Bianchi type-I cosmological models in $f(R)$ gravity

A. Swart



orcid.org/0000-0002-1347-135X

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Supervisor: Prof. A. Abebe

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Declaration

This is written so as to hereby declare that this dissertation has not been submitted, either in the same or different form, to this or any other university for any degree and that it represents my own work.

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Abstract

The Standard Model of cosmology makes an assumption of homogeneity and isotropy throughout the Universe. However, recent observations have led us to believe that the Universe is somewhat anisotropic on large scales. This has led us to investigate an anisotropic cosmological model in order to possibly solve this anisotropy problem. We also know that the Universe was almost perfectly homogeneous at early times, but there needed to be some slight inhomogeneity for the large-scale structures to form. To include these inhomogeneities, we include cosmological perturbations in our model. Another problem that is famous in cosmology is related to the mysterious dark energy that causes the accelerated expansion of our universe. There exist several methods to insert this dark energy into cosmological models, among them the cosmological constant, and modified gravity theories. We attempt to use an $f(R)$ gravity model to get around this problem. In this dissertation, we study the Bianchi-type I model. We find numerical solutions to the field equations and we give plots of these solutions, along with plots of the deceleration parameter. We then move on to study this model in $f(R)$ gravity by first deriving the field equations, and then finding solutions with the use of a power-law relation. We then linearize the propagation and constraint equations in the energy frame of the matter, and examine the background of Bianchi-type I models in both the GR and the $f(R)$ gravity contexts. We then derive the first set of perturbation equations using the gauge-invariant 1+3 covariant formalism, and we offer some conclusions and future outlooks for this dissertation.

Keywords: cosmology; Bianchi-I; spacetime; dark energy; dark matter; homogeneity; anisotropy; perturbations; $f(R)$ gravity.

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Table of abbreviations

A table containing a list of abbreviations and acronyms that will be used throughout this text.

BB	Big Bang
BI	Bianchi Type-I
CDM	cold dark matter
CMB	cosmic microwave background
EEP	Einstein Equivalence Principle
EFE	Einstein Field Equation
EMT	energy-momentum tensor
EoS	equation of state
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	General Relativity
HP	Hubble parameter
IC	Initial conditions
LHS	left-hand side
LRF	local rest frame
LRS	locally-rotationally-symmetric
PSTF	projected symmetric trace free
RHS	right-hand side
SEP	Strong Equivalence Principle
SM	Standard Model
WEP	Weak Equivalence Principle
WMAP	Wilkinson Microwave Anisotropy Probe

Chapter 1

Introduction

Cosmology, as a branch of astronomy, studies the evolution of the Universe - from the theorised initial singularity that might have caused the Big Bang, followed by inflation and the subsequent formation of isotopes in the case of Big Bang Nucleosynthesis, to the expansion of the Universe, formation of large-scale structures and the ultimate fate of our universe. The philosophical and simultaneously scientific question of, "where do we come from and why do we exist?" has persisted through the ages, causing various hypotheses to be formed. Many theories have been disputed, and many have been accepted until a later theory comes to take the old one's place.

Einstein initially suggested a homogeneous, static, spatially curved universe [88] in 1916, but this static-universe theory was discarded shortly after the time of its first formulation. With the discovery of the Universe's expansion by Edwin Hubble in 1925 [82], a question formed: will the Universe expand for an infinite time, or will it eventually start contracting and ultimately cause a collapse? This was called the Big Crunch scenario. Along with this, Big Rip theories and Big Bounce theories followed [68]. Luckily, this is a question that can be answered by knowing the contents of the Universe. It was found that we have a set amount of baryonic matter and relativistic matter in the Universe, accompanied by some radiation. All of this forms the known energy content of the Universe.

A method of observing the expansion of the Universe is by observing the explosions of giant, dying stars: Type Ia supernovae. Seeing as Type Ia supernova explosions always release nearly the same amount of energy, the particular peak brightness of these supernovae can serve as remarkably precise standard candles. The color, spectrum, and an empirical correlation observed between the timescale of the supernova light curve and peak luminosity also aid in providing additional information [77]. Thus, if we can measure the luminosity of Type Ia Supernovae over time, we can determine the distances that they are from us [68, 77]. But the theorised distances of the oldest supernovae were found to be inconsistent with the measurements - they were observed to be further away than theorised by a large margin, and thus the Universe had to be expanding at an increasing rate. This discovery lit flames in the scientific community; the current energy content that

was accounted for could not explain this accelerated expansion, and thus, the conclusion had to be that there was a gap in the theory.

This gap could possibly be filled in several different ways: either, we had missing energy (coining the term dark energy), or perhaps the General Relativity model formed by Einstein needed to be modified. This missing link between the observed universe and the theory behind it needed to be found.

According to the observed analysis, approximately 5% of the energy content of the Universe is said to be taken up by baryonic matter, relativistic matter and radiation - the observable content. But roughly 27% of the energy content of the Universe is believed to be dark matter, and the remaining 68% is thus dark energy [1, 77]. This dark energy is believed to be the source of the accelerated expansion of the Universe. Consequently, this has become one of the most widely studied topics in astronomy.

Some candidates that can possibly explain dark energy are the cosmological constant (which is believed to be a constant energy density that fills space up homogeneously), exotic scalar fields (that form a dynamical energy density that is not homogeneous), and geometrical dark energy [6]. The scalar fields are studied with models like quintessence, while the geometrical dark energy is studied with modified gravity models.

When studying inhomogeneous models, generally dark energy is not required, and thus this is a way around adding a mysterious energy to the model. Cosmological perturbation theory is a way of inserting inhomogeneities into a homogeneous model, and thus being able to study the growth of the large-scale structures in a homogeneous model. This theory can also be applied to modified gravity models.

The standard cosmological model makes an assumption of isotropy and homogeneity throughout the Universe. The problem with this, however, is that there has been evidence that has led us to believe that the Universe is not completely isotropic at large scales - the accelerated expansion of the Universe is also found to be more anisotropic [67]. Thus, the studies of anisotropic cosmological models are advantageous - perhaps there is a model that would better describe the evolution of the Universe than the standard model. The assumption of homogeneity also has some issues - if the Universe was completely homogeneous in early times, we would not have had any large-scale structure formation.

In this dissertation, we do the following: In Chapter 2, we give an overview of General Relativity (GR) and the Standard Model (SM) of cosmology. We also discuss the problems with GR and the SM, and then we suggest a possible solution to these problems. Chapter 3 entails a discussion of the 1+3 Covariant formalism, which is a gauge-invariant formalism. We also discuss cosmological perturbations, and the use of this covariant formalism in perturbation theory. In the fourth chapter we introduce the Bianchi Models as a possible solution to the anisotropy problem, and then we focus on the Bianchi type-I (BI) model. We give the field equations and background dynamics of this model and we also solve the field equations numerically. After this, we introduce $f(R)$ gravity in Chapter 5, stating

the propagation and constraint equations and the conditions for viability of $f(R)$ models, and we give some examples of different $f(R)$ models. In Chapter 6, we then apply this $f(R)$ gravity model to our BI metric and find some solutions for this model by applying a power-law relation. The 1+3 covariant formalism is applied to our BI model in $f(R)$ gravity, and we give the first set of perturbation equations. We then conclude by pointing our the implications of our work, as well as stating the problems and future outlook of this work.

We use the (-,+,+,+) spacetime signature with natural unit conventions, i.e.,$\hbar = c = \kappa = 8\pi G = 1$, or the normalized reduced Planck's constant, speed of light, Boltzmann constant and Newton's gravitational constant [46]. We can define a *covariant tensor* as $T_{ab\ldots}$ and a *contravariant tensor* as $T^{ab\cdots}$. Some standard notations for this dissertation are given by:

1. Symmetrization over the indices a and b of the rank-two tensor T_{ab}:
$$T_{(ab)} := \frac{1}{2}(T_{ab} + T_{ba}), \tag{1.1}$$
2. Anti-symmetrization over the indices a and b of the rank-two tensor T_{ab}:
$$T_{[ab]} := \frac{1}{2}(T_{ab} - T_{ba}), \tag{1.2}$$
3. Furthermore
$$T_{ab} = T_{(ab)} + T_{[ab]}, \tag{1.3}$$
and so on also for higher rank tensors.

Angled brackets denote orthogonal projections of vectors (represented by V_a) and the projected symmetric trace-free (PSTF) parts of tensors (represented by T_{ab}) given by
$$V_{\langle a\rangle} = h^b_a V_b, \quad T_{\langle ab\rangle} = \left(h^c_{(a}h^d_{b)} - \frac{1}{3}h_{ab}h^{cd}\right)T_{cd}, \tag{1.4}$$
where h_{ab} is the projection tensor, defined in Chapter 2.1.

The symmetric *metric tensor* is represented by g_{ab} and it is a covariant tensor of order (or rank) 2.

The covariant and contravariant metrics are symmetric matrices which imply that
$$g_{ca}g^{ab} = \delta^b_c, \tag{1.5}$$
where δ^b_c is the Kronecker Delta function, i.e.,
$$\delta^b_c = \begin{cases} 1 & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Wherever we have g, this can be assumed to represent the determinant of the metric, $g = \det g_{ab}$.

Chapter 2

The Standard Cosmological Model

Modern cosmology has been influenced by many renowned astronomers and scientists: from the Middle Ages with Nicolaus Copernicus, Johannes Kepler, Galileo Galilei and Isaac Newton, to the more recent Edwin Hubble, Albert Einstein, Stephen Hawking and many more. These great minds have changed the way we perceive the universe.

In this chapter, we will review the widely accepted Standard Model (SM) of cosmology which uses a Friedmann-Lemaître-Robertson-Walker (FLRW) metric, and thereafter we will discuss the successes and the problems of the standard model. We will also introduce a way out, considering some of the problems we face with the SM (Standard Model) of cosmology.

2.1 General Relativity

General Relativity (GR) is based on the General Principle of Relativity which states that the laws of physics are the same in all reference frames [7]. Modern cosmology is greatly influenced by Einstein's theory of GR formulated between 1907 and 1915. Thus, the following section will describe this theory with its applications.

We start by defining a few variables:

1. **Comoving coordinates** are used in spacetime geometry and are adapted to the fundamental worldines [42]. A comoving coordinate is a coordinate that expands in the same way as the space expands, for example, as space expands, the galaxies in the Universe would always remain in the same position if measured in comoving coordinates.

2. The **4-velocity** vector (or fluid flow vector) is another important quantity. Ellis et al., describes the fluid flow vector in [37] as

$$u = e_0, \quad u^a = \delta^a_0, \quad u_a = -\delta^0_a, \tag{2.1}$$

where e_0 is the timelike vector (or basis vector) that is chosen as the fluid flow vector, the index a runs from 0 to 3, and δ^i_j is known as the Kronecker delta symbol given by

$$\delta^i_j = \begin{cases} 0, & (i \neq j), \\ 1, & (i = j). \end{cases} \tag{2.2}$$

Geometrically speaking, the 4-velocity vector can be depicted as an arrow pointing along the fundamental worldlines. Mathematically, we write it as

$$u^a = \frac{dx^a}{d\tau}, \tag{2.3}$$

where x^a are the local coordinates of the worldlines, given by $x^a = x^a(\tau)$, and τ is the proper time along the worldlines. This 4-velocity vector is the relativistic counterpart of velocity. We state that $u^a u_a = -1$. We then have a projection tensor U^a_b which satisfies

$$U^a{}_b = -u^a u_b \qquad \Rightarrow \qquad U^a{}_c U^c{}_b = U^a{}_b, \quad U^a{}_a = 1, \quad U_{ab}u^b = u_a. \tag{2.4}$$

3. The **time derivative** of any tensor $S^{a\cdots}{}_{b\cdots}$ along the fluid flow lines is [42]

$$\dot{S}^{a\cdots}_{b\cdots} = u^c \nabla_c S^{a\cdots}{}_{b\cdots}. \tag{2.5}$$

In particular, we define the time differentiation of the derivative of the 4-velocity vector itself in its own direction, which results in the acceleration vector, and is represented by

$$\dot{u}^a = u^b \nabla_b u^a \quad \Rightarrow \quad \dot{u}^a u_a = 0. \tag{2.6}$$

This acceleration vanishes if and only if the flow lines are geodesic [42]. The relativistic acceleration vector represents the degree to which matter moves under forces other than gravity plus inertia [36]. For the rest of this dissertation, we will denote the time derivative with an overdot.

4. We have an induced effective metric tensor for locally defined surfaces of simultaneity for fundamental observers (due to the existence of a preferred rest frame at each point where there is also a preferred velocity). This metric tensor is defined as

$$h_{ab} = g_{ab} + u_a u_b. \tag{2.7}$$

Given the definition for the 4-velocity, equation (2.3), we can also write

$$h^a{}_b h^b{}_c = h^a{}_c, \quad h^a{}_a = 3, \quad h^a{}_b u^b = 0, \tag{2.8}$$

which has an implication that $h^a{}_b$ is a **projection tensor** projecting into the three-dimensional tangent plane orthogonal to u^a [42].

Some identities to note:

$$u^a u_a = -1, \quad h^{ab}u_a = 0, \quad g^{ab}g_{ab} = 4. \tag{2.9}$$

Certain evidence-based assumptions are made in order to have a starting point for our cosmological model. Two of these assumptions are known as the Cosmological Principle. The Cosmological Principle states that the Universe is both homogeneous (the same at every point) and isotropic (the same in every direction). Now, this is not true if we look at the small structures in the Universe, but if we approximate and look at the large-scale structures, then this principle becomes true, for the most part. This also means that we do not have preferred points or preferred directions in the Universe [7], and this helps us constrain the metric of the spacetime.

If the Cosmological Principle holds in the Universe, then we make the assumption that matter moves along irrotational and shear-free geodesic curves with tangent vector u^a [46].

The Principle of Equivalence is one of the major principles that Einstein based his field equations on. This principle states that both gravitation and inertial forces are indistinguishable and of a similar nature.

The principle can be divided into three parts:

1. **Weak Equivalence Principle (WEP):** when a test-particle is in free fall, its trajectory is independent of its internal structure and composition [7].
2. **Strong Equivalence Principle (SEP):** assumes that all non-gravitational phenomena locally take the same form in GR as in special relativity [42].
3. **Einstein Equivalence Principle (EEP):** assumes the WEP holds, and it states that the outcome of any local non-gravitational experiment is independent of the velocity, as well as the where and when it is performed of the freely-falling reference frame in which it is performed [7]. In essence, this principle is a stronger form of the SEP, and it implies that all physics in freely falling systems is locally the same as in special relativity [42].

The EEP is the fundamental pillar of the theory of GR [7]. The equivalence principle loosely translates to 'a particle moves on a spacetime geodesic if in free fall, i.e., if it moves under gravity and inertia only', as stated by Ellis et al., in [42].

Riemannian Geometry

We will be dealing with Riemannian geometry quite often, leading us to define the Riemannian space - a curved 4-dimensional space which is immersed in a flat space of larger number of dimensions. When we think about a smaller region in a Riemannian space, this will be considered an almost flat space. We also consider the invariant space ds between a point x^a and a neighbouring point $x^a + dx$, and is sometimes called the *line element*:

$$ds^2 = g_{ab}dx^a dx^b. \tag{2.10}$$

The next important quantity to define is the *affine connection*, Γ^a_{bc}, also known as the Christoffel symbol of the second kind. This quantity can be written as

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(g_{db,a} + g_{ad,b} - g_{ab,d} \right), \tag{2.11}$$

where the comma in the indices represents a partial derivative with respect to the component following the comma (or $T_{ab,c} = \partial_c T_{ab}$). Note that the connection is not a tensor since the partial derivative of a tensor is also not a tensor. However, the combination of partial derivatives and connection terms is a tensor [42].

We can use the metric tensor, g_{ab}, to contract tensors, e.g. $\Gamma_{cab} = g_{cd}\Gamma^d_{ab}$.

The *covariant derivative*, ∇_a, of a tensor is denoted in this dissertation by a semicolon and is usually written as

$$\nabla_b V_a = V_{a;b} = V_{a,b} - \Gamma^c_{ab} V_c, \tag{2.12}$$

$$\nabla_b V^a = V^a_{;b} = V^a_{,b} + \Gamma^a_{cb} V^c. \tag{2.13}$$

For a tensor with multiple indices, it is given as

$$T_{ab;c} = T_{ab,c} - \Gamma^d_{ac} T_{db} - \Gamma^e_{bc} T_{ae}, \tag{2.14}$$

and so on. Since the covariant derivative of a tensor is itself also a tensor [42], it can also be covariantly differentiated, and that can be written as

$$\nabla_d \nabla_c T_{ab} = T_{ab;cd}. \tag{2.15}$$

Curvature arises from the *Riemann curvature tensor*, which is defined as

$$R^d_{abc} = \Gamma^d_{ac,b} - \Gamma^d_{ab,c} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{ab}\Gamma^d_{ec}. \tag{2.16}$$

We also define two important identities that involve the Riemann tensor, namely the Bianchi identities [7]:

$$\begin{aligned} R^d_{abc} + R^d_{cab} + R^d_{bca} = 0 = R^d_{[abc]} \qquad & \text{(the first Bianchi identity)}, \\ R^a_{bcd;e} + R^a_{bde;c} + R^a_{bec;d} = 0 = R^a_{b[cd;e]} \qquad & \text{(the second Bianchi identity)}. \end{aligned} \tag{2.17}$$

The *second* Bianchi identity is also called *the* Bianchi identities.

The *Ricci tensor* is a symmetric tensor of second-order which arises from the contraction of the curvature tensor, and can hence be defined as

$$R^c_{acb} = R_{ab} = R_{ba} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^e_{ab}\Gamma^c_{ec} - \Gamma^e_{ac}\Gamma^c_{eb}. \tag{2.18}$$

The *Ricci scalar* is obtained by contracting the indices of the Ricci tensor and is given as

$$g^{ab} R_{ab} = R^a_a = R. \tag{2.19}$$

This is the total curvature of the spacetime. It is defined in such a way that $R > 0$ for the surface of a 3 dimensional sphere.

If we know what the Ricci tensor and the Ricci scalar are, we define the Einstein tensor as

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R. \tag{2.20}$$

In this tensor, we can see that both R_{ab} and g_{ab} are second-order tensors, and thus we can state that the Einstein tensor is also a symmetric second-order tensor. All of these quantities can be computed by hand, but for this dissertation we used the *GRTensor* package on Maple. This package makes computations quick and easy if you specify the appropriate spacetime.

From the second Bianchi identity, we find that the divergence of the Einstein tensor vanishes, i.e.

$$G^{ab}_{\ ;a} = 0. \tag{2.21}$$

The volume element for the 3-restspaces orthogonal to u^a is given by

$$\eta_{abc} = u^d \eta_{dabc} \quad \Rightarrow \quad \eta_{abc} = \eta_{[abc]}, \quad \eta_{abc} u^c = 0, \tag{2.22}$$

where η_{abcd} is the 4-dimensional volume element, also known as the Levi-Civita 4-form, that has components

$$\eta_{abcd} = -\sqrt{|g|}\epsilon_{abcd}, \tag{2.23}$$

where ϵ_{abcd} is the totally skew tensor density whose components are fixed by $\epsilon_{0123} = 1$ [42]. This 4-dimensional volume element satisfies some conditions [46]:

$$\eta_{abcd} = \eta_{[abcd]} = 2\epsilon_{ab[c} u_{d]} - 2u_{[a}\epsilon_{b]cd}. \tag{2.24}$$

In a Riemannian space, $\eta_{abcd;e} = 0$, and thus this volume element can be freely moved in covariant differentiation [42]. We also have the volume element for the 3-restspaces as defined by [46]:

$$\eta_{abc} = u^d \eta_{dabc} \Rightarrow \eta_{abc} = \eta_{[abc]}, \ \eta_{abc} u^c = 0. \tag{2.25}$$

We can define $\eta_{0123} = \sqrt{|det\ g_{ab}|}$. The quantity η_{abc} satisfies the following identities:

$$\eta^{abc}\eta_{def} = 3! h^{[a}_{\ \ d} h^{b}_{\ \ e} h^{c]}_{\ \ f}, \tag{2.26}$$

$$\eta^{abc}\eta_{cef} = 3! h^{[a}_{\ \ e} h^{b]}_{\ \ f}, \tag{2.27}$$

$$\eta^{abc}\eta_{abc} = 3!. \tag{2.28}$$

The trace of the Ricci tensor (which has 10 independent components) forms the Ricci scalar. All leftover components are contained in the Weyl tensor [42],

$$C^{ab}_{\ \ cd} = R^{ab}_{\ \ cd} - 2g^{[a}_{\ \ [c} R^{b]}_{\ \ d]} + \frac{R}{3} g^{[a}_{\ \ [c} g^{b]}_{\ \ d]}. \tag{2.29}$$

The Weyl tensor is hence trace-free, which means that

$$C^{a}_{\ bad} = 0, \tag{2.30}$$

and it can be split into its 'electric' and 'magnetic' parts, denoted by E_{ab} and H_{ab} respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \frac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \tag{2.31}$$

Our electric part represents the free gravitational field, or tidal forces and the magnetic part is responsible for the gravitational waves. This magnetic part has no Newtonian counterpart, but the electric part can be clarified by its Newtonian analogue [46]:

$$E_{ab} = \phi_{,ab} - \frac{1}{3}h_{ab}\phi^d{}_{,d} \tag{2.32}$$

and ϕ is the usual Newtonian gravitational potential [30]. We can say that these tensors are both symmetric and trace-free in the local rest frame (LRF) of u^a [46]:

$$\begin{aligned} E_{ab} &= E_{(ab)}, & H_{ab} &= H_{(ab)}, \\ E^a{}_a &= 0, & H^a{}_a &= 0, \\ E_{ab}u^b &= 0, & H_{ab}u^b &= 0. \end{aligned} \tag{2.33}$$

We can thus write our Weyl tensor as

$$\begin{aligned} C_{abcd} \quad &= (4g_{a[p}g_{q]b}g_{c[r}g_{s]d} - \eta_{abpq}\eta_{cdrs})u^p u^r E^{qs} \\ &\quad + 2(\eta_{abpq}g_{c[r}g_{s]d} + g_{a[p}g_{q]b}\eta_{cdrs})u^p u^r H^{qs}. \end{aligned} \tag{2.34}$$

Einstein's Field Equations

The field equations that Einstein proposed in 1915 are still accepted as true, and is the best description so far of the behaviour of space-time on macroscopic scales.

We can derive the field equations with the use of an appropriate action. The Principle of Least Action allows us to derive the equations of motion of a system once its action is known. This action is related to the Lagrangian of that system [7]. The Einstein-Hilbert action that allows us to derive the Einstein Field Equations (EFE's) is given by (assuming normalized units)

$$\mathcal{A}_{EH} = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}_m + \Lambda\right), \tag{2.35}$$

where $\mathcal{L}_m$ is the matter Lagrangian and Λ is the cosmological constant.

Using the Einstein tensor given in equation (2.20), the field equations can be written as

$$G_{ab} + \Lambda g_{ab} = T_{ab}, \tag{2.36}$$

where Λ is the cosmological constant, and T_{ab} is the energy-momentum tensor. Einstein had first inserted this cosmological constant into his equations in order to describe his proposed static universe, but then later removed it stating that it was the "biggest blunder" of his career. But later on, when it was discovered that the expansion of the Universe is accelerating, this constant was again included in the EFE's. Now, we use it as a placeholder for dark energy in the Standard Cosmological Model.

The field equations are a set of ten equations which are contained in the tensors shown above. These equations are vital because they contain information about the expansion of the Universe, the behaviour of black holes, the propagation of gravitational waves, and the formation of structures in the Universe. They describe gravity as a result of spacetime being curved by both mass and energy.

The left-hand side (LHS) of equation (2.36) describes the curvature of spacetime dependent on the spacetime metric, and the right-hand side (RHS) describes the matter-energy content of spacetime. As John Wheeler had said, "Space-time tells matter how to move; matter tells space-time how to curve" [62].

These field equations are non-linear, but quasi-linear, second-order partial differential equations for the gravitational potentials, g_{ab}, and they yield the Newtonian limit in the weak-field and slow approximation. They also guarantee that the energy-momentum tensor is conserved [42]. GR is considered a second-order theory of gravity due to the highest derivatives contained in the field equations being second-order. On small scales, GR is thought to be a good theory, but it is inadequate on large scales.

We can re-write these field equations to read

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = T_{ab}. \tag{2.37}$$

Throughout this dissertation, we use the term 'matter' to refer to all components that contribute to the cosmic fluid, except for curvature, including baryonic matter and cold dark matter (CDM), with the combination commonly being referred to as "dust" [46], and radiation.

The energy-momentum tensor (EMT), T_{ab}, describes the matter-energy content of the Universe, and can be decomposed relative to u^a, yielding

$$T_{ab} = \rho u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab}, \tag{2.38}$$

where we have

$$\rho = T_{ab}u^a u^b, \quad q^a = -T_{bc}u^b h^{ca}, \quad p = \frac{1}{3}(T_{ab}h^{ab}), \quad \pi_{ab} = T_{cd}h^c{}_{\langle a}h^d{}_{b\rangle} \tag{2.39}$$

as the relativistic energy density relative to u^a, the relativistic momentum density, or the energy flux relative to u^a, the isotropic pressure, and the trace-free anisotropic pressure (stress) respectively. This is the general EMT, not written for any specific component. Note that $q_a u^a = 0$, $\pi^a{}_a = 0$, $\pi_{ab} = \pi_{(ab)}$, and $\pi_{ab}u^b = 0$.

We also have the trace of the EMT which is given by

$$T = T^a{}_a = 3p - \rho. \tag{2.40}$$

The trace equation can be obtained by substituting the trace of the EMT, equation (2.40), into the EFE's, equation (2.37), and this yields the trace of the EFE's, which is also the trace of the Ricci tensor, or just the Ricci scalar:

$$R = \rho - 3p + 4\Lambda. \tag{2.41}$$

Some restrictions are commonly imposed to define a perfect fluid (in a perfect cosmological medium, both q^a and π_{ab} vanish, i.e.,$q^a = \pi_{ab} = 0$), which results in the EMT which defines a perfect fluid

$$T_{ab} = \rho u_a u_b + p h_{ab}. \tag{2.42}$$

Perfect fluids have a linear equation of state (EoS)

$$p = w\rho, \tag{2.43}$$

where w is the EoS parameter which can take on values

$$w = \begin{cases} 0, & \text{for non-relativistic matter, or dust, including baryonic and dark matter,} \\ \frac{1}{3}, & \text{for radiation,} \\ -1, & \text{for dark energy, or the cosmological constant } \Lambda. \end{cases} \tag{2.44}$$

When $p = 0$, we have pressure-free matter, which is the simplest case.

