On the gluing of quasi-pseudometric spaces

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On the gluing of quasi-pseudometric spaces

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Abstract

A classical problem that arises in geometry is how to glue a family of metric spaces such that the resulting space preserves their properties. In this MSc dissertation, we generalise the concept of gluing a family of metric spaces to the framework of quasi-pseudometric spaces. In particular, we will look at gluing a family of q-hyperconvex quasi-pseudo metric spaces along externally q-hyperconvex subsets and along weakly externally q-hyperconvex subsets such that the resulting space preserves the q-hyperconvexity structure. We relate these results to the well-known results in the literature. The notion of externally q-hyperconvex quasi-pseudometric spaces and weakly externally q-hyperconvex spaces are revisited and some original results are presented. Moreover, we introduce the concept of gated subsets of a quasi-pseudometric space and extend the notion of strong convexity in our context.

Preface

The work described in this dissertation was carried out under the supervision of Prof. Olivier Olela Otafudu, Department of Mathematical Sciences, North-West University, Mafikeng, from March 2016 to March 2017.

The Msc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

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I.Y. Mutemwa (Student)

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Prof. Olivier Olela Otafudu (Supervisor)

Dedication

I dedicate this work to my family, Pastor K Mutemwa, Mrs P.S. Mutemwa, Dr Muyunda Buwa, Mbulelo Buwa, Mebelo Mutemwa and Muyowa Mutemwa, my boyfriend Lourens Wessels, my unborn daughter Kiara Danil Wessels and friend Hope Sabao, for his support.

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Notation and Conventions

(M, ρ)	Metric space
(M, q)	Quasi-pseudometric space
$C_\rho(u, r)$	Closed ball in metric space (M, ρ)
$C_q(u, r)$	Closed ball in quasi-pseudometric space (M, q)
$\mathcal{H}(M)$	The collection of hyperconvex subsets of a metric space (M, ρ)
$\mathcal{E}(M)$	The collection of externally hyperconvex subsets of (M, q)
$\mathcal{A}(M)$	The collection of admissible hyperconvex subsets of (M, q)
$\mathcal{A}_q(M)$	The collection of q-admissible subsets of a quasi-pseudometric space (M, q)
$\mathcal{E}_q(M)$	The collection of externally q-hyperconvex subsets of a quasi-pseudometric space (M, q)
$\mathcal{H}_q(M)$	The collection of q-hyperconvex subsets of a quasi-pseudometric space (M, q)

Contents

Introduction

The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [2] who defined hyperconvexity in the following way:

Definition 0.0.1. *([2, Definition 1]) Let (M, ρ) be a metric space. Then (M, ρ) is* hyperconvex *if for any family $(u_i)_{i \in I}$ of points in M and any family $(r_i)_{i \in I}$ of nonnegative real numbers satisfying*

$$\rho(u_i, u_j) \leq r_i + r_j \quad \textit{whenever} \quad i, j \in I,$$

then

$$\bigcap_{i \in I} C_\rho(u_i, r_i) \neq \emptyset.$$

They proved that a hyperconvex space is a nonexpansive absolute retract, meaning that a hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically embedded. They also characterized hyperconvex metric spaces as injective spaces in the category of metric spaces as objects and nonexpansive maps as morphisms. Since then, many other authors such as Piatek [22], Espinola [7], Khamsi and Kirk [15], Miesch [18], Isbell [11] and Sine [23] have also proved some classical fixed point theorems for nonexpansive self mapping on hyperconvex spaces. Furthermore, Isbell in [11] proved that every metric space has an envelope (injective hull) which is hyperconvex, whilst Miesch [18] studied how to glue a family of hyperconvex metric spaces such that the resulting space remains hyperconvex. The concept of gluing metric spaces is well-known in metric geometry. For a good overview we recommend the reader to [5].

In [18], Miesch gave an answer to the question that arises in metric geometry: How to glue a family of hyperconvex metric spaces such that the resulting space remains hyperconvex. Miesch succeeded in gluing a family of hyperconvex metric spaces along gated subsets and externally hyperconvex subsets such that the resulting space is hyperconvex. Gated subsets and externally hyperconvex subsets are defined as follows:

Definition 0.0.2. *([6, p.112]) Let (M, ρ) be a metric space and A be a subset of M. Then A is gated if for all $u \in M$, there is some $\overline{u} \in A$ such that for all $a \in A$ we have the following:*

$$\rho(u, a) = \rho(u, \overline{u}) + \rho(\overline{u}, a).$$

If such a point $\overline{u}$ exists, it is called the gate *of u in A.*

It is easy to see that, if $\overline{u}$ is the gate of u in A and $\overline{\overline{u}}$ is another gate of $u \in A$, then

$$\rho(u, \overline{\overline{u}}) = \rho(u, \overline{u}) + \rho(\overline{u}, \overline{\overline{u}}) = \rho(u, \overline{\overline{u}}) + \rho(\overline{\overline{u}}, \overline{u}) + \rho(\overline{u}, \overline{\overline{u}}),$$

that is $\rho(\overline{u}, \overline{\overline{u}}) = 0$. Hence the gate $\overline{u}$ is unique.

Remark 0.0.1. *Under different motivations, gated sets have been investigated by other authors. For instance in [24] gated sets are called prefibers , in [9, 11] gated sets are called Chebychev sets and in [25] gated sets are called J-convex sets.*

Definition 0.0.3. *([7, Definition 3.5]) Let (M, ρ) be a metric space and H a subset of M. Then H is* externally hyperconvex *if for any family of points $\{u_\alpha\}_{\alpha \in \Gamma}$ in M and any family of real numbers $\{r_\alpha\}_{\alpha \in \Gamma}$ satisfying $\rho(u_\alpha, u_\beta) \le r_\alpha + r_\beta$, and $dist(u_\alpha, H) \le r_\alpha$, for all $\alpha, \beta \in \Gamma$, then it follows that $\bigcap C_\rho(u_\alpha, r_\alpha) \cap H \neq \emptyset$.*

Furthermore, Piatek [22] independently answered the above question by gluing the family of hyperconvex metric spaces along a unique metric interval such that the resulting space preserves hyperconvexity.

The theory of quasi-metric spaces was introduced by Wilson ([26]) and was further studied and improved by many authors such as Kelly ([13]), Künzi and Olela Otafudu [16], Kazeem et al. [12] etc.

The main goal of this MSc work is to introduce the concept of amalgamating (gluing) a family of quasi-pseudometric spaces. In particular, we study the concept of gluing a family of q-hyperconvex quasi-pseudometric spaces along an externally hyperconvex subset and a weakly externally q-hyperconvex such that the resulting space is q-hyperconvex. Moreover, we define a gated subset in the framework of quasi-metric spaces.
It should be pointed out that there are many unknown results about gluing quasi-pseudometric spaces, for example, it is not yet known how one can glue two q-hyperconvex spaces such that the resulting space is q-hyperconvex too. Hence this MSc dissertation will also try to present some of these unknown results.
The application of hyperconvex metric spaces in graph theory and combinatorics is a great motivation for the generalization of results about hyperconvex metric space from symmetric settings to asymmetric point of view.

Outline of chapters

Chapter 1

In this chapter, we present some well-known concepts that will be used throughout this dissertation. We first recall the definition of a quasi-pseudometric space and it's conjugate quasi-pseudometric space. Then, we provide the definitions of isometric mapping, non-expansive mapping, bounded subsets in quasi-pseudometric spaces. Finally, we recall the definitions of disjoint unions and disjoint unions in terms of topology.

Chapter 2

In this chapter, we summarize the well-known concept of hyperconvex metric spaces and present the results of Miesch ([18]) on the gluing of a family of hyperconvex metric spaces such that the resulting space preserves hyperconvexity. In section one, we present the concept of hyperconvex metric spaces. In section two, we study the notion of gated subsets of a metric space. In section three, we discuss how one goes about gluing along gated subsets such that the resulting space is hyperconvex. In the fourth section, we discuss the theory of gluing a family of hyperconvex metric spaces along an externally hyperconvex subset such that the resulting space is hyperconvex and in the fifth section, we present the concept of gluing along weakly hyperconvex subsets.

Chapter 3

In this chapter, we discuss q-hyperconvexity and look at some important subsets of pseudometric spaces. In section one, we summarize the theory q-hyperconvex quasi-pseudometric spaces and we provide some examples. In section two, we introduce the concept of in-gated subsets and out-gated subsets of a quasi-pseudometric space. We prove that if an in-gate (or out-gate) exists, it is unique whenever the quasi-pseudometric space is T_0 (Lemma 3.2.1). Moreover, we prove that if a subset of a T_0-quasi-metric is in-gated, then it is strongly convex (Proposition 3.2.1). In the third section, we present the concept of externally q-hyperconvexity in the framework of quasi-pseudometric spaces and look at some characteristics thereof. We also explore the notion of weakly externally q-hyperconvex spaces.

Chapter 4

This chapter is our main and own work. We introduce the concept of gluing a family of quasi-pseudometric spaces (called amalgamating) see Proposition 4.1.1. In particular, we will present results on gluing such a family of q-hyperconvex quasi-pseudometric spaces along externally q-hyperconvex subsets and along weakly externally q-hyperconvex quasi-pseudometric subsets such that the resulting space is q-hyperconvex.

Chapter 5

In this chapter, we summarize the results obtained and present some open problems encountered throughout our study.

1
Preliminaries

In this chapter, we present some well known concepts that will be used throughout this dissertation and we give some interesting examples.

We begin by summarising the notion of quasi-pseudometric spaces and we provide some examples.

Definition 1.0.4. *([14, Definition 1]) Let M be a nonempty set and let q be a function that maps $M \times M$ into the set $[0, \infty)$ of nonnegative real numbers, that is $q : M \times M \to [0, \infty)$. Then, q is a* quasi-pseudometric *on M if*

i) $q(u, u) = 0$, *whenever* $u \in M$.

ii) $q(u, v) \leq q(u, w) + q(w, v)$, *whenever* $u, v, w \in M$.

The pair (M, q) is said to be a quasi-pseudometric space.

We further say that q is quasi-metric *or* T_0-quasi-metric *if it satisfies the following conditions: for $u, v \in M$,*

$$q(u, v) = 0 = q(v, u) \text{ implies that } u = v.$$

Next, we recall the definition of the conjugate quasi-pseudometric of a quasi-pseudometric.

Definition 1.0.5. *([14, Remark 1]) Let q be a quasi-pseudometric (or T_0-quasi-metric) on a set M, then the* conjugate quasi-pseudometric $q^{-1} : M \times M \to [0, \infty)$ *of q is defined by $q^{-1}(u, v) = q(v, u)$ whenever $u, v \in M$. If q is a quasi-pseudometric on M such that $q = q^{-1}$, then q is said to be a pseudometric on M. For any T_0-quasi-metric q, $q^s = max\{q, q^{-1}\} = q \vee q^{-1}$ is a pseudometric.*

Example 1.0.1. *The space* $(\mathbb{R}, \rho)$, *defined by* $\rho(u,v) = u\dot{-}v = \max\{u-v, 0\}$ *is a* T_0*-quasi-metric space. The conjugate* ρ^{-1} *of this* T_0*-quasi-metric space* q *is defined as follows,*

$$\rho^{-1}(u,v) = v\dot{-}u = \max\{v-u, 0\}.$$

Furthermore, the symmetrized ρ^{-1} *of* ρ,

$$\rho^s(u,v) = \mid u - v \mid$$

is the usual metric on $\mathbb{R}$.

Example 1.0.2. *The subspace* $[0,\infty)$ *of* $\mathbb{R}$ *endowed with the quasi-pseudometric* $\rho(u,v) = u\dot{-}v$ *is a* T_0*-quasi-metric space.*

We now recall the definitions of a closed and open ball of a quasi-pseudometric space.

Definition 1.0.6. *Let* (M,q) *be a quasi-pseudometric space. For any point* $u \in M$ *and nonnegative real number* ϵ, $B_q(u,\epsilon) = \{v \in M : q(u,v) < \epsilon\}$ *denotes an open ball centered at* u *with radius* ϵ. *The collection of all "open" balls yields a base for a topology* $\tau(q)$ *and is called the topology induced by* q *on* M.

Definition 1.0.7. *Let* (M,q) *be a quasi-pseudometric space. For any point* $u \in M$ *and nonnegative real number* ϵ, *the set* $C_q(u,\epsilon) = \{v \in M : q(u,v) \leq \epsilon\}$ *is a closed ball centered at* u *with radius* ϵ.

Remark 1.0.2. *Note that this set is* $\tau(q^{-1})$*-closed, but not* $\tau(q)$*-closed in general.*

Definition 1.0.8. *Let* (M,q) *be a quasi-pseudometric space. Let* A *be a subset of* M *and* $\epsilon \geq 0$, *we denote* $C_q(A,\epsilon)$ *by*

$$C_q(A,\epsilon) = \{v \in M : dist_q(v,A) \leq \epsilon\},$$

where

$$dist_q(v,A) = \inf\{q(v,w) : u \in A\}$$

and

$$dist_{q^{-1}}(v,A) = dist_q(A,v) = \inf\{q(w,v) : w \in A\}.$$

We recall the well-known definition of a retraction of a quasi-metric space to a subset.

Definition 1.0.9. *Let* (M,q) *be a quasi-metric space and let* Y *be a subset of* M. *Then a continuous map* $f : M \to Y$ *is said to be a retraction if* f *preserves the position of all points in* Y, *that is for* $u \in M$ *and* $v \in Y$

(i) $f(u) \in Y$ *and*
(ii) For each $v \in Y$, $f(v) = v$, *that is,* f *is the identity function on it's own image.*

Next, we recall the following useful definitions.

Definition 1.0.10. *Let (M_1, q_1) and (M_2, q_2) be quasi-pseudometric spaces. The map $T : (M_1, q_1) \to (M_2, q_2)$ is said to be an* isometry *if $q_2(T(u), T(v)) = q_1(u, v)$, where $u, v \in M_1$. Two quasi-pseudometric spaces (M_1, q_1) and (M_2, q_2) will be called* isometric *provided that there exists a bijective isometry between them.*

Definition 1.0.11. *Let (M_1, q_1) and (M_2, q_2) be quasi-pseudometric spaces. The map $T : (M_1, q_1) \to (M_2, q_2)$ is said to be* nonexpansive *if $q_2(T(u), T(v)) \leq q_1(u, v)$, where $u, v \in M_1$.*

Definition 1.0.12. *Let (M, q) be a quasi-pseudometric space and A a subset of M. Then A is said to be* bounded *if there is a real number $K > 0$ such that $q(u, v) < K$ for $u, v \in A$. Equivalently one could say that A is bounded if*

$$A \subseteq C_q(u, r) \cap C_{q^{-1}}(v, s),$$

whenever $u, v \in M$ and r and s are nonnegative real numbers.

Definition 1.0.13. *Let (M, q) be a quasi-pseudometric space. Then (M, q) is said to be* injective *if it has the following* extension property*: Whenever P is a subspace of a quasi-pseudometric space (N, q) and $f : P \to M$ is nonexpansive, then f has a nonexpansive extension $F : N \to M$.*

In the following, we recall the definition of disjoint unions.

Definition 1.0.14. *([5, p.64]) Let $\{A_i : i \in I\}$ be a family of sets indexed by I. Then the disjoint union of this family of indexed sets is defined as follows:*

$$\bigsqcup_{i \in I} A_i = \bigcup A_i^* \quad \text{where } A_i^* = \{(x, i) : x \in A_i\}.$$

Remark 1.0.3. *The essence of disjoint unions is to give us a way to take the union of sets while still remembering from which set each element comes from.*

Example 1.0.3. *Let $A_0 = \{a, b, c\}$ and $A_1 = \{e, f\}$*
$A_0^* = \{(a, 0), (b, 0), (c, 0)\}$ and $A_1^* = \{(e, 1), (f, 1)\}$
$A_0 \bigsqcup A_1 = A_0^* \bigcup A_1^* = \{(a, 0), (b, 0), (c, 0), (e, 1), (f, 1)\}$

We also define disjoint unions in terms of topology.

Definition 1.0.15. *Let $\{X_i : i \in I\}$ be a family of topological spaces indexed by I. Let*

$$X = \coprod_{i \in I} X_i$$

be the disjoint union of the underlying sets. For each $i \in I$, let

$$\varphi_i : X_i \to X,$$

defined by $\varphi_i(x) = (x, i)$, be a canonical injection.

The disjoint union topology on X *is defined as the finest topology on X for which the canonical injections $\{\varphi_i : i \in I\}$ are continuous.*

We now recall the definition of a geodesic.

Definition 1.0.16. *[5, Definition 1.3] Let (M, q) be a quasi-metric space and let $u, v \in M$. Then a geodesic path joining u to v is a map $c : [0, l] \to M$ where $[0, l] \subset \mathbb{R}$ such that $c(0) = u$, $c(l) = v$ and $q(c(t), c(t')) = \mid t - t' \mid$ whenever $t, t' \in [0, l]$ that is $q(u, v) = l$.*

2

Gluing hyperconvex metric spaces

In this chapter, we summarize the well-known concept of hyperconvex metric spaces. Then, we present the results of Miesch ([18]) on the gluing of a family of hyperconvex metric spaces such that the resulting space preserves hyperconvexity. In particular, we study the notion of gated subsets of a metric space, how one goes about gluing along these gated subsets and also study how to glue along externally hyperconvex subsets and along weakly externally hyperconvex subsets.

2.1. Hyperconvex metric spaces

In this section, we discuss important results on hyperconvex metric spaces. We will also look at some examples and some characterizations on this theory.

We begin by recalling the concept of metric convexity.

Definition 2.1.1. *([15, Definition 4.1]) Let (M, ρ) be a metric space. Then (M, ρ) is said to be* metrically convex *if for any two points $u, v \in M$ and any two positive real numbers r_1 and r_2 with $\rho(u, v) \leq r_1 + r_2$, there exists $z \in M$ such that $\rho(u, z) \leq r_1$ and $\rho(z, v) \leq r_2$ or equivalently $z \in C_\rho(u, r_1) \cap C_\rho(v, r_2)$, where $C_\rho(u, r_1)$ and $C_\rho(v, r_2)$ denote closed balls with center $u, v \in M$ and radius $r_1, r_2 \geq 0$.*

Remark 2.1.1. *([15, Definition 4.1]) Applying the Triangle Inequality, we obtain that $C_\rho(u, r_1) \cap C_\rho(v, r_2) \neq \emptyset$, which implies that $\rho(u, v) \leq r_1 + r_2$ for any $u, v \in M$ and positive real numbers r_1, r_2. The converse is true on the real line and corresponds to* Menger convexity *in metric spaces.*

We now recall the definition of a hyperconvex metric space which was first introduced by Aronszajn and Panitchpakdi in ([2]).

Definition 2.1.2. *([2, Definition 1]) Let (M,ρ) be a metric space. Then (M,ρ) is* hyperconvex *if for any family of $(u_i)_{i\in I}$ of points in M and any family $(r_i)_{i\in I}$ of positive real numbers satisfying*

$$\rho(u_i,u_j) \leq r_i + r_j \quad \text{whenever} \quad i,j \in I,$$

then

$$\bigcap_{i\in I} C_\rho(u_i,r_i) \neq \emptyset.$$

Aronszajn and Panitchpakdi were the first to study the notion of hyperconvexity in [2] and since then other authors have also studied this notion.

Theorem 2.1.1. *([15, p.82]) Let (M,ρ) be a metric space. Then (M,ρ) is injective if and only if it is hyperconvex.*

Proof. We refer the reader to ([15, p.82]). □

Remark 2.1.2. *Theorem 2.1.1 above, tells us that hyperconvex spaces are the same as injective spaces.*

Hyperconvex metric spaces are metrically convex. To see this, let (M,ρ) be a hyperconvex metric space and $u,v \in M$ such that $u \neq v$. Let $r_1 = \alpha\rho(u,v)$ and $r_2 = (1-\alpha)\rho(u,v)$, for any $\alpha \in [0,1]$. Then $\rho(u,v) = r_1 + r_2$ and since (M,ρ) is hyperconvex, it follows that $C_\rho(u,r_1) \cap C_\rho(v,r_2) \neq \emptyset$. Let $z \in C_\rho(u,r_1) \cap C_\rho(v,r_2)$, then we have

$$\rho(u,z) \leq r_1 \quad \text{and} \quad \rho(z,v) \leq r_2.$$

Moreover, by the Triangle Inequality we have

$$\rho(u,z) = r_1 \quad \text{and} \quad \rho(z,v) = r_2.$$

Therefore, (M,ρ) is metrically convex.

Proposition 2.1.1. *([15, Proposition 4.4]) Let (M,ρ) be any metric space. If (M,ρ) is hyperconvex then it is complete.*

Proof: See ([15, Proposition 4.4]). □

We move on to the concept of binary ball intersection property in metric spaces.

Definition 2.1.3. *([15, P.79]) Let (M, ρ) be a metric space. For any family of $(u_i)_{i \in I}$ of points in M and any family $(r_i)_{i \in I}$ of positive real numbers, a family of closed balls $C_\rho(u_i, r_i)_{i \in I}$ in M, where each two intersect, is said to have a* binary ball intersection property *if*

$$\bigcap_{i \in I} C_\rho(u_i, r_i) \neq \emptyset, \quad for\ all\ i \in I.$$

Now suppose (M, ρ) is a complete metric space which has the binary ball intersection property and suppose (M, ρ) is metrically convex. If $(C_\rho(u_i, r_i))_{i \in I}$ is a family of balls in M such that $\rho(u_i, u_j) \leq r_i + r_j$, whenever $i, j \in I$, then there is a line segment joining u_i and u_j and some point of this line segment lies in $C_\rho(u_i, r_i) \cap C_\rho(u_j, r_j)$. Thus, $\bigcap_{i \in I} C_\rho(u_i, r_i) \neq \emptyset$ by binary ball intersection property.

The following lemma makes a connection between hyperconvexity and metric convexity (see [15, p.79]).

Lemma 2.1.1. *If (M, ρ) is a* complete metric space, *then the following are equivalent:*

(i) M is hyperconvex.

(ii) M is metrically convex and has the binary ball intersection property.

We shall denote the collection of subsets of a metric space (M, ρ) which are hyperconvex by $\mathcal{H}(M)$. We now provide examples of hyperconvex metric spaces.

Example 2.1.1. *The set $\mathbb{R}$ of real numbers equipped with the usual metric $\rho(u, v) = |u - v|$ is hyperconvex. Also, any closed interval in $\mathbb{R}$ is hyperconvex.*

Example 2.1.2. *([15, Theorem 4.3]) Let l_∞ be the set of all bounded sequences $(u_i)_{i \in I}$ of real numbers. If we equip l_∞ with the metric ρ_∞ defined by:*

$$\rho_\infty(u, v) = \sup_{1 \leq i \leq \infty} |u_i - v_i|, \ where\ u = (u_i)_{i \in I}\ and\ v = (v_i)_{i \in I},$$

then (l_∞, ρ_∞) is a hyperconvex space.

Let us turn our attention to some important subsets of hyperconvex spaces. We begin by discussing admissible subsets, but before we do that we recall the definition of a cover.

Definition 2.1.4. *([16, p.4]) Let (M, ρ) be a metric space and P a subset of M. The* cover *of P, denoted by* $\text{cov}(P)$, *is the set*

$$\text{cov}(P) = \bigcap \{C_\rho(u, r) : P \subseteq C_\rho(u, r), u \in M\ and\ r\ is\ a\ real\ positive\ number\}.$$

In the following, we recall the definition of admissible subset of a metric space.

Definition 2.1.5. *([7, Definition 3.4]) Let (M,ρ) be a metric space and P a subset of M. Then P is an* admissible subset *of M if it is the intersection of closed balls contained in M, that is for $P \subseteq M$ we have $P = cov(P)$. We denote the set of all admissible subsets of M by $\mathcal{A}(M)$, thus $\mathcal{A}(M) = \{P \subseteq M : P = \text{cov}(P)\}$.*

Remark 2.1.3. *Since $\mathcal{A}(M)$ contains all the closed balls in M and is stable by intersection, that is, the intersection of any collection of elements from $\mathcal{A}(M)$ is also in $\mathcal{A}(M)$. From Definition 2.1.5, if $P \in \mathcal{A}(M)$, then we have*

$$C(P) = \bigcap_{p\in P} C_\rho(p, R(P)) \bigcap P \in \mathcal{A}(M). \tag{2.1}$$

Furthermore, $diam(C(P)) \leq \dfrac{diam(P)}{2}$. Hence, $P = C(P)$ if and only if $P \in \mathcal{A}(M)$ and $diam(P) = 0$, that is, P is reduced to one point.

