Weighted noncommutative Banach function spaces

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Summary

For a semifinite von Neumann algebra $\mathcal{M}$ with τ-measurable operators $\widetilde{\mathcal{M}}$, weighted noncommutative Banach function spaces, denoted $L^{\rho}_{x}(\widetilde{\mathcal{M}})$, were first introduced by Labuschagne and Majewski as a generalisation of noncommutative Banach function spaces. This thesis presents the first investigation into certain key aspects of the theory of these spaces.

Along with the concept of weighted noncommutative Banach function spaces Labuschagne and Majewski also introduced a pseudo tracial map τ_x. In our investigation, we start by translating some basic concepts of noncommutative integration theory into weighted analogues by letting the map τ_x take the place of the trace. In particular, we explore the weighted analogues of τ-measurability and the topology of convergence in measure. Crucially we also show that the weighted noncommutative decreasing rearrangement is related to the tracial noncommutative decreasing rearrangement through a classical decreasing rearrangement. An alternative definition to the one introduced by Labuschagne and Majewski is formulated and we show that these definitions define the same class of spaces.

Next, we investigate weighted noncommutative Orlicz spaces. We first show that both definitions of weighted noncommutative Banach function spaces render the same weighted noncommutative Orlicz space for a given Young function. Next, we investigate Köthe duality of weighted noncommutative Orlicz spaces. For a certain class of Young functions, we can recover the tracial result that the Köthe dual of a weighted space generated by a Young function is, up to an equivalent norm, the weighted space generated by the convex conjugate of the Young function.

Finally, we develop the theory of real interpolation for the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$. The so-called K-functional is crucial to the real interpolation of these spaces. As such an important task that we undertake is to investigate the nature of the K-functional. Using these results we follow the theory of real interpolation of an abstract Banach couple to show that every weighted noncommutative Banach function space generated by a monotone Riesz-Fischer norm, is an exact interpolation space. Conversely every exact monotone interpolation space is generated by a monotone Riesz-Fischer norm.

Key Words

von Neumann algebras, noncommutative Banach spaces, weighted, noncommutative integration theory, Köthe duality, real interpolation theory

Samevatting

Vir 'n semi-eindige von Neumann algebra $\mathcal{M}$ met τ-meetbare operatore $\widetilde{\mathcal{M}}$, was geweegde niekommutatiewe Banach funksie ruimtes, aangedui deur $L^{\rho}_{x}(\widetilde{\mathcal{M}})$, eerste bekendgestel deur Labuschagne en Majewski as 'n veralgemening van niekommutatiewe Banach funksie ruimtes. Hierdie tesis bied die eerste ondersoek aan in sekere sleutel aspekte van die teorie van die ruimtes.

Saam met die konsep van geweegde niekommutatiewe Banach funksie ruimtes het Labuschagne en Majewski ook 'n pseudo spoor afbeelding τ_x voorgestel. In ons ondersoek begin ons deur om sekere basiese concepte te vertaal na hulle geweegde analoë deur om die afbeelding τ_x die plek te neem van die spoor. Ons ondersoek die geweegde analoë van τ-meetbaarheid en die topologie van konvergensie in maat. Krities wys ons ook dat die geweegde niekommutatiewe afnemende herrangskikking is verwant aan die niekommutatiewe afnemende herrangskikking met betrek tot die loop deur 'n klassieke afnemende herrangskikking. 'n Alternatiewe definisie tot die een voorgestel deur Labuschagne en Majewski word geformuleer en ons wys dat die definisies definieer dieselfde ruimtes.

Volgende ondersoek ons geweegde Orlicz ruimtes. Eers wys ons dat beide definisies van geweegde niekommutatiewe Banach funksie ruimtes dieselfde Orlicz ruimtes gee vir 'n gegewe Young funksie. Daarna ondersoek ons Köthe dualiteit van geweegde Orlicz ruimtes. Vir 'n sekere klas van Young funksies kan ons die loop resultate herkry dat die Köthe duaal van 'n geweegde ruimte gegenereer deur 'n Young funksie is, tot op 'n ekwivalent norm, die geweegde ruimte gegenereer deur die konvekse toegevoegde van die young funksie.

Ten laaste ontwikkel ons die teorie van reële interpolasie vir die Banach koppel $(L^{1}_{x}(\widetilde{\mathcal{M}}), \mathcal{M})$. Die sogenaamde K-funksionaal is krities tot die reële interpolasie van die ruimtes. As sulks 'n belangrike taak wat ons onderneem is om die natuur van die K-funksionaal te ondersoek. Ons gebruik dan hierdie resultate en volg die teorie van reële interpolasie van 'n abstrakte Banach koppel om te wys dat elke geweegde niekommutatiewe Banach funksie ruimte gegenereer deur 'n monotone Riesz-Fischer norm 'n eksakte interpolasie ruimte is. Omgekeerd word elke eksakte monotone interpolasie ruimte gegenereer deur 'n Riesz-Fischer norm.

Sleutelwoorde

von Neuman algebras, niekommutatiewe Banach ruimtes, geweegde, niekommutatiewe integrasie teorie, Köthe dualiteit, reële interpolasie

Contents

Introduction

The origin of noncommutative Banach function spaces in some way can be traced back to von Neumann when he published his paper [**19**] in 1937. It was several more decades, however, before further progress were made. First by Ovcinnikov in his 1970 paper [**13**], followed by Dodds, Dodds and de Pagter in their paper [**3**] in 1989 which they explicitly formulated the concept of a noncommutative Banach function space, often denoted $L^{\rho}(\widetilde{\mathcal{M}})$ for a Banach function norm ρ. The development of the theory of noncommutative decreasing rearrangements by Fack and Kosaki in their 1986 paper [**6**] was key to the work of Dodds, Dodds and de Pagter. Shortly thereafter Dodds, Dodds and de Pagter published two followup papers in which they achieved a robust theory of Köthe duality [**4**] and a refined theory of real interpolation [**5**]. Since then the theory of noncommutative Banach function spaces has seen further wide-ranging research efforts that developed the theory even further. Among several other important achievements within the theory of noncommutative Banach function spaces was the result of Kalton and Sukochev [**9**] where they showed that one only needs to assume a symmetric function norm, the most general version of the principle, for the construction by Dodds, Dodds and de Pagter in [**3**] to work.

In [**10**] Labuschagne and Majewski aimed to describe the regular random observables with respect to a state $x \in L^1_+(\widetilde{\mathcal{M}})$, a quantisation of the concept of regular random variables. Here $L^1(\widetilde{\mathcal{M}})$ refers to the noncommutative Banach function space corresponding to $L^1([0,\infty))$. They found that the natural space to describe these observables were the so-called weighted noncommutative Banach function spaces, a concept they introduced in the same paper. The set of regular random observables, they found is a closed subset of the weighted noncommutative Orlicz space $L^{\cosh -1}_x(\widetilde{\mathcal{M}})$. In proving these results a pseudo tracial map $\tau_x : \widetilde{\mathcal{M}} \mapsto [0,\infty] : a \mapsto \int_0^\infty \mu_s(a)\mu_s(x)\mathrm{d}s$ was introduced. It is of some interest to note that in the case when $\mu(x)$ is the identity function on $[0,\tau(1))$, allowing for $\tau(1)=\infty$, then τ_x is simply the trace in the sense as it was extended to $\widetilde{\mathcal{M}}_+$ in [**6**]. In this sense, one can think of τ_x as a weighted version of the trace. In our further development, we will take this idea a step further when we define weighted analogues of various concepts in the tracial theory by letting the map τ_x take the place of the trace. In this way, we will define concepts such as τ_x-measurable operators, a weighted topology of convergence in measure, a weighted noncommutative decreasing rearrangement and indeed an alternative definition for weighted Banach function spaces. Later we will also define a concept of a weighted Köthe dual.

In this thesis, $\mathcal{M}$ will always denote a semi-finite von Neumann algebra equipped with a semi-finite, normal, faithful trace τ. The topological *-algebra of τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. For the classical theory, we will refer to a Banach function norm as a map that satisfies the conditions in [**2**, Definition 1.1.1].

This thesis assumes a basic understanding of the theory of von Neumann algebras and (noncommutative) Banach function spaces. In the first chapter, named Preliminaries, we present the basic concepts and results necessary for the rest of the thesis, all without proof. First, we cover the concepts of the theory of von Neumann algebras that will be most important to the thesis. The second section in the chapter covers the construction of classical Banach function spaces, and the final section in the chapter shows how one can construct noncommutative Banach function spaces given a semi-finite von Neumann algebra and a classical Banach function space. This chapter should not be seen as a comprehensive exposition of these subjects. For readers who are unfamiliar with any of these topics we highly advise using the various books and papers mentioned in the chapter to gain some familiarity.

In the second chapter, we start our development of weighted spaces. As mentioned previously Labuschagne and Majewski introduced a pseudo tracial map τ_x. The philosophy we follow in this chapter, and in fact, the entire thesis is to let τ_x take the place of the trace in developing the theory. To do this, we first investigate some of the basic properties of τ_x. We then define a concept of a τ_x-measurable operator. We denote the set of all such operators as $\widetilde{\mathcal{M}}_x$. We also define a weighted topology of convergence in measure, which we show is indeed a topology. In fact we show that $\widetilde{\mathcal{M}}_x$ equipped with the weighted topology of convergence in measure is exactly $\widetilde{\mathcal{M}}$ equipped with the usual topology of convergence in measure. This provides the setting in which we shall develop our theory. Next, we define a weighted noncommutative decreasing rearrangement $\mu(a, x)$ and explore some of its properties, including a classical connection with the tracial noncommutative decreasing rearrangement. The main aim of this chapter, however, is to provide an alternative definition for weighted noncommutative Banach function spaces and to show that our definition is equivalent to that of Labuschagne and Majewski.

Weighted noncommutative Orlicz spaces are the subject of investigation in the third chapter. First, we show that the equivalence provided by the previous chapter can be refined even further in the case of weighted noncommutative Orlicz spaces. In particular, we can show that for a given Young function, both approaches will render the same space. This is of particular importance since, as the reader may recall, the weighted noncommutative Orlicz space $L_x^{\cosh -1}(\widetilde{\mathcal{M}})$ was shown to be the home of the regular random observables. The second subject of investigation in this chapter is that of the Köthe duality of Orlicz spaces. The reader may recall that the theory of Köthe duality of noncommutative Banach function spaces developed by Dodds, Dodds and de Pagter encompassed noncommutative Banach function spaces in general. Unfortunately when applying their strategy to weighted noncommutative Banach function spaces one encounters significant obstacles at points where additivity of the trace plays a vital role due to the subadditive nature of τ_x. We do however manage to obtain results for weighted noncommutative Orlicz spaces generated by a certain class of Young functions.

The final chapter uses the theory of monotone interpolation spaces of an abstract Banach couple to investigate the real interpolation theory of the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$. Our development of the theory is based on the K-method of interpolation. We find that every weighted Banach function space generated by a monotone Riesz-Fischer norm is an exact interpolation space. In the converse, we also find that every exact monotone interpolation space is generated by a monotone Riesz-Fischer norm. Whether every exact space is monotone is not known at this point.

CHAPTER 1

Preliminaries

1.1. Von Neumann algebras

In this section we review the necessary theory on von Neumann algebras. There are numerous books written on von Neumann algebras, in particular the books *Theory of Operator Algebras (vol. I and II)* by Takesaki, *Fundamentals of the Theory of Operator Algebras* by Kadison and Ringrose and *C^*-algebras and W^*-algebras* by Sakai can each serve as a resource on the theory of von Neumann algebras.

Let $\mathcal{H}$ be a Hilbert space. The *strong operator topology* is the topology induced by the seminorms on $B(\mathcal{H})$ given by $a \mapsto \|a\xi\|$ for each $\xi \in \mathcal{H}$. So the strong operator topology has a neighbourhood basis of zero of sets of the form

$$V_{SO}(\xi_1, \xi_2, \dots, \xi_n, \epsilon) = \{a \in B(\mathcal{H}) : \|a\xi_i\| < \epsilon, i \in \{1, 2, \dots n\}\}.$$

Similarly the *weak operator topology* is the topology induced by the seminorms given by $a \mapsto |\langle \xi_1, a\xi_2 \rangle|$ and has a neighbourhood basis of zero of sets of the form

$$V_{WO}(\xi_1, \xi_2, \dots, \xi_n, \epsilon) = \{a \in B(\mathcal{H}) : |\langle \xi_i, a\xi_j \rangle| < \epsilon, i, j \in \{1, 2, \dots n\}\}.$$

It is well known that the strong operator and weak operator closures of a convex subset of $B(\mathcal{H})$ coincide.

DEFINITION 1.1. A subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ is called a von Neumann algebra if $\mathcal{M}$ is a unital subalgebra of $B(\mathcal{H})$ that is closed under taking adjoints, i.e. if $a \in \mathcal{M}$ then $a^* \in \mathcal{M}$, and closed in the strong (or weak) operator topology.

Note that $\mathcal{M}$ being closed in the strong operator topology implies that $\mathcal{M}$ is also closed under the norm topology of $B(\mathcal{H})$, thus making it a C*-algebra.

For any subset $S \subset B(\mathcal{H})$ the commutant of S, denoted S', is given by $S' = \{a \in B(\mathcal{H}) : ab = ba, \forall b \in S\}$. With this we can give a characterisation of von Neumann algebras.

THEOREM 1.2. *Let $\mathcal{M}$ be a unital subalgebra of $B(\mathcal{H})$ that is closed under taking adjoints. Then $\mathcal{M}$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$.*

The positive elements of a von Neumann algebra are those elements $a \in \mathcal{M}$ such that the $a = b^*b$ for some $b \in \mathcal{M}$. Equivalently the positive elements are those elements with real positive spectrum. We will denote the cone of positive elements of a von Neumann algebra $\mathcal{M}$ as $\mathcal{M}^+$.

DEFINITION 1.3. A map $\tau : \mathcal{M}^+ \mapsto [0, \infty]$ is a *trace* if for all $a, b \in \mathcal{M}^+$ and $\lambda \geq 0$

- $\tau(a + b) = \tau(a) + \tau(b)$
- $\tau(\lambda a) = \lambda\tau(a)$
- $\tau(a^*a) = \tau(aa^*)$.

Additionally, a trace is

- *faithful* if $\tau(a) = 0$ implies $a = 0$
- *normal* if for a net $\{a_i\} \subset \mathcal{M}^+$ with $a_i \uparrow_{SO} a$ in $\mathcal{M}^+$, then $\tau(a_i) \uparrow \tau(a)$
- *semi-finite* if for all $a \in \mathcal{M}^+$ there exist a nonzero b with $b \leq a$ such that $\tau(b) < \infty$. Equivalently there exists a net $a_i \uparrow a$ such that $\tau(a_i) < \infty$.

It is well known that a von Neumann $\mathcal{M}$ algebra allows a semifinite, faithful, normal (sfn) trace if and only if $\mathcal{M}$ is semifinite. Due to this we will from now constrain our attention to semifinite von Neumann algebras.

Let a be a self-adjoint densely defined operator on $\mathcal{H}$, then there exists a unique spectral family $\{e_t : t \in \mathbb{R}\}$ such that

(1) e_t is a projection for each $t \in \mathbb{R}$
(2) $t_1 \leq t_2$ implies $e_{t_1} \leq e_{t_2}$
(3) $e_{t+\epsilon} \downarrow e_t$ in the strong operator topology
(4) $e_t \uparrow 1$ as $t \uparrow \infty$
(5) $e_t \downarrow 0$ as $t \downarrow -\infty$
(6) $a = \int_{-\infty}^{\infty} t de_t$,

known as the spectral decomposition of a. If a is positive, then $e_t = 0$ for all $t < 0$ and hence $a = \int_0^{\infty} t de_t$.

Any closed densely defined operator a can be written in its polar decomposition as $a = u|a|$ where u is a partial isometry.

DEFINITION 1.4. A closed densely defined operator a is *affilaited* with a von Neumann algebra $\mathcal{M}$ if either of the following equivalent conditions hold

- For all $b \in \mathcal{M}'$, $ba \subset ab$
- If $a = u|a|$ is the polar decomposition of a and $|a| = \int_0^{\infty} t de_t$ is the spectral decomposition of a, then $u, e_t \in \mathcal{M}$ for all $t \geq 0$.

If a is affiliated with $\mathcal{M}$ and bounded, then $a \in \mathcal{M}$.

If $\mathcal{M}$ is a commutative von Neumann algebra, then $\mathcal{M}$ is isomorphic to some $L^\infty(X, \Sigma, \nu)$ for a localisable measure space (X, Σ, ν). Conversely, for a localisable measure space, $L^\infty(X, \Sigma, \nu)$ is a von Neumann algebra over the Hilbert space $L^2(X, \Sigma, \nu)$, where the operator action is taken as multiplication. In this case the map $\tau : f \mapsto \int f d\nu$ is a sfn trace. This leads to the general philosophy of noncommutative integration theory. Specifically we regard a general semifinite von Neumann algebra $\mathcal{M}$ as a noncommutative analogue of an L^∞ with the trace taking the role of a noncommutative integral.

1.2. Classical Banach function spaces

As was mentioned in the previous section, any commutative von Neumann algebra is isomorphic to $L^\infty(X, \Sigma, \nu)$ for some measure space (X, Σ, ν). The study of noncommutative Banach function spaces uses this as a starting point. In order to understand noncommutative Banach function spaces we must first review the necessary theory of commutative Banach function spaces. For a measure space (X, Σ, ν), we will denote the set of all measurable functions that are finite almost everywhere by $L^0(X, \Sigma, \nu)$ and $L^0(X, \Sigma, \nu)^+$ as the subset of all positive functions. In this chapter and throughout this thesis we will denote the Lebesgue measure by m.

The concept of a function norm plays an important role in the development of Banach function spaces. For this text we will follow Bennett and Sharpley as in [**2**] for our definition. We do however acknowledge that there are a number of different approaches, some of which are more general than the one presented here.

Definition 1.5. Let (X, Σ, ν) be a semi-finite measure space. Then a *Banach function norm* on $L^0(X, \Sigma, \nu)^+$ is a map $\rho : L^0(X, \Sigma, \nu)^+ \mapsto [0, \infty]$ that satisfies the following conditions for all $f, g \in L^0(X, \Sigma, \nu)^+$ and $\lambda \geq 0$

(1) $\rho(f) = 0$ if and only if $f = 0$ almost everywhere
    $\rho(f + g) \leq \rho(f) + \rho(g)$
    $\rho(\lambda f) = \lambda \rho(f)$
(2) $0 \leq g \leq f$ almost everywhere implies $\rho(g) \leq \rho(f)$
(3) $f_i \uparrow f$ almost everywhere implies $\rho(f_i) \uparrow \rho(f)$
(4) $\nu(E) < \infty$ implies $\rho(\chi_E) < \infty$
(5) $\nu(E) < \infty$ implies there exist a constant C_E dependent on E and ρ but independent of f such that $\int_E f d\nu \leq C_E \rho(f)$.

Of the above assumptions, only (1) and (2) can be considered universal for function norms. We will, however, continue with the additional assumptions and use the term Banach function norm to indicate this.

For a (Banach) function norm ρ one can extend the domain of ρ to all of $L^0(X,\Sigma,\nu)$ using the prescription $\rho(f)=\rho(|f|)$ for all $f \in L^0(X,\Sigma,\nu)$.

DEFINITION 1.6. The Banach function space associated with the Banach function norm is the space $L^\rho(X,\Sigma,\nu)=\{f\in L^0 : \rho(f)<\infty\}$. The space $L^\rho(X,\Sigma,\nu)$ is a Banach space under the norm $\|f\|_\rho=\rho(f)$ for all $f\in L^\rho(X,\Sigma,\nu)$.

In the more general approach a Banach function space is the space generated by a function norm as above that is complete with respect to the function norm.

DEFINITION 1.7. Let ρ be a Banach function space on $L^0(X,\Sigma,\nu)$. The *associate norm* of ρ, denoted ρ', is the Banch function norm given by

$$\rho'(f)=\sup\{\int fg\mathrm{d}\nu : f\in L^\rho(X,\Sigma,\nu),\rho(f)\leq 1\}.$$

The *associate space* of $L^\rho(X,\Sigma,\nu)$ is $L^{\rho'}(X,\Sigma,\nu)$.

THEOREM 1.8 (Hölder's inequality). [**2**, Theorem 1.2.4] *Let $L^\rho(X,\Sigma,\nu)$ be a Banach function space with associate space $L^{\rho'}(X,\Sigma,\nu)$. Then for $f\in L^\rho(X,\Sigma,\nu)$ and $g\in L^{\rho'}(X,\Sigma,\nu)$,*

$$\int fg d\nu \leq \|f\|_\rho\|g\|_{\rho'}.$$

DEFINITION 1.9. The distribution function of $f\in L^0(X,\Sigma,\nu)$ is the map $d(f):[0,\infty)\mapsto [0,\infty]: s\mapsto \nu\{x\in X:|f(x)|\geq s\}$. We denote $d(f)$ evaluated at s as $d_s(f)$.

The decreasing rearrangement of f is the map $\mu(f):[0,\infty)\mapsto[0,\infty]: t\mapsto \inf\{s\geq 0: d_s(f)\leq t\}$. We denote $\mu(f)$ evaluated at t as $\mu_t(f)$.

A function norm ρ for which $\rho(f)=\rho(g)$ whenever $\mu(f)=\mu(g)$ is called symmetric. The Banach space generated by such a function norm is then referred to as a symmetric Banach function space. In this text we will exclusively be concerned with symmetric Banach function norms.

We provide the following lemma and theorem due to their importance throughout this text.

LEMMA 1.10 (Hardy's lamma). *Let f and g be nonnegative measurable functions on $(0,\infty)$ and suppose that $\int_0^t f(s)ds\leq\int_0^t g(s)ds$ for all $t>0$. Let h be any nonnegative decreasing function on $(0,\infty)$, then $\int_0^\infty f(s)h(s)ds\leq\int_0^\infty g(s)h(s)ds$.*

A totally σ-finite measure (X,Σ,ν) space is called *resonant* if $\int\mu(f)\mu(g)\mathrm{dm}=\sup\int|fh|\mathrm{d}\nu$ where the supremum is taken over all functions h equimeasuarable with g(i.e. $d(h)=d(g)$) on X.

THEOREM 1.11 (Luxemburg representation theorem). [**2**, Theorem 2.4.10] *Let ρ be a symmetric Banach function norm on (X, Σ, ν) for a resonant measure space (X, Σ, ν). Then there exists a (not necessarily unique) symmetric Banach function norm $\bar{\rho}$ on $L^0_+(\mathbb{R}^+, m)$, where m is the Lebesgue measure, such that $\rho(f) = \bar{\rho}(\mu(f))$ for all $f \in L^0(X, \Sigma, \nu)$.*

1.3. Noncommutative Banach function spaces

Throughout $\mathcal{M}$ will be a semifinite von Neumann algebra equipped with a semifinite normal faithful (snf) trace τ. The general philosophy of noncommutative integration theory is to treat $\mathcal{M}$ as a noncommutative L^∞ space, and to treat the trace as a noncommutative integral. Following this philosophy, we will formulate the noncommutative analogues of the concepts presented in the previous section and recall the results that will be most relevant to our development of weighted noncommutative Banach function spaces.

DEFINITION 1.12. For a trace τ on a semifinite von Neumann algebra $\mathcal{M}$, we say that τ is

(1) *semifinite* if for all $a \in \mathcal{M}^+$ there exist a nonzero $0 \le b \le a$ such that $\tau(b) < \infty$.
(2) *normal* if for any net $\{a_i\}$ with $a_i \uparrow a$ in the strong operator topology, then $\tau(a_i) \uparrow \tau(a)$.
(3) *faithful* if for any $a \in \mathcal{M}^+$, $\tau(a) = 0$ if and only if $a = 0$.

DEFINITION 1.13. A closed operator a affiliated with $\mathcal{M}$ is τ-measurable ($a \in \widetilde{\mathcal{M}}$) if and only if for all $\delta > 0$ there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset D(a)$, $\|ap\| < \infty$ and $\tau(1-p) \le \delta$.

The space $\widetilde{\mathcal{M}}$ is a *-algebra with respect to strong sum and strong multiplication, i.e. for $a, b \in \widetilde{\mathcal{M}}$ the sum and product of a and b in $\widetilde{\mathcal{M}}$ is taken to be the closures of the operators $a + b$ and ab respectively.

DEFINITION 1.14. The noncommutative decreasing rearrangement (or generalised singular value function) of $a \in \widetilde{\mathcal{M}}$ is the map $\mu(a) : [0, \infty) \mapsto [0, \infty]$ where the value of $\mu(a)$ at $t \ge 0$, denoted $\mu_t(a)$, can be calculated using either of the following prescriptions [**6**]

(1) $\mu_t(a) = \inf\{\|ae\| : e \in \mathbb{P}(\mathcal{M}), \tau(1-e) \le t\}$.
(2) $\mu_t(a) = \inf\{s \ge 0 : \tau(1-e_s) \le t\}$.

