Conservation laws and solutions of the Drinfel'd-Sokolov-Wilson system, the Boussinesq system and the complex modified KdV equation

by

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Contents

Declaration

I CATHERINE MATJILA student number 20981325, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Ms CATHERINE MATJILA

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF C.M. KHALIQUE

Date: ....

Dedication

To my family.

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Abstract

In this dissertation the conservation laws for the Drinfel'd-Sokolov-Wilson, modified Korteweg-de Vries and the Boussinesq system will be derived using the Noether approach. Noether approach requires the knowledge of a Lagrangian. Since these systems are of third order, they do not have a Lagrangian and therefore we will increase the order of the systems by one. The new systems obtained have Lagrangians and so Neother approach can be used to find the conservation laws. The inverse transformation will then be used to obtain the conservation laws for the underlying systems.

Moreover the exact solutions of the Drinfel'd-Sokolov-Wilson and modified Korteweg-de Vries systems will be obtained using the (G'/G)-expansion method. The Lie point symmetries for the Boussinesq system will be calculated.

List of Acronyms

PDEs:	Partial differential equations
ODEs:	Ordinary differential equations
NLPDEs:	Nonlinear partial differential equations
DSW:	Drinfel'd-Sokolov-Wilson
CAS:	Computer algebra system
mKdV:	modified Korteweg-de Vries
NRF:	National Research Fund

Introduction

Conservation laws are important in the solution process and reductions of partial differential equations (PDEs) [1, 2]. Many methods have been developed for the construction of conservation laws such as the Laplace direct method [3], characteristic form introduced by Stuedel [4], the multiplier approach [5, 6] and Noether approach [7].

In this study Noether approach will be used to obtain the conservation laws of three systems, namely Drinfeld-Sokolov-Wilson (DSW), modified Korteweg-de Vries (mKdV) and the Boussinesq system. Noether approach requires the knowledge of a Lagrangian.

Nonlinear partial differential equations (NLPDEs) are widely used to describe physical, chemical and biological phenomena, and their use has spread into economics, finance, image processing, medicine and other fields. In the past few decades a number of new methods have been proposed to get the exact solutions. Some of these methods include the exp-function method, the homogeneous balance method, the sine-cosine method and the hyperbolic tangent function expansion method (See, for example, [8–13]).

In this work the (G'/G)-expansion method [14] is used to obtain the exact solutions of DSW and mKdV systems. This method was first introduced by Wang et al. [15] and has been used by many researchers to find exact solutions of nonlinear equations [16–20]. The main goal of this method is that the traveling wave solutions of nonlinear equations can be expressed by a polynomial in (G'/G), where $G = G(z)$ satisfies the second order linear differential equation $G''(z) + \lambda G'(z) + \mu G(z) = 0$, where $z = x - ct$ and c is arbitrary constant.

An invertible transformation of the dependent and independent variables that leaves the equation unchanged is called a Lie point symmetry of a differential equation. It is an unachievable task to construct all the symmetries of a differential equation. Nevertheless, in the middle of the nineteenth century Sophus Lie (1842-1899) recognized that we can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries if we limit ourselves to symmetries that depend continuously on a small parameter and that form a continuous one-parameter group of transformations. In the past few decades a substantial progress has been made in symmetry methods for differential equations (see, for example, [21–24]). For this research we will calculate the Lie point symmetries of the Boussinesq system.

In this research three nonlinear problems will be studied. Firstly, the DSW system [25] which was introduced as a model of water waves, given by

$$\begin{aligned} u_t + \beta v v_x &= 0, \\ v_t + \alpha v_{xxx} + \beta v u_x + \beta u v_x &= 0, \end{aligned} \tag{1}$$

where β and α are nonzero constants and $u(x,t)$ and $v(x,t)$ are velocity component along the x-axis and the y-axis respectively.

Secondly we consider the mKdV system [26] given by

$$\begin{aligned} u_t - 3u^2 u_x - u_x v^2 - 2uvv_x - u_{xxx} &= 0, \\ v_t - 3v^2 v_x - v_x u^2 - 2vuu_x - v_{xxx} &= 0. \end{aligned} \tag{2}$$

This system describes the interaction of two orthogonally polarized transverse waves.

Lastly we study the Boussinesq system [27] given by

$$\begin{aligned} v_t + u_x + (uv)_x &= 0, \\ u_t + v_x + uu_x + v_{ttx} &= 0, \end{aligned} \tag{3}$$

which is an approximation of the two-dimensional Euler equations that models two-way propagation of longwaves of small amplitude on the surface of an incompressible, inviscid fluid in a uniform horizontal channel of finite depth.

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations and Noether approach are presented.

In Chapter two, Noether approach is employed to obtain conservation laws of the DSW system. Exact solutions of the DSW system are obtained using the (G'/G)-expansion method.

In Chapter three, conservation laws of the mKdV system are derived using Noether approach. The (G'/G)-expansion method is used to obtain exact solutions of the mKdV system.

In Chapter four, conservation laws of the Boussinesq system are constructed. Lie point symmetries for the Boussinesq system are obtained.

In Chapter five, a summary of the results of the dissertation is given and future work is discussed.

Bibliography is given at the end of this dissertation.

Chapter 1

Definition of concepts

In this chapter, brief introduction of methods of Lie symmetry analysis of differential equations is given. We also give the algorithm to determine the Lie point symmetries of PDEs and some definitions concerning Noether approach are presented.

1.1 Introduction

More than a hundred years ago, the Norwegian mathematician Sophus Lie (1842-1899) developed a new method, known as Lie group analysis, for solving differential equations. He developed a symmetry-based approach to obtaining exact solutions of differential equations. Several books have been written on this topic. We list a few of them here, Ovsiannikov [28], Olver [5], Bluman and Kumei [29], Stephani [30], Ibragimov [31], [32], Cantwell [33] and Mahomed [34]. The definitions and results presented in this chapter are taken from the books mentioned above.

1.2 Continuous one-parameter groups

Suppose $x = (x^1, ..., x^n)$ is the independent variable with coordinates x^i and $u = (u^1, .., u^m)$ is the dependent variable with coordinates u^α (n and m finite). We consider a change of the variables x and u:

$$T_a : \bar{x}^i = f^i(x, u, a), \ \bar{u}^\alpha = \phi^\alpha(x, u, a), \tag{1.1}$$

where a is a real parameter which continuously takes values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u is a set G of transformations (1.1) which satisfies the following:

(i) If T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) There exists $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that from (i) the associativity property is satisfied. The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law.* A group parameter a is called *canonical* if the group composition law is additive, i.e. $\phi(a, b) = a + b$.

Theorem 1.1 For any composition law $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)},$$

where

$$w(s) = \frac{\partial \phi(s, b)}{\partial b}|_{b=0}.$$

1.3 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots , \tag{1.3}$$

where the operator of total differentiation is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots , \quad i = 1, ..., n \tag{1.4}$$

The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$,$u_{(5)}$ $\cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Therefore the u_s^α are called differential variables.

Considering a pth-order PDE, namely

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

1.3.1 Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u}^\alpha = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, finding the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x}^i \approx x^i + a\,\xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Consequently, we have

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x, u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

We now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^\alpha \approx (1 + a\,X)u,$$

where the differential operator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

is known as the infinitesimal operator or generator of the group G. We say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5), if the group G is admitted by (1.5).

We now show how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}^\alpha_i = \bar{D}_j(u^\alpha), \qquad \bar{u}^\alpha_{ij} = \bar{D}_j(\bar{u}^\alpha_i) = \bar{D}_i(\bar{u}^\alpha_j), \cdots.$$

Let us now apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}^\alpha_j. \end{aligned} \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, a for small a, ie., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (this can be proved) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

1.3.2 Prolonged generators

Using (1.11) together with (1.7) and (1.14) we obtain

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j). \qquad (1.15)
\end{aligned}$$

This is called the first prolongation formula. Similarly, one can obtain the second prolongation

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} = D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}j} D_{i_p}(\xi^j), \quad \text{(sum on } j). \qquad (1.17)$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$X^{[1]} = X + \zeta^{\alpha}_{i}\frac{\partial}{\partial u^{\alpha}_{i}} \quad \text{(sum on } i, \alpha),$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$X^{[p]} = X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p}\frac{\partial}{\partial u^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1, \qquad (1.18)$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}. \qquad (1.19)$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \qquad (1.20)$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E) = 0 \qquad (1.21)$$

whenever $E = 0$. This can also be written as

$$X^{[p]}\, E\big|_{E=0} = 0, \qquad (1.22)$$

where the symbol $\big|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 An equation (1.21) that determines all the infinitesimal symmetries of (1.5) is called the *determining equation.*

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{u}, u_{\bar{(1)}}, \cdots, u_{\bar{(p)}}) = 0, \tag{1.23}$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x,u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x,u,a),\, \phi^\alpha(x,u,a)) = F(x,u), \tag{1.24}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x,q)$ to be an invariant is that

$$X\,F \equiv \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.25}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants. One can take, as basic invariants the left-hand side $n-1$ first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.26}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \ [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Essential relationship concerning the Noether theorem

In this section, some definitions and concepts concerning Noether approach are presented. More details are given in [7,35].

