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ORIGINAL PAPER

Power law slip boundary condition for Navier-Stokes equations: Discontinuous Galerkin schemes

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Abstract

This study deals with the numerical analysis of several discontinuous Galerkin (DG) methods for the resolution of the Navier-Stokes equations with power law slip boundary condition. The physical context corresponding to this problem is the description of a flow when a position and the direction slip boundary condition is taken into consideration. The main goal in this work is to examine the solvability, convergence of several DG methods, and to discuss their practical resolution by means of fixed point iterative algorithm. Theoretical findings are backed up by solid computational results.

Keywords Power law · Anisotropic slip boundary condition · DG approximations · Error estimates

Mathematics Subject Classification (2010) 65N30 · 35J85

1 Introduction

We consider the incompressible Navier-Stokes equations

$$-2\nu \operatorname{div} D(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega\,, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ is a bounded domain with the Lipschitz boundary $\partial\Omega$. where; $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \sum_{i=1}^{d} v_i \dfrac{\partial \boldsymbol{v}}{\partial x_i}$, is the convection term, ν is a positive parameter and represents the viscosity of the fluid. $\boldsymbol{u}$ is the velocity, the pressure is p and $\boldsymbol{f}$ is the external force acting on the fluid. The quantity $D\boldsymbol{u} = 1/2\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ is the symmetric part of the velocity gradient. For the boundary condition, we assume $\partial\Omega = \overline{\Gamma_0}\cup\overline{\Gamma}$, with $\Gamma_0\cap\Gamma = \emptyset$, where Γ_0 and Γ are nonempty open subsets of $\partial\Omega$. We note that with $\overline{A}$ being the closure of the set A. We assume the adhesive (homogeneous Dirichlet condition) on Γ_0, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{1.3}$$

while along Γ, the following nonlinear slip boundary condition is imposed

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0\,, \quad (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K^2\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{g} \quad \text{on } \Gamma\,, \tag{1.4}$$

where $\boldsymbol{n} : \Gamma \longrightarrow \mathbb{R}^d$ is the normal outward unit vector to Γ, and the Cauchy stress tensor is $\boldsymbol{T}(u, p) = -p\boldsymbol{I} + 2\nu D\boldsymbol{u}$, with $\boldsymbol{I}$ being the d-dimensional identity matrix. Thus $\boldsymbol{T}\boldsymbol{n}$ represents the stress vector, that is the product of the matrix $\boldsymbol{T} = (T_{ij})$ and the vector $\boldsymbol{n} = (n_i)$. Hence the tangential stress is $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} = \boldsymbol{T}\boldsymbol{n} - (\boldsymbol{T}\boldsymbol{n})\boldsymbol{n}$ while $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$. Here $|\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$ is the Euclidean norm and K is the anisotropic tensor, assumed to be uniformly positive definite, symmetric, $s \geq 1$ is a real number and $\boldsymbol{g}$ is the tangential surface traction. Such a boundary condition arises when the contact surface is

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lubricated with a thin layer of a non-Newtonian fluid. If $s = 2$ and $K = \boldsymbol{I}$ then Eq. 1.4 reduces to the usual slip boundary condition. So Eq. 1.4 can be regarded as a generalization of Navier's slip boundary condition previously studied in [1–3]. Although Navier-Stokes equations are associated in general with Navier's slip boundary condition, they have been considerable effort put in place to develop nonlinear response between $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ and $\boldsymbol{u}_{\boldsymbol{\tau}}$ (see [4–7]). We note that sometimes $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ and $\boldsymbol{u}_{\boldsymbol{\tau}}$ are given implicitly (see [8]), given rise to more difficult equations to analyse. In [9, 10], the solvability of Stokes and Navier-Stokes systems are presented without Eqs. 1.3 but 1.4 replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ and } \quad (\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}} + H(\boldsymbol{u}_{\boldsymbol{\tau}})\boldsymbol{u}_{\boldsymbol{\tau}} = g \quad \text{on } \partial\Omega\,,$$

and H having specific conditions. The situation in our hands is different because the response function defined in $Eq.1.4_2$ does not verify the properties listed in the work of [9, 10]. On the other hand, we observe that Eq. 1.4 has been used in solid mechanics for bonded structure (see [11–13]). Before embarking on the discrete scheme, we review the results presented in the study of the continuous problems [14]. A weak form for Eqs. 1.1…1.4 is the following variational equation:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V} \times M \text{ such that for all } (\boldsymbol{v}, q) \in \mathbb{V} \times M, \\ a(\boldsymbol{u}, \boldsymbol{v}) + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau}} \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} ds + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = \ell(\boldsymbol{v}), \\ b(\boldsymbol{u}, q) = 0\,, \end{cases} \tag{1.5}$$

where, we have used the standard notation of Lebesgue and Sobolev spaces [15], we define the function spaces and functionals which appear above as follows

$$\begin{aligned} \mathbb{V} &= \left\{ \boldsymbol{v} \in H^1(\Omega)^2 \; : \; \boldsymbol{v}|_{\Gamma_0} = 0 \;,\; \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0 \right\}, \\ M &= L_0^2 = \left\{ p \in L^2(\Omega) : \quad (p, 1) = 0 \right\} \quad , \end{aligned} \tag{1.6}$$

we set $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega)^2}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)^2}$, and

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\nu \int_\Omega \boldsymbol{D}\boldsymbol{u} : \boldsymbol{D}\boldsymbol{v} dx\;,\quad b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v} dx\;,$$

$$\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx + \int_\Gamma g \boldsymbol{v}_{\boldsymbol{\tau}} ds\;,$$

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} dx\;, \text{ and } \quad \boldsymbol{A} : \boldsymbol{B} = \sum_{1 \le i,j \le d} A_{ij} B_{ij}\;.$$

It is worth mentioning that by deriving (1.5), we make use of the identity $\sum_{1 \le i,j \le d} D_{ij} \boldsymbol{u} \dfrac{\partial v_i}{\partial x_j} = \sum_{1 \le i,j \le d} D_{ij} \boldsymbol{u} D_{ij} \boldsymbol{v}$ which is the consequence of the symmetry of $\boldsymbol{D}\boldsymbol{u}$. The variational problem (1.5) is nonlinear of monotone type, the following existence result is established in [14]

Theorem 1.1 *Let $(\boldsymbol{f}, g) \in L^2(\Omega)^d \times L^\infty(\Gamma)$ and $s \in (1, 2)$. The variational problem (1.5) has at least one solution solution $(\boldsymbol{u}, p) \in \mathbb{V} \times M$, moreover there is a constant C such that*

$$\nu C \|\boldsymbol{u}\|_1^2 + \int_\Gamma |K\boldsymbol{u}_{\boldsymbol{\tau}}|^s ds \le \frac{C}{\nu} \left(\|\boldsymbol{f}\|^2 + \|\boldsymbol{g}\|^2_{L^\infty(\Gamma)} \right),$$

$$\|p\| \le \frac{1}{\beta} \Big(2\nu \|\boldsymbol{u}\|_1 + \|K\|^s_{L^\infty(\Gamma)} \|\boldsymbol{u}_{\boldsymbol{\tau}}\|^{s-1}_{L^{s-1}(\Gamma)} + C\|\boldsymbol{u}\|_1^2 + \|\boldsymbol{f}\| + \|\boldsymbol{g}\|_{L^\infty(\Gamma)} \Big)\,.$$

If moreover

$$\|\boldsymbol{f}\| + \|\boldsymbol{g}\|_{L^\infty(\Gamma)} < \frac{\nu^2}{C}\,,$$

then the solution is unique.

Remark 1.1 *When $2 < s$, we have $\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^2(\Gamma)} \le C\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{L^s(\Gamma)}$. Then the velocity is prescribed in*

$$\mathbb{V} = \left\{ \boldsymbol{v} \in H^1(\Omega)^d,\; \boldsymbol{v}_{\boldsymbol{\tau}} \in L^s(\Gamma)^d \;:\; \textit{with } \boldsymbol{v}|_{\Gamma_0} = 0 \textit{ and } \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0 \right\}$$

equipped with the norm

$$|||\boldsymbol{v}|||^2 = \|\boldsymbol{v}\|_1^2 + \|\boldsymbol{v}_{\boldsymbol{\tau}}\|^2_{L^s(\Gamma)}\,.$$

In this setting proving the existence of solution of Eq. 1.5 *remain open.*

The purpose of this work is to propose a discretization of Eq. 1.5 based on discontinuous Galerkin schemes and establish its well-posedness and convergence. In the case of conforming approximation (see [14]), establishing the convergence of the finite element discretization has proven to be a non trivial analysis, the principal hurdles being, the difficulties in the usual Navier-Stokes equations and the nonlinearity in the power law slip boundary condition. The analyses in [1, 2] are concerned with penalized formulation when Navier's slip is considered and conforming finite element approximation is employed, while in [3] finite element approximation with velocity and pressure discretized with low order Crouziex-Ravirat's element associated with Navier's slip is considered on a domain with a curved boundary.

The first objective here is to extend the results of [14] by considering several discontinuous Galerkin schemes. In order to do so, we formulate the discontinuous Galerkin schemes by making use of the stabilizations terms introduced in [16–19], whereas the convective term is approximated following the discretization proposed in [20, 21] for similar term. It should be mentioned that the resulting finite element schemes is non conforming and differ enormously to the conforming approach. The DG schemes obtained is then analysed in two steps (see Sections 3 and 4). We establish the solvability of the DG-schemes by making use of Brouwer's fixed point, while uniqueness is derived with usual procedure. We note at this step that the norm used in the DG setting is locally defined, and takes into account the discontinuity imposed in the finite element approximations. Hence the properties that are true in the continuous description do not necessarily hold in the discrete setting. The second step in our analysis is concerned with the a priori error analysis (see Section 4). Using the inequality of the triangle and techniques based on mixed formulations, we derive the convergence of the DG-solution by estimating the error and subsequently obtaining the rate of convergence. At this juncture, it is important to note that the consistency of the DG schemes is key in the analysis of the error. Next, making use of the properties of the operators involved, we establish that the convergence is uniform with respect to the parameter s in the constitutive equation between $(\boldsymbol{T}\boldsymbol{n})_{\boldsymbol{\tau}}$ and $\boldsymbol{u}_{\boldsymbol{\tau}}$. This work differ from the previous work [14]. Indeed, in this work, the norm used to analyse the existence and the convergence is locally defined and take into account the discontinuity. In the DG schemes the inter-element continuity requirements are imposed weakly by penalty terms, and discrete spaces are spanned by fully discontinuous piecewise polynomials on the mesh and there are no shared degrees of freedom. In general an important distinction between the DG method and the classical finite element method is that in the DG approach the resulting equations are local to the generating element. In this work, the solution within each element is not reconstructed by looking the neighboring elements (we do not have continuity). Thus, each element may be thought of as a separate entity that merely needs to obtain some boundary data from its neighbors. The compact form of the discontinuous Galerkin method makes it well suited for parallel computations. This compactness also allows a heterogeneous treatment of problems. That is, the element topology, the degree of approximation and even the choice of governing equations can vary from element to element and in time over the course of a computation without loss of rigor in the method.

The second objective in this work is to validate the theoretical findings. Indeed, the DG schemes to be solved are nonlinear, hence iterative/incremental scheme should be considered for their effective resolution. We consider the fixed point approach. This approach entails the linearization of nonlinear expression (see paragraph 5). This approach is very popular for solving nonlinear problems. One of the principal attributes of the fixed point approach is that it converges to the unique fixed point of the problem for any starting point, and it is first order accurate. In this work, we present a complete analysis of the method and validate it by means of presenting the results from numerical simulations (see paragraph 5.2). One of the disadvantages of the method is its high storage and high computational requirements. We note that the implementation of the discontinuous Galerkin (DG) discretization relies on evaluation of integrals which implies the use of quadrature rules. An alternative formulation of the DG method is possible under the flux reconstruction strategy, where the equations are solved in differential form and the discretization is free from quadrature rules, resulting in computationally efficient algorithms (see [22, 23]).

The outline of the paper is as follows. The discontinuous Galerkin schemes are formulated in Section 2. The existence of DG solution is treated in Sections 3 and 4 deals with the a priori error estimates of the DG schemes. In Section 5, we formulate and analyse the iterative scheme for solving the non linear finite element problem. We also implement the iterative scheme and exhibit some numerical results.

2 Discontinuous galerkin approximations

In this section, we develop several discontinuous Galerkin finite approximations for the numerical solution of Eq. 1.5.

2.1 Notation and approximation properties

We assume that Ω is a polygonal domain of $\mathbb{R}^2$. Let $\mathcal{T} = \{K\}$ be a regular family of triangulations of $\overline{\Omega}$, consisting of triangles of maximum diameter h. Let h_K be a diameter of a triangle K and ρ_K the diameter of its inscribed circle. With abuse of notation we denote the triangulation $\mathcal{T}_h$ instead of $\mathcal{T}$. By regular we mean that there exists a parameter $\sigma > 0$ such that

$$\text{for all } \ K \in \mathcal{T}_h, \ \ \frac{h_K}{\rho_K} = \sigma_K \leq \sigma \, . \tag{2.1}$$

If e is a face of $K \in \mathcal{T}_h$, then h_e is the length of e. Let $\mathcal{E}_h = \{e\}$ denote the set of the edges of $\mathcal{T}_h$, and $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \overline{\Gamma}$ the set of all edges not lying on $\overline{\Gamma}$, and $\mathcal{E}_h^{\gamma}$ the set of all edges lying on $\overline{\Gamma}$. $\mathcal{E}_h^I$ is the set of all interior edges. We associate with each edge of an element K_i, the outward unit normal vector $\boldsymbol{n}_i$. For an edge that lies on the boundary, $\boldsymbol{n}_i$ is defined to be the outward normal to $\partial\Omega$. For a positive integer m, we set

$$H^m(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega), \ \ v|_K \in H^m(K), \quad \text{for all } K \in \mathcal{T} \Big\},$$

$$T(\mathcal{E}_h) = \prod_{K \in \mathcal{T}} L^2(\partial K) \, .$$

In this work, for each edge e, we associate a normal unit vector $\boldsymbol{n}$. If the e is on the boundary, the unit vector $\boldsymbol{n}$ is taken to be the outward normal to $\partial\Omega$. Let e_k be an edge shared by elements K_i and K_j with $\boldsymbol{n}_k$ exterior to K_i. We define the jump $[v]$ and average $\{v\}$ of a function v by

$$[v]|_{e_k} = v|_{\partial K_i} - v|_{\partial K_j}, \quad \{v\}|_{e_k} = \frac{1}{2}\left(v|_{\partial K_i} + v|_{\partial K_j}\right).$$

If $e = K_1 \cap \partial\Omega$, then the jump and average of v coincides with its trace on e.
The following identity obtained by re-arranging terms relates the scalar product of two quantities to the products of their jumps and averages