We move on to the concept of externally hyperconvex subsets of a metric space. Note that the following definition strengthens the concept of a hyperconvex subset of a metric space (M,ρ).

Definition 2.1.6. *([7, Definition 3.5]) Let (M,ρ) be a metric space and H a subset of M. Then H is* externally hyperconvex *if for any family of points $(u_i)_{i\in I}$ in M and any family of real numbers $(r_i)_{i\in I}$ satisfying $\rho(u_i,u_j) \leq r_i + r_j$, and $dist(u_i, H) \leq r_i$, for all $i,j \in I$, then it follows that $\bigcap C_\rho(u_i,r_i) \cap H \neq \emptyset$.*

Remark 2.1.4. *In this chapter, we denote the set of all externally hyperconvex subsets of the metric space (M,ρ) by $\mathcal{E}(M)$.*

Definition 2.1.7. *([7, Definition 3.7]) Let (M,ρ) be a metric space. A subset P of M is said to be* proximal *(with respect to M) if*

$$P \cap C_\rho(u, \text{dist}(u,P)) \neq \emptyset \text{ for each } u \in M.$$

Remark 2.1.5. *Note that proximal subsets are closed.*

The next theorem makes connections between $\mathcal{A}(M)$, $\mathcal{E}(M)$ and $\mathcal{H}(M)$.

Theorem 2.1.2. *([7, Theorem 3.10]) Let (M,ρ) be a hyperconvex metric space, then*

$$\mathcal{A}(M) \subseteq \mathcal{E}(M) \subseteq \mathcal{H}(M).$$

Proof: Firstly, we show that $\mathcal{A}(M) \subseteq \mathcal{E}(M)$. *Let* P *be an admissible subset of* M *and let* $(u_i)_{i\in I}$ *be a family of points in* M *and* $(r_i)_{i\in I}$ *be a family of nonnegative real numbers satisfying:* $\rho(u_i, u_j) \leq r_i + r_j$ *and* $dist(u_i, P) \leq r_i$ *for any* $i, i \in I$. *Since* P *is proximinal, for any* $i \in I$, *there exists* $a_i \in P$ *such that* $\rho(u_i, a_i) = dist(u_i, P)$, *which gives* $P \cap C_\rho(u_i, r_i) \neq \emptyset$. *Since* M *is hyperconvex, the conditions on both families imply* $\bigcap_{i\in I} C_\rho(u_i, r_i) \neq \emptyset$. *Since* P *is admissible and* $P \cap C_\rho(u_i, r_i) \neq \emptyset$, *it follows that*

$$P \bigcap \left(\bigcap_{i\in I} C_\rho(u_i, r_i) \right) \neq \emptyset,$$

which proves the first inclusion.
The second inclusion $\mathcal{E}(X) \subseteq \mathcal{H}(X)$ *follows directly from the definition of an externally hyperconvex subset of* (M, ρ). □

Definition 2.1.8. *([2, Definition 1]) Let* (M, ρ) *be a metric space. Then* (M, ρ) *is* m-hyperconvex *if for any family* $(u_i)_{i\in I}$ *of points in* M *and any family* $(r_i)_{i\in I}$ *of positive real numbers satisfying* $\rho(u_i, u_j) \leq r_i + r_j$ *and* $|I| \leq m$, *we have*

$$\bigcap_{i\in I} C_\rho(u_i, r_i) \neq \emptyset.$$

Definition 2.1.9. *Let* (M, ρ) *be a metric space. Then for each pair* $u, v \in M$ *the* metric interval $I(u, v)$ *is defined as follows:* $I(u, v) = \{z \in M : \rho(u, z) + \rho(z, v) = \rho(u, v)\}$.

Definition 2.1.10. *Let* (M, ρ) *be a metric space. If for* $u, v, z \in M$ *satisfying the following condition:*

$$I(u, v) \cap I(v, z) \cap I(z, u) \neq \emptyset,$$

then we say that (M, ρ) *is a* modular.

The following result states that any 3-hyperconvex metric space is modular.

Lemma 2.1.2. *Let* (M, ρ) *be a 3-hyperconvex metric space. For* $u, v, z \in M$, *we have*

$$I(u, v) \cap I(v, z) \cap I(z, u) \neq \emptyset.$$

Moreover, the 3-hyperconvex metric is modular.

Proof. By Definition 2.1.9 we have the following: If for any $u, v, z \in M$ there is a $t \in M$ such that $t \in I(u, v) \cap I(v, z) \cap I(z, u) \neq \emptyset$, then

$$\rho(u, v) = \rho(u, t) + \rho(t, v),$$

$$\rho(v,z) = \rho(v,t), +\rho(t,z),$$
$$\rho(z,u) = \rho(z,t) + \rho(t,u).$$

Let $\alpha, \beta, \gamma \geq 0$ be the unique solution of the linear system such that

$$\rho(u,t) = \rho(t,u) = \alpha,$$
$$\rho(v,t) = \rho(t,v) = \beta,$$

and

$$\rho(z,t) = \rho(t,z) = \gamma.$$

Hence

$$\rho(u,v) = \rho(u,t) + \rho(t,v) = \alpha + \beta,$$
$$\rho(v,z) = \rho(v,t) + \rho(t,z) = \beta + \gamma$$

and

$$\rho(z,u) = \rho(z,t) + \rho(t,u) = \gamma + \alpha.$$

From Definition 2.1.8 we can deduce that,

$$\text{if } \alpha + \beta = \rho(u,v), \text{ then } C_\rho(u,\alpha) \cap C_\rho(v,\beta) \neq \emptyset;$$
$$\text{if } \beta + \gamma = \rho(v,z) \text{ then } C_\rho(v,\beta) \cap C_\rho(z,\gamma) \neq \emptyset$$

and

$$\text{if } \gamma + \alpha = \rho(z,u), \text{ then} C_\rho(z,\gamma) \cap C_\rho(u,\alpha) \neq \emptyset.$$

Since $C_\rho(u,\alpha) \cap C_\rho(v,\beta) \neq \emptyset$ and $C_\rho(v,\beta) \cap C_\rho(z,\gamma) \neq \emptyset$, then $C_\rho(u,\alpha) \cap C_\rho(v,\beta) \cap C_\rho(z,\gamma) \neq \emptyset$.

But $C_\rho(u,\alpha) \cap C_\rho(v,\beta) \cap C_\rho(z,\gamma) = I(u,v) \cap I(v,z) \cap I(z,u)$. Therefore, $I(u,v) \cap I(v,z) \cap I(z,u) \neq \emptyset$. □

2.2. Gated sets in metric spaces

In this section, we present the concept of gated subsets of a metric space which has been studied by many authors for different purposes (see [6], [10], [24] etc.).

Definition 2.2.1. *([6, p.112]) Let (M, ρ) be a metric space and A be a subset of M. Then A is gated if for all $u \in M$, if there is some $\overline{u} \in A$ such that for all $a \in A$, we have the following:*

$$\rho(u, a) = \rho(u, \overline{u}) + \rho(\overline{u}, a).$$

If such a $\overline{u}$ exists, it is called the gate *of u in A.*

It is easy to see that, if $\overline{u}$ is the gate of u in A and $\overline{\overline{u}}$ is another gate, then

$$\rho(u, \overline{\overline{u}}) = \rho(u, \overline{u}) + \rho(\overline{u}, \overline{\overline{u}}) = \rho(u, \overline{\overline{u}}) + \rho(\overline{\overline{u}}, \overline{u}) + \rho(\overline{u}, \overline{\overline{u}}),$$

that is, $\rho(\overline{u}, \overline{\overline{u}}) = 0$. Hence the gate $\overline{u}$ is unique.

Remark 2.2.1. *Under different motivations, gated sets have been defined by other authors as prefibers see ([24, Definition 2.1]), Chebychev sets ([9, 11]) and J-convex sets in ([25]).*

Example 2.2.1. *Consider the three point set $X = \{1, 2, 3\}$. If we equip X with metric ρ defined by the distance matrix*

$$M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

that is, $\rho_{i,j} = \rho(i, j)$ whenever $i, j \in X$. Then (X, ρ) is a metric space. It is readily checked that $A = \{1, 2\}$ is a gated subset of X, where the point $\bar{i} = 2$ is the gate of A in (X, ρ) with respect to $i = 2$ and $i = 3$ while $\bar{i} = 1$ is the gate of A in (X, ρ) with respect to $i = 1$.

We now recall the definition of a projection of a gated subset of metric spaces.

Definition 2.2.2. *Let (M, ρ) be a metric space and A be a gated subset of M. We define the projection map $pr_A \colon (M, \rho) \to (A, \rho)$ as follows:*
For any point $u \in M$, there is some unique gate $\overline{u} \in A$ of A with $pr_A(u) = \overline{u}$ such that

$$\rho(u, a) = \rho(u, pr_A(u)) + \rho(pr_A(u), a),$$

for all $a \in A$.

Next, we recall the definition of a strongly convex subset.

Definition 2.2.3. *Let (M, ρ) be a metric space and A a subset of M. Then A is* strongly convex *if for each pair $u, v \in A$, the metric interval $I(u, v) = \{z \in M : \rho(u, z) + \rho(z, v) = \rho(u, v)\}$ is contained in A.*

Proposition 2.2.1. *([6, Proposition 1]) Let (M, ρ) be a metric space and A be a subset of M. If A is gated in M, then A is also strongly convex in M.*

Proof. Let us assume that e and f are points in A and b is a point in M such that $\rho(e, f) = \rho(e, b) + \rho(b, f)$. If $\bar{b} := pr_A(b)$, that is, $\bar{b}$ is the projection from $b \in M$ onto A. Then

$$\rho(e, f) = \rho(e, \bar{b}) + \rho(\bar{b}, b) + \rho(b, \bar{b}) + \rho(\bar{b}, f),$$

but by the symmetry property of a metric space $\rho(\bar{b}, b) = \rho(b, \bar{b})$ and $\rho(e, f) = \rho(e, \bar{b}) + \rho(\bar{b}, f)$, hence $\rho(e, f) \geq \rho(e, f) + 2\rho(b, \bar{b})$. Thus, $\rho(b, \bar{b}) = 0$, implying that, $b = \bar{b} \in A$. Therefore A is strongly convex in M. ☐

Proposition 2.2.2. *([6, Proposition 2]) Let (M, ρ) be a metric space. If A is a gated subset of M and B is a gated subset of A, then B is a gated subset of M. Also $pr_{B^\rho} = pr^A_{B^\rho} \circ pr_{A^\rho}$, where $pr^A_{B^\rho}$ is a projection map from A onto B.*

Proof. Suppose $c \in M$ and $b \in B$. Then

$$\rho(c, b) = \rho(c, pr_A(c)) + \rho(pr_A(c), b) = \rho(c, pr_A(c)) + \rho(pr_A(c), pr^A_B(pr_A(c))) + \rho(pr^A_B(pr_A(c)), b)$$

$$= \rho(c, pr^A_B(pr_A(c))) + \rho(pr^A_B(pr_A(c)), b).$$

☐

The following result states that the projection map into a gated subset is a nonexpanding map.

Lemma 2.2.1. *([6, Lemma 1]) Let (M, ρ) be a metric space and let A be any gated subset of M. Then the projection map onto A is nonexpanding, that is, for all $c, d \in M$, the following holds:*

$$\rho(pr_A(c), pr_A(d)) + \mid \rho(c, pr_A(c)) - \rho(d, pr_A(d)) \mid \leq \rho(c, d).$$

Proof. We assume that $\bar{c} := pr_A(c)$, $\bar{d} := pr_A(d)$ and thus we have

$$\rho(c, \bar{c}) + \rho(\bar{c}, \bar{d}) = \rho(c, \bar{d}) \leq \rho(c, d) + \rho(d, \bar{d}),$$

that is,

$$\rho(\overline{c},\overline{d}) + (\rho(c,\overline{c}) - \rho(d,\overline{d})) \leq \rho(c,d),$$

but $\overline{c} := pr_A(c)$ and $\overline{d} := pr_A(d)$, thus

$$\rho(pr_A(c), pr_A(d)) + \mid \rho(c, pr_A(c)) - \rho(d, pr_A(d)) \mid \leq \rho(c,d).$$

□

Lemma 2.2.2. *([6, Lemma 2]) Let (M,ρ) be a metric space. If A is a gated subset of M, then for any $v \in I(u, pr_A(u))$, whenever $u \in M$, we have*

$$pr_A(v) = pr_A(u).$$

Proof. Assume $\overline{u} := pr_A(u)$ and $\overline{v} := pr_A(v)$. By successively using the assumption that $\overline{u} := pr_A(u)$, the assumption on v and the assumption that $\overline{v} := pr_A(v)$, we obtain the following:

$$\rho(u,\overline{v}) = \rho(u,\overline{u}) + \rho(\overline{u},v),$$

but $\rho(u,\overline{u}) = \rho(u,v) + \rho(v,\overline{u})$, hence

$$\rho(u,\overline{v}) = \rho(u,v) + \rho(v,\overline{u}) + \rho(\overline{u},\overline{v}) \quad (2.2)$$

$$= \rho(u,v) + \rho(v,\overline{v}) + \rho(\overline{v},\overline{u}) + \rho(\overline{u},\overline{v}). \quad (2.3)$$

$$= \rho(u,v) + \rho(v,\overline{v}) + 2\rho(\overline{u},\overline{v}). \quad (2.4)$$

Hence $2\rho(\overline{u},\overline{v}) = 0$ and thus $\overline{u} = \overline{v}$ and hence, $pr_A(v) = pr_A(u)$.

□

Theorem 2.2.1. *([6, p.116]) Let (M,ρ) be a metric space. Let A_1 and A_2 be gated subsets of M. If we set $B_1 := p_1A_2$ and $B_2 := p_2A_1$, where $p_i := pr_{A_i}$ denotes the projection A_i, then*

(i) p_1 and p_2 induce isometries, inverses to each other, between B_1 and B_2.

(ii) For $u_1 \in A_1$ and $u_2 \in A_2$, the following are equivalent:

(a) $\rho(u_1,u_2) = \rho(A_1,A_2)$

(b) $u_1 = p_1u_2$, $u_2 = p_2u_1$.

(iii) B_1 and B_2 are gated and $pr_{A_1} = p_1 \circ p_2$, $pr_{A_2} = p_2 \circ p_1$.

Proof.

(i) Since p_1 and p_2 are non-expanding maps, this shows that $p_1 \circ p_2$ induces an identity on B_1 and similarly $p_2 \circ p_1$ induces an identity on B_2.
We consider $b_1 = p_1 a$, where $a \in A_2$ and assume $b_2 = p_2 b_2$. By Lemma 2.2.2, we obtain that $b_1 = p_1 b_2$ since $b_2 \in I(a, b_1)$. Since $b_2 = p_2 b_1$, this holds trivially for all $a \in A_2$.

(ii) Since (a) above implies that $\rho(u_1, u_2) = \rho(A_1, u_2) = \rho(u_1, A_2)$, it is trivial that $(a) \Rightarrow (b)$. In order to show that $(b) \Rightarrow (a)$, we show that $\rho(u_1, u_2) = \rho(v_1, v_2)$, whenever $v_1 \in A_1$, $v_2 \in A_2$ such that $v_1 = p_1 v_2$, $v_2 = p_2 v_1$.
Applying Lemma 2.2.1 symmetrically we obtain that:

$$\rho(u_1, v_1) + \mid \rho(v_1, v_2) - \rho(u_1, u_2) \mid \leq \rho(u_2, v_2)$$

$$\rho(u_2, v_2) + \mid \rho(v_1, v_2) - \rho(u_1, u_2) \mid \leq \rho(u_1, v_1).$$

Thus for $u_1 = p_1 u_2$, $u_2 = p_2 u_1$ implies that $\rho(u_1, u_2) = \rho(A_1, A_2)$.

(iii) By Proposition 2.2.2, we can show that B_1 is gated in A_1, with projection $p_1 \circ p_2$. Let a be a point in A_1, b a point in B_1 and we assume that $\overline{\overline{a}} := p_1 p_2 a$. Thus we show that

$$\rho(a, b) = \rho(a, \overline{\overline{a}}) + \rho(\overline{\overline{a}}, b)$$

which can be written as

$$\rho(a, b) = \rho(a, \overline{a}) + \rho(\overline{a}, \overline{b}), \ \overline{a} := p_2 a = p_2 \overline{\overline{a}}, \ \overline{b} = p_2 b, \tag{2.5}$$

by part i).
By part ii), if we add $\rho(b, \overline{b})$ to the left hand side of Equation 2.5, we obtain that

$$\rho(\overline{b}, b) + \rho(b, a) = \rho(\overline{b}, a),$$

using the assumption that $b = p_1 \overline{b}$. Similarly, by part 2.2 above, if we add $\rho(\overline{a}, \overline{\overline{a}})$ to the right hand side of Equation 2.5, we obtain that

$$(\rho(a, \overline{\overline{a}}) + \rho(\overline{\overline{a}}, \overline{a}) + \rho(\overline{a}, \overline{b}) = \rho(a, \overline{a}) + \rho(\overline{a}, \overline{b}) = \rho(a, \overline{b}),$$

By the symmetry property of a metric,

$$\rho(\overline{b}, a) = \rho(a, \overline{b}).$$

Hence

$$\rho(a, b) = \rho(a, \overline{\overline{a}}) + \rho(\overline{\overline{a}}, b).$$

□

We now recall a characterization of gated subsets of a 3-Hyperconvex metric space.

Lemma 2.2.3. *Let (M, ρ) be a 3-hyperconvex metric space and A a subset of M. Then A is strongly convex and closed if and only if it is gated.*

Proof. Let us assume that A is strongly convex and closed. Let u be a fixed point in (M, ρ) and (u_n) a sequence of points in A with $\rho(u, u_n) \leq \rho(u, A) + \frac{1}{n}$. For $n, k \in \mathbb{N}$ take $m_{n,k} \in I(u, u_n) \cap I(u, u_k) \cap I(u_n, u_k)$.
By the definition of strong convexity,

$$\rho(u, m_{n,k}) + \rho(m_{n,k}, u_n) = \rho(u, u_n).$$

$$\rho(u, m_{n,k}) + \rho(m_{n,k}, u_k) = \rho(u, u_k).$$

$$\rho(u_n, m_{n,k}) + \rho(m_{n,k}, u_k) = \rho(u_n, u_k).$$

$$I(u, u_n) \cap I(u, u_k) \cap I(u_n, u_k) = C_\rho(u, \alpha) \cap C_\rho(u_n, \alpha_n) \cap C_\rho(u_k, \alpha_k) \neq \emptyset \ \ for\ all\ n, k \in \mathbb{N},$$

but $m_{n.k} \in I(u, u_n) \cap I(u, u_k) \cap I(u_n, u_k) \ \ for\ all\ n, k \in \mathbb{N}$ from above, therefore

$$m_{n.k} \in C_\rho(u, \alpha) \cap C_\rho(u_n, \alpha_n) \cap C_\rho(u_k, \alpha_k) \neq \emptyset \ and\ is\ contained\ in\ A \ \ for\ all\ n, k \in \mathbb{N}.$$

Hence $m_{n,k} \in A$.
Since A is gated we have the following:

$$\rho(u, u_n) = \rho(u, m_{n,k}) + \rho(m_{n,k}, u_n)\ for\ all\ n, k \in \mathbb{N}.$$

By the symmetry property of a metric $\rho(m_{n,k}, u_n) = \rho(u_n, m_{n,k})\ for\ all\ n, k \in \mathbb{N}$.
Hence

$$\rho(u, u_n) = \rho(u, m_{n,k}) + \rho(u_n, m_{n,k})\ for\ all\ n, k \in \mathbb{N}$$

$$\rho(u_n, m_{n,k}) = \rho(u, u_n) - \rho(u, m_{n,k})\ for\ all\ n, k \in \mathbb{N}.$$

Since $\rho(u, u_n) \leq \rho(u, A) + \frac{1}{n}$, therefore we have

$$\rho(u_n, m_{n,k}) = \rho(u, u_n) - \rho(u, m_{n,k}) \leq \rho(u, A) + \frac{1}{n} - \rho(u, m_{n,k}),$$

but $m_{n,k} \in A$.
Thus

$$\rho(u_n, m_{n,k}) \leq \rho(u, A) + \frac{1}{n} - \rho(u, A)$$

and hence

$$\rho(u_n, m_{n,k}) \leq \frac{1}{n}.$$

Similarly for u_k,

$$\rho(u_k, m_{n,k}) \leq \frac{1}{k}.$$

Hence

$$\rho(u, u_n) = \rho(u, m_{n,k}) + \rho(u_n, m_{n,k}) \leq \frac{1}{n} + \frac{1}{k}.$$

We can thus deduce that (u_n) is a cauchy sequence and since A is closed it converges to some $\overline{u}$ in A. Furthermore, $\rho(u, \overline{u}) = \rho(u, A)$.
We claim that $\overline{u}$ is a gate for u in A. Let $v \in A$. By Lemma 2.1.2, there is some $z \in M$ such that $z \in I(u, \overline{u}) \cap I(\overline{u}, v) \cap I(v, u)$.
By convexity

$$I(u, \overline{u}) = \{z \in M : \rho(u, z) + \rho(z, \overline{u}) = \rho(u, \overline{u})\} \subseteq A;$$

$$I(\overline{u}, v) = \{z \in M : \rho(\overline{u}, z) + \rho(z, v) = \rho(\overline{u}, v)\} \subseteq A$$

and

$$I(v, u) = \{z \in M : \rho(v, z) + \rho(z, u) = \rho(v, u)\} \subseteq A.$$

Therefore, $z \in I(u, \overline{u}) \cap I(\overline{u}, v) \cap I(v, u) \subseteq A$. Thus $z \in A$ and $\rho(u, z) \geq \rho(u, A) = \rho(u, \overline{u})$. Since $z \in I(u, \overline{u})$, this means that $z = \overline{u}$ and since $I(u, \overline{u}) \cap I(\overline{u}, v) \cap I(v, u) \neq \emptyset$ we have $\overline{u} \in I(v, u)$. This concludes that A is gated.
Conversely, let us assume A is gated. Hence for all $u, v \in A$ and $z \in I(u, v)$ we have $z = \overline{z}$ and hence $I(u, v) \subset A$. For all $u \in W$, we have $\rho(u, A) = \rho(u, \overline{u})$ and hence $\overline{u} \subset C_\rho(u, \rho(u, A)) \cap A$, that is, A is proximal and hence closed.

□

Lemma 2.2.4. *([18, Lemma 3.3]) Let (M, ρ) be a m-hyperconvex metric space and A a gated subset of (M, ρ). Then (A, ρ) is also m-hyperconvex.*

Proof. We assume (M, ρ) is a m-hyperconvex metric space and let A be a gated subset of (M, ρ). Let $\{C_\rho(u_i, r_i)\}_{i \in I}$ be a family of closed balls in M with $\rho(u_i, u_j) \leq r_i + r_j$, for all $i, j \in I$ and centers $(u_i)_{i \in I}$ in A.
Since (M, ρ) is m-hyperconvex, then by convexity there exists $z \in M$ such that $z \in \bigcap C_\rho(u_i, r_i) \neq \emptyset$. Since A is a gated subset of (M, ρ), there exists $\overline{z} \in A$ such $\overline{z}$ is a gate of z in A. By convexity $\rho(u_i, z) \leq r_i$ and by Definition 2.2.1, $\rho(z, u_i) = \rho(z, \overline{z}) + \rho(\overline{z}, u_i)$.
Hence $\rho(u_i, \overline{z}) = \rho(u_i, z) - \rho(z, \overline{z}) \leq r_i$ and therefore, $\rho(u_i, \overline{z}) \leq r_i$. Thus $\overline{z} \in \bigcap C_\rho(u_i, r_i) \cap A$.