Where e_s are the spectral projections of $|a|$.

If the integral is taken as a trace on a classical L^∞ space, then the above formulation is exactly that presented in section 1.2. The nature of the functions $\mu(a)$ was extensively investigated in articles such as [**6**], and subsequently in [**3**] and [**4**]. In the subsequent chapters we will prove weighted analogues of many of their results. In many cases our proofs for our weighted version of the noncommutative decreasing rearrangements do not vary greatly from the original proofs for the non-weighted versions. We do wish to draw the attention of the reader to Lemma 2.5 of [**6**], of which we will prove a weighted analogue in Chapter 2.

Definition 1.15. Let ρ be a symmetric Banach function norm. The noncommutative Banach function space associated with ρ and the von Neumann algebra $\mathcal{M}$ is the space

$$L^\rho(\widetilde{\mathcal{M}}) = \{a \in \widetilde{\mathcal{M}} : \rho(\mu(a)) < \infty\}$$

which is a Banach space under the norm given by $\|a\|_\rho = \rho(\mu(a))$ for all $a \in L^\rho(\widetilde{\mathcal{M}})$.

In the above definition we assumed that ρ is a Banach function norm, so it satisfies all the assumptions of 1.5. To ensure that $L^\rho(\widetilde{\mathcal{M}})$ is a Banach space it is possible to relax some of the assumptions. In particular Kalton and Sukochev showed in [**9**] that one only needs the function norm to be symmetric, the most general version of this principle.

CHAPTER 2

Weighted Banach Function Spaces

2.1. A first definition of weighted non-commutative Banach function spaces

Noncommutative weighted Banach function spaces were first defined and shown to be Banach spaces that inject continuously into $\widetilde{\mathcal{M}}$ in [**10**], in particular, [**10**, Definition 3.6] and [**10**, Theorem 3.7] respectively. In [**10**] Labuschagne and Majewski had in mind that x should be a state in the quantum mechanical sense, and as such considered $x \in L^1(\widetilde{\mathcal{M}})$. This requirement was entirely motivated by the physical considerations that lead to Labuschagne and Majewski defining weighted noncommutative Banach function spaces. If one were to only consider the mathematics of the situation, one could relax this assumption to $0 \leq x \in L^1_+(\widetilde{\mathcal{M}}) + \mathcal{M}$. We now state the definition of a weighted non-commutative Banach function space proposed by Labuschagne and Majewski under our relaxed assumption.

Definition 2.1. [**10**, Definition 3.6] *Let $0 \leq x \in L^1(\widetilde{\mathcal{M}}, \tau) + \mathcal{M}$, and let ρ be a rearrangement-invariant Banach function norm on $L^0((0,\infty), \mu_t(x)dt)$. Then the weighted non-commutative Banach function space is defined as $L^\rho_x(\widetilde{\mathcal{M}}, \tau) = \{a \in \widetilde{\mathcal{M}} : \mu(a) \in L^\rho((0,\infty), \mu_t(x)dt)\}$.*

Comparing this definition with that of the non-commutative Banach function spaces introduced by Dodds, Dodds and de Pagter [**3**], we can see a clear parallel. However, the proof that the spaces $L^\rho_x(\widetilde{\mathcal{M}})$ are Banach spaces under the norm induced by the Banach function norm deviates from the proofs of Dodds, Dodds and de Pagter non-trivially, but was proven by Labuschagne and Majewski in [**10**]. The proof as given in [**10**] is provided in the appendix with no alteration.

Theorem 2.2. [**10**, Theorem 3.7] *Let $0 \leq x \in L^1(\widetilde{\mathcal{M}}, \tau) + \mathcal{M}$, and let ρ be a rearrangement-invariant Banach function norm on $L^0((0,\infty), \mu_t(x)dt)$. Then $L^\rho_x(\widetilde{\mathcal{M}})$ is a linear space and $\|\cdot\|_\rho : a \mapsto \rho(a)$ is a norm on $L^\rho_x(\widetilde{\mathcal{M}})$. Equipped with the norm $\|\cdot\|_\rho$ the space $L^\rho_x(\widetilde{\mathcal{M}})$ is a Banach space that injects continuously into $\widetilde{\mathcal{M}}$.*

To justify this we should show that the proof of [**10**, Theorem 3.7] holds for a positive $x \in L^1(\widetilde{\mathcal{M}}, \tau) + \mathcal{M}$. To see this first note that the function

$$F_x(t) = \int_0^t \mu_s(x)ds$$

is continuous and strictly increasing on $[0, t_x)$, where $t_x = \inf\{s > 0 : \mu_s(x) = 0\}$, and constant on $[t_x, \infty)$ (when $t_x < \infty$). In the context of our assumption that $0 \leq x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$, we have that

F_x is a homeomorphism from $[0, t_x)$ to $[0, \tau(x))$, and we allow for $\tau(x) = \infty$. We remind the reader that ν was the measure given by $\mu_t(x)\mathrm{dt}$.

Furthermore we need the measure ν to be non-atomic. As pointed out in [**10**], ν is mutually absolutely continuous to the Lebesgue measure m. Let E be a Borel set. We now need to consider the two possibilities $\nu(E) < \infty$ and $\nu(E) = \infty$. Suppose that $\nu(E) < \infty$. Then we argue exactly as in [**10**] to find a measurable subset F of E with $0 < \nu(F) < \nu(E)$. Now suppose that $\nu(E) = \infty$. Then we choose an interval $[0, a]$ such that for $F = E \cap [0, a]$, $\nu(F) > 0$. It is clear that $0 < \nu(F) = \int_0^a \chi_E \mu_t(x)dt < \infty$ and therefore $0 < \nu(F) < \nu(E)$. This is enough to show that ν is non-atomic, from which the rest of the proof found in [**10**] follows.

The proof of [**10**, Theorem 3.7] made implicit use of the map $\tau_x : \widetilde{\mathcal{M}} \mapsto \mathbb{R} : a \mapsto \int_0^\infty \mu_t(a)\mu_t(x)dt$. This map will be of primary concern to us in the development of the theory of weighted non-commutative Banach function spaces.

2.2. The map τ_x

Our tasks in this section are to show that τ_x is a reasonable candidate as a substitution for the trace and to establish the appropriate setting within which we should develop our theory. For the former, we will recall Proposition 2.3, proved by Labuschagne and Majewski in [**10**], which shows the "trace-like" nature of τ_x and inspired us to develop a theory of weighted spaces based on τ_x taking the place of the trace. For the latter, we must answer two questions. The first is what kind of operators are to be considered in our theory. More specifically, we need to find a weighted analogue for the τ-measurable operators, the τ_x-measurable operators. We answer this question by showing that our concept of τ_x-measurability is equivalent to τ-measurability. The second question we must answer is what is the appropriately weighted analogue of the topology of convergence in measure. We will define a neighbourhood basis of zero for a weighted analogue of the topology of convergence in measure. Much like for the first question we will discover that the topology this neighbourhood basis of zero generates is nothing other than the topology of convergence in measure. Together with the answer to the first question, this shows that the topological *-algebra of τ-measurable functions is the appropriate setting for our theory.

For the map $\tau_x : \widetilde{\mathcal{M}} \mapsto \mathbb{R} : a \mapsto \int_0^\infty \mu_t(a)\mu_t(x)dt$ we will say that τ_x is *normal* if for every net $\{a_\alpha\}$ with $a_\alpha \uparrow a$ in $\widetilde{\mathcal{M}}$, we have that $\tau_x(a_\alpha) \uparrow \tau_x(a)$. We say that τ_x is *semi-finite* if for every $a \in \widetilde{\mathcal{M}}$ there exists a non-zero $b \in \widetilde{\mathcal{M}}$ such that $b \leq a$ and $\tau(b) < \infty$.

We begin by citing the result from [**10**] giving us the foundational properties of the map τ_x. The proof as given in [**10**] is provided in appendix B.

PROPOSITION 2.3. [**10**, Proposition 3.10] *For any non-zero* $0 \leq x \in L^1(\widetilde{\mathcal{M}}, \tau) + \mathcal{M}$*, the map* $\tau_x : \widetilde{\mathcal{M}} \mapsto \mathbb{R} : a \mapsto \int_0^\infty \mu_t(a)\mu_t(x)dt$ *has the following properties for all* $0 \leq a \in \widetilde{\mathcal{M}}$*:*

(1) τ_x *is subadditive, homogeneous and satisfies* $\tau_x(a^*a) = \tau_x(aa^*)$*;*

(2) $\tau_x(a) = 0$ *implies* $a = 0$*;*

(3) τ_x *is normal.*

(4) *if* $x \in L^1(\widetilde{\mathcal{M}}, \tau)$*, then* $\tau_x(1) < \infty$*, otherwise* τ_x *is semi-finite.*

It is an exercise showing that property (3) holds using the normality of the map $f \mapsto \int f \mathrm{dm}$ in the von Neumann algebra $L^\infty([0,\infty))$ and [**4**, Proposition 1.7].

Note that, as before, our assumption is that $a \in L^1(\widetilde{\mathcal{M}})+\mathcal{M}$ as opposed to $x \in L^1(\widetilde{\mathcal{M}})$, which influences property (4). To see that our assumption causes τ_x to be semifinite, recall that by [**4**, Proposition 2.6] $\int_0^t \mu(x)\mathrm{dm}$ is finite for all t. Now let $a \in \mathcal{M}^+$ with spectral decomposition $a = \int_0^\infty t\mathrm{d}e_t$. Since $a \in \widetilde{\mathcal{M}}$ there exists a spectral projection e_t such that $\tau(1-e_t) < \infty$. Clearly $(1-e_t)a \leq a$ and $\tau_x((1-e_t)a) = \int_o^\infty \mu((1-e_t)a)\mu(x)\mathrm{dm} \leq \int_0^\infty \|a\|\mu(1-e_t)\mu(x)\mathrm{dm} = \|a\| \int_0^{\tau(1-e_t)} \mu(x)\mathrm{dm} < \infty$, where we made use of [**6**, Lemma 2.5(i) and (vii)].

We can see that τ_x closely resembles a trace. Recall that for any positive τ-measurable operator a we have that $\tau(a) = \int \mu_t(a)\mathrm{dt}$ [**6**, Proposition 2.7]. Comparing this with $\tau_x(a) = \int \mu_t(a)\mu_t(x)\mathrm{dt}$, we may be motivated to consider τ_x as a "weighted trace" acting on $\widetilde{\mathcal{M}}$. Inspired by this, the philosophy we will follow in the development of the theory of weighted Banach function spaces will be to let τ_x take the place of the trace. As such we will define our weighted spaces using the classical spaces $L^\rho((0,\infty), m)$ as was done by Dodds, Dodds and de Pagter and τ_x as the "trace".

In order to develop our theory using τ_x, we will need to develop some of the basic properties of τ_x. We will see that for these properties τ_x mimics the well-known results for traces. For the most part, their proofs follow exactly as in the tracial case. The subadditivity of τ_x does play a small role in 2.7, but in such a way that the desired result still follows.

LEMMA 2.4. *If* $p, q \in \mathbb{P}(\mathcal{M})$ *and* $p \sim^v q$*, then* $\tau_x(p) = \tau_x(q)$*.*

PROOF. $\tau_x(p) = \tau_x(v^*v) = \tau_x(vv^*) = \tau_x(q)$. □

LEMMA 2.5. *For* $p, q \in \mathbb{P}(\mathcal{M})$*, if* $p \wedge q = 0$*, then* $\tau_x(p) \leq \tau_x(1-q)$*.*

PROOF. If $p \wedge q = 0$, then $p = 1 - p^\perp = (p\wedge q)^\perp - p^\perp = p^\perp \vee q^\perp - p^\perp \sim q^\perp - q^\perp \wedge p^\perp \leq q^\perp = 1 - q$. Therefore by Lemma 2.4 $\tau_x(p) \leq \tau_x(1-q)$. □

LEMMA 2.6. *Let* $a, b \in \widetilde{\mathcal{M}}^+$ *and* $\tau_x(b) < \infty$*, then* $\tau_x(a-b) \geq \tau_x(a) - \tau_x(b)$*. If* $b \leq a$ *then* $\tau_x(a-b) \geq \tau_x(a) - \tau_x(b) \geq 0$*.*

PROOF. From the subadditivity of τ_x we have that $\tau_x(a) = \tau_x(a - b + b) \leq \tau_x(a - b) + \tau_x(b)$. Upon rearrangement this becomes $\tau_x(a - b) \geq \tau_x(a) - \tau_x(b)$. □

LEMMA 2.7. *For* $p_1, p_2, ..., p_n \in \mathcal{M}_P$,

$$\tau_x(\vee_{i=1}^n p_i) \leq \sum_{i=1}^n \tau_x(p_i).$$

PROOF. Observe that if $\tau_x(p_i) = \infty$ for any $i \leq n$, then the result will follow trivially, so we can assume that $\tau_x(p_i) < \infty$ for all $i = 1, 2, \ldots, n$. For $n = 1$, the result is trivial. Suppose then that the lemma holds for the case $n = k$, i.e.

$$\tau_x(\vee_{i=1}^k p_i) \leq \sum_{i=1}^k \tau_x(p_i).$$

Then by the Kaplansky formula $\vee_{i=1}^{k+1} p_i - p_{k+1} \sim \vee_{i=1}^k p_i - (p_{k+1} \wedge \vee_{i=1}^k p_i)$, whence

$$\begin{aligned} \tau_x(\vee_{i=1}^{k+1} p_i) - \tau_x(p_{k+1}) &\leq \tau_x(\vee_{i=1}^{k+1} p_i - p_{k+1}) \\ &= \tau_x(\vee_{i=1}^k p_i - (p_{k+1} \wedge \vee_{i=1}^k p_i)) \\ &\leq \tau_x(\vee_{i=1}^k p_i) \end{aligned}$$

Therefore $\tau_x(\vee_{i=1}^{k+1} p_i) \leq \tau_x(\vee_{i=1}^k p_i) + \tau_x(p_{k+1})$, and so by the induction hypothesis we have that $\tau_x(\vee_{i=1}^n p_i) \leq \sum_{i=1}^n \tau_x(p_i)$ for all $n \in \mathbb{N}$. □

There are however certain properties held by traces not shared by τ_x. As an example, let $0 \leq a \in \widetilde{\mathcal{M}}$ and let $a = \int_0^\infty t de_t$ be the spectral decomposition of a. It is then well known that τ induces a measure on $[0, \infty)$ through the assignment $B \mapsto \tau(e_B)$ where B is a Borel set and e_B is the spectral projection associated with B.

Now τ_x is a positive, unitarily invariant functional on $\mathcal{M}$, but is clearly not a trace, or even a weight in the sense of von Neumann algebras.

Furthermore τ_x does not induce a measure in the same way τ does. Let $\mathcal{M} = L^\infty([0, 2], dt)$ be equipped with the usual trace and $x = f(t) = \exp(-t)$. Note that $\mu_t(x) = \exp(-t)$.

Then $\tau_x(\chi_{[0,2)}) = 1 - e^{-2} < 2(1 - e^{-1}) = \tau_x(\chi_{[0,1)}) + \tau_x(\chi_{[1,2)})$, and therefore τ_x does not induce a measure on $[0, \infty)$ in this way.

The *-algebra of τ-measurable operators play an important role in the construction of non-commutative Banach function spaces. If we are to use τ_x as a substitute for a trace, we will need to develop the concept of τ_x-"measurability". First, recall the definition of τ-measurability.

DEFINITION 2.8. A closed operator a affiliated with $\mathcal{M}$ is τ-measurable ($a \in \widetilde{\mathcal{M}}$) if and only if for all $\delta > 0$ there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset D(a)$, $\|ap\| < \infty$ and $\tau(1 - p) \leq \delta$.

We will now substitute τ_x into the role of the trace and thereby define a concept of τ_x-measurability for operators.

DEFINITION 2.9. A closed operator a affiliated with $\mathcal{M}$ is τ_x-measurable if and only if for all $\delta > 0$ there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset D(a)$, $\|ap\| < \infty$ and $\tau_x(1-p) \leq \delta$. We denote the set of all τ_x measurable operators $\widetilde{\mathcal{M}}_x$.

In order to follow the same development done by Dodds, Dodds and de Pagter, we must work within the context of $\widetilde{\mathcal{M}}_x$. We would, therefore, like to know whether $\widetilde{\mathcal{M}}_x$ is a topological *-algebra. We would also like to know how $\widetilde{\mathcal{M}}_x$ is related to $\widetilde{\mathcal{M}}$. That $\widetilde{\mathcal{M}}_x$ is a *-algebra and the relationship it has with $\widetilde{\mathcal{M}}$ can be shown simultaneously.

THEOREM 2.10. *An operator a is τ-measurable if and only if a is τ_x-measurable.*

PROOF. We first assume $a \in \widetilde{\mathcal{M}}$ and then let $\delta > 0$. Since $0 \leq x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ we have that $\int_0^t \mu_s(x)ds < \infty$ for every $t > 0$ [**4**, Proposition 2.6]. As a function of t, the quantity $\int_0^t \mu_s(x)ds$ is continuous, increasing and $\int_0^0 \mu_s(x)ds = 0$. Therefore we can find an $\epsilon > 0$ such that $\int_0^\epsilon \mu_s(x)ds \leq \delta$.

Since $a \in \widetilde{\mathcal{M}}$ there exists $p \in \mathbb{P}(\mathcal{M})$ such that $p\mathcal{H} \subset D(a)$, $\|ap\| < \infty$ and $\tau(1-p) \leq \epsilon$. Then

$$\begin{aligned} \tau_x(1-p) &= \int_0^{\tau(1-p)} \mu_s(x)ds \\ &\leq \int_0^\epsilon \mu_s(x)ds \\ &\leq \delta. \end{aligned}$$

It follows that a is τ_x measurable and hence $\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}}_x$.

Now assume that $a \in \widetilde{\mathcal{M}}_x$ and again let $\delta > 0$. For each $n \in \mathbb{N}$ we let $\epsilon_n = \frac{1}{2^n}$ and then let p_n be a projection such that $p_n\mathcal{H} \subset D(a)$, $\|ap_n\| < \infty$ and $\tau_x(1-p_n) \leq \frac{1}{2^n}$. Clearly $\tau_x(1-p_n) \downarrow 0$ as $n \to \infty$.

If there exists an n such that $\tau(1-p_n) = 0$, then by the faithfulness of τ, $p_n = 1$ and $a \in \mathcal{M} \subset \widetilde{\mathcal{M}}$. In light of this we may assume that $\tau(1-p_n) > 0$ for all $n \in \mathbb{N}$.

The measure $\nu = \mu_t(x)dt$ is mutually absolutely continuous to the Lebesgue measure on the interval $[0, t_x)$, when $t_x = \inf\{t > 0 : \mu_t(x) = 0\}$ (where $t_x = \infty$ is allowed as a possibility), as was pointed out in the proof of [**10**, Theorem 3.7]. Since we assumed that $\tau(1-p_n) > 0$ for each $n \in \mathbb{N}$, it follows that $\int_0^{\tau(1-p_n)} \mu_t(x)dt > 0$ for all $n \in \mathbb{N}$.

We will aim for a contradiction by supposing that $\tau(1-p_n) \geq k > 0$ for some $k > 0$ and all $n \in \mathbb{N}$. Then

$$\tau_x(1-p_n) = \int_0^{\tau(1-p_n)} \mu_s(x)ds \geq \int_0^k \mu_s(x)ds = \beta > 0$$

for all $n > 0$. Thus we have a contradiction. It follows that $\inf\{\tau(1-p_n) : n \in \mathbb{N}\} = 0$. So there exists an n such that $p_n\mathcal{H} \subset D(a)$, $\|ap_n\| < \infty$ and
$\tau(1-p_n) \leq \delta$. Therefore we have that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_x$. □

Since $\widetilde{\mathcal{M}}_x = \widetilde{\mathcal{M}}$, at least setwise, and the concepts of measurability is equivalent, we can use either $\widetilde{\mathcal{M}}$ or $\widetilde{\mathcal{M}}_x$. In general we will refer to $\widetilde{\mathcal{M}}$, but may refer to $\widetilde{\mathcal{M}}_x$ if doing so is more convenient.

Since $\widetilde{\mathcal{M}}$ is a *-algebra it follows immediately from this result that $\widetilde{\mathcal{M}}_x$ is also a *-algebra. From the previous result we know that $\widetilde{\mathcal{M}}_x$ would be a topological *-algebra under the topology of convergence in measure with respect to τ, which we will denote γ_{cm}. As before though, if we are to develop the theory of weighted noncommutative Banach spaces in the spirit of Dodds, Dodds and de Pagter, the appropriate topology should be a topology in convergence in measure with respect to τ_x. Our first task will be to show that such a topology can be defined and does indeed exist and then to show its relationship to the topology of convergence in measure with respect to τ.

Recall that γ_{cm} has a neighbourhood basis of zero consisting of sets of the form

$$\mathcal{N}(\epsilon,\delta) = \{a \in \widetilde{\mathcal{M}} : \exists p \in \mathbb{P}(M), p\mathcal{H} \subset D(a), \|ap\| \leq \epsilon, \tau(1-p) \leq \delta\}$$

for $\epsilon, \delta > 0$. If we follow our prescription of replacing τ with τ_x, we can similarly define a family of sets $\{\mathcal{N}_x(\epsilon,\delta) : \epsilon > 0, \delta > 0\}$ where

$$\mathcal{N}_x(\epsilon,\delta) = \{a \in \widetilde{\mathcal{M}} : \exists p \in \mathbb{P}(M), p\mathcal{H} \subset D(a), \|ap\| \leq \epsilon, \tau_x(1-p) \leq \delta\}.$$

In the hopes that this family of sets will be a neighbourhood basis of zero for a vector topology, we will denote this family of sets by $\gamma_{cm,x}$. We can show that $\gamma_{cm,x}$ is a vector topology using the same approach used when showing γ_{cm} is a vector topology.

LEMMA 2.11. *For all $\epsilon, \epsilon_1, \epsilon_2, \delta, \delta_1, \delta_2 > 0$ and $\lambda \in \mathbb{C}$, we have*

(1) $\mathcal{N}_x(|\lambda|\epsilon,\delta) = \lambda\mathcal{N}_x(\epsilon,\delta)$
(2) $\epsilon_1 < \epsilon_2$ *and* $\delta_1 < \delta_2$ *implies that* $\mathcal{N}_x(\epsilon_1,\delta_1) \subset \mathcal{N}_x(\epsilon_2,\delta_2)$
(3) $\mathcal{N}_x(\epsilon_1 \wedge \epsilon_2, \delta_1 \wedge \delta_2) \subset \mathcal{N}_x(\epsilon_1,\delta_1) \cap \mathcal{N}_x(\epsilon_2,\delta_2)$
(4) $\mathcal{N}_x(\epsilon_1,\delta_1) + \mathcal{N}_x(\epsilon_2,\delta_2) \subset \mathcal{N}_x(\epsilon_1+\epsilon_2, \delta_1+\delta_2)$.

PROOF. All the above claims follow fairly directly from the definition of the neighbourhoods $\mathcal{N}_x(\epsilon,\delta)$. For the convenience of the reader we will include the proofs of each claim.
(1) Let $\epsilon, \delta > 0$ and let $\lambda \in \mathbb{C}$. First suppose that $a \in \mathcal{N}_x(|\lambda|\epsilon,\delta)$, that is to say that there exist a $p \in \mathbb{P}(\mathcal{M})$ such that $p\mathcal{H} \subset D(a)$, $\|ap\| \leq |\lambda|\epsilon$ and $\tau_x(1-p) \leq \delta$. Clearly we then have that $\|\frac{1}{\lambda}ap\| \leq \epsilon$, which shows that $\frac{1}{\lambda}a \in \mathcal{N}_x(\epsilon,\delta)$, or equivalently $a \in \lambda\mathcal{N}_x(\epsilon,\delta)$.

Now if $a \in \lambda\mathcal{N}_x(\epsilon,\delta)$, then $a = \lambda b$ for some $b \in \mathcal{N}_x(\epsilon,\delta)$. Then there exist $p \in \mathbb{P}(\mathcal{M})$ such that $p\mathcal{H} \subset D(b)$, $\|bp\| \leq \epsilon$ and $\tau_x(1-p) \leq \delta$. Then we clearly also have that $\|\lambda bp\| \leq |\lambda|\epsilon$, i.e.

$\|ap\| \le |\lambda|\epsilon$, and therefore $a \in \mathcal{N}_x(|\lambda|\epsilon, \delta)$.

(2) Suppose $\epsilon_1 < \epsilon_2$ and $\delta_1 < \delta_2$, then it is clear that if for an operator a such that there exist $p \in \mathbb{P}(\mathcal{M})$ such that $p\mathcal{H} \subset D(a)$, $\|ap\| \le \epsilon_1$ and $\tau_x(1-p) \le \delta_1$, that it is the case that $\|ap\| \le \epsilon_2$ and $\tau_x(1-p) \le \delta_2$, from which the desired result follows.