Definition 1.8 A function $L(x, u, u_{(1)}, u_{(2)}, ...u_{(s)}) \in \mathcal{A}$ (space of differential functions) is called a Lagrangian of

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0, \quad \alpha = 1, 2, ..., N, \tag{1.27}$$

if the system (1.27) is equivalent to the Euler-Lagrange differential equations

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, ..., N, \tag{1.28}$$

where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geqslant 1} (-1)^s D_{i_1} ... {}_{i_s} \frac{\partial}{\partial {u^\alpha}_{i_s}}, \quad s = 1, 2, \cdots . \tag{1.29}$$

Definition 1.9 A Lie-Bäcklund operator X is a Noether symmetry generator associated with a Lagrangian L if there exists a vector $B = (B^1, B^2, ..., B^n)$, $B^i \in \mathcal{A}$ called the gauge function, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i), \tag{1.30}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n. \tag{1.31}$$

For each Noether symmetry generator X associated with a given Lagrangian L corresponding to the Euler-Lagrange differential equations, there corresponds a vector $T = (T^1, T^1, ..., T^n)$ with T^i defined by [35]

$$T^i = B^i - N^i L = B^i - \xi^i L - W^\alpha \frac{\delta L}{\delta u_i^\alpha} - \sum_{s \geqslant 1} D_{i_1 \ldots i_s}(W^\alpha) \frac{\delta L}{\delta u^\alpha_{i_1 \ldots i_s}}, \tag{1.32}$$

which is a conserved vector for the Euler-Lagrange differential equations (1.28). Noether's approach requires the knowledge of $L(x, u, ... u_{(k-1)})$ and (1.30) is used to determine Noether symmetries. Finally, (1.32) yields the corresponding Noether conserved vectors. The characteristics W^α of the Noether symmetry generator are the characteristics of the conservation law where W^α is defined by

$$W^\alpha = \eta^i - \xi^i u_j^\alpha, \ \ \alpha = 1, ..., N.$$

1.8 Conclusion

In this chapter, a brief introduction to the Lie group analysis of PDEs and Noether approach has been presented and some results which will be used throughout this project have been given.

Chapter 2

Conservation laws and exact solutions of the Drinfel'd-Sokolov-Wilson system

2.1 Introduction

The classical DSW system is given by [25]

$$\begin{aligned} u_t + pvv_x &= 0, \\ v_t + qv_{xxx} + svu_x + ruv_x &= 0, \end{aligned} \tag{2.1}$$

where p, q, r and s are non-zero constants. Some explicit expressions of solutions for the system (2.1) were obtained by using the bifurcation method and qualitative theory of dynamical systems in [25]. These solutions contained solitary wave solutions, blow-up solutions, periodic solutions, periodic blow-up solutions and kink-shaped solutions. The exact solutions of the DSW system (2.1) have also been obtained in [36, 37] by using a direct algebra method. Several other authors also studied the DSW system (2.1) when $p = 3$, $q = r = 2$ and $s = 1$ (see, for example [38–43]).

In this chapter we construct the conservation laws and exact solutions of the DSW system given by

$$\begin{aligned} u_t + \beta v v_x &= 0, \\ v_t + \alpha v_{xxx} + \beta v u_x + \beta u v_x &= 0, \end{aligned} \tag{2.2}$$

where we have set $p = r = s = \beta$ and $q = \alpha$ in (2.1).

This work has been submitted for publication (see [44]).

2.1.1 Conservation laws of the DSW system

In this section we construct the conservation laws of the DSW system (2.2) using Noether approach. Noether approach requires the knowledge of a Lagrangian. Since the third order DSW system (2.2) does not have a Lagrangian, we use the transformation $u = U_x$, $v = V_x$ to transform the third order DSW system (2.2) to a fourth order system. Thus the fourth order system is given by

$$\begin{aligned} U_{tx} + \beta V_x V_{xx} &= 0, \\ V_{tx} + \alpha V_{xxxx} + \beta V_x U_{xx} + \beta U_x V_{xx} &= 0. \end{aligned} \tag{2.3}$$

This system has a second order Lagrangian L given by

$$L = \frac{1}{2}(\alpha V_{xx}^2 - \beta U_x V_x^2 - U_x U_t - V_x V_t). \tag{2.4}$$

The Lagrangian (2.4) satisfy

$$\frac{\delta L}{\delta U} = 0, \; \frac{\delta L}{\delta V} = 0, \tag{2.5}$$

where the Euler operators $\delta/\delta U$ and $\delta/\delta V$ are defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_t^2\frac{\partial}{\partial U_{tt}} + D_x^2\frac{\partial}{\partial U_{xx}} + D_xD_t\frac{\partial}{\partial U_{tx}} - \cdots, \tag{2.6}$$

and

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} + D_t^2\frac{\partial}{\partial V_{tt}} + D_x^2\frac{\partial}{\partial V_{xx}} + D_xD_t\frac{\partial}{\partial V_{tx}} - \cdots. \tag{2.7}$$

Consider the vector field

$$X = \quad \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U} + \eta^2(t,x,U,V)\frac{\partial}{\partial V}, \tag{2.8}$$

which has the second order prolongation $X^{[2]}$ for system (2.3) defined by

$$X^{[2]} = \quad \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U} + \eta^2(t,x,U,V)\frac{\partial}{\partial V} + \zeta_t^1\frac{\partial}{\partial U_t} + \zeta_t^2\frac{\partial}{\partial V_t} + \zeta_x^1\frac{\partial}{\partial U_x} + \zeta_x^2\frac{\partial}{\partial V_x} + \ldots, \tag{2.9}$$

where

$$\zeta_t^1 = D_t(\eta^1) - U_tD_t(\xi^1) - U_xD_t(\xi^2),$$

$$\zeta_t^2 = D_t(\eta^2) - V_tD_t(\xi^1) - V_xD_t(\xi^2),$$

$$\zeta_x^1 = D_x(\eta^1) - U_tD_x(\xi^1) - U_xD_x(\xi^2),$$

$$\zeta_x^2 = D_x(\eta^2) - V_tD_x(\xi^1) - V_xD_x(\xi^2),$$

and

$$D_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + V_t\frac{\partial}{\partial V} + U_{tt}\frac{\partial}{\partial U_t} + V_{tt}\frac{\partial}{\partial V_t} + U_{tx}\frac{\partial}{\partial U_x} + V_{tx}\frac{\partial}{\partial V_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + V_x\frac{\partial}{\partial V} + U_{xx}\frac{\partial}{\partial U_x} + V_{xx}\frac{\partial}{\partial V_x} + U_{tx}\frac{\partial}{\partial U_t} + V_{tx}\frac{\partial}{\partial V_t} + \cdots.$$

We recall that the Lie-Bäcklund operator X defined in equation (2.8) is a Noether operator corresponding to the Lagrangian L if it satisfies

$$X(L) + L[D_t(\xi^1) + D_x(\xi^2)] = D_t(B^1) + D_x(B^2), \tag{2.10}$$

where $B^1(t,x,U,V)$, $B^2(t,x,U,V)$ are the gauge terms. Expansion of (2.10) with the second order Lagrangian (2.4) yields

$$\begin{aligned}
&-\frac{1}{2}U_x[\eta^1_t + U_t\eta^1_U + V_t\eta^1_V - U_t\xi^1_t - U_t^2\xi^1_U - U_tV_t\xi^1_V - U_x\xi^2_t - U_tU_x\xi^2_U - U_xV_t\xi^2_V] \\
&-\frac{1}{2}V_x[\eta^2_t + U_t\eta^2_U + V_t\eta^2_V - V_t\xi^1_t - U_tV_t\xi^1_U - V_t^2\xi^1_V - V_x\xi^2_t - U_tV_x\xi^2_U - V_tV_x\xi^2_V] \\
&-\frac{1}{2}\left(\beta V_x^2 + U_t\right)[\eta^1_x + U_x\eta^1_U + V_x\eta^1_V - U_t\xi^1_x - U_tU_x\xi^1_U - U_tV_x\xi^1_V - U_x\xi^2_x - U_x^2\xi^2_U \\
&-U_xV_x\xi^2_V] - \left(\frac{1}{2}V_t + \beta U_xV_x\right)[\eta^2_x + U_x\eta^2_U + V_x\eta^2_V - V_t\xi^1_x - U_xV_t\xi^1_U - V_tV_x\xi^1_V \\
&-V_x\xi^2_x - U_xV_x\xi^2_U - V_x^2\xi^2_V] + \alpha V_{xx}[D_x^2\eta^2 - V_tD_x^2\xi^1 - V_xD_x^2\xi^2 - 2V_{tx}(\xi^1_x + U_x\xi^1_U \\
&+V_x\xi^1_V) - 2V_{xx}(\xi^2_x + U_x\xi^2_U + V_x\xi^2_V)] + \frac{1}{2}[\alpha V_{xx}^2 - \beta U_xV_x^2 - U_xU_t - V_xV_t][\xi^1_t + U_t\xi^1_U \\
&+V_t\xi^1_V + \xi^2_x + U_x\xi^2_U + V_x\xi^2_V] = B^1_t + U_tB^1_u + V_tB^1_v + B^2_x + U_xB^2_u + V_xB^2_v. \qquad (2.11)
\end{aligned}$$

