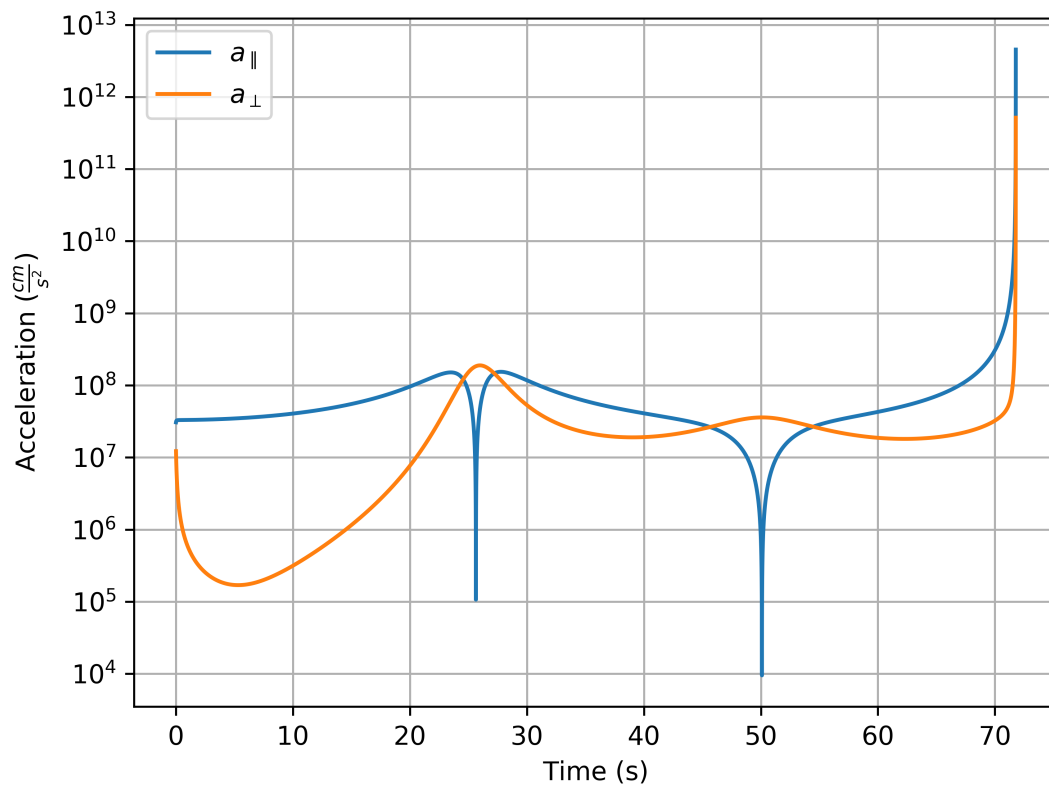


Fig. 4.4 Acceleration components as a function of time for particle 3, with an initial position of $\vec{r}_3 = (3.318 \times 10^9, 1.356 \times 10^{10}, 6.008 \times 10^9) \text{ cm}$ and initial velocity $\vec{v}_3 = (-23.09, -49.19, -6.877) \times 10^{-6} c$.