(3) Is a direct consequence of (2).

(4) Let $a \in \mathcal{N}_x(\epsilon_1, \delta_1)$ and $b \in \mathcal{N}_x(\epsilon_2, \delta_2)$ be given. Then there exists a projection $p_1 \in \mathcal{M}$ such that $p_1\mathcal{H} \subset D(a)$, $\|ap_1\| \le \epsilon_1$, and $\tau_x(1-p_1) \le \delta_1$. Similarly there exists a projection $p_2 \in \mathcal{M}$ such that $p_2\mathcal{H} \subset D(b)$, $\|bp_2\| \le \epsilon_2$, and $\tau_x(1-p_2) \le \delta_2$. Let $p = p_1 \wedge p_2$. Then

$$p\mathcal{H} \subset p_1\mathcal{H} \cap p_2\mathcal{H} \subset D(a) \cap D(b) = D(a+b),$$

$$\|(a+b)p\| \le \|ap\| + \|bp\| \le \epsilon_1 + \epsilon_2,$$

and

$$\tau_x(1-p) = \tau_x((p_1 \wedge p_2)^\perp) = \tau_x(p_1^\perp \vee p_2^\perp) \le \tau_x(1-p_1) + \tau_x(1-p_2) \le \delta_1 + \delta_2.$$

It follows that $a + b \in \mathcal{N}_x(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$ from which the result follows. □

As in the tracial case showed in [**18**], it follows from the above that the family of sets $\{\mathcal{N}_x(\epsilon, \delta) : \epsilon, \delta\}$ form a neighbourhood base of zero for a vector topology on $\mathcal{M}$. We, therefore, have that $\widetilde{\mathcal{M}}_x$ is a topological *-algebra. We can be more precise though, as we will see in our next result.

THEOREM 2.12. *The family of sets $\{\mathcal{N}_x(\epsilon, \delta) : \epsilon > 0, \delta > 0\}$ is a neighbourhood basis at zero for γ_{cm}, i.e. $\gamma_{cm,x} = \gamma_{cm}$.*

PROOF. Let $\mathcal{N}(\epsilon, \delta)$ be given. Let λ_0 be given with $\lambda_0 < \int_0^\delta \mu_t(x)\mathrm{dt}$. We will show that then $\mathcal{N}_x(\epsilon, \lambda_0) \subset \mathcal{N}(\epsilon, \delta)$. Given $a \in \mathcal{N}_x(\epsilon, \lambda_0)$, there exists a projection $q \in \mathbb{P}(\mathcal{M})$ such that $q\mathcal{H} \subset D(a)$, $\|aq\| \le \epsilon$, and $\tau_x(1-q) \le \lambda_0$. Now since

$$\begin{aligned}\int_0^{\tau(1-q)} \mu_t(x)\mathrm{dt} &= \int_0^\infty \chi_{(0,\tau(1-q))}(t)\mu_t(x)\mathrm{dt} \\ &= \int_0^\infty \mu_t(1-q)\mu_t(x)\mathrm{dt} \\ &= \tau_x(1-q) \le \lambda_0 < \int_0^\delta \mu_t(x)\mathrm{dt},\end{aligned}$$

it is clear that $\tau(1-q) < \delta$. Therefore $a \in \mathcal{N}(\epsilon, \delta)$, whence $\mathcal{N}_x(\epsilon, \lambda_0) \subset \mathcal{N}(\epsilon, \delta)$ as claimed.

Let $\mathcal{N}_x(\epsilon, \delta)$ be given. Select $\lambda > 0$ such that $\int_0^\lambda \mu_t(x)\mathrm{dt} < \delta$. We show that then $\mathcal{N}(\epsilon, \lambda) \subset \mathcal{N}_x(\epsilon, \delta)$. Let $a \in \mathcal{N}(\epsilon, \lambda)$ be given. Then there exists a projection $q \in \mathbb{P}(\mathcal{M})$ such that $q\mathcal{H} \subset D(a)$, $\|aq\| \le \epsilon$, $\tau(1-q) \le \lambda$ and therefore $\tau_x(1-q) = \int_0^{\tau(1-q)} \mu_t(x)\mathrm{dt} \le \int_0^\lambda \mu_t(x)\mathrm{dt} < \delta$. Therefore $a \in \mathcal{N}_x(\epsilon, \delta)$, or equivalently $\mathcal{N}(\epsilon, \lambda) \subset \mathcal{N}_x(\epsilon, \delta)$ as claimed. □

So we have that $\widetilde{\mathcal{M}}_x$ is not only the same *-algebra as $\widetilde{\mathcal{M}}$ but that equipped with the topology $\gamma_{cm,x}$ it is the same topological *-algebra as $\widetilde{\mathcal{M}}$. In light of this, we can truly interchange $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_x$ depending on convenience. This also shows that we are justified in working in the context of $\widetilde{\mathcal{M}}$ both in an algebraic sense and in a topological sense. In light of this, we will always refer to $\widetilde{\mathcal{M}}$ rather than $\widetilde{\mathcal{M}}_x$, unless referring to $\widetilde{\mathcal{M}}_x$ is more beneficial.

The proof of the following lemma 2.13 is nearly identical to the proofs of the equivalent statement in the tracial case, but we will present the proof nonetheless for the benefit of the reader.

Lemma 2.13. *$a \in \mathcal{N}_x(\epsilon, \delta)$ if and only if $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq \delta$.*

Proof. Let $a \in \widetilde{\mathcal{M}}$ be given with $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq \delta$ for some $\epsilon, \delta > 0$. Take $p = e_{(\epsilon,\infty)}(|a|)$. Then $p\mathcal{H} \subset D(a)$, $\||a|p\| \leq \epsilon$, and $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq \delta$. Hence $a \in \mathcal{N}_x(\epsilon, \delta)$.

Conversely suppose $a \in \mathcal{N}_x(\epsilon, \delta)$. Then of course $|a| \in \mathcal{N}_x(\epsilon, \delta)$. So there exists a projection $p \in \mathbb{P}(\mathcal{M})$ such that $p\mathcal{H} \subset D(|a|)$, $\||a|p\| \leq \epsilon$, and $\tau_x(1-p) \leq \delta$. Now for all $\xi \in p\mathcal{H}$, $\||a|\xi\| = \||a|p\xi\| \leq \epsilon\|\xi\|$. But for all $\xi \in e_{(\epsilon,\infty)}(|a|)\mathcal{H}$ we have that $\||a|\xi\| = \||a|e_{(\epsilon,\infty)}(|a|)\xi\| \geq \epsilon\|\xi\|$. It follows that $p \wedge e_{(\epsilon,\infty)}(|a|) = 0$ and therefore that $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq \tau_x(1-p) \leq \delta$. □

Section Notes. Proposition 2.3 was proved in [**10**] by Labuschagne and Majewski. Theorem 2.10 was proved in [**15**] by the author and Theorem 2.12 was proved in [**12**] by Labuschagne and the author.

2.3. Weighted non-commutative decreasing rearrangements

The theory of noncommutative Banach function spaces, as developed by Dodds, Dodds and de Pagter [**3**], starts by using the results from Fack and Kosaki in [**6**] for the generalised singular value function and using said function as a noncommutative decreasing rearrangement. The noncommutative Banach function spaces are then defined in terms of the generalised singular value function. Our task in this section is to define a weighted analogue of the generalised singular value function, which we will call the weighted noncommutative decreasing rearrangement, and explore the foundational properties of this function. We will find that the function enjoys many of the same properties held by the generalised singular value function. We will also show the relationship between our weighted noncommutative decreasing rearrangement and the generalised singular value function. Specifically that the classical decreasing rearrangement of the generalised singular value function with respect to the measure given by $\nu = \mu_t(x)\mathrm{d}t$, one would end up with the weighted noncommutative decreasing rearrangement. To conclude this section we will apply this result to demonstrate it's usefulness.

Recall that in the tracial case the noncommutative decreasing rearrangement (or generalised singular value function), can be defined in either of the two equivalent ways.

DEFINITION 2.14. [**6**, Definition 2.1 and Proposition 2.2] For $a \in \widetilde{\mathcal{M}}$, the noncommutative decreasing rearrangement is defined to be the function $\mu(a) : [0,\infty) \mapsto [0,\infty] : t \mapsto \mu(a)$ by either of the following expressions

(1) $\mu_t(a) = \inf\{\|ae\| : e \in \mathbb{P}(\mathcal{M}), \tau(1-e) \le t\}$.

(2) $\mu_t(a) = \inf\{s \ge 0 : d_s(a) \le t\}$,

where $d(a) : [0,\infty) \mapsto [0,\infty] : s \mapsto d_s(a)$ is the function given by $\tau(1-e_s) = \tau(e_{(s,\infty)})$ for the spectral projection e_s of $|a|$. The map $d(a)$ is the noncommutative distribution function of a.

Note that in the above definition, if we treat the trace as an integral and the spectral projections as characteristic functions of a Borel set, (2) is a direct noncommutative analogue of the classical decreasing rearrangement.

Since we have a concept of τ_x-measurability and a topology of convergence in measure with respect to τ_x, which happens to be equivalent to the corresponding tracial concepts, we are justified in defining a weighted noncommutative decreasing rearrangement (or weighted generalised singular value function). We will use (1) in Definition 2.14 as the tracial blueprint from which we will define the weighted analogue, i.e. we will replace the trace in Definition 2.14 with τ_x.

DEFINITION 2.15. For $a \in \widetilde{\mathcal{M}}$, we define the function $\mu(a,x) : [0,\infty) \mapsto [0,\infty] : t \mapsto \mu_t(a,x)$ by

$$\mu_t(a,x) = \inf\{\|ae\| : e \in \mathbb{P}(\mathcal{M}), \tau_x(1-e) \le t\}.$$

We also define $d(a,x) : [0,\infty) \mapsto [0,\infty] : t \mapsto d_t(a,x)$ by

$$d_t(a,x) = \tau_x(e_{(t,\infty)}(|a|)).$$

The fundamental properties of the generalised singular value function, $\mu(a)$, was demonstrated by Fack and Kosaki in [**6**]. If we are to use the weighted noncommutative decreasing rearrangement $\mu(a,x)$ in place of $\mu(a)$ we will need to demonstrate analogues weighted results, which will be done in the remainder of this section. We found that for these results the proofs in our setting differs from the proofs in [**6**] only superficially. For the sake of the reader, we will nevertheless provide the proofs.

LEMMA 2.16. *$d(a,x)$ is decreasing.*

PROOF. Let $t_1 \le t_2$. Then $e_{(t_1,\infty)}(|a|) \ge e_{(t_2,\infty)}(|a|)$.

By the monotonicity of τ_x, it follows that

$$d_{t_1}(a,x) = \tau_x(e_{(t_1,\infty)}(|a|)) \ge \tau_x(e_{(t_2,\infty)}(|a|)) = d_{t_2}(a,x).$$

□

LEMMA 2.17. *$d(a,x)$ is right continuous.*

PROOF. Suppose $t_i \downarrow t$. Then $e_{t_i}(|a|) \downarrow_{SO} e_t(|a|)$ in the strong operator topology. So $e_{(t_i,\infty)}(|a|) \uparrow e_{(t,\infty)}(|a|)$. Since τ_x is normal, it follows that
$d_{t_i}(a,x) = \tau_x(e_{(t_i,\infty)}(|a|)) \uparrow \tau_x(e_{(t,\infty)}(|a|)) = d_t(a,x)$. □

PROPOSITION 2.18. *For $a \in \widetilde{\mathcal{M}}$, we have that $\mu_t(a,x) = \inf\{s \geq 0 : d_s(a,x) \leq t\}$. Moreover $\mu_t(a,x)$ is non-increasing and right continuous. Also $d_{\mu_t(a,x)}(a,x) \leq t$ for all $t \geq 0$.*

PROOF. Let $t \geq 0$ and $\alpha_t = \inf\{s \geq 0 : d_s(a,x) \leq t\}$. Let (s_n) be a sequence in $\{s \geq 0 : d_s(a,x) \leq t\}$ such that $s_n \downarrow \alpha_t$. Then $d_{s_n}(a,x) \leq t$ for all $n \in \mathbb{N}$. Since $d(a,x)$ is right continuous it follows that $d_{\alpha_t}(a,x) \leq t$.

Let $a = v|a|$ be the polar decomposition of a.

The inequality $d_{\alpha_t}(a,x) \leq t$ can be written as $\tau_x(1 - e_{(0,\alpha_t)}(|a|)) \leq t$. Since $\|ae_{(0,\alpha_t)}(|a|)\| = \||a|e_{(0,\alpha_t)}(|a|)\| \leq \alpha_t$, it follows that $\mu_t(a,x) \leq \alpha_t$.

It follows from the definition of $\mu(a,x)$ that for an arbitrary $\epsilon > 0$ there is a $p \in \mathcal{M}_P$ such that $\tau_x(1-p) \leq t$ and $\|ap\| < \mu_t(a,x) + \epsilon$.

If $\xi \in pH \cap e_{(\mu_t(a,x)+\epsilon,\infty)}(|a|)H$ with $\|\xi\| = 1$, then $\langle a^*a\xi,\xi\rangle \geq (\mu_t(a,x)+\epsilon)^2$ since $\xi \in e_{(\mu_t(a,x)+\epsilon,\infty)}(|a|)H$.

We also have that$\langle a^*a\xi,\xi\rangle < (\mu_t(a,x)+\epsilon)^2$ since $\xi \in pH$ and $\|ap\| < \mu_t(a,x)+\epsilon$, a contradiction and therefore $pH \cap e_{(\mu_t(a,x)+\epsilon,\infty)}(|a|)H = \{0\}$. Equivalently $p \wedge e_{(\mu_t(a,x)+\epsilon,\infty)}(|a|) = 0$. So $\tau_x(e_{(\mu_t(a,x)+\epsilon,\infty)}(|a|)) \leq \tau_x(1-p) \leq t$ by lemma 2.5. But then $d_{\mu_t(a,x)+\epsilon}(a,x) \leq t$ and therefore $\alpha_t \leq \mu_t(a,x)+\epsilon$. But since ϵ was arbitrary, $\alpha \leq \mu_t(a,x)$. Hence $\mu_t(a,x) = \alpha_t$.

To prove the entirety of the proposition, we now need to show that $\mu_t(a,x)$ is right continuous. Note that the first paragraph in this proof therefore shows that

$$d_{\mu_t(a,x)}(a,x) \leq t \tag{2.1}$$

for all $t \geq 0$.

If $t_1 \leq t_2$, then

$$\{\|ae\| : e \in \mathcal{M}_P, \tau_x(1-e) \leq t_1\} \subset \{\|ae\| : e \in \mathcal{M}_P, \tau_x(1-e) \leq t_2\}$$

and hence $\mu_{t_2}(a,x) \leq \mu_{t_1}(a,x)$.

Suppose $\mu(a,x)$ is not right continuous at some $t \in [0,\infty)$. Then, since $\mu(a,x)$ is decreasing, there exist $c > 0$ such that $\mu_t(a,x) > c \geq \mu_{t+\epsilon}(a,x)$ for all $\epsilon > 0$.

Then $d_c(a,x) \leq d_{\mu_{t+\epsilon}(a,x)}(a,x) \leq t+\epsilon$. Since ϵ was arbitrary, $d_c(a,x) \leq t$, and it follows that $\mu_t(a,x) \leq c$, a contradiction. □

LEMMA 2.19. *For each $t \geq 0$, let $R_t(x)$ be the set of all τ-measurable operators b such that $\tau_x(\mathrm{supp}(|b|)) \leq t$. For $a \in \widetilde{\mathcal{M}}$, we have that*

$$\mu_t(a, x) = \inf\{\|a - b\| : b \in R_t(x)\}.$$

PROOF. Let $t \geq 0$ and as before let $a = u|a|$ be the polar decomposition of a. Now let where $\alpha = \mu_t(a, x)$ and let b be the operator

$$b = u \int_{\alpha}^{\infty} s de_s(|a|).$$

Then $\|a - b\| \leq \alpha = \mu_t(a, x)$ and $\tau_x(\mathrm{supp}(|b|)) = d_\alpha(a, x) = d_{\mu_t(a,x)}(a, x) \leq t$. It follows that $\inf\{\|a - b\| : b \in R_t(x)\} \leq \mu_t(a, x)$.

Now let $b \in R_t(x)$ and set $e = 1 - \sup(|b|)$. Then $\|ae\| = \|(a-b)e\| \leq \|a-b\|$. Since $\tau_x(1-e) \leq t$, it follows that $\mu_t(a, x) \leq \|ae\| \leq \|a - b\|$ and hence $\mu_t(a, x) \leq \inf\{\|a - b\| : b \in R_t(x)\}$.
Therefore $\mu_t(a, x) = \inf\{\|a - b\| : b \in R_t(x)\}$. □

LEMMA 2.20. *For $a \in \widetilde{\mathcal{M}}$ and $x \in L^1(\widetilde{\mathcal{M}})$,*

(1) $\lim_{t\downarrow 0} \mu_t(a, x) = \|a\|$.
(2) $\mu_t(a, x) = \mu_t(|a|, x) = \mu_t(a^*, x)$ *and* $\mu_t(\lambda a, x) = |\lambda|\mu_t(a, x)$.
(3) $\mu_{t+s}(a + b, x) \leq \mu_t(a, x) + \mu_s(b, x)$.
(4) $\mu_{t+s}(ab, x) \leq \mu_t(a, x)\mu_s(b, x)$.

PROOF. (1) Clearly $\|a\| \geq \mu_t(a, x)$ for all $t > 0$. Suppose that for all $\epsilon > 0$, it is the case that $\|a\| > \alpha \geq \mu_\epsilon(a, x)$. Then $d_{\mu_\epsilon(a,x)}(a, x) \leq \epsilon$. But then since $d(a, x)$ is decreasing we have that $0 \leq d_\alpha(a, x) \leq d_{\mu_\epsilon(a,x)}(a, x) \leq \epsilon$. Now since this is true for all $\epsilon > 0$, we must have that $d_\alpha(a, x) = 0$. By the faithfulness of τ_x it must be the case that $e_{(\alpha,\infty)}(|a|) = 0$. But this implies that $\||a|\| = \|a\| \leq \alpha$, a contradiction. It follows that $\lim_{t\downarrow 0} \mu_t(a, x) = \|a\|$.

(2) That $\mu(a, x) = \mu(|a|, x)$ is clear from lemma 2.18. So to show the second part of the claim let $T = v|a|$ be the polar decomposition of a. Then $aa^* = va^*av^*$. Hence aa^* restricted to the range of a is unitarily equivalent to a^*a restricted to the range of a^*. It follows that $|a^*| = v|a|v^*$. By the uniqueness of the spectral decomposition it follows that $e_{(t,\infty)}(|a^*|) = ve_{(t,\infty)}(|a|)v^*$ for all $t > 0$. From this we have that for $t > 0$

$$\begin{aligned}
\tau_x(e_{(t,\infty)}(|a^*|)) &= \tau_x(ve_{(t,\infty)}(|a|)v^*) \\
&= \tau_x(ve_{(t,\infty)}(|a|)(ve_{(t,\infty)}(|a|))^*) \\
&= \tau_x((ve_{(t,\infty)}(|a|))^*(ve_{(t,\infty)}(|a|))) \\
&= \tau_x(e_{(t,\infty)}(|a|)v^*ve_{(t,\infty)}(|a|)) \\
&= \tau_x(e_{(t,\infty)}(|a|)).
\end{aligned}$$

Therefore $d(a,x) = d(a^*,x)$ which proves that $\mu(a,x) = \mu(a^*,x)$.

(3) Let $\epsilon > 0$. From lemma 2.19, we can find $f, g \in \widetilde{\mathcal{M}}$ such that

$$\|a - f\| \leq \mu_t(a,x) + \epsilon, \quad \tau_x(\text{supp}(|f|)) \leq t$$

$$\|b - g\| \leq \mu_s(b,x) + \epsilon, \quad \tau_x(\text{supp}(|g|)) \leq s.$$

We have that

$$\|(a+b) - (f+g)\| \leq \|a - f\| + \|b - g\| \leq \mu_t(a,x) + \mu_s(b,x) + 2\epsilon.$$

Further we have that

$$\begin{aligned} \tau_x(\text{supp}(|f+g|)) &= \tau_x(\text{supp}(|f|) \vee \text{supp}(|g|)) \\ &\leq \tau_x(\text{supp}(|f|)) + \tau_x(\text{supp}(|g|)) \\ &\leq t + s \end{aligned}$$

It follows that $\mu_{t+s}(a+b,x) \leq \mu_t(a,x) + \mu_s(b,x) + 2\epsilon$, and since ϵ was arbitrary, that $\mu_{t+s}(a+b,x) \leq \mu_t(a,x) + \mu_s(b,x)$.

(4) Let $\epsilon > 0$ and $f, g \in \widetilde{\mathcal{M}}$ be as in (3). Then set $h = (a-f)g + fb$. We have

$$\begin{aligned} \|ab - h\| &= \|ab - (a-f)g - fb\| \\ &= \|(a-f)(b-g)\| \\ &\leq \|(a-f)\|\|(b-g)\| \\ &\leq (\mu_t(a,x) + \epsilon)(\mu_s(b,x) + \epsilon). \end{aligned}$$

Furthermore we also have the following:

$$\begin{aligned} \text{supp}(|h|) &\leq \text{supp}(|(a-f)g|) \vee \text{supp}(|fb|) \\ \text{supp}(|(a-f)g|) &\leq \text{supp}(|g|) \\ \text{supp}(|fb|) \sim \text{supp}(|b^* f^*|) &\leq \text{supp}(|f^*|) = \text{supp}(|f|). \end{aligned}$$

It follows from this that $\tau_x(\text{supp}(|h|)) \leq ts$. Therefore $\mu_{t+s}(ab,x) \leq (\mu_t(a,x)+\epsilon)(\mu_s(b,x)+\epsilon)$. Since ϵ was arbitrary, we have that $\mu_{t+s}(ab,x) \leq \mu_t(a,x)\mu_s(b,x)$.

□

LEMMA 2.21. *For* $a \in \widetilde{\mathcal{M}}$, $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq t \Leftrightarrow \mu_t(a,x) \leq \epsilon$.

PROOF. If $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq t$ then $\mu_t(a,x) = \inf\{s > 0 : \tau_x(e_{(s,\infty)}(|a|) \leq t\} \leq \epsilon$. Conversely let $\mu_t(a,x) = \inf\{s > 0 : \tau_x(e_{(s,\infty)}(|a|) \leq t\} \leq \epsilon$, then by Lemma 2.16 and Proposition 2.18, we have that $\tau_x(e_{(\epsilon,\infty)}(|a|)) \leq \tau_x(e_{(\mu_t(a,x),\infty)}(|a|)) \leq t$. □

LEMMA 2.22. *$a \in \mathcal{N}_x(\epsilon, t)$ if and only if $\mu_t(a, x) \leq \epsilon$.*

PROOF. $a \in \widetilde{\mathcal{M}}_x(\epsilon, t) \Leftrightarrow \tau_x(e_{(\epsilon,\infty)}(|a|)) \leq t \Leftrightarrow \mu_t(a, x) \leq \epsilon$. □

LEMMA 2.23. *Let (a_i) be a sequence in $\widetilde{\mathcal{M}}$ and $a \in \widetilde{\mathcal{M}}$. Then the following are equivalent:*

(1) *$a_i \to a$ in $\widetilde{\mathcal{M}}$*
(2) *$\mu_t(a_i - a, x) \to 0$ for all $t > 0$*
(3) *$\mu_t(a_i - a) \to 0$ for all $t > 0$.*

PROOF. That (1) is equivalent to (3) is a well-known result and proved in [**6**, Lemma 3.1]. The proof of the equivalence of (1) and (2) is identical to that of [**6**, Lemma 3.1], though for the sake of the reader we reiterate it here in somewhat more detail. The proof essentially follows by observing that for any net $(a_i) \subset \widetilde{\mathcal{M}}$, we have that

$$\begin{aligned} a_i \to a \text{ in } \widetilde{\mathcal{M}} &\Leftrightarrow \text{for all } \epsilon, t > 0 \text{ there exists an } i_0 \in I \text{ such that } a_i - a \in \mathcal{N}_x(\epsilon, t) \text{ whenever } i \geq i_0 \\ &\Leftrightarrow \text{for all } \epsilon, t > 0 \text{ there exists an } i \in I \text{ such that } \mu_t(a_i - a, x) \leq \epsilon \text{ whenever } i \geq i_0 \\ &\Leftrightarrow \mu_t(a_i - a, x) \to 0. \end{aligned}$$

□

LEMMA 2.24. *Let $a \in \widetilde{\mathcal{M}}$, $x \geq 0$ and $\tau(x) < \infty$. Then $\mu_t(a, x) = 0$ when $t > \tau(x)$.*

PROOF. Let $|a| = \int_0^\infty s e_{(s,\infty)}(|a|)$ be the spectral decomposition of $|a|$ and $t > \tau(x) = \int_0^\infty \mu_s(x)\mathrm{ds}$. Then $\mu_t(a, x) = \inf\{s \geq 0 : \tau_x(e_{(s,\infty)}(|a|)) \leq t\}$. But we have that $\tau_x(e_{(s,\infty)}(|a|)) \leq \tau_x(\mathbf{1}) = \tau(x) \leq t$ for all $s \geq 0$. Therefore $\mu_t(a, x) = 0$. □

We can see that $\mu(a, x)$ is well behaved and displays all of the fundamental properties we would expect from a noncommutative decreasing rearrangement. In fact, this section was very much a demonstration that the fundamental theory developed in the first section 2 of [**6**] still holds in our case. We remind the reader that in the case where $\mu(x) = \mu(\mathbf{1})$, that $\tau_x = \tau$, and then trivially $\mu(a) = \mu(a, x)$. In light of this, we can say that this section has been a generalisation of the analogous tracial results in [**6**]. This both justifies naming it the weighted noncommutative decreasing rearrangement and shows that $\mu(a, x)$ is an excellent weighted analogue to substitute for $\mu(a)$ in the further development of the theory.

