

Portfolio optimisation under the tracking error constraint

MJ Maxwell

 **orcid.org/000-0002-3372-8191**

Dissertation submitted in fulfilment of the requirements for
the degree [Master of Commerce](#) in [Risk Management](#) at
the North-West University

Supervisor: Prof GW van Vuuren

Co-Supervisor: Prof PMS van Heerden

Graduation ceremony: October 2019

Student number: 30034078

Preface

This dissertation has been assembled and completed under the article format.

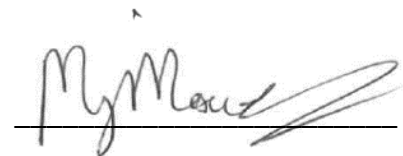
The theoretical work described in this dissertation was carried out whilst in the employ of Investec (Cape Town, South Africa) and Bloomberg (London, UK). Some theoretical and practical work was carried out in collaboration with the Department of Risk Management, School of Economics, North-West University (South Africa) under the supervision of Prof Gary van Vuuren.

These studies represent the original work of the author and have not been submitted in any form to another university. Where use was made of the work of others, this has been duly acknowledged in the text.

Unless otherwise stated, all data were obtained from Bloomberg™, non-proprietary internet sources, and non-proprietary financial databases of Investec, Cape Town, South Africa. Discussions with personnel from this institution also provided invaluable insight into current investment trends and challenges faced in the investment risk and portfolio management arena.

The results associated with the work presented in Chapter 3 (Tracking error-constrained portfolios) has been published in *Applied Economics*. The work described in Chapter 4 (Optimal portfolios on the tracking error frontier) has been published in *Journal of International Advances in Economic Research* (August 2019). Both articles have been published under an open access agreement; no copyright is required.

The results obtained from these articles and the contributions they make to the existing body of knowledge are summarised in Chapter 5 which also assesses future research opportunities.

A handwritten signature in dark ink, appearing to read 'Michael Maxwell', is written over a horizontal line.

MICHAEL MAXWELL

03 October 2019

Acknowledgements

I acknowledge an enormous debt of gratitude to everyone who has contributed in some way or other to the completion of this dissertation.

In particular, I would like to thank:

- My supervisor, Gary van Vuuren, for the endless encouragement, monitoring and selfless sacrifice of his time and resources over this last year, and
- Investec, for their interest, conversations and provision of resources invaluable to this research.

Abstract

Active portfolio managers are judged on their ability to outperform agent's benchmarks, hence optimising fund returns is critically important. Maximising fund outperformance is, however, non-trivial because active portfolios are subject to tracking error (TE) (and other) constraints. Portfolios constrained by a TE are fenced by an elliptical frontier in mean/variance space and may not be efficient. The ellipse's flat shape suggests an additional constraint which improves the performance of the active portfolio. Some attempts have been made to identify optimal portfolios subject to the restrictions imposed by TEs, i.e. to locate these on the frontier. Although subsequent work isolated and explored different portfolios subject to these constraints, absolute portfolio risk has been consistently ignored.

First a different restriction – maximisation of the traditional Sharpe ratio on the constant tracking error frontier in absolute risk/return space – is added to the existing constraint set, and a method to generate this portfolio is explained. The resultant portfolio has a lower volatility and higher return than the benchmark, it satisfies the tracking error constraint *and* the ratio of excess absolute return to risk is maximised (i.e. maximum Sharpe ratio in absolute space). Second, we review these portfolio assemblies and introduce more possibilities using the previously derived method: portfolios which are maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum risk. Such portfolios behave differently to those which are part of the efficient set, i.e. populate the efficient frontier and are TE-unconstrained, giving managers constrained by TEs portfolio selection options based on risk preferences and/or investment strategies.

Key words: Active management, tracking error, benchmarks, portfolio asset allocation.

Table of contents

Preface	1
Acknowledgements.....	2
Abstract.....	3
Table of contents	4
List of figures.....	6
List of tables	7
Chapter 1: Introduction	8
1.1 Background	8
1.2 Problem statement	9
1.3 Research question.....	10
1.4 Study motivation.....	10
1.5 Dissertation structure	10
1.6 General objective	12
1.7 Specific objectives.....	12
1.8 Research design	12
1.8.1 Literature review	14
1.8.2 Data.....	14
1.8.3 Research output	14
1.9 Conclusion	15
Chapter 2: Literature study	16
2.1 Absolute portfolio optimisation.....	16
2.2 Relative portfolio optimisation	19
2.3 Active investment strategies under the tracking error constraint	25
Chapter 3: Optimising tracking error-constrained portfolios.....	27
3.1 Introduction	27
3.2 Literature review.....	29
3.3 Data and methodology.....	34
3.3.1 Data.....	34

3.3.2	<i>Methodology</i>	34
3.4	Results and discussion.....	38
3.4.1	<i>CML line and maximum Sharpe portfolio</i>	39
3.4.2	<i>Vector of portfolio deviations from benchmark: x</i>	41
3.4.3	<i>Weights</i>	43
3.4.4	<i>Sharpe ratio portfolio and Jorion's (2003) proposal</i>	44
3.4.5	<i>Maximum Sharpe ratio portfolio locus</i>	44
3.5	Conclusions	45
Chapter 4:	Active investment strategies under tracking error constraints	48
4.1	Introduction	48
4.2	Literature survey	50
4.3	Data and methodology.....	54
4.3.1	<i>Data</i>	54
4.3.2	<i>Methodology</i>	55
4.4	Results and discussion.....	64
4.4.1	<i>Maximum return</i>	66
4.4.2	<i>Maximum Sharpe ratio (risk-adjusted return)</i>	66
4.4.3	<i>Jorion benchmark risk</i>	67
4.4.4	<i>Minimum variance</i>	67
4.4.5	<i>Minimum intra-correlation</i>	67
4.4.6	<i>Maximum diversification</i>	68
4.4.7	<i>Risk parity (inverse volatility)</i>	68
4.5	Conclusions and suggestions	69
Chapter 5:	Conclusions and suggestions for future research	72
5.1	Summary and conclusions	72
5.1.1	<i>Paper 1: Optimising tracking error-constrained portfolios</i>	73
5.1.2	<i>Paper 2: Active investment strategies under TE constraints</i>	74
5.2	Suggestions for future research	75
Bibliography		76
Appendix – collaboration affidavit		79

List of figures

Chapter 2: Literature study

Figure 2.1. TE frontier and TE-constrained portfolio	22
Figure 2.2. TE frontier, TE-constrained portfolio and constant TE frontier (with TE =5%)	23

Chapter 3: Optimising tracking error-constrained portfolios

Figure 3.1. TE frontier and TE-constrained portfolio	31
Figure 3.2. TE frontier, TE-constrained portfolio and constant TE frontier (with TE =5%)	32
Figure 3.3. T Constant TE frontier (TE =5%) and intersection with CML for increasing values of σ_P from (a) through (d).....	40
Figure 3.4. (a) TE-constrained portfolio, constant TE frontier and CML with optimal portfolio and (b) enlarged view showing all three portfolios.....	41
Figure 3.5. Deviations from the benchmark determined using (7) for the three portfolios: maximum return on TE frontier, position on TE frontier where $\sigma_P = \sigma_B$ (Jorion, 2003) and position on TE frontier corresponding to a maximum Sharpe ratio	43
Figure 3.6. Locus of maximum Sharpe ratio portfolios for $2\% \leq TE \leq 10\%$ ($r_f=2\%$). ...	44
Figure 3.7. Locus of maximum Sharpe ratio portfolios for $2\% \leq r_f \leq 20\%$ (TE=5%)	45

Chapter 4: Active investment strategies under tracking error constraints

Figure 4.1. Efficient frontier, TE frontier and constant TE frontier in mean/standard deviation space.	59
Figure 4.2. TE frontier, constant TE frontier and CML (associated with the constant TE frontier) in mean/standard deviation space.....	60
Figure 4.3. Intra-correlation and diversification ratio subject to a TE=7% using (20) and (18) respectively.....	64
Figure 4.4. TE frontier, constant TE frontier and positions of optimised portfolios on constant TE frontier	65
Figure 4.5. Loci of relevant portfolios in mean/risk space for $1\% \leq TE \leq 12\%$	66
Figure 4.6. Positions of global minimum intra-correlation, maximum diversification and inverse volatility portfolios in mean/risk space (on the same scale as that given in Figure 14).	69

List of tables

Chapter 1: Introduction

Table 1.1: Data requirements, frequency and source.	15
--	----

Table 1.2: Research output.	15
----------------------------------	----

Chapter 3: Optimising tracking error-constrained portfolios

Table 3.1: Change in μ_P and σ_P for various levels of Δ_1 (constraints: TE prescribed, $\sigma_P = \sigma_B$) and $r_f = 2.0\%$	38
--	----

Table 3.2: Change in μ_P and σ_P for various levels of Δ_1 (constraints: TE prescribed, $SR = SR_{MAX}$) and $r_f = 2.0\%$	42
--	----

Table 3.3: The effect of increasing r_f on μ and σ for the TE-constrained and maximum Sharpe portfolios	42
---	----

Chapter 4: Active investment strategies under tracking error constraints

Table 6: Stylised input data.	54
------------------------------------	----

Chapter 1

Introduction

1.1 Background

Tracking error (TE) is an active risk measure that reflects a portfolio manager's decision to deviate from weights of benchmark positions to achieve returns in excess of the benchmark. This deviation introduces a measure of benchmark risk – called TE – defined as the standard deviation of the difference between portfolio and benchmark returns. Generally, TE is not a metric used in isolation when determining manager performance and is best used in combination with other performance evaluators such as the information ratio, Value at Risk, etc. The optimal level of TE is dependent on the portfolio's investment policy, i.e. outperformance target, risk-return profile and growth strategy (value or growth). Thomas, Rottschäfer and Zvingelis (2013) outline several causes of TE (fees, transaction costs, taxes, factor tilts, cash management and general market volatility), but a principal objective that research has aimed to solve is optimisation: maximising excess returns while maintaining a prescribed TE.

Thomas *et al.*, (2013) noted that TE does not provide a directional metric since it only measures excess return volatility, thus it provides no information about the direction of the return differences (i.e. whether they are positive or negative). Portfolio managers generally pursue low TEs and positive excess returns, because although higher TEs may indeed translate into higher returns, this is not always the case.

Standard portfolio construction allows portfolio managers to select assets and determine their exposures (weights) based on factor tilts, volatilities, expected returns and correlations with one another. This information provides the basis for the calculation of an expected TE relative to the benchmark and it is this forecast which enables portfolio managers to remain within the TE bounds, set by investors, rebalancing (if necessary) to prevent punitive charges from mandate violations.

Roll (1992) argued that the dual goals of benchmark return outperformance and TE reduction were equivalent to standard Markowitz mean-variance objectives with subtle alterations. Instead of identifying a Markowitz-efficient portfolio (i.e. a portfolio with the smallest total re-

turn volatility for a given expected total return), fund managers were obliged to isolate a portfolio with a minimum TE for a given performance relative to the benchmark. Roll (1992) labelled this the "tracking error criterion": minimisation of TE for given expected benchmark outperformance, and formalised this criterion along with its consequences for portfolio management. Fund managers pursuing the TE criterion intentionally did not generate mean/variance Markowitz efficient portfolios under many – indeed most – circumstances. Roll (1992) speculated that the inherent flaw in TE optimisation could be mitigated by diversifying among managers, but Jorion (2003) demonstrated that this was not the case.

By imposing additional constraints on the active portfolio (whilst maintaining the usual TE constraint), Jorion (2003) extended this analysis to determine the constant TE frontier, namely the frontier in mean/variance space on whose boundary all portfolios have the same TE. Jorion (2003) asserted that, because the TE constraint was so widely used in the industry, it could be taken as given, even though the restriction did not lead to optimal portfolio construction. Jorion (2003) proposed an additional constraint: that of maximising portfolio excess return for a given TE, whilst restricting the portfolio volatility to be the same as that of the benchmark. Because of the 'flatness' of the TE ellipse, the lower portfolio volatility compensated for the slight reduction in excess return.

Alternative 'optimal' portfolios are proposed which satisfy the TE constraint (i.e. the proposed portfolio still lies on the constant TE ellipse), but which have different levels of absolute risk (and thus different levels of return) relative to the benchmark risk. These portfolios reflect the portfolio manager's risk sensitivity, so different levels of portfolio return can be expected depending on the risk aversion level and investment strategy.

1.2 Problem statement

Portfolio managers, constrained by a tracking error to a benchmark and seeking only to maximise returns, naïvely select sub-optimal portfolios with unnecessarily high volatility. 'Optimal' does not necessarily mean maximum portfolio return, or minimum portfolio variance: a combination of these factors (such as a maximal risk-adjusted return), under the tracking error constraint, may better reflect optimality. Identifying, isolating and characterising tracking error-constrained portfolios which deliver such optimal investment performance are non-trivial endeavours which have, to date, eluded the industry.

1.3 Research question

What is the procedure for establishing the optimal portfolio (that which generates the maximum possible excess return per unit risk taken) weights for tracking error-constrained portfolios given a risk preference/investment strategy?

1.4 Study motivation

Tracking error (TE) is an active risk measure that reflects a portfolio manager's decision to deviate from the weights of benchmark positions to achieve returns in excess of a benchmark. This deviation introduces a measure of benchmark risk – called TE – defined as the standard deviation of the difference between portfolio and benchmark returns. Portfolio managers have historically selected the maximum return portfolio under these constraints (or, more recently, that portfolio which has the same risk as the benchmark) as the optimal portfolio, but both these choices consider portfolio attributes (risk and return) in isolation.

The combination of these features may offer fresh insights into constrained portfolio optimisation. Such alternative 'optimal' portfolios would reflect the portfolio manager's risk sensitivity, so different levels of portfolio return are expected depending on the level of risk aversion and the investment strategy chosen.

1.5 Dissertation structure

This dissertation is structured as follows: Chapter 2 presents the literature governing the history of asset allocation and optimal portfolio assembly and construction. The frameworks governing such activities evolved as a natural consequence of market efficiency and serve as tests for its validity. Accepting the tenets of market efficiency, this chapter also investigates the history of constrained portfolio optimisation – a far wider, and more relevant, field than unconstrained optimisation. Performance of actively-managed portfolios is assessed and rewarded based on outperformance of a prescribed benchmark, and portfolio managers are constrained by a plethora of investment restrictions. These include ranges of permissible constituent asset weights, ceilings and floors of portfolio betas, maximum risk allowed relative to the benchmark, and so on. The simultaneous observation of these curbs increase the complexity of portfolio management enormously and the underlying mathematics describing these activities is non-trivial. All these issues are presented in this chapter.

Chapter 3 sets out Article 1: *Optimising tracking error-constrained portfolios*. Active portfolios subject to tracking error constraints are the typical setup for active managers tasked with outperforming a benchmark. The risk and return relationship of such constrained portfolios is described by an ellipse in traditional mean-variance space and the ellipse's flat shape suggests an additional constraint which improves the performance of the active portfolio. Although subsequent work isolated and explored different portfolios subject to these constraints, absolute portfolio risk has been consistently ignored. A different restriction – maximisation of the traditional Sharpe ratio on the constant tracking error frontier in absolute risk/return space – is added here to the existing constraint set, and a method to generate this portfolio is explained. The resultant portfolio has a lower volatility and higher return than the benchmark, it satisfies the tracking error constraint and the ratio of excess absolute return to risk is maximised (i.e. maximum Sharpe ratio in absolute space).

Chapter 4 presents Article 2: *Active investment strategies under tracking error constraints*. Active portfolio managers are judged on their ability to outperform agent's benchmarks, so optimising fund returns is critically important. Maximising fund outperformance is, however, non-trivial because active portfolios are subject to tracking error (TE) (and other) constraints. Portfolios constrained by a TE are fenced by an elliptical frontier in mean/variance space and may not be efficient. Some attempts have been made to identify optimal portfolios subject to the restrictions imposed by TEs, i.e. to locate these on the frontier. We review these portfolio assemblies and introduce more possibilities: portfolios which are maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum risk. Such portfolios behave differently to those which are part of the efficient set, i.e. populate the efficient frontier and are TE-unconstrained.

Chapter 5 concludes the dissertation by summarising the findings of the entire study and proposing suggestions for future research.

1.6 General objective

The general objective of this research is to establish the portfolio weights required for various investment strategies' risk/return profiles, subject to industry constraints such as tracking error and β .

1.7 Specific objectives

Specific objectives of this research are:

1. identify and calculate the allowable investment universe in risk/return space when portfolios are subject to a tracking error constraint
2. isolate on this frontier the position (in risk/return space) of various investment strategy portfolios such as the
 - a. minimum variance
 - b. maximum return
 - c. benchmark return
 - d. maximum diversification
 - e. risk parity
 - f. minimum intra-asset correlation and
 - g. maximum Sharpe ratio.
3. Establish the relative investment weights of these portfolios: how do they differ
 - a. from each other and
 - b. under different economic conditions?

1.8 Research design

The research design of this dissertation follows in the outline below:

Pose research problem statement and question: Portfolio optimality (in whatever form) as well as constrained portfolio optimality are complex pursuits. How may component asset weights be determined for actively managed portfolios subject to tracking-error constraints?

Critical literature review: Critical literature reviews are conducted through Chapters 2 through 4 by consulting and considering existing literature. Adjustments to existing risk management procedures, techniques and methodologies to solve problems are documented and highlighted in the literature studies. The existing literature for this research theme is copious. Where an entirely new approach to risk practices is required, the literature was less obliging, but this was not a constraint in this study, because popular, well-established mathematical

techniques are almost always available for research endeavours and again, abundant literature exists to address and divulge these.

Theory building/adapting/testing: Adaptation of existing financial tools and mathematical techniques for practical implementation enjoys rich precedent. The bulk of the results reported in this dissertation were from empirical analyses of simulated data derived using both known and innovative risk metrics.

Data collection: Data used were either simulated or from third-party, internet-based, electronic databases (e.g. McGregor BFA,¹ Opendata and Bloomberg™ for historic index prices). Adequate data were available for all the chapters, so sample error was minimised. Data in this study comprised several published, historical time series, available from both proprietary (e.g. Bloomberg™) and non-proprietary sources (e.g. internet databases).

Conceptual development and empirical investigation: This research is intended to provide robust, but practical, solutions for use by investors and traders. As a direct result, the primary source of analytical work was Microsoft Excel™ since this tool is used by most financial institutions. These spreadsheet-based models use visual basic programming language (a flexible, functional desktop tool available to all quantitative analysts and risk managers) to develop macros to replace onerous and repetitive computing tasks. The empirical study comprises the practical implementation of the research method, using techniques and models developed in Microsoft Excel™.

The variables employed are data assembled from various historical time series. All data are available in the public domain. Some pricing data were simulated for illustration.

Illustrate and reason findings: Having analysed the data, obtained meaningful results and displayed these appropriately, the findings were written up into article-style reports for peer review and publication. The articles which comprise Chapter 3 and 4 have been published as detailed in Table 1.2.

¹ McGregor BFA was acquired by (and renamed as) IRESS in late 2016.

Further work: To complement major findings of and ensure the continuation of much needed work not addressed in this dissertation, future work regarding the many consequences of optimal constrained portfolios is proposed for active fund managers and academics.

1.8.1 Literature review

The literature reviews focus on the origin, development, history and applications of the issues identified through problem statements and research questions, in this case the development and identification of optimal portfolio selection. These literature studies explain and clarify the problem of portfolio optimality and elucidate how previous studies have addressed the problem. Alternative ideas – explored for the first time here – are also presented.

1.8.2 Data

Data requirements, frequency and source are shown in Table 1.1 below.

Table 1.1: Data requirements, frequency and source.

#	Topic	Data required	Frequency	Sources
1	Optimising tracking error-constrained portfolios	Historical asset returns and covariance matrices (including individual asset volatilities and correlations). Portfolio weights calculated.	Monthly	Bloomberg, McGregor BFA, Opendata, non-proprietary internet databases
2	Active investment strategies under tracking error constraints			

1.8.3 Research output

The research output is shown in Table 1.2 below.

Table 1.2: Research output.

#	Topic	Mathematics	Research methodology
1	Maxwell, M., Daly, M., Thomson, D. and van Vuuren, G. 2017. Optimising tracking error-constrained portfolios. <i>Applied Economics</i> , 50(54): 5846–5858.	Proprietary, bespoke Microsoft Excel models	Portfolio optimisation approaches
2	Maxwell, M. and van Vuuren, G. 2019. Active investment strategies under tracking error constraints. <i>International Advances in Economic</i>	Calculus (differentiation techniques to	Asset selection under prescribed

	Research, https://doi.org/10.1007/s11294-019-09746-3 : 1 – 14.	identify optimal solutions) Linear algebra Lagrangian dynamics	portfolio constraints
--	---	--	-----------------------

1.9 Conclusion

The conclusion presents a summary of the findings of both topics, providing details of recommendations for possible future research. The next chapter presents a literature survey governing the background information relevant to the dissertation.

Chapter 2

Literature study

2.1 Absolute portfolio optimisation

Inherent in the success of a portfolio manager is the skill of portfolio optimisation, forming part of Modern Portfolio Theory, or Portfolio Selection Theory (PST). The principal objective of PST (in addition to maximising returns) is to find the optimal allocation of investments between different assets, given the investor's risk profile/preference, i.e. to diversify away as much risk (volatility of returns) as possible. This 'optimal' allocation results in a portfolio of selected assets whose risk matches the investors risk preference, thus maximising their utility (Ghosh & Mahanti, 2014). This suggests that the trade-off between risk and return (mean/variance trade-off) is different for each investor, but Markowitz (1952) indicated that, although this may be the case, the preferences of all investors lie on a curve, namely *the efficient frontier*. The efficient frontier comprises efficient (diversified) portfolios which have the lowest risk for a given level of return or, equivalently, the highest return for a given level of risk, i.e. the set of the best risk/return combinations forms this frontier.

Portfolio optimisation falls in as one of the phases of what is known as the *investment management process*. This process encapsulates the general procedure followed by portfolio managers when selecting an 'optimal' portfolio for the investor. The other phases involve capturing the risk preference/profile of an investor, recorded in their investment policy along with; the investment objectives, constraints, permitted investment allocations in asset classes and/or sectors, the investment strategy (e.g. passive/active, value/growth), and the performance measures and evaluators (Fabozzi & Markowitz, 2011). These form part of portfolio planning and optimisation models used to solve the "portfolio optimisation (or selection) problem".