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h}\int_{\partial K}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}ds &= \sum_{e\in\mathcal{E}_h}\int_e[\boldsymbol{v}]\cdot\{\boldsymbol{T}\}\boldsymbol{n}_e ds + \sum_{e\in\mathcal{E}_h^I}\int_e\{\boldsymbol{v}\}\cdot[\boldsymbol{T}]\boldsymbol{n}_e ds\,,\\
\sum_{K\in\mathcal{T}_h}\int_{\partial K}q\boldsymbol{v}\cdot\boldsymbol{n}ds &= \sum_{e\in\mathcal{E}_h}\int_e[\boldsymbol{v}]\cdot\boldsymbol{n}_e\{q\}ds + \sum_{e\in\mathcal{E}_h^I}\int_e\{\boldsymbol{v}\}\cdot\boldsymbol{n}_e[q]ds\,.
\end{aligned} \tag{2.2}$$

The velocity and pressure will be approximated by $\mathbb{V}_h$ and M_h given as follows

$$\widetilde{\mathbb{V}}_h = \left\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega): \quad \boldsymbol{v}|_K\in\mathcal{P}_k(K)^2 \text{ for all } K\in\mathcal{T}_h\right\},$$

$$\mathbb{V}_h = \Big\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega): \quad \boldsymbol{v}|_K\in\mathcal{P}_k(K)^2 \text{ for all } K\in\mathcal{T}_h\,, \text{ and } \boldsymbol{v}\cdot\boldsymbol{n}_e = 0 \text{ for all } e\in\mathcal{E}_h^\gamma\Big\},$$

$$M_h = \left\{q\in L_0^2(\Omega): \quad q|_K\in\mathcal{P}_{k-1}(K) \quad \text{for all } K\in\mathcal{T}_h\right\},$$

with $\mathcal{P}_k(K)$ the space of polynomials for degree at most k in $K\in\mathcal{T}_h$. In the sequel, we denote by $C, C', \ldots$ generic constants that can vary from line to line but are always independent of all discretization parameter. Having in mind that $\mathcal{T}_h$ is regular (see Eq. 2.1), for each $k\geq 1$, one can construct an operator $r_h\in\mathcal{L}(L_0^2(\Omega); M_h)$ (see [24, 25]) such that

$$\begin{cases}\text{for any } \ K\in\mathcal{T}_h \text{ and for all } \ (q,p)\in\mathcal{P}_{k-1}(K)\times L_0^2(\Omega),\\ \displaystyle\int_K q\,(r_h p - p)\,d\boldsymbol{x} = 0\,,\end{cases}$$

moreover

$$\begin{cases}\text{for any real } s\in[0,k] \text{ and for all } \ q\in H^s(\Omega)\cap L_0^2(\Omega)\\ \|r_h q - q\|_K \leq C h_K^s |q|_{s,K}\,,\end{cases} \tag{2.3}$$

with $|\cdot|_{s,K}$ the semi-norm. The interpolation of functions with values belonging to $\mathbb{V}_h$ is a bit delicate because the normal component of the velocity must be zero for all edges $e\in\mathcal{E}_h^\gamma$.

So, we take the interpolation operator of Raviart-Thomas [24, 26] $\Pi_h\in\mathcal{L}(H^1(\Omega)^2;\widetilde{\mathbb{V}}_h)$ satisfying the following properties:

$$\begin{aligned}
&\forall(\boldsymbol{v},T,q_h)\in H^1(\Omega)^2\times\mathcal{T}_h\times\mathcal{P}_{k-1}(T)\,, \int_T q_h \operatorname{div}(\Pi_h\boldsymbol{v}-\boldsymbol{v})dx = 0\,,\\
&\forall(\boldsymbol{v},e,q_h)\in H^1(\Omega)^2\times\mathcal{E}_h^0\times\mathcal{P}_{k-1}(e)\,, \int_e q_h(\Pi_h\boldsymbol{v}-\boldsymbol{v})\cdot\boldsymbol{n}_e d\sigma = 0\,,\\
&\forall\boldsymbol{v}\in H^1(\Omega)^2\,, \ |||\Pi_h\boldsymbol{v}|||_h \leq c\|\nabla\boldsymbol{v}\|\,,\\
&\forall(\boldsymbol{v},T)\in H^1(\Omega)^2\times\mathcal{T}_h\,, \ \|\nabla(\boldsymbol{v}-\Pi_h\boldsymbol{v})\|_T + \frac{1}{h_T}\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_T \leq c\|\nabla\boldsymbol{v}\|_T\,.
\end{aligned} \tag{2.4}$$

Remark 2.1 *In this text,we do not dealt with domain with curved boundary, and the readers interested in that direction may consult the work of Ibrahima Dione* [27].

2.2 DG models

In this paragraph, we present DG schemes for solving the mixed variational problem (1.5). These methods can be found in [16–20].

The discontinuous Galerkin method for Eq. 1.5 is as follows:

$$\begin{cases}\text{Find } (\boldsymbol{u}_h,p_h)\in\mathbb{V}_h\times M_h, \text{ such that for all } (\boldsymbol{v}_h,q_h)\in\mathbb{V}_h\times M_h,\\ A(\boldsymbol{u}_h,\boldsymbol{v}_h) + B(\boldsymbol{v}_h,p_h) + H(\boldsymbol{u}_h)\boldsymbol{v}_h + \widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) = F(\boldsymbol{v}_h)\,,\\ B(\boldsymbol{u}_h,q_h) = 0\,,\end{cases} \tag{2.5}$$

where

$$F(\boldsymbol{v}) = \sum_{K\in\mathcal{T}_h}\int_K\boldsymbol{f}\cdot\boldsymbol{v}d\boldsymbol{x} + \sum_{e\in\mathcal{E}_h^\gamma}\int_e g\boldsymbol{v}_{\boldsymbol{\tau}}ds \ ,$$

$$H(\boldsymbol{v})\boldsymbol{w} = \sum_{e\in\mathcal{E}_h^\gamma}\int_e|K\boldsymbol{v}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{v}_{\boldsymbol{\tau}}\cdot K\boldsymbol{w}_{\boldsymbol{\tau}}ds\,.$$

The bilinear form $B(\cdot,\cdot)$ is given by

$$B(\boldsymbol{v},q) = -\sum_{\kappa\in\mathcal{T}_h}\int_T q\operatorname{div}\boldsymbol{v}dx + \sum_{e\in\mathcal{E}_h^0}\int_e\{q\}[\boldsymbol{v}]\cdot\boldsymbol{n}_e ds$$

while the convective term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}$ is approximated by (see [20, 21])

$$\begin{aligned}
\widetilde{d}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa\left((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} + \frac{1}{2}\operatorname{div}\boldsymbol{u}\ \boldsymbol{v}\cdot\boldsymbol{w}\right)dx\\
&-\frac{1}{2}\sum_{e\in\mathcal{E}_h^0}\int_e[\boldsymbol{u}]\boldsymbol{n}_e\{\boldsymbol{v}\cdot\boldsymbol{w}\}ds + \sum_{\kappa\in\mathcal{T}_h,\kappa\cap\mathcal{E}_h^\gamma=\emptyset}\int_{\partial\kappa_-}\\
&\times|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_{\partial\kappa}|\left(\boldsymbol{v}^{int}-\boldsymbol{v}^{ext}\right)\cdot\boldsymbol{w}^{int}ds,
\end{aligned}$$

with $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ given such that the integrals defined above make sense. The inflow boundary is

$$\partial\kappa_- = \{\boldsymbol{x} \in \partial\kappa : \quad \{\boldsymbol{u}\} \cdot \boldsymbol{n}_{\partial\kappa} < 0\},$$

and the superscript int (resp., ext) stands to the trace of the function on a side of κ coming from the interior of κ (resp., coming from the exterior of κ on that side). It is noted that $\widetilde{d}(\cdot,\cdot,\cdot)$ is not linear with respect to its first argument, but is linear with respect to its second and third argument. As usual, when the inflow boundaries $\partial\kappa_-$ are defined with respect to the velocity $\{\boldsymbol{u}\}$, we let $\widetilde{d}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \equiv \widetilde{d}_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$. It is noted that $\widetilde{d}(\cdot,\cdot,\cdot)$ is consistent with $d(\cdot,\cdot,\cdot)$ in the sense that

$$\text{for all } (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \mathbb{V}_{\text{div}} \times H^1(\Omega) \times H^1(\Omega)\ ,\ \widetilde{d}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\,.$$

Now, from [20, Lemma 6.1], one deduces that

$$\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{v}_h,\boldsymbol{v}_h) \geq 0 \quad \text{for all} \quad \boldsymbol{v}_h, \boldsymbol{u}_h \in \mathbb{V}_h\,. \tag{2.6}$$

The bilinear form $A(\cdot,\cdot)$ is given by

$$A(\boldsymbol{u}_h,\boldsymbol{v}_h) = a(\boldsymbol{u}_h,\boldsymbol{v}_h) + \widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h)$$

and the bilinear form $\widetilde{a}_h(\cdot,\cdot)$ takes into account all the consistency and stability expressions. In [16] several DG schemes are formulated and analysed for solving elliptic equations. In this work, we just select a few of them. Let r_0, and r_e denote the global and local lifting operators respectively (see [16]) given as follows

$$r_e : L^1(e) \longrightarrow \Sigma_h = \Big\{\boldsymbol{v}_h \in L^2(\Omega)^2 : \boldsymbol{v}_h \in \mathcal{P}_k(K)^2 \quad \text{for all} \quad K \in \mathcal{T}_h\Big\}$$

such that for all $\boldsymbol{w}_h \in \Sigma_h$

$$\int_\Omega r_e([\boldsymbol{v}_h]) \cdot \boldsymbol{w}_h dx = -\int_e [\boldsymbol{v}_h] \cdot \{\boldsymbol{w}_h\} ds \quad \text{for all } e \in \mathcal{E}_h^0$$
$$\text{and} \quad r_0([\boldsymbol{v}_h]) = \sum_{e \in \mathcal{E}_h^0} r_e([\boldsymbol{v}_h])\,.$$

We now present various possibilities for $\widetilde{a}_h(\cdot,\cdot)$. **SIPG method** [17].

$$\widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = -2\nu \sum_{e \in \mathcal{E}_h^0} \int_e \Big(\{D\boldsymbol{u}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_h] + \{D\boldsymbol{v}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_h]\Big) ds + \sum_{e \in \mathcal{E}_h^0} \int_e \frac{\alpha}{h_e} [\boldsymbol{u}_h] \cdot [\boldsymbol{v}_h] ds\,,$$

for $(\boldsymbol{v}_h,\boldsymbol{u}_h)$ elements of $\mathbb{V}_h$ and $\alpha \geq \alpha_0 > 0$. This scheme is called the symmetric interior penalty Galerkin scheme. We easily check that $\widetilde{a}_h(\cdot,\cdot)$ is symmetric. The stability of this scheme depends on the choice of parameter α. **NIPG method** [18, 20].

$$\widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = -2\nu \sum_{e \in \mathcal{E}_h^0} \int_e \Big(\{D\boldsymbol{u}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_h] - \{D\boldsymbol{v}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_h]\Big) ds + \sum_{e \in \mathcal{E}_h^0} \int_e \frac{\alpha}{h_e} [\boldsymbol{v}_h] \cdot [\boldsymbol{u}_h] ds\,,$$

for $(\boldsymbol{v}_h,\boldsymbol{u}_h)$ elements of $\mathbb{V}_h$ and $\alpha > 0$. This scheme is called the non symmetric interior penalty Galerkin scheme. We easily check that $\widetilde{a}_h(\cdot,\cdot)$ is non symmetric. This scheme is stable for all values of α. With obvious notation, we observe that

$$A^{SIPG}(\boldsymbol{v}_h,\boldsymbol{w}_h) = A^{NIPG}(\boldsymbol{v}_h,\boldsymbol{w}_h) - 2\nu \sum_{e \in \mathcal{E}_h^0} \int_e \{D\boldsymbol{w}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_h] ds\,.$$

IIPG method [19].

$$\widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \sum_{e \in \mathcal{E}_h^0} \int_e \frac{\alpha}{h_e} [\boldsymbol{v}_h] \cdot [\boldsymbol{u}_h] ds\,,$$

for $(\boldsymbol{v}_h,\boldsymbol{u}_h)$ elements of $\mathbb{V}_h$ and $\alpha > 0$. This scheme is symmetric and unconditionally stable. It is also called over-stabilization approach or incomplete interior penalty scheme. It has less expressions than the SIPG and NIPG. Bassi et al [28].

$$\widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \sum_{e \in \mathcal{E}_h^0} \int_e \eta_e r_e([\boldsymbol{u}_h]) \cdot r_e([\boldsymbol{v}_h]) ds - 2\nu \times \sum_{e \in \mathcal{E}_h^0} \int_e \Big(\{D\boldsymbol{u}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_h] + \{D\boldsymbol{v}_h\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_h]\Big) ds,$$

for $(\boldsymbol{v}_h,\boldsymbol{u}_h)$ elements of $\mathbb{V}_h$ and $\eta_e \geq \eta_0 > 0$. It is a symmetric scheme and conditionally stable. This scheme can be regarded as a variant of the symmetric interior penalty scheme (SIPG), the difference being that $r_e([\boldsymbol{u}_h]) \neq \frac{1}{h_e^{1/2}}[\boldsymbol{u}_h]$
LDG method [29, 30].

$$\widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \int_\Omega r_0([\boldsymbol{u}_h]) \cdot r_0([\boldsymbol{v}_h]) dx + \sum_{e \in \mathcal{E}_h^0} \int_e \frac{\alpha}{h_e} [\boldsymbol{v}] \cdot [\boldsymbol{w}] ds - 2\nu \sum_{e \in \mathcal{E}_h^0} \int_e \Big(\{D\boldsymbol{v}\}\boldsymbol{n}_e \cdot [\boldsymbol{w}] + \{D\boldsymbol{w}\}\boldsymbol{n}_e \cdot [\boldsymbol{v}]\Big) ds,$$

for $(\boldsymbol{v}_h,\boldsymbol{u}_h)$ elements of $\mathbb{V}_h$ and $\alpha > 0$.It is a symmetric scheme and unconditionally stable. Again, this approach can be regarded as a variant of the symmetry interior penalty method (SIPG), with the difference that it has an extra term

$\int_\Omega r_0([\boldsymbol{u}_h]) \cdot r_0([\boldsymbol{v}_h])dx$ which forced the scheme to be unconditionally stable. Now, we state that

Lemma 2.1 (consistency). *Let $(\boldsymbol{u}, p)$ be the solution of Eq. 1.2 with $(\boldsymbol{u}, p) \in \boldsymbol{H}^2 \times H^1(\Omega)$, then*

$$\begin{cases} \text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h, \\ A(\boldsymbol{u}, \boldsymbol{v}_h) + B(\boldsymbol{v}_h, p) + H(\boldsymbol{u})\boldsymbol{v}_h + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) = F(\boldsymbol{v}_h)\,, \\ B(\boldsymbol{u}, q_h) = 0\,. \end{cases}$$

Proof The regularity assumption $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ together with Eq. 2.2 and a result due to Evans and Gariepy (see [31, Section 4.9.2, Theorem 2]), suffice to obtain the desired result. We discuss next the solvability of Eq. 2.5.