For the final result in this section, we will show that we can find yet another way of computing $\mu_t(a, x)$. This result was proved in [**15**, Theorem 3.7].

THEOREM 2.25. *Let $a \in \widetilde{\mathcal{M}}$ and consider $\mu_t(a) \in L^0([0, \infty), \nu)$, where ν is the measure given by $\mu_t(x)dt$. Then the decreasing rearrangement of $\mu(a)$ with respect to ν is $\mu(a, x)$.*

PROOF. We denote the distribution function of a function $f \in L^0([0,\infty),\nu)$ with respect to ν by $d(f,\nu)$ and the decreasing rearrangement with respect to ν by $\mu(f,\nu)$. We will calculate $\mu(a,x)$ and $\mu_t(\mu(a),\nu)$ using the prescriptions in Proposition 2.18 and [**2**, Definition 2.1.5] respectively.

It is well known that $d_t(\mu(|a|)) = d_t(|a|)$. Since $\mu(a)$ is decreasing and therefore $\chi_{(t,\infty)}(\mu(a)) = \chi_{[0,d_t(\mu(|a|)))}$, it follows that

$$\begin{aligned} d_t(\mu(a),\nu) &= \nu(\chi_{(t,\infty)}(\mu(a))) \\ &= \int_0^\infty \chi_{(t,\infty)}(\mu(a))(s)\mu_s(x)ds \\ &= \int_0^\infty \chi_{[0,d_t(\mu(|a|)))}(s)\mu_s(x)ds \\ &= \int_0^{d_t(\mu(|a|))} \mu_s(x)ds \\ &= \int_0^{d_t(|a|)} \mu_s(x)ds \\ &= \int_0^\infty \chi_{[0,\tau(e_{(t,\infty)}(|a|))}(s)\mu_s(x)ds \\ &= d_t(a,x). \end{aligned}$$

By using Proposition 2.18 to calculate $\mu_t(a,x)$ and [**2**, Definition 2.1.5] to calculate $\mu(\mu(a),\nu)$, it is clear that $\mu_t(a,x) = \mu(\mu(a),\nu)$. □

Immediately from this, we have the following corollary.

COROLLARY 2.26. *For* $a \in \widetilde{\mathcal{M}}$,

$$\int_0^\infty \mu_t(a,x)dt = \int_0^\infty \mu_t(a)\mu_t(x)dt,$$

i.e.

$$\int_0^\infty \mu_t(a,x)dt = \tau_x(a).$$

PROOF. Since $x \in L^1(\widetilde{\mathcal{M}})+\mathcal{M}$, we have that $\int_0^n \mu_t(x)dt < \infty$ for all $n \in \mathbb{N}$. It follows that ν is a σ-finite measure. The corollary then follows from Theorem 2.25 and [**2**, Chapter 2, Proposition 1.8] with $p = 1$. □

Note that the corollary shows that $L^1_x(\widetilde{\mathcal{M}}) = \{a \in \widetilde{\mathcal{M}} : \mu(a,x) \in L^1(\widetilde{\mathcal{M}})\}$.

In light of Theorem 2.25 it is then no surprise that $\mu(a,x)$ displays all the characteristics of a decreasing rearrangement since, in fact, it is one. Theorem 2.25 will be fundamental in many of our further results and we will make use of it often. The usefulness of this result is that it gives a connection between the weighted noncommutative decreasing rearrangement and the generalised singular value function through the classical theory. A strategy that we will often employ is applying a known result for classical decreasing rearrangements to $\mu(a)$ and to show that a similar result holds for $\mu(a,x)$ than for $\mu(a)$. To demonstrate an application of this strategy we end this section with the following lemma.

LEMMA 2.27. *Let* $a, b \in \widetilde{\mathcal{M}}$, *then* $\int \mu(ab,x)\,dm \leq \int \mu(a,x)\mu(b,x)\,dm$.

PROOF. By [**6**, Theorem 4.2] we have that $\int_0^t \mu(ab)\mathrm{dm} \leq \int_0^t \mu(a)\mu(b)\mathrm{dm}$ for all $t > 0$. Then by Hardy's Lemma ([**2**, Proposition 2.3.6]), we have that $\int \mu(ab)\mu(x)\mathrm{dm} \leq \int \mu(a)\mu(b)\mu(x)\mathrm{dm}$. We can write this as

$$\begin{aligned}\int \mu(ab,x)\mathrm{dm} &= \int \mu(ab)\mu(x)\mathrm{dm} \\ &\leq \int \mu(a)\mu(b)\mu(x)\mathrm{dm} \\ &= \int \mu(\mu(a)\mu(b),\nu)\mathrm{dm} \\ &\leq \int \mu(\mu(a),\nu)\mu(\mu(b),\nu)\mathrm{dm} \\ &= \int \mu(a,x)\mu(b,x)\mathrm{dm}\end{aligned}$$

□

In this section, we have seen that we can define a weighted noncommutative decreasing rearrangement. Using Fack and Kosaki's approach in [**6**] as a guide we found that $\mu(a,x)$ displays many important properties that we would expect from a decreasing rearrangement type function, which in fact it is, as shown in Theorem 2.25. The next step will be to use $\mu(a,x)$ to construct weighted noncommutative Banach function spaces similar to the construction of Dodds, Dodds and de Pagter in [**3**] for noncommutative Banach function spaces.

SECTION NOTES. The majority of the results in this section first appeared in [**15**]. Exceptions are Lemmas 2.21, 2.22 and 2.23. Theorem 2.25 was proved by the author in [**15**].

2.4. Equivalence of weighted spaces

Having developed $\mu(a,x)$ (for a τ-measurable operator a) as a weighted noncommutative decreasing rearrangement, we can now use Dodds Dodds and de Pagter's prescription to define a type of weighted noncommutative Banach function space. The advantage of this is that we would have a well behaved decreasing rearrangement type function at the heart of our weighted spaces. Then we can use $\mu(a,x)$ to develop the theory further, as can be seen in later chapters. Our task in this section, however, is to show that the weighted spaces defined as previously alluded to are Banach spaces and to show some correspondence with the weighted noncommutative Banach function space as defined by Labuschagne and Majewski in [**10**]. In particular, we will show that our definition is equivalent to the definition in [**10**] in the sense that given a weighted space in the sense of [**10**], we can find a Banach function space on $L^0(0,\infty)$ that generates the same Banach space via our definition. Recall the definition given in [**10**].

DEFINITION 2.28. [**10**, Definition 3.6] Let $0 \leq x \in L^1(\widetilde{\mathcal{M}},\tau)+\mathcal{M}$, and let ρ be a rearrangement-invariant Banach function norm on $L^0((0,\infty),\mu_t(x)dt)$. Then the weighted non-commutative Banach function space is defined as $L^\rho_x(\widetilde{\mathcal{M}},\tau) = \{a \in \widetilde{\mathcal{M}} : \mu(a) \in L^\rho((0,\infty),\mu_t(x)dt)\}$.

We now give an alternative definition. Notice that the given definition is nearly identical to the definition of a tracial noncommutative Banach function space in the sense of Dodds, Dodds and de Pagter. The difference is that we have $\mu(a,x)$ taking the place of $\mu(a)$.

DEFINITION 2.29. Let $0 \leq x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$ and ρ a rearrangement-invariant Banach function function norm on $L^0([0,\infty))$. The space $L^\rho(\widetilde{\mathcal{M}},\tau_x)$ is the space of all $a \in \widetilde{\mathcal{M}}$ such that $\mu(a,x) \in L^\rho([0,\infty))$.

In the previous section we saw that we can write $L^1_x(\widetilde{\mathcal{M}}) = \{a \in \widetilde{\mathcal{M}} : \mu(a,x) \in L^1(\widetilde{\mathcal{M}})\} = L^1(\widetilde{\mathcal{M}},\tau_x)$. As mentioned earlier our task is to show that the above definition defines a Banach space and to show its correspondence with the spaces defined by Labuschagne and Majewski. While it is possible to show, for a rearrangement-invariant Banach function norm ρ on $L^0(0,\infty)$, that $L^\rho(\widetilde{\mathcal{M}},\tau_x)$ is a Banach space using [**2**, Theorem 5.1.19], it is not necessary. Instead we will show there exists a rearrangement-invariant Banach function norm $\bar{\rho}$ on $L^0((0,\infty),\nu)$ such that $L^\rho(\widetilde{\mathcal{M}},\tau_x)$ is identical to $L^{\bar{\rho}}_x(\widetilde{\mathcal{M}})$ as Banach spaces. We do this in the following theorem and corollary. In addition we also show the converse, that $L^{\bar{\rho}}_x(\widetilde{\mathcal{M}})$ corresponds to $L^\rho(\widetilde{\mathcal{M}},\tau_x)$ for some ρ. In so doing we show that the two definitions define the same spaces.

THEOREM 2.30. *Let $L^\rho_x(\widetilde{\mathcal{M}})$ be a weighted non-commutative Banach function space. Then there exists a rearrangement invariant Banach function norm $\bar{\rho}$ in the sense of* [**2**] *on $L^0([0,\infty))$ such that $L^\rho_x(\widetilde{\mathcal{M}}) = L^{\bar{\rho}}(\widetilde{\mathcal{M}},\tau_x)$.*

Conversely, for the space $L^\rho(\widetilde{\mathcal{M}},\tau_x)$, there exists a weighted non-commutative Banach function space $L^{\bar{\rho}}_x(\widetilde{\mathcal{M}})$ such that $L^\rho(\widetilde{\mathcal{M}},\tau_x) = L^{\bar{\rho}}_x(\widetilde{\mathcal{M}})$.

PROOF. Let ρ be a Banach function norm over $L^0([0,\infty),\nu)$. We have that ρ is rearrangement invariant and that ν is nonatomic from the proof of [**10**, Theorem 3.7], and therefore $L^0([0,\infty),\nu)$ is a resonant measure space. It then follows from the Luxemburg representation theorem [**2**, Theorem 2.4.10] that there exists a rearrangement invariant Banach function norm $\bar{\rho}$ over $L^0([0,\infty))$ such that for all $f \in L^0([0,\infty),\nu)$

$$\rho(f) = \bar{\rho}(\mu(f,\nu)),$$

where $\mu(f,\nu)$ is the decreasing rearrangement of f with respect to the measure ν.

Since for all $a \in \widetilde{\mathcal{M}}$ it was shown that $\mu(a,x)$ is the decreasing rearrangement of $\mu(a)$ with respect to ν, it follows that

$$\rho(\mu(a)) = \bar{\rho}(\mu(a,x)).$$

Therefore $a \in L^{\rho}_x(\widetilde{\mathcal{M}})$ if and only if $a \in L^{\bar{\rho}}(\widetilde{\mathcal{M}}, \tau_x)$, and we also have that $\|a\|_\rho = \|a\|_{\bar{\rho}}$.

Given a Banach function norm ρ on $L^0([0,\infty))$ (in the sense of [**2**]), the second part follows from setting $\bar{\rho}(f) = \rho(\mu(f,\nu))$ for all $f \in L^0([0,\infty),\nu)^+$. We must show that $\bar{\rho}$ is a rearrangement invariant Banach function norm on $L^0([0,\infty),\nu)$ in the sense of [**2**]. Now for $0 \leq f \in L^0([0,\infty),\nu)$ we have that $f = 0$ ν-a.e. if and only if $\mu(f,\nu) = 0$ m-a.e. if and only if $\bar{\rho}(f) = \rho(\mu(f,\nu)) = 0$.

Let $0 \leq g \leq f$ ν-a.e. in $L^0([0,\infty),\nu)$. Then $\mu(g,\nu) \leq \mu(f,\nu)$ m-a.e. and therefore $\bar{\rho}(g) = \rho(\mu(g,\nu)) \leq \rho(\mu(f,\nu)) = \bar{\rho}(f)$.

Suppose we have that $0 \leq f_n \uparrow f$ ν-a.e., then $\mu(f_n,\nu) \uparrow \mu(f,\nu)$ by [**2**, Proposition 2.1.7] and therefore $\bar{\rho}(f_n) = \rho(\mu(f_n,\nu)) \uparrow \rho(\mu(f,\nu)) = \bar{\rho}(f)$.

Now suppose we have a Borel set $E \subset [0,\infty)$ with $\nu(E) < \infty$. We want to show that $\bar{\rho}(\chi_E) < \infty$. But $\bar{\rho}(\chi_E) = \rho(\mu(\chi_E)) = \rho(\chi_{[0,\nu(E))}) < \infty$ since $\nu(E) < \infty$ and therefore $\int_{[0,\nu(E))} dt < \infty$. Furthermore we have that there exists $C_E > 0$, dependent on $[0,\nu(E))$ and ρ, such that $\int_0^{\nu(E)} \mu(f,\nu)dt \leq C_E \rho(\mu(f,\nu))$ for all $0 \leq f \in L^0([0,\infty),\nu)$. Then it follows from [**2**, Theorem 2.2.2], with $g = \chi_E$, that $\int_E f d\nu \leq \int_0^{\nu(E)} \mu_t(f,\nu)dt \leq C_E \bar{\rho}(f)$.

We also need to show that $\bar{\rho}$ is subadditive. Let $f,g \in L^0([0,\infty),\nu)$. Then if we consider the commutative von Neumann algebra $L^\infty([0,\infty),\nu)$, we can conclude from [**3**, Theorem 3.4] that $|\mu(f+g,\nu) - \mu(g,\nu)| \prec\prec \mu(f,\nu)$. Then it is routine to show that $\rho(\mu(f+g,\nu)) \leq \rho(\mu(f,\nu)) + \rho(\mu(g,\nu))$, i.e. $\bar{\rho}(f+g) \leq \bar{\rho}(f) + \bar{\rho}(g)$ (see [**2**, Theorem 2.4.6]). So we have that $\bar{\rho}$ is a Banach function norm satisfying the conditions in Theorem 2.2.

Let $f,g \in L^0([0,\infty),\nu)$ such that $\mu(f,\nu) = \mu(g,\nu)$. Then since $\bar{\rho}(f) = \rho(\mu(f,\nu)) = \rho(\mu(g,\nu)) = \bar{\rho}(g)$ we have that $\bar{\rho}$ is rearrangement invariant.

For all $a \in \widetilde{\mathcal{M}}$ it follows from Theorem 2.25 that $\bar{\rho}(a) = \bar{\rho}(\mu(a)) = \rho(\mu(a,x))$, and therefore $L^\rho(\widetilde{\mathcal{M}}, \tau_x) = L^{\bar{\rho}}_x(\widetilde{\mathcal{M}})$. □

This theorem shows that the two definitions of weighted spaces are equivalent set-wise. Thus far we have not equipped $L^\rho(\widetilde{\mathcal{M}}, \tau_x)$ with a norm. Intuitively we can do so by defining $\|a\|_{\rho,x} = \rho(\mu(a,x))$. That $L^\rho(\widetilde{\mathcal{M}}, \tau_x)$ is a Banach space will now follows from 2.30 as a corollary.

COROLLARY 2.31. *Let ρ be a rearrangement-invariant Banach function norm. The space $L^\rho(\widetilde{\mathcal{M}}, \tau_x)$ is a Banach space with respect to the norm $a \mapsto \rho(\mu(a,x))$ for all $a \in L^\rho(\widetilde{\mathcal{M}}, \tau_x)$ that injects continuously into $\widetilde{\mathcal{M}}$.*

PROOF. To see that $a \mapsto \rho(\mu(a,x)) = \rho(a)$ is a norm, let $\bar{\rho}$ be the Banach function norm for which $L^{\rho}(\widetilde{\mathcal{M}}, \tau_x) = L_x^{\bar{\rho}}(\widetilde{\mathcal{M}})$ setwise. Then for all $a \in L^{\rho}(\widetilde{\mathcal{M}}, \tau_x)$, we have that $\rho(a) = \bar{\rho}(a)$. Since $\bar{\rho}$ acts as a norm on $L_x^{\bar{\rho}}(\widetilde{\mathcal{M}})$, it follows that ρ acts as a norm on $L^{\rho}(\widetilde{\mathcal{M}}, \tau_x)$. Furthermore we know that $L_x^{\bar{\rho}}(\widetilde{\mathcal{M}})$ is a Banach space with respect to the norm $a \mapsto \bar{\rho}(a)$ that injects continuously into $\widetilde{\mathcal{M}}$, and hence the corollary follows. □

We have managed to show that we can construct the same spaces as defined in [**10**] by Labuschagne and Majewski using a weighted noncommutative decreasing rearrangement. Given a weighted space $L^{\rho}(\widetilde{\mathcal{M}}, \tau_x)$, it is clear from the proofs of the previous two sections which Banach function norm $\bar{\rho}$ causes $L^{\rho}(\widetilde{\mathcal{M}}, \tau_x) = L_x^{\bar{\rho}}(\widetilde{\mathcal{M}})$. Now given a weighted space $L_x^{\rho}(\widetilde{\mathcal{M}})$ the Luxemburg representation theorem gives us a corresponding Banach function norm. If f^* and g^* denotes the classical decreasing rearrangements of functions $f \in L^{\rho}((0,\infty),\nu)$ and $g \in L^{\rho'}((0,\infty),\nu$ where ρ' is the associate norm of ρ, then $\bar{\rho}(f) = \sup\{\int f^* g^* \mathrm{dm} : \rho'(g) \leq 1\}$. While this gives us a concrete Banach function norm, it may not always be convenient in the given form. We will see in the next chapter the classical Banach spaces generating weighted noncommutative Orlicz spaces are simply the classical Orlicz spaces generated by the same function, i.e. that $L_x^{\psi}(\widetilde{\mathcal{M}}) = L^{\psi}(\widetilde{\mathcal{M}}, \tau_x)$ for an Orlicz function ψ.

In this chapter we have seen that we can use the map $\tau_x : \widetilde{\mathcal{M}} \mapsto [0,\infty] : a \mapsto \int \mu(a)\mu(x)\mathrm{dm}$ as a weighted replacement of the trace τ. In particular, we have seen that we can define τ_x-measurability and a topology $\tau_{cm,x}$, which corresponds exactly with τ-measurability and the topology of convergence in measure respectively. We have also seen that we can define a decreasing rearrangement type function $\mu(a,x)$ and use these functions to construct weighted noncommutative Banach function spaces which correspond with the weighted spaces of Labuschagne and Majewski. Essentially we have managed to construct noncommutative Banach function spaces using a map τ_x which does not have all the structure of a trace. One could view the work we have done in this chapter as a step away from a purely tracial setting. Of course, the existence of a trace is still needed since we still rely on the generalised singular value functions to define τ_x, but regardless we have shown that there exists at least one non-tracial map with which one could construct noncommutative Banach function spaces. Further exploring this line of thought is left for future projects.

SECTION NOTES. The results of this chapter were first presented by the author in [**15**].

CHAPTER 3

Weighted Orlicz Spaces

3.1. Young functions and Orlicz spaces

An important class of Banach function spaces are the Orlicz spaces. Other than being an interesting generalisation of L^p spaces, they also play an important role in probability in Banach function spaces. In this chapter we want to eventually investigate Köthe duality of Orlicz spaces. For classical Orlicz spaces the theory is well established. Chapter 4 section 5 of [**2**] is an excellent resource for the theory of classical Orlicz spaces. Köthe duality of noncommutative Banach function spaces is also well established as a consequence of the work done in [**4**]. Ultimately we will show that under certain conditions the previously mentioned results also hold for weighted noncommutative Orlicz spaces.

To start, let's define what we mean by a Young function.

Definition 3.1. A function $\Phi : [0, \infty) \mapsto [0, \infty]$ is called a Young function if

- Φ is convex and increasing with $\Phi(0) = 0$.
- Φ is continuous on $[0, b_\Phi]$ where $b_\Phi = \sup\{t \geq 0 : \Phi(t) < \infty\}$.
- Φ is neither identically zero or infinite valued on all of $[0, \infty)$. If $a_\Phi = \inf\{t \geq 0 : \Phi(t) > 0\}$, then neither $a_\Phi = 0$ or $a_\Phi = \infty$ is the case.

Let Φ be a Young function. Then Φ has the property that there exists some function $\phi : [0, \infty) \mapsto [0, \infty]$ for which $\Phi(t) = \int_{[0,t]} \phi \mathrm{dm}$. The function ϕ is a left-continuous, nondecreasing function that is infinite valued on (b_Φ, ∞), but is neither identically zero nor infinite on all of $[0, \infty)$.

For a Young function Φ we can define the conjugate Young function Φ^* by setting $\Phi^*(s) = \sup\{t > 0 : st - \Phi(t)\}$. The conjugate Young function is itself a Young function and $\Phi^{**} = \Phi$. Furthermore the pair (Φ, Φ^*) satisfies the Hausdorff-Young inequality, i.e. $st \leq \Phi(t) + \Phi^*(s)$ for all $s, t \geq 0$. Equality is reached when $s = \phi(t)$ (or $t = \psi(s)$ where $\Phi^*(s) = \int_{[0,s]} \psi \mathrm{dm}$).

To build intuition we give a few examples of some Young functions.

- $\Phi(t) = t^p$.
- $\Phi(t) = \cosh(t) - 1$
- $t \log(t + 1)$

Our interest in Young functions are in their use to build Banach function spaces. Generally Orlicz spaces are defined over more general measure spaces, but for our purposes we only need to define them over $(\mathbb{R}^+, \Sigma, \sigma)$ where σ could either be the Lebesgue measure, or the measure ν given by $\mu_t(x)$dt.

Definition 3.2. An Orlicz space associated with a Young function Φ, denoted by $L^\Phi(\mathbb{R}^+, \sigma)$, is the space of all functions $f \in L^0(\mathbb{R}^+, \sigma)$ such that there exists a constant $\alpha > 0$ such that $\phi(\alpha|f|) \in L^1(\mathbb{R}^+, \sigma)$.

$L^\Phi(\mathbb{R}^+, \sigma)$ is a Banach function space under the Luxemburg-Nakano norm given by:

$$\|f\|_\Phi = \inf\{\lambda > 0 : \|\Phi(\lambda^{-1}|f|)\|_1 \leq 1\}.$$

The Orlicz norm, given by

$$\|f\|_\Phi^O = \sup\{\|fg\|_1 : g \in L^{\Phi^*}(\mathbb{R}^+, \sigma), \|g\|_{\Phi^*} \leq 1\}.$$

Note that the Orlicz norm is the associate norm of the Luxemburg-Nakano norm generated by Φ^*. In fact the associate space of $L^{\Phi^*}(\mathbb{R}^+, \sigma)$ is in fact $L^\Phi(\mathbb{R}^+, \sigma)$ equipped with the Orlicz norm. Furthermore the Orlicz norm and the Nakano-Luxemburg norm are equivalent for σ-finite measures, so in general we can say that $L^\Phi(\mathbb{R}^+, \sigma)$ is the associate space of $L^{\Phi^*}((\mathbb{R}^+, \sigma)$.

Given these classical Orlicz spaces we follow the usual procedure to construct non-commutative Orlicz spaces. In particular we can define the following three types of non-commutative Orlicz spaces:

(1) $L^\Phi(\mathcal{M}) = \{a \in \mathcal{M} : \mu(a) \in L^\Phi([0, \infty))\}$;
(2) $L_x^\Phi(\mathcal{M}) = \{a \in \mathcal{M} : \mu(a) \in L^\Phi([0, \infty), \mu_t(x)dt)\}$;
(3) $L^\Phi(\mathcal{M}, \tau_x) = \{a \in \mathcal{M} : \mu(a, x) \in L^\Phi([0, \infty))\}$.

