SYMMETRY SOLUTIONS AND CONSERVATION LAWS FOR CERTAIN PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

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Thesis submitted for the degree *Doctor of Philosophy* in Applied Mathematics at the Mafikeng Campus of the North-West University

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Contents

Declaration vi

Declaration of Publications vii

Dedication ix

Acknowledgements x

Abstract xi

Introduction **1**

1 Preliminaries **5**

1.1 Introduction 5

1.2 Continuous group of transformations 6

1.3 Prolongations 7

1.4 Group admitted by a partial differential equation 11

1.5 Infinitesimal criterion of invariance 12

1.6 Exact solutions 13

1.6.1 Description of $(G'/G)-$expansion method 13

1.6.2 The simplest equation method 14

1.6.3 The Kudryashov method 16
1.7 Conservation laws 17
1.7.1 Fundamental operators and their relationship 17
1.7.2 Multiplier method 19
1.7.3 Ibragimov method for conservation laws 19
1.7.4 Noether theorem 21
1.8 Conclusion 22

2 A study of a generalized Benney-Luke equation with time dependent coefficients 23

2.1 Equivalence transformations 25
2.2 Principal Lie algebra 28
2.3 Lie group classification of (2.6) 29
2.4 Symmetry reductions and group invariant solutions of (2.6) 30
2.4.1 Case 2 31
2.4.2 Case 3 31
2.4.3 Case 4 32
2.4.4 Case 5 35
2.4.5 Case 6 36
2.5 Conservation laws of (2.6) 38
2.6 Conclusion 40

3 Solutions and conservation laws of the symmetric regularized long

wave equation **41**

3.1 Exact solutions of (3.1) 42

3.1.1 Lie point symmetries of (3.1) and symmetry reduction . . . 42

3.1.2 Exact solutions of (3.1) using simplest equation method . . . 43

3.1.3 Solution of (3.1) using the exp-function method 46

3.2 Conservation laws of (3.1) 47

3.3 Conclusion 50

4 Exact solutions for Klein-Gordon-Zakharov equations **51**

4.1 Solution of (4.1) using the travelling wave variable approach 52

4.2 Solutions of (4.1) using the simplest equation method 54

4.3 Conclusion 57

5 Travelling wave solutions and conservation laws of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation **58**

5.1 Exact solutions of (5.1) 59

5.1.1 Exact solutions of (5.1) using Lie point symmetries 59

5.1.2 Exact solutions of (5.1) using the (G'/G)-expansion method 64

5.2 Conservation laws of (5.1) 66

5.3 Conclusion 69

6 Traveling wave solutions and Conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity **70**

6.1 Similarity reduction and exact solutions 71

6.1.1 Similarity reduction of (6.1) 71

6.1.2 Exact solutions of (6.6) using Kudryashov approach 73

6.2 Construction of conservation laws for (6.1) 74

6.3 Conclusion 76

7 Noether symmetries, conservation laws and exact solutions of a four parameter Boussinesq system 77

7.1 Conservation laws of (7.1) using Noether approach 79

7.2 Exact solutions of (7.1) using Kudryashov method 84

7.3 Conclusion 87

8 Exact solutions and conservation laws of an integrable coupling with Korteweg-de Vries equation 88

8.1 Lie point symmetries and symmetry reductions of (8.2) 89

8.2 Exact solutions of (8.2) using the $(G'/G)-$expansion method 91

8.3 Conservation laws of (8.2) using the multiplier approach 94

8.4 Conclusion 98

9 Exact solutions and conservation laws for the coupled Benjamin-Bona-Mahony equations 99

9.1 Exact solutions of (9.1) using Kudryashov method 100

9.2 Conservation laws of (9.1) 102

9.3 Conclusion 104

10 Conservation laws and exact solutions of the convective Cahn-

Hilliard Equation **105**

10.1 Conservation laws of (10.1) 106

10.2 Travelling wave solutions of (10.1) 107

10.2.1 Lie point symmetries and symmetry reduction of (10.1) . . 107

10.2.2 Exact solutions of (10.10) using simplest equation method . 108

10.3 Conclusion 111

11 Concluding remarks **112**

Bibliography **114**

Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

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Date: 15-06-18 ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed: ....