The portfolio optimisation problem, formulated by Markowitz (1952), consists of two criteria, namely expected return (mean) and risk (standard deviation), measuring the volatility/variability of returns. Markowitz (1952) formulated this problem into a single investment period model, in which the investor allocates the capital amongst several assets. Over the course of the investment period, a random rate of return is generated by the portfolio resulting in

greater or lower capital value at the end of the period (relative to the principal amount). Subsequent research has extended this model to multiple periods and it remains the foundation upon which Modern Portfolio theory is based (Mansini, Ogryczak & Speranza, 2014).

Markowitz (1952) stated that the portfolio selection process can be divided into two main phases. The first phase couples experience and observation in order to forecast the future performance of the assets of interest, and the second uses these forecasts in choosing the most suitable portfolio. The first stage depends largely on the ability of the fund manager, the models used for forecasting and the way estimation error is dealt with. There has been extensive research done on this and it is beyond the scope of this dissertation. Markowitz (1952) concludes that investors should focus on return and risk in conjunction when selecting desired portfolios. In doing so, they will most likely choose a portfolio aligned with their preferences, thereby maximising their utility.

Utility maximisation is ultimately the desired outcome for the investor, and forms part of an extensively researched field of economics called Utility Theory (Nawrocki & Viole, 2014). The utility of an investor is the total satisfaction received from consumption or investment of capital. It is described by a utility function, which assigns numeric values to all possible choices faced by the investor where the higher the numeric value of a choice, the greater the satisfaction derived from it (Fabozzi & Markowitz, 2011). As such, PST sets out to find the optimal choice (portfolio) resulting in the maximum possible utility, given a set of investor constraints. This is achieved using indifference curves. An indifference curve represents a set of choices (in this case portfolios with different risk/return combinations) for which the investor derives the same level of utility from each and is therefore indifferent to which is chosen (Carlsson, Fullér & Majlender, 2002). These indifference curves can be mapped out in the same space (mean/variance) as the efficient frontier, enabling the portfolio manager to select the optimal portfolio from the point where the maximum indifference curve is tangential to the efficient frontier (Fabozzi & Markowitz, 2011). This results in the selection of an efficient portfolio which is optimal for the investor, i.e. satisfies their risk profile/preferences.

Markowitz (1952, 1959) formulated mean variance optimisation into a quadratic programming model, providing a quantitative tool to use when making the investment allocation decision by considering the trade-off between risk and return (mean and variance) of a portfolio

of assets (Ghosh & Mahanti, 2014). Markowitz (1959) extended his 1952 work and transformed it into the Markowitz model (aforementioned), in which this optimal allocation of holdings/investments is determined through the solution of this quadratic programming model (Ghosh & Mahanti, 2014). This mean variance model has been altered in various ways since its inception, namely; the single index/market model which ignores the covariance between asset returns, the CAPM (Capital Asset Pricing Model) as an extension of the single index model (considering the returns of securities to depend on the market index and not the covariance between asset pairs), and the multiple period Mean Variance model.

Ghosh and Mahanti (2014) restated that an important implication of Modern Portfolio Theory (based on the work of Markowitz (1952, 1959)) is that when selecting an asset for a portfolio, the risk and return of an asset should not be considered in isolation but rather in conjunction with the correlation of that asset with the other constituents. This co-movement with other assets, if negligible or in the opposite direction, can reduce the risk (volatility of returns) of the portfolio significantly, whilst maintaining the same overall portfolio return. The process of adding additional uncorrelated or negatively correlated assets to reduce the overall risk of portfolio is known as diversification.

Once the optimal portfolio is selected, its performance (and hence the manager's) must be measured and evaluated, a fundamental issue in portfolio management. Various performance measures and attribution models (performance evaluation) have been proposed, two of which are used in this paper to 'reverse engineer' the optimal portfolio. The most noteworthy performance attribution model is the Fama Decomposition of Total Return (Fabozzi & Markowitz, 2011), which identifies the sources of the portfolio's return, indicating how much of the return can be attributed to the manager and how and why he/she earned that return. Notable performance measures include: the Treynor ratio ("ratio of excess returns, above risk-free rate, to Beta (systematic risk)") indicating the manager's skill in market timing; the Jensen index, as an absolute measure, indicating the ability of the manager in forecasting returns and portfolio diversification against risk (Ghosh & Mahanti, 2014); the Information ratio ("ratio of excess return, above the benchmark, to the TE of the portfolio"), not only highlighting the manager's ability to generate excess returns but also the consistency of those returns. Lastly, the Sharpe ratio ("ratio of excess return, above the risk free rate, to total portfolio risk"), indicating the manager's aptitude in security selection; and the TE, measuring the

volatility of the excess returns above the benchmark. These are used in the final phase of the Investment management process and give the investor an overall picture of the ability of the manager, the performance of the portfolio, and the level of satisfaction the investor has derived (whether performance is aligned with their preferences).

2.2 Relative portfolio optimisation

Portfolio Selection Theory, although widely used, is based on several assumptions which are often not observed in practice. Fund managers, who often use this theory as a base, are classified into two broad types: passive and active. Passive managers primarily buy and hold securities, closely corresponding to mandated benchmark weights (thereby keeping fees to a minimum and somewhat replicating benchmark returns), while active managers tend to deviate from the prescribed benchmark, trading securities dynamically, with the goal of maximising return through asset allocation, stock selection and market timing. A continuous dispute over which approach (active versus passive) provides superior returns causes healthy contention in the industry, with a resolution on which is *more optimal*² never widely accepted. While it is true that many active portfolios underperform their benchmark, all truly passive funds underperform theirs. It is only with an element – however small – of *active* management (or zero management, administration and trading costs) that a 'passive' fund can match its benchmark performance.

Active and passive managers generally optimise their portfolio returns in relative, rather than absolute, return space, leading to higher unsystematic risk and greater absolute portfolio volatility (often used as a measure of risk, however the terminology contentiously debated as volatility is directionless and risk refers to capital loss). As asset managers are assessed on total return performance relative to prescribed benchmarks – which may comprise broad, diversified indices of assets or be specific, often limited to only two or three asset classes such as "cash" and "local equity" for example (Roll, 1992) - the services of active managers will be worth retaining if the performance contribution is above that of the benchmark. To satisfy the absolute return performance condition of an investor's utility function, choosing component weights close to the benchmark in bull market conditions is sufficient. In bear market

² Protagonists also argue over the definition of *optimal*.

conditions, however, when the benchmark is generally *inefficient*, fund managers must materially outperform the benchmark. This requires superior *relative* return performance, only possible if the manager takes on active risk, but taking substantial bets which breach mandates incurs corresponding punitive penalty charges. To limit these costs, constraints are imposed on the active portfolio (Jorion, 2003).

Asset returns are, however, noisy and reported infrequently, thus many months of return data are required before reliable average performance is known. Minimising the TE – the standard deviation of the differences between portfolio and benchmark returns – is a universal gauge of active manager performance. Consequently, active fund managers are constrained by (amongst other agency-mandated requirements) TE, also an active risk measure reflecting portfolio managers' decisions to deviate from benchmark positions to achieve positive excess returns (note, that TE is directionless so deviations from benchmark positions could equally result in *negative* 'excess' returns). When used to determine manager competence, TE is not used in isolation and is best applied in combination with other performance evaluators. The optimal level of TE depends on the portfolio's investment policies, i.e. outperformance target, risk-return profile, growth strategy (value or growth), etc. Thomas, Rottschäfer & Zvingelis, (2013) outline several causes of TE (fees, transaction costs, taxes, factor tilts, cash management), however the prime driver is return maximisation. In the process of portfolio construction, not all assets which constitute the benchmark will be held, and those that are will be held in different proportions. Hence, benchmark and portfolio returns differ from month to month, resulting in a TE.

Outperformance of the benchmark by the fund is synonymous with the generation of a positive expected TE. Roll (1992) asserted that these criteria amounted to mean/variance analysis, but rather than establishing portfolios with the smallest total return volatility for a given expected total return, fund managers constrained by a TE must seek portfolios with a minimum TE for a given expected performance relative to a benchmark.

Performance fee incentives compel fund managers to sometimes adopt unnecessary risk to achieve high excess returns, but as asset returns are noisy the average manager's performance can only be accurately assessed after sufficient performance data have been accumulated. As such, the value added by such managers is not obvious, so instead investors focus

on TE as an important performance measure, leading to portfolio selection in the relative space.

The principal problem associated with this approach is that absolute portfolio risk is neglected. These portfolios are not, then, optimal in a mean-variance sense and they are *riskier* than the benchmark. Fund managers must balance their investment strategy such that they simultaneously maximise benchmark outperformance (i.e. increase TE) while minimising excess return variance (i.e. decrease TE) - a seemingly contradictory endeavor. By applying an additional constraint on portfolio β ,³ Roll (1992) set out the description, established the constraints and obtained the solution for the "TE frontier" (Figure 1), and found that all managed portfolios (under the TE constraint) with positive expected performance will have $\beta > 1$, while portfolios that have higher expected returns and lower total volatility have $\beta < 1$. Roll (1992) generated TE frontiers with a constraint on β and found that minimising the TE did not result in more efficiently-managed portfolios. Roll (1992) proved that it is impossible to produce a portfolio that is simultaneously constrained by a TE, a given expected performance and a specified β . Roll (1992) also recommended that, because portfolio analysis is subject to significant expected return estimation errors, estimating the *expected TE* rather than these individual asset returns could be used as a more feasible manager goal.

Roll (1992) constructed portfolios that maximised excess returns over a benchmark for a given TE and illustrated that a TE frontier comprises all such portfolios, one at each level of TE volatility. This frontier has a similar shape to the standard efficient frontier but is shifted to the right in mean-variance space. These portfolios are mean/variance inefficient if the benchmark is inefficient (which is most often the case), i.e. TE managed portfolios are dominated by other portfolios that have both lower volatility and higher total return.

In Figure 2.1, the solid black curve represents the 'universal' efficient frontier, i.e. the frontier that would result if *all* assets (and all *combinations* of assets) were available to the fund manager. Since the benchmark is prescribed, it will not always be efficient and it is, indeed, unusual for a benchmark to lie on the universal efficient frontier. Fund returns, constrained further by a mandated TE target, lie on a TE frontier – the grey curve in Figure 2.1. The TE frontier,

³ Where β (a measure of systematic risk) has the standard definition, namely $\beta = \text{covar}(r_P, r_B) / \text{var}(r_B)$ where r_P are portfolio returns and r_B are benchmark returns.

then, does *not* reflect the maximum total return of portfolios comprising deviations from the benchmark, but rather the maximum total return for a given TE.

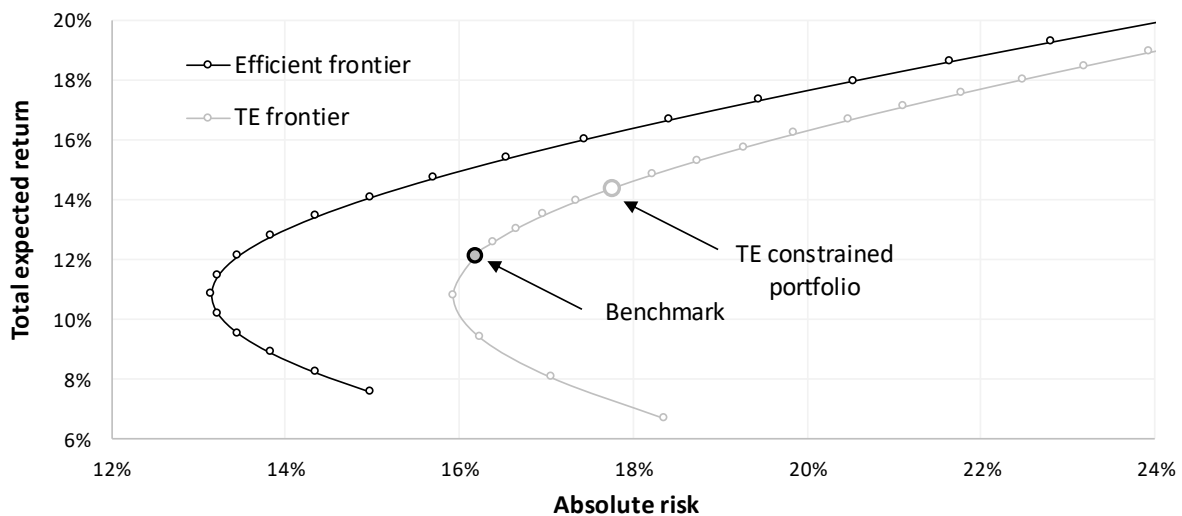


Figure 2.1: TE frontier and TE-constrained portfolio. In this example, $TE = 5\%$.

Source: Roll (1992) and own calculations.

Each point/marker on the TE curve is the maximum total return possible for a given level of TE relative to the benchmark. Markers are placed at intervals of 1%, so the TE-constrained portfolio point indicated represents the maximum excess return possible for a fund relative to the indicated benchmark with a TE constraint of 5%.

Bertrand, Prigent and Sobotka (2001) considered the problem of mean-variance maximisation under a TE constraint and thus reintroduced both absolute and relative risk (i.e. TE) aversion into the optimisation program. Larsen & Resnick (2001) considered a range of optimisation and holding periods, but did not consider transaction cost constraints. These rebalancing costs must be offset against the gains in risk control, but Larsen & Resnick (2001) argued that, beyond certain (threshold) risk levels, these would be untenable. El-Hassan & Kofman (2003) found that frequent rebalancing was necessary to maintain control over total risk (though not necessarily TE risk) when actively managing portfolios (see also Plaxco & Arnott, 2002), but this did not necessarily lead to optimal portfolios.

Rather than using the TE to value the risks undertaken by the asset manager, overall portfolio volatility may be a better metric to compare with benchmark volatility. The asset management industry, however, maintains emphasis on TE risk control, so Jorion (2003) returned to the problem by first establishing the shape of constant TE portfolios, finding that these are

described by an ellipse on the traditional mean-variance plane, but not in the efficient frontier (μ/σ) plane, where μ represents the portfolio return and σ the portfolio volatility – usually a measure of the portfolio's risk. The shape of the constant TE frontier in μ/σ space is a *distorted* ellipse in which the bi-axial symmetry associated with ellipses is lost. In this work, "ellipse" will be used when referring to the shape in either space.

In the discussion that follows, σ_P is the portfolio volatility, σ_B the benchmark volatility, r_P the return on the portfolio and r_B the return on the benchmark.

Figure 2.2 plots the constant TE frontier for a TE of 5% where all portfolios on this ellipse have a TE of 5%. The maximum excess return portfolio subject to this TE constraint is *above* the benchmark (higher return than the benchmark) and *to the right of* the benchmark (higher risk than the benchmark: Figure 2.2a). The flat shape of the ellipse, however, led Jorion (2003) to suggest the addition of a constraint on total portfolio volatility, namely the selection of a portfolio with the same total risk as that of the benchmark ($\sigma_P = \sigma_B$: Figure 2.2b).

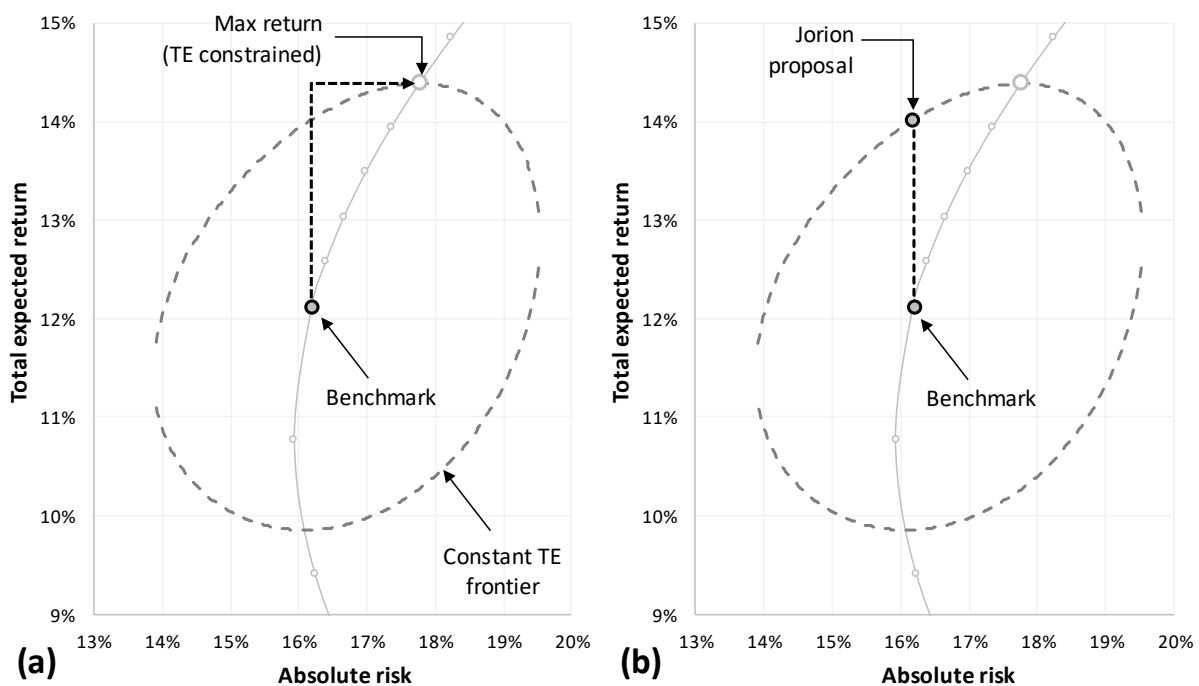


Figure 2.2: TE frontier, TE-constrained portfolio and constant TE frontier (with TE = 5%). (a) shows the naïve portfolio: excess return is maximised for a given TE constraint. (b) shows Jorion's (2003) suggestion: observe constraints from (a) but restrict portfolio risk to that of the benchmark.

Source: Roll (1992), Jorion (2003) and own calculations.

This portfolio volatility constraint leads to portfolios with lower excess returns than are possible (but still higher than the benchmark return) and the same volatility as the benchmark, as shown in Figure 2b. Jorion (2003) found that this constraint substantially improved managed portfolio performance, in particular, for those with lower TEs and less efficient benchmarks. For these portfolios, the information ratio (IR), given by:

$$IR = \frac{r_P - r_B}{TE} \quad (1)$$

is not maximised.

Bertrand (2009) examined the TE minimisation problem from another perspective. Instead of considering constant TE frontiers, he allowed the TE to vary and fixed risk aversion, thereby establishing an optimal portfolio with several desirable attributes and which lies on what he termed as iso-aversion frontier. All portfolios on this frontier have the same IR, allowing the selection of portfolio to satisfy volatility preference and/or TE preference. Bertrand (2010) analyses and discusses the IR decomposition proposed by Menchero and Hu (2006) in the light of the analysis of consistency between the risk-adjusted performance attribution process and portfolio optimisation under TE constraints, developed in Bertrand (2005, 2009). Consistency was only attained when optimising the TE constraint in isolation: additional constraints, on total risk for example, distorts the information ratio of the entire portfolio (component IRs are no longer uniform nor equal to the IR of the entire portfolio) indicating that no equilibrium between expected return and relative risk has been reached.

Bertrand (2010) also found that, to increase TE, portfolio managers tend to decrease portfolio IRs with the result that extra TE is not rewarded by sufficient extra return. Basing decisions on IR alone, portfolio managers have no incentive to move away from the benchmark and, since $\sigma_p = \sigma_b$, taking total risk into account does not provide more incentive to move away from the benchmark: a "disturbing feature" (Bertrand, 2010).

Wu and Jakshoj (2011) decompose excess returns into selection and allocation effects and then apply this attribution approach for an unconstrained portfolio and a TE-constrained optimal portfolio. Wu and Jakshoj (2011) found that the TE optimised weights do improve the expected IR for a given portfolio, a result confirmed over several time periods. TE-optimised portfolios were also easier to implement.

Chow (1995) suggested a utility function, that measures return, variance and TE, was a more appropriate performance metric than standard portfolio performance evaluators (such as return and variance) because it accounts for both total risk and TE. Chow (1995) found that the implementation of this utility function generated a set of efficient portfolios which included the mean-variance efficient set, the mean tracking-error efficient set and all convex combinations of both sets.

Stowe (2014) noted that the conventional practices of β constraints, studied in Roll (1992), and TE volatility constraints, studied in Jorion (2003), assure utility improvements for the investor. If these constraints are implemented properly (optimisation over a TE-parameterised utility function), they force the delegated manager to buy a more efficient portfolio than the benchmark. Thus, even though relative utility maximisation is sub-optimal and inefficient, if the delegated manager is more skilful than the investor, delegated portfolio management is still likely preferred to naïvely holding the benchmark.

2.3 Active investment strategies under the Tracking error constraint

In the early 2000s, an extended bear market shifted the investment focus from naïve TE minimisation, to outperformance of the benchmark whilst remaining within an (agency-imposed) TE limit. El-Hassan & Kofman (2003) proposed a strategy of dynamic portfolio allocation using Jorion's (2002) TE frontier and pointed out that TE can either be an investment goal or investment constraint, leading to two interpretations: one in the passive, the other in the active sense. While both approaches incur incidental risk, only active management incurs 'intentional' risk (asset specific or idiosyncratic). Since there is an element (however small) of active management in passive investment, both strategies are susceptible to an increase in overall idiosyncratic portfolio risk. This *additional* portfolio risk is overlooked when optimising in the relative, TE, space.

Bajeux-Besnainou, Portait & Tergny (2011) investigated active portfolios subject to TE and component weight constraints (a common agency-mandated feature of active portfolios, e.g. no more than 10% equity, only investment grade sovereign bonds, and so on) and found these restrictions to be simultaneously binding.

Riccetti (2010) derived analytical methods that indicated whether managers using active strategies could simultaneously generate a positive excess return large enough to cover performance fees and limit the portfolio's variance to be less than that of the benchmark. This formulation was a required, yet insufficient, condition to outperform the benchmark's returns, without increasing portfolio variance.

Previous work highlights the limitations of the TE constraint on portfolio management, although taken as a given by the industry, and the difficulty in optimising under such conditions. Outperforming the benchmark whilst adequately managing overall portfolio volatility is task not deliberately achieved amongst fund managers.

The ultimate objective of portfolio management is to select a portfolio which best describes an investor's goals and satisfies their utility desires. With the entire unconstrained investable universe to choose from, i.e. the efficient frontier, there are a number of investment strategies at a manager's disposal that can satisfy the requirements set forth by the investor. While this list is not all-encompassing, the more commonly used strategies are set forth; namely, maximum return, minimum variance, maximum diversification, risk parity/inverse volatility, minimum intra-portfolio correlation and Maximum Sharpe ratio portfolio.