For a barotropic fluid, the isentropic speed of sound of the fluid can be defined as

$$c_s^2 = \frac{dp}{d\rho}, \tag{2.45}$$

which is also known as the adiabatic formula. Here, we need to constrain the acoustic speed in order to preserve the matter stability and causality:

$$0 \leq c_s^2 \leq 1. \tag{2.46}$$

When $c_s^2 < 0$, matter is unstable, and causality breaks beyond $c_s^2 = 1$ [46].

There are certain energy conditions that are required to hold when specifying an equation of state that relates p and ρ to define other cases. The energy conditions put constraints on the energy density, ρ, and the pressure, p, of any cosmological model [89]:

1. Null Energy Condition:
$$\rho + p \geq 0, \tag{2.47}$$

2. Weak Energy Condition:
$$\rho \geq 0, \quad \rho + p \geq 0, \tag{2.48}$$

3. Strong Energy Condition (which inflationary models typically violate):
$$\rho + p \geq 0, \quad \rho + 3p \geq 0, \tag{2.49}$$

4. Dominant Energy Condition:
$$\rho \geq |p|. \tag{2.50}$$

From the EFE's, we can obtain the generalized Friedmann equation, as given by [42]:

$$\tilde{R} = 2(\sigma^2 - \frac{1}{3}\Theta^2 + \rho + \Lambda), \tag{2.51}$$

and the generalized Raychaudhuri-Ehlers equation (or the acceleration equation):

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - \nabla_a \dot{u}^a + \frac{1}{2}(\rho + 3p) - \Lambda = 0, \tag{2.52}$$

with the ρ and p quantities being quantities for the total fluid [42], $\tilde{R}$ is the 3-Ricci scalar (discussed in Chapter 6.4), σ is the shear scalar, Θ is the expansion scalar and ω is the vorticity scalar (a detailed discussion on these quantities is given in Chapter 3.2). These two equations contain all ten of the EFE's, and are strictly speaking both known as Friedmann equations, but we will use the aforementioned names in this dissertation. The expansion or contraction of the Universe is inherent to these equations. In the first equation, we observe that the rate of expansion, $\dot{a}$, increases proportionally to the energy density of matter. The second equation shows us that the pressure and energy density slow down the expansion of the Universe [77].

We also have a parameter called the deceleration parameter, q, which is related to the Friedmann and acceleration equations. It is given as

$$q \equiv -\frac{a\ddot{a}}{\dot{a}^2}. \tag{2.53}$$

This deceleration parameter describes the rate of expansion of the Universe. When $q > 0$, the Universe is expanding at a decelerating rate, but when $q < 0$, the expansion is accelerated.

The symmetric energy-momentum tensor, T_{ab}, that describes the relativistic energy, momentum and stresses of whichever matter fields are present, is conserved due to the conservation of energy and momentum. This yields

$$T^{ab}{}_{;b} = 0 \Rightarrow \dot{\rho} = -\Theta(\rho + p). \tag{2.54}$$

This equation is known as the conservation equation. The Friedmann equation, equation (2.51), controls the expansion of the Universe, whereas the acceleration equation, equation (2.52), gives the evolution of the expansion, Θ, along the fluid flow lines. It is known as the fundamental equation of gravitational attraction [42]. The conservation equation, equation (2.54), describes the density of matter as the Universe expands.

Classical Tests of GR

The celestial mechanics within the Solar System serves as a classical testing ground for theories of GR [77]. Einstein proposed three tests in the year 1916 to verify his theory:

1. The gravitational redshift of light. This phenomenon was first measured in 1960 by Robert Pound and Glen Rebka [34]. First observed in a cloud of plasma ejected by the Sun, it was found (as predicted) that atomic clocks ran slower in the presence of a strong gravitational field [77].

2. The perihelion precession of Mercury. This was verified as early as 1859 already by Urbain Le Verrier who first pointed out an anomaly in the perihelion precession of Mercury with the use of Newton's theory [34]. Mercury, being the planet closest to the Sun, feels the solar gravitation strongly, but the other planets perturb its orbit [77]. The calculations that Le Verrier did using Newton's theory on how much the orbit is advancing yielded a result that was too little. The use of GR enabled the calculations to be re-done, and hence the discrepancy in the calculations were successfully explained [77].

3. The deflection of light by the Sun. Arthur Eddington led a group in 1919, and was the first to test this theory. He did so by observing the deflection of light rays by the Sun. However, radio observations today provide more reliable and precise measurements [34]. This deflection of light is observed, for example, when the Sun is fully eclipsed by the Moon to shut out its intense direct light. Then, if there is a very bright star behind the Sun, we observe the deflection of light from stars at lines of vision near the Sun [77].

We can also mention Shapiro's Effect, which is also called the fourth classical test of GR. This test isbased on the measurement of time delay of an electromagnetic signal to move from one point to another in the solar system with respect to the time that the same signal would take in a flat spacetime [80]. This test was proposed by Irwin Shapiro in 1964. Theoretical predictions correspond very well with current measurements.

The radio observations of pulsars that are members of binary pairs have aided in carrying out the most important tests of GR [77].

2.2 The Standard Cosmological Model

2.2.1 Cosmological Equations

A line element describes the spacetime using a given metric. This line element is a function of the metric tensor, and is denoted by ds. If we are in a three dimensional Euclidean space, our line element is given as

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{2.55}$$

and ds essentially describes the miniscule distance from the origin to the point with coordinates $(dx,\ dy,\ dz)$. The line element is invariant under a change of coordinates, and we can thus write the line element for an arbitrary coordinate system as

$$ds^2 = g_{ab}dx^a dx^b. \tag{2.56}$$

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric was found to be the only background compatible with our current assumption of the Cosmological Principle to be

true [7]. This FLRW metric is given in spherical coordinates, $(r,\ \Theta,\ \phi)$, as

$$ds^2 = -dt^2 + a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right), \tag{2.57}$$

where $a = a(t)$ is the scale factor that is dependent on time t (this is a parameter that describes the expansion of the Universe), k can be positive, zero or negative and is determined by the curvature of the Universe, r is the radial coordinate, and Θ and ϕ are the angles as determined by spherical coordinates. We can write k as

$$k = \begin{cases} -1 & \text{open,} \\ 0 & \text{flat, and} \\ 1 & \text{closed.} \end{cases} \tag{2.58}$$

The open, flat and closed cases can be better explained with an image:

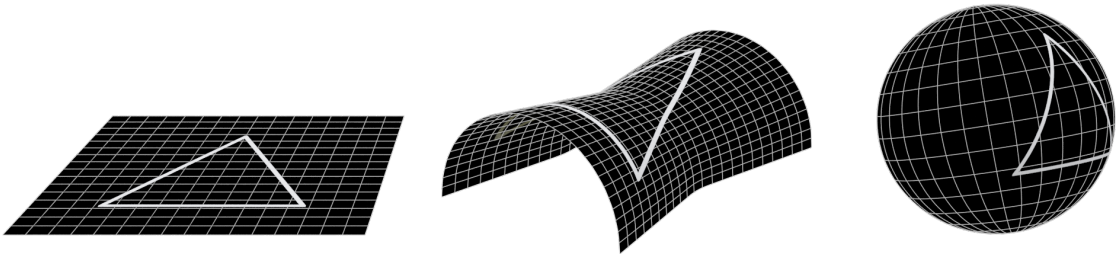


Figure 2.1: The first image depicts a flat curvature, the second an open curvature and the third a closed curvature. The triangle shows how the same angle seen in different curvatures can change. Even though the angles of an equilateral triangle in flat space are all 60°, this will not be the case if the curvature is different. [Source: made in *Blender* by K. Haynes.]

For the FLRW metric, we have that the shear and the vorticity vanishes, as well as the acceleration, and $\Theta = 3\frac{\dot{a}}{a}$. Thus, the Friedmann equation for the FLRW metric can be written as

$$\tilde{R} = -3\left(\frac{\dot{a}}{a}\right)^2 + 2\rho + 2\Lambda, \tag{2.59}$$

where $\tilde{R} = \frac{6k}{a^2}$ (k being the spatial curvature). The Raychaudhuri equation for the FLRW metric is found to be

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.60}$$

The energy conservation equation (2.54) for the FLRW metric can be used to solve for the scale factor, using the barotropic EoS, equation (2.43), as well as the relation between the expansion Θ and the scale factor a. Both the energy density and the scale factor depend on time, which leads to

$$a(t) = a_0 t^{\frac{2}{3(w+1)}}, \tag{2.61}$$

where a_0 is an integration constant which is to be found with an appropriate choice of initial conditions. When the EoS parameter $w = 0$, we are in a matter-dominated universe,

and thus $a(t) \propto t^{2/3} \Rightarrow \rho(a) \propto a^{-3}$. When we substitute $w = 1/3$, as for a radiation-dominated universe, we find that $a(t) \propto t^{1/2} \Rightarrow \rho(a) \propto a^{-4}$. In a dark energy-dominated universe, we have $\rho(t) \propto C$, where C is a constant, with an exponentially growing scale factor.

We cannot directly measure the size of our expanding universe, however the scale factor is a convenient way to express the size of the Universe at different epochs. We can take the size of the Universe at present times to be $a(t_0) \equiv a_0 = 1$. When combined with a subscript 0, we refer to the constant that is obtained through integrating over initial conditions (i.e.,the values of the energy density at the present time).

In reality, the Universe is filled with a combination of all of these components, and thus we can write the total energy density as the sum of the components:

$$\rho = \rho_d + \rho_r + \rho_\Lambda = \rho_{d0} a^{-3} + \rho_{r0} a^{-4} + \Lambda, \tag{2.62}$$

where the subscripts d, r and Λ represent dust, radiation and the cosmological constant, respectively, and the last term represents the value of the constant dark-energy density.

There exists a quantity called the Hubble parameter, denoted by H, that depicts the rate of change of the scale factor as

$$H \equiv \frac{\dot{a}}{a}, \tag{2.63}$$

where a dot denotes a derivative with respect to time. This parameter describes the expansion of the Universe.

2.2.2 Initial Conditions

At an arbitrary time t_0, which could be today, we can write initial data related to the Hubble parameter as

$$H(t = 0) \equiv H_0. \tag{2.64}$$

The Friedmann equation for the FLRW metric at time $t = 0$ can be written as

$$H_0^2 = \frac{1}{3}\left(\rho_{d0} a_0^{-3} + \rho_{r0} a_0^{-4}\right) + \frac{1}{3}\Lambda - \frac{k}{a_0^2}. \tag{2.65}$$

If we take $a_0 = 1$, this equation can be re-written in the form

$$1 = \frac{\rho_{d0}}{3H_0^2} + \frac{\rho_{r0}}{3H_0^2} + \frac{\Lambda}{3H_0^2} - \frac{k}{H_0^2}. \tag{2.66}$$

There thus exists a dimensionless normalized density parameter given by

$$\Omega_{i0} = \frac{\rho_{i0}}{3H_0^2}, \tag{2.67}$$

for $i = \{d,\ r\}$. For a non-vanishing cosmological constant we also have the fractional energy density given by

$$\Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2}. \tag{2.68}$$

We can thus write Ω_0 as

$$\Omega_0 = \Omega_{d0} + \Omega_{r0} + \Omega_{\Lambda 0}. \tag{2.69}$$

This Ω_0 is the total density parameter given by $\Omega_0 = \frac{\rho}{\rho_c}$, where ρ_c is the critical density. When $\Omega_0 = 1$, we call it the critical density parameter. If this condition is met, the consequence is that the Universe would continue to expand forever.

The deceleration parameter at this initial time is given as

$$q_0 = -\left[\frac{\ddot{a}}{a}\right]\Bigg|_{t=0} H_0^{-2}. \tag{2.70}$$

If the pressure term in the Raychaudhuri equation is negligible relative to the energy density, then we can write

$$q_0 = \frac{1}{2}\Omega_{m0} - \Omega_{\Lambda 0}. \tag{2.71}$$

From this equation, we can notice that, if the cosmological constant is the dominant component ($\Omega_{m0} < \Omega_{\Lambda 0}$), then we have $q_0 < 0$. This means that a dominant cosmological constant has the effect that the expansion of the Universe will accelerate, whereas, the matter would decelerate the expansion given that it is the dominant component ($\Omega_{m0} > \Omega_{\Lambda 0} \;\Rightarrow\; q_0 = \frac{1}{2}\Omega_{m0}$).

Equation (2.66) can enable us to evaluate the spatial curvature at the initial time:

$$1 = \Omega_{d0} + \Omega_{r0} + \Omega_{\Lambda 0} - \frac{K_0}{H_0^2} \tag{2.72}$$

$$\Rightarrow \qquad K_0 \equiv \frac{k}{a_0^2} = H_0^2\left(\Omega_0 - 1\right). \tag{2.73}$$

When $K_0 = 0$, we have $\Omega_0 = 1$, and the Universe is said to be flat. An open universe is obtained when $K_0 < 0$, and we have that $\Omega_0 < 1$, and a closed universe leads us to $K_0 > 0$ and $\Omega_0 > 1$. Thus, we can completely define the geometry of a homogeneous and isotropic universe with the normalized, dimensionless density parameter Ω_k, in such a way that [1]

$$1 = \Omega_m + \Omega_\Lambda + \Omega_k, \tag{2.74}$$

where

$$\Omega_m = \frac{\rho_{d0}}{3H^2}a^{-3} + \frac{\rho_{r0}}{3H^2}a^{-4}, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}, \quad \Omega_k = -\frac{k}{H^2}a^{-2}. \tag{2.75}$$

From equation (2.74), we can conclude that the total energy of the Universe in terms of the density parameters is always 1.

A useful thing to do when modeling equations with the Hubble parameter is to re-write this Hubble parameter as a non-dimensional quantity. We would do this by dividing the Hubble parameter with the Hubble constant to obtain

$$h \equiv \frac{H}{H_0}. \tag{2.76}$$

Next, a change from time to redshift would be appropriate, seeing as we cannot directly measure the time that the expansion of the Universe has taken, but we can directly measure

the redshift of the galaxies. We have a relation that relates the scale factor, a and the redshift, z:

$$1 + z = \frac{a_0}{a} \Rightarrow 1 + z = \frac{1}{a}. \tag{2.77}$$

If we apply equation (2.77) to the Friedmann equation, and divide throughout by H_0^2, we obtain the normalized, dimensionless Hubble parameter in terms of redshift:

$$h(z) = \sqrt{\Omega_{d0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{k0}(1+z)^2 + \Omega_\Lambda}. \tag{2.78}$$

2.2.3 Observational Values

The Planck 2018 survey found the value of the dimensionless density parameter for the matter content of our Universe to be $\Omega_{m0} = 0.3153 \pm 0.0073$, and for the dark-energy to be $\Omega_{\Lambda 0} = 0.6847 \pm 0.0073$. They also found that the Universe was spatially flat to high accuracy, since $\Omega_{k0} = 0.0007 \pm 0.0019$, consistent with predictions of simple inflationary models. They found a value for the Hubble constant to be $H_0 = (67.36 \pm 0.54)\ km\ s^{-1}\ Mpc^{-1}$ [2]. This value is obtained from the observations of Type Ia supernova explosions.

2.2.4 Distance Measure

It is not possible to measure the comoving distance from us to an observable galaxy, but it is possible to observe the light that that galaxy emits. Thus the varying luminosity of a galaxy as the Universe expands is an important quantity in cosmology.

If we consider the proper distance to the galaxy, d_P, we can write a relation that works for small redshifts:

$$d_P(z) \approx \frac{c}{H_0}\left(z - \frac{1}{2}(1+q_0)z^2\right). \tag{2.79}$$

Considering an astronomical object with a luminosity, L, we have a quantity called the luminosity distance, denoted by d_L. This luminosity distance is related to the object's surface brightness by

$$B = \frac{L}{4\pi d_L^2}. \tag{2.80}$$

Now, if the Universe was just stationary and did not expand, we would have a proper distance d_P. But since the Universe's expansion has been proven, we consider the scale factor, $a(t)$, and we can state that the energy of photons emitted at time t_e is redshifted by a factor of $1 + z = \frac{1}{a(t_e)}$. If we take time dilation into account as well, then we can arrive at $d_L = d_P(1+z)$. Now, if we replace the luminosity L by the absolute magnitude M of a luminous object, and the surface brightness B by the apparent magnitude m, then the replacement rule is given by

$$m - M = -5 + \log_{10} d_L, \tag{2.81}$$

where d_L has units of parsecs (pc) [77]. This formula is very important in observational cosmology as it helps us relate our observations to theory.

2.3 Successes of the Standard Model

One of the successes of the Standard Model is the prediction of light element nucleosynthesis [42]. In a hot and dense early universe, the conditions were that of a nuclear reactor. But seeing as the expansion and cooling down period was very rapid, only the lightest elements were synthesized before the density and reaction rates dropped too low for other nuclear expansion rates to compete with the universal expansion rate [93]. And thus, our most common elements were born - hydrogen and a few isotopes of helium, as well as some other less common isotopes, like lithium.

We have several components that make up the Universe. Baryonic matter, cold dark matter and dark energy all come together to form our Universe as we know it. Baryonic matter, also referred to as 'matter', includes protons, electrons, atoms and molecules. These particles are all observed in non-gravitational experiments. This type of matter forms stars, galaxies, gas and clusters under the effect of gravity. Cold dark matter has not yet been detected with gravitational experiments, but it is required indirectly by astrophysical and cosmological dynamics. Several candidates for dark matter have been proposed, but none have yet been confirmed. The last component, dark energy, is dependent on the cosmological model used, and is deduced purely from cosmological and gravitational effect [42].

The Standard Model has also enabled us to predict the expansion of the Universe. Now, the isotropy and homogeneity in the past is thought to be greater than today, since the Universe is expanding and has a finite age. With the use of observations of relic radiation from the Big Bang (BB), i.e.,the cosmic microwave background (CMB), we have managed to confirm the fluctuations of the density [84]. These fluctuations in the density play a significant role in the evolution of the Universe.

Inflation theory assists in putting these primordial cosmological perturbations in place at the time of the Big Bang [84]. In addition to setting up initial conditions necessary for the BB, inflation has found that the Universe has slight departures from homogeneity, and these are what we define as perturbations. During the time of inflation, the Universe is a vacuum and this, along with scalar fields, is believed to be responsible for the rapid expansion of the Universe. After inflation, the energy from the vacuum is transferred to the matter and energy via the process of reheating. The primordial cosmological perturbations thus lead to the formation of all structure in the Universe.

2.4 Problems with the Standard Model

The Standard Model, albeit largely successful, has some problems that are still under scrutiny in the scientific community today. Some of these problems stem from a discrepancy between the theory and observations, while others stem from fine-tuning. This section discusses some of these problems.

2.4.1 Evidence for the Anisotropy of the Universe

Our current Cosmological model is based on the Cosmological principle which states that our universe is both homogeneous and isotropic on large scales. But we have not yet come upon evidence to support this principle. CMB and Large-scale structure data has thoroughly tested this assumption. Analysis of the CMB data, most notably from the WMAP (Wilkinson Microwave Anisotropy Probe) [12] and Planck [2] experiments, has not provided any conclusive evidence for the Cosmological Principle hypothesis as of yet [27, 45]. Fosalba and Gaztanaga [45] finds that the probability of the cosmological parameters being consistent with a Gaussian fluctuation is about 10^{-9}, which is the largest reported evidence for the violation of the Cosmological principle.

The Standard Model is said to suffer from anomalous features at large scales, which in turn signals a deviation from the usual geometry of the Universe [97]. A slight red shift from the exact scale of invariance of the primordial power spectrum is observed from the Planck data [2], which makes it clear that the Standard Model does not fit well to the temperature power spectrum at low multipoles [97]. The WMAP also found precise measurements which predict asymmetric expansion, where one direction is not expanding in the same way as other two transverse directions [20, 97].

2.4.2 The Horizon Problem

If the age of the Universe is finite, then a particle horizon needs to exist. This particle horizon grows with the spatial width in proportion to the longer time perspective of the past light cone. This implies that what we can observe spatially is larger than what our past light cone encloses today. In time, as our horizon expands, we will have more causally connected regions [77].

Now, the Horizon Problem is a cosmological fine-tuning problem that deals with the cosmic microwave background (CMB). When detecting radiation from the CMB in different areas of the Universe, it is found that the amount of radiation is the same throughout, irrespective of which direction we are looking - we are in thermal equilibrium. The problem is that, if the Universe is expanding, the radiation in different areas of the Universe should not have been able to balance each other out so as to cause thermal equilibrium since they could not have communicated with each other (they are causally disconnected regions). The way scientists got around this was by introducing inflation.

2.4.3 The Flatness Problem

The flatness problem also arises from extreme fine-tuning. The initial conditions for the Universe seem to have to be very precise in order for us to have the Universe we have today. One of these fine-tuned quantities is the curvature of the Universe.

The total density parameter today is almost 1, with a very slight error. Thus this error accounts for our curvature density parameter, making it almost zero in the present epoch. Now, at early times, this total density parameter was even closer to one, constraining our curvature parameter closer to zero, implying a very flat curvature. This seems unlikely that the density in the Universe is almost exactly the critical density, since it could have taken any value. If it was slightly more than it is, we would have undergone a collapse that would have led to the Big Crunch, and if it was less than the critical density, the Universe would have expanded so quickly that gravity would not have been able cause the formation of complex structures [50, 77].

2.4.4 The Structure Problem

We know that the Universe is not absolutely homogeneous. The structures, like galaxies and clusters, that we see in the Universe today are caused by matter clumping. In order for these structures to exist, there needed to be inhomogeneities in the Universe at early times. From perturbation theory, however, we know that any slight inhomogeneities would have an enormous impact on the Universe today. Thus, at early times, these inhomogeneities needed to be almost zero, i.e., almost immediately after the BB, the Universe needed to be almost perfectly smooth, but not absolutely smooth, or else our structures today would not have existed. This extreme smoothness that is expected in the primordial matter is not accounted for in the Standard Model, and hence this problem is called the smoothness problem, the homogeneity problem or the structure problem in different texts [52, 66].

2.4.5 The Magnetic Monopole Problem

The BB model predicts that the Universe was very hot at early times. Given that this is true, then a large number of stable magnetic monopoles would have been produced, and it should be possible to detect these monopoles observationally. The ratio of monopoles to nucleons should be around one-to-one [65] , yet direct experimental searches for monopoles on Earth find less than 10^{-30} monopoles per nucleon [101]. This is a widely studied topic.

2.4.6 Accelerated Expansion

Observations of type-Ia supernovae, as well as the CMB, have led researchers to conclude that the Universe's expansion is accelerating. However, the current GR model as is (without including the cosmological constant) does not predict this phenomenon. Thus, this problem is one of the most researched problems in cosmology today. If the expansion is accelerating, then a universe consisting entirely of matter is rendered inadequate [84], and we come to speculating what it is that causes the accelerated expansion.

The measurements of matter in the Universe have found that approximately 31.5% of the

total energy in the Universe exists in the form of non-relativistic matter. Only about 4.9% of this content is known to exist in baryonic matter form [1]. The remaining matter has not yet been properly understood, which is why theories of *dark matter* have been formulated (to account for the missing matter that our measurements predict that there needs to be).

It is also necessary to mention that 68.5% of the total energy in the Universe is unaccounted for, but it is believed to exist in an exotic, unclustered and invisible form that permeates all of space. This is what we call *dark energy* [1]. This is also an entirely theoretical concept - one that would explain the accelerated expansion quite well.

Another possible alternative to dark energy is that the acceleration is caused by a known component of the Universe whose effects on the cosmic expansion have not fully been examined yet, for example, the inhomogeneities in a matter dominated Universe could be influencing the expansion rate at late times [84].

2.4.7 The Coincidence Problem

The coincidence problem stems from the fact that, in our current epoch, the ratio of the dark-energy density to the dark matter density of the same order of magnitude. This seems like too big of a coincidence.

This coincidence problem goes hand-in-hand with the Anthropic Principle - "We see the Universe the way it is because we exist" [10, 49]. The Anthropic Principle is an attempt at answering the philosophical question of why the Universe appears the way that it does. With this principle we can see the true roots of Cosmology being planted firmly in philosophy, as it was before Cosmology became a science.

2.5 The Way Out

With all of these problems that persist, we have to formulate an approach to find solutions to the persisting problems. A possible solution to the anisotropy problem is to use different models that are anisotropic in nature, like the Bianchi models formulated by Luigi Bianchi. If the large-scale anisotropy is accounted for in the model, we might be able to have a model that better predicts the evolution of the Universe. Another option is to use a modified gravity model to cancel out the need for a cosmological constant, One can also insert some inhomogeneity into the model by applying perturbation theory in an attempt to model the large-scale structure evolution of the Universe caused by the inhomogeneities.

2.5.1 Anisotropic Cosmology - Bianchi Models

The Bianchi models are useful in the sense that they may be a possible solution to the anisotropy problem stated earlier. These models are defined as spatially homogeneous and anisotropic models, and they were theorised by Luigi Bianchi (1856 - 1928) [56]. These models were first used to study a real cosmological problem in 1933 by Lemaître - he explored the possibility of the Big Bang only being a consequence of the assumed symmetry of the FLRW model by using a Bianchi type-I model. His conclusion was that this was not the case.

The Bianchi models can be split into two classes, with nine models in total. The first class, A, contains the I, II, VI, VII, $VIII$ and IX models, and the second class, B, contains the III, IV, V, VI_h and the VII_h models [97]. These classes are determined by their solutions for the Lie identities. Ellis and MacCallum [35] were the ones who introduced this classification for the Bianchi models, and these classes are widely in use at the present time.

None of the Bianchi models possess any rotation or acceleration. These models are of interest as they provide generalisations of the FLRW model, with spatial sections of constant curvature [38]. The closed FLRW universe is obtained from the Bianchi type-IX model, the open FLRW universe from the Bianchi type-V and the Bianchi type-VII$_h$ models, and the flat FLRW universe from the Bianchi type-I and the Bianchi type-VII$_0$ models [38].

Spatially homogeneous cosmological models are vital in helping us understand the solutions of the EFE's. They help us with understanding the structure and properties of the space of these solutions. Bianchi models are exact solutions of the EFE's, and they are more general than the FLRW model. They can provide interesting results with regards to the anisotropy of the Universe [97]. Ellis gives a comprehensive review of the structure of Bianchi models in [38]. Khalatnikov and Lifshitz studied the singularity problem by applying the theory of Bianchi universes from the year 1963 [55].

MacCallum [60] suggests that the Bianchi models can be relevant to cosmology only as asymptotic models in the early or late universe, or over long time scales, because of the fact that the present anisotropy of the Universe is not large enough to be able to readily detect the shear and vorticity.