Each of these spaces can be equipped with the Luxemburg-Nakano norm and the Orlicz norm. It is of interest to note that we can write the Orlicz norm on $L_x^\Phi(\mathcal{M})$ as $\|a\|_\Phi^O = \sup\{\tau(|ab|) : b \in L^{\Phi^*}(\widetilde{\mathcal{M}}), \|b\|_\Phi \leq 1\}$.

As can be seen in the book by Bennett and Sharpley, it is well known that in the classical setting the associate space of $L^\Phi(\mathbb{R}^+, \sigma)$ is $L^{\Phi^*}(\mathbb{R}^+, \sigma)$ under the Orlicz norm. It then follows by the work done by Dodds, Dodds and de Pagter in [**4**] that the Köthe dual of $L^\Phi(\widetilde{\mathcal{M}})$ is $L^{\Phi^*}(\widetilde{\mathcal{M}})$ under the Orlicz norm.

Weighted Orlicz spaces are significant in the motivation for the importance of weighted noncommutative Banach function spaces. In particular Labuschagne and Majewski showed the following result.

THEOREM 3.3. [**10**, Theorem 3.8] *Given some* $x \in (\mathcal{M}_{*,0}^{+,1})$, $f \in \widetilde{\mathcal{M}}$ *is a regular quantum observable in the sense that* $t \to \int_0^\infty \exp(t\mu_s(f))\mu_s(x)\,ds$ *exists in a neighbourhood of 0 if and only if* $f \in L_x^{\cosh -1}(\widetilde{\mathcal{M}})$. *Moreover* $\{f \in L_x^{\cosh -1} : \tau(fx) = 0\}$ *is a closed subspace of* $L_x^{\cosh -1}(\widetilde{\mathcal{M}})$.

We have two main objectives in this chapter. In the previous chapter we managed to show that the two approaches to defining weighted noncommutative Banach function spaces are equivalent. We will take this further by showing that for any Young function Φ, we have that $L_x^{\Phi}(\widetilde{\mathcal{M}}) = L^{\Phi}(\widetilde{\mathcal{M}}, \tau_x)$. This of course implies that in fact the regular quantum observables correspond to a subspace of $L^{\cosh -1}(\widetilde{\mathcal{M}}, \tau_x)$.

Our second objective will be explore the Köthe duals of weighted noncommutative Orlicz spaces. We will manage to show that at least when Φ is such that, when written as $\Phi(t) = \int_{[0,t]} \phi \mathrm{dm}$, the function ϕ is continuous, we can then recover the tracial results. More specifically we will show that for such Young functions we have that the Köthe dual of $L_x^{\Phi^*}(\widetilde{\mathcal{M}})$ is $L_x^{\Phi}(\widetilde{\mathcal{M}})$ under a norm that is equivalent to the Luxemburg-Nakano and Orlicz norm on $L_x^{\Phi}(\widetilde{\mathcal{M}})$.

3.2. Equivalence of weighted Orlicz spaces

In the previous chapter we managed to show the equivalence of weighted noncommutative Banach function spaces. While the Luxemburg representation theorem does provide us with a Banach function norm under which the two approaches define the same spaces, we can go further when dealing with Orlicz spaces. In particular we will show that a Young function Φ will generate the same space regardless of which approach is used. That is to say $L_x^{\Phi}(\widetilde{\mathcal{M}}) = L^{\Phi}(\widetilde{\mathcal{M}}, \tau_x)$.

The results in Lemma 3.4 and Corollary 3.5 below can be arrived at by an application of Theorem 2.25. We believe, however, that the arguments given in [**15**] are more revealing of the underlying nature of these results. As such we will first give the proofs as they appeared in [**15**], after which we will show how one can arrive at the these results by simply applying Theorem 2.25.

LEMMA 3.4. *Let* Φ *be an increasing continuous function on* $[0, \infty)$ *that possibly takes on infinite values on an interval* (b_Φ, ∞) *and* $a \in \widetilde{\mathcal{M}}$ *such that* $\Phi(|a|) \in \widetilde{\mathcal{M}}$. *Set* $\Phi(\infty) = \lim_{t\to\infty} \Phi(t)$, *which may be infinity, to extend* Φ *to a function on* $[0, \infty]$. *Then* $\Phi(\mu_t(a, x)) = \mu_t(\Phi(|a|), x)$.

PROOF. As was noted by [**6**, Remark 2.3.1] in their context, the proof of Lemma 2.18 easily adapts to show that when computing weighted decreasing rearrangements according to the prescription given in Definition 2.15, we may restrict our attention to the commutative von Neumann algebra generated by the spectral projections of $|a|$.

Using the Borel functional calculus, we can follow the proof of [**10**, Lemma 2.1] verbatim to show that $\Phi(\||a|e\|) = \|\Phi(|a|)e\|$, where values of infinity are allowed. Therefore

$$\begin{aligned}\mu_t(\Phi(a), x) &= \inf\{\|\Phi(|a|)e\| : e \in \mathcal{M}_P, \tau_x(1-e) \leq t\}\\ &= \inf\{\Phi(\||a|e\|) : e \in \mathcal{M}_P, \tau_x(1-e) \leq t\}\\ &= \Phi(\inf\{\||a|e\| : e \in \mathcal{M}_P, \tau_x(1-e) \leq t\})\\ &= \Phi(\mu_t(a, x)).\end{aligned}$$

□

Since an Orlicz function satisfies the above conditions, we immediately have the following corollary.

Corollary 3.5. *Let Φ be an Orlicz function on $[0, \infty)$ and $a \in \widetilde{\mathcal{M}}$ such that $\Phi(|a|) \in \widetilde{\mathcal{M}}$. Set $\Phi(\infty) = \infty$ to extend Φ to a function on $[0, \infty]$. Then $\Phi(\mu_t(a, x)) = \mu_t(\Phi(|a|), x)$.*

We now show how one can arrive at these results by an application of Theorem 2.25. In the next lemma we will say that a function F commutes with noncommutative decreasing rearrangements if for any semi-finite von Neumann algebra $\mathcal{M}$ equipped with a nfs trace τ, we have that $\mu(F(a)) = F(\mu(a))$ for all $a \in \widetilde{\mathcal{M}}$.

Lemma 3.6. *Let F be any function that commutes with noncommutative decreasing rearrangements. Then $\mu(F(a), x) = F(\mu(a, x))$.*

Proof. Let $a \in \widetilde{\mathcal{M}}$. Then $\mu(F(a)) = F(\mu(a))$. Now recall that, with ν being the measure given by $\mu_t(x)\mathrm{dt}$, we have that $\mu(a, x) = \mu(\mu(a), \nu)$ for all $a \in \widetilde{\mathcal{M}}$. It follows that $\mu(F(a), x) = \mu(\mu(F(a)), \nu) = \mu(F(\mu(a)), \nu) = F(\mu(\mu(a), \nu)) = F(\mu(a, x))$, where we used the fact that F commutes with noncommutative decreasing rearrangements twice. □

Theorem 3.7. *Let Φ be a Young function. Then $L^\Phi_x(\widetilde{\mathcal{M}}) = L^\Phi(\widetilde{\mathcal{M}}, \tau_x)$.*

Proof. Recall that

$$L^\Phi_x(\widetilde{\mathcal{M}}) = \{a \in \widetilde{\mathcal{M}} : \mu(a) \in L^\Phi([0, \infty), \nu)\}.$$

Therefore $a \in L^\Phi_x(\widetilde{\mathcal{M}})$ if and only if $\mu(a) \in L^\Phi([0,\infty),\nu)$ if and only if there exist an $\lambda > 0$ such that $\Phi(\lambda\mu(a)) \in L^1([0,\infty),\nu)$. So we have that

$$\begin{aligned} a \in L^\Phi_x(\widetilde{\mathcal{M}}) &\Leftrightarrow \text{ there exists } \lambda > 0 \text{ such that } \int_0^\infty \Phi(\lambda\mu_t(a))\mu_t(x)dt < \infty \\ &\Leftrightarrow \text{ there exists } \lambda > 0 \text{ such that } \int_0^\infty \mu_t(\Phi(\lambda a))\mu_t(x)dt < \infty \\ &\Leftrightarrow \text{ there exists } \lambda > 0 \text{ such that } \int_0^\infty \mu_t(\Phi(\lambda a), x)dt < \infty \\ &\Leftrightarrow \text{ there exists } \lambda > 0 \text{ such that } \int_0^\infty \Phi(\lambda\mu_t(a, x))dt < \infty \\ &\Leftrightarrow a \in L^\Phi(\widetilde{\mathcal{M}}, \tau_x). \end{aligned}$$

Similarly for the norm of the weighted Orlicz space we have that

$$\begin{aligned} \|a\|_{x,\Phi} &= \inf\{\lambda > 0 : \|\Phi(\mu(a)/\lambda)\|_{1,\nu} \leq 1\} \\ &= \inf\{\lambda > 0 : \int_0^\infty \Phi(\mu(a)/\lambda)\mu_t(x)dt \leq 1\} \\ &= \inf\{\lambda > 0 : \int_0^\infty (\Phi(\mu_t(a, x)/\lambda))dt \leq 1\} \\ &= \|a\|_{\tau_x,\Phi}. \end{aligned}$$

□

From Theorem 3.7 it follows that $L^p_x(\widetilde{\mathcal{M}}) = L^p(\widetilde{\mathcal{M}}, \tau_x)$ for $1 \leq p \leq \infty$ and that $L^{\cosh -1}_x(\widetilde{\mathcal{M}}) = L^{\cosh -1}_x(\widetilde{\mathcal{M}}, \tau_x)$. Recall that $L^{\cosh -1}_x(\widetilde{\mathcal{M}})$ coincides with the set of regular observables, and therefore so does $L^{\cosh -1}(\widetilde{\mathcal{M}}, \tau_x)$. Since $L^\Phi(\widetilde{\mathcal{M}}, \tau_x) = L^\Phi_x(\widetilde{\mathcal{M}})$ for all Young functions Φ, we will always denote a weighted Orlicz space generated by Φ by $L^\Phi_x(\widetilde{\mathcal{M}})$. We will do so even when, directly or indirectly, using ideas from the construction of weighted spaces in the sense of Definition 2.29.

The prescription in defining weighted noncommutative Banach function spaces seen in Definition 2.29 is not only a useful alternative, in the sense that it has a decreasing rearrangement in $\mu(a, x)$ at it centre in a way more reminiscent of tracial noncommutative decreasing rearrangements. We have also seen that our formulation agrees on a large class of weighted spaces. More precisely we have seen that any Young function Φ will generate the same weighted space regardless of the approach one uses.

Section Notes. Section 3.1 consists of known results. Lemma 3.4 and Lemma 3.5 appeared in [**15**] as a single lemma. Theorem 3.7 was also first proved in [**15**] by the author.

3.3. The Köthe dual of weighted noncommutative Orlicz spaces

In this section we will investigate Köthe duality of weighted noncommutative Orlicz spaces. A significant challenge when developing Köthe duality for weighted noncommutative Banach function spaces is the inability to extend τ_x to a linear map on $L^1(\widetilde{\mathcal{M}})$. In the tracial case the linearity of τ is a key component to a significant number of results leading up to a description of Köthe duality for noncommutive Banach function spaces. For this reason we must restrict ourselves to weighted noncommutative Orlicz spaces.

The differences with the tracial theory first become evident when we define three different norms in Definition 3.9. In the tracial setting these three norms reduce to two, since as seen in [**5**] and noted previously, the Orlicz norm on $L^\Phi(\widetilde{\mathcal{M}})$ can be written in the form $\|a\|_\Phi^O = \sup\{\tau(ab) : b \in L^{\Phi^*}(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*} \leq 1\}$.

The main result of this section will be a description of the Köthe duals of a class of Orlicz spaces. Unfortunately the subadditivity of τ_x forces us to only consider Young functions Φ, and say $\Phi(t) = \int_{[0,t]} \phi \mathrm{dm}$, where ϕ is continuous. For a general Young function this need not be the case, but we will need continuity in the proof of Theorem 3.17.

We should first make it clear what is meant by the Köthe dual of a weighted noncommutative Banach function space.

Definition 3.8. The non-commutative Köthe dual of a weighted noncommutative Banach function space $L_x^\rho(\widetilde{\mathcal{M}})$ is $L_x^{\rho'}(\widetilde{\mathcal{M}}) = \{a \in \widetilde{\mathcal{M}} : ab \in L_x^1(\widetilde{\mathcal{M}}), b \in L_x^\rho(\widetilde{\mathcal{M}})\}$
with norm $\|a\|_{\rho'} = \sup\{\tau_x(ab) : b \in L_x^{\rho'}(\widetilde{\mathcal{M}}), \|b\|_\rho \leq 1\}$.

Here we followed our general philosophy by replacing the trace with the map τ_x. As mentioned before, we can define three norms on $L_x^\Phi(\widetilde{\mathcal{M}})$.

Definition 3.9. We define the following quantities on $\widetilde{\mathcal{M}}$

$$\|a\|_{\Phi,x} = \inf\{\lambda > 0 : \int (\Phi(\mu(a,x)/\lambda))\mathrm{dm} \leq 1\} \tag{3.1}$$

$$|||a|||_{\Phi,x}^O = \sup\{\tau_x(ab) : b \in L_x^{\Phi^*}(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*,x} \leq 1\} \tag{3.2}$$

$$\|a\|_{\Phi,x}^O = \|\mu(a,x)\|_\Phi^O = \sup\{\int \mu(a,x) g \mathrm{dm} : g \in L^{\Phi^*}(\mathbb{R}^+, \mathrm{dm}), \|g\|_{\Phi^*} \leq 1\} \tag{3.3}$$

Clearly 3.1 and 3.3 are norms on $L_x^\Phi(\widetilde{\mathcal{M}})$. In fact they are the Banach function norms on $\widetilde{\mathcal{M}}$ generated by the classical Luxemburg-Nakano norm and Orlicz norm respectively. Furthermore from [**2**, Theorem 4.8.14] we have that $\|a\|_{\Phi,x} \leq \|a\|_{\Phi,x}^O \leq 2\|a\|_{\Phi,x}$ and therefore these are equivalent norms on $L_x^\Phi(\widetilde{\mathcal{M}})$.

The question remains as to the relationship 3.2 has with the other norms. If τ_x is a trace it is clearly equal to the Orlicz norm, but a general answer to this is unknown at the moment. The demonstration in [**4**] that the Orlicz norm can be written using the trace fails for τ_x thanks to the subadditve nature of τ_x. As such whether 3.2 and 3.3 are in fact the same norms are unknown at the time of writing.

It is important to at least establish that 3.2 is at least a norm. This is not difficult and is done in the next proposition. The proof is essentially a translation of the proof of [**2**, Theorem 1.2.2] into our setting.

PROPOSITION 3.10. *The quantity* $|||a|||^O_{\Phi,x}$ *defines a norm on* $L^\Phi_x(\widetilde{\mathcal{M}})$.

PROOF. Clearly if $a = 0$ then $\tau_x(ab) = 0$ for all $b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*,x} \leq 1$ and therefore $|||a|||^O_{\Phi,x} = 0$. Now suppose $|||a|||^O_{\Phi,x} = 0$. This implies that $\tau_x(ab) = 0$ for all $b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*,x} \leq 1$. Consider $supp(a)$, the support projection of a. If $supp(a) = 0$ then $b = 0$. Otherwise, since τ is semifinite, we can choose a nonzero projection $p \leq supp(a)$ such that $\tau(p) < \infty$. Since p is a nonzero subprojection of the support of a, then $ap = 0$ if and only if $a = 0$, so we will proceed by showing that $ap = 0$.

To make sense of the following step we make the observation that $\nu([0,\tau_x(p))) = \int_0^{\tau(p)} \mu(x)\mathrm{dm} < \infty$ since $0 \leq x \in L^1(\widetilde{\mathcal{M}}) + \mathcal{M}$. We make the further observation that $\tau_x(p) = \int_0^{\tau(p)} \mu(x)\mathrm{dm}$ and is therefore finite. Then we have that $\mu(p,x) = \mu(\mu(p),\nu) = \mu(\chi_{[0,\tau(p))},\nu) = \chi_{[0,\nu([0,\tau(p)))} = \chi_{[0,\tau_x(p))}$. This ensures that $\|p\|_{\Phi,x} = \|\mu(p,x)\|_\Phi = \|\chi_{[0,\tau_x(p))}\|_\Phi < \infty$ since the Luxemburg-Nakano norm is a Banach function norm. Now let $b = \frac{p}{\alpha}$ where $\alpha = \|p\|_{\Phi,x}$. Then clearly $\|b\|_{\Phi,x} = 1$. This implies that $0 = \tau_x(ab) = \frac{1}{\alpha}\tau_x(ap) = 0$ which implies that $ap = 0$. As observed earlier this shows that $ap = 0$, which is a contradiction unless $b = 0$.

From the properties of τ_x it is clear that $\tau_x(\lambda ab) = |\lambda|\tau_x(ab)$, and therefore $|||\lambda a|||^O_{\Phi,x} = |\lambda||||a|||^O_{\Phi,x}$.

For $a_1, b_1 \in L^\Phi_x(\widetilde{\mathcal{M}})$ we have that $\tau_x((a_1 + a_2)b) = \tau_x(a_1 b + a_2 b) \leq \tau_x(a_1 b) + \tau_x(a_2 b)$. It follows that $|||a_1 + a_2|||^O_{\Phi,x} \leq |||a_1|||^O_{\Phi,x} + |||a_2|||^O_{\Phi,x}$, which completes the proof.

□

Having established that 3.2 defines a norm, we can continue to the main objective of this section. We will from now refer to the norm defined in 3.2 as the weighted noncommutative Orlicz norm. In order to describe Köthe duality for weighted noncommutative Orlicz spaces, we will need to show the equivalence of the following statements.

(1) $|||a|||^O_{\Phi,x} < \infty$
(2) $\|a\|^O_{\Phi,x} = \|\mu(a,x)\|^O_\Phi < \infty$

(3) $a \in L^{\Phi}_x(\widetilde{\mathcal{M}})$, i.e. there exists a $\lambda > 0$ such that $\tau_x(\Phi(\lambda a)) < \infty$.

PROPOSITION 3.11. *For the above statements,* $3 \Rightarrow 2 \Rightarrow 1$.

PROOF. The implication $3 \Rightarrow 2$ is clear from the existing theory of classical Orlicz spaces, so we only need to show $2 \Rightarrow 1$.

We therefore suppose that $\sup\{\int \mu(a,x) g \mathrm{dm} : g \in L^{\Phi^*}(\mathbb{R}^+, \mathrm{dm}), \|g\|_{\Phi^*} \leq 1\} < \infty$. Then we have that

$$\begin{aligned}
|||a|||^{O}_{\Phi,x} &= \sup\{\tau_x(ab) : b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*} \leq 1\} \\
&= \sup\{\int \mu(ab,x)\mathrm{dm} : b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*} \leq 1\} \\
&\leq \sup\{\int \mu(a,x)\mu(b,x)\mathrm{dm} : b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*} \leq 1\} \\
&\leq \sup\{\int \mu(a,x) g \mathrm{dm} : g \in L^{\Phi^*}(\mathbb{R}^+, \mathrm{dm}), \|g\|_{\Phi^*} \leq 1\} \\
&< \infty.
\end{aligned}$$

□

We will spend the rest of this chapter on showing that under certain conditions the implication $1 \Rightarrow 3$ holds, which will show that under those conditions the three statements are equivalent. This will be enough to show that the Köthe dual of $L^{\Phi^*}_x(\widetilde{\mathcal{M}})$ is $L^{\Phi}_x(\widetilde{\mathcal{M}})$ under the norm $|||\cdot|||^{O}_{\Phi,x}$. To do this we will prove a number of weighted noncommutative analogues of some classical results. Our general strategy will be to follow the approach laid out in [**2**, Section 4.8].

LEMMA 3.12. *Let* $a \in L^{\Phi}_x(\widetilde{\mathcal{M}})$ *be non-zero.*

(1) *Then* $\tau_x(\Phi((\|a\|_{\Phi,x})^{-1}|a|)) \leq 1$.

(2) *If* $\|a\|_{\Phi,x} \leq 1$, *then* $\tau(\Phi(|a|)) \leq \|a\|_{\Phi,x}$.

(3) *If* $\|a\|_{\Phi,x} > 1$, *then* $\tau(\Phi(|a|)) \geq \|a\|_{\Phi,x}$.

Therefore $\tau_x(\Phi(|a|)) \leq 1$ *if and only if* $\|a\|_{\Phi,x} \leq 1$.

PROOF. (1) Let $(\epsilon_n) \subset (\|a\|_{\Phi,x}, \infty)$ be a sequence decreasing to $\|a\|_{\Phi,x}$. By the monotone convergence theorem we have that $\int \Phi(\epsilon_n^{-1}\mu(a,x))\mathrm{dm} \uparrow \int \Phi((\|a\|_{\Phi,x})^{-1}\mu(a,x))\mathrm{dm}$. It follows from the definition of $\|\mu(a,x)\|_{\Phi,x}$, and $\|a\|_{\Phi,x} = \|\mu(a,x)\|_{\Phi,x}$, that $\int \Phi(\epsilon_n^{-1}\mu(a,x))\mathrm{dm} \leq 1$ and therefore that $\int \Phi((\|a\|_{\Phi,x})^{-1}\mu(a,x))\mathrm{dm} \leq 1$. Now observe that due to Corollary 3.5 we have that $\int \Phi((\|a\|_{\Phi,x})^{-1}\mu(a,x))\mathrm{dm} = \int \Phi\big(\mu((\|a\|_{\Phi,x})^{-1}a,x)\big)\mathrm{dm} = \int \mu\big(\Phi((\|a\|_{\Phi,x})^{-1}a),x\big)\mathrm{dm} = \tau_x(\Phi((\|a\|_{\Phi,x})^{-1}|a|))$. So we can restate our previous inequality as $\tau_x(\Phi((\|a\|_{\Phi,x})^{-1}|a|)) \leq 1$.

(2) Suppose that $\|a\|_{\Phi,x} \leq 1$. From (1) we know that $\tau_x(\Phi((\|a\|_{\Phi,x})^{-1}|a|)) \leq 1$, and therefore $\Phi((\|a\|_{\Phi,x})^{-1}|a|) \in \widetilde{\mathcal{M}}_x = \widetilde{\mathcal{M}}$. Recall that for any $t \geq 0$ and $\gamma \geq 1$ we have that $\Phi(\gamma t) \geq \gamma\Phi(t)$.

From this we have that $\Phi((\|a\|_{\Phi,x})^{-1}t) \geq (\|a\|_{\Phi,x})^{-1}\Phi(t)$, and therefore that $0 \leq (\|a\|_{\Phi,x})^{-1}\Phi(|a|) \leq \Phi((\|a\|_{\Phi,x})^{-1}|a|)$ from which it follows that $(\|a\|_{\Phi,x})^{-1}\Phi(|a|) \in \widetilde{\mathcal{M}}$ and $\tau_x((\|a\|_{\Phi,x})^{-1}\Phi(|a|)) \leq \tau_x(\Phi((\|a\|_{\Phi,x})^{-1}|a|))$.

(3) Suppose that $\|a\|_{\Phi,x} > 1$. If $\tau_x(\Phi(|a|)) = \infty$, then the inequality follows trivially. So suppose that $\tau_x(\Phi(|a|)) < \infty$. Then $\Phi(|a\|) \in \widetilde{\mathcal{M}}$. Since $\|a\|_{\Phi,x} > 1$, there exist an $\epsilon > 0$ such that $\|a\|_{\Phi,x} - \epsilon > 1$. Now since $\|a\|_{\Phi,x} - \epsilon < \|a\|_{\Phi,x}$, we can conclude from the definition of the Luxemburg-Nakano norm that $\int \Phi((\|a\|_{\Phi,x} - \epsilon)^{-1}\mu(a,x))\mathrm{dm} > 1$. Using the same argument we used in part (1), we can write this as $\tau_x(\Phi((\|a\|_{\Phi,x} - \epsilon)^{-1}|a|)) = \int \Phi((\|a\|_{\Phi,x} - \epsilon)^{-1}\mu(a,x))\mathrm{dm} \leq 1$. The convexity of Φ ensures that for any $t \geq 0$ and $\gamma \leq 1$, we have that $\Phi(\gamma t) \leq \gamma\Phi(t)$. From this we have that $\Phi((\|a\|_{\Phi,x} - \epsilon)^{-1}t) \leq (\|a\|_{\Phi,x} - \epsilon)^{-1}\Phi(t)$, and arguing similarly as in (2) we have that $1 \leq \tau_x(\Phi((\|a\|_{\Phi,x} - \epsilon)^{-1}|a|) \leq (\|a\|_{\Phi,x} - \epsilon)^{-1}\tau_x(\Phi(|a|))$. Now $\epsilon > 0$ can be made arbitrarily small, so we therefore have that $\tau(\Phi(|a|)) \geq \|a\|_{\Phi,x}$. □

Following from the previous lemma, we have that $|||a|||^{O}_{\Phi,x} = \sup\{\tau_x(ab) : b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}}), \tau_x(\Phi^*(|a|)) \leq 1\}$.