PROF CM KHALIQUE

Date: 15 June 2018 ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

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Chapter 3

IE Mhlanga, CM Khalique, Exact Solutions of the Symmetric Regularized Long Wave Equation and the Klein-Gordon-Zakharov Equations, *Abstract and Applied Analysis*, Volume 2014, Article ID 679016, 7 pages

Chapter 4

IE Mhlanga, CM Khalique, On the exact solutions of the Klein-Gordon-Zakharov equations, *Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science*, ISBN 978-3-319-12306-6. Series: Springer Proceedings in Mathematics & Statistics, Vol. 117. Cojocaru, M., Kotsireas, I.S., Makarov, R.N., Melnik, R., Shodiev, H. (Eds.) 2015, I, 479 p. 145 illus., 102 illus. in color. Copyright Springer International Publishing Switzerland 2015, 301–307

Chapter 5

IE Mhlanga, CM Khalique, Travelling wave solution and conservation laws of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation, *AIP Conference Proceedings* 1863, 280006 (2017); DOI: 10.1063/1.4992437

Chapter 6

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Chapter 7

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IE Mhlanga, CM Khalique, Exact solutions and conservation laws for a coupled Benjamin-Bona-Mahony equations, Proceedings of the International Conference on Numerical Analysis and Applied Mathematics 2017 (ICNAAM-2017), accepted and to appear in *AIP Conference Proceedings of ICNAAM 2017*

Chapter 10

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Dedication

I dedicate this work to my children

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Abstract

In this thesis we study some nonlinear partial differential equations which appear in many physical phenomena of science and engineering. Exact solutions and conservation laws are obtained for such equations using various methods. In this work we study a generalized Benney-Luke equation with time dependent coefficients, the symmetric regularized long wave equation, the Klein-Gordon-Zakharov equations, a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation, the Korteweg-de Vries-Burgers equation with power law nonlinearity, a four parameter Boussinesq system, an integrable coupling with Korteweg-de Vries equation, the coupled BBM equations and the convective Cahn-Hilliard equation.

A complete Lie group classification is performed on a generalized Benney-Luke equation with time dependent coefficients. This equation models an approximation of the full water wave equation and is formally suitable for describing two-way water wave propagation in the presence of surface tension. The Lie group classification of this equation provides us with a two-dimensional principal Lie algebra and has several possible extensions. Six distinct cases arise in classifying the time dependent coefficients. They include amongst others the power and exponential functions. Group-invariant solutions are obtained for all cases.

Exact solutions of two nonlinear evolution equations, namely, the symmetric regularized long wave equation and the Klein-Gordon-Zakharov Equations are obtained using Lie symmetry analysis along with the simplest equation method and the exp-function method. Conservation laws are constructed using the multiplier approach.

The $(G'/G)-$expansion method is used to obtain solutions of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation and a (2+1)-dimensional integrable coupling system with the Korteweg-de Vries equation. Exact travelling wave solutions of three types are obtained and these are the solitary waves, peri-

odic and rational functions. Conservation laws in both cases are obtained using the new conservation theorem.

Travelling wave solutions and conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity are derived via Lie symmetry analysis along with Kudryashov approach and the new conservation theorem, respectively.

Conservation laws for a four parameter Boussinesq system are derived using the Noether approach and its exact travelling wave solutions are obtained using the Kudryashov method.

Exact solutions of the third-order coupled Benjamin-Bona-Mahony equations are derived by employing the Kudryashov method and conservation laws are obtained using the multiplier approach.

Conservation laws and travelling wave solutions of the fourth-order Cahn-Hilliard equation are obtained. The conservation laws are constructed using the multiplier approach while Lie symmetry analysis along with the simplest equation method are used to obtain exact solutions.

Introduction

A large variety of real-world physical systems are governed by nonlinear partial differential equations. Such equations are very important because they are able to describe the real features in various fields of applications, for example, fluid mechanics, gas dynamics, combustion theory, relativity, thermodynamics, biology, and many others. Nonlinear partial differential equations of real life problems are difficult to solve analytically. Finding exact solutions of the nonlinear partial differential equations is a very important task and plays an important role in nonlinear science. There has recently been much attention devoted to the search for better and more efficient solution methods for determining solutions to nonlinear partial differential equations [1–29].

In the last few decades, a variety of effective methods for finding exact solutions were discovered. These include the homogeneous balance method [3], the ansatz method [4, 5], the variable separation approach [6], the inverse scattering transform method [7], the Bäcklund transformation [8], the Darboux transformation [9], the Hirota bilinear method [10], the $(G'/G)-$expansion method [11–13], the Kudryashov method [14–19] and Lie group analysis [20–26]. Such methods were successfully applied to nonlinear partial differential equations in obtaining their exact solutions.