The work discussed above describes optimal portfolios on the *efficient* frontier and introduces new optimal investment suggestions. Our contribution applies optimal investment strategies to portfolios constrained by TE. We connect these ideas and adapt these *efficient frontier* strategies to the constant TE constrained frontier. No work to our knowledge has, to date, established such optimal investment strategies subject to constant TE constraints. This work develops (mathematically where possible) approaches for determining the requisite optimal portfolio asset weights for various investment strategies constrained by the TE.

Chapter 3

Optimising tracking error-constrained portfolios

Michael Maxwell,⁴ Michael Daly,⁵ Daniel Thomson⁶ and Gary van Vuuren⁷

Abstract

Active portfolios subject to tracking error constraints are the typical setup for active managers tasked with outperforming a benchmark. The risk and return relationship of such constrained portfolios is described by an ellipse in traditional mean-variance space and the ellipse's flat shape suggests an additional constraint which improves the performance of the active portfolio. Although subsequent work isolated and explored different portfolios subject to these constraints, absolute portfolio risk has been consistently ignored. A different restriction – maximisation of the traditional Sharpe ratio on the constant tracking error frontier in absolute risk/return space – is added here to the existing constraint set, and a method to generate this portfolio is explained. The resultant portfolio has a lower volatility and higher return than the benchmark, it satisfies the tracking error constraint *and* the ratio of excess absolute return to risk is maximised (i.e. maximum Sharpe ratio in absolute space).

Keywords *Tracking error frontier, optimisation, Sharpe ratio, risk-adjusted returns*

JEL classification C52, G11

3.1 Introduction

Tracking error (TE) is an active risk measure that reflects a portfolio manager's decision to deviate from weights of benchmark positions to achieve returns in excess of the benchmark. This deviation introduces a measure of benchmark risk – called TE – defined as the standard deviation of the difference between portfolio and benchmark returns. Generally, TE is not a metric used in isolation when determining manager performance and is best used in combination with other performance evaluators such as the information ratio, Value at Risk, etc. The optimal level of TE is dependent on the portfolio's investment policy, i.e. outperformance target, risk-return profile and growth strategy (value or growth). Thomas, Rottschäfer and Zvingelis (2013) outline several causes of TE (fees, transaction costs, taxes, factor tilts, cash management and general market volatility), but a principal objective that research has aimed to solve is optimisation: maximising excess returns while maintaining a prescribed TE.

⁴ Postgraduate student, School of Economics, Department of Risk Management, North West University, Potchefstroom, South Africa.

⁵ Postgraduate student, North West University, Potchefstroom, South Africa.

⁶ Postgraduate student, North West University, Potchefstroom, South Africa.

⁷ Extraordinary professor, North West University, Potchefstroom, South Africa.

Thomas *et al.*, (2013) noted that TE does not provide a directional metric since it only measures excess return volatility, thus it provides no information about the direction of the return differences (i.e. whether they are positive or negative). Portfolio managers generally pursue low TEs and positive excess returns, because although higher TEs may indeed translate into higher returns, this is not always the case.

Standard portfolio construction allows portfolio managers to select assets and determine their exposures (weights) based on factor tilts, volatilities, expected returns and correlations with one another (Clarke, de Silva & Thorley, 2002). This information provides the basis for the calculation of an expected TE relative to the benchmark and it is this forecast which enables portfolio managers to remain within the TE bounds, set by investors, rebalancing (if necessary) to prevent punitive charges from mandate violations (Ammann & Zimmerman, 2001).

Roll (1992) argued that the dual goals of benchmark return outperformance and TE reduction were equivalent to standard Markowitz mean-variance objectives with subtle alterations. Instead of identifying a Markowitz-efficient portfolio (i.e. a portfolio with the smallest total return volatility for a given expected total return), fund managers were obliged to isolate a portfolio with a minimum TE for a given performance relative to the benchmark. Roll (1992) labelled this the "tracking error criterion": minimisation of TE for given expected benchmark outperformance, and formalised this criterion along with its consequences for portfolio management. Fund managers pursuing the TE criterion *intentionally* did not generate mean/variance Markowitz efficient portfolios under many – indeed most – circumstances. Roll (1992) speculated that the inherent flaw in TE optimisation could be mitigated by diversifying among managers, but Jorion (1992, 2003) demonstrated that this was not the case.

By imposing additional constraints on the active portfolio (whilst maintaining the usual TE constraint), Jorion (2003) extended this analysis to determine the *constant* TE frontier, namely the frontier in mean/variance space on whose boundary all portfolios have the same TE. Jorion (2003) asserted that, because the TE constraint was so widely used in the industry, it could be taken as given, even though the restriction did not lead to optimal portfolio construction. Jorion (2003) proposed an additional constraint: that of maximising portfolio excess return for a given TE, whilst restricting the portfolio volatility to be the same as that of the benchmark. Because of the 'flatness' of the TE ellipse, the lower portfolio volatility compensated for the slight reduction in excess return.

In this article, an alternative 'optimal' portfolio is proposed which satisfies the TE constraint (i.e. the proposed portfolio still lies on the constant TE ellipse), but which has lower *absolute* risk relative to the benchmark risk and which has the highest Sharpe ratio – i.e. is located on a capital market line (CML) tangential to the constant TE frontier so it generates the highest excess return per unit of *absolute* risk. This portfolio exhibits these benefits by sacrificing some excess return over the benchmark but, again, because the constant TE ellipse is more or less flat, the resulting excess return reduction is not substantial but the reduction in total portfolio volatility is substantial. This leads to significantly improved performance (i.e. ratio of return to risk).

The remainder of this article proceeds as follows: Section 3.2 provides a literature study which covers the evolution of ideas regarding TE frontiers and *constant* TE frontiers as well as other proposals of portfolio optimisation in relative and absolute risk/return space. Data used and methodology adopted – including relevant mathematics – are provided in Section 3.3. Section 3.4 presents and discusses results of existing and new proposals. Section 3.5 concludes.

3.2 Literature survey

Asset managers are assessed on total return performance relative to prescribed benchmarks – which may comprise broad, diversified indices of assets or be specific, often limited to only two or three asset classes such as "cash" and "local equity" for example (Roll, 1992). The services of active managers will be worth retaining if the performance contribution is – on average – positive. Asset returns are, however, noisy and reported infrequently, thus many months of return data are required before reliable average performance is known. Minimising the TE – the volatility of the difference between managed fund and benchmark returns – is an important criterion for active manager performance. Outperformance of the benchmark by the fund is synonymous with the generation of a positive expected TE. Roll (1992) asserted that these criteria amounted to mean/variance analysis, but rather than establishing portfolios with the smallest total return volatility for a given expected total return, fund managers constrained by a TE must seek portfolios with a minimum TE for a given expected performance relative to a benchmark.

The principal problem associated with this approach is that absolute portfolio risk is neglected. These portfolios are not, then, optimal in a mean-variance sense and they are riskier

than the benchmark. By applying an additional constraint on portfolio β ,⁸ Roll (1992) set out the description, established the constraints and obtained the solution for the "TE frontier" (Figure 3.1), and found that all managed portfolios (under the TE constraint) with positive expected performance will have $\beta > 1$, while portfolios that have higher expected returns and lower total volatility have $\beta < 1$. Roll (1992) generated TE frontiers with a constraint on β and found that minimising the TE did not result in more efficiently-managed portfolios. Roll (1992) proved that it is impossible to produce a portfolio that is simultaneously constrained by a TE, a given expected performance and a specified β .

The location of the global total return efficient frontier portfolio is not known with absolute certainty, since it depends on individual asset expected returns and these are beset with large noise components. Roll (1992) recommended that, because portfolio analysis is subject to these considerable estimation errors, estimating the *expected TE* rather than individual asset returns could be used as a more feasible manager goal.

In Figure 3.1, the solid black curve represents the 'universal' efficient frontier, i.e. the frontier that would result if *all* assets (and all *combinations* of assets) were available to the fund manager. Since the benchmark is prescribed, it will not always be efficient. Indeed, it is unusual for a benchmark to lie on the universal efficient frontier. Fund returns, constrained further by a prescribed TE target, lie on a TE frontier – the grey curve in Figure 3.1. The TE frontier, then, does *not* reflect the maximum return of portfolios comprising deviations from the benchmark, but rather the maximum total return for a given TE.

⁸ Where β (a measure of systematic risk) has the standard definition, namely $\beta = \text{covar}(r_P, r_B) / \text{var}(r_B)$ where r_P are portfolio returns and r_B are benchmark returns.

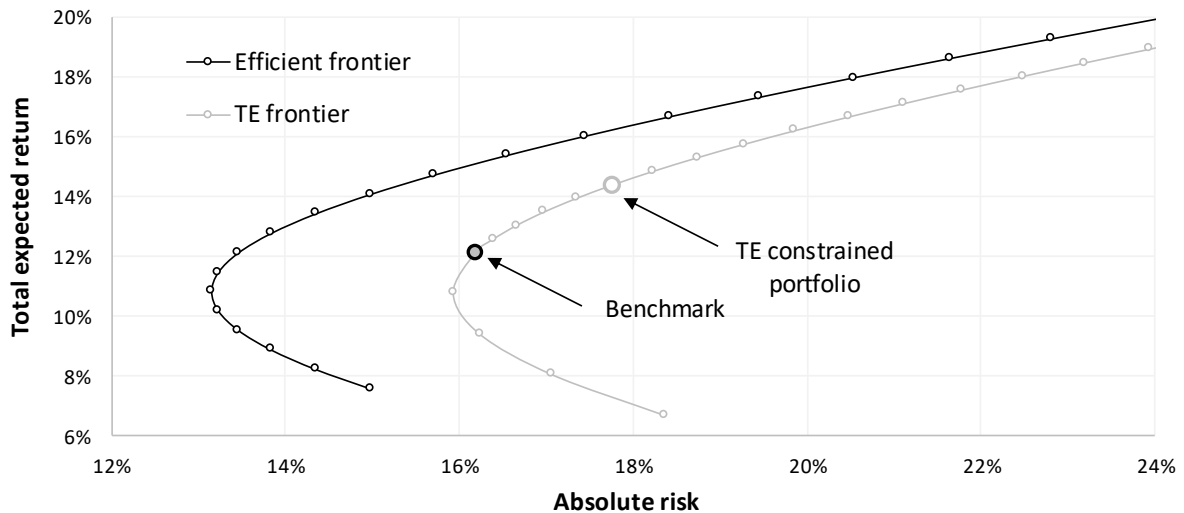


Figure 3.1: TE frontier and TE-constrained portfolio. In this example, $TE = 5\%$.

Source: Roll (1992) and own calculations.

Each point/marker on the TE curve is the maximum total return possible for a given level of TE relative to the benchmark. Markers are placed at intervals of 1%, so the TE-constrained portfolio point indicated represents the maximum excess return possible for a fund relative to the benchmark indicated with a TE constraint of 5%.

Bertrand, Prigent and Sobotka (2001) considered the problem of mean-variance maximisation under a TE constraint and thus reintroduced both absolute and relative risk (i.e. TE) aversion into the optimisation program. Larsen & Resnick (2001) considered a range of optimisation and holding periods, but did not consider transaction cost constraints. These rebalancing costs must be offset against the gains in risk control, but Larsen & Resnick (2001) argued that, beyond certain (threshold) risk levels, these would be untenable. El-Hassan & Kofman (2003) found that frequent rebalancing was necessary to maintain control over total risk (though not necessarily TE risk) when actively managing portfolios (see also Plaxco & Arnott, 2002), but this did not necessarily lead to optimal portfolios.

Rather than using the TE to value the risks undertaken by the asset manager, overall portfolio volatility may be a better metric to compare with benchmark volatility. The asset management industry, however, maintains emphasis on TE risk control, so Jorion (2003) returned to the problem by first establishing the shape of constant TE portfolios, finding that these are described by an ellipse on the traditional mean-variance plane, but not in the efficient frontier

(μ/σ) plane, where μ represents the portfolio return and σ the portfolio risk – usually a measure of the portfolio's volatility. The shape of the constant TE frontier in μ/σ space is a *distorted* ellipse in which the bi-axial symmetry associated with ellipses is lost. In this article, "ellipse" will be used when referring to the shape in either space.

In the discussion that follows, σ_P is the portfolio volatility, σ_B the benchmark volatility, r_P the return on the portfolio and r_B the return on the benchmark.

Figure 3.2 plots the constant TE frontier for a TE of 5%. All portfolios on the ellipse have a TE of 5%. The maximum excess return portfolio subject to this TE constraint is *above* the benchmark (higher return than the benchmark) and *to the right of* the benchmark (higher risk than the benchmark: Figure 3.2a). The flat shape of the ellipse, however, led Jorion (2003) to suggest the addition of a constraint on total portfolio volatility, namely the selection of a portfolio with the same total risk as that of the benchmark ($\sigma_P = \sigma_B$: Figure 3.2b).

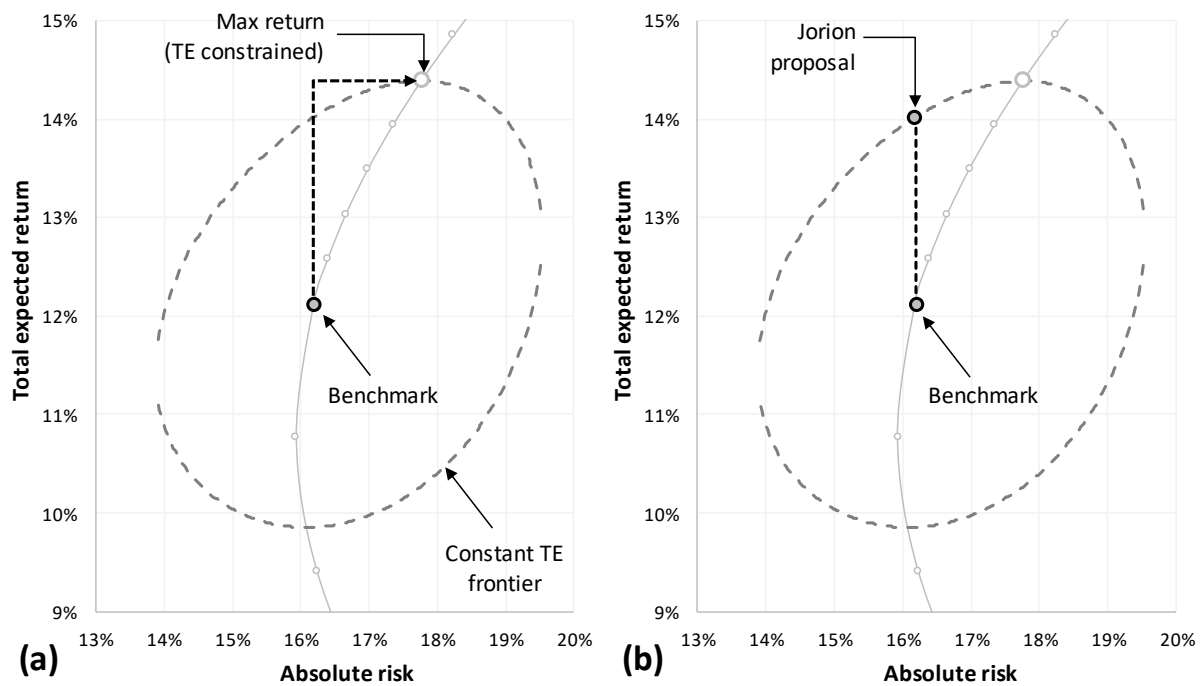


Figure 3.2: TE frontier, TE-constrained portfolio and constant TE frontier (with TE = 5%). (a) shows the naïve portfolio: excess return is maximised for a given TE constraint. (b) shows Jorion's (2003) suggestion: observe constraints from (a), but restrict portfolio risk to that of the benchmark.

Source: Roll (1992), Jorion (2003) and own calculations.

This portfolio volatility constraint leads to portfolios with lower excess returns than are possible (but still higher than the benchmark return) and the same volatility as the benchmark as

shown in Figure 3.2b. Jorion (2003) found that this constraint substantially improved managed portfolio performance, in particular for those with lower TEs and less efficient benchmarks. For these portfolios, the information ratio (IR), given by:

$$IR = \frac{r_P - r_B}{TE} \quad (1)$$

is not maximised.

Bertrand (2009) allowed the TE to vary and fixed the risk aversion, rather than consider only constant TE frontier-constrained portfolios. The resulting optimal portfolios were found to exhibit several desirable properties. They all have the same IR, allowing a portfolio to be selected on a TE frontier according to desired volatility and/or to desired TE. Bertrand (2010) analyses and discusses the IR decomposition proposed by Menchero and Hu (2006) in the light of the analysis of risk-adjusted performance attribution developed in Bertrand (2005). Bertrand (2009) showed that only optimisation under the TE constraint alone is consistent with the risk-adjusted performance attribution process. When additional constraints (e.g. on total risk) are introduced, component IRs are no longer uniform nor equal to the IR of the entire portfolio, meaning that no equilibrium between expected return and relative risk has been reached.

Bertrand (2010) also found that, to increase TE, portfolio managers tend to decrease portfolio IRs with the result that extra TE is not rewarded by sufficient extra return. Basing decisions on IR alone, portfolio managers have no incentive to move away from the benchmark and, since $\sigma_p = \sigma_b$, taking total risk into account does not provide more incentive to move away from the benchmark: a "disturbing feature" (Bertrand, 2010).

Stowe (2014) noted that the conventional practices of β constraints, studied in Roll (1992), and TE volatility constraints, studied in Jorion (2003), assure utility improvements for the investor. If these constraints are implemented properly, they force the delegated manager to buy a more efficient portfolio than the benchmark. Thus, even though relative utility maximisation is sub-optimal, if the delegated manager is more skilful than the investor, delegated portfolio management is still likely preferred to naïvely holding the benchmark.

The existing literature ignores *absolute* portfolio risk, so an optimal portfolio is proposed which embraces the mandated TE, but which also maximises the Sharpe ratio (i.e. the ratio of excess return to absolute risk taken).

3.3 Data and methodology

3.3.1 Data

The data comprised simulated realistic weights, returns, volatilities and correlations for a small benchmark comprising three assets. Portfolio constituents were derived only from the benchmark universe (including short-selling of benchmark components).

Note that the "assets" which constitute the portfolio could easily be asset *classes* (such as equity, bonds and cash) or specific industry *sectors* within an asset class (e.g. an industrial equity index, a banking and finance index, etc.) or individual *assets* such as single name stocks or bonds.

3.3.2 Methodology

To set out the methodologies required for the various frontiers, some definitions are necessary. These are recreated below in line with the notation developed by Roll (1992) and perpetuated by Jorion (2003).

Fund managers, tasked with outperforming benchmarks, must take positions in assets which may or may not be components of the benchmark (depending on the fund mandate).

Define

$\mathbf{q}$: vector of benchmark weights for a sample of N assets

$\mathbf{x}$: vector of deviations from the benchmark

$\mathbf{q}_P (= \mathbf{q} + \mathbf{x})$: vector of portfolio weights

$\mathbf{E}$: vector of expected returns, and

$\mathbf{V}$: covariance matrix of asset returns.

Net short sales *are* allowed in this formulation, so the total active weight $\mathbf{q}_i + \mathbf{x}_i$ may be negative for any individual asset, i . The universe of assets can generally exceed the components of the benchmark, but for Roll's (1992) methodology, assets in the benchmark *must* be included.

Expected returns and variances are expressed in matrix notation as:

$\mu_B = \mathbf{q}'\mathbf{E}$: expected benchmark return

$\sigma_B^2 = \mathbf{q}'\mathbf{V}\mathbf{q}$: variance of benchmark return

$\mu_\varepsilon = \mathbf{x}'\mathbf{E}$: expected excess return and

$\sigma_\varepsilon^2 = \mathbf{x}'\mathbf{V}\mathbf{x}$: TE variance (TE^2).

The active portfolio expected return and variance is given by:

$$\mu_P = (\mathbf{q} + \mathbf{x})'\mathbf{E} = \mu_B + \mu_\varepsilon \quad (2)$$

$$\begin{aligned} \sigma_P^2 &= (\mathbf{q} + \mathbf{x})'\mathbf{V}(\mathbf{q} + \mathbf{x}) = \sigma_B^2 + 2\mathbf{q}'\mathbf{V}\mathbf{x} + \mathbf{x}'\mathbf{V}\mathbf{x} \\ &= \sigma_B^2 + 2\mathbf{q}'\mathbf{V}\mathbf{x} + \sigma_\varepsilon^2 \end{aligned} \quad (3)$$

The portfolio must be fully invested, defined as:

$$(\mathbf{q} + \mathbf{x})'\mathbf{1} = 1 \quad (4)$$

where $\mathbf{1}$ represents an N -dimensional vector of 1s.

Using Merton's (1972) terminology, the following parameters are also defined:

$$a = \mathbf{E}'\mathbf{V}^{-1}\mathbf{E}$$

$$b = \mathbf{E}'\mathbf{V}^{-1}\mathbf{1}$$

$$c = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

$$d = a - \frac{b^2}{c}$$

Note that Jorion's (2003) paper contains an erroneous definition of c , here corrected.

Jorion (2003) also defined the quantities:

$$\Delta_1 = \mu_B - \frac{b}{c} \quad (5)$$

where $b/c = \mu_{MV}$ and

$$\Delta_2 = \sigma_B^2 - \frac{1}{c} \quad (6)$$

where $1/c = \sigma_{MV}^2$.

With these definitions in place, the frontiers are developed and distinguished by constraints imposed upon them.

3.3.2.1 Mean variance frontier in absolute return space

Minimise $\mathbf{q}_P'\mathbf{V}\mathbf{q}_P$ subject to:

$$\mathbf{q}_P' \mathbf{1} = 1$$

$$\mathbf{q}_P' \mathbf{E} = G$$

where G is the target return.