2.5.2 Modified Gravity

Modified gravity is a way to possibly solve the inflation, dark matter and dark energy problems that persist. There are various modified gravity theories - from scalar field theories (where a scalar field is implemented), quasilinear theories and tensor theories (like Starobinsky gravity, Gauss-Bonnet gravity and $f(R)$ theories), to scalar-tensor theories, vector-tensor theories and non-metric theories. All of these theories attempt to solve some of the critical problems using the context that gravity follows GR at the present epoch,

but perhaps the past epochs and future epochs might be very different to GR.

Higher-order theories of gravity involve the modification of the Einstein-Hilbert action with the purpose of including higher-order curvature invariants with respect to the Ricci scalar [91]. There are many ways to deviate from GR, which include the well-known Brans-Dicke theory [15], braneworld gravity [58] and others. There are more comprehensive reviews of general modified gravity models given by [16, 28, 98].

2.5.3 Cosmological Perturbations

Our currently accepted mode, the FLRW model, does provide a relatively good description of the large-scale dynamics of the Universe, however, at the smaller scales, this model does not hold up. This is because the Universe contains inhomogeneities such as small over-and under-densities at early times, and stars and galaxies at later times [42]. Cosmological perturbation theory is an invaluable tool that enables us to investigate how these small fluctuations grow and form and then hence form the large-scale structures of the Universe.

The cosmological principle is valid for large-scale structures. In the early universe, it is believed that the degree of inhomogeneity in the density was very small - the tiny imperfections in the early universe were slightly less than perfect with only thirty parts per million [84]. But these imperfections had an enormous effect on the large-scale structure formation of the Universe. If we take the average energy density to be $\overline{\rho}$, then the deviation from this energy density can be written as $\delta\rho$. The average departure from inhomogeneity can then be written as

$$\delta \equiv \frac{\delta\rho}{\overline{\rho}} \equiv \frac{\rho - \overline{\rho}}{\overline{\rho}}. \tag{2.82}$$

This $\delta\rho$ tends to be taken to be very small, but in actuality, it can take on any assigned value. A gauge-transformation reflects the freedom of choosing different coordinates on the spacetime manifold, but it leads to unphysical modes that need to be identified and eliminated when they arise [46] - this approach is called the Gauge-Invariant Perturbation Theory. It was pioneered by Lifshitz in 1946, and later influenced by the developments of Bardeen in 1980, and Kodama and Sasaki in 1984. This approach is based on the foliation of the background of spacetime with hypersurfaces and perturbing away from it. This theory is coordinate dependent, which means that one needs to fix the metric first, and it is a non-local, linear theory [46]. This approach's main shortcoming is that the physical meaning of the gauge variables are difficult to extract [33].

In order to avoid this gauge-dependence, there are several perturbation formalisms that can be chosen to reach our objective. The Bardeen metric based approach, for example, is such an approach that guarantees gauge-independence. This approach does have some flaws, though, since the introduction of the necessary variables only have a clear physical meaning in certain gauges. This is not a problem in GR, but it could lead to misinterpretation of results in higher order theories [9, 17, 24].

And thus this led to the formulation of a gauge-invariant perturbation theory that uses variables with simple geometrical and physical meanings - the covariant formalism. The approach that will be used in this dissertation is the covariant and gauge-invariant approach that was developed for GR in [17, 18, 31, 32, 39, 41], and later adapted for higher order gravity in [24]. This approach uses the 1+3 covariant formalism.

In this dissertation we will investigate the Bianchi-Type I universe in $f(R)$ gravity, and apply gauge-invariant perturbations using the 1+3 covariant formalism to our model in an attempt to find a solution that better portrays the evolution of the Universe.

Chapter 3

The 1+3 Covariant Formalism

In a covariant formalism, one describes spacetime with covariantly defined variables with respect to a partial frame formalism such as 1+3, 3+1 or 1+1+2 [46]. In this dissertation, we discuss the 1+3 formalism. This formalism has the advantage that it is valid in all coordinate systems because the physics and geometry is described by tensor quantities and relations. This theory is local and covariant, and it threads spacetimes with frames. It was first formulated by Ellis and Bruni, and further developed by Bruni, Dunsby and Ellis [17, 32, 40].

The splitting of spacetime into a temporal and three spatial components is the approach involved when applying the 1+3 covariant formalism, i.e., 1+3 to indicate the dimensions of every slice [46]. This is done in order to be able to investigate any deviations from isotropy and homogeneity of the Universe. The difference between this approach and the standard approach is that this one starts from the theory and reduces to linearities in a particular background [46], as opposed to starting from a particular metric. A 1+3 split relative to a 4-velocity field u^a enables us to relate the physics and geometry of preferred worldlines generated by this 4-velocity field to the observations [42]. The 1+3 covariant approach "provides an alternative description of spacetime in terms of scalars, 3-vectors and projected symmetric trace free (PSTF) 3-tensors and their corresponding equations using the Ricci and Bianchi identities" [14, 30, 36, 46].

3.1 Covariant Variables

Given a 1+3 time-space split, we can take the covariant time derivative of a tensor, as given in equation (2.5), and project it to obtain a three-dimensional derivative operator. This projected covariant derivative can hence be written as

$$\tilde{\nabla}_e T^{a\ldots b}{}_{c\ldots d} \equiv h^f{}_e h^a{}_g \ldots h^b{}_i h^t{}_c \ldots h^m{}_d \nabla_f T^{g\ldots i}{}_{t\ldots m}, \tag{3.1}$$

for any tensor $T^{a\ldots b}{}_{c\ldots d}$. We have to mention here that, unless u^a is vorticity-free, $\tilde{\nabla}$ is not the same as the 3-dimensional (spatial) covariant derivative ${}^3\nabla$ [1]. This spatial derivative

has other different notations that authors use, and they are given by ${}^{(3)}\nabla_a$ (as used in [32]), $\hat{\nabla}_a$, $\overrightarrow{\nabla}_a$ and D_a (as used in [24]).

We can define the covariant spatial curl of a vector V_a and a tensor T_{ab} as

$$\text{curl } V_a = \eta_{abc}\tilde{\nabla}^b V^c, \quad \text{curl } T_{ab} = \eta_{cd(a}\tilde{\nabla}^c T_{b)}{}^d. \tag{3.2}$$

These relations will be used in the linearization process for the propagation and constraint equations when we apply our model to these equations. These identities can also be found in Appendix D.

A few identities, for a scalar f, which portray the covariant generalization of Newtonian vector calculus identities, that are needed for this dissertation are given by [42]

$$\nabla_a f = -u_a \dot{f} + \tilde{\nabla}_a f, \tag{3.3}$$

$$(\tilde{\nabla}_a f)^{\cdot} = \tilde{\nabla}_a \dot{f} + (\dot{u}^b \tilde{\nabla}_b f)u_a + \dot{u}_a \dot{f} - \frac{1}{3}\Theta\tilde{\nabla}_a f - \sigma_{ab}\tilde{\nabla}^b f + \eta_{abc}\omega^b \tilde{\nabla}^c f. \tag{3.4}$$

These identities are important for commuting time and space derivatives of gradient quantities and will be used regularly when deriving the perturbation equations for our model in a later chapter.

3.2 Kinematical Quantities

The most important quantities when dealing with the kinematics of a cosmological model are the expansion, the shear, the vorticity and the acceleration [24]. These components tell us about the overall spacetime kinematics by splitting $\nabla_a u_b$ into its irreducible parts defined by their symmetry properties [1]:

$$\nabla_a u_b = \tilde{\nabla}_a u_b - u_a \dot{u}_b = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_a \dot{u}_b. \tag{3.5}$$

This equation provides the definitions for each of our kinematical quantities:

1. The volume rate of expansion, Θ, defined earlier as $\Theta \equiv 3H$ (and thus relating Θ to the linear Hubble expansion H), is given by

$$\Theta \equiv \tilde{\nabla}_a u^a = h^{ab}\nabla_a u_b, \tag{3.6}$$

and the expansion tensor is given by

$$\Theta_{ab} \equiv \frac{1}{3}\Theta h_{ab} + \sigma_{ab}. \tag{3.7}$$

2. The symmetric, trace-free rate of shear tensor is found to be

$$\sigma_{ab} \equiv \tilde{\nabla}_{\langle a} u_{b\rangle} = h^c{}_a h^d{}_b \nabla_{(c} u_{d)} - \frac{1}{3}\Theta h_{ab}. \tag{3.8}$$

There are certain properties that the rate of shear tensor has, namely

$$\sigma_{ab} = \sigma_{(ab)}, \qquad \sigma_{ab}u^b = 0, \qquad \sigma^a{}_a = 0. \tag{3.9}$$

This rate of shear tensor describes the rate of distortion of the fluid flow. It is the PSTF part of Θ_{ab}.

3. We also have the skew-symmetric vorticity tensor can be written as

$$\omega_{ab} \equiv \tilde{\nabla}_{[a} u_{b]} = h^{c}{}_{a} h^{d}{}_{b} \nabla_{[d} u_{c]}. \tag{3.10}$$

We also have some identities that the skew-symmetric vorticity tensor (which describes the rotation of the fluid relative to a non-rotating frame), complies to

$$\omega_{ab} = \omega_{[ab]}, \qquad \omega_{ab} u^{b} = 0. \tag{3.11}$$

In addition to the vorticity tensor, we also have a vorticity vector ω^{a}, which is defined to be

$$\omega^{a} = \frac{1}{2} \eta^{abc} \omega_{bc} \quad \Rightarrow \quad \omega_{a} u^{a} = 0, \ \omega_{ab} \omega^{b} = 0. \tag{3.12}$$

Now, the magnitudes of the vorticity and the shear can be written as

$$\omega^{2} = \frac{1}{2} (\omega_{ab} \omega^{ab}) \geq 0, \quad \sigma^{2} = \frac{1}{2} (\sigma_{ab} \sigma^{ab}) \geq 0. \tag{3.13}$$

Spatial Gradients

We can define the spatial gradient of any scalar function f (orthogonal to u_{a}) in the local rest frame (LRF) of fundamental observers as

$$f_{a} = \tilde{\nabla}_{a} f. \tag{3.14}$$

This is useful, as we will need to use the spatial gradient of the thermodynamical quantities in order to find our perturbation equations for our model. Now, the thermodynamical quantities that we will use for this dissertation are ρ, Θ and σ, and thus we can write their spatial gradients as [30]

$$\mathcal{D}^{m}_{a} = \frac{a}{\rho^{m}} \tilde{\nabla}_{a} \rho^{m}, \quad Z_{a} = a \tilde{\nabla}_{a} \Theta, \quad \Sigma_{a} = a \tilde{\nabla}_{a} \sigma, \tag{3.15}$$

where the first quantity is the comoving fractional density gradient, the second is the gradient of the expansion, and the last is the spatial variation of the shear scalar. This last quantity is important for the discussion of isocurvature perturbations, not discussed in this dissertation. In principle, these quantities obey exact non-linear evolution equations and measurements for these quantities are done by observers at rest with respect to the fluid [29].

3.3 Kinematical Effects

To understand the effect of our kinematical variables, we can consider the effect of pure expansion, shear and vorticity on the relative positions of theoretical galaxies [42]:

1. **Pure expansion ($\omega_{ab} = \sigma_{ab} = 0$):** a distortion-free expansion without any rotation. This means that, if we consider a sphere of galaxies, they expand outward from the centre without their directions changing, meaning that the galaxies just form a larger sphere with each galaxy still lying in the same direction as before.

2. **Pure shear ($\omega_{ab} = \Theta = 0$):** a pure distortion without any rotation or change of volume. If we consider an expansion in one direction, for the volume to stay constant, we need to have a contraction in at least one other direction. Thinking about a sphere of galaxies - when part of the sphere expands, then the other part needs to contract. Thus, the galaxies end up forming an ellipsoid, expanded in one direction, but contracted in another. Each of the galaxies lying in a shear eigen-direction (the direction in which the contractions and expansions happened) are still lying in the same direction as before, while the other galaxies will appear to have rotated. But the average rotation over the whole sky for all of the galaxies will be zero.

3. **Pure vorticity ($\sigma_{ab} = \Theta = 0$):** no distortion or expansion occurring - only pure rotation. In the case of pure vorticity, the distances between galaxies stay preserved, and there exists an axis of rotation. All of these galaxies that do not fall on the axis of rotation will rotate around the axis of rotation, without changing the distances between them.

These quantities are non-zero in a general fluid flow, which means that a combination of the different effects, volume change, rotation and distortion, will occur.

3.4 Conservation Equations

The energy-momentum conservation equation (2.54) can be decomposed in terms of the 1+3 decomposition to obtain the generalized energy conservation equation that determines the rate of change of relativistic energy along the worldlines. This equation is obtained by projecting along u^a, and results in

$$\dot{\rho} + (\rho + p)\Theta + \pi^{ab}\sigma_{ab} + \tilde{\nabla}_a q^a + 2\dot{u}_a q^a = 0. \tag{3.16}$$

The projection orthogonal to u^a gives the momentum conservation equation which determines the acceleration caused by various pressure components [42]:

$$\dot{q}_{\langle a\rangle} + \frac{4}{3}\Theta q_a + (\rho + p)\dot{u}_a + \tilde{\nabla}_a p + \tilde{\nabla}^b \pi_{ab} = -\sigma_{ab}q^b + \eta_{abc}\omega^b q^c - \dot{u}^b \pi_{ab}. \tag{3.17}$$

From this last equation, we find that the inertial mass density of matter is $\rho + p$, which can be interpreted as: "Any form of internal energy (e.g. heat or chemical energy) contributes to the effective inertial mass both directly (by increasing ρ) and indirectly (by contributing to p)" [42]. In the case of the total matter EMT, these conservation laws are valid. However, if we have several components, then the EMT for each component might been to be modified when interactions between the components are taken into account [42], as we will see in Chapter 5.

3.5 Propagation and Constraint Equations

To be able to completely describe any given spacetime using the covariant approach, the geometrical and thermodynamical quantities need to form irreducible sets [46]:

$$\{\Theta,\ \sigma_{ab},\ \omega_{ab},\ \dot{u}^a,\ E_{ab},\ H_{ab}\} \quad \text{and} \quad \{\rho,\ p,\ q_a,\ \pi_{ab}\}\,. \tag{3.18}$$

We also require a prescribed EoS of the cosmological fluid which will then, in addition to these irreducible sets, allow us to derive the evolution equations for the kinematic quantities. This is generally done by separating the parallel projected part of the Ricci identity for the fundamental timelike 4-velocity u^a into trace, symmetric trace-free and skew symmetric parts [46]. These equations are given for a general spacetime in [42]. We will present them in the section below. Note that the quantities ρ, p, q_a and π_{ab} are all total quantities. The propagation equations can be written as:

1. **Expansion propagation (generalized Raychaudhuri equation):**

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(\rho + 3p) - \tilde{\nabla}^a \dot{u}_a = -\sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a + \dot{u}_a\dot{u}^a, \tag{3.19}$$

   which determines the evolution of the volume expansion Θ along the fluid flow lines [29]. This is the basic equation of gravitational attraction. The active gravitational mass density is represented by the $(\rho+3p)$-term in this equation . Here, we also have that the pressure term appears as a general-relativistic effect. The average distance of the worldlines is affected by the divergence of the acceleration which represents spatial pressure gradients [46].

2. **Vorticity propagation:**

$$\dot{\omega}_{\langle a\rangle} + \frac{2}{3}\Theta\omega_a + \frac{1}{2}\ \text{curl}\ \dot{u}_a = \sigma_{ab}\omega^b, \tag{3.20}$$

   which is obtained from the twice-projected skew-symmetric part of the Ricci identities.

3. **Shear propagation:**

$$\dot{\sigma}_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} = -\sigma_{c\langle a}\sigma_{b\rangle}{}^c - \omega_{\langle a}\omega_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle}, \tag{3.21}$$

   which is obtained through the twice-projected symmetric part of the Ricci identities. This equation represents the evolution of the shear, and it shows how the anisotropic pressure and the electric part of the Weyl tensor induce distortion in the surrounding fluid flow [46].

The constraint equations are subsequently obtained from the Ricci identity through orthogonal projection, index contraction and by taking the PSTF parts [46]. For this set of equations, we notice that no time derivatives of the kinematical quantities are involved. The constraint equations must be satisfied at each hypersurface during propagation, and are given as:

1. **Vorticity constraint:**

$$\tilde{\nabla}^a \omega_a = \dot{u}^a \omega_a, \tag{3.22}$$

also known as the vorticity divergence identity, the second term on the LHS is second order, and hence we can observe the decoupling of vector and tensor modes from the scalar modes in the linearized background [46].

2. **Shear constraint:**

$$\tilde{\nabla}^b \sigma_{ab} - \text{ curl } \omega_a - \frac{2}{3}\tilde{\nabla}_a \Theta + q_a = -2\eta_{abc}\omega^b \dot{u}^c, \tag{3.23}$$

which shows the relation between the spatial inhomogeneity of the expansion and the momentum flux. For a perfect fluid, we have a momentum flux (or momentum density) that is zero.

3. **Gravito-magnetic constraint:**

$$\text{curl } \sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = -2\dot{u}_{\langle a}\omega_{b\rangle}, \tag{3.24}$$

where the distortion of the vorticity and the curl of the shear are what the magnetic Weyl curvature is constructed from.

Covariant decomposition of the Weyl tensor and the Bianchi identities yield Maxwell-like gravitational equations, which are the following evolution and constraint equations:

1. **Gravito-electro propagation:**

$$\begin{aligned}\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - \text{curl } H_{ab} + \frac{1}{2}\left[(\rho + p)\sigma_{ab} + \dot{\pi}_{\langle ab\rangle} + \frac{1}{4}\Theta\pi_{ab} + \tilde{\nabla}_{\langle a}q_{b\rangle}\right] \\ = -\dot{u}_{\langle a}q_{b\rangle} + 2\dot{u}^c\eta_{cd(a}H_{b)}{}^d + 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c\eta_{cd(a}E_{b)}{}^d \\ -\frac{1}{2}\left(\sigma^c{}_{\langle a}\pi_{b\rangle c} - \omega^c\eta_{cd(a}\pi_{b)}{}^d\right),\end{aligned} \tag{3.25}$$

2. **Gravito-magnetic propagation:**

$$\begin{aligned}\dot{H}_{\langle ab\rangle} + \Theta H_{ab} + \text{curl } E_{ab} - \frac{1}{2}\text{ curl } \pi_{ab} = 3\sigma_{c\langle a}H_{b\rangle}{}^c - \omega^c\eta_{cd(a}H_{b)}{}^d \\ -2\dot{u}^c\eta_{cd(a}E_{b)}{}^c + \frac{1}{2}\left(\sigma^c{}_{(a}\eta_{b)cd}q^d - 3\omega_{\langle a}q_{b\rangle}\right),\end{aligned} \tag{3.26}$$

Equations (3.25) and (3.26) describe how gravitational radiation arises, and are analogues to the Maxwell's equations for electromagnetic radiation [30].

3. **Gravito-electric divergence:**

$$\begin{aligned}\tilde{\nabla}^b E_{ab} + \frac{1}{6}\left(\tilde{\nabla}^b\pi_{ab} - 2\tilde{\nabla}_a\rho + 2\Theta q_a\right) = \eta_{abc}\sigma^b{}_d H^{cd} - 3H_{ab}\omega^b \\ +\frac{1}{2}(\sigma_{ab} + 3\eta_{abc}\omega^c)q^b,\end{aligned} \tag{3.27}$$

where the spatial gradient of the energy density here is a source term.

4. **Gravito-magnetic divergence:**

$$\tilde{\nabla}^b H_{ab} + \frac{1}{2}\left[\text{curl } q_a - 2(\rho + p)\omega_a\right] = 3E_{ab}\omega^b - \eta_{abc}\sigma^b{}_d E^{cd} - \frac{1}{2}\left(\eta_{abc}\sigma^b{}_d \pi^{cd} + \pi_{ab}\omega^b\right), \tag{3.28}$$

where we have the fluid vorticity and shear that act as source terms [94].

From equation (3.27), it is deduced that scalar modes are coupled to the divergence of the electric Weyl tensor. Equation (3.28) shows how vector (vorticity) modes are coupled to the divergence of the magnetic Weyl tensor [14].

If we consider the trace equation, equation (2.41), we can write a 1+3 split of the Ricci tensor, equation (2.18), to obtain

$$R_{ab} = \frac{1}{2}(\rho + 3p)u_a u_b + \frac{1}{2}(\rho - p)h_{ab} + 2u_{(a}q_{b)} + \pi_{ab}. \tag{3.29}$$

This equation enables us to write the EFE's in a 1+3 covariant form. We can hence write the 3-space Ricci tensor as (in the form of the Gauss equation [42])

$$\tilde{R}_{\langle ab\rangle} = (R_{ab})_\perp - \Theta\tilde{\nabla}_b u_a + \tilde{\nabla}_c u_a \tilde{\nabla}^c u_b, \tag{3.30}$$

where $(R_{ab})_\perp$ is the orthogonal part of the Ricci tensor. This leads to the following two constraint equations for the 3-space Ricci tensor and Ricci scalar (or contracted 3-space Ricci tensor), respectively:

$$\tilde{R}_{\langle ab\rangle} = E_{ab} + \frac{1}{2}\pi_{ab} - \frac{1}{3}\Theta(\sigma_{ab} + \omega_{ab}) + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + \omega_{c\langle a}\omega_{b\rangle}{}^c - 2\sigma_{c[a}\omega_{b]}{}^c, \tag{3.31}$$

$$\tilde{R} = 2\rho - \frac{2}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - \omega_a\omega^a. \tag{3.32}$$

Now that we understand how the kinematical quantities affect the evolution of the Universe, and we have introduced the propagation and constraint equations that will govern the dynamics of our model, we can introduce our anisotropic metric for the Bianchi type-I universe. The following chapter will give an overview of the background dynamics of the BI model in GR.

Chapter 4

The Bianchi Type-I Cosmological Model in GR

An overview of the general Bianchi Classification is given in Chapter 2.5.1. The simplest Bianchi model is the Bianchi Type-I (BI) universe. This model is almost isotropic at all late times [42], and there exists only one essential dynamical coordinate (the time, t). When finding the EFE's for this spacetime, we notice that they reduce to ordinary differential equations, and this is because the inhomogeneous degrees of freedom have been 'frozen out' [36].

Some authors have studied the BI model by using variable G and Λ [5, 11, 61, 74, 99]. Tiwari and Singh [96] also investigated the BI model with a varying Λ term. The BI string cosmological model with bulk viscosity and time-dependent Λ term was investigated by Tiwari and Sharma [95]. Other papers that investigate the BI model from different viewpoints are given in [29, 43, 69, 71, 72, 73, 78]. We also have what is called a locally-rotationally-symmetric (LRS) BI model that has been studied by [3, 86, 44, 87].

The BI universe is not shear-free, which means that we will have to take shear into account in all calculations done. The other Bianchi models are noted to often have phases in their evolution when we can neglect the terms in spatial curvature, and their dynamics then approximate the BI case, and thus the BI metric is important. The simplicity of the BI model enables this metric to be used with more complicated background content. Thus, the BI model is often also used to model other processes by considering it the background for these processes [42]. The BI metric is given by

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \tag{4.1}$$

where $A(t)$, $B(t)$ and $C(t)$ are the cosmological scale factors that depend on time, t, only and will henceforth be written as A, B and C. When the scale factors are equal, i.e.,$A = B = C = a$, we obtain the FLRW metric in flat space.

The average scale factor can be written as

$$a^3 = ABC \quad \Rightarrow \quad a = \sqrt[3]{ABC}, \tag{4.2}$$

which leads us to now define the average Hubble parameter, H, as

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{4.3}$$

$$= \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{4.4}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are directional Hubble parameters, and a dot denotes a derivative with respect to time. We can also write the first time-derivative of the average Hubble parameter as

$$\dot{H} = \frac{1}{3}\left[\frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2 + \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2 + \frac{\ddot{C}}{C} - \left(\frac{\dot{C}}{C}\right)^2\right], \tag{4.5}$$

$$= \frac{1}{3}\left(\dot{H}_1 + \dot{H}_2 + \dot{H}_3\right), \tag{4.6}$$

where $\dot{H}_1 = \frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2$, $\dot{H}_2 = \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2$ and $\dot{H}_3 = \frac{\ddot{C}}{C} - \left(\frac{\dot{C}}{C}\right)^2$.

From the average Hubble parameter we can obtain an expression for the expansion scalar, Θ:

$$\Theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}. \tag{4.7}$$

The Ricci scalar, R, can be written as

$$R = 2\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C}\right], \tag{4.8}$$

$$= 2\left[3\dot{H} + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_1H_3 + H_2H_3\right]. \tag{4.9}$$

The matter energy-momentum tensor is found to be

$$T^m_{ab} = \begin{bmatrix} \rho^m & 0 & 0 & 0 \\ 0 & A^2p^m & 0 & 0 \\ 0 & 0 & B^2p^m & 0 \\ 0 & 0 & 0 & C^2p^m \end{bmatrix}, \tag{4.10}$$

where ρ^m is the total matter energy density, and p^m is the total matter pressure of the perfect fluid. Now, we can assume that the momentum conservation of standard matter holds, i.e.,$T^{m;b}_{ab} = 0$, which yields the conservation equation:

$$\dot{\rho}^m = -\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)(\rho^m + p^m). \tag{4.11}$$

The shear scalar, σ^2, can be obtained using the definition of the shear tensor, equation (3.8), and the identity for the shear scalar given by equation (3.13). It is found to be

$$\sigma^2 = \frac{2}{3}\left(\frac{\dot{A}^2}{A^2}+\frac{\dot{B}^2}{B^2}+\frac{\dot{C}^2}{C^2}-\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\frac{\dot{C}}{C}\right), \tag{4.12}$$

$$= 6H^2 - 2(H_1H_2 + H_1H_3 + H_2H_3). \tag{4.13}$$

We can thus write our Ricci scalar in terms of our average Hubble parameter and shear:

$$R = 6\dot{H} + 12H^2 + \sigma^2. \tag{4.14}$$

We can also find $\dot{\sigma}$ as

$$\dot{\sigma} = \frac{H_1\dot{H}_1 + H_2\dot{H}_2 + H_3\dot{H}_3 - 3H\dot{H}}{6H^2 - 2(H_1H_2 + H_1H_3 + H_2H_3)}. \tag{4.15}$$

4.1 Field Equations for BI in GR

The normalized field equations for the BI metric in GR are found from the EFE's given by equation (2.36) to be

$$H_1H_2 + H_1H_3 + H_2H_3 = \rho^m, \tag{4.16}$$

$$\dot{H}_2 + \dot{H}_3 + H_2^2 + H_2H_3 + H_3^2 = -p^m, \tag{4.17}$$

$$\dot{H}_1 + \dot{H}_3 + H_1^2 + H_1H_3 + H_3^2 = -p^m, \tag{4.18}$$

$$\dot{H}_1 + \dot{H}_2 + H_1^2 + H_1H_2 + H_2^2 = -p^m. \tag{4.19}$$

These equations yield solutions of the form

$$\frac{B}{A} = m_1 e^{c_4\int\frac{1}{a^3}dt}, \tag{4.20}$$

$$\frac{C}{B} = m_2 e^{c_5\int\frac{1}{a^3}dt}, \tag{4.21}$$

$$\frac{A}{C} = m_3 e^{c_6\int\frac{1}{a^3}dt}, \tag{4.22}$$

where c_4, c_5, c_6, m_1, m_2 and m_3 are integration constants, and

$$c_4 + c_5 + c_6 = 0, \quad \text{and} \quad m_1m_2m_3 = 1 \tag{4.23}$$

need to hold for these constants. Equations (4.20) to (4.22) can be used to finally obtain

$$A = ap_1e^{q_1\int\frac{1}{a^3}dt},\ B = ap_2e^{q_2\int\frac{1}{a^3}dt} \text{ and } C = ap_3e^{q_3\int\frac{1}{a^3}dt}, \tag{4.24}$$

where a is the average scale factor, and

$$p_1 = \sqrt[3]{m_1^{-2}m_2^{-1}},\ p_2 = \sqrt[3]{m_1m_2^{-1}},\ p_3 = \sqrt[3]{m_1m_2^2}, \tag{4.25}$$

and

$$q_1 = \frac{2c_4 + c_5}{3},\ q_2 = \frac{c_4 - c_5}{3},\ q_3 = \frac{c_4 + 2c_5}{3}. \tag{4.26}$$

We notice here that

$$q_1 + q_2 + q_3 = 0, \quad \text{and} \quad p_1 p_2 p_3 = 1. \tag{4.27}$$

The problem with these solutions is that they are all dependent on each other, seeing as they contain the average scale factor in the solution. A way around this would be to assume a power-law relation between the average scale factor and the average Hubble parameter, and hence obtain a solution for the average scale factor in terms of time. We can then substitute this into these expressions for our directional scale factors and obtain analytical solutions.