One of our objectives is to show that the weighted noncommutative Orlicz norm is equivalent to the other norms defined on $L^{\Phi}_x(\widetilde{\mathcal{M}})$. Half of this is achieved in the following proposition.

Proposition 3.13. *For any $a \in L^{\Phi}_x(\widetilde{\mathcal{M}})$ we have that $|||a|||^{O}_{\Phi,x} \leq 2\|a\|_{\Phi}$.*

Proof. Let $a \in L^{\Phi}_x(\widetilde{\mathcal{M}})$ be non-zero. Recall that, if Φ^* is the conjugate Orlicz function of Φ, then for $u, v > 0$ we have that $uv = \Phi(u) + \Phi^*(v)$. Now let $b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}})$ and $\alpha \geq 0$ be given. Then

$$\begin{aligned}
\tau_x(\alpha|a|b) &= \int \mu(|a|b,x)\mathrm{dm} \\
&\leq \int \alpha\mu(a,x)\mu(b,x)\mathrm{dm} \\
&\leq \int \Phi(\alpha\mu(a,x))\mathrm{dm} + \int \Phi^*(\mu(b,x))\mathrm{dm} \\
&= \tau_x(\Phi(\alpha|a|)) + \tau_x(\Phi^*(|b|)).
\end{aligned}$$

Now set $\alpha = (\|a\|_{\Phi,x})^{-1}$, and recall that $\tau_x(\Phi((\|a\|_{\Phi,x})^{-1}|a|)) \leq 1$. We showed in Lemma 3.12 that $\|b\|_{\Phi^*,x} \leq 1$ is equivalent to $\tau_x(\Phi^*(b)) \leq 1$, so if have that this is the case, then we can conclude that $\tau_x((\|a\|_{\Phi,x})^{-1}|a|b) \leq 2$ and therefore that $(\|a\|_{\Phi,x})^{-1}\|a\|^{O}_{\Phi,x} \leq 2$, which proves the proposition. □

In the proof of our main result, we will rely heavily on approximating positive elements of $L^{\Phi}_x(\widetilde{\mathcal{M}})$ by increasing nets. To clarify, for an increasing net $\{a_\alpha\}$, we write $a_\alpha \uparrow a$ to indicate that

$a \in \widetilde{\mathcal{M}}$ is the minimal element that dominates all a_α. As such we will need to investigate the behaviour of the weighted noncommutative Orlicz norm with respect to increasing nets.

LEMMA 3.14. *Let $a_\alpha \uparrow a$ in $\widetilde{\mathcal{M}}$. Then $\mu_t(a_\alpha, x) \uparrow \mu_t(a, x)$ for all $t \geq 0$.*

PROOF. Let $a_\alpha \uparrow a$ in $\widetilde{\mathcal{M}}$. Then we have from [**4**, Proposition 1.7] that $\mu_t(a_\alpha) \uparrow \mu_t(a)$ for all $t \geq 0$. Now consider $f_\alpha = \mu(a_\alpha)$ to be a net in $L^\infty([0,\infty),\nu)$ that increases to $f = \mu(a)$. We can consider $\mathcal{N} = L^\infty([0,\infty),\nu)$ to be a commutative von Neumann algebra with a trace given by $\int \cdot \mathrm{d}\nu$. A second application of [**4**, Proposition 1.7] in the context of $\mathcal{N}$ gives us $\mu_t(f_\alpha,\nu) \uparrow \mu_t(f,\nu)$ for all $t \geq 0$, i.e. $\mu_t(a_\alpha, x) \uparrow \mu_t(a, x)$. □

Again we have seen the usefulness of Theorem 2.25. A modified proof of [**4**, Proposition 1.7] might still be possible even without Lemma 2.25, though it would certainly have been significantly more challenging had we not had access to Theorem 2.25. Given the continued utility of Theorem 2.25, it could be said that this theorem is perhaps the most important technology in studying weighted noncommutative Banach function spaces we have developed thus far.

We continue with our investigation of increasing nets. Ultimately we want to show how the weighted noncommutative Orlicz norm behaves with respect to such nets. Suppose that $\{a_\alpha\} \subset \{a \in \widetilde{\mathcal{M}} : ab \in L^1_x(\widetilde{\mathcal{M}}), b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}})\}$ is a net of positive elements such that $a_\alpha \uparrow a$ in $\{a \in \widetilde{\mathcal{M}} : ab \in L^1_x(\widetilde{\mathcal{M}}), b \in L^{\Phi^*}_x(\widetilde{\mathcal{M}})\}$. This implies that $\mu(a_\alpha, x) \uparrow \mu(a, x)$. From the properties of the Luxemburg-Nakano norm this implies that $\|a_\alpha\|_{\Phi,x} = \|\mu(a_\alpha, x)\|_\Phi \uparrow \|\mu(a,x)\|_\Phi = \|a\|_{\Phi,x}$. In the case where the net $\{a_\alpha\}$ increases to a in some abelian subalgebra of $\mathcal{M}$, we can show that the above property also holds for the norm $|||\cdot|||^O_{\Phi,x}$. We will need this fact later in the proof of Theorem 3.17.

LEMMA 3.15. *Let $a_1, a_2, b \in \widetilde{\mathcal{M}}$ be commuting elements such that $0 \leq a_1 \leq a_2$. Then $\tau_x(a_1 b) \leq \tau_x(a_2 b)$. Furthermore suppose $\{a_\alpha\} \subset \widetilde{\mathcal{M}}$ is a net of positive operators such that $a_\alpha \uparrow a$ and $b \in \widetilde{\mathcal{M}}$. If a and all the a_α's commute, then $\tau_x(a_\alpha b) \uparrow \tau_x(ab)$.*

PROOF. Since $0 \leq a_1 \leq a_2$ and a_1 commutes with a_2, we have that $a_1^2 \leq a_2^2$. Then we have that for every $t \geq 0$

$$\begin{aligned}
\mu_t(a_1 b, x) &= \mu_t(|a_1 b|^2, x)^{\frac{1}{2}} \\
&= \mu_t(b^* a_1^2 b, x)^{\frac{1}{2}} \\
&\leq \mu_t(b^* a_2^2 b, x)^{\frac{1}{2}} \\
&= \mu_t(|a_1 b|^2, x)^{\frac{1}{2}} \\
&= \mu_t(a_1 b, x)
\end{aligned}$$

From this we have that $\tau_x(a_1b) = \int_0^\infty \mu(a_1b, x)\mathrm{dm} \leq \int_0^\infty \mu(a_2b, x)\mathrm{dm} = \tau_x(a_2b)$, which concludes the first part of the proof.

Now suppose that $a_\alpha \uparrow a$ as described in the hypothesis. Then $a_\alpha^2 \uparrow a^2$ in $\widetilde{\mathcal{M}}$ and in turn, using [**4**, Proposition 1.3], we have that $b^*a_\alpha^2 b \uparrow b^*a^2b$ in $\widetilde{\mathcal{M}}$. We then have by Lemma 3.14 that

$$\mu_t(a_\alpha b, x) = \mu_t(b^*a_\alpha^2 b, x)^{\frac{1}{2}} \uparrow \mu_t(b^*a^2b, x)^{\frac{1}{2}} = \mu_t(ab, x).$$

Finally by the dominated convergence theorem we have that $\lim_\alpha \tau_x(a_\alpha b) = \lim_\alpha \int \mu(a_\alpha b, x)\mathrm{dm} = \int \mu(ab, x)\mathrm{dm} = \tau_x(ab)$, i.e. $\tau(a_\alpha b) \uparrow \tau_x(ab)$. □

Corollary 3.16. *Let $a_\alpha \uparrow a$ in $\widetilde{\mathcal{M}}$ with $a \in \{f \in \widetilde{\mathcal{M}} : fb \in L_x^1(\widetilde{\mathcal{M}}), b \in L_x^{\Phi^*}(\widetilde{\mathcal{M}})\}$, where all the a_α's commute with a and each other. Then we have that $|||a_i|||_{\Phi,x}^O \uparrow |||a|||_{\Phi,x}^O$.*

Proof. First, since $\tau_x(a_\alpha b) \leq \tau_x(ab)$ for all α, we have that $a_\alpha \in \{f \in \widetilde{\mathcal{M}} : fb \in L_x^1(\widetilde{\mathcal{M}}), b \in L_x^{\Phi^*}(\widetilde{\mathcal{M}})\}$. Since $\tau_x(a_{\alpha_1}b) \leq \tau_x(a_{\alpha_2}b)$ for $\alpha_1 \leq \alpha_2$ and all $b \in L_x^{\Phi^*}(\widetilde{\mathcal{M}})$ such that $a_{\alpha_1}b, a_{\alpha_2}b \in L_x^1(\widetilde{\mathcal{M}})$, we have that $|||a_{\alpha_1}|||_{\Phi,x}^O \leq |||a_{\alpha_2}|||_{\Phi,x}^O$. Furthermore since $\tau_x(a_\alpha b) \uparrow \tau_x(ab)$, we have that $|||a_\alpha|||_{\Phi,x}^O \uparrow |||a|||_{\Phi,x}^O$. □

This last corollary is a key component that we will need in the proof of the main theorem of this section. In this section thus far we have formulated what we mean by the Köthe dual of a weighted noncommutatvie Banach function space, then established three important norms on weighted noncommutative Orlcicz spaces. Following this we demonstrated some preliminary results regarding these norms, after which we investigated the behaviour of the weighted noncommutative Orlicz norm with respect to increasing nets of commuting elements. We now move on to the main theorem of this section.

Theorem 3.17. *Let Φ be an Orlicz function with $\Phi(t) = \int_0^t \phi(s)ds$ for $t \geq 0$ such that ϕ is increasing and continuous on $[0, b_\Phi)$. For $a \in \widetilde{\mathcal{M}}$ we have that $|||a|||_{\Phi,x}^O = \sup\{\tau_x(ab) : b \in L_x^{\Phi^*}(\widetilde{\mathcal{M}}), \|b\|_{\Phi^*} \leq 1\} < \infty$ if and only if $a \in L_x^\Phi(\widetilde{\mathcal{M}})$. If a meets this criterion then $\|a\|_{\Phi,x} \leq |||a|||_{\Phi,x}^O \leq 2\|a\|_\Phi$.*

Proof. We have already shown that $a \in L_x^\Phi(\widetilde{\mathcal{M}})$ implies that $|||a|||_{\Phi,x}^O < \infty$ and that $|||a|||_{\Phi,x}^O \leq 2\|a\|_{\Phi,x}$. We therefore only need to show that $|||a|||_{\Phi,x}^O < \infty$ implies $a \in L_x^\Phi(\widetilde{\mathcal{M}})$ and that $\|a\|_{\Phi,x} \leq |||a|||_{\Phi,x}^O$.

So let $a \in \widetilde{\mathcal{M}}$ be such that $|||a|||_{\Phi,x}^O < \infty$. Now since $\tau_x(ab) = \tau_x(|ab|) = \tau_x(||a|b|) = \tau_x(|a|b)$ we may assume that $a \geq 0$. Consider a maximal abelian von Neumann algebra containing the spectral projections of a, say $\mathcal{N}$. First we note that a is the limit of an increasing sequence of bounded elements of $\mathcal{N}$ of the form $a_n = a\chi_{[0,n]}(a)$. But each such element is the limit of an increasing sequence of simple functions in $\mathcal{N}$ (see [**1**, §2.2 Theorem 6]). So a is the limit of an increasing sequence of functions of the form $a_n = \sum_{k=1}^m \alpha_k e_k$ where each α_k is a positive real number and the e_k's are mutually orthogonal projections. By the previous argument we have that there exist a sequence $\{a_n\}$ such that $a_n \uparrow a$. Each a_n commutes with every member of the sequence, so we can apply 3.16 to see that $|||a_n|||_{\Phi,x}^O \uparrow |||a|||_{\Phi,x}^O$. Since $|||a|||_{\Phi,x}^O$ is finite we have that $|||a_n|||_{\Phi,x}^O$ is finite for each n. So without any loss of generality we may assume that $a = \sum_{k=1}^m \alpha_k e_k$.

Let $\{p_\alpha\}$ be a maximal set of mutually orthogonal subprojections of e_1 with finite trace. The projection $p = \sum_\alpha p_\alpha$ is clearly a subprojection of e_1. Suppose that $p \neq e_1$. Then the projection $e_1 - p \neq 0$. If $\tau(e_1 - p) < \infty$, then $\{p_\alpha\}$ is not maximal. But if $\tau(e_1 - p) = \infty$, then by the semi-finiteness of $\mathcal{M}$, there exists a non-zero projection $p' \leq e_1 - p$ such that $\tau(p') < \infty$, in which case $\{p_\alpha\}$ is again not maximal. Hence $p = e_1$, so we can write e_1 as the supremum of a net $\{e_\beta\}$ of projections of finite trace. Define the operator $a_\beta = \alpha_1 e_\beta + \sum_{k=2}^m \alpha_k e_k$. Then the net of operators a_β increases to a in the strong operator topology, and therefore $|a_\beta b|$ increases to $|ab|$. If we recall that τ_x is normal, then we get that $\tau_x(a_\beta b) \uparrow \tau_x(ab)$. So $|||a_\alpha|||_{\Phi,x}^O \uparrow |||a|||_{\Phi,x}^O$. Since $|||a|||_{\Phi,x}^O$ is finite each $|||a_\alpha|||_{\Phi,x}^O$ will also be finite. So we may assume that e_1 is finite. Following the same argument for every e_k inductively we see that we can assume that each e_k is finite. Note that if $\tau(e_k) < \infty$, then $\tau_x(e_k) = \int_0^\infty \mu(e_k)\mu(x)\mathrm{dm} = \int_0^\infty \chi_{[0,\tau(p))}\mu(x)\mathrm{dm} = \int_0^{\tau(e_k)} \mu(x)\mathrm{dm} < \infty$. So we have that $\tau_x(e_k) < \infty$ for each k.

We also rescale a so that $|||a|||_{\Phi,x}^O = 1$.

Now we want to show that $a \leq b_\Phi \sum_{k=1}^m e_k$. If $b_\Phi = \infty$, then the inequality holds trivially, so suppose $b_\Phi < \infty$. If $\alpha_k \leq b_\Phi$ for all k, then the inequality will hold, so therefore assume that there exists a k_0 such that $\alpha_{k_0} \geq b_\Phi + \epsilon$ for some $\epsilon > 0$. Let $h = (b_\Phi \tau_x(e_{k_0}))^{-1} e_{k_0}$. Now recall that if $b_\Phi < \infty$, we then have that $\Phi^*(s) \leq b_\Phi s$ for any $s > 0$. Using this we have that $\tau_x(\Phi^*(h)) \leq \tau_x(b_\Phi h) = \tau_x(b_\phi (b_\phi \tau_x(e_{k_0}))^{-1} e_{k_0}) \leq 1$. Then

$$|||a|||_{\Phi,x}^O \geq \tau_x(ah) = (b_\Phi \tau_x(e_{k_0}))^{-1} \tau_x(\alpha_{k_0} e_{k_0}) = \frac{\alpha_{k_0}}{b_\Phi} \geq \frac{b_\Phi + \epsilon}{b_\Phi} > 1.$$

This contradicts the assumption that $|||a|||_{\Phi,x}^O = 1$, and therefore the operator inequality must hold. For any $0 < \gamma < 1$ we have that $\Phi(\gamma b_\Phi) < \infty$, and therefore we have that $\tau_x(\Phi(\gamma a)) \leq \Phi(\gamma b_\Phi)\tau_x(\sum_{k=1}^m e_k) < \infty$.

Let $0 < \gamma < 1$. The function Φ as described in the hypothesis is a bounded and increasing function on $[0, \gamma b_\Phi]$. Recall that Φ can be written in the form $\Phi(s) = \int_{[0,s)} \phi \mathrm{dm}$ where ϕ is an increasing and continuous function on $[0, b_\Phi)$ by the hypothesis. Since ϕ is bounded and increasing on $[0, \gamma b_\Phi]$, we have that $\phi(\gamma a)$ is a well defined element of $\widetilde{\mathcal{M}}$ supported on $\sum_{k=1}^n e_k$. Now for $s, t \in [0, \infty)$ we have by the Hausdorff-Young inequality that $st \leq \Phi(t) + \Phi^*(s)$. If in fact $s = \phi(t)$

then by [**2**, Theorem 4.8.12] we have that

$$st = \Phi(t) + \Phi^*(s). \tag{3.4}$$

We also observe that since ϕ is bounded on $[0, \gamma b_\Phi]$, so too is $\Phi^* \circ \phi$ by 3.4 above. In particular, since $\gamma\|a\|_\infty < b_\Phi$, that $\Phi^* \circ \phi$ will be bounded on $[0, \gamma\|a\|_\infty]$. If we set $b = \phi(\gamma a)$, then we have that $\Phi^*(\phi(\gamma a)) = \Phi^*(b) \in \widetilde{\mathcal{M}}$. Now it follows from the equality condition for the Hausdorff-Young inequality mentioned earlier that $\gamma ab = \Phi(\gamma a) + \Phi^*(b)$. Clearly γab is of the form $F(a)$ for the measurable function $F : t \mapsto \gamma t\phi(\gamma t)$, which is an increasing function, continuous on $[0, b_\Phi]$ and infinite valued on (b_Φ, ∞). We now have by Lemma 3.4 that for every $t > 0$,

$$\begin{aligned}
\mu_t(F(a), x) = F(\mu_t(a, x)) &= \gamma\mu_t(a, x)\phi(\gamma\mu_t(a, x)) \\
&= \Phi(\gamma\mu_t(a, x)) + \Phi^*(\phi(\gamma\mu_t(a, x))) \\
&= \Phi(\gamma\mu_t(a, x)) + \Phi^*(\mu_t(\gamma\phi(a), x) \\
&= \Phi(\gamma\mu_t(a, x)) + \Phi^*(\mu_t(b, x)).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
\tau_x(\gamma ab) &= \int \mu(\gamma ab, x)\mathrm{dm} \\
&= \int \Phi(\gamma\mu_t(a, x))\mathrm{dm} + \int \Phi^*(\mu_t(b, x))\mathrm{dm} \\
&= \tau_x(\Phi(\gamma a)) + \tau_x(\Phi^*(b)). \qquad (3.5)
\end{aligned}$$

Now $\tau_x(\Phi(\gamma a)) < \infty$. Furthermore we have that $\phi(\gamma a) \leq r\sum_{k=1}^m e_k$ for some $r > 0$, and therefore we have that $0 \leq \gamma ab \leq r\gamma\|a\|_\infty \sum_{k=1}^m e_k$, and therefore $\tau_x(\gamma ab) \leq r\gamma\|a\|_\infty \sum_{k=1}^m \tau_x(e_k) < \infty$. We can conclude from this that $\tau_x(\Phi^*(b)) < \infty$, and so $b \in L_x^\Phi(\widetilde{\mathcal{M}})$. Furthermore $\tau_x(\gamma ab) \leq |||\gamma a|||_{\Phi,x}^O \|b\|_{\Phi^*,x} \leq \|b\|_{\Phi^*,x}$. We can therefore say that

$$\tau_x(\gamma ab) \leq \|b\|_{\Phi^*,x} \leq \max\{1, \tau_x(\Phi^*(b)))\} \leq 1 + \tau(\Phi^*(b)). \tag{3.6}$$

Combining 3.5 and 3.6 we get that $\tau_x(\Phi(\gamma a)) \leq 1$. This is enough to show that $a \in L_x^\Phi(\widetilde{\mathcal{M}})$. Furthermore, since $0 < \gamma < 1$ was arbitrary, we can conclude that $\|a\|_{\Phi,x} \leq 1$. This is enough to show that $\|a\|_{\Phi,x} \leq |||a|||_{\Phi,x}^O$.

□

It is important to note the significance of equation 3.4. Because τ_x is subadditive one encounters difficulties when investigating Köthe duality of weighted noncommutative Banach function spaces. In particular the linear nature of the trace plays an important role in the theory as developed by Dodds, Dodds and de Pagter in [**4**]. In the case of the weighted noncommutative Orlicz spaces generated by Young functions with continuous derivatives, equation 3.4 allows us to achieve some degree of linearity even when dealing with τ_x.

The implications toward Köthe duality in Theorem 3.17 should be stated explicitly.

COROLLARY 3.18. *The non-commutative Köthe dual of* $L_x^{\Phi^*}(\widetilde{\mathcal{M}})$ *is* $L_{\Phi,x}(\widetilde{\mathcal{M}})$*, i.e.* $L_x^{\Phi^*}(\widetilde{\mathcal{M}})$ *under* $|||\cdot|||_{\Phi,x}^O$.

We can say even more regarding the relationship between $|||a|||_{\Phi,x}^O$ and $\|a\|_{\Phi,x}^O$. In particular we have that $\frac{1}{2}\|a\|_{\Phi,x}^O \leq \|a\|_{\Phi,x} \leq |||a|||_{\Phi,x}^O \leq 2\|a\|_\Phi \leq 2\|a\|_{\Phi,x}^O$, i.e.

$$\frac{1}{2}\|a\|_{\Phi,x}^O \leq |||a|||_{\Phi,x}^O \leq 2\|a\|_{\Phi,x}^O.$$

This shows that $|||\cdot|||_{\Phi,x}^O$ is an equivalent norm to $\|\cdot\|_{\Phi,x}^O$ on $L_x^\Phi(\widetilde{\mathcal{M}})$. It may be possible that these two norms are in fact identical, but this is not known in general yet, apart from some trivial cases, for example when τ_x is a trace.

SECTION NOTES. The results obtained in this section have not been published previously.

CHAPTER 4

Interpolation Spaces of Weighted Banach Function Spaces

We now turn our attention to the real interpolation theory of weighted Banach function spaces. Having a well-developed interpolation theory is a useful tool in showing that weighted noncommutative Banach function spaces allow sensible dynamics. Our main tool will be the K-method of interpolation of an abstract Banach couple (X_0, X_1), in particular, the theory of monotone interpolation for Banach couples. In our case, the Banach couple will be given by $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$. Our aim will be to describe the exact monotone interpolation spaces of this Banach couple. We will see that every space $L^\rho_x(\widetilde{\mathcal{M}})$, where ρ is a monotone Riesz-Fischer norm, is an exact interpolation space and conversely every exact monotone interpolation space is generated by a monotone Riesz-Fischer norm. A follow-up question that we do not answer here is whether every exact interpolation space is automatically monotone. In the tracial case, this question was answered by Dodds, Dodds and de Pagter in [**5**] where they showed that all such spaces are indeed exact. Their paper [**5**] relied heavily on results from [**4**]. As in the case of Köthe duality, the subadditivity of τ_x prevents us from using the same approach as was taken in [**5**]. The results in this section are, however, enough to ensure that we obtain sensible quantum dynamics.

To ease the reading of this text we will at times cite known results from real interpolation theory, but leave out the proofs. The book by Bennet and Sharpley [**2**, Chapter 5] can serve as an excellent resource on the K-method of interpolation.

To start, we give the definition of a Banach couple.

Definition 4.1. [**2**, Definition 3.1.1] If X_0 and X_1 are Banach spaces, then they are referred to as a Banach couple, denoted (X_0, X_1), if there exists a Hausdorff topological vector space X in which X_0 and X_1 are continuously embedded.

Since both X_0 and X_1 exist in some larger Hausdorff topological vector space, we can make sense of the spaces $X_0 + X_1$ and $X_0 \cap X_1$. The space $X_0 + X_1$ is a Banach space under the norm given by

$$\|a\|_{X_0+X_1} = \inf\{\|a_0\|_{X_0} + \|a_1\|_{X_1} : a = a_0 + a_1, a_0 \in X_0, a_1 \in X_1\},$$

and the space $X_0 \cap X_1$ is a Banach space under the norm given by

$$\|a\|_{X_0\cap X_1} = \max\{\|a\|_{X_0}, \|a\|_{X_1}\}.$$

To describe the objective of interpolation theory we will need to provide some definitions before proceeding.

DEFINITION 4.2. A bounded linear map T on $X_0 + X_1$ is called admissible if T restricted to either X_0 or X_1 is a bounded linear map. The admissible norm of such a map is then given by $\|T\|_{\mathcal{A}} = \max\{\|T\|_0, \|T\|_1\}$.

DEFINITION 4.3. Let (X_0, X_1) be a Banach couple. A Banach space X is called an *intermediate space* if $X_0 \cap X_1$ embeds continuously into X and X embeds continuously into $X_0 + X_1$. This can be written symbolically as

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$$

X is called an *interpolation space* of the Banach couple (X_0, X_1) if every admissible map on the couple (X_0, X_1) induces a bounded map on X. If additionally the norm on X of any admissible map T is majorised by $\|T\|_{\mathcal{A}}$, we say that the space X is an exact interpolation space.