The separation of (2.11) with respect to different combinations of derivatives of U and V results in the following over determined system of linear PDEs for $\xi^1, \xi^2, \eta^1, \eta^2, B^1$ and B^2:

$$\xi^1_V = 0, \tag{2.12}$$

$$\xi^1_U = 0, \tag{2.13}$$

$$\xi^1_x = 0, \tag{2.14}$$

$$\xi^1_t = 0, \tag{2.15}$$

$$\xi^2_V = 0, \tag{2.16}$$

$$\xi^2_U = 0, \tag{2.17}$$

$$\xi^2_x = 0, \tag{2.18}$$

$$\xi^2_t = 0, \tag{2.19}$$

$$\eta^1_V = 0, \tag{2.20}$$

$$\eta^1_U = 0, \tag{2.21}$$

$$\eta^1_x = 0, \tag{2.22}$$

$$\eta^2_U = 0, \tag{2.23}$$

$$\eta^2_V = 0, \tag{2.24}$$

$$\eta^2_x = 0, \tag{2.25}$$

$$
\begin{aligned}
B^2_U &= -\frac{1}{2}\eta^1_t, && (2.26)\\
B^2_V &= -\frac{1}{2}\eta^2_t, && (2.27)\\
B^1_U &= 0, && (2.28)\\
B^1_V &= 0, && (2.29)\\
B^1_t + B^2_x &= 0. && (2.30)
\end{aligned}
$$

The above system of linear PDEs is now solved for $\xi^1, \xi^2, \eta^1, \eta^2, B^1$ and B^2. Solving (2.12)-(2.15), we get

$$\xi^1 = c_1, \tag{2.31}$$

where c_1 is an arbitrary constant. From equations (2.16)-(2.19) we obtain

$$\xi^2 = c_2, \tag{2.32}$$

where c_2 is an arbitrary constant. Solving equation (2.20)-(2.22) gives

$$\eta^1 = f(t), \tag{2.33}$$

where $f(t)$ is an arbitrary function of t. From equations (2.23)-(2.25) we obtain

$$\eta^2 = g(t), \tag{2.34}$$

where $g(t)$ is an arbitrary function of t. Solving (2.28) and (2.29) we obtain

$$B^1 = A(t, x), \tag{2.35}$$

where $A(t, x)$ is an arbitrary function of t and x. Differentiating (2.33) with respect to t and substituting the result into (2.26), we obtain

$$B^2_U = -\frac{1}{2}f'(t). \tag{2.36}$$

Integrating (2.36) with respect to U we have

$$B^2 = -\frac{1}{2}f'(t)U + D(t, x, V), \tag{2.37}$$

where $D(t,x,V)$ is an arbitrary function of t, x and V. Differentiating (2.34) and (2.37) with respect to t and V respectively and substituting the results in (2.27) we obtain

$$D_V = -\frac{1}{2}g'(t). \tag{2.38}$$

The integration of (2.38) with respect to V yields

$$D = -\frac{1}{2}g'(t)V + H(t,x), \tag{2.39}$$

where $H(t,x)$ is an arbitrary function of t and x. Substituting (2.39) into (2.37) we obtain

$$B^2 = -\frac{1}{2}f'(t)U - \frac{1}{2}g'(t)V + H(t,x). \tag{2.40}$$

Differentiating (2.35) and (2.40) with respect to t and x respectively and substituting the results into (2.30) we get

$$A_t + H_x = 0. \tag{2.41}$$

The solutions for the system (2.12)-(2.30) are given by

$$\begin{aligned}
&\xi^1 = c_1,\\
&\xi^2 = c_2,\\
&\eta^1 = f(t),\\
&\eta^2 = g(t),\\
&B^1 = A(t,x),\\
&B^2 = \frac{-1}{2}f'(t) - \frac{1}{2}g'(t) + H(t,x),\\
&A_t + H_x = 0.
\end{aligned}$$

We can set $H = 0$ and $A = 0$ as they contribute to the trivial part of the conserved vector. Thus we obtain the following Noether symmetries and gauge terms:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\ B^1 = B^2 = 0,\\
X_2 &= \frac{\partial}{\partial x},\ B^1 = B^2 = 0,\\
X_{f(t)} &= f(t)\frac{\partial}{\partial U},\ B^1 = 0,\ B^2 = -\frac{1}{2}f'(t),\\
X_{g(t)} &= g(t)\frac{\partial}{\partial V},\ B^1 = 0,\ B^2 = -\frac{1}{2}g'(t).
\end{aligned}$$

We now use the above results to find the components of the conserved vectors for the second order Lagrangian (2.4). The conserved vector for the second order Lagrangian L is defined by [45]

$$\begin{aligned} T^1 &= -B^1 + \xi^1 L + W^1\left[\frac{\partial L}{\partial U_t} - D_t\frac{\partial L}{\partial U_{tt}} - D_x\frac{\partial L}{\partial U_{tx}}\cdots\right] \\ &\quad + W^2\left[\frac{\partial L}{\partial V_t} - D_t\frac{\partial L}{\partial V_{xt}} - D_x\frac{\partial L}{\partial V_{tt}}\cdots\right] \\ &\quad + D_t(W^1)\frac{\partial L}{\partial U_{tt}} + D_t(W^2)\frac{\partial L}{\partial V_{tt}}, \end{aligned} \tag{2.42}$$

$$\begin{aligned} T^2 &= -B^2 + \xi^2 L + W^1\left[\frac{\partial L}{\partial U_x} - D_t\frac{\partial L}{\partial U_{xt}} - D_x\frac{\partial L}{\partial U_{xx}}\cdots\right] \\ &\quad + W^2\left[\frac{\partial L}{\partial V_x} - D_t\frac{\partial L}{\partial V_{xt}} - D_x\frac{\partial L}{\partial V_{xx}}\cdots\right] \\ &\quad + D_x(W^1)\frac{\partial L}{\partial U_{xx}} + D_x(W^2)\frac{\partial L}{\partial V_{xx}}, \end{aligned} \tag{2.43}$$

where $W^1 = \eta^1 - U_t\xi^1 - U_x\xi^2$ and $W^2 = \eta^2 - V_t\xi^1 - V_x\xi^2$ are the Lie characteristic functions. The conserved vectors T^1 and T^2 must satisfy

$$D_t(T^1) + D_x(T^2)|_{system} = 0. \tag{2.44}$$

Utilizing equations (2.42), (2.43) together with X_1 we obtain the following independent conserved vector

$$T_1^1 = \frac{1}{2}(\alpha V_{xx}^2 - \beta U_x V_x^2), \tag{2.45}$$

$$T_1^2 = \frac{1}{2}\beta V_x^2 U_t + \frac{1}{2}U_t^2 + \beta U_x V_x V_t + \frac{1}{2}V_t^2 + \alpha V_t V_{xxx} - \alpha V_{tx} V_{xx}. \tag{2.46}$$

Using the inverse transformation $U = \int u dx$ and $V = \int v dx$ into (2.45) we obtain the following nonlocal conserved vector for the DSW system (2.2)

$$\begin{aligned} T_1^1 &= \frac{1}{2}(\alpha v_x^2 - \beta u v^2), \\ T_1^2 &= -\frac{1}{2}\beta\int u_t dx + \frac{1}{2}\int u_t dx\int u_t dx + \beta uv\int v_t dx + \frac{1}{2}\int v_t dx\int v_t dx + \\ &\quad \alpha u_{xx}\int v_t dx - \alpha v_t v_x. \end{aligned} \tag{2.47}$$

Similarly, we obtain the following conserved vectors for symmetries X_2, $X_{f(t)}$ and $X_{g(t)}$ for the DSW system (2.2):

$$\begin{aligned}
T_2^1 &= \frac{1}{2}(u^2+v^2), \\
T_2^2 &= -\frac{1}{2}\alpha v_x^2 + \alpha v v_{xx} + \beta u v^2, && (2.48) \\
T_{(f,g)}^1 &= -\frac{1}{2}uf(t) - \frac{1}{2}vg(t), \\
T_{(f,g)}^2 &= \frac{1}{2}f'(t)\int u dx + \frac{1}{2}g'(t)\int v dx + f(t)\Big[-\frac{1}{2}\beta v^2 - \frac{1}{2}\int u_t dx\Big] && (2.49) \\
&\quad + g(t)\Big[-\frac{1}{2}\beta u v - \frac{1}{2}\int v_t dx - \alpha v_{xx}\Big].
\end{aligned}$$

The conserved vector (2.48) is a local conserved vector. From the conserved vector (2.49) we extract two particular cases by choosing $f(t)=1$ and $g(t)=0$ which gives a nonlocal conserved vector

$$\begin{aligned}
T_3^1 &= -\frac{1}{2}u, \\
T_3^2 &= \frac{1}{2}\int u dx - \frac{1}{2}\beta v^2 - \frac{1}{2}\int u_t dx, && (2.50)
\end{aligned}$$

and by setting $f(t)=0$ and $g(t)=1$ we obtain a nonlocal conserved vector

$$\begin{aligned}
T_4^1 &= -\frac{1}{2}v, \\
T_4^2 &= \frac{1}{2}\int v dx - \beta u v - \frac{1}{2}\int v_t dx - \alpha v_{xx}. && (2.51)
\end{aligned}$$

Infinitely many nonlocal conservation laws exist for system (2.2) for arbitrary values of $f(t)$ and $g(t)$.