Hence, let us assume there exists an exact power-law solution to the field equations as was done in [47]:

$$a(t) = a_0 t^m, \tag{4.28}$$

where $m > 0$ is a fixed real number. This will yield

$$A = a_0 t^m p_1 e^{r_1 \int \frac{1}{t^{3m}} dt}, \tag{4.29}$$

$$B = a_0 t^m p_2 e^{r_2 \int \frac{1}{t^{3m}} dt}, \tag{4.30}$$

$$C = a_0 t^m p_3 e^{r_3 \int \frac{1}{t^{3m}} dt}, \tag{4.31}$$

where $r_1 = \frac{q_1}{a_0^3}$, $r_2 = \frac{q_2}{a_0^3}$ and $r_3 = \frac{q_3}{a_0^3}$. This gives final solutions of the form

$$A = a_0 t^m p_1 \exp\left[r_1(-3m+1)^{-1} t^{-3m+1}\right], \tag{4.32}$$

$$B = a_0 t^m p_2 \exp\left[r_2(-3m+1)^{-1} t^{-3m+1}\right], \tag{4.33}$$

$$C = a_0 t^m p_3 \exp\left[r_3(-3m+1)^{-1} t^{-3m+1}\right], \tag{4.34}$$

if $m \neq \frac{1}{3}$, and

$$A = a_0 p_1 t^{r_1+m}, \; B = a_0 p_2 t^{r_2+m} \; \text{and} \; C = a_0 p_3 t^{r_3+m}, \tag{4.35}$$

if $m = \frac{1}{3}$. These solutions can then be used to further describe the dynamics of the Universe.

Another method of obtaining solutions is described by Cáceres et al., in [21]: here they use the general solutions that we obtained to find a relation for the shear tensor in terms of the average scale factor, and hence they use this shear tensor in the generalized Friedmann equation to obtain a solution for the average scale factor. Some other papers that have different methods of finding solutions are [85], who finds exact solutions to a time-varying G and Λ, while [100] uses an LRS BI model (where $B = C$) and obtains solutions with the use of a power-law relation. [53] studies the BI model with a varying Λ and [76] studies the BI model with a quadratic equation of state.

In the GR case, it is possible to find solutions for the EFE's numerically as well. We can do that by making our EFE's for BI dimensionless by writing them in terms of redshift and dividing with the normalized Hubble parameter. We use equation (2.77) to convert from cosmic time to redshift (as shown in Appendix A), and then obtain the non-dimensional

EFE's for the BI model in GR:

$$h_1h_2 + h_1h_3 + h_2h_3 = 3\Omega_{d0}(1+z)^3 + 3\Omega_{r0}(1+z)^4, \tag{4.36}$$

$$h_2^2 + h_3^2 + h_2h_3 - (1+z)hh_2' - (1+z)hh_3' = -\Omega_{r0}(1+z)^4, \tag{4.37}$$

$$h_1^2 + h_3^2 + h_1h_3 - (1+z)hh_1' - (1+z)hh_3' = -\Omega_{r0}(1+z)^4, \tag{4.38}$$

$$h_2^2 + h_1^2 + h_2h_1 - (1+z)hh_2' - (1+z)hh_1' = -\Omega_{r0}(1+z)^4. \tag{4.39}$$

These equations are ordinary differential equations and their solutions can be computed using numerical methods. In this case, as we do not have a dark energy component, we can conclude that $\Omega_d + \Omega_r = 1$. And the value for Ω_r today is approximately 10^{-5}. Thus, $\Omega_d = 1 - \Omega_r = 1 - 10^{-5}$. These constants can be put into the above equations to possibly assist us with plotting the solutions to these functions.

To be able to obtain initial conditions for solving our ordinary differential equations, At the present time, we can write $h(0) = 1$. Now, from observations, we have noticed that the Universe is mostly isotropic at present, and thus we can conclude that the directional Hubble parameters are approximately equal. Thus, using the relation between the directional Hubble parameters and the average Hubble parameter, equation (4.4), we can conclude that

$$h_1(0) \approx h_2(0) \approx h_3(0) \approx 1, \tag{4.40}$$

where $h_i = \frac{H_i}{H_0}$, with $i = 1, 2, 3$. Now, the problem is that when we make $h_1(0) = h_2(0) = h_3(0)$, then our solutions are exactly symmetrical, and we are left with a completely isotropic model. Thus we want to vary our initial conditions slightly in order to get a more anisotropic result, given that there is observed anisotropy in the Universe's expansion.

We can use the Planck 2018 Survey [2] to obtain the error in the Hubble constant and use this error to help us roughly determine some initial conditions for our directional scale factors by making the assumption that this error could be due to the anisotropy in the Universe. The Planck base-ΛCDM cosmology requires a Hubble constant $H_0 = (67.4 \pm 0.5)\ km\ s^{-1}$ [2]. Thus we can take this error and divide it by three (for our three directions) and then obtain the error to be ± 0.1667 per direction:

$$h(0) \equiv \frac{H(0)}{H_0} = \frac{1}{3}\left[\frac{(H_1(0) \pm 0.1667)}{H_0 \pm 0.5} + \frac{(H_2(0) \pm 0.1667)}{H_0 \pm 0.5} + \frac{(H_3(0) \pm 0.1667)}{H_0 \pm 0.5}\right] = 1. \tag{4.41}$$

Thus, we will use these values as the limits for our initial conditions. These initial conditions can aid in finding the solutions to equations (4.36) to (4.39) numerically with the use of Maple. We wrote a simple code to obtain the solutions of the non-dimensional directional Hubble constants, and consequently used these solutions to plot h, h_1, h_2 and h_3 versus z. Thus, we obtain Figures 4.1 to 4.7. These figures show the solutions obtained by assuming different initial conditions.

In Figure 4.5, we observe that the BI Hubble parameter does not quite correspond to the FLRW Hubble parameter. This was the case for many different initial conditions. Perhaps with the use of simulations such as Markov chain Monte Carlo, it would be possible to find some initial conditions that would enable the BI plot to correspond to the FLRW plot.

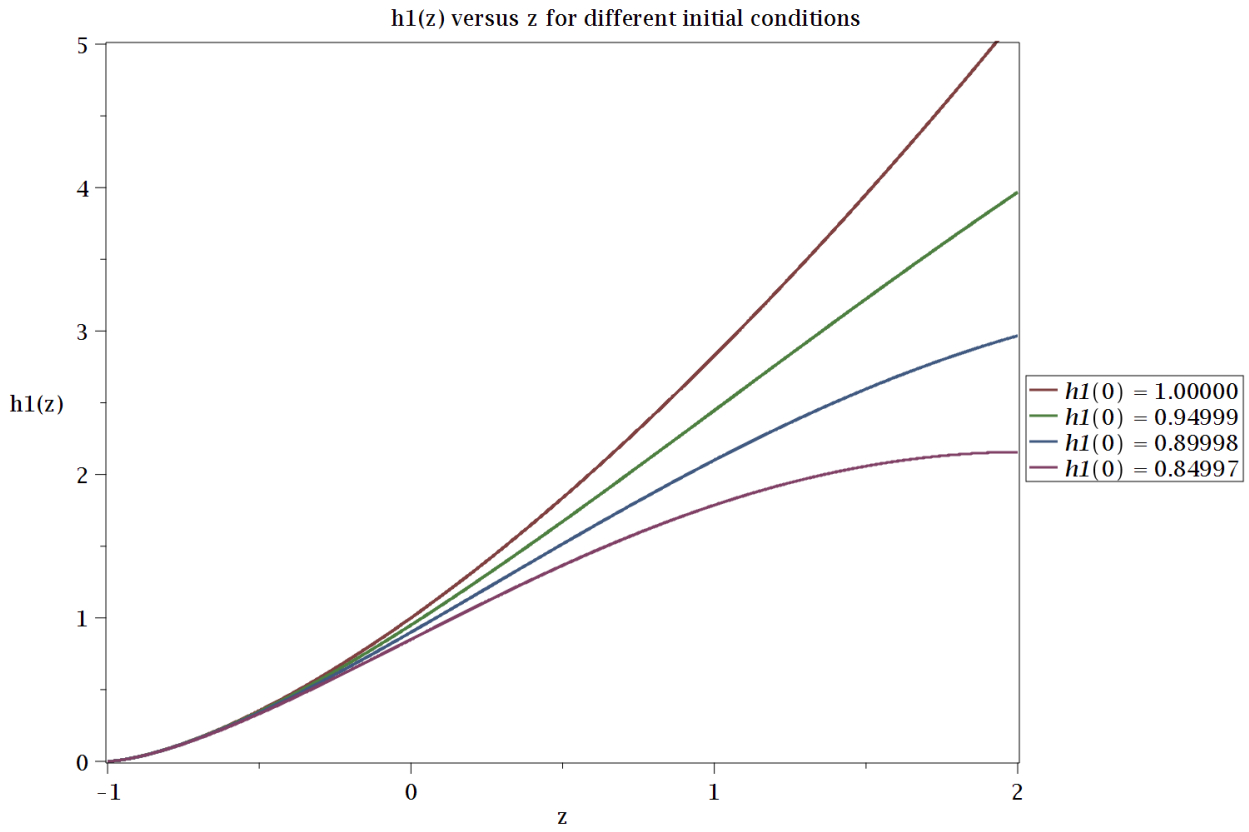


Figure 4.1: Different initial conditions were applied when solving for h_1, h_2 and h_3. These initial conditions resulted in different solutions for our parameters. On the x-axis, the redshift z is given, and on the y-axis we have the Hubble parameter as a function of the redshift for the first direction, $h_1(z)$.

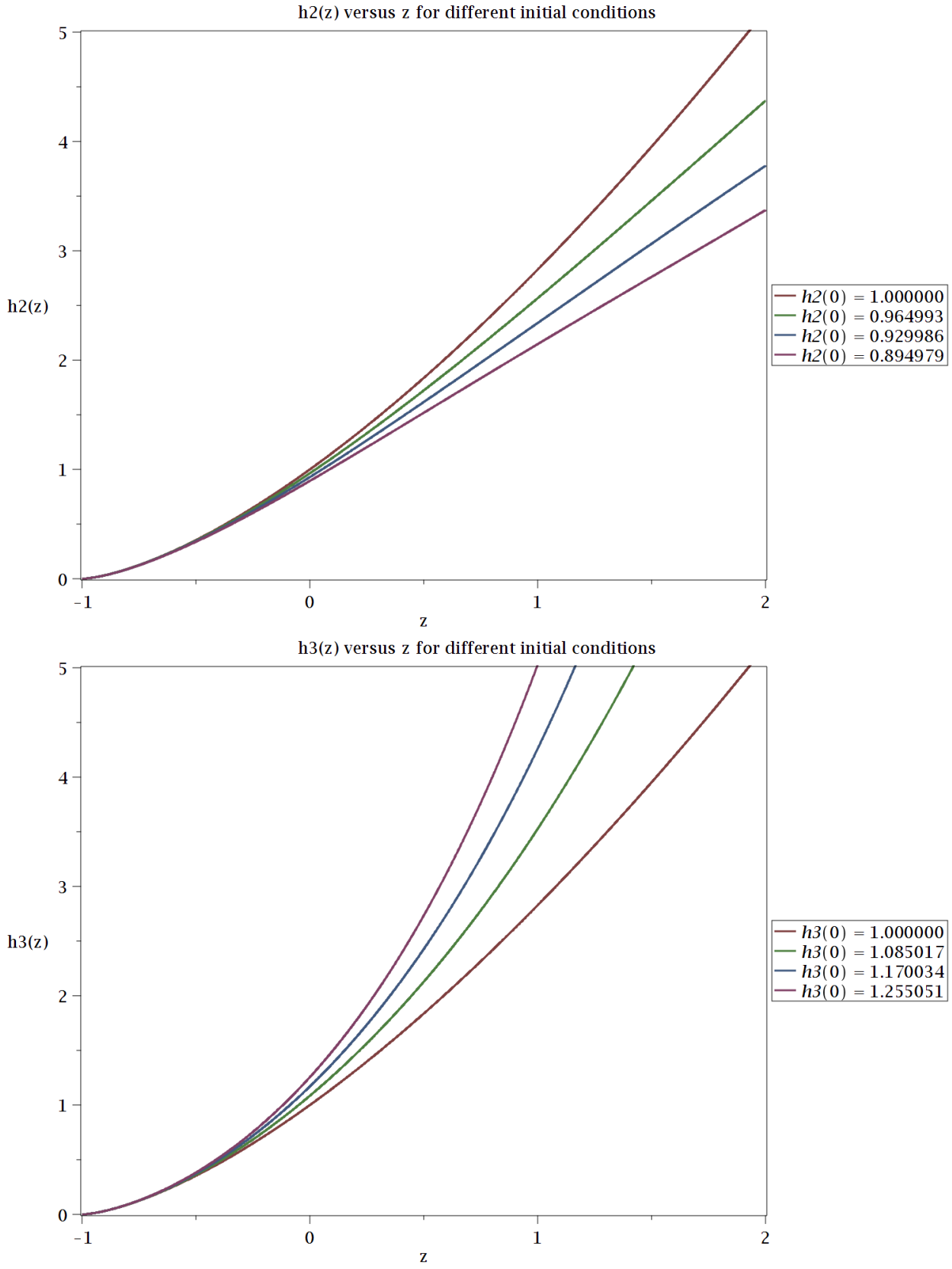


Figure 4.2: On the x-axis, the redshift z is given, and on the y-axis we have the Hubble parameter as a function of the redshift. The first plot depicts the second direction, $h_2(z)$, and the second plot depicts the third direction, $h_3(z)$.

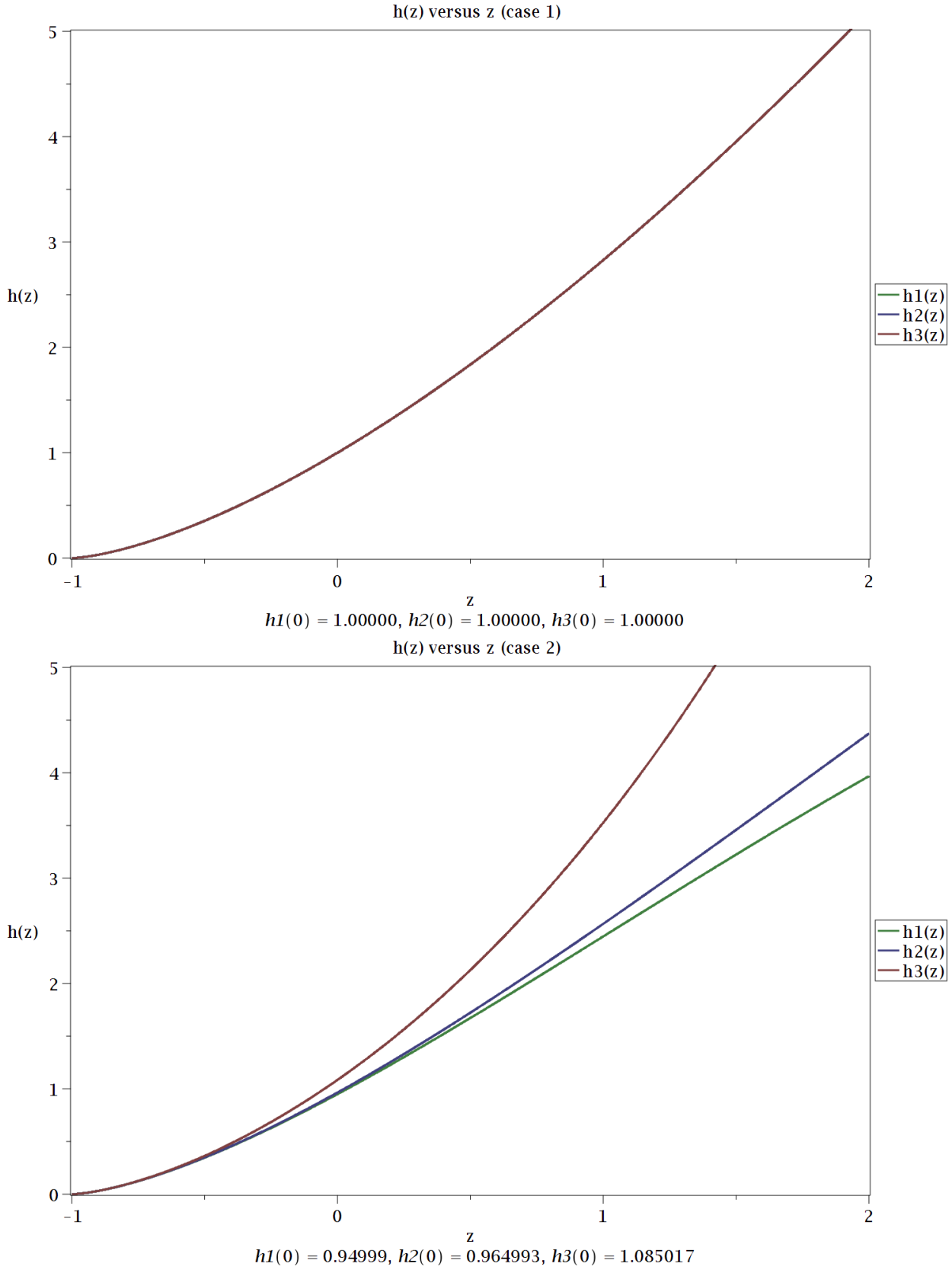


Figure 4.3: On the x-axis, the redshift z is given, and on the y-axis we have the Hubble parameter as a function of the redshift. On these plots, we have included all directional Hubble parameters for specific sets of initial conditions. Case 1 shows an instance where all three directional Hubble parameters have the same initial conditions (which is what is assumed by FLRW cosmology).

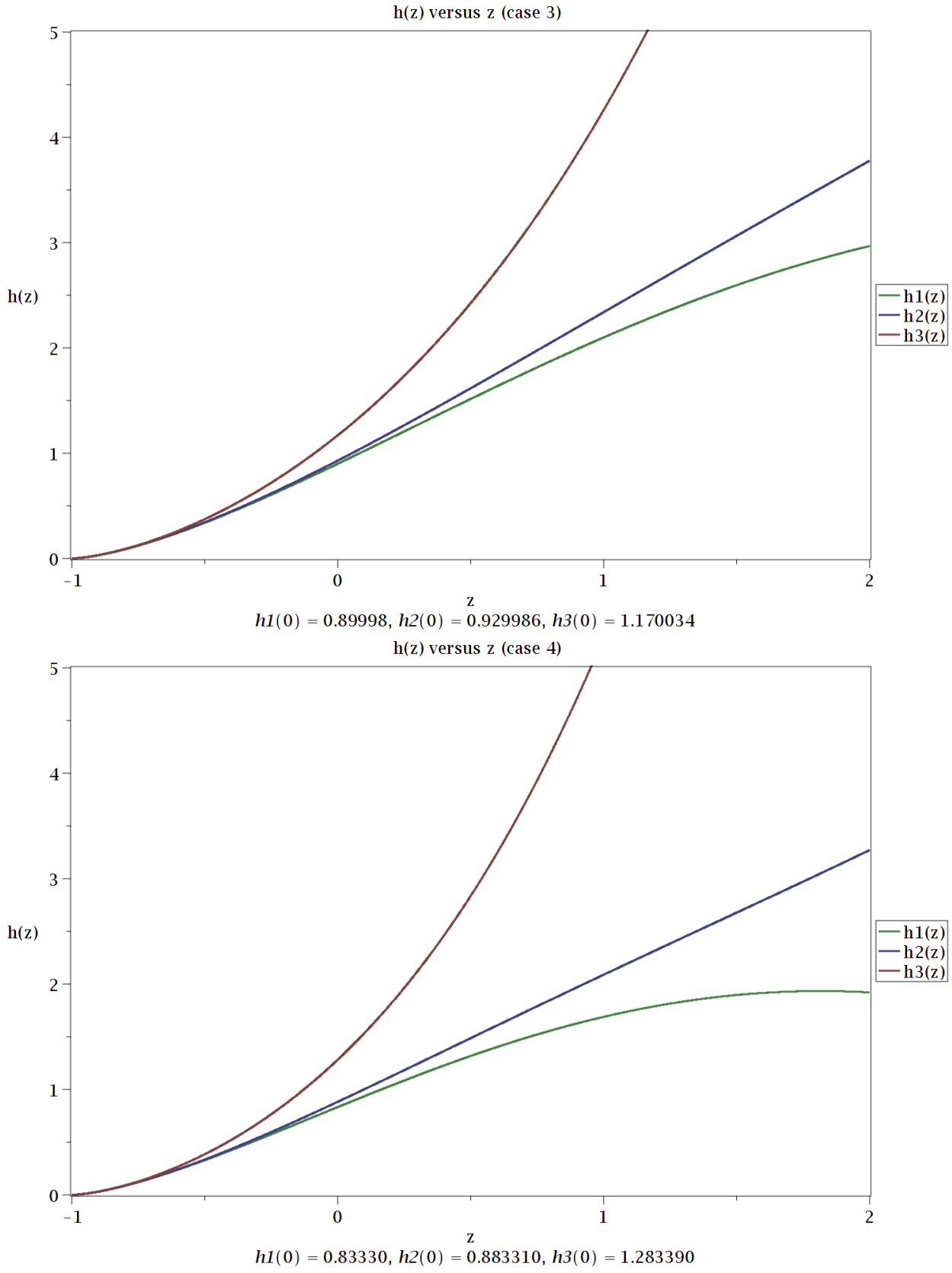


Figure 4.4: On the x-axis, the redshift z is given, and on the y-axis we have the Hubble parameter as a function of the redshift. On these plots, we have included all directional Hubble parameters for specific sets of initial conditions.

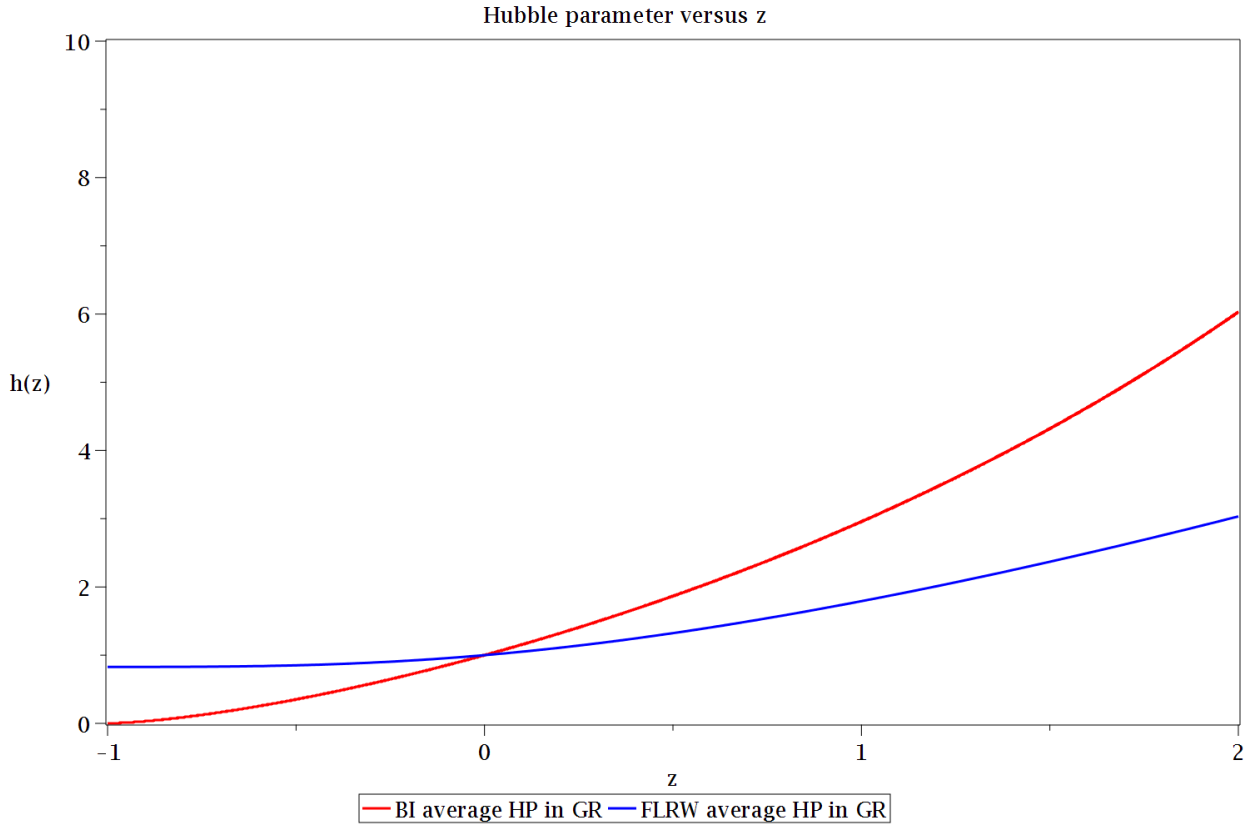


Figure 4.5: On this plot, we observe both the FLRW Hubble parameter (HP), and the average BI Hubble parameter for arbitrary initial conditions ($h_1(0) = 1,\ h_2(0) = 0.8$ and $h_3(0) = 1.2$).

