Symmetry classification and conservation laws for some nonlinear partial differential equations

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SYMMETRY CLASSIFICATION AND CONSERVATION LAWS FOR SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR TSHEPO EDWARD MOGOROSI

Date: ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed:..

PROF B MUATJETJEJA

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

TE Mogorosi, IL Freire, B Muatjetjeja, CM Khalique, Group analysis of a hyperbolic Lane-Emden system, Applied Mathematics and Computation, 292, (2017) 156-164

Chapter 3

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Chapter 4

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Chapter 5

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Chapter 6

B Muatjetjeja, TE Mogorosi, Lie reductions and conservation laws of a coupled Jaulent-Miodek system, submitted to Journal of Mathematical analysis and applications

Dedication

I dedicate this work to my late Grandparents, Motswaing and Seonyana Mogorosi, whose memories and support kept me going.

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Abstract

In this thesis we study some nonlinear partial differential equations which appear in several physical phenomena of the real world. Exact solutions and conservation laws are obtained for such equations using various methods. The equations which are studied in this work are: a hyperbolic Lane-Emden system, a generalized hyperbolic Lane-Emden system, a coupled Jaulent-Miodek system and a (2+1)-dimensional Jaulent-Miodek equation power-law nonlinearity.

We carry out a complete Noether and Lie group classification of the radial form of a coupled system of hyperbolic equations. From the Noether symmetries we establish the corresponding conserved vectors. We also determine constraints that the nonlinearities should satisfy in order for the scaling symmetries to be Noetherian. This led us to a critical hyperbola for the systems under consideration. An explicit solution is also obtained for a particular choice of the parameters.

We perform a complete Noether symmetry analysis of a generalized hyperbolic LaneEmden system. Several constraints for which Noether symmetries exist are derived. In addition, we construct conservation laws associated with the admitted Noether symmetries. Thereafter, we briefly discuss the physical meaning of the derived conserved vectors.

We carry out a complete group classification of a generalized coupled hyperbolic Lane-Emden system. It is shown that the underling system admits six-dimensional equivalence Lie algebra. We further show that the principle Lie algebra which is one dimensional extends in several cases. We also carry out Lie reductions for some cases.

Symmetry analysis is performed on a coupled Jaulent-Miodek system, which arises in many branches of physics such as particle physics and fluid dynamics. The similarity reductions and new exact solutions are constructed. Subsequently, conservation laws are derived using the multiplier approach.

We study complete Noether symmetry classification of a (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity. Conservation laws for several cases which admit Noether point symmetries are established.

Introduction

Many physical phenomena of the real world are governed by nonlinear partial differential equations (NLPDEs). Therefore, it is imperative to study these NLPDEs from different points of view. One important aspect of studying NLPDEs is to find their exact explicit solutions. However, this is a very difficult task because there are no specific tools or techniques which can be used to find exact solutions of NLPDEs.

Nevertheless in recent years many scientists have developed various methods of finding exact solutions of NLPDEs. Some of these methods are variable separation approach [1], the ansatz method [2, 3], inverse scattering transform method [4], homogeneous balance method [5], Bäcklund transformation [6], Darboux transformation [7] and Hirota's bilinear method [8], Kudryashov method [9] and the Lie symmetry method [10–21].

Lie symmetry method, also called Lie group method, is one of the most powerful methods to determine solutions of nonlinear partial differential equations. It is based upon the study of the invariance under one parameter Lie group of point transformations. Lie symmetry method was developed by Sophus Lie (1842-1899) in the latter half of the nineteenth century and is highly algorithmic. These methods systematically unify and extend well known ad hoc techniques to construct explicit solutions for differential equations, especially for nonlinear differential equations.

In the study of differential equations (DEs), conservation laws are of undisputed importance. They are the keystone for every fundamental theory of nature. They can provide valuable physical information about the complicated behavior non-linear systems. From the mathematical point of view, when analyzed, they can detect integrability; they can also be employed to check accuracy of numerical methods and they provide an insight into the development of good discretizations technique. In fact, the existence of a large number of conservation laws of a partial differential equation (PDE) is a strong indication of its integrability [10–21]. An association among symmetries and conservation laws for differential equations is set up through Noether theorem [22]. In addition to Lie point symmetries, Noether symmetries are also widely studied and are associated, in particular, with those differential equations which possess Lagrangians. The Noether symmetries, which are symmetries of the Euler-Lagrange systems, have interesting applications in the study of properties of particles moving under the influence of gravitational field. Noether theorem allows construction of conservation laws systematically. However, it can only be applied to differential equations with a Lagrangian. In order to overcome this limitation, several works have been done. See for example [23–29],

Recently, in [29] the conserved quantity was used to determine the unknown exponent in the similarity solution which cannot be obtained from the homogeneous boundary conditions. Thus, it is essential to study conservation laws of differential equations.

This thesis is structured as follows:

In Chapter one we present the preliminaries that are going to be needed in our study.

In Chapter two we carry out a complete Noether and Lie group classification of the radial form of a coupled system of hyperbolic equations. As a result, the arbitrary constants which appear in the system are specified.

Chapter three deals with the Noether symmetry classification of a generalized hyperbolic Lane-Emden system and conservation laws are constructed for various cases.

In Chapter four we perform a complete Lie group classification of the generalized coupled hyperbolic Lane-Emden system, which is studied in chapter 3.

In Chapter five Lie reductions and conservation laws are obtained for a coupled Jaulent-Miodek system, which is encountered in fluid dynamics, particle physics and many other areas of physics and mathematical sciences.

Chapter six studies the (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity. Noether symmetry classification is performed and thereafter conservation laws are constructed for various cases that arise.

Finally in Chapter seven a summary of the results of the thesis is presented and future work is suggested.

Bibliography is given at the end.

Chapter 1

Preliminaries

In this chapter, we present some preliminaries on Lie symmetry analysis and conservation laws of differential equations, which will be used throughout this work and are based on references [10–21].

1.1 One-parameter group of continuous transformations

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a), \;\; \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 (Lie group) A set G of transformations (1.1) is called a continuous one-parameter (local) Lie group of transformations in the space of variables

x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a,b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a,b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a,b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if $\phi(a,b) = a + b$.

Theorem 1.1 For any $\phi(a,b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\,\phi(s,b)}{\partial b}\right|_{b=0}.$$

1.2 Prolongations

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\cdots$.

In group analysis, all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [14].

We now consider a pth-order partial differential equations, namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a, about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x,u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an admitted operator of (1.5) or X is an infinitesimal symmetry of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots.$$

Applying (1.6) and (1.10), we obtain

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha, \end{aligned} \tag{1.11}$$

and so

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

Letting

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

to be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j\text{)}. \qquad (1.15)
\end{aligned}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^{\alpha} = D_j(\eta_i^{\alpha}) - u_{ik}^{\alpha} D_j(\xi^k), \quad (\text{sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^{\alpha} = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^{\alpha}) - u_{i_1,i_2,\ldots,i_{p-1}\,j}^{\alpha} D_{i_p}(\xi^j), \quad (\text{sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} \quad (\text{sum on } i, \alpha), \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^{\alpha} \frac{\partial}{\partial u_{i_1,\ldots,i_p}^{\alpha}} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}.$$

1.3 Group admitted by a partial differential equation

Definition 1.2 (Point symmetry) The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \tag{1.18}$$

is a point symmetry of the pth-order partial differential equation (1.5), if

$$X^{[p]}(E_{\alpha}) = 0 \tag{1.19}$$

whenever $E_{\alpha} = 0$. This can also be written as

$$X^{[p]} \, E_{\alpha}\big|_{E_{\alpha}=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 (Determining equation) Equation (1.19) is called the determining equation of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.4 Infinitesimal criterion of invariance

Definition 1.5 (Invariant) A function $F(x, u)$ is called an invariant of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\, \phi^\alpha(x, u, a)) = F(x, u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 (Lie equations) If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u^\alpha}}{da} = \eta^\alpha(\bar{x},\bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0} = x, \quad \bar{u^\alpha}\big|_{a=0} = u\,.$$

1.5 Conservation laws

1.5.1 Fundamental operators and their relationship

Consider a pth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, given by equation (1.5).

Definition 1.6 (Euler-Lagrange operator) The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1,\ldots,m. \tag{1.25}$$

Definition 1.7 (Lagrangian) If there exists a function $\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)})$, $s \leq p$, p being the order of equation (1.5), such that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0 \quad \alpha = 1, \cdots, m \tag{1.26}$$

then $\mathcal{L}$ is called a Lagrangian of equation (1.5). Equation (1.26) is known as the Euler-Lagrange equation.

Definition 1.8 (Lie-Bäcklund operator) The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.27}$$

where $\mathcal{A}$ is the space of differential functions [14]. The operator (1.27) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.28}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.29}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{1.30}$$

One can write the Lie-Bäcklund operator (1.28) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.31}$$

Definition 1.9 (Conservation law) The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a *conserved vector* of (1.5) if T^i satisfies

$$D_i T^i|_{(1.5)} = 0. \tag{1.32}$$

The equation (1.32) defines a local conservation law of system (1.5).

1.5.2 Multiplier method

The multiplier approach is an effective algorithmic for finding the conservation laws for partial differential equations with any number of independent and dependent variables. Authors in [23] gave this algorithm by using the multipliers presented in [15]. A local conservation law of a given differential system arises from a linear combination formed by local multipliers (characteristics) with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the given differential equation system.

The advantage of this approach is that it does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.33}$$

hold identically, where E_α, D_i are defined by equations (1.5), (1.4) and T^i is defined in definition (1.9).

The right hand side of (1.33) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.34}$$

Once the multipliers are obtained the conserved vectors are constructed by invoking the homotopy operator [23].

1.5.3 Preliminaries on Noether symmetry

In this section we give some salient features of Noether symmetries concerning the system of two second-order partial differential equations (PDEs). For more details see for example [22, 30].

We now consider the vector field

$$
\begin{aligned}
X = \quad & \tau(t,r,u,v)\frac{\partial}{\partial t} + \xi(t,r,u,v)\frac{\partial}{\partial r} + \eta^1(t,r,u,v)\frac{\partial}{\partial u} \\
& + \eta^2(t,r,u,v)\frac{\partial}{\partial v}.
\end{aligned} \tag{1.35}
$$

The first-order prolongation of X is given by

$$
X^{[1]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_t^2\frac{\partial}{\partial v_t} + \zeta_r^1\frac{\partial}{\partial u_r} + \zeta_r^2\frac{\partial}{\partial v_r}, \tag{1.36}
$$

where

$$
\zeta_t^1 = D_t(\eta^1) - u_t D_t(\tau) - u_r D_t(\xi),\ \zeta_r^1 = D_r(\eta^1) - u_t D_r(\tau) - u_r D_r(\xi) \tag{1.37}
$$

$$
\zeta_t^2 = D_t(\eta^2) - v_t D_t(\tau) - v_r D_t(\xi),\ \zeta_r^2 = D_r(\eta^2) - v_t D_r(\tau) - v_r D_r(\xi), \tag{1.38}
$$

and

$$
D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_r} + v_{tt}\frac{\partial}{\partial v_r} + u_{tr}\frac{\partial}{\partial u_r} + v_{tr}\frac{\partial}{\partial v_r} + \cdots, \tag{1.39}
$$

$$
D_r = \frac{\partial}{\partial r} + u_r\frac{\partial}{\partial u} + v_r\frac{\partial}{\partial v} + u_{rr}\frac{\partial}{\partial u_r} + v_{rr}\frac{\partial}{\partial v_r} + u_{tr}\frac{\partial}{\partial u_t} + v_{tr}\frac{\partial}{\partial v_t} + \cdots \tag{1.40}
$$

Recall that the Euler-Lagrange operators are defined by

$$
\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_r\frac{\partial}{\partial u_r} + D_t^2\frac{\partial}{\partial u_{tt}} + D_r^2\frac{\partial}{\partial u_{rr}} + \cdots, \\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_r\frac{\partial}{\partial v_r} + D_t^2\frac{\partial}{\partial v_{tt}} + D_r^2\frac{\partial}{\partial v_{rr}} + \cdots.
\end{aligned}
$$

Definition 1.10 A function $L(t, r, u, v, u_t, v_t, u_r, v_r, \cdots)$ is said to be a Lagrangian of the system of two PDEs of two independent variables (t, r) and two dependent variables (u, v), viz.,

$$\psi_1(t, r, u, v, u_r, v_r, v_{tt}, u_{tt}, u_{rr}, v_{rr}, \cdots) = 0, \tag{1.41}$$

$$\psi_2(t, r, u, v, u_r, v_r, v_{tt}, u_{tt}, u_{rr}, v_{rr}, \cdots) = 0, \tag{1.42}$$

if (1.41)-(1.42) are equivalent to the Euler-Lagrange equations

$$\frac{\delta L}{\delta u} = 0, \quad \frac{\delta L}{\delta v} = 0. \tag{1.43}$$

Definition 1.11 The vector field X, given by (1.35), is called a Noether point symmetry generator associated with a Lagrangian $L(t, r, u, v, u_t, v_t, u_r, v_r)$ of Eqs. (1.41)-(1.42) if there exists gauge functions $B^1(t, r, u, v)$ and $B^2(t, r, u, v)$ such that

$$X^{[1]}(L) + \{D_t(\tau) + D_r(\xi)\}L = D_t(B^1) + D_r(B^2). \tag{1.44}$$

Theorem 1.4 *(Noether [30]) If X given by* (1.35) *is a Noether point symmetry generator corresponding to a Lagrangian $L(t, r, u, v, u_t, v_t, u_r, v_r)$ of Eqs.* (1.41) − (1.42)*, then the vector $T = (T^1, T^2)$ with components*

$$\begin{aligned} T^1 &= \tau L + W^1 \frac{\partial L}{\partial u_t} + W^2 \frac{\partial L}{\partial v_t} - B^1, \\ T^2 &= \xi^1 L + W^1 \frac{\partial L}{\partial u_r} + W^2 \frac{\partial L}{\partial v_r} - B^2, \end{aligned} \tag{1.45}$$

is a conserved vector for Eqs. (1.41) − (1.42) *associated with the operator X. Here W^1 and W^2 are the characteristic functions, given by $W^1 = \eta^1 - u_t\tau - u_r\xi$ and $W^2 = \eta^2 - v_t\tau - v_r\xi$.*

Remark 1.1 We recall that if (T^1, T^2) is a conserved vector for the system (1.41)−(1.42), then $D_tT^1 + D_rT^2 \equiv 0$ on the solutions of (1.41) − (1.42). Therefore this provides a conservation law for such system. If $D_tT^1 + D_rT^2$ provides a vanishing divergency, then we say that (T^1, T^2) is a trivial conservation law. If (T^1, T^2) and $(\tilde{T}^1, \tilde{T}^2)$ are two conserved vectors such that $(T^1 - \tilde{T}^1, T^2 - \tilde{T}^2)$ is a trivial one, then they are said to be equivalent and provide the same conservation law. For further details, see [31] and Chapter 5 of [15].

From the above observations, if the components B^1 and B^2 in (1.45) − (1.45) provides a vanishing divergency $D_t B^1 + D_r B^2 \equiv 0$, the components T^1 and T^2 can therefore be simplified to

$$\begin{aligned} T^1 &= \tau L + W^1 \frac{\partial L}{\partial u_t} + W^2 \frac{\partial L}{\partial v_t}, \\ T^2 &= \xi^1 L + W^1 \frac{\partial L}{\partial u_r} + W^2 \frac{\partial L}{\partial v_r}. \end{aligned} \tag{1.46}$$

1.6 Concluding remarks

In this chapter we presented a brief introduction to the Lie group analysis and conservation laws of partial differential equations and gave some results which will be used throughout this thesis. We also presented methods to determine conservation laws of differential equations.

Chapter 2

Group analysis of a hyperbolic Lane-Emden system

In this chapter we carry out a complete Noether and Lie group classification of the radial form of a coupled system of hyperbolic equations. From the Noether symmetries we establish the corresponding conserved vectors. We also determine constraints that the nonlinearities should satisfy in order for the scaling symmetries to be Noetherian. This led us to a critical hyperbola for the systems under consideration. An explicit solution is also obtained for a particular choice of the parameters.