Lie group analysis is one of the most powerful and systematic methods to determine solutions of nonlinear differential equations. It was originally developed by Marius Sophus Lie (1842-1899). His study gave rise to the modern theory of what is now universally known as Lie groups. Ever since, a large amount of work has been published in the literature on the subject of Lie groups applied to differential equations in terms of the Lie point symmetries admitted by the equation under study. Lie point symmetry of a differential equation is a one parameter point transformation which leaves the differential equation invariant. Lie theory enables one to reduce the order of ordinary differential equations. The reduction of a partial differential equation with respect to $r-$dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

It is well-known that conservation laws play an important role in the study of differential equations. Conservation laws describe physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion [23, 30, 31]. They have been used in investigating the existence, uniqueness, and stability of solutions of nonlinear partial differential equations [32, 33]. Also, they have been used in the development and use of numerical methods [34, 35]. Recently, conservation laws were used to obtain exact solutions of some partial differential equations [36–40]. Thus, it is essential to study conservation laws of differential equations.

Sophus Lie's work had influence on many mathematicians including Emmy Noether (1882-1935). A connection between symmetries and conservation laws for differential equations is established via Noether theorem [41, 42]. In addition to Lie point symmetries, Noether symmetries are also widely studied and are associated, in particular, with those differential equations which possess Lagrangians. The Noether symmetries, which are symmetries of the Euler-Lagrange systems, have interesting applications in the study of properties of particles moving under the influence of gravitational fields.

Noether theorem [41, 42] allows construction of conservation laws systematically. However, it can only be applied to differential equations with a Lagrangian. In order to overcome this limitation, several works have been done. See for example, [43–48]. Further developments have been made in this direction and the concepts of quasi self-adjoint, weak self-adjoint and nonlinear self-adjoint were introduced in [49–54].

The outline of this thesis is as follows:

Chapter one is devoted to definitions, theorems and other preliminaries that form the basis of this study.

In Chapter two, a complete Lie group classification is performed on a generalized Benney-Luke equation with time dependent coefficients. As a result, the arbitrary functions which appear in the system are specified.

In Chapter three exact travelling wave solutions of the symmetric regularized long wave equation using the simplest equation method and the exp-function method are obtained. Conservation laws for this equation are constructed using the multiplier approach.

In Chapter four, exact solutions for the Klein-Gordon-Zakharov equations are found using the travelling wave variable approach and the simplest equation method.

Chapter five studies the exact solutions and conservation laws of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation. Exact solutions are obtained using direct integration and also by the $(G'/G)-$expansion method. The new conservation theorem due to Ibragimov is used to find conservation laws.

In Chapter six, the exact solutions and conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity are obtained using the Lie symmetry method along with Kudryashov method and the new conservation theorem due to Ibragimov, respectively.

Chapter seven investigates a four parameter Boussinesq system. Conservation laws of the system are derived using the Noether approach and the Kudryashov method is employed to obtain exact solutions of the system.

Chapter eight looks at the exact solutions and conservation laws of an integrable coupling with Korteweg-de Vries equation. The $(G'/G)-$expansion method and multiplier approach are employed for finding the exact solutions and for the construction of conservation laws respectively.

Chapter nine presents the exact solutions and conservation laws for a coupled Benjamin-Bona-Mahony equations. The Kudryashov method is used to obtain the exact solutions and conservation laws are derived using the multiplier method.

In Chapter ten, conservation laws for a convective Cahn-Hillard equation are constructed by applying the multiplier approach and exact solutions obtained using the simplest equation method.

Finally, in Chapter eleven, a summary of the results of the thesis is presented and future work is suggested.

Bibliography is given at the end.

Chapter 1

Preliminaries

In this chapter, we present some salient features on Lie symmetry analysis, conservation laws of differential equations and solution methods for differential equations, which are used throughout this work and are based on references [11, 18, 20–26, 30, 41].

1.1 Introduction

Lie group analysis was developed in the 1870s by an eminent mathematician of the nineteenth century Sophus Lie (1842-1899). He showed that the majority of known methods of integration of ordinary differential equations, which until then had seemed artificial, could be derived in a unified manner using his theory of continuous transformation groups. In particular, Lie reduced the classical four hundred types of ordinary differential equations to four types only. Lie group analysis is one of the few universal and effective methods for solving nonlinear differential equations analytically.

1.2 Continuous group of transformations

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a), \ \ \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 (Lie group) A set G of transformations (1.1) is called a continuous one-parameter (local) Lie group of transformations in the space of variables x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a,b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\,\phi(s,b)}{\partial b}\right|_{b=0}.$$

1.3 Prolongations

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)} \cdots$.

In group analysis, all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [24].