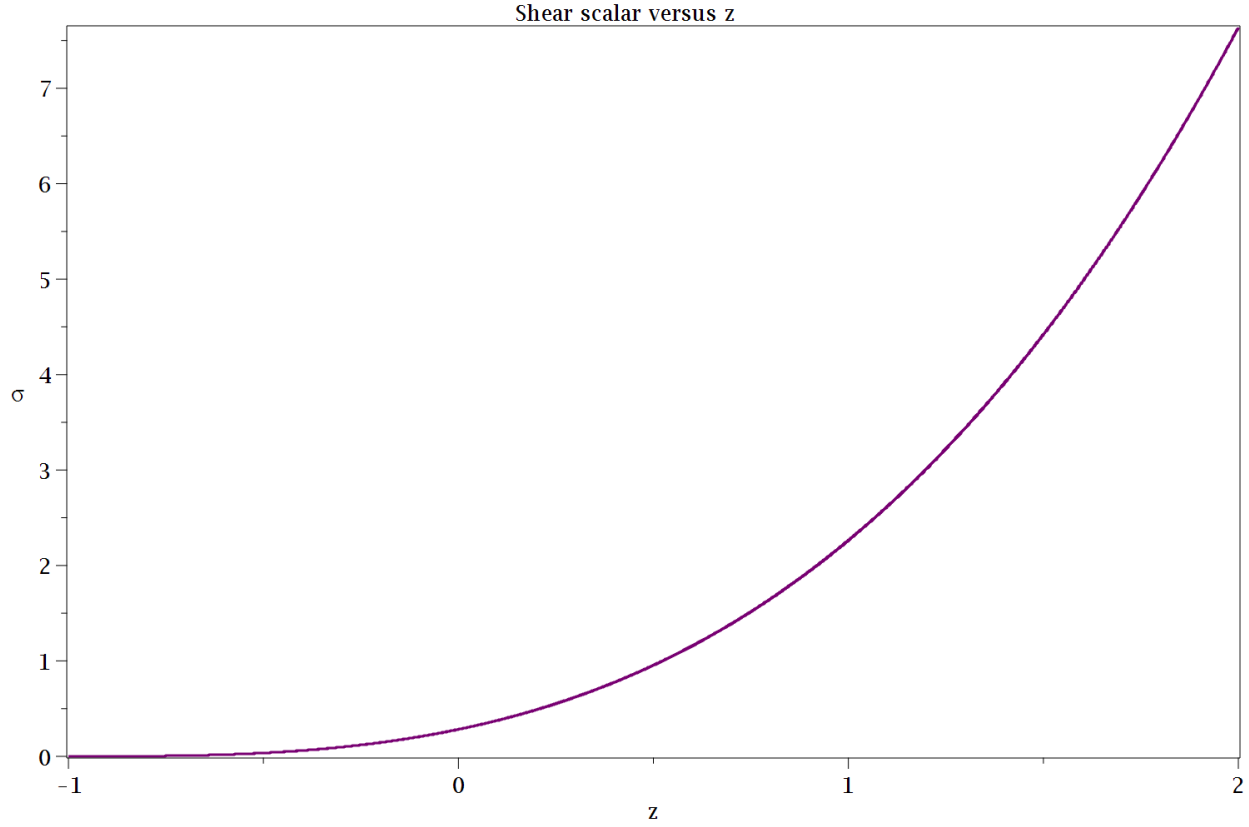


Figure 4.6: This plot depicts the average shear scalar's growth versus redshift for arbitrary initial conditions ($h_1(0) = 1,\ h_2(0) = 0.8$ and $h_3(0) = 1.2$).

Figure 4.6 shows the shear scalar versus redshift for BI. Here we can observe at late times (negative redshift), the shear dies down which would mean that the anisotropy becomes zero at negative redshift for the BI case in GR.

4.2 Deceleration Parameter

We can also define the average deceleration parameter for the BI model as follows:

$$\begin{aligned} q &= -\frac{\dot{H}}{H^2} - 1, && (4.42)\\ &= \frac{(1+z)H'}{H} - 1, && (4.43)\\ &= -\left(\frac{1}{3} - \frac{(1+z)H_1'}{3H}\right) - \left(\frac{1}{3} - \frac{(1+z)H_2'}{3H}\right) - \left(\frac{1}{3} - \frac{(1+z)H_3'}{3H}\right), && (4.44)\\ &= q_1 + q_2 + q_3, && (4.45) \end{aligned}$$

where

$$q_i = -\left(\frac{1}{3} - \frac{(1+z)H_i'}{3H}\right) \tag{4.46}$$

is the directional deceleration parameter for $i = 1, 2, 3$, respectively.

Hence, we can obtain plots for our directional deceleration parameters (as well as the average deceleration parameter) using our numerical solutions that we obtained for the EFE's, as shown in Figures 4.7 to 4.8. In these plots, we notice that this model predicts a decelerating universe. This does not correspond to the FLRW model's deceleration parameter. This could be due to the need for a cosmological constant in this model (as is needed in the FLRW case for GR), and hence why we choose to modify our form of gravity in future chapters.

4.3 Background Dynamics of BI in GR

In the background for the BI model, we have that the vorticity and the spatial gradient of any scalar f vanishes:

$$\omega_{ab} = 0, \quad \tilde{\nabla}_a f = 0, \tag{4.47}$$

which means that the gradient of the energy density $\mathcal{D}_a$, the gradient of the pressure Y_a, the gradient of the expansion Z_a and the gradient of the spatial variation of the shear scalar Σ_a vanish in a BI model. Note that neither σ_{ab}, nor E_{ab} vanishes, as opposed to the FLRW model where both of these quantities are zero in the background.

The consequence of $Y_a \equiv \tilde{\nabla}_a p$ vanishing is that $\dot{u}_a$ also vanishes since

$$\dot{u}_a = -\frac{\tilde{\nabla}_a p}{(\rho + p)}. \tag{4.48}$$

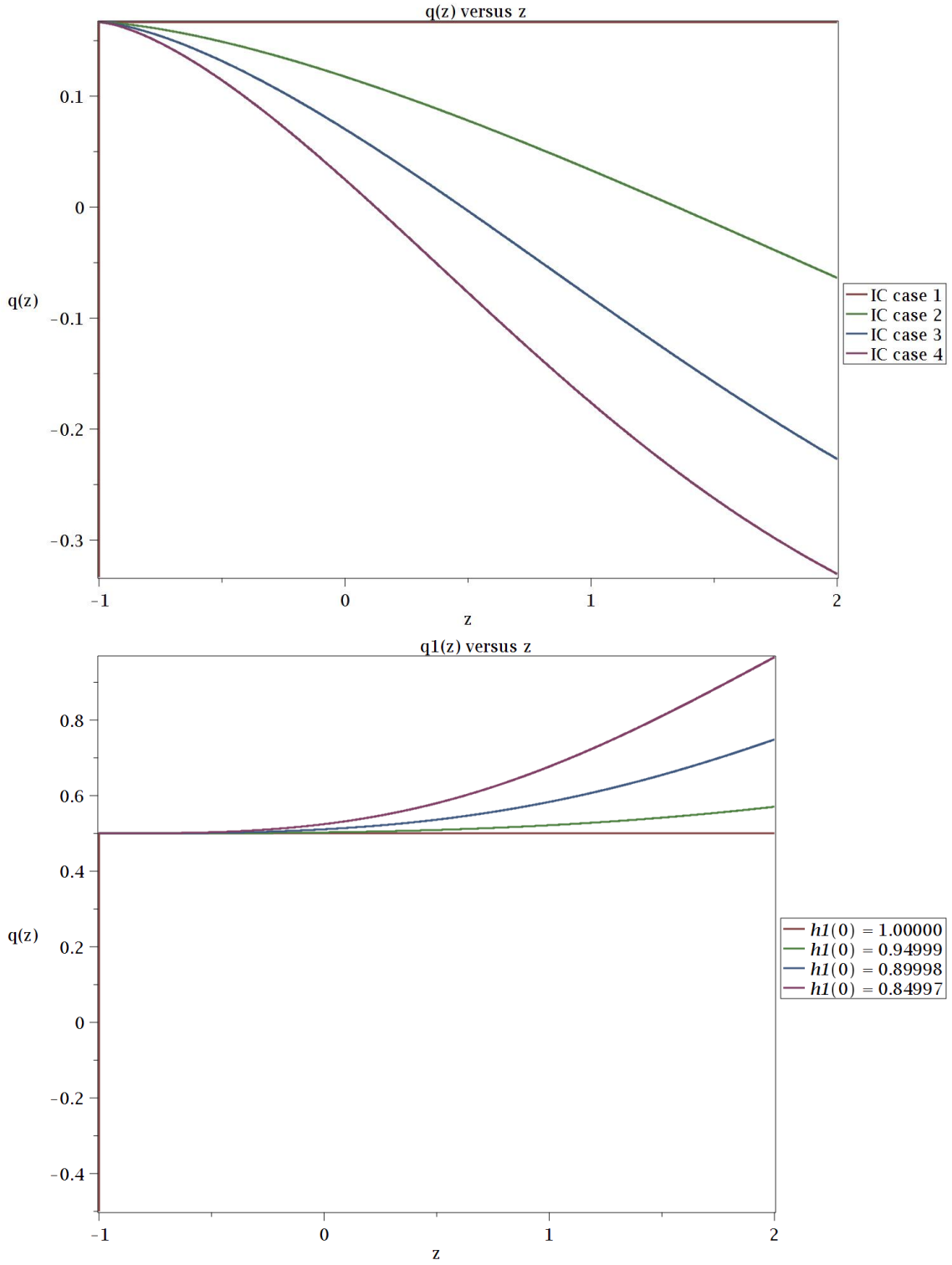


Figure 4.7: On the x-axis, the redshift z is given, and on the y-axis we have the deceleration parameter as a function of the redshift. The first plot is for the average deceleration parameter, and the second shows the directional deceleration parameter related to A, namely q_1. A few instances with different IC's are shown. The IC's for the cases in the top plot are: case 1 - $h_1(z) = 1.00000$, $h_2(z) = 1.00000$ and $h_3(z) = 1.00000$, case 2 - $h_1(z) = 0.94999$, $h_2(z) = 0.964993$ and $h_3(z) = 1.085017$, case 3 - $h_1(z) = 0.89998$, $h_2(z) = 0.929986$ and $h_3(z) = 1.170034$ and case 4 - $h_1(z) = 0.84997$, $h_2(z) = 0.894979$ and $h_3(z) = 1.255051$.

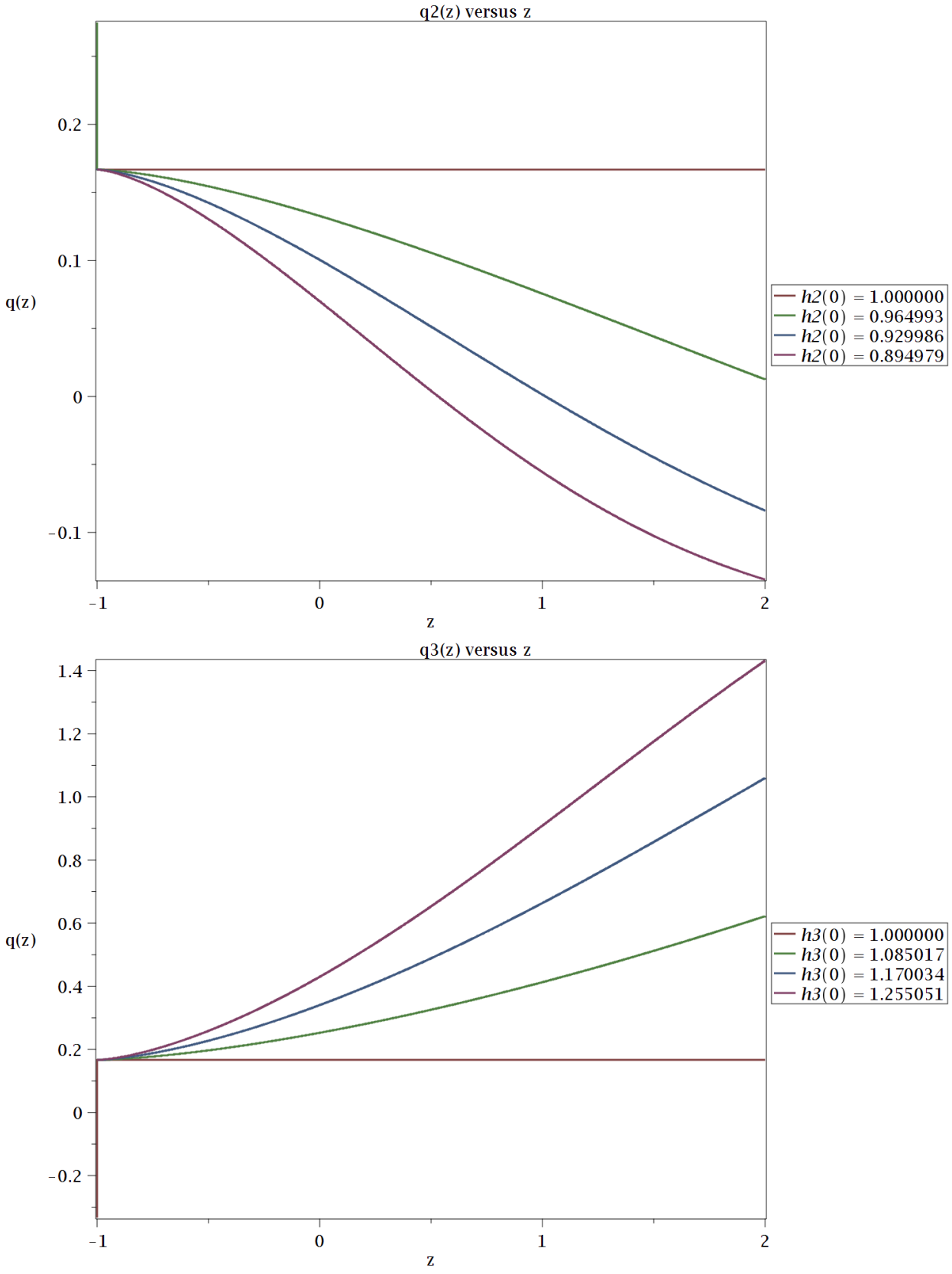


Figure 4.8: On the x-axis, the redshift z is given, and on the y-axis we have the deceleration parameter $q(z)$. The first plot shows the directional deceleration parameter related to B, namely q_2, as a function of the redshift, and the second plot shows the directional deceleration parameter related to C, namely q_3. A few instances with different IC's are shown.

At any constant value of t, we notice that our three-space metric, $d\sigma^2 = A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2$, has constant coefficients. This means that it is simply a flat three-space, and thus we can conclude that $\tilde{R}_{ab} = 0$ in these spaces [29, 43, 59].

The background evolution and constraint equations are given as

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(\rho + 3p) = -2\sigma^2, \tag{4.49}$$

$$\dot{\sigma}_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} = -\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}, \tag{4.50}$$

$$\tilde{\nabla}^b\sigma_{ab} = 0, \tag{4.51}$$

$$\text{curl } \sigma_{ab} = 0, \tag{4.52}$$

$$\dot{E}_{\langle ab\rangle} + \Theta E_{ab} + \frac{1}{2}(\rho + p)\sigma_{ab} = 3\sigma_{c\langle a}E_{b\rangle}{}^{c}, \tag{4.53}$$

$$\text{curl } E_{ab} = 0, \tag{4.54}$$

$$\tilde{\nabla}^b E_{ab} = 0, \tag{4.55}$$

$$0 = -\eta_{abc}\sigma^b{}_d E^{cd}, \tag{4.56}$$

$$E_{ab} = \frac{1}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c}, \tag{4.57}$$

$$\tilde{R} = 2\rho - \frac{2}{3}\Theta^2 + 2\sigma^2. \tag{4.58}$$

The Gauss equation and the Ricci identities for u^a enable us to write the linearized 3-space Ricci tensor as [94]

$$\tilde{R}_{ab} = -\dot{\sigma}_{\langle ab\rangle} - \Theta\sigma_{ab} + \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} + \pi_{ab} + \frac{1}{3}h_{ab}\left(2\rho - \frac{2}{3}\Theta^2 + 2\sigma^2\right). \tag{4.59}$$

We can write this equation in the background as

$$0 = -\left(-\frac{2}{3}\Theta\sigma_{ab} - E_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c}\right) - \Theta\sigma_{ab} + \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} + \pi_{ab} + \frac{1}{3}h_{ab}\left(2\rho - \frac{2}{3}\Theta^2 + 2\sigma^2\right). \tag{4.60}$$

In the background, we also have that

$$H_{ab} = 0, \quad \tilde{\nabla}_b E^{ab} = 0. \tag{4.61}$$

We can write the background Raychaudhuri equation as

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + \frac{1}{2}(\rho + 3p) = 0, \tag{4.62}$$

which can be re-written in terms of the scale factor (refer to [29]) to yield

$$\frac{\ddot{a}}{a} = -2\sigma^2 - \frac{1}{2}(\rho + 3p). \tag{4.63}$$

Another important equation for the BI universe is the background shear evolution equation, which, when linearized, is given as

$$\dot{\sigma}_{ab} + \frac{2}{3}\Theta\sigma_{ab} - \frac{2}{3}\sigma^2 h_{ab} + E_{ab} + \sigma_{ac}\sigma^c{}_b = 0. \tag{4.64}$$

We also have $\tilde{R}_{ab} = 0$ in the background. Using this, and the Gauss-Codazzi relation, equation (4.80), Dunsby [29] finds an expression for the background electric part of the Weyl tensor as

$$E_{ab} = \frac{2}{3}\sigma^2 h_{ab} - \sigma_{ac}\sigma^c{}_b + \frac{1}{3}\Theta\sigma_{ab}. \tag{4.65}$$

If we take the definition of the shear scalar, $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$, then we can write the evolution of the shear scalar as [29]

$$\dot{\sigma} = \frac{1}{\sigma}\left[\frac{1}{2}\sigma^{ab}\tilde{\nabla}_{(a}\dot{u}_{b)} - \frac{1}{2}\sigma^{ab}\sigma_{af}\sigma^{f}{}_{b} - \frac{2}{3}\sigma^{ab}E_{ab}\right]. \tag{4.66}$$

4.4 Linearized Propagation and Constrain Equations for BI in GR

In order to linearize our propagation and constraint equations, we need to take into account what order the terms in our equations are. We can do this in the same way as [29]: those quantities that do not vanish in the background (like ρ, p, Θ, σ and E_{ab}, as well as their time derivatives) are treated as zero order and those that do vanish in the background (i.e.,π_{ab}, $\mathcal{D}_a$, Y_a, Z_a, ω_{ab}, $\dot{u}_a$, $\tilde{R}_{ab}$, H_{ab}, q_a and $\tilde{\nabla}_b E^{ab}$) are treated as first order. We then neglect all terms that are second order, or higher - this happens when at least two first order terms are acting on each other. Note that a zero order term acting on a first order term gives first order, for example $\sigma^b_a\mathcal{D}_b$, and can thus not be neglected.

The linearized equations are given by

$$\dot{\rho} + (\rho + p)\Theta + \pi^{ab}\sigma_{ab} + \tilde{\nabla}_a q^a = 0, \tag{4.67}$$

$$\dot{q}_{\langle a\rangle} + \frac{4}{3}\Theta q_a + (\rho + p)\dot{u}_a + \tilde{\nabla}_a p + \tilde{\nabla}^b\pi_{ab} = -\sigma_{ab}q^b - \dot{u}^b\pi_{ab}, \tag{4.68}$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(\rho + 3p) - \tilde{\nabla}^a\dot{u}_a = -\sigma_{ab}\sigma^{ab}, \tag{4.69}$$

$$\dot{\omega}_{\langle a\rangle} + \frac{2}{3}\Theta\omega_a + \frac{1}{2}\ \mathrm{curl}\ \dot{u}_a = \sigma_{ab}\omega^b, \tag{4.70}$$

$$\dot{\sigma}_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} = -\sigma_{c\langle a}\sigma_{b\rangle}{}^c, \tag{4.71}$$

$$\tilde{\nabla}^a\omega_a = 0, \tag{4.72}$$

$$\tilde{\nabla}^b\sigma_{ab} - \ \mathrm{curl}\ \omega_a - \frac{2}{3}\tilde{\nabla}_a\Theta + q_a = 0, \tag{4.73}$$

$$\mathrm{curl}\ \sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0, \tag{4.74}$$

$$\begin{aligned}\dot{E}_{\langle ab\rangle} + \Theta E_{ab} - \mathrm{curl}\ H_{ab} + \frac{1}{2}\left[(\rho + p)\sigma_{ab} + \dot{\pi}_{\langle ab\rangle} + \frac{1}{4}\Theta\pi_{ab} + \tilde{\nabla}_{\langle a}q_{b\rangle}\right]\\ = 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c\eta_{cd(a}E_{b)}{}^d - \frac{1}{2}\sigma^c{}_{\langle a}\pi_{b\rangle c},\end{aligned} \tag{4.75}$$

$$\dot{H}_{\langle ab\rangle} + \Theta H_{ab} + \mathrm{curl}\ E_{ab} - \frac{1}{2}\ \mathrm{curl}\ \pi_{ab} = 3\sigma_{c\langle a}H_{b\rangle}{}^c - 2\dot{u}^c\eta_{cd(a}E_{b)}{}^c + \frac{1}{2}\sigma^c{}_{(a}\eta_{b)cd}q^d, \tag{4.76}$$

$$\tilde{\nabla}^b E_{ab} + \frac{1}{6}\left(\tilde{\nabla}^b\pi_{ab} - 2\tilde{\nabla}_a\rho + 2\Theta q_a\right) = \eta_{abc}\sigma^b{}_d H^{cd} + \frac{1}{2}\sigma_{ab}q^b, \tag{4.77}$$

$$\tilde{\nabla}^b H_{ab} + \frac{1}{2}[\ \mathrm{curl}\ q_a - 2(\rho + p)\omega_a] = 3E_{ab}\omega^b - \eta_{abc}\sigma^b{}_d E^{cd} - \frac{1}{2}\eta_{abc}\sigma^b{}_d\pi^{cd}, \tag{4.78}$$

$$\tilde{R}_{\langle ab\rangle} = E_{ab} + \frac{1}{2}\pi_{ab} - \frac{1}{3}\Theta(\sigma_{ab} + \omega_{ab}) + \sigma_{c\langle a}\sigma_{b\rangle}{}^c - 2\sigma_{c[a}\omega_{b]}{}^c, \tag{4.79}$$

$$\tilde{R} = 2\rho - \frac{2}{3}\Theta^2 + 2\sigma^2. \tag{4.80}$$

If we were to perturb the BI metric in GR, we would have used these equations. But, since we are introducing $f(R)$ gravity in the next chapter, we will also introduce a new set of propagation and constraint equations for our modified gravity model that will aid us in finding the perturbation equations.

Chapter 5

$f(R)$ Theories of Gravity

The $f(R)$ theories of gravity have been formulated with the purpose of addressing the problems and shortcomings of the SM and from a lack of a final theory of Quantum Gravity [23]. This form of modified gravity was first proposed by Hans Adolph Buchdahl in 1970 [19], and has been greatly influenced by Starobinsky [92]. This model is a fourth order theory, which means that it gives fourth order equations. There are other variants that are similar to $f(R)$ gravity models, including $f(R,T)$ models, and scalar-tensor theories. For reviews on $f(R)$ gravity, consult [16, 26, 28, 98]. $f(R)$ theories are believed to be good toy models because of several reasons - one of these reasons is that $f(R)$ theories avoid the 'long known and fatal' Ostrogradski instability [90].

The $f(R)$ gravity theories are obtained by modifying the Einstein-Hilbert action given by equation (2.35), and obtaining a new action given as

$$\mathcal{A} = \int d^4x\sqrt{-g}\,[\frac{1}{2}f(R) + \mathcal{L}_m], \tag{5.1}$$

where $f(R)$ is an arbitrary function of the Ricci scalar, R [91], and the $\mathcal{L}_m$ and R terms show that the Lagrangian contains matter and curvature contributions, respectively [46].

In order to obtain the EFE's, the variational principle is applied to the action. There are different ways of doing this based on which variational principle is used. The most commonly used standard approach is the variation of the action $\mathcal{A}$ with respect to the metric g_{ab}, and this is known as the metric formalism. There is also what is known as the Palatini formalism, in which the action is varied with respect to the connection and the metric, depending on if one assumes two independent variables [91].

The three different formalisms are:

1. **The Metric Formalism:**
   This is a second order formalism where the metric g_{ab} is the only independent variable

with respect to the action. The variation of the action then leads to

$$f'(R)G_{ab} = f'(R)(R_{ab} - \frac{1}{2}g_{ab}R) = T^m_{ab} + \frac{1}{2}g_{ab}\left[f(R) - Rf'(R)\right] + \nabla_b\nabla_a f'(R) - g_{ab}\nabla_c\nabla^c f'(R), \tag{5.2}$$

where $f'(R) \equiv \frac{df(R)}{dR}$ and the matter energy-momentum tensor can be defined in terms of the Lagrangian as

$$T^m_{ab} = g_{ab}\mathcal{L}_m - \frac{\partial \mathcal{L}_m}{\partial g^{ab}}. \tag{5.3}$$

This formalism introduces fourth order derivatives of the metric in the last two terms of the right-hand side of equation (5.2), and this has the implication that this formalism is also sometimes referred to as a *fourth order theory of gravity*.

2. **The Palatini Formalism:**

   As we have already mentioned, the Palatini formalism treats both the metric and the affine connection $\Gamma^a_{\ bc}$ as independent variables, which means that the action gets varied with respect to both of these symbols to obtain the EFE's. When the Palatini formalism is used, the EFE's obtained are given by

$$\begin{aligned} G_{ab} \quad &= \frac{T_{ab}}{f'(R)} - \frac{1}{2}\left[R - \frac{f(R)}{f'(R)}\right]g_{ab} + \frac{1}{f'(R)}\left(\nabla_a\nabla_b - g_{ab}\Box\right)f'(R) \\ &\quad - \frac{3}{2\left(f'(R)\right)^2}\left[\nabla_a f'(R)\nabla_b f'(R) - \frac{1}{2}g_{ab}\nabla_c f'(R)\nabla^c f'(R)\right]. \end{aligned} \tag{5.4}$$

3. **The Metric-Affine Formalism:**
   In this formalism, the matter part of the Lagrangian in the action depends only on the affine connection. For an extensive analysis of this formalism, consult [22, 89].

