The influence of worldview on the first foundational crisis in mathematics

CP Hattingh

orcid.org/0000-0002-0479-048X

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Supervisor: Prof DFM Strauss

Co-Supervisor: Prof AH Verhoef

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Ageometretos medeis eisito

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Abstract: The early philosophy of mathematics among the Greeks

The aim of this study is to determine what the ontic status of mathematical entities (sometimes called mathematicals) is in the three main phases of ancient Greek philosophy: the Presocratic stage, Plato and his early Academy, and Aristotle. It turns out that the basic assumptions that a person makes about the structure of reality (which are also collectively known as the person's worldview) have an impact on what his view is on the philosophy of mathematics. This will also be examined throughout.

In order to specify exactly what aspect of the philosophy of mathematics we are interested in, it is maybe a good idea to first have a look at what aspects of the philosophy of mathematics can be discerned in this time period. The following questions were treated by Plato and some of the other early Greek philosophers (Wedberg, 1955:9-10):

1. Where are mathematicals located in a classification of the universe?
2. How are mathematicals generated?
3. Are mathematicals Ideas?
4. What relationship is there between mathematicals and the external world?
5. How should mathematics be done?

Although each of the other points are certainly important and have an influence on how we answer the first question, the point of this study is to answer the first question. We will certainly at times in this study briefly look at some of the other questions in terms of how they help us to answer the question of the ontic status of mathematical entities. For instance, if the answer to the fifth question includes saying that observation (naively understood) is part of the methodology of mathematics, then clearly our answer to the first question could not possibly be that mathematical entities have no perceptible existence.

It turns out that asking what the ontic status of mathematical entities is, in the context of Greek philosophy, immediately leads to the necessity of classifying the different kinds of mathematical existence. The ancient Greeks did not think that numbers and geometrical configurations had the same kind of existence. In fact, three phases can be discerned in how they approached the existence of arithmetical entities (sometimes called arithmeticals) and geometrical entities (sometimes called geometricals). These are as follows:

A. Thales (and probably also the later Milesians) thought of arithmeticals and geometricals as having coordinate places in the ontic scheme: neither could numbers be reduced to figures, nor could figures be reduced to numbers.
B. The Pythagoreans thought of arithmeticals as subordinate to geometricals, that is, numbers were just special kinds of geometrical arrangements.

C. From Plato onwards, there is a coordinated effort to place arithmeticals and geometricals on an equal footing, with equal ontic status. Both numbers and figures are fundamental kinds of existences, and neither could be reduced to the other. Plato himself seems to have been still very undecided about this issue.

It will be the aim of this study to flesh out the details of these claims and to suggest why the only reasonable solution of the problem of the ontic grounding of mathematics is to say that both arithmetical and geometrical entities have a fundamental kind of existence.

The early development of pre-Socratic philosophy of mathematics

It has often, quite confidently, been asserted that the Greeks learnt their arithmetic from the Babylonians and their geometry from the Egyptians. (Freeman, 1946:49-50; Fowler, 1999:279, 313, 364; Fowler says that the Greeks learnt "the measurement of space from Egypt and the measurement of time from Babylon" on p. 281, a tantalizing hint of how geometry and arithmetic had their origins.) The story runs as follows: Thales, who is considered to be the first real philosopher (at least, in the Western account of the history of philosophy), took it upon himself to learn as much as he could from other civilizations. He visited the Babylonian empire as well as the Egyptian kingdom. While there, he was inducted into the Egyptian mysteries of the cultic religion by their priests, and this would include some knowledge of geometry. The geometry of Egypt was a rough-and-ready practical set of rules of thumb that were used to reconstruct the boundaries of the farms adjacent to the Nile which were flooded by the annual breaking of the banks by the Nile. This is the kind of geometry that Thales would have learnt. (Cornford, 1991:142, footnote)

On his other expedition, he would have learnt the calendrical methods of the Babylonians, which served as their tool for keeping record of astronomical and astrological observations (these two were at the time still very much the same science). In order to be able to develop their calendrical methods, the Babylonians needed to develop a sophisticated system of calculation, which purportedly enabled them to predict solar and lunar eclipses. Thales himself is reported (by Diogenes Laertius) to have predicted a solar eclipse that he managed to use as leverage to dissuade a battle from occurring. In modern times, much doubt has been cast on his ability to make such a prediction, since the method the Babylonians apparently used would not have worked.

The Rhind papyrus seems to indicate that it would not have been necessary for Thales to have learnt arithmetic from the Babylonians, since Egypt herself had developed a rather sophisticated arithmetical system by the time of Thales, and doubtless if he had learnt geometry from the priests, there would have been no impediment to their teaching him arithmetic as well. (Burnet, 1920:11-12; Fowler, 1999:373) So it seems entirely possible that both arithmetic and geometry were introduced into Ionia from the same source. However the historical case may be, it seems pretty clear that the form of mathematics that would have reached Ionia was practical rules, not a developed scientific apparatus. If we wish to distinguish between calculation and mathematics proper (as Plato did), then what Thales would have learned would fall strictly in the category of calculation. It might be useful later to discuss whether such a distinction is necessary and helpful in understanding what mathematical entities are.

Thales is reported to have developed proofs for five geometrical propositions (according to Proclus and Eudemus of Rhodes). What can we say about the form of proof that Thales would have employed? Certainly it would not be the rigorous form of proof that we find in modern mathematics. It would not even be the slightly less rigorous form of proof that we find in Euclid's *Stoicheia* (the *Elements*). What we find in reports of Thales' proofs are indefinite, approximate words. Instead of saying that the angles opposite equal sides of a triangle are equal, we find the report that he proved them to be similar. It is very difficult to decide what to make of this report, since we do not know the terminological conventions used for geometry of this time.

Heath seems to think that we credit Thales with too much if we say that he knew what equality of angles means, and instead that he meant to prove that they always have the same rough shape. (Heath, 1921:I 131) Szabo makes a rather compelling case that Thales' method of proof, and any method of proof before the discovery of incommensurability for that matter, would have to be a method of geometrical superposition. Crucial to the understanding of mathematical proof, however, is the word *always*. We at least see something in Thales' proofs of mathematical propositions that was unknown before his time: The idea that something is true about geometrical configurations that holds true for all the geometrical configurations of that type. All isosceles triangles share the same common characteristic that the shapes of the angles at their bases are the same.

Introducing universality in the discussion of mathematical statements is not a trivial development. It is really at the heart of what a mathematical proof must minimally be. As such, his proofs of propositions (however it was done and with however much success) at least deserves the name of proto-proof. This is exactly what Burnet also points out: "What the tradition really points to is that Thales applied this empirical rule to practical problems which the Egyptians had never faced, and that he was thus the original of *general* methods. That is a sufficient title to fame." (Burnet, 1920:34-35, my emphasis)

From Thales we move on to Pythagoras. Our sources that speak of Pythagoras are scanty and unclear. Aristotle wrote three treatises about Pythagoras and the Pythagoreans, but none of them have survived to our day. None of the writings by individual early Pythagoreans have survived either; and therefore, our knowledge of Pythagoras comes from sources centuries later, sometimes complicated by a decidedly hagiographical character. What we can deduce, we need to deduce from trends in the development of Greek philosophy, together with carefully read secondary sources. Even so, it is almost impossible to determine what would have been the teachings of Pythagoras himself and what would have been the teachings of his disciples. (Freeman, 1946:81)

Some things, however, seem to be clear about Pythagoras. Freeman seems to be correct in saying that the idea of numbers as the ontological ground of being was a result of the teaching of Pythagoras himself and not the product of a research group. (Freeman, 1946:82) His famous dictum "all is number" has been misunderstood, in my opinion. Some experts on ancient Greek philosophy consider this dictum to mean that arithmetic is the fundamental science, since arithmetic is the science of number. However, considering the way the Pythagoreans thought of numbers (square numbers, oblong numbers, triangular numbers, the even and odd as gnomons, the tetraktys, etc.), it is very clear that they thought of numbers as geometrical arrangements. Further evidence of this is the geometrical shapes of Eurytos that gave the number of man, horse, and so on. (Burnet, 1920:71-72) Therefore, the dictum "all is number" would be better understood as a proto-materialistic credo (not yet fully materialistic because the distinction between spirit and matter was not yet made, *pace* Copleston) that everything is geometrical, where geometry here should not be understood as some idealistic collection of thought-patterns, but as something concrete, something that has its being in the same spatial world in which we live. Here I have a disagreement with Stamatellos (he says that they were definitely non-materialistic), but he is right in also pointing to the central role of the monad, to which I will now turn. (Stamatellos, 2012:23; see Freeman, 1946:246-247)

Whereas everything in the observable universe could be derived from number according to the Pythagoreans, number itself was not the ultimate existence in this philosophy. Instead, numbers themselves sprung out of the monad, the One, which was not considered to be a number, but to be the ontic ground of numbers. (Stamatellos, 2012:23) The One was identified with the hearth fire at the centre of the universe, and numbers and everything else was seen as emanations from the One. This clearly had its origin in the creation myths that were prevalent among the ancient Greeks. Burnet uses this proximity in ideas to ancient myths as a criterion to know that we are dealing here with what Pythagoras himself, as distinguished from his disciples, taught. (Burnet, 1920:71-72) What comes out of all this, however, is that the first resolution in the foundational crisis of mathematics (the question whether arithmetic or geometry is more fundamental) is that geometry is the foundation of everything, and arithmetic is based upon geometry.

Later Pythagoreans, however, thought of arithmetic as a science independent of geometry. (Burnet, 1920:71-72) When we consider these later Pythagoreans in their turn, we shall see what lead to a renewed crisis in the foundations of mathematics. For now, we have a good opportunity to consider the philosophical conclusions Pythagoras came to on the basis of his number-mysticism by looking at the teachings about the *tetraktys*.

> We also gather that the dots were supposed to represent pebbles (*psephoi*), and this throws light on early methods of what we still call *calculation*. That Aristotle refers to this seems clear, and is confirmed by the tradition that the great revelation made by Pythagoras to mankind was precisely a figure of this kind, the *tetraktys*, by which the Pythagoreans used to swear, and we have the authority of Speusippos for holding that the whole theory was Pythagorean. In later days there were many kinds of *tetraktys*, but the original one, that by which the Pythagoreans swore, was the "tetraktys of the dekad". It was a figure of ten dots in an equilateral triangle with a base of four dots, and represented the number ten as the triangle of four. It showed at a glance that 1+2+3+4=10. Speusippos tells us of several properties which the Pythagoreans discovered in the dekad. It is, for instance, the first number that has in it an equal

> number of prime and composite numbers. How much of this goes back to Pythagoras himself, we cannot tell; but we are probably justified in referring to him the conclusion that it is "according to nature" that all Hellenes and barbarians count up to ten and then begin over again. (Burnet, 1920:72-74)

The way Pythagoras thought of numbers as dots arranged in geometrical shapes, made it possible for him to see connections between harmony in nature and form in geometry. This he extended to numerical symbols for experientially encountered abstracts, like the "right time" being seven, justice being four, marriage being three, and so on. Thinking of numbers in terms of geometrical shapes might also have suggested an early solution to the problem of the incommensurability of $\sqrt{2}$, namely thinking of it in terms of *commensurability in square*. Consider what Burnet says about "fields":

> The dots which stand for the pebbles are regularly called "boundary-stones" (*horoi*, *termini*, "terms"), and the area they mark out is the "field" (*chora*). This is evidently an early way of speaking, and may be referred to Pythagoras himself. Now it must have struck him that "fields" could be compared as well as numbers, and it is likely that he knew the rough methods of doing this traditional in Egypt, though certainly these would fail to satisfy him. (Burnet, 1920:72-74)

Comparing this with Szabo's excellent study of the word *dynamis* (for the complete discussion, see the first hundred pages or so of Szabo, 1978), it seems plausible that the idea of thinking about the square on the diagonal rather than the length of the diagonal of a square came to Pythagoras rather early in his attempts to explain how the square root of 2 could be measurable.

I should mention that my interpretation of first-generation Pythagoreanism, although consistent with the facts available to us, is not shared by everyone. The idea that geometry is in fact more fundamental than arithmetic according to the Pythagoreans, which I propounded above, is disputed by many others who have thought about this topic. Sweeney, for instance, interprets it as follows: arithmeticals (numbers) produce geometricals (shapes). (Sweeney, 1972:85) Although it is the case that the shapes are formed to give us understanding of numbers, I think it is significant that it is always the same shapes that are connected to the numbers. Ten, when considered as a number on its own and not as part of a series, always is arranged in the form of the *tetraktys*. Four is always a square. Six is always a rectangle. Although four is a gnomon when moving from the two-rectangle to the six-rectangle, here it is viewed as part of an infinite series of gnomons, not as a number in its own right. It seems, therefore, that shape comes first, and then number "fills in the blanks" of the shape.

Next we will look at some individual members of the early Pythagorean school. The main member we consider is Philolaus of Tarentum. With him, we already have a philosophical refinement: no longer *is* everything number, but rather everything *has* number. (Freeman, 1948, fragment 3)

Everything now can be expressed in terms of numbers, but they do not have to have their ontic grounding in numbers themselves. Instead, our epistemic access to things is through their possessing number. It has to do with our knowledge of the things, no longer the inner essence of the things. (Freeman, 1946:230-231) We see here also that the theory of odd and even, which Plato later repeatedly says is the basis for arithmetic, starts taking shape. (Freeman, 1948, fragment 4) Because numbers are timeless, and therefore immortal according to Greek understanding, they are divine. Whatever was immortal and deathless was also, as a result, divine in the Greek worldview; and therefore numbers had a divine nature. (Freeman, 1948, fragment 11)

The theory of the even and the odd led the Pythagoreans to believe in a fundamental multiplicity in nature as well. There is a famous table of opposites that described the diversity in existence: male and female, limit and unlimited (which was related to even and odd via the gnomons), good and evil, just and unjust, etc. At the head of this table was the even and the odd. The idea clearly was that there is a balance in nature, an "attunement of the lyre" between what can be found in the one column under even and what can be found in the other column under odd, and that wisdom lay in understanding this balance. The Pythagoreans were, in this, much closer in philosophy to Heraclitus than to Parmenides. However, since they derived all of this, via numbers, from the One, there was also some kinship with the philosophy of Parmenides. It seems that the Eleatic philosophy was developed at least in part in response to the philosophy of the Pythagoreans.

The early Pythagoreans did not distinguish strictly between something having properties and something existing. It is due to this that numbers were hypostatized. Consider this excerpt from Horky (Horky, 2013:142-144):

> What, then, does Philolaus mean by "number"? If I am right that the mathematical Pythagoreans sought to "demonstrate" in some fashion the doctrines of Pythagoras, then it may be relevant to consider whether Philolaus was attempting to explain the *acusma* concerning number: "What is the wisest? Number". The evidence from Philolaus goes beyond the Euclidean definition of "number" as a "multitude composed of units", since it demonstrates what we might consider a protodialectical concern with classification of number according to "proper" kinds (odd and even) and a derivative kind (odd-even). As Huffman has argued, when Philolaus refers to "number", he implies an ordered plurality *that is counted*. This interpretation of early Pythagorean "number" is corroborated by Epicharmus's usage in Fragment B 2, where "number" is understood to be composite, can be either even or odd, and loses or retains its identity according to whether a pebble is added or subtracted or not. The concerns with identity and analogy exhibited in Epicharmus's Fragment B 2 (as discussed above), as well as with classification and epistemology in Philolaus's Fragments 4 and 5, both exhibit non-Euclidean aspects and may even go beyond the Hippocratic and, more generally, Presocratic practice of conflating an object's existence and its possession of properties.

> Indeed, when Philolaus refers to known objects *having number*, he seems to be adopting an Eleatic linguistic and conceptual apparatus, but to different ends: in Fragment 5, Philolaus says that particular objects “themselves give as signs” the “shapes” of each “kind” of number, that is, even or odd. This constitutes an adaptation, or rather a reversal of sorts, of Parmenides’s argument in Fragment 8, where mortals who make distinctions bestow signs and decide to “name shapes” of What-Is. For Philolaus, to be sure, it is *each object in and of itself* that signifies by giving forth many shapes of each kind of number. Thus, in Philolaus’s epistemology, the objects in nature, which are knowable, are not subject to the problem of the status of human discourse as formulated by Parmenides and others, on the grounds that they themselves give reliable *signs* or *tokens* of their numerical shape. Indeed, it should be noted that the extant fragments of Philolaus – by contrast with both early natural philosophers and Sophists – never speculate about human language as a potentially “deceitful” medium for communication about what we in fact know. It is possible that Philolaus did not seek to engage in discussing the potentially problematic relationship between the objects of (human) knowledge, our perceptions of them, and our ways of speaking about them. Be that as it may, Philolaus’s classification of “kinds” of number in Fragment 5 is abstract enough to accommodate Archytas’s description of Eurytus’s approach to definition of a human being by pebble-arithmetic – even in the rather cryptic explanation given by Ps.-Alexander.

We see here that the idea of having knowledge of something for the early Pythagoreans implied that the something had to have an independent, non-ideal, external existence. (Freeman, 1946:247-248) Not only are numbers *that which is counted*, which acknowledges that we interact with them cognitively, but they have proper kinds (odd and even), which means that they have independent natures, not reliant on our cognitive classification. The semiotic dimension of mathematicals already starts to be observed, but they are not symbols of something else. Instead, they signify what their own inner nature is.