We now consider a pth-order partial differential equations, namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations :

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a, about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x, u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the symbol of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an admitted operator of (1.5) or X is an infinitesimal symmetry of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots .$$

Applying (1.6) and (1.10), we obtain

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha, \end{aligned} \tag{1.11}$$

and so

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

Letting

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

to be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$
\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j\text{)}. \qquad (1.15)
\end{aligned}
$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$
\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k\text{)}. \qquad (1.16)
$$

By induction (recursively)

$$
\zeta_{i_1,i_2,\dots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\dots,i_{p-1}}^\alpha) - u_{i_1,i_2,\dots,i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j\text{)}. \qquad (1.17)
$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$
\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha\text{)}, \\
&\vdots \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\dots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\dots,i_p}^\alpha} \quad p \geq 1,
\end{aligned}
$$

where

$$
X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.
$$

1.4 Group admitted by a partial differential equation

Definition 1.2 (Point symmetry) The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{1.18}$$

is a Lie point symmetry of the pth-order partial differential equation (1.5), if

$$X^{[p]}(E_\alpha) = 0 \tag{1.19}$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]}\, E_\alpha\big|_{E_\alpha=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 (Determining equation) Equation (1.19) is called the determining equation of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.5 Infinitesimal criterion of invariance

Definition 1.5 (Invariant) A function $F(x, u)$ is called an invariant of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\, \phi^\alpha(x, u, a)) = F(x, u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \cdots, J_{n-1}(x, u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^n}{\eta^n(x, u)}.$$

Theorem 1.3 (Lie equations) If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Exact solutions

In this section we provide some solution methods for finding exact solutions of differential equations.

1.6.1 Description of $(G'/G)-$expansion method

The $(G'/G)-$expansion method for finding exact solutions of nonlinear differential equations was introduced in [11]. It provides a very effective and powerful mathematical tool for solving nonlinear differential equations (see, for example, papers [11–13].

Let us consider a nonlinear partial differential equation of two independent variables x and t given by

$$P(u, u_x, u_t, u_{tt}, u_{xt}, u_{xx} \cdots) = 0, \tag{1.25}$$

where $u(x,t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The $(G'/G)-$expansion method is described in the following steps.

Step 1. The substitution $u(x,t) = U(z), \quad z = x - \nu t$ transforms equation (1.25) into the ordinary differential equation

$$P(U, -\nu U', U', \nu^2 U'', -\nu U'', U'' \cdots) = 0. \tag{1.26}$$

Step 2. It is assumed that the solution of equation (1.26) can be expressed by a polynomial in (G'/G) as follows:

$$U(z) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{1.27}$$

where $G = G(z)$ satisfies the second-order linear ordinary differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \tag{1.28}$$

with α_i, $i = 0, 1, 2, \cdots, m$, λ and μ being constants to be determined. The positive integer m is determined by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in ordinary differential equation (1.26).

Step 3. Substituting (1.27) into (1.26) and using the second-order ordinary differential equation (1.28), collecting all terms with same order of (G'/G) together, the left-hand side of (1.26) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_0, \cdots, \alpha_m\, \nu$, λ, μ.

Step 4. Finally, the constants may be obtained by solving the algebraic equations in Step 3, since the general solution of (1.28) is known, then substituting the constants and the general solutions of (1.28) into (1.27) we obtain solutions of the nonlinear partial differential equation (1.25).

1.6.2 The simplest equation method

The simplest equation method for finding exact solutions of nonlinear partial differential equations was developed by Kudryashov [14, 15] and has been applied to various nonlinear partial differential equations (see, for example, papers [16–19]). The simplest equation method can be described as follows:

Consider the nonlinear partial differential equation of the form

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{yy} \cdots) = 0. \tag{1.29}$$

Using the following transformation

$$u(t,x,y) = F(z), \quad z = k_1t + k_2x + k_3y + k_4 \tag{1.30}$$

reduces equation (1.29) to an ordinary differential equation

$$E_2[F(z), k_1F'(z), k_2F'(z), k_3F'(z), k_1^2F''(z), k_2^2F''(z), k_3^2F''(z), \cdots] = 0. \tag{1.31}$$

The simplest equations that we use here are the Bernoulli equation:

$$H'(z) = aH(z) + bH^2(z), \tag{1.32}$$

and the Riccati equation:

$$G'(z) = aG^2(z) + bG(z) + c, \tag{1.33}$$

where a, b and c are constants [14, 19]. We look for solutions of the nonlinear ordinary differential equation (1.31) that are of the form

$$F(z) = \sum_{i=0}^{M} A_i(G(z))^i, \tag{1.34}$$

where $G(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

The solution of Bernoulli Equation (1.32) we use here is given by:

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}$$

where C is a constant of integration. For the Riccati Equation (1.33), the solutions to be used are:

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right] \tag{1.35}$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \tag{1.36}$$

with $\theta = \sqrt{b^2 - 4ac}$ and C is a constant of integration.

1.6.3 The Kudryashov method

This method due to Kudryashov is used for finding exact solutions of nonlinear differential equations [18] and has been applied to various nonlinear differential equations (see, for example, papers [55, 56]).