The vector of portfolio weights is determined using

$$\mathbf{q}_P = \left(\frac{a - bG}{d} \right) \mathbf{q}_{MV} + \left(\frac{bG - \frac{b^2}{c}}{d} \right) \mathbf{q}_{TG}$$

where $\mathbf{q}_{MV}$ is the vector of asset weights for the minimum variance portfolio given by:

$$\mathbf{q}_{MV} = \mathbf{V}^{-1} \frac{\mathbf{1}}{c}$$

and $\mathbf{q}_{TG}$ is the vector of asset weights for the tangent (optimal) portfolio given by:

$$\mathbf{q}_{TG} = \mathbf{V}^{-1} \frac{\mathbf{E}}{b}$$

3.3.2.2 TE frontier

Maximise $\mathbf{x}' \mathbf{E}$ subject to:

$$\mathbf{x}' \mathbf{1} = 0$$

$$\mathbf{x}' \mathbf{V} \mathbf{x} = \sigma_\varepsilon^2$$

The solution for the vector of deviations from the benchmark, $\mathbf{x}$, is:

$$\mathbf{x} = \pm \sqrt{\frac{\sigma_\varepsilon^2}{d}} \mathbf{V}^{-1} \left(\mathbf{E} - \frac{b}{c} \mathbf{1} \right)$$

3.3.2.3 Constant TE frontier

Maximise $\mathbf{x}' \mathbf{E}$ subject to:

$$\mathbf{x}' \mathbf{1} = 0$$

$$\mathbf{x}' \mathbf{V} \mathbf{x} = \sigma_\varepsilon^2$$

$$(\mathbf{q} + \mathbf{x})' \mathbf{V} (\mathbf{q} + \mathbf{x}) = \sigma_P^2$$

The solution for the vector of deviations from the benchmark, $\mathbf{x}$, is:

$$\mathbf{x} = -\frac{1}{\lambda_2 + \lambda_3} \mathbf{V}^{-1} (\mathbf{E} + \lambda_1 + \lambda_3 \mathbf{V} \mathbf{q}) \quad (7)$$

where

$$\lambda_1 = -\frac{\lambda_3 + b}{c} \quad (8)$$

$$\lambda_2 = \pm(-2) \sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2\Delta_2 - y^2}} - \lambda_3 \quad (9)$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2} \sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2\Delta_2 - y^2}} \quad (10)$$

Jorion (2003) defined

$$z = \mu_P - \mu_B \quad (11)$$

and

$$y = \sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2 \quad (12)$$

and established that the relationship between y and z is:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4\sigma_\varepsilon^2(d\Delta_2 - \Delta_1^2) = 0$$

which is a quadratic equation in both y and z .

Solving for z :

$$\begin{aligned} (4\Delta_2)z^2 - 4\Delta_1 yz + dy^2 - 4\sigma_\varepsilon^2(d\Delta_2 - \Delta_1^2) &= 0 \\ z &= \frac{4\Delta_1 y \pm \sqrt{(4\Delta_1 y)^2 - 4(4\Delta_2)(dy^2 - 4\sigma_\varepsilon^2(d\Delta_2 - \Delta_1^2))}}{8\Delta_2} \\ &= \frac{\Delta_1 y \pm \sqrt{(\Delta_1 y)^2 - \Delta_2(dy^2 - 4\sigma_\varepsilon^2(d\Delta_2 - \Delta_1^2))}}{2\Delta_2} \\ z &= \frac{\Delta_1 y \pm \sqrt{(\Delta_1^2 - d\Delta_2) \cdot (y^2 - 4\Delta_2\sigma_\varepsilon^2)}}{2\Delta_2} \end{aligned} \quad (13)$$

which describes an ellipse – a *constant* TE frontier – in return/risk space (Figure 3.4) once the definitions of y and z have been reinstalled. In Figure 3.4, each point on the ellipse represents a portfolio with TE = 5%. The point on the ellipse corresponding to the largest outperformance of the portfolio over the benchmark is common to both the TE frontier and the *constant* TE frontier. Managers attempting to maximise excess return need to move up and to

the right of the benchmark in the μ/σ plane – so the portfolio will always exhibit higher risk than that of the benchmark. This led Jorion (2003) to propose a constraint on total risk. Jorion (2003) suggested that the portfolio risk could be constrained to equal that of the benchmark (i.e. that $\sigma_P = \sigma_B$), which implies that $2\mathbf{q}'\mathbf{V}\mathbf{x} = -\sigma_\varepsilon^2$ (from 3).

The shape of the constant TE frontier in Figure 3.2a suggests that the reduction in return from this restriction may not be large because the upper hemisphere of the ellipse is flat. The effects of a constraint on total volatility are illustrated in Table 3.1, which reports the reduction in expected return and total portfolio risk (relative to the maximum return portfolio) for various levels of Δ_1 . This philosophy is also used in the subsequent development.

Table 3.1: Change in μ_P and σ_P for various levels of Δ_1 (constraints: TE prescribed, $\sigma_P = \sigma_B$) and $r_f = 2.0\%$. All values in %.

	TE →	2%		4%		6%		8%	
		$\Delta\mu_P$	$\Delta\sigma_P$	$\Delta\mu_P$	$\Delta\sigma_P$	$\Delta\mu_P$	$\Delta\sigma_P$	$\Delta\mu_P$	$\Delta\sigma_P$
$\Delta_1 = 0\%$	$\sigma_{MV} = 6\%$	0.00	-0.11	-0.03	-0.43	-0.09	-0.96	-0.21	-1.68
	$\sigma_{MV} = 8\%$	0.00	-0.11	-0.02	-0.42	-0.07	-0.94	-0.17	-1.64
	$\sigma_{MV} = 10\%$	0.00	-0.10	-0.02	-0.41	-0.06	-0.92	-0.14	-1.60
$\Delta_1 = 1\%$	$\sigma_{MV} = 6\%$	-0.02	-0.22	-0.06	-0.65	-0.16	-1.27	-0.33	-2.07
	$\sigma_{MV} = 8\%$	-0.02	-0.25	-0.06	-0.70	-0.15	-1.34	-0.30	-2.16
	$\sigma_{MV} = 10\%$	-0.02	-0.27	-0.06	-0.75	-0.15	-1.40	-0.28	-2.22
$\Delta_1 = 2\%$	$\sigma_{MV} = 6\%$	-0.03	-0.33	-0.11	-0.86	-0.25	-1.58	-0.47	-2.46
	$\sigma_{MV} = 8\%$	-0.04	-0.39	-0.12	-0.98	-0.25	-1.74	-0.46	-2.66
	$\sigma_{MV} = 10\%$	-0.04	-0.44	-0.13	-1.07	-0.26	-1.87	-0.46	-2.82

3.4 Results and discussion

The maximum return possible for a portfolio subject to a TE constraint is represented in Figure 3.2a by the open white circle (above the benchmark return). The associated absolute risk for such a constrained portfolio is also greater than that of the benchmark (i.e. to the right of the benchmark in Figure 3.2a). Jorion's (2003) additional constraint generates portfolios with both diminished return and risk relative to the maximum return portfolio. Because of the flatness of the constant TE ellipse, the reduction of the former is not substantial, and this return reduction represents the price for the reduction in risk (see results in Table 3.2).

The question arises as to whether Jorion's (2003) constraint represents an "optimal portfolio" in some sense. Portfolios subject to a TE constraint have been demonstrated to be non-optimal, but given the industry's emphasis on TE, like Jorion (2003), it is taken here as a given. Other definitions of "optimality" were considered, among them the possibility of determining portfolios which satisfy the original Sharpe ratio criterion: namely that the quotient of the portfolio's excess return (over the risk-free rate, r_f) and absolute portfolio volatility is maximised:

$$SR = \frac{\mu_P - r_f}{\sigma_P} \quad (14)$$

To characterise such a portfolio (i.e. the determination of x , the vector of deviations from the benchmark and thus ultimately, q_P , the vector of portfolio weights) requires some convoluted algebra.

The equation of the capital market line (CML) in this case, the line linking r_f and the tangent portfolio on the constant TE ellipse rather than the efficient frontier in traditional mean-variance analysis, must first be established. Once this has been determined, the relevant weights may be backed out of the resultant tangent portfolio's return and risk coordinates using the results obtained from Jorion's (2003) analysis.

3.4.1 CML line and maximum Sharpe portfolio

To obtain the intersection point between the ellipse and the CML line, note that the equation of the upper half of the ellipse in Figure 3.2 may be calculated using (11) and (13):

$$\begin{aligned} z &= \mu_P - \mu_B \\ \mu_P &= \frac{\Delta_1(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2) + \sqrt{(\Delta_1^2 - d\Delta_2) \cdot [(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)^2 - 4\Delta_2\sigma_\varepsilon^2]}}{2\Delta_2} + \mu_B \end{aligned} \quad (15)$$

The equation of the CML linking r_f and the optimal portfolio on the constant TE ellipse is:

$$\mu_P = m\sigma_P + r_f \quad (16)$$

where m is the slope of the CML line and the coordinates of the optimal portfolio are (σ_P, μ_P) .

Setting (15) and (16) equal and rearranging terms to make m the subject of the formula:

$$m = \frac{\mu_B - r_f}{\sigma_P} + \frac{\Delta_1(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2) + \sqrt{(\Delta_1^2 - d\Delta_2) \cdot [(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)^2 - 4\Delta_2\sigma_\varepsilon^2]}}{2\Delta_2\sigma_P} \quad (17)$$

As σ_P increases, m (which intersects the constant TE frontier in two places) increases until it reaches a maximum (when there is only one intersection point) and then decreases (again with two intersection points) as shown in Figure 3.3.

Differentiating (17) gives

$$\frac{dm}{d\sigma_P} = \frac{(r_f - \mu_B)}{\sigma_P^2} + \frac{\frac{(\Delta_1^2 - d\Delta_2) \cdot (\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)}{\sqrt{(\Delta_1^2 - d\Delta_2)[(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)^2 - 4\Delta_2\sigma_\varepsilon^2]} + \Delta_1}{\Delta_2} \quad (18)$$

$$- \frac{\sqrt{(\Delta_1^2 - d\Delta_2)[(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)^2 - 4\Delta_2\sigma_\varepsilon^2]} + \Delta_1 \cdot (\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)}{2\Delta_2\sigma_P^2}$$

Setting (18) = 0, i.e. the point at which the change in the slope with respect to σ_P reverses direction, and solving for μ_P and σ_P establishes the maximum Sharpe ratio portfolio coordinates.

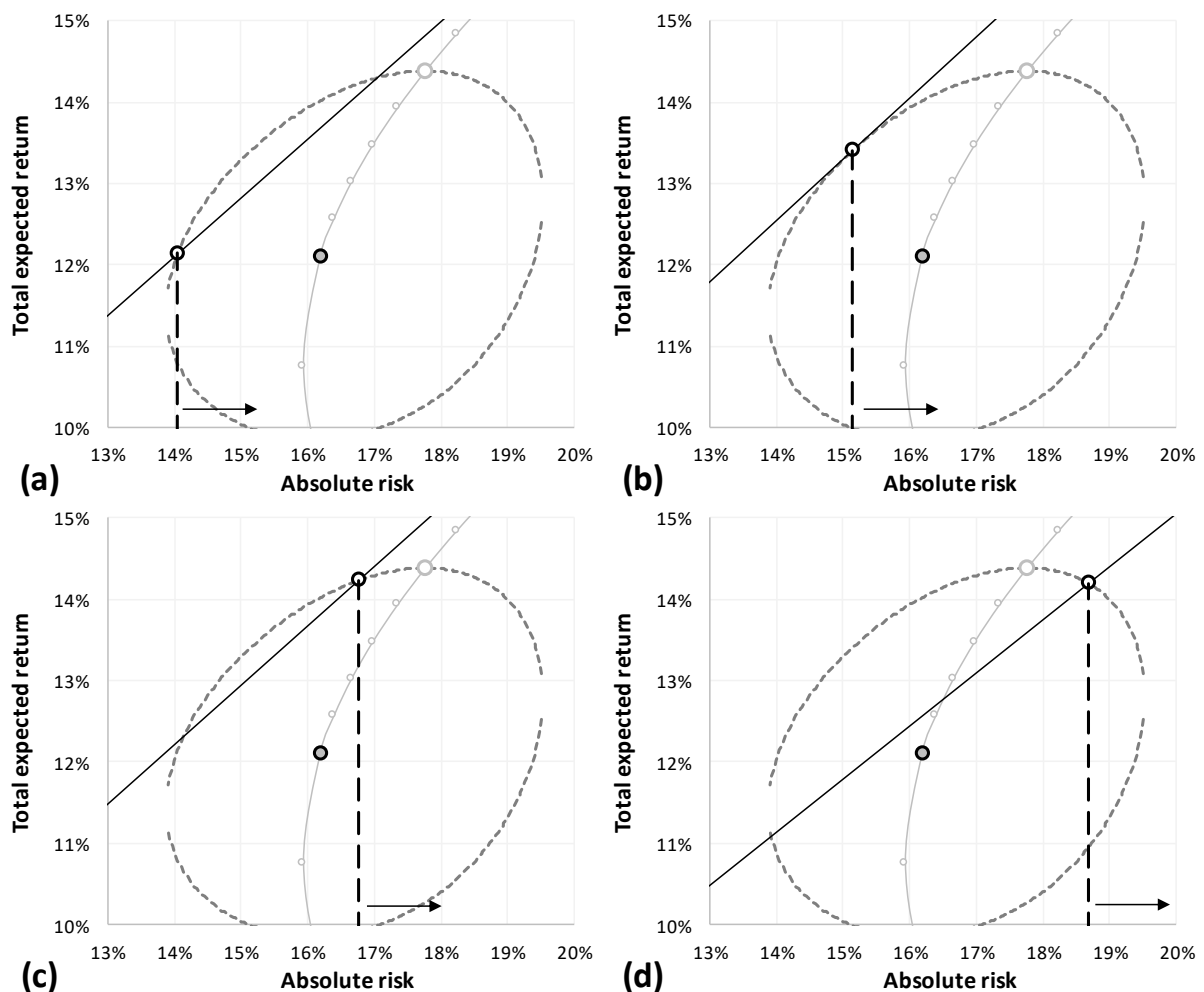


Figure 3.3: Constant TE frontier (TE = 5%) and intersection with CML for increasing values of σ_P from (a) through (d). The slope *increases* from (a) → (b) and then *decreases* from (b) → (d)

as σ_P increases. Where the slope changes direction, $dm/d\sigma_P = 0$, gives the tangent portfolio. Note the two intersection points in (a), (c) and (d), but only the single (tangent) point in (b).

Source: Own calculations

3.4.2 Vector of portfolio deviations from benchmark: x

Jorion (2003) solved for x , the vector of deviations from the benchmark and used these to determine the relevant portfolio on the constant TE frontier (with TE known).

In this work, the process is reversed. The return and risk of the desired (maximum Sharpe ratio) portfolio are *known* (from 18), so the x s are *calculated* using (7) through (10). Recall that the inputs for λ_1 , λ_2 and λ_3 as well as the values for σ_ε , σ_P and μ_P are all known, so determining the x s is a trivial exercise. The results are shown in Figure 3.4.

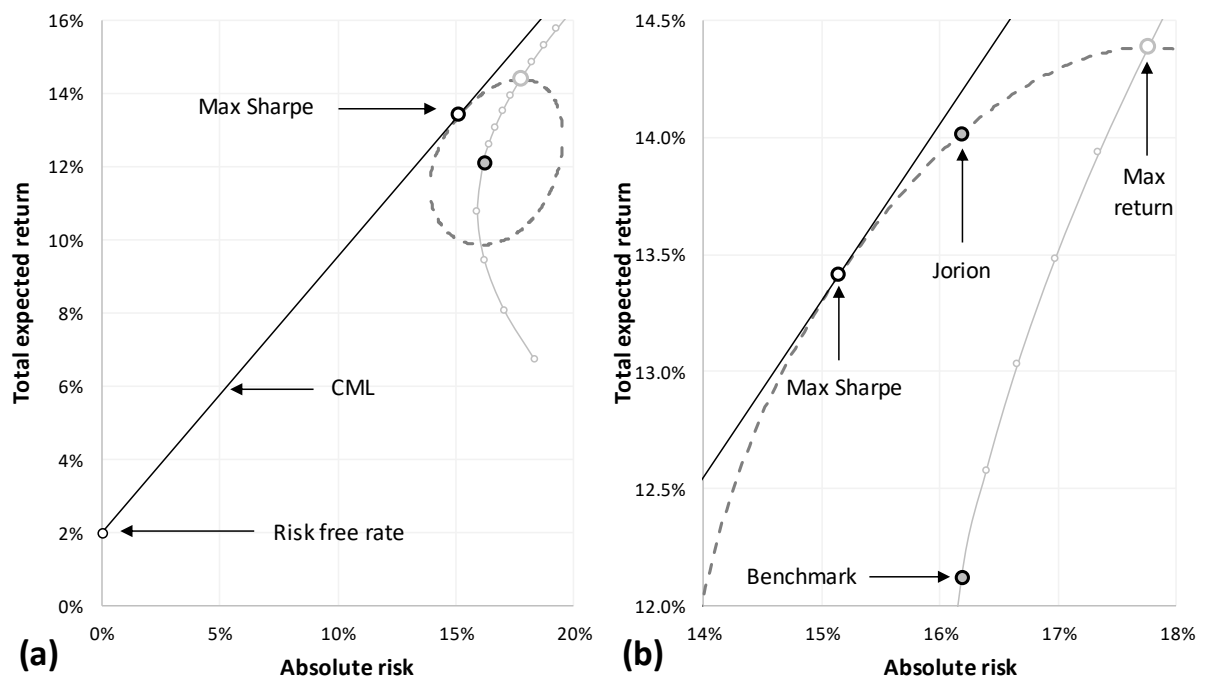


Figure 3.4: (a) TE-constrained portfolio, constant TE frontier and CML with optimal portfolio and (b) enlarged view showing all three portfolios. TE = 5% and $r_f = 2\%$.

Source: Jorion (2003) and own calculations.

Continuing the philosophy detailed in Table 3.2, the effects of a constraint on the portfolio's Sharpe ratio are illustrated in Table 3.2, which reports the reduction in expected return and total portfolio risk (again relative to the maximum return portfolio) for various levels of Δ_1 . While the reduction in expected return is greater than that for Jorion's (2003) proposal, the reduction in total volatility is significantly greater, so the relative performance improvement is considerably better.

Table 3.2: Change in μ_P and σ_P for various levels of Δ_1 (constraints: TE prescribed, $SR = SR_{\max}$) and $r_f = 2.0\%$. All values in %.

	TE →	2%		4%		6%		8%	
		$\Delta\mu_P$	$\Delta\sigma_P$	$\Delta\mu_P$	$\Delta\sigma_P$	$\Delta\mu_P$	$\Delta\sigma_P$	$\Delta\mu_P$	$\Delta\sigma_P$
$\Delta_1 = 0\%$	$\sigma_{MV} = 6\%$	-0.37	-1.14	-1.03	-2.73	-2.10	-4.76	-3.62	-7.19
	$\sigma_{MV} = 8\%$	-0.35	-1.21	-0.93	-2.80	-1.78	-4.73	-2.89	-6.92
	$\sigma_{MV} = 10\%$	-0.32	-1.21	-0.81	-2.72	-1.48	-4.48	-2.32	-6.41
$\Delta_1 = 1\%$	$\sigma_{MV} = 6\%$	-0.38	-1.16	-1.07	-2.77	-2.18	-4.83	-3.78	-7.31
	$\sigma_{MV} = 8\%$	-0.38	-1.26	-1.00	-2.91	-1.90	-4.90	-3.11	-7.18
	$\sigma_{MV} = 10\%$	-0.35	-1.28	-0.89	-2.87	-1.63	-4.73	-2.55	-6.77
$\Delta_1 = 2\%$	$\sigma_{MV} = 6\%$	-0.39	-1.20	-1.10	-2.84	-2.23	-4.93	-3.86	-7.45
	$\sigma_{MV} = 8\%$	-0.40	-1.32	-1.05	-3.03	-2.00	-5.09	-3.27	-7.44
	$\sigma_{MV} = 10\%$	-0.39	-1.36	-0.97	-3.04	-1.76	-4.99	-2.75	-7.12

The benefits of this approach are most pronounced for lower levels of TE (also found by Jorion, 2003), higher levels of absolute risk (σ_{MV}), higher r_f (see Table 3.3 which shows the improvement in the max Sharpe portfolio as r_f increases, where the return decrease is lower for higher r_f) and benchmark returns close to minimum variance portfolio levels ($\Delta_1 = 0\%$).

Table 3.3: The effect of increasing r_f on μ and σ for the TE-constrained and maximum Sharpe portfolios. $\Delta_1 = 1.0\%$, $\sigma_{MV} = 8\%$. All μ and σ values in %.

		TE →	2%	4%	6%	8%
r_f ↓	TE constrained	μ	13.60	15.07	16.55	18.02
		σ	17.72	18.21	18.89	19.76
2%	Max Sharpe	μ	13.20	14.01	14.54	14.76
		σ	16.44	15.26	13.94	12.54
		$\Delta\mu$	-0.40	-1.06	-2.01	-3.26
		$\Delta\sigma$	-1.28	-2.95	-4.96	-7.22
		Cost: $\Delta(\mu/\sigma)$	0.31	0.36	0.41	0.45
6%		μ	13.40	14.49	15.35	15.93
		σ	16.81	15.99	15.02	13.92
		$\Delta\mu$	-0.20	-0.58	-1.20	-2.10
		$\Delta\sigma$	-0.91	-2.21	-3.88	-5.84
		Cost: $\Delta(\mu/\sigma)$	0.22	0.26	0.31	0.36

3.4.3 Weights

The maximum return possible for a portfolio subject to a TE constraint *and* a maximum Sharpe ratio constraint is, again, lower than the maximum return possible for only a TE-constrained portfolio. The Sharpe ratio constraint added here results in portfolios with, like Jorion's (2003) work, both diminished return and risk relative to only TE-constrained portfolios *and* Jorion's (2003) TE-constrained and absolute risk-constrained portfolios. The flatness of the constant TE ellipse again means that the reduction of portfolio return is not substantial, again paying only a small price for the reduction in risk (see results in Table 3.3). The maximum Sharpe ratio portfolio is below and to the *left* of Jorion's (2003) proposed portfolio.

The weights of a three-asset portfolio for the three portfolios are shown in Figure 3.5.