If things are derived from numbers, and numbers are derived from the One, then we end up with a system that is ultimately monistic, and therefore the Parmenidean system seems to be a close consequence of this set-up. However, the Pythagoreans were not Parmenideans. Why not? Instead of having the monad as the only fundamental principle, they also had the dyad as the fundamental principle of the unlimited (remember, their philosophy was intimately intertwined with the concepts of *peras* and *apeiron*). Therefore, with the monad as principle of the limited and the dyad as the principle of the unlimited, they had a solution to the problem of the one and the many that reduces to dualism.

The dyad was the cause of multiplicity, and so also of division, spatial separation, space itself, and generation. Since the even and the odd as species of number was very important to Pythagorean philosophy, the fact that even numbers could be divided into two equal parts was considered very important. This division would take place in space (according to their visualization), and therefore even numbers participated in the dyad. For this reason, among others, the dyad was also called the "infinite dyad". (Freeman, 1946:247-248) Since numbers came from the monad and space from the dyad, it is clear that arithmetic had as its ontic basis the monad, and geometry had as its ontic basis the dyad.

Earlier, I said that the early Pythagoreans reduced all of mathematics to geometry. What I said just above would seem to contradict that statement. However, as Freeman points out, there seems to have been some disagreement (Freeman implies that this disagreement was unintentional) about whether there were one or two fundamental principles. (Freeman, 1946:247-248) It seems to me, however, based on the dictum "all is number", that the dualistic ontology that takes both the monad and the dyad as fundamental is a later development, possibly in response to the Eleatics. The monad was the principle of number; and if all is number, then the monad is the principle of all things. Therefore, the monad seems to have been originally the ontic foundation of all things in Pythagorean philosophy, only later to be replaced by the monad and the dyad. Unfortunately, the evidence is scanty either way, so we have to rely on some conjecturing. I find some support for my position, however, in Burnet. (Burnet, 1920:211-216)

It is probably at this time a good idea to say something about harmony. In the classical quadrivium there were four subjects: arithmetic, geometry, harmony, and sphaeric. These four subjects formed the basis of education. All four of these, seen in their ancient Greek context, are mathematical subjects. Arithmetic and geometry are well known to readers today, but harmony and sphaeric are somewhat less familiar. Sphaeric was the study of astronomy specifically with regards to the observation and the mathematical explanation of stellar events, celestial movements, and risings and settings. A lot of spherical trigonometry, studies of conic sections, and three-dimensional mechanics has its origin here. To the Pythagoreans, sphaeric was probably not quite so important. (After all, Plato complains that this was a criminally neglected field of study.)

On the other hand, harmony was tremendously important to the Pythagoreans. By harmony was meant the study of music as a mathematical discipline. It developed as a study of the different harmonic intervals that could be created using a single string. A string was taken (I'm not sure what the string was made of, but this isn't really relevant), attached to two fixtures at its ends, and laid down next to a straight-edge ruler. The ruler was then marked at thirteen and labeled at twelve equally spaced points (forming twelve intervals; the Greeks made no use of the number zero) with the outer two points being at the positions of the fixtures. Now the string was held down with a finger at a position, say position 8, and plucked. Thereafter, the string was let loose at position 8, while still fixed at the fixtures, and plucked again. This would form two notes in succession, and these two notes would be in the interval of a fifth. Holding the string down at other positions, like position 6 or position 9, would give other intervals between the notes. At one position you could get an octave, at another a fourth, and so on.

Using this method, many musical harmonies could be associated with mathematical ratios. The fifth, for example, would be associated with the fraction $\frac{12}{8}$, because the whole length of the string is twelve line segments, whereas the string being held down had a length of eight line segments.

The relationship that music had with mathematics confirmed to the Pythagoreans the idea that mathematics was underlying everything in the universe. Harmony could be seen as a mathematical property, and since everything had to have a degree of harmony (there has to be a measure of *peras* in anything for that thing to be a thing, and the Erinyes would assure that the harmony of *peras* always existed in everything that is a thing), everything had to be numerable.

One thing we need to be careful of, however, is the tendency to think of mathematics as a discipline clearly cut off from everything else with a hatchet. This is not how the Pythagoreans thought of mathematics. Brumbaugh gives us this caution by means of some examples that make it clear how mathematics was seen as intertwined with all other reality:

> Contemporary references, later tradition, mathematical vocabulary, and collateral material from the history of science all indicate something about the notion of "mathematics" that the Pythagoreans held. What they indicate appears quite clearly to be that mathematics was not sharply separated from other branches of natural science. The statements made about the relations of things and numbers by the Pythagoreans make sense only on the assumption (which their vocabulary confirms) that the language of this mathematics was intended to be metaphorical. By this I mean that the genera and differentiae into and by which numbers were classified were not regarded as peculiar to numbers, but were classifications derived from and shared by nonmathematical subject matters; thus, when a surface was called a "skin", or a line not amenable to measurement was called "irrational", that naming reflected a conviction that there is some common class including both skins and geometric surfaces, both moral stubbornness and incommensurability. Insofar as members of the same class have common properties, theorems about surfaces are also theorems about skins, and theorems about incommensurables are also theorems about moral behaviour. Conversely, problems involving skins or morality may have analogues in pure mathematics, and it is these analogues that are crucial in the use of mathematics as a technique of research and of pedagogy. (Brumbaugh, 1954:5-7)

Even the heavens obeyed the laws of mathematics, since the constellations themselves each had a number associated with them, so number pervaded not only our earthly realm, but all of the universe everywhere. This means that once one could decode the behaviour of number, one would understand all of the universe and everything it contains. This is of course an attractive epistemological position, but it has consequences for ontology also. If the furthest reaches of the world obey mathematical laws, then mathematical laws cannot be geographically restricted to our town, our province, our country, or even our planet. They have to have ubiquity in place, and since these laws always apply, they also must have ubiquity in time. To the Greek mind, there is nothing that comes closer to divinity than that. It is no wonder that the Pythagoreans worshipped the *tetraktys*, swore oaths by it, entered into binding contracts in its name, and made an intensive study of arithmetic and geometry.

We must remember, however, that the Pythagoreans did not just say that everything is number, but also that the whole heaven is a musical scale and a number. Number can clearly be compared with the central idea of limit (*peras*). Musical scale, on the other hand, points more directly to the harmony that there must be in all things. Of course, harmony in the form of musical scale referred back to number for the Pythagoreans.

It is worth noting that even in their most arithmetizing, monistic moments, the Pythagoreans could not refrain from using geometrical analogies. Consider the language that they used to describe everything as coming from number, and number from the One: “But again we are told that the numbers are not separable from the things, but that existing things, even perceptible substances, are made up of numbers; that the substance of all things is number, that things are numbers, that numbers are made up from the unit, and that the whole heaven is numbers.” (Heath, 1921:I 67-69) Here we see 1) that numbers cannot be *separated* (a spatial analogy) from the things, 2) that perceptible substances are *made up* of (a historical analogy relying on the spatial analogy of bringing together into one place) numbers, 3) that numbers are *made up* from the unit. So, while saying that arithmetic is the foundational reality, they have to move away from arithmetic to express themselves. I think it is also necessary to point out that when the Pythagoreans said that everything was made up of number, they are implicitly making reference to the whole-parts relationship, which again has its meaning in the spatial modality.

Heath gives us another example of this: “Again, according to them, abstractions and immaterial things are also numbers, and they place them in different regions; for example, in one region they place opinion and opportunity, and in another, a little higher up or lower down, such things as injustice, sifting, or mixing.” (Heath, 1921:I 67-69) Clearly, abstract entities did not have to be non-spatial in the philosophy of the early Pythagoreans. This, taken together with the pebble-arithmetic of Eurytus, makes it clear that numbers themselves were considered to be spatial. Perhaps this, together with the impossibility to express their views on arithmetic in completely space-devoid language, was a contributing factor that led later Pythagoreans to accept the dyad, the principle of space, as another fundamental reality besides the monad. It is clearly not the only factor (the discovery of incommensurability is another large factor and philosophical language was not yet used so very precisely).

> It was the great merit of Pythagoras himself (apart from any particular geometrical or arithmetical theorems which he discovered) that he was the first to take this view of mathematics; it is characteristic of him that, as we are told, 'geometry was called by Pythagoras *inquiry* or *science*' (*ekaleito de e geometria pros Pythagorou historia*). Not only did he make geometry a liberal education; he was the first to attempt to explore it down to its first principles; as part of the scientific basis which he sought to lay down he 'used definitions'. A point was, according to the Pythagoreans, a 'unit having position'; and, if their method of regarding a line, a surface, a solid, and an angle does not amount to a definition, it at least shows that they had reached a clear idea of the *differentiae*, as when they said that 1 was a point, 2 a line, 3 a triangle, and 4 a pyramid. (Heath, 1921:I 165-166)

It must be noticed in this quote of Heath that Pythagoras introduced the idea of first defining terms in mathematics. As far as mathematical innovations that have stood the test of time goes, this innovation's antiquity is only surpassed by Thales' innovation to prove statements in mathematics. If this was the only claim to fame that Pythagoras had, it would still make him one of the most influential mathematicians of all time. It should be noted how he defined mathematical concepts. Numbers were defined in terms of what we in modern mathematics would call dimensions. With only one point, you could not define any dimensions. Two points can define a line, which is one-dimensional (and Pythagoras said that 2 is a line). Three points that are not collinear define a triangle, which is the most simple two-dimensional closed figure. Four points that are not coplanar and that has no subset of three collinear points defines a tetrahedron, which is the most simple three-dimensional closed figure (Pythagoras calls this tetrahedron a pyramid, harking back to Egyptian influence).

Some more confirmation of this view comes from Sweeney, who says that:

> A Pythagorean philosopher views all things as somehow involving numbers: they either imitate or, even, are numbers. What does this involvement mean? Having no simple form of numerical notation, the early Pythagoreans used patterns of dots or alphas to express numbers – for instance, the number 10 was represented by ten dots arranged in an equilateral triangle (KR 230, n. 2). The result of representing numbers by such diagrams was that these philosophers thought of numbers as spatially extended and came to confuse the arithmetical unit with the geometrical point and, finally, with the primitive atoms which constitute physical matter. This confusion had two consequences. "All things are numbers" is equivalent to saying, "All things are literally composed of units-points-atoms" (KR 245-50). Secondly, the generation of numbers is at once the generation of geometrical figures and of physical bodies. The origin of mathematical items is also a cosmogony. (Sweeney, 1972:76-77)

Underlying the reduction that the Pythagoreans made of everything to the monad and the dyad lies the ground motive of form and matter. Everything material is composed of number-atoms, which are material, but they at the same time participate in the *peras*/*apeiron* dualism, and as we saw, there is a parallelism between this dualism and the monad/dyad dualism. The result of this is, of course, that there is a fundamental conflict in the philosophical system between two ultimate principles. This struggle is clearly seen in the table of opposites that the Pythagoreans constructed, with limit/unlimited at the heads of the columns, and with even/odd, male/female, monad/dyad, and so on considered to be polar opposites that counteract one another. Despite the attempt to assuage the feeling of disharmony that the philosophical adherent feels by a *fiat* “there is no harmony without struggle”, the tension is never fully relieved in this system, and finally we can see it bursting out into the two philosophies that are now associated with Parmenides and Heraclitus. Considering that the set of words associated with “odd” contains the concept “stasis”, whereas the set of words associated with “even” contains the concept “change”, one can see that Parmenides basically decided to focus his philosophy on the “odd” column, while Heraclitus decided to focus his philosophy on the “even” column. Both absolutized their respective central notion, and so two new philosophies were born out of Pythagoreanism.

The theory of odd and even can hardly be seen in its full importance today. For the Pythagoreans, it was central to the way they did mathematics, philosophy, and how they understood the world. One consequence of the theory of odd and even that we still see today in mathematics is the proof of the incommensurability of the diagonal of a square with its side. The main way it is proved to university students to this day is by showing that if the diagonal is commensurable with the side, that is, if the square root of 2 is rational, then we can start by assuming that the numerator and denominator has no common factors, and then proceeding to show that they must both be even, meaning that they must have a common factor. This is a direct application of the concept of evenness, which Szabo argues dates back to Pythagoras. (Szabo, 1978:25-29)

Szabo traces the beginnings of mathematics back to the musical investigations of Pythagoras and the first Pythagoreans:

> I believe I can show that the theory of odd and even presupposed the basic definitions of Euclidean arithmetic. Not only does the oldest Pythagorean *mathema* go back to the first half, if not to the beginning, of the fifth century B.C. but this is also the time at which the theoretical foundations of arithmetic were laid. Only the theoretical foundations of geometry are perhaps later, since Euclid’s first three postulates are attributable to Oenopides (around the middle of the fifth century). Becker’s second and third stages are omitted in my reconstruction. In view of the interpretation proposed in Part 1 of this book for the relevant passage of Plato, it cannot be shown that Theodorus and Theaetetus (as they appear in Plato) made any new contributions to mathematics. On the contrary, since the concept of *dynamis* is used in this passage, I have come to the

> conclusion that the discoveries which were once attributed to the characters Theodorus and Theaetetus in Plato's dialogue date in fact from pre-Platonic times and in all likelihood antedate Hippocrates of Chios. I have nothing to say about Becker's fourth stage, since my investigations are not directly concerned with the mathematical discoveries which modern research attributes to Eudoxus. Instead of Becker's chronology, the following investigations enable us to divide up the development of early Greek mathematics into different stages. These will be briefly outlined here. The oldest stage in the development of Greek mathematics which is still accessible to us is the musical theory of proportion. All the technical terms of the later general theory originated in the musical one. (This stage is dealt with in chapters 2.3-2.19.) I believe it would be a hopeless undertaking to try and pin down this stage as having taken place in some particular century or half-century. Even if the fragments are included, our oldest texts touching on musical questions scarcely antedate the age of Plato. Yet those terms from the theory of music which had already been taken over into geometry at the time of Hippocrates of Chios must have been coined much earlier. It seems to me much more important that within this stage two periods can be clearly distinguished. In the earlier period experiments with the monochord took place (although these did not as yet involve a measuring rod). Also, some important terms in the theory of music originated at this time, namely *displasion diastema* (2:1), *hemiolion diastema* (1½=3:2) and *epitriton diastema* (1⅓=4:3). These of course were later carried over into the theory of proportions. Finally, the method of the 'Euclidean algorithm' (successive subtraction, *antanairesis* or *anthyphairesis*) was developed during this period. The later period was characterized by the introduction of the measuring rod which was divided into twelve intervals. This brought into being the new musical/mathematical concept of *logos* (the relation between two numbers). (Szabo, 1978:25-29)

Although it does seem very plausible that the investigations into music was indeed the historical origin of the theory of the odd and the even, from there the theory of *anthyphairesis*, and from there the theory of proportions and the theory of linear incommensurability, we must be careful about what this means. If this historical reconstruction is correct, what it would mean is that the study of harmony came first in the chronological order, but it does not mean that the Pythagoreans considered harmony to be more fundamental than the spatial numbers in terms of the ontic order. So, despite being important for the history of mathematics and the history of ideas, Szabo's insights into the development of Pythagorean mathematics does not seem to be as important for understanding their philosophical system. Of course, it does highlight the need to fit the mathematical theory of music into whatever understanding we eventually have of their ontology. This is an important point, but it serves more as a corrective than as a directive.

One aspect of Pythagorean-era Greek mathematics that makes it difficult to get a proper grasp of their ontology is the discovery, made by Zeuthen, that many of the geometrical propositions that can safely be dated to mathematicians before Plato can be rephrased in an algebraic form. Zeuthen called this *geometric algebra*. He could, however, not answer the priority question: Were these results first algebraic and then put into geometric garb, or first geometric and then later discovered to have algebraic implications? Neugebauer, the discoverer of “Babylonian algebra”, later was able to utilize Zeuthen’s discovery to support his own idea, that the Greeks “geometrized algebra”. This fitted with his theory of how the discovery of linear incommensurability fitted into Greek history. (Szabo, 1978:30-31) We have seen that it is more likely that geometry was first considered to be the foundation of mathematics and therefore of everything and that many of the Presocratics thought in geometric rather than arithmetical terms, so that it seems more likely to me that the geometrical propositions came first, maybe with arithmetic subconsciously smuggled in, and then later the algebraic interpretations were made explicit.

As Fraenkel et al. points out, the existence of incommensurable lines led to a crisis in the foundations of mathematics (Fraenkel et al., 1973:12-14) which made it necessary to reconsider the foundations of mathematics. This lies at the root of the struggle between different schools in the philosophy of mathematics among the pre-Socratics. This struggle was not resolved by them, or by either Plato or Aristotle, but a lot of the tension eventually just wore out as conventional work-arounds were developed and then gradually accepted as if they solved the problem. This is a pattern that we see in later foundational crises in the philosophy of mathematics again, and it is worthwhile to investigate this in a later study.

Taking it as reasonably well established by now that geometrical entities are the foundational entities in the Pythagorean ontological system, we are not yet done. We have not yet considered where these geometrical entities fit in terms of our universe. Did they exist *within* our universe, or were they relegated to some exterior, untouchable universe like the universe of Plato’s Ideas?

I think we can make a good case for the position that the geometrical entities as Pythagoras and his school saw them, existed in our universe and were intimately involved in those parts of the universe that we can perceive. We never read about the Pythagoreans that they held such positions as “these figures that we draw are not the *real* figures” or “we cannot trust our eyes to tell us the truth about reality”. Instead, we have reliable evidence, documented by Heath for example, that mathematical proof in the early Pythagorean school was empirical, evidential, comparative, and visual. Superposition was an important tool in the mathematician’s tool belt.

We also see that the entities in the universe are considered to be made of geometrical shapes. Specifically, a traditional view of the *Timaeus* is that it is based in part on the cosmology of the early Pythagoreans. The idea that the elements were shaped like cubes, tetrahedra, octahedra, and icosahedra seems to date back to the Pythagoreans.