The study of the coupled elliptic equations

$$\begin{cases} \Delta u + v^q = 0, \\ \Delta v + u^p = 0, \end{cases} \tag{2.1}$$

called Lane-Emden systems, is an active branch in nonlinear analysis. In (2.1), $u = u(x)$, $v = v(x)$ and $x \in \mathbb{R}^n$. Such a system, particularly when $n \geq 3$ and $p, q > 0$, has been widely investigated from different point of views. See for example [32–35].

System (4.1) can be considered as a natural generalisation of the celebrated Lane-Emden equation

$$\left\{ \Delta u + u^p = 0, \right. \tag{2.2}$$

where $u = u(x)$, $x \in \mathbb{R}^n$ [34]. Equation (2.2) has a "natural hyperbolic partner", given by the following nonlinear wave equation:

$$\left\{ u_{tt} - \Delta u - u^p = 0, \right. \tag{2.3}$$

where $(t, x) \in \mathbb{R}^{1+n}$ and $u = u(t, x)$. In this case, t can be interpreted as a time variable, while x corresponds to the spatial ones. It is then natural to consider the hyperbolic generalisation of (2.1), which we shall refer as hyperbolic Lane-Emden system, given by

$$\left\{ \begin{array}{l} \tilde{u}_{tt} - \Delta\tilde{u} - \tilde{v}^q = 0, \\ \\ \tilde{v}_{tt} - \Delta\tilde{v} - \tilde{u}^p = 0, \end{array} \right. \tag{2.4}$$

If we define $r := \|x\|$, $\tilde{u}(t, x) = u(t, r)$ and $\tilde{v}(t, x) = v(t, r)$, system (2.4) can therefore be rewritten in its *radial* form as

$$\left\{ \begin{array}{l} u_{tt} - u_{rr} - \frac{n-1}{r} u_r - v^q = 0, \\ \\ v_{tt} - v_{rr} - \frac{n-1}{r} v_r - u^p = 0, \end{array} \right. \tag{2.5}$$

A simple generalisation of (2.5) can easily be obtained if we replace the integer $n-1$, related with the dimension on the space of spatial coordinates, by an arbitrary real parameter ν.

Thus, in this chapter we consider the following hyperbolic version of the Lane-Emden system:

$$\begin{cases} u_{tt} - u_{rr} - \frac{\nu}{r}u_r - v^q = 0, \\ \\ v_{tt} - v_{rr} - \frac{\nu}{r}v_r - u^p = 0, \end{cases} \tag{2.6}$$

from the point of view of Lie group analysis.

As far as we know, it was the PhD thesis of Gilli Martins [36], and the works arisen from there (see [37] and references therein), that started the investigation of symmetry properties of the Lane-Emden systems in the sense of S. Lie symmetry theory [11, 13, 15, 20, 38]. Since then several works have been done in this direction. See for example [30, 39–45].

If at least one of the powers in (2.4) is 0 we obtain a coupled system such that one of the equations satisfies $w_{tt} - \Delta w - 1 = 0$, which does not have any dependence with respect to the other variable and leads us to a not interesting case. On the other hand, if at least one of them is 1, say q, we can consequently obtain the biwave equation $\Box^2 w - w^p = 0$, where $\Box^2 := \partial_{tt} - \Delta$, a case already investigated, from the point of view of Lie symmetries, in [46]. For this reason, in this chapter we assume that $p, q \neq 0, 1$. The later condition is the only one to be assumed regarding the nonlinearities.

With regard to the parameter ν, we only assume that it is different from 0. Actually, the case $\nu = 0$ can either be obtained from [40], under the complex transformation $(x, y, u, v) \mapsto (t, ir, u, v)$ into the original variables of the mentioned reference, or from [45], making use of projections on the $(t, x)-$space once it is assumed that in [45] the functions u and v depend only on (t, x) instead of (t, x, y). In this case, the symmetries for system (2.6), with $\nu = 0$, for any p and q, are given by

$$\begin{cases} T = \frac{\partial}{\partial t}, \;\; R = \frac{\partial}{\partial r}, \;\; H = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}. \end{cases} \tag{2.7}$$

For other generators, depending on the powers p and q, see [45].

The work of this chapter has been published in [47].

2.1 Noether symmetries and conservation laws of the system (2.6)

We first find the Noether symmetries of the hyperbolic Lane-Emden system (2.6), viz.,

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{\nu}{r}u_r - v^q = 0, \\ v_{tt} - v_{rr} - \dfrac{\nu}{r}v_r - u^p = 0. \end{cases}$$

We need to study four cases separately.

2.1.1 $p \neq -1, q \neq -1$

It can be seen that the hyperbolic Lane-Emden system (2.6) has a variational structure. This is given in the following Lemma.

Lemma.

The hyperbolic Lane-Emden system (2.6) *constitutes of the Euler-Lagrange equations with the functional*

$$J(u,v) = \int_0^\infty \int_0^\infty L(t, r, u, v, u_t, v_t, u_r, v_r)dtdr,$$

where the corresponding function of Lagrange is given by

$$L = r^\nu u_t v_t - r^\nu u_r v_r + \frac{r^\nu}{q+1}v^{q+1} + \frac{r^\nu}{p+1}u^{p+1}, \;\; p \neq -1, \;\; q \neq -1. \tag{2.8}$$

Proof. Follows from (1.43).

$$
\begin{aligned}
\frac{\delta L}{\delta u} &= \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} - D_r \frac{\partial L}{\partial u_r} + D_t^2 \frac{\partial L}{\partial u_{tt}} + D_r^2 \frac{\partial L}{\partial u_{rr}} + \cdots, \\
&= r^\nu u^p - D_t(r^\nu v_t) - D_r(-r^\nu v_r) \\
&= r^\nu u^p - r^\nu v_{tt} + \nu r^{\nu-1} v_r + r^\nu v_{rr} \\
&= -r^\nu (v_{tt} - v_{rr} - \nu r^{-1} v_r - u^p) \\
&= 0 \\
\frac{\delta L}{\delta v} &= \frac{\partial L}{\partial v} - D_t \frac{\partial L}{\partial v_t} - D_r \frac{\partial L}{\partial v_r} + D_t^2 \frac{\partial L}{\partial v_{tt}} + D_r^2 \frac{\partial L}{\partial v_{rr}} + \cdots, \\
&= r^\nu v^q - D_t(r^\nu u_t) - D_r(-r^\nu u_r) \\
&= r^\nu v^q - r^\nu u_{tt} + \nu r^{\nu-1} u_r + r^\nu u_{rr} \\
&= -r^\nu (u_{tt} - u_{rr} - \nu r^{-1} u_r - v^q) \\
&= 0.
\end{aligned}
$$

We now substitute the value of L from (2.8) into Eq. (1.44) and split the resulting equation with respect to derivatives of u and v. This yields the following linear overdetermined system of PDEs:

$$\tau_v = 0, \tag{2.9}$$

$$\tau_u = 0, \tag{2.10}$$

$$\xi_u = 0, \tag{2.11}$$

$$\xi_v = 0, \tag{2.12}$$

$$\eta_u^2 = 0, \tag{2.13}$$

$$\eta_v^1 = 0, \tag{2.14}$$

$$\nu r^{\nu-1}\xi + r^\nu \eta_u^1 - r^\nu \tau_t + r^\nu \eta_v^2 + r^\nu \xi_r = 0, \tag{2.15}$$

$$-\nu r^{\nu-1}\xi - r^\nu \eta_u^1 + r^\nu \xi_r - r^\nu \eta_v^2 - r^\nu \tau_t = 0, \tag{2.16}$$

$$
\begin{aligned}
r^{\nu}\tau_r - r^{\nu}\xi_t &= 0, && (2.17)\\
r^{\nu}\eta^1_t &= B^1_v, && (2.18)\\
r^{\nu}\eta^2_t &= B^1_u, && (2.19)\\
-r^{\nu}\eta^1_r &= B^2_v, && (2.20)\\
-r^{\nu}\eta^2_r &= B^2_u, && (2.21)\\
\frac{\nu}{q+1}r^{\nu-1}\xi v^{q+1} + \frac{\nu}{p+1}r^{\nu-1}\xi u^{p+1} + & \\
r^{\nu}\eta^1 u^p + r^{\nu}\eta^2 v^q + \frac{r^{\nu}}{q+1}\tau_t v^{q+1} + & \\
\frac{r^{\nu}}{p+1}\tau_t u^{p+1} + \frac{r^{\nu}}{q+1}\xi_r v^{q+1} + \frac{r^{\nu}}{p+1}\xi_r u^{p+1} &= B^1_t + B^2_r. && (2.22)
\end{aligned}
$$

The above system is now solved for $\tau, \xi, \eta^1, \eta^2, B^1$ and B^2.

Eqs. (2.9) and (2.10) imply that

$$\tau = a(t, r), \qquad (2.23)$$

where $a(t, r)$ is an arbitrary function of t and r. Eqs.(2.11) and (2.12), imply that

$$\xi = b(t, r), \qquad (2.24)$$

where $b(t, r)$ is an arbitrary function of t and r. Eq. (2.14), gives

$$\eta^1 = c(t, r, u), \qquad (2.25)$$

where $c(t, r, v)$ is an arbitrary function of t, r and v. Solving Eq. (2.13), we get

$$\eta^2 = d(t, r, v), \qquad (2.26)$$

where $d(t, r, v)$ is an arbitrary function of t, r and v. Substituting the values of τ, ξ, η^1 and η^2 into (2.15) and solving for $c(t, r, u)$ gives

$$\eta^1 = c(t, r, u) = (a_t(t, r) - b_r(t, r) - d_v(t, r, v) - \frac{\nu}{r}b(t, r))u + e(t, r). \qquad (2.27)$$

Substituting the value of η^1 into (2.16) gives

$$b_r(t,r) - a_t(t,r) = 0. \tag{2.28}$$

Solving Eq. (2.17), we obtain

$$a_r(t,r) - b_t(t,r) = 0. \tag{2.29}$$

Replacing the values of τ, ξ, η^1 and η^2 back into Eqs. (2.18) and (2.19), we get

$$B^1 = r^\nu d_t u + r^\nu e_t v + g(t,r). \tag{2.30}$$

Similarly by solving Eqs. (2.20) and (2.21), yield

$$B^2 = -r^\nu d_r u - r^\nu e_t v + k(t,r). \tag{2.31}$$

Now substituting these values of $\tau, \xi, \eta^1, \eta^2, B^1$ and B^2 into (2.22) and simplifying yields

$$\begin{aligned}
\tau &= a(t,r), \\
\xi &= b(t,r), \\
\eta^1 &= -d_v u - \frac{\nu}{r} bu + e(t,r), \\
\eta^2 &= d(t,r,v), \\
B^1 &= r^\nu d_t u + r^\nu e_t v + g(t,r), \\
B^2 &= -r^\nu d_r u - r^\nu e_t v + k(t,r),
\end{aligned}$$

$$\begin{aligned}
&\frac{\nu}{q+1} r^{\nu-1} bv^{q+1} + \frac{\nu}{p+1} r^{\nu-1} bu^{p+1} + \frac{r^\nu}{q+1} a_t v^{q+1} + r^\nu dv^q + \\
&r^\nu u^p(-d_v u - \frac{\nu}{r} bu + e(t,r)) + \frac{r^\nu}{p+1} a_t u^{p+1} + \frac{r^\nu}{q+1} b_r v^{q+1} + \\
&\frac{r^\nu}{p+1} b_r u^{p+1} = r^\nu d_{tt} u - r^\nu d_{rr} u - \nu r^{\nu-1} d_r u + r^\nu e_{tt} v - r^\nu e_{rr} v \\
&-\nu r^{\nu-1} e_r v + g_t + k_r.
\end{aligned} \tag{2.32}$$

The examination of Eq.(2.32) gives rise to the four cases. In what follows, $g = g(t, r)$ and $k = k(t, r)$ are arbitrary functions.

Case 1: ν arbitrary but not in the form in Case 2 – Case 4

This case gives only one Noether point symmetry, namely

$$X_1 = \frac{\partial}{\partial t}, \tag{2.33}$$

with $B^1 = g$, $B^2 = k$ and $g_t + k_r = 0$. The use of Theorem 1.4 and Remark 1.1 yield the following nontrivial conserved vector associated with this Noether point symmetry:

$$\begin{aligned} T_1^1 &= -r^\nu u_t v_t - r^\nu u_r v_r + \frac{r^\nu}{q+1} v^{q+1} + \frac{r^\nu}{p+1} u^{p+1}, \\ T_2^1 &= r^\nu u_t v_r + r^\nu u_r v_t. \end{aligned} \tag{2.34}$$

Case 2: $p \neq q, \ \nu = \dfrac{2q + 2p + 4}{pq - 1}$

In this case we obtain two Noether point symmetries, viz., X_1 given by (2.33) and

$$X_{p,q} = (1 - pq) r \frac{\partial}{\partial r} + (1 - pq) t \frac{\partial}{\partial t} + 2(1 + q) u \frac{\partial}{\partial u} + 2(1 + p) v \frac{\partial}{\partial v}, \tag{2.35}$$

with $B^1 = g$, $B^2 = k$ and $g_t + k_r = 0$. The application of Noether conserved vectors (1.46) gives the following two nontrivial conserved vectors corresponding to the two Noether symmetries. The first one, established from X_1, was already obtained in (2.34), while the new one, obtained by using (2.35) is given by

$$\begin{aligned} T_1^2 &= -tr^\nu u_t v_t - tr^\nu u_r v_r - r^{\nu+1} u_r v_t - r^{\nu+1} u_t v_r - \frac{2(q+1)}{pq-1} r^\nu u v_t \\ &- \frac{2(p+1)}{pq-1} r^\nu u_t v + \frac{t}{q+1} r^\nu v^{q+1} + \frac{t}{p+1} r^\nu u^{p+1}, \\ T_2^2 &= r^{\nu+1} u_t v_t + r^{\nu+1} u_r v_r + tr^\nu u_t v_r + tr^\nu u_r v_t + \frac{2(q+1)}{pq-1} r^\nu u v_r \\ &+ \frac{2(p+1)}{pq-1} r^\nu u_r v + \frac{r^{\nu+1}}{q+1} v^{q+1} + \frac{r^{\nu+1}}{p+1} u^{p+1}. \end{aligned} \tag{2.36}$$

Case 3: $p = q$, $\nu = \dfrac{4}{q-1}$

For this case we obtain three Noether point symmetry operators, viz., X_1 given by (2.33), X_p defined by

$$X_p \equiv \frac{1}{1+p} X_{p,p} = (1-p)r\frac{\partial}{\partial r} + (1-p)t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} \tag{2.37}$$

and

$$X_3 = \frac{1}{2}(t^2+r^2)\frac{\partial}{\partial t} + rt\frac{\partial}{\partial r} + \frac{2}{1-p}ut\frac{\partial}{\partial u} + \frac{2}{1-p}vt\frac{\partial}{\partial v} \tag{2.38}$$

with $B^1 = \dfrac{2}{1-p}r^\nu uv + g$, $B^2 = k$ and $g_t + k_r = 0$.