3 Solvability

The following monotonicity and continuity properties (see [32, 33]) will be helpful: there exists a constant C independent of $\boldsymbol{x}, \boldsymbol{y}$ such that for $1 \le s < 2$

$$(|\boldsymbol{y}| + |\boldsymbol{x}|)^{2-s}\left(|\boldsymbol{y}|^{s-2}\boldsymbol{y} - |\boldsymbol{x}|^{s-2}\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}\right) \ge C|\boldsymbol{x} - \boldsymbol{y}|^2 \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\,, \tag{3.1}$$

and

$$\left| |\boldsymbol{x}|^{s-2}\boldsymbol{x} - |\boldsymbol{y}|^{s-2}\boldsymbol{y} \right| \le C|\boldsymbol{x} - \boldsymbol{y}|^{s-1} \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\,. \tag{3.2}$$

We endowed $\mathbb{V}_h$ with the norm $|||\cdot|||_h$ given as follows

$$|||\boldsymbol{v}_h|||_h^2 = \sum_{\kappa\in\mathcal{T}_h} \|D\boldsymbol{v}_h\|_\kappa^2 + \sum_{e\in\mathcal{E}_h^0} \frac{1}{h_e}\|[\boldsymbol{v}_h]\|_e^2\,.$$

Remark 3.1 *Thus from Korn's type inequality (see [34]), we deduce that for any $\boldsymbol{v}_h \in \mathbb{V}_h$*

$$\sum_{T\in\mathcal{T}_h} \|\nabla\boldsymbol{v}_h\|_T^2 \le C\Big(\sum_{T\in\mathcal{T}_h} \|D\boldsymbol{v}_h\|_T^2 + \sum_{e\in\mathcal{E}_h^0} \frac{1}{h_e}\|[\boldsymbol{v}_h]\|_e^2\Big)\,.$$

Hence on $\mathbb{V}_h$, $|||\cdot|||_h$ is equivalent to the norm

$$\boldsymbol{v} \longrightarrow \Big(\sum_{T\in\mathcal{T}_h} \|\nabla\boldsymbol{v}\|_T^2 + \sum_{e\in\mathcal{E}_h^0} \frac{1}{h_e}\|[\boldsymbol{v}_h]\|_e^2\Big)^{1/2}\,.$$

From [20, Lemma 6.2], for each number $p \in [2, \infty)$, there exists a constant $C(p)$ such that

$$\text{for all } \boldsymbol{v} \in \mathbb{V}_h\,,\ \|\boldsymbol{v}\|_{L^p(\Omega)} \le C(p)|||\boldsymbol{v}|||_h\,. \tag{3.3}$$

Finally, we recall some standard trace and inverse inequalities, which hold true on each element $\kappa \in \mathcal{T}_h$, with diameter h_κ, and for any face e of κ (see [24, 35]):

$$\|\boldsymbol{v}\|_{L^p(e)} \le C\left(h_\kappa^{-1/p}\|\boldsymbol{v}\|_{L^p(\kappa)} + h_\kappa^{1-1/p}\|\nabla\boldsymbol{v}\|_{L^p(\kappa)}\right) \text{ for all } p \in [1, \infty] \text{ and } \boldsymbol{v} \in W^{1,p}(\kappa) \tag{3.4a}$$

$$\|\boldsymbol{v}_h\|_{L^p(e)} \le Ch_\kappa^{-\frac{1}{p}+d\left(\frac{1}{p}-\frac{1}{r}\right)}\|\boldsymbol{v}_h\|_{L^r(\kappa)} \text{ for all } r, p \in [1, \infty] \text{ and } (\boldsymbol{v}_h, \kappa) \in \mathcal{P}_k(\kappa) \times \mathcal{T}_h\,. \tag{3.4b}$$

With Cauchy Schwarz's inequality, Hölder's inequality, Young's inequality, Eqs. 3.3, and 3.4b, we can show that:

Lemma 3.1 *(continuity of $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$). The bilinear forms $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are continuous on $\mathbb{V}_h \times \mathbb{V}_h$ and $\mathbb{V}_h \times M_h$ respectively. More precisely, there exists C_I independent of the mesh size h such that*

$$A(\boldsymbol{v}, \boldsymbol{w}) \le C_I|||\boldsymbol{v}|||_h|||\boldsymbol{w}|||_h \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_h\,,$$
$$B(\boldsymbol{v}, q) \le C_I|||\boldsymbol{v}|||_h\|q\| \quad \text{for all } (\boldsymbol{v}, q) \in \mathbb{V}_h \times M_h\,.$$

The coercivity of the bilinear forms $A(\cdot,\cdot)$ are proved in the literature, it is not the place here to repeat these proofs, we refer the interested readers to [16–19, 29, 30] for more details. We state that

Lemma 3.2 *(coercivity of $A(\cdot,\cdot)$). There exists a constant C, independent of h such that*

$$A(\boldsymbol{v}, \boldsymbol{v}) \ge C|||\boldsymbol{v}|||_h^2 \quad \text{for all } \boldsymbol{v} \in \mathbb{V}_h\,.$$

Remark 3.2 *The coercivity for the SIPG, Bassi et al approaches are conditional, while no condition is required for NIPG, IIPG and LDG approaches.*

With Cauchy-Schwarz's inequality, Hölder's inequality and Eq. 3.3, we deduce that

Lemma 3.3 *There exists a positive constant C independent of h such that for all $\boldsymbol{v}_h \in \mathbb{V}_h$*

$$F(\boldsymbol{v}_h) \le \left(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{f}\|_K^2\right)^{1/2} |||\boldsymbol{v}_h|||_h + C\,|\Gamma|^{1/2}\,\|\boldsymbol{g}\|_{L^\infty(\Gamma)}|||\boldsymbol{v}_h|||_h\,.$$

The following inf-sup condition holds.

Lemma 3.4 *Let the triangulation $\mathcal{T}_h$ satisfies (2.1). There exists $\beta^* > 0$, independent of h such that*

$$\forall q_h \in M_h, \quad \beta^* \|q_h\| \leq \sup_{\boldsymbol{v}_h \in \widetilde{\mathbb{V}}_h} \frac{B(\boldsymbol{v}_h, q_h)}{|||\boldsymbol{v}_h|||_h}. \tag{3.5}$$

Proof The proof is not new and we refer the readers to [20].

The solution of the nonlinear variational problem (2.5) can be constructed by using Brouwer's fixed point theorem. To this end, we have that

Theorem 3.1 *The variational problem (2.5) has a solution $(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h \times M_h$, which furthermore satisfy*

$$\begin{aligned} C\nu|||\boldsymbol{u}_h|||_h^2 + \sum_{e \in \mathcal{E}_h^\gamma} \int_e \left|K\boldsymbol{u}_{\boldsymbol{\tau},h}\right|^s ds &\leq \frac{1}{\nu}\Bigg(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2 \\ &\quad + C\sum_{e \in \mathcal{E}_h^\gamma} \|g\|_{L^\infty(e)}^2\Bigg), \\ \|p_h\| \leq C|||\boldsymbol{u}_h|||_h + C\|K\|_{L^\infty(\Gamma)}^s |||\boldsymbol{u}_h|||_h^{s-1} \\ + C\sum_{e \in \mathcal{E}_h^\gamma} \|g\|_{L^\infty(e)} + c\left(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2\right)^{1/2}. \end{aligned}$$

Moreover there exists C such that if ν, $\boldsymbol{f}$, g satisfies the condition

$$\|\boldsymbol{f}\| + \|g\|_{L^\infty(\Gamma)} < \frac{\nu^2}{C}$$

then the solution is unique.

Proof The discrete kernel of the bilinear form $B(\cdot,\cdot)$ is the set

$$\mathcal{Z}_h = \left\{\boldsymbol{v}_h \in \mathbb{V}_h : \quad \text{for all } q_h \in M_h, \;\; B(\boldsymbol{v}_h, q_h) = 0\right\}.$$

For fixed $\boldsymbol{w}_h \in \mathcal{Z}_h$, we define $\mathcal{H}(\boldsymbol{w}_h)$ in $\mathcal{Z}_h$ by

for all $\boldsymbol{v}_h \in \mathcal{Z}_h$, $\quad \mathcal{H}(\boldsymbol{w}_h)\boldsymbol{v}_h = A(\boldsymbol{w}_h, \boldsymbol{v}_h) + H(\boldsymbol{w}_h)\boldsymbol{v}_h + \widetilde{d}(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{v}_h) - F(\boldsymbol{v}_h)$.

- $\mathcal{H}$ defines a continuous mapping from $\mathcal{Z}_h \longrightarrow \mathcal{Z}_h$. Indeed, Let $\boldsymbol{w}^1$, $\boldsymbol{w}^2$ and $\boldsymbol{v}$ elements of $\mathcal{Z}_h$. We have from Lemma 3.1

$$\begin{aligned} (\mathcal{H}(\boldsymbol{w}^1) - \mathcal{H}(\boldsymbol{w}^2)\boldsymbol{v} &= A(\boldsymbol{w}^1 - \boldsymbol{w}^2, \boldsymbol{v}) + \left(H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2)\right)\boldsymbol{v} \\ &+ \widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v}) \leq C|||\boldsymbol{w}^1 - \boldsymbol{w}^2|||_h |||\boldsymbol{v}|||_h \\ &+ \left(H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2)\right)\boldsymbol{v} + \widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v}). \end{aligned} \tag{3.6}$$

It is manifest that w need to bound $\left(H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2)\right)\boldsymbol{v}$ and $\widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v})$. Starting with $\big(H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2)\big)\boldsymbol{v}$. Using Cauchy-Schwarz's inequality, Hölder's inequality, Eqs. 3.2, 3.4b and 3.3, the trace's theorem, one gets

$$\begin{aligned} \left(H(\boldsymbol{w}^1) - H(\boldsymbol{w}^2)\right)\boldsymbol{v} &= \sum_{e \in \mathcal{E}_h^\gamma} \int_e \Bigg(\left|K\boldsymbol{w}_{\boldsymbol{\tau}}^1\right|^{s-2} K\boldsymbol{w}_{\boldsymbol{\tau}}^1 \\ &\quad - \left|K\boldsymbol{w}_{\boldsymbol{\tau}}^2\right|^{s-2} K\boldsymbol{w}_{\boldsymbol{\tau}}^2\Bigg) \cdot K\boldsymbol{v}_{\boldsymbol{\tau}} ds \\ &\leq C\|K\|_{L^\infty(\Gamma)}^s \sum_{e \in \mathcal{E}_h^\gamma} \int_e \left|\boldsymbol{w}_{\boldsymbol{\tau}}^1 - \boldsymbol{w}_{\boldsymbol{\tau}}^2\right|^{s-1} |\boldsymbol{v}_{\boldsymbol{\tau}}| ds \\ &\leq C\|K\|_{L^\infty(\Gamma)}^s \left\|\boldsymbol{w}^1 - \boldsymbol{w}^2\right\|_\Gamma^{s-1} \|\boldsymbol{v}_{\boldsymbol{\tau}}\|_\Gamma \\ &\leq C|||\boldsymbol{w}^1 - \boldsymbol{w}^2|||_h^{s-1} |||\boldsymbol{v}|||_h. \end{aligned} \tag{3.7}$$

Next, we need to bound $\widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v})$. But we first observe that

$$\begin{aligned} \widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v}) &= \widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1 - \boldsymbol{w}_2, \boldsymbol{v}) \\ &\quad + \widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v}). \end{aligned}$$

Since $\boldsymbol{v}$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ are in $\mathbb{V}_h$, [21, Proposition 4.1] applies, thus

$$\widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_1 - \boldsymbol{w}_2, \boldsymbol{v}) \leq C|||\boldsymbol{w}_1|||_h |||\boldsymbol{w}_1 - \boldsymbol{w}_2|||_h |||\boldsymbol{v}|||_h. \tag{3.8}$$

To analyse the term $\widetilde{d}(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{v}) - \widetilde{d}(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v})$, we first write $\widetilde{d} = \widetilde{d}^L + \widetilde{d}^{NL}$ with

$$\begin{aligned} \widetilde{d}^L(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \left((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} + \frac{1}{2}\operatorname{div}\boldsymbol{u}\;\boldsymbol{v}\cdot\boldsymbol{w}\right) dx \\ &\quad - \frac{1}{2}\sum_{e \in \mathcal{E}_h^0} \int_e [\boldsymbol{u}]\{\boldsymbol{v}\cdot\boldsymbol{w}\} ds \\ \widetilde{d}^{NL}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \sum_{\kappa \in \mathcal{T}_h, \kappa \cap \mathcal{E}_h^\gamma = \emptyset} \int_{\partial\kappa_-} |\{\boldsymbol{u}\}\cdot\boldsymbol{n}| \left(\boldsymbol{v}^{int} - \boldsymbol{v}^{ext}\right) \\ &\quad \cdot \boldsymbol{w}^{int} ds. \end{aligned}$$

Clearly $\widetilde{d}^L(\cdot,\cdot,\cdot)$ is linear with respect to the first entry, hence

$$\widetilde{d}^L(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{v}) - \widetilde{d}^L(\boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v}) = \widetilde{d}^L(\boldsymbol{w}_1 - \boldsymbol{w}_2, \boldsymbol{w}_2, \boldsymbol{v}).$$

Now using Cauchy-Schwarz's inequality, Hölder inequality, Eqs. 3.4b, 3.3 and 3.3, we have

$$\begin{aligned}
&\widetilde{d}^L(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{w}_2,\boldsymbol{v})\\
&\le \sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^4(\kappa)}\|\nabla\boldsymbol{w}_2\|_\kappa\|\boldsymbol{v}\|_{L^4(\kappa)}+\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{1,\kappa}\\
&\quad\times\|\boldsymbol{w}_2\|_{L^4(\kappa)}\|\boldsymbol{v}\|_{L^4(\kappa)}+C\sum_{e\in\mathcal{E}_h^0}h_e^{-1/2}\|[\boldsymbol{w}_1-\boldsymbol{w}_2]\|_e h_e^{1/2}\|\\
&\quad\times\boldsymbol{w}_2\|_{L^4(e)}\|\boldsymbol{v}\|_{L^4(e)}\le C|||\boldsymbol{w}_1-\boldsymbol{w}_2|||_h|||\boldsymbol{w}_2|||_h|||\boldsymbol{v}|||_h\\
&\quad+c\left(\sum_{e\in\mathcal{E}_h^0}h_e^{-1}\|[\boldsymbol{w}_1-\boldsymbol{w}_2]\|_e^2\right)^{1/2}\times\left(\sum_{e\in\mathcal{E}_h^0}h_e\|\boldsymbol{w}_2\|_{L^4(e)}^4\right)^{1/4}\\
&\quad\times\left(\sum_{e\in\mathcal{E}_h^0}h_e\|\boldsymbol{v}\|_{L^4(e)}^4\right)^{1/4}\le C|||\boldsymbol{w}_1-\boldsymbol{w}_2|||_h|||\boldsymbol{w}_2|||_h|||\boldsymbol{v}|||_h\\
&\quad+C\left(\sum_{e\in\mathcal{E}_h^0}h_e^{-1}\|[\boldsymbol{w}_1-\boldsymbol{w}_2]\|_e^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h^0}h_e^{1/4}\|\boldsymbol{w}_2\|_{L^4(e)}\right)\\
&\quad\times\left(\sum_{e\in\mathcal{E}_h^0}h_e^{1/4}\|\boldsymbol{v}\|_{L^4(e)}\right)\le C|||\boldsymbol{w}_1-\boldsymbol{w}_2|||_h|||\boldsymbol{w}_2|||_h|||\boldsymbol{v}|||_h\,. \qquad (3.9)
\end{aligned}$$

For the nonlinear term, we have

$$\begin{aligned}
&\widetilde{d}^{NL}(\boldsymbol{w}_1,\boldsymbol{w}_2,\boldsymbol{v})-\widetilde{d}^{NL}(\boldsymbol{w}_2,\boldsymbol{w}_2,\boldsymbol{v})\\
&\quad\le\sum_{\kappa\in\mathcal{T}_h,\kappa\cap\mathcal{E}_h^\gamma=\emptyset}\int_{\partial\kappa_-}|\{\boldsymbol{w}_1-\boldsymbol{w}_2\}|\left|\boldsymbol{w}_2^{int}-\boldsymbol{w}_2^{ext}\right|\left|\boldsymbol{v}^{int}\right|ds\\
&\quad\le\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^4(\partial\kappa)}\,\|[\boldsymbol{w}_2]\|_{L^2(\partial\kappa)}\,\|\boldsymbol{v}\|_{L^4(\partial\kappa)}\\
&\quad\le C|||\boldsymbol{w}_1-\boldsymbol{w}_2|||_h|||\boldsymbol{w}_2|||_h|||\boldsymbol{v}|||_h, \qquad (3.10)
\end{aligned}$$

which together with Eqs. 3.9, 3.8 and 3.7 in Eq. 3.6 gives

$$\begin{aligned}
(\mathcal{H}(\boldsymbol{w}^1)-\mathcal{H}(\boldsymbol{w}^2)\boldsymbol{v}\ &\le C|||\boldsymbol{w}^1-\boldsymbol{w}^2|||_h|||\boldsymbol{v}|||_h\\
&+c|||\boldsymbol{w}^1-\boldsymbol{w}^2|||_h^{s-1}|||\boldsymbol{v}|||_h+C|||\boldsymbol{w}_1-\boldsymbol{w}_2|||_h\\
&\times(|||\boldsymbol{w}_2|||_h+|||\boldsymbol{w}_1|||_h)\,|||\boldsymbol{v}|||_h\,.
\end{aligned}$$

Hence, $\mathcal{H}$ is continuous.