□

2.3. Gluing along gated subsets

In this section, we recall the concept of gluing a family of metric spaces. We point out that this concept is well-known, but for an overview, we refer the reader to [5].

Definition 2.3.1. *([5, Definition 5.23]) Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a family of metric spaces and let U_α denote closed subspaces in M_α whenever $\alpha \in \Gamma$. Suppose that for each $\alpha \in \Gamma$, we have an isometry $\varphi_\alpha : U \to U_\alpha$. Let M denote the quotient of the disjoint union $\bigsqcup_U M_\alpha$ by the equivalence relation generated by $[\varphi_\alpha(u) \sim \varphi_{\alpha'}(u)\ \forall u \in U, \alpha, \alpha' \in \Gamma]$. We identify each M_α with it's image in M and write*

$$M = \bigsqcup_U M_\alpha.$$

M is called the gluing *of the M_α along U.*

We deduce that M admits a natural metric. Thus from ([5, Lemma 5.24]), for $v \in M_\alpha$ and $w \in M_{\alpha'}$ we have,

$$\rho(v,w) = \begin{cases} \rho_\alpha(v,w), & \text{if } \alpha = \alpha', \\ \inf_{u\in U}\{\rho_\alpha(v,\varphi_\alpha(u)) + \rho_{\alpha'}(w,\varphi_{\alpha'}(u))\}, & \text{if } \alpha \neq \alpha'. \end{cases}$$

Example 2.3.1. *Suppose (M_1, ρ_1) and (M_2, ρ_2) are metric spaces, and $f_i : (Y, d) \to (M_i, \rho_i)$ is an isometry. Then the* ***union*** *of X_1 and X_2 along Y, written $X_1 \cup_Y X_2$ is*

$$X_1 \sqcup X_2/\sim,$$

where, $x_1 \sim x_2$ if there exists $y \in Y$ with $f_i(y) = x_i$.

Lemma 2.3.1. *([18, Lemma 3.4]) Let $(M_\alpha, \rho_\alpha)_{\alpha\in A}$ be a collection of m-hyperconvex metric spaces and (M, ρ) be a metric obtained by gluing $(M_\alpha, \rho_\alpha)_{\alpha\in A}$ along some set B. If B is gated in M_α for all α. Then for $u \in M_\alpha, v \in M_{\alpha'}$ we have:*

$$\rho(u,v) = \rho(u,\overline{u}) + \rho(\overline{u},\overline{v}) + \rho(\overline{v},v)$$

with $\alpha \neq \alpha'$.

Proof. Since B is gated, there exists $\overline{u} \in B$ the gate of u in B. Similarly, since B is gated, there exists $\overline{v} \in B$ the gate of v in B. Hence for $b \in B$,

$$\rho(u,b) = \rho(u,\overline{u}) + \rho(\overline{u},b) \tag{2.6}$$

and

$$\rho(v,b) = \rho(v,\overline{v}) + \rho(\overline{v},b) \tag{2.7}$$

Applying the symmetry property to Equation 2.7, we obtain :

$$\rho(b,v) = \rho(b,\overline{v}) + \rho(\overline{v},v) \tag{2.8}$$

Adding Equation 2.6 and Equation 2.8, we obtain:

$$\rho(u,b) + \rho(b,v) = \rho(u,\overline{u}) + \rho(\overline{u},b) + \rho(b,\overline{v}) + \rho(\overline{v},v)$$

By the triangular property of a metric: $\rho(\overline{u},b) + \rho(b,\overline{u}) \geq \rho(\overline{u},\overline{v})$. Therefore

$$\rho(u,v) \leq \rho(u,b)+\rho(b,v) = \rho(u,\overline{u})+\rho(\overline{u},b)+\rho(b,\overline{v})+\rho(\overline{v},v) \geq \rho(u,\overline{u})+\rho(\overline{u},\overline{v})+\rho(\overline{v},v) \geq \rho(u,v).$$

Thus

$$\rho(u,v) = \rho(u,\overline{u}) + \rho(\overline{u},\overline{v}) + \rho(\overline{v},v).$$

□

Proposition 2.3.1. *([18, Proposition 3.5]) Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a collection of m-hyperconvex metric spaces. If (M,ρ) is a metric obtained by gluing $(M_\alpha,\rho_\alpha)_{\alpha\in\Gamma}$ along some set B, where B is closed and strongly convex in all M_α, then (M,ρ) is also m-hyperconvex.*

Proof. Let(M,ρ) be a metric space and $\{C_\rho(u_\alpha,r_\alpha)\}_{\alpha\in\Gamma}$ a collection of closed balls in M with $\rho(u_\alpha,u_\beta) \leq r_\alpha + r_\beta \;\; for \; all \; \alpha,\beta \in \Gamma$. In order to show that (M,ρ) is m-hyperconvex, we show that the $\bigcap C_\rho(u_\alpha,r_\alpha) \neq \emptyset$. Let us assume that B is strongly convex and closed. From Lemma 2.2.3, if B is strongly convex and closed, then it is gated, that is, for any $u \in M$, there is $\overline{u} \in B$ such that for all $a \in B$ $\rho(u,a) = \rho(u,\overline{u}) + \rho(\overline{u},a)$ and hence $\overline{u}$ is a gate of u in B. Hence for $u_\alpha \in M$ and $\overline{u}_\alpha, a_\alpha \in B$ $\rho(u_\alpha,a_\alpha) = \rho(u_\alpha,\overline{u_\alpha}) + \rho(\overline{u_\alpha},a_\alpha)$. If $\rho(u_\alpha,\overline{u_\alpha}) \leq r_\alpha$, we define $\overline{r_\alpha} = r_\alpha - \rho(u_\alpha,\overline{u_\alpha})$. Since $\overline{u_\alpha} \subset B$, then $C_\rho(\overline{u_\alpha},\overline{r_\alpha}) \subset B$, but $B \subset M$. Therefore $C_\rho(u_\alpha,r_\alpha) \subset M$ and hence $u_\alpha \subset M$. Then $C_\rho(u_\alpha,r_\alpha) \subset M$ and thus $C_\rho(\overline{u_\alpha},\overline{r_\alpha}) \subset C_\rho(u_\alpha,r_\alpha)$. We look at three cases

Case 1: Let $\rho(u_\alpha,\overline{u}_\alpha) \leq r_\alpha$ and $C_\rho(\overline{u}_\alpha,\overline{r}_\alpha) \cap C_\rho(u_\alpha,r_\alpha) \neq \emptyset$
But both $C_\rho(\overline{u}_\alpha,\overline{r}_\alpha)$ *and* $C_\rho(\overline{u}_\beta,\overline{r}_\beta)$ are contained in B. Therefore B must be hyperconvex. If B is hyperconvex and a subset of M, then

$$\bigcap_{\alpha\in\Gamma} C_\rho(\overline{u}_\alpha,\overline{r}_\alpha) \neq \emptyset.$$

Hence M is also hyperconvex and therefore

$$\bigcap_{\alpha\in\Gamma} C_\rho(u_\alpha,r_\alpha) \neq \emptyset.$$

Case 2: We assume $\rho(u_\alpha, \overline{u}_\alpha) \leq r_\alpha$ for all $\alpha \in \Gamma$ and

$$C_\rho(\overline{u}_\alpha, \overline{r}_\alpha) \cap C_\rho(u_\alpha, r_\alpha) = \emptyset \tag{2.9}$$

This implies that $\rho(u_\alpha, u_\beta) \leq r_\alpha + r_\beta$.
By Lemma 2.3.1:

$$\rho(u_\alpha, u_\beta) = \rho(u_\alpha, \overline{u}_\alpha) + \rho(\overline{u}_\alpha, \overline{u}_\beta) + \rho(\overline{u}_\beta, u_\beta).$$

Hence

$$\rho(u_\alpha, u_\beta) = \rho(u_\alpha, \overline{u}_\alpha) + \rho(\overline{u}_\alpha, \overline{u}_\beta) + \rho(\overline{u}_\beta, u_\beta) > r_\alpha + r_\beta \ \ for\ all\ \alpha, \beta \in \Gamma.$$

By Lemma 2.3.1, we also have: $C_\rho(\overline{u}_\alpha, \overline{r}_\alpha) \cap C_\rho(\overline{u}_\beta, \overline{r}_\beta) \neq \emptyset$ if $u \in M_\alpha,\ v \in M'_\alpha\ with\ \alpha \neq \alpha'$. Therefore, for any u_α, u_β such that $C_\rho(\overline{u}_\alpha, \overline{r}_\alpha) \cap C_\rho(\overline{u}_\beta, \overline{r}_\beta) = \emptyset$, u_α, u_β must be in some M_α. We now claim that there is only one M_{α_0} containing such pairs. We assume that $u_1, u_2 \in M_{\alpha_0}$ such that $C_\rho(\overline{u}_1, \overline{r}_1) \cap C_\rho(\overline{u}_2, \overline{r}_2) = \emptyset$ and $u_3, u_4 \in M_\alpha$ for some $\alpha \neq \alpha_0$.
We define

$$r := \frac{\rho(\overline{u}_1, \overline{u}_2) - r_1 - r_2}{2}.$$

Thus we have $C_\rho(\overline{u}_1, \overline{u}_1 + r) \cap C_\rho(\overline{u}_\beta, \overline{r}_\beta + r) = \emptyset$. Since M_α is hyperconvex, there exists $z \in C_\rho(\overline{u}_1, \overline{r}_1) \cap C_\rho(\overline{u}_2, \overline{r}_2) \cap C_\rho(\overline{u}_3, \overline{r}_3 \cap C_\rho(\overline{u}_4, \overline{r}_4)$. By strong convexity: $I(u_1, u_2) = \{z \in M : \rho(u_1, z) + \rho(z, u_2) = \rho(u_1, u_2)\} \subset B$. Hence $z \in I(u_1, u_2) \subset B$. Therefore, for $\alpha = 3{,}4$ $\rho(u_\alpha, z) = \rho(u_\alpha, \overline{u}_\alpha) + \rho(\overline{u}_\alpha, z) \leq r_\alpha$. Then $\rho(u_\alpha, z) - \rho(u_\alpha, \overline{u}_\alpha) = \rho(\overline{u}_\alpha, z) \leq r_\alpha - \rho(u_\alpha, \overline{u}_\alpha)$, but $r_\alpha - \rho(u_\alpha, \overline{u}_\alpha) = \overline{r}_1$, therefore, $\rho(u_\alpha, z) - \rho(u_\alpha, \overline{u}_\alpha) = \rho(\overline{u}_\alpha, z) = \overline{r}_\alpha$. Hence $z \in C_\rho(\overline{u}_\alpha, \overline{r}_\alpha)$ and thus, $z \in C_\rho(\overline{u}_3, \overline{r}_3) \cap C_\rho(\overline{u}_4, \overline{r}_4) \neq \emptyset$. Denote $\Gamma_0 = \{\alpha \in \Gamma : u_\alpha \in M_{\lambda_0}\}$. Implying that $\{C_\rho(u_\alpha, r_\alpha)\}_{\alpha \in \Gamma_0} \cup \{C_\rho(\overline{u}_\alpha, \overline{r}_\alpha)\}_{\alpha \in \Gamma \setminus \Gamma_0}$ is a family of pairwise intersecting balls in M_{λ_0} and hence has a non-empty intersection

$$\bigcap_{\alpha \in \Gamma_0} C_\rho(u_\alpha, r_\alpha) \cap \bigcap_{\alpha \in \Gamma \setminus \Gamma_0} C_\rho(\overline{u}_\alpha, \overline{r}_\alpha).$$

This implies that

$$\bigcap_{\alpha \in \Gamma} C_\rho(u_\alpha, r_\alpha).$$

Case3: We assume $\rho(u_\alpha, \overline{u}_\alpha) > r_\alpha\ for\ \alpha \in \Gamma$.
From Lemma 2.5.1 we can conclude that all u_α are contained in some M_{λ_0}. If we fix $u_{\alpha_0} \in M_{\lambda_0} \in M_{\lambda_0}$ with $d(u_{\alpha_0}, \overline{u}_{\alpha_0}) \leq r_{\alpha_0}$, then for $u_\alpha \notin M_{\lambda_0}$ we have $\overline{u}_0 \in C_\rho(\overline{u}_\alpha, \overline{r}_\alpha)$ and hence $C_\rho(\overline{u}_\alpha, \overline{r}_\alpha) \cap C_\rho(\overline{u}_\beta, \overline{r}_\beta)\ for\ some\ u_\alpha, u_\beta \notin M_{\lambda_0}$. Thus

$$\bigcap_{\alpha \in \Gamma} C_\rho(u_\alpha, r_\alpha) \neq \emptyset.$$

Since from all three cases,

$$\bigcap_{\alpha\in\Gamma} C_\rho(u_\alpha, r_\alpha) \neq \emptyset.$$

Thus we can conclude that (M,ρ) is also m-hyperconvex.

□

Corollary 2.3.1. *Proposition 2.3.1 still holds for m-hyperconvexity.*

2.4. Gluing along externally hyperconvex subsets

In this section, we discuss the theory of gluing a family of hyperconvex metric spaces along an externally hyperconvex subset as studied by Miesch in ([18]).

We begin by recalling the following lemma.

Lemma 2.4.1. *([18, Lemma 4.1]) Let (M,ρ) be a metric space. If A is an admissible subset of M and E is an externally hyperconvex subset of M, then $A\cap E\in\mathcal{E}(M)$. If (M,ρ) is a hyperconvex metric space, then $\mathcal{A}(M)\subseteq\mathcal{E}(M)$.*

Proof. We assume that A is an admissible subset and hence can written as a non-empty intersection of closed balls $C_\rho(u_i,r_i)_{i\in I}$ such that

$$A=\bigcap_{i\in I} C_\rho(u_i, r_i) \neq \emptyset.$$

Let $\{C_\rho(u_j,r_j)\}_{j\in I}$ be a family of closed balls such that $\rho(u_j,u_k)\leq r_j+r_k$ and $\rho(u_j,A\cap E)\leq r_j$, then $\rho(u_j,A\cap E)\leq r_j$ implies that $A\cap E\in C_\rho(u_j,r_j)$.
But $A=\bigcap_{i\in I} C_\rho(u_i,r_i)\neq\emptyset$,
therefore, $\bigcap_{i\in I} C_\rho(u_i,r_i)\cap E\in\{C_\rho(u_j,r_j)\}_{j\in I}$
and hence $\bigcap_{i\in I} C_\rho(u_i,r_i)\cap\{C_\rho(u_j,r_j)\}_{j\in I}\neq\emptyset$.
Thus $\rho(u_i,u_j)\leq r_i+r_j$ *for all* $i,j\in I$ and $\rho(u_i,A\cap E)\leq r_i$ *for all* $i\in I$.

Since $E\in\mathcal{E}(M)$, we have

$$A\cap E\cap\bigcap_{j\in I} C_\rho(u_j,r_j)\neq\emptyset.$$

But

$$A=\bigcap_{i\in I} C_\rho(u_i,r_i).$$

Thus

$$A \cap E \cap \{C_\rho(u_j, r_j)\}_{j \in I} = \bigcap_{i \in I} C_\rho(u_i, r_i) \cap E \cap \bigcap_{j \in I} C_\rho(u_j, r_j) \neq \emptyset.$$

If (M, ρ) is a hyperconvex metric space, then $M \in \mathcal{E}(M)$. But $\mathcal{A}(M) \subset M$, hence $\mathcal{A}(M) \subseteq \mathcal{E}(M)$. □

We now define the concept of closed r-neighborhood of a subset.

Definition 2.4.1. *Let (M, ρ) be a hyperconvex metric space and $C_\rho(u, r) = \{a \in M : \rho(a, u) \leq r\}$ denote a closed ball with center u and radius r such that $u \in M$ and r a nonnegative real number. For any subset A in M the closed r-neighborhood of A is defined as follows:*

$$C_\rho(A, r) = \{p \in M : \rho(p, A) := inf_{q \in A} \rho(p, q) \leq r\}.$$

Lemma 2.4.2. *([18, Lemma 4.2]) Let (M, ρ) be a metric space. If $A \in \mathcal{E}(M)$. Then $C_\rho(A, r) \in \mathcal{E}(M)$.*

Proof. Let $\{C_\rho(u_i, r_i)\}_{i \in I}$ be a family of closed balls with $\rho(u_i, u_j) \leq r_i + r_j$ and $\rho(u_i, C_\rho(A, r)) \leq r_i \; for \; all \; i, j \in I$.
By Definition 2.4.1 above $C_\rho(A, r) = \{p \in M : \rho(u, A) := inf_{v \in A} \rho(u, v) \leq r\} \; for \; all \; i \in I$.
Therefore $\rho(u, A) \leq r_i$ and hence $\rho(u_i, A) \leq r_i + r \quad for \; all \; i \in I$. Since $A \in \mathcal{E}(M)$, there is some $v \in \bigcap C_\rho(u_i, r_i) \cap A \quad for \; all \; i, j \in I$. Hence v is contained in A and therefore $\rho(u_i, v) \leq r_i + r \quad for \; all \; i, j \in I$. Thus implying that $\bigcap C_\rho(u_i, r_i) \cap C_\rho(v, r) \neq \emptyset \quad for \; all \; i, j \in I$.
Since $C_\rho(v, r) \subset C_\rho(v, A)$ *and* (M, ρ) *is hyperconvex*, then

$$\emptyset \neq \bigcap C_\rho(u_i, r_i) \cap C_\rho(v, r) \subset \bigcap C_\rho(u_i, r_i) \cap C_\rho(v, A) \quad for \; all \; i, j \in I.$$

Therefore, $C_\rho(v, r) \in \mathcal{E}(M)$.

□

Lemma 2.4.3. *([18, Lemma 4.3]) Let (M, ρ) be a hyperconvex metric space. If $A, A' \in \mathcal{E}(M)$ with $v \in A \cap A' \neq \emptyset$ and $u \in M$ with $\rho(u, A) \leq r$, $\rho(u, A') \leq r$. Denote $\rho := \rho(u, v)$ and $s := \rho - r$.*
Then $A \cap A' \cap C_\rho(u, r) \cap C_\rho(v, s) \neq \emptyset$, $s \geq 0$ and hence in any case of s

$$A \cap A' \cap C_\rho(u, r) \neq \emptyset.$$

Proof. Case 1: $s \leq 0$ implies $\rho - r \geq 0 \Rightarrow \rho \leq r \Rightarrow \rho(u, v) \leq r$, hence $v \in C_\rho(u, r)$. Since $v \in A \cap A'$, therefore $v \in A \cap A' \cap C_\rho(u, r)$.

Case 2: For $s > 0$ we have $0 < l \leq s$ such that for $a \in A$ and $a' \in A'$ such that $\rho(a, a' \leq)$ and $a, a' \in C_\rho(u, r) \cap C_\rho(v, s)$. Since $s \geq l > 0$ then, $C_\rho(v, l) \subseteq C_\rho(v, s)$, $l \leq s := \rho - r \Rightarrow l \leq \rho - r \Rightarrow r \leq \rho - l$. Hence $C_\rho(u, r) \subseteq C_\rho(u, \rho - l)$. Since $a \in C_\rho(v, l) \subseteq C_\rho(v, s) \cap C_\rho(u, r) \subseteq C_\rho(u, \rho - l)$ and $a \in A$, then $a \in C_\rho(v, l) \cap C_\rho(u, \rho - l) \cap A$. From $\rho(a, a') \leq l$, $a' \in C_\rho(a, l)$ and also from above, $a \in C_\rho(v, l) \cap C_\rho(u, \rho - l)$, therefore $a' \in C_\rho(v, l) \cap C_\rho(u, \rho - l)$. But $a' \in A'$, therefore $a' \in C_\rho(v, l) \cap C_\rho(u, \rho - l) \cap A'$

Choosing

$$a_1 \in C_\rho(v, l) \cap C_\rho(u, \rho - l) \cap A$$

and

$$a_1' \in C_\rho(v, l) \cap C_\rho(u, \rho - l) \cap C_\rho(a, l) \cap A'.$$

We can deduce that

$$a_n \in C_\rho(v, nl) \cap C_\rho(u, \rho - nl) \cap C_\rho(a_{n-1}', l) \cap A$$

and

$$a_n' \in C_\rho(v, nl) \cap C_\rho(u, \rho - nl) \cap C_\rho(a_n, l) \cap A',$$

upon the condition that $n \leq [\frac{s}{l}] =: n_0$.

Hence,

$$a \in C_\rho(v, s) \cap C_\rho(u, r) \cap C_\rho(a_{n_0}', l) \cap A$$

and

$$a' \in C_\rho(v, s) \cap C_\rho(u, r) \cap C_\rho(a, l) \cap A'.$$

We now construct converging sequences $(a_n) \subset A$ and $(a_n') \subset A'$ which are said to be recursive, such that $a_n, a_n' \in C_\rho(u, r) \cap C_\rho(v, s)$

with

$$\rho(a_n, a_n') \leq \frac{1}{2^{n+1}}, \ \rho(a_{n-1}, a_n) \leq \frac{1}{2^n} \text{ and } \rho(a_{n-1}', a_n') \leq \frac{1}{2^n}.$$

For $a_0, a_0' \in C_\rho(u, r) \cap C_\rho(v, s)$, we will have $\rho(a_0, a_0') \leq \frac{1}{2}$ *based on the claim above*. Given $a_{n-1}, a_{n-1}' \in C_\rho(u, r) \cap C_\rho(v, s)$ with $\rho(a_{n-1}', a_n) \leq \frac{1}{2^n}$, there is some $u_n \in C_\rho(a_{n-1}, \frac{1}{2^{n+1}}) \cap C_\rho(u, r - \frac{1}{2^{n+1}})$. Applying the claim above to u_n *and* v, we have

$$a_n, a_n' \in C_\rho(v, s) \cap C_\rho(u_n, 2^{-(n+1)}) \subset C_\rho(v, s) \cap C_\rho(u, r)$$

with $\rho(a_n, a_n^{'}) \leq \frac{1}{2^{n+1}}$.
Furthermore

$$\rho(a_{n-1}, a_n) \leq \rho(a_{n-1}, u_n) + \rho(u_n, a_n) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

For $m \geq n$ we obtain the following:

$$\rho(a_n, a_m) \leq \sum_{k=n+1}^{m} \rho(a_{k-1}, a_k) \leq \sum_{k=n+1}^{\infty} \left(\frac{1}{2^k}\right) = \frac{1}{2^n}.$$

Similarly, for $a_n^{'}$

$$\rho(a_n^{'}, a_m) \leq \sum_{k=n+1}^{m} \rho(a_{k-1}, a_k) \leq \sum_{k=n+1}^{\infty} \left(\frac{1}{2^k}\right) = \frac{1}{2^n}.$$

Hence both (a_n) *and* $(a_n^{'})$ converge and since $\rho(a_n, a_n^{'}) \to 0$, then they converge to the same limit point $a \in C_\rho(v, s) \cap C_\rho(u, r) \cap A \cap A^{'}$.