Remark 1.1 yields the three nontrivial conserved vectors. Those obtained from X_1 and X_p are given, respectively, by (2.34) and (2.36) with $p = q$. Using Theorem 1.4, we obtain the following conserved vector corresponding to the Noether operator (2.38):

$$\begin{aligned}
T_1^3 &= -\frac{1}{2}t^2r^\nu u_r v_r - \frac{1}{2}r^{\nu+2}u_r v_r - \frac{1}{2}t^2 r^\nu u_t v_t - \frac{1}{2}r^{\nu+2}u_t v_t - tr^{\nu+1}u_r v_t - tr^{\nu+1}u_t v_r \\
&\quad + \frac{1}{2(q+1)}t^2r^\nu v^{q+1} + \frac{1}{2(q+1)}t^2 r^\nu u^{q+1} + \frac{1}{2(q+1)}r^{\nu+2}v^{q+1} + \frac{1}{2(q+1)}r^{\nu+2}u^{q+1} \\
&\quad - \frac{2}{q-1}tr^\nu u v_t - \frac{2}{q-1}tr^\nu u_t v + \frac{2}{q-1}r^\nu uv, \\
T_2^3 &= tr^{\nu+1}u_t v_t + \frac{1}{q+1}tr^{\nu+1}v^{q+1} + \frac{1}{q+1}tr^{\nu+1}u^{q+1} + \frac{2}{q-1}tr^\nu u v_r + \frac{1}{2}t^2 r^\nu u_t v_r.
\end{aligned}$$

Case 4: $p = q$, $\nu \neq \dfrac{4}{q-1}$

One Noether point symmetry operator is obtained in this case, which is X_1 given by (2.33) and so the use of Theorem 1.4 gives again the nontrivial conserved vector (2.34) with $p = q$.

2.1.2 $p = -1, q = -1$

When $p = -1$ and $q = -1$ the hyperbolic Lane-Emden system (2.6) becomes

$$u_{tt} - u_{rr} - \frac{\nu}{r}u_r - \frac{1}{v} = 0, \tag{2.39}$$

$$v_{tt} - v_{rr} - \frac{\nu}{r}v_r - \frac{1}{u} = 0. \tag{2.40}$$

Then the associated function of Lagrange of system (2.39)-(2.40) is defined by

$$L = r^\nu u_t v_t - r^\nu u_r v_r + r^\nu \ln|v| + r^\nu \ln|u|. \tag{2.41}$$

Following the above procedure as in Case 1, one obtains two Noether point symmetries, viz., X_1 given by (2.33) and

$$X_2 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \tag{2.42}$$

with $B^1 = g$, $B^2 = k$ and $g_t + k_r = 0$.

The components (1.46) therefore gives, respectively, the following conserved vectors associated with these Noether point symmetries:

$$\begin{aligned} T_1^1 &= -r^\nu u_r v_r - r^\nu u_t v_t + r^\nu \ln|v| + r^\nu \ln|u|, \\ T_2^1 &= r^\nu u_t v_r + r^\nu u_r v_t \end{aligned}$$

and

$$\begin{aligned} T_1^2 &= r^\nu u v_t - r^\nu v u_t, \\ T_2^2 &= r^\nu u v_r - r^\nu v u_r. \end{aligned}$$

2.1.3 $p = -1, q \neq -1$

For the case when $p = -1$ and $q \neq -1$, the hyperbolic Lane-Emden system (2.6) becomes

$$u_{tt} - u_{rr} - \frac{\nu}{r} u_r - v^q = 0, \tag{2.43}$$

$$v_{tt} - v_{rr} - \frac{\nu}{r} v_r - \frac{1}{u} = 0. \tag{2.44}$$

The corresponding Lagrange of system (2.43)-(2.44) is given by

$$L = r^\nu u_t v_t - r^\nu u_r v_r + \frac{1}{q+1} r^\nu v^{q+1} + r^\nu \ln |u|, \ q \neq -1. \tag{2.45}$$

This case provide us with one Noether point symmetry, viz., X_1 given by (2.33). The application of Theorem 1.4 gives the nontrivial conserved vector

$$\begin{aligned} T_1^1 &= -r^\nu u_r v_r - r^\nu u_t v_t + r^\nu \ln |v| + r^\nu \ln |u|, \\ T_2^1 &= r^\nu u_t v_r + r^\nu u_r v_t \end{aligned}$$

associated with the Noether operator (2.33).

2.1.4 $p \neq -1, q = -1$

In this case the hyperbolic Lane-Emden system (2.6) becomes

$$u_{tt} - u_{rr} - \frac{\nu}{r} u_r - \frac{1}{v} = 0, \tag{2.46}$$

$$v_{tt} - v_{rr} - \frac{\nu}{r} v_r - u^p = 0 \tag{2.47}$$

The assosiated Lagrangian for system (2.46)-(2.47) is

$$L = r^\nu u_t v_t - r^\nu u_r v_r + r^\nu \ln |v| + \frac{1}{p+1} r^\nu u^p, \ p \neq -1. \tag{2.48}$$

Following the above proceedure, we obtain one Noether point symmetry viz., X_1 given by (2.33) and making use of Theorem 1.4 we obtain the associated nontrivial conserved vector

$$\begin{aligned} T_1^1 &= -r^\nu u_r v_r - r^\nu u_t v_t + r^\nu \ln|v| + r^\nu + \frac{1}{p+1}u^{p+1}, \\ T_2^1 &= r^\nu u_t v_r + r^\nu u_r v_t. \end{aligned}$$

2.2 Comparison of Lie and Noether symmetries of (2.6)

Here we carry out a complete group classification of system (2.6). According to the Lie symmetry theory, a differential operator X, given by (1.35) generates a one-parameter group of transformations

$$T_\varepsilon(t,r,u,v) = e^{\varepsilon X}(t,r,u,v) \tag{2.49}$$

preserving the solutions (symmetries) of the system (2.6) if and only if

$$X^{[2]}(u_{tt} - u_{rr} - \frac{\nu}{r}u_r - v^q)\Big|_{(2.6)} \equiv 0, \quad X^{[2]}(v_{tt} - v_{rr} - \frac{\nu}{r}v_r - u^p)\Big|_{(2.6)} \equiv 0, \tag{2.50}$$

where

$$X^{[2]} = X^{[1]} + \zeta^1_{rr}\frac{\partial}{\partial u_{rr}} + \zeta^1_{tt}\frac{\partial}{\partial u_{tt}} + \zeta^2_{rr}\frac{\partial}{\partial v_{rr}} + \zeta^2_{tt}\frac{\partial}{\partial v_{tt}}$$

is the extension of X to the jet space $(t, r, u, v, u_r, u_{rr}, u_{tt})$, where $X^{[1]}$ is given by (1.36) and

$$\zeta^1_{rr} = D_r(\zeta^1_r) - u_{rt}D_r(\tau) - u_{rr}D_r(\xi), \tag{2.51}$$

$$ \tag{2.52} $$

$$\zeta^1_{tt} = D_t(\zeta^1_t) - u_{tt}D_t(\tau) - u_{tr}D_t(\xi), \tag{2.53}$$

$$ \tag{2.54} $$

$$\zeta^2_{rr} = D_r(\zeta^2_r) - v_{rt}D_r(\tau) - v_{rr}D_r(\xi), \tag{2.55}$$

$$ \tag{2.56} $$

$$\zeta^2_{tt} = D_t(\zeta^2_t) - v_{tt}D_t(\tau) - v_{tr}D_t(\xi). \tag{2.57}$$

The reader is referred to [11, 13, 15, 20, 38] for further details.

The conditions (2.50) lead us to the system of the determining equations:

$$\eta^2_{ru} = 0, \tag{2.58}$$

$$\eta^1_{rv} = 0, \tag{2.59}$$

$$\eta^2_{tu} = 0, \tag{2.60}$$

$$\eta^1_{tv} = 0, \tag{2.61}$$

$$\eta^2_{uu} = 0, \tag{2.62}$$

$$\eta^1_{uu} = 0, \tag{2.63}$$

$$\eta^2_{uv} = 0, \tag{2.64}$$

$$\eta^1_{uv} = 0, \tag{2.65}$$

$$\tau_u = 0, \tag{2.66}$$

$$\xi_u = 0, \tag{2.67}$$

$$\eta^2_{vv} = 0 \tag{2.68}$$

$$\eta^1_{vv} = 0, \tag{2.69}$$

$$\tau_v = 0, \tag{2.70}$$

$$\xi_v = 0, \tag{2.71}$$

$$\xi_t - \tau_r = 0, \tag{2.72}$$

$$\tau_t - \xi_r = 0, \tag{2.73}$$

$$r\left(\nu\xi_r + 2r\eta^2_{rv}\right) - \nu\xi^1 = 0, \qquad (2.74)$$

$$r\left(\nu\xi_r + 2r\eta^1_{ru}\right) - \nu\xi = 0, \qquad (2.75)$$

$$2r\eta^1_{tu} + \nu\tau_r + r\left(\tau_{rr} - \xi_{rt}\right) = 0, \qquad (2.76)$$

$$2r\eta^2_{tv} + \nu\tau_r + r\left(\tau_{rr} - \xi_{rt}\right) = 0, \qquad (2.77)$$

$$u\left(ru^p\eta^2_v + rv^q\eta^2_u + r\eta^2_{tt} - \nu\eta^2_r - 2ru^p\xi_r - r\eta^2_{rr}\right) - pru^p\eta^1 = 0, \qquad (2.78)$$

$$v\left(ru^p\eta^1_v + rv^q\eta^1_u + r\eta^1_{tt} - \nu\eta^1_r - 2rv^q\xi_r - r\eta^1_{rr}\right) - qrv^q\eta^2 = 0. \qquad (2.79)$$

The solution of system (2.6) leads the following theorem.

Theorem 2.1 Let (1.35) be a Lie point symmetry generator of (2.6). Then

1. For any values of p, q and ν, X is a linear combination of the generators (2.33) and (2.35).

2. If $p = q$ and $\nu \neq \frac{4}{p-1}$, the generators are (2.33) and (2.37).

3. If $p = q$, $q \neq -1$, and $\nu = \frac{4}{q-1}$, the generators are (2.33), (2.37) and (2.38).

4. If $p = q = -1$ and $\nu = -2$, the generators are (2.33), (2.38), (2.42) and

$$Y = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}. \qquad (2.80)$$

5. If $p = q = -1$ and $\nu \neq -2$, the generators are (2.33), (2.42) and (2.80).

Remark 2.1 We observe that $X_{p,p} = (1+p)X_p$.

Remark 2.2 From Section 2.1 we observe that (2.35) is a Noether symmetry depending on certain values of the parameters ν, p and q. According to Theorem 2.2, (2.35) is always a Lie point symmetry generator. Thus we shall now look for when it is also a Noether symmetry.

The left hand side of (1.44), namely

$$X^{[1]}(L) + \{D_t(\tau) + D_r(\xi)\}L = [(1 - pq)(\nu + 2) + 2(1 + p + q + pq)]L, \quad (2.81)$$

for all Lagrangian L given in Section 2.1. This implies that (2.35) is a Noether symmetry if and only if

$$(1 - pq)(\nu + 2) + 2(1 + p + q + pq) = 0. \quad (2.82)$$

If we assume that $p, q \neq -1$, (2.82) is equivalent to the hyperbola

$$\frac{\nu + 2}{p + 1} + \frac{\nu + 2}{q + 1} = \nu. \quad (2.83)$$

Additionally, according to the results of Case 3 and those presented in Theorem 2.2, all Lie point symmetries are Noether symmetries. Then the condition (2.83) can be considered as a critical hyperbola for the system (2.6). Such condition was already observed in Lane-Emden systems in [36, 37, 40, 41, 48]. For further discussions, see [48, 49]. References [48, 50] provides enough and interesting discussions about this fact.

In particular, if $\nu = n - 1$ in (2.6), (2.83) reads

$$\frac{n + 1}{p + 1} + \frac{n + 1}{q + 1} = n - 1,$$

a well known result, see [48], Theorem 10.

Remark 2.3

If we consider $\nu = 0$ in (2.82) we obtain the constraint

$$p + q + 2 = 0. \tag{2.84}$$

Condition (2.84) was called critical straight line in [40] with respect to the system

$$\begin{aligned} u_{tt} + u_{yy} + v^q &= 0, \\ v_{tt} + v_{yy} + u^p &= 0 \end{aligned} \tag{2.85}$$

on $\mathbb{R}^2$. In fact, according to the results obtained in [40], if $p, q \neq -1$ satisfy (2.84), then all Lie point symmetries are also Noether symmetries. Combining the results of Case 1, Case 2 and Theorem 2.2, we recover the same conclusion.

Additionaly, from (2.5) and (2.6) we observe that $\nu = 0$ correspond to the case $n = 1$. This implies that we have just the system (2.4) in $\mathbb{R}^{1+1}$. In this case, generators (2.7) correspond to the generators of the isometry group of $(\mathbb{R}^2, ds^2)$, with $ds^2 = dt^2 - dr^2$.

Remark 2.4 From (2.6) we conclude that $u(t,r) = \phi(t)$ and $v(r,t) = \psi(t)$, where ϕ and ψ satisfy

$$\begin{aligned} \phi'' + \psi^q &= 0, \\ \psi'' + \phi^p &= 0. \end{aligned} \tag{2.86}$$

From the results obtained in [42], we conclude that

$$\psi(t) = \sqrt{2t}, \;\; \phi(t) = \sqrt{2t} \tag{2.87}$$

are solutions of (2.86) with $p = q = -3$.

Using that the hyperbolic rotation $(t, r, u, v) \mapsto (t\cosh\varepsilon + r\sinh\varepsilon, r\cosh\varepsilon + t\sinh\varepsilon, u, v)$ is a symmetry of

$$u_{tt} - u_{rr} - v^{-3} = 0,$$
$$v_{tt} - v_{yy} - u^{-3} = 0. \tag{2.88}$$

Then, from (2.87) we obtain a one-parameter family of solutions to (2.88) given by

$$u_\varepsilon(t, r) = v_\varepsilon(t, r) = \sqrt{2(t\cosh\varepsilon + r\sinh\varepsilon)},$$

provided that $t\cosh\varepsilon + r\sinh\varepsilon > 0$.

2.3 Concluding remarks

In this chapter we carry out a complete Noether and Lie group classication of the radial form of a coupled system of hyperbolic equations. From the Noether symmetries we establish the corresponding conserved vectors. We also determine constraints that the non-linearities should satisfy in order for the scaling symmetries to be Noetherian. This led us to a critical hyperbola for the systems under consideration. An explicit solution is also obtained for a particular choice of the parameters.

Chapter 3

Variational principle and conservation laws of a generalized hyperbolic Lane-Emden system

In this chapter we study the coupled generalized hyperbolic Lane-Emden system

$$
\begin{aligned}
&u_{tt} - u_{rr} - \frac{m}{r}u_r + f(v) = 0, \\
&v_{tt} - v_{rr} - \frac{m}{r}v_r + g(u) = 0,
\end{aligned}
\tag{3.1}
$$

with the spatial dimensions $m \neq 0$ and $f(v)$, $g(u)$ are non-zero arbitrary self-interaction functions of v and u respectively. The parameter m is assumed to be different from 0. Actually if $m = 0$, system (3.1) can be obtained from [30], under the complex transformation $(x, y, u, v) \longmapsto (t, ir, u, v)$ into the original variables of the mentioned reference.

Systems of this type arise in many physical applications, see for example [35,37,40,47,48,51] and references therein. System (3.1) can also be considered as a natural two-component generalization of the nonlinear wave equation:

$$u_{tt} - u_{rr} - \frac{m}{r}u_r - u^p = 0, \tag{3.2}$$

where $u = u(t, r)$ is a real-valued function, with p denoting the interaction power and (t, r) represent time and radial coordinates respectively in $m \neq 0$ dimensions.

The methods of modern group analysis have be used to study equations (3.1)-(3.2). However, to the authors knowledge, the method of Noether symmetry analysis has not been used in the study of the generalized hyperbolic Lane-Emden system (3.1). Hence the aim of this chapter is to compensate this absence by performing a complete Noether symmetry classification of system (3.1) and construct conservation laws of system (3.1). Conservation laws are mathematical expressions of the physical laws, such as conservation of energy, mass, momentum and so on. They play a very crucial role in the solution and reduction of partial differential equations. Conservation laws have been extensively used in studying the existence, uniqueness and stability of solutions of nonlinear partial differential equations. See for example [52] and references therein. They have also been used in the development and use of numerical methods. Noether theorem [22] gives us an elegant way to derive conservation laws provided a Lagrangian is known for an EulerLagrange equation. Thus, the knowledge of a Lagrangian is important in this work.

The work of this chapter has been published in [53].

3.1 Noether symmetries and conservation laws

The hyperbolic Lane-Emden system (3.1) admits a general variational structure. This yields the following Lemma.