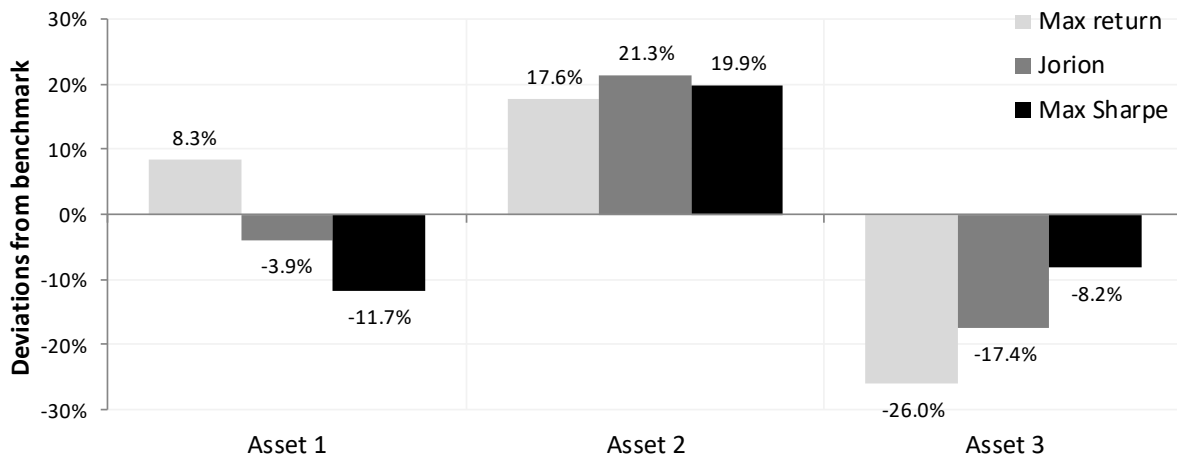


Figure 3.5: Deviations from the benchmark determined using (7) for the three portfolios: maximum return on TE frontier, position on TE frontier where $\sigma_P = \sigma_B$ (Jorion, 2003) and position on TE frontier corresponding to a maximum Sharpe ratio.

Source: Jorion (2003) and own calculations.

Although this is a stylised example, it demonstrates that differences can be considerable: in this example, even changing sign (asset 1). These large differences could potentially represent substantial changes in portfolio construction (rather than minor tweaks to the existing proposals).

3.4.4 Sharpe ratio portfolio and Jorion's (2003) proposal

The maximum Sharpe ratio portfolio coincides with Jorion's (2003) proposal when $\sigma_P = \sigma_B$. Substituting this condition into (17) gives the value of m when the two portfolios are equivalent:

$$m = \frac{\mu_B - r_f}{\sigma_B} + \frac{\sigma_\varepsilon \cdot \sqrt{(\Delta_1^2 - d\Delta_2) \cdot (\sigma_\varepsilon^2 - 4\Delta_2)} - \Delta_1\sigma_\varepsilon^2}{2\Delta_2\sigma_B}$$

But it is also known that, at this point, the slope of the CML is

$$m = \frac{\mu_P - r_f}{\sigma_B}$$

since at this point, $\sigma_B = \sigma_P$.

Equating the two slopes:

$$\begin{aligned} \frac{\mu_P - r_f}{\sigma_B} &= \frac{\mu_B - r_f}{\sigma_B} + \frac{\sigma_\varepsilon \cdot \sqrt{(\Delta_1^2 - d\Delta_2) \cdot (\sigma_\varepsilon^2 - 4\Delta_2)} - \Delta_1\sigma_\varepsilon^2}{2\Delta_2\sigma_B} \\ \mu_P &= \mu_B + \underbrace{\frac{\sigma_\varepsilon}{2\Delta_2} \left(\sqrt{(\Delta_1^2 - d\Delta_2) \cdot (\sigma_\varepsilon^2 - 4\Delta_2)} - \Delta_1\sigma_\varepsilon \right)}_{\text{Excess return over benchmark}} \end{aligned}$$

3.4.5 Maximum Sharpe ratio portfolio locus

Figure 3.6 presents the locus of the maximum Sharpe ratio portfolio as a function of TE (in this case, with static $r_f = 2\%$).

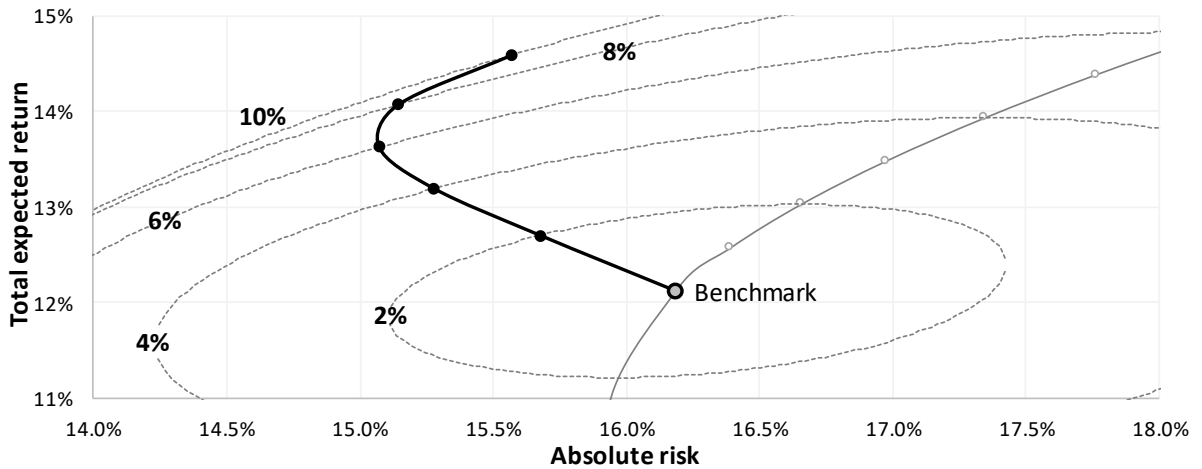


Figure 3.6: Locus of maximum Sharpe ratio portfolios for $2\% \leq TE \leq 10\%$ ($r_f = 2\%$).

Source: Jorion (2003) and own calculations.

As TE increases, the portfolio first moves up (increased return) and to the left (decreased risk) of the benchmark. At higher levels of TE, the portfolio continues to move up but then moves to the right, i.e. absolute risk increases. At some value of TE, the maximum Sharpe ratio portfolio will be coincident upon Jorion's (2003) proposal (i.e. where $\sigma_B = \sigma_P$, see Section 3.4.4)

and for higher TE values, the portfolio will continue to move up and right – eventually coincident with the maximum return portfolio. The maximum Sharpe portfolio, therefore, is best suited for low levels of TE, but even for portfolios with $\sigma_P > \sigma_B$, these are still maximum Sharpe ratio portfolios and thus, still optimal.

Figure 3.7 shows the locus of the maximum Sharpe ratio portfolio as a function of r_f – in this case with static $TE = 5\%$. As r_f increases, the maximum Sharpe ratio portfolio traverses the constant TE frontier ellipse in the clockwise direction. At a certain level of r_f , the maximum Sharpe portfolio will be coincident with Jorion's (2003) proposal, and the portfolio's maximum return = r_f when the CML line is horizontal. At higher levels of r_f , the maximum Sharpe ratio portfolio moves along the ellipse with reduced returns and increased risk. This is not unexpected: at such high levels of r_f , optimal portfolios should comprise the risk-free asset only.

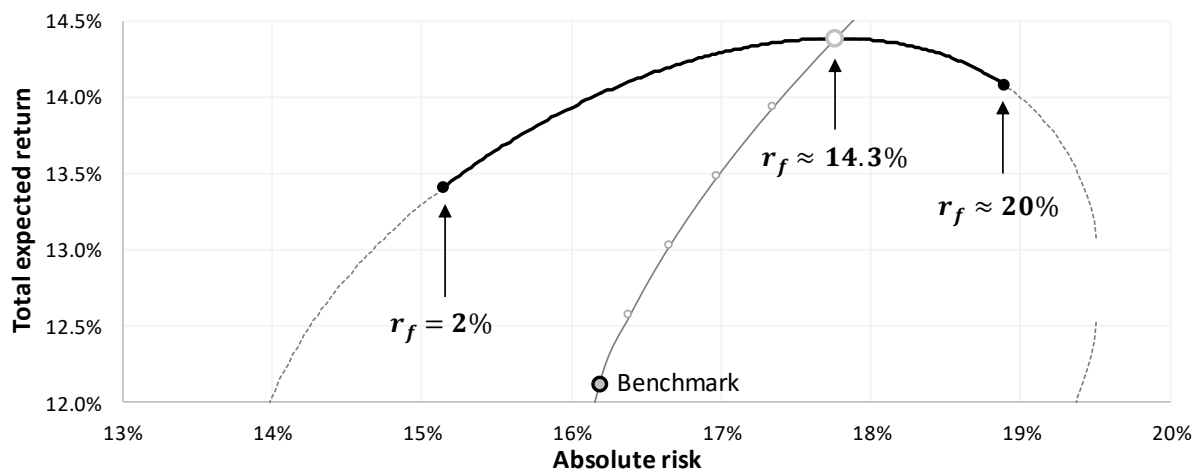


Figure 3.7: Locus of maximum Sharpe ratio portfolios for $2\% \leq r_f \leq 20\%$ ($TE = 5\%$).

Source: Jorion (2003) and own calculations.

3.5 Conclusions

Using benchmarks to gauge relative performance allows investors to evaluate value-added relative to risks taken. Benchmarks generally comprise a combination of disparate assets, dependent on investors' risk preference/profiles, but they tend to be inefficient as both the components of – and weights within – the benchmark are somewhat arbitrarily chosen. Optimisation in excess return space with an inefficient benchmark for performance comparison and performance measures such as the information ratio, results in inefficient/sub-optimal portfolio selection.

Applying tracking error constraints result in more inefficiency if fund managers naïvely pursue maximum excess returns as a sole investment objective, as this leads to higher total risk which can be diversified away. Accepting the TE mandate as given (since it is widely used in the industry), optimisation efforts have concentrated on various performance ratios and features of the TE frontier. In all previous work, absolute risk is consistently ignored.

Maximising the Sharpe ratio with respect to the constant TE frontier selects portfolios which achieve higher excess returns and lower absolute risk relative to the benchmark. Several interesting (and important) aspects of this approach have been identified and explored. The work should prove useful as it allows fund managers to calculate deviations from the benchmark (and thus, ultimately, actual portfolio weights) explicitly. These deviations have been shown to vary considerably depending on the desired portfolio selected (i.e. maximum return, benchmark risk or maximum Sharpe ratio tracking error-constrained portfolios). While this new approach does not account for tax considerations, regulatory constraints or rebalancing costs, these should be simple to implement in practice. As such, the usual due diligence should be observed in selecting assets and forecasting expected returns.

REFERENCES

- Ammann, M. and Zimmermann, H. 2001. Tracking error and tactical asset allocation. *Financial Analysts Journal*, 57(2): 32 – 43.
- Bertrand, P. 2005. A note on portfolio performance attribution: taking risk into account. *Journal of Asset Management*, 5(6): 428 – 437.
- Bertrand, P. 2009. Risk-adjusted performance attribution and portfolio optimisations under tracking-error constraints. *Journal of Asset Management*, 10(2): 75 – 88.
- Bertrand, P. 2010. Another look at portfolio optimization under tracking error constraints. *Financial Analysts Journal*, 66(3): 78 – 90.
- Bertrand, P., Prigent, J-L. and Sobotka, R. 2001. Optimisation de portefeuille sous contrainte de variance de la tracking-error. *Banque & Marchés*, 54(1): 19 – 28.
- Clarke, R., de Silva, H. and Thorley, S. 2002. Portfolio constraints and the fundamental law of active management. *Financial Analysts Journal*, 58(5): 48 – 66.
- El-Hassan, N. and Kofman, P. 2003. Tracking error and active portfolio management. *Australian Journal of Management*, 28(2): 183 – 207.
- Jansen, R. and van Dijk, R. 2002). Optimal benchmark tracking with small portfolios. *Journal of Portfolio Management*, 28(2): 33 – 39.
- Jorion, P. 1992. Portfolio optimization in practice. *Financial Analysts Journal*, 48(1): 68 – 74.

- Jorion, P. 2003. Portfolio optimization with tracking-error constraints. *Financial Analysts Journal*, 59(5): 70 – 82.
- Larsen, G. A. and Resnick, B. G. 2001. Parameter estimation techniques, optimization frequency, and portfolio return enhancement. *Journal of Portfolio Management*, 27(4): 27 – 34.
- Menchero, J. and Hu, J. 2006. Portfolio risk attribution. *The Journal of Performance Measurement*, 10(3): 22 – 33.
- Merton, R. 1972. An analytic derivation of the efficient portfolio frontier. *The Journal of Financial and Quantitative Analysis*, 7(4): 1851 – 1872. doi: 10.2307/2329621.
- Plaxco, L. M. and Arnott, R. D. 2002. Rebalancing a global policy benchmark. *Journal of Portfolio Management*, 28(2): 9 – 22.
- Riccetti, L. 2010. Minimum tracking error volatility. *Quaderno di Ricerca n°340 del Dipartimento di Economia dell'Università Politecnica delle Marche.*, Working paper, 340.
- Roll, R. 1992. A mean/variance analysis of tracking error. *The Journal of Portfolio Management*, 18(4): 13 – 22.
- Stowe, D. L. 2014. Tracking error volatility optimization and utility improvements. *Working paper*. Online: [http://swfa2015.uno.edu/F_Volatility & Risk Exposure/paper_221.pdf](http://swfa2015.uno.edu/F_Volatility_%20Risk_Exposure/paper_221.pdf), accessed 14 June 2016.
- Thomas, B., Rottschäfer, D. and Zvingelis, J. 2013. A tracking error primer. Envestnet PMC. *White paper*. Online: <http://www.envestnet.com/sites/default/files/documents/A%20Tracking%20Error%20Primer%20-%20White%20Paper.pdf>, accessed 22 Jul 2016.
- Wu, M. and Jakshoj, C. 2011. Risk-adjusted performance attribution and portfolio optimisation under tracking-error constraints for SIAS Canadian Equity Fund. *Masters dissertation*, Simon Fraser University, Canada. Online: <http://summit.sfu.ca/item/13058>, accessed 8 Aug 2016.

Chapter 4

Active investment strategies under tracking error constraints

Michael Maxwell⁹ and Gary van Vuuren¹⁰

Abstract

Active portfolio managers are judged on their ability to outperform agent's benchmarks, so optimising fund returns is critically important. Maximising fund outperformance is, however, non-trivial because active portfolios are subject to tracking error (TE) (and other) constraints. Portfolios constrained by a TE are fenced by an elliptical frontier in mean/variance space and may not be efficient. Some attempts have been made to identify optimal portfolios subject to the restrictions imposed by TEs, i.e. to locate these on the frontier. We review these portfolio assemblies and introduce more possibilities: portfolios which are maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum risk. Such portfolios behave differently to those which are part of the efficient set, i.e. populate the efficient frontier and are TE-unconstrained.

Keywords

Tracking error frontier, optimal portfolios, investment constraints, benchmark

JEL classification

C52, G11

4.1 Introduction

Modern portfolio theory (MPT) treats portfolios of assets as single, connected entities whose mean returns are a linear combination of the individual component returns and whose return variance embraces not only individual constituent variances, but correlations between them as well (Markowitz, 1952). MPT introduced the notion of an efficient frontier in mean/variance space: portfolios which lie on the frontier are efficient, i.e. their locus traces out a path of maximum return for given variance levels.¹¹ Using MPT, Sharpe (1966) established the capital market line (CML) whose origin on the return axis is the risk-free rate and whose tangential intersection with the efficient frontier is the portfolio with the highest risk-adjusted return for the associated level of risk, i.e. the maximum Sharpe ratio.

⁹ Postgraduate student, North West University, Potchefstroom Campus, South Africa.

¹⁰ Extraordinary Professor, North-West University, South Africa.

¹¹ Henceforth, the standard deviation (or the square root of the variance) is used as the risk metric.

Portfolio optimisation on the efficient frontier has generated copious research: the literature is replete with different examples of 'optimised' efficient frontier portfolios, their position in mean/risk space and the calculations required to determine the asset weights which constitute the portfolio. Example portfolios include the maximum risk-adjusted (Sharpe, 1966), traditional 60% stock/40% bond (Cheng, 1971), maximum diversified (Choueifaty & Coignard, 2008), risk-parity (Maillard, Roncalli & Teiletche, 2010), minimum intra-portfolio correlation (Livingstone, 2013) and global minimum variance (Bodnar, Parolya, & Schmid, 2018). The focus in this work adapts the portfolio optimisation techniques, mentioned above, to TE constrained portfolios. Other approaches – which will not be considered here – exist to accomplish this, such as taking liquidity and higher moments of asset returns into account (Beardsley, Field & Xiao, 2012) and using momentum (Mattei, 2017).

The efficient frontier and the efficient portfolios on it, represent an idealised world in which component positions may be long or short (of any size) and which are not constrained by agent mandates. The reality is that active portfolio construction is restricted by many agent requirements, for example: portfolio performance constrained by a range of permissible portfolio β s or α s and/or lower than a tracking error (the standard deviation of the difference between portfolio and benchmark returns). Violating these constraints can have serious legal and reputational consequences for fund managers (Penhall, 2015).

A prominent restriction imposed on active fund managers is that of the TE. Roll (1992) explored the relationship between fund returns and variance when subjected to different levels of TEs and formulated the TE frontier: the locus of maximum portfolio returns (and associated risk) for different TE levels. Jorion (2003) examined the set of portfolios which satisfied TE constraints and found that, for a given TE level, the set of possible mean/risk values was an ellipse, which at low TE levels enclosed the benchmark. Jorion (2003) argued that a possible 'optimal' portfolio subject to a TE should be that which has the same risk as the benchmark, rather than that which has the maximum return for a prescribed TE. Maxwell, Daly, Thomson & van Vuuren (2018) continued this work and determined the position of a maximum Sharpe ratio portfolio on this constant TE frontier, i.e. where the CML tangentially intersects the constant TE ellipse.

As investment strategies shift from passive back to active management approaches (Gilreath, 2017; *Financial Times*, 2017), portfolio optimisation is once again becoming important (while

still constrained by TEs, β s and α s). No work to our knowledge has, to date, established optimal investment strategies subject to the constant TE constraint. The contribution of this work then is to establish, mathematically if possible, methodologies for determining the requisite portfolio asset weights for various investment strategies which are subject to TE constraints.

The remainder of this article proceeds as follows. Section 4.2 provides a literature survey which establishes the timeline and development of portfolio optimisation techniques. Section 4.3 presents the data used for the analysis and sets out the relevant quantitative techniques and mathematics used for the determination of the various strategy portfolio weights. In Section 4.4, the results obtained from the analysis are presented and discussed and Section 4.5 concludes.

4.2 Literature survey

Portfolio theory governs the process by which managers construct portfolios to optimise returns, given investors' risk tolerance. Although widely used, portfolio selection theory is based on several assumptions which are often not observed in practice.

Fund managers are classified into two broad types: passive and active: the former tend to buy and hold securities, closely matching mandated benchmark weights (thereby keeping transaction costs and fees low and closely matching benchmark returns) while the latter trade securities dynamically, seeking to maximise profits through stock selection and market timing. There is a contentious debate over which approach (active versus passive) provides optimal¹² returns. While it is true that many active portfolios underperform their benchmark, all truly passive funds underperform theirs. It is only with an element – however small – of *active* management (or zero management plus administration and trading costs) that a 'passive' fund can match its benchmark performance.

Both types of fund manager – active and passive – generally optimise their portfolio returns in relative (rather than absolute) return space, which leads to higher unsystematic risk and greater absolute portfolio volatility. To satisfy the absolute return performance condition of an investor's utility function, choosing constituent weights close to the benchmark in bull market conditions is sufficient. In bear market conditions, however, when the benchmark is generally *inefficient*, fund managers must outperform the benchmark. This requires superior

¹² Protagonists also argue over the definition of *optimal*.

relative return performance, only possible if the manager takes on active risk, but taking substantial bets which breach mandates incurs corresponding punitive penalty charges. To limit these costs, constraints are imposed on the active portfolio (Jorion, 2003).

Active fund managers are constrained by (amongst other agency-mandated requirements) TE (the standard deviation of the differences between portfolio and benchmark returns), an active risk measure that reflects portfolio manager decisions to deviate from benchmark positions to achieve positive excess returns (note, that TE is directionless so deviations from benchmark positions could equally result in *negative* 'excess' returns). When used to determine manager competence, TE is not used in isolation and is best applied in combination with other performance evaluators. The optimal level of TE depends on the portfolio's investment policies, i.e. outperformance target, risk-return profile, growth strategy (value or growth), etc. Thomas, Rottschäfer & Zvingelis, (2013) outline several causes of TE (fees, transaction costs, taxes, factor tilts, cash management), however the prime driver is return maximisation. In the process of portfolio construction, not all assets which constitute the benchmark will be held, and those that are will be held in different proportions. Consequently, benchmark and portfolio returns differ from month to month, resulting in a TE.

Performance fee incentives compel fund managers to sometimes adopt unnecessary risk to achieve high excess returns, but asset returns are noisy and the average manager's performance can only be accurately assessed after sufficient performance data have been accumulated. The value added by such managers is not obvious, so instead investors focus on TE as an important performance measure. Fund managers must balance their investment strategy such that they simultaneously maximise benchmark outperformance (i.e. increase TE) while minimising excess return variance (i.e. decrease TE).

Roll (1992) constructed portfolios that maximised excess returns over a benchmark for a given TE and illustrated that a TE frontier comprises all such portfolios, one at each level of TE volatility. This frontier has a similar shape to the standard efficient frontier shifted to the right in mean-variance space.¹³ These portfolios are mean/variance inefficient if the benchmark is

¹³ The true position of the global total return efficient frontier is unknown as this depends on individual asset expected returns whose estimation is fraught with error because of the large component of noise in observed returns (Rudolf, Wolter & Zimmermann, 1999).

inefficient (which is most often the case), i.e. TE managed portfolios are dominated by other portfolios that have both lower volatility and higher total return.

Jorion (2003) proposed an approach to optimise the risk/return relationship of TE-constrained active portfolios by imposing an additional, absolute risk, constraint. Jorion (2003) derived the *constant tracking-error frontier* in the original mean-variance plane. This frontier is an ellipse, and the flatness of this ellipse allowed Jorion (2003) to impose an additional constraint of requiring that total portfolio volatility must be the same as the benchmark volatility. This additional restriction substantially improved active portfolio performance.

Rudolf, Wolter & Zimmermann (1999) compared four linear models for minimising the TE of a portfolio's returns, arguing that because performance fees are linear, linear deviations provide a more accurate representation of investors' risk preferences (as opposed to non-linear, squared deviations).