□

Lemma 2.4.4. *([18, Lemma 4.4]) Let (M, ρ) be a hyperconvex metric space and let A_0, A_1, A_2 be externally hyperconvex subspaces intersecting pairwise. Then $A_0 \cap A_1 \cap A_2 \neq \emptyset$.*

Proof. Let u_0 be a point in $A_1 \cap A_2$ and $r := \rho(u_0, A_0)$. Since $r := \rho(u_0, A_0)$, then $A_0 \in C_\rho(u_0, r) \Rightarrow A_0 \cap C_\rho(u_0, r)$. But $u_0 \in A_1 \cap A_2 \Rightarrow u_0 \in A_1$, therefore by Lemma 2.4.3 there exists $v_0 \in A_0 \cap A_1 \cap C_\rho(u_0, r)$. Let $A_0^{'}$ be defined by $A_0 \cap C_\rho(v_0, r) \in \mathcal{E}(M)$. Then, again by Lemma 2.4.3 $A_0^{'} \cap A_2 = A_0 \cap A_2 \cap C_\rho(v_0, r)$. But $A_0 \in C_\rho(u_0, r)$, therefore $A_0^{'} \cap A_2 = A_0 \cap A_2 \cap C_\rho(u_0, r) \cap C_\rho(v_0, r)$. Hence there exists a point $z_0 \in A_0^{'} \cap A_2 = A_0 \cap A_2 \cap C_\rho(u_0, r) \cap C_\rho(v_0, r)$ and since $A_0 \in \mathcal{E}(M)$, that is the subspace is externally hyperconvex, then there exists $\tilde{u}_0 \in C_\rho(u_0, r) \cap C_\rho(v_0, \frac{r}{2}) \cap C_\rho(z_0, \frac{r}{2}) \cap A_0$. By Lemma 2.4.3, there is some point $u_1 \in A_1 \cap A_2 \cap C_\rho(\tilde{u}_0, \frac{r}{2}) \cap C_\rho(u_0, \frac{r}{2})$. Proceeding this way, we get some sequence $(u_n) \in A_1 \cap A_2$ with $\rho(u_n, A_0) \leq \frac{r}{2^n}$ and $\rho(u_{n-1}, u_n) \leq \frac{r}{2^n}$.
Hence for $m \geq n$ we obtain

$$\rho(u_n, u_m) \leq \sum_{k=n+1}^{m} \frac{r}{2^k} \leq \frac{r}{2^n}$$

Hence the sequence u_n converges to some point $u \in A_0 \cap A_1 \cap A_2$.

□

Lemma 2.4.5. *([18, Lemma 4.5]) Let (M, ρ) be a hyperconvex metric space. If we assume A_0 and A_1 are externally hyperconvex subspaces of M and $A_0 \cap A_1 \neq \emptyset$. Then the intersection of A_0 and A_1 is also externally hyperconvex.*

Proof. By Lemma 2.4.2, if $\{C_\rho(u_i, r_i)\}_{i\in I}$ is a family of closed balls with $\rho(u_i, u_j) \leq r_i + r_j$ and $\rho((u_i, A_1) \cap A_2) \leq r_i$, for all $i \in I$ then we define A as an intersection of closed balls $C_\rho(u_i, r_i)$, that is

$$A := \bigcap_{i\in I} C_\rho(u_i, r_i).$$

Since A_k is said to be externally hyperconvex, then

$$A \cap A_k = \bigcap_{i\in I} C_\rho(u_i, r_i) \cap A_k.$$

Since by lemma 2.4.1, admissible sets are said to be externally hyperconvex, then by Lemma 2.4.4

$$A_0 \cap A_1 \cap A \neq \emptyset.$$

□

We obtain the following proposition through the process of induction.

Proposition 2.4.1. *([18, Proposition 4.6]) Let (M, ρ) be a hyperconvex metric space. If $A_0, ..., A_n$ are externally hyperconvex subspaces with $A_i \cap A_j \neq \emptyset$, for all $i, j \in I$, then*

$$\bigcap_{k=0}^{n} A_k \neq \emptyset \text{ and } \bigcap_{k=0}^{n} A_k \in \mathcal{E}(M).$$

Theorem 2.4.1. *([7, Theorem 5.1]) If (M, ρ) is a metric space that is bounded and $(H_i)_{i\in I}$ is a decreasing collection of non-empty hyperconvex subsets of (M, ρ), where I is totally ordered. Then $\bigcap_{i\in I} H_i$ is non-empty and hyperconvex.*

Proof. See ([7, Theorem 5.1]). □

As a result of Theorem 2.4.1, we have the following theorem which has been proven by Espinola and Khamsi (see [7]).

Theorem 2.4.2. *([7, Theorem 5.4]) Let (M, ρ) be a bounded hyperconvex metric space and let $\{A_i\}_{i\in I}$ be a descending chain of non-empty externally hyperconvex subsets of (M, ρ). Then the intersection of the descending chain $\{A_i\}_{i\in I}$ is non-empty and externally hyperconvex in (M, ρ).*

Proof. By Theorem 2.4.1 there is $D = \bigcap_{i\in I} A_i \neq \emptyset$. We now show that D is externally hyperconvex. Let $\{u_i\}_{i\in I} \subset M$ and $\{r_i\}_{i\in I} \subset \mathbb{R}$ such that $\rho(u_i, u_j) \leq r_i + r_j$ and $\rho(u_i, D) \leq r_i$, whenever $i, j \in I$. Since (M, ρ) is hyperconvex , then $A = \bigcap_{i\in I} C_\rho(u_i, r_i) \neq \emptyset$ and since

$\rho(u_i, D) \leq r_i$ and $D = \bigcap_{i\in I} A_i \neq \emptyset$, then $\rho(u_i, \cap_{i\in I} A_i) \leq r_i$. Due to the fact that A_i is externally hyperconvex, then $A_i \cap C_\rho(u_i, r_i) \neq \emptyset$
but $A = \bigcap_{i\in I} C_\rho(u_i, r_i) \neq \emptyset$, therefore $A_i \cap A \neq \emptyset$.
By Lemma 2.4.1 $\{A_i \cap A \neq \emptyset\}_{i\in I}$ is a descending chain of non-empty hyperconvex subsets of M. Therefore again by Theorem 2.4.1 $\bigcap(A_i \cap A) = \bigcap A_i \cap \bigcap A = \bigcap A_i \cap A$
but $D = \bigcap_{i\in I} A_i \neq \emptyset$, therefore $A \cap D \neq \emptyset$
Hence D is externally hyperconvex.

□

We recall Zorn's Lemma.

Theorem 2.4.3. *Let S be a partially ordered set. If S has the property that every chain (that is, every totally ordered subset of S) has an upper bound, then S contains a maximal element.*

Corollary 2.4.1. *([18, Corollary 4.8]) Let (M, ρ) be a bounded hyperconvex metric space and let $\{A_i\}_{i\in I}$ be a collection of externally hyperconvex subsets of (M, ρ) intersecting pairwise. Then*

$$\bigcap_{i\in I} A_i \neq \emptyset$$

and

$$\bigcap_{i\in I} A_i \text{ is externally hyperconvex in } M. \tag{2.10}$$

Proof. We consider the following set: $\mathcal{F} = \{J \subset I$ such that for $F \subset I\}$, by Proposition 2.4.1 we have

$$\bigcap_{i\in I\cup J} A_i \neq \emptyset \text{ is said to be externally hyperconvex.}$$

Since any set is comprised of $\emptyset$, therefore $\emptyset \in \mathcal{F}$.
Let J_k be a decreasing chain in the set $\mathcal{F}$ and $F \in I$, then the sets

$$A_{J_k} = \bigcap_{i\in J\cup F} A_i$$

build a decreasing chain of non-empty externally hyperconvex sets. Let us define J to be $\bigcup_{k\in I} J_k$, then by Theorem 2.4.2 $A = \bigcap_{i\in J\cup F} A_i = \bigcap_{k\in I} A_{J_k} \neq \emptyset$ and is said to externally hyperconvex.
Therefore, the set $\mathcal{F}$ satisfies the hypothesis in Theorem 2.4.3 and hence there is some maximal element $J_0 \in \mathcal{F}$. But for $i \in I$ we have $J_0 \cup \{i\} \in \mathcal{F}$ and since J_0 is the maximal element we can conclude that $I = J_0 \in \mathcal{F}$.

□

Proposition 2.4.2. *([18, Proposition 1.2]) Let (M,ρ) be a hyperconvex metric space and let $\{A_i\}_{i\in I}$ be a collection of pairwise intersecting externally hyperconvex subsets of (M,ρ) such that one of them is bounded. Then the intersection of these family of $\{A_i\}_{i\in I}$ is non-empty, that is*

$$\bigcap_{i\in I} A_i \neq \emptyset$$

Proposition 2.4.3. *([18, Proposition 4.9]) Let (M,ρ) be a metric space. If Y is an externally hyperconvex subset of (M,ρ) and B is an externally hyperconvex subset of Y, then B is also said to be externally hyperconvex in (M,ρ).*

Proof. Let (M,ρ) be a metric space, $\{C_\rho(u_i,r_i)\}_{i\in I}$ a family of closed balls with $\rho(u_i,u_j)\leq r_i+r_j$ and $\rho(u_i,A)\leq r_i$, whenever $i,j\in I$. We then define $A_i := C_\rho(u_i,r_i)\cap Y$, where A_i is a set of externally hyperconvex subsets in (M,ρ) and hence in Y. Since $\rho(u_i,A)\leq r_i$, then $A\cap C_\rho(u_i,r_i)\neq\emptyset$, but $A_i := C_\rho(u_i,r_i)\cap Y$, hence $A_i\cap A\neq\emptyset$. Since Y is externally hyperconvex in (M,ρ), we obtain that $A_i\cap A_j = C_\rho(u_i,r_i)\cap C_\rho(u_j,r_j)\cap Y\neq\emptyset$. Hence we have a family of pairwise intersecting externally hyperconvex subsets of Y and by Proposition 2.4.2 the family of pairwise intersecting externally hyperconvex subsets of Y is non-empty, that is, since $A\cap C_\rho(u_i,r_i)\neq\emptyset$ and $A_i := C_\rho(u_i,r_i)\cap Y$, then $A\cap A_i\neq\emptyset$.

□

In the following result a closed ball in the family of hyperconvex spaces $(M_\alpha,\rho_\alpha)_{\alpha\in\Gamma}$ will be denoted by $C^\alpha_\rho(u,r)$.

Lemma 2.4.6. *([18, Lemma 4.10]) Let $(M_\alpha,\rho_\alpha)_{\alpha\in\Gamma}$ be a collection of hyperconvex metric spaces. Let (M,ρ) be the metric obtained by gluing the collection of hyperconvex metric spaces $(M_\alpha,\rho_\alpha)_{\alpha\in\Gamma}$ along some set A, where $A\in\mathcal{E}(M_\alpha)$. If we pick $u\in M_\alpha$ and $v\in M_{\alpha'}$ with $\alpha\neq\alpha'$, then for $s=\rho(u,A)$, there is some $a\in A\bigcap C^\alpha_\rho(u,s)$ such that*

$$\rho(u,v)=\rho(u,a)+\rho(a,v).$$

Proof. Let $A' = A\bigcap C^\alpha_\rho(u,s)\neq\emptyset$. For $a\in A$ there is some point $a'\in C^\alpha_\rho(u,s)\bigcap C^\alpha_\rho(a,\rho(a,u)-s)\bigcap A$ such that $\rho(u,a) = \rho(u,a')+\rho(a',a)$. Hence $\rho(u,a)+\rho(a,v) = \rho(u,a')+\rho(a',a)+\rho(a,v)$, but by triangular property of a metric $\rho(a',a)+\rho(a,v)\geq\rho(a',v)$, therefore

$$\rho(u,a)+\rho(a,v)=\rho(u,a')+\rho(a',a)+\rho(a,v)\geq\rho(u,a')+\rho(a',v).$$

Since

$$s = \rho(u, A) = \inf_{a \in A'} \rho(u, a),$$

therefore

$$\rho(u, v) = \inf_{a \in A'} \rho(u, a) + \rho(a, v) = s + \rho(A', v).$$

By Lemma 2.4.1 and Proposition 2.4.3 we can deduce that $A' \in \mathcal{E}(M_\alpha)$. Thus for a point $u \in A'$ there is some point $a \in A' \bigcap C_\rho^{\alpha'}(v, \rho(A', v))$ such that

$$\rho(u, v) = \rho(u, a) + \rho(a, v).$$

□

Lemma 2.4.7. *([18, Lemma 4.11]) Let $(M_\alpha, \rho_\alpha)_{\alpha \in \Gamma}$ be a collection of hyperconvex metric spaces. Let (M, ρ) be the metric obtained by gluing the collection of hyperconvex metric spaces $(M_\alpha, \rho_\alpha)_{\alpha \in \Gamma}$ along some set A, where $A \in \mathcal{E}(M_\alpha)$. If $u \in M_\alpha$ and $r \geq s := \rho(u, A)$. Then for $\alpha_0 \neq \alpha$, we have*

$$C_\rho(u, r) \cap M_{\alpha_0} = C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A, r - s).$$

Furthermore, $C_\rho(u, r) \cap M_{\alpha_0} \in \mathcal{E}(M_{\alpha_0})$.

Proof. Since $C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A, r - s)$ denotes closed balls in $(M_{\alpha_0}, r_{\alpha_0})$, then $C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A), r - s) \subset M_{\alpha_0}$, therefore

$$C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A), r - s) \subset M_{\alpha_0} \cap C_\rho(u, r).$$

Let $y \in C_\rho(u, r) \cap M_{\alpha_0}$. Then there is some $a \in C_\rho^{\alpha}(u, s) \cap A$ such that $\rho(u, v) = \rho(u, a) + \rho(a, v)$. By Lemma 2.4.6 $\rho(a, v) \leq r - s$, then $v \in C_\rho(a, r - s)$. But $a \in A$ and $A \cap C_\rho^{\alpha}(u, s)$, therefore $y \in C_\rho^{\alpha}(u, s) \cap A \Rightarrow v \in C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A, r - s)$. Hence $M_{\alpha_0} \cap C_\rho(u, r) \subset C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A), r - s)$. Since

$$C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A), r - s) \subset M_{\alpha_0} \cap C_\rho(u, r)$$

and

$$M_{\alpha_0} \cap C_\rho(u, r) \subset C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A), r - s).$$

Therefore,

$$M_{\alpha_0} \cap C_\rho(u, r) = C_\rho^{\alpha_0}(C_\rho^{\alpha}(u, s) \cap A), r - s).$$

By Lemma 2.4.1 $C_\rho^\alpha(u,s) \cap A \in \mathcal{E}(M)$. Hence, $A \in \mathcal{E}(M_{\alpha_0})$.
Since $A \in \mathcal{E}(M_{\alpha_0})$, by Proposition 2.4.3, we also obtain that $C_\rho^\alpha(u,s) \cap A \in \mathcal{E}(M_{\alpha_0})$.
By Lemma 2.4.2

$$C_\rho^{\alpha_0}(C_\rho^\alpha(u,s) \cap A), r-s) \in \mathcal{E}(M_{\alpha_0})$$

Since we have shown from above that

$$M_{\alpha_0} \cap C_\rho(u,r) = C_\rho^{\alpha_0}(C_\rho^\alpha(u,s) \cap A), r-s).$$

Therefore,

$$M_{\alpha_0} \cap C_\rho(u,r) \in \mathcal{E}(M_{\alpha_0}).$$

□

Theorem 2.4.4. *([18, Theorem 1.3]) Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a collection of hyperconvex metric spaces and let (M,ρ) be the metric obtained by gluing the collection of hyperconvex metric spaces $(M_\alpha,\rho_\alpha)_{\alpha\in\Gamma}$ along some set A, where $A \in \mathcal{E}(M_\alpha)$. Then (M,ρ) is hyperconvex and the set A is hyperconvex in M.*

Proof. Let (M,ρ) be a metric space and $\{C_\rho(u_i,r_i)\}_{i\in I}$ be a family of closed balls in (M,ρ) with $\rho(u_i,u_j) \leq r_i + r_j$, for all $i,j \in I$. We note that there is at most $\alpha_0 \in \Gamma$ such that $\rho(u_i, A) > r_i$ for some $u_i \in M_{\alpha_0}$. If there is no such $\alpha_0 \in \Gamma$ such that $\rho(u_i,A) > r_i$ for some $u_i \in M_{\alpha_0}$, then we fix any $\alpha_0 \in \Gamma$. We then define $A_i = C_\rho(u_i,r_i) \cap M_{\alpha_0} \neq \emptyset$ and assume that $A_i \cap A_j \neq \emptyset$ for all $i,j \in I$. Let $u_i \in M_\alpha$ and $u_j \in M_{\alpha_0}$. For $\alpha, \alpha' \neq \alpha_0$ and $\alpha = \alpha'$, we will have $A_i \cap A_j \neq \emptyset$ for all $i,j \in I$, since $A \in \mathcal{E}(M)$. For $\alpha,\alpha' \neq \alpha_0$ and $\alpha \neq \alpha'$, there is some $a \in C_\rho(u, \rho(u,A))$ such that $\rho(u_i,u_j) = \rho(u_i,a) + \rho(a,u_j)$. Since A is externally hyperconvex in M_α, therefore

$$C_\rho^{\alpha'}(a, r_i - \rho(u_i,A)) \cap C_\rho^{\alpha'}(u_i,r_i) \cap A \neq \emptyset$$

and

$$C_\rho^{\alpha'}(a, \beta_i - \rho(u_i,A)) \cap C_\rho^{\alpha'}(u_i,r_i) \cap A \subset A_j \cap A_j$$

For $\alpha' = \alpha_0$ and $\alpha = \alpha'$, we obtain

$$C_\rho^{\alpha_0}(u_i,r_i) \cap C_\rho^{\alpha_0}(u_j,r_j) \neq \emptyset.$$

For $\alpha' = \alpha_0$ and $\alpha \neq \alpha'$ and since, M_{α_0} is hyperconvex, we obtain that

$$C_\rho^{\alpha_0}(a, r_i - \rho(u_i,A)) \cap C_\rho^{\alpha_0}(u_i,r_i) \subset A_j \cap A_j \text{ for all } i,j \in I$$

By Lemma 2.4.7 the sets A_i are a family of bounded pairwise interesting and externally hyperconvex subsets of $M_{\alpha_0}, \rho_{\alpha_0}$. Therefore by Proportion 2.4.2 $\bigcap A_i \neq \emptyset$ and hence $\bigcap C_\rho(u_i,r_i) \neq \emptyset$. Hence (M,ρ) is hyperconvex. □

2.5. Gluing along weakly externally hyperconvex subsets

In this section, we summarise the gluing of a family of hyperconvex metric spaces along weakly externally hyperconvex subsets. We begin by recalling the definition of a weakly externally hyperconvex subset.

Definition 2.5.1. *[19, Lemma 2.3] Let (M, ρ) be a metric space and let A be a non-empty subset of M. Then A is weakly externally hyperconvex relative to M if for any $u \in M$, A is externally hyperconvex relative to $A \cup \{u\}$. Particularly, for any family of points $(u_i)_{i\in i}$ in M, where at most one of the points lies in M and a family of nonnegative real numbers $(r_i)_{i\in i}$ satisfying, $\rho(u_i, u_j) \leq r_i + r_j$, with dist$(u_i, A) \leq r_i$ if $u_i \notin A$ whenever $i, j \in I$ then,*

$$\bigcap_{i\in I} C_\rho(u_i, r_i) \cap A \neq \emptyset.$$

We denote the family of all weakly externally hyperconvex subsets of M by $\mathcal{W}(M)$.

Lemma 2.5.1. *[19, Lemma 3.4] Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a family of hyperconvex metric spaces and (M, ρ) be the metric space obtained by gluing $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ along some weakly externally hyperconvex subset A in M_α. For $u \in M_\alpha$ and $v \in M_{\alpha'}$, there are then points in $a \in C_\rho(u, \rho(u, A)) \cap A$ and $v \in C_\rho(v, \rho(v, A)) \cap A$ such that*

$$\rho(u, v) = \rho(u, a) + \rho(a, a') + \rho(a', v).$$

Proof. Given that A is weakly externally hyperconvex in each M_λ, by [19, Lemma 2.3 (ii)], there are for every $z \in A$, points $a \in C_\rho(u, \rho(u, A)) \cap A$ and $a' \in C_\rho(v, \rho(v, A)) \cap A$ such that the following hold:

$$\rho(u, z) = \rho(u, a) + \rho(a, v)$$

and

$$\rho(z, v) = \rho(z, a') + \rho(a', v).$$

Hence,

$$\rho(u, v) = \rho(u, A) + \rho(C_\rho(u, \rho(u, A)) \cap A, C_\rho(v, \rho(v, A)) \cap A) + \rho(v, A).$$

But the sets $C_\rho(u, \rho(u, A)) \cap A$ and $C_\rho(v, \rho(v, A)) \cap A$ are externally hyperconvex in A and therefore by [19, Lemma 2.13] there are $a, a' \in A$ with

$$\rho(a, a') = \rho(C_\rho(u, \rho(u, A)) \cap A), C_\rho(v, \rho(v, A)) \cap A).$$

□

Lemma 2.5.2. *[19, Lemma 3.5]*

Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a family of hyperconvex metric spaces and (M, ρ) be the metric space obtained by gluing $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ along some weakly externally hyperconvex subset A in M_α. Then, if $\alpha \neq \alpha'$, $u \in M_\alpha$ and $r \geq s := \rho(u, A)$ where s and r are nonnegative real numbers, we have

$$C_\rho(u,r) \cap M_{\alpha'} = C_\rho^{\alpha'}(C_\rho^{\alpha}(u,s) \cap A, r-s).$$

Thus, if $C_\rho^\alpha(u,s) \cap A$ is externally hyperconvex in $M_{\alpha'}$, then $C_\rho(u,r) \cap M_{\alpha'}$ is also externally hyperconvex in $M_{\alpha'}$.

Proof. We assume that $v \in C_\rho(u,r) \cap M_{\alpha'}$.
By Lemma 2.5.1, there is some point $a \in C_\rho^\alpha(u,s) \cap M_{\alpha'}$ such that

$$\rho(u,v) = \rho(u,a) + \rho(a,v).$$

Since we have $\rho(a,v) \leq r - s$ then,

$$v \in C_\rho^{\alpha'}(C_\rho^{\alpha}(u,s) \cap M, r-s).$$

□

Proposition 2.5.1. *[19, Proposition 3.6] Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a family of hyperconvex metric spaces and (M, ρ) be the metric space obtained by gluing $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ along some weakly externally hyperconvex subset A in M_α. If M is hyperconvex, then for all $\alpha \in \Gamma$ and all $u \in M \setminus M_\alpha$ the set $C_\rho(u, \rho(u,A)) \cap A$ is externally hyperconvex in M_α.*

Proof. Let s be denoted by $\rho(u,A)$ that is $s := \rho(u,A)$. Let $(u_i)_{i\in I}$ be a family of points in M_α and let $(r_i)_{i\in I}$ be a family of nonnegative real numbers such that

$$\rho(u_i, u_j) \leq r_i + r_j \text{ and } \rho(u_i, C_\rho(u,s) \cap A) \leq r_i.$$

Since M is hyperconvex there is some point $v \in C_\rho(u,s) \cap C_\rho(u_i, r_i)$. Because $v \in C_\rho(u,s)$, we obtain that $v \in M_{\alpha'}$ for some $\alpha \neq \alpha'$. Thus by Lemma 2.5.1 there is $v_i \in C_\rho(u_i, \rho(u_i, A)) \cap A$ with $\rho(v, u_i) = \rho(v, v_i) + \rho(v_i, u_i)$ for $i \in I$. We define $r_i' = r_i - \rho(v_i, u_i)$. Thus we have

$$\rho(v_i, v_j) \leq \rho(v_i, v) + \rho(v, v_j) \leq r_i' + r_j'$$

and

$$\rho(u, v_i) \leq s + r_i'.$$

Since A is weakly externally hyperconvex in $M_{\alpha'}$, there is some point $z \in \bigcap_{i\in I} C_\rho(v_i, r_i^{'}) \cap C_\rho(u,s) \cap A$. Thus,

$$\bigcap_{i\in I} C_\rho(v_i, r_i^{'}) \cap C_\rho(u,s) \cap A \neq \emptyset.$$

□

The following proposition generalizes the results for gluing along a gated subset and along an externally hyperconvex subset, and in both cases, the gluing is weakly externally hyperconvex.

Proposition 2.5.2. *[19, Proposition 3.7] Let $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ be a family of hyperconvex metric spaces and let (M, ρ) be the metric space obtained by gluing $(M_\alpha, \rho_\alpha)_{\alpha\in\Gamma}$ along some weakly externally hyperconvex subset A in M_α. For $u \in M \setminus M_\alpha$ the intersection $C_\rho(u, \rho(u, A)) \cap A$ is externally hyperconvex in M_i. Then M is hyperconvex and M_α is weakly externally hyperconvex for every $\alpha \in \Gamma$.*

Proof. Let $(u_i)_{i\in I}$ be a family of points in M and let $(r_i)_{i\in I}$ family of positive real numbers such that we have a family of closed balls $\{C_\rho(u_i, r_i)\}_{i\in I}$ in M with $\rho(u_i, u_j) \leq r_i + r_j$ for every $i \in I$.