For our investigation we need to confirm that $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ is in fact a Banach couple. To see that this is the case we remind the reader that the space $(L^1 + L^\infty)_x(\widetilde{\mathcal{M}})$ is described by all $a \in \widetilde{\mathcal{M}}$ such that $\int_0^1 \mu_s(a, x)\, ds < \infty$. It then becomes clear that the spaces $L^1_x(\mathcal{M})$ and $\mathcal{M}$ both contractively map into $(L^1 + L^\infty)_x(\widetilde{\mathcal{M}})$.

Having established that $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ is indeed a Banach couple, we can now start investigating their interpolation theory. Our main tool will be the K-method. The key component of this method, and the origin of the method's name, is the so-called K-functional. For a Banach couple (X_0, X_1) the K-functional is defined for all $a \in X_0 + X_1$ as

$$K(a, t) = \inf\{\|a_1\|_{x_0} + t\|a_2\|_{X_1} : t > 0\}$$

where the infimum is taken over all decompositions $a = a_1 + a_2$ such that $a_0 \in X_0$ and $a_1 \in X_1$. If we write this for the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$, we get

$$K(a, t) = \inf\{\|a_1\|_{1,x} + t\|a_2\|_\infty\}.$$

Now $t \mapsto K(a, t)$ is a concave function, so we can write it in the form

$$K(a, t) = K(a, 0+) + \int_0^t k(a, s)\mathrm{ds}.$$

The function $t \mapsto k(a, t)$ is a unique non-negative, right-continuous, non-increasing function and is known as the k-functional.

For the K-method of interpolation we are particularly interested in the so-called monotone interpolation spaces. These spaces are defined as follows.

DEFINITION 4.4. Let (X_0, X_1) be a compatible couple of Banach spaces. An intermediate space X of (X_0, X_1), is said to be *monotone* if and only if the condition

$$K(g,t) \leq K(f,t) \text{ for each } t > 0 \text{ where } f \in X, g \in X_0 + X_1,$$

ensures that $g \in X$ with $\|g\|_X \leq \|f\|_X$.

It is well known that in the tracial case the K-functional can be written as $K(a,t) = \int_0^t \mu_s(a)\mathrm{d}s$ for all $a \in \widetilde{\mathcal{M}}$ [**6**, page 289] and therefore that $k(a,t) = \mu_t(a)$. For our investigation reproducing this result will be a key component of attaining our desired outcome. In particular we would like to show that, for the Banach couple $(L^1 + L^\infty)_x(\widetilde{\mathcal{M}})$ the k-functional is similarly given by $k(a,t) = \mu_t(a,x)$ for all $t \geq 0$ and that $K(a, 0+) = 0$. In order to apply the K-method of interpolation, we must know more about the structure of the K- and k-functionals.

To show this, recall the results from Lemma 2.20.

For $a, b \in \widetilde{\mathcal{M}}$ and $x \in (L^1 + L^\infty)(\widetilde{\mathcal{M}})$,

(1) $\lim_{t\downarrow 0} \mu_t(a,x) = \|x\|$.
(2) $\mu_t(a,x) = \mu_t(|a|,x) = \mu_t(a^*,x)$ and $\mu_t(\lambda a,x) = |\lambda|\mu_t(a,x)$.
(3) $\mu_{t+s}(a+b,x) \leq \mu_t(a,x) + \mu_s(b,x)$.
(4) $\mu_{t+s}(ab,x) \leq \mu_t(a,x)\mu_s(b,x)$.

The above results provide all the ingredients we need to prove the desired theorem. The proof of the theorem is simply a repetition of the proof in [**6**]. As before we still provide the proof of the theorem, because of its importance and for the sake of completeness of this text.

THEOREM 4.5. *Let $a \in L^1_x(\widetilde{\mathcal{M}}) + \mathcal{M}$. Then*

$$\int_0^t \mu_s(a,x)ds = \inf\{\|a_1\|_{1,x} + t\|a_2\|_\infty\}$$

where the infimum is taken over all decompositions $a = a_1 + a_2$ for where $a_1 \in L^1_x(\widetilde{\mathcal{M}})$ and $a_2 \in \mathcal{M}$.

PROOF. Fix $t > 0$. Let $a = a_1 + a_2$ be a decomposition of a where $a_1 \in L^1_x(\widetilde{\mathcal{M}})$ and $a_2 \in \mathcal{M}$, and let $0 < \alpha < 1$ be given. Then $\mu_s(a,x) \leq \mu_{\alpha s}(a_1,x) + \mu_{(1-\alpha)s}(a_2,x) \leq \mu_{\alpha s}(a,x) + \|a_2\|$, whence

$$\begin{aligned}\int_0^t \mu_s(a,x)ds &\leq \int_0^t \mu_{\alpha s}(a_1,x)ds + t\|a_2\|_\infty \\ &\leq \int_0^\infty \mu_{\alpha s}(a_1,x)ds + t\|a_2\|_\infty \\ &= \alpha^{-1}\int_0^\infty \mu_s(a_1,x)ds + t\|a_2\|_\infty.\end{aligned}$$

On letting $\alpha^{-1} \uparrow 1$, we then get that $\int_0^t \mu_s(a,x)\mathrm{d}t \leq \|a_1\|_{1,x} + t\|a_2\|_\infty$.

It remains to prove the reverse inequality. Again we fix $t > 0$. Let $a = u|a|$ be the polar decomposition of a and $|a| = \int_0^\infty \lambda\, de_t$ be the spectral decomposition of $|a|$. Set $\alpha = \mu_t(a,x)$, $a_1 = u\int_\alpha^\infty(\lambda - \alpha)de_t$ and $a_2 = a - a_1$. For the function

$$f(\lambda) = \begin{cases} 0 & 0 \le \lambda \le \alpha \\ \lambda - \alpha & \lambda \ge \alpha \end{cases}$$

we then have that $|a_1| = f(|a|)$. It then follows that

$$\mu_s(a_1,x) = f(\mu_s(a,x)) = \begin{cases} \mu_s(a,x) - \alpha & 0 < s < t \\ 0 & s \ge t \end{cases}.$$

We also have that $\|a_2\| \le \alpha$. Therefore

$$\begin{aligned} \|a_1\|_{1,x} + t\|a_2\| &\le \int_0^\infty \mu_s(a_1,x)ds + t\alpha \\ &= \int_0^t (\mu_s(a,x) - \alpha)ds + t\alpha \\ &= \int_0^t \mu_s(a,x)ds, \end{aligned}$$

from which the result then follows. □ □

The above theorem shows that the k-functional for the Banach pair $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ is $t \to \mu_t(a,x)$, as it is in the tracial case. Achieving this result is the most important step for using the K-method of interpolation. In fact attaining the desired result will now nearly be a straightforward application of the K-method.

The next concept we will need to describe is that of a Riesz-Fischer norm. The Riesz-Fischer norms are very similar to the concept of a Banach function norm we have been using thus far. In fact, all the relevant Banach function norms we have encountered have also been Riesz-Fischer norms.

Definition 4.6. A Banach function norm ρ on $L^{0+}([0,\infty),m)^+$, where m is Lebesgue measure, is called a *Riesz-Fischer* norm on $([0,\infty),m)$ if ρ satisfies the following conditions for all $f,g \in L^{0+}([0,\infty),m)^+$, $\alpha > 0$ and E a Borel set:

(1) If f and g are equimeasurable, then $\rho(f) = \rho(g)$.
(2) $\rho(\chi_E) < \infty$ whenever $m(E) < \infty$.
(3) Whenever $m(E) < \infty$, there exists a constant C_E dependent only on ρ and E such that $\int_E f\mathrm{dm} \le C_E\rho(f)$ for all $f \in L^{0+}([0,\infty),dt)^+$.
(4) $\rho(\sum_n f_n) \le \sum_n \rho(f_n)$ for each sequence (f_n) in $L^{0+}([0,\infty),dt)$.

If ρ also satisfies the condition that $\rho(f) \leq \rho(g)$ whenever $\int_0^t f^*(s)\,dm(s) \leq \int_0^t g^*(s)\,dm(s)$ for any $t > 0$ (where f^* and g^* denote the rearrangements of f and g), then ρ is said to be a *monotone* Riesz-Fischer norm.

As can be seen, we only require 4 as an additional condition to those we have already required for a Banach function norm. Before we used a Banach function norm to define various noncommutative Banach function spaces. For a Banach couple (X_0, X_1) we can define an analogous concept if we allow the k-functional to take the role that the decreasing rearrangement has taken previously.

DEFINITION 4.7. Let (X_0, X_1) be a compatible couple of Banach spaces, and let ρ be a monotone Riesz-Fischer norm on $([0,\infty), \mathscr{B}, m)$. We denote the completion of $X_0 \cap X_1$ with respect to the norm in X_0 by $\overline{(X_0 \cap X_1)}^{X_0}$. The space $(X_0, X_1)_\rho$, is defined to be the space of all $f \in \overline{(X_0 \cap X_1)}^{X_0} + X_1$ for which $\rho(k(f, \cdot; X_0, X_1)) < \infty$.

In the case when the Banach couple is $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$, if we also consider the density of $\mathcal{M} \cap L^1(\widetilde{\mathcal{M}})$ in $L^1(\widetilde{\mathcal{M}})$, it becomes apparent that this describes the spaces $L^\rho(\widetilde{\mathcal{M}})$ for a Riesz-Fischer norm ρ.

In order to arrive at a similar conclusion for the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$, we will need to show that $\mathcal{M} \cap L^1(\widetilde{\mathcal{M}})$ is dense in $L^1(\widetilde{\mathcal{M}})$. But for all $a \in L^1_x(\widetilde{\mathcal{M}}) + \mathcal{M}$ we have that $K(a, t) = \int_0^t \mu_s(a, x)\mathrm{ds}$, so it follows from the general theory of the K-method of interpolation that $L^1_x(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ is dense in $L^1_x(\widetilde{\mathcal{M}})$ [**2**, Theorem V.1.15]. Additionally observe that it follows from condition 3 of Definition 4.6 that $\int_0^1 f^*(s)\,dm(s) < \infty$ for all $f \in L^\rho([0,\infty), m)$, or equivalently that $f \in (L^1 + L^\infty)([0,\infty), m)$ if $f \in L^\rho([0,\infty), m)$. If we consider weighted noncommutative Banach function spaces, this ensures that for any Riesz-Fischer norm ρ, the requirement that $\mu(a, x) \in L^\rho([0,\infty), m)$ forces $a \in L^1_x(\widetilde{\mathcal{M}}) + \mathcal{M}$. So we have that the prescription in Definition 4.7 encompasses all the spaces $L^\rho_x(\widetilde{\mathcal{M}})$ as described in Definition 2.1. From this it follows that for the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$, we also have that the spaces defined using Definition 4.7 are the spaces $L^\rho_x(\widetilde{\mathcal{M}})$.

The next step is to introduce the concept of a Gagliardo couple.

DEFINITION 4.8. A Banach couple (X_0, X_1) is a *Gagliardo* couple if:

- for all $f \in X_0 + X_1$ the quantity $\lim_{t\to\infty} K(f, t)$ is finite if and only if $f \in X_0$. If so, then $\lim_{t\to\infty} K(f, t) = \|f\|_{X_0}$.
- for all $f \in X_0 + X_1$ the quantity $\lim_{t\to 0} t^{-1}K(f, t)$ is finite if and only if $f \in X_1$. If so, then $\lim_{t\to 0} t^{-1}K(f, t) = \|f\|_{X_1}$.

For the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ observe that for any $a \in L^1_x(\widetilde{\mathcal{M}})$, we have by Theorem 4.5 that $\lim_{t\to\infty} K(f, t) = \int_0^\infty \mu_s(a, x)\mathrm{ds} = \tau_x(a) = \|a\|_{1,x}$, so the first condition of Definition 4.8 is satisfied. Furthemore we have that for any $a \in \widetilde{\mathcal{M}}$ we have that $\|a\|_\infty = \lim_{t\downarrow 0} \mu_t(a, x)$. Now recall that $\mu(a,x)$ is right continuous, particularly $\mu(a,x)$ is right continuous at 0. This ensures that we can make sense of $\lim_{t\downarrow 0} t^{-1}K(f,t) = \lim_{t\downarrow 0} t^{-1}\int_0^t \mu_s(a,x)\mathrm{ds}$. If we recognise that this is simply the value of the derivative of the function $t \mapsto \int_0^t \mu_s(a,x)\mathrm{ds}$ at zero, then we clearly have that $\lim_{t\downarrow 0} t^{-1}K(f,t) = \lim_{t\downarrow 0} t^{-1}\int_0^t \mu_s(a,x)\mathrm{ds} = \lim_{t\to 0}\mu_t(a,x) = \|a\|_\infty$. For the converse suppose that for some $a \in (L^1_x(\widetilde{\mathcal{M}}) + \mathcal{M})$ we have that $\lim_{t\downarrow 0} t^{-1}K(f,t) = \lim_{t\downarrow 0} t^{-1}\int_0^t \mu_s(a,x)\mathrm{ds} < \infty$. For each $t > 0$, the fact that $s \to \mu_s(a,x)$ is non-decreasing, ensures that $t^{-1}\int_0^t \mu_s(a,x)\mathrm{ds} \geq t^{-1}\int_0^t \mu_t(a,x)\mathrm{ds} = \mu_t(a,x)$. Since then $\|a\|_\infty = \sup_{t>0}\mu_t(a,x) = \lim_{t\downarrow 0}\mu_t(a,x) \leq \lim_{t\downarrow 0} t^{-1}\int_0^t \mu_s(a,x)\mathrm{ds} < \infty$, we have that $a \in \mathcal{M}$ as required.

With $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ established as a Gagliardo couple, we can proceed to use the known interpolation results for such couples. In particular, we will use the following theorem.

THEOREM 4.9. *Let (X_0, X_1) be a Gagliardo couple. Then X is a monotone interpolation space for the couple (X_0, X_1) if it's one of the following forms:*

(1) *if $X \subset \overline{(X_0 \cap X_1)}^{X_0} + X_1$, then there is a monotone Banach function norm ρ such that X appears as an equivalent renorming of $(X_0, X_1)_\rho$.*
(2) *if $X \subset \overline{(X_0 \cap X_1)}^{X_1} + X_0$, then there is a monotone Banach function norm ρ such that X appears as an equivalent renorming of $(X_1, X_0)_\rho$.*
(3) *$X = X_0 + X_1$ otherwise.*

Now considering that $(L^1_x(\widetilde{\mathcal{M}}) \cap \mathcal{M})$ is dense in $L^1_x(\widetilde{\mathcal{M}})$, and recalling that we have already observed that $L^\rho_x(\widetilde{\mathcal{M}}) \subset L^1_x(\widetilde{\mathcal{M}}) + \mathcal{M}$, we then see that

$$L^\rho_x(\widetilde{\mathcal{M}}) \subset L^1_x(\widetilde{\mathcal{M}}) + \mathcal{M} = \overline{(L^1_x(\widetilde{\mathcal{M}}) \cap \mathcal{M})}^{1,x} + \mathcal{M}.$$

So for the Gagliardo couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$, it is clear that 1 is applicable. In light of this we can conclude with the desired result.

THEOREM 4.10. *For the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ every space of the form $L^\rho_x(\widetilde{\mathcal{M}})$, where ρ is a monotone Riesz-Fischer norm on $L^{0+}([0,\infty), m)^+$, is an exact interpolation space. Conversely if X is an exact monotone interpolation space, then there exists a monotone Riesz-Fischer norm ρ on $L^{0+}([0,\infty), m)^+$, such that $X = L^\rho_x(\widetilde{\mathcal{M}})$.*

The above theorem is the promised result we have been aiming for. A further question is whether every exact space is automatically monotone. As of the writing of this text, this remains an open question. The approach used by Dodds, Dodds and de Pagter in [**5**] to show this for the spaces $L^\rho(\widetilde{\mathcal{M}})$ presents significant challenges, in particular with the reliance on the linearity of the trace.

To end this chapter we will show an application of the technology developed above to show that quantum dynamical maps associated with the system described by $\mathcal{M}$, have a canonical action on each of the weighted Banach function spaces. These quantum dynamical maps will consist of the maps described by the definition below.

DEFINITION 4.11. We define a positive map $T : \mathcal{M} \to \mathcal{M}$ to be a (sub-)Markov map, if T is normal, $T(\mathbb{1}) \leq \mathbb{1}$, and $\tau \circ T \leq \tau$.

Let $a, b \in \widetilde{\mathcal{M}}$. We will say a is *submajorised* by b, denoted $a \prec\prec b$ if $\int_0^t \mu_s(a)\mathrm{ds} \leq \int_0^t \mu_s(b)\mathrm{ds}$ for all $t > 0$. We will denote the case $\int_0^t \mu_s(a,x)\mathrm{ds} \leq \int_0^t \mu_s(b,x)\mathrm{ds}$ for all $t > 0$ by $a \prec\prec_x b$.

REMARK 4.12. *It is known that all maps of the form described above, extend canonically to maps with a contractive action on* $L^1(\mathcal{M},\tau)$ [**7**, Theorem 5.1]. *As can be seen from* [**4**, Proposition 2.1] *and the discussion preceding it, this in turn ensures that T extends canonical to a contractive map on the space* $(L^1 + L^\infty)(\widetilde{\mathcal{M}})$ *which is "admissible" for the Banach couple* $(L^1(\widetilde{\mathcal{M}}), \mathcal{M})$*, and that for this canonical extension we have that* $T(a) \prec\prec a$ *for every* $a \in (L^1 + L^\infty)(\widetilde{\mathcal{M}})$.

For this application, we will first need the following results.

LEMMA 4.13. *Let* $0 \leq f \in L^0((0,\infty), dt)$ *be non-increasing and* $t > 0$. *Then* $\mu(\chi_{(0,t)}f, \nu) = \chi_{(0,t')}\mu(f,\nu)$ *where* $t' = \int_0^t \mu_s(x)ds$.

PROOF. If $\theta < f(t)$, then since f is non-increasing, we have that $d_\theta(\chi_{(0,t)}f,\nu) = \nu\{s \geq 0 : (\chi_{(0,t)}f)(s) > \theta\} = \nu(0,t) = \int_0^t \mu_s(x)\mathrm{ds} = t'$. If however $\theta > f(t)$, then $d_\theta(\chi_{(0,t)}f,\nu) = d_\theta(f,\nu)$. So

$$d_\theta(\chi_{(0,t)}f,\nu) = \begin{cases} t' & 0 < \theta < f(t) \\ d_\theta(f,\nu) & \theta > f(t). \end{cases} \tag{4.1}$$

Recall that $\mu_s(\chi_{(0,t)}f,\nu) = \inf\{\theta \geq 0 : d_\theta(\chi_{(0,t)}f,\nu) \leq s\}$. First let $0 < s < t'$. Now $\theta \mapsto d_\theta(\chi_{(0,t)}f,\nu)$ is non-increasing. From equation 4.1 it is clear that if $\psi \in \{\theta \geq 0 : d_\theta(\chi_{(0,t)}f,\nu) \leq s < t'\}$, then $d_\psi(f,\nu) = d_\psi(\chi_{(0,t)}f,\nu) \leq s$, and so $\psi \in \{\theta \geq 0 : d_\theta(f,\nu) \leq s\}$. From this it follows that $\mu_s(\chi_{(0,t)}f,\nu) \geq \mu_s(f,\nu)$, since $\mu_s(f,\nu) = \inf\{\theta > 0 : d_\theta(f,\nu) \leq s\}$. If on the other hand $\psi \in \{\theta \geq 0 : d_\theta(f,\nu) \leq s < t'\}$, then of course $d_\psi(f,\nu) \leq s < t'$, which by equation 4.1 and the fact that $r \to d_r(\chi_{(0,t)}f,\nu)$ is decreasing, can only be true if $\psi > f(t)$, in which case it will then follow that $d_\psi(\chi_{(0,t)}f,\nu) = d_\psi(f,\nu) \leq s$. Therefore $\psi \in \{\theta \geq 0 : d_\theta(\chi_{(0,t)}f,\nu) \leq s < t'\}$. From this we have that $\mu_s(f,\nu) \geq \mu_s(\chi_{(0,t)}f,\nu)$, and hence that $\mu_s(\chi_{(0,t)}f,\nu) = \mu_s(f,\nu)$.

Suppose $s \geq t'$, then $d_\theta(\chi_{(0,t)}f,\nu) \leq t' \leq s$ for all $\theta \geq 0$, and so $\mu_s(\chi_{(0,t)}f,\nu) = 0$. Therefore $\mu(\chi_{(0,t)}f,\nu) = \chi_{(0,t')}\mu(f,\nu)$. ☐

PROPOSITION 4.14. *There exists a non-decreasing, continuous map* $(0,\infty) \mapsto (0,\tau(x)] : t \mapsto \int_0^t \mu_s(x)\,ds = t'$ *such that for all non-increasing* $0 \le f \in L^0((0,\infty), ds)$ *and* $t > 0$, *we have that*

$$\int_0^t f(s)\mu_s(x)\,ds = \int_0^{t'} \mu_s(f,\nu)\,ds.$$

PROOF. We have that

$$\begin{aligned}\int_0^t f(s)\mu_s(x)\mathrm{ds} &= \int_0^\infty \chi_{(0,t)}(s)f(s)\mu_s(x)\mathrm{ds}\\ &= \int_0^\infty \mu_s(\chi_{(0,t)}f,\nu)\mathrm{ds}\\ &= \int_0^\infty \chi_{(0,t')}(s)\mu_s(f,\nu)\mathrm{ds}\\ &= \int_0^{t'} \mu(f,\nu)\mathrm{ds}.\end{aligned}$$

□

COROLLARY 4.15. *If* $a, b \in \widetilde{\mathcal{M}}$ *are given such that* $a \prec\prec b$ *then* $a \prec\prec_x b$ *in the sense that for all* $t > 0$, $\int_0^t \mu_s(a,x)\,ds \le \int_0^t \mu_s(b,x)\,ds$.

PROOF. We have that for all $t > 0$, $\int_0^t \mu_s(a)\mathrm{ds} \le \int_0^t \mu_s(b)\mathrm{ds}$. By Hardy's lemma [**2**, Proposition 2.3.6] we then have that for all $t > 0$, $\int_0^t \mu_s(a)\mu_s(x)\mathrm{ds} \le \int_0^t \mu_s(b)\mu_s(x)\mathrm{ds}$.

Now let $t' > 0$. Suppose that $t' > \tau(x)$. From Lemma 2.24 we have that $\mu_s(a,x) = 0 = \mu_s(b,x)$ for all $s > \tau(x)$. Therefore $\int_0^{t'} \mu_s(a,x)\mathrm{ds} = \int_0^\infty \mu_s(a,x)\mathrm{ds} = \tau_x(a) \le \tau_x(b) = \int_0^\infty \mu_s(b,x)\mathrm{ds} = \int_0^{t'} \mu_s(b,x)\mathrm{ds}$.

Now suppose that $0 < t' < \tau(x)$. The map $t \mapsto \int_0^t \mu_s(x)\mathrm{ds}$ is non-decreasing and continuous, with 0 mapping onto 0, and $\lim_{t\to\infty} \int_0^t \mu_s(x)\mathrm{ds} = \int_0^\infty \mu_s(x)\mathrm{ds} = \tau(x)$. So by the intermediate value theorem, there exists $t > 0$ such that $t \mapsto t'$. On considering Theorem 2.25 alongside Proposition 4.14, it now follows that $\int_0^t \mu_s(a)\mu_s(x)\mathrm{ds} = \int_0^{t'} \mu_s(\mu(a),\nu)\mathrm{ds} = \int_0^{t'} \mu_s(a,x)\mathrm{ds}$, and similarly that $\int_0^t \mu_s(b)\mu_s(x)\mathrm{ds} = \int_0^{t'} \mu_s(b,x)\mathrm{ds}$. The known inequality $\int_0^t \mu_s(a)\mu_s(x)\mathrm{ds} \le \int_0^t \mu_s(b)\mu_s(x)\mathrm{ds}$, therefore corresponds to the statement that

$$\int_0^{t'} \mu_s(a,x)\mathrm{ds} \le \int_0^{t'} \mu_s(b,x)\mathrm{ds}.$$

□

With these results achieved, we can now come back to the promised application.

THEOREM 4.16. *For any Riesz-Fischer norm* ρ *on* $L^{0+}([0,\infty),m)^+$, *each Markov map* $T : \mathcal{M} \to \mathcal{M}$ *will canonically induce a contractive map on the space* $L^\rho_x(\widetilde{\mathcal{M}})$.