- $\mathcal{H}(\boldsymbol{v}_h)\boldsymbol{v}_h$ is bounded from below. Indeed for $\boldsymbol{v}_h\in\mathbb{V}_h$, and having in mind (2.6), one gets

$$\begin{aligned}
\mathcal{H}(\boldsymbol{v}_h)\boldsymbol{v}_h\ &=\ A(\boldsymbol{v}_h,\boldsymbol{v}_h)+H(\boldsymbol{v}_h)\boldsymbol{v}_h-F(\boldsymbol{v}_h)\\
&\ge\ C\nu|||\boldsymbol{v}_h|||_h^2+\sum_{e\in\mathcal{E}_h^\gamma}\int_e\left|K\boldsymbol{v}_{\boldsymbol{\tau},h}\right|^s ds\\
&\quad-\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa^2\right)^{1/2}|||\boldsymbol{v}_h|||_h-C\,|\Gamma|^{1/2}\,\|\boldsymbol{g}\|_{L^\infty(\Gamma)}|||\boldsymbol{v}_h|||_h\\
&\ge\ |||\boldsymbol{v}_h|||_h\Bigg(c\nu|||\boldsymbol{v}_h|||_h-C\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa^2\right)^{1/2}\\
&\quad-C\|\boldsymbol{g}\|_{L^\infty(\Gamma)}|\Gamma|^{1/2}\Bigg).
\end{aligned}$$

Thus for $\nu|||\boldsymbol{v}_h|||_h=r$ with $r>C\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa^2\right)^{1/2}+C\|\boldsymbol{g}\|_{L^\infty(\Gamma)}|\Gamma|^{1/2}$, $\mathcal{H}(\boldsymbol{v}_h)\boldsymbol{v}_h$ is non negative.

By Brouwer's Fixed point theorem, this proves the existence of at least one solution $\boldsymbol{u}_h\in\mathcal{Z}_h$ of $\mathcal{H}(\boldsymbol{u}_h)\boldsymbol{v}_h=0$ for all $\boldsymbol{v}_h\in\mathcal{Z}_h$. The pressure is constructed in the usual way thank to the inf-sup condition (3.5). The estimates are obtained by taking $\boldsymbol{v}_h=\boldsymbol{u}_h$ is Eq. 2.5, utilization of Eq. 2.6 and the inf-sup condition (3.5).
We next check under (if any) which condition (2.5) has a unique solution. For that purpose, we assume that Eq. 2.5 has two solutions, say, $(\boldsymbol{u}^1,p^1)$ and $(\boldsymbol{u}^2,p^2)$. Then

$$\begin{cases}
\text{for all } (\boldsymbol{v}_h,q_h)\in\mathbb{V}_h\times M_h,\\
A(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{v}_h)+B(\boldsymbol{v}_h,p^1-p^2)+\left(H(\boldsymbol{u}^1)-H(\boldsymbol{u}^2)\right)\\
\boldsymbol{v}_h=\widetilde{d}(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{v}_h)-\widetilde{d}(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{v}_h)\,,\\
B(\boldsymbol{u}^1-\boldsymbol{u}^2,q_h)=0\,.
\end{cases} \qquad (3.11)$$

We take $\boldsymbol{v}_h=\boldsymbol{u}^1-\boldsymbol{u}^2$ and obtain

$$\begin{aligned}
&A(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{u}^1-\boldsymbol{u}^2)+\underbrace{\left(H(\boldsymbol{u}^1)-H(\boldsymbol{u}^2)\right)(\boldsymbol{u}^1-\boldsymbol{u}^2)}_{\ge 0}\\
&=\widetilde{d}(\boldsymbol{u}^2,\boldsymbol{u}^2,\boldsymbol{u}^1-\boldsymbol{u}^2)-\widetilde{d}(\boldsymbol{u}^1,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)\,,
\end{aligned}$$

which implies that

$$\begin{aligned}
&C|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2+\widetilde{d}(\boldsymbol{u}^2,\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{u}^1-\boldsymbol{u}^2)\\
&\qquad\le\widetilde{d}(\boldsymbol{u}^2,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)-\widetilde{d}(\boldsymbol{u}^1,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)\,. \qquad (3.12)
\end{aligned}$$

Now, we bound the right hand side of Eq. 3.12. Using the decomposition of $\widetilde{d}$ as linear and nonlinear term, we have (see Eqs. 3.9 and 3.10)

$$\begin{aligned}
\widetilde{d}^L(\boldsymbol{u}^2,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)-\widetilde{d}^L(\boldsymbol{u}^1,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)&=\widetilde{d}^L(\boldsymbol{u}^2-\boldsymbol{u}^1,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)\\
&\le C|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2|||\boldsymbol{u}^1|||_h\\
&\le\frac{C}{\nu}\left(\|\boldsymbol{f}\|+\|g\|_{L^\infty(\Gamma)}\right)|||\boldsymbol{u}^1\\
&\quad-\boldsymbol{u}^2|||_h^2\,,
\end{aligned} \qquad (3.13)$$

and

$$\begin{aligned}\widetilde{d}^{NL}(\boldsymbol{u}^2,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)-&\widetilde{d}^{NL}(\boldsymbol{u}^1,\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2)\\ &\le C|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2|||\boldsymbol{u}^1|||_h\\ &\le \frac{C}{\nu}(\|\boldsymbol{f}\|+\|\boldsymbol{g}\|)\,|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2\,.\end{aligned} \tag{3.14}$$

Thus returning to Eqs. 3.12 with 3.13 and 3.14 and knowing that the second term on the left hand side of Eq. 3.12 is non-negative (see Eq. 2.6) , one has

$$\begin{aligned}C\nu^2|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2 \le& C\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa^2+|\Gamma|\,\|\boldsymbol{g}\|^2_{L^\infty(\Gamma)}\right)^{1/2}\\ &\times|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2\,,\end{aligned}$$

which implies that $|||\boldsymbol{u}^1-\boldsymbol{u}^2|||_h^2=0$, so $\boldsymbol{u}^1=\boldsymbol{u}^2$. Returning to Eq. 3.11, one obtains that

$$\begin{cases}\text{for all } \boldsymbol{v}_h\in\mathbb{V}_h,\\ B(\boldsymbol{v}_h,p^1-p^2)=0\,,\end{cases}$$

together with the inf-sup condition (3.5) leads to $p^1=p^2$. □

Remark 3.3 *It is worth mentioning that the constants obtained for the uniqueness of solution in Theorems 1.1 and 3.1 are not the same.*

We discuss next the convergence of the DG-solution.

4 A priori error estimate

We assume that $(\boldsymbol{u},p)$ is the unique solution constructed in Theorem 1.1, while $(\boldsymbol{u}_h,p_h)$ is the DG solution constructed in Theorem 3.1. In this section, our goal is to bound the errors $|||\boldsymbol{u}-\boldsymbol{u}_h|||_h$ and $\|p-p_h\|$. The next result is important in our analysis (see also [20, 36]):

Lemma 4.1 *There exists an approximation operator $R_h\in\mathcal{L}(H^1(\Omega)^2;\widetilde{\mathbb{V}}_h)$ satisfying*

$$\begin{aligned}&\forall\,(\boldsymbol{v},q_h)\in H^1(\Omega)^2\times M_h,\quad B(R_h\boldsymbol{v}-\boldsymbol{v},q_h)=0,\\ &\forall\,(s,\boldsymbol{v})\in[1,k+1]\times H^s(\Omega)^2\,,\ |||R_h\boldsymbol{v}-\boldsymbol{v}|||_h\\ &\le ch^{s-1}|\boldsymbol{v}|_{H^s(\Omega)}\,,\\ &\forall\,\boldsymbol{v}\in H^1(\Omega)^2\,,\ |||R_h\boldsymbol{v}|||_h\le c\|D\boldsymbol{v}\|\,.\end{aligned}$$

Proof Let $\pi_h\in\mathcal{L}(H^1(\Omega)^2;\mathbb{V}\cap\widetilde{\mathbb{V}}_h)$ be the Scott-Zhang's interpolation operator (see [37]), and set

$$R_h\boldsymbol{v}=\pi_h\boldsymbol{v}+\boldsymbol{x}_h(\boldsymbol{v})\ \text{ for all } \boldsymbol{v}\in\mathbb{V}$$

with $\boldsymbol{x}_h(\boldsymbol{v})\in\widetilde{\mathbb{V}}_h$ taken (hopefully) such that

$$\text{for all } (\boldsymbol{v},q_h)\in\mathbb{V}\times M_h\,,\quad B(\pi_h\boldsymbol{v}+\boldsymbol{x}_h(\boldsymbol{v})-\boldsymbol{v},q_h)=0\,. \tag{4.1}$$

Ones observes that Eq. 4.1 is equivalent to;

$$\begin{cases}\text{for all } (\boldsymbol{v},q_h)\in H^1(\Omega)^2\times M_h,\\ B(\boldsymbol{x}_h(\boldsymbol{v}),q_h)=B(\boldsymbol{v}-\pi_h\boldsymbol{v},q_h)=\displaystyle\int_\Omega \operatorname{div}(\boldsymbol{v}-\pi_h\boldsymbol{v})q_h dx\\ +\displaystyle\sum_{e\in\mathcal{E}_h^0}\int_e\{q_h\}\underbrace{[\boldsymbol{v}-\pi_h\boldsymbol{v}]}_{=0}\cdot\boldsymbol{n}_e d\sigma\,.\end{cases} \tag{4.2}$$

From Babuska-Brezzi's approach for mixed problems (see [24]), Eq. 4.2 has at least solution $\boldsymbol{x}_h(\boldsymbol{v})\in\widetilde{\mathbb{V}}_h$, which is unique in $(\mathcal{Z}_h)^\perp$ (the orthogonal being taken with respect to $|||\cdot|||_h$). From Lemma 3.4, there exists β^* independent of h such that

$$|||\boldsymbol{x}_h(\boldsymbol{v})|||_h\le\frac{1}{\beta^*}\|\operatorname{div}(\boldsymbol{v}-\pi_h\boldsymbol{v})\|\,. \tag{4.3}$$

Then, the first assertion in Lemma 4.1 follows from Eq. 4.2.
The second assertion is a consequence of the approximation properties of π_h.
Using the triangle's inequality, approximation properties of π_h, Eq. 4.3 one gets

$$\begin{aligned}&|||R_h\boldsymbol{v}|||_h\le|||\pi_h\boldsymbol{v}|||_h+|||x_h(\boldsymbol{v})|||_h\le|||\pi_h\boldsymbol{v}|||_h\\ &+\frac{1}{\beta^*}\|\operatorname{div}(\boldsymbol{v}-\pi_h\boldsymbol{v})\|\le c\|\nabla\boldsymbol{v}\|\end{aligned}$$

this ends the proof. □

We let

$$\boldsymbol{e}_u=\boldsymbol{u}-\boldsymbol{u}_h\ \text{ and }\ e_p=p-p_h\,.$$

We adopt the following decomposition of the errors

$$\boldsymbol{e}_u=\boldsymbol{u}-R_h\boldsymbol{u}+R_h\boldsymbol{u}-\boldsymbol{u}_h \text{ and } e_p=p-r_hp+r_hp-p_h\,.$$

We note that the terms $\boldsymbol{u}-R_h\boldsymbol{u}$ and $p-r_hp$ are bounded in Eqs. 2.4, 2.3 respectively. It is then manifest that we need to find the bounds for $R_h\boldsymbol{u}-\boldsymbol{u}_h$ and r_hp-p_h. In the next result we show that the convergence of the DG solution is uniform with respect to s. To be more precise, we have the following a priori error estimates

Theorem 4.1 *Let $(\boldsymbol{u},p)$ be the unique solution of Eq. 1.5 given in Theorem 1.1 and satisfying $(\boldsymbol{u},p)\in\boldsymbol{H}^2(\Omega)\times$*

$H^1(\Omega)$. *In addition, assume that there exists C independent of h such that*

$$\|\boldsymbol{f}\| + \|\boldsymbol{g}\|_{L^\infty(\Gamma)} < \frac{\nu^2}{C}\,.$$

Then, the DG solution $(\boldsymbol{u}_h, p_h)$ constructed in Theorem 3.1 satisfy

$$\begin{aligned}|||\boldsymbol{u}-\boldsymbol{u}_h|||_h + \|\boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma \le C\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2 \\ + h_T^2\|\nabla(p-r_hp)\|_T^2\Big)^{1/2} \\ + C|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h \\ + C\Big(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau}} - (R_h\boldsymbol{u})_{\boldsymbol{\tau}}\|_e\Big)^{1/2},\end{aligned}$$

$$\begin{aligned}\|p-p_h\| \le C\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2 \\ + h_T^2\|\nabla(p-r_hp)\|_T^2\Big)^{1/2} \\ + C|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h \\ + C\left(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau}} - (R_h\boldsymbol{u})_{\boldsymbol{\tau}}\|_e\right)^{1/2}.\end{aligned}$$