The formalism that we will use for this dissertation is the metric formalism. We can rewrite the EFE's for $f(R)$ in a similar fashion as for the GR form, as given in [91]:

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R = \tilde{T}^m_{ab} + T^R_{ab}, \tag{5.5}$$

where T^R_{ab} is the curvature stress-energy tensor given by

$$T^R_{ab} = g_{ab}\frac{f(R) - R\ f'(R)}{2f'(R)} + \frac{\nabla_a\nabla_b f'(R) - g_{ab}\Box f'(R)}{f'(R)}, \tag{5.6}$$

and $\tilde{T}^m_{ab} = T^m_{ab}/f'(R)$. We can thus write the effective energy-momentum tensor for the total fluid as

$$T_{ab} = \tilde{T}^m_{ab} + T^R_{ab}. \tag{5.7}$$

Henceforth, we will write $f \equiv f(R)$, $f' \equiv f'(R)$ and $f'' \equiv f''(R)$.

The energy-momentum of the standard matter and that of the total fluid are conserved, meaning that

$$T^{m;b}_{ab} = 0, \quad T^{;b}_{ab} = 0, \tag{5.8}$$

but the individual components, $\tilde{T}^m_{ab}$ and T^R_{ab} are not individually conserved. We have that (from [24])

$$\tilde{T}^{m;b}_{ab} = \frac{T^{m;b}_{ab}}{f'} - \frac{f''}{f'^2} T^m_{ab} R^{;b}, \tag{5.9}$$

$$T^{R;b}_{ab} = \frac{f''}{f'^2} \tilde{T}^m_{ab} R^{;b}. \tag{5.10}$$

5.1 Conditions for the Viability of $f(R)$ Models

There are some conditions that the function $f(R)$ must satisfy in order to be an acceptable function to replace the Ricci scalar [4, 46]:

1. To ensure that the gravity will remain attractive, we need to impose

$$f' > 0 \; \forall \; R. \tag{5.11}$$

2. For stable matter-dominated and high-curvature cosmological regimes (nontachyonic scalaron) [64], we need to have:

$$f'' > 0 \; \forall \; R \gg f''. \tag{5.12}$$

3. We need to have a GR-like law of gravitation in the early universe (for Big Bang nucleosynthesis and CMB constraints) [64], which implies that

$$\lim_{R \to \infty} \frac{f(R)}{R} = 1 \Rightarrow f' < 1. \tag{5.13}$$

4. At recent epochs, we need to have that

$$|f' - 1| \ll 1. \tag{5.14}$$

5.2 Covariant Decomposition and Projection

For this section, we will use a general frame since the choice of frame allows us to obtain an irreducible decomposition of the stress energy momentum tensor [24]. Given the general EMT, equation (2.38), we can write the dynamical quantities of the effective total EMT for $f(R)$ gravity as

$$\rho = T_{ab} u^a u^b = \tilde{\rho}^m + \rho^R, \tag{5.15}$$

$$p = \frac{1}{3} T_{ab} h^{ab} = \tilde{p}^m + p^R, \tag{5.16}$$

$$q_a = -T_{bc} h^b_a u^c = \tilde{q}^m_a + q^R_a, \tag{5.17}$$

$$\pi_{ab} = T_{cd} h^c_{\langle a} h^d_{b\rangle} = \tilde{\pi}^m_{ab} + \pi^R_{ab}, \tag{5.18}$$

were

$$\tilde{\rho}^m = \frac{\rho^m}{f'}, \quad \tilde{p}^m = \frac{p^m}{f'}, \quad \tilde{q}_a^m = \frac{q_a^m}{f'}, \quad \tilde{\pi}_{ab}^m = \frac{\pi_{ab}^m}{f'}. \tag{5.19}$$

The last two quantities, q_a^m and π_{ab}^m, both vanish in a perfect cosmological medium, and since we assume that the standard matter is a perfect fluid, we can neglect these quantities.

We can write the generalized effective thermodynamical quantities for the curvature fluid as

$$\rho^R = T_{ab}^R u^a u^b = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right], \tag{5.20}$$

$$p^R = \frac{1}{3}T_{ab}^R h^{ab} = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\big(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\big)\right], \tag{5.21}$$

$$q_a^R = -T_{bc}^R h_a^b u^c = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right], \tag{5.22}$$

$$\pi_{ab}^R = T_{cd}^R h_{\langle a}^c h_{b\rangle}^d = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle} R + f'''\tilde{\nabla}_{\langle a} R\tilde{\nabla}_{b\rangle} R + \sigma_{ab}\dot{R}\right]. \tag{5.23}$$

We can also write the $f(R)$ trace equation as

$$3(f'''\dot{R}^2 + f''\ddot{R}) + 3\Theta f''\dot{R} - 3\nabla^2 f' = f'R - 2f + \rho^m - 3p^m. \tag{5.24}$$

5.3 Conservation Equations

We can obtain the energy and momentum equations (evolution equations for ρ^m, ρ^R and q_a^R) by taking the parallel and orthogonal projection of the twice-contracted Bianchi identities with respect to the flow lines [46]:

$$\dot{\rho}^m = -\Theta(\rho^m + p^m), \tag{5.25}$$

$$\dot{\rho}^R + \tilde{\nabla}^a q_a^R = -\Theta(\rho^R + p^R) - 2\dot{u}^a q_a^R - \sigma^{ab}\pi_{ba}^R + \rho^m\frac{f''\dot{R}}{f'^2}, \tag{5.26}$$

$$\dot{q}_{\langle a\rangle}^R + \tilde{\nabla}^a p^R + \tilde{\nabla}^b \pi_{ab}^R = -\frac{4}{3}\Theta q_a^R - \sigma^b{}_a q_b^R - (\rho^R + p^R)\dot{u}_a - \dot{u}^b\pi_{ab}^R - \eta^{bc}{}_a\omega_b q_c^R + \rho^m\frac{f''\tilde{\nabla}_a R}{f'^2}. \tag{5.27}$$

It can be noted from the RHS of the last two equations, respectively, that energy and momentum is exchanged between the curvature and effective matter fluids. Energy density and isotropic pressure can be related to the relativistic acceleration using the conservation of momentum of standard matter, and thus we can write

$$\tilde{\nabla}^a p^m = -(\rho^m + p^m)\dot{u}^a. \tag{5.28}$$

The standard GR equations can be obtained by substituting $f(R) = R$.

5.4 Propagation and Constraint Equations

The full non-linear 1+3 propagation and constraint equations, as given in Section 3.5, can also be found for higher order gravity using 1+3 covariant splitting of the Bianchi and Ricci identities. This results in the general equations, as given by [24]

1. **Expansion propagation (generalized Raychaudhuri equation):**

$$\dot{\Theta}+\frac{\Theta^2}{3}+\sigma_{ab}\sigma^{ab}-2\omega_a\omega^a-\tilde{\nabla}^a\dot{u}_a+\dot{u}_a\dot{u}^a+\frac{1}{2}(\tilde{\rho}^m+3\tilde{p}^m)=-\frac{1}{2}(\rho^R+3p^R), \tag{5.29}$$

2. **Vorticity propagation:**

$$\dot{\omega}_{\langle a\rangle}+\frac{2\Theta}{3}\omega_a+\frac{1}{2}\text{curl }\dot{u}_a-\sigma_{ab}\omega^b=0, \tag{5.30}$$

3. **Shear propagation:**

$$\dot{\sigma}_{\langle ab\rangle}+\frac{2\Theta}{3}\sigma_{ab}+E_{ab}-\tilde{\nabla}_{\langle a}\dot{u}_{b\rangle}+\sigma_{c\langle a}\sigma_{b\rangle}{}^c+\omega_{\langle a}\omega_{b\rangle}-\dot{u}_{\langle a}\dot{u}_{b\rangle}=\frac{1}{2}\pi^R_{ab}, \tag{5.31}$$

4. **Gravito-electro propagation:**

$$\begin{aligned}\dot{E}_{\langle ab\rangle}+\Theta E_{ab}-\text{curl }H_{ab}+\frac{1}{2}(\tilde{\rho}^m+\tilde{p}^m)\sigma_{ab}-2\dot{u}^c\eta_{cd(a}H_{b)}{}^d-3\sigma_{c\langle a}E_{b\rangle}{}^c+\omega^c\eta_{cd(a}E_{b)}{}^d\\ =-\frac{1}{2}(\rho^R+p^R)\sigma_{ab}-\frac{1}{2}\dot{\pi}^R_{\langle ab\rangle}-\frac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle}-\frac{\Theta}{6}\pi^R_{ab}-\frac{1}{2}\sigma^c{}_{\langle a}\pi^R_{b\rangle c}-\frac{1}{2}\omega^c\eta^d_{c(a}\pi^R_{b)d},\end{aligned} \tag{5.32}$$

5. **Gravito-magnetic propagation:**

$$\begin{aligned}\dot{H}_{\langle ab\rangle}+\Theta H_{ab}+\text{curl }E_{ab}-3\sigma_{c\langle a}H_{b\rangle}{}^c+\omega^c\eta_{cd(a}H_{b)}{}^d+2\dot{u}^c\eta_{cd(a}E_{b)}{}^d\\ =\frac{1}{2}\text{curl }\pi^R_{ab}-\frac{3}{2}\omega_{\langle a}q^R_{b\rangle}+\frac{1}{2}\sigma^c{}_{(a}\eta_{b)c}{}^dq^R_d,\end{aligned} \tag{5.33}$$

6. **Vorticity constraint:**

$$\tilde{\nabla}^a\omega_a-\dot{u}^a\omega_a=0, \tag{5.34}$$

7. **Shear constraint:**

$$\tilde{\nabla}^b\sigma_{ab}-\text{curl }\omega_a-\frac{2}{3}\tilde{\nabla}_a\Theta+2[\omega,\dot{u}]_a=-q^R_a, \tag{5.35}$$

8. **Gravito-magnetic constraint:**

$$\text{curl }\sigma_{ab}+\tilde{\nabla}_{\langle a}\omega_{b\rangle}-H_{ab}+2\dot{u}_{\langle a}\omega_{b\rangle}=0, \tag{5.36}$$

9. **Gravito-electric divergence:**

$$\tilde{\nabla}^bE_{ab}-\frac{1}{3}\tilde{\nabla}_a\tilde{\rho}^m-[\sigma,H]_a+3H_{ab}\omega^b=\frac{1}{2}\sigma^b_aq^R_b-\frac{3}{2}[\omega,q^R]_a-\frac{1}{2}\tilde{\nabla}^b\pi^R_{ab}+\frac{1}{3}\tilde{\nabla}_a\rho^R-\frac{1}{3}\Theta q^R_a, \tag{5.37}$$

10. **Gravito-magnetic divergence:**

$$\tilde{\nabla}^b H_{ab} - (\tilde{\rho}^m + \tilde{p}^m)\omega_a + [\sigma, E]_a - 3E_{ab}\omega^b = -\frac{1}{2}\text{curl } q_a^R + (\rho^R + p^R)\omega_a - \frac{1}{2}[\sigma, \pi^R]_a - \frac{1}{2}\pi_{ab}^R\omega^b. \tag{5.38}$$

When we set $f(R) = R$ in order to obtain the GR equations, we will notice that the RHS for all of these equations vanishes.

We have the normalized 3-space Ricci tensor and Ricci scalar in $f(R)$ gravity:

$$\tilde{R}_{\langle ab\rangle} = E_{ab} + \frac{1}{2}\pi_{ab} - \frac{1}{3}\Theta(\sigma_{ab} + \omega_{ab}) + \sigma_{c\langle a}\sigma_{b\rangle}{}^c + \omega_{c\langle a}\omega_{b\rangle}{}^c - 2\sigma_{c[a}\omega_{b]}{}^c, \tag{5.39}$$

$$\tilde{R} = 2(\tilde{\rho}^m + \rho^R) - \frac{2}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - \omega_a\omega^a. \tag{5.40}$$

All of these equations can be used in the derivations of the perturbation equations for our model. We will linearize these equations around our BI model in the following chapter.

5.5 Some $f(R)$ Models

There are some general models that are often in use. These include R^n models, the Hu-Sawicki Model, and the Starobinsky model.

R^n models are oftentimes named toy-models. The most common way of writing this model is

$$f(R) = \beta R^n, \tag{5.41}$$

where β and n are arbitrary free parameters. We can also write a second toy model as

$$f(R) = \alpha R + \beta R^n, \tag{5.42}$$

where α is also an arbitrary free parameter [51].

These models are not models that closely describe the dynamics of our universe, but they do act as a convenient way to test the $f(R)$ models, seeing as it is simpler than the other models.

The Hu-Sawicki model is an $f(R)$-gravity model of the form

$$f(R) = R - \frac{\beta_1 m^2 \left(\frac{R}{m^2}\right)^n}{1 + \beta_2 m^2 \left(\frac{R}{m^2}\right)^n}, \tag{5.43}$$

where β_1, β_2 and n (where $n\rangle 0$) are the arbitrary free parameters, and $m^2 = \frac{\rho^m}{3}$ is a curvature scale that depends on the value of the matter-energy density.

The Starobinsky $f(R)$-gravity model is given as

$$f(R) = R + \beta R_c \left[\left(1 + \frac{R^2}{R_c^2}\right)^{-1} - 1\right], \tag{5.44}$$

where $\beta > 0$ and $n > 0$. In the special case where either of these parameters are equal to zero, this model would reduce back to the GR model. R_c parametrises the curvature scale. Both the Hu-Sawicki and the Starobinsky models are more realistic that the R^n model [51].

Another approach taken in many papers is to assume a power-law relation between f' and a (this is often done when the solutions to a specific model's EFE's are difficult to obtain) [47, 48]. This is an approach that we will also apply in the following chapter.

5.6 Dynamics of Scalar Perturbations in $f(R)$ Gravity

In addition to the quantities defined by equation (3.15), we also need to define some dimensionless gradients that describe the inhomogeneity in the Ricci scalar:

$$\mathcal{R}_a = a\tilde{\nabla}_a R, \quad \mathfrak{R}_a = a\tilde{\nabla}_a \dot{R}. \tag{5.45}$$

If we take all of these quantities that describe the dynamics of the scalar perturbations and determine their evolution equations (find the derivatives with respect to time), then we can examine the perturbations.

Chapter 6

Cosmological Solutions for Bianchi Type-I in $f(R)$ Gravity

The study of the BI model in $f(R)$ gravity is not new. Several authors have done studies on BI in $f(R)$ gravity [8, 75, 79]. Scalar-tensor theories as a different form of modified gravity have been applied to BI by [70]. An LRS BI metric as a phantom energy model in $f(R)$ gravity is studied by [63]. In this chapter, we will find the EFE's, and then look at some studies done in $f(R)$ gravity. We will also find the perturbation equations for this model.

6.1 EFE's for BI in $f(R)$ Gravity

As we have already discussed the importance of f(R) gravity in the previous chapter, we can now apply this theory to the BI model using the EFE's obtained for $f(R)$ gravity, equation (5.5). The full process of the derivation of these $f(R)$ EFE's is shown in Appendix B.1. Our general EFE's for the BI metric in $f(R)$ gravity are found to be

$$-\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right)f'+\frac{1}{2}f+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)f''\dot{R}=\rho^m, \tag{6.1}$$

$$-\left[\frac{\ddot{A}}{A}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]f'-\frac{1}{2}f-f'''\dot{R}^2-f''\ddot{R}-\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)f''\dot{R}=-p^m, \tag{6.2}$$

$$-\left[\frac{\ddot{B}}{B}+\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)\right]f'-\frac{1}{2}f-f'''\dot{R}^2-f''\ddot{R}-\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)f''\dot{R}=-p^m, \tag{6.3}$$

$$-\left[\frac{\ddot{C}}{C}+\frac{\dot{C}}{C}\left(\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)\right]f'-\frac{1}{2}f-f'''\dot{R}^2-f''\ddot{R}-\left(\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)f''\dot{R}=-p^m. \tag{6.4}$$

These equations are subsequently our general EFE's. To determine if these equations are correct, we can see if they reduce to the GR EFE's when we substitute $f(R)=R$. When $f=R$, then $f'=1$, and f'' as well as f''' are zero. Substituting these, as well

as the Ricci scalar, into equations (6.1) to (6.4), we obtain the GR results, as given by equations (4.17) to (4.19) in Chapter 4.1. From equations (6.1) to (6.4), we can obtain our generalized Raychaudhuri equation in $f(R)$ gravity as

$$(6H^2 - \sigma^2)f' - 3Hf''\dot{R} - 3f'''\dot{R}^2 - 3f''\ddot{R} - f = \rho^m + 3p^m. \tag{6.5}$$

In order find exact solutions for the EFE's, we can first subtract equation (6.3) from (6.2), equation (6.4) from (6.3), and equation (6.2) from (6.4). We also now write $F \equiv f'$, and hence we obtain:

$$\frac{\dot{H}_1 - \dot{H}_2}{H_1 - H_2} + H_1 + H_2 + H_3 + \frac{\dot{F}}{F} = 0, \tag{6.6}$$

$$\frac{\dot{H}_2 - \dot{H}_3}{H_2 - H_3} + H_1 + H_2 + H_3 + \frac{\dot{F}}{F} = 0, \tag{6.7}$$

$$\frac{\dot{H}_3 - \dot{H}_1}{H_3 - H_1} + H_1 + H_2 + H_3 + \frac{\dot{F}}{F} = 0. \tag{6.8}$$

These equations can be integrated to find the solutions to these equations as

$$\frac{B}{A} = m_1 e^{c_4 \int \frac{1}{a^3 F} dt}, \tag{6.9}$$

$$\frac{C}{B} = m_2 e^{c_5 \int \frac{1}{a^3 F} dt}, \tag{6.10}$$

$$\frac{A}{C} = m_3 e^{c_6 \int \frac{1}{a^3 F} dt}, \tag{6.11}$$

where $c_4,\ c_5,\ c_6,\ k_1,\ k_2,\ k_3,\ m_1,\ m_2$ and m_3 are integration constants, and

$$c_4 + c_5 + c_6 = 0, \quad \text{and} \quad m_1 m_2 m_3 = 1 \tag{6.12}$$

need to hold for these constants. Equations (6.9) to (6.11) can be used to obtain

$$A = a p_1 e^{q_1 \int \frac{1}{a^3 F} dt},\ B = a p_2 e^{q_2 \int \frac{1}{a^3 F} dt},\ C = a p_3 e^{q_3 \int \frac{1}{a^3 F} dt}, \tag{6.13}$$

where a is the average scale factor, and

$$p_1 = \sqrt[3]{m_1^{-2} m_2^{-1}},\ p_2 = \sqrt[3]{m_1 m_2^{-1}},\ p_3 = \sqrt[3]{m_1 m_2^2}, \tag{6.14}$$

and

$$q_1 = \frac{2c_4 + c_5}{3},\ q_2 = \frac{c_4 - c_5}{3},\ q_3 = \frac{c_4 + 2c_5}{3}. \tag{6.15}$$

We notice here that

$$q_1 + q_2 + q_3 = 0, \quad \text{and} \quad p_1 p_2 p_3 = 1. \tag{6.16}$$

The solutions that we obtained are written in a similar fashion as [48, 75, 81, 83]. In [48], they obtained solutions by assuming a vacuum, and hence we note that our solutions and their solutions are of a similar form. These solutions can then be applied to whichever model we choose for $f(R)$ in order to find an analytical solution to the scale factors. However, this does prove to be a complicated method, since the solutions themselves depend on the average scale factor. Hence, there have been a few different ways that others have finished this solution: Hasmani and Al-Haysah [48] obtain further exact solutions of

the EFE's by assuming a power-law relation between F and a, and then furthermore using a relation by Berman [13] for the average Hubble parameter and the scale factor to obtain a relation for a. This then leads them to two different solutions for the $f(R)$ function, instead of using a specific function for $f(R)$ from the beginning.

Shri Ram [83] presents a special form of the average scale factor after finding the solutions that we presented earlier. They assume a specific power-law relation between F and a, after which they then use a few different models: in the first, they set the deceleration parameter equal to a function of a, and then they find a solution to a that hence enables them to find solutions to the directional deceleration parameters. In the second model, they assume a power-law relation for the scale factor and the time, and then they obtain different relations to all of the variables, as well as the $f(R)$ function, using the solutions obtained.

We can obtain exact solutions by applying a combination of these methods.

6.2 Applications of the Bianchi Type-I Cosmological Model

The BI model can be modified to in other ways as well, for example, Katore et al., [54] finds the BI equations for a string cloud, while Singh and Beesham [87] use an LRS model where $B = C$ in $f(R, T)$ gravity. Chakraborty et al., [25] use a dynamical variables approach to find solutions in $f(R) = R^{1+\delta}$ modified gravity, and they show that the dynamics of this model in a vacuum is exactly solvable.

One paper that we can look at is by Moraes et al., [63]. They examine a plane symmetric, or LRS, BI model [44, 63, 87] in Big Rip $f(R)$ cosmology. In this paper, they show that the phantom energy-dominated era may be transient, or that the Universe does not necessarily end at the Big Rip. We will attempt to obtain similar solutions to [63] in this section.

We take the BI metric to be

$$ds^2 = -dt^2 + A^2(dx^2 + dy^2) + B^2 dz^2, \tag{6.17}$$

thus meaning that there are only two scale factors instead of three, with $B = C$. We compare solutions given in [63] with the solutions we find using Maple and manual calculations. [63] uses a (+, -, -, -) sign convention, but we will continue with our (-, +, +, +) convention. The Ricci scalar in this case is

$$R = 2\left(2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\ddot{B}}{B}\right), \tag{6.18}$$

and the general $f(R)$ EFE's for this model are found to be:

$$-\left(2\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}\right)f'+\left(2\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\dot{f}'+\frac{f}{2} \quad =\rho^m, \tag{6.19}$$

$$\left(\frac{\ddot{A}}{A}+\frac{\dot{A}^2}{A^2}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right)f'-\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\dot{f}'-\ddot{f}'-\frac{f}{2} \quad =p^m, \tag{6.20}$$

$$\left(\frac{\ddot{B}}{B}+2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right)f'-2\frac{\dot{A}}{A}\dot{f}'-\ddot{f}'-\frac{f}{2} \quad =p^m. \tag{6.21}$$

These EFE's can be written in terms of their directional Hubble parameters:

$$-\left(2\dot{H}_1+2H_1^2+\dot{H}_2+H_2^2\right)f'+\left(2H_1+H_2\right)\dot{f}'+\frac{f}{2} \quad =\rho^m, \tag{6.22}$$

$$\left(\dot{H}_1+2H_1^2+H_1H_2\right)f'-\left(H_1+H_2\right)\dot{f}'-\ddot{f}'-\frac{f}{2} \quad =p^m, \tag{6.23}$$

$$\left(\dot{H}_2+H_2^2+2H_1H_2\right)f'-2H_1\dot{f}'-\ddot{f}'-\frac{f}{2} \quad =p^m. \tag{6.24}$$

Here, the average Hubble parameter is given by

$$H=\frac{1}{3}(2H_1+H_2). \tag{6.25}$$

We can use this average Hubble parameter and add the previous equations to obtain both the Friedmann and the Raychaudhuri equations:

$$-(3\dot{H}+2H_1^2+H_2^2)f'+3H\dot{f}'+\frac{f}{2} \quad =\rho^m, \tag{6.26}$$

$$-(6\dot{H}+2H_1^2+H_2^2+H^2)f'+9H\dot{f}'+3\ddot{f}'+2f \quad =\rho^m-3p^m. \tag{6.27}$$

When we subtract equation (6.23) from equation (6.24), we obtain a solution of the form

$$\frac{A}{B}=e^{c_1\int\frac{1}{a^3f'}dt}, \tag{6.28}$$

where c_1 is an integration constant. This means that we can re-write our solutions to be in a similar form as in [63]:

$$A=ac_2^{1/3}e^{\frac{c_1}{3}\int\frac{dt}{a^3}}, \tag{6.29}$$

$$B=ac_2^{-2/3}e^{-\frac{2c_1}{3}\int\frac{dt}{a^3}}, \tag{6.30}$$

where c_2 is a constant, and a represents the average or mean scale factor, such that the spatial volume is defined as $V=A^2B=a^3$. From [63], we see that they decide to use an expression for the Hubble parameter that is related to Big Rip cosmology. This means that the scale factor of the Universe becomes infinite at a finite time scale, which is what we call a 'big rip singularity' [63]. The suggested mean Hubble parameter is given as

$$H=\frac{h_0}{t_0-t}, \tag{6.31}$$

where h_0 and t_0 are positive, dimensionless constants, and H diverges at t_0 with a singularity appearing at this t_0, and $0<t<t_0$ in order for the Hubble parameter to have a positive real value. However, if we apply an appropriate $f(R)$ model (as will be done

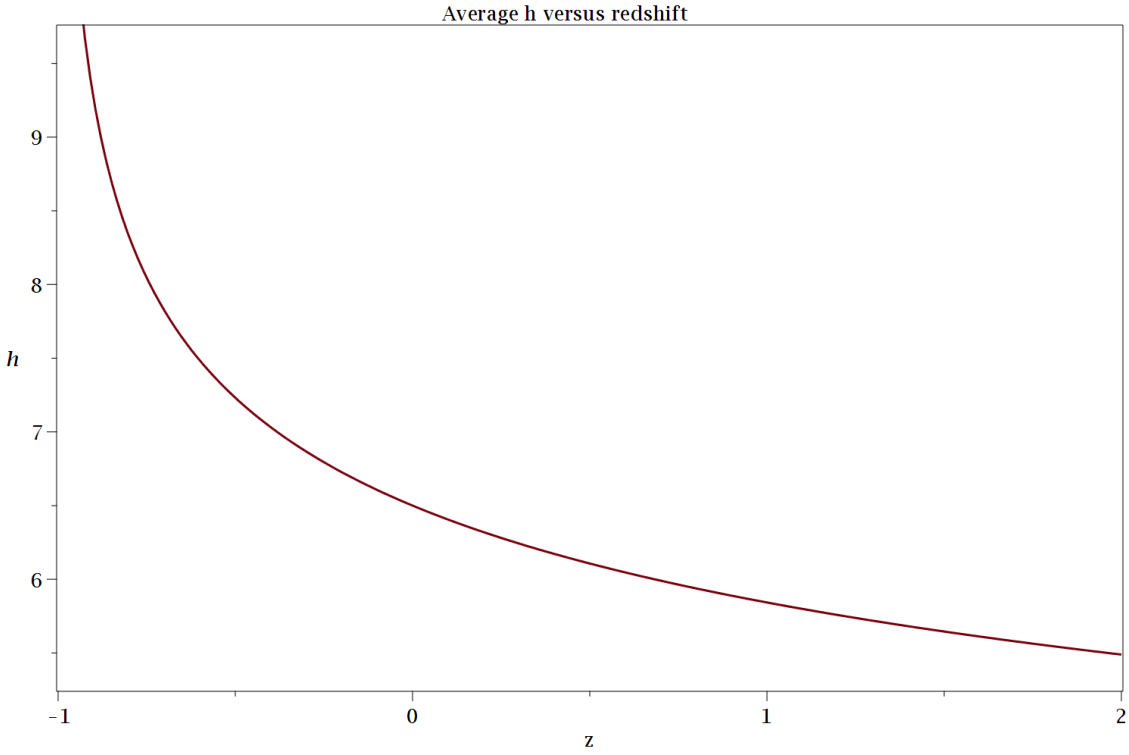


Figure 6.1: The average non-dimensional Hubble parameter for this model versus redshift, given that we assume the constants as stated below.

later), then the Universe does not end at the Big Rip singularity that will be observed. It will instead return to a dark energy-dominated era after this singularity appears. We can plot this Hubble parameter as shown in Figure 6.1.