Case 1: We show that the intersection $C_\rho(u, \rho(u, A)) \cap A$ is a single point and thus externally hyperconvex in M. Whenever $i, j \in I$, we have

$$C_\rho(u_i, r_i) \cap C_\rho(u_j, r_j) \cap A \neq \emptyset.$$

If we set $C_i := A \cap C_\rho(u_i, r_i)$, we get that the family $\{C_i\}_{i\in I}$ is pairwise intersecting. Furthermore, because A is weakly externally hyperconvex, we have that $\{C_i\}_{i\in I}$ is contained in $\mathcal{E}(A)$. By Proposition [19, Proposition 2.1] we get that $\bigcap_{i\in I} C_i \neq \emptyset$ and therefore,

$$\bigcap_{i\in I} C_\rho(u_i, r_i) \neq \emptyset.$$

Case 2: We show that if $C_\rho(u, \rho(u, A)) \cap A \in \mathcal{E}(A)$, then by [18, Proposition 4.9], we obtain that $C_\rho(u, \rho(u, A)) \cap A \in \mathcal{E}(M_\alpha)$.

Otherwise, we have $u_{i_0}, u_{j_0} \in M_{\alpha_0}$ whenever $i_0, i_0 \in I$, such that

$$C_\rho(u_{i_0}, r_{i_0}) \cap C_\rho(u_{j_0}, r_{j_0}) \cap A \neq \emptyset.$$

Indeed, there is some $i_0 \in I$ with $\rho(u_{i_0}, A) > r_{i_0}$ and we may assume that $i_0 = j_0$ or if $C_\rho(u, \rho(u, A)) \cap A \neq \emptyset$ such that $\rho(u_{i_0}, A) \leq r_{i_0}$ and $\rho(u_{j_0}, A) \leq r_{i_0}$, we obtain that $u_{i_0}, u_{j_0} \in M_{\alpha_0}$ by Lemma 2.5.1. We see that in both cases if $u_i \in M_\alpha \neq M_{\alpha_0}$ we have $\rho(u_i, A) \leq r_i \leq r_i$.
If we define $A_i^{\alpha^0} = C_\rho(u_i, r_i) \cap M_{\alpha 0}$, we show that for every $i, j \in I$, we have $A_i^{\alpha^0} \cap A_j^{\alpha^0} \neq \emptyset$. Then by Lemma 2.5.2, we have $A_i^{\alpha^0} \in \mathcal{E}(M_{\alpha_0})$ and by [19, Proposition 2.1], we obtain that

$$\bigcap_{i\in I} C_\rho(u_i, r_i) \cap M_{\alpha 0} = \bigcap_{i\in I} A_i^{\alpha^0} \neq \emptyset.$$

In order to prove the above, we consider the following two steps

- If $u_i, u_j \in M_{\alpha_0}$, then by hyperconvexity of M_{α_0}, we are done.
- If $u_i \in M_\alpha \neq M_{\alpha_0} \not\ni u_j$, we have that $C_\rho(u_i, r_i) \cap C_\rho(u_j, r_j) \cap M_{\alpha 0} \neq \emptyset$ by Lemma 2.5.1.

Step 1: We set

$$A' = C_\rho(u_{i_0}, r_{i_0}) \cap C_\rho(u_{j_0}, r_{j_0}) = C_\rho^{\alpha^0}(u_{i_0}, r_{i_0}) \cap C_\rho^{\alpha^0}(u_{j_0}, r_{j_0})$$

and

$$s := \rho(A, A').$$

By [19, Corollary 2.13], we obtain that $C_\rho(A', s) \cap A \neq \emptyset$.
Moreover, we obtain that

$$C_\rho(A', s) = C_\rho^{\alpha^0}(u_{i_0}, r_{i_0} + s) \cap C_\rho^{\alpha^0}(u_{j_0}, r_{j_0} + s),$$

and therefore,

$$C_\rho^{\alpha^0}(u_{i_0}, r_{i_0} + s) \cap C_\rho^{\alpha^0}(u_{j_0}, r_{j_0} + s) \subset M_{\alpha_0}.$$

To illustrate this, we observe that by [19, Lemma 2.2 (i)], we have

$$C_\rho(A', s) = C_\rho^{\alpha^0}(u_{i_0}, r_{i_0} + s) \cap C_\rho^{\alpha^0}(u_{j_0}, r_{j_0} + s) \subset M_{\alpha_0}$$

and thus,

$$(C_\rho(u_{i_0}, r_{i_0} + s) \cap C_\rho(u_{j_0}, r_{j_0} + s)) \setminus C_\rho(A', s) \subset M \setminus M_{\alpha_0}.$$

Hence, we assume that there is some

$$v \in C_\rho(u_{i_0}, r_{i_0} + s) \cap C_\rho(u_{j_0}, r_{j_0} + s) \cap (M_\alpha \setminus M_{\alpha_0}).$$

Now since

$$C_\rho(u_{i_0}, r_{i_0} + s) \cap C_\rho(u_{j_0}, r_{j_0} + s) \cap A \neq \emptyset$$

and

$$C_\rho(u_{i_0}, r_{i_0} + s) \cap C_\rho(u_{j_0}, r_{j_0} + s) \cap M_\alpha$$

is externally hyperconvex in M_α and thus path-connected, there is some

$$v' \in C_\rho(u_{i_0}, r_{i_0} + s) \cap C_\rho(u_{j_0}, r_{j_0} + s) \cap (M_\alpha \setminus M_{\alpha 0})$$

such that $\rho(v', A) \leq s$.
But then, by Lemma 2.5.1 we have

$$C_\rho(u_{i_0}, r_{i_0} + s) \cap C_\rho(v', s) \cap M_{\alpha_0} \neq \emptyset$$

and

$$C_\rho(u_{j_0}, r_{j_0} + s) \cap C_\rho(v', s) \cap M_{\alpha_0} \neq \emptyset.$$

Thus,

$$C_\rho(u_{i_0}, r_{i_0}) \cap C_\rho(u_{j_0}, r_{j_0}) \cap C_\rho(v', s) \cap M_{\alpha_0} \neq \emptyset$$

i.e. $v' \in C_\rho(v', s)$ contradicting $v' \not\ni M_{\alpha_0}$.

Step 2: We now show that the family

$$\mathcal{F} := \{C_\rho(u_{i_0}, r_{i_0} + s) \cap M_\alpha, C_\rho(u_{j_0}, r_{j_0} + s) \cap M_\alpha, C_\rho^\alpha(u_i, r_i), C_\rho^\alpha(u_j, r_j)\}$$

is pairwise intersecting.
From step 1 we have seen that

$$(C_\rho(u_{i_0}, r_{i_0} + s) \cap M_\alpha) \cap (C_\rho(u_{j_0}, r_{j_0} + s) \cap (M_\alpha) \neq \emptyset.$$

Moreover, since $u_{i_0} \in M_{\alpha_0} \neq M_\alpha \ni u_i$, by Lemma 2.5.1, we have

$$(C_\rho(u_{i_0}, r_{i_0}) \cap M_\alpha) \cap C_\rho^\alpha(u_i, r_i) \neq \emptyset.$$

Similarly, for (i_0, i) replaced by (i_0, j) we have

$$(C_\rho(u_{i_0}, r_{i_0}) \cap M_\alpha) \cap C_\rho^\alpha(u_j, r_j) \neq \emptyset$$

and also for (i_0, i) replaced by (j_0, i) and (j_0, j) we have

$$(C_\rho(u_{j_0}, r_{j_0}) \cap M_\alpha) \cap C_\rho^\alpha(u_i, r_i) \neq \emptyset$$

and

$$(C_\rho(u_{j_0}, r_{j_0}) \cap M_\alpha) \cap C_\rho^\alpha(u_j, r_j) \neq \emptyset.$$

Finally, by hyperconvexity of M_α

$$C_\rho^\alpha(u_i, r_i) \cap C_\rho^\alpha(u_j, r_j) \cap M_\alpha \neq \emptyset.$$

Thus, the family $\mathcal{F}$ is pairwise intersecting. Since $\mathcal{F} \subset \mathcal{E}(M_\alpha)$ it follows by [19, Proposition 2.1] that

$$C := C_\rho^\alpha(u_{i_0}, r_{i_0} + s) \cap C_\rho^\alpha(u_{j_0}, r_{j_0} + s) \cap C_\rho^\alpha(u_i, r_i) \cap C_\rho^\alpha(u_j, r_j) \neq \emptyset.$$

Because $C_\rho(u_{i_0}, r_{i_0} + s) \cap (C_\rho(u_{j_0}, r_{j_0} + s) \cap M_\alpha \subset A$, we obtain that $C \subset A$.
Therefore,

$$C_\rho^\alpha(u_i, r_i) \cap C_\rho^\alpha(u_j, r_j) \cap A \supset C \cap A = C \neq \emptyset.$$

To show that M_α is weakly externally hyperconvex,
we use that for $u \in M$, $r \leq \rho(u, M_\alpha)$, $u_i \in M_\alpha$ such that $\rho(u, u_i) \leq r + r_i$, $\rho(u_i, u_j) \leq r_i + r_j$ we have

$$C_\rho(u, r) \cap C_\rho(u_i, r_i) \cap M_\alpha \neq \emptyset$$

by Lemma 2.5.1 and thus, $\{C_\rho(u, r) \cap M_\alpha, C_\rho(u_i, r_i)\}$ is a family of pairwise intersecting externally hyperconvex subsets of M_α.

□

The following theorem is a combination of Proposition 2.5.1 and Proposition 2.5.2.

Theorem 2.5.1. *[19, Theorem 1.1] Let $(M_\alpha, \rho_\alpha)_{\alpha \in \Gamma}$ be a family of hyperconvex metric spaces and (M, ρ) be the metric space obtained by gluing $(M_\alpha, \rho_\alpha)_{\alpha \in \Gamma}$ along some weakly externally hyperconvex subset A in M_α. Then, M is hyperconvex if and only if for all $\alpha \in \Gamma$ and for all $u \in M \setminus M_\alpha$, the set $C_\rho(u, \rho(u, A)) \cap A$ is externally hyperconvex in M_α.*
Furthermore, if M is hyperconvex, the subspaces M_α are weakly externally hyperconvex.

Theorem 2.5.2. *[19, Theorem 3.8] Let (M_0, ρ_0) be a hyperconvex metric space and let $\{M_\alpha\}_{\alpha \in \Gamma}$ be a family of hyperconvex metric spaces with weakly externally hyperconvex subsets $A_\alpha \in \mathcal{W}(M_\alpha)$ such that for all $\alpha \in \Gamma$, there is some isometry copy $A_\alpha \in \mathcal{W}(M_0)$ and $A_\alpha \cap A_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. If for every $u_\alpha \in M_\alpha$ and for every $u \in M_0$ we have*

$$C_\rho(u_\alpha, \rho(u_\alpha, A_\alpha)) \cap A_\alpha \in \mathcal{E}(M_0)$$

and

$$C_\rho(u, \rho(u, A_\alpha)) \cap A_\alpha \in \mathcal{E}(M_\alpha),$$

then $M = M_0 \bigsqcup_{\{A_\alpha : \alpha \in \Gamma\}} M_\alpha$ is hyperconvex.

Proof. By Theorem 2.5.1, we obtain that $N_\alpha = M_0 \bigsqcup_{\{A_\alpha:\alpha\in\Gamma\}} M_\alpha$ is hyperconvex and $M_0 \in \mathcal{W}(N_\alpha)$. We can see that M can be obtained by gluing the spaces N_α along M_0, i.e. $M = \bigsqcup_{M_0} N_\alpha$. Hence, what remains is to show that for $\alpha \neq \alpha'$ and $u \in N_\alpha$ the intersection

$$B := C_\rho(u, \rho(u, M_0)) \cap M_0 \in \mathcal{E}(N_{\alpha'}).$$

Applying [19, Corollary 2.12], we obtain that $B \in \mathcal{W}(N_{\alpha'})$. Without loss of generality we may assume that $u \notin M_0$. Thus, we have $\rho(u, M_0) = \rho(u, A_\alpha)$ and hence $B = C_\rho(u, \rho(u, A_\alpha)) \cap A_\alpha \in \mathcal{E}(M_0)$, especially $B \subset A_\alpha$. Therefore, $\rho(B, A_{\alpha'}) > 0$ by [19, Corollary 2.13], that is, there is some $s > 0$ such that $C_\rho^{\alpha'}(B, s) \subset M_0$. Hence, $B \in \mathcal{E}(C_\rho^{\alpha'}(B, s))$ and $B \in \mathcal{E}(N_{\alpha'})$ by [19, Lemma 2.16]. □

Proposition 2.5.3. *[19, Proposition 3.9] Let (M, ρ) be a metric space and let A be a subset of M such that $M \bigsqcup_A M$ is hyperconvex. Then the following hold:*

i) A is weakly externally hyperconvex in M.
ii) For every $u \in M$, the intersection $C_\rho(u, \rho(u, A)) \cap A$ is externally hyperconvex in M.

Proof. i) We denote the second copy of M by M' and for and $v \in M$, we denote it's corresponding copy by v'. Choosing a point $u_0 \in M$ and $r \geq 0$ with $\rho(u_0, A)) \leq r_0$ and if we let $\{(u_i, r_i)\}_{i\in I} \subset A \times [0, \infty)$ be such that $\rho(u_i, u_j) \leq r_i + r_j$ for every $i, j \in I \bigcup\{0\}$. It follows that $\rho(u_0, u_0') \leq 2r_0$ and because $M \bigsqcup_A M$ is hyperconvex we obtain that

$$B := \bigcap_{i\in I} C_\rho(u_i, r_i) \cap C_\rho(u_0, r_0) \cap C_\rho(u_0', r_0) \neq \emptyset.$$

By the symmetric property of a metric, there are $v, v' \in B$ with $v \in M$ and $v' \in M'$. Therefore, since the intersections of balls are hyperconvex, there is some geodesic $[v, v'] \subset B$, which must intersect with A.
Hence, we obtain that

$$\bigcap_{i\in I} C_\rho(u_i, r_i) \cap C_\rho(u_0, r_0) \cap A \neq \emptyset,$$

and therefore, A is weakly externally hyperconvex.

ii) We can see that

$$C_\rho(u, \rho(u, A)) \cap A = C_\rho(u, \rho(u, A)) \cap C_\rho(u', \rho(u, A)) \in \mathcal{A}(M \bigsqcup_A M).$$

□

Example 2.5.1. *Let M_1 and M_2 be two copies of l^3_∞. Consider the gluing $M := M_1 \bigsqcup_A M_2$, where*

$$A := \{u \in l^3_\infty : u_1 = u_2 \text{ and } u_3 = 0\}$$

and where the gluing maps are given by the inclusion maps for A. To show that M is hyperconvex, assume

$$b_1 := (0,0,1) \in M_1$$

as well as

$$b_2 := (2,0,0) \in M_2$$

and

$$b_2' := (0,-2,0) \in M_2.$$

Consider that $C_\rho(b_1,1) \cap M_2 = \{(t,t,0) : t \in [-1,1]\}$ and therefore $C_\rho(b_1,1) \cap C_\rho(b_2,1) = \{(1,1,0)\} \in A$ as well as $C_\rho(b_1,1) \cap C_\rho(b_2',1) = \{(-1,-1,0)\} \in A$. Furthermore, $C_\rho(b_2,1) \cap C_\rho(b_2',1) = \{(1\} \times \{-1\} \times [-1,1] \subset M_2$. Therefore,

$$C_\rho(b_1,1) \cap C_\rho(b_2,1) \cap C_\rho(b_2',1) = \emptyset.$$

Although as a consequence of Proposition 2.5.2 and Proposition 2.5.3 we get the necessary and sufficient condition stated in Theorem 2.5.1.

Example 2.5.2. *Let (M,ρ) be a hyperconvex metric space and let A be a strongly convex subset of M. Let $r \geq 0$. Then, the gluing along $C_\rho(A,r)$ is weakly externally hyperconvex and for every $u \in M$ the set*

$$C_\rho(u, \rho(u, C_\rho(A,r))) \cap C_\rho(A,r)$$

is externally hyperconvex in M by [19, Lemma 2.5]. Therefore, gluing along $C_\rho(A,r)$ preserves hyperconvexity.

3

Some quasi-pseudometric spaces

In this chapter, we present some interesting classes of quasi-pseudometric spaces and look at characterisations of the externally q-hyperconvex and weakly externally q-hyperconvex quasi-pseudometric subsets. We begin by summarising the notion of q-hyperconvexity in the framework of quasi-pseudometric spaces. Moreover, we introduce the concept of gated subsets of a quasi-pseudometric space and study the q_3-hyperconvex space.

3.1. Q-hyperconvex quasi-pseudometric spaces

In this section, we summarize the theory of hyperconvexity in the setting of quasi-pseudometric spaces called q-hyperconvex quasi-pseudometric spaces and we provide some useful examples.

Definition 3.1.1. *(Compare Definition 2.1.1) Let (M, q) be a quasi-pseudometric space. Then M is said to be* quasi-pseudometrically convex *if for any two points $u, v \in M$ and positive real numbers r and s such that*

$$q(u, v) \leq r + s,$$

there is some $w \in M$ such that $q(u, w) \leq r$ and $q(w, v) \leq s$.

We now recall the definition of a q-hyperconvex quasi-pseudometric space.

Definition 3.1.2. *Let (M, q) a be quasi-pseudometric space, then M is q-*hyperconvex *if for each collection of of points $(u_i)_{i \in I}$ in M and families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ satisfying $q(u_i, u_j) \leq r_i + s_j$ whenever $i, j \in I$, the following holds:*

$$\bigcap_{i \in I} (C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \neq \emptyset.$$

Next, we look at some basic examples of q-hyperconvex spaces.

Example 3.1.1. *[14, Example 1] Let the set of all real numbers $\mathbb{R}$ be equipped with the T_0 space $q(u,v) = max\{u-v, 0\} = u\dot{-}v$, for all $u, v \in M$. Then the pair $(\mathbb{R}, q)$ is q-hyperconvex.*

Proof. see [14, Example 1]. □

Example 3.1.2. *The subspace $[0,\infty)$ of the set of all real numbers $\mathbb{R}$ equipped with the T_0 space $q(u,v) = max\{u-v, 0\} = u\dot{-}v$, whenever $u, v \in M$ is q-hyperconvex.*

Proof. see [14, Corrolary 1]. □

Example 3.1.3. *If we consider the product of two quasi-pseudometric spaces $(\mathbb{R}, q)$ and $(\mathbb{R}, q^{-1})$, that is, $\mathbb{R}^2$ equipped with the T_0- quasi metric space $D((\alpha,\beta),(\alpha',\beta')) = (\alpha - \alpha') \vee (\beta' - \beta)$, for all $(\alpha,\beta),(\alpha',\beta') \in \mathbb{R}^2$. Hence the diagonal $\{(\alpha,\alpha) : \alpha \in \mathbb{R}\}$ in this product T_0- quasi metric space is isometric to $(\mathbb{R}, q^s)$. It is easily proven that $(\mathbb{R}^2, q \times q^{-1})$ is q-hyperconvex.*

Example 3.1.4. *Let $u, v \in (0,\infty)$ and $Y = [0,u] \times [0,v]$. Let $D((\alpha,\beta),(\alpha',\beta')) = (\alpha\dot{-}\alpha') \vee (\beta'\dot{-}\beta)$, for all $(\alpha,\beta),(\alpha',\beta') \in Y$. Then Y is identified with the q-hyperconvex subspace $X = \{(0,u),(0,v)\}$ of Y.*

We now prove some results on q-hyperconvexity.

Proposition 3.1.1. *[16, Proposition 1] Let (M,q) be a q-hyperconvex quasi-pseudometric space. Let $(u_i)_{i\in I}$ be a family of nonempty points in M and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative real numbers with $q(u_i,u_j) \leq r_i + s_i$. If we set $A = \bigcap_{i\in I}(C_q(u_i,r_i) \cap C_{q^{-1}}(u_i,s_i))$. Then A is nonempty and q-hyperconvex.*

Proof. Firstly, A is nonempty by q-hyperconvexity of M. Whenever $\alpha \in S$ let $u_\alpha \in A$ and let r_α, s_β be nonnegative real numbers such that $q(u_\alpha,u_\beta) \leq r_\alpha + s_\beta$, for all $\alpha,\beta \in S$. We prove that the family

$$[(C_q(u_\alpha,r_\alpha))_{\alpha\in S},(C_q(u_k,r_k))_{k\in I};(C_{q^{-1}}(u_\alpha,s_\alpha))_{\alpha\in S},(C_{q^{-1}}(u_k,s_k))_{k\in I}]$$

satisfies the notion of q-hyperonvexity. Whenever $\alpha \in S$ and $k \in I$ we have

$$q(u_\alpha,u_k) \leq s_k \leq r_\alpha + s_k$$

and

$$q(u_k,u_\alpha) \leq r_k \leq r_k + s_\alpha.$$

Thus, by q-hyperconvexity on M we have

$$\emptyset \neq \bigcap_{k\in I}(C_q(u_k, r_k) \cap (C_{q^{-1}}(u_k, s_k)) \cap \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha)) \cap (C_{q^{-1}}(u_\alpha, s_\alpha)) =$$

$$A \cap (C_q(u_\alpha, r_\alpha)) \cap (C_{q^{-1}}(u_\alpha, s_\alpha)).$$

Therefore, the subspace A of M is q-hyperconvex. □

Theorem 3.1.1. *[16, Theorem 2] Let (M, q) be a bounded T_0-quasi metric space and let $(P_i)_{i\in I}$ be a nonempty family of descending q-hyperconvex subsets of M, for all I totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ holds if and only if $P_{i_2} \subseteq P_{i_1}$. Then $\bigcap_{i\in I} P_i$ is nonempty and q-hyperconvex.*

Proof. Firstly, we show that $\bigcap_{i\in I} P_i \neq \emptyset$. We take into account that (M, q^s) is a bounded metric space and by [3, Proporsition 2] $(P_i)_{i\in I}$ is a descending chain of hyperconvex sets in (M, q^s). By [3, Theorem 7], we can deduce that $\bigcap_{i\in I} P_i \neq \emptyset$ and hyperconvex in (M, q^s).

Secondly, we show that $P = \bigcap_{i\in I} P_i$ is q-hyperconvex. Let $(u_\alpha)_{\alpha\in S}$ be a nonempty family of points in P and $(r_\alpha)_{\alpha\in S}$ and $(S_\beta)_{\beta\in S}$ be families of nonnegative real numbers such that $q(u_\alpha, u_\beta) \leq r_\alpha + s_\beta$, for all $\alpha, \beta \in S$. We assume $i \in I$ is fixed. Since P_i is a q-hyperconvex space and since $u_\alpha \in P_i$ for all $\alpha \in S$, thus $A_i = \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \cap P_i \neq \emptyset$ and q-hyperconvex by the proof of Proposition 3.1.1 and therefore by [3, Proposition 2] a hyperconvex subset of (M, q^s).

Therefore, by the first part of our current proof

$$\emptyset \neq \bigcap_{i\in I} A_i = \bigcap_{i\in I}[\bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \cap P_i]$$

$$= \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \cap \bigcap_{i\in I} P_i,$$

since $(A_i)_{i\in I}$ is descending. This proves that $P = \bigcap_{i\in I} P_i$ is q-hyperconvex.

□

Lemma 3.1.1. *[16, Lemma 1] Let (M, q) be a T_0-quasi-metric space. If $(P_\alpha)_{\alpha\in S}$ is a collection of bounded q-hyperconvex subsets of M such that $\bigcap_{\alpha\in D} P_\alpha$ is nonempty and q-hyperconvex whenever $D \subseteq S$ is finite, then the intersection $\bigcap_{\alpha\in S} P_\alpha$ is nonempty and q-hyperconvex.*

Proof. Consider $\mathcal{D} = \{I \subseteq S :$ for all J finite, $J \subseteq S, \bigcap_{I\cup J} P_\alpha$ is nonempty and q-hyperconvex$\}$. Obviously, $\emptyset \in \mathcal{D}$ and satisfies the notion of Zorn's Lemma because of Theorem 3.1.1. Let

I be a maximal in $\mathcal{D}$. Then $I \cup \{\alpha\} \in \mathcal{D}$, whenever $\alpha \in S$. Due to the maximality of I, we have $\alpha \in I$ whenever $\alpha \in S$.