Proof Inf-sup condition on $B(\cdot,\cdot)$. We recall that since $p_h - r_hp$ is an element of M_h, and the couple $(M_h, \widetilde{\mathbb{V}}_h \cap \{\boldsymbol{v}_h|_{\partial\Omega} = \boldsymbol{0}\})$ is inf-sup stable. Then

$$\begin{aligned}\beta^*\|p_h - r_hp\| &\le \sup_{\boldsymbol{v}_h\in\mathbb{V}_h\cap\{\boldsymbol{v}_h|_\Gamma=\boldsymbol{0}\}} \frac{B(\boldsymbol{v}_h, p_h-r_hp)}{|||\boldsymbol{v}_h|||_h} \\ &= \sup_{\boldsymbol{v}_h\in\mathbb{V}_h\cap\{\boldsymbol{v}_h|_\Gamma=\boldsymbol{0}\}} \frac{B(\boldsymbol{v}_h, p_h-p) + B(\boldsymbol{v}_h, p-r_hp)}{|||\boldsymbol{v}_h|||_h}\,.\end{aligned} \tag{4.4}$$

From the consistency equation (see Lemma 2.1), we deduce that

$$\begin{cases}\text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h\times M_h,\\ A(\boldsymbol{u}_h-\boldsymbol{u}, \boldsymbol{v}_h) + B(\boldsymbol{v}_h, p_h-p) + \widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)\\ -\widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) + (H(\boldsymbol{u})-H(\boldsymbol{u}_h))\boldsymbol{v}_h = 0\\ B(\boldsymbol{u}_h-\boldsymbol{u}, q_h) = 0\,.\end{cases} \tag{4.5}$$

Solving for $B(\boldsymbol{v}_h, p_h-p)$ in Eq. 4.5 and replacing in Eq. 4.4, one gets

$$\begin{aligned}\beta^*\|p_h - r_hp\| \le \sup_{v_h\in\mathbb{V}_h\cap\{v_h|_\Gamma=0\}} \\ \frac{A(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + B(\boldsymbol{v}_h, p-r_hp) + \widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) - \widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h}\,.\end{aligned} \tag{4.6}$$

From Cauchy-Schwarz's inequality, Hölder's inequality, the first expression is bounded as follows

$$A(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) \le C|||\boldsymbol{u}-\boldsymbol{u}_h|||_h|||\boldsymbol{v}_h|||_h\,.$$

The second term on the right hand side of Eq. 4.6 is treated with Cauchy-Schwarz's inequality, Hölder's inequality, and Eq. 3.4a, as follows

$$\begin{aligned}B(\boldsymbol{v}_h, p-r_hp) &= -\sum_{T\in\mathcal{T}_h}\int_T (p-r_hp)\,\mathrm{div}\,\boldsymbol{v}_h dx \\ &+ \sum_{e\in\mathcal{E}_h^0}\int_e \{p-r_hp\}[\boldsymbol{v}_h]\cdot\boldsymbol{n}_e ds \\ &\le \sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T\|\nabla\boldsymbol{v}_h\|_T + \sum_{e\in\mathcal{E}_h^0}\|\{p-r_hp\}\|_e\|[\boldsymbol{v}_h]\|_e \\ &\le \Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}_h\|_T^2\Big)^{1/2} \\ &+ C\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2 + h_T^2\|\nabla(p-r_hp)\|_T^2\Big)^{1/2} \\ &\times\Big(\sum_{e\in\mathcal{E}_h^0}\frac{1}{h_e}\|[\boldsymbol{v}_h]\|_e^2\Big)^{1/2} \le C\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2 \\ &+ h_T^2\|\nabla(p-r_hp)\|_T^2\Big)^{1/2}|||\boldsymbol{v}_h|||_h\,.\end{aligned}$$

Now, for the term $\widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) - \widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)$, we have

$$\begin{aligned}&\widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) - \widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) \\ &\quad= \widetilde{d}(\boldsymbol{u},\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{v}_h) + \widetilde{d}(\boldsymbol{u}_h, R_h\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) \\ &\quad\le C||\boldsymbol{u}||_1\,|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h\,|||\boldsymbol{v}_h|||_h + C|||\boldsymbol{u}_h|||_h \\ &\quad\times|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h|||\boldsymbol{v}_h|||_h \le \frac{C}{\nu}\left(\|\boldsymbol{f}\| + \|\boldsymbol{g}\|_{L^\infty(\Gamma)}\right) \\ &\quad\times(|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h + |||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h)\,|||\boldsymbol{v}_h|||_h\,.\end{aligned}$$

So, returning to Eq. 4.6, one obtains

$$\begin{aligned}\beta^*\|p_h - r_hp\| &\le C|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h + C|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h \\ &+ \frac{C}{\nu}\left(\|\boldsymbol{f}\| + \|\boldsymbol{g}\|_{L^\infty(\Gamma)}\right)(|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h \\ &+ |||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h) + \Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2 \\ &+ h_T^2\|\nabla(p-r_hp)\|_T^2\Big)^{1/2}.\end{aligned} \tag{4.7}$$

coercivity of $A(\cdot,\cdot)$. We recall that $R_h\boldsymbol{u} - \boldsymbol{u}_h \in \widetilde{\mathbb{V}}_h$ and $A(\cdot,\cdot)$ is coercive in that space, thus one has

$$\begin{aligned}C\nu|||\boldsymbol{u}_h - R_h\boldsymbol{u}|||_h^2 &\le A(\boldsymbol{u}_h-R_h\boldsymbol{u}, \boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &= A(\boldsymbol{u}_h-\boldsymbol{u}, \boldsymbol{u}_h-R_h\boldsymbol{u})\end{aligned}$$

$$+A(\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ = T_1+T_2\,, \tag{4.8}$$

with $T_1 = A(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u})$ and $T_2 = A(\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u})$. In the lines that follows, we bound from above T_1 and T_2. We begin with T_2, From Cauchy-Schwarz's inequality, and Hölder's inequality, we have we have

$$T_2 = A(\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \le C|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h|||\boldsymbol{u}_h-R_h\boldsymbol{u}|||_h\,. \tag{4.9}$$

From Lemma 2.1 (consistency equation), one deduces that (taking $\boldsymbol{v}_h=\boldsymbol{u}_h-R_h\boldsymbol{u}$)

$$A(\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) = F(\boldsymbol{u}_h-R_h\boldsymbol{u}) - B(\boldsymbol{u}_h-R_h\boldsymbol{u},p) \\ -\widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) - H(\boldsymbol{u})(\boldsymbol{u}_h-R_h\boldsymbol{u})\,. \tag{4.10}$$

But we recall that $\boldsymbol{u}_h$ satisfies

$$A(\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u}) = F(\boldsymbol{u}_h-R_h\boldsymbol{u}) - B(\boldsymbol{u}_h-R_h\boldsymbol{u},p_h) \\ -\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u}) - H(\boldsymbol{u}_h)(\boldsymbol{u}_h-R_h\boldsymbol{u}). \tag{4.11}$$

Thus putting together (4.10) and (4.11), one has

$$\begin{aligned} T_1 &= A(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &= A(\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u}) - A(\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &= B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-p_h) + \widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &\quad -\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &\quad +(H(\boldsymbol{u})-H(\boldsymbol{u}_h))\,(\boldsymbol{u}_h-R_h\boldsymbol{u})\,. \end{aligned} \tag{4.12}$$

Using the second equation in Eq. 4.5 and the first relation in Lemma 4.1, we deduce that

$$B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-p_h) = B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-r_hp)\,. \tag{4.13}$$

We replace (4.13) in (4.12), and obtain

$$T_1 = B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-r_hp) + \widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ -\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u}) + (H(\boldsymbol{u})-H(\boldsymbol{u}_h))(\boldsymbol{u}_h-R_h\boldsymbol{u})\,,$$

which is re-written as follows

$$\begin{aligned} &T_1+(H(\boldsymbol{u})-H(\boldsymbol{u}_h))\,(\boldsymbol{u}-\boldsymbol{u}_h)+\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &= B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-r_hp) + (H(\boldsymbol{u})-H(\boldsymbol{u}_h))\,(\boldsymbol{u}-R_h\boldsymbol{u}) \\ &\quad +\widetilde{d}(\boldsymbol{u},\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &\quad +\widetilde{d}(\boldsymbol{u},R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) - \widetilde{d}(\boldsymbol{u}_h,R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u})\,. \end{aligned} \tag{4.14}$$

Using Eq. 3.1, and the arguments in [33], one can find α_1,α_2 two positive constants such that

$$\alpha_1\frac{\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{\alpha_2+\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}} \le (H(\boldsymbol{u})-H(\boldsymbol{u}_h))\,(\boldsymbol{u}-\boldsymbol{u}_h).$$

Next, Eq. 2.6 implies that $\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \ge 0$. Thus returning to Eq. 4.14, one gets

$$\begin{aligned} &T_1+\alpha_1\frac{\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^2}{\alpha_2+\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}} \le T_1+(H(\boldsymbol{u})-H(\boldsymbol{u}_h))\,(\boldsymbol{u}-\boldsymbol{u}_h) \\ &+\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \le B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-r_hp) \\ &+(H(\boldsymbol{u})-H(\boldsymbol{u}_h))\,(\boldsymbol{u}-R_h\boldsymbol{u}) + \widetilde{d}(\boldsymbol{u},\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &+\widetilde{d}(\boldsymbol{u},R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) - \widetilde{d}(\boldsymbol{u}_h,R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u})\,. \end{aligned} \tag{4.15}$$

We now bound the right hand side of Eq. 4.15. Now from Cauchy-Schwarz's inequality, Hölder's inequality, and Eq. 3.4a, we have

$$B(\boldsymbol{u}_h-R_h\boldsymbol{u},p-r_hp) \le C\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2+h_T^2\|\nabla \\ \times(p-r_hp)\|_T^2\Big)^{1/2}|||\boldsymbol{u}_h-R_h\boldsymbol{u}|||_h\,. \tag{4.16}$$

The third and fourth expressions on the right hand side of Eq. 4.15 are treated with the same arguments presented in [21, Section 4], and one obtains

$$\begin{aligned} \widetilde{d}(\boldsymbol{u},\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) &\le C||\boldsymbol{u}||_1|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h|||\boldsymbol{u}_h-R_h\boldsymbol{u}|||_h \\ &\le \frac{C}{\nu}\left(\|\boldsymbol{f}\|+\|g\|_{L^\infty(\Gamma)}\right)|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h \\ &\quad\times|||\boldsymbol{u}_h-R_h\boldsymbol{u}|||_h\,, \end{aligned}$$

and

$$\begin{aligned} &\widetilde{d}(\boldsymbol{u},R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) - \widetilde{d}(\boldsymbol{u}_h,R_h\boldsymbol{u},\boldsymbol{u}_h-R_h\boldsymbol{u}) \\ &\le C|||\boldsymbol{u}-\boldsymbol{u}_h|||_h|||R_h\boldsymbol{u}|||_h|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h \\ &\le C\left(|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h+|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h\right) \\ &\quad\times||\boldsymbol{u}||_1\,|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h \\ &\le \frac{C}{\nu}\left(\|\boldsymbol{f}\|+\|g\|_{L^\infty(\Gamma)}\right)(|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h \\ &\quad+|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h)\,|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h\,. \end{aligned} \tag{4.17}$$

The second term on the right hand side of Eq. 4.15 is treated with the help of Cauchy-Schwarz's and Hölder's inequality, and Eq. 3.2 as follows

$$(H(\boldsymbol{u})-H(\boldsymbol{u}_h))(\boldsymbol{u}_h-R_h\boldsymbol{u})=\sum_{e\in\mathcal{E}_h^\gamma}\int_e\Big(|K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau}}$$
$$-\left|K\boldsymbol{u}_{\boldsymbol{\tau},h}\right|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}\Big)\cdot K(\boldsymbol{u}_{\boldsymbol{\tau},h}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}})ds$$
$$\leq\sum_{e\in\mathcal{E}_h^\gamma}\int_e\left||K\boldsymbol{u}_{\boldsymbol{\tau}}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau}}-\left|K\boldsymbol{u}_{\boldsymbol{\tau},h}\right|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}\right|$$
$$\times|K(\boldsymbol{u}_{\boldsymbol{\tau},h}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}})|ds\leq C\|K\|^s_{L^\infty(\Gamma)}\sum_{e\in\mathcal{E}_h^\gamma}\int_e\left|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\right|^{s-1}$$
$$\times|\boldsymbol{u}_{\boldsymbol{\tau},h}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}}|ds\leq C\|K\|_{L^\infty(\Gamma)}\sum_{e\in\mathcal{E}_h^\gamma}\left(\int_e\left|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\right|^{2(s-1)}\right)^{1/2}$$
$$\times\left(\int_e|\boldsymbol{u}_{\boldsymbol{\tau},h}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}}|^2\right)^{1/2}\leq C\|K\|^s_{L^\infty(\Gamma)}\sum_{e\in\mathcal{E}_h^\gamma}\left\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\right\|_e^{s-1}$$
$$\times\left\|\boldsymbol{u}_{\boldsymbol{\tau},h}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}}\right\|_e. \tag{4.18}$$

So, returning to Eqs. 4.15 with 4.18, 4.17 and 4.16, one obtains

$$T_1+\alpha_1\frac{\|\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|^2_\Gamma}{\alpha_2+\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_\Gamma^{2-s}+\|\boldsymbol{u}_{\boldsymbol{\tau},h}\|_\Gamma^{2-s}}$$
$$\leq C\Big[\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2\Big)^{1/2}+\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2$$
$$+h_T^2\|\nabla(p-r_hp)\|_T^2\Big)^{1/2}\Big]|||\boldsymbol{u}_h-R_h\boldsymbol{u}|||_h$$
$$+C\left(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_e^{s-1}+\left\|\boldsymbol{u}_{\boldsymbol{\tau},h}\right\|_e^{s-1}\right)\left(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau}}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}}\|_e^2\right)^{1/2}$$
$$+\frac{C}{\nu}\left(\|\boldsymbol{f}\|+\|g\|_{L^\infty(\Gamma)}\right)\left(|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h+|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h\right)$$
$$\times|||R_h\boldsymbol{u}-\boldsymbol{u}_h|||_h. \tag{4.19}$$

We replace Eqs. 4.19 and 4.9 in 4.8, using Eq. 3.4b and Young's inequality, one obtains

$$\left(\nu-\frac{C}{\nu}\left(\|\boldsymbol{f}\|+\|g\|_{L^\infty(\Gamma)}\right)\right)|||\boldsymbol{u}_h-R_h\boldsymbol{u}|||_h^2$$
$$\leq C\Big[\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2+h_T^2\|\nabla(p-r_hp)\|_T^2\Big]+C|||R_h\boldsymbol{u}-\boldsymbol{u}|||_h^2$$
$$+C\left(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_e^{s-1}+\left\|\boldsymbol{u}_{\boldsymbol{\tau},h}\right\|_e^{s-1}\right)\left(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau}}-(R_h\boldsymbol{u})_{\boldsymbol{\tau}}\|_e^2\right)^{1/2}. \tag{4.20}$$

Using the inequality of the triangle ends the proof. We recall that the a priori error for the pressure is obtained by using Eqs. 4.7 and 4.20. □

Remark 4.1 *From Theorem 4.1, its appears that the DG solution converges strongly to the solution* $(\boldsymbol{u}, p)$ *when* h *tends to zero. Hence the DG schemes (2.5) are good approximation of* Eq. 1.5.