Lemma.

The generalized hyperbolic wave system (3.1) *constitutes of the Euler-Lagrange equations with the functional*

$$J(u,v)=\int_0^{\infty}\int_0^{\infty}L(t,r,u,v,u_t,v_t,u_r,v_r)dtdr,$$

where the corresponding function of Lagrange is given by

$$L=r^m u_t v_t - r^m u_r v_r - r^m\int f(v)dv - r^m\int g(u)du. \tag{3.3}$$

Substituting L in the Euler-Lagrange equations (1.43) yields

$$\frac{\delta L}{\delta u}=v_{tt}-\Delta v+g(u)=0,\ \frac{\delta L}{\delta v}=u_{tt}-\Delta u+f(v)=0. \tag{3.4}$$

Note that these Euler-Lagrange equations are twisted in the sense that the variational derivative of L with respect to u, v yields the hyperbolic Lane-Emden system (3.1) for v, u respectively.

We now insert the expression of L from (3.3) into Eq.(1.44) and following the Noether algorithm, yields the symmetries determining equations:

$$\begin{aligned}
\xi^1_v &= 0,\\
\xi^1_u &= 0,\\
\xi^2_u &= 0,\\
\xi^2_v &= 0,\\
\eta^2_u &= 0,\\
\eta^1_v &= 0,\\
mr^{m-1}\xi^2+r^m\eta^1_u-r^m\xi^1_t+r^m\eta^2_v+r^m\xi^2_r &= 0,\\
-mr^{m-1}\xi^2-r^m\eta^1_u+r^m\xi^2_r-r^m\eta^2_v-r^m\xi^1_t &= 0,\\
r^m\xi^1_r-r^m\xi^2_t &= 0,
\end{aligned}$$

$$
\begin{aligned}
r^m\eta^1_t &= B^1_v,\\
r^m\eta^2_t &= B^1_u,\\
-r^m\eta^1_r &= B^2_v,\\
-r^m\eta^2_r &= B^2_u,\\
-mr^{m-1}\xi^2\int f(v)dv - mr^{m-1}\xi^2\int g(u)du - r^m\eta^1 g(u) - r^m\eta^2 f(v) &\\
-r^m\xi^1_t\int f(v)dv - r^m\xi^1_t\int g(u)du - r^m\xi^2_r\int f(v)dv - r^m\xi^2_r\int g(u)du &= B^1_t + B^2_r.
\end{aligned}
$$

After some substantial algebra, the above system of PDEs yields

$$
\begin{aligned}
\xi^1 &= a(t,r),\\
\xi^2 &= b(t,r),\\
\eta^1 &= -c_v u - \frac{m}{r}bu + d(t,r),\\
\eta^2 &= c(t,r,v),\\
B^1 &= r^m c_t u + r^m d_t v + w(t,r),\\
B^2 &= -r^m c_r u - r^m d_r v + z(t,r),
\end{aligned}
$$

$$
\begin{aligned}
&-mr^{m-1}b\int f(v)dv - mr^{m-1}b\int g(u)du + r^m u c_v g(u) + mr^{m-1}bug(u) - r^m d g(u)\\
&-r^m c f(v) - r^m a_t\int f(v)dv - r^m a_t\int g(u)du - r^m b_r\int f(v)dv - r^m b_r\int g(u)du\\
&= r^m u c_{tt} - r^m u c_{rr} - mr^{m-1}uc_r + r^m v d_{tt} - r^m v d_{rr} - mr^{m-1}vd_r + w_t + z_r. \quad (3.5)
\end{aligned}
$$

A complete analysis of Eq. (3.5) prompts the following cases.

Case 1. $m \neq 0, f(v),\ g(u)$ are arbitrary functions but not in the form contained in Cases 2–11.

Here the generalized hyperbolic Lane-Emden system (3.1) admits a one-dimensional Noether algebra, viz.,

$$X_1 = \frac{\partial}{\partial t},\ B^1 = w, B^2 = z,\ w_t + z_x = 0. \quad (3.6)$$

The application of Theorem 1.4, yields the nontrivial conserved vector associated with this Noether point symmetry as

$$
\begin{aligned}
T_1^1 &= -r^m u_t v_t - r^m u_r v_r - r^m \int f(v)dv - r^m \int g(u)du, \\
T_2^1 &= r^m u_t v_r + r^m u_r v_t.
\end{aligned} \tag{3.7}
$$

We have set $w = 0$ and $z = 0$ as they contribute to the trivial part of the conserved vector. This observations will be used in the latter cases without further discussion.

Case 2. $f(v) = \alpha v + \beta, g(u) = \gamma u + \lambda$, with $\alpha, \gamma, \beta, \lambda$ are constants, $\alpha, \gamma \neq 0$ and m arbitrary.

In this case the generalized hyperbolic Lane-Emden system (3.1) admits two Noether symmetries, viz., X_1 given by (3.6), and

$$
X_2 = d(t,r)\frac{\partial}{\partial u} + e(t,r)\frac{\partial}{\partial v},\ B^1 = r^m u e_t + r^m u d_t, B^2 = r^m u e_r - r^m u d_r, \tag{3.8}
$$

Employing Theorem 1.4, the two nontrivial conserved vectors associated with these Noether point symmetries is (3.7) established from X_1 while the new one is

$$
\begin{aligned}
T_2^1 &= r^m e u_t + r^m d v_t - r^m u e_t - r^m v d_t, \\
T_2^2 &= r^m u e_r + r^m v d_r - r^m e u_r - r^m d v_r
\end{aligned} \tag{3.9}
$$

obtained by using (3.8) where $d(t,r)$ and $e(t,r)$ are any solutions of the system $d_{tt} - d_{rr} - \frac{m}{r}d_r + \alpha e = 0$, $e_{tt} - e_{rr} - \frac{m}{r}e_r + \gamma d = 0$, $\lambda d + \beta e = 0$.

Here we observe that due to the presence of the arbitrary functions $d(t,r)$ and $e(t,r)$, one obtains infinitely many local conserved vectors for system (3.1).

Case 3. $f(v) = \alpha v^q$, $g(u) = \gamma u^p$, $\alpha, \gamma \neq 0$.

Here we have two subcases:

Case 3.1. $p \neq q, m = \dfrac{2q + 2p + 4}{pq - 1}$.

Here system (3.1) admits two Noether generators, namely X_1 given by (3.6) and

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2(q+1)}{pq-1}u\frac{\partial}{\partial u} - \frac{2(p+1)}{pq-1}v\frac{\partial}{\partial v},\ B^1 = 0, B^2 = 0, \quad (3.10)$$

and the associated nontrivial conserved vectors are (3.7) obtained from X_1 while the extra one is

$$\begin{aligned} T_2^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - \frac{2(q+1)}{pq-1} r^m u v_t \\ &\quad - \frac{2(p+1)}{pq-1} r^m u_t v + \frac{\alpha t}{q+1} r^m v^{q+1} + \frac{\gamma t}{p+1} r^m u^{p+1}, \\ T_2^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + \frac{2(q+1)}{pq-1} r^m u v_r \\ &\quad + \frac{2(p+1)}{pq-1} r^m u_r v + \frac{\alpha r^{m+1}}{q+1} v^{q+1} + \frac{\gamma r^{m+1}}{p+1} u^{p+1} \end{aligned} \quad (3.11)$$

obtained from (3.10).

It should be noted that when $pq = 1$, we get only the one-dimensional Noether algebra (3.6).

Case 3.2. $p = q, m = \dfrac{4}{q-1}$.

In this case system (3.1) admits X_1 given by (3.6) and two extra Noether operators, namely

$$X_2 = (1-p)r\frac{\partial}{\partial r} + (1-p)t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v},\ B^1 = 0, B^2 = 0, \quad (3.12)$$

$$\begin{aligned} X_3 &= \frac{1}{2}(t^2 + r^2)\frac{\partial}{\partial t} + rt\frac{\partial}{\partial r} + \frac{2}{1-p}ut\frac{\partial}{\partial u} + \frac{2}{1-p}vt\frac{\partial}{\partial v}, \\ &\quad B^1 = -\frac{2}{q-1} r^m uv, B^2 = 0 \end{aligned} \quad (3.13)$$

and the resulting nontrivial conserved vectors are (3.7) obtained from X_1 and (3.11) established from X_2 with $p = q$, while the extra one is

$$\begin{aligned} T_3^1 &= -\frac{1}{2}t^2 r^m u_r v_r - \frac{1}{2} r^{m+2} u_r v_r - \frac{1}{2} t^2 r^m u_t v_t - \frac{1}{2} r^{m+2} u_t v_t - tr^{m+1} u_r v_t - tr^{m+1} u_t v_r \\ &\quad + \frac{\alpha}{2(q+1)} t^2 r^m v^{q+1} + \frac{\gamma}{2(q+1)} t^2 r^m u^{q+1} + \frac{\alpha}{2(q+1)} r^{m+2} v^{q+1} + \frac{\gamma}{2(q+1)} r^{m+2} u^{q+1} \end{aligned}$$

$$-\frac{2}{q-1}tr^m uv_t - \frac{2}{q-1}tr^m u_t v + \frac{2}{q-1}r^m uv,$$

$$\begin{aligned}
T_3^2 &= tr^{m+1}u_t v_t - \frac{\alpha}{q+1}tr^{m+1}v^{q+1} - \frac{\gamma}{q+1}tr^{m+1}u^{q+1} + \frac{2}{q-1}tr^m uv_r + \frac{1}{2}t^2 r^m u_t v_r \\
&+ \frac{1}{2}r^{m+2}u_t v_r + tr^{m+1}u_r v_r + \frac{2}{q-1}tr^m u_r v + \frac{1}{2}t^2 r^m u_r v_t + \frac{1}{2}r^{m+2}u_r v_t \qquad (3.14)
\end{aligned}$$

obtained by using (3.13).

Note that when $q = 1$, then this case is subsumed in case 6.

Case 4. $f(v)$ arbitrary, $g(u) = \gamma u^{-1}$, $\gamma \neq 0$.

In this case we have two subcases:

Case 4.1. m arbitrary.

Here the generalized hyperbolic wave system (3.1) has two Noether generators, namely X_1 given by (3.6) and

$$X_2 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \;\; B^1 = w, \;\; B^2 = z, \;\; w_t + z_r = r^m v f(v) - \gamma r^m, \qquad (3.15)$$

and the use of the components (1.45) give the nontrivial conserved vectors; (3.7) derived from X_1 and X_2 yields

$$\begin{aligned}
T_2^1 &= r^m uv_t - r^m vu_t - w, \\
T_2^2 &= r^m u_r v - r^m uv_r - z. \qquad (3.16)
\end{aligned}$$

Case 4.2. $m = -2$.

In this case system (3.1) admits X_1, X_2 given by (3.6), (3.15) and extra two new Noether operators, namely

$$\begin{aligned}
X_3 &= t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2u\frac{\partial}{\partial u}, \;\; B^1 = w, \;\; B^2 = z, \;\; w_t + z_r = -2\gamma r^m, \qquad (3.17) \\
X_4 &= t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2v\frac{\partial}{\partial v}, \;\; B^1 = w, \;\; B^2 = z, \;\; w_t + z_r = -2r^m v f(v), \qquad (3.18)
\end{aligned}$$

and the invocation of the components (1.45) give the nontrivial conserved vectors;

(3.7), (3.16) and

$$\begin{aligned} T_3^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1}u_r v_t - r^{m+1}u_t v_r + 2r^m u v_t - \gamma t r^m \ln u \\ &\quad -tr^m \int f(v)dv - w, \\ T_3^2 &= r^{m+1}u_t v_t + r^{m+1}u_r v_r + tr^m u_t v_r + tr^m u_r v_t - 2r^m u v_r - \gamma r^{m+1} \ln u \\ &\quad -r^{m+1} \int f(v)dv - z; \end{aligned} \tag{3.19}$$

$$\begin{aligned} T_4^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1}u_r v_t - r^{m+1}u_t v_r + 2r^m u_t v - \gamma t r^m \ln u \\ &\quad -tr^m \int f(v)dv - w, \\ T_4^2 &= r^{m+1}u_t v_t + r^{m+1}u_r v_r + tr^m u_t v_r + tr^m u_r v_t - 2r^m u_r v - \gamma r^{m+1} \ln u \\ &\quad -r^{m+1} \int f(v)dv - z \end{aligned} \tag{3.20}$$

derived from X_3 and X_4 respectively.

Case 5. $g(u)$ arbitrary, $f(v) = \alpha v^{-1}$, $\alpha \neq 0$.

Here we have two subcase:

Case 5.1. m arbitrary.

In this case system (3.1) has two Noether operators, namely X_1 given by (3.6) and

$$X_2 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad B^1 = w, \quad B^2 = z, \quad w_t + z_r = \alpha r^m - r^m u g(u), \tag{3.21}$$

thus the components (1.45) yields the nontrivial conserved vectors; (3.7) given by X_1 and X_2 yields

$$\begin{aligned} T_2^1 &= r^m u v_t - r^m v u_t - w, \\ T_2^2 &= r^m u_r v - r^m u v_r - z. \end{aligned} \tag{3.22}$$

Case 5.2. $m = -2$.

Here system (3.1) admits four Noether operators, X_1, X_2 given by (3.6), (3.15) respectively and

$$X_3 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2u\frac{\partial}{\partial u}, \quad B^1 = w, \quad B^2 = z, \quad w_t + z_r = -2r^m u g(u), \tag{3.23}$$

$$X_4 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2v\frac{\partial}{\partial v}, \quad B^1 = w, \quad B^2 = z, \quad w_t + z_r = -2\alpha r^m, \tag{3.24}$$

and the associated nontrivial conserved vectors are: (3.7), (3.22) and

$$\begin{aligned} T_3^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r + 2r^m u v_t \\ &\quad -\alpha t r^m \ln v - tr^m \int g(u) du - w, \\ T_3^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t - 2r^m u v_r \\ &\quad -\alpha r^{m+1} \ln v - r^{m+1} \int g(u) du - z; \end{aligned} \tag{3.25}$$

$$\begin{aligned} T_4^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r + 2r^m u_t v \\ &\quad -\alpha t r^m \ln v - tr^m \int g(u) du - w, \\ T_4^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t - 2r^m u_r v \\ &\quad -\alpha r^{m+1} \ln v - r^{m+1} \int g(u) du - z \end{aligned} \tag{3.26}$$

obtained from X_3 and X_4 respectively.

Case 6. $f(v) = \alpha v$, $g(u) = \gamma u^p$ with $\alpha, \gamma \neq 0$, $m = \dfrac{2p+6}{p-1}$, $p \neq \pm 1$.

In this case system (3.1) admits two Noether generators, namely X_1 given by (3.6) and

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{4u}{p-1}\frac{\partial}{\partial u} - \frac{2v(p+1)}{p-1}\frac{\partial}{\partial v}, \; B^1 = 0, \; B^2 = 0, \tag{3.27}$$

and the components (1.45) give the nontrivial conserved vectors; (3.7) derived from X_1 and X_2 yields

$$\begin{aligned} T_2^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - \frac{2(p+1)}{p-1} r^m u_t v \\ &\quad -\frac{4}{p-1} r^m u v_t - \frac{\alpha t}{2} r^m v^2 - \frac{\gamma t}{p+1} r^m u^{p+1}, \\ T_2^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + \frac{2(p+1)}{p-1} r^m u_r v \\ &\quad +\frac{4}{p-1} r^m u v_r - \frac{\alpha}{2} r^{m+1} v^2 - \frac{\gamma}{p+1} r^{m+1} u^{p+1}. \end{aligned} \tag{3.28}$$

It is worthy mentioning that if $p = 1$, we recover case 2 and for $p = -1$, we obtain case 5. The analysis will also be encountered in case 7.

Case 7. $f(v) = \alpha v^q$, $g(u) = \gamma u$ with $\alpha, \gamma \neq 0$ and $m = \dfrac{2q+6}{q-1}$, $q \neq \pm 1$.