□

We define a bicover of a subset.

Definition 3.1.3. *[16, Proposition 1]*

Let (M, q) be a quasi-pseudometric space. For any bounded nonempty subset A of M, we define the bicover of A as follows:

$$bicov(A)_+ := \bigcap\{C_q(u,r) : A \subseteq C_q(u,r), u \in M, r \geq 0\},$$
$$bicov(A)_- := \bigcap\{C_{q^{-1}}(u,s) : A \subseteq C_{q^{-1}}(u,s), u \in M, s \geq 0\},$$
$$bicov(A) := bicov(A)_+ \cap bicov(A)_-.$$

We next define q-admissible subsets in quasi-pseudometric spaces and it can be compared to Definition 2.1.5.

Definition 3.1.4 (Compare Definition 2.1.5). *Let (M, q) be T_0-quasi-pseudometric space and A a non-empty bounded subset of M. Then A is* q-admissible *if $A = bicov(A)$. The set of all q-admissible subsets of M will be denoted by $\mathcal{A}_q(M)$.*

We now give the definition of a q_m-hyperconvex quasi-pseudometric space.

Definition 3.1.5. *Let (M, q) be quasi-pseudometric space, then M is* q_m-hyperconvex *if for each collection of of points $(u_i)_{i\in I}$ in M, families of non-negative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ and for such classes with $|I| \leq m$, satisfying $q(u_i, u_j) \leq r_i + s_j$ whenever $i, j \in I$, the following hold:*

$$\bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \neq \emptyset.$$

Remark 3.1.1. *[1, Remark 4.2] From the definition of q_m-hyperconvexity above we can deduce that q-hyperconvexity is stronger than q_m-hyperconvexity and q_m-hyperconvexity is stronger than q_n-hyperconvexity whenever $n \leq m$.*
Notice that q_3-hyperconvexity is equivalent to total convexity and we prove it below.

Proof. Let (M, q) be a totally convex quasi-pseudometric space and $I = \{1, 2\}$. Thus we have the balls $C_q(u_1, r_1)$, $C_{q^{-1}}(u_1, s_1)$, $C_q(u_2, r_2)$, $C_{q^{-1}}(u_2, s_2)$ with $u_1, u_2 \in M$ and $r_i, s_i \geq 0$, $i \in I$, satisfying

$$q(u_1, u_2) \leq r_1 + s_2$$

and

$$q(u_2, u_1) \leq r_2 + s_1.$$

Let

$$q(u_1, z) = \frac{r_1}{r_1 + s_2} q(u_1, u_2),$$
$$q(z, u_1) = \frac{s_1}{s_1 + r_2} q(u_2, u_1),$$
$$q(u_2, z) = \frac{r_2}{s_1 + r_2} q(u_2, u_1),$$

and

$$q(z, u_2) = \frac{s_2}{r_1 + s_2} q(u_1, u_2).$$

Therefore, we obtain that

$$C_q(u_1, r_1) \cap C_{q^{-1}}(u_1, s_1) \cap C_q(u_2, r_2) \cap C_{q^{-1}}(u_2, s_2) \neq \emptyset.$$

Thus (M, q) is q_3-hyperconvex.

Let us assume that (M, q) is q_3-hyperconvex, then for the decomposition $q(u_1, u_2) \leq \alpha_1 + \alpha_2$, $\alpha_1, \alpha_2 > 0$, we take any $u \in C_q(u_1, \alpha_1) \cap C_{q^{-1}}(u_2, \alpha_2)$, we obtain that $q(u_1, u) \leq \alpha_1$, $q(u, u_2) \leq \alpha_2$ and thus $q(u_1, u) + q(u, u_2) \geq q(u_1, u_2) = \alpha_1 + \alpha_2$.

□

We provide an example of q_m-hyperconvexity

Example 3.1.5. *Let $M = [0, \infty\}$ and we define $q(u, v) = u \dot{-} v$ whenever $u, v \in M$. Then (M, q) is said to be a T_0-quasi-metric space. By Remark 3.1.1 above we have that (M, q) is q_3-hyperconvex.*

The proof of the following result is straight forward.

Proposition 3.1.2. *Let (M, q) be a T_0-quasi-metric space.*

(i) If (M, q) is q_m-hyperconvex, then (M, q^{-1}) is also q_m-hyperconvex.

(ii) If (M, q) is q_m-hyperconvex, then (M, q^{-1}) is m-hyperconvex metric space.

Proof. (i) The proof follows from Definition 3.1.5.

(ii) Let us assume that (M, q) is q_m-hyperconvex. Suppose $(u_i)_{i\in I}$ is a family of points in M and suppose $(r_i)_{i\in I}$ a family of nonnegative real numbers. Assume that $q^{-1}(u_i, u_j) \leq r_i + r_j$ for all $i, j \in I$. Since (M, q) is q_m-hyperconvex, we have that

$$\emptyset \neq \bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, r_i)) \subseteq \bigcap_{i\in I} C_{q^{-1}}(u_i, r_i).$$

It follows that the metric space (M, q^s) is m-hyperconvex.

□

We now recall the definition of a quasi-pseudometric space with mixed binary intersection property.

Definition 3.1.6. *[14, Definition 7] Let us assume that (M, q) is a quasi-pseudometric space. A collection of double balls*

$$(C_q(u_i, r_i); C_{q^{-1}}(u_i, s_i))_{i\in I}$$

with, $r_i, s_i \geq 0$ and $u_i \in M$ whenever $i \in I$ is said to have the mixed binary intersection property *if for all indices $i, j \in I$, we have*

$$(C_q(u_i, r_i) \cap C_{q^{-1}}(u_j, s_j)) \neq \emptyset.$$

We recall the definition of hypercompleteness in the framework of quasi-pseudometric spaces.

Definition 3.1.7. *Let (M, q) be quasi-pseudometric space, then M is* q-hypercomplete *if for any collection if for every collection of double balls*

$$C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)_{i\in I},$$

where $r_i, s_i \geq 0$ and $u_i \in M$ whenever $i \in I$ possessing the mixed binary intersection property satisfies

$$\bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \neq \emptyset, \qquad \textit{for all } i \in I.$$

We recall that a quasi-pseudometric space (M, q) is called bicomplete provided the pseudometric space (M, q^s) is complete.

Remark 3.1.2. *Every q_m-hyperconvex T_0-*quasi-pseudometric space *(M, q) is bicomplete. Since (M, q^s) is complete m-hyperconvex (see [2, Theorem 1]), then by Proposition 3.1.2 above we have that (M, q) is bicomplete.*

Proposition 3.1.3. *[1, Proposition 4.4] Let (M, q) be a q_m-hyperconvex space. Let $(u_i)_{i\in I}$ be a collection of of points in M and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of non-negative real numbers and for such classes with $|I| \leq m$, satisfying $q(u_i, u_j) \leq r_i + s_j$ whenever $i, j \in I$. If we set $A = \bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \neq \emptyset$ whenever $i \in I$, then A is nonempty and q_m-hyperconvex.*

Proof. By the q_m-hyperconvexity of M, we can conclude that $A \neq \emptyset$.
For each $\alpha \in S$, we let $u_\alpha \in A$ and let r_α, s_α be non-negative real numbers with $q(u_\alpha, u_\beta) \leq r_\alpha + s_\beta$, whenever $\alpha, \beta \in S$. For $i \in I$ and each $\alpha \in S$, we have $q(u_\alpha, u_i) \leq s_i \leq r_\alpha + s_i$ and $q(u_i, u_\alpha) \leq r_i \leq r_i + s_\alpha$. Hence the family

$$[(C_q(u_i, r_i))_{i\in I}, (C_q(u_\alpha, r_\alpha))_{\alpha\in S}; (C_{q^{-1}}(u_i, s_i))_{i\in I}, (C_{q^{-1}}(u_\alpha, s_\alpha))_{\alpha\in S}]$$

satisfies the notion of q_m-hyperconvexity. By q_m-hyperconvexity of M, we have

$$\emptyset \neq [\bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \cap \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha))]$$

$$= A \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)).$$

Thus the subspace A is q_m-hyperconvex.

□

3.2. Gated sets in quasi-pseudometric spaces

In this section, we introduce the concept of gated subsets of a quasi-pseudometric space. By lack of symmetry in a quasi-pseudometric space we define an in-gate subset and an out-gate subset. Furthermore, we prove that if a subset is in-gated (or out-gated), then the in-gate (or out-gate) point is unique if it exists. Moreover, we prove that if a subset of a T_0-quasi-metric is in-gated (or out-gated), then it is strongly convex.

Definition 3.2.1. *Let (M, q) any quasi-pseudometric space. For any $u, v \in M$, in the sequel we denote $\langle u, v\rangle_q$ by*

$$\langle u, v\rangle_q := \{w \in M : q(u, v) = q(u, w) + q(w, v)\}.$$

The following definition can be compared to Definition 2.2.1.

Definition 3.2.2. *Let (M, q) be a quasi-pseudometric space and A a subset of M.*

i) We say that A is in-gated *in (M,q) if for all $u \in M$, there is some $i \in A$ such that $i \in \langle u,v\rangle_q$, that is $q(u,v) = q(u,i) + q(i,v)$ for all $v \in A$. Moreover, if such a point i exists, it is called the* in-gate of A for q with respect to u.

ii) We say that A is *in-gated* in (M,q^{-1}) if for all $u \in M$, there is some $o \in A$ such that $o \in \langle v,u\rangle_q$, that is $q(v,u) = q(v,o) + q(o,u)$ for all $v \in A$. Moreover, if such a point o exists, it is called the *out-gate of A for q with respect to u.*

Remark 3.2.1. *Let (M,q) be a quasi-pseudometric space and $A \subseteq M$. We observe that a point $z \in A$ is an in-gate of A for q with respect to u if and only if z is an out-gate of A for q^{-1} with respect to $u \in M$.*

Proof. Let $z \in A$ be an in-gate of A for q with respect to u. Then $q(u,a) + q(a,b) = q(u,b)$ for all $b \in A$. It follows that $q^{-1}(b,u) = q^{-1}(b,a) + q^{-1}(a,u)$ for all $b \in A$. Thus $z \in \langle b,u\rangle_{q^{-1}}$. The converse can be obtained by similar arguments.

□

In the next lemma, we show that if an in-gate of a gated subset of a quasi-pseudometric exists, then it is unique if the quasi-pseudometric is T_0.

Lemma 3.2.1. *Let (M,q) be a T_0-quasi-metric space and A be an in-gated subset in (M,q). If a point $z \in A$ is an in-gate of A for q with respect to $u \in M$, then z is unique.*

Proof. Let us assume that A is an in-gated subset of (M,q) and u a point in M. If z is an in-gate of A for q with respect to u and z' is another in-gate of A for q with respect to the same point u. Then we have to prove that $z = z'$.

Since $z' \in A$ and z is an in-gate of A for q with respect to u we have

$$q(u,z') = q(u,z) + q(z,z').$$

Furthermore, $z \in A$ and z' is an in-gate of A for q with respect to u, it follows that

$$q(u,z') = q(u,z') + q(z',z) + q(z,z').$$

Then we have $q(z',z) + q(z,z') = 0$. Hence $q(z',z) = 0 = q(z,z')$. Therefore $z = z'$ by T_0 property of q.

□

The following lemma can be proven by analogy to the above lemma.

Lemma 3.2.2. *Let (M, q) be a T_0-quasi-metric space and A be a in-gated subset in (M, q^{-1}). If a point $z \in A$ is an out-gate of A for q (or an in-gate of A for q^{-1}) with respect to $u \in M$, then z is unique.*

Example 3.2.1. *Consider the four point set $M = \{1, 2, 3, 4\}$. If we equip M with T_0-quasi-pseudometric ρ defined by the distance matrix*

$$Q = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

that is, $q_{i,j} = q(i, j)$ whenever $i, j \in X$. Then (M, ρ) is a T_0-quasi-pseudometric space. It is readily checked that $A = \{1, 2\}$ is an in-gated subset of (M, q), where the point $i' = 2$ is the in-gate of A for q with respect to $i = 2$, $i = 3$ and $i = 4$, while $i' = 1$ is the in-gate of A for q with respect to $i = 1$. Furthermore, $B = \{2, 3\}$ is not an in-gated subset of M since the point $i = 4$ does not have an in-gate in B, that is,

$$1 = q(4, 3) \neq q(4, 2) + q(2, 3) = 1 + 1$$

and

$$1 = q(4, 2) \neq q(4, 3) \neq q(3, 2) = 1 + 1.$$

Next, we give an example of a subset which is in-gated in the T_0-quasi-metric space but is not in-gated in the conjugate T_0-quasi-metric.

Example 3.2.2. *If we equip $M = \{u, v, w\}$ with the T_0-quasi-metric q defined by $q(u, u) = q(v, v) = q(w, w) = 0$, $q(u, v) = q(v, u) = q(v, w) = q(w, v) = q(w, u) = 1$ and $q(u, w) = 2$. Then $A = \{v, w\}$ is an in-gated subset of (M, q) and v is the in-gate of A for q with respect to u. On the other hand, A is not in-gated in (M, q^{-1}) since v is not an in-gate for q^{-1} with respect to u*

Definition 3.2.3. *[24, Definition] Let (M, q) be a T_0-metric space and $A \subseteq M$ be an in-gate (M, q). We define the* projection map *$pr_{A^d} : (M, q) \to (A, q)$ on A as follows: for any point $u \in M$, there is a unique in-gate $z \in A$ of A on (M, q) with $pr_{A^q}(u) = z$ such that for all $a \in A$*

$$q(u, a) = q(u, pr_{A^q}) + q(pr_{A^q}, a).$$

Example 3.2.3. *Consider the T_0-quasi-pseudometric q on $M = \{1,2,3,4\}$, as given in Example 3.2.1. If $A = \{1,2\}$ is an in-gated subset of M, then $pr_{A^q}(3) = 1 = pr_{A^q}(4)$, $pr_{A^q}(1) = 1$ and $pr_{A^q}(2) = 2$.*

We next define strong convexity in the context of quasi-pseudometric spaces and compare our definition to Definition 2.2.3.

Definition 3.2.4. *Let (M,q) be a quasi-pseudometric space and A a subset of (M,q). Then A is* strongly convex *in M if for $u,v \in M$, $\langle u,v\rangle_q \subseteq A$ whenever $u,v \in A$.*

We now give an example of a strongly convex subset in a quasi-pseudometric space.

Example 3.2.4. *Let the set of real numbers $\mathbb{R}$ be endowed with the T_0-quasi-metric $q(u,v) = max\{u-v,0\}$, whenever $u,v \in M$. Then $A = [0,\infty)$ is strongly convex in $(\mathbb{R},q)$. For any any two points $a,b \in A$. Let $t \in \langle a,b\rangle_q$, the we show that $t \in A$. If $a \leq b$, then we have three cases.*

Case 1: If $a \leq t \leq b$, then the equation $q(a,b) = q(a,t) + q(t,b)$ is satisfied, since $0 = 0+0$. So $t \in A$.

Case 2: If $a \leq b \leq t$, then the equation $q(a,b) = q(a,t) + q(t,b)$ implies that $0 = 0+t-b$. So $t = b \in A$.

Case 3: If $t \leq a \leq b$, then the equation $q(a,b) = q(a,t) + q(t,b)$ implies that $0 = a-t+0$, Hence $t = a \in A$.

If $a < b$, then we have the following three cases.

Case 4: If $b \leq t \leq a$, it follows that $b-a = b-t+t-a$, so the equation $q(a,b) = q(a,t)+q(t,b)$ is satisfied. Thus $t \in A$.

Case 5: If $b < a \leq t$, then the equation $q(a,b) = q(a,t)+q(t,b)$ implies that $a-b = 0+t-b$. Thus $t = b \in A$.

Case 6: If $t \leq b < a$, then we have $a-b = a-t+0$ from equation $q(a,b) = q(a,t)+q(t,b)$. Hence $t = b \in A$.

The following result can be compared to Proposition 2.2.1.

Proposition 3.2.1. *Let (M,q) be a T_0-quasi-metric space and $A \subseteq M$. If A is in-gated in (M,q), then A is also strongly convex in (M,q).*

Proof. Let A be an in-gated subset of (M,q). Then we have to show that A is strongly convex in (M,q), that is, for $u,v \in M$

$$\langle u,v\rangle_q = \{t \in M : q(u,v) = q(u,t) + q(t,v)\} \subseteq A$$

whenever $u, v \in A$.
Let $z \in \langle u, v\rangle_q$, then $z \in M$ with $q(u,v) = q(u,z) + q(z,v)$.
Let $z^{'}$ be an in-gate in A for q with respect to z, then

$$q(u,v) = q(u,z^{'}) + q(z^{'},z) + q(z,v).$$

It follows that

$$q(u,v) \leq q(u,z^{'}) + q(z^{'},z) + q(z,z^{'}) + q(z^{'},v).$$

Applying the triangular inequality, we obtain that

$$q(u,v) = q(z^{'},z) + q(z,z^{'}) + q(u,z^{'}) + q(z^{'},v) \geq q(u,v) + q(z^{'},z) + q(z,z^{'}) \geq q(u,v).$$

Hence

$$q(u,v) = q(z^{'},z) + q(z,z^{'}) = q(u,v).$$

Thus $q(z^{'},z) + q(z,z^{'}) = 0$, since (M,q) is a T_0-quasi-metric space, then $z = z^{'} \in A$. $\square$

3.3. Externally q-hyperconvex quasi-pseudometric spaces

In this section, we present the concept of external q-hyperconvexity in quasi-pseudometric spaces and we provide some characterizations thereof.

We begin by recalling the definition of an externally q-hyperconvex quasi-pseudometric space.

Definition 3.3.1. *(Compare to Definition 2.1.6) Let (M,q) be quasi-pseudometric space and E a non-empty bounded subset of M. Then E is said to be* externally q-hyperconvex *if for any family of points $(u_i)_{i\in I}$ in M and family of real positive numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:*

If $q(u_i,u_j) \leq r_i + s_j$ for all $i,j \in I$, dist$(u_i,E) \leq r_i$ and dist$(E,u_i) \leq s_i$ for all $i \in I$, then

$$\bigcap_{i\in I} C_q(u_i,r_i) \cap C_{q^{-1}}(u_i,s_i) \cap E \neq \emptyset, \text{ for all } i \in I.$$

The set of all externally q-hyperconvex subsets of M will be denoted by $\mathcal{E}_q(M)$.

We give an example of an externally q-hyperconvex quasi-pseudometric space.

Example 3.3.1. *Let (M,q) be a quasi-pseudometric space. If P is a nonempty externally q-hyperconvex subset of M and m is any point in M such that dist$(m,P) = r$ and dist$(P,m) = s$. Then by application of externally q-hyperconvexity of P to the double ball $(C_q(u,r); C_{q^{-1}}(u,s))$, we say that there is a point $z \in C_q(u,r) \cap C_{q^{-1}}(u,s) \cap P$. Therefore $q(u,z) =$ dist(u,P) and $q(z,u) =$ dist(P,u).*

Lemma 3.3.1. *[16, Lemma 4] Let (M,q) be a q-hyperconvex space and let u be a point in M. Moreover, let $\emptyset \neq B = \bigcap_{i\in I}(C_q(u_i,r_i) \cap C_{q^{-1}}(u_i,s_i)$ where $(u_i)_{i\in I}$ is a nonempty family of points in M and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ are two families of nonnegative real numbers. Then there is a point $z \in B$ such that dist$(u,B) = q(u,z)$ and dist$(B,u) = q(z,u)$.*

Proof. Evidently $[(C_q(u_i,r_i)_{i\in I}, (C_q(u, \text{dist}(u,B)+\epsilon)_{\epsilon>0}; (C_{q^{-1}}(u_i,s_i)_{i\in I}, (C_{q^{-1}}(u, \text{dist}(B,u)+\epsilon)_{\epsilon>0})]$ satisfies the mixed binary intersection property. Hence by the q-hyperconvexity of (M,q) there is

$$z \in B \cap (C_q(u, \text{dist}(u,B)) \cap (C_q(u, \text{dist}(B,u)).$$

Thus there is a point $z \in B$ such that $\text{dist}(u,B) = \text{dist}(u,z)$ and $\text{dist}(B,u) = \text{dist}(z,u)$.

□

Lemma 3.3.2. *[16, Lemma 5] Let (M,q) be a q-hyperconvex quasi-pseudometric space. Assume that P is an externally q-hyperconvex subset of M and assume that B is a q-admissible subset of (M,q) such that $P \cap B \neq \emptyset$. Then $P \cap B$ is externally q-hyperconvex relative to M.*

Proof. Suppose $(u_\alpha)_{\alpha\in S}$ is a family of nonempty points in M and $(r_\alpha)_{\alpha\in S}$ and $(s_\alpha)_{\alpha\in S}$ are two families of nonnegative real numbers such that $q(u_\alpha, u_\beta) \leq r_\alpha + s_\beta$, $\text{dist}(u_\alpha, B\cap P) \leq r_\alpha$ and $\text{dist}(B\cap P, u_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$. Since B is q-admissible, then $\bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i))$ with $u_i \in M$ and $r_i, s_i \geq 0$ for all $i \in I$. Due to the fact that $\text{dist}(u_\alpha, B) \leq \text{dist}(u_\alpha, B\cap P) \leq r_\alpha$ and $\text{dist}(B, u_\alpha) \leq \text{dist}(B\cap P, u_\alpha) \leq s_\alpha$ for all $\alpha \in S$, it follows that for each $i \in I$, $p \in B$ and $u \in M$ such that $\text{dist}(u, B) = q(u,p)$ and $\text{dist}(B, u) = q(p, u)$ we have

$$q(u_\alpha, u_i) \leq q(u_\alpha, p) + q(p, u_i) \leq r_\alpha + s_i$$

and

$$q(u_i, u_\alpha) \leq q(u_i, p) + q(p, u_\alpha) \leq r_i + s_\alpha.$$

Furthermore, since for each $i \in I$, $B \subseteq C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)$, and because $B \cap P \neq \emptyset$, we have that $q(u_i, u_j) \leq r_i + s_i$ whenever $i, j \in I$. Trivially we have $\text{dist}(u_\alpha, P) \leq r_\alpha$ and $\text{dist}(P, u_\alpha) \leq s_\alpha$ for all $\alpha \in S$.
Thus by external q-hyperconvexity of P, we can deduce that

$$[\bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i))] \cap [\bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \cap P]$$

$$= \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \cap (P \cap B) \neq \emptyset.$$

□

In the following theorem, we show that the intersection of a descending family of externally q-hyperconvex nonempty subspaces of a bounded q-hyperconvex T_0-quasi-metric space is nonempty and externally q-hyperconvex.

Theorem 3.3.1. *[16, Theorem 5] Let (M, q) be a bounded q-hyperconvex T_0-quasi-metric space. Furthermore, let $(M_i)_{i\in I}$ be a nonempty descending family of externally q-hyperconvex subsets of M, where I is assumed to be totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ if and only if $M_{i_2} \subseteq M_{i_2}$. Then $\bigcap_{i\in I} M_i$ is nonempty and externally q-hyperconvex relative to M.*

Proof. Theorem 3.1.1 implies that $H = \bigcap_{i\in I} M_i \neq \emptyset$. To show that H is externally q-hyperconvex, we assume $(u_i)_{i\in I}$ is a family of nonempty points in M and assume $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ are two families of nonnegative real numbers such that $q(u_\alpha, u_\beta) \leq r_\alpha + s_\beta$, and $\text{dist}(u_\alpha, H) \leq r_\alpha$ and $\text{dist}(H, u_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$. Given that M is q-hyperconvex, we know that $B := \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \neq \emptyset$.