Ammann and Zimmermann (2001) investigated the relationship between TE and restrictions on deviations of asset weights from the benchmark. Constraints were imposed on these deviations as it is more convenient for investors to specify limits on non-tactical deviations from the benchmark weights defined for various asset classes rather than on the portfolio manager.

Chow (1995) suggested a utility function that measures return, variance and TE was a more appropriate performance metric than standard portfolio performance evaluators (such as return and variance) because it accounts for both *total risk and TE*. Chow (1995) found that the implementation of this utility function generated a set of efficient portfolios which included the mean-variance efficient set, the mean tracking-error efficient set and all convex combinations of both sets.

Bertrand (2005) examined the consistency between the risk-adjusted performance attribution process and portfolio optimisation under TE constraints. Consistency was only attained when optimising the TE constraint in isolation: additional constraints, on total risk for example, distorts the information ratio of the entire portfolio indicating that no equilibrium between expected return and relative risk has been achieved.

Bertrand (2009) examined the TE minimisation problem from another perspective. Instead of considering constant TE frontiers, he allowed the TE to vary and fixed risk aversion, thereby

establishing an optimal portfolio with several desirable attributes and which lies on what he termed as iso-aversion frontier. All portfolios on this frontier have the same IR, allowing the selection of portfolio to satisfy volatility preference and/or TE preference.

Riccetti (2010) derived analytical methods that indicated whether managers using active strategies could simultaneously generate a positive excess return large enough to cover performance fees and limit the portfolio's variance to be less than that of the benchmark. This formulation was a required, yet insufficient, condition to outperform the benchmark's returns, without increasing portfolio variance.

Wu and Jakshoj (2011) decompose excess returns into selection and allocation effects and then apply this attribution approach for an unconstrained portfolio and a TE-constrained optimal portfolio. Wu and Jakshoj (2011) found that the TE optimised weights do improve the expected IR for a given portfolio, a result confirmed over several time periods. TE-optimised portfolios were also easier to implement.

Bajeux-Besnainou, Portait & Tergny (2011) investigated active portfolios subject to TE and component weight constraints (a common agency-mandated feature of active portfolios, e.g. no more than 10% equity, only investment grade sovereign bonds, and so on) and found that these restrictions to be simultaneously binding.

Zibri (2014) examined the relative risk reduction of minimum TE portfolios for global assets (equities and bonds) and found that weight constraints have no effect on out-of-sample TE reduction. The result was robust when applied to different scenarios of transaction costs and revision frequencies.

Stowe (2014) demonstrated that active managers optimise over a TE-parameterised utility function (a commonly-applied framework in active management and a consequence of performance incentives). Stowe (2014) showed that relative optimisation over TE is inefficient and sub-optimal portfolios are selected by active managers (i.e. active managers are inefficient and inferior for investors when constrained by TEs, even though they may have more skill, information, expertise and opportunity).

Maxwell, Daly, Thomson & van Vuuren (2018) applied a different (rather than an additional) restriction to that of Jorion (2003) to the existing constraint set on the constant TE frontier in absolute risk/return space. Maxwell et al., (2018) derived a methodology for maximising the

traditional Sharpe ratio on the constant TE frontier which generated portfolios with a lower volatility but higher return than the benchmark, satisfied the TE constraint and maximised the ratio of excess absolute return to risk.

This work describes optimal portfolios on the *efficient* frontier and introduces new optimal investment suggestions. Our contribution applies optimal investment strategies to portfolios constrained by TE. We connect these ideas and adapt these *efficient frontier* strategies to the constant TE constrained frontier. No work to our knowledge has, to date, established such optimal investment strategies subject to constant TE constraints. This work develops (mathematically where possible) approaches for determining the requisite optimal portfolio asset weights for various investment strategies constrained by the TE.

4.3 Data and methodology

4.3.1 Data

The data comprised simulated realistic weights, returns, volatilities and correlations for a small, standardised benchmark comprising three assets with the stylised description as given in Table 4.1.

Table 4.1: Stylised input data.

Assets	A	B	C
Mean annual return	15%	19%	6%
Annual volatility	28%	25%	18%
Correlation matrix	1	0.09	0.16
	0.09	1	0.12
	0.16	0.12	1
Benchmark weights	50%	22%	28%

Portfolio constituents were derived only from the benchmark universe (and short-selling of benchmark constituents was permitted). This may constitute an unrealistic representation of what usually occurs in practice: most managed, TE-constrained portfolios are low-risk pension funds. The mandates governing investments in such funds will usually stipulate conservative investment strategies, so the high risks which arise from short selling will not be permitted. See Daly (2018) for a discussion on the differences between unconstrained and constrained (in the long/short sense) constant TE frontiers.

Note that the "assets" which constitute the portfolio in the examples which follow could be asset *classes* (such as equity, bonds and cash), specific industry *sectors* within an asset class (e.g. an industrial equity index, a banking and finance index, etc.) or individual *assets* such as single name stocks or bonds.

4.3.2 Methodology

To establish the methodologies required for the various frontiers, some definitions are first required. This section proceeds by introducing and describing the relevant variables and algebraic components. The mathematics governing the generation of the efficient frontier is then set out, followed by the algebra which defines the TE frontier and then the *constant* TE frontier. Having built these foundations, the algebraic methodology which is required to extract the portfolio weights for each possible strategy, is presented. This section follows Jorion's (2003) terminology and approach closely.

4.3.2.1 Components

Active fund managers are tasked with outperforming specified benchmarks and the active asset positions they take may or may not be benchmark components (depending on the mandate governing the fund). The algebra required to derive the relevant investment strategy weights uses the same underlying variables, matrices and matrix notation, defined below.

$\mathbf{q}$: vector of benchmark weights for a sample of N assets

$\mathbf{x}$: vector of deviations from the benchmark

$\mathbf{q}_P (= \mathbf{q} + \mathbf{x})$: vector of portfolio weights

$\mathbf{E}$: vector of expected returns,

$\boldsymbol{\sigma}$: vector of benchmark component volatilities

ρ : benchmark correlation matrix

$\mathbf{V}$: covariance matrix of asset returns and

r_f : is the risk-free rate.

Net short sales *are* allowed in this formulation, so the total active weight $\mathbf{q}_i + \mathbf{x}_i$ may be negative for any individual asset, i . The universe of assets can generally exceed the components of the benchmark, but for Roll's (1992) methodology, assets in the benchmark *must* be included.

Expected returns and variances are expressed in matrix notation as:

$$\mu_B = \mathbf{q}'\mathbf{E}: \quad \text{expected benchmark return}$$

$$\sigma_B^2 = \mathbf{q}'\mathbf{V}\mathbf{q}: \quad \text{variance of benchmark return}$$

$$\mu_\varepsilon = \mathbf{x}'\mathbf{E}: \quad \text{expected excess return; and}$$

$$\sigma_\varepsilon^2 = \mathbf{x}'\mathbf{V}\mathbf{x}: \quad \text{TE variance (i.e. } TE^2\text{)}.$$

The active portfolio expected return and variance is given by:

$$\mu_P = (\mathbf{q} + \mathbf{x})'\mathbf{E} = \mu_B + \mu_\varepsilon \quad (1)$$

$$\begin{aligned} \sigma_P^2 &= (\mathbf{q} + \mathbf{x})'\mathbf{V}(\mathbf{q} + \mathbf{x}) = \sigma_B^2 + 2\mathbf{q}'\mathbf{V}\mathbf{x} + \mathbf{x}'\mathbf{V}\mathbf{x} \\ &= \sigma_B^2 + 2\mathbf{q}'\mathbf{V}\mathbf{x} + \sigma_\varepsilon^2 \end{aligned} \quad (2)$$

The portfolio must be fully invested, defined as:

$$(\mathbf{q} + \mathbf{x})'\mathbf{1} = 1 \quad (3)$$

where $\mathbf{1}$ represents an N -dimensional vector of 1s.

Using Merton's (1972) terminology, the following parameters are also defined:

$$\begin{aligned} a &= \mathbf{E}'\mathbf{V}^{-1}\mathbf{E}, b = \mathbf{E}'\mathbf{V}^{-1}\mathbf{1}, c = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}, d = a - \frac{b^2}{c} \\ \Delta_1 &= \mu_B - \frac{b}{c} \end{aligned} \quad (4)$$

where $b/c = \mu_{MV}$ and

$$\Delta_2 = \sigma_B^2 - \frac{1}{c} \quad (5)$$

where $1/c = \sigma_{MV}^2$.

With these definitions in place, the frontiers are developed and distinguished by constraints imposed upon them. Note that, where the algebra allows for deviations from the benchmark to be calculated ($\mathbf{x}$) these are presented and the total portfolio component weights required

are simply $\mathbf{q} + \mathbf{x}$ ($= \mathbf{q}_P$). In other cases, where the investment strategy is unaffected by the imposition of a TE constraint, the relevant portfolio weights, w , are used instead.

4.3.2.2 Mean variance frontier in absolute return space

Minimise $\mathbf{q}_P' \mathbf{V} \mathbf{q}_P$ subject to:

$$\mathbf{q}_P' \mathbf{1} = 1$$

$$\mathbf{q}_P' \mathbf{E} = G$$

where G is the target return.

The vector of portfolio weights is determined using

$$\mathbf{q}_P = \left(\frac{a - bG}{d} \right) \mathbf{q}_{MV} + \left(\frac{bG - \frac{b^2}{c}}{d} \right) \mathbf{q}_{TG} \quad (6)$$

where $\mathbf{q}_{MV}$ is the vector of asset weights for the minimum variance portfolio given by:

$$\mathbf{q}_{MV} = \mathbf{V}^{-1} \frac{\mathbf{1}}{c}$$

and $\mathbf{q}_{TG}$ is the vector of asset weights for the tangent (optimal) portfolio given by:

$$\mathbf{q}_{TG} = \mathbf{V}^{-1} \frac{\mathbf{E}}{b}$$

4.3.2.3 TE frontier

Maximise $\mathbf{x}' \mathbf{E}$ subject to:

$$\mathbf{x}' \mathbf{1} = 0$$

$$\mathbf{x}' \mathbf{V} \mathbf{x} = \sigma_\varepsilon^2$$

The solution for the vector of deviations from the benchmark, $\mathbf{x}$, is:

$$\mathbf{x} = \pm \sqrt{\frac{\sigma_\varepsilon^2}{d}} \mathbf{V}^{-1} \left(\mathbf{E} - \frac{b}{c} \mathbf{1} \right) \quad (7)$$

4.3.2.4 Constant TE frontier

Maximise $\mathbf{x}' \mathbf{E}$ subject to:

$$\mathbf{x}' \mathbf{1} = 0$$

$$\mathbf{x}' \mathbf{V} \mathbf{x} = \sigma_\varepsilon^2$$

$$(\mathbf{q} + \mathbf{x})' \mathbf{V}(\mathbf{q} + \mathbf{x}) = \sigma_P^2$$

The solution for the vector of deviations from the benchmark, $\mathbf{x}$, is:

$$\mathbf{x} = -\frac{1}{\lambda_2 + \lambda_3} \mathbf{V}^{-1}(\mathbf{E} + \lambda_1 + \lambda_3 \mathbf{V} \mathbf{q}) \quad (8)$$

where

$$\lambda_1 = -\frac{\lambda_3 + b}{c} \quad (9)$$

$$\lambda_2 = \pm(-2) \sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2 \Delta_2 - y^2}} - \lambda_3 \quad (10)$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2} \sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2 \Delta_2 - y^2}} \quad (11)$$

Jorion (2003) defined

$$z = \mu_P - \mu_B \quad (12)$$

and

$$y = \sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2 \quad (13)$$

and established that the relationship between y and z is:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4\sigma_\varepsilon^2 (d\Delta_2 - \Delta_1^2) = 0$$

which describes an ellipse – a *constant* TE frontier – in return/risk space Solving for z :

$$z = \frac{\Delta_1 y \pm \sqrt{(\Delta_1^2 - d\Delta_2) \cdot (y^2 - 4\Delta_2 \sigma_\varepsilon^2)}}{2\Delta_2} \quad (14)$$

The constant TE frontier shown in Figure 10 establishes the boundary of possible risk-return combinations of permissible (i.e. satisfy the TE constraint imposed by the mandate), active portfolios. The upper segment of this frontier, shown in Figure 4.1, is bounded on the left by the minimum variance portfolio and above by the maximum return portfolio; both portfolios are TE-constrained. This segment of the constant TE frontier represents the efficient set of portfolios in a TE-constrained milieu. The question is: which of these portfolios represents the *optimal* portfolio? Several possibilities are considered, each satisfying various investment objectives and utility requirements.

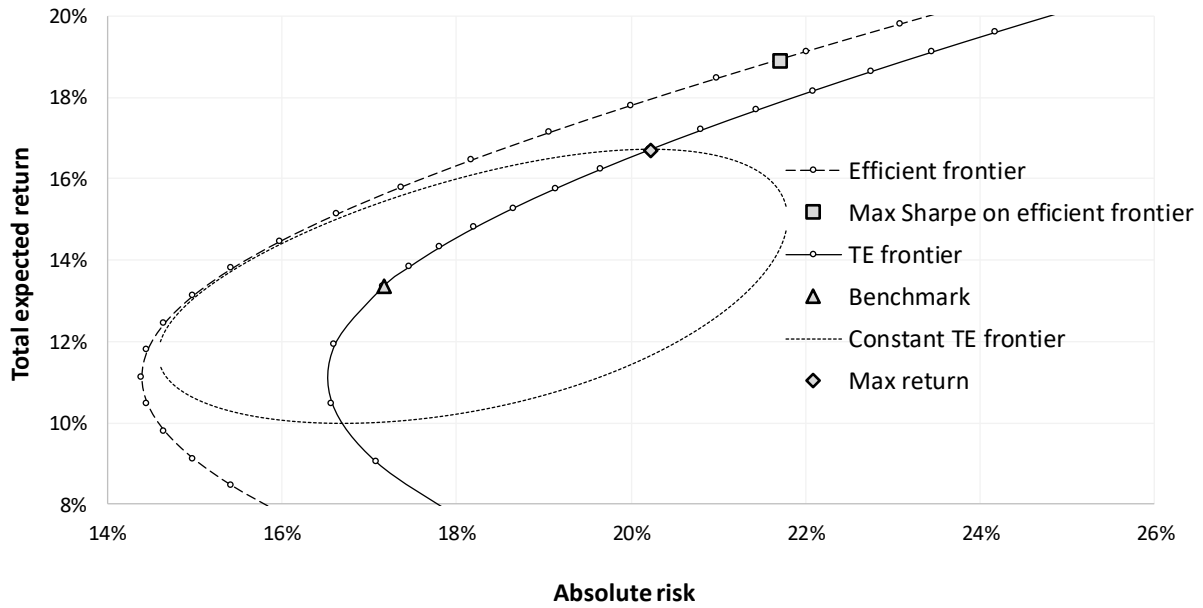


Figure 4.1: Efficient frontier, TE frontier and constant TE frontier in mean/standard deviation space. The square marker indicates the maximum Sharpe ratio on the global efficient frontier with no constraints imposed. $TE = 7\%$ and $r_f = 5\%$.

A highly risk-averse manager may opt for a minimum variance portfolio (extreme left end of the ellipse), foregoing potential higher returns in exchange for the lowest possible risk, while a risky manager may select the maximum return portfolio at the expense of associated high risk. Managers seeking optimal risk/reward trade-offs would choose tangent (maximum Sharpe ratio) portfolios (Maxwell, Daly, Thomson & van Vuuren, 2018), shown in Figure 4.2, while others may seek portfolios for which the diversification ratio is at a maximum, and still others might desire a minimum intra-correlation portfolio, or one which exhibited risk parity, etc. Note that component volatilities, correlations, expected returns and benchmark weights are all fixed, the variables in this exposition are the active portfolio weights. This is a major contribution of this paper – describing and defining these weights for active portfolios *subject to a TE constraint*.

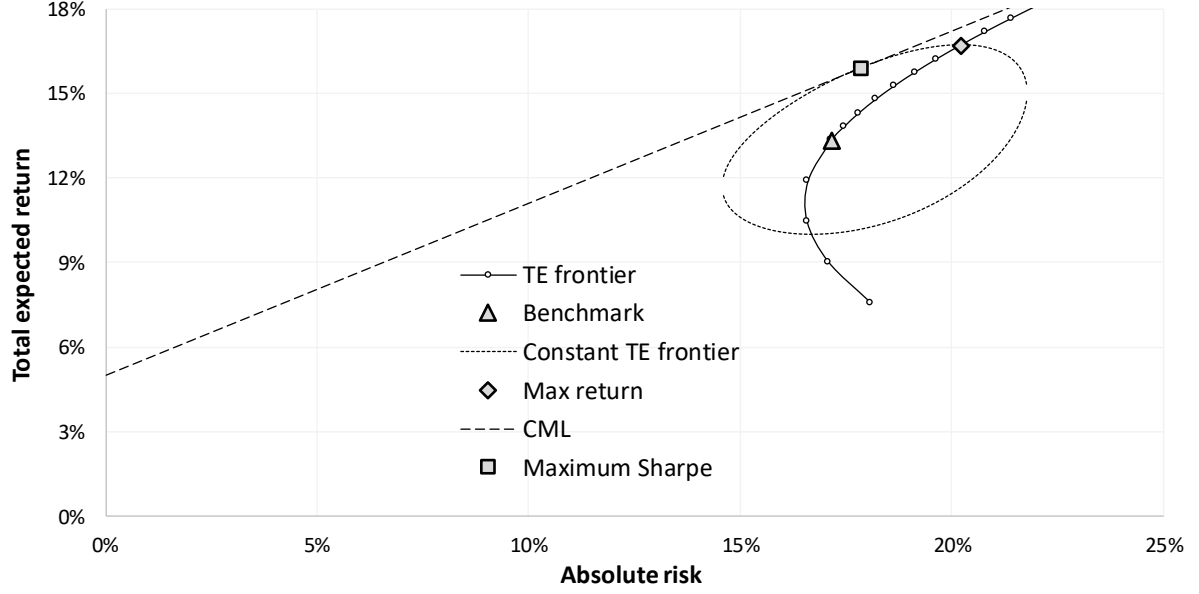


Figure 4.2: TE frontier, constant TE frontier and CML (associated with the constant TE frontier) in mean/standard deviation space. The square marker in this figure indicates the maximum Sharpe ratio on the constant TE frontier. $TE = 7\%$ and $r_f = 5\%$.

4.3.2.5 Maximum return

Jorion (2003) showed that the absolute maximum return on the constant TE frontier would be reached at the intersection of the TE frontier with the constant TE frontier, i.e. where

$$\mu_P = \mu_B + \sqrt{dT}.$$

The associated portfolio volatility is calculated using (2)

$$\sigma_P^2 = \sigma_B^2 + 2\Delta_1 \sqrt{\frac{T}{d}} + \sigma_\varepsilon^2.$$

With the component quantities in (13) now known, y is substituted into (9), (10) and (11) which then yields the vector of portfolio weight deviations from the benchmark, x , in (8). Note that this portfolio is equivalent to maximising the information ratio, IR, given by:

$$IR = \frac{\text{Excess return}}{TE}.$$

With a fixed TE, the maximum IR is reached when the numerator reaches a maximum, i.e. where $\mu_P = \mu_B + \sqrt{dT}$.

4.3.2.6 Benchmark risk

For this constraint, Jorion (2003) set $\sigma_P^2 = \sigma_B^2$ so that, from (2), this constraint implies that

$$2\mathbf{q}'\mathbf{V}\mathbf{x} = -\sigma_\varepsilon^2 \quad (15)$$

From (15), the vector of portfolio weight deviations from the benchmark, $\mathbf{x}$, in (8) may be determined.

4.3.2.7 Maximum Sharpe ratio (risk-adjusted return)

Using the approach devised by Maxwell *et al.*, (2018), the tangent portfolio (maximum Sharpe ratio portfolio) on the constant TE frontier occurs where the slopes of the tangent line and the constant TE frontier are equal, i.e. where:

$$\begin{aligned} \frac{(r_f - \mu_B)}{\sigma_P^2} &= \frac{\sqrt{(\Delta_1^2 - d\Delta_2)[(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)^2 - 4\Delta_2\sigma_\varepsilon^2]} + \Delta_1 \cdot (\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)}{2\Delta_2\sigma_P^2} \\ &\quad - \left(\frac{\frac{(\Delta_1^2 - d\Delta_2) \cdot (\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)}{\sqrt{(\Delta_1^2 - d\Delta_2)[(\sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2)^2 - 4\Delta_2\sigma_\varepsilon^2]} + \Delta_1}}{\Delta_2} \right) \end{aligned} \quad (16)$$

Solving for μ_P and σ_P establishes the maximum Sharpe ratio portfolio coordinates in return/risk space. Because these coordinates are unique, the portfolio weight deviations from the benchmark, $\mathbf{x}$, are easily reverse engineered from (16). Recall that the Sharpe ratio is

$$SR = \frac{\mu_P - r_f}{\sigma_P} \quad (17)$$

4.3.2.8 Minimum variance

Jorion (2003) showed that, using (3) and (7), the active portfolio volatility is:

$$\sigma_P^2 = \sigma_B^2 \pm 2 \sqrt{\frac{T}{d}} (\mu_B - \mu_{MV}) + \sigma_\varepsilon^2$$

and that the absolute minimum variance (of a portfolio subject to a TE constraint) is:

$$\sigma_P^2 = \sigma_B^2 - 2 \sqrt{T(\sigma_B^2 - \sigma_{MV}^2)} + \sigma_\varepsilon^2.$$

As before, with the component quantities in (13) now known, $\mathbf{y}$ is substituted into (9), (10) and (11) which then yields the vector of portfolio weight deviations from the benchmark in (8). The associated expected return is calculated using (1).