Consider a nonlinear partial differential equation of two independent variables t and x given by

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0, \tag{1.37}$$

where $u(x, t)$ is an unknown function, E is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. The algorithm of Kudryashov method consists of the following five steps:

Step 1. The transformation $u(x, t) = U(z)$, $z = kx + \omega t$, where k and ω are constants, reduces equation (1.37) to the ordinary differential equation

$$E_2(U, \omega U_z, kU_z, \omega^2 U_{zz}, k^2 U_{zz}, \cdots) = 0. \tag{1.38}$$

Step 2. It is assumed that the exact solution of equation (1.38) can be expressed by a polynomial in Q as follows:

$$U(z) = \sum_{n=0}^{N} a_n Q^n(z), \tag{1.39}$$

where the coefficients a_n $(n = 0, 1, 2, \cdots, N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(z)$ is the solution of the first-order nonlinear ordinary differential equation

$$Q_z(z) = Q^2(z) - Q(z). \tag{1.40}$$

We note that the equation (1.40) has the solution given by

$$Q(z) = \frac{1}{1 + e^z}. \tag{1.41}$$

The positive integer N is determined by taking the pole order of general solution for equation (1.38). Substituting $U(z) = z^{-p}$, $p > 0$ into monomials of equation

(1.38) and comparing the two or more terms with smallest powers in equation we find the value for N.

Step 3. We substitute the derivatives of $U(z)$ with respect to z and the expression for $U(z)$ into equation (1.38) and as a result we obtain the equation that has the function Q, coefficients a_n $(n = 0, 1, \cdots, N)$ and parameters k, ω of equation (1.38).

Step 4. The method now transforms the problem of finding the exact solution of ordinary differential equation (1.38) into the problem of looking for solutions of the system of algebraic equations. Equating expressions at the different powers of Q to zero, we obtain the system of algebraic equations in the form

$$P_n(a_N, a_{N-1}, \cdots, a_0, k, \omega, \cdots) = 0, \quad (n = 0, \cdots, N). \tag{1.42}$$

Step 5. Solving the system of algebraic equations, we obtain values of coefficients $a_N, a_{N-1}, \cdots, a_0$ and relations for parameters of equation (1.38). As a result, we obtain exact solutions of equation (1.38) in the form (1.39).

1.7 Conservation laws

1.7.1 Fundamental operators and their relationship

Consider a pth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, given by equation (1.5).

Definition 1.6 (Euler-Lagrange operator) The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{1.43}$$

Definition 1.7 (Lagrangian) If there exists a function $\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)})$, $s \leq p$, p being the order of equation (1.5), such that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0 \quad \alpha = 1, \cdots, m \tag{1.44}$$

then $\mathcal{L}$ is called a Lagrangian of equation (1.5). Equation (1.44) is known as the Euler-Lagrange equation.

Definition 1.8 (Lie-Bäcklund operator) The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.45}$$

where $\mathcal{A}$ is the space of differential functions [24]. The operator (1.45) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.46}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.47}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{1.48}$$

One can write the Lie-Bäcklund operator (1.46) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.49}$$

Definition 1.9 (Conservation law) The n-tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (1.5) if T^i satisfies

$$D_i T^i|_{(1.5)} = 0. \tag{1.50}$$

The equation (1.50) defines a local conservation law of system (1.5).

1.7.2 Multiplier method

The algorithm of finding the conservation laws for differential equations is given in [25, 45]. The advantage of this approach is that it does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.51}$$

hold identically, where E_α, D_i are defined by equations (1.5), (1.4) and T^i is defined in definition (1.9). The right hand side of (1.51) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.52}$$

Once the multipliers are obtained the conserved vectors are constructed by invoking the homotopy operator [45].

1.7.3 Ibragimov method for conservation laws

A new conservation theorem by Ibragimov [48] provides the procedure for computing the conserved vector associated with all symmetries of the system of the pth-order differential equation (1.5).

Definition 1.10 (Adjoint equations) Consider a system of pth-order partial differential equations given by (1.5). We introduce the differential functions

$$E^*_\alpha(x, u, v, \cdots, u_{(p)}, v_{(p)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \cdots, m, \tag{1.53}$$

where $v = (v^1, \cdots, v^m)$ are new dependent variables, $v = v(x)$, and define the system of *adjoint equations* to equation (1.5) by

$$E^*_\alpha(x, u, v, \cdots, u_{(p)}, v_{(p)}) = 0, \quad \alpha = 1 \cdots, m. \tag{1.54}$$

Theorem 1.4 Any system of pth-order differential equations (1.5) considered together with its adjoint equation (1.54) has a Lagrangian. Namely, the Euler-Lagrange equations (1.44) with the Lagrangian

$$\mathcal{L} = v^\beta E^*_\beta(x, u, v, \cdots, u_{(p)}) \tag{1.55}$$

provide the simultaneous system of equations (1.5) and (1.53)$-$(1.54) with $2m$ dependent variables $u = u(u^1, \cdots, u^m)$ and $v = (v^1, \cdots, v^m)$.