2.1.2 Exact solutions of the DSW system (2.2)

In this section, preliminaries on the (G'/G)-expansion method are given and used to obtain the exact solutions of the DSW system (2.2). We note that the following information is found in [16, 46].

Description of the (G'/G)-expansion method

Assume that the given nonlinear partial differential equation for $u(t,x)$ can be of the form

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0, \tag{2.52}$$

where P is a polynomial in its arguments. The principle of the(G'/G)-expansion method can be presented in the following steps:

Step 1. Use the travelling wave transformation $u(t,x) = U(z)$ where $z = x - ct$ to transform (2.52) into the ODE

$$Q(U, U', U'', \cdots) = 0, \tag{2.53}$$

where prime denotes the derivative with respect to z.

Step 2. If possible, integrate (2.52) term by term one or more times. This yields constant(s) of integration. The integration constant(s) are set to zero for simplicity.

Step 3. Suppose that the solution of (2.53) can be expressed by a polynomial in (G'/G) as follows

$$U(z) = \sum_{i=1}^{N} \phi_i (\frac{G'}{G})^i + \phi_0, \tag{2.54}$$

where ϕ_i are real constants with $\phi_n \neq 0$ to be determined and N is a positive integer to be determined. The function $G(z)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(z) + \lambda G'(z) + \mu G(z) = 0, \tag{2.55}$$

where λ and μ are real constants to be determined.

Step 4. The positive integer N is determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in (2.53).

Step 5. Substituting (2.54) together with (2.55) into (2.53) yields an algebraic equation involving powers of (G'/G). Collecting all terms with the same powers of (G'/G) together and then equating each coefficients of the resulting polynomial to zero yields a set of algebraic equations for ϕ_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system (CAS), such as Mathematica, to determine these constants. Since the general solutions of (2.55) are well known to us, then the substitution of ϕ_i, λ, μ, c and the general solutions (2.55) into (2.54) we obtain the travelling wave solutions for (2.52).

Application of the (G'/G)-expansion method for system (2.2)

We assume that the travelling wave solution for system (2.2) can be written in the following form:

$$\begin{aligned} u(x,t) &= U(z), \\ v(x,t) &= V(z), \end{aligned} \tag{2.56}$$

where $z = x - ct$ and c is a constant. The above travelling wave variables permit us to convert (2.2) into the following nonlinear ODEs

$$\begin{aligned} -cU'(z) + \beta V(z)V'(z) = 0, \\ -cV'(z) + \alpha V'''(z) + \beta V(z)U'(z) + \beta U(z)V'(z) = 0. \end{aligned} \tag{2.57}$$

Integrating the above system with respect to z, we obtain

$$\begin{aligned} -cU + \frac{\beta}{2}V^2 &= 0, \\ -cV + \alpha V'' + \beta VU &= 0, \end{aligned} \tag{2.58}$$

where the integration constant is set to be zero. Making U the subject of the formula from the first equation of the system, we have

$$U(z) = \frac{\beta}{2c}V^2, \tag{2.59}$$

Substituting (2.59) into the second equation of system (2.58), we obtain the following ODE

$$-cV + \alpha V'' + \frac{\beta^2}{2c}V^3 = 0. \tag{2.60}$$

Before we apply the (G'/G)-expansion method to (2.60) we note that we can integrate (2.60) to obtain an exact solution. Multiplying the above equation by V' and integrating while taking the constant of integration to be zero, we arrive at a first-order variables separable equation. Integrating this equation and reverting back to our original variables yields

$$v(x,t) = \frac{4c^2 \exp\left[\sqrt{\frac{c}{\alpha}}\Big(x - ct + A\Big)\right]}{1 + \beta c^2 \exp\left[2\sqrt{\frac{c}{\alpha}}\Big(x - ct + A\Big)\right]}, \tag{2.61}$$

where A is an arbitrary constant of integration. Since $U = \beta V^2/(2c)$, we have

$$u(x,t) = \frac{\beta}{2c}\left\{\frac{4c^2 \exp\left[\sqrt{\frac{c}{\alpha}}\Big(x - ct + A\Big)\right]}{1 + \beta c^2 \exp\left[2\sqrt{\frac{c}{\alpha}}\Big(x - ct + A\Big)\right]}\right\}^2. \tag{2.62}$$

We now apply the (G'/G)-expansion method to (2.60) to obtain more exact solutions. Suppose that the solution of the nonlinear ODE (2.60) can be expressed by a polynomial in (G'/G) as follows:

$$V(z) = \sum_{i=1}^{n} \phi_i (\frac{G'}{G})^i + \phi_0, \tag{2.63}$$

where $\phi_n \neq 0$ and n is called the balancing number and is a constant to be determined. The function $G(z)$ satisfies the second order linear ODE

$$G''(z) + \lambda G'(z) + \mu G(z) = 0, \tag{2.64}$$

where λ and μ are arbitrary constants. Considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in nonlinear ODE (2.60), we get $n = 1$. Substituting equations (2.63) and (2.64) into equation (2.60) we obtain

$$\begin{aligned} &\frac{\alpha\beta_1\lambda^2 G'(z)}{G(z)} + \frac{3\alpha\beta_1\lambda G'(z)^2}{G(z)^2} + \frac{2\alpha\beta_1\mu G'(z)}{G(z)} + \frac{2\alpha\beta_1 G'(z)^3}{G(z)^3} + \alpha\beta_1\lambda\mu - c\beta_0 \qquad (2.65)\\ &+\frac{3\beta^2\beta_1\beta_0^2 G'(z)}{2cG(z)} + \frac{3\beta^2\beta_1^2\beta_0 G'(z)^2}{2cG(z)^2} + \frac{\beta^2\beta_1^3 G'(z)^3}{2cG(z)^3} + \frac{\beta^2\beta_0^3}{2c} - \frac{c\beta_1 G'(z)}{G(z)} = 0. \end{aligned}$$

Collecting all the terms with the same power of (G'/G) together and equating each coefficient to zero, we obtain the following set of algebraic equations:

$$\begin{aligned}
\alpha\phi_1\lambda^2 + 2\alpha\phi_1\mu + \frac{3\beta^2\phi_0^2\phi_1}{2c} - c\phi_1 &= 0,\\
3\alpha\phi_1\lambda + \frac{3\beta^2\phi_0\phi_1^2}{2c} &= 0,\\
2\alpha\phi_1 + \frac{\beta^2\phi_1^3}{2c} &= 0,\\
\alpha\phi_1\lambda\mu + \frac{\beta^2\phi_0^3}{2c} - c\phi_0 &= 0.
\end{aligned}$$

Solving the algebraic equations above with the aid of mathematica, we obtain the following results

$$\begin{aligned}
\alpha &= -\frac{2c}{(\lambda^2-4\mu)},\\
\phi_0 &= \pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta},\\
\phi_1 &= \pm\frac{2\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta},
\end{aligned}$$

where λ, μ, β and c are arbitrary constants. Substituting the above values into (2.63) gives

$$V(z) = \pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left(\frac{G'}{G}\right). \tag{2.66}$$

The substitution of the general solution of (2.64) into (2.66), gives the following two types of travelling wave solutions for equation (2.60)

When $\lambda^2-4\mu>0$, we obtain the hyperbolic functions travelling wave solutions

$$\begin{aligned}
V_1(z) &= \pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left[\frac{-\lambda}{2}\right. \qquad (2.67)\\
&\left.+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z}\right)\right].
\end{aligned}$$

Substituting (2.67) into (2.59), we get

$$\begin{aligned}
U_1(z) &= \frac{\beta}{2c}\left\{\pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left[\frac{-\lambda}{2}\right.\right. \qquad (2.68)\\
&\left.\left.+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z + c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z + c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)z}\right)\right]\right\}^2.
\end{aligned}$$

Substituting the value of z into (2.67) and (2.68), we obtain the hyperbolic function travelling wave solution for the DSW system (2.2)

$$\begin{aligned} v_1(t,x) &= \pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left[\frac{-\lambda}{2}\right. \\ &\quad \left. + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}\right)\right]. \end{aligned}$$

$$\begin{aligned} u_1(t,x) &= \frac{\beta}{2c}\left\{\pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left[\frac{-\lambda}{2}\right.\right. \\ &\quad \left.\left. + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}\right)\right]\right\}^2. \end{aligned}$$

Also, when $\lambda^2 - 4\mu < 0$, we obtain the following trigonometric function travelling wave solutions for the DSW system (2.2)

$$\begin{aligned} v_2(t,x) &= \pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left[\frac{-\lambda}{2}\right. \\ &\quad \left. + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}\right)\right], \end{aligned}$$

$$\begin{aligned} u_2(t,x) &= \frac{\beta}{2c}\left\{\pm\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\beta} \pm 2\frac{\sqrt{2}\sqrt{c^2-2\alpha c\mu}}{\lambda\beta}\left[\frac{-\lambda}{2}\right.\right. \\ &\quad \left.\left. + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}\right)\right]\right\}^2. \end{aligned}$$

2.2 Conclusion

In this chapter the third order DSW system (2.2) was studied. In order to apply Noether approach we used the transformation $u = U_x$, $\imath = V_x$ since the third order DSW system does not have a Lagrangian. The DSW system (2.2) transformed to the fourth order system in U, V variables which has a Lagrangian. Conservation laws were then obtained in U, V variables. The inverse transformation was then used to obtain the conservation laws for the DSW system (2.2) in u and v variables. The conservation laws for the third order DSW system (2.2) consist of a local and infinite number of nonlocal conserved vectors. Preliminaries were given for the (G'/G)-expansion method. Finally we used the (G'/G)-expansion method to determine the exact solution of the DSW system (2.2).