Also, since $\mathrm{dist}(u_\alpha, H) \leq r_\alpha$ and $\mathrm{dist}(H, u_\alpha) \leq s_\alpha$ for all $\alpha \in S$, we have $\mathrm{dist}(u_\alpha, M_i) \leq r_\alpha$ and $\mathrm{dist}(M_i, u_\alpha) \leq s_\alpha$ whenever $i \in I$, hence by external q-hyperconvexity of M_i, we conclude that $B \cap M_i \neq \emptyset$ for all $i \in I$.

By Lemma 3.3.2 $(B \cap M_i)_{i\in I}$ is a descending chain of nonempty externally q-hyperconvex subsets of (M, q) such that by applying Theorem 3.1.1 again we obtain that $\bigcap_{i\in I}(B \cap M_i) = (B \cap H) \neq \emptyset$.

□

Proposition 3.3.1. *Let (M, q) be a q-hyperconvex T_0-quasi-metric space and $A \subseteq M$. Let $(u_i)_{i\in I}$ be a family of points in M and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative real numbers . If $A = \bigcap_{i\in I} C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)$ and $r, s \geq 0$, then*

$$C_q(A, s) \cap C_{q^{-1}}(A, r) = C_q(u_i, r_i + s) \cap C_{q^{-1}}(u_i, s_i + r) \in \mathcal{A}_q.$$

Proof. Let $v \in C_q(A, s) \cap C_{q^{-1}}(A, r)$, then $\mathrm{dist}(A, v) \leq s$ and $\mathrm{dist}(v, A) \leq r$. Furthermore, we obtain that $q(t, v) \leq s$ and $q(v, t) \leq r$ for some $t \in A$. Hence

$$q(u_i, v) \leq q(u_i, t) + q(t, v) \leq r_i + s$$

and

$$q(v, u_i) \leq q(v, t) + q(t, u_i) \leq r + s_i$$

whenever $i \in I$.
Therefore,

$$v \in \bigcap_{i\in I} C_q(u_i, r_i + s) \cap C_{q^{-1}}(u_i, s_i + r).$$

Conversely, if we let $w \in \bigcap_{i\in I} C_q(u_i, r_i + s) \cap C_{q^{-1}}(u_i, s_i + r)$, then

$$q(u_i, w) \leq r_i + s$$

and

$$q(w, u_i) \leq s_i + r$$

whenever $i \in I$.
By the q-hyperconvexity of (M, q), we obtain that

$$\emptyset \neq C_q(w, r) \cap C_{q^{-1}}(w, s) \cap C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i) = C_q(w, r) \cap C_{q^{-1}}(w, s) \cap A.$$

Let $t \in C_q(w, r) \cap C_{q^{-1}}(w, s) \cap A$, then $q(w, t) \leq r$ and $q(t, w) \leq s$. Hence,

$$w \in C_q(A, s) \cap C_{q^{-1}}(A, r).$$

□

Proposition 3.3.2. *(Compare [19, Lemma 2.3]) Let (M, q) be a q-hyperconvex T_0-quasi-metric space and $A \subseteq M$. Then A is weakly externally q-hyperconvex if and only if for any $u \in M$ and if we set dist$(A, u) = s$ and dist$(u, A) = r$, the following conditions hold:*

(i) The set $C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A$ is externally q-hyperconvex in A.
(ii) For any $v \in A$, there exists $a, a' \in C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A$ with

$$q(u, v) = q(u, a) + q(a, v)$$

and

$$q(v, u) = q(v, a') + q(a', u).$$

Proof. Let us assume that A is weakly externally hyperconvex. Thus, $C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A$ is externally q-hyperconvex in A as an intersection of a q-admissible and an externally q-hyperconvex subsets of a q-hyperconvex T_0-quasi-metric space (M, q).
Moreover, for some $v \in A$ and because dist$(A, u) = s$ and dist$(u, A) = r$, we obtain that $q(v, u) \geq s$ and $q(u, v) \geq r$.
Therefore,

$$q(u, v) = q(u, v) - r + r$$

and

$$q(v, u) = s + q(v, u) - s.$$

By weak externally q-hyperconvex of A, there is some

$$a \in C_q(u, q(u, v) - r) \cap C_{q^{-1}}(v, r) \cap A$$

and some

$$a' \in C_{q^{-1}}(u, q(v, u) - s) \cap C_q(v, s) \cap A.$$

Hence

$$q(u, v) \leq q(u, a) + q(a, v) \leq q(u, v) - r + r = q(u, v)$$

and

$$q(v, u) \leq q(v, a') + q(a', u) \leq s + q(v, u) - s = q(v, u).$$

Thus,

$$q(u, v) = q(u, a) + q(a, v)$$

and

$$q(v, u) = q(v, a') + q(a', u).$$

Conversely, if we assume that (i) and (ii) are satisfied, then we show that A is weakly externally q-hyperconvex. Let $(u_i)_{i \in I}$ be a family of points in M and $(r_i)_{i \in I}$ and $(s_{i \in I})_{i \in I}$ be two families of nonnegative real numbers, such that $q(u_i, u_j) \leq r_i + s_i$, with $\text{dist}(u_i, A) = r_i$, and $\text{dist}(A, u_i) = s_i$. Then by q-hyperconvexity of M we have

$$\bigcap_{i \in I} (C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \neq \emptyset.$$

If $u \in \bigcap_{i \in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i))$, then for $u_i \in A$, we have

$$\text{dist}(u, C_q(u, \text{dist}(u, A)) \cap C_{q^{-1}}(u, \text{dist}(A, u)) \cap A) \leq r_i$$

and

$$\text{dist}(C_q(u, \text{dist}(u, A)) \cap C_{q^{-1}}(u, \text{dist}(A, u)) \cap A, u) \leq s_i.$$

It follows that

$$\emptyset \neq \Big(A \cap C_q(u, \text{dist}(u, A)) \cap C_{q^{-1}}(u, \text{dist}(A, u))\Big) \cap \Big(\bigcap_{i \in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i))\Big).$$

If $u \in M$, $\text{dist}(A, u) = s \leq r_1$ and $\text{dist}(u, A) = r \leq r_2$, then

$$C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A = C_q(C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A, s - r_1) \cap C_{q^{-1}}(C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A, r - r_1).$$

Thus, $C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A$ is externally q-hyperconvex by Proposition 3.3.1 and Lemma 3.4.2. Moreover, by (ii) for $u_i \in A$, $q(u, u_i) \leq s + r_i$ and $q(u_i, u) \leq r_i + s_i$, we have

$$\text{dist}(u_i, C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A) \leq s_i$$

and

$$\text{dist}(C_q(u, s) \cap C_{q^{-1}}(u, r) \cap A, u_i) \leq r_i.$$

Hence

$$A \cap \Big(C_q(u, s) \cap C_{q^{-1}}(u, r)\Big) \cap \Big(\bigcap_{i \in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i))\Big) \neq \emptyset.$$

□

3.4. Weakly externally q-hyperconvex subsets

In this section, we study the notion of weakly externally q-hyperconvex spaces which will be useful in the following chapter when amalgamating (gluing) q-hyperconvex quasi-pseudometric spaces.

We begin by recalling the definition of a weakly externally q-hyperconvex space and give some examples.

Definition 3.4.1. *[21, Definition 4.1] Let (M, q) be a quasi-pseudometric space and P a subspace of (M, q). Then P is said to be* weakly externally q -hyperconvex *relative to M if P is externally q-hyperconvex relative to $P \cup \{z\}$ whenever $z \in M$. For any family $(u_i)_{i\in I}$ of points in M with at most one of the points lying in P, and two families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ such that $q(u_i, u_j) \leq r_i + s_j$, with dist$(u_i, P) \leq r_i$ and dist$(P, u_i) \leq s_i$ upon the condition that u_i does not lie in P, thus it follows that $\bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \cap P \neq \emptyset$.*

We denote the collection of all weakly externally q-hyperconvex subsets of M by $\mathcal{W}_q(M)$.

Example 3.4.1. *Let $M = [0, \infty)$ be the set of nonnegative real numbers equipped with the T_0-quasi-pseudometric space $w(u, v) = \max\{u - v, 0\}$. Then $H = [0, 1] \subseteq M$ is weakly externally q-hyperconvex relative to M.*

Proof. We know that from [14, Corollary 1], (M, w) is a q-hyperconvex space. Note that $C_w(u, \delta) = [u - \delta, \infty) \cap [0, \infty)$ and $C_{w^{-1}}(u, \delta) = (\infty, u + \delta] \cap [0, \infty)$ for all $u \in [0, \infty)$ and $\delta \geq 0$. For any point $u \in M$, let us assume that r_u and s_u are nonnegative real numbers such that $w(u, v) \leq r_u + r_v$ whenever $u, v \in M$ and dist$(u, H) \leq r_u$ and dist$(H, u) \leq s_u$ for all $u > 1$. If $\bigcap_{u\in U}(C_w(u, r_u) \cap C_{w^{-1}}(u, s_u)) \cap H \neq \emptyset$ for some nonempty finite subset U of M. Then we obtain that

$$\max\{u - r_u : u \in U\} > \min\{u + r_u : u \in U\}.$$

Hence, there are points $u, v \in U$ such that $u - r_u > v + s_v$, that is, $(C_w(u, r_u) \cap C_{w^{-1}}(v, s_u)) \cap H \neq \emptyset$. Furthermore, $u > v$. Thus $w(u, v) = u - v > r_u + s_v$ is a contradiction. Moreover, $\bigcap_{u\in U}(C_w(u, r_u) \cap C_{w^{-1}}(v, s_u)) \cap H \neq \emptyset$ whenever U is a finite nonempty subset of M. Because for any $u \in M$, $C_w(u, r_u) \cap C_{w^{-1}}(u, s_u) \cap H$ is compact with respect to the topology $\tau(w^s)$ of $\mathbb{R}$. Thus $\bigcap_{u\in M}(C_w(u, r_u) \cap C_{w^{-1}}(u, s_u)) \cap H \neq \emptyset$.

□

Example 3.4.2. *Consider the q-hyperconvex T_0-quasi-metric space $(\mathbb{R}, w)$ where $w(u,v) = \max\{u-v, 0\}$. Then $A = C_w(2,0) \cap C_{w^{-1}}(2,0)$ is not weakly externally q-hyperconvex relative to $[0,\infty)$ but it is externally q-hyperconvex relative to M.*

Proof. Firstly we consider that $A = C_w(2,0) \cap C_{w^{-1}}(2,0)$. It can easily be shown that A is externally q-hyperconvex relative to M. Furthermore, for any $u \in \mathbb{R}$ with $u \neq 2$, we set $r_u = \frac{1}{4}$ and $s_u = \frac{3}{4}$. Then $w(u,v) \leq 1 = \frac{1}{4} + \frac{3}{4}$ and $\mathrm{dist}(u, \{2\}) \leq r_u$ for all $u \in \mathbb{R}$ with $u \neq 2$. However,

$$\bigcap_{u \in \mathbb{R}, u \neq 2} (C_w(u, r_u) \cap C_{u^{-1}}(u, s_u)) \cap A \subseteq \left(C_w\left(-\frac{7}{4}, \frac{1}{4}\right) \cap C_{w^{-1}}\left(-\frac{9}{4}, \frac{3}{4}\right)\right) \cap A = \emptyset,$$

because

$$\left(C_w\left(-\frac{7}{4}, \frac{1}{4}\right) \cap C_{w^{-1}}\left(-\frac{9}{4}, \frac{3}{4}\right) \cap A = [-2, \infty) \cap (-\infty, -\frac{3}{2}\right] \cap A = \left[-2, -\frac{3}{2}\right] \cap \{2\} \neq \emptyset.$$

Hence A is not weakly externally q-hyperconvex relative to $\mathbb{R}$.

□

Lemma 3.4.1. *[16, Lemma 6] Let (M,q) be a q-hyperconvex quasi-pseudometric space. Suppose that E is an externally q-hyperconvex subset of (M,q) and suppose that A is a q-admissible subset of (M,q) such that $E \cap A \neq \emptyset$. Then $E \cap A$ is externally q-hyperconvex relative to M.*

Lemma 3.4.2. *Let (M,q) be a q-hyperconvex quasi-pseudometric space. If $A \in \varepsilon_q(M)$, then $C_q(A,r) \cap C_{q^{-1}}(A,s) \in \varepsilon_q(M)$ with $r, s \geq 0$.*

Proof. Let A be an externally q-hyperconvex subset of M and $C := C_q(A,r) \cap C_{q^{-1}}(A,s)$. Suppose a given family $(u_i)_{i \in I}$ of points in M and families of nonnegative real numbers $(ri)_{i \in I}$ and $(s_i)_{i \in I}$ satisfy $q(u_i, u_j) \leq r_i + s_j$, whenever $i, j \in I$ and $\mathrm{dist}(u_i, C) \leq r_i$ and $\mathrm{dist}(C, u_i) \leq s_i$, whenever $i \in I$.
Then we observe for $a \in A$, we have

$$\mathrm{dist}(u_i, A) \leq q(u_i, a) + \mathrm{dist}(a, A) \leq r_i + s$$

and

$$\mathrm{dist}(A, u_i) \leq \mathrm{dist}(A, a) q(a, u_i) \leq s_i + r,$$

whenever $i \in I$. By external q-hyperconvexity of A we have

$$\bigcap_{i \in I} (C_q(u_i, r_i + s) \cap C_{q^{-1}}(u_i, s_i + r)) \cap A \neq \emptyset.$$

Let $v \in \bigcap_{i\in I}(C_q(u_i, r_i + s) \cap C_{q^{-1}}(u_i, s_i + r)) \cap A$, then

$$q(u_i, v) \leq r_i + s$$

and

$$q(v, u_i) \leq s_i + r.$$

Hence

$$\emptyset \neq \bigcap_{i\in I}(C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i) \cap (C_q(v, r) \cap C_{q^{-1}}(v, s))) \subseteq (C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i) \cap C.$$

□

Lemma 3.4.3. *(Compare [19, Lemma 2.4]) Let (M, q) be a q-hyperconvex T_0-quasi-metric space and let A be a weakly externally q-hyperconvex subset of M. Let $r, s \geq 0$. Then there exists a retraction $\varphi : C_q(A, s) \cap C_{q^{-1}}(A, r) \to A$ with $q(u, \varphi(u)) \leq s$ and $q(\varphi(u), u) \leq r$ whenever $u \in C_q(A, s) \cap C_{q^{-1}}(A, r)$.*

Proof. Consider
$\mathcal{F} = \{(C, \varphi) : A \subseteq C \subseteq C_q(A, s) \cap C_{q^{-1}}(A, r)$ and $\varphi : C \to A$ is a retraction such that $q(u, \varphi(u)) \leq s$ and $q(\varphi(u), u) \leq r$ whenever $u \in C\}$. Let Id denote the identity map on A. Then $(A, Id) \in \mathcal{F}$. Thus $\mathcal{F} \neq \emptyset$. If we partially ordered $\mathcal{F}$ by $((C_1, \varphi_1) \preceq (C_2, \varphi_2)$ if and only if $C_1 \subseteq C_2$ and φ_2 is an extension of $\varphi_1)$, then each chain of $(\mathcal{F}, \preceq)$ is bounded above. Hence, by Zorn's lemma, $\mathcal{F}$ has maximum element. Let the maximum element of $\mathcal{F}$ be $(\overline{C}, \overline{\varphi})$. We have to illustrate that $\overline{C} = C_q(A, s) \cap C_{q^{-1}}(A, r)$.
Suppose that there is some $u \in \overline{C}$ with $u \notin \overline{C}$. Then for any $v \in \overline{C}$ if we set $r_v = q(u, v)$ and $s_v = q(v, u)$.
Then

$$q(u, \overline{\varphi}(v)) \leq q(u, v) + q(v, \overline{\varphi}(v)) \leq r_v + s$$

and

$$q(\overline{\varphi}(v), u) \leq q(\overline{\varphi}(v), v) + q(v, u) \leq r + s_v.$$

By the weakly externally q-hyperconvexity of A, we obtain that

$$u \in C_q(u, s) \cap C_{q^{-1}}(u, r) \cap [\bigcap_{v\in\overline{C}} C_q(\overline{\varphi}(v), s_v) \cap C_{q^{-1}}(\overline{\varphi}(v), r_v)] \cap A.$$

Let $v \in \overline{C}$, we define $\varphi' : \overline{C} \cup \{w\} \to A$ by $\varphi'(v) = \overline{\varphi}(v)$ if $v \in \overline{C}$ and $\varphi'(v) = w$.
Hence, for $v \in \overline{C}$, we have

$$q(\varphi'(v), v) = q(\overline{\varphi}(v), v) \leq r$$

and

$$q(v, \varphi'(v)) = q(v, \overline{\varphi}(v) \leq s.$$

Thus, the pair $(\overline{C} \cup \{w\}, \varphi')$ contradicts the maximality of $(\overline{C}, \overline{\varphi})$ in $\mathcal{F}, \preceq$. Therefore $\overline{C} = C_q(A, s) \cap C_{q^{-1}}(A, r)$.

□

Lemma 3.4.4. *Let (M, q) be a q-hyperconvex T_0-quasi-metric space and let A be a weakly externally q-hyperconvex subset of M. Then for any $r, s \geq 0$, and for every $u \in M$, we have that $C_q(u, dist(A, u)) \cap C_{q^{-1}}(u, dist(u, A)) \cap A$ is an externally q-hyperconvex subset of M. Furthermore, the set $C_q(u, dist(u, C)) \cap C_{q^{-1}}(u, dist(C, u)) \cap C$ is also externally q-hyperconvex, where $C := C_q(A, s) \cap C_{q^{-1}}(A, r)$, $r = dist(u, C)$ and $s = dist(C, u)$.*

Proof. Let $u \in M$. Consider a family of points $(u_i)_{i \in I}$ in $C_q(A, s) \cap C_{q^{-1}}(A, r)$ and two families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ with $q(u_i, u_j) \leq r_i + s_j$, $q(u_i, u) \leq s + r_i$ and $q(u, u_i) \leq r + s_i$ such that

$$\text{dist}(C_q(A, s) \cap C_{q^{-1}}(A, r), u) = s$$

and

$$\text{dist}(u, C_q(A, s) \cap C_{q^{-1}}(A, r)) = r.$$

By Lemma 3.4.3, there is a retraction $\varphi : C_q(A, s) \cap C_{q^{-1}}(A, r) \to A$ such that $q(v, \varphi(v)) \leq s$ and $q(\varphi(v), v) \leq r$ whenever $v \in C_q(A, s) \cap C_{q^{-1}}(A, r)$. Then it follows that

$$q(\varphi(u_i), u) \leq q(\varphi(u_i), u_i) + q(u_i, u) \leq r + s + r_i$$

and

$$q(u, \varphi(u_i)) \leq q(u, u_i) + q(u_i, \varphi(u_i)) \leq r + s_i + s$$

whenever $i \in I$. Therefore, there exists

$$v \in C_{q^s}(u, r + s) \cap \left[\bigcap_{i \in I} C_q(\varphi(u_i), r_i) \cap C_{q^{-1}}(\varphi(u_i), s_i) \right] \cap A.$$

Hence, $q(u, v) \leq r + s$, $q(v, u) \leq r + s$ and

$$q(u_i, v) \leq q(u_i, \varphi(u_i)) + q(\varphi(u_i), v) \leq s + r_i$$

and

$$q(v, u_i) \leq q(v, \varphi(u_i)) + q(u_i, \varphi(u_i)) \leq r + s_i$$

whenever $i \in I$.
By the q-hyperconvexity of (M,q), we have

$$\emptyset \neq C_q(u,r) \cap C_{q^{-1}}(u,s) \cap \left[\bigcap_{i\in I} C_q(u_i,r_i) \cap C_{q^{-1}}(u_i,s_i)\right] \cap C_q(v,s) \cap C_{q^{-1}}(v,r)$$

$$\subseteq C_q(u,r) \cap C_{q^{-1}}(u,s) \cap \left[\bigcap_{i\in I} C_q(u_i,r_i) \cap C_{q^{-1}}(u_i,s_i)\right] \cap C_q(A,s) \cap C_{q^{-1}}(A,r)$$

Therefore, $C_q(A,s) \cap C_{q^{-1}}(A,r)$ is externally q-hyperconvex in M.
Moreover, we have
$C_q(u,r) \cap C_{q^{-1}}(u,s) \cap C_q(A,s) \cap C_{q^{-1}}(A,r) = C_q(u,r) \cap C_{q^{-1}}(u,s) \cap \Big[C_q[C_q(u,\text{dist}(u,A)) \cap$
$C_{q^{-1}}(u,\text{dist}(A,u)) \cap A, r] \cap C_{q^{-1}}[C_q(u,\text{dist}(u,A)) \cap C_{q^{-1}}(u,\text{dist}(A,u)) \cap A, s]\Big]$.
If $C_q(u,\text{dist}(u,A)) \cap C_{q^{-1}}(u,\text{dist}(A,u)) \cap A$ is externally q-hyperconvex, then $C_q(u,r) \cap C_{q^{-1}}(u,s) \cap C_q(A,s) \cap C_{q^{-1}}(A,r)$ is also externally q-hyperconvex. □

In the following theorem, we show that the intersection of a descending family of externally q-hyperconvex nonempty subspaces of a bounded q-hyperconvex T_0-quasi-metric space is nonempty and externally q-hyperconvex.

Theorem 3.4.1. *[16, Theorem 5] Let (M,q) be a bounded q-hyperconvex T_0-quasi-metric space. Furthermore, let $(M_i)_{i\in I}$ be a nonempty descending family of externally q-hyperconvex subsets of M, where I is assumed to be totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ if and only if $M_{i_2} \subseteq M_{i_2}$. Then $\bigcap_{i\in I} M_i$ is nonempty and externally q-hyperconvex relative to M.*

Proof. Theorem 3.1.1 implies that $H = \bigcap_{i\in I} M_i \neq \emptyset$. To show that H is externally q-hyperconvex, we assume $(u_i)_{i\in I}$ is a family of nonempty points in M and assume $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ are two families of nonnegative real numbers such that $q(u_\alpha, u_\beta) \leq r_\alpha + s_\beta$, and $\text{dist}(u_\alpha, H) \leq r_\alpha$ and $\text{dist}(H, u_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$.
Given that M is q-hyperconvex, we know that $B := \bigcap_{\alpha\in S}(C_q(u_\alpha, r_\alpha) \cap C_{q^{-1}}(u_\alpha, s_\alpha)) \neq \emptyset$.

Also, since $\text{dist}(u_\alpha, H) \leq r_\alpha$ and $\text{dist}(H, u_\alpha) \leq s_\alpha$ for all $\alpha \in S$, we have $\text{dist}(u_\alpha, M_i) \leq r_\alpha$ and $\text{dist}(M_i, u_\alpha) \leq s_\alpha$ whenever $i \in I$, hence by external q-hyperconvexity of M_i, we conclude that $B \cap M_i \neq \emptyset$ for all $i \in I$.

By Lemma 3.3.2 $(B \cap M_i)_{i\in I}$ is a descending chain of nonempty externally q-hyperconvex subsets of (M,q) such that by applying Theorem 3.1.1 again we obtain that $\bigcap_{i\in I}(B \cap M_i) = (B \cap H) \neq \emptyset$.

□

Lemma 3.4.5. *Let (M,q) be a T_0-quasi-metric space. If $N \in \mathcal{W}_q(M)$ and $A \in \mathcal{E}_q(M)$, then $A \in \mathcal{W}_q(M)$.*

Lemma 3.4.6. *Let (M,q) be a T_0-quasi-metric space. Let A be an externally q-hyperconvex subset of M and let Y be a weakly externally q-hyperconvex subset of M such that $A \cap Y \neq \emptyset$. If $(u_i)_{i\in I}$ is a family of points in Y with $q(u_i, uj) \leq r_i + s_j$ and dist$(u_i, A) \leq r_i$ and dist$(A, u_i) \leq s_i$ whenever $i \in I$. Then there is some*

$$a \in A \cap \bigcap_{i\in I} C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)$$

and

$$v \in Y \cap \bigcap_{i\in I} C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)$$

with

$$q(a,v) \leq s \leq dist\Big(v_0, \bigcap_{i\in I} C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)\Big)$$

and

$$q(v,a) \leq r \leq dist\Big(\bigcap_{i\in I} C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i), v_0\Big)$$

whenever $v_0 \in A \cap Y$.