One particularly striking piece of evidence in this line is the story of Hippasos being drowned for revealing the construction of the dodecahedron, the “sphere formed out of twelve pentagons” (it looks sort of like a soccer ball). (Burnet, 1920:216) It seems that the problem with revealing this construction was not that mathematics was taught to the uninitiated, but rather that this very specific piece of mathematical knowledge was divulged. The significance of this piece of mathematics was that the heavens were thought to participate in the shape of the dodecahedron, and that therefore a particularly sacred pearl was thrown to the swine.

It is clear therefore that mathematics not only was fundamental to the structure of the universe, but also that it was not some transcendent pattern for perceptible reality to aim at, but rather an intensely immanent underlying reality. Echoes of this idea of the immanence of the true reality can be seen in the philosophy of Aristotle, which we shall later consider, but of course Aristotle would have had serious issues with the idea that the *real* reality is imperceptible.

An obvious problem (in hindsight, which gives us some advantage over the Pythagoreans) with the conception that mathematical entities are immanent in our universe is how changeability then arises. The Pythagoreans did not seem to have any problem with the idea that the mathematical entities in their universe could exhibit motion and change. In fact, they would express their cosmogenesis as a motion out of the One (very reminiscent of how the neo-Platonists would later come to view it in the third to fifth centuries AD) and would speak of space as coming out of the Dyad.

It seems that they had a very organic view of mathematical entities, as if they were imbued with a life-force, with the power of procreation. Their figure-numbers (see my discussion of the tetraktys above) did not merely act as if there is harmony and peace between them. There actually was harmony between them (at least, talking in their idiom). Justice was not just similar to four in certain parallelisms; no, justice *was* four. (I am aware of Cleary’s statement that justice is four “on account of the latter’s reciprocity”, but considering how careful Philolaus was to distinguish *being* number from *being similar to* number, I think my point still stands; cf. Cleary, 1995:32.) This is a very foreign way of thinking to us; but if we want to stand any chance of honestly and correctly evaluating Pythagorean philosophy, we need to attempt to place ourselves in their shoes and see numbers as they did.

I think it is just this foreignness that has plagued the attempts philosophers have made to compare the concepts of participation in Pythagorean philosophy on the one hand, and imitation in Platonic philosophy on the other. I believe it is nearly impossible to understand the distinction between saying that things participate in number and that things imitate number without taking this broader view of the cosmologies of Pythagoras and Plato. (Cleary seems to agree with this assessment; Cleary, 1995:41, footnote 81.)

When one does keep in mind these different cosmologies, with the concomitant importance of immanence of numbers as compared to transcendence of numbers, what comes out is very interesting. Specifically, it becomes immediately clear that in the Pythagorean system, mathematical objects suffuse the entire universe, and that participation in number then must mean something like “a little piece of number is in this thing”. This is not quite right, but it comes close. Maybe it is a little more like saying that the predominant atomic make-up of this thing (say, for instance, earth) is this mathematical figurate number (in this example, cubes, if it is correct to identify this part of *Timaeus* with early Pythagorean teaching).

I’m not sure that the atomic model is entirely right here, but it is not easy to see what the precise content of participation would be. This is not to say that it is not easy to draw the line between participation and imitation. Whatever participation in number means, it certainly has to mean this one thing: number, as immanent in the universe, has a direct causal effect on the things that participate in number, which is all things (remember, the whole heaven is a musical scale and a number). Imitation, on the other hand, could never mean this, because number in the Platonic scheme is fully transcendent and not immanent. (Aristotle – Metaphysics, 987b29-33)

This preview of the Platonic philosophy of mathematics, which we will discuss later, has been a little out of place, but it would have been detrimental to the discussion of the Pythagorean system to not say anything about the concept of participation here, because it is central to their philosophy. However, a better and fuller discussion of imitation will have to follow when Plato’s philosophy is discussed in due turn, since this concept is fundamental to the philosophy of Ideas.

In order to make sure that the foundational argument of this section is as clear as possible, I shall briefly return to what my main aim is: to show that geometry was considered to be more foundational to the pre-Socratics than arithmetic. One corroborating piece of evidence comes from an unexpected quarter: the etymology of the earliest terms of Greek mathematics. I mean the word *deiknymi*. This word which means *proof*, Szabo points out (Szabo, 1978:186-190), meant making something visible or, as Plato put it, putting something before the eyes.

Clearly, if the point of a proof, in the earliest understanding of the term, meant to make something visible to visual inspection, then there must have been a diagrammatical (I use this word on purpose; *diagramma* also meant *proof*), schematical exemplification of it in a geometrical way. The simplest of arithmetical theorems, such as the commutativity of addition for example, would have been carried out by placing two arrays with *m* and *n* dots next to one another and counting the resulting number *m+n* of dots, and then swapping the arrays and repeating the experiment.

This shows that what was clear to the eyes, as geometrical figures, was clear to the mind as well, as being a reflection of reality. Therefore, reality was seen as fundamentally geometrical, and this dialectical method shows that that was how the early Greek philosophers saw it. Admittedly, *in vacuo*, this argument is not particularly strong, but in concert with the other arguments that have been provided, it points to the same conclusion, namely, that pre-Socratic philosophy of mathematics was reductionistic, and that in particular it had the tendency to reduce all of mathematics to geometry and applications of geometry.

Other evidence for this last statement can be seen in how the other branches of mathematics (at least, the other branches known to early Greek mathematicians) were treated. Harmony was treated as a branch of applied geometry, and it serves as a particularly interesting treatment of what can be achieved in one-dimensional geometry. The theory of proportions and the method of *anthyphairesis* seems to me to have had its origin in the search for the array of means that were discovered by Archytas and Hippasos. Astronomy, also called sphaeric, was another branch of applied geometry (in this case three-dimensional geometry), and the treatment it received at the hands of Plato and the early Academy makes it clear that there was a strong tradition of viewing it as rather a field of geometry than an observational science of the actual heavens.

This tendency to reduce all of mathematics was also noticed by Laugwitz, who wrote (Laugwitz, 1986:649):

> Ausgerechnet am Ordanssymbol der Pythagoräer, dem regelmäßigen Fünfeck, bilden Seite und Diagonale ein solches Paar inkommensurabler Strecken. Jedes Zahlenverhältnis läßt sich geometrisch darstellen, aber nicht jedes Streckenverhältnis arithmetisch. Das begründet einen Vorrang der Geometrie vor der Arithmetik, und die Konsequenz sind die Bücher des Euklid: Die Theorie der Zahlen ist ein Teil der Geometrie. [The regular pentagon, fully a sacramental symbol of the Pythagoreans, shows in its edge and diagonal one such pair of incommensurable line segments. Every numerical relation can be represented geometrically, but not every geometrical relation can be represented arithmetically. This shows the foundation of arithmetic to be geometry, and the consequence is the books of Euclid, where the theory of numbers is a part of geometry. – My translation]

The fact that geometrical relations can be represented arithmetically but not the other way around, is really a reflection of the fact that the modality of number is cosmically earlier (a Dooyeweerdian term) than the spatial modality. The fact that the spatial modality cannot be reduced to anything else is an indication that we have here a fundamental modality of the world, whereas the failed attempts of the Pythagoreans to reduce arithmetic to geometry shows that the arithmetical modality is also a fundamental mode in which the things in the world function.

This reductionistic approach to mathematics could not last indefinitely. In mathematics, there are distinct, irreducible concepts that militate against reductionism. Zeno of Elea was an expert at pointing out these *aporia*. He saw clearly the problem with trying to construct a continuous, infinitely divisible universe from indivisible, discrete points of space.

Geometry before the time of Parmenides and Zeno made frequent use of analogies from motion to describe geometrical configurations. For instance, a geometer might say something like “take this line segment and extend it” or “this point moves from this line to the center of the opposite side of the triangle to form the median”. Zeno’s paradoxes of motion made it clear to subsequent geometers that in order to avoid dealing with the objections that Eleatics might bring forward, it would be better to eliminate language of motion from the discussion of geometry altogether. This became standard practice both in philosophy of geometry (consider Plato’s discussion in the *Timaeus* on tensed language for geometry) and in geometry itself (consider the *Elements* of Euclid, where language of motion has almost completely been eliminated).

A word or two should also be said about how Zeno dealt with the whole and parts. Fundamentally, Zeno wanted to prove that the idea of multiplicity is inherently contradictory, and thereby, to establish the metaphysical idea of oneness. (Strauss, 2012:62) If it were the case that a whole could be divided into multiple parts (as is the case with the continuous line segment), then multiplicity would have a foothold in his philosophy. It is at this problem that he set his sights in producing his famous paradoxes. All of the paradoxes aim at making division into parts seem illogical. Two of his paradoxes are basically equivalent: The paradox of Achilles and the tortoise has a race between these two where Achilles gives the tortoise a head start, after which he has to go through an infinite task of progressively catching up more and more to the tortoise (halfway closer in each step). But this infinite task can never be completed, because there is always a next step in the task. In another paradox, Achilles has to finish a race, where he first finishes half of the race, then the next quarter, then the next eighth, and so on. Again, this is an infinite task, which can never be completed. Essentially, this latter paradox is again Achilles and the tortoise, but now with a stationary tortoise (Maybe the tortoise has died? We haven’t checked up on the welfare of the tortoise in a while).

B Fragment 3 has Zeno presenting an argument against the possibility of forming parts. For, he says, if there were parts making up a whole spatial continuity, then there would have to be finitely many of them, because just so many (a precise amount) can be put together to form the whole. On the other hand, there would have to be infinitely many parts, because whenever you subdivide the continuity, you can again subdivide the parts. So we end up with the number of parts being both finite and infinite, which is impossible.

The whole-parts relation of space turns infinity in upon itself, that is, it focuses on the successively infinite division of the continuum into smaller and smaller parts. Zeno’s paradoxes follow from his inability to make sense of the mutual coherence of the aspects of reality that requires that the spatial, quantitative, and kinematic aspects be interpreted not in isolation from one another, but as an unbreakable whole! In trying to avoid multiplicity, Zeno inevitably has to introduce multiplicity in his philosophy (there is now a world of truth, where the One is all, and another world of perception), because reality is not reducible to only one aspect. Simply denying the reality of the multiplicity inherent in his own philosophy by denying the trustworthiness of perception does not actually get rid of the multiplicity in his philosophy.

Geometry could easily survive a change in terminology that eliminates motion. It is simple to change “extend this line segment” to “this line segment is part of a longer, eternally existing line”. But Zeno’s impact could not be confined just to creating new elocutions of old geometrical terms. No, the Eleatics also held to the idea that plurality is a figment of our imagination. This had a profound impact on the branch of applied geometry known as arithmetic. The foundational concept in arithmetic is multiplicity. If plurality is only a figment of our imagination, then does Eleatic philosophy imply that all numbers are merely figments of our imagination too?

I doubt the Eleatic philosophers would have said that numbers are *merely* figments of our imagination, but their philosophy makes it difficult to explain multiplicity in mathematics. Several attempts were made to resolve this difficulty. The most famous of these, surely, is the separation (*chorismos*) that Plato made between the realm of perceptibles in which we live, and the ideal realm of imperceptibles where the Ideal Numbers exist. In this inaccessible realm, there is only one 2, only one 3, and arithmetic does not apply, so that it is not true to say in this realm that 2+3=5 or that 2<5. We shall return to this attempted solution later when we focus our attention on Plato.

Plato was not the only philosopher to try to solve the problem of mathematical multiplicity in an Eleatic context, though. Another attempted solution, proposed by the atomistic school of thought led by Leucippus and Democritus, was to deny part of the Eleatic assumptions, namely continuity. If there was a problem caused by accepting both continuity and discreteness at the same time, then maybe the problem could be solved by denying discreteness (this is what Cleary seems to think Aristotle did) or by denying continuity (this is what the atomistic school did). If Achilles needs to run smaller and smaller distances to get to the finish line, then the atomists proposed that there was a final minimum distance that could no longer be halved, and at that level of movement, Achilles’ run would turn into a series of discrete jumps.

In this atomistic philosophical system, it seems clear now that numbers would gain far more prominence than in the Pythagorean or Eleatic systems. The unit would be an absolute, and the unit corresponds much more naturally to numbers than to figures. Cornford gives us a clue as to how the unit was viewed differently by the Pythagoreans and the atomists:

> In discussing the segregation of opposites and its possible origin, as reflecting the exogamous segmentation of the undifferentiated human herd, we have already pointed out that the Pythagorean one, or monad, splits into two principles, male and female, the even and the odd, which are the elements of all numbers and so of the universe. The analogy reminds us that the one is not simply a numerical unit, which gives rise to other numbers by a process of addition. That conception belongs to the later atomistic number-doctrine, presently to be considered. In the earlier Pythagoreanism, we must

> think of the one (which is not itself a number at all) as analogous to Anaximander's *apeiron*. It is the primary, undifferentiated group-soul, or *physis*, of the universe, and numbers must arise from it by a process of differentiation or 'separating out' (*apokrisis*). Similarly, each of these numbers is not a collection of units, built up by addition, but itself a sort of minor group-soul – a distinct 'nature', with various mystical properties. In the same way, it is by dividing up the whole interval of the octave that the harmonic proportions are determined. (Cornford, 1991:210)

What we are seeing here is the beginning of the trend in the foundations of mathematics to place geometry and arithmetic on an equal footing. However, exactly at this stage, such a coordinate position was not yet possible. It might be quite reasonable to expect that numbers must be measurable by the unit, but it seems a lot less intuitive that there are line segments that cannot be bisected. A finitist, atomistic approach to geometry is certainly possible (modern day mathematicians have constructed such geometries), but it is at the same time less natural. For the atomistic solution to the Eleatic problem to work, our natural intuitions and inclinations have to be ignored. Continuity has to be banished or else Zeno's paradoxes lose none of their sting.

We see some traces of this atomistic approach to geometry in Democritus' description of the cone. I think it is obvious that what he had in mind was a cone made of layers of atoms. The following quote gives his description:

> If a cone were cut by a plane parallel to the base, what ought one to think of the surfaces resulting from the section: are they equal or unequal? If they are unequal, they will make the cone have many steplike indentations and unevennesses; but if they are equal, the sections will be equal, and the cone will appear to have the same property as a cylinder, being made up of equal, not unequal, circles, which is most absurd. (Freeman, 1948:106)

Equal sections would result from the assumption of continuity, and Democritus constructed this argument to demolish that assumption by a *reductio ad absurdum*. What is then left, the assumption of discreteness, was according to Democritus the only possibility; and therefore, however unlikely, must be the truth. It leads to the conclusion that the outer surface of a cone must consist of many, many steplike indentations and unevennesses, the height of each step being the length of an atom. The insistence that there are many, many indentations also at the same time overthrows the Eleatic rejection of plurality. The price to be paid for this interpretation of mathematical reality is the loss of intuitiveness. To the atomists, this was an acceptable price.

A more serious problem that they seem not to have considered, is that once atomism is introduced into geometry, the cone completely disappears. There are no circles from which to construct the cones with steplike indentations, since the "pixelated" circles that could be formed from atoms are not true circles, unless the "pixels" are infinitely small. If the "pixels" are infinitely small, then the problem in the Democritus quote is not resolved. We have no way of knowing what the atomists would have done with this consideration produced by hindsight, but Plato certainly protected himself from this problem by projecting “real” geometrical figures into a distant realm.

Heath gives evidence (Heath, 1921:I 178-181) that Democritus was indeed conscious of the concept of infinitesimals. However, Heath goes too far in assuming that Democritus didn’t postulate indivisible lines. Heath bases this conclusion on an interpretation of the Democritus quote above which is incorrect. He interprets it as saying that there is an impossible choice: discreteness leads to steplike indentations (Heath thinks Democritus rejects this possibility) while continuity leads to the conclusion that cones are cylinders. In fact, as the quote shows, Democritus only says that the second choice is impossible. To conclude that Democritus thought that the first choice is also impossible is surely to read one’s own preconceptions into the quote. One hesitates to disagree with a scholar like Heath, but I think that here the facts speak against him.

What emerges in this whole discussion is that the concepts of continuity and discreteness, as well as the disciplines of geometry and arithmetic, are seen as inimical to one another and co-destructive. Despite the protestations of Szabo to the contrary (Szabo, 1978:94-97), I really don’t see how this can be seen as anything other than a foundational crisis. It might be that the distance of time between us and the ancient Greeks has smoothed over the ragged edges of the crisis, but there surely must have been tension between the Eleatics, the atomists, and the Pythagoreans on the question of the foundations of mathematics.

Here we must turn to the real question then: *are* the disciplines of arithmetic and geometry enemies? To criticize the early Greeks for coming to this conclusion is to pre-determine the answer to this question. One must first know that these disciplines are not enemies before deciding that anyone who says that they are enemies must be wrong.

Let us look at the question of the incommensurability of the diagonal of the square to its side. If one asks the question “is $\sqrt{2}$ a rational number?”, the answer is no. In the Greek interpretation of arithmetic, this question sounds like this, “is $\sqrt{2}$ a number?”. In arithmetic, we answer “no”. However, if one asks the question “is there a line with length $\sqrt{2}$?”, the answer is yes. The diagonal of the unit square is this line. So in geometry, we answer “yes”. If the questions “is $\sqrt{2}$ a number?” and “is there a line with length $\sqrt{2}$?” are really the same question, then there is enmity between arithmetic and geometry.