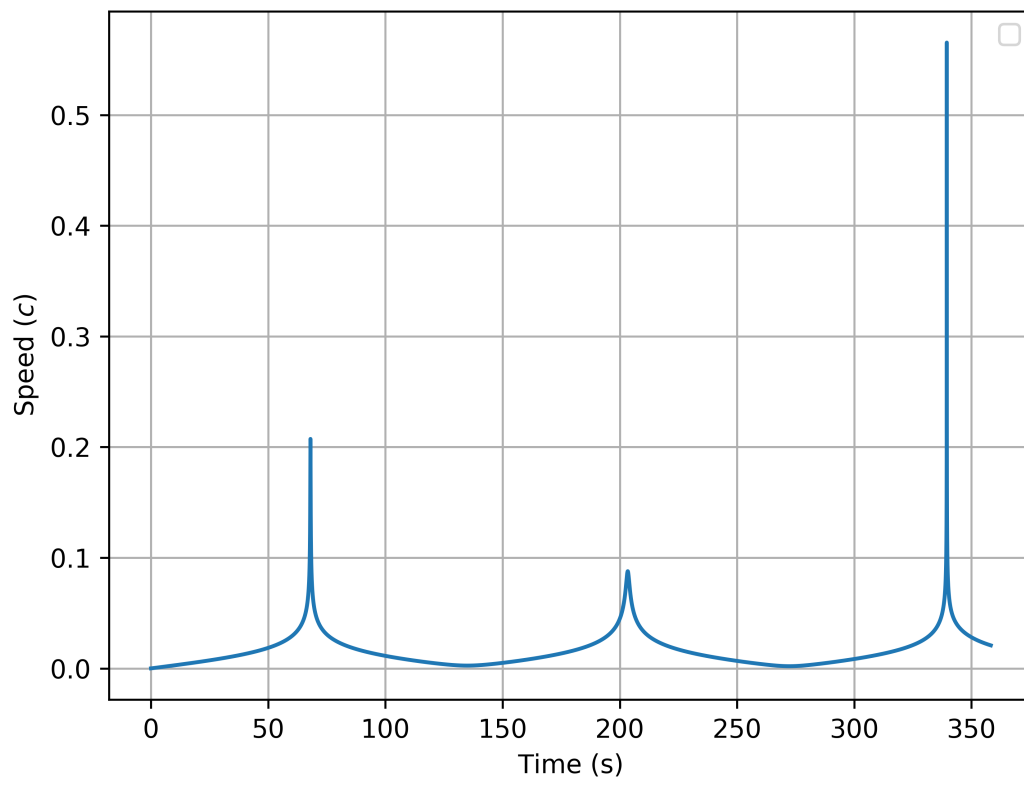


Fig. 4.5 Speed as a function of time for particle 1, with an initial position of $\vec{r}_1 = (-2.734 \times 10^{10}, -7.215 \times 10^{10}, 1.740 \times 10^{10}) \text{ cm}$ and initial velocity $\vec{v}_1 = (1.124, 4.636, -1.395) \times 10^{-5} c$.