Here system (3.1) admits two Noether operators, X_1 given by (3.6) and X_2

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2u(q+1)}{q-1}\frac{\partial}{\partial u} - \frac{4v}{q-1}\frac{\partial}{\partial v}, \quad B^1 = 0, \quad B^2 = 0, \tag{3.29}$$

and the corresponding nontrivial conserved vectors are (3.7) and

$$\begin{aligned} T_2^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - \frac{2(q+1)}{q-1} r^m u v_t \\ &\quad - \frac{4}{q-1} r^m u_t v - \frac{\gamma t}{2} r^m u^2 - \frac{\alpha t}{q+1} r^m v^{q+1}, \\ T_2^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + \frac{2(q+1)}{q-1} r^m u v_r \\ &\quad + \frac{4}{q-1} r^m u_r v - \frac{\gamma}{2} r^{m+1} u^2 - \frac{\alpha}{q+1} r^{m+1} v^{q+1} \end{aligned} \tag{3.30}$$

derived from X_1 and X_2 respectively.

Case 8. $f(v) = \alpha v$, $g(u) = \gamma e^{\lambda u}$, $\alpha, \gamma, \lambda \neq 0$, with $m = 2$.

Here system (3.1) has two Noether generators, namely X_1 given by (3.6) and

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{4}{\lambda}\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}, \; B^1 = 0, B^2 = 0, \tag{3.31}$$

and the nontrivial conserved vectors are (3.7) obtained from X_1 while the new one is

$$\begin{aligned} T_2^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - 2r^m u_t v \\ &\quad - \frac{4}{\lambda} r^m v_t - \frac{\alpha t}{2} r^m v^2 - \frac{\gamma t}{\lambda} r^m e^{\lambda u}, \\ T_2^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + 2r^m u_r v \\ &\quad + \frac{4}{\lambda} r^m v_r - \frac{\alpha}{2} r^{m+1} v^2 - \frac{\gamma}{\lambda} r^{m+1} e^{\lambda u} \end{aligned} \tag{3.32}$$

obtained from (3.31).

Case 9. $f(v) = \alpha e^{\beta v}$, $g(u) = \gamma u$, $\alpha, \gamma, \beta \neq 0$, with $m = 2$.

In this case system (3.1) admits two Noether operators, namely X_1 given by (3.6) and

$$X_2 \;=\; t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - 2u\frac{\partial}{\partial u} - \frac{4}{\beta}\frac{\partial}{\partial v},\; B^1 = 0, B^2 = 0, \tag{3.33}$$

and the resulting nontrivial conserved vectors are (3.7) while the new one is

$$\begin{aligned} T_2^1 \;&=\; -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - 2r^m u v_t \\ &\quad -\frac{4}{\beta} r^m u_t - \frac{\gamma t}{2} r^m u^2 - \frac{\alpha t}{\beta} r^m e^{\beta v}, \\ T_2^2 \;&=\; r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + 2r^m u v_r \\ &\quad +\frac{4}{\beta} r^m u_r - \frac{\gamma}{2} r^{m+1} u^2 - \frac{\alpha}{\beta} r^{m+1} e^{\beta v} \end{aligned} \tag{3.34}$$

established from X_2.

Case 10. $f(v) = \alpha e^{\beta v}$, $g(u) = \gamma u^p$, $\alpha, \gamma, \beta \neq 0$ and $m = \frac{2}{p}$, $p \neq \pm 1$

Here system (3.1) provides us with two Noether generators, namely X_1 given by (3.6) and

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2u}{p}\frac{\partial}{\partial u} - \frac{2(p+1)}{\beta p}\frac{\partial}{\partial v},\;\; B^1 = 0,\; B^2 = 0, \tag{3.35}$$

and the components (1.45) give the nontrivial conserved vectors; (3.7) and X_2 gives

$$\begin{aligned} T_2^1 \;&=\; -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - \frac{2(p+1)}{\beta p} r^m u_t \\ &\quad -\frac{2u}{p} r^m u v_t - \frac{\alpha t}{\beta} r^m e^{\beta v} - \frac{\gamma t}{p+1} r^m u^{p+1}, \\ T_2^2 \;&=\; r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + \frac{2(p+1)}{\beta p} r^m u_r \\ &\quad +\frac{2u}{p} r^m u v_r - \frac{\alpha}{\beta} r^{m+1} e^{\beta v} - \frac{\gamma}{p+1} r^{m+1} u^{p+1}. \end{aligned} \tag{3.36}$$

Case 11. $f(v) = \alpha v^q$, $g(u) = \gamma e^{\lambda u}$, $\alpha, \gamma, \lambda \neq 0$, $m = \frac{2}{q}$, $q \neq \pm 1$.

In this case system (3.1) has two Noether operators, X_1 given by (3.6) and X_2

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2(q+1)}{\lambda q}\frac{\partial}{\partial u} - \frac{2v}{q}\frac{\partial}{\partial v}, \quad B^1 = 0,\ B^2 = 0, \tag{3.37}$$

then the corresponding nontrivial conserved vectors are (3.7) and

$$\begin{aligned} T_2^1 &= -tr^m u_t v_t - tr^m u_r v_r - r^{m+1} u_r v_t - r^{m+1} u_t v_r - \frac{2(q+1)}{\lambda q} r^m v_t \\ &\quad -\frac{2}{q} r^m u_t v - \frac{\gamma t}{\lambda} r^m e^{\lambda u} - \frac{\alpha t}{q+1} r^m v^{q+1}, \\ T_2^2 &= r^{m+1} u_t v_t + r^{m+1} u_r v_r + tr^m u_t v_r + tr^m u_r v_t + \frac{2(q+1)}{\lambda q} r^m v_r \\ &\quad +\frac{2}{q} r^m u_r v - \frac{\gamma}{\lambda} r^{m+1} e^{\lambda u} - \frac{\alpha}{q+1} r^{m+1} v^{q+1} \end{aligned} \tag{3.38}$$

derived from X_1 and X_2 respectively.

Remark 3.1

It should be noted that all the cases for which the functional forms of the arbitrary elements do not extended the one-dimensional Noether algebra (3.6) have been excluded in the preceding classification. This includes amongst others, the logarithmic case and the exponential case (analyzed at the same time). The cases when the functions are constants are also excluded.

We observe that the Lagrangian (3.3) is invariant under the time translation symmetry (3.6), and this yields energy conserved vectors. We further notice that the scaling symmetries e.g., (3.10), (3.12) result in boost momentum conservation laws. It is interesting to is see that if we set $w = 0, z = 0, f(v) = \gamma v^{-1}, g(u) = \alpha u^{-1}$ respectively, then the divergence infinitesimal scaling symmetries (3.15) and (3.21) on space (u, v) become variational symmetries and this yields charge conserved vectors. We further observe that the time-space inversion symmetry (3.13) can never be variational in this context [22, 54].

3.2 Concluding remarks

We have performed a complete Noether symmetry classification of the generalized hyperbolic Lane-Emden system (3.1). We obtained several cases for the arbitrary elements $f(v)$, $g(u)$ and m which resulted in Noether point symmetries. Furthermore, we constructed the associated conservation laws for the admitted Noether point symmetry. The results of the problem under study were initiated by the recent work in [47].

Chapter 4

Group classification of a generalized coupled hyperbolic Lane-Emden system

The coupled hyperbolic Lane-Emden system

$$
\begin{aligned}
&u_{tt} - \Delta u + v^q = 0, \\
&v_{tt} - \Delta v + u^p = 0,
\end{aligned} \tag{4.1}
$$

where the radial Laplacian $\Delta = r^{-m}\frac{\partial}{\partial r}r^m\frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + mr^{-1}\frac{\partial}{\partial r}$ with the spatial dimensions $m \neq 0$ was studied in [47]. The authors in [47] investigated both Lie and Noether point symmetries classification of (4.1) with the arbitrary constants $p, q \notin \{0, 1\}$ so as to bring truly nonlinearity to the system. Motivated by the work in [30, 47], we study the generalized coupled hyperbolic Lane-Emden system

$$
\begin{aligned}
&u_{tt} - \Delta u + f(v) = 0, \\
&v_{tt} - \Delta v + g(u) = 0,
\end{aligned} \tag{4.2}
$$

where $f(v)$, $g(u)$ are non-zero arbitrary self-interaction functions of v and u respectively. The parameter m is assumed to be different than 0. In fact, if we take $m = 0$, system (4.2) can be obtained from [30], under the complex transformation $(x, y, u, v) \longmapsto (t, ir, u, v)$ into the original variables of the aforementioned reference. No further restrictions will be placed on m (even allowing negative and non-integer values). System (4.1) and (4.2) can also be considered as natural two-component extension of the nonlinear wave equation:

$$u_{tt} - \Delta u - u^p = 0, \tag{4.3}$$

where $u = u(t, r)$ is a real-valued function, p symbolizes the interaction power and (t, r) denote time and radial coordinates respectively in $m \neq 0$ dimensions. System (4.1) and (4.2) are commonly encountered in many physical phenomena, see for example [35, 37, 40, 45, 47, 48] and reference therein.

The organization of this chapter is as follows. In Section 4.1, we compute the equivalent transformations of the generalized coupled hyperbolic Lane-Emden system (4.2). In Section 4.2, we determine the principal Lie algebra and carry out the Lie group classification of the underlying system. In Section 4.3, we perform some symmetry reductions of system (4.2). Finally, concluding remarks are summarized in Section 4.4.

The work in this chapter has been published in [55].

4.1 Equivalence transformations

The vector field

$$\begin{aligned} Y &= \xi^1(t, r, u, v)\frac{\partial}{\partial t} + \xi^2(t, r, u, v)\frac{\partial}{\partial r} + \eta^1(t, r, u, v)\frac{\partial}{\partial u} + \eta^2(t, r, u, v)\frac{\partial}{\partial v} \\ &\quad + \mu^1(t, r, u, v, f, g)\frac{\partial}{\partial f} + \mu^2(t, r, u, v, f, g)\frac{\partial}{\partial g}, \end{aligned} \tag{4.4}$$

is said to be the generator of the equivalence group of (4.2) provided it is admitted by the extended system [1, 56]

$$u_{tt} - u_{rr} - \frac{m}{r}u_r + f(v) = 0,\ v_{tt} - v_{rr} - \frac{m}{r}v_r + g(u) = 0, \tag{4.5}$$

$$f_t = f_r = f_u = 0,\ g_t = g_r = g_v = 0. \tag{4.6}$$

The prolongation of the generator (4.4) for the extended system (4.5)-(4.6) is

$$\widetilde{Y} = Y^{[2]} + \omega_t^1 \frac{\partial}{\partial f_t} + \omega_r^1 \frac{\partial}{\partial f_r} + \omega_u^1 \frac{\partial}{\partial f_u} + \omega_t^2 \frac{\partial}{\partial g_t} + \omega_r^2 \frac{\partial}{\partial g_r} + \omega_v^2 \frac{\partial}{\partial g_v}, \tag{4.7}$$

where $Y^{[2]}$ is the second-prolongation of (4.4) given by

$$\begin{aligned} Y^{[2]} &= Y + \zeta_t^1 \frac{\partial}{\partial u_t} + \zeta_r^1 \frac{\partial}{\partial u_r} + \zeta_t^2 \frac{\partial}{\partial v_t} + \zeta_r^2 \frac{\partial}{\partial v_r} + \zeta_{tt}^1 \frac{\partial}{\partial u_{tt}} \\ &\quad + \zeta_{rr}^1 \frac{\partial}{\partial u_{rr}} + \zeta_{tt}^2 \frac{\partial}{\partial v_{tt}} + \zeta_{rr}^2 \frac{\partial}{\partial v_{rr}} + \cdots . \end{aligned}$$

Here the variables ζ's and ω's are defined by the prologation formlae

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_r D_t(\xi^2), \\ \zeta_r^1 &= D_r(\eta^1) - u_t D_r(\xi^1) - u_r D_r(\xi^2), \\ \zeta_t^2 &= D_t(\eta^2) - v_t D_t(\xi^1) - v_r D_t(\xi^2), \\ \zeta_r^2 &= D_r(\eta^2) - v_t D_r(\xi^1) - v_r D_r(\xi^2), \\ \zeta_{tt}^1 &= D_t(\zeta_t^1) - u_{tt} D_t(\xi^1) - u_{tr} D_t(\xi^2), \\ \zeta_{rr}^1 &= D_r(\zeta_r^1) - u_{tr} D_r(\xi^1) - u_{rr} D_r(\xi^2), \\ \zeta_{tt}^2 &= D_t(\zeta_t^2) - v_{tt} D_t(\xi^1) - v_{tr} D_t(\xi^2), \\ \zeta_{rr}^2 &= D_r(\zeta_r^2) - v_{tr} D_r(\xi^1) - v_{rr} D_r(\xi^2) \end{aligned}$$

and

$$\omega_t^1 = \widetilde{D}_t(\mu^1) - f_t \widetilde{D}_t(\xi^1) - f_r \widetilde{D}_t(\xi^2) - f_u \widetilde{D}_t(\eta^1),$$

$$\omega_r^1 = \widetilde{D}_r(\mu^1) - f_t\widetilde{D}_r(\xi^1) - f_r\widetilde{D}_r(\xi^2) - f_u\widetilde{D}_r(\eta^1),$$
$$\omega_u^1 = \widetilde{D}_u(\mu^1) - f_t\widetilde{D}_u(\xi^1) - f_r\widetilde{D}_u(\xi^2) - f_u\widetilde{D}_u(\eta^1),$$
$$\omega_t^2 = \widetilde{D}_t(\mu^2) - g_t\widetilde{D}_t(\xi^1) - g_r\widetilde{D}_t(\xi^2) - g_v\widetilde{D}_t(\eta^2),$$
$$\omega_r^2 = \widetilde{D}_r(\mu^2) - g_t\widetilde{D}_r(\xi^1) - g_r\widetilde{D}_r(\xi^2) - g_v\widetilde{D}_r(\eta^2),$$
$$\omega_v^2 = \widetilde{D}_v(\mu^2) - g_t\widetilde{D}_v(\xi^1) - g_r\widetilde{D}_v(\xi^2) - g_v\widetilde{D}_v(\eta^2),$$

respectively, where

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + \cdots,$$
$$D_r = \frac{\partial}{\partial r} + u_r\frac{\partial}{\partial u} + v_r\frac{\partial}{\partial v} + \cdots,$$

are the usual total differentiation operators and

$$\widetilde{D}_t = \frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + g_t\frac{\partial}{\partial g} + \cdots,$$
$$\widetilde{D}_r = \frac{\partial}{\partial r} + f_r\frac{\partial}{\partial f} + g_r\frac{\partial}{\partial g} + \cdots,$$
$$\widetilde{D}_u = \frac{\partial}{\partial u} + f_u\frac{\partial}{\partial f} + g_u\frac{\partial}{\partial g} + \cdots,$$
$$\widetilde{D}_v = \frac{\partial}{\partial v} + f_v\frac{\partial}{\partial f} + g_v\frac{\partial}{\partial g} + \cdots,$$

are the new total differentiation operators for the extended system. The invocation of the generator (4.7) and the invariance conditions of system (4.5)-(4.6) yields the following equivalence generators:

$$X_1 = \frac{\partial}{\partial t},\ \ X_2 = \frac{\partial}{\partial u},\ \ X_3 = \frac{\partial}{\partial v},\ \ X_4 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f},\ \ X_5 = v\frac{\partial}{\partial v} + g\frac{\partial}{\partial g},$$
$$X_6 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}.$$