Chapter 3

Conservation laws and exact solutions of the modified Korteweg-de Vries system

3.1 Introduction

Consider the complex modified KdV equation [26] given by

$$\omega_t - (|\omega|^2\omega)_x - \omega_{xxx} = 0, \tag{3.1}$$

where ω is a complex valued function. We let

$$\omega = u + iv, \tag{3.2}$$

where $u = u(t,x)$ and $v = v(t,x)$ are real valued functions. Substituting (3.2) and its derivatives into (3.1) and splitting with real and imaginary functions, we obtain the following system

$$\begin{aligned} u_t - 3u^2u_x - u_xv^2 - 2uvv_x - u_{xxx} &= 0, \\ v_t - 3v^2v_x - v_xu^2 - 2vuu_x - v_{xxx} &= 0, \end{aligned} \tag{3.3}$$

where the two coupled nonlinear equations portray the interaction of two orthogonally polarized transverse waves.

Hizel [47] solved this system using the classical Lie group method where he gave all possible group invariant solutions by using one-dimensional optimal system of Lie symmetry generators of mKdV system, and also the transformation groups which generate those vector fields. Ismail [48] derived the collocation method using quintic B-spline to solve the complex mKdV.

In this chapter Noether approach is used to construct the conservation laws of mKdV system. We also use the (G'/G)-expansion method to find the exact solutions of the mKdV system.

3.1.1 Conservation laws of the mKdV system

In this section Noether approach is used to obtain the conservation laws of the mKdV system (3.3). In order to apply Noether approach we increase the order of the mKdV system by one, since the third order mKdV does not have a Lagrangian. Then the system (3.3) becomes the fourth order system given by

$$\begin{aligned} U_{tx} - 3U_x^2 U_{xx} - U_{xx} V_x^2 - 2U_x V_x V_{xx} - U_{xxxx} &= 0, \\ V_{tx} - 3V_x^2 V_{xx} - V_{xx} U_x^2 - 2V_x U_x U_{xx} - V_{xxxx} &= 0. \end{aligned} \tag{3.4}$$

The fourth order system (3.4) has a second order Lagrangian given by

$$L = \frac{1}{2}(-U_x U_t - V_x V_t + U_x^2 V_x^2 + \frac{1}{2}U_x^4 + \frac{1}{2}V_x^4 - U_{xx}^2 - V_{xx}^2). \tag{3.5}$$

The expansion of (2.10) with the second order Lagrangian (3.5) yields

$$
\begin{aligned}
&-\frac{1}{2}U_x[\eta^1_t + U_t\eta^1_U + V_t\eta^1_V - U_t\xi^1_t - U_t^2\xi^1_U - U_tV_t\xi^1_V - U_x\xi^2_t - U_tU_x\xi^2_U - U_xV_t\xi^2_V]\\
&-\frac{1}{2}V_x[\eta^2_t + U_t\eta^2_U + V_t\eta^2_V - V_t\xi^1_t - U_tV_t\xi^1_U - V_t^2\xi^1_V - V_x\xi^2_t - U_tV_x\xi^2_U - V_tV_x\xi^2_V]\\
&+\left(-\frac{1}{2}U_t + U_xV_x^2 + U_x^3\right)[\eta^1_x + U_x\eta^1_U + V_x\eta^1_V - U_t\xi^1_x - U_tU_x\xi^1_U - U_tV_x\xi^1_V - U_x\xi^2_x\\
&-U_x^2\xi^2_U - U_xV_x\xi^2_V] + \left(-\frac{1}{2}V_t + U_x^2V_x + V_x^3\right)[\eta^2_x + U_x\eta^2_U + V_x\eta^2_V - V_t\xi^1_x - U_xV_t\xi^1\\
&{}_U - V_tV_x\xi^1_V - V_x\xi^2_x - U_xV_x\xi^2_U - V_x^2\xi^2_V] - U_{xx}[D_x^2\eta^1 - U_tD_x^2\xi^1 - U_xD_x^2\xi^2\\
&-2U_{tx}(\xi^1_x + U_x\xi^1_U + V_x\xi^1_V) - 2U_{xx}(\xi^2_x + U_x\xi^2_U + V_x\xi^2_V)] - V_{xx}[D_x^2\eta^2 - V_tD_x^2\xi^1\\
&-V_xD_x^2\xi^2 - 2V_{tx}(\xi^1_x + U_x\xi^1_U + V_x\xi^1_V) - 2V_{xx}(\xi^2_x + U_x\xi^2_U + V_x\xi^2_V)] + \frac{1}{2}[-U_xU_t\\
&-V_xV_t + U_x^2V_x^2 + \frac{1}{2}U_x^4 + \frac{1}{2}V_x^4 - U_{xx}^2 - V_{xx}^2][\xi^1_t + U_t\xi^1_U + V_t\xi^1_V + \xi^2_x + U_x\xi^2_U + V_x\xi^2_V]\\
&= B^1_t + U_tB^1_u + V_tB^1_v + B^2_x + U_xB^2_u + V_xB^2_v. \qquad (3.6)
\end{aligned}
$$

Splitting (3.6) with respect to different combinations of derivatives of U and V results in the following over determined system of linear PDEs:

$$\xi^1_U = 0, \tag{3.7}$$

$$\xi^1_V = 0, \tag{3.8}$$

$$\xi^1_x = 0, \tag{3.9}$$

$$\xi^2_U = 0, \tag{3.10}$$

$$\xi^2_V = 0, \tag{3.11}$$

$$\xi^2_t = 0, \tag{3.12}$$

$$\eta^1_U = 0, \tag{3.13}$$

$$\eta^1_V = 0, \tag{3.14}$$

$$\eta^1_x = 0, \tag{3.15}$$

$$\eta^2_U = 0, \tag{3.16}$$

$$\eta^2_V = 0, \tag{3.17}$$

$$\eta^2_x = 0, \tag{3.18}$$

$$
\begin{aligned}
\xi^1_t - 3\xi^2_x &= 0, && (3.19)\\
B^2_U &= -\frac{1}{2}\eta^1_t, && (3.20)\\
B^2_V &= -\frac{1}{2}\eta^2_t, && (3.21)\\
B^1_U &= 0, && (3.22)\\
B^1_V &= 0, && (3.23)\\
\xi^2_{xx} &= 0, && (3.24)\\
B^1_t + B^2_x &= 0. && (3.25)
\end{aligned}
$$

Solving the above system of linear PDEs, we obtain the following results

$$
\begin{aligned}
\xi^1 &= 3c_1 t + c_3,\\
\xi^2 &= c_1 x + c_2,\\
\eta^1 &= E(t),\\
\eta^2 &= F(t),\\
B^1 &= G(t,x),\\
B^2 &= \frac{-1}{2}E'(t) - \frac{1}{2}F'(t) + I(t,x),
\end{aligned}
$$

$$G_t + I_x = 0,$$

where we have set $G = 0$ and $I = 0$ as they contribute to the trivial part of the conserved vector. Hence the Noether symmetries and gauge term are

$$
\begin{aligned}
X_1 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \;\; B^1 = B^2 = 0,\\
X_2 &= \frac{\partial}{\partial x}, \;\; B^1 = B^2 = 0,\\
X_3 &= \frac{\partial}{\partial t}, \;\; B^1 = B^2 = 0,\\
X_{E(t)} &= E(t)\frac{\partial}{\partial U}, \;\; B^1 = 0, \;\; B^2 = -\frac{1}{2}E'(t),\\
X_{F(t)} &= F(t)\frac{\partial}{\partial V}, \;\; B^1 = 0, \;\; B^2 = -\frac{1}{2}F'(t).
\end{aligned}
$$

We then use equations (2.42), (2.43) together with $u = U_x$, $v = V_x$ and the above symmetries, which yields the following independent conserved vectors for system