PROOF. Let ρ and T be as in the hypothesis. We have already noted in the preceding remark that T canonically extends to a map on $(L^1 + L^\infty)(\widetilde{\mathcal{M}})$ for which we have that $T(a) \prec\prec a$ for each $a \in (L^1 + L^\infty)(\widetilde{\mathcal{M}})$. But by Corollary 4.15 we will then have that $T(a) \prec\prec_x a$ for each $a \in \mathcal{M}$. Since by Theorem 4.10, the space $L^1_x(\widetilde{\mathcal{M}})$ is a monotone intermediate space of the pair $(L^1_x(\mathcal{M}), \mathcal{M})$ in the sense of Definition 4.4, this in turn ensures that $\|T(a)\|_{1,x} \leq \|a\|_{1,x}$ for each $a \in \mathcal{M}$. Thus T canonically extends to a contractive map on $L^1_x(\widetilde{\mathcal{M}})$. But by Theorem 4.5, contractivity of T on both $\mathcal{M}$ and $L^1_x(\widetilde{\mathcal{M}})$, ensure that $T(a) \prec\prec_x a$ for each $a \in L^1_x(\widetilde{\mathcal{M}})$. The fact that $\|T(a)\|_{\rho,x} \leq \|a\|_{\rho,x}$ for each $a \in L^\rho_x(\widetilde{\mathcal{M}})$, is then a direct consequence of Theorem 4.10. □

In this chapter, we have seen that the Banach couple $(L^1_x(\widetilde{\mathcal{M}}), \mathcal{M})$ allows a robust theory of real interpolation. The K-method of interpolation was particularly fruitful in the development of the theory. While there are still some open questions, the results we have achieved are strong enough to allow us to use the interpolation theory in applications to quantum dynamics.

Throughout this text, we have seen how the weighted spaces first described by Labuschagne and Majewski in [**10**] has lead to an approach to describing noncommutative Banach function spaces that rely on a map τ_x that, while it is very similar to a trace, is not generally one. As such the spaces that were the central objects of our study are novel in the sense that they represent a step away from an explicit reliance on a trace. Admittedly we still implicitly rely on the trace, as our map τ_x is defined in terms of noncommutative decreasing rearrangements. In Chapter 2 our task was to develop the basic results upon which we would build our theory. Importantly we showed that we could define a weighted noncommutative decreasing rearrangement and use these functions to define the same spaces that were defined in [**10**]. An important result was Theorem 2.25, which showed the connection between $\mu(a)$ and $\mu(a, x)$ through the classical world. This result proved to be a key factor in obtaining many of the results that followed.

Furthermore, we showed that for a class of weighted noncommutative Orlicz spaces, we could describe their Köthe duals and in so doing recreate some of the results from the tracial theory.

The theory of these spaces are still in their infancy, especially compared to that of tracial noncommutative Banach function spaces. Even concerning the topics that were the focus of our study, there remain some open questions.

SECTION NOTES. The results in this section, including Theorem 4.10, were first published in [**12**] by the present author and Labuschagne.

APPENDIX A

Noncommutative regular random variables

As was mentioned, weighted noncommutative Banach function spaces were motivated by Labuschagne and Majewski in response to certain physical considerations. In particular in [**10**], the authors were interested in finding the correct space to contain states with well defined entropy and to formulate the concept of regular random variables in the quantum setting. It is the second consideration that lead to the formulation of weighted noncommutative Banach function spaces. The formulation of quantum regular random variables naturally introduces a weight in the form of $\mu(x)$ for some state x. Furthermore in attempting to find a noncommutative analogue of the Pistone-Sempi theorem A.3, the authors found that the natural space for quantum regular random variables is a weighted noncommutative Orlicz space.

We start by reviewing the classical statistical model and recalling the definition of a regular random variable in this setting. The review given here closely follows the review given in [**10**].

Let $\{\Omega, \Sigma, \nu\}$ be a measure space. We define the set

$$\mathcal{S}_\nu = \{f \in L^1(\nu) : f > 0 \quad \nu - a.s., \int f \cdot g \mathrm{d}\nu = 1\}, \tag{A.1}$$

i.e. the densities of all probability measures equivalent to ν. $\mathcal{S}_\nu$ is referred to as the state space.

Definition A.1. The classical statistical model consists of the measure space $\{\Omega, \Sigma, \nu\}$, state space $\mathcal{S}_\nu$, and the set of measurable functions $L^0(\Omega, \Sigma, \nu)$.

Regular random variables are those variables having all finite moments. For a fixed $f \in \mathcal{S}_\nu$, we define the moment generating function as

$$\hat{u}_f(t) = \int exp(tu) f d\nu, \qquad t \in \mathbb{R}. \tag{A.2}$$

The function $\hat{u}$ has the following properties [**20**].

(1) $\hat{u}$ is analytic in the interior of its domain,
(2) its derivatives are obtained by differentiating under the integral sign.

With this we can formulate the following definition.

Definition A.2. The set of all random variables such that

(1) $\hat{u}_f$ is well defined in a neighborhood of the origin 0,
(2) the expectation of u is zero,

will be denoted by L_f and called the regular random variables.

Finally we present the Pistone-Sempi theorem which shows that the appropriate space to model regular random variables is a classical Orlicz space with a weighted measure.

THEOREM A.3. *(Pistone-Sempi,* [**14**]*)* L_f *is the closed subspace of the Orlicz space* $L^{\cosh -1}(f \cdot \nu)$ *of zero expectation random variables.*

We now move to the noncommutative setting. The strategy employed by Labuschagne and Majewski was the standard procedure of quantising the classical definitions.

As such we first review the noncommutative statistical model.

Recall the following definitions [**16**].

(1) $n_\tau = \{x \in \mathcal{M} : \tau(x^*x) < +\infty\}$.
(2) $m_\tau = \{xy : x, y \in n_\tau\}$. known as the *definition ideal of the trace* τ.
(3) $\omega_x(y) = \tau(xy), \quad x \geq 0$.

It is well known that

(1) if $x \in m_\tau$, and $x \geq 0$, then $\omega_x \in \mathcal{M}_*^+$.
(2) $L^1(\mathcal{M}, \tau)$ is isometrically isomorphic to $\mathcal{M}_*$.
(3) $\mathcal{M}_{*,0} \equiv \{\omega_x : x \in m_\tau\}$ is norm dense in $\mathcal{M}_*$.

By $\mathcal{M}_*^{+,1}$ ($\mathcal{M}_{*,0}^{+,1}$) we denote the set of all normalized normal positive functionals in $\mathcal{M}_*$ (in $\mathcal{M}_{*,0}$ respectively). In this context if we translate the definition of a regular random variable into the noncommutative setting, we arrive at the following definition.

DEFINITION A.4. The noncommutative statistical model consists of a quantum measure space $(\mathcal{M}, \tau)$, "quantum densities with respect to τ" in the form of $\mathcal{M}_{*,0}^{+,1}$, and the set of τ-measurable operators $\widetilde{\mathcal{M}}$.

In the framework of the noncommutative statistical model the noncommutative regular random variables can be defined in the following way:

DEFINITION A.5.

$$L_x^{quant} = \{g \in \widetilde{\mathcal{M}} : \quad 0 \in D(\widehat{\mu_x^g(t)})^0, \quad x \in m_\tau^+\}, \tag{A.3}$$

where $D(\cdot)^0$ stands for the interior of the domain $D(\cdot)$ and

$$\widehat{\mu_x^g(t)} = \int \exp(t\mu_s(g))\mu_s(x)ds, \qquad t \in \mathbb{R}. \tag{A.4}$$

Here we can already see the "weight" $\mu(x)$ appearing. If we restrict ourselves to this specific setting in which $x \in x \in m_\tau^+$, then we have that $\int_0^\infty \mu_s(x)\mathrm{ds} = \tau(x) = 1$. So $\mu(x)$ acts as a density of a probability measure.

From this it we are motivated to consider $\mu(x)$ as a weight. In this way Labuschagne and Majewski were motivated to define weighted noncommutative Banach function spaces. We now recall their definition.

DEFINITION A.6. Let $x \in L^1_+(\mathcal{M}, \tau)$ and let ρ be a Banach function norm on $L^0((0,\infty), \mu_t(x)dt)$. We formally define the weighted noncommutative Banach function space $L^\rho_x(\widetilde{\mathcal{M}})$ to be the collection of all $a \in \widetilde{\mathcal{M}}$ for which $\mu(a)$ belongs to $L^\rho((0,\infty), \mu_t(x)dt)$. For any such a we write $\|a\|_\rho = \rho(\mu(a))$.

To show the connection between weighted noncommutative Banach function spaces and noncommutative regular random variables more explicitly, we recall the result proved by Labuschagne and Majewski.

THEOREM A.7. *The set* L_x^{quant} *coincides with the closed subspace of the weighted Orlicz space* $L_x^{\cosh -1}(\widetilde{\mathcal{M}}) \equiv L_x^\psi(\widetilde{\mathcal{M}})$ *(where* $\psi = \cosh -1$*) of noncommutative random variables with a fixed expectation.*

While this result isn't exactly a direct translation of the Pistone-Sempi theorem into noncommutative language, it is clear that it does play the role of the noncommutative analogue thereof. The fact that it is a weighted noncommutative Orlicz space that is the natural home of noncommutative regular random variables is what initially motivated this study.

APPENDIX B

Additional Proofs

We give the proof of the following theorem and proposition as they appeared in [**10**] without alterations.

THEOREM B.1. [**10**, Definition 3.6] *Let $x \in L^1_+(\widetilde{\mathcal{M}})$. If ρ is a rearrangement-invariant Banach function norm on $L^0((0,\infty),\mu_t(x)dt)$ which satisfies the Fatou property, then $L^\rho_x(\widetilde{\mathcal{M}})$ is a linear space and $\|\cdot\|_\rho$ a norm. Equipped with the norm $\|\cdot\|_\rho$, $L^\rho_x(\widetilde{\mathcal{M}})$ is a Banach space which injects continuously into $\widetilde{\mathcal{M}}$.*

PROOF. We will not give a detailed proof, but only indicate how the argument in Section 4 of [**3**] may be adapted to the present context. For the sake of convenience, we will assume that $\tau(x) = 1$.

Since $t \to \mu_t(x)$ is decreasing, right-continuous on $[0,\infty)$, and finite-valued on $(0,\infty)$, it is actually Riemann-integrable on any bounded sub-interval of $(0,\infty)$, and zero-valued on $[t_x,\infty)$ where $t_x = \inf\{t > 0 : \mu_t(x) = 0\}$. These facts enable us to conclude that the function

$$F_x(t) = \int_0^t \mu_s(x)ds \qquad t \geq 0$$

is continuous and strictly increasing on $[0,t_x)$, and constant on $[t_x,\infty)$. So F_x is actually a homeomorphism from $[0,t_x)$ onto $[0,1)$. For any measurable function $g : [0,\infty) \to \mathbb{R}$ and any $t > 0$ we therefore have

$$\int_0^{F_x(t)} g(s)ds = \int_0^t g(F_x(s))\mu_s(x)ds$$

by the change of variables formula (see for example p 155 of [**17**]).

For ease of notation we write ν_x for the Borel measure

$$\nu_x(E) = \int \chi_E(t)\mu_t(x)dt.$$

Since $\mu(x)$ is non-zero on $[0,t_x)$, it is a simple matter to conclude that on $[0,t_x)$, $\nu_x \ll \lambda$ and $\lambda \ll \nu_x$ (here λ denotes Lebesgue measure). Since ν_x is a finite measure, it is in fact $\epsilon - \delta$ absolutely continuous with respect to λ. Using these facts, one is now able to show that ν_x is non-atomic. (If $\nu_x(E) \neq 0$ then so too $\lambda(E) \neq 0$. Now set $\epsilon = \frac{1}{2}\nu_x(E)$, and select $F \subset E \cap [0,t_x)$ with $0 < \lambda(F) \leq \delta(\epsilon)$ to see that ν_x is non-atomic.) Thus by [**2**, Theorem 2.2.7], ν_x is a resonant measure.

In view of the fact that $\mu(x)$ is decreasing, we have that $F_x(t) = \nu_x([0,t]) \geq \nu_x([s, s+t])$ for any $s, t > 0$. More generally by approximating with intervals, one can show that for any $t > 0$ and any Borel set E in $[0, \infty)$ with $\lambda(E) = t$, we have that $\nu_x(E) \leq \nu_x([0,t]) = F_x(t)$. Given some measurable function f on $[0, \infty)$, these facts ensure that

$$\inf\{\|f\chi_E\|_\infty : \lambda(E^c) \leq t\} \geq \inf\{\|f\chi_E\|_\infty : \nu_x(E^c) \leq F_x(t)\}.$$

In other words

$$\widetilde{\mu}_t(f, \lambda) \geq \widetilde{\mu}_{F_x(t)}(f, \nu_x).$$

(The centered expressions above respectively denote the decreasing rearrangement of f computed using λ and ν_x.)

Now if h is decreasing and right-continuous on $[0, \infty)$, and finite valued on $(0, \infty)$, then more can be said. It is an exercise to see that in this case $\inf\{\|h\chi_E\|_\infty : \lambda(E^c) \leq t\} = \|h\chi_{(t,\infty)}\|_\infty$. (To see that "$\leq$" holds is trivial. For the converse note that if $\lambda(E \cap [0,t]) \neq 0$, then $\|h\chi_E\|_\infty \geq \|h\chi_{(t,\infty)}\|_\infty$ by the fact that h is decreasing.) The right-continuity of h combined with the fact that it is decreasing, ensures that $\|h\chi_{(t,\infty)}\|_\infty = h(t)$. A similar argument to the above shows that for any $0 < t < t_x$, we have that $\inf\{\|h\chi_E\|_\infty : \nu_x(E^c) \leq F_x(t)\} = \|h\chi_{(t,\infty)}\|_\infty$. For functions such as these, we therefore have

$$h(t) = \widetilde{\mu}_t(h, \lambda) = \widetilde{\mu}_{F_x(t)}(h, \nu_x) \qquad \text{for all} \qquad 0 < t < t_x.$$

Finally let $a, b \in L^\rho_x(\widetilde{\mathcal{M}})$ be given. We first show that then $a + b \in L^\rho_x(\widetilde{\mathcal{M}})$, and hence that $L^\rho_x(\widetilde{\mathcal{M}})$ is linear, before going on to conclude that $\|\cdot\|_\rho$ is a norm. By Theorem 3.4 of [**3**] we have that

$$\int_0^t \widetilde{\mu}_s(\mu(a+b) - \mu(b), \lambda) ds \leq \int_0^t \widetilde{\mu}_s(\mu(a), \lambda) ds = \int_0^t \mu_t(a) ds$$

for any $t > 0$. If now we apply Hardy's Lemma [**2**, Proposition 2.3.6] to the decreasing function $\mu(x)\chi_{[0,t]}$, we may conclude that

$$\int_0^t \widetilde{\mu}_s(\mu(a+b) - \mu(b), \lambda)\mu_s(x) ds \leq \int_0^t \mu_s(a)\mu_s(x) ds$$

for any $t > 0$. Next use the facts that $\mu_s(a) = \widetilde{\mu}_{F_x(s)}(\mu(a), \nu_x)$ for all $0 < s < t_x$, and $\widetilde{\mu}_s(\mu(a+b) - \mu(b), \lambda) \geq \widetilde{\mu}_{F_x(s)}(\mu(a+b) - \mu(b), \nu_x)$, to get

$$\int_0^t \widetilde{\mu}_{F_x(s)}(\mu(a+b) - \mu(b), \nu_x)\mu_s(x) ds \leq \int_0^t \widetilde{\mu}_{F_x(s)}(\mu(a), \nu_x)\mu_s(x) ds$$

for any $t_x \geq t > 0$. Since F_x is a homeomorphism from $[0, t_x)$ to $[0, 1)$, the change of variables formula now ensures that

$$\int_0^r \widetilde{\mu}_s(\mu(a+b) - \mu(b), \nu_x) ds \leq \int_0^r \widetilde{\mu}_s(\mu(a), \nu_x) ds$$

for any $1 > r > 0$. (Simply let $F_x(t) = r$.) Since ν_x is a probability measure, we in fact have that $\widetilde{\mu}_s(\mu(a+b) - \mu(b), \nu_x) = \widetilde{\mu}_s(\mu(a), \nu_x) = 0$ for all $s \geq 1$. Hence the previous centered inequality actually holds for all $r > 0$. We may now finally apply [**2**, Theorem 2.4.6] to conclude that $\rho(\mu(a+b) - \mu(b)) \leq \rho(\mu(a))$. But since $\mu(a), \mu(b) \in L^\rho((0,\infty), \mu_t(x)dt)$, this inequality surely forces $\mu(a+b) - \mu(b) \in L^\rho((0,\infty), \mu_t(x)dt)$, and hence $\mu(a+b) \in L^\rho((0,\infty), \mu_t(x)dt)$. Thus by definition $a + b \in L^\rho_x(\widetilde{\mathcal{M}})$, ensuring that $L^\rho_x(\widetilde{\mathcal{M}})$ is linear. (The fact that $\alpha a \in L^\rho_x(\widetilde{\mathcal{M}})$ whenever $a \in L^\rho_x(\widetilde{\mathcal{M}})$ is easy to verify.) But then this same inequality also ensures that $\|a+b\|_\rho \leq \|a\|_\rho + \|b\|_\rho$, and hence that $\|\cdot\|_\rho$ is a semi-norm on $L^\rho_x(\widetilde{\mathcal{M}})$. Now observe that if $\|a\|_\rho = \rho(\mu(a)) = 0$, then $\mu(a) = 0$ ν_x-ae. But since we have that $\lambda \ll \nu_x$ on $[0, t_x)$, this fact forces $\mu_t(a) = 0$ for λ-almost every t in $[0, t_x)$. The right-continuity of $t \to \mu_t(a)$ then ensures that $\|a\| = \lim_{t\downarrow 0} \mu_t(a) = 0$, and hence that $a = 0$. Thus $\|\cdot\|_\rho$ is in fact a norm.

The rest of the proof runs along similar lines as the argument in Section 4 of [**3**]. □

Proposition B.2. [**10**, Proposition 3.10] *For any* $0 \neq x \in L^1_+(\widetilde{\mathcal{M}})$, *the quantity*

$$\tau_x(f) = \int_0^\infty \mu_t(f)\mu_t(x)dt \quad f \in \mathcal{M}^+$$

is almost *a normal finite faithful trace in the sense that*

- τ_x *is subadditive, positive-homogeneous, and satisfies* $\tau_x(a^*a) = \tau_x(aa^*)$ *for every* $a \in \mathcal{M}$;
- $\tau_x(\mathbb{1}) < \infty$, *and for any* $a \in \mathcal{M}^+$ *the situation* $\tau_x(a) = 0$ *forces* $a = 0$;
- $\sup_n \tau_x(f_n) = \tau_x(f)$ *for every sequence* $\{f_n\}$ *in* $\mathcal{M}^+$ *increasing to some* $f \in \mathcal{M}^+$.

Proof. Note that by [**6**, Lemma 2.5], we have that $\mu_t(f^*f) = \mu_t(|f|)^2 = \mu_t(f)^2 = \mu_t(f^*)^2 = \mu_t(|f^*|)^2 = \mu_t(ff^*)$ and also that $\mu_t(\alpha f) = |\alpha|\mu_t(f)$ for each $t > 0$. This is enough to ensure that τ_x is positive-homogeneous, and satisfies the trace property $\tau_x(a^*a) = \tau_x(aa^*)$. Next let $a, b \in \mathcal{M}^+$ be given. From the proof of [**3**, Theorem 3.4] it is then clear that $\int_0^t |\mu_s(a+b) - \mu_s(a)|ds \leq \int_0^t \mu_s(b)ds$ for any $t > 0$ (simply apply what is proved there to the set $T = [0, t]$). In view of the fact that $t \to \mu_t(x)$ is decreasing, we may then apply Hardy's Lemma [**2**, Theorem 2.3.6], to conclude that

$$\begin{aligned}
\tau_x(a+b) - \tau_x(a) &= \int_0^\infty (\mu_t(a+b) - \mu_t(a))\mu_t(x)dt \\
&\leq \int_0^\infty |\mu_t(a+b) - \mu_t(a)|\mu_t(x)dt \\
&\leq \int_0^\infty \mu_t(b)\mu_t(x)dt \\
&= \tau_x(b).
\end{aligned}$$

Let $\alpha = \tau(\mathbb{1})$. Using [**6**, Lemma 2.6], the fact τ_x is finite, is then a simple consequence of the observation that $\tau_x(\mathbb{1}) = \int_0^\alpha \mu_t(\mathbb{1})\mu_t(x)dt = \int_0^\alpha \mu_t(x)dt = \tau(x) < \infty$.

Given any $f \in \mathcal{M}^+$, it is clear that if $0 = \tau_x(f) = \int_0^\infty \mu_t(f)\mu_t(x)dt$, then $\mu_t(f)\mu_t(x) = 0$ for all $t > 0$. (Use the fact that $t \to \mu_t(f)\mu_t(x)$ is decreasing.) Since $t \to \mu_t(x)$ is decreasing, we may conclude from the inequality $0 < \tau(x) = \int_0^\infty \mu_t(x)dt$, that there exists some $\delta > 0$ so that $0 \neq \mu_t(x)$ for all $0 \leq t < \delta$. But then we must have $0 = \mu_t(f)$ for all $0 < t < \delta$. The fact that $t \to \mu_t(f)$ is decreasing, ensures that $\mu_t(f) = 0$ for all $t > 0$, and hence that $\|f\| = \lim_{t \to 0^+} \mu_t(f) = 0$.

It remains to verify the claim about increasing sequences. To this end suppose that we are given a sequence $\{f_n\} \subset \mathcal{M}$ increasing to some $f \in \mathcal{M}$. Since $\mu_t(f_n) \leq \mu_t(f)$ for each n and each t, it is a simple matter to conclude from this that $\limsup_n \tau_x(f_n) \leq \tau_x(f)$. On the other hand [**5**, Proposition 1.7] ensures that $\mu_t(f) = \liminf_n \mu_t(f_n)$. By the usual Fatou's lemma, this in turn enables us to conclude that $\tau_x(f) = \int_0^\infty \liminf_n \mu_t(f_n)\mu_t(x)dt \leq \liminf_n \int_0^\infty \mu_t(f_n)\mu_t(x)dt = \liminf_n \tau_x(f_n)$. We then clearly have that $\tau_x(f) = \lim_n \tau_x(f_n) = \sup_n \tau_x(f_n)$. □

Bibliography

1. M. Adams, V. Guillemin, *Measure Theory and Probability*, Birkhäuser, 1996.
2. C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Orlando, 1988.
3. P. G. Dodds, T. K. -Y. Dodds and B. de Pagter, Non-commutative Banach function spaces *Math. Z.* **201** (1989), 583-597
4. P. G. Dodds, T. K. -Y. Dodds and B. de Pagter, Noncommutative Köthe duality, *Trans Am Math Soc.* **339** (1993), 717-750
5. P. G. Dodds, T. K. -Y. Dodds and B. de Pagter, Fully symmetric operator spaces, *Integr equ oper th.* **15** (1992), 942-972.
6. T. Fack and H. Kosaki, Generalized s-numbers of τ-measurable operators, *Pacific J Math* **123** (1986), 269-300.
7. U. Haagerup, M. Junge and Q. Xu, A reduction method for noncommutative L^p-spaces and applications, *TAMS*, **362**, (2010) 2125-2165
8. R. V. Kadison, J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras Vol I*, Academic Press, San Diego, 1983
9. N. J. Kalton, F. A. Sukochev, Symmetric norms and spaces of operators, *J. reine angew. Math.* **621** (2008), 81121, DOI 10.1515/CRELLE.2008.059
10. L. E. Labuschagne and W. A. Majewski, Maps on Noncommutative Orlicz Spaces, *Illinois J Math* **55**(3) (2011), 1053-1081.
11. L. E. Labuschagne and W. A. Majewski, On Applications of Orlicz Spaces to Statistical Physics. *Ann. Henri Poincaré* **15** (2014), 1197-1221
12. L. E. Labuschagne and C. Steyn, Weighted noncommutative Banach function spaces, *Positivity and Noncommutative Analysis: Festschrift in Honour of Ben de Pagter on the Occasion of his 65th Birthday*
13. V. I. Ovcinnikov, Symmetric spaces of measurable operators, Dokl. Akad. Nauk SSSR, **191** (1970), 769771
14. G. Pistone, C. Sempi, An infinite-dimensional geometric structure on the space of all the probability measures equivalent to a given one, *The Annals of Statistics* **23** (1995), 1543-1561.
15. C. Steyn, An alternative approach to weighted non-commutative Banach function spaces, *C. Complex Anal. Oper. Theory* (2019), https://doi.org/10.1007/s11785-019-00909-y.
16. M. Takesaki, *Theory of Operator Algebras, Vol I,II,III*, Springer, New York, 2003.
17. S. J. Taylor, *Introduction to Measure and Integration*, Cambridge University Press, Cambridge, 1966.
18. M. Terp, *L^p spaces associated with von Neumann algebras.* Københavs Universitet, Mathematisk Institut, Rapport No 3a (1981)
19. J. von Neumann, Some matrix inequalities and metrization of matrix space, Tomsk Univ Rev 1(1937), 286-300.
20. D. V. Widder, *The Laplace transform*, Princeton University Press, 1946

Subject Index

Notation Index