Proposition 3.4.1. *Let (M,q) be a T_0-quasi-metric space. Let A be an externally q-hyperconvex subset of M and let Y be a weakly externally q-hyperconvex subset of M such that $A \cap Y \neq \emptyset$. Then $A \cap Y$ is externally q-hyperconvex in Y. Furthermore, $A \cap Y$ is weakly externally q-hyperconvex in M.*

Corollary 3.4.1. *Let (M,q) be a T_0-quasi-metric space. Let A be an externally q-hyperconvex subset of M and let Y be a weakly externally q-hyperconvex subset of M such that $A \cap Y \neq \emptyset$. Then*

$$dist(u, A) = dist(u, A \cap Y)$$

and

$$dist(A, u) = dist(A \cap Y, u)$$

whenever $u \in Y$.

Lemma 3.4.7. *Let (M,q) be a T_0-quasi-metric space. Let A be an externally q-hyperconvex subset of M and let Y be a weakly externally q-hyperconvex subset of M such that $A \cap Y \neq \emptyset$. Then there exists $a \in A$ and $\imath \in Y$ such that*

$$dist(a, v) = dist(A, Y)$$

and

$$dist(v, a) = dist(Y, A).$$

Proof. Let $v \in Y$. We set $s := \text{dist}(A, v)$ and $r := \text{dist}(v, A)$. For ant $n \in \mathbb{N}$, one can see that

$$C_q(A, s + 2^{-n}) \cap C_{q^{-1}}(A, r + 2^{-n})$$

is externally q-hyperconvex by Lemma 3.4.2 and

$$C_q(A, s + 2^{-n}) \cap C_{q^{-1}}(A, r + 2^{-n}) \cap Y$$

is weakly externally q-hyperconvex by Proposition 3.4.1.

If (v_n) is a sequence of points in $C_q(A, s+2^{-n}) \cap C_{q^{-1}}(A, r+2^{-n}) \cap Y$ by Lemma 3.4.1, there is some $v_{n+1} \in C_q(v_n, s + 2^{-(n+)}) \cap C_{q^{-1}}(v_n, r + 2^{-(n+1)}) \cap C_q(A, s + 2^{-(n+1)}) \cap C_{q^{-1}}(A, r + 2^{-(n+1)}) \cap Y$. Then we obtain that

$$\text{dist}(v_n, A) \leq s + 2^{-n}$$

and

$$\text{dist}(A, v_n) \leq r + 2^{-n}.$$

Furthermore, we obtain that

$$q(v_n, v_{n+1}) \leq 2^{-(n+1)}$$

and

$$q(v_{n+1}, v_n) \leq 2^{-(n+1)}.$$

Hence (v_n) is a q^s-Cauchy sequence. Thus, v_n is a convergent sequence in Y by q-hyperconvexity. If we assume that the sequence (v_n) converges to $v \in Y$, then $\text{dist}(v, A) \leq s$ and $\text{dist}(A, v) \leq r$. Since A externally q-hyperconvex is proximinal, there is $a \in A$ such that $q(a, v) \leq r = \text{dist}(v, A)$ and $q(v, a) \leq s = \text{dist}(A, v)$.

$\square$

4

On gluing of q-hyperconvex quasi-pseudometric spaces

In this chapter, we start our own investigations on gluing a family of a quasi-pseudometric space such that the resulting space preserves the same properties. We extend the results of Miesch [18] to the framework of q-hyperconvex quasi-pseudometric spaces. In particular, we will introduce the concept of gluing a family of q-hyperconvex quasi-pseudometric spaces while preserving their property of being q-hyperconvex. We study gluing a family of q-hyperconvex spaces along externally q-hyperconvex and weakly externally q-hyperconvex quasi-pseudometric subsets.

4.1. Gluing of q-hyperconvex quasi-pseudometric spaces along externally q-hyperconvex subsets

The theory of amalgamation of two finite T_0-quasi-metric spaces has been introduced in [17]. We introduce the concept of amalgamation of a family of quasi-pseudometric spaces in the following sense.

The following is the symmetric version of Definition 2.3.1.

Proposition 4.1.1. *Let (M_α, q_α) be a family of T_0-quasi-metric spaces and (A, q_A) be a T_0-quasi-metric space. If $A_\alpha \subseteq M_\alpha$ and we fix some isometry $\varphi_\alpha : A \to A_\alpha$ whenever $\alpha \in \Gamma$, then there exists a T_0-quasi-metric space $M = \bigsqcup_A M_\alpha$, the co-product of M_α amalgamated along A or φ_α such that for all $a \in A$, $\varphi_a(a)$ coincides with a in M. For $u \in M_\alpha$ and $v \in M_{\alpha'}$ with $\alpha \neq \alpha'$, we set the T_0-quasi-metric q on M by*

$$q(u, v) = \inf_{a \in A} \{q_\alpha(u, \varphi_\alpha(a)) + q_{\alpha'}(\varphi_{\alpha'}(a), v)\}$$

and

$$q(v,u)=\inf_{a\in A}\{q_{\alpha'}(\varphi_{\alpha'}(a),v)+q_\alpha(u,\varphi_\alpha(a)))\},$$

the subspaces M_α of M carry their T_0-quasi-metrics q_α, respectively.

Proof. For any $u,v,w\in M$, it is easily checked that $q(u,v)\le q(u,w)+q(w,v)$. Fundamentally, since all other nontrivial cases are analogous, we have to consider two cases where the path u,v,w crosses into the other subspaces twice in a row, or only once.
Case 1: If $u,v\in M_\alpha$ and $w\in M_{\alpha'}$ with $\alpha\neq\alpha'$, then we prove that

$$q_\alpha(u,v)\le q(u,w)+q(w,v).$$

In fact, we have

$$q(u,w)=q_\alpha(u,\varphi_\alpha(a))+q_{\alpha'}(\varphi_{\alpha'}(a),w)$$

and

$$q(w,v)=q_{\alpha'}(w,\varphi_{\alpha'}(b))+q_\alpha(\varphi_\alpha(b),w).$$

Then

$$q_\alpha(u,v)\le q_\alpha(u,\varphi_\alpha(a))+q_\alpha(\varphi_\alpha(a),\varphi_\alpha(b))+q_\alpha(\varphi_\alpha(b),v).$$

Moreover,

$$q_\alpha(u,v)\le q_\alpha(u,\varphi_\alpha(a))+q_{\alpha'}(\varphi_{\alpha'}(a),\varphi_{\alpha'}(b))+q_\alpha(\varphi_\alpha(b),v),$$

since

$$q_\alpha(\varphi_\alpha(a),\varphi_\alpha(b))=q_A(a,b)=q_{\alpha'}(\varphi_{\alpha'}(a),\varphi_{\alpha'}(b)).$$

It follows that

$$q_\alpha(u,v)\le q_\alpha(u,\varphi_\alpha(a))+q_{\alpha'}(\varphi_{\alpha'}(a),w)+q_{\alpha'}(w,\varphi_{\alpha'}(b))+q_\alpha(\varphi_\alpha(b),v).$$

Hence the inequality is satisfied.
Case 2: If $u\in M_\alpha$ and $v,w\in M_{\alpha'}$ with $\alpha\neq\alpha'$, then we are going to show that

$$q(u,v)\le q(u,w)+q'(w,v).$$

Suppose that

$$q(u,w)=q_\alpha(u,\varphi_\alpha(a))+q_{\alpha'}(\varphi_{\alpha'}(a),w).$$

Then

$$q(u,v)\le q_\alpha(u,\varphi_\alpha(a))+q_{\alpha'}(\varphi_{\alpha'}(a),v),$$

hence it follows that

$$q(u,v)=q_\alpha(u,\varphi_\alpha(a))+q_{\alpha'}(\varphi_{\alpha'}(a),w)+q_{\alpha'}(w,v)=q(u,w)+q_{\alpha'}(w,v).$$

Thus the inequality is satisfied, as well in this case. Obviously q satisfies the T_0 property. □

Remark 4.1.1. *If $\Gamma = \{1,2\}$ from Proposition 4.1.1, then we have [17, Example 5].*

Definition 4.1.1. *(Compare Definition 2.3.1) Let $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$ be a family of T_0-quasi-metric spaces and (A, q_A) be a T_0-quasi-metric spaces. If $A_\alpha \subseteq M_\alpha$ and we fix some isometry $\varphi_\alpha : A \to A_\alpha$ whenever $\alpha \in \Gamma$. If (M,q) is the coproduct of M_α amalgamated along A or φ_α, then we call the T_0-quasi-metric space (M,q) the gluing of $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$ along A or φ_α.*

Definition 4.1.2. *Let (M,q) be a quasi-pseudometric space. For any $A \subseteq M$ and $r \in [0,\infty)$ let*

$$C_q(A,r) = \{v \in M : dist(A,v) = \inf_{u\in A} q(u,v) \leq r\}$$

be the bicomplete-closed r-neighborhood of A.

The following theorem is the asymmetric version of a well-known theorem due to Bailon [3].

Theorem 4.1.1. *[16, Theorem 4.1] Let (M,q) be a bounded q-hyperconvex T_0-quasi-metric space. Moreover, let $(M_i)_{i\in I}$ be a descending family of nonempty externally q-hyperconvex subsets of M, where I is assumed to be totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ if and only if $M_{i_2} \subseteq M_{i_1}$. Then $\bigcap_{i\in I} M_i$ is nonempty and externally q-hyperconvex relative to M.*

Lemma 4.1.1. *If $(M_i)_{i\in I}$ is a family of pairwise intersecting bounded externally q-hyperconvex subsets of T_0-quasi-metric space (M,q) such that $\bigcap_{i\in J} M_i$ is nonempty and externally q-hyperconvex, whenever $J \subseteq I$ is finite, then the intersection $\bigcap_{i\in I} M_i$ is nonempty and externally q-hyperconvex.*

Proof. Consider the following set:
$A = \{K \subseteq I :$ for all J finite, J $\subseteq I$, $\bigcap_{i\in K\cup J} M_i$ is nonempty and externally q-hyperconvex$\}$.
Obviously, $\emptyset \in A$ and satisfies the hypothesis of Zorn's lemma because of Theorem 4.1.1. Let K be maximal in A. Then $K \cup \{i\} \in A$, whenever $i \in I$. Due to the maximality of J, we therefore, have $i \in K$, whenever $i \in I$. $\square$

Proposition 4.1.2. *Let (M,q) be a T_0-quasi-metric space. If Y is an externally q-hyperconvex subset of (M,q) and A is externally q-hyperconvex relative to Y, then A is also externally q-hyperconvex relative to M.*

Proof. Let $(u_i)_{i\in I}$ be a family of points in M and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of nonnegative real numbers such that $q(u_i,u_j) \leq r_i + s_j$, whenever $i,j \in I$ and $\text{dist}(u_i, A) \leq r_i$ and $\text{dist}(A, u_i) \leq s_i$, whenever $i \in I$. Then for $i \in I$, the set $A_i = (C_q(u_i,r_i) \cap C_{q^{-1}}(u_i,s_i) \cap Y)$

is an externally q-hyperconvex subset of M and Y by Lemma 3.4.1.
It is easy to see that $A_i \cap A \neq \emptyset$, whenever $i \in I$ and

$$A_i \cap A = (C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i)) \cap ((C_q(u_j, r_j) \cap C_{q^{-1}}(u_j, s_j)) \cap Y \neq \emptyset$$

by external q-hyperconvexity of Y.
Hence $A \bigcap_{i\in I} (C_q(u_i, r_i) \cap C_{q^{-1}}(u_i, s_i) = A \bigcap_{i\in I} A_i \neq \emptyset$ by Lemma 4.1.1. □

Proposition 4.1.3. *Let $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$ be a family of q-hyperconvex T_0-quasi-metric spaces and let A be an externally q-hyperconvex subset relative to $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$. If (M, q) is the T_0-quasi-metric obtained by gluing the metric spaces $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$ along the set A, then for $dist_\alpha(u, A) = r$ and $dist_\alpha(A, u) = s$ there exists $a \in A \cap [C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s)]$ such that*

$$q(u, v) = q(u, a) + q(a, v)$$

and

$$q(v, u) = q(v, a) + q(a, u),$$

whenever $u \in M_\alpha$ and $v \in M_{\alpha'}$ with $\alpha \neq \alpha'$.

Proof. Consider the set $A' = A \cap [C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s)] \neq \emptyset$.
Since for any $\alpha \in \Gamma$, we have

$$q_\alpha(u, a) = q_\alpha(u, a) - r + r$$

and

$$q_\alpha(a, u) = q_\alpha(u, a) - s + s,$$

then

$$A'' = A \cap [C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s)] \cap [C_{q_\alpha}(a, q_\alpha(a, u) - s) \cap C_{q_\alpha^{-1}}(a, q_\alpha(u, a) - r)] \neq \emptyset$$

by external q-hyperconvex of A.
Let $a' \in A''$. Then

$$q_\alpha(u, a') \leq q_\alpha(u, a) - r \text{ and } q_\alpha(a', a) \leq r \tag{4.1}$$

and

$$q_\alpha(a, a') \leq s \text{ and } q_\alpha(a', u) \leq q_\alpha(a, u) - s. \tag{4.2}$$

Moreover, for $a \in A' = A \cap [C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s)]$ and $a' \in A''$. From inequalities on 4.1 we have

$$q_\alpha(u, a') + q_{\alpha'}(a', v) \leq q_\alpha(u, a') + q_{\alpha'}(a', a) + q_{\alpha'}(a, v) \leq q_\alpha(u, a) + q_{\alpha'}(a, v).$$

Therefore,

$$q(u,v) = \inf_{a \in A'} \{q_\alpha(u,a) + q_{\alpha'}(a,v)\} = \text{dist}_\alpha(u, A') + \text{dist}_{\alpha'}(A', v).$$

Hence

$$q(u,v) = r + \text{dist}_{\alpha'}(A', v)$$

since $\text{dist}_{\alpha'}(u, A') = \text{dist}_\alpha(u, A) = r$.
By similar arguments and inequalities on 4.2 we have

$$q(u,v) = \text{dist}_{\alpha'}(v, A') + s.$$

By Lemma 3.4.1, it follows that A' is externally q-hyperconvex and A is externally q-hyperconvex relative to M_α, whenever $\alpha \in \Gamma$. Moreover, since $A' \subseteq A$, then by Proposition 4.1.2 we have A' is externally q-hyperconvex relative to M_α, whenever $\alpha \in \Gamma$. Thus

$$\emptyset \neq [C_{q_\alpha}(u,r) \cap C_{q_\alpha^{-1}}(u,s)] \cap [C_{q_\alpha}(v, \text{dist}_{\alpha'}(A', v)) \cap C_{q_\alpha^{-1}}(v, \text{dist}_{\alpha'}(v, A'))].$$

If $a \in C$, then

$$q(u,v) \leq q_\alpha(u,a) + q_\alpha(a,v) \leq \text{dist}_\alpha(u, A') + \text{dist}_{\alpha'}(A', v) \geq q(u,v).$$

Hence $q(u,v) = q(u,a) + q(a,v)$. Similarly, we have $q(v,u) = q(v,a) + q(a,u)$.

□

Corollary 4.1.1. *Let $(M_\alpha, q_\alpha)_{\alpha \in \Gamma}$ be a family of q-hyperconvex T_0-quasi-metric spaces and let A be an externally q-hyperconvex subset relative to $(M_\alpha, q_\alpha)_{\alpha \in \Gamma}$. Let $u \in M_\alpha$ and $dist_\alpha(u, A) = r \leq t_1$ and $dist_\alpha(A, u) = s \leq t_2$.*
Then for $\beta \neq \alpha$, we have

$$C_q(u,t_1) \cap C_{q^{-1}}(u,t_2) \cap M_\beta = C_{q_\beta}(C_{q_\alpha} \cap C_{q_\alpha^{-1}}(u,s) \cap A, r - t_1) \cap C_{q_\beta^{-1}}(C_{q^{-1}}(u,r) \cap C_{q_\alpha^{-1}}(u,s) \cap A, s - t_2).$$

Furthermore, $C_q(u,t_1) \cap C_{q^{-1}}(u,t_2) \cap M_\beta$ is externally q-hyperconvex relative to M_β.

Proof. Let $v \in C_{q_\beta}(C_{q_\alpha} \cap C_{q_\alpha^{-1}}(u,s) \cap A, r - t_1) \cap C_{q_\beta^{-1}}(C_{q^{-1}}(u,r) \cap C_{q_\alpha^{-1}}(u,s) \cap A, s - t_2)$. Then for $w \in C_{q^{-1}}(u,r) \cap C_{q_\alpha^{-1}}(u,s) \cap A$, we have $\text{dist}_\beta(w,v) \leq t_1 - r$ and $q_\alpha(u,w) \leq r$. Furthermore, we have

$$q(u,v) = \inf_{w \in C_{q_\alpha}(u,r) \cap A} \{q_\alpha(u,w) + q_\beta(w,v)\} \leq r + t_1 r = t_1.$$

So $v \in C_q(u,t_1) \cap M_\beta$. Similarly, it can be shown that $v \in C_{q^{-1}}(u,t_2) \cap M_\beta$. Hence $v \in C_q(u,t_1) \cap C_{q^{-1}}(u,t_2) \cap M_\beta$.

Let $v \in C_q(u, t_1) \cap C_{q^{-1}}(u, t_2) \cap M_\beta$. Then there exists $a \in C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s) \cap A$ such that $q(u, v) = q(u, a) + q(a, v)$ by Proposition 4.1.3. Thus

$$q_\beta(a, v) = q(a, v) \leq q(u, v) - q(u, a) \leq t_1 - r,$$

it follows that

$$\text{dist}_\beta(A, v) = \inf\{q_\beta(a, v) : a \in C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s) \cap A\} \leq t_1 + r.$$

Hence $v \in C_{q_\beta}(C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s) \cap A, r - t_1)$. Analogously, one sees that $v \in C_{q_\beta^{-1}}(C_{q_\alpha}(u, r) \cap C_{q_\alpha^{-1}}(u, s) \cap A, s - t_2)$.

□

4.2. Gluing of q-hyperconvex quasi-pseudometric spaces along weakly externally q-hyperconvex subsets

In this section, we study some properties on gluing q-hyperconvex quasi-pseudometric spaces along weakly externally q-hyperconvex subsets such that the resulting space preserves the property of q-hyperconvexity.

Proposition 4.2.1. *Let $(M_\alpha, q_\alpha)_{\alpha \in \Gamma}$ be a family of q-hyperconvex T_0-quasi-metric spaces and let A be a weakly externally q-hyperconvex subset relative to $(M_\alpha, q_\alpha)_{\alpha \in \Gamma}$. If (M, q) is the T_0-quasi-metric space obtained by gluing the family $(M_\alpha, q_\alpha)_{\alpha \in \Gamma}$ of q-hyperconvex T_0-quasi-metric spaces along the set A, then there is some point*

$$a \in C_q(u, \mathit{dist}(u, A)) \cap C_{q^{-1}}(u, \mathit{dist}(A, u)) \cap A$$

and

$$a' \in C_q(u', \mathit{dist}(u', A)) \cap C_{q^{-1}}(u', \mathit{dist}(A, u')) \cap A$$

such that

$$q(u, u') = q(u, a) + q(a, a') + q(a', u')$$

whenever $u \in M_\alpha$ and $M_{\alpha'}$.

Proof. Assume that A is weakly externally q-hyperconvex in M_α whenever $\alpha \in \Gamma$. Then there is some point

$$a \in C_q(u, \text{dist}(u, A)) \cap C_{q^{-1}}(u, \text{dist}(A, u)) \cap A$$

and

$$a' \in C_q(u', \text{dist}(u', A)) \cap C_{q^{-1}}(u', \text{dist}(A, u')) \cap A$$

such that

$$q(u,v) = q(u,a) + q(a,y))$$

and

$$q(v,u^{'}) = q(v,a^{'}) + q(a^{'},u^{'}))$$

by Proposition 3.3.2.
Let

$$C := C_q(u, \text{dist}(u,A)) \cap C_{q^{-1}}(u, \text{dist}(A,u)) \cap A$$

and

$$C^{'} := C_q(u^{'}, \text{dist}(u^{'},A)) \cap C_{q^{-1}}(u^{'}, \text{dist}(A,u^{'})) \cap A.$$

Then

$$q(u,u^{'}) = q(u,a) + q(a,a^{'}) + q(a^{'},u^{'}) \leq \text{dist}(u,A) + \text{dist}(C,C^{'}) + \text{dist}(A,u^{'}). \tag{4.3}$$

Moreover,

$$\text{dist}(u,A) + \text{dist}(C,C^{'}) + \text{dist}(A,u^{'}) \geq q(u,u^{'}) \tag{4.4}$$

By triangular inequality and taking the infimum on C and $C^{'}$.
Combining 4.3 and 4.4 we obtain

$$q(u,u^{'}) = q(u,a) + q(a,a^{'}) + q(a^{'},u^{'}).$$

□

Corollary 4.2.1. *Let $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$ be a family of q-hyperconvex T_0-quasi-metric spaces and let A be a weakly externally q-hyperconvex subset relative to $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$. If (M,q) is the T_0-quasi-metric space obtained by gluing the family of $(M_\alpha, q_\alpha)_{\alpha\in\Gamma}$ q-hyperconvex T_0-quasi-metric spaces along the set A, then for $\alpha \neq \alpha^{'}$, $u \in M_\alpha$, $u \in M_{\alpha'}$, dist$(u,A) \leq s$ and dist$(A,u) \leq r$, we have*
$C_q(u,s)) \cap C_{q^{-1}}(u,r) \cap M_{\alpha'} = C_{q'}[C_\alpha(u,s) \cap C_{\alpha'}(u,r) \cap A, s - dist(u,A)] \cap C_{q_{\alpha'}^{-1}}[C_q(u,s) \cap C_{\alpha'}(u,r) \cap A, r - dist(A,u)]$. Therefore $C_q(u,s)) \cap C_{q^{-1}}(u,r) \cap M_{\alpha'}$ is externally q-hyperconvex in $M_{\alpha'}$ whenever $C_{q_{\alpha'}}[C_\alpha(u,s) \cap C_{\alpha'}(u,r) \cap A, s - dist(u,A)] \cap C_{q_{\alpha'}^{-1}}[C_q(u,s) \cap C_{\alpha'}(u,r) \cap A, r - dist(A,u)]$ is externally q-hyperconvex in $M_{\alpha'}$.

5
Conclusion

In this MSc dissertation, we have successfully introduced the concept of gluing a family of pseudometric spaces. In particular, we have studied the gluing of a family of q-hyperconvex quasi-pseudometric along an externally q-hyperconvex and weakly externally q-hyperconvex quasi-pseudometric subset. Also, we introduced the concept of in-gated(out-gated) sets in quasi-pseudometric. In this part of the work, we present a summary of our results.

In the first section of our results, we presented the results of gluing q-hyperconvex quasi-pseudometric spaces along externally q-hyperconvex subsets, while preserving their properties as compared to that done in terms of metric spaces. We began by introducing the concept of gluing a family of q-hyperconvex quasi-pseudometric spaces while preserving their property of being q-hyperconvex. We then presented the results on gluing such a family of spaces along externally q-hyperconvex quasi-pseudometric subsets. While many ideas can be generalised from the metric to quasi-pseudometric setting, these generalisations were not easy due to the asymmetry property which required new variations of old arguments.

In the second section, we presented the concept of gluing q-hyperconvex quasi-pseudometric spaces along weakly externally q-hyperconvex subsets, while preserving their properties and extend the results from metric point of view to quasi-pseudometric settings.

Our conclusion leads us to some open problems encountered throughout the present investigations which we hope to study in future work.

Problem 1. *How can we characterize the concept of in-gated sets and out-gated sets on q-hyperconvex spaces?*

Problem 2. *How we can glue a family of q-hyperconvex spaces along in-gated subsets (or out-gated subsets) such that the resulting space preserves q-hyperconvexity?*

Problem 3. *What are the connections between metric gated sets and quasi-pseudometric in-gated (or out-gated) sets?*

Problem 4. *How can one extend the concept of gluing quasi-pseudometric spaces to the ultra-quasi-pseudometric space?*

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