Consequently, the six-parameter equivalence group is

$$X_1 \ : \ \bar{t} = a_1 + t,\ \bar{r} = r, \bar{u} = u,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,$$
$$X_2 \ : \ \bar{t} = t,\ \bar{r} = r, \bar{u} = u + a_2,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,$$

$$X_3 \ : \ \bar{t} = t, \ \bar{r} = r, \ \bar{u} = u, \ \bar{v} = v + a_3, \ \bar{f} = f, \ \bar{g} = g,$$

$$X_4 \ : \ \bar{t} = t, \ \bar{r} = r, \ \bar{u} = ue^{a_4}, \ \bar{v} = v, \ \bar{f} = fe^{a_4}, \bar{g} = g,$$

$$X_5 \ : \ \bar{t} = t, \ \bar{r} = r, \ \bar{u} = u, \ \ \bar{v} = ve^{a_5}, \ \bar{f} = f, \bar{g} = ge^{a_5},$$

$$X_6 \ : \ \bar{t} = te^{a_6}, \ \bar{r} = re^{a_6}, \ \bar{u} = u, \ \bar{v} = v, \ \bar{f} = fe^{-2a_6}, \ \bar{g} = ge^{-2a_6},$$

and the associated composition of the aforementioned transformations is

$$\begin{aligned} \bar{t} &= e^{a_6}(t + a_1), \\ \bar{r} &= re^{a_6}, \\ \bar{u} &= e^{a_4}(u + a_2), \\ \bar{v} &= e^{a_5}(v + a_3), \\ \bar{f} &= e^{a_4 - 2a_6} f, \\ \bar{g} &= e^{a_5 - 2a_6} g. \end{aligned} \tag{4.8}$$

4.2 Principal Lie algebra and group classification

Following the classical approach of group classification [1], system (4.2) is invariant under the group with the generator

$$\begin{aligned} X &= \xi^1(t,r,y,u,v)\frac{\partial}{\partial t} + \xi^2(t,r,y,u,v)\frac{\partial}{\partial x} + \xi^3(t,r,y,u,v)\frac{\partial}{\partial y} \\ &\quad + \eta^1(t,r,y,u,v)\frac{\partial}{\partial u} + \eta^2(t,r,y,u,v)\frac{\partial}{\partial v}, \end{aligned} \tag{4.9}$$

if and only if

$$\begin{aligned} X^{[2]}\left(u_{tt} - u_{rr} - \frac{m}{r}u_r + f(v) = 0\right)\Big|_{(4.2)} &= 0, \\ X^{[2]}\left(v_{tt} - v_{rr} - \frac{m}{r}v_r + g(u) = 0\right)\Big|_{(4.2)} &= 0, \end{aligned} \tag{4.10}$$

where $X^{[2]}$ symbolizes the second prolongation of the generator (4.9). The expansion of (4.10) and separate the monomials give rise to linear overdetermined system of partial differential equations:

$$\begin{aligned}
\xi^1_u &= 0, && (4.11)\\
\xi^1_v &= 0, && (4.12)\\
\xi^2_u &= 0, && (4.13)\\
\xi^2_v &= 0, && (4.14)\\
\eta^1_{uu} &= 0, && (4.15)\\
\eta^1_{uv} &= 0, && (4.16)\\
\eta^1_{vv} &= 0, && (4.17)\\
\eta^1_{vt} &= 0, && (4.18)\\
\eta^1_{vr} &= 0, && (4.19)\\
\eta^2_{uu} &= 0, && (4.20)\\
\eta^2_{uv} &= 0, && (4.21)\\
\eta^2_{vv} &= 0, && (4.22)\\
\eta^2_{ut} &= 0, && (4.23)\\
\eta^2_{ur} &= 0, && (4.24)\\
\xi^2_r - \xi^1_t &= 0, && (4.25)\\
\xi^1_r - \xi^2_t &= 0, && (4.26)\\
\xi^1_{rr} - \xi^1_{tt} + \frac{m}{r}\xi^1_r + 2\eta^1_{tu} &= 0, && (4.27)\\
\xi^1_{rr} - \xi^1_{tt} + \frac{m}{r}\xi^1_r + 2\eta^2_{vt} &= 0, && (4.28)\\
\xi^2_{rr} - \xi^2_{tt} - \frac{m}{r}\xi^2_r + \frac{m}{r^2}\xi^2 - 2\eta^2_{vr} &= 0, && (4.29)\\
\xi^2_{rr} - \xi^2_{tt} - \frac{m}{r}\xi^2_r + \frac{m}{r^2}\xi^2 - 2\eta^1_{ru} &= 0, && (4.30)\\
\eta^1_{tt} - \eta^1_{rr} - \frac{m}{r}\eta^1_r - f\eta^1_u - g\eta^1_v + 2f\xi^1_t + f'(v)\eta^2 &= 0, && (4.31)\\
\eta^2_{tt} - \eta^2_{rr} - \frac{m}{r}\eta^2_r - f\eta^2_u - g\eta^2_v + 2g\xi^2_r + g'(u)\eta^1 &= 0. && (4.32)
\end{aligned}$$

Solving the above system of partial differential equations for arbitrary $f(v)$ and $g(u)$, we conclude that the system (4.2) admits the one-dimensional principal Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial t},$$

4.3 Lie group classification

Solving the system (4.11)-(4.32), we obtain the following classifying relations:

$$(\sigma v + \phi) f''(v) + (\theta + \alpha + \lambda) f'(v) = 0, \tag{4.33}$$
$$(\omega u + \beta) g''(u) + (\psi + \tau + \rho) g'(u) = 0,$$

where $\alpha, \beta, \gamma, \delta, \theta, \lambda, \varphi, \tau, \rho$ and ω are constants. The aforementioned classifying relations are invariant under the equivalence transformations (4.8) if

$$\bar{\alpha} = \alpha, \quad \bar{\lambda} = \lambda, \quad \bar{\omega} = \omega, \quad \bar{\beta} = \omega a_2 + \beta e^{-a_4}, \quad \bar{\psi} = \psi, \quad \bar{\phi} = \sigma a_3 + \phi e^{-a_5},$$
$$\bar{\sigma} = \sigma, \quad \bar{\theta} = \theta, \quad \bar{\tau} = \tau, \quad \bar{\rho} = \rho.$$

The invocation of the classifying relations (4.33) prompted the following cases for the functional forms of the arbitrary elements $f(v)$, $g(u)$ and m together with the their associated extra generators.

Case 1. $f(v)$, $g(u)$ and m arbitrary but not of the form in Cases 2-11 given below

In this case, we obtain the principal Lie algebra

$$X_1 = \frac{\partial}{\partial t}.$$

Case 2. $f(v) = \bar{a}_3 v + \bar{a}_4$, $g(u) = \bar{a}_1 u + \bar{a}_2$ and m arbitrary, $\bar{a}_1, \bar{a}_3 \neq 0$ where $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ and $\bar{a}_4$ are constants.

This case extends the principal Lie algebra by three symmetries, namely

$$
\begin{aligned}
X_2 &= \bar{a}_3 v\frac{\partial}{\partial u} + (\bar{a}_1 u + \bar{a}_2)\frac{\partial}{\partial v},\\
X_3 &= \bar{a}_3 u\frac{\partial}{\partial u} + (\bar{a}_3 v + \bar{a}_4)\frac{\partial}{\partial v},\\
X_e &= \bar{a}_3 e(t,r)\frac{\partial}{\partial u} + (e_{rr} - e_{tt} + \frac{m}{r}e_r)\frac{\partial}{\partial v},\\
&\quad \text{with } 2e_{ttrr} - e_{tttt} - e_{rrrr} + \frac{2m}{r}e_{ttr} - \frac{2m}{r}e_{rrr}\\
&\quad -\frac{(m-2)m}{r}e_{rr} - \frac{(m-2)m}{r}e_r + \bar{a}_1\bar{a}_3 e = 0.
\end{aligned}
$$

Case 3. $f(v) = \bar{a}_3 v + \bar{a}_4$, $g(u) = \bar{a}_1 u + \bar{a}_2$ and $m = 2$, $\bar{a}_1, \bar{a}_3 \neq 0$, where $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ and $\bar{a}_4$ are constants

This case extends the principal Lie algebra by five symmetries, namely

$$
\begin{aligned}
X_2 &= \bar{a}_3 v\frac{\partial}{\partial u} + (\bar{a}_1 u + \bar{a}_2)\frac{\partial}{\partial v},\\
X_3 &= \bar{a}_3 u\frac{\partial}{\partial u} + (\bar{a}_3 v + \bar{a}_4)\frac{\partial}{\partial v},\\
X_4 &= \bar{a}_3 r\frac{\partial}{\partial t} + \bar{a}_3 t\frac{\partial}{\partial r} - \frac{t}{r}\bar{a}_3 u\frac{\partial}{\partial u} - \frac{t}{r}(\bar{a}_3 v + \bar{a}_4)\frac{\partial}{\partial v},\\
X_5 &= \bar{a}_3\frac{\partial}{\partial r} - \frac{1}{r}\bar{a}_3 u\frac{\partial}{\partial u} - \frac{1}{r}(\bar{a}_3 v + \bar{a}_4)\frac{\partial}{\partial v},\\
X_e &= \bar{a}_3 e(t,r)\frac{\partial}{\partial u} + (e_{rr} - e_{tt} + \frac{2}{r}e_r)\frac{\partial}{\partial v},\\
&\quad \text{with } 2e_{ttrr} - e_{tttt} - e_{rrrr} + \frac{4}{r}e_{ttr} - \frac{4}{r}e_{rrr} + \bar{a}_1\bar{a}_3 e = 0.
\end{aligned}
$$

Case 4. $f(v) = \bar{a}_3 v^q$ and $g(u) = \bar{a}_1 u^p$, $\bar{a}_1, \bar{a}_3 \neq 0$. This case reduces to the system in [47].

Case 5. $f(v) = \bar{a}_3 v^{-1}$ and $g(u)$, m arbitrary, $\bar{a}_3 \neq 0$.

In this case the principal Lie algebra extends by one symmetry:

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2v\frac{\partial}{\partial v}.$$

Case 6. $f(v)$ is arbitrary, $g(u) = \bar{a}_1 u^{-1}$ and m arbitrary, $\bar{a}_1 \neq 0$.

This case the principal Lie algebra extends by one symmetry, namely

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + 2u\frac{\partial}{\partial u}.$$

Case 7. m arbitrary, $f(v) = \bar{a}_3 e^{qv}$ and $g(u) = \bar{a}_1 e^{pu}$, $\bar{a}_1, \bar{a}_3 \neq 0$, where $\bar{a}_1$, $\bar{a}_3$ are constants.

In this case the principal Lie algebra extends by

$$X_2 = t\frac{\partial}{\partial t} + r\partial_r - \frac{2}{p}\frac{\partial}{\partial u} - \frac{2}{q}\frac{\partial}{\partial v}.$$

Case 8. $f(v) = \bar{a}_3 v^q$ and $g(u) = \bar{a}_1 e^{pu}$, $\bar{a}_1, \bar{a}_3 \neq 0$, where $\bar{a}_1$, $\bar{a}_3$ are constants and m arbitrary.

Here the principal Lie algebra is extended by

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2(q+1)}{pq}\frac{\partial}{\partial u} - \frac{2v}{q}\frac{\partial}{\partial v}.$$

Case 9. $f(v) = \bar{a}_3 e^{qv}$ and $g(u) = \bar{a}_1 u^p$, $\bar{a}_1, \bar{a}_3 \neq 0$, where $\bar{a}_1$, $\bar{a}_3$ are constants and m arbitrary.

This case extends the principal Lie algebra by one symmetry

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2u}{p}\frac{\partial}{\partial u} - \frac{2(q+1)}{pq}\frac{\partial}{\partial v}.$$

Case 10. $f(v) = \bar{a}_3 v$ and $g(u) = \bar{a}_1 u^p$, $\bar{a}_1, \bar{a}_3 \neq 0$, where $\bar{a}_1$, $\bar{a}_3$ are constants and m arbitrary.

The principal Lie algebra is extended by

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{4u}{p-1}\frac{\partial}{\partial u} - \frac{2(p+1)v}{p-1}\frac{\partial}{\partial v}.$$

Case 11. $f(v) = \bar{a}_3 v^q$ and $g(u) = \bar{a}_1 u$, $\bar{a}_1, \bar{a}_3 \neq 0$, where $\bar{a}_1$, $\bar{a}_3$ are constants and m arbitrary.

This case extends the principal Lie algebra by one symmetry

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \frac{2(q+1)u}{q-1}\frac{\partial}{\partial u} - \frac{4v}{q-1}\frac{\partial}{\partial v}.$$

Remark 4.1 All the cases that do not extend the principal Lie algebra have been excluded in the preceding classification. The cases when the functions are constants are also excluded.

4.4 Symmetry reduction of system (4.2)

This section aims to transform system (4.2) into a system of ordinary differential equations through the invariant surface condition [13, 57], by making use of some of the generators obtained in section 4.3. First we begin with the generator $X = \partial_t$ with arbitrary $g(u)$ and $f(v)$ and we get the two general group invariant solutions of system (4.2) as

$$u(t,r) = \phi(r), v(t,r) = \psi(r), \tag{4.34}$$

where $\phi(r)$ and $\psi(r)$ satisfy the ordinary differential system

$$\begin{aligned} \phi''(r) + \frac{m}{r}\phi'(r) - f(\psi) = 0, \\ \psi''(r) + \frac{m}{r}\psi'(r) - g(\phi) = 0. \end{aligned} \tag{4.35}$$

Remark 4.2 Observe that when $m = n-1$, $-f(\psi) = \psi^q$ and $-g(\phi) = \phi^p$ system (4.35) reduces to the well-known Lane-Emden system

$$\begin{aligned} \phi''(r) + \frac{n-1}{r}\phi'(r) + \psi^q = 0, \\ \psi''(r) + \frac{n-1}{r}\psi'(r) + \phi^p = 0, \end{aligned} \tag{4.36}$$

where $n-1$ is an integer related to the dimension on the space of spatial coordinates and $p, q > 0$ are arbitrary constants. System (4.36) has been extensively studied for both Lie and Noether symmetries. See for example [**?**, 37, 58, 59].

Other general group invariant solution of system (4.2) will be obtained from the generator $X = t\partial_t + r\partial_r + 2u\partial_u$ with $g(u) = \bar{a}_1 u^{-1}$ and $f(v)$ arbitrary. Here we obtain the three invariants

$$\xi = t/r, u = r^2\phi, v = \psi, \tag{4.37}$$

and consequently employing these invariants, system (4.2) reduces to

$$\begin{aligned} &(1-\xi^2)\phi''(\xi) + (m+2)\xi\phi'(\xi) - 2(m+1)\phi(\xi) + f(\psi) = 0, \\ &(1-\xi^2)\psi''(\xi) + (m-2)\xi\psi'(\xi) + \frac{\bar{a}_1}{\phi(\xi)} = 0. \end{aligned} \tag{4.38}$$

Thus, we conclude that

$$u(r,t) = r^2\phi(\xi), v(r,t) = \psi(\xi), \tag{4.39}$$

is a general group invariant solution of system (4.2) where ϕ and ψ are any solutions of (4.38).

We now work with the generator $X = (t^2 + r^2)\partial_t + 2rt\partial_r + 2ut\partial_u + 2vt\partial_v$ with $g(u) = \bar{a}_1 u^{-1}$ and $f(v) = \bar{a}_3 v^{-1}$ and we obtain the three invariants

$$\xi = \frac{(t^2 - r^2)}{r}, u = r\phi, v = r\psi. \tag{4.40}$$

Invoking these invariants, system (4.2) transforms to

$$\xi^2\phi''(\xi) + 2\xi\phi'(\xi) - 2\phi(\xi) - \frac{\bar{a}_3}{\psi(\xi)} = 0,$$
$$\xi^2\psi''(\xi) + 2\xi\psi'(\xi) - 2\psi(\xi) - \frac{\bar{a}_1}{\phi(\xi)} = 0. \quad (4.41)$$

Consequently, the general group invariant solution of system (4.2) is

$$u(r,t) = r\phi(\xi), v(r,t) = r\psi(\xi), \quad (4.42)$$

where ϕ and ψ are any solutions of the ordinary differential system (4.41).