Are they the same question, though? It is certainly popular to look at geometry with post-Cartesian lenses and then to conclude that, yes, obviously, concepts like length, area, and volume are fundamental concepts in geometry that cannot be understood in any other way than as part of arithmetic. This is definitely a temptation, but this temptation must be resisted if we are to answer this question as an ancient Greek philosopher would.

There is no doubt that concepts like length (measured in units), area, and volume were considered to be a part of mathematics. The discussion between Socrates and the slave boy in the *Meno* makes it clear that thinking of line segments as having lengths of 1, 2, etc. units, and of enclosed regions as having areas of 1, 2, etc. units squared, was a common phenomenon, so common in fact that slave boys with only a very rudimentary education were aware of these concepts.

But we cannot simply come to the conclusion that these would have been thought of as concepts falling within the purview of true geometry. Plato himself makes the distinction between pure and applied arithmetic without causing a stir, and we may safely assume that this same distinction would have held true for geometry. But if this is the case, then the *Meno* passage could easily be understood as an appeal to applied geometry as a means of showing the theory of reminiscence. It seems to me a little dubious to say that Socrates would have expected the slave boy to be able to follow a discussion about the Ideas. Plato writes that Diotima was unsure that Socrates would be able to follow her into the abstractions of the deep theory of the Forms. If Socrates was not certain to be able to follow through all the abstruse, mystical argumentation, then surely Plato would not have depicted the slave boy as being able to follow the conversation. Therefore, I think it is purposeful for the discussion to be centred on what would have been very practical knowledge in geometry that would be necessary for performing everyday tasks.

If we were therefore trying to answer the question whether geometry and arithmetic intersected in ancient Greek thought, the most obvious line of approach (lengths, areas, volumes) would be easily blocked by claiming that this is not *real* geometry. I think the Greeks would have argued that geometry as such deals with configurations, but that these configurations only acquire measurements once we try to apply them, otherwise they only have relative positions. This would mean that the existence of the diagonal line of a unit square does not contradict the impossibility of providing that line with an arithmetically determined length.

I think this instinct is correct. The implication is that we can say things about a figure in geometry without necessarily being able to say something about it in arithmetic. Here is an example in the opposite direction: Suppose I have a point and I want to draw a unit line segment from that point to the left or to the right. If the digit in the expansion of $\sqrt{2}$ at position $10^{10^{10}}+1$ is even, then we draw the line to the left. If the digit in that expansion at position $10^{10^{10}}+1$ is odd, then we draw the line to the right. Now we can say with certainty (in any philosophical framework that would have been familiar to the ancient Greeks) that the length of the line must be equal to 1 (this is our arithmetical piece of knowledge). However, we don't know, and we probably never will know, whether the line goes to the left or to the right (this is our geometrical piece of ignorance).

I think that this would have been pretty intelligible to the ancient Greeks (after some explanation of the decimal system) and would also reflect their philosophical framework.

If my thought experiment would succeed, and I think that it would if only it were possible to verify it given everything we know about ancient Greek philosophy, then the obvious next step would be to ask them the question “Is $\sqrt{2}$ the same thing as the diagonal line of a unit square?”. The answer would clearly be no. But if this is the case, then geometry and arithmetic do not deal with the same questions and objects, and therefore they do not come in conflict. They are not enemies. They are in fact different modalities, and the Greek intuition here is perfectly right.

Since the pre-Socratics, and specifically the Pythagoreans, said that the sensible reality around us is composed of mathematicals as a kind of “matter”, we can infer that they thought the matter-theme in the form-matter dialectical ground motive is more fundamental and more important.

Plato’s view of the ontic status of mathematical entities

We turn now to consider the philosophy of Plato with regard to the ontic status of mathematical objects. Plato was greatly influenced by the Pythagoreans. He is also the most influential philosopher of mathematics in all of history. This speaks much to the credit of the Pythagoreans, which is why so much attention was given to them in the previous section.

Plato’s philosophy of mathematics must be understood in the greater context of his philosophy of the Forms or Ideas (the terms are used synonymously). The theory of Forms has its origin in the desire to have fixed, unshakable knowledge of reality. Plato, just like many of the pre-Socratics before him, believed that observation is not a reliable way of coming to know reality. Our senses give us information about sensible things, but these things are in constant development, change, and flux. Plato’s epistemology of sensible things is deeply rooted in the Heraclitean theory of flux. He postulated that we can have no real knowledge of sensible reality.

If it is true that we can have knowledge while it is also true that we cannot have knowledge of sensible reality, then it becomes clear that the only things that we can have knowledge about is the insensible, or in Plato’s scheme, the supersensible things. There is supposedly a realm of things that exist outside of the reach of changeability, therefore outside of our universe, which experiences no change ever. These things, because of their stability, can be known, and once known, will remain known because they do not change.

These things (which Plato called the Forms or Ideas) are reflected in our sensible world like a mirror image reflects the real object in front of the mirror. The comparison is apt: if the object is not in front of the mirror, there is no mirror image; and if the Idea does not exist, then the sensible thing in our world also cannot exist. At least, this is Cornford’s interpretation:

> The logical theory is not distinguished from a metaphysical doctrine, which may be stated thus: ‘This beautiful thing *exists* (or begins to exist, *gignetai*) for no other reason than that Beauty exists, and this thing partakes (or comes to partake) of its nature’. On this interpretation, the existence of Beauty Itself is asserted to be the *cause* of the existence of all particular beautiful things in the world of sense. The relation called *methexis* is not here the logical relation of subject to predicate in a proposition, but a causal relation. The Idea is to be, somehow, the supersensible *ground* of the existence of sensible things which become and perish in time. (Cornford, 1991:256-257)

There are some problems with Cornford’s interpretation. Plato directly asks and answers the question about whether there are Forms for mud, hair, and so on, and his answer is no. But for the most part, if things exist in our world, they reflect things in the world of Forms. I think it was an act of kindness on Cornford's part to make Plato consistent with Plato that caused him to not remember the exceptions to the rule of imitation.

The analogy of the line describes Plato's simultaneous approach to ontology and epistemology. At the bottom rung of being and knowledge are the reflections we see in ponds and mirrors, paintings depicting natural scenes, actors representing characters, and so on. If one takes the last example I gave, one sees the point. You can study Laurence Olivier as Hamlet for years without getting one jot more knowledge about the real Hamlet. In this case, it seems there was no real Hamlet. But if there were, studying Laurence Olivier's performance would never teach you anything about the historical figure. You could only learn about the depiction of the historical figure. In essence, what you would learn is an opinion.

The next rung in the ladder of being and knowledge is occupied by the sensible things around us. About these things, which have a transitory nature as we already said, you can come to know some things, but you would have to include the knowledge of their transitoriness and your ability to someday be wrong about the thing if you wanted to have accurate knowledge of these things.

Following this level of existence, there is another level, although there is some debate about whether Plato recognized this level or not (Wedberg, 1955:12-14). This level is the level of the Intermediates, something below the Ideas but yet not part of the sensible existence. There is some similarity between this level, where the mathematical objects of the geometer exists, for instance, and the level of depictions and reflections. As the level of the depictions gives us knowledge not of the real existences but only of what we see, so the figures of the geometer gives him some knowledge of supersensible particulars, but not of the real Forms.

To give an example of what this would mean: in geometrical figures, we can identify various kinds of triangles, like isosceles triangles, equilateral triangles, scalene triangles, right-angled triangles, and so on. We can manipulate these triangles, we can reduplicate a triangle, and we can add triangles together along a common side to form a bigger triangle. This can be distinguished, however, from the Idea of the Triangle. There is only one Form of the Triangle, and this Form cannot be reduplicated, added, or manipulated. It lies outside of our sphere of influence.

We will return later to the mathematical objects in the level of the Intermediates when we consider how Wedberg schematized the philosophy of geometry and the philosophy of arithmetic which Plato developed. For now, however, we will first continue with the last ontic level in the Platonic ontology.

The highest level of Plato's ontology is the level of Forms or Ideas. This level is populated by unique, perfect models of the things in the lower levels. For instance, there is an ideal table, which models the concept of table in such a way that it not only exemplifies to the artisan what he should be aiming at when he builds a table, but when the visible table has been built, that table is a table by participating in the meaning of being a table in that it is caused by the ideal table.

Cornford says that there is a mythical outflowing of the soul of the ideal table to the soul of the instantiated table (Cornford, 1991:256-257). This sounds like a neo-Platonic interpretation of the philosophy of Plato until one remembers that Aristotle also writes about Thales' explanation of the magnet that "everything is full of gods". This seems to have been a current idea in early Greek philosophy, and it would not be surprising if Plato borrowed the idea of panpsychism to explain the correlation between the Ideas and their sensible counterparts.

That gives us a preliminary picture of the ontological classification that Plato employed. For our purposes, the lower two levels of the classification are not particularly important, but if we want a coherent view of the world we cannot entirely ignore them. It seems to me that that was also the main criticism that Aristotle had for Plato's philosophy. However, we can place them in the background while we consider in more detail what Plato had to say about the objects of mathematics.

Wedberg gives us a pretty full picture of Plato's philosophy of mathematics. We get the following summary of Plato's philosophy of geometry from him (Wedberg, 1955:61-62):

1. There are two kinds of geometry, the popular kind and the philosophical kind. Popular geometry deals with visible figures, and is always approximate. Philosophical geometry deals with the supersensible realm of the ideas and the intermediates, namely Euclidean entities, and it makes absolute and exact pronouncements of truth.
2. The entities of philosophical geometry share some properties with the Ideas:
    a. They share the mode of existence of the Forms.
    b. These entities are perfect exemplars.
    c. There is a multiplicity of mathematical entities (such as triangles, squares, etc.)
3. The ideal geometrical entities (the geometrical intermediates) lie between the geometrical Ideas and the sensible world in the classification of ontology. The geometrical Ideas are perfectly exemplified by the ideal geometrical entities and imperfectly exemplified by the visible diagrams. There is an imperfect resemblance between the ideal geometrical entities and the visible diagrams since the visible diagrams are only approximations of the ideal geometrical entities.
4. Plato's argument for the existence of these geometrical intermediates is as follows:
    a. Geometry is true.
    b. Geometry can only be true if we can identify geometrical entities of which we can make true statements.
    c. We cannot make true statements about the sensible world because
        i. Everything in the sensible world is in a constant state of flux (note the dependence of change on the prior presence of constancy), and
        ii. Everything in the sensible world is a mixture of opposites.

   d. Considering statements (a) to (c), we may conclude that the geometrical entities must exist, but not in the sensible world, therefore they are in the supersensible world.
5. The statements of geometry must be phrased in a way that reflects its ontological truth, which means that there can be no flux-language used in them. Any geometrical statement that contains reference to movement or influence by the geometer is false, because the concepts and entities of geometry are eternal and unchangeable.
6. Philosophical geometry is a deductive science. There must be foundational hypotheses upon which the rest of the science must be based. This is so in order to avoid an infinite regression to more and more fundamental assumptions.
7. In philosophical geometry, the most fundamental concepts must be derived from the most certain principles, namely the Ideas that can be found in the science of dialectic. Ultimately, the foundation of geometry is the Idea of the Good. This has the implication for geometry that an argument from appropriateness would be a valid argument. This is very strange for a modern-day mathematician, but before this implication is thoroughly understood, some of the discussions in Plato's dialogues, like the discussion of the colours of the celestial spheres in the *Timaeus* and the discussion of the ideal city size in the *Laws*, just cannot be properly interpreted.

A large part of the contribution that Plato made to the development of mathematics was a direct outflow of this systematic philosophy of mathematics. Several philosophers have pointed out that Plato had a tremendous impact on the course of mathematical methodology. As pointed out in the previous section on the pre-Socratics, it was accepted mathematical practice to make use of sensible demonstration by means of superposition, for instance, in order to prove mathematical theorems. Plato insisted that mathematics ought to be done using the principles of dialectic.

As a result, we see differences between the way that Autolycus approaches geometrical demonstration and the method of the *Elements*. In the *Elements*, all reference to perception and to motion is systematically purged and replaced by argumentation that requires the reader to either imagine or, in case that is too difficult for the reader, to supply a diagram himself. Likewise, instead of extending lines, the language implies that the line segments in diagrams are part of already existing longer line segments (the at once infinite line is not considered to be a well-formed concept).

It should be noted, however, that constructive proof is not co-extensive with empirical proof. After Aristotle, but still in the Academic line of thinking, Euclid gave his famous five postulates, all of which demand (the original meaning of *aitema* (Szabo, 1978:276)) that the postulated constructions deal with things that exist. Szabo's interpretation of this, that mathematical entities are said to be existing if they can be obtained from construction, makes sense in this context. Although Plato's ontology of mathematics requires that the mathematical intermediates exist more "really" than as a result of construction, I think that his epistemology of mathematics aligns very well with this constructive approach to mathematics. However, the movement of his approach is definitely away from empirical proof, so that what must be understood under construction here is dialectical construction, not visible construction.

Interpreting *aitemata* as a dialectical term that has been adopted by mathematicians has some interesting implications. It seems that there was a debate raging about the methodology of mathematics, and that *aitemata* were demands made by one party to another party in this debate. Plato retains this terminology, and the tension between the Eleatic demand for changelessness and the Pythagorean view of flowing, living figurate numbers can be detected in Plato's different points of departure for his ontology of mathematics and his epistemology of mathematics with regard to construction. Plato's intuition that change and constancy are both fundamental to the way the world works was a good intuition, but in the absence of a theory of coherence between different modalities, he was not able to successfully account for these two aspects of the world.

Of course, this dialectical tension becomes inevitable as soon as geometry is understood to be the science of space, since the Eleatics rejected the existence of space. One cannot have a science of space if there is no space, since knowledge is always of something, as Plato himself points out. Therefore, in order to say anything at all about geometry, Plato must make some dialectical demands. At the same time, there is clear evidence that Plato took the Eleatics seriously and tried to incorporate their philosophy into his own (the changelessness of the Ideas, for instance, is a sign of Eleatic influence). Therefore, Plato tried to incorporate aspects of Eleatic philosophy with aspects that he received from the Pythagoreans, and these two sources are fundamentally in conflict with one another.

Another point of internal conflict in Plato's philosophy of mathematics lies at the point of constructive proof. The idea that mathematical proof cannot be based on observation can be traced back to the Eleatic distrust of the senses; whereas, the idea that construction can be done dialectically so that the paper or parchment or whatever medium they were using does not "sully" any pristine truths that can be constructed mentally, must have had its origin in the earlier tradition of demonstrative proof. For more on the objections the Eleatics had to the existence of space, consult Szabo. (Szabo, 1978:312-316)

Part of the method of dialectic is the technique of *reductio ad absurdum*. This began having a felt influence on the methodology of mathematics as well, roughly around the time of Plato. It seems that the first application of this method might have been related to the discovery of linear incommensurability of the diagonal of the square to its side, which at the same time would have weakened the confidence of the ancient Greek mathematicians in the empirical method of proof. (Szabo, 1978:215-216)

The need to examine fundamental assumptions would also come into clearer focus as a result of the incommensurability problem. Plato writes in his dialogues of a Socrates who would always examine the initial assumptions of discussions, mainly on ethical subjects (but not exclusively). He also seems to have realized that this same examination of foundations would be necessary for geometry, which has started to indicate that it was not based on rock solid foundations. As Szabo points out:

> These words, which imply that 'first assumptions' have to be *certain*, inevitably bring to mind the *protai archai* of mathematics. As far as we know, Aristotle was the first author to discuss explicitly the nature and function of first principles in mathematics. Yet Socrates' remarks about the 'first assumptions' of a dialectical argument give the impression that Plato was also fully aware of the problem of 'first assumptions' in mathematics. (Szabo, 1978:237-238)

Szabo, of course, goes further and shows that the development of mathematics in the dialectical direction has a history that precedes the time of Plato, and provides quite convincing evidences of this, such as the assumptions in the *Elements* that are never exploited to prove propositions. (Szabo, 1978:253-257) This indicates that there was a history to the assumptions that preceded Euclid and, Szabo (probably rightly) assumes, also Plato. However, none of this proves that there was a concentrated effort before the time of Plato to lay mathematics on the foundations of dialectic. It seems to me, from the available material, that this concentrated effort can be traced back to Plato and no further.

I will systematically move through the written works of Plato so that the places where he writes about his philosophy of mathematics can be examined one by one until we have confidence that we are indeed getting a proper view of his philosophy and to make sure that we are interpreting him accurately.

In Plato's *Euthyphro* 12, we find the first reference by Plato to mathematical objects. His first mention already shows that he was influenced by the Pythagoreans: saying that the odd and the even are the parts of numbers indicates that the concepts of odd and even are frontmost in his thoughts when he thinks of mathematics. We already considered the important place that the odd and the even had in the philosophy of the Pythagoreans, and here we see that odd and even also were important to Plato.

In the *Phaedo*, Plato discusses the creation of the universe from fire and earth, and he says that the binding material between fire and earth is proportion. This looks like a Pythagorean concept to me, similar to saying that the unifying principle of the universe is number. An interesting point to notice here, is that proportion is treated almost as if it is a material substance. This definitely is a Pythagorean idea that harkens back to the syncretistic attempts of the Pythagoreans to unify the religious ground-motives of form and matter.

Also in the *Phaedo*, Plato gives an account of the origin of numbers. He explains that the Creator made the world in accordance with a perfect exemplar, setting the celestial bodies in motion around the earth. This motion of the starry heavens gives origin to time, and time is the foundational building block of number. Here, the telluric religions that Dooyeweerd talks about finds their home in the philosophy of Plato, since time only leads to number when there is a perceiver that can count the revolutions of the celestial bodies (to be fair, Plato points out that for the most part only the revolutions of the sun and the moon are important to most people and that the other revolutions have never been named).