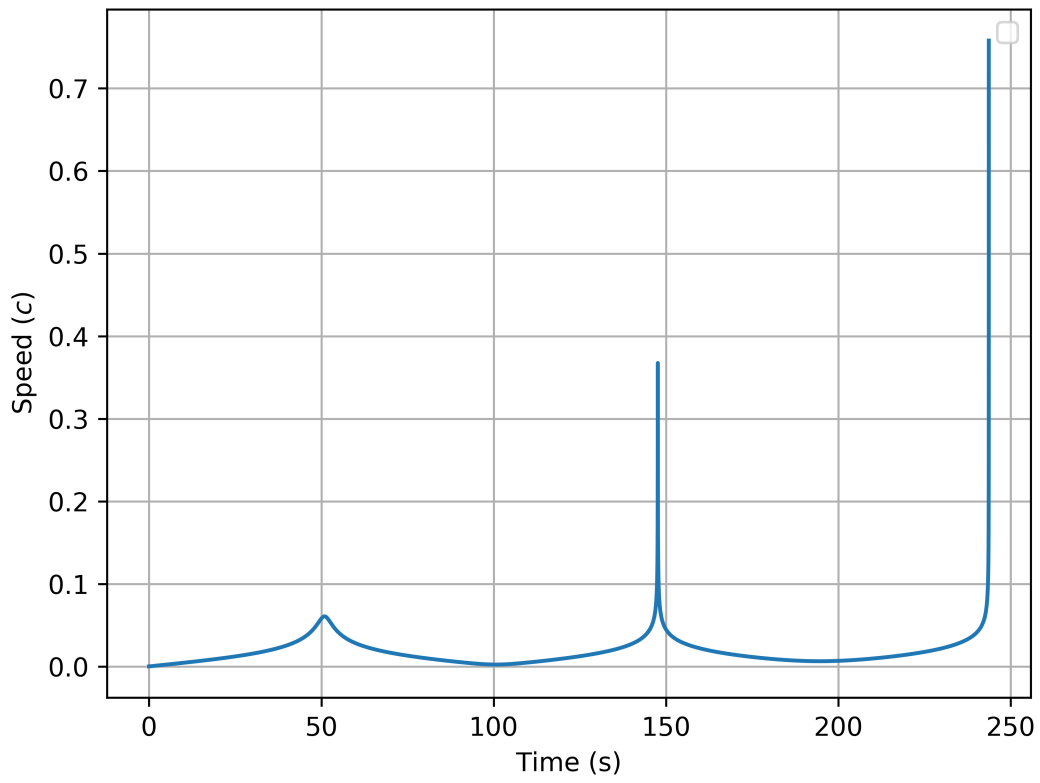


Fig. 4.6 Speed as a function of time for particle 2, with an initial position of $\vec{r}_2 = (8.811 \times 10^9, -9.305 \times 10^9, -2.213 \times 10^{10}) \text{ cm}$ and initial velocity $\vec{v}_2 = (-47.04, 32.97, 1.657) \times 10^{-6} c$.

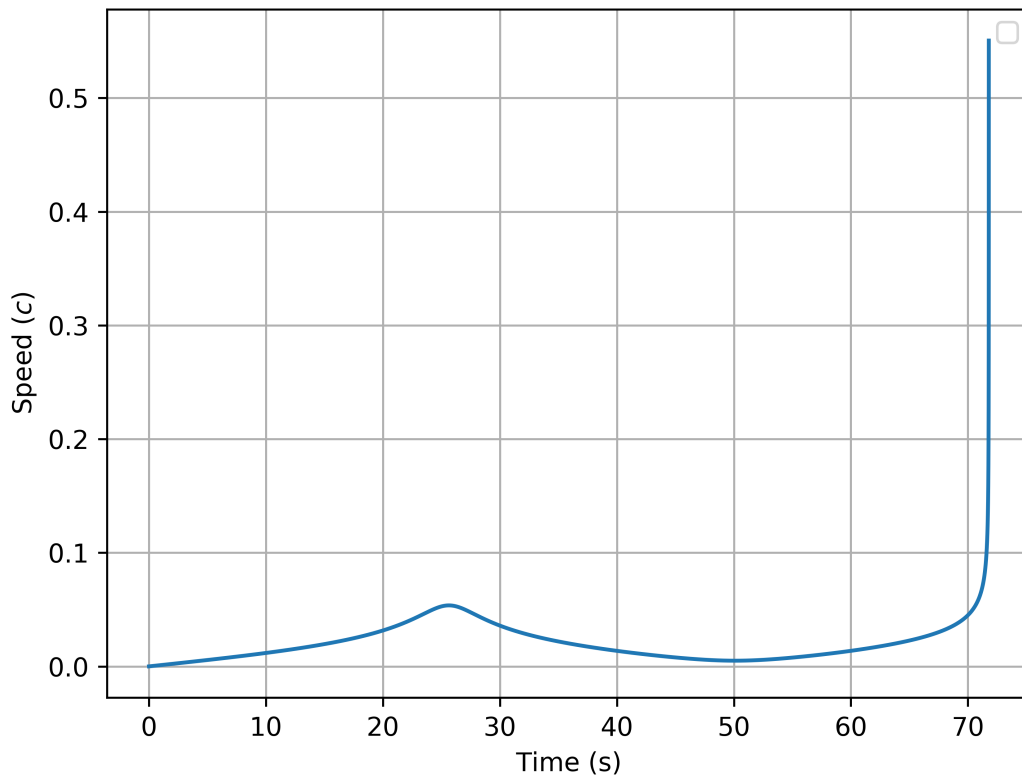


Fig. 4.7 Speed as a function of time for particle 3, with an initial position of $\vec{r}_3 = (3.318 \times 10^9, 1.356 \times 10^{10}, 6.008 \times 10^9) \text{ cm}$ and initial velocity $\vec{v}_3 = (-23.09, -49.19, -6.877) \times 10^{-6} c$.