Remark 4.2 *It is manifest from Theorem 4.1 that the error on the velocity and the pressure are dominated by the interpolation error on the slip zone* Γ*, and uniform with respect the parameter* s*. Using* Eq. 3.4a*, one obtains*

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_h+\|p-p_h\|\leq C\Big(\sum_{T\in\mathcal{T}_h}\|p-r_hp\|_T^2+h_T^2\|$$
$$\times\nabla(p-r_hp)\|_T^2\Big)^{1/2}$$
$$+C|||\boldsymbol{u}-R_h\boldsymbol{u}|||_h$$
$$+C\Big(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{-1/2}\|\boldsymbol{u}-R_h\boldsymbol{u}\|_\kappa$$
$$+h_\kappa^{1/2}\|\nabla(\boldsymbol{u}-R_h\boldsymbol{u})\|_\kappa\Big)^{1/2}. \tag{4.21}$$

The fact that the convergence is dominated by the interpolation error on the slip zone is not new, and symptomatic to these problems (see [1–3, 14]). Thus if $k=1$, $(\boldsymbol{u},p)\in\boldsymbol{H}^2(\Omega)\times H^1(\Omega)$*, then* Eq. 4.21 *gives*

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_h+\|p-p_h\|\leq Ch^{3/4}.$$

If moreover $\boldsymbol{u}_{\boldsymbol{\tau}}\in\boldsymbol{H}^2(\Gamma)$*, then*

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_h+\|p-p_h\|\leq Ch.$$

In [14], first order convergence rate is obtained with the velocity and pressure approximated by Mini-element, while in [20, 21] first order convergence rate is obtained for Navier-Stokes equations with Dirichlet boundary condition.

Since the discrete system of Eq. 2.5 is nonlinear, it can be solved by some iterative schemes. Considering that $1<s<2$, the nonlinearity on Γ has a poor differentiability, implying that iterative methods such as Newton's or conjugate gradient will encounter convergence difficulties. We discuss next an iterative method for the resolution of Eq. 2.5.

5 Iterative scheme and simulations

5.1 Iterative scheme

We solve DG schemes by adopting the following strategy. Let $\boldsymbol{u}_h^0\in\mathbb{V}_h$ be a given initial guess. we compute the sequence

$(\boldsymbol{u}^n, p_h^n)$ for $n \geq 1$, until an adequate stopping condition is satisfied, by computing $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ solution of

$$\begin{aligned}
&\text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h\,,\\
&A(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + \sum_{e \in \mathcal{E}_h^\gamma} \int_e |K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} \cdot K\boldsymbol{v}_{\boldsymbol{\tau},h} ds\\
&+ \widetilde{d}(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - B(\boldsymbol{v}_h, p_h^{n+1}) = F(\boldsymbol{v}_h)\,,\\
&B(\boldsymbol{u}_h^{n+1}, q_h) = 0\,.
\end{aligned} \tag{5.1}$$

After the convergence of the system (5.1), we get the numerical solution $(\boldsymbol{u}_h, p_h)$ of the DG schemes. One of the principal interest of the fixed point scheme (5.1) is that it converges to the unique fixed point for any starting point, but that convergence is slow. About the feasibility of Eq. 5.1, we observe that it is linear problem, and from Babuska-Brezzi's theory for mixed problems (see [24, 25]). $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ is well defined. Continuing on the qualitative analysis on Eq. 2.5, we now have the following result

Lemma 5.1 *(see [38]) The algorithm (5.1) is consistent with* Eq. 2.5 *in the sense that if $(\boldsymbol{u}_h^1, p_h^1) = (\boldsymbol{u}_h, p_h)$ with $(\boldsymbol{u}_h, p_h)$ solution of* Eq. 2.5*, then for all $n \geq 2$, $(\boldsymbol{u}_h^n, p_h^n) = (\boldsymbol{u}_h, p_h)$.*

Proof It is done by indiction on n. We assume that $(\boldsymbol{u}_h^n, p_h^n) = (\boldsymbol{u}_h, p_h)$, and we will show that $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) = (\boldsymbol{u}_h, p_h)$.

We use the induction hypothesis, take the difference between Eqs. 5.1 and 2.5. This gives

$$\begin{cases}
\text{for all } (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h\,,\\
A(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, \boldsymbol{v}_h) + \sum_{e \in \mathcal{E}_h^\gamma} \Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}\\
-|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} K\boldsymbol{u}_{\boldsymbol{\tau},h}, K\boldsymbol{v}_{\boldsymbol{\tau},h}\Big)_e\\
+\widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + B(\boldsymbol{v}_h, p_h^{n+1} - p_h) = 0\,,\\
B(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, q_h) = 0\,.
\end{cases} \tag{5.2}$$

We take $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h$ and $q_h = p_h^{n+1} - p_h$ in Eq. 5.2, and obtain

$$\begin{aligned}
&A(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)\\
&+ \sum_{e \in \mathcal{E}_h^\gamma} \int_e \underbrace{|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2} \left(K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - K\boldsymbol{u}_{\boldsymbol{\tau},h}\right) \cdot (K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1} - K\boldsymbol{u}_{\boldsymbol{\tau},h})}_{\geq 0}\\
&+ \underbrace{\widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h, \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h)}_{\geq 0} = 0
\end{aligned}$$

from which we deduce after utilization of the coercivity of $A(\cdot,\cdot)$ that $\boldsymbol{u}_h^{n+1} = \boldsymbol{u}_h$. Thus the system of Eq. 5.2 becomes

$$\begin{cases}
\text{for all } \boldsymbol{v}_h \in \mathbb{V}_h\,,\\
B(\boldsymbol{v}_h, p_h^{n+1} - p_h) = 0\,.
\end{cases} \tag{5.3}$$

The inf-sup condition on $B(\cdot,\cdot)$ and utilization of Eq. 5.3 implies that $p_h^{n+1} = p_h$. □

In order to justify why we use the iterative scheme Eq. 5.1 (implemented in paragraph 5.2) to compute the solution of Eq. 2.5, we shall prove a convergence result. For that purpose, we first have the following result

Lemma 5.2 *Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ the solution of the algorithm (5.1). Then there exists a constant C independent of h and n such that*

$$|||\boldsymbol{u}_h^{n+1}|||_h \leq \frac{C}{\nu}\left(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2\right)^{1/2} + \frac{C}{\nu}\left(\sum_{e \in \mathcal{E}_h^\gamma} ||\boldsymbol{g}||_e^2\right)^{1/2},$$

and

$$\begin{aligned}
||p_h^{n+1}|| \leq{}& C\left(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2\right)^{1/2} + C\left(\sum_{e \in \mathcal{E}_h^\gamma} ||g||_e^2\right)^{1/2}\\
&+\frac{C}{\nu^2}\left(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2\right) + \frac{C}{\nu^2}\left(\sum_{e \in \mathcal{E}_h^\gamma} ||\boldsymbol{g}||_e^2\right).
\end{aligned}$$

Proof We take $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$ in Eq. 5.1 and obtain

$$\begin{aligned}
A(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}) &+ \widetilde{d}(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1})\\
&+ \sum_{e \in \mathcal{E}_h^\gamma} \int_e |K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2} \left|K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}\right|^2 ds\\
&= F(\boldsymbol{u}_h^{n+1}) \leq \left(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2\right)^{1/2} |||\boldsymbol{u}_h^{n+1}|||_h + c\\
&\times \left(\sum_{e \in \mathcal{E}_h^\gamma} ||\boldsymbol{g}||_e^2\right)^{1/2} |||\boldsymbol{u}_h^{n+1}|||_h
\end{aligned}$$

which implies that

$$|||\boldsymbol{u}_h^{n+1}|||_h \leq \frac{C}{\nu}\left(\sum_{\kappa \in \mathcal{T}_h} \|\boldsymbol{f}\|_\kappa^2\right)^{1/2} + \frac{C}{\nu}\left(\sum_{e \in \mathcal{E}_h^\gamma} ||\boldsymbol{g}||_e^2\right)^{1/2}.$$

From the inf-sup condition on $B(\cdot,\cdot)$ (see Lemma 3.4), we obtain that

$$\begin{aligned}\beta^*||p_h^{n+1}|| &\le \sup_{\boldsymbol{v}_h\in\mathbb{V}_h\cap\{\boldsymbol{v}|_\Gamma=0\}} \frac{B(\boldsymbol{v}_h, p_h^{n+1})}{|||\boldsymbol{v}_h|||_h} \\ &\le \sup_{\boldsymbol{v}_h\in\mathbb{V}_h\cap\{\boldsymbol{v}|_\Gamma=0\}} \frac{\left|-F(\boldsymbol{v}_h)+A(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)+\widetilde{d}(\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)\right|}{|||\boldsymbol{v}_h|||_h} \\ &\le C\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa^2\right)^{1/2} + C\left(\sum_{e\in\mathcal{E}_h^\gamma}||\boldsymbol{g}||_e^2\right)^{1/2} + \nu C|||\boldsymbol{u}_h^{n+1}|||_h \\ &\quad + C|||\boldsymbol{u}_h^n|||_h|||\boldsymbol{u}_h^{n+1}|||_h\,.\end{aligned}$$

We replace the estimate on the velocity to obtain the estimate on the pressure. □

Based on Lemma 5.2 and following similar result in [14], one can show that

Proposition 5.1 *Let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ be the solution of* Eq. 5.1, *and $(\boldsymbol{u}_h, p_h)$ the solution of* Eq. 2.5. *Then*

$$\lim_{n\to\infty}\boldsymbol{u}_h^n = \boldsymbol{u}_h \quad \textit{weakly} \quad \textit{in} \quad \mathbb{V}\,.$$

The main result on this paragraph is the following

Proposition 5.2 *Set $s\in(1,2)$, let $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ be the solution of* Eq. 5.1, *and $(\boldsymbol{u}_h, p_h)$ the solution of* Eq. 2.5. *Then there exists C independent of both h and n such that*

$$\begin{aligned}|||\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h|||_h^2 \le& C\|K\|_{L^\infty(\Gamma)}^s \sum_{\kappa\in\mathcal{T}_h} h_k^{-s/2} \\ &\times \|\boldsymbol{u}_h-\boldsymbol{u}_h^n\|_\kappa^{s-1}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h\|_\kappa\,.\end{aligned}$$

If moreover

$$C\|K\|_{L^\infty(\Gamma)}^s \max_{k\in\mathcal{T}_h} h_k^{-s/2} < 1\,,$$

then the iterative scheme (5.1) converge with a rate $s-1$.

Proof The difference between Eqs. 5.1 and 2.5 gives

$$\begin{cases}\text{for all } (\boldsymbol{v}_h,q_h)\in\mathbb{V}_h\times M_h\,,\\ A(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h,\boldsymbol{v}_h)+\sum_{e\in\mathcal{E}_h^\gamma}\Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}\\ -|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h},\,K\boldsymbol{v}_{\boldsymbol{\tau},h}\Big)_e\\ +\widetilde{d}(\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)-\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)-B(\boldsymbol{v}_h,p_h^{n+1}-p_h)=0,\\ B(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h,q_h)=0\,.\end{cases} \tag{5.4}$$

For $(\boldsymbol{v}_h,q_h)=(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h, p_h^{n+1}-p_h)$ in Eq. 5.4, one has

$$\begin{aligned}&A(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)+\widetilde{d}(\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)\\ &+\sum_{e\in\mathcal{E}_h^\gamma}\int_e|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}\left|K\boldsymbol{u}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}\right|^2dx\\ =&\,\widetilde{d}(\boldsymbol{u}_h-\boldsymbol{u}_h^n,\boldsymbol{u}_h,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)+\sum_{e\in\mathcal{E}_h^\gamma}\Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}\\ &-|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^n,\,K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-K\boldsymbol{u}_{\boldsymbol{\tau},h}\Big)_e\\ &+\sum_{e\in\mathcal{E}_h^\gamma}\Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}\left(K\boldsymbol{u}_{\boldsymbol{\tau},h}^n-K\boldsymbol{u}_{\boldsymbol{\tau},h}\right),K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-K\boldsymbol{u}_{\boldsymbol{\tau},h}\Big)_e\\ =&\,X_1+X_2+X_3\,.\end{aligned} \tag{5.5}$$

We observe that the left hand side of Eq. 5.5 is non negative. To estimate the right hand side of Eq. 5.5, and starting with $\widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)-\widetilde{d}(\boldsymbol{u}_h^n,\boldsymbol{u}_h,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)$, we follow to the line the estimates obtained when studying the continuity of $\mathcal{H}$, and deduce that

$$\begin{aligned}X_1 &= \widetilde{d}(\boldsymbol{u}_h-\boldsymbol{u}_h^n,\boldsymbol{u}_h,\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h)\\ &\le C|||\boldsymbol{u}_h-\boldsymbol{u}_h^n|||_h|||\boldsymbol{u}_h|||_h|||\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h|||_h\\ &\le \frac{C}{\nu}\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa+\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{g}\|_e\right)|||\boldsymbol{u}_h-\boldsymbol{u}_h^n|||_h\\ &\quad\times|||\boldsymbol{u}_h-\boldsymbol{u}_h^{n+1}|||_h\,.\end{aligned}$$

Secondly

$$\begin{aligned}X_2 =& \sum_{e\in\mathcal{E}_h^\gamma}\Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}-|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}K\boldsymbol{u}_{\boldsymbol{\tau},h}^n,\,K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-K\boldsymbol{u}_{\boldsymbol{\tau},h}\Big)_e\\ &\le C\|K\|_{L^\infty(\Gamma)}\sum_{e\in\mathcal{E}_h^\gamma}|K\boldsymbol{u}_{\boldsymbol{\tau},h}-K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-1}\left|\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-\boldsymbol{u}_{\boldsymbol{\tau},h}\right|ds\\ &\le C\|K\|_{L^\infty(\Gamma)}^s\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau},h}^n\|_{L^{2(s-1)}(e)}^{s-1}\|\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_e\\ &\le C\|K\|_{L^\infty(\Gamma)}^s\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_{\boldsymbol{\tau},h}-\boldsymbol{u}_{\boldsymbol{\tau},h}^n\|_e^{s-1}\|\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-\boldsymbol{u}_{\boldsymbol{\tau},h}\|_e\\ &\le C\|K\|_{L^\infty(\Gamma)}^s\sum_{\kappa\in\mathcal{T}_h}h_k^{-s/2}\|\boldsymbol{u}_h-\boldsymbol{u}_h^n\|_\kappa^{s-1}\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h\|_\kappa\end{aligned}$$

where we have used Eq. 3.4b in the last inequality. Thirdly,

$$\begin{aligned}X_3 =& \sum_{e\in\mathcal{E}_h^\gamma}\Big(|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}\left(K\boldsymbol{u}_{\boldsymbol{\tau},h}^n-K\boldsymbol{u}_{\boldsymbol{\tau},h}\right),K\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-K\boldsymbol{u}_{\boldsymbol{\tau},h}\Big)_e\\ &\le \|K\|_{L^\infty(\Gamma)}^2\sum_{e\in\mathcal{E}_h^\gamma}\int_e|K\boldsymbol{u}_{\boldsymbol{\tau},h}^n|^{s-2}|\boldsymbol{u}_{\boldsymbol{\tau},h}^n-\boldsymbol{u}_{\boldsymbol{\tau},h}|\,|\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-\boldsymbol{u}_{\boldsymbol{\tau},h}|ds\\ &\le C\|K\|_{L^\infty(\Gamma)}^2K_m^{s-2}\Big(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_h^n\|_{L^\infty(e)}^{s-2}\Big)\Big(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{-1}\left\|\boldsymbol{u}_h-\boldsymbol{u}_h^n\right\|_\kappa\\ &\quad\times\left\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h\right\|_\kappa\Big)\,,\end{aligned}$$

with K_m the smallest eigen-value of K. Using the coercivity of $A(\cdot,\cdot)$ (see Lemma 3.2), and the bounds obtained for X_1, X_2 and X_3, Eq. 5.5 becomes

$$\begin{aligned}|||\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h|||_h^2 \leq \frac{C}{\nu}&\left(\sum_{\kappa\in\mathcal{T}_h}\|\boldsymbol{f}\|_\kappa+\sum_{e\in\mathcal{E}_h^\gamma}\|g\|_e\right)\\
&\times|||\boldsymbol{u}_h-\boldsymbol{u}_h^n|||_h|||\boldsymbol{u}_h-\boldsymbol{u}_h^{n+1}|||_h\\
&+C\|K\|_{L^\infty(\Gamma)}^s\sum_{\kappa\in\mathcal{T}_h}h_k^{-s/2}\|\boldsymbol{u}_h-\boldsymbol{u}_h^n\|_\kappa^{s-1}\\
&\times\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h\|_\kappa\\
&+C\|K\|_{L^\infty(\Gamma)}^2K_m^{s-2}\Big(\sum_{e\in\mathcal{E}_h^\gamma}\|\boldsymbol{u}_h^n\|_{L^\infty(e)}^{s-2}\Big)\\
&\times\Big(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{-1}\left\|\boldsymbol{u}_h-\boldsymbol{u}_h^n\right\|_\kappa\left\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h\right\|_\kappa\Big).\end{aligned}$$