4.3.2.9 Maximum diversification

The diversification ratio was introduced by Choueifaty (2006) and is defined as:

$$DR = \frac{(\mathbf{q} + \mathbf{x})' \boldsymbol{\sigma}}{\sqrt{(\mathbf{q} + \mathbf{x})' \mathbf{V} (\mathbf{q} + \mathbf{x})}} \quad (18)$$

Choueifaty (2006) asserted that portfolios with maximum DR provided an efficient alternative to market cap-weighted portfolios. Maximum diversification portfolios have the following Lagrange function (Pemberton & Rau, 2007):

$$L(\mathbf{q} + \mathbf{x}, \lambda) = \frac{1}{2} (\mathbf{q} + \mathbf{x})' \mathbf{V} (\mathbf{q} + \mathbf{x}) - \lambda ((\mathbf{q} + \mathbf{x})' \boldsymbol{\sigma} - 1)$$

where λ is a Lagrange multiplier and L is the Lagrangian function which, at its maximum, satisfies:

$$\begin{aligned} \frac{\partial L(\mathbf{q} + \mathbf{x}, \lambda)}{\partial (\mathbf{q} + \mathbf{x})} &= \frac{1}{2} \cdot 2\mathbf{V}(\mathbf{q} + \mathbf{x}_{MD}) - \lambda \boldsymbol{\sigma} = 0 \\ \frac{\partial L(\mathbf{q} + \mathbf{x}, \lambda)}{\partial \lambda} &= (\mathbf{q} + \mathbf{x}_{MD})' \boldsymbol{\sigma} - 1 = 0 \end{aligned}$$

where $\mathbf{x}_{MD}$ is the vector of active portfolio weights for the maximum diversification ratio. Solving these simultaneous equations gives:

$$\mathbf{x}_{MD} = \frac{\mathbf{V}^{-1} \boldsymbol{\sigma}}{\boldsymbol{\sigma}' \mathbf{V}^{-1} \boldsymbol{\sigma}} - \mathbf{q}.$$

Note that these weights generate a universal, non-TE-constrained, MD portfolio (so it is not necessarily on, or inside, the constant TE frontier). A closed-form solution for a TE-constrained MD portfolio has not been identified, but such a portfolio may be identified empirically. Using (18), active portfolio weights, $\mathbf{x}$, which define the efficient TE-constrained set (from (8)) were used to calculate the DR at various σ_P values. The active portfolio weights which generate the maximum DR are easily identified.

4.3.2.10 Risk parity/inverse volatility

Portfolios in which the risk *contribution* from each component is made equal is a form of diversification maximisation because such portfolios are like minimum variance portfolios subject to diversification constraints on component weights (Maillard, Roncalli & Teiletche, 2010). The components weights for such portfolios are:

$$w_i = \frac{1}{n\beta_i}$$

where β_i are the component β s and n is the number of assets of which the portfolio comprises. The problem of endogeneity arises here because since w_i is a function of the component β_i which in turn depends on the portfolio composition (i.e. w_i). Various iterative numerical solutions are used (Maillard, Roncalli & Teiletche, 2010).

The TE constraint does not affect these weights – the constituents of the weights are affected by the number of constituents and their respective β . Neither of these are altered by imposing a TE constraint. These portfolios are included for comparison only. Inverse volatility portfolios are similar in construction. The portfolio weights are assembled in proportion to the inverse of their volatility, so

$$w_i = \frac{\frac{1}{\sigma_i}}{\sum_i^n \frac{1}{\sigma_i}} \quad (19)$$

where σ_i are the individual component volatilities. This approach ignores correlation between asset components and again, the TE constraint does not affect these weights: they are shown for comparison only.

4.3.2.11 Minimum intra-portfolio correlation

There are competing definitions of intra-portfolio correlation, but the one used here avoids most of the problems associated with the measure (Hedge Fund Consistency Index, 2011 and Livingstone, 2013):

$$\begin{aligned} IPC &= \frac{\sum_i \sum_j (\mathbf{q} + \mathbf{x})_i (\mathbf{q} + \mathbf{x})_j \rho_{ij}}{\sum_i \sum_j (\mathbf{q} + \mathbf{x})_i (\mathbf{q} + \mathbf{x})_j} \\ &= [(\mathbf{q} + \mathbf{x})'(\mathbf{q} + \mathbf{x})]^{-1} (\mathbf{q} + \mathbf{x})' [\rho - \text{diag}(\rho)] (\mathbf{q} + \mathbf{x}) \end{aligned} \quad (20)$$

where $\text{diag}(\rho)$ is the matrix of the diagonal elements of ρ .

As with the maximum diversification portfolio, the IPC as defined above is a universal, non-TE-constrained, IPC portfolio (so, again, it is not necessarily on or inside the constant TE frontier). A closed-form solution for a TE-constrained IPC portfolio must be found empirically. Using (20), active portfolio weights, $\mathbf{x}$, which define the efficient TE-constrained set (from (8)) are used to calculate the IPC at different values for σ_P . The active portfolio weights which

generate the minimum IPC are easily identified. For a constant $TE = 7\%$ the intra-correlation and maximum diversification portfolio returns/standard deviations are shown in Figure 4.3.

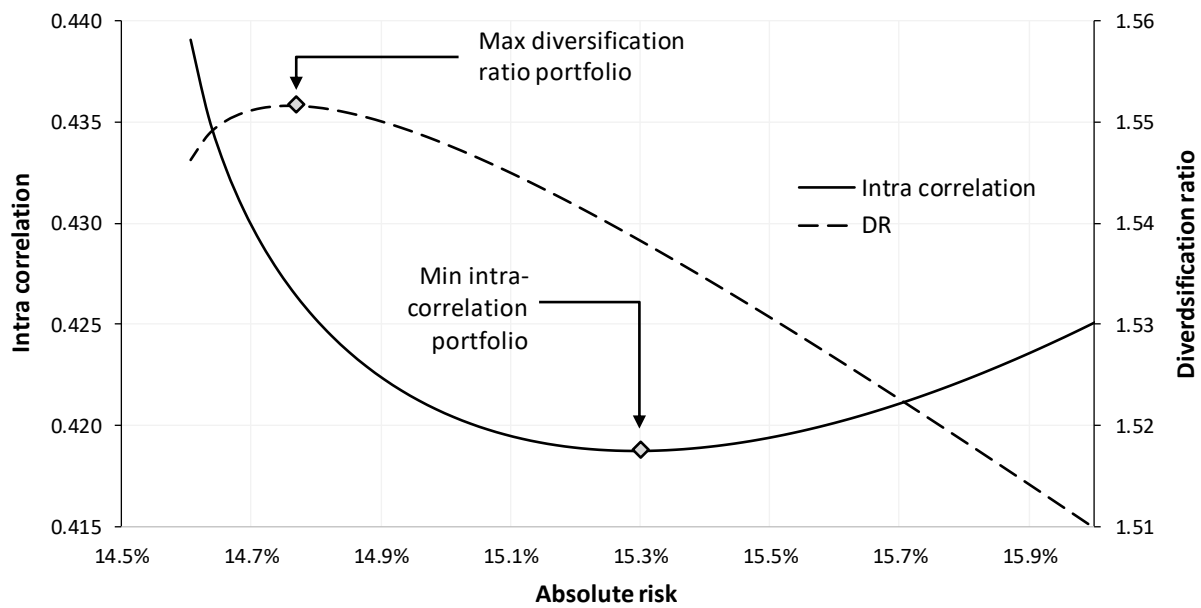


Figure 4.3: Intra-correlation and diversification ratio subject to a $TE = 7\%$ using (20) and (18) respectively.

4.4 Results and discussion

The weights for the optimal portfolios of the various strategies discussed in Section 4.3.2 were calculated and identified on the constant TE frontier (if possible). These are shown in Figure 4.4 for the stylised example as described in Table 4.1. Although the *absolute* positions of the portfolios in mean/risk space depend on the level of TE, risk-free rate, and underlying asset components, the *relative* positions are not strongly dependent on these factors. The maximum Sharpe ratio portfolio approaches (and can be co-incident with) the maximum return portfolio at high risk-free rates – but these, along with Jorion's (2003) benchmark risk portfolio, are clustered in the upper right quadrant of the constant TE ellipse while the minimum variance, minimum intra-correlation, risk parity, inverse volatility and maximum diversification portfolios all occupy the upper left quadrant.

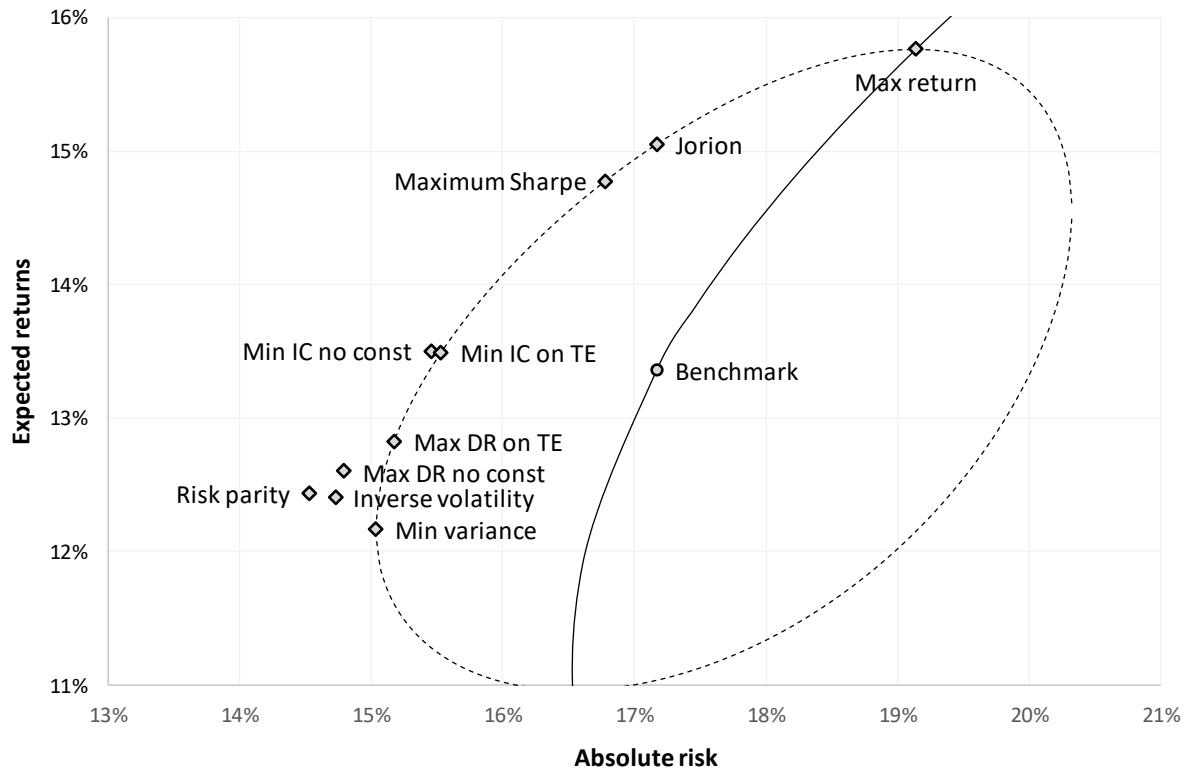


Figure 4.4: TE frontier, constant TE frontier and positions of optimised portfolios on constant TE frontier (a $TE = 7\%$ and $r_f = 5\%$). At these values, the inverse volatility, unconstrained maximum diversification, and minimum intra-correlation portfolios lie inside the constant TE ellipse.

The loci of the portfolios in mean/risk space as the TE is increased are shown in Figure 4.5 for $1\% \leq TE \leq 12\%$. The constant TE ellipses are shown as faint, grey dotted lines, in 1% steps.

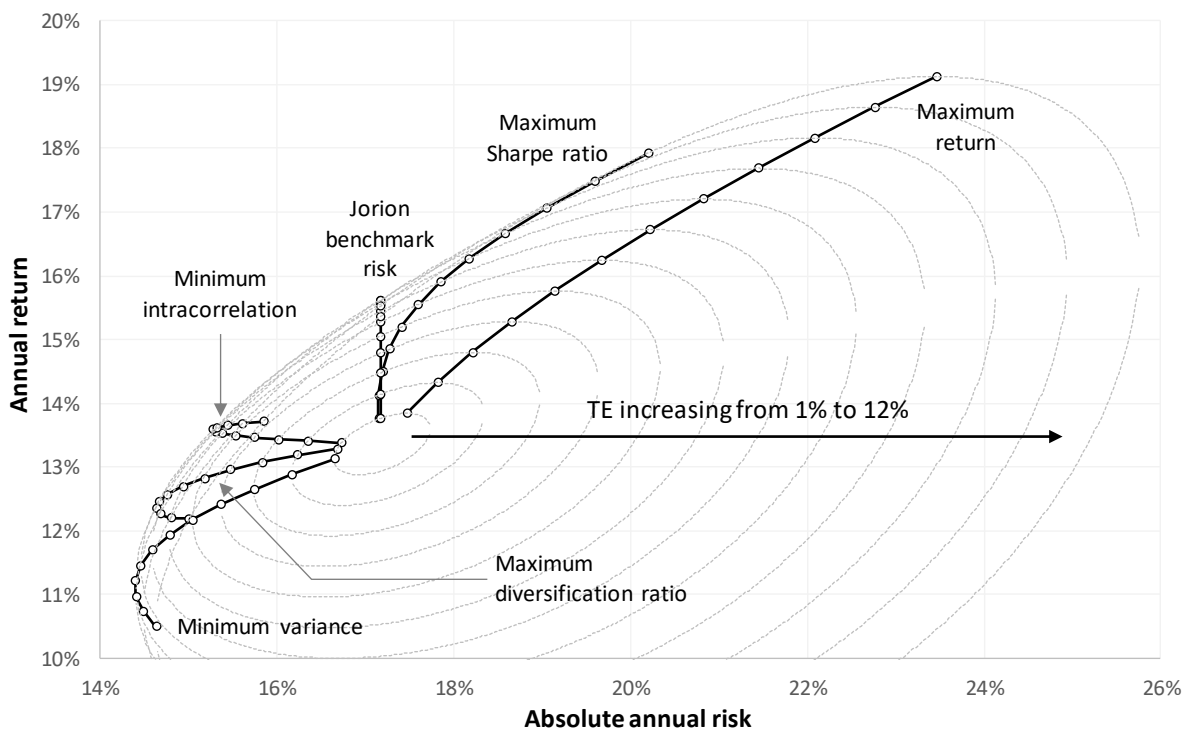


Figure 4.5: Loci of relevant portfolios in mean/risk space for $1\% \leq TE \leq 12\%$.

4.4.1 Maximum return

The locus of the maximum return portfolio in mean/risk space is monotonically upwards and to the right as TE increases – so increasing return, but with increasing risk. Unless it is the investment objective of the investor or mandate, active managers naïvely select this portfolio by focusing on relative (as opposed to absolute) performance metrics, incur too much risk per unit of return, which results in suboptimal portfolios. Much of this risk can be diversified away to achieve more optimal portfolios.

4.4.2 Maximum Sharpe ratio (risk-adjusted return)

As expected, the locus of the maximum Sharpe ratio portfolio is also monotonically upwards and to the right in mean/risk space as TE increases. The slope of this increase is greater than that for maximum return portfolio, demonstrating the superiority (in maximum Sharpe ratio terms) of this strategy over all others. The maximum Sharpe ratio portfolio is the maximum risk-adjusted return portfolio attainable: these portfolios also maximise utility. However, this is not observed in practice and the literature is almost silent on this point (but see Daly, Maxwell, van Vuuren, 2018).

4.4.3 Jorion benchmark risk

The locus of the benchmark risk portfolio increases vertically upwards in mean/risk space for increasing TE until $TE \leq 6\%$. For $TE \geq 6\%$ the locus decreases vertically – i.e. diminishing return for the same level of (benchmark) risk. Recall that the constant TE frontier pulls away (downwards) from the efficient frontier and to the right (Jorion, 2003; Maxwell *et al.*, 2018), so the benchmark risk portfolio return increases for $TE \leq 6\%$ then diminishes as TE increases. For low TEs, the flatness of the ellipse allows for a significant risk reduction for a minor reduction in return (representing the price for risk reduction through efficient strategic/tactical asset allocation). This does not hold for higher TEs as the slope of the ellipse steepens considerably at the ellipse's left and right extremes.

4.4.4 Minimum variance

The locus of this portfolio is downwards and to the left (and for higher values of TE, continues downwards, but then to the right – i.e. with increasing absolute risk, the constant TE frontier

pulls downwards and to the right). In terms of the Sharpe ratio, these portfolios are *always* suboptimal. This portfolio functions more as a placeholder for the constant tracking error frontier's turning point and should not be considered as a portfolio for selection. Although the frontier's flatness benefits most portfolios (small reduction in return for large reductions in risk), this is less true at the left and right extremes of the ellipse where the slope steepens. As the minimum variance portfolio is at the left extreme of the ellipse, significant positive returns can always be achieved for small risk increases so the minimum variance will always be a sub-optimal portfolio in more than just Sharpe ratio terms (i.e. relative to other selections along the frontier).

4.4.5 Minimum intra-correlation

The minimum intra-correlation portfolio locus is slightly upwards and to the left with increasing TE up to $TE = 7\%$ and then slightly upwards and to the right for $TE \geq 7\%$. These portfolios have low absolute risk, but the associated low returns may not compensate for these risks. This portfolio does, however, offer a more stable solution across the TE spectrum, where the risk remains relatively stable (slightly improves for smaller TEs) for a marginal improvement in return. Consequently, this may be a satisfactory selection to satisfy certain risk averse utility requirements. The position of the global minimum intra-correlation portfolio (i.e. with no TE constraints imposed) is shown in Figure 15.

4.4.6 Maximum diversification

The locus of the maximum diversification ratio portfolio is slightly downwards to the left for $TE \leq 9\%$ and then slightly downwards to the right for $TE \geq 9\%$. Again, as with the minimum intra-correlation portfolios, these have low absolute risk, but the associated low returns do not compensate for this. This is more prevalent in this case as the associated returns for taking on more TE are relatively lower than *others* (e.g. minimum intra-correlation and Jorion benchmark risk) albeit for lower risk. The position of the global maximum diversification ratio portfolio, with no TE constraints imposed, is shown in Figure 15.

4.4.7 Risk parity (inverse volatility)

This portfolio has only one position – it is not influenced by the TE (see 19) as the weights are a function of the equality of asset volatility contributions to the portfolio (i.e. each asset contributing equal volatility to the portfolio and the weights are adjusted accordingly to achieve

this). Although this approach reduces overall portfolio volatility versus other portfolio selections (e.g. maximum return), it reflects the notion that volatility is risk and that it is symmetrical (i.e. as many positive as negative returns). This may often not be the case. The global risk parity portfolio is shown in Figure 4.6.

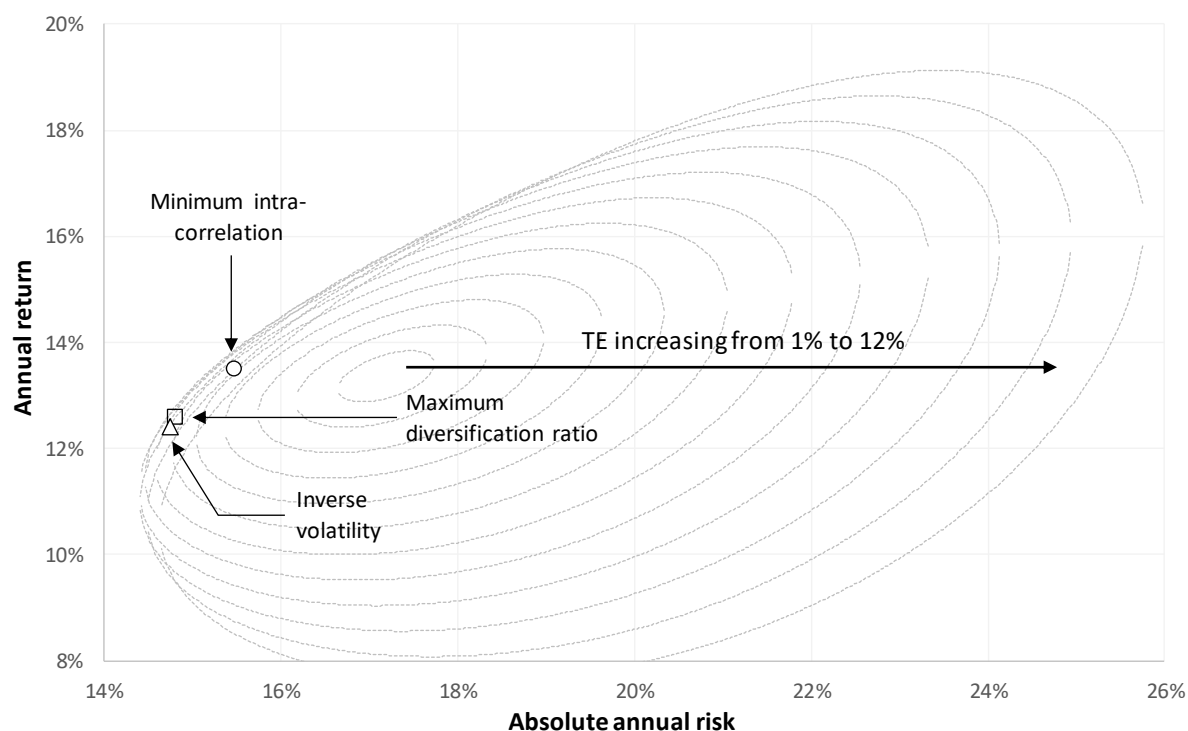


Figure 4.6: Positions of global minimum intra-correlation, maximum diversification and inverse volatility portfolios in mean/risk space (on the same scale as that given in Figure 14).

4.5 Conclusions and suggestions

It has been well established that benchmarks are generally inefficient and any relative performance measure to these benchmarks provides an inaccurate measure of true investor satisfaction. Optimising under such performance measures, in this excess/relative space, results in sub-optimal portfolios and failure to capture the best result for the investor. As it is more difficult to change investor mind-sets than to navigate these restrictions, the latter has been researched extensively.

Under the awareness of the detriment caused by ignoring absolute risk (when subject to TE restrictions), there are various investment selections available to the investor depending on their objectives. These investment solutions cannot be conclusively established as optimal or sub-optimal versus one another, as optimality is dependent on various factors over and above risk and return. All else equal, a portfolio that is optimal to one investor may not be the same

as that for another investor, as it depends on whether their utility has been maximised (i.e. maximum satisfaction gained from the portfolio selection). The difference lies in the investor policy statement, where their objectives and goals are laid out. As the investor is often naïve in the investment realm, so are their goals and objectives, many of which are subjective by nature (e.g. realising large gains vs avoiding large losses). Consequently, the full benefits gained from portfolio optimisation cannot be captured and thus determining which the most optimal portfolio is, rendered moot. Without this much needed change in comprehension of optimality, risk and utility (and their association), a range of selections have been analysed on individual merits and a depicted argument provided upon which the reader can decide. Here risk is associated with volatility, however, *there* lies another important differentiation. Volatility is directionless and comprises of both positive and negative returns, whereas risk is the permanent loss of capital. Often these terms are used interchangeably where they shouldn't be but, as with TE, it is industry standard to use volatility as a risk measure and this has been carried through into this research (see Daly, Maxwell & van Vuuren, 2018).