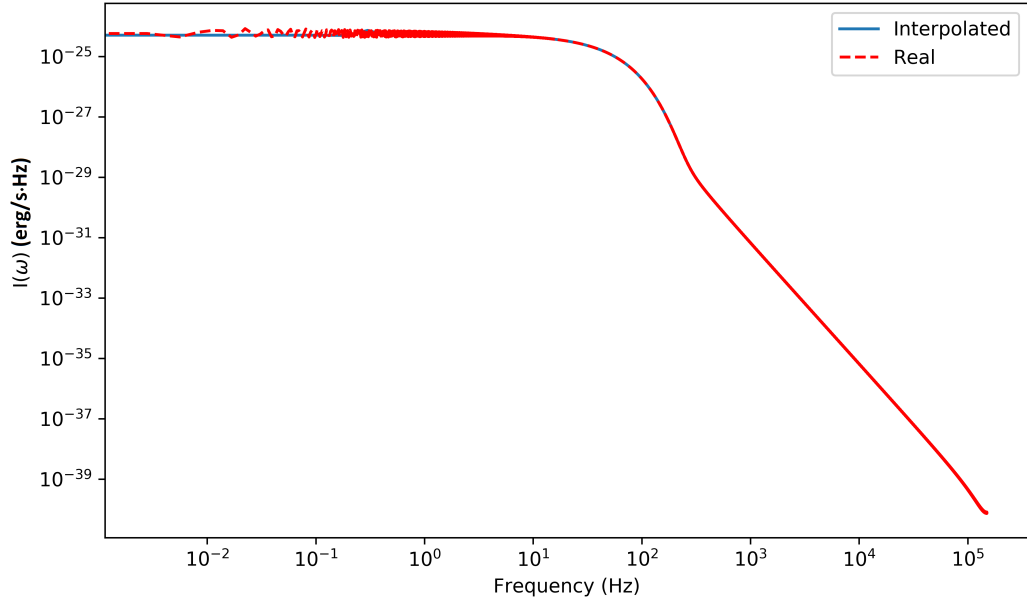


Fig. 4.8 Both the numerically calculated and interpolated SEDs as a function of frequency for particle 1, with initial position of $\vec{r}_1 = (-2.734 \times 10^{10}, -7.215 \times 10^{10}, 1.740 \times 10^{10}) \text{ cm}$ and initial velocity $\vec{v}_1 = (1.124, 4.636, -1.395) \times 10^{-5} c$.