We now choose the operator $X = t\partial_t + r\partial_r - \frac{2}{p}\partial_u - \frac{2}{q}\partial_v$ with $g(u) = \bar{a}_1 e^{pu}$ and $f(v) = \bar{a}_3 e^{qv}$ and this leads to the following invariants

$$\xi = \frac{r}{t}, u = \phi - \frac{2}{p}\ln t, v = \psi - \frac{2}{q}\ln t. \quad (4.43)$$

Making use of these invariants system (4.2) reduces to

$$(\xi^2 - 1)\xi^2\phi''(\xi) + (2\xi^2 - m)\xi\phi'(\xi) + \frac{2}{p}\xi^2 + \xi^2\bar{a}_3 e^{q\psi} = 0,$$
$$(\xi^2 - 1)\xi^2\psi''(\xi) + (2\xi^2 - m)\xi\psi'(\xi) + \frac{2}{q}\xi^2 + \xi^2\bar{a}_1 e^{p\phi} = 0. \quad (4.44)$$

Thus, the general group invariant solution of system (4.2) is

$$u(r,t) = \phi(\xi) - \frac{2}{p}\ln t, v(r,t) = \psi(\xi) - \frac{2}{q}\ln t, \quad (4.45)$$

with ϕ and ψ be any solutions of (4.44).

To construct another general group invariant solution of system (4.2), we will use the generator $X = pqt\partial_t + pqr\partial_r - 2qu\partial_u - 2(p+1)\partial_v$ with $g(u) = \bar{a}_1 u^p$ and $f(v) = \bar{a}_3 e^{qv}$. This provides us with the following invariants

$$\xi = \frac{r}{t}, u = t^{-\frac{2}{p}}\phi(\xi), v = \psi(\xi) - \frac{2}{q}\ln t - \frac{2}{pq}\ln t, \quad (4.46)$$

and making use of them, system (4.2) transforms to a second ordinary differential system

$$(\xi^2 - 1)\phi''(\xi) + (\frac{4}{p} + 2)\xi\phi'(\xi) - \frac{m}{\xi}\phi'(\xi) + \left(\frac{2}{p} + \frac{4}{p^2}\right)\phi(\xi) + \bar{a}_3 e^{q\psi} = 0,$$

$$(\xi^2-1)\xi^2\psi''(\xi)+(2\xi^2-m)\xi\psi'(\xi)+\frac{2}{q}\left(1+\frac{1}{p}\right)\xi^2+\xi^2\bar{a}_1\phi^p(\xi)=0. \quad (4.47)$$

Thus, we conclude that

$$u(r,t)=t^{-\frac{2}{p}}\phi(\xi), v(r,t)=\psi(\xi)-\frac{2}{q}\ln t-\frac{2}{pq}\ln t, \quad (4.48)$$

is a general group invariant solution of system (4.2) where ϕ and ψ are any solutions of (4.47).

Remark 4.3 Likewise, one can construct more general group invariants solutions of the generalized coupled hyperbolic Lane-Emden system (4.2) using the other symmetries.

4.5 Concluding remarks

In this chapter we considered the generalized coupled hyperbolic Lane-Emden system (4.2) from the point of view of classical Lie symmetry analysis. We found the non-equivalent forms of the functions $f(v)$ and $g(u)$ for which the one dimensional principal Lie algebra extends. Some general group invariant solutions for the underlying system were constructed.

Chapter 5

Lie reductions and conservation laws of a coupled Jaulent-Miodek system

5.1 Introduction

In this chapter, we study the coupled Jaulent-Miodek system [60]

$$u_t + 3uu_x - 2v_x = 0, \tag{5.1}$$

$$v_t + 2u_x v + uv_x - \frac{1}{2}u_{xxx} = 0. \tag{5.2}$$

This system is associated with the Jaulent-Miodek spectral problem [60,61]. These type of systems are encountered in fluid dynamics, particle physics and many other areas of physics and mathematical sciences. Integrability of nonlinear evolution equations has attracted certain attention in the mathematical and physical communities [62–68]. Recent studies have been made on the Jaulent-Miodeks equations. For example, in [60] exact solutions of system (5.1) were obtained using the Darboux transformation method based on the Lax pairs. In [69, 70], Wazwaz obtained exact solutions of the Jaulent-Miodek system using the Hereman-Nuseir form, a simplified form of the Hirota's method, the tanh-coth method and the sech method. Many other powerful methods have been used in examining solutions on such systems. See [71–73] and references therein.

The objective of this work is twofold. Firstly, we seek to establish new exact solutions of the coupled Jaulent-Miodek system (5.1) using the Lie symmetry method. Thereafter, we aim to derive local conservation laws of system (5.1) using the invariance and multiplier approach based on the well known results that the Euler-Lagrange operator annihilates the total divergence.

The work of this chapter has been published in [74].

5.2 Symmetry reductions and exact solutions of (5.1)

The symmetry generator of the coupled Jaulent-Miodek system (5.1) will be generated by the vector field

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}. \quad (5.3)$$

The invocation of the third prolongation $\mathrm{pr}^{(3)}X$ [15] to (5.1) and split the monomials leads to linear overdetermined system partial differential equations. These are

$$\xi^1_u = 0, \quad (5.4)$$

$$\xi^1_v = 0, \quad (5.5)$$

$$\xi^2_u = 0, \quad (5.6)$$

$$\xi^1_x = 0, \quad (5.7)$$

$$\eta^1_{uu} = 0, \quad (5.8)$$

$$\eta^1_v = 0, \quad (5.9)$$

$$\eta^1_{xu} - \xi^2_{xx} = 0, \quad (5.10)$$

$$\eta^1_t - 2\eta^2_x + 3u\eta^1_x = 0, \quad (5.11)$$

$$\xi^2_x - \xi^1_t + \eta^1_u - \eta^2_v = 0, \quad (5.12)$$

$$\xi^1_t - 3\xi^2_x + \eta^1_u - \eta^2_v = 0, \quad (5.13)$$

$$\eta^1_{xxx} - 4v\eta^1_x - 2\eta^2_t - 2u\eta^2_x = 0, \quad (5.14)$$

$$3u\xi^1_t - \xi^2_t - 3u\xi^2_x + 3\eta^1 - 2u\eta^2_u = 0, \quad (5.15)$$

$$\xi^2_t - 2u\xi^2_x - \eta^1 + u\eta^1_u - u\eta^2_v - 2\eta^2_u = 0, \quad (5.16)$$

$$3\eta^1_{xxu} + 4u\eta^2_u - 4\eta^2 - 8v\xi^2_x - \xi^2_{xxx} = 0. \quad (5.17)$$

Solving the above system of partial differential equations we obtain

$$\xi^1(t,x,u,v) = C_1 + 2tC_4,$$
$$\xi^2(t,x,u,v) = C_2 + 4tC_3 + xC_4,$$
$$\eta^1(t,x,u,v) = 2C_3 - uC_4,$$
$$\eta^2(t,x,u,v) = uC_3 - 2vC_4.$$

The above general solution contains four arbitrary constants. Thus, the infinitesimal symmetries of system (5.1) form the four-dimensional Lie algebra spanned by the following linearly independent generators:

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = 4t\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},$$
$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}.$$

5.2.1 Symmetry reductions of (5.1)

In order to construct symmetry reductions and exact solutions, we need to implement the associated Lagrange equations

$$\frac{dt}{\xi^1(t,x,u,v)} = \frac{dx}{\xi^2(t,x,u,v)} = \frac{du}{\eta^1(t,x,u,v)} = \frac{dv}{\eta^2(t,x,u,v)}. \tag{5.18}$$

We study the following cases.

Case 1. $C_1 = 1, C_2 = 0, C_3 = 0, C_4 = 0$

Here we get the generator X_1. Solving the characteristics equations (5.18) with respect to X_1 yields the three invariants $z = x$, $u = \phi$ and $v = \psi$. Thus, system (5.1) reduces to the third order ordinary differential equations

$$3\phi(z)\phi^{'}(z) - 2\psi^{'}(z) = 0, \tag{5.19}$$

$$2\phi^{'}(z)\psi(z) + \phi(z)\psi^{'}(z) - \frac{1}{2}\phi^{'''}(z) = 0. \tag{5.20}$$

After solving system (5.19), we get $\phi(z) = A\mathrm{sn}(ix + B, i)$ and $\psi(z) = \frac{3}{4}A^2\mathrm{sn}^2(ix + B, i)$. Consequently, the invariant solution for system (5.1) is

$$u(t,x) = A\,\mathrm{sn}(ix + B, i), \quad v(t,x) = \frac{3}{4}A^2\mathrm{sn}^2(ix + B, i), \tag{5.21}$$

where A and B are arbitrary constants.

Case 2. $C_1 = 0, C_2 = 0, C_3 = 1, C_4 = 0$

In this case, the operator is X_3. Employing the characteristics equations (5.18) with respect to X_3, yields the following invariants

$$z = x,\ u = \frac{x}{2t} + \phi,\ v = \frac{xu}{4t} - \frac{x^2}{16t^2} + \psi.$$

Implementing these invariants, system (5.1) transforms to the first order ordinary differential equations

$$z\phi^{'}(z) + \phi(z) = 0, \tag{5.22}$$

$$\phi^2(z) + 4\psi(z) + 4z\psi'(z) = 0. \tag{5.23}$$

Solving system (5.22), we obtain $\phi(z) = \frac{A}{z}$ and $\psi(z) = \frac{A^2}{4z^2} + \frac{B}{z}$. Thus, the group invariant solution for system (5.1) is

$$u(t,x) = \frac{x}{2t} + \frac{A}{t}, \quad v(t,x) = \frac{x^2}{16t^2} + \frac{A}{4t^2}(x+A) + \frac{B}{t}, \tag{5.24}$$

with A and B being arbitrary constants. A profile of the rational solution is given in Figure 1.

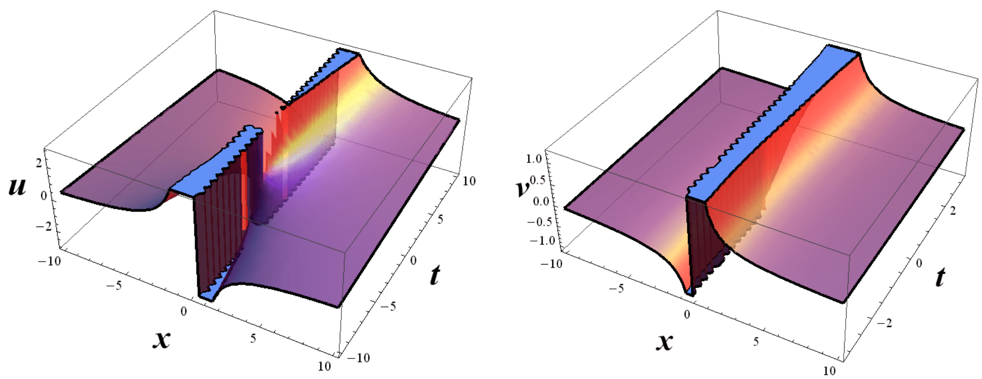

Figure 5.1: A profile of the rational solution (2.8).

Case 3. $C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 1$

In this case by solving the corresponding Lagrange system for the symmetry X_4, one obtains an invariant $z = \frac{t}{x^2}$, and the associated group-invariant solution is of the form

$$u(t,x) = \frac{1}{x}\phi(z), \quad v(t,x) = \frac{1}{x^2}\psi(z), \tag{5.25}$$

where ϕ and ψ are any solutions of the third order ordinary differential equations

$$\phi'(z) - 3\phi^2(z) - 6z\phi(z)\phi'(z) + 4\psi(z) + 4z\psi'(z) = 0, \tag{5.26a}$$

$$\psi'(z) + 3\phi(z) - 4\phi(z)\psi(z) - 4z\phi'(z)\psi(z) - 2z\phi(z)\psi'(z)$$
$$+27z\phi'(z) + 24z^2\phi''(z) + 4z^3\phi'''(z) = 0. \tag{5.26b}$$

Remark 5.1 It should be noted that the symmetry X_2 ($C_2 = 1, C_1 = C_3 = C_4 = 0$) leads to a constant solution for system (5.1).

5.3 Conservation laws

A conservation law of system (5.1) is a total space-time divergence expression that vanishes on the solution space ε of system (5.1),

$$D_tT^t + D_xT^x|_{\varepsilon} = 0, \tag{5.27}$$

where D_t and D_x are the total derivative operators while T^t is a conserved density and T^x is a spatial flux. To determine the conservation law for system (5.1), we will implement the multiplier method. Since the joint Euler operator annihilates the total divergence, we get

$$\frac{\delta}{\delta u}\left((u_t + 3uu_x - 2v_x)\Lambda_1 + (v_t + 2u_xv + uv_x - \frac{1}{2}u_{xxx})\Lambda_2\right) = 0, \tag{5.28}$$

$$\frac{\delta}{\delta v}\left((u_t + 3uu_x - 2v_x)\Lambda_1 + (v_t + 2u_xv + uv_x - \frac{1}{2}u_{xxx})\Lambda_2\right) = 0, \tag{5.29}$$

where Λ_1 and Λ_2 are the zeros-order multipliers to be determined. The expansion of equations (5.28)-(5.29) and split the monomial and simplify yields

$$\begin{aligned}
\Lambda_{2xx} &= 0,\\
\Lambda_{1vv} &= 0,\\
2\Lambda_1t &= (3u^2 - 4v)\Lambda_{2x},\\
\Lambda_{2t} &= -2u\Lambda_{2x},\\
2\Lambda_1x &= -u\Lambda_{2x},\\
2\Lambda_{1u} &= -\Lambda_2 - 2u\Lambda_{1v},\\
\Lambda_{2u} &= \Lambda_{1v},\\
\Lambda_{2v} &= 0.
\end{aligned}$$

$$(5.30)$$

The solution of the above overdetermined systems of partial differential equations gives the following multipliers

$$\Lambda_1 = \frac{1}{4}C_1\left(xu - 3tu^2 + 4tv\right) + \frac{1}{4}C_2(4v - 3u^2) - \frac{C_3 u}{2} + C_4, \quad (5.31)$$

$$\Lambda_2 = \frac{1}{2}C_1\left(2tu - x\right) + C_2 u + C_3, \quad (5.32)$$

where C_i, $i = 1, 2, 3, 4$ are arbitrary constants. The multipliers Λ_1 and Λ_2 of system (5.1) has the property

$$D_t T^t + D_x T^x = (u_t + 3uu_x - 2v_x)\Lambda_1 + (v_t + 2u_x v + uv_x - \frac{1}{2}u_{xxx})\Lambda_2, \quad (5.33)$$

for the arbitrary functions $u(t, x)$, $v(t, x)$ [27], where the predetermined arguments of T^t, T^x are of some order in derivatives of the field variables u and v. The computations for T^t and T^x from equation (5.33) reveal that corresponding to the above multipliers we have the following four conserved vectors for system (5.1):

$$T_1^t = \frac{1}{8}\left\{xu^2 + 8tuv - 2tu^3 - 4xv\right\},$$

$$T_1^x = \frac{1}{16}\left\{4xu^3 - 9tu^4 + 4tx^2u - 18tv^2 + 40tu^2v - 16xuv - 8uu_{xxt} + 2u_{xxx} - 2u_x\right\};$$

$$T_2^t = \frac{1}{4}\left\{4uv - v^3\right\},$$

$$T_2^x = \frac{1}{16}\left\{40u^2v - 8uu_{xx} - 9u^4 + 4x^2u - 16v^2\right\};$$

$$T_3^t = \frac{1}{4}\left\{4v - u^2\right\},$$

$$T_3^x = \frac{1}{2}\left\{4uv - u^3 - u_{xx}\right\};$$

$$T_4^t = u,$$

$$T_4^x = \frac{1}{2}\left\{3u^2 - 4v\right\}.$$

Remark 5.2 It is important to observe that the last three conserved quantities yields a conserved momentum whereas the first one is typically connected with integrability. We also noticed that the last two conserved quantities can be derived directly from system (5.1) by inspection. It is worth mentioning that higher-order conservation laws can be derived systematically by increasing the order of the multiplier.

5.4 Concluding remarks

In this chapter we employed the Lie symmetry method to study the coupled Jaulent-Miodek system. We constructed similarity reductions and exact solutions for the coupled Jaulent-Miodek system. The correctness of the solutions obtained were checked by inserting them back into the coupled system (5.1) with the aid of Maple. The conservation laws of the coupled Jaulent-Miodek system were derived using the multiplier method.