Theorem 1.5 Consider a system of m equations (1.5). The adjoint system given by (1.54), inherits the symmetries of the system (1.5). Namely, if the system (1.5) admits a point transformation group with a generator (1.45), then the adjoint system (1.54) admits the operator (1.45) extended to the variables v^α by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha} \tag{1.56}$$

with appropriately chosen coefficients $\eta^\alpha_* = \eta^\alpha_*(x, u, v, \cdots)$.

Definition 1.11 (Nonlinearly self-adjoint) A system (1.5) is said to be nonlinearly self-adjoint if the adjoint system (1.54) is satisfied for all the solutions of (1.5) after some substitution of v^α given by

$$v^\alpha = \phi^\alpha(x, u, u_{(1)}, \cdots), \quad \alpha = 1, \cdots, m, \tag{1.57}$$

under the condition that not all ϕ^α vanish identically [51].

Theorem 1.6 (Ibragimov theorem) Any infinitesimal symmetry (Lie point,Lie-Bäcklund , nonlocal) given by (1.45) of a nonlinearly self-adjoint system (1.5) leads to a conservation law $D_i(C^i) = 0$ for the system (1.50). The components of the conserved vector are given by the formula

$$\begin{aligned} C^i &= \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \cdots\right] \\ &\quad + D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_jD_k(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots\right], \end{aligned} \tag{1.58}$$

where W^α is is the Lie characteristic function given by (1.48)and $\mathcal{L}$ is the formal Lagrangian (1.55) [48].

1.7.4 Noether theorem

Definition 1.12 (Noether operator) The Noether operators associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha\frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\dots D_{i_s}(W^\alpha)\frac{\delta}{\delta u_{ii_1i_2\dots i_s}^\alpha}, \quad i = 1,\dots,n, \tag{1.59}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.43) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\dots D_{j_s}\frac{\partial}{\partial u_{ij_1j_2\dots j_s}^\alpha}, \quad i = 1,\dots,n, \quad \alpha = 1,\dots,m, \tag{1.60}$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity [48]

$$X + D_i(\xi^i) = W^\alpha\frac{\delta}{\delta u^\alpha} + D_iN^i. \tag{1.61}$$

Definition 1.13 (Noether symmetry) A Lie-Bäcklund operator X of the form (1.45) is called a Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$, if there exists a vector $B^i = (B^1, \cdots, B^n)$, $B^1 \in \mathcal{A}$ such that

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i) \tag{1.62}$$

if $B^i = 0$ $(i = 1, \cdots, n)$, then X is called a strict Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$.

Theorem 1.7 (Noether Theorem) For any Noether symmetry generator X associated with a given Lagrangian $\mathcal{L} \in \mathcal{A}$, there corresponds a vector $T = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, given by

$$T^i = N^i(\mathcal{L}) - B^i, \quad i = 1, ..., n, \tag{1.63}$$

which is a conserved vector of the Euler-Lagrange differential equations (1.44).

In the Noether approach, we find the Lagrangian $\mathcal{L}$ and then equation (1.62) is used to determine the Noether symmetries. Then, equation (1.63) will yield the corresponding Noether conserved vectors.

1.8 Conclusion

In this chapter we provided a brief introduction to the Lie group analysis and various methods for finding exact solutions of differential equations. We also presented different approaches for deriving conservation laws of partial differential equations.

Chapter 2

A study of a generalized Benney-Luke equation with time dependent coefficients

The one-dimensional Benney-Luke equation given by

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0, \tag{2.1}$$

models an approximation of the full water wave equation and is formally suitable for describing two-way water wave propagation in the presence of surface tension [57,58]. The coefficients a and b are arbitrary positive constants such that $a - b = \sigma - 1/3$. The dimensionless parameter σ is named the Bond number, which captures the effects of surface tension and gravity force and is a formally valid approximation for describing two-way water wave propagation in the presence of surface tension [58–60].

In [57], González examined the question of the minimal Sobolov regularity required to construct local solutions to the Cauchy problem for the Benney-Luke equation (2.1). Quintero and Muñoz Grajales [58] studied asymptotic stability of solitary wave solutions of (2.1). Gözükızıl and Akçağıl [59] used the tanh-coth method and obtained some travelling wave solutions of equation (2.1) while Akter and Akbar [60] implemented the modified simple equation method to find exact travelling wave and solitary wave solutions of the same equation.