I'm not certain that Plato believes humans have always existed. If he doesn't believe that, then a form of panpsychism must be involved in order to account for the pre-existence of number. He certainly makes it seem later on that human observers are not necessary, since the entire celestial world is itself a living thing. He says "This, then, is how as well as why those stars were begotten which, on their way through the universe, would have turnings. The purpose was to make this living thing as like as possible to that perfect and intelligible Living Thing, by way of imitating its sempiternity." (Plato – *Timaeus*, 39d-e)

It seems that Plato was uncertain about the eternity of number. In some places of his dialogues, he implies that all the mathematical objects must be eternal, but in the *Timaeus* he says that it is a human invention, that "As it is, however, our ability to see the periods of day-and-night, of months and of years, of equinoxes and solstices, has led to the invention of number, and has given us the idea of time and opened the path to inquiry into the nature of the universe." (Plato – *Timaeus*, 47a) If this is the case, then number could not be eternal. I suspect that in the course of his philosophical career, Plato changed his mind on this subject. Another interpretation would be that Plato is here speaking only about our epistemic access to number, but in this case the word "invent" would be a strange word choice.

Another problem with the interpretation that says that numbers are only invented epistemologically, though they exist outside of our epistemic access to them, is that Plato makes a big deal of the idea that we cannot learn the real truth of something from our senses. In the context of our quote, Plato is talking about the good effects that the creation of our eyes have had upon us. He is therefore quite definitely thinking about our sense-perceptions, and so it would be strange if he would then make no distinction between what we learn about numbers through observation and subsequent invention, as opposed to the true nature of numbers. It therefore seems much more likely to me that Plato has in mind here the Pythagorean conception of number as a flowing, unfixed reality which we actually can invent.

In the *Timaeus* we also find some support for the idea that Aristotle also supported that there are two different kinds of objects in the supersensible world, namely the forms and the intermediates. In describing the origin of the four elements, Plato says "Indeed, it is a fact that before this took place the four kinds all lacked proportion and measure, and at the time the ordering of the universe was undertaken, fire, water, earth and air initially possessed certain traces of what they are now. They were indeed in the condition one would expect thoroughly god-forsaken things to be in. So, finding them in this natural condition, the first thing the god then did was to give them their distinctive shapes, using forms and numbers." (Plato – *Timaeus*, 53a-b) In differentiating between the forms and the numbers, Plato seems to be saying that these two categories of things are not the same, but they are underlying to the sensible world. Here we also learn about the triangles that make up the bodies of the elements, triangles which are forms in their uniqueness and numbers in their multiplicity. Plato is trying to straddle the gap between Heraclitus and Parmenides.

In order to understand the philosophy of mathematics as we find it with the Presocratics, it is necessary to understand first of all that one cannot with right speak of the (unique) philosophy of mathematics. The Pythagorean and the Eleatic approaches to the philosophy of mathematics must always be kept in sharp focus. When analyzing Plato's philosophy of mathematics, it is probably a good idea to keep these two main schools of thought in mind as well, and not to assume too readily that there is a unique philosophy of mathematics that arises in his writings. As we have shown so far, there are elements of both Pythagorean and Eleatic philosophies to be found in Plato. There are therefore seams where these two philosophical systems come together but never fully blend into one another. These seams are the scars left by trying to reconcile two fundamentally opposed religious ground motives into one overarching philosophical system.

One place where these seams become visible is in the insistence of Plato on using dialectical proofs such as proofs by contradiction (an Eleatic contribution to the methodology of mathematics) to prove propositions in geometry, the science of space (a Pythagorean branch of study which has its origin in the original dyad according to the Pythagoreans, which is therefore incompatible with a system like that of the Eleatics that denies plurality). (Szabo, 1978:317-322)

Part of the consequences of the change in methodology that Plato brought about is what Fowler calls *arithmeticisation*. It results in a mathematics that is dominated by the concept of number, where the idea of assigning number to mathematical concepts becomes ubiquitous. (Fowler, 1999:8-13) What we find in Plato is that the Pythagorean trend to try to resolve everything into geometry is balanced by the Eleatic trend to try to resolve everything to number, and ultimately to One. This balance is hardly achieved and when disturbed ever so slightly, shows itself to be a precarious balance that leads to a sudden downfall.

In the *Philebus*, Plato tells us that there are different kinds of sciences. There are sciences of the ideal objects, which deal with unalterable truth, while there are also sciences that deal with practical, sensible things. It is rather surprising to hear Plato call the latter sciences as well, considering his Parmenidean epistemology; but he makes it clear that the practical sciences of mensuration and practical arithmetic are less noble, by which it seems that he means they do not get to the real truth of matters. Here is the relevant passage:

> SOCRATES: And let's take those among them as most accurate that we called primary just now.
> PROTARCHUS: I suppose you mean arithmetic and the other disciplines you mentioned after it.
> SOCRATES: That's right. But don't you think we have to admit that they, too, fall into two kinds, Protarchus?
> PROTARCHUS: What two kinds do you mean?
> SOCRATES: Don't we have to agree, first, that the arithmetic of the many is one thing, and the philosophers' arithmetic is quite another?
> PROTARCHUS: How could anyone distinguish these two kinds of arithmetic?
> SOCRATES: The difference is by no means small, Protarchus. First there are those who compute sums of quite unequal units, such as two armies or two herds of cattle, regardless whether they are tiny or huge. But then there are the others who would not follow their example, unless it were guaranteed that none of those infinitely many units differed in the least from any of the others.
> PROTARCHUS: You explain very well the notable difference among those who make numbers their concern, so it stands to reason that there are those two different kinds of arithmetic.
> SOCRATES: Well, then, what about the art of calculating and measuring as builders and merchants use them and the geometry and calculations practiced by philosophers – shall we say there is one sort of each of them or two?
> PROTARCHUS: Going by what was said before, I ought to vote for the option that they are two of each sort. (Plato – *Philebus*, 55a-59e)

In the *Meno* we find one of the most famous passages in all of the works of Plato on the objects of mathematics. In his conversation with the slave boy (*Meno*, 81-85), he tries to prove to Meno that knowledge is recollection. His paradigm case is to show that the slave boy, despite having no formal training in geometry, is able to deduce that a square *B* with base on the diameter of square *A* has twice the area of square *A*. This slave boy definitely did not get this knowledge in his current lifetime. Now whether Plato has the doctrine of metempsychosis in mind here or instead a pre-existence in an ideal realm is not clear. However, what is clear is that Plato clearly thinks that the recollection of the slave boy is recollection of true knowledge. Considering that Plato teaches that knowledge is always of something, and true knowledge is always of something unchanging and eternal (Plato – *Republic* 479e), this corroborates the proposition that mathematical objects have an unchanging and eternal existence.

What is particularly interesting about this passage, and what I have not seen anyone else noticing, is that Plato does something very strange with this doctrine of recollection. He finishes off the passage by saying that it doesn't matter whether the truth about geometrical figures was recollected or learnt (*Meno*, 87b). In *Meno*, 89, however, he makes it clear that he does not think that virtue can be learnt because one can find nobody to teach it. If mathematical statements could maybe be learnt, whereas truth about virtue definitely cannot be learnt, is this a nod to the doctrine that mathematicals and Ideas occupy different ontic levels? I think it may be, but the evidence is very difficult to interpret.

In *Phaedo*, 73b, we find a similar discussion of recollection. What I find interesting here is that once again recollection is spoken of in the context of diagrams (geometrical objects). I have seen several researchers (Frederick Copleston, for instance) speak of the epistemology of Plato as an epistemology of recollection, but it seems to me that just as we can possess different kinds of knowledge about different kinds of objects in different ontic levels, Plato is also saying in these passages that there are different methodologies of acquiring these different kinds of knowledge. If this is the case, then there is a lot of work that can be done on the topic of the interplay between these different kinds of epistemological methodologies; however, this is not the place to pursue that line of research.

Phaedo, 74a-75d, shows that the things that we have recollection of are not only geometrical objects, but also concepts that occur in arithmetic as well. This passage does cause the above idea a bit of difficulty, considering that here some things that would be considered as part of the realm of Ideas are nevertheless spoken of as being recollected (things like the Beautiful and the Good). I don't know what to make of it that virtue cannot be learnt but the Idea of the Good can be recollected. It seems like Plato vacillated in his opinion on whether Ideas can be recollected or not.

The problem with really understanding what Plato thought about how we acquire knowledge is that he couched his discussions in very symbolic language. For instance, if one looks at Book VII of the *Republic*, one sees the cave-dwellers analogy where knowledge of the realities of the world outside the cave is acquired through observation (looking at the people carrying the things that cast their shadows into the cave, looking at the things themselves, and ultimately looking at the sun). If one were to take this literally, one would come to the conclusion that we learn about the Ideas and about the mathematicals through direct observation. However, Plato also thinks that our sense perceptions are not reliable, whereas we can get reliable knowledge about the things in these ontic levels. So, clearly, we are not to interpret it literally. But are we justified in then interpreting recollection literally? These are difficult questions for the epistemology of Plato. Fortunately, we are busy with considering the ontic status of mathematical entities, and however we want to interpret the epistemological methodology of Plato, we come to the same conclusion about the mathematical entities: they must be separately existing entities that have an independent existence that does not depend on our interaction with them.

It is important to note that the astronomy of the *Phaedo* is not in the main observational astronomy of the kind that we have today. Plato talks about the movement of the heavenly bodies as perfect circles, although actually matching up observation with the hypothesis of perfect circles would cause severe problems to the observational astronomer. The observations, however, are not Plato’s motivation for saying that these movements are circles. Instead, it is the assumption that movement in a circle is the most perfect kind of movement while the heavenly bodies are the most perfect kind of bodies and therefore would exhibit the most perfect kind of movement. So, ultimately, what Plato is really talking about are ideal stellar and planetary entities that move in perfect ways, and sometimes he hints at it that the observed stellar and planetary entities approximate these ideal entities in some way or other. In distinguishing these different kinds of entities, it is interesting to note what Plato says in his letters.

> A passage in one of the *Letters* (No. 7, to the friends of Dion) is interesting in this connexion. Speaking of a circle by way of example, Plato says there is (1) something called a circle and known by that name; next there is (2) its definition as that in which the distances from its extremities in all directions to the centre are always equal, for this may be said to be the definition of that to which the names ‘round’ and ‘circle’ are applied; again (3) we have the circle which is drawn or turned: this circle is perishable and perishes; not so, however, with (4) *autos ho kyklos*, the essential circle, or the idea of circle: it is by reference to this that the other circles exist, and it is different from each of them. (Heath, 1921:288-289)

There are therefore four different kinds of circles according to this classification, although types 1 and 3 really devolve into one category of circle, whereas types 2 and 4 are sometimes considered to be one and the same thing.

In *Theaetetus*, 195d-198e, we have an account of how we know arithmetical propositions. The analogy is to a cage full of birds. The man who owns the cage can be said to possess all the birds inside the cage. However, we would only say that he has a particular bird if the bird is in the cage and he catches the bird that we are talking about. Knowledge of arithmetical truths is like this. If we were to ask for the value of 7+5, then the man would enter the cage and start chasing around the bird he has his eye on, bird 12. Oftentimes he will catch the right bird, but it can happen that although he possesses knowledge of all the numbers, he catches bird 11 instead. This is how Plato accounts for errors in arithmetic. Again we see here that the numbers are independent objects, that there is a unique bird 12, and that although bird 12 exists independently, we nevertheless can interact with it.

Another thing we can notice from this passage is that the numbers exist discretely, that is, bird 12 is not in continuity with birds close by. Although this is not the point of the *Theaetetus* passage, it is indicative of the way that Plato (as well as all the other Greek philosophers of mathematics) thought about numbers. In noticing the fundamental property of numbers that they are discrete, the Greeks isolated a very important aspect of understanding mathematics, one that has in modern times been downplayed as part of an effort to arithmeticize geometry. (See Fowler, 1999:172-173.)

In *Theaetetus*, 201e-205e, Plato approaches the relationship between parts and whole from the standpoint of questions about knowledge. He says that in order for us to be able to know something about a whole, we need to know sufficiently about the parts of that whole. In the process of discussing this, he makes a distinction between a whole and a sum, saying that a sum is necessarily a sum of parts, but a whole can be viewed as an uncompounded thing. It seems that the point here is that a continuous whole cannot be compounded of atomic parts. We can infer that Plato had some understanding of the difference between continuity and discreteness, and therefore we are not unjustified in saying that if Plato thinks of numbers as discrete, he distinguishes that from the concept of continuity.

If arithmetic is therefore the science of the discrete and geometry is the science of continuous space, then it must be impossible to reduce arithmetic to geometry or geometry to arithmetic. Because what is discrete is not continuous and what is continuous is not discrete. Therefore, arithmetic is not a branch of geometry and geometry is not a branch of arithmetic.

Plato's philosophy of arithmetic has many parallels to his philosophy of geometry. Because of his attempts at synthesizing the form and matter ground-motives, he viewed continuity and discreteness as two equally fundamental principles, and therefore it is not surprising that much of what he says about geometrical figures can also be found in his statements about numbers. The following is a short summary of what his philosophy of arithmetic says (largely based on the work of Wedberg):

1. There are two kinds of arithmetic, the popular kind and the philosophical kind. Popular arithmetic deals with sensible entities, and its truths are always approximate. Philosophical arithmetic deals with the supersensible realm of the ideas and the intermediates, namely abstract number, and it makes absolute and exact pronouncements of truth.
2. There are two kinds of ideal arithmetical entities. Aristotle calls them "Mathematical Numbers" and "Ideal Numbers", but he adds that actually believing that there are two kinds of numbers strains credulity, since nobody can give a proper account of how they can differ.
3. The mathematical numbers have the following properties:
    a. They are made up of idealized units. The mathematical number *N* is made up of *N* units. This is reminiscent of the Pythagorean figurate numbers that appear as an array of dots.
    b. There is an inexhaustible supply of units. (Wedberg says an infinite supply, but Aristotle would hardly have reported that Plato said there is an actual infinite supply of units without having a word or two to say about the non-existence of an actual infinite.)
    c. Units cannot be distinguished from one another, just like the dots or alphas of the Pythagoreans could not be distinguished from one another.

    d. An ideal unit is an indivisible whole that has no parts. It is similar to the Euclidean point, except that it does not have any position.
    e. The mathematical numbers are not unique. There are many twos, many threes, and so on. This is specifically true of the mathematical numbers and not of the ideal numbers.
    f. Since the mathematical numbers are conglomerates of units, arithmetical operations can be performed on them using operations that resemble what we now know as set-theoretical operations. Addition, for instance, is akin to set union.
4. The ideal entities of philosophical arithmetic share some properties with the Ideas:
    a. They share the mode of existence of the Forms.
    b. Ideal numbers are simple, uncomposed entities.
    c. In particular, ideal numbers are not conglomerates of ideal units.
    d. Arithmetic does not deal with ideal numbers.
    e. There is an ordering to the ideal numbers, although not related to the relation of greater than or less than which is strictly an arithmetical relation and therefore only applicable to the mathematical numbers. The ordering relation of the ideal numbers instead relies on the whole-parts relation, which is an antecipation of the spatial aspect.
    f. The ideal numbers are the subject of study of Dialectic and not mathematics.
5. The mathematical numbers (the arithmetical intermediates) lie between the ideal numbers and the sensible world in the classification of ontology. The ideal numbers are perfectly exemplified by the mathematical numbers and imperfectly exemplified by the collections of visible things. There is an imperfect resemblance between the mathematical numbers and the collections of visible things, since the collections of visible things are only approximations of the mathematical numbers. The mathematical numbers are collections of pure units, whereas with the collections of visible things we only treat the things as units, these “visible units” partake in both the one and the many.
6. Plato’s argument for the existence of these arithmetical intermediates is as follows:
    a. Arithmetic is true.
    b. Arithmetic can only be true if we can identify arithmetical entities of which we can make true statements.
    c. We cannot make true statements about the sensible world because
        i. Everything in the sensible world is in a constant state of flux, and
        ii. Everything in the sensible world is a mixture of opposites.
    d. Considering statements (a) to (c), we may conclude that the arithmetical entities must exist, but not in the sensible world, therefore they are in the supersensible world.

7. The statements of arithmetic must be phrased in a way that reflects its ontological truth, which means that there can be no flux-language used in them. Any arithmetical statement that contains reference to movement or influence by the mathematician is false, because the concepts and entities of arithmetic are eternal and unchangeable. Plato uses the analogy that if we talk about bringing one and one together, we get two from the process of bringing together, whereas if we split two into two ones, we get two from the process of taking apart. Two opposite processes are then the cause of being two. This makes no sense to Plato, and he says so. The solution, in Plato's philosophy, is to deny that a process leads to there being two. Either there was two eternally, or there was never any two.
8. Philosophical arithmetic is a deductive science. There must be foundational hypotheses upon which the rest of the science must be based. This is so in order to avoid an infinite regression to more and more fundamental assumptions.
9. In philosophical arithmetic, the most fundamental concepts must be derived from the most certain principles, namely the Ideas that can be found in the science of dialectic. Ultimately, the foundation of arithmetic is also the Idea of the Good. This has the implication for arithmetic that an argument from appropriateness would be a valid argument. Aristotle says that this is ridiculous and that nobody ever used this test for validity, but Aristotle is working with a different truth-criterion and often spoke of previous philosophers as mere precursors to his own philosophy.