Chapter 6

Conservation laws of a (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity

The (2+1)-dimensional Jaulent-Miodek equation reads [75]

$$aw_t - w_{xxx} + bw^2 w_x - cw_x \partial_x^{-1} w_y - dww_y - e\partial_x^{-1} w_{yy} = 0, \tag{6.1}$$

where a, b, c, d and e are free constants and $\partial_x^{-1} = \int dx$ is the inverse transformation. The author in [75] studied Eq.(6.1) and obtained some solutions based on symmetry method. Equation (6.1) is a natural three-components extension of the (1+1)-dimensional Jaulent-Miodek hierarchy. In [76] some special quasi-periodic solutions for some $(2 + 1)$-dimensional integrable models generated by the Jaulent-Miodek hierarchy were obtained.

In this chapter we study a generalization of the (2+1)-dimensional Jaulent-Miodek equation (6.1), namely,

$$aw_t - w_{xxx} + bw^n w_x - cw_x \partial_x^{-1} w_y - dww_y - e\partial_x^{-1} w_{yy} = 0, \tag{6.2}$$

which we shall refer to as the (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity. Employing the transformation $w = u_x$ in conjunction with the inverse scattered relation and letting $d = 2c$ equation (6.2) transforms to

$$au_{tx} - u_{xxxx} + bu_x^n u_{xx} - cu_{xx} u_y - 2cu_x u_{xy} - eu_{yy} = 0. \tag{6.3}$$

In this chapter we perform Noether symmetry classification and construct conservation laws of eqution (6.3).

This work has been submitted for publication [77].

6.1 Conservation laws

We observe that neither equation (6.1) nor equation (6.2) do admit any Lagrangian formulation in their present form. However, the (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity (6.3) possess a Lagrangian structure. The standard Lagrangian of the (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity (6.3) is

$$\mathcal{L} = \frac{1}{2} c {u_x}^2 u_y - M(u_x) - \frac{1}{2} a u_t u_x - \frac{1}{2} e {u_y}^2 - \frac{{u_{xx}}^2}{2} \tag{6.4}$$

where

$$M(u_x) = \begin{cases} \dfrac{b u_x^{n+2}}{(n+1)(n+2)}, & \text{if} \quad n \neq -1,\, n \neq -2 \\ \\ b u_x \ln|u_x|, & \text{if} \quad n = -1 \\ \\ -b \ln|u_x|, & \text{if} \quad n = -2 \end{cases}$$

Thus, the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\delta u} = eu_{yy} + bu_x^n u_{xx} - 2cu_x u_{xy} - cu_{xx}u_y - u_{xxxx} - au_{tx} = 0. \tag{6.5}$$

Consider the infinitesimal generator of a point symmetry of the form

$$\begin{aligned} X = \quad & \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} \\ & + \eta^1(t,x,y,u)\frac{\partial}{\partial u}, \end{aligned} \tag{6.6}$$

such that the Killing-type equation

$$X^{(1)}\mathcal{L} + D_i(\xi^i)\mathcal{L} = D_i B^i, \tag{6.7}$$

holds for some point-dependent gauge terms $B = (B^i)$ where $B^i = B^i(t,x,y,u)$, $i = 1,2,3$ and $X^{(1)}$ is the first prolongation of the generator X. We now recall the Noether Theorem [22,51,78–80], that is, corresponding to each Noether symmetry, there exist a vector $T = (T^i)$ with components

$$T^i = \xi^i \mathcal{L} + \frac{\partial \mathcal{L}}{\delta u_i^j}(\eta_j - u_s^j \xi^s) - B^i, \tag{6.8}$$

which is a conserved vector of equation (6.3). The expansion of (6.7) leads to an overdetermining systems of PDEs. Solving the resulting systems of PDEs, prompt the results in Table 6.1.

The insertion of the components of the Noether symmetries in Table 6.1 above together with the Langragian $\mathcal{L}$ from (6.4) into (6.8) and perform some computations yield the following non trivial conserved vectors:

Case 1. $n \neq -1, -2$

$$\begin{aligned} T_1^1 \quad = \quad & \frac{1}{2(n+2)(n+1)}(-2bu_x^{n+2} - cn^2u_x^2u_y - 3cnu_x^2u_y - 2cu_x^2u_y + \\ & en^2u_y^2 + 3enu_y^2 + 2eu_y^2 + n^2u_{xx}^2 + 3nu_{xx}^2 + 2u_{xx}^2), \end{aligned}$$

Table 6.1: Classification results: a, b, c, n are non-zero constants, with symbol $\forall$ = arbitrary

$a, b, c\ n$	Noether symmetry generators	Gauge terms
$\forall$	$X_1 = \partial_t$	$B^1 = 0, B^2 = 0, B^3 = 0$
	$X_2 = \partial_y$	$B^1 = 0, B^2 = 0, B^3 = 0$
	$X_3 = f_1(t)\partial_x - \frac{ay}{c} f_1'(t)\partial_u$	$B^1 = 0, B^2 = \frac{a^2}{2c} yuf_1''(t)$,
		$B^3 = \frac{ae}{c} uf_1'(t)$
	$X_4 = f_2(t)\partial_u$	$B^1 = 0, B^2 = -\frac{a}{2} uf_2'(t)$,
		$B^3 = 0$

$$T_1^2 = -\frac{1}{2(n+1)}(-anu_t^2 - au_t^2 - 2bu_tu_x^{n+1} + 2cnu_tu_xu_y + 2cu_tu_xu_y + 2nu_tu_{xxx} - 2nu_{xx}u_{tx} + 2u_tu_{xxx} - 2u_{xx}u_{tx}),$$

$$T_1^3 = \frac{1}{2}\left(2eu_tu_y - cu_tu_x^2\right);$$

$$T_2^1 = \frac{1}{2}au_xu_y,$$

$$T_2^2 = -\frac{1}{2(n+1)}(-anu_tu_y - au_tu_y - 2bu_y{u_x}^{n+1} + 2cnu_xu_y^2 + 2cu_x{u_y}^2 - 2nu_{xx}u_{xy} + 2nu_{xxx}u_y - 2u_{xx}u_{xy} + 2u_{xxx}u_y),$$

$$T_2^3 = \frac{1}{2(n+2)(n+1)}(-an^2u_tu_x - 3anu_tu_x - 2au_tu_x - 2b{u_x}^{n+2} + en^2{u_y}^2 + 3enu_y^2 + 2eu_y^2 - n^2u_{xx}^2 - 3nu_{xx}^2 - 2u_{xx}^2);$$

$$T_3^1 = \frac{1}{2c}(a^2yf(t)_1'u_x + acf_1(t)u_x^2),$$

$$T_3^2 = \frac{1}{2c(n+1)(n+2)}(-a^2n^2yf(t)_1''u - 3a^2nyf(t)_1''u - 2a^2yf(t)_1''u + a^2n^2yf(t)_1'u_t + 3a^2nyf(t)_1'u_t + 2a^2yf(t)_1'u_t + 4abyf(t)_1'{u_x}^{n+1} + 2abnyf_1'u_x^{n+1} - 2acn^2yf(t)_1'u_xu_y - 6acnyf(t)_1'u_xu_y - 4acyf(t)_1'u_xu_y - 2an^2yf(t)_1'u_{xxx}6anyf(t)_1'u_{xxx} - 4ayf(t)_1'u_{xxx} + 2bcf_1(t)u_x^{n+2} + 2bcnf_1(t){u_x}^{n+2} - c^2n^2f_1(t)u_x^2u_y - 3c^2nf_1(t)u_x^2u_y - 2c^2f_1(t)u_x^2u_y - cen^2f_1(t)u_y^2 - 3cenf_1(t)u_y^2 - 2cef_1(t)u_y^2 - 2cn^2f_1(t)u_{xxx}u_x + cn^2f_1(t)u_{xx}^2 - 6cnf_1(t)u_{xxx}u_x + 3cnf_1(t)u_{xx}^2 - 4cf_1(t)u_{xxx}u_x + 2cf_1(t){u_{xx}}^2),$$

$$T_3^3 = \frac{1}{2c}(-2aef(t)_1'u - acyf(t)_1'u_x^2 + 2aeyf(t)_1'u_y - c^2f_1(t)u_x^3 + 2cef_1(t)u_xu_y);$$

$$T_4^1 = -\frac{1}{2}af_2(t)u_x,$$

$$T_4^2 = \frac{1}{2(n+1)}(anf_2'u + af_2'u - anf_2(t)u_t - af_2(t)u_t - 2bf_2(t)u_x^{n+1} + 2cnf_2(t)u_xu_y + 2cf_2(t)u_xu_y + 2nf_2(t)u_{xxx} + 2f_2(t)u_{xxx}),$$

$$T_4^3 = \frac{1}{2}\left(cf_2(t)u_x^2 - 2ef_2(t)u_y\right);$$

Case 2. $n = -1$

$$
\begin{aligned}
T_1^1 &= \frac{1}{2}\left(-2bu_x \ln u_x + cu_x^2 u_y - eu_y^2 - {u_{xx}}^2\right), n = -1,\\
T_1^2 &= \frac{1}{2}\left(au_t^2 + 2bu_t \ln u_x + 2bu_t - 2cu_t u_x u_y - 2u_t u_{xxx} + 2u_{xx} u_{tx}\right),\\
T_1^3 &= \frac{1}{2}\left(2eu_t u_y - cu_t u_x^2\right);\\
T_2^1 &= \frac{1}{2} a u_x u_y,\\
T_2^2 &= \frac{1}{2}\left(au_t u_y + 2bu_y \ln u_x + 2bu_y - 2cu_x u_y^2 - 2u_{xxx} u_y + 2u_{xx} u_{xy}\right),\\
T_2^3 &= \frac{1}{2}\left(-au_t u_x - 2bu_x \ln u_x + eu_y^2 - u_{xx}^2\right);\\
T_3^1 &= \frac{1}{2c}\left(a^2 y f_1' u_x + acf_1(t) u_x^2\right),\\
T_3^2 &= \frac{1}{2c}(-a^2 y f_1'' u + a^2 y f_1' u_t + 2aby f_1' \ln u_x - 2acy f_1' u_x u_y\\
&\quad -2ay f_1' u_{xxx} + 2bcf_1(t) u_x - c^2 f_1(t) u_x^2 u_y - cef_1(t) u_y^2 + cf_1(t) u_{xx}^2\\
&\quad -2cf_1(t) u_x u_{xxx} + 2aby f_1'),\\
T_3^3 &= \frac{1}{2c}(-2aef_1' u - acy f_1' u_x^2 + 2aey f_1' u_y - c^2 f_1(t) u_x^3 + 2cef_1(t) u_x u_y);\\
T_4^1 &= -\frac{1}{2} a f_2(t) u_x,\\
T_4^2 &= \frac{1}{2}(af_2' u - af_2(t) u_t - 2bf_2(t) \ln u_x + 2cf_2(t) u_x u_y + 2f_2(t) u_{xxx}\\
&\quad -2bf_2(t)),\\
T_4^3 &= \frac{1}{2}\left(cf_2(t) u_x^2 - 2ef_2(t) u_y\right);
\end{aligned}
$$

Case 1. $n = -2$

$$
\begin{aligned}
T_1^1 &= \frac{1}{2}\left(2b \ln u_x + cu_x^2 u_y - eu_y^2 - u_{xx}^2\right),\\
T_1^2 &= \frac{1}{2}(au_t^2 u_x - 2bu_t - 2cu_t u_x^2 u_y - 2u_t u_{xxx} u_x + 2u_{xx} u_x u_{tx}) u_x^{-1},
\end{aligned}
$$

$$
\begin{aligned}
T_1^3 &= \frac{1}{2}\left(2eu_tu_y - cu_tu_x^2\right);\\
T_2^1 &= \frac{1}{2}au_xu_y,\\
T_2^2 &= \frac{1}{2}\left(au_tu_xu_y - 2bu_y - 2cu_x^2u_y^2 + 2u_{xx}u_xu_{xy} - 2u_{xxx}u_xu_y\right)u_x^{-1},\\
T_2^3 &= \frac{1}{2}\left(-au_tu_x + 2b\ln u_x + eu_y^2 - u_{xx}^2\right);\\
T_3^1 &= \frac{1}{2c}\left(a^2yf_1'u_x + acf_1(t)u_x^2\right),\\
T_3^2 &= \frac{1}{2cu_x}(-a^2yf_1''u_xu + a^2yf_1'u_tu_x - 2acyf_1'u_x^2u_y - 2ayf_1'u_{xxx}u_x -\\
&\quad 2bcf_1(t)u_x + 2bcf_1(t)u_x\ln u_x - c^2f_1(t)u_x^3u_y - cef_1(t)u_xu_y^2 -\\
&\quad 2cf_1(t)u_{xxx}u_x^2 + cf_1(t)u_{xx}^2u_x - 2abyf_1'),
\end{aligned}
$$

$$
\begin{aligned}
T_3^3 &= \frac{1}{2c}\left(-2aef_1' - acyf_1'u_x^2 + 2aeyf_1'u_y - c^2f_1(t)u_x^3 + 2cef_1(t)u_xu_y\right); \quad (6.9)\\
T_4^1 &= -\frac{1}{2}af_2(t)u_x,\\
T_4^2 &= \frac{u_x^{-1}}{2}\left(af_2'u_xu - af_2(t)u_tu_x + 2cf_2(t)u_x^2u_y + 2f_2(t)u_{xxx}u_x + 2bf_2(t)\right),\\
T_4^3 &= \frac{1}{2}\left(cf_2(t)u_x^2 - 2ef_2(t)u_y\right) \quad (6.10)
\end{aligned}
$$

respectively.

Remark 6.1 It is interesting to observe that although different Lagrangian (see (6.4)) may result in same Noether operators (see Table 6.1), this does not hold with the conservation laws. This can be seen from the above derived conserved vectors. It is worthy mentioning that due to the presence of the arbitrary elements in the conserved vectors, one can obtain infinitely local conserved vectors of the (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity (6.3).

6.2 Concluding remarks

In this chapter we carried out a complete Noether symmetry classification of a (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity. Conservation laws for several cases which admitted Noether point symmetries were established.

Chapter 7

Conclusions

Obtaining exact solutions for nonlinear partial differential equations that model real world problems are important to understand the physical aspects of the problem. The aim of this work was to determine exact solutions and conservation laws of some nonlinear partial differential equations by using various methods.

Chapter one provided basic background material, such as, the definitions and theorems of important concepts that were used to carry out the calculations in this work.

In Chapter two we carried out a complete Noether and Lie group classification of the radial form of a coupled system of hyperbolic equations. From the Noether symmetries we derived the corresponding conserved vectors. We also determined constraints that the non-linearities should satisfy in order for the scaling symmetries to be Noetherian. An explicit solution was also obtained for a particular choice of the parameters.

A complete Noether symmetry analysis of a generalized hyperbolic Lane-Emden system was performed in Chapter three. Several constraints for which Noether symmetries exist were derived. Furthermore, we constructed conservation laws associated with the admitted Noether symmetries. Thereafter, we briefly discussed the physical meaning of the derived conserved vectors.

In Chapter four we carried out a complete group classification of a generalized coupled hyperbolic Lane-Emden system. It was shown that the underling system admitted six-dimensional equivalence Lie algebra. We further showed that the principle Lie algebra which is one dimensional extended in several cases. We also carried out Lie reductions for some cases.

Chapter five dealt with the symmetry analysis of a coupled Jaulent-Miodek system, which arises in many branches of physics such as particle physics and fluid dynamics. The similarity reductions and new exact solutions were constructed. Thereafter, conservation laws were derived using the multiplier approach.

In Chapter six we studied complete Noether symmetry classification of a (2+1)-dimensional Jaulent-Miodek equation with power-law nonlinearity. Conservation laws for several cases which admitted Noether point symmetries were established.

In future, we plan to use the conservation laws obtained in this study to find exact solutions for the corresponding nonlinear partial differential equations studied here.

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