The relationship along the constant efficient frontier between various possibilities, provides the portfolio manager with a range of selections based on the investment goals. Although some of these look fairly similar in risk return space, their allocations of the portfolio constituents to achieve the desired outcome can differ substantially. In many cases, the choice between different portfolios can determine whether one goes overweight or underweight the benchmark for a particular (or multiple) assets (i.e. change in sign). This represents substantial changes in portfolio construction (rather than minor tweaks) which could differ substantially from the house view or market sentiment around certain asset classes. This provides value in many different ways (e.g. consequent change in market view, alpha opportunity against general market sentiment, etc.). In such cases, where the proposed asset allocation for a selected portfolio does not align with the house view, it can also be used as an opportunity to revisit the investment policy statement and re-adjust the investors' constraints (objectives) on the portfolio and start a shift in mind-set away from inefficient performance measures, constraints and, ultimately, benchmarks.

This work developed the loci, in risk/return space, of various optimal portfolio strategies subject to TE constraints. Because an infinite number of portfolios exist (dependent on the num-

ber of portfolio constituents, the benchmark selected, benchmark constituent weights, restrictions on benchmark and portfolio weights, and component volatilities and correlations) a simple example portfolio (with fixed benchmark weights, expected returns, volatilities and correlations) was employed to demonstrate possible stylised outcomes. A more thorough investigation of the loci of TE-constrained optimal portfolio strategies deserves attention. Future work could involve studies which interrogate the impact of changing these factors on portfolio performance. While we have demonstrated the variation of optimal TE-constrained portfolio performance dependent on the level of TE, results from work which investigates other aspects of portfolio construction would be of great benefit to active portfolio managers seeking optimal performance.

REFERENCES

- Ammann, M. and Zimmermann, H. 2001. Tracking error and tactical asset allocation. *Financial Analysts Journal*, 57(2): 32 – 43.
- Bajeux-Besnainou, I., Portait, R. and Tergny, G. 2011. Optimal portfolio allocations with tracking error volatility and stochastic hedging constraints. *Quantitative Finance*, 13(10): 1 – 14.
- Beardsley, X., Field, B. and Xiao, M. 2012. Mean-variance-skewness-kurtosis portfolio optimization with return and liquidity. *Communications in Mathematical Finance*, 1(1): 13 – 49.
- Bertrand, P. 2005. A note on portfolio performance attribution: taking risk into account. *Journal of Asset Management*, 5(6): 428 – 437.
- Bertrand, P. 2009. Risk-adjusted performance attribution and portfolio optimisations under tracking-error constraints. *Journal of Asset Management*, 10(2): 75 – 88.
- Bodnar, T., Parolya, N. and Schmid, W. 2018. Estimation of the global minimum variance portfolio in high dimensions. *European Journal of Operational Research*, 266(1): 371 – 390.
- Cheng, P. 1971. Efficient portfolio selections beyond the Markowitz Frontier. *Journal of Financial and Quantitative Analysis*, 6(5): 1207 – 1234.
- Choueifaty, Y. and Coignard, Y. 2008. Toward maximum diversification. *The Journal of Portfolio Management*, 35(1): 40 – 51.
- Chow, G. 1995. Portfolio selection based on return, risk, and relative performance. *Financial Analysts Journal*, 51(2): 54 – 60.
- Daly, M., Maxwell, M., and van Vuuren, G. 2018. Feasible portfolios under tracking error, β , α and utility constraints. *The Journal of Investment Management and Financial Innovations*, 15(1): 1 – 13.
- El-Hassan, N. and Kofman, N. 2003, Tracking error and active portfolio management. *Australian Journal of Management*, 28(2): 183 – 207.

- Financial Times, September 2017. Active management stages a comeback. Online: <https://www.ft.com/content/7f9b5cd6-7d00-11e7-ab01-a13271d1ee9c>, accessed 15 May 2018.
- Gilreath, D. 2017. The tide has turned: Active outpacing passive investing. CNBC. Online: <https://www.cnbc.com/2017/09/18/the-tide-has-turned-active-outpacing-passive-investing.html>, accessed 15 May 2018.
- Hedge Fund Consistency Index, 2011. Online: http://www.hedgefund-index.com/d_diversification.asp, accessed 23 January 2018.
- Jorion, P. 2003. Portfolio optimization with tracking-error constraints. *Financial Analysts Journal*, 59(5): 70 – 82.
- Livingstone, L. 2013. Intra-portfolio correlation: an application for investments students. *Business Education and Accreditation*, 5(1): 91 – 105.
- Maillard, S., Roncalli, T. and Teiletche, J. 2010. On the properties of equally-weighted risk contributions portfolios. *The Journal of Portfolio Management*, 36(4): 60 – 70.
- Markowitz, H. 1952. Portfolio selection. *The Journal of Finance*, 7(1): 77 – 91.
- Mattei, M. 2017. Enhanced portfolio performance using a momentum approach to annual rebalancing. *International Journal of Financial Studies*, 6(15): 1 – 9.
- Maxwell, M., Daly, M., Thomson, D. and van Vuuren, G. 2018. Optimising tracking error-constrained portfolios. *Applied Economics*, 50(54): 5846 – 5858.
- Penhall, W. 2015. Fiduciary duties of UK investment managers and conflicts: the arch financial products case. *Journal of Investment Compliance*, 16(3): 43 – 48.
- Riccetti, L. 2010. Minimum tracking error volatility. *Quaderno di Ricerca n°340 del Dipartimento di Economia dell'Università Politecnica delle Marche., Working paper*, 340.
- Roll, R. 1992. A mean/variance analysis of tracking error. *The Journal of Portfolio Management*, 18(4): 13 – 22.
- Rudolf, M., Wolter, H. and Zimmermann, H. 1999. A linear model for tracking error minimization. *Journal of Banking and Finance*, 23(1): 85 – 103.
- Sharpe, W. 1966. Mutual fund performance. *The Journal of Business*, 39(1): 119 – 138.
- Stowe, D. 2014. Tracking error volatility optimization and utility improvements. *Working paper*. Online: http://swfa2015.uno.edu/F_Volatility_Risk_Exposure/paper_221.pdf, accessed 14 June 2016.
- Thomas, B., Rottschäfer, D. and Zvingelis, J. 2013. A tracking error primer. Envestnet. PMC. Online: <https://www.envestnet.com/sites/default/files/documents/A%20Tracking%20Error%20Primer%20-%20White%20Paper.pdf>, accessed 22 January 2018.
- Wu, M. and Jakshoj, C. 2011. Risk-adjusted performance attribution and portfolio optimisation under tracking-error constraints for SIAS Canadian Equity Fund. *Masters dissertation, Simon Fraser University, Canada*. Online: <http://summit.sfu.ca/item/13058>, accessed 8 Aug 2016.
- Zibri, A. 2014. Constraint effect on minimum tracking error volatility: a global asset approach. *Interdisciplinary Journal of Research and Development*, 1(1): 1 – 6.

Chapter 5

Conclusions and suggestions for future research

5.1. Summary and conclusions

The investment nous of active managers is judged on their ability to outperform their benchmarks. Using benchmarks to gauge relative performance allows investors to evaluate value added relative to risks taken. Benchmarks generally comprise a combination of disparate assets, dependent on investors' risk preferences/profiles, but they tend to be inefficient as both the components of – and weights within – the benchmark are somewhat arbitrarily chosen (a well-established flaw). Any relative performance measure to these benchmarks provides an inaccurate measure of true investor satisfaction. Optimising under such performance measures, in this excess/relative space, results in sub-optimal portfolio selection and failure to capture the best result for the investor. As it is far more difficult to change investor mind-set than to find a solution around these restrictions, the latter has been researched extensively and resulted in interesting observations.

Applying tracking error constraints result in more inefficiency if fund managers naïvely pursue maximum excess returns as a sole investment objective, as this leads to higher total risk which can be diversified away. Accepting the TE mandate as given (since it is widely used in the industry), optimisation efforts have concentrated on various performance ratios and features of the TE frontier. In all previous work, absolute risk is consistently ignored.

The work of Jorion (2003) led to the awareness of the detriment caused by ignoring absolute risk (when subject to TE restrictions), which spurred testing of various additional *other* restrictions to the existing constraint set (previously only tested on the efficient frontier). These restrictions provide positions on the frontier satisfying differing investment strategies, improving investor utility. The resulting various investment selections now available to the investor, dependent on their objectives, offers a dynamic to portfolio optimization further to the simple generic maximise return, reduce volatility by offering a goal-based approach to the same ultimate objective, providing a platform to satisfy investor utility maximisation via more precise and robust asset allocation through LaGrangian reverse engineering. It addresses the

fundamental meaning of an “optimal portfolio”, where the investors’ objectives are quantitatively considered in addition to risk-adjusted return maximisation.

The relationship along the constant efficient frontier between various possibilities, provides the portfolio manager with a range of selections based on the investment goals. Although some of these look fairly similar in risk return space, their allocations of the portfolio constituents to achieve the desired outcome can differ substantially. In many cases, the choice between different portfolios can determine whether one goes overweight or underweight the benchmark for a particular (or multiple) assets (i.e. change in sign). This represents substantial changes in portfolio construction (rather than minor tweaks) which could differ considerably from the house view or market sentiment around certain asset classes. This provides value in many different ways (e.g. consequent change in market view, alpha opportunity against general market sentiment, etc.). In such cases, where the proposed asset allocation for a selected portfolio does not align with the house view, it can also be used as an opportunity to revisit the investment policy statement and re-adjust the investors’ constraints (objectives) on the portfolio and start a shift in mind-set away from inefficient performance measures, constraints and, ultimately, benchmarks.

5.1.1. Paper 1: *Optimising tracking error-constrained portfolios*

Jorion (2003) proposed an approach to optimise the risk/return relationship of TE-constrained active portfolios by imposing an additional, absolute risk, constraint. Jorion (2003) derived the constant tracking-error frontier in the original mean-variance plane. This frontier is an ellipse, and the flatness of this ellipse allowed Jorion (2003) to impose this additional constraint of requiring that total portfolio volatility must be the same as the benchmark volatility. This additional restriction substantially improved active portfolio performance.

Applying a different (rather than additional) restriction to that of Jorion (2003) to the existing constraint set on this constant TE frontier, in absolute risk/return space, offered a more efficient solution. Maximising the traditional Sharpe ratio with respect to the constant TE frontier selects portfolios which achieve higher excess returns and lower absolute risk relative to the benchmark, whilst satisfying the TE constraint and maximising the ratio of excess absolute return to risk.

5.1.2. Paper 2: Active investment strategies under tracking error constraints

Maximising benchmark outperformance, whilst desirable, is a complex task as most active portfolios are subject to tracking error (and other) constraints. Given these restrictions, theoretically possible portfolios are enclosed by a tracking error frontier – an ellipse in risk/return space. The portfolios which constitute the frontier are not necessarily efficient. Some attempts have been made to identify optimal portfolios subject to the restrictions imposed by tracking errors, i.e. to locate these on the frontier. Examples include portfolios having the maximum return, the same risk as the benchmark and the maximum Sharpe ratio (risk-adjusted returns). Extending these portfolio assemblies and introducing more possibilities, provides the manager with a suite of options for different investors, namely: portfolios which are maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum variance. Such portfolios behave differently to those which are part of the efficient set, i.e. populate the efficient frontier and which are not constrained by tracking errors.

These investment solutions cannot be conclusively established as optimal or sub-optimal versus one another, as optimality is dependent on various factors over and above risk and return. All else equal, a portfolio that is optimal to one investor may not be the same as that for another investor, as it depends on whether their utility has been maximised (i.e. maximum satisfaction gained from the portfolio selection). The difference lies in the investor policy statement, where their objectives and goals are laid out. As the investor is often naïve in the investment realm, so are their goals and objectives, many of which are subjective by nature (e.g. realising large gains vs avoiding large losses). Consequently, the full benefits gained from portfolio optimisation cannot be captured and thus determining which the most optimal portfolio is, rendered moot. Without this much needed change in comprehension of optimality, risk and utility (and their association), a range of selections have been analysed on individual merits and a depicted argument provided upon which the reader can decide. Here risk is associated with volatility, however, there lies another important differentiation. Volatility is directionless and comprises of both positive and negative returns, whereas risk is the permanent loss of capital. Often these terms are used interchangeably where they shouldn't be, but, as with TE, it is industry standard to use volatility as a risk measure and this has been carried through into this research (see Daly *et al.*, 2018).

The relationship along the constant efficient frontier between various possibilities, although seemingly similar visually (in risk return space), their allocations of the portfolio constituents to achieve the desired outcome can differ substantially. In many cases, the choice between different portfolios can determine whether one goes overweight or underweight the benchmark for a particular (or multiple) assets (i.e. change in sign). This represents substantial changes in portfolio construction (rather than minor tweaks), providing value-add to the industry.

5.2. Suggestions for future research

Several interesting (and important) aspects of the various approaches have been identified and explored. The work should prove useful as it allows fund managers to calculate deviations from the benchmark (and thus, ultimately, actual portfolio weights) explicitly. While the strategies do not account for tax considerations, regulatory constraints or rebalancing costs, these should be simple to implement in practice (a possible further study). In so doing, the usual due diligence should be observed in selecting assets and forecasting expected returns.

This work developed the loci, in risk/return space, of various optimal portfolio strategies subject to TE constraints. Because an infinite number of portfolios exist (dependent on the number of portfolio constituents, the benchmark selected, benchmark constituent weights, restrictions on benchmark and portfolio weights, and component volatilities and correlations) a simple example portfolio (with fixed benchmark weights, expected returns, volatilities and correlations) was employed to demonstrate possible stylised outcomes. A more thorough investigation of the loci of TE-constrained optimal portfolio strategies deserves attention.

Future work could involve studies which interrogate the impact of changing these factors on portfolio performance. While we have demonstrated the variation of optimal TE-constrained portfolio performance dependent on the level of TE, results from work which investigates other aspects of portfolio construction would be of great benefit to active portfolio managers seeking optimal performance.

Bibliography

- Ammann, M. and Zimmermann, H. 2001. Tracking error and tactical asset allocation. *Financial Analysts Journal*, 57(2): 32 – 43.
- Bajeux-Besnainou, I., Portait, R. and Tergny, G. 2011. Optimal portfolio allocations with tracking error volatility and stochastic hedging constraints. *Quantitative Finance*, 13(10): 1 – 14.
- Beardsley, X., Field, B. and Xiao, M. 2012. Mean-variance-skewness-kurtosis portfolio optimization with return and liquidity. *Communications in Mathematical Finance*, 1(1): 13 – 49.
- Bertrand, P. 2005. A note on portfolio performance attribution: taking risk into account. *Journal of Asset Management*, 5(6): 428 – 437.
- Bertrand, P. 2009. Risk-adjusted performance attribution and portfolio optimisations under tracking-error constraints. *Journal of Asset Management*, 10(2): 75 – 88.
- Bertrand, P. 2010. Another look at portfolio optimization under tracking error constraints. *Financial Analysts Journal*, 66(3): 78 – 90.
- Bertrand, P., Prigent, J-L. and Sobotka, R. 2001. Optimisation de portefeuille sous contrainte de variance de la tracking-error. *Banque & Marchés*, 54(1): 19 – 28.
- Bodnar, T., Parolya, N. and Schmid, W. 2018. Estimation of the global minimum variance portfolio in high dimensions. *European Journal of Operational Research*, 266(1): 371 – 390.
- Carlsson, C., Fullér, R. and Majlender, P. 2002. A possibilistic approach to selecting portfolios with highest utility score. *Fuzzy Sets and Systems*, 131(1): 13 – 21.
- Cheng, P. 1971. Efficient portfolio selections beyond the Markowitz Frontier. *Journal of Financial and Quantitative Analysis*, 6(5): 1207 – 1234.
- Choueifaty, Y. and Coignard, Y. 2008. Toward maximum diversification. *The Journal of Portfolio Management*, 35(1): 40 – 51.
- Chow, G. 1995. Portfolio selection based on return, risk, and relative performance. *Financial Analysts Journal*, 51(2): 54 – 60.
- Clarke, R., de Silva, H. and Thorley, S. 2002. Portfolio constraints and the fundamental law of active management. *Financial Analysts Journal*, 58(5): 48 – 66.
- Daly, M., Maxwell, M., and van Vuuren, G. 2018. Feasible portfolios under tracking error, β , α and utility constraints. *The Journal of Investment Management and Financial Innovations*, 15(1): 1 – 13.
- El-Hassan, N. and Kofman, P. 2003. Tracking error and active portfolio management. *Australian Journal of Management*, 28(2): 183 – 207.
- Fabozzi, F. J. and Markowitz, H. M. 2011. "The theory and practice of investment management: asset allocation, valuation, portfolio construction, and strategies". 2nd edition. 2011 John Wiley & Sons, New York.
- Financial Times, September 2017. Active management stages a comeback. Online: <https://www.ft.com/content/7f9b5cd6-7d00-11e7-ab01-a13271d1ee9c>, accessed 15 May 2018.

- Ghosh, A. and Mahanti, A. 2014. Investment Portfolio Management: A Review from 2009 to 2014. *Proceedings of 10th Global Business and Social Science Research Conference*, 23 -24 June 2014, Radisson Blu Hotel, Beijing, China. Online: https://wbiworldconpro.com/uploads/china-conference-2014/finance/1402555460_305-Amitava.pdf, accessed 12 Aug 2017.
- Gilreath, D. 2017. The tide has turned: Active outpacing passive investing. CNBC. Online: <https://www.cnbc.com/2017/09/18/the-tide-has-turned-active-outpacing-passive-investing.html>, accessed 15 May 2018.
- Hedge Fund Consistency Index, 2011. Online: http://www.hedgefund-index.com/d_diversification.asp, accessed 23 January 2018.
- Jansen, R. and van Dijk, R. 2002. Optimal benchmark tracking with small portfolios. *Journal of Portfolio Management*, 28(2): 33 – 39.
- Jorion, P. 1992. Portfolio optimization in practice. *Financial Analysts Journal*, 48(1): 68–74.
- Jorion, P. 2003. Portfolio optimization with tracking-error constraints. *FAJ*, 59(5): 70 – 82.
- Larsen, G. A. and Resnick, B. G. 2001. Parameter estimation techniques, optimization frequency, and portfolio return enhancement. *Journal of Portfolio Management*, 27(4): 27 – 34.
- Livingstone, L. 2013. Intra-portfolio correlation: an application for investments students. *Business Education and Accreditation*, 5(1): 91 – 105.
- Maillard, S., Roncalli, T. and Teiletche, J. 2010. On the properties of equally-weighted risk contributions portfolios. *The Journal of Portfolio Management*, 36(4): 60 – 70.
- Mansini, R., Ogryczak, W. and Speranza, M. G. 2014. Twenty years of linear programming-based portfolio optimization. *European Journal of Operational Research*, 234(2): 518 – 535.
- Markowitz, H. 1952. Portfolio selection. *The Journal of Finance*, 7(1): 77 – 91.
- Mattei, M. 2017. Enhanced portfolio performance using a momentum approach to annual rebalancing. *International Journal of Financial Studies*, 6(15): 1 – 9.
- Maxwell, M., Daly, M., Thomson, D. and van Vuuren, G. 2018. Optimising tracking error-constrained portfolios. *Applied Economics*, 50(54): 5846 – 5858.
- Menchero, J. and Hu, J. 2006. Portfolio risk attribution. *The Journal of Performance Measurement*, 10(3): 22-33.
- Nawrocki, D. and Viole, F. 2014. Behavioral finance in financial market theory, utility theory, portfolio theory and the necessary statistics: A review. *Journal of Behavioral and Experimental Finance*, 2(6): 10 – 17.
- Penhall, W. 2015. Fiduciary duties of UK investment managers and conflicts: the arch financial products case. *Journal of Investment Compliance*, 16(3): 43 – 48.
- Plaxco, L. M. and Arnott, R. D. 2002. Rebalancing a global policy benchmark. *Journal of Portfolio Management*, 28(2): 9 – 22.
- Riccetti, L. 2010. Minimum tracking error volatility. *Quaderno di Ricerca n°340 del Dipartimento di Economia dell'Università Politecnica delle Marche.*, Working paper, 340.

- Roll, R. 1992. A mean/variance analysis of tracking error. *The Journal of Portfolio Management*, 18(4): 13 – 22.
- Rudolf, M., Wolter, H. and Zimmermann, H. 1999. A linear model for tracking error minimization. *Journal of Banking and Finance*, 23(1): 85 – 103.
- Sharpe, W. 1966. Mutual fund performance. *The Journal of Business*, 39(1): 119 – 138.
- Stowe, D. L. 2014. Tracking error volatility optimization and utility improvements. *Working paper*. Online: [http://swfa2015.uno.edu/F_Volatility & Risk Exposure/paper_221.pdf](http://swfa2015.uno.edu/F_Volatility_%20Risk_Exposure/paper_221.pdf), accessed 14 June 2016.
- Thomas, B., Rottschäfer, D. and Zvingelis, J. 2013. A tracking error primer. Envestnet PMC. *White paper*. Online: <http://www.envestnet.com/sites/default/files/documents/A%20Tracking%20Error%20Primer%20-%20White%20Paper.pdf>, accessed 22 Jul 2016.
- Wu, M. and Jakshoj, C. 2011. Risk-adjusted performance attribution and portfolio optimisation under tracking-error constraints for SIAS Canadian Equity Fund. *Masters dissertation*, Simon Fraser University. Online: <http://summit.sfu.ca/item/13058>, accessed 8 Aug 2016.
- Zibri, A. 2014. Constraint effect on minimum tracking error volatility: a global asset approach. *Interdisciplinary Journal of Research and Development*, 1(1): 1 – 6.

Appendix

Collaboration affidavit

This serves to advise that I, as co-author of the article listed below, concur that Michael Maxwell undertook the bulk of the work. I was involved with the original setting up of the problem only.

Mr Maxwell has my permission to include this article in his dissertation.



Daniel Thomson

This serves to advise that I, as co-author of the article below, concur that Michael Maxwell undertook the bulk of the work. I was involved with only a small part of the data analysis and I provided some minimal assistance with the construction of the Excel macros.

Mr Maxwell has my permission to include this article in his dissertation.



Michael Daly

Maxwell, M., Daly, M., Thomson, D. and van Vuuren, G. 2017. Optimising tracking error-constrained portfolios. *Applied Economics*, 50(54): 5846–5858.