(3.3):

$$\begin{aligned}
T_1^1 &= \frac{3}{2}t(u^2v^2+\frac{1}{2}u^4+\frac{1}{2}v^4-u_x^2-v_x^2)+\frac{1}{2}x(u^2+v^2),\\
T_1^2 &= -\frac{3}{2}xu^2v^2-\frac{3}{4}xu^4-\frac{3}{4}xv^4+\frac{1}{2}xu_x^2+\frac{1}{2}xv_x^2+3t\Big[\frac{1}{2}\Big(\int u_t dx\Big)^2-uv^2\int u_t dx\\
&\quad -u^3\int u_t dx-u_{xx}\int u_t dx\Big]-xuu_{xx}+3t\Big[\frac{1}{2}\Big(\int v_t dx\Big)^2-u^2v\int v_t dx\\
&\quad -v^3\int v_t dx-v_{xx}\int v_t dx\Big]-xvv_{xx}+3tu_tu_x+uu_x+3tv_xv_t+vv_x, \qquad (3.26)\\
T_2^1 &= \frac{1}{2}(u^2+v^2),\\
T_2^2 &= -\frac{3}{2}u^2v^2-\frac{3}{4}u^4-\frac{3}{4}v^4+\frac{1}{2}u_x^2+\frac{1}{2}v_x^2-vv_{xx}-uu_{xx}, \qquad (3.27)\\
T_3^1 &= \frac{1}{2}(u^2v^2+\frac{1}{2}u^4+\frac{1}{2}v^4-u_x^2-v_x^2),\\
T_3^2 &= \frac{1}{2}\Big(\int u_t dx\Big)^2-uv^2\int u_t dx-u^3\int u_t dx-u_{xx}\int u_t dx+\frac{1}{2}\Big(\int v_t dx\Big)^2\\
&\quad -u^2v\int v_t dx-v^3\int v_t dx-v_{xx}\int v_t dx+u_xu_t+v_xv_t, \qquad (3.28)\\
T_{(E,F)}^1 &= -\frac{1}{2}uE(t)-\frac{1}{2}vF(t),\\
T_{(E,F)}^2 &= \frac{1}{2}E'(t)\int u dx+\frac{1}{2}F'(t)\int v dx+E(t)[-\frac{1}{2}\int u_t dx+uv^2+u^3+u_{xx}]\\
&\quad +F(t)[-\frac{1}{2}\int v_t dx+u^2v+v^3+v_{xx}]. \qquad (3.29)
\end{aligned}$$

The vectors (3.26), (3.28) and (3.29) are nonlocal conserved vectors and vector (3.27) is a local conserved vector. We consider two particular cases for the conserved vector (3.29). For $E(t)=1$ and $F(t)=0$, we obtain the nonlocal conserved vector

$$\begin{aligned}
T_4^1 &= -\frac{1}{2}u,\\
T_4^2 &= \frac{1}{2}\int u dx-\frac{1}{2}\int u_t dx+uv^2+u^3+u_{xx}, \qquad (3.30)
\end{aligned}$$

and for $E(t)=0$ and $F(t)=1$ we get the nonlocal conserved vector

$$\begin{aligned}
T_5^1 &= -\frac{1}{2}v,\\
T_5^2 &= \frac{1}{2}\int v dx-\frac{1}{2}\int v_t dx+u^2v+v^3+v_{xx}. \qquad (3.31)
\end{aligned}$$

Infinitely many nonlocal conservation laws exist for system (3.4) for arbitrary values of $f(t)$ and $g(t)$.

3.1.2 Exact solutions of the mKdV system

In this section we apply the (G'/G)-expansion method to obtain the exact solutions of the mKdV system (3.3). We consider the travelling wave transformation $u(x,t) = U(z)$, $v(x,t) = V(z)$ and transform system (3.3) into a system of ODEs

$$\begin{aligned}
&-cU'(z) - 3U^2(z)U'(z) - U'(z)V^2(z) - 2U(z)V(z)V'(z) - U'''(z) = 0, \\
&-cV'(z) - 3V^2(z)V'(z) - V'(z)U^2(z) - 2V(z)U(z)U'(z) - V'''(z) = 0. \qquad (3.32)
\end{aligned}$$

The integration of the above nonlinear ODEs with respect to z yields

$$\begin{aligned}
-cU - U^3 - V^2U - U'' &= 0, \\
-cV - V^3 - U^2V - V'' &= 0, \qquad (3.33)
\end{aligned}$$

where the integration constant is set to be zero. The solutions of the nonlinear ODEs (3.33) can be expressed by a polynomial in (G'/G) by

$$U(z) = \sum_{i=1}^{m} \alpha_i (\frac{G'}{G})^i + \alpha_0, \qquad (3.34)$$

$$V(z) = \sum_{i=1}^{n} \beta_i (\frac{G'}{G})^i + \beta_0, \qquad (3.35)$$

where $\beta_n \neq 0$, $\alpha_m \neq 0$ and n and m are constants to be determined. Consider the homogeneous balance between the highest order derivatives and nonlinear term appearing in nonlinear ODE (3.34), we get $n = 1$ and $m = 1$. The substitution of equations (3.34), (3.35) and (2.64) into equation (3.33) results in the following equations,

$$\begin{aligned}
&-\frac{c\alpha_1 G'(z)}{G(z)} - c\alpha_0 - \frac{\alpha_0\beta_1^2 G'(z)^2}{G(z)^2} - \frac{2\alpha_0\beta_0\beta_1 G'(z)}{G(z)} - \frac{\alpha_1\beta_1^2 G'(z)^3}{G(z)^3} - \frac{2\alpha_1\beta_0\beta_1 G'(z)^2}{G(z)^2} \\
&-\frac{\alpha_1\beta_0^2 G'(z)}{G(z)} - \frac{\alpha_1\lambda^2 G'(z)}{G(z)} - \frac{3\alpha_1\lambda G'(z)^2}{G(z)^2} - \frac{2\alpha_1\mu G'(z)}{G(z)} - \frac{3\alpha_1\alpha_0^2 G'(z)}{G(z)} - \frac{3\alpha_1^2\alpha_0 G'(z)^2}{G(z)^2} \\
&-\frac{\alpha_1^3 G'(z)^3}{G(z)^3} - \frac{2\alpha_1 G'(z)^3}{G(z)^3} - \alpha_0\beta_0^2 - \alpha_1\lambda\mu - \alpha_0^3 = 0,
\end{aligned}$$

$$-\frac{c\beta_1 G'(z)}{G(z)} - c\beta_0 - \frac{\alpha_1^2\beta_0 G'(z)^2}{G(z)^2} - \frac{2\alpha_0\alpha_1\beta_0 G'(z)}{G(z)} - \frac{\alpha_1^2\beta_1 G'(z)^3}{G(z)^3} - \frac{2\alpha_0\alpha_1\beta_1 G'(z)^2}{G(z)^2}$$
$$-\frac{\alpha_0^2\beta_1 G'(z)}{G(z)} - \frac{\beta_1\lambda^2 G'(z)}{G(z)} - \frac{3\beta_1\lambda G'(z)^2}{G(z)^2} - \frac{2\beta_1\mu G'(z)}{G(z)} - \frac{3\beta_1\beta_0^2 G'(z)}{G(z)} - \frac{3\beta_1^2\beta_0 G'(z)^2}{G(z)^2}$$
$$-\frac{\beta_1^3 G'(z)^3}{G(z)^3} - \frac{2\beta_1 G'(z)^3}{G(z)^3} - \alpha_0^2\beta_0 - \beta_1\lambda\mu - \beta_0^3 = 0.$$

Splitting all the terms with the same powers of (G'/G) and equating each coefficient to zero, yields the following algebraic equations:

$$\begin{aligned}
-c\alpha_0 - \alpha_0\beta_0^2 - \alpha_1\lambda\mu - \alpha_0^3 &= 0, \qquad (3.36)\\
-c\alpha_1 - \alpha_1\beta_0^2 - 2\alpha_0\beta_0\beta_1 - \alpha_1\lambda^2 - 2\alpha_1\mu - 3\alpha_0^2\alpha_1 &= 0,\\
-2\alpha_1\beta_0\beta_1 - \alpha_0\beta_1^2 - 3\alpha_1\lambda - 3\alpha_0\alpha_1^2 &= 0,\\
-\alpha_1\beta_1^2 - \alpha_1^3 - 2\alpha_1 &= 0,\\
-c\beta_0 - \alpha_0^2\beta_0 - \beta_1\lambda\mu - \beta_0^3 &= 0,\\
-c\beta_1 - 2\alpha_0\alpha_1\beta_0 - \alpha_0^2\beta_1 - \beta_1\lambda^2 - 2\beta_1\mu - 3\beta_0^2\beta_1 &= 0,\\
-\alpha_1^2\beta_0 t - 2\alpha_0\alpha_1\beta_1 - 3\beta_1\lambda - 3\beta_0\beta_1^2 &= 0,\\
-\alpha_1^2\beta_1 - \beta_1^3 - 2\beta_1 &= 0.
\end{aligned}$$

Solving the above algebraic equation, we obtain the following results

$$\begin{aligned}
c &= \frac{1}{2}(\lambda^2 - 4\mu),\\
\alpha_1 &= \frac{2\alpha_0}{\lambda},\\
\beta_0 &= \pm\frac{\sqrt{-\lambda^2 - 2\alpha_0^2}}{\sqrt{2}},\\
\beta_1 &= \frac{2\beta_0}{\lambda},
\end{aligned}$$

where λ, μ and α_0 are arbitrary constants. Substituting the above values into (3.34) and (3.35) gives

$$U(z) = \alpha_0 + \frac{2\alpha_0}{\lambda}\left(\frac{G'}{G}\right), \qquad (3.37)$$
$$V(z) = \pm\frac{\sqrt{-\lambda^2 - 2{\alpha_2}^2}}{\sqrt{2}} \pm 2\frac{\sqrt{-\lambda^2 - 2{\alpha_2}^2}}{\sqrt{2}\lambda}\left(\frac{G'}{G}\right). \qquad (3.38)$$