The conclusion follows after application of Cauchy-Schwarz's inequality and Eq. 3.3. □

5.2 Numerical experiments

In this section, we show numerical simulations using the FreeFem++ code (see [39]). We validate the theoretical results obtained in Section 4. For that purpose, we evaluate the performance of the algorithm Eq. 5.1 by solving two problems. First, we assess the convergence properties of the method by using the standard mesh refinement analysis. The second example is concerned with the lid-Driven cavity, a well studied problem in computational mechanics. In this paragraph, we do not considered Bassi et al or LDG approaches because of the similarity with SIPG scheme. Thus the bilinear form $A(\cdot,\cdot)$ we considered are

$$\begin{aligned}A^{SIPG}(\boldsymbol{v},\boldsymbol{w})=2\nu&\sum_{K\in\mathcal{T}_h}\int_K D\boldsymbol{v}:D\boldsymbol{w}dx\\
&-\nu\sum_{e\in\mathcal{E}_h^0}\int_e\left(\{D\boldsymbol{v}\}\boldsymbol{n}_e\cdot[\boldsymbol{w}]+\{D\boldsymbol{w}\}\boldsymbol{n}_e\cdot[\boldsymbol{v}]\right)ds\\
&+\nu\sum_{e\in\mathcal{E}_h^0}\int_e\frac{\alpha_s}{h_e}[\boldsymbol{v}]\cdot[\boldsymbol{w}]ds\,,\\
A^{NIPG}(\boldsymbol{v},\boldsymbol{w})=2\nu&\sum_{K\in\mathcal{T}_h}\int_K D\boldsymbol{v}:D\boldsymbol{w}dx\\
&-\nu\sum_{e\in\mathcal{E}_h^0}\int_e\left(\{D\boldsymbol{v}\}\boldsymbol{n}_e\cdot[\boldsymbol{w}]-\{D\boldsymbol{w}\}\boldsymbol{n}_e\cdot[\boldsymbol{v}]\right)ds\\
&+\nu\sum_{e\in\mathcal{E}_h^0}\int_e\frac{\alpha_n}{h_e}[\boldsymbol{v}]\cdot[\boldsymbol{w}]ds\,,\\
A^{IIPG}(\boldsymbol{v},\boldsymbol{w})=2\nu&\sum_{K\in\mathcal{T}_h}\int_K D\boldsymbol{v}:D\boldsymbol{w}dx\\
&+\nu\sum_{e\in\mathcal{E}_h^0}\int_e\frac{\alpha_I}{h_e}[\boldsymbol{v}]\cdot[\boldsymbol{w}]ds\,,\end{aligned}$$

with the stabilization parameters α_s, α_n and α_I being bounded below by sufficiently large positive reals α_{0s}, α_{0n} and α_{0I}. We denote by $SIGP$, $NIPG$ and $IIPG$ the iterative problem (5.1) with the above corresponding bilinear forms. The algebraic representation of Eq. 5.1 reads

$$\begin{aligned}&\mathcal{A}\boldsymbol{u}_h^{n+1}+\mathcal{G}(\boldsymbol{u}_{\boldsymbol{\tau},h}^n)\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}+\mathcal{D}(\boldsymbol{u}_h^n)\boldsymbol{u}_h^{n+1}-\mathcal{B}^Tp_h^{n+1}=\mathcal{F}\,,\\
&\mathcal{B}\boldsymbol{u}_h^{n+1}=0\,,\end{aligned}\tag{5.6}$$

where $\mathcal{A}$ corresponds to the velocity terms, $\mathcal{G}(\boldsymbol{u}_{\boldsymbol{\tau},h}^n)$ is the linearization of the nonlinear slip on Γ, $\mathcal{D}(\boldsymbol{u}_h^n)$ is the linearization of the convective term, $\mathcal{F}$ is the vector representing the external forces on the system, $\mathcal{B}$ and $\mathcal{B}^T$ the coupling between velocity and pressure.

Remark 5.1 *We note that the stiffness matrix in* Eq. 5.6 *has a sparse block structure, consisting of two types of matrix sub blocks: (i) diagonal blocks, which describe interaction of an element degrees of freedom with itself, (ii) off diagonal blocks, which describe interactions between the degrees of freedom of K_+ and the degrees of freedom of K_- through edge.*

Remark 5.2 *The iterative scheme* (5.6) *is easy to implement in practice, the only difficulty being the choice of the turning parameter α. If α is too big then the discontinuity flavor will disappear, but taking α too small may be detrimental for SIPG (which may be unstable). Nevertheless, using FreeFem++ which is a powerful and flexible tool for numerical computations, the formulation has: (i) optimal a priori estimates with DG norm, (ii) the nonlinear slip occurs only on Γ, so it has little impact on the size of the problem.*

We consider $\boldsymbol{u}_h^0=\boldsymbol{0}$ and we approximate the velocity and pressure by $\mathcal{P}_1(\kappa)^2$ and $\mathcal{P}_0(\kappa)$ respectively. We stopped the computation if Eq. 5.7 is satisfied

$$\frac{|||\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n|||_h+||\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}-\boldsymbol{u}_{\boldsymbol{\tau},h}^n||_\Gamma+||p_h^{n+1}-p_h^n||}{|||\boldsymbol{u}_h^{n+1}|||_h+||\boldsymbol{u}_{\boldsymbol{\tau},h}^{n+1}||_\Gamma+||p_h^{n+1}||}\leq 10^{-5}.\tag{5.7}$$

5.2.1 Navier-Stokes flow in square domain

We begin by checking numerically the convergence properties of the method by means of comparison with a manufactured solution on fine mesh in a square geometry.

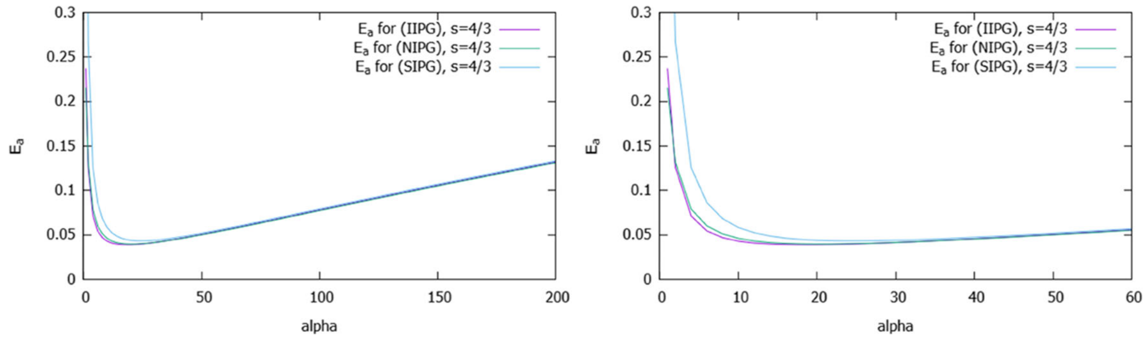

Fig. 1 E_α associated to the iterative schemes $SIGP$, $NIPG$ and $IIPG$ for $s = 4/3$ and $\alpha \in [1, 200]$ (zoom on the right)

The geometry is $\Omega = (0, 1)^2$, and each edge of the domain is divided into N segments of equal length, thus the corresponding mesh contains $2N^2$ elements. We set $\nu = 1$, $K = I$ and $\boldsymbol{f} = (yx^3 + xy^3, xy^2)^T$, $\boldsymbol{g} = (0, 0)^T$. We take the isotropic slip boundary conditions (1.4) on $\Gamma = \Big\{(x, y)|\ x = 1 \text{ or } y = 1\Big\} = \overline{\Gamma_2 \cup \Gamma_3}$, and $\boldsymbol{u} = \boldsymbol{0}$ on $S = \overline{\Gamma_1 \cup \Gamma_4}$ (bottom and left boundaries).

We consider the exact solution to be $(\boldsymbol{u}_{ref}, p_{ref})$, a finite element solution computed using $P_2(\kappa)^2 \times P_1(\kappa)$ element for $N = 100$. Thus, we test the algorithms $SIGP$, $NIPG$ and $IIPG$ for $N = 40$ when the stabilization parameters α_I, α_n and α_s are in $[0, 200]$. Figures 1, 2 and 3 (with a zoom on the right) show the following error

$$E_a = \frac{|||\boldsymbol{u}_{ref} - \boldsymbol{u}_h|||_h + ||\boldsymbol{u}_{\boldsymbol{\tau},ref} - \boldsymbol{u}_{\tau,h}||_\Gamma + ||p_{ref} - p_h||}{|||\boldsymbol{u}_{ref}|||_h + ||\boldsymbol{u}_{\boldsymbol{\tau},ref}||_\Gamma + ||p_{ref}||} \tag{5.8}$$

where $\boldsymbol{u}_h$ is computed by using the iterative schemes $SIGP$, $NIPG$ and $IIPG$. We can deduce that the schemes $NIPG$ and $IIPG$ give similar results while the results obtained with the scheme $SIGP$ is less precise (at least for $s = 7/4$). Concerning the number of iterations corresponding to the convergence of Algorithm (5.1), for $s = 4/3$ (resp. $s = 3/2$ and $s = 7/4$) all the schemes $SIGP$, $NIPG$ and $IIPG$ converge after 13 to 15 iterations (resp. 10 to 12 for $s = 3/2$ and 5 to 6 for $s = 7/4$).

Figure 4 show comparisons the second component of the velocity (u_2) between the reference solution and those obtained with the iterative schemes $SIGP$, $NIPG$ and $IIPG$ on the horizontal line $y = 0.5$ (left) and the vertical line $x = 1$ (right) for $s = 7/4$. We can clearly see that the schemes $NIPG$ and $IIPG$ gives better results compare to $SIPG$. On the theoretical explanation of the performance of the $SIPG$ scheme, it is well documented (see [18, 20]) that the bilinear form $A(\cdot, \cdot)$ associated to that scheme is conditionally elliptic. Therefore the results obtained here maybe due to the fact that the stabilization parameter used is not in the right interval.

Next, we analyse by means of simulations the rate of convergence of Eq. 2.5. We denote by E_{rr} the same expression as E_a defined in Eq. 5.8 but with $\alpha = 24$ for each algorithm ($IIPG$, $NIPG$ or $SIPG$). To get the *a priori* error estimates corresponding to our schemes, we compute the numerical solution $(\boldsymbol{u}_h, p_h)$ for $N = 10, 12, 14, \ldots, 24$. The coarse mesh is successively refined uniformly. Figure 5 shows the graphs of the relative error E_{rr} with respect to $h = \dfrac{1}{N}$ in

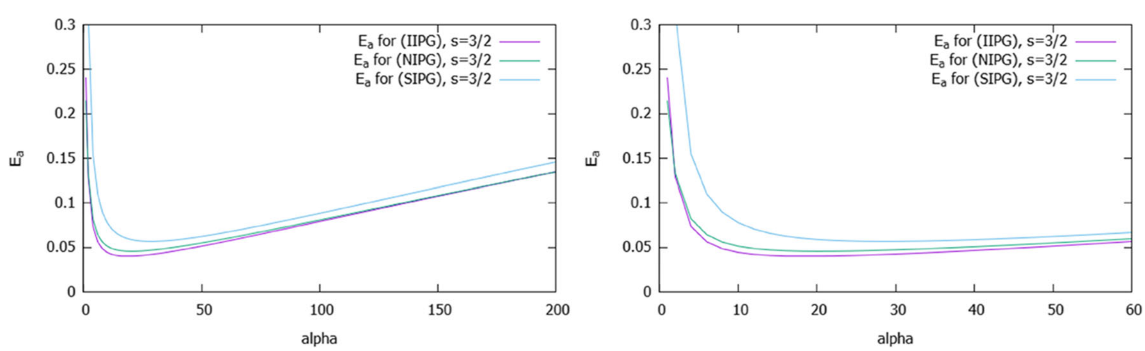

Fig. 2 E_α associated to the iterative schemes $SIGP$, $NIPG$ and $IIPG$ for $s = 3/2$ and $\alpha \in [1, 200]$ (zoom on the right)

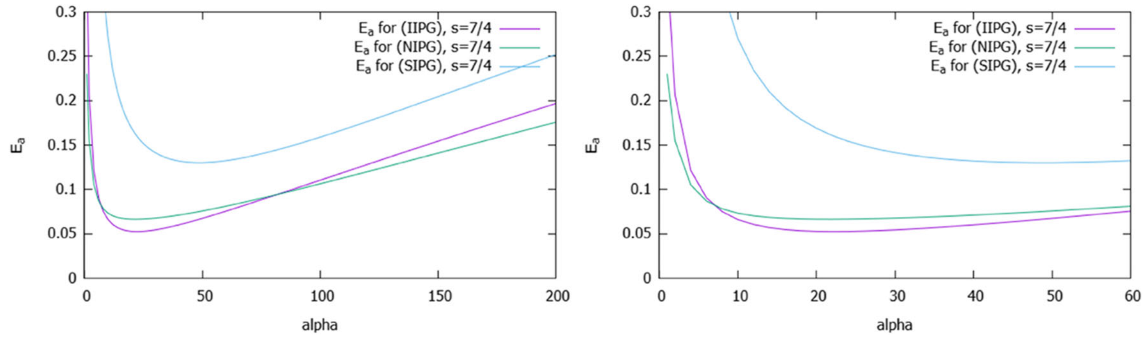

Fig. 3 E_α associated to the iterative schemes $SIGP$, $NIPG$ and $IIPG$ for $s = 7/4$ and $\alpha \in [1, 200]$ (zoom on the right)

logarithmic scale. Table 1 shows the slopes of these lines for the proposed schemes and $s = 4/3, 3/2, 7/4$.