Other scholars have considered different generalizations of the Benney-Luke equation. For example, Quintero [61] proved the existence and analyticity of lump solutions for the generalized Benney-Luke equation

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \varepsilon F(\Phi_t, \nabla\Phi, \nabla\Phi_t, \Delta\Phi) = 0, \qquad (2.2)$$

where ε, μ, a and b are positive numbers. In [62], Quintero and Muñoz Grajales, studied linear instability of solitary wave solutions for the one-dimensional generalized Benney-Luke equation

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + n\Phi_t(\Phi_x)^{p-1}\Phi_{xx} + 2(\Phi_x)^n\Phi_{xt} = 0. \qquad (2.3)$$

Wang et al. [63] considered the generalized Benney-Luke of the form

$$\begin{aligned}&\Phi_{tt} - \Delta\Phi + a\Delta^2\Phi - b\Delta\Phi_{tt} + p\Phi_t(\Phi_x)^{p-1}(\Phi_{xx} + \Phi_{yy})u_t u_{xx}\\ &+2[(\Phi_x)^p\Phi_{xt} + (\Phi_y\Phi_{yt})] = 0 \end{aligned} \qquad (2.4)$$

and employed an auxiliary equation algorithm to obtain travelling wave solutions.

Most recently, Bruzon [64] applied the Lie group method and the nonclassical method to deduce symmetries of the generalized Benney-Luke equation of the form

$$u_{tt} - k^2u_{xx} + au_{xxxx} - bu_{xxtt} + cu_tu_{xx} + du_xu_{xt} = 0 \qquad (2.5)$$

and obtained several new solutions which were not obtainable by Lie classical symmetries. More different generalizations of the Benney-Luke equations can be seen in [65–68].

In this chapter we study a generalization of (2.1) by replacing the constants a and b by nonzero arbitrary functions of time, $f(t)$ and $g(t)$ respectively. Thus we consider the generalized Benney-Luke (GBL) equation of the form

$$u_{tt} - u_{xx} + f(t)u_{xxxx} - g(t)u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{2.6}$$

The outline of the chapter is as follows. In Section 2.1, equivalence group of transformations for (2.6) is obtained. The principal Lie algebra is computed in Section 2.2 by solving the overdetermined system of linear partial differential equations generated by the symmetry group of (2.6) for arbitrary time dependent coefficients. In Section 2.3 we provide the classifying relations and then use the equivalence transformations to obtain six distinct cases for the time dependent coefficients $f(t)$ and $g(t)$ for which the principal Lie algebra extends. Then in Section 2.4, we present symmetry reductions and exact group-invariant solutions corresponding to all five cases of the group classification for which the principal Lie algebra extends. Conservation laws are obtained in Section 2.5 for two cases by using the multiplier method. Finally, concluding remarks are made in Section 2.6.

The work presented in this chapter has been published in [69].

2.1 Equivalence transformations

An equivalence transformation [24] of (2.6) is an invertible transformation involving the variables t, x and u that map (2.6) into itself. Just as symmetries of a differential equation transform solutions of the differential equation to other solutions of the same differential equation, point equivalence transformations transform differential equations in some specified class C to other differential equations in the same class C. The operator

$$Y = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u + \mu^1(t,x,u,f,g)\partial_f$$

$$+\mu^2(t,x,u,f,g)\partial_g \tag{2.7}$$

is the generator of the continuous group of equivalence transformations for equation (2.6) provided it is admitted by the extended system

$$\begin{aligned}
&u_{tt} - u_{xx} + f(t)u_{xxxx} - g(t)u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0,\\
&f_x = 0, \quad f_u = 0, \quad g_x = 0, \quad g_u = 0.
\end{aligned} \tag{2.8}$$

The prolonged operator for the extended system (2.8) has the form

$$\widetilde{Y} = Y^{[4]} + \omega^1_x\partial_{f_x} + \omega^1_u\partial_{f_u} + \omega^2_x\partial_{g_x} + \omega^2_u\partial_{g_u}, \tag{2.9}$$

where $Y^{[4]}$ is the fourth-prolongation of (2.7) given by

$$\begin{aligned}
Y^{[4]} &= \tau\partial_t + \xi\partial_x + \eta\partial_u + \mu^1\partial_f + \mu^2\partial_g + \zeta_t\partial_{u_t} + \zeta_x\partial_{u_x} + \zeta_{tt}\partial_{u_{tt}}\\
&\quad + \zeta_{xx}\partial_{u_{xx}} + \zeta_{xt}\partial_{u_{xt}} + \zeta_{xxxx}\partial_{u_{xxxx}} + \zeta_{ttxx}\partial_{u_{ttxx}}.
\end{aligned}$$