It is important to note, both in Plato's philosophy of arithmetic and his philosophy of geometry, that Plato is struggling to solve the problem of the one and the many. Take triangles for instance. Plato wants to be able to say that there is really only one triangle, the triangle given by the dialectician's definition. However, in practice, he knows that there are many triangles of different kinds, some equilateral, some scalene, some isosceles. The only way he can solve this is by doubling the world (this is Aristotle's critique of Plato). He says that there is a world where there is a multitude of triangles and another world where there is only one triangle. This latter world with the single triangle later caused John Locke some difficulty, which is why Locke rejected the existence of this world and became an empiricist (among other reasons, of course).

These two worlds with different characteristics, the one grounded in unity and the other in plurality, get their conception from the dialectical clash between two different ground-motives. The "world of unity" comes from the form principle in the Olympian culture religion where difference (read "disharmony") is anathema. The "world of plurality" comes from the matter principle in the Dionysian religion where harmony has to be enforced by keeping diverse opposing forces in check by means of *dike* (just measures; in Louw and Nida, 1989, entry 34.47, *dikaios* is defined as being in the right relation, which is the idea I'm referring to). Plato is trying to straddle both of these worldviews simultaneously, and the result is his levels of existence.

This dual world theory, which Aristotle heavily critiqued, is clearly a fundamental problem to the philosophy of Plato. But this philosophical problem does not confine it just to the field of abstract ontology. It clearly has an impact also on the foundations of mathematics. It is therefore difficult to see how Fowler could come to the conclusion that there is no evidence at all for any foundational crisis in mathematics (Fowler, 1999:362). Even if Fowler does not agree with the reformational school of thought on the solution to the foundational crisis, the problem is something that many philosophers see clearly. Perhaps the evidence is not convincing to Fowler, but denying evidence exists at all seems like a puzzling position to take.

As long as there are contrary principles in dialectical tension within a philosophy, a philosopher will always have to choose between one of the principles and set his choice up as the fundamental, "real" reality. This is exactly what we find with Plato. After setting up the two worlds, in the *Timaeus* he shows definite preference for the world of ideas which is firmly located in the form principle of the form-matter ground-motive. He makes it clear that what should be considered as truly real is the world where change (which is characteristic of the matter principle) has no power and where stability (characteristic of the form principle) is king.

In making this choice, Plato gives up the gig. He is only interested in maintaining the world of change in order to save the appearances, but when this seems to be in conflict with saving the proprieties, he abandons this world and calls it unreal. To Plato, what is fitting is much more important than what is perceived, and if the movement of the stars refuse to conform perceptibly to what is fitting, then so much the worse for our perceptions: the stars are perfect and will conform to what is fitting despite what our eyes tell us.

This preference for the form motive, once we are aware of it, can be traced throughout all of Plato's philosophy. His ontology is of course the part that we have already discussed. His epistemology favours our internal intuitions, recollections, and reasonings, and disfavours our external observations and perceptions. His logic is turned towards the fitting, the honorable, rather than what accords with external truth. His aesthetics is also affected by the form principle. It is not the beautiful flower, the beautiful maiden, the beautiful song that is the embodiment of beauty. Rather it is the world of harmonious perfection that is constructed by the mind in dialectics and in mathematics.

We see also in his ethics how the form principle rules supreme. Only someone who had a strong hold on the harmony of the form principle would think that his rather more bizarre recommendations of conjugal communism and communal non-parental childrearing are good ideas. These ideas fit perfectly with his adaptation of the form principle, however, and indicate that mathematical and dialectical principles were more important to him than what he could learn from his own experience and that of those around him.

Another interesting side to the philosophy of mathematics that Plato espoused is his understanding of mathematical entities as objects. Plato did not doubt that mathematical entities were subject to dialectical and numerical or spatial laws, but he failed to see that this entails that they are also subjects. Serving in a subject function means being a subject. Mathematical entities can still of course be objects, when considered within the context of theoretical abstraction; but when they are considered as dependent upon and correlated with their subject functions, they occupy the role of subjects. However, in the exalted positions that Plato assigned to the Ideas, they could hardly be seen as subject to higher laws, since in his philosophy there was nothing higher than the dialectical realm of Ideas.

This exalted position of course requires that mathematical (and dialectical, but this is not our focus) objects be as far from fickle as can be. If the harmonious rule of the universe depends on these objects to be stable, then stable they ought to be. As a result, they have to be cut off as with a hatchet from the changing world of the physical and kinematic aspects. But, as Dooyeweerd writes, "an indissoluble inner coherence binds the numerical to the spatial aspect, the latter to the aspect of mathematical movement, the aspect of movement to that of physical energy, which itself is the necessary basis of the aspect of organic life". (Dooyeweerd, 1997:I 3)

In the *Euthydemus*, 290, Plato again makes mention of the analogy of hunting birds to refer to how mathematicians come by the mathematical objects that they use in mathematics. Again, it is clear that what Plato has in mind is that mathematical entities have an independent existence. In short, Plato was no constructionist. But in asserting the total independence of mathematical entities, he misses the fact that they are in fact also subjects that are subject to a modal law order that God has created in Christ Jesus, and therefore they are not in fact independent. They can only be rightly understood as they disclose themselves as part of the coherent modal totality of creation.

In *Republic*, 476a, we have a turning point in the dialogue. Here we see a significant sign of the Pythagorean influence on Plato. Plato connects the forms with the problem of the many and the one, which is ultimately a problem about the interconnectedness of arithmetic with geometry and discreteness with continuity. Justice (as it relates to statesmanship), which is ultimately the topic of the *Republic*, is connected with mathematics, which is a signal to us that Plato will incorporate the philosophy of the Pythagoreans into his own philosophy of statescraft.

We find some interesting seeming contradictions in the *Republic*, like in this quote:

> Reason it out from what was said before. If the one is adequately seen itself by itself or is so perceived by any of the other senses, then, as we were saying in the case of fingers, it wouldn't draw the soul towards being. But if something opposite to it is always seen at the same time, so that nothing is apparently any more one than the opposite of one, then something would be needed to judge the matter. The soul would then be puzzled, would look for an answer, would stir up its understanding, and would ask what

> the one itself is. And so this would be among the subjects that lead the soul and turn it around towards the study of that which is.
> But surely the sight of the one does possess this characteristic to a remarkable degree, for we see the same thing to be both one and an unlimited number at the same time.
> Then, if this is true of the one, won't it also be true of all numbers?
> Of course.
> Now, calculation and arithmetic are wholly concerned with numbers.
> That's right.
> Then evidently they lead us towards truth. (Plato – *Republic*, 524d-525a)

The point here is that in order to "leave the cave", to see the realities of the world outside, which represents the world of true existence (the world of the Ideas), one needs to have an entry point, almost a "gateway drug", that at the same time accustoms one to the abstract world and which draws one in deeper with a desire to know more. This "gateway drug" is the numbers, which, this quote points out, leads one towards truth. The reason this seems like a contradiction is that Plato surely thought of arithmetical propositions as eternal, fixed truths (we already mentioned that when we talked about Wedberg's analysis of Plato's philosophy of mathematics) and as a result these arithmetical propositions should surely qualify as truth already in Plato's system. But here, Plato says it only *leads to* truth.

I think that here we again see the struggle between two fundamental principles in a dialectical religious ground-motive. Plato clearly opted for the form principle of the form-matter religious ground-motive, and in so doing subordinated the matter principle to it as productive of lesser truths. Since Plato seems to have gotten his arithmetic and geometry from the Pythagoreans, who in my opinion subjected the form principle to the matter principle, it seems that to Plato arithmetic and geometry would be inextricably connected with the matter principle. If this is the case, then it would make sense that Plato thought that the propositions of arithmetic and geometry are not the truly true, the dialectical propositions which are based on the form principle.

We see this same problem crop up a little later in the *Republic* again, where Plato writes:

> Therefore, if geometry compels the soul to study being, it's appropriate, but if it compels it to study becoming, it's inappropriate.
> So we've said, at any rate.
> Now, no one with even a little experience of geometry will dispute that this science is entirely the opposite of what is said about it in the accounts of its practitioners.
> How do you mean?
> They give ridiculous accounts for it, though they can't help it, for they speak like practical men, and all their accounts refer to doing things. They talk of "squaring", "applying", "adding", and the like, whereas the entire subject is pursued for the sake of knowledge.

> Absolutely.
>
> And mustn't we also agree on a further point?
>
> What is that?
>
> That their accounts are for the sake of knowing what always is, not what comes into being and passes away.
>
> That's easy to agree to, for geometry *is* knowledge of what always is.
>
> Then it draws the soul towards truth and produces philosophic thought by directing upwards what we now wrongly direct downwards. (Plato – *Republic*, 526e-527b)

So, geometry also leads one to the truly true truths, but because of the way geometers talk about geometrical entities as things that experience change (which is a central theme in the matter-principle inspired Pythagorean way of thinking of geometrical subjects), Plato does not consider the geometricals to be something that we can have real knowledge about.

A sort of throw-away comment later also shows the form-directedness of Plato's thought. In commenting on the appropriate children to train up to become the rulers, Plato says the following:

> Then, as for those children of yours whom you're rearing and educating in theory, if you ever reared them in fact, I don't think that you'd allow them to rule in your city or be responsible for the most important things while they are as irrational as incommensurable lines. (Plato – *Republic*, 534d)

Oftentimes, mathematicians assume that the word *irrational* comes from a root which means *ratio*, meaning a fraction made from an integer numerator and a non-zero integer denominator. This comment of Plato's shows that in fact, originally, this was not the intended meaning of irrational (of course, not of the English word, but of the concept of irrationality). Instead, it seems that there was a moral interpretation of the concept as meaning something stubborn, unwilling to yield to reason (*ratio* in Latin). This seems like just a piece of interesting etymological trivia until one examines it in the context of the religious ground-motives. In this context, one soon sees the connection that this term has to the form-principle of the religious ground-motive, expressed in the Olympian culture religion. Here, obedience (as opposed to stubborn unwillingness to yield) to the norms of reason characterizes the true and the good, and incommensurable lines do not conform to these norms. Therefore, one sees that it is an attempt to solve the problem of linear incommensurability that led Plato to construct a world of Ideas, where the form principle reigns supreme and where, no doubt, incommensurable lines would have been banished.

One hears again and again from commentator after commentator, that Plato thought of the Ideas as immutable, static, fixed objects. I think it is a signpost to the multi-modal nature of the world that when Plato presents his theory of the Ideas in the cave analogy, he wants to present a world of changelessness while introducing motion into that world. The entities whose images are cast on the back wall of the cave by the fire, are not static entities. They are carried to and fro by "people" (I'm not sure what role the people carrying the entities play in this context). The fact is, the Ideas are in motion. This is particularly interesting because a fire casting shadows on a wall would already have been providing motion in the perceptual world. So why is it necessary for the entities to be carried as well?

I think that here Plato (consciously or not, I don't know) was forced to face up to the mutual coherence of the physical aspect (the meaning-seat of motion) and the kinematic aspect (the meaning-seat of constancy). In order to speak of the constancy that he wished to incorporate into his philosophy, he had to tacitly admit its coherence with the physical aspect. His commitment to the form principle of the form-matter religious ground-motive makes it so that he has to downplay it by making the reference to motion a passing remark that effectively gets shut down later; but, because he lives in a multi-modal world which he wants to accurately represent in his philosophy, he is forced to admit the multi-modality of this world. Again, we see that reductionism is no proper staying-place for good philosophy.

Aristotle's view of the ontic status of mathematical subjects

Aristotle deservedly holds a position among the most influential philosophers of all time. He characterized science by means of a systematic elaboration of the theoretical attitude, and this characterization has remained a major influence on the philosophy of science for over two millennia. He also developed a philosophy of mathematics, but his philosophy of mathematics has not been as successful as his philosophy of science. Although there are present-day platonist philosophers of mathematics, there are not correspondingly many present-day aristotelian philosophers of mathematics. Despite this strange lack of representation in the philosophy of mathematics today, he nevertheless made important contributions to the discipline, and therefore warrants a treatment in this study.

My main source for understanding Aristotle's philosophy of mathematics is of course Aristotle himself. We have a relatively large amount of his works still available to us today, and I will presently turn to considering what he says there in more detail. Another important source that presents Aristotle's view is the book by Cleary (Cleary, 1995) where he focuses specifically on the passages from the works of Aristotle that he reckons are relevant to understanding what Aristotle said about mathematics. I will first have a look at what Cleary had to say, then turn to Aristotle himself to see whether what Cleary says is justified and to see if he left anything out, and then give some systematic considerations evaluating Aristotle's view.

Now that my method has been laid out, let us turn to Aristotle's philosophy of mathematics. Cleary identifies three basic problems that have a direct or indirect influence on how to answer the question concerning the ontic status of mathematical entities (Cleary, 1995:xxxiii-xxxiv). Firstly, in response to the philosophy of Plato, Aristotle asked whether there really are intermediate mathematical entities between the sensible world and the Forms (it seems that instead of rejecting the Forms, Aristotle rather made them immanent in the world). Secondly, Aristotle tries to determine how mathematics coheres with his own philosophy of substances and accidents by posing the question whether mathematical entities are substances or not. The last *aporia* that Cleary detects in Aristotle's work is the question whether the One and Being are the substance of things. This is clearly a reference to the philosophy that came from the Pythagoreans via Plato to Aristotle during his time in the Academy.

A major contribution Aristotle made to philosophy is located in his development of his system of causes, which he also applies to the One itself and Being itself. (Cleary, 1995:xxxv) If the One itself and Being itself were the causes of the sensible things, then they have to be classifiable as one of Aristotle's causes for Aristotle to accept them as causes. However, they cannot serve as any of the causes other than the final cause (we will consider how Aristotle establishes this argument later) and therefore it makes no sense to call them causes in any sense of the word. It follows that the

One itself and Being itself is independent, unconnected, and disjoint from our world and ends up being the Prime Mover. On the other hand, the Forms and the intermediates in the supersensible realm of Plato do not have any kind of causal link to the sensible things, and therefore Aristotle rejects their supersensible existence. Of course, he does not reject their existence entirely; instead, he embeds them in the world.

Aristotle made a special study of continuity. In this regard, Cleary describes the relationship between the continuous and the discrete particularly well, and makes it clear that the root of the meaning of continuity lies in the spatial aspect:

> Contrary to the Pythagorean assumption, a conjunction of points does not form a continuum because their limits cannot be united in the way that Aristotle’s definition requires. The reason is simply that indivisible things do not have limits as distinct parts, so that if two points are places in contact they must coalesce into one. Therefore they are either indistinguishable from each other or they leave space for a line between them on which more points can be taken. Thus points cannot be next-door neighbours because other points can always be placed between them, and so they cannot form a continuum. (Cleary, 1995:76)

What is clear here is what Hilbert rediscovered in the 20th century: the connection between two points is always a line. A line is not just the shortest distance between two points (which is a retrocipation to the quantitative aspect) but in order for two points to be connected, there must be a connection called a line (and this explanation remains in the spatial aspect).That points contain no parts and therefore that part of one point cannot serve as the end of the one point and the beginning of the next point (as is necessary in Aristotle’s definition of continuity) is a typically Greek way of defending the notion that two points can never form a continuum. What Aristotle did not see here, however, is that the meaning-nucleus of distinctness lies in the arithmetical aspect, but we can surely excuse this. Philosophy as a discipline was still in its inceptive stage.

Responding to the Eleatics, he made a distinction between actual infinity and potential infinity (Aristotle’s terms). Any spatial continuum can be divided into parts, and this can happen again and again *ad infinitum*. Here we see an infinity that proceeds stepwise, but instead of each time adding another term, it instead turns inward in a finite line-stretch. (Weyl, 1932:57) Since it proceeds stepwise, Aristotle would classify this as a potential infinity, and what is particularly interesting is that if you start with a finite continuous line segment, then you end up with a potential infinity confined in a finite space. In order to make sense of this seeming paradox, Aristotle further developed a concept that was already familiar at the time: the distinction between parts and the whole.

Cleary points out that Aristotle also made an important distinction between the question of whether mathematical objects exist or not and how mathematical objects exist if they do exist (Cleary, 1995:276). After these questions are addressed, one can ask whether mathematical entities are Forms, intermediates, or sensibles, but not before. One detects in Cleary a definite post-Cartesian bias in favour of keeping the subject-object relation in sharp focus and thinking of mathematicals as objects rather than as subjects or entities, and therefore immediately and constantly regarding mathematical entities by means of the theoretical orientation, but this is not necessary for the discussion of their existence. Cleary makes his post-Cartesian attitude clear in the following quote:

> So I prefer to formulate Aristotle's question in terms of being, given that his categories are different modes of being. Speaking in those terms, we can say that the question is whether or not mathematical objects are entities, and what is their mode of being. (Cleary, 1995:276)

It does not seem as if this invariably theoretical attitude is characteristic of Aristotle's own orientation towards mathematical entities. In fact, considering the fundamental position that Aristotle takes towards the sciences, that they study things which have a prior existence independent of their theoretical examination, seems to directly contradict Cleary's characterization. Therefore, in this one respect, I think that Cleary made a mistake.