Our numerical rates confirm the theoretical rates. As the mesh is uniformly refined, increasing the degrees of freedom in the mesh improves the accuracy on a given mesh. The convergence is first order and independent of s, which is in line with Remark 4.2. The first order convergence for the schemes under consideration is acceptable because on one hand we have a nonlinearity at the boundary with a poor differentiability for $1 < s < 2$. Hence one cannot used Newton's method because it might fail. Secondly, the velocity is approximated by piecewise linear function while the pressure is constant. Hence one cannot get a rate of convergence greater than one. We note that the numerical results are sightly better for $IIPG$ and $NIPG$ compare to the $SIPG$. Again, this can be explained by the fact that the bilinear form associated to the schemes $IIPG$ and $NIPG$ are unconditionally elliptic, while the bilinear form associated with the scheme $SIPG$ is conditionally elliptic.

5.2.2 lid Driven cavity flow

This paragraph is devoted to the numerical simulations of the Navier-Stokes flows with power law slip boundary condition in the lid Driven cavity by using the scheme (5.1). This a classical example in fluid and it as been studied by many authors (see for instance [40, 41] among others).

The domain is the xy-plane with $(x, y) \in (0, 1)^2$, and the boundary condition for the velocity field is given by

$$\boldsymbol{u} = \begin{cases} (16x^2(x-1)^2, 0)^T & \text{on } \; y = 1, \;\; 0 < x < 1, \\ (0, 0)^T & \text{elsewhere.} \end{cases}$$

Thus one has a smooth start of the flow, and the lid velocity attains its maximum, $\boldsymbol{u} = (1, 0)^T$, at the center $x = 1/2$. This boundary condition has the effect of avoiding local singularities at the top-right and top-left corners as it ensures both the velocity and velocity gradient vanish at the corners. Furthermore, we consider the isotropic slip boundary condition on the other parts of the boundary. In all the numerical results considered in this section, we set $\boldsymbol{f} = \boldsymbol{0}$, $\boldsymbol{g} = \boldsymbol{0}$, $\nu = 1$ and

$$K = \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix}.$$

From the results showed in the previous section, we will present only the numerical results corresponding to the schemes $IIPG$ and $NIPG$ (we will omit the those of the scheme $SIPG$.) The algorithm is allowed to move forward, until convergence is obtained. As the focus here is on slip

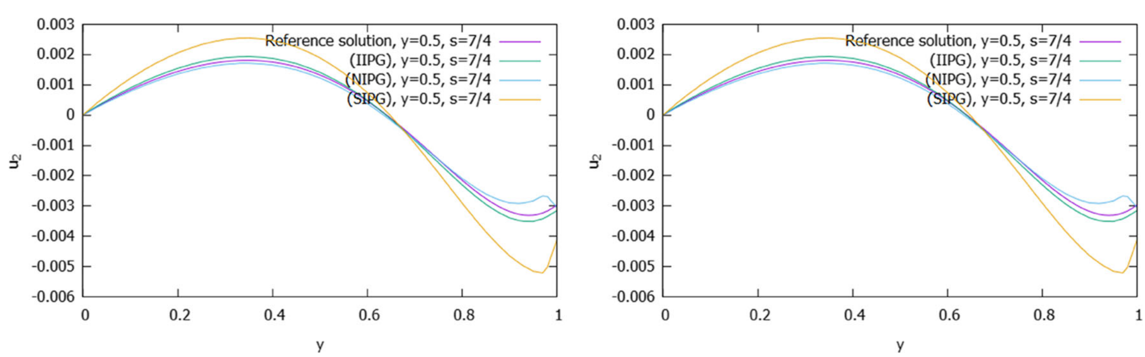

Fig. 4 Comparison of u_2 on the horizontal line $y = 0.5$ (left) and vertical line $x = 1$ (right) between the reference solution and those obtained with $SIGP$, $NIPG$ and $IIPG$ for $\alpha = 24$

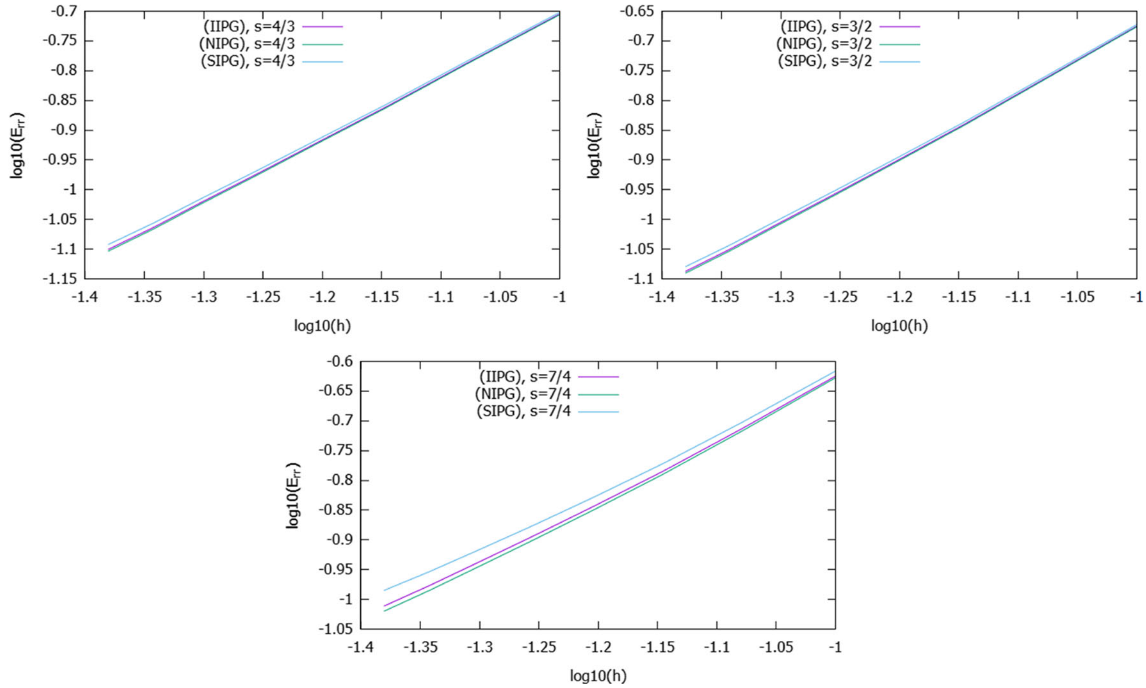

Fig. 5 The *a priori error* estimates with respect to the mesh step h in logarithmic scale for $s = 4/3$ (up left), $s = 3/2$ (up right) and $s = 7/4$ (down), and associated to the schemes $IIGP$, $NIPG$ and $SIPG$

boundary condition, we assume a mesh with $N = 100$. In Fig. 6, the tangential component of the velocity on $\Gamma = S_1 = S_2$ are presented for $IIPG$ taking $s = 3/2$, and $\alpha = 24$. This result is obtained with less than 30 iterations, and throughout the computations, the solution do not diverge even near the top left corner where the largest gradients are observed. The results are similar for $NIPG$. We highlight the interest of imposing the nonlinear slip boundary condition in this context: the current formulation allows to observe a nonlinear effect on the boundary Γ, which is physically meaningful. We note a good correlation between solutions with different stabilization parameter $\alpha = \{24, 30, 50\}$. For $N > 100$, large mesh-sensitivity is observed, but the solution remain stable for both schemes and consequently the tangential component of the velocity on the slip zone Γ is nonzero.

In Fig. 7, we display the distribution of the velocity associated to the schemes $IIPG$ and $NIPG$ for $s = 3/2$ and $\alpha = 24$. We note that these graphs have the same behavior, implying that the schemes $IIPG$ and $NIPG$ gives almost similar results. Although each solution converges with respect to mesh refinement, it is observed when the stabilization parameter $\alpha > 100$, the scheme NIPG takes more times to produce satisfactorily results.

Next, Fig. 8 shows comparisons of the first and second components of the velocity on the horizontal line $y = 0.5$ between $IIPG$ and $NIPG$ for $\alpha = 24$. Here also $IIPG$ and $SIPG$ gives close results.

Table 1 The slopes of the error lines corresponding to the schemes $IIPG$, $NIPG$ and $SIPG$, and for $s = 4/3, 3/2, 7/4$

	$s = 4/3$	$s = 3/2$	$s = 7/4$
IIPG	1.04	1.08	1.01
NIPG	1.04	1.08	1.
SIPG	1.02	1.07	0.96

5.3 Conclusions

The goal of the present work is to propose some new formulations for the Navier-Stokes equations with partly direction dependent slip on the boundary. In particular, we have; (a)-studied several DG methods for the resolution of the Navier-Stokes system with a power law slip boundary condition. We have shown uniform convergence with respect to s and obtained optimal convergence when the tangential part of the velocity on the slip zone has enough regularity. (b)-formulated and studied a fixed point method for the practical resolution of the nonlinear DG problems. (c)-verified computationally the theoretical findings and finally we showed the robustness of our method by considering varying permeability fields. The current methodology should be further extended, in particular by (i)-devising an adaptive mesh strategy based on a posteriori error estimate; (ii)-extending the

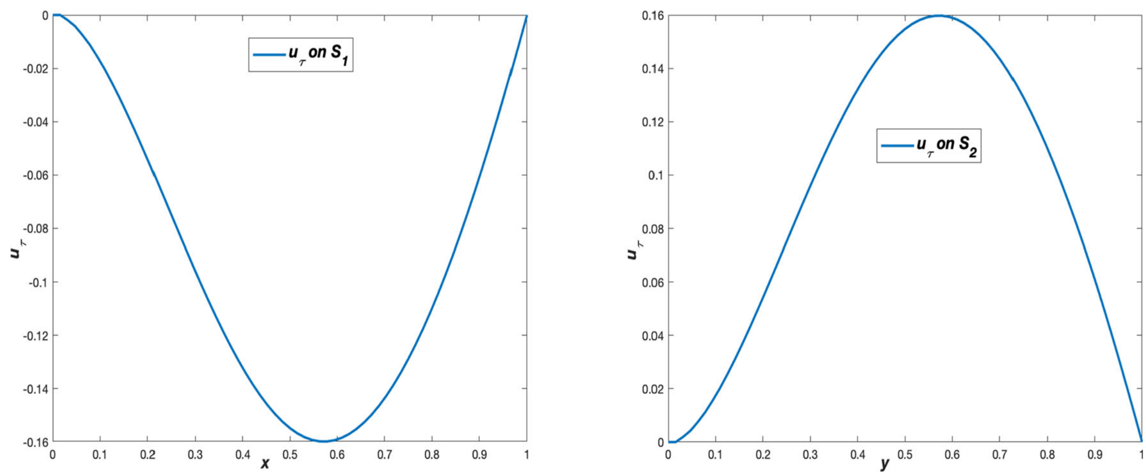

Fig. 6 Tangential component of the velocity on the slip zone

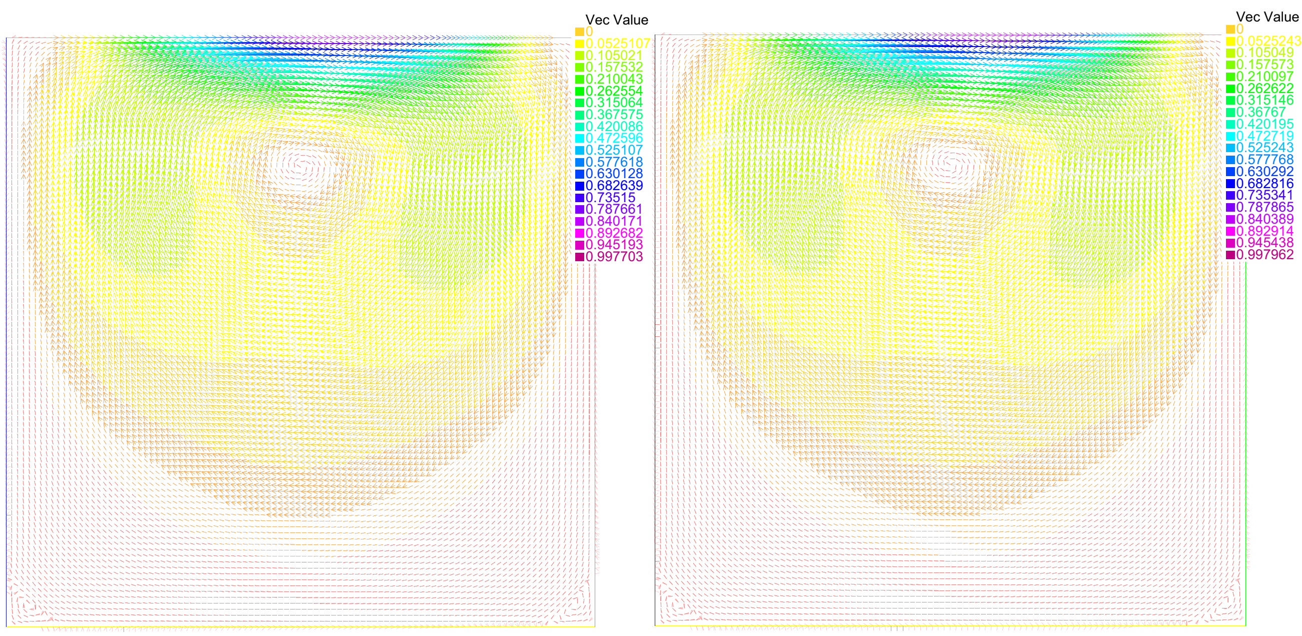

Fig. 7 The distribution of the velocity for the schemes $IIPG$ on the left and $NIPG$ on the right with $\alpha = 24$, $N = 100$ and $s = 3/2$

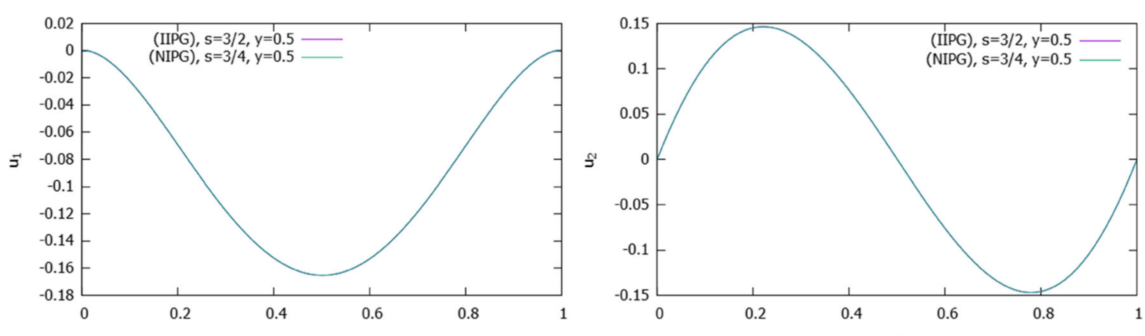

Fig. 8 Comparisons of the first and second components of the velocity on the horizontal line $y = 0.5$

present analysis to generalised non-Newtonian models; (iii)-extending the present analysis to time dependent problem.

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Declarations

Conflicts of interest The authors declare no conflict of interest

Informed Consent Not applicable.

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