Substituting the general solution of (2.64) into (3.37) and (3.38) and reverting back to original variables, we obtain two types of traveling wave solutions of the mKdV system (3.3) as follows:

For $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic functions travelling wave solutions

$$\begin{aligned} u_1(t,x) &= \alpha_0 + \frac{2\alpha_0}{\lambda}\left[\frac{-\lambda}{2} + \right. \\ & \left. \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct) + c_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}\right)\right], \end{aligned}$$

$$\begin{aligned} v_1(t,x) &= \pm\frac{\sqrt{-\lambda^2-{\alpha_2}^2}}{\sqrt{2}} \pm \frac{2\sqrt{-\lambda^2-2{\alpha_2}^2}}{\sqrt{2}\lambda}\left[\frac{-\lambda}{2} + \right. \\ & \left. \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct) + c_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)(x-ct)}\right)\right]. \end{aligned}$$

For $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling solutions

$$\begin{aligned} u_2(t,x) &= \alpha_0 + \frac{2\alpha_0}{\lambda}\left[\frac{-\lambda}{2} + \right. \\ & \left. \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct) + c_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}{c_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct) + c_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}\right)\right], \end{aligned}$$

$$\begin{aligned} v_2(t,x) &= \pm\frac{\sqrt{-\lambda^2-{\alpha_2}^2}}{\sqrt{2}} \pm \frac{2\sqrt{-\lambda^2-2{\alpha_2}^2}}{\sqrt{2}\lambda}\left[\frac{-\lambda}{2} + \right. \\ & \left. \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct) + c_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}{c_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct) + c_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)(x-ct)}\right)\right]. \end{aligned}$$

3.2 Conclusion

Noether approach was used to obtain the conservation laws of the mKdV system (3.3). But the mKdV system did not have a Lagrangian. In order to apply Noether approach we increased the order of the third order mKdV system by one. The mKdV system transformed to the fourth order system which has a Lagrangian. The conservation laws for the mKdV system (3.3) were then obtained. Infinitely many nonlocal conserved vectors exist for the third order mKdV system. Finally we used the (G'/G)-expansion method to determine the exact solution of the mKdV system.

Chapter 4

Conservation laws and Lie point symmetries of the Boussinesq system

4.1 Introduction

Boussinesq originally derived a system of two first order equations for weakly nonlinear surface waves in shallow water. Boussinesq system was formally derived from the Euler equations, in the appropriate parameter regime [49, 50]. In this chapter we study the Boussinesq system given by [27]

$$\begin{aligned} v_t + u_x + (uv)_x &= 0, \\ u_t + v_x + uu_x + v_{ttx} &= 0. \end{aligned} \tag{4.1}$$

This system (4.1) is a member of a general family of Boussinesq systems derived in [27] which are approximations to the Euler equations of the same order and written in non-dimensional, unscaled variables.

In this chapter conservation laws of the Boussinesq system (4.1) are constructed using Noether approach. We also find the Lie point symmetries of the Boussinesq system (4.1).

4.1.1 Conservation laws of the Boussinesq system

In this section we employ Noether approach to derive conservation laws for the third order Boussinesq system (4.1). For Noether approach to be applied, we transform the third order coupled Boussinesq system (4.1) to fourth order coupled system using the transformation $u = U_x$ and $v = V_x$. Then the coupled system (4.1) becomes

$$\begin{aligned} V_{tx} + U_{xx} + U_{xx}V_x + U_xV_{xx} &= 0, \\ U_{tx} + V_{xx} + U_xU_{xx} + V_{ttxx} &= 0. \end{aligned} \tag{4.2}$$

The fourth order coupled system (4.2) have a Lagrangian given by

$$L = -\frac{1}{2}(U_tV_x + U_xV_t + U_x^2V_x + U_x^2 + V_x^2 - V_{tx}^2). \tag{4.3}$$

The insertion of (4.3) into the determining equation (2.10) yields

$$\begin{aligned} &-\frac{1}{2}V_x[\eta_t^1 + U_t\eta_U^1 + V_t\eta_V^1 - U_t\xi_t^1 - U_t^2\xi_U^1 - U_tV_t\xi_V^1 - U_x\xi_t^2 - U_tU_x\xi_U^2 - U_xV_t\xi_V^2] \\ &-\frac{1}{2}U_x[\eta_t^2 + U_t\eta_U^2 + V_t\eta_V^2 - V_t\xi_t^1 - U_tV_t\xi_U^1 - V_t^2\xi_V^1 - V_x\xi_t^2 - U_tV_x\xi_U^2 - V_tV_x\xi_V^2] \\ &-\left(\frac{1}{2}V_t + U_xV_x + U_x\right)[\eta_x^1 + U_x\eta_U^1 + V_x\eta_V^1 - U_t\xi_x^1 - U_tU_x\xi_U^1 - U_tV_x\xi_V^1 - U_x\xi_x^2 \\ &-U_x^2\xi_U^2 - U_xV_x\xi_V^2] + \left(-\frac{1}{2}U_t + \frac{1}{2}U_x^2 + V_x\right)[\eta_x^2 + U_x\eta_U^2 + V_x\eta_V^2 - V_t\xi_x^1 - U_xV_t\xi^1 \\ &-V_tV_x\xi_V^1 - V_x\xi_x^2 - U_xV_x\xi_U^2 - V_x^2\xi_V^2] + V_{tx}[D_x^2\eta^2 - V_tD_xD_t\xi^1 - V_xD_xD_t\xi^2 - V_{tx}(\xi_t^1 \\ &+U_t\xi_U^1 + V_t\xi_V^1 + \xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)] - V_{xx}[\eta_t^2 + U_t\xi^2 + V_t\xi_v^2] - V_{tt}\Big[\xi_x^1 + U_x\xi_u^1 \\ &+V_x\xi_v^1\Big] - \frac{1}{2}[U_tV_x + U_xV_t + U_x^2V_x + U_x^2 + V_x^2 - V_{tx}^2][\xi_t^1 + U_t\xi_U^1 + V_t\xi_V^1 + \xi_x^2 \\ &+U_x\xi_U^2 + V_x\xi_V^2] = B_t^1 + U_tB_u^1 + V_tB_v^1 + B_x^2 + U_xB_u^2 + V_xB_v^2. \end{aligned} \tag{4.4}$$

The separation of (4.4) with respect to different combinations of derivatives of U and V results in an overdetermined system of linear PDEs:

$$\xi_x^1 = 0, \tag{4.5}$$

$$\xi_t^1 = 0, \tag{4.6}$$

$$\xi_U^1 = 0, \tag{4.7}$$

$$\xi_V^1 = 0, \tag{4.8}$$

$$\xi^2_x = 0, \tag{4.9}$$

$$\xi^2_t = 0, \tag{4.10}$$

$$\xi^2_U = 0, \tag{4.11}$$

$$\xi^2_V = 0, \tag{4.12}$$

$$\eta^1_U = 0, \tag{4.13}$$

$$\eta^1_V = 0, \tag{4.14}$$

$$\eta^2_U = 0, \tag{4.15}$$

$$\eta^2_V = 0, \tag{4.16}$$

$$\eta^2_x = 0, \tag{4.17}$$

$$\eta^1_x = 0, \tag{4.18}$$

$$B^2_V = -\frac{1}{2}\eta^1_t, \tag{4.19}$$

$$B^2_U = -\frac{1}{2}\eta^2_t, \tag{4.20}$$

$$B^1_V = 0, \tag{4.21}$$

$$B^1_U = 0, \tag{4.22}$$

$$B^1_t + B^2_x = 0. \tag{4.23}$$

Solving the above system for $\xi^1, \xi^2, \eta^1, \eta^2, B^1$ and B^2 gives the following solutions

$$\begin{aligned}
&\xi^1 = c_1,\\
&\xi^2 = c_2,\\
&\eta^1 = A(t),\\
&\eta^2 = D(t),\\
&B^1 = E(t,x),\\
&B^2 = -\frac{1}{2}A'(t)V - \frac{1}{2}D'(t)U + G(t,x),\\
&E_t + G_x = 0.
\end{aligned}$$

We can set $E = 0$ and $G = 0$ as they contribute to the trivial part of the conserved vector. Therefore, the Noether symmetries and gauge terms are

$$X_1 = \frac{\partial}{\partial t}, \;\; B^1 = B^2 = 0,$$

$$X_2 = \frac{\partial}{\partial x}, \ \ B^1 = B^2 = 0,$$

$$X_{A(t)} = A(t)\frac{\partial}{\partial U}, \ \ B^1 = 0, \ \ B^2 = -\frac{1}{2}A'(t)V,$$

$$X_{D(t)} = D(t)\frac{\partial}{\partial V}, \ \ B^1 = 0, \ \ B^2 = -\frac{1}{2}D'(t)U.$$