In order to determine whether mathematical entities exist or not, Aristotle points to four possibilities:

1. Mathematical entities might be in the sensible things (this is the Pythagorean position).
2. Mathematical entities might be separate from the sensible things (this is the Academic position).
3. Mathematical entities might not exist (it is unclear whether Aristotle had anyone in mind when he mentioned this possibility, but he does not take this possibility very seriously).
4. Mathematical entities might exist in a way that prohibits us from speaking of them as either in or separate from the sensible things (this is the position Aristotle in the final analysis takes). (Cleary, 1995:278)

What is at issue in the first two possibilities, according to Aristotle in harmony with his wider philosophy, is whether mathematical entities exist in the sense of being substances. Since the substance-accidents distinction only comes into sharp focus in the philosophy of Aristotle, it is unlikely that he is here thinking of the positions of the Pythagoreans and the Academics in their original formulation. Instead, he is trying to test whether either of these two positions would fit into his larger scheme of philosophy. Contrasted with this, the last two possibilities contemplate the ways mathematical entities might be said to exist if they are not substances (here, of course, I include non-existence as a way of existing, which is strictly speaking not correct, but it does reflect the way Aristotle articulates the problem). Again, Cleary expresses this clearly:

> Since the first two possibilities cover the ways in which mathematical objects can exist as substances, the last two possibilities must be about alternative modes of being: (iii)

> either mathematical objects do not exist (*e ouk eisin*) or (iv) they exist in some other way (*e allon tropon eisin*). The third possibility is included only for the sake of logical completeness, since Aristotle does not subsequently return to it. His apparent oversight can be explained in terms of the argument from the sciences, whose fundamental assumption is that any genuine science must have a real or existent object. Since Aristotle shares that epistemological assumption, he cannot accept the non-existence of the objects of mathematics as that would leave his paradigmatic sciences without foundations. Hence, if the first two possibilities are to be denied and the third is ruled out, the remaining option assumes a new importance. As stated, this is the possibility that mathematical objects exist in some other manner which lies between complete non-being and being in the primary sense as substance.
>
> Having outlined all the possibilities with respect to the existence of mathematical objects, Aristotle concludes ([*Metaphysics* - CPH] 1076a36-37) that the dispute will not be about their being (*peri tou einai*) but rather about their mode of being (*peri tou tropou*). Since this implicitly excludes the possibility that mathematical objects do not exist in any way, there remains only their two possible modes of existence as substances and some other unspecified mode of being. As these are merely different ways in which mathematical objects can exist, Aristotle can conclude that the dispute is not really about whether these objects exist but rather about their mode of existence. In his own terms, it is not whether they are (*ei eisi*) but rather how they are (*pos eisi*) which is in dispute. (Cleary, 1995:279-280)

Aristotle's argument against the first possibility seems to hinge upon his definition of continuity. If a point is indivisible, then clearly a line cannot be divided by dividing its point of division. Therefore, under this hypothesis, a line is also indivisible. On the other hand, if the mathematical entities are in the sensible things, then the line in a sensible thing must be divisible at the point in a divisible thing where the actual division takes place (which is possible for sensible things). Therefore, a point has to be both indivisible and divisible at the same time and in the same respect for the first possibility above to hold. Since this leads to a contradiction, Aristotle concludes that this possibility is in fact no possibility. (Cleary, 1995:282-283)

Aristotle next turns to the second possibility, which represents the position of the Platonists. The position of the Platonists seems plausible at first: separate the mathematical entities from the sensible things, then the problem of variability does not present itself. But Aristotle points out that this separation (*chorismos*) leads to an absurd multiplication of potentially mathematical entities: in terms of kinds, there are now *two* solids, *three* planes, *four* lines, and *five* points! Aristotle demonstrates that the principle of separation necessitates this multiplication of entitary types, and that therefore the problem is not with a particular kind of separation, but rather with separation *per se*.

The Academics could of course respond to this objection by appealing to their doctrine of the intermediates. A multiplicity of points is of course possible, but this is because we are speaking of the intermediate point that lies below the Form of the point in the Platonic ontology structure. Once we speak of the Form of the point, there will only be one. Aristotle cuts off this counter-argument by pointing out the inconsistent way it is applied. He points out that a consistent application of the concept of intermediates would lead to intermediates of all the Forms. There should be, besides the sensible man and the Form of man, a third man. Since the Platonists did not postulate the existence of a third man, it seems that they were not fully committed to this solution, and used it only as a stop-gap measure to fill in the gaping hole in their philosophy.

A key principle in Aristotle's approach to the sciences, that Cleary hints at (Cleary, 1995:287), is the maxim that nature creates nothing in vain (Aristotle – *On the soul*, 434a31). In multiplying the mathematical entities, the Academics violated this principle by rejecting parsimony (this is how Cleary articulates it) while at the same time it turns out that mathematics is not about any of these kinds of mathematical entity (at least, it is impossible to point out one kind of point that is supposed to be the object of mathematical investigation rather than any other). This serves to show that the second possibility is also no possibility.

Another argument Aristotle makes against the second possibility is that a criterion for substantiality is perceptibility. A substance is either something that exists physically, bodily in our three dimensions (in which case planes, lines, and points do not have enough dimensions to exist as complete substances) or something that is ensouled (which he argues cannot be said of mathematical entities on the basis of sense experience). Therefore, mathematical entities cannot be substances, which again refutes the position of the second possibility. (Cleary, 1995:298-300)

Considering the third possibility, Aristotle concludes that since the science of mathematics deals with mathematical entities, and since no science functions without entities that it studies, it follows that we have to say that mathematical entities exist and that therefore the third possibility is also not a possibility. We only have one possibility left: Mathematical entities exist in a way that prohibits us from speaking of them as either in or separate from the sensible things.

In order to explain what kind of existence this last kind of existence is, Aristotle talks about abstraction (not using this specific term, but this is what he means when he says that we think of things *qua* stuff). Cleary discusses the situation of this abstraction in the quote below:

> In fact, it is quite clear from the whole passage that priority in substance is being denied to the so-called 'result of subtraction'. The ancient Greek commentators assume that Aristotle is here referring specifically to mathematical objects, though they give no adequate explanation of why mathematical objects are referred to as 'the results of abstraction' or of what implications this terminology may have for their ontological status. This is a lacuna even in modern Aristotelian scholarship, which needs to be filled

> by explaining how such terminology applies to mathematical objects, given the peculiar fact that it is not used at all by Aristotle in XIII.3 for his positive account of the mode of being of mathematical objects. (Cleary, 1995:305)

It is clear that in order to think of the spatial aspect of the sun, for instance, what we need to do is lift out its geometrical side while disregarding its heat, physicality, ability to produce light, and so on. In short, we need to use the method of analysis, which is the flip side of the process of abstraction. (See Strauss, 2009:14) The idea, as Cleary points out (Cleary, 1995:308), is that geometry is about sensible magnitudes considered as spatial only, not as sensible. This is almost modal abstraction, except Aristotle expands the scope of this "subtraction" (Aristotle's term) to abstraction that distinguishes between scalene, isosceles, and equilateral triangles, which are types, not modalities. It seems that Aristotle did not distinguish between the different levels of abstraction.

Cleary explains the identification of the arithmetical aspect and the use of the arithmetical aspect in the following passage:

> Furthermore, the arithmetician can count sensible bodies while treating them as indivisibles only (*he adiaireta monon*). The additional step here, over and above treating bodies as indivisibles with position, is the removal of the aspect of relative position so that bodies are treated simply as interchangeable units to be counted by the arithmetician. Later (1078a23-25) he gives a different justification in terms of counting sensible things as instances of natural kinds, but the notion of treating a body as a unit without position fits better into the kind of schema which is typically presupposed by the method of subtraction: solid, plane, line, point, and unit. While this schema may have been Pythagorean in origin, it seems to have been a conventional way of treating mathematical entities; cf. *Met*. 1016b24-31. Thus, in order to show that such entities are not ontologically separated, Aristotle must establish that it is logically possible to treat sensible things under these descriptions; e.g. as units or as points. (Cleary, 1995:317-318)

It is therefore clear that though Aristotle thinks of substances as in some way sensible, he definitely thinks of mathematical entities as distinct, but not in a substantial way. Rather, they are logically abstracted from the substances that have a prior existence, and therefore mathematical entities have a posterior kind of existence. Mathematical entities have a kind of existing that is similar to how attributes exist in that they exist depending on sensible substance existing prior.

It seems that a large part of the dispute between Aristotle and Plato is to determine which science is the most fundamental. In the eyes of Plato (and, by implication, also the Pythagoreans), the most fundamental science and the science that determines the contours of our metaphysics, should be mathematics. Therefore, mathematical entities lie higher in the ontological scheme of

Plato than sensible things. To Aristotle, the most fundamental science is the science that deals most directly with our experiences. As we can experience biotic things both within ourselves and outside of ourselves, the most complete view of the world would come from the science of biology, and Aristotle correspondingly makes biology the measure of the sciences. As a consequence, the sensible become the most real (they exist in substantial form) whereas mathematics takes the back-stage, since it deals with objects with a lesser form of existence (mathematical entities exist in the same form as the accidents).

This brings us to the end of our consideration of Cleary's interpretation. We have seen here that Cleary certainly interprets Aristotle with post-Cartesian goggles. This is not merely a chronological observation (I too live after Descartes, after all), but rather a methodological observation. Cleary does not seem to be aware of his Cartesian influence when he interprets Aristotle's metaphysics of mathematical entities. When this possible influence is kept in mind, a more precise view of Aristotle's ontology comes to the fore. We shall now turn to the consideration of Aristotle's own words.

In the *Categories*, we find the first mention of the definition of continuity, and in this context, Aristotle already distinguishes between the arithmetical and the spatial aspects:

> It is, therefore, distinctive of substance that what is numerically one and the same is able to receive contraries. This brings to an end our discussion of substance. Of quantities some are discrete, others continuous; and some are composed of parts which have position in relation to one another, others are not composed of parts which have position. Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and place. For the parts of a number have no common boundary at which they join together. For example, if five is a part of ten the two fives do not join together at any common boundary but are separate; nor do the three and the seven join together at any common boundary. Nor could you ever in the case of a number find a common boundary of its parts, but they are always separate. (Aristotle – *Categories*, 4b20-31)

Again, in *Categories*, 3b24-27, Aristotle makes it clear that substance has a certain property: there is nothing contrary to substance. He goes on to say, however, that *other* kinds of things, such as quantity, also does not have contraries (thereby implying that quantity is not a substance).

Aristotle again says in *Categories*, 5a1-2, that something is continuous at a point when two parts meet at the point in such a way that the one part ends at that point (as a point in the first part) and the other part starts at that point (as a point in the second part). The point where the parts join is therefore counted twice.

Of course, Aristotle realized the problem that results from saying that mathematical entities are abstracted from sensible things. What might be concluded, is that since the line that the geometer draws is never perfectly straight, the geometer is lying when he says he is talking about a straight line that was abstracted from the imperfectly drawn line that can be seen. Aristotle says, in *Posterior Analytics*, 76b35-77a9, that the geometer is not lying, but instead that the line the geometer draws is not the actual line that he is talking about. But if this is the case, Aristotle gets right back into the problem of the Academics: there is now again a separation (*chorismos*) between the line the geometer has drawn and the line he is speaking about. Is the line an aspect of the physically drawn line? Then it is not perfectly straight. Is the line not an aspect of the physically drawn line? Then how many lines are there? Are there four? Aristotle falls into his own trap and as far as I can see, doesn't get back out of it.

In *Posterior Analytics*, 87a31-37, Aristotle returns to the topic of distinctions between sciences, and he clearly sees the difference between the arithmetical and the spatial aspect when he distinguishes between a unit (a substance without position he calls it) and the point (a substance with a position). In doing so, he accurately identifies the first two modalities. In his own words:

> One science is more precise than another and prior to it both if it is at the same time of the fact and of the reason why and not of the fact separately from the science of the reason why; and if it is not said of an underlying subject and the other is said of an underlying subject (e.g. arithmetic and harmonics); and if it depends on fewer items and the other on an additional posit (e.g. arithmetic and geometry). (I mean by on an additional posit, e.g. a unit is a positionless substance, and a point a substance having position – the latter depends on an additional posit.) A science is one if it is of one genus – of whatever things are composed from the primitives and are parts or attributes of these in themselves. (Aristotle – *Posterior analytics*, 87a31-37)

A study of the philosophy of mathematics can hardly progress very far without a mention of the infinite. Aristotle already, in *Physics*, 203a1-4, said that in order for someone to properly study nature, he needs to consider the infinite and whether the infinite exists or not. Interestingly, Aristotle does make a distinction between a mathematician and a student of nature, and he nevertheless does not restrict the study of the infinite to the mathematician, but says that also the student of nature needs to consider the question of its existence. He also reports that anyone who made a contribution to the science of nature that is worth thinking about (such as Pythagoras, Heraclitus, Parmenides, and Plato), all had things to say about the infinite, and all in their own way came to the conclusion that the infinite is the principle of existence. Let us consider a few of these. Pythagoras said that the principle of everything is the odd and the even, and the even is the principle of unboundedness (*apeiron*). Therefore, to Pythagoras, the infinite was inherent in the existence of the world. Heraclitus made the statement that everything is in an *eternal* state of flux.

Parmenides said that the One, which is his central principle of being, is a sphere, which means that it doesn't have borders and doesn't end (which is the root concept of infinity). Plato said that the *absolutely* good, true, and beautiful is the principle of origination of everything (the use of the *absolute* is a reference to the concept of the infinite). In *Physics*, 203a10, Aristotle points out that Plato implied that the infinite is in the sensible things and also in the Forms.

In order to make sense of the seeming need to incorporate the infinite into our philosophy while rejecting the existence of a completed, actual infinite, Aristotle says the following:

> It is plain from these arguments that there is no body which is actually infinite. But on the other hand to suppose that the infinite does not exist in any way leads obviously to many impossible consequences: there will be a beginning and an end of time, a magnitude will not be divisible into magnitudes, number will not be infinite. If, then, in view of the above considerations, neither alternative seems possible, an arbiter must be called in; and clearly there is a sense in which the infinite exists and another in which it does not. Now things are said to exist both potentially and in fulfilment. Further, a thing is infinite either by addition or by division. (Aristotle – *Physics*, 206a8-14)

Aristotle goes on to make the case that saying something exists potentially is not equivalent to saying that this something does not exist. He gives the example of the Olympic games which is said to exist because another Olympic games event may occur in the future (Aristotle – *Physics*, 206a25-27). There is therefore the potentiality of the Olympic games, which we find sufficient to say that there are Olympic games. In the same way, when we say that the infinite exists potentially, we are at the same time justified in saying that the infinite exists. The point is that when the mode of existence of the infinite is asked for, our answer must be that it exists *in potentia*.

He illustrates this concept of the potential infinite by directing our attention to the bisection of the continuum. When a magnitude is bisected, we have not reached a point where bisection is now suddenly impossible. Instead, we can bisect again. And then we can bisect again. This potentiality to bisect never ends. There is a successive infinity of potential bisections. (Aristotle – *Physics*, 207b10-15) But what we cannot conclude is that at some point, this infinity can be completed. This seems to be Aristotle's answer to the paradoxes of Zeno, in particular the paradox of the arrow. The paradox of the arrow relies on the existence of points in time that are not intervals of time, but rather true indivisible points. Aristotle comes to the conclusion that time has the nature of a continuum, and therefore we can never reach the point of the process of subdivision of a time interval where we are left with an indivisible instant. He says as much in the quote below:

> Now for the reason given above they cannot be continuous; and one thing can be in contact with another only if whole is in contact with whole or part with part or part with whole. But since indivisibles have no parts, they must be in contact with one another as

> whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous; for that which is continuous has distinct parts, and these parts into which it is divisible are different in this way, i.e. spatially separate. Nor, again, can a point be *in succession* to a point or a now to a now in such a way that length can be composed of points or time of nows; for things are in succession if there is nothing of their own kind intermediate between them, whereas intermediate between points there is always a line and between nows a period of time. (Aristotle – *Physics*, 231b2-7)

Again, in *Physics*, 231b15-16, Aristotle reminds us of what is necessary for something to be continuous: if something is divided into parts and those parts cannot be divided any further, we are not dealing with a continuum. If on the other hand, whenever we divide something into parts and those parts can always be divided again into parts and so on, then what we are dealing with is a continuum. So a continuum has two essential properties, and these properties make up what we still today in mathematics use to identify a continuum:

1. If a point in a continuum is used to split up the continuum, then the two parts that result from dividing at this point both must contain the point within them.
2. If a continuum is divided, the parts can always be subdivided.

Think for instance of a continuous interval (a,b) in the real numbers. If we pick a point c between a and b to divide the interval up, then the interval is only continuous if we get the two intervals (a,c] and [c,b). If not, for instance if we get the intervals (a,c) and (c,b), then we didn't have a continuous interval to start with, but instead the two disjoint intervals taken in union. So, in *Physics* explicitly stated,

> In the act of dividing the continuous distance into two halves one point is treated as two, since we make it a beginning and an end; and this same result is produced by the act of counting halves as well as by the act of dividing into halves. (Aristotle – *Physics*, 263a25)

In *On the Heavens*, 271b5-12, Aristotle says that a minimum magnitude could not exist (appealing to the successively infinite divisibility of the continuum, no doubt), and that if a minimum magnitude *did* exist, the "greatest truths of mathematics" would totter. This is essentially an argument against the use of infinitesimals. Interestingly enough, it was during a period of tottering of the greatest truths of mathematics, namely the third foundational crisis of mathematics, that Abraham Robinson introduced the non-standard analysis, which explicitly and unapologetically makes use of infinitesimals in analysis. This non-standard approach has permeated into many branches of mathematics. Sometimes it is also known as non-Archimedean analysis (the standard branch of the study of Riesz spaces, for instance, is also known as the study of Archimedean Riesz spaces).

It is worth noting that Aristotle already saw the relation between the existence of a minimum magnitude and the very roots of the philosophy of mathematics.