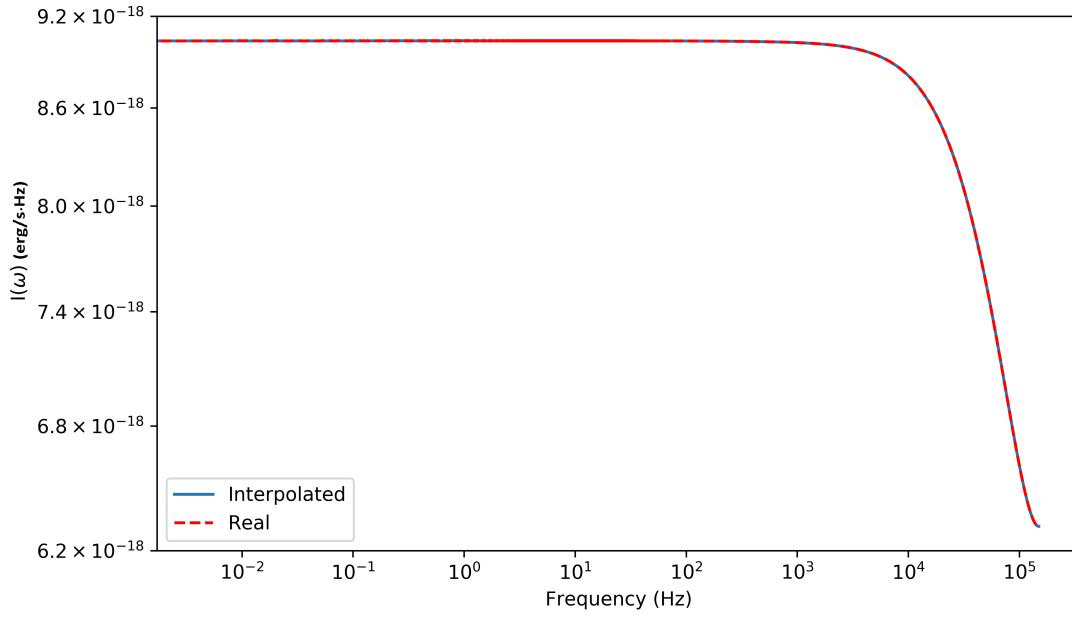


Fig. 4.9 Both the numerically calculated and interpolated SEDs as a function of frequency for particle 2, with initial position of $\vec{r}_2 = (8.811 \times 10^9, -9.305 \times 10^9, -2.213 \times 10^{10})$ cm and initial velocity $\vec{v}_2 = (-47.04, 32.97, 1.657) \times 10^{-6}c$.

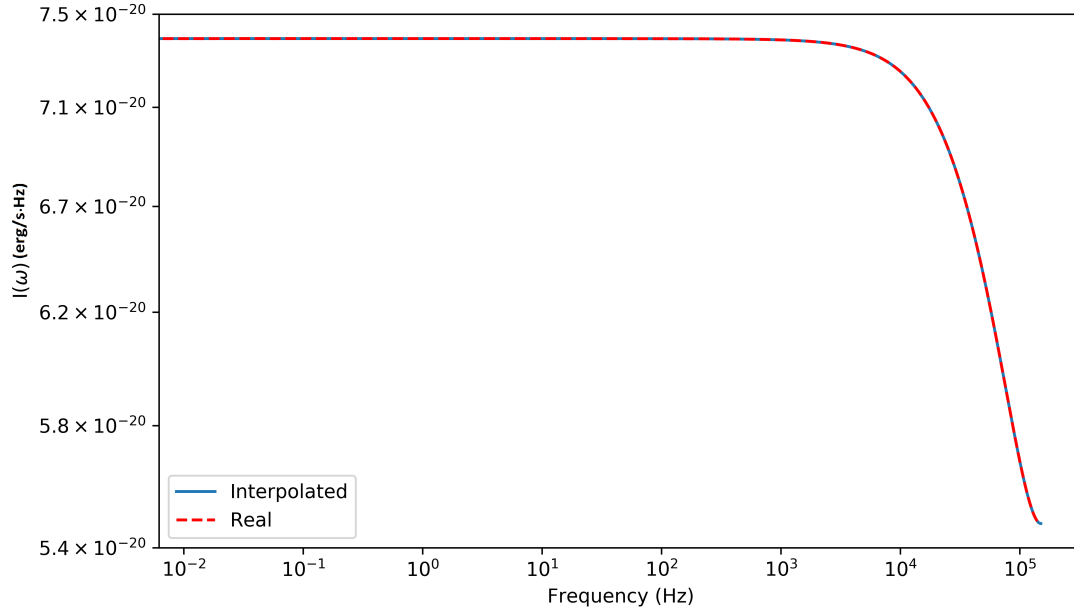


Fig. 4.10 Both the numerically calculated and interpolated SEDs as a function of frequency for particle 3, with initial position of $\vec{r}_3 = (3.318 \times 10^9, 1.356 \times 10^{10}, 6.008 \times 10^9) \text{ cm}$ and initial velocity $\vec{v}_3 = (-23.09, -49.19, -6.877) \times 10^{-6} c$.