Using equations (2.42), (2.43) together with the above symmetries and gauge terms, and the transformations $u = U_x$ and $v = V_x$ yield the following independent conserved vectors for system (4.1):

$$\begin{aligned} T_1^1 &= -\frac{1}{2}(u^2v + u^2 + v^2 - v_t^2) + v_{tt}\int v_t dx, \\ T_1^2 &= uv\int u_t dx + u\int u_t dx + \frac{1}{2}u^2\int v_t dx + v\int v_t dx + v_{tt}\int v_t dx \\ &\quad + \int u_t dx \int v_t dx, \end{aligned} \tag{4.24}$$

$$\begin{aligned} T_2^1 &= uv + vv_{tt}, \\ T_2^2 &= -\frac{1}{2}u^2 + \frac{1}{2}v_t + u^2v + vv_{tt} + \frac{1}{2}v^2, \end{aligned} \tag{4.25}$$

$$\begin{aligned} T_{(A,D)}^1 &= -\frac{1}{2}A(t)v - \frac{1}{2}D(t)u, \\ T_{(A,D)}^2 &= \frac{1}{2}A'(t)\int v dx + \frac{1}{2}D'(t)\int u dx + D(t)[-\frac{1}{2}\int u_t dx - \frac{1}{2}u^2 - v - v_{tt}] \\ &\quad + A(t)[-\frac{1}{2}\int v_t dx - uv - u]. \end{aligned} \tag{4.26}$$

The above vectors do not satisfy (2.44). We use some manipulations to obtain the conservation laws of the Boussinesq system which satisfy (2.44). The substitution of (4.24) into (2.44) gives

$$D_t(T^1) + D_x(T^2)|_{(4.2)} = D_t(\frac{1}{2}v_t^2 + v_{tt}\int v_t dx). \tag{4.27}$$

Substituting T_1^1 and T_1^2 in (4.27), we have

$$\begin{aligned} &D_t\Big(-\frac{1}{2}(u^2v + u^2 + v^2)\Big) + D_x\Big(uv\int u_t dx + u\int u_t dx + \frac{1}{2}u^2\int v_t dx \\ &\quad + v\int v_t dx + v_{tt}\int v_t dx + \int u_t dx\int v_t dx\Big)\Big|_{(4.2)} = 0. \end{aligned} \tag{4.28}$$

Thus we now have the new nonlocal conserved vector for the Boussinesq system (4.2)

$$
\begin{aligned}
\widetilde{T}_1^1 &= -\frac{1}{2}(u^2v+u^2+v^2),\\
\widetilde{T}_1^2 &= uv\int u_t dx + u\int u_t dx + \frac{1}{2}u^2\int v_t dx + v\int v_t dx + v_{tt}\int v_t dx\\
&\quad + \int u_t dx \int v_t dx. \qquad (4.29)
\end{aligned}
$$

Following the same manner, we obtain the following conserved vectors for the Boussinesq system (4.2)

$$
\begin{aligned}
\widetilde{T}_2^1 &= uv + v_x v_t,\\
\widetilde{T}_2^2 &= -\frac{1}{2}u^2 + \frac{1}{2}v_t^2 + u^2v + vv_{tt} + \frac{1}{2}v^2, \qquad (4.30)\\
\widetilde{T}_{(A,D)}^1 &= -\frac{1}{2}A(t)v - \frac{1}{2}D(t)u,\\
\widetilde{T}_{(A,D)}^2 &= \frac{1}{2}A'(t)\int v dx + \frac{1}{2}D'(t)\int u dx + D(t)[-\frac{1}{2}\int u_t dx - \frac{1}{2}u^2 - v - v_{tt}]\\
&\quad + A(t)[-\frac{1}{2}\int v_t dx - uv - u]. \qquad (4.31)
\end{aligned}
$$

Which satisfies (2.44). The conserved vector (4.30) is local and the conserved vector (4.31) is nonlocal conserved vector. We now consider two particular cases for the conserved vector (4.31). For $A(t) = 1$ and $D(t) = 0$ we get a nonlocal conserved vector

$$
\begin{aligned}
\widetilde{T}_3^1 &= -\frac{1}{2}v,\\
\widetilde{T}_3^2 &= -\frac{1}{2}\int v_t dx - uv - u. \qquad (4.32)
\end{aligned}
$$

And for $A(t) = 0$ and $D(t) = 1$, we have a nonlocal conserved vector

$$
\begin{aligned}
\widetilde{T}_4^1 &= -\frac{1}{2}u, \qquad (4.33)\\
\widetilde{T}_4^2 &= -\frac{1}{2}\int u_t dx - \frac{1}{2}u^2 - v - v_{tt}.
\end{aligned}
$$

Infinitely many nonlocal conservation laws exist for system (4.2) for arbitrary values of $A(t)$ and $D(t)$.

4.1.2 Lie symmetries of the Boussinesq system

The Lie point symmetries of the Boussinesq system (4.1) will be generated by the vector field given by

$$\begin{aligned} X = \quad & \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U} \\ & + \eta^2(t,x,U,V)\frac{\partial}{\partial V}. \end{aligned} \tag{4.34}$$

Applying the third-order prolongation $pr^{[3]}X$ to (4.1) results in an overdetermined system given by

$$\begin{aligned}
& \xi^1_t = 0, \\
& \xi^2_x = 0, \\
& \xi^1_v = 0, \\
& \xi^1_u = 0, \\
& \xi^2_u = 0, \\
& \xi^2_v = 0, \\
& \eta^2_u = 0, \\
& \eta^2_{vx} = 0, \\
& \eta^2_{vv} = 0, \\
& 2\eta^2_{vt} - \xi^2_{tt} = 0, \\
& v\eta^1_x + \eta^1_x + \eta^2_t + u\eta^2_x = 0, \\
& \eta^2_x + \eta^1_t + \eta^2_{xtt} + u\eta^1_x = 0, \\
& \eta^2_v - \xi^1_x - \xi^2_t - \eta^1_u = 0, \\
& \eta^1 - u\xi^1_x + u\xi^2_t + v\eta^1_v + \eta^1_v = 0, \\
& \eta^1 - u\xi^1_x + u\xi^2_t - \eta^1_v - v\eta^1_v = 0, \\
& \eta^2_{vtt} + \eta^2_v + \xi^2_t - \eta^1_u - \xi^1_x = 0, \\
& \eta_2 - \xi^1_x - v\xi^1_x + v\xi^2_t + \xi^2_t + v\eta^1_u + \eta^1_u - \eta^2_v - v\eta^2_v = 0.
\end{aligned}$$

Solving the above overdetermined system we obtain the following three Lie point symmetries

$$X_1 = \frac{\partial}{\partial x}, \tag{4.35}$$

$$X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + (2+2v)\frac{\partial}{\partial v}, \tag{4.36}$$

$$X_3 = \frac{\partial}{\partial t}. \tag{4.37}$$

We now compute the commutation relations for the above symmetry generators. By the definition of the Lie bracket, we have

$$\begin{aligned}
[X_1, X_2] &= X_1X_2 - X_2X_1 \\
&= \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + (2+2v)\frac{\partial}{\partial v}\right) - \left(x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + (2+2v)\frac{\partial}{\partial v}\right)\frac{\partial}{\partial t} \\
&= X_1.
\end{aligned}$$

Similarly, we obtain the commutation relations between other vector fields. We present the commutator table of the Lie algebra of system (4.2) in **Table 1.**. Here the entry in row i and column j is represented by $[X_i, X_j]$:

Table 1.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	X_1	0
X_2	$-X_1$	0	0
X_3	0	0	0

4.2 Conclusion

In this chapter, Noether approach was employed to obtain the conservation laws of the Boussinesq system (4.1). In order to apply Noether's approach, we increased the third order system (4.1) to a fourth order system (4.2). Conservation laws were then obtained for the Boussinesq system (4.1), however the conserved vectors did not satisfy (2.44). Some manipulations were then done and the conserved vectors which satisfy (2.44) were obtained. The conservation laws for the third order Boussinesq system consist of local and infinite number of nonlocal conserved vectors. Lie point symmetries of system (4.1) were constructed.

Chapter 5

Concluding remarks

In this research project some important definitions and results from Lie group theory were first recalled and were later used in the dissertation. In Chapter 2, the DSW system was studied. We used the transformation $u = U_x$ and $v = V_x$ in order to apply Noether approach. The system then turned into a fourth order system in U, V variables which has a Lagrangian. The inverse transformation was then used to obtain the conservation laws of the DSW system in u, v variables. The exact solutions of the DSW system was obtained using the (G'/G)-expansion method. In Chapter 3, the third order mKdV was transformed to a fourth order system in order to apply Noether approach. The conservation laws for the third order mKdV system were then obtained. We also used the (G'/G)-expansion method to find the exact solutions of the mKdV. In Chapter 4, we studied the third-order Boussinesq system. In order to apply Noether's approach, we increased the order of the system by one which admits a Lagrangain. Noether approach was used to derive the conservation laws in u, v variables, however the conserved vectors did not satisfy (2.44). More manipulations were done in order for us to obtain the conservation laws of the third order Boussinesq system which satisfies (2.44). Finally the Lie point symmetries of the Boussinesq system were obtained.

In future the conservation laws obtained in this dissertation should be used to obtain the solution of these nonlinear systems.

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