Assuming the Hubble parameter as stated in equation (6.31), we hence write the following:

$$a = (t_0 - t)^{-h_0} = t_s^{-h_0}, \tag{6.32}$$

$$A = \frac{c_2^{1/3}}{t_s^{h_0}} e^{\frac{-c_1}{3}\frac{t_s^{3h_0+1}}{(3h_0+1)}}, \tag{6.33}$$

$$B = \frac{c_2^{-2/3}}{t_s^{h_0}} e^{\frac{2c_1}{3}\frac{t_s^{3h_0+1}}{(3h_0+1)}}, \tag{6.34}$$

where $t_s \equiv t_0 - t$. We can hence plot these two Hubble constants. They are depicted in Figure 6.2. These equations can be given in terms of redshift if we apply the relation in equations (6.32) and (2.77). We can plot the directional Hupple parameters with the use of the directional scale factors. Here, we have assumed $a = 7$, $b = -2$, $c_1 = -2.3$, $h_0 = 6.5$ and $t_0 = 5$, similar to [63].

Our deceleration parameter depends on H, and hence, if we assume the relation in equation (6.31), then we can conclude that $q = -1 - \frac{1}{h_0}$. This shows that, if h_0 is positive, then $q < -1$, and this describes a super exponential expansion of the Universe [63], which leads to the occurrence of a future Big Rip singularity.

Using the values obtained for A and B in equations (6.33) and (6.34), we can rewrite our

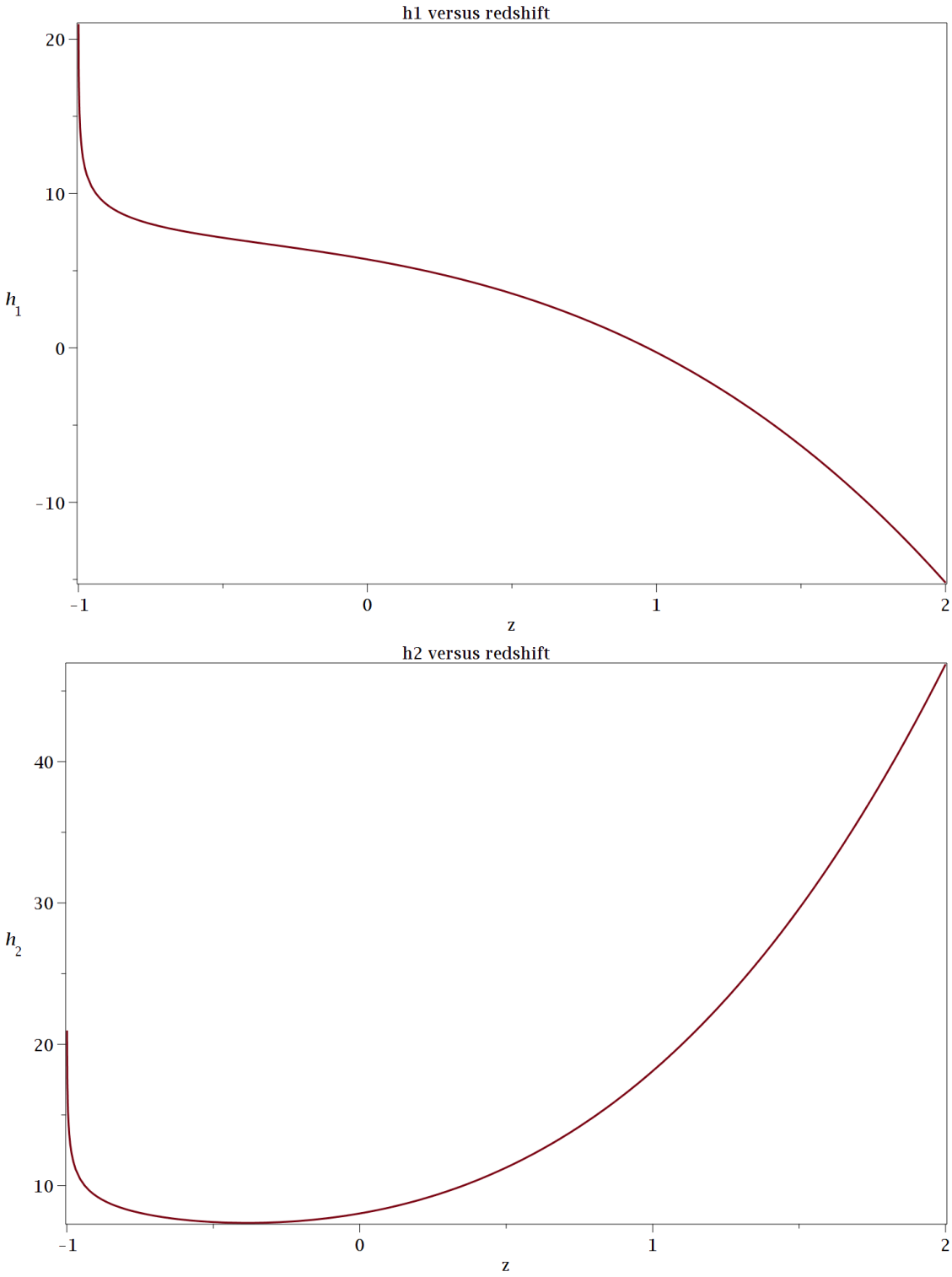


Figure 6.2: These plots show the directional non-dimensional Hubble parameters for this model versus redshift, given that we assume the constants as stated above.

Ricci scalar from equation (6.18) as

$$R = \frac{2}{3}\left[\frac{c_1^2 t_s^{2+6h_0} + 9(2h_0^2 + h_0)}{t_s^2}\right]. \tag{6.35}$$

To verify the correctness of equation (6.35), we substituted $A = a = B$ and obtained the correct Ricci scalar for an FLRW metric, and hence this confirmed our equation's validity/correctness.

This paper uses a Nojiri and Odintsov functional form of the $f(R)$ function, which is given as

$$f(R) = R - \frac{a}{R} - bR^2, \tag{6.36}$$

where a and b are real numbers. We can attempt to obtain the solutions to the energy density, ρ and pressure, p, by substituting this and the solutions for A and B into the EFE's. The derivative of our $f(R)$ function then becomes

$$f'(R) = 1 + \frac{a}{R^2} - 2bR. \tag{6.37}$$

Substituting all appropriate relations into equations (6.19) and (6.20), we obtain expressions for the energy density and pressure, respectively. These expressions can be used to plot the EoS parameter, equation (2.43), as given in Figure 6.3. Here, we have assumed $a = 7$, $b = -2$, $c_1 = -2.3$, $h_0 = 1.5$ and $t_0 = 5$, as done in [63]. We notice that our plot differs to this paper's plot. This is most probably because of the chosen spacetime signature being different, and hence causing the Ricci scalar to be different. In this figure, the EoS parameter converges to a value of 0.33 at high redshifts (or early times), which is consistent with a radiation-dominated era. At late times, or low redshifts, the value of the EoS parameter converges to -1.2, which is consistent with a dark energy-dominated era.

6.3 Background Evolution and Constraint Equations for BI in $f(R)$ Gravity

In the background for the BI model in $f(R)$ gravity the same quantities as for the GR case vanish. Thus, all quantities that disappear in Chapter 4.3 also disappear here in the matter frame. The background conservation equation is hence given as

$$\dot{\rho}^m = -\Theta(\rho^m + p^m). \tag{6.38}$$

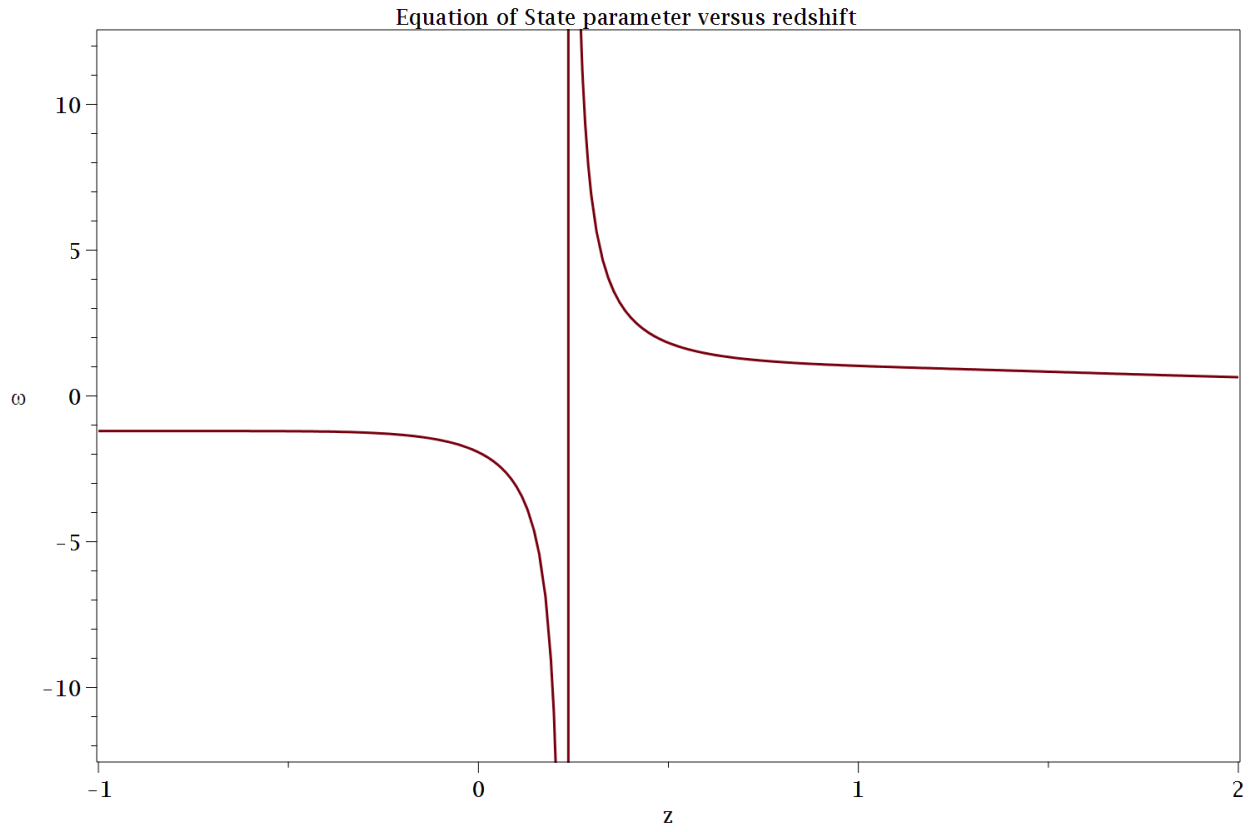


Figure 6.3: The EoS parameter versus redshift plot depicts a Big Rip singularity happening.

Our background propagation and constraint equations can then be written as

$$\dot{\Theta}+\frac{\Theta^2}{3}+2\sigma^2+\frac{1}{2}(\tilde{\rho}^m+3\tilde{p}^m)=-\frac{1}{2}(\rho^R+3p^R), \tag{6.39}$$

$$\dot{\sigma}_{\langle ab\rangle}+\frac{2\Theta}{3}\sigma_{ab}+E_{ab}+\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}=\frac{1}{2}\pi^R_{ab}, \tag{6.40}$$

$$\dot{E}_{\langle ab\rangle}+\Theta E_{ab}+\frac{1}{2}(\tilde{\rho}^m+\tilde{p}^m)\sigma_{ab}-3\sigma_{c\langle a}E_{b\rangle}{}^{c}=-\frac{1}{2}(\rho^R+p^R)\sigma_{ab} -\frac{1}{2}\dot{\pi}^R_{\langle ab\rangle}-\frac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle}-\frac{\Theta}{6}\pi^R_{ab}-\frac{1}{2}\sigma^{c}{}_{\langle a}\pi^R_{b\rangle c}, \tag{6.41}$$

$$\text{curl } E_{ab}=\frac{1}{2}\text{curl } \pi^R_{ab}+\frac{1}{2}\sigma^{c}{}_{(a}\eta_{b)c}{}^{d}q^R_d, \tag{6.42}$$

$$\tilde{\nabla}^b\sigma_{ab}=-q^R_a, \tag{6.43}$$

$$\text{curl } \sigma_{ab}=0, \tag{6.44}$$

$$\tilde{\nabla}^b E_{ab}=\frac{1}{2}\sigma^b_a q^R_b-\frac{1}{2}\tilde{\nabla}^b\pi^R_{ab}+\frac{1}{3}\tilde{\nabla}_a\rho^R-\frac{1}{3}\Theta q^R_a, \tag{6.45}$$

$$[\sigma,E]_a=-\frac{1}{2}\text{curl } q^R_a-\frac{1}{2}[\sigma,\pi^R]_a, \tag{6.46}$$

$$0=E_{ab}+\frac{1}{2}\pi^R_{ab}-\frac{1}{3}\Theta\sigma_{ab}+\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}, \tag{6.47}$$

$$\tilde{R}=2(\tilde{\rho}^m+\rho^R)-\frac{2}{3}\Theta^2+2\sigma^2. \tag{6.48}$$

6.4 Linearized Propagation and Constraint Equations

Here, we follow the same procedure as given in Chapter 4.4: those quantities that do not vanish in the background (like ρ, p, Θ, σ_{ab}, π^R_{ab}, q^R_{ab} and E_{ab}, as well as their time derivatives) are treated as zero order and those that do vanish in the background (i.e.,π^m_{ab}, $\mathcal{D}_a$, Y_a, Z_a, ω_{ab}, $\dot{u}_a$, $\tilde{R}_{ab}$, H_{ab}, and $\tilde{\nabla}_b E^{ab}$) are treated as first order. We then neglect all terms that are second order, or higher. Thus, our linearized equations can be written as

$$\dot{\Theta}+\frac{\Theta^2}{3}+2\sigma^2-\tilde{\nabla}^a\dot{u}_a+\frac{1}{2}(\tilde{\rho}^m+3\tilde{p}^m)=-\frac{1}{2}(\rho^R+3p^R), \tag{6.49}$$

$$\dot{\omega}_{\langle a\rangle}+\frac{2\Theta}{3}\omega_a+\frac{1}{2}\text{curl } \dot{u}_a-\sigma_{ab}\omega^b=0, \tag{6.50}$$

$$\dot{\sigma}_{\langle ab\rangle}+\frac{2\Theta}{3}\sigma_{ab}+E_{ab}-\tilde{\nabla}_{\langle a}\dot{u}_{b\rangle}+\sigma_{c\langle a}\sigma_{b\rangle}{}^{c}=\frac{1}{2}\pi^R_{ab}, \tag{6.51}$$

$$\dot{E}_{\langle ab\rangle}+\Theta E_{ab}-\text{curl } H_{ab}+\frac{1}{2}(\tilde{\rho}^m+\tilde{p}^m)\sigma_{ab}-3\sigma_{c\langle a}E_{b\rangle}{}^{c}+\omega^c\eta_{cd(a}E_{b)}{}^{d}=-\frac{1}{2}(\rho^R+p^R)\sigma_{ab} -\frac{1}{2}\dot{\pi}^R_{\langle ab\rangle}-\frac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle}-\frac{\Theta}{6}\pi^R_{ab}-\frac{1}{2}\sigma^{c}{}_{\langle a}\pi^R_{b\rangle c}-\frac{1}{2}\omega^c\eta^d_{c(a}\pi^R_{b)d}, \tag{6.52}$$

$$\dot{H}_{\langle ab\rangle}+\Theta H_{ab}+\text{curl } E_{ab}-3\sigma_{c\langle a}H_{b\rangle}{}^{c}+2\dot{u}^c\eta_{cd(a}E_{b)}{}^{d}=\frac{1}{2}\text{curl } \pi^R_{ab}-\frac{3}{2}\omega_{\langle a}q^R_{b\rangle} +\frac{1}{2}\sigma^{c}{}_{(a}\eta_{b)c}{}^{d}q^R_d, \tag{6.53}$$

$$\tilde{\nabla}^a\omega_a = 0, \tag{6.54}$$

$$\tilde{\nabla}^b\sigma_{ab} - \text{curl}\ \omega_a - \frac{2}{3}\tilde{\nabla}_a\Theta = -q^R_a, \tag{6.55}$$

$$\text{curl}\ \sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0, \tag{6.56}$$

$$\tilde{\nabla}^b E_{ab} - \frac{1}{3}\tilde{\nabla}_a\tilde{\rho}^m - [\sigma, H]_a = \frac{1}{2}\sigma^b_a q^R_b - \frac{3}{2}[\omega, q^R]_a - \frac{1}{2}\tilde{\nabla}^b\pi^R_{ab} + \frac{1}{3}\tilde{\nabla}_a\rho^R - \frac{1}{3}\Theta q^R_a, \tag{6.57}$$

$$\tilde{\nabla}^b H_{ab} - (\tilde{\rho}^m + \tilde{p}^m)\omega_a + [\sigma, E]_a - 3E_{ab}\omega^b = -\frac{1}{2}\text{curl}\ q^R_a + (\rho^R + p^R)\omega_a - \frac{1}{2}[\sigma, \pi^R]_a - \frac{1}{2}\pi^R_{ab}\omega^b, \tag{6.58}$$

$$\tilde{R}_{\langle ab\rangle} = E_{ab} + \frac{1}{2}\pi_{ab} - \frac{1}{3}\Theta(\sigma_{ab} + \omega_{ab}) + \sigma_{c\langle a}\sigma_{b\rangle}{}^c - 2\sigma_{c[a}\omega_{b]}{}^c, \tag{6.59}$$

$$\tilde{R} = 2(\tilde{\rho}^m + \rho^R) - \frac{2}{3}\Theta^2 + 2\sigma^2. \tag{6.60}$$

These equations can help us to derive the perturbation equations in the next section.

6.5 Linear Gradient Perturbation Equations of BI in $f(R)$ Gravity

Given that our fluid has a barotropic equation of state as defined in equation (2.43) and considering the conservation of the matter EMT, then our acceleration in the frame of matter, defined by equation (5.28), becomes

$$\dot{u}^a = -\frac{\tilde{\nabla}^a p^m}{\rho^m + p^m} = -\frac{w\tilde{\nabla}^a\rho^m}{\rho^m(1+w)} = -\frac{w\mathcal{D}^m_a}{a(1+w)}. \tag{6.61}$$

Given that $\tilde{R}$ is our three-curvature of the surface, as stated previously, then we can write the inhomogeneities in the 3-curvature scalar, equation (6.60), as

$$C_a = a\tilde{\nabla}\tilde{R}. \tag{6.62}$$

The evolution of the quantities on the RHS of this equation thus govern the inhomogeneities in our model [94]. Taking the definitions given in equation (3.15) and applying this to the above equation, we can subsequently write C_a as (full derivation shown in Appendix C.2):

$$C_a = \frac{2f''}{f'}\left[\left(\frac{f''}{f'} - \frac{f'''}{f''}\right)\Theta\dot{R} - \frac{1}{2f'}(f'R - f) + \frac{R}{2} - \frac{2\rho^m}{f'} + \tilde{\nabla}^2\right]\mathcal{R}_a - 2\left(\frac{2\Theta}{3} + \frac{f''\dot{R}}{f'}\right)Z_a + 4\sigma\Sigma_a + \frac{2\rho^m}{f'}\mathcal{D}^m_a - \frac{2f''\Theta}{f'}\mathfrak{R}_a - \frac{2f''\tilde{R}}{f'}\mathcal{R}_b. \tag{6.63}$$

To obtain the evolution equations of the quantities given in equation (3.15), it is customary to differentiate with respect to cosmic time, and then apply the commutation relations as found in Appendix D and the linearized forms of the propagation and constraint equations. We thus find the propagation equation for the matter density gradient of a vorticity-free,

perfect matter fluid with a constant equation of state w by taking the time derivative of $\mathcal{D}_a^m$. Equation (3.4) aids in finding a relation for this quantity, as shown in Appendix C.3. The evolution equation is found to be

$$\dot{\mathcal{D}}_a^m = \Theta w^m \mathcal{D}_a^m - (1+w^m)Z_a - \sigma_a^b \mathcal{D}_b^m. \tag{6.64}$$

This equation depicts how the expansion hinders the growth of the density perturbations [46].

The evolution of the comoving spatial gradient of the expansion is found in the same way as the previous evolution equation, and the derivation is shown in Appendix C.4. This expression yields

$$\begin{aligned}\dot{Z}_a \quad &= \left(\frac{f''\dot{R}}{f'} - \frac{2\Theta}{3}\right)Z_a + \frac{f''\Theta}{f'}\mathfrak{R}_a - f\sigma\Sigma_a - \frac{f''\tilde{R}}{f'}\mathcal{R}_b - \sigma_a^b Z_b \\ &\quad - \frac{w^m}{(1+w^m)}\left(\frac{f''\Theta\dot{R}}{f'} - 2\sigma^2 - \frac{\Theta^2}{3} + \frac{f}{2f'} + \frac{\rho^m}{w^m f'} + \tilde{\nabla}^2\right)D_a^m \\ &\quad + \frac{1}{f'}\left(f'''\Theta\dot{R} - \frac{(f'')^2\Theta\dot{R}}{f'a} + \frac{f'}{2} - \frac{ff''}{2f'} + \frac{f''\rho^m}{f'} - f''\tilde{\nabla}^2\right)\mathcal{R}_a. \end{aligned}\tag{6.65}$$

We found this equation by taking the time derivative of the comoving spatial gradient and applying the covariant identities (Appendix D) in addition to the trace equation (2.41) wherever needed.

The spatial variation of the shear scalar is covariantly defined and gauge-invariant since it vanishes in the background of our BI model. The evolution of our spatial variation of the shear scalar (as shown in Appendix C.5) gives

$$\begin{aligned}\dot{\Sigma}_a \quad &= -\frac{1}{2\sigma^2}\left(\frac{2\Theta}{3}\sigma^2 - \sigma^{cb}E_{cb} - \frac{w^m\sigma^{cb}}{a(1+w^m)}\tilde{\nabla}_c\mathcal{D}_b^m - \sigma^{cb}\sigma_{dc}\sigma_b{}^d + \frac{f''\sigma^{cb}}{2f'}\tilde{\nabla}_{\langle c}\tilde{\nabla}_{b\rangle}R - \frac{\sigma^2\dot{R}}{f'}\right)\Sigma_a \\ &\quad + \frac{a}{2\sigma}\tilde{\nabla}_a\left(-\sigma^{cb}E_{cb} - \frac{w^m\sigma^{cb}}{a(1+w^m)}\tilde{\nabla}_c\mathcal{D}_b^m - \sigma^{cb}\sigma_{dc}\sigma_b{}^d + \frac{f''\sigma^{cb}}{2f'}\tilde{\nabla}_{\langle c}\tilde{\nabla}_{b\rangle}R\right) \\ &\quad - \frac{\sigma Z_a}{3} + \frac{\sigma}{2f'}\mathfrak{R}_a - \frac{\sigma\dot{R}f''}{2(f')^2}\mathcal{R}_a - \frac{\sigma}{2}\left(-\frac{2\Theta}{3} + \frac{\dot{R}}{f'}\right)\frac{w^m\mathcal{D}_a^m}{(1+w^m)} - \sigma_a^c\Sigma_c. \end{aligned}\tag{6.66}$$

In this equation, we needed a definition for the time derivative of the shear scalar. We used the definition of the shear scalar given by equation (3.13) and the evolution of the shear tensor, equation (6.51) (as shown in the Appendix) to obtain the final solution.

The derivation of the evolution of $\mathcal{R}_a$ and $\mathfrak{R}_a$ yields

$$\dot{\mathcal{R}}_a = \mathfrak{R}_a - \left(\frac{w^m}{1+w^m}\right)\dot{R}\mathcal{D}_a^m - \sigma_a^b\mathcal{R}_b, \tag{6.67}$$

and

$$\begin{aligned}\dot{\mathfrak{R}}_a \quad &= \frac{f'''}{f''}\left[\left(\frac{f'''}{f''} + \frac{f''''}{f'''}\right)\dot{R}^2 + 2\Theta\dot{R} - \frac{\rho^m(1-3w^m)}{3f''} - \frac{f'R}{3f''} + \frac{2f}{3f''} - \frac{f'}{3f'''} + \frac{f''}{f'''}\left(\frac{2}{3} - \frac{R}{3} - \tilde{\nabla}^2\right)\right]\mathcal{R}_a \\ &\quad - \frac{1}{3f''}\left[\rho^m(1-3w^m) + \frac{w^m}{(1+w^m)}\left(3f'''\dot{R}^2 + 3\Theta f''\dot{R} - \rho^m(1-3w^m) - Rf' + 2f\right)\right]\mathcal{D}_a^m \\ &\quad + \tilde{R}\mathcal{R}_b + \left(\frac{2f'''\dot{R}}{f''} + \Theta\right)\mathfrak{R}_a - \sigma_a^b\mathfrak{R}_b + \dot{R}Z_a. \end{aligned}\tag{6.68}$$

These linear gradient equations together form a closed system that would be possible to find solutions to.

Chapter 7

Conclusions and Future Outlook

The Standard Cosmological Model assumes a homogeneous and isotropic universe. However, recent studies have shown that the Universe is less isotropic in the large-scale structures than we initially thought. The accelerated expansion of the Universe has also been found to be anisotropic. In order to solve this anisotropy problem, this dissertation discusses the anisotropic Bianchi type-I model that assumes an anisotropic metric that approximates to the flat FLRW case when the scale factors are equal. This dissertation also investigated the $f(R)$ gravity model as a possible solution to the dark energy problem presented. In order to model the large-scale structure evolution of the Universe, we applied perturbations to our BI metric in $f(R)$ gravity.