To counter the Academic view of mathematical entities, Aristotle again in *On the Soul*, 431b13-432a14), makes it clear that in order to talk of mathematical entities, we have to abstract, thinking of them as separate, but they are not in fact separate. Therefore, Aristotle is implying that the concept of separation which Platonists employ is not entirely useless, but it is at the same time not entirely according to truth. So it is a useful fiction. One is reminded of Wittgenstein's "ultimately significant nonsense". There is clearly a problem here, and it seems as if Aristotle does not have a solution to this problem.

Thus far we have considered works of Aristotle that only peripherally deal with the existence of mathematical entities. Our next work, however, is the *Metaphysics*, where Aristotle sets out his philosophy of mathematics in some detail. It is therefore worthwhile to note the beginning of our discussion of this important work, in order to be clear as to where the meat of his philosophy of mathematics will be encountered.

It is in the *Metaphysics*, 983b23-986a25, that we find Aristotle's famous treatment of the Pythagoreans and their number-philosophy, which also is the most famous quote about the Pythagoreans. It runs as follows:

> Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being – more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity – and similarly almost all other things being numerically expressible); since, again, they saw that the attributes and the ratios of the musical scales were expressible in numbers; since, then, all other things seemed in their whole nature to be modelled after numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number.

As I pointed out before, this has often been interpreted as putting arithmetic in the foundational position and then building geometry on top of it as a derivative science. I believe there are some serious issues with this interpretation, which I have pointed out in the first part of this study. We also find in this context the table of opposites, which again highlights the importance of the infinite in Pythagorean philosophy: the first pair of opposites that is mentioned is the limit (*to peras*) and the unlimited (*ton apeiron*). The unlimited has an indisputable connection with the infinite in Greek philosophy.

In *Metaphysics*, 987a32-b29, Aristotle discusses the Platonic scheme of ontology, and it is here that we learn about the intermediates. Aristotle gives us the important information that the intermediates serve as an explanation for the multiplicity in mathematical entities, which contradicts the unity of the Forms. The Forms are of course seen as the reason why everything in the sensible realm exists, so that it seems that the main change from the Pythagoreans to Plato is that the maxim "All is number" is changed to "All is Form". That this is how Aristotle interpreted it is anyhow shown by the parallel in the language Aristotle uses to describe the Pythagoreans and the Academics. Aristotle says of the Pythagoreans that they thought that numbers were the fundamental things and that "their elements were the elements of all things". On the other hand, Aristotle says of Plato and his followers that they thought that the Forms were the causes of everything and (here is the parallelism) that "their elements were the elements of all things".

So clearly, in Aristotle's estimation, the Forms merely took the place of the numbers in the shift from Pythagoreanism to Platonism, and this seems to be a rather fair assessment of the situation. Aristotle does point out some points of contention between the Platonists and the Pythagoreans:

> But he agreed with the Pythagoreans in saying that the One is substance and not a predicate of something else; and in saying that the numbers are the causes of the substance of other things, he also agreed with them; but positing a dyad and constructing the infinite out of great and small, instead of treating the infinite as one, is peculiar to him; and so is his view that the numbers exist apart from sensible things, while *they* say that the things themselves are numbers, and do not place the objects of mathematics between Forms and sensible things. (Aristotle – *Metaphysics* 987a32-b29)

I don't quite know what to make of Aristotle's fourth kind of existence in *Metaphysics*, 990a30-b18, but it seems like it would be worthwhile to return to this argument at a later stage to see how this critique of the Platonic system works. It seems that Aristotle is saying that the Platonic ontological classification is incomplete and that because of motion, there is need for another level in Plato's system, somewhere between the intermediates and the sensible things.

We get some objections to the doctrine of the intermediates in *Metaphysics*, 997a34-998a19, where Aristotle considers two possibilities: either the intermediates are not in the sensible things or they are in the sensible things. He says that some philosophers hold that the intermediates are not in the sensible things and that they are not there because the sensible things change whereas the intermediates cannot change. In this case, there is motion and perishability in the sensible things and neither motion nor perishability in the intermediates. As a result, Aristotle says that in order to account for the motion, we need to construct another level of intermediates (inter-intermediates?) between the intermediates and the sensible things that share in the imperishability (the eternity) of the intermediates while sharing in the motion of the sensible things. According to Aristotle, if we don't make this step, we won't be able to account for motion.

On the other hand, other philosophers hold that the intermediates are in the sensible things. Now Aristotle says that if this is the case, then there must be sun, moon, and stars visible to us, and another sun, moon, and stars not visible to us, but eternal and unchanging, and more problematically, that the visible sun and the invisible sun must be in the same place, and correspondingly for the visible and invisible moon, and the visible and invisible stars. This is completely unfathomable to Aristotle, so he advocates that we reject this silly state of affairs.

Supposing that Plato's classification would be complete if instead we took away the level of the Forms and just restrict ourselves to the perceptible things and the intermediates, also leads to some problems, according to *Metaphysics*, 1000b21-1003a2. According to the Platonists, we need the top level of the scheme, specifically because otherwise we do not have a principle of unification for the plurality of things in the level of the intermediates. Instead of having a Form of triangle, we will instead have many triangles of all different sorts, with different shapes and different sizes. Then we encounter the problems that the Eleatics made us aware of; and without the Forms, the Platonists have no answer to the Eleatics. But Aristotle points out that he has already dealt with the problems that arise if one does assume that the Forms exist. For our purposes, the relevant objection is that the Forms of mathematical entities cannot be either in or separate from the sensible things, which means that we end up with insubstantial substances.

In *Metaphysics*, 1023b9-25, we have a nice categorization of the different kinds of parts there can be in a whole, which I think is worth quoting in full:

> We call a part (1) that into which a quantity can in any way be divided; for that which is taken from a quantity *qua* quantity is always called a part of it, e.g. two is called in a sense a part of three. – (2) It means, of the parts in the first sense, only those which measure the whole; this is why two, though in one sense it is, in another is not, a part of three. – (3) The elements into which the kind might be divided apart from the quantity, are also called parts of it; for which reason we say the species are parts of the genus. – (4) The elements into which the whole is divided, or of which it consists – 'the whole' meaning either the form or that which has the form; e.g. of the bronze sphere or of the bronze cube both the bronze – i.e. the matter in which the form is – and the characteristic angle are parts. – (5) The elements in the formula which explains a thing are parts of the whole; this is why the genus is called a part of the species, though in another sense the species is part of the genus. (Aristotle – *Metaphysics* 1023b9-25)

One of the main themes of the *Metaphysics* is the study of substance. In *Metaphysics*, 1042a20-31, Aristotle explains how we can distinguish between a substance and something that is not a substance. His criterion, broadly speaking, depends on asking oneself whether the thing one is speaking about is a 'this'. In scholastic times, this was rephrased as asking whether something has haecceity (derived from the Latin for "thisness"). Three kinds of things have haecceity: the material that sensible substances are made from (you can point to a wooden chair and say "this wood"), the form of a thing (you can indicate a blueprint and say "I'll pay you to make me this chair"), and the individual thing that is a composite of the matter and the form (after getting the wood together and using it to make the entity in the blueprint, you can finally say "this chair"). In order to determine if mathematical entities exist as substances, Aristotle needs to compare them to these three categories of things with haecceity, and see if mathematical entities can be said to have haecceity.

Only the third kind of haecceity brings with it the possibility of independent existence. The wood as potentially a chair cannot exist as a chair independently of the pattern that will make it a chair. Aristotle is far from saying that matter cannot exist independently as that kind of matter, but even speaking of a kind of matter is already to impose form on the matter, which is exactly his point. Without the form, the matter cannot be a kind of matter, and therefore cannot exist in the world. Similarly, a pattern (his translators call it a formula) cannot exist independently of being used to form matter in a specified way. Free-floating pattern might be a concept, but a concept is not a substance. It seems, however, that mathematical entities are very similar to this latter kind of thing: they are not considered *qua* formed matter. As Aristotle said concerning mathematics,

> Nor does the science which we are now seeking treat of the objects of mathematics; for none of them can exist separately. But again it does not deal with perceptible substances; for they are perishable. (Aristotle – *Metaphysics* 1059b10-15)

Aristotle is gradually working through the four possible ways mathematical entities could exist (I mentioned these four ways above). He has now established that mathematical entities cannot exist separately as substance, and he moves on to the next possibility in *Metaphysics*, 1064b6-16. He will return to the topic of substance later when he considers the fourth possibility, because this last possibility is distinct from, but easily confused with, the second possibility. In *Metaphysics* 1064b6-16, he points out that the mathematical sciences deal with a subject matter and study objects, and therefore it is not possible that mathematical entities have no existence at all. That deals with the third possibility. Aristotle moves on swiftly to the fourth possibility, that mathematical entities are not substances, but instead exist as abstractions from the sensible substances in the world.

Again in *Metaphysics*, 1070a9-12, Aristotle repeats the three kinds of substance, with examples. He calls the three kinds matter (which is perceptible), nature (which is the teleological end of the matter), and particular substance (which is composed of matter and nature).

In *Metaphysics*, 1075b34-1078b6, Aristotle launches into the most protracted discussion of the kind of existence of mathematical entities in all of his works. He points out that when some say that everything is number (as the Pythagoreans did) they then deprive the universe of all goal-directedness or cause-and-effect, because number is never the cause of anything. Since it is central to Aristotle's philosophy that there is a cause-and-effect structure to the world, he rejects this Pythagorean explanation of the world as infeasible and not corresponding to the world that we know exists.

Cleary has pointed out Aristotle's approach to previous philosophers several times, and it is in this section of the *Metaphysics* that we find this approach laid out:

> Now since our inquiry is whether there is or is not besides the sensible substances any which is immovable and eternal, and, if there is, what it is, we must first consider what is said by others, so that, if there is anything which they say wrongly, we may not be liable to the same objections, while, if there is any opinion common to them and us, we shall not quarrel with ourselves on that account; for one must be content to state some points better than one's predecessors, and others no worse.

After saying what his approach will be, Aristotle points out that some philosophers have said that the objects of mathematics are substances. He distinguishes two different groups: those who assimilate the intermediates and the Ideas, and those who distinguish them. But since both camps would say that the objects of mathematics are substances, Aristotle says that it is better to first determine whether mathematical entities exist and then to determine whether they are Ideas or not. If we have established that the mathematical entities do exist, then we can enquire how they exist, but if we don't even know if they exist or not, asking what their mode of existence is, is a meaningless question.

I don't think Aristotle has said anything explicitly about the four different ways mathematical entities could exist before this, even though it was implicit in his discussion in the *Metaphysics* thus far. However, here he explicitly lays out the different possibilities:

> If the objects of mathematics exist, then they must exist either in sensible objects, as some say, or separate from sensible objects (and this also is said by some), or if they exist in neither of these ways, either they do not exist, or they exist in some other way.

In fact, in his very next sentence of this section, he admits that he has been pursuing this line of inquiry all along. The existence of the mathematical entities in the sensible things is impossible according to his earlier argument that this would require the superposition of solids in the same space, which is impossible, and also because this would make sensible things incapable of division, which is against the appearances (knives exist because sensible things can be divided; this isn't Aristotle's argument, but it might as well be).

The second possibility is also dismissed here, as pointed out earlier, because of the unnecessary multiplication of solids, planes, lines, and points. Aristotle asks

> With which of these, then, will the mathematical sciences deal?

None of the kinds of lines, for instance, have any more right to be studied by mathematics than any of the other kinds. But in Aristotle's view, a science always deals with one kind of thing, so mathematics will have to make the impossible choice of which kind of line to study. Not only geometry has this problem: the number of kinds of numbers will also be multiplied, for the same reason.

Aristotle in the end does give us his own view of the mode of existence of mathematical entities, in the following part of the section under discussion:

> For 'exist' has many senses. Just as the universal part of mathematics deals not with objects which exist separately, apart from magnitudes and from numbers, but with magnitudes and numbers, not however *qua* such as to have magnitude or to be divisible, clearly it is possible that there should also be both formulae and demonstrations about sensible magnitudes, not however *qua* sensible but *qua* possessed of certain definite qualities.

It is therefore in saying that number and magnitude exists as a property of substance that we finally hear from Aristotle what the mode of existence of mathematical entities is. They exist not as things in themselves, but as abstractions from the substances. This view has its problems, as Gottlob Frege pointed out in the nineteenth century, but Bernays seems to have held this view as well.

Aristotle was a good teacher. As such, he knew that difficult and controversial ideas need to be conveyed in a way that shows sympathy for the learners. When he introduced his view of mathematical entities, he knew that Platonists might find it difficult that the good and the beautiful does not play as central a role in his philosophy as it did in Plato's philosophy. Therefore, at the end of *Metaphysics*, 1075b34-1078b6, he sets the Platonists at ease by saying that the mathematical sciences have lost none of its beauty or its goodness in his philosophy, but that his point is just that these attributes cannot be the determiners of existence. Mathematical entities exist because they are properties inherent in the substances, not because they are good or beautiful, but they are no less beautiful or good for all that.

Let's consider again the problems that arise in Aristotle's philosophy. Aristotle sees the task of science very clearly as considering substances from certain perspectives, analysing some of their properties as ways for those substances to exist. Now this is very good, but it leaves something out of the picture. Aristotle was convinced that every science has its own objects of study. This is problematic.

Consider geometry for instance. Geometry is the science of space, which means that it studies the spatial properties of things. Are there only certain substances that have spatial properties? In Aristotle's view, there must be, since geometry only studies some substances, not all of them. But in the sensible world, everything has a spatial dimension. Therefore, when we study geometry, we are not just abstracting from a select few things, we are abstracting the modal aspect from all things.

Another problem that Aristotle does not overcome, is that there is a difference between the aspect of things that they have a spatial extension, and a type of spatial extension. Saying that something has a spatial aspect is saying that it can be used to serve as a study object for geometry. Saying that something is an isosceles triangle, on the other hand, is saying that there is a particular kind of geometrical configuration that the object of study exhibits. Aristotle equates these two kinds of abstraction (or, at least, does not distinguish between them). But in order to do geometry accurately, in order to give a true picture of the spatial aspect of things, these two kinds of abstraction have to be distinguished.

Conclusion

The ancient Greeks made tremendous strides in developing mathematics. These developments were not only in the fields of arithmetic and geometry and their applications, but also in the methodological and philosophical understanding of mathematics. Nevertheless, they were only human, and therefore they did not exhaust what could be said about the philosophy of mathematics.

In evaluating the contribution that the Greeks made to the understanding of mathematical entities, it is important to keep in mind their historical position. In critiquing what they managed to do, one must fight the temptation to be over-critical and therefore not see the truly amazing and important discoveries that they made.

However, the Greeks are not above reproach. They did make mistakes in their understanding of mathematical entities, and it is useful to know where these mistakes were so that we do not fall into the same traps. Let us consider their contributions in the order that we find them in history.

The Pythagoreans were correct in identifying two fundamental modal spheres that are studied by mathematics: they realized that mathematics studies both the arithmetical and the spatial aspects of reality. Unfortunately, they both subordinated the arithmetical to the spatial aspect, and tried to reduce all the other aspects to the spatial aspect. This is unfortunate, because it inevitably leads to problems that can then no longer be solved. For instance, civil law is determined by people, not by numbers. But for the purposes of this study, it is even more important to note, as Laugwitz did (Laugwitz, 1986:649), that there are geometrical truths that cannot be framed in terms of arithmetic, and this shows that a reductionism that tries to reduce everything to the spatial aspect will always fail.

The Eleatics made the important contribution of identifying the crucial concept of whole and parts, and they correctly placed this relation in the spatial aspect. Unfortunately, they used it to deny the existence of the at-once infinite, which is also a part of the correct understanding of the spatial aspect. Our modern understanding of set theory relies on the at-once infinite, as Strauss has demonstrated (Strauss, 2009).

Plato tried to give a solution to the foundational crisis in mathematics (the question: is arithmetic or geometry more foundational?) by postulating a world where number-Forms are the foundations of everything else. This solution, though beautiful and good, is not a true solution since it still encounters the same problem that Laugwitz pointed out (Laugwitz, 1986:649), namely that arithmetic and geometry are inter-irreducible.

Lastly, Aristotle also tried to solve the foundational crisis. He realized that Plato's solution does not work. Postulating an Ideal world where the mathematical problems of the Eleatics do not crop up, only duplicates the real foundational problem that we encounter already in the sensible realm. If arithmetic and geometry cannot be reduced the one to the other in the sensible realm, then they still cannot be reduced the one to the other in the Ideal world. Aristotle also importantly made a distinction between two different kinds of infinity (the successive infinite and the at-once infinite) and gave us an accurate understanding of the continuum. Unfortunately, his pre-conceived bias against the existence of the at-once infinite made a full picture of the spatial aspect impossible; and therefore, he also was not able to fully answer the problem of the foundational crisis. In order to understand the place of the spatial and the arithmetical aspects in the world, and therefore in order to understand what mathematical entities are, it is necessary to give a true and full picture of what the spatial and arithmetical aspects are.

We can conclude therefore that we have to look elsewhere to solve the problem of the first foundational crisis in mathematics. Another philosophical system is necessary that relies on a different kind of worldview, a non-reductionist worldview. That is, in order to put arithmetic and geometry in their true places in our world, we need to make foundational assumptions that do not require geometry to be reducible to arithmetic, or arithmetic to geometry. This study has not reached such a philosophy yet, as it was focused on expounding the philosophy of the Greeks, and it definitely has not arrived at a point where such a philosophy can be proved to solve the problems that we have considered, but I venture here to identify a philosophy that serves as a candidate to investigate in a future study. I think that the problems that we encounter in the first foundational crisis of mathematics would be solved in the greater context of the reformational philosophy as it was propounded by Dooyeweerd. The reader can look forward to a subsequent study where I will attempt to prove that this is the case.

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