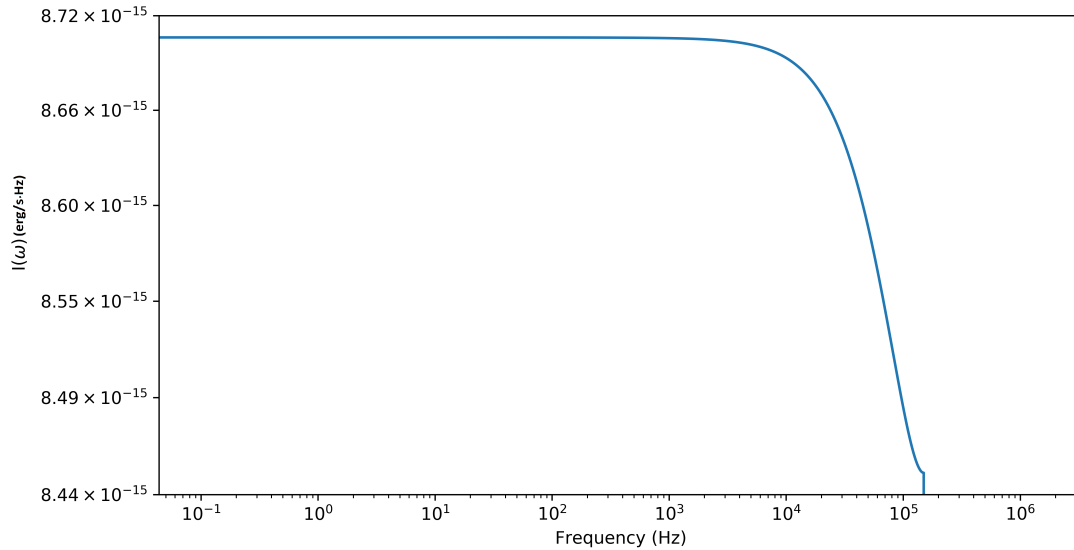


Fig. 4.11 Total SED, determined by integrating over all 40 particles distributed in the system.

Chapter 5

Discussion, Conclusion and Future Work

5.1 Discussion and Conclusion

From the results given in the previous chapter, it becomes evident that the bulk of possible EM radiation that originates from charged particles, accelerated in near-inspiral BBH merger systems (determined from our model), is distributed at frequencies well below the operating ranges of current radio telescopes. The current operating range of the Low-Frequency Array (LOFAR) is $10 - 240\text{MHz}$ (van Haarlem et al., 2013). Figure 4.11 illustrates how the total SED calculated from the model drops off at 10^5Hz , which lies below the frequency range of LOFAR. If we now consider that the inter stellar medium (ISM) plasma frequency is $\sim 2\text{kHz}$ ¹⁷, we know that radiation emitted at frequencies below this threshold will be absorbed by the ISM, and, therefore, the bulk of radiation emitted by the system will be absorbed by the ISM. This means that only radiation in a range of $\sim 3\text{kHz}$ to $\sim 100\text{kHz}$ will emerge from the system and reach the earth. Furthermore, we see an additional sharp cut-off at 10^6Hz . This cut-off occurs due to the limitations of current computational resources available to us. For this reason, the code will have to be applied using larger frequency resolutions, once more computational resources become available to us. However, from the physical drop-off that occurs at 10^5Hz , we do not expect significant changes in the spectral behaviour at higher frequencies.

5.2 Future Work

The model will be extended to include inspiral phase BBH orbits where the possible particle trajectories and resulting gravitational acceleration is expected to become much more chaotic

¹⁷This is for an assumed electron density of 0.03cm^{-3}

for which the model and code implementation will necessarily include simulations with a larger number of particles and smaller step sizes, in order to probe possible higher frequency elements. For this reason, it will become necessary to implement more realistic, fully general relativistic equations of motion for charged particles distributed through the system. We have also assumed that neither of the BHs have a Magnetosphere, with only gravity acting in on the particles distributed within the system. If at least one of the BHs are significantly magnetized, particle dynamics within the system will likely be dominated by the resulting Lorentz forces, rather than the gravitational forces.

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I would also like to thank Frankie Beilfeld for his friendship and assistance in making sure that my writing is up to standard.

Finally, I would like to thank all of the professors, students and staff at the North-West University (NWU) for their support, especially Tannie Petro and Lendl for always helping to cut through the crippling admin. For their financial assistance, I would also like to thank the Center for Space Research (CSR), the National Research Foundation (NRF) and the National Astrophysics and Space Science Programme (NASSP).

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