In Chapter 1, we reviewed the history of cosmology. Here, we introduced dark energy as a concept, and explained where the need for a cosmological constant arose, and how the cosmological constant was inserted into GR. We also gave some standard notations for this dissertation. We then moved on to discuss the Standard Model of cosmology in Chapter 2. In this chapter, we gave a summary of important concepts in GR, and we reviewed the FLRW cosmological model. The initial conditions were derived, and the currently observed values for the cosmological parameters were discussed. We also stated some of the successes of the SM, and then we mentioned a few of the problems with the SM. The chapter ended with a review of Bianchi models, cosmological perturbation theory and modified gravity.

Chapter 3 was dedicated to discussing the 1+3 covariant formalism. This formalism was used because it enables us to find the perturbation equations in a gauge-invariant way. It involves a time-space split which was enforced with the use of a spatial covariant derivative throughout the rest of the dissertation. The kinematical quantities and their effects were discussed. We stated the conservation and propagation equations for a general model in GR along with their significance, and then we introduced the spatial derivatives that enable us to find the linear gradient perturbation equations.

In Chapter 4, we introduced our BI model and gave a review of the existing literature

for this model. The BI universe was found not to be shear-free, and hence our dynamical equations contained shear terms in the background. We derived the EFE's for this model in GR, which proved to be a system of ordinary differential equations. We then found some solutions to the EFE's with the use of power-law relations. We also solved these equations numerically and plotted the solutions in Maple with the use of initial conditions. The deceleration parameter of this model was observed not to agree with that of the FLRW metric, starting at early times with an observed acceleration, and ending at later times with a decelerated expansion. We then stated the background dynamics for this model, and went on to linearize the propagation and constraint equations in GR.

The next chapter, Chapter 5, dealt with a review of $f(R)$ gravity. We first introduced the EFE's for the Einstein-Hilbert action in $f(R)$ gravity, and then discussed the EMT for this modified gravity. The viability conditions for $f(R)$ models were given, and then the propagation and constraint equations in $f(R)$ gravity were stated. After giving a brief summary of some of the possible $f(R)$ models, we also introduced some quantities that define the inhomogeneities in the Ricci scalar that would aid us in finding the perturbation equations in the following chapter.

Chapter 6 was a combination of all the previous chapters: the EFE's for BI was derived in $f(R)$ gravity, and a review of different methods for finding solutions to these equations was given. We then mentioned some applications of this model, including different forms of the metric. The background dynamics were obtained, and then used in the linearization process for the constraint and propagation equations in $f(R)$ gravity. It was observed that the shear has a big influence on the background dynamics of this model. We then used the linearized equations to derive the linear gradient perturbation equations.

The future outlook of this work includes solving the system of perturbation equations, seeing as this involves a detailed analysis with certain assumptions of the initial conditions that are currently not well understood. We could also possibly find more detailed solutions to the EFE's with the help of a dynamical systems approach, since these equations are quite difficult to solve in $f(R)$ gravity without assuming a power-law relation.

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Appendix A

Dimensionless EFE's for BI in GR

This Appendix describes the process of making the EFE's given by equations (4.17) to (4.19) dimensionless. The final, non-dimensional equations are given in Chapter 4.1.

If ρ^m and p^m are for the matter fluid, we can substitute $\rho^m = \rho_{0_d}(ABC)^{-1} + \rho_{0_r}(ABC)^{-4/3}$ and $p^m = \frac{1}{3}\rho_{0_r}(ABC)^{-4/3}$ seeing as $a^3 = ABC$. This leads to

$$\begin{aligned} H_1H_2 + H_1H_3 + H_2H_3 &= \rho_{0_d}(ABC)^{-1} + \rho_{0_r}(ABC)^{-4/3}, \\ H_2^2 + H_3^2 + H_2H_3 + \dot{H}_2 + \dot{H}_3 &= -\frac{1}{3}\rho_{0_r}(ABC)^{-4/3}, \\ H_1^2 + H_3^2 + H_1H_3 + \dot{H}_1 + \dot{H}_3 &= -\frac{1}{3}\rho_{0_r}(ABC)^{-4/3}, \\ H_2^2 + H_1^2 + H_2H_1 + \dot{H}_2 + \dot{H}_1 &= -\frac{1}{3}\rho_{0_r}(ABC)^{-4/3}, \end{aligned} \tag{A.1}$$

We can also change the differentiation with respect to time to differentiation with respect to redhsift using the fact that $H = (da/dt)/a \Rightarrow da/dt = aH$, and $z = \frac{1}{a} - 1 \Rightarrow \frac{dz}{da} = -\frac{1}{a^2}$:

$$\dot{H} = -(1+z)HH', \tag{A.2}$$

where a prime denotes differentiation with respect to z. We can also do the same for the directional Hubble parameters to obtain

$$\begin{aligned} \dot{H}_1 &= -(1+z)HH_1', \\ \dot{H}_2 &= -(1+z)HH_2', \\ \dot{H}_3 &= -(1+z)HH_3'. \end{aligned} \tag{A.3}$$

Substituting these equations into the EFE's, we obtain

$$\begin{aligned} &H_1H_2 + H_1H_3 + H_2H_3 = \rho_{0_d}(ABC)^{-1} + \rho_{0_r}(ABC)^{-4/3}, \\ &H_2^2 + H_3^2 + H_2H_3 - (1+z)HH_2' - (1+z)HH_3' = -\frac{1}{3}\rho_{0_r}(ABC)^{-4/3}, \\ &H_1^2 + H_3^2 + H_1H_3 - (1+z)HH_1' - (1+z)HH_3' = -\frac{1}{3}\rho_{0_r}(ABC)^{-4/3}, \\ &H_2^2 + H_1^2 + H_2H_1 - (1+z)HH_2' - (1+z)HH_1' = -\frac{1}{3}\rho_{0_r}(ABC)^{-4/3}. \end{aligned} \tag{A.4}$$

Given that the average scale factor and the redshift are related by equation (2.77), and the average scale factor is related to the directional scale factors by $a = \sqrt[3]{ABC}$, we can write

$$ABC = (1+z)^{-3}\,. \tag{A.5}$$

We can make the EFE's dimensionless by dividing throughout with $3H_0^2$. If we then also substitute $\frac{\rho_{0_d}}{3H_0^2} = \Omega_{d0}$, $\frac{\rho_{0_r}}{3H_0^2} = \Omega_{r0}$, and the non-dimensional Hubble parameters, $h = \frac{H}{H_0}$, $h_1 = \frac{H_1}{H_0}$, $h_2 = \frac{H_2}{H_0}$ and $h_3 = \frac{H_3}{H_0}$, and change the scale factors to redshift with the use of equation (A.5), then we are left with the non-dimensional EFE's for the BI model in GR:

$$\begin{aligned}
h_1h_2 + h_1h_3 + h_2h_3 &= 3\Omega_{d0}(1+z)^3 + 3\Omega_{r0}(1+z)^4,\\
h_2^2 + h_3^2 + h_2h_3 - (1+z)hh_2' - (1+z)hh_3' &= -\Omega_{r0}(1+z)^4,\\
h_1^2 + h_3^2 + h_1h_3 - (1+z)hh_1' - (1+z)hh_3' &= -\Omega_{r0}(1+z)^4,\\
h_2^2 + h_1^2 + h_2h_1 - (1+z)hh_2' - (1+z)hh_1' &= -\Omega_{r0}(1+z)^4.
\end{aligned} \tag{A.6}$$

Appendix B

BI in $f(R)$

B.1 Derivations of the EFE's

This Appendix describes the process of finding the EFE's for BI in $f(R)$ gravity. The non-zero components of the Einstein-tensor is given as:

$$\begin{aligned} G_{tt} &= \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C}, \\ G_{xx} &= A^2\left(-\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}}{B}\frac{\dot{C}}{C}\right), \\ G_{yy} &= B^2\left(-\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\frac{\dot{C}}{C}\right), \\ G_{zz} &= C^2\left(-\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} - \frac{\dot{B}}{B}\frac{\dot{A}}{A}\right). \end{aligned} \tag{B.1}$$

The matter energy-momentum tensor, $\tilde{T}^m_{ab}$, can be written as

$$\begin{aligned} \tilde{T}^m_{tt} &= \frac{1}{f'}\rho^m, \\ \tilde{T}^m_{xx} &= \frac{A^2}{f'}p^m, \\ \tilde{T}^m_{yy} &= \frac{B^2}{f'}p^m, \\ \tilde{T}^m_{zz} &= \frac{C^2}{f'}p^m. \end{aligned} \tag{B.2}$$

The covariant derivative of a scalar, S, can be found with the use of

$$\nabla_a\nabla_b S = \partial_a\partial_b S - \Gamma^c_{ab}\partial_c S. \tag{B.3}$$

The derivatives of the function $f(R)$ can be written as

$$\begin{aligned} \dot{f}' &= f''\dot{R}, \\ \ddot{f}' &= f'''\dot{R}^2 + f''\ddot{R}, \end{aligned} \tag{B.4}$$

and thus we can write the second covariant derivatives as

$$\begin{aligned}\nabla_t\nabla_t f' &= f'''\dot{R}^2 + f''\ddot{R},\\ \nabla_x\nabla_x f' &= -f''\dot{R}A\dot{A},\\ \nabla_y\nabla_y f' &= -f''\dot{R}B\dot{B},\\ \nabla_z\nabla_z f' &= -f''\dot{R}C\dot{C}.\end{aligned} \tag{B.5}$$

The D'Alembertian is found to yield

$$\Box f' = -f'''\dot{R}^2 - f''\ddot{R} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R}. \tag{B.6}$$

The curvature energy-momentum tensor, T^R_{ab}, for each component is found to be

$$\begin{aligned}T^R_{tt} &= -\frac{1}{f'}\left[\frac{1}{2}(f - R\ f') + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R}\right],\\ T^R_{xx} &= \frac{A^2}{f'}\left[\frac{1}{2}(f - R\ f') - f''\dot{R}\frac{\dot{A}}{A} + f'''\dot{R}^2 + f''\ddot{R} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R}\right],\\ T^R_{yy} &= \frac{B^2}{f'}\left[\frac{1}{2}(f - R\ f') - f''\dot{R}\frac{\dot{B}}{B} + f'''\dot{R}^2 + f''\ddot{R} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R}\right],\\ T^R_{zz} &= \frac{C^2}{f'}\left[\frac{1}{2}(f - R\ f') - f''\dot{R}\frac{\dot{C}}{C} + f'''\dot{R}^2 + f''\ddot{R} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R}\right].\end{aligned} \tag{B.7}$$

Combining all of these expressions with the use of equation (5.5) and simplifying, we arrive at the EFE's for the BI model in f(R) gravity

$$\begin{aligned}&-\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) f' + \frac{1}{2}f + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R} = \rho^m,\\ &-\left[\frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] f' - \frac{1}{2}f - f'''\dot{R}^2 - f''\ddot{R} - \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) f''\dot{R} = -p^m,\\ &-\left[\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\right] f' - \frac{1}{2}f - f'''\dot{R}^2 - f''\ddot{R} - \left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) f''\dot{R} = -p^m,\\ &-\left[\frac{\ddot{C}}{C} + \frac{\dot{C}}{C}\left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right)\right] f' - \frac{1}{2}f - f'''\dot{R}^2 - f''\ddot{R} - \left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right) f''\dot{R} = -p^m.\end{aligned} \tag{B.8}$$

These used in Chapter 6.

B.2 The Trace Equation

The trace equation, equation (2.41), yields

$$3(f'''\dot{R}^2 + f''\ddot{R}) + 3\Theta f''\dot{R} - 3\nabla^2 f' = f'R - 2f + \rho^m - 3p^m. \tag{B.9}$$

Making $\ddot{R}$ the subject of the formula, we obtain

$$\ddot{R} = \frac{1}{3f''}\left(-3f'''\dot{R}^2 - 3\Theta f''\dot{R} + 3f''\tilde{\nabla}^2 R + \rho^m - 3p^m + Rf' - 2f\right). \tag{B.10}$$

This equation is used throughout Chapter 6.

Appendix C

Perturbation Variables

C.1 Some Spatial Gradients

In this section, we find some different spatial gradients for use in the following sections.

The spatial gradient of $\tilde{\rho}^m$ is given as

$$\begin{aligned}\tilde{\nabla}_a\tilde{\rho}^m &= \tilde{\nabla}_a\left(\frac{\rho^m}{f'}\right),\\ &= \frac{\rho^m}{af'}\mathcal{D}^m_a - \frac{\rho^m f''}{a(f')^2}\mathcal{R}_a.\end{aligned} \tag{C.1}$$

Given equations (D.7), (D.8) and (B.10), we can write

$$\rho^R + 3p^R = \frac{1}{f'}\bigg(-2f''\Theta\dot{R} + 2f''\tilde{\nabla}^2 R + (1+3w^m)\rho^m - f\bigg). \tag{C.2}$$

We can subsequently find the spatial derivative of this quantity by using equation (D.3) where necessary :

$$\begin{aligned}\tilde{\nabla}_a(\rho^R + 3p^R) = \frac{1}{f'a}\bigg[&-\frac{f''}{(f')^2}\bigg(-2f''\Theta\dot{R} + 2f''\tilde{\nabla}^2 R + (1-3w^m)\rho^m - f\bigg) - 2f'''\Theta\dot{R} - 1\\ &+ 2f''\tilde{\nabla}^2\bigg]\mathcal{R}_a + \frac{(1+3w^m)\rho^m}{f'a}\mathcal{D}^m_a - \frac{2f''\dot{R}}{f'a}Z_a - \frac{2f''\Theta}{f'a}\mathfrak{R}_a - \frac{2f''\tilde{R}}{f'a}\mathcal{R}_b.\end{aligned} \tag{C.3}$$

We can also find the spatial gradient of the curvature energy density:

$$\begin{aligned}\tilde{\nabla}_a\rho^R = \frac{1}{f'a}\bigg[&-\frac{f''}{f'}\bigg(\frac{1}{2}(f'R - f) - f''\Theta\dot{R} + f''\tilde{\nabla}^2 R\bigg) + \frac{f''R}{2} - f'''\Theta\dot{R} + f''\tilde{\nabla}^2\bigg]\mathcal{R}_a\\ &- \frac{f''\dot{R}}{f'a}Z_a - \frac{f''\Theta}{f'a}\mathfrak{R}_a - \frac{f''\tilde{R}}{f'a}\mathcal{R}_b.\end{aligned} \tag{C.4}$$

Given that

$$\dot{\Theta} = \frac{f''\Theta\dot{R}}{f'} - 2\sigma^2 + \tilde{\nabla}^a\dot{u}_a - \frac{\Theta^2}{3} - \frac{f''}{f'}\tilde{\nabla}^2 R + \frac{f}{2f'} - \frac{\rho^m}{f'}, \tag{C.5}$$

we can find $(\tilde{\nabla}_a\Theta)^{\cdot}$:

$$\begin{aligned}(\tilde{\nabla}_a\Theta)^{\cdot} &= \tilde{\nabla}_a\dot{\Theta} - \frac{1}{3}\Theta\tilde{\nabla}_a\Theta + \dot{\Theta}\dot{u}_a - \sigma_a^b\tilde{\nabla}_b\Theta,\\ &= \frac{1}{a}\left(\frac{f''\dot{R}}{f'} - \Theta\right)Z_a + \frac{f''\Theta}{f'a}\mathfrak{R}_a - \frac{f\sigma}{a}\Sigma_a - \frac{f''\tilde{R}}{f'a}\mathcal{R}_b - \frac{\sigma_a^b}{a}Z_b\\ &\quad + \frac{1}{f'a}\left(f'''\Theta\dot{R} - \frac{(f'')^2\Theta\dot{R}}{f'a} + \frac{f'}{2} - \frac{ff''}{2f'} + \frac{f''\rho^m}{f'} - f''\tilde{\nabla}^2\right)\mathcal{R}_a\\ &\quad - \frac{w^m}{a(1+w^m)}\left(\frac{f''\Theta\dot{R}}{f'} - 2\sigma^2 - \frac{\Theta^2}{3} + \frac{f}{2f'} - \frac{\rho^m}{f'} + \frac{(1+w^m)\rho^m}{w^m f'} + \tilde{\nabla}^2\right)D_a^m.\end{aligned} \tag{C.6}$$

C.2 Inhomogeneities in the 3-Curvature Scalar: C_a

From equation (6.60), we can write C_a as

$$\begin{aligned}C_a &= a\tilde{\nabla}_a\tilde{R},\\ &= \frac{2f''}{f'}\left[\left(\frac{f''}{f'} - \frac{f'''}{f''}\right)\Theta\dot{R} - \frac{1}{2f'}(f'R - f) + \frac{R}{2} - \frac{2\rho^m}{f'} + \tilde{\nabla}^2\right]\mathcal{R}_a - 2\left(\frac{2\Theta}{3} + \frac{f''\dot{R}}{f'}\right)Z_a\\ &\quad + 4\sigma\Sigma_a + \frac{2\rho^m}{f'}\mathcal{D}_a^m - \frac{2f''\Theta}{f'}\mathfrak{R}_a - \frac{2f''\tilde{R}}{f'}\mathcal{R}_b.\end{aligned} \tag{C.7}$$

C.3 Evolution of Comoving Fractional Density: $\dot{\mathcal{D}}_a^m$

We find the expression for the evolution of the comoving fractional density as

$$\begin{aligned}\dot{\mathcal{D}}_a^m &= \left(\frac{a}{\rho^m}\tilde{\nabla}_a\rho^m\right)^{\cdot},\\ &= \Theta w^m\mathcal{D}_a^m - (1+w^m)Z_a - \sigma_a^b\mathcal{D}_b^m.\end{aligned} \tag{C.8}$$

C.4 Evolution of Expansion Gradient: $\dot{Z}_a$

We can substitute equation (C.6) into the equation for $\dot{Z}_a$:

$$\dot{Z}_a = \left(a\tilde{\nabla}_a\Theta\right)^{\cdot}, \tag{C.9}$$

$$\begin{aligned}&= \left(\frac{f''\dot{R}}{f'} - \frac{2\Theta}{3}\right)Z_a + \frac{f''\Theta}{f'}\mathfrak{R}_a - f\sigma\Sigma_a - \frac{f''\tilde{R}}{f'}\mathcal{R}_b - \sigma_a^b Z_b\\ &\quad - \frac{w^m}{(1+w^m)}\left(\frac{f''\Theta\dot{R}}{f'} - 2\sigma^2 - \frac{\Theta^2}{3} + \frac{f}{2f'} - \frac{\rho^m}{f'} + \frac{(1+w^m)\rho^m}{w^m f'} + \tilde{\nabla}^2\right)D_a^m\\ &\quad + \frac{1}{f'}\left(f'''\Theta\dot{R} - \frac{(f'')^2\Theta\dot{R}}{f'a} + \frac{f'}{2} - \frac{ff''}{2f'} + \frac{f''\rho^m}{f'} - f''\tilde{\nabla}^2\right)\mathcal{R}_a.\end{aligned} \tag{C.10}$$

C.5 Evolution of the Spatial Gradient of the Shear Scalar

Given the evolution equation, equation (6.51), and the definition of the shear scalar, equation (3.13), we can obtain a definition for the time derivative of the shear scalar:

$$\begin{aligned}\dot{\sigma} &= \frac{1}{2\sigma}\sigma^{cb}\dot{\sigma}_{cb},\\ &= \frac{1}{2\sigma}\bigg(-\frac{2\Theta}{3}\sigma^2 - \sigma^{cb}E_{cb} + \sigma^{cb}\tilde{\nabla}_c\dot{u}_b - \sigma^{cb}\sigma_{dc}\sigma_b{}^d + \frac{f''\sigma^{cb}}{2f'}\tilde{\nabla}_{\langle c}\tilde{\nabla}_{b\rangle}R + \frac{\sigma^2\dot{R}}{f'}\bigg).\end{aligned} \tag{C.11}$$

Using equations (D.1) and (C.11), we can find the evolution of the spatial gradient of the shear scalar:

$$\begin{aligned}\dot{\Sigma}_a &= (a\tilde{\nabla}_a\sigma)^{\cdot},\\ &= -\frac{1}{2\sigma^2}\bigg(\frac{2\Theta}{3}\sigma^2 - \sigma^{cb}E_{cb} - \frac{w^m\sigma^{cb}}{a(1+w^m)}\tilde{\nabla}_c\mathcal{D}^m_b - \sigma^{cb}\sigma_{dc}\sigma_b{}^d + \frac{f''\sigma^{cb}}{2f'}\tilde{\nabla}_{\langle c}\tilde{\nabla}_{b\rangle}R - \frac{\sigma^2\dot{R}}{f'}\bigg)\Sigma_a\\ &\quad - \frac{\sigma Z_a}{3} + \frac{\sigma}{2f'}\mathfrak{R}_a - \frac{\sigma\dot{R}f''}{2(f')^2}\mathcal{R}_a - \frac{\sigma}{2}\bigg(-\frac{2\Theta}{3} + \frac{\dot{R}}{f'}\bigg)\frac{w^m\mathcal{D}^m_a}{(1+w^m)} - \sigma^c_a\Sigma_c\\ &\quad + \frac{a}{2\sigma}\tilde{\nabla}_a\bigg(-\sigma^{cb}E_{cb} - \frac{w^m\sigma^{cb}}{a(1+w^m)}\tilde{\nabla}_c\mathcal{D}^m_b - \sigma^{cb}\sigma_{dc}\sigma_b{}^d + \frac{f''\sigma^{cb}}{2f'}\tilde{\nabla}_{\langle c}\tilde{\nabla}_{b\rangle}R\bigg).\end{aligned} \tag{C.12}$$

C.6 Evolution of $\mathcal{R}_a$

We can find $\dot{\mathcal{R}}_a$:

$$\dot{\mathcal{R}}_a = \Big(a\tilde{\nabla}_a R\Big)^{\cdot}, \tag{C.13}$$

$$= \mathfrak{R}_a - \bigg(\frac{w^m}{1+w^m}\bigg)\dot{R}\mathcal{D}^m_a - \sigma^b_a\mathcal{R}_b. \tag{C.14}$$

C.7 Evolution of $\mathfrak{R}_a$

We can find $\dot{\mathfrak{R}}_a$:

$$\begin{aligned}\dot{\mathfrak{R}}_a &= \Big(a\tilde{\nabla}_a\dot{R}\Big)^{\cdot},\\ &= \bigg(\frac{2f'''\dot{R}}{f''} + \Theta\bigg)\mathfrak{R}_a + \dot{R}Z_a + \tilde{R}\mathcal{R}_b - \sigma^b_a\mathfrak{R}_b + \frac{f'''}{f''}\bigg[\frac{f'''\dot{R}^2}{f''} + \Theta\dot{R} - \frac{\rho^m(1-3w^m)}{3f''}\\ &\quad - \frac{f'R}{3f''} + \frac{2f}{3f''} + \frac{\dot{R}^2f''''}{f'''} + \Theta\dot{R} - \frac{f'}{3f'''} - \frac{f''R}{3f'''} + \frac{2f''}{3f'''} - \frac{f''}{f'''}\tilde{\nabla}^2\bigg]\mathcal{R}_a - \frac{1}{3f''}\bigg[\rho^m(1-3w^m)\\ &\quad - \Big(-3f'''\dot{R}^2 - 3\Theta f''\dot{R} + \rho^m(1-3w^m) + Rf' - 2f\Big)\frac{w^m}{(1+w^m)}\bigg]\mathcal{D}^m_a.\end{aligned} \tag{C.15}$$

Appendix D

Useful Relations

D.1 Covariant Identities

Identities for covariant spatial derivatives for any scalar f are found in [29, 57] and given as:

$$(\tilde{\nabla}_a f)^{\cdot} = \tilde{\nabla}_a \dot{f} - \frac{1}{3}\Theta\tilde{\nabla}_a f + \dot{f}\dot{u}_a - \sigma_a^b\tilde{\nabla}_b f, \tag{D.1}$$

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]} f = 0, \tag{D.2}$$

$$\tilde{\nabla}_a(\tilde{\nabla}^2 f) = \tilde{\nabla}^2(\tilde{\nabla}_a f) - \tilde{R}\tilde{\nabla}_b f. \tag{D.3}$$

The spatial curl of a vector V_a and a tensor S_{ab} is given as

$$\text{curl } V_a = \eta_{abc}\tilde{\nabla}^b V^c, \quad \text{curl } S_{ab} = \eta_{cd(a}\tilde{\nabla}^c S_{b)}{}^d. \tag{D.4}$$

D.2 Useful Relations in $f(R)$ Gravity

Some relations for the derivatives of $f(R)$:

$$\begin{aligned}
&\frac{df'(R)}{dx} = f''(R)\frac{dR}{dx}, \\
&\frac{d^2 f'(R)}{dx^2} = \frac{d}{dx}\left(f''(R)\frac{dR}{dx}\right) = f'''(R)\left(\frac{dR}{dx}\right)^2 + f''(R)\frac{d^2R}{dx^2},
\end{aligned} \tag{D.5}$$

where x represents any general variable. The spatial covariant derivative of a function $f \equiv f(R)$ can thus be written as

$$\begin{aligned}
\tilde{\nabla}_a f &= f'\tilde{\nabla}_a R = \frac{f'}{a}\mathcal{R}_a, \\
\tilde{\nabla}_a f' &= f''\tilde{\nabla}_a R = \frac{f''}{a}\mathcal{R}_a, \\
\tilde{\nabla}_a f'' &= f'''\tilde{\nabla}_a R = \frac{f'''}{a}\mathcal{R}_a, \\
\tilde{\nabla}^2 f' &= f''\tilde{\nabla}^2 R.
\end{aligned} \tag{D.6}$$

Other useful relations in $f(R)$ gravity are given by

$$\rho^R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right], \tag{D.7}$$

$$p^R = \frac{1}{f'}\Bigg[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\big(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\big)\Bigg], \tag{D.8}$$

$$\rho = \frac{\rho^m}{f'} + \rho^R, \tag{D.9}$$

$$p = \frac{p^m}{f'} + p^R, \tag{D.10}$$

$$R = \rho - 3p = \frac{\rho^m}{f'} - 3\frac{p^m}{f'} + \rho^R - 3p^R, \tag{D.11}$$

$$3(f'''\dot{R}^2 + f''\ddot{R}) + 3\Theta f''\dot{R} - 3\nabla^2 f' = f'R - 2f + \rho^m - 3p^m. \tag{D.12}$$