The variables ζ's and ω's are defined by the prolongation formulae

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\tau) - u_{xt} D_t(\xi),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\
\zeta_{xt} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{xt} D_x(\xi),\\
\zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{txxx} D_x(\tau) - u_{xxxx} D_x(\xi),\\
\zeta_{ttxx} &= D_x(\zeta_{ttx}) - u_{ttxx} D_x(\tau) - u_{xtxx} D_x(\xi)
\end{aligned}$$

and

$$\omega^1_x = \widetilde{D}_x(\mu^1) - f_t\widetilde{D}_x(\tau) - f_x\widetilde{D}_x(\xi) - f_u\widetilde{D}_x(\eta),$$

$$\omega_u^1 = \widetilde{D}_u(\mu^1) - f_t\widetilde{D}_u(\tau) - f_x\widetilde{D}_u(\xi) - f_u\widetilde{D}_u(\eta),$$
$$\omega_x^2 = \widetilde{D}_x(\mu^2) - g_t\widetilde{D}_x(\tau) - g_x\widetilde{D}_x(\xi) - g_u\widetilde{D}_x(\eta),$$
$$\omega_u^2 = \widetilde{D}_u(\mu^2) - g_t\widetilde{D}_u(\tau) - g_x\widetilde{D}_u(\xi) - g_u\widetilde{D}_u(\eta),$$

respectively, where D_t and D_x are the total differential operators defined by

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots ,$$
$$D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots$$

and the total differential operators for the extended system are given by

$$\widetilde{D}_x = \partial_x + f_x\partial_f + g_x\partial_g,$$
$$\widetilde{D}_u = \partial_u + f_u\partial_f + g_u\partial_g.$$

The system of determining equations is obtained by applying the prolonged operator (2.9) on the extended system (2.8). Thus,

$$\widetilde{Y}(u_{tt} - u_{xx} + f(t)u_{xxxx} - g(t)u_{xxtt} + u_tu_{xx} + 2u_xu_{xt})|_{(2.8)} = 0,$$
$$\widetilde{Y}(f_x)|_{(2.8)} = 0,\ \widetilde{Y}(f_u)|_{(2.8)} = 0,\ \widetilde{Y}(g_x)|_{(2.8)} = 0,\ \widetilde{Y}(g_u)|_{(2.8)} = 0.$$

The solution of the above determining equations leads to the following generators:

$$Y_1 = \frac{\partial}{\partial t},$$
$$Y_2 = \frac{\partial}{\partial x},$$
$$Y_3 = \frac{\partial}{\partial u},$$
$$Y_4 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial u} + g\frac{\partial}{\partial f},$$
$$Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2f\frac{\partial}{\partial f} + 2g\frac{\partial}{\partial g}.$$

Thus the five-parameter equivalence group associated to these equivalence generators is given by

$$Y_1 \ : \ \bar{t} = a_1 + t,\ \bar{x} = x,\ \bar{u} = u,\ \bar{f} = f,\ \bar{g} = g,$$

$$
\begin{aligned}
Y_2 &: \ \bar{t} = t,\ \bar{x} = a_2 + x,\ \bar{u} = u,\ \bar{f} = f,\ \bar{g} = g, \\
Y_3 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{u} = a_3 + u,\ \bar{f} = f,\ \bar{g} = g, \\
Y_4 &: \ \bar{t} = te^{a_4},\ \bar{x} = xe^{\frac{1}{2}a_4},\ \bar{u} = u - t + te^{a_4},\ \bar{f} = fe^{a_4},\ \bar{g} = g, \\
Y_5 &: \ \bar{t} = te^{a_5},\ \bar{x} = xe^{a_5},\ \bar{u} = ue^{a_5},\ \bar{f} = fe^{2a_5},\ \bar{g} = ge^{2a_5}.
\end{aligned}
$$

The composition of these gives

$$
\begin{aligned}
\bar{t} &= a_1 + te^{a_4+a_5}, \\
\bar{x} &= a_2 + xe^{\frac{1}{2}a_4+a_5}, \\
\bar{u} &= a_3 + (u - t + te^{a_4})e^{a_5}, \\
\bar{f} &= fe^{a_4+2a_5}, \\
\bar{g} &= ge^{2a_5}.
\end{aligned} \tag{2.10}
$$

2.2 Principal Lie algebra

The symmetry group of equation (2.6) is generated by the vector field of the form

$$
\Gamma = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.11}
$$

if and only if

$$
\Gamma^{[4]}\big(u_{tt} - u_{xx} + f(t)u_{xxxx} - g(t)u_{xxtt} + u_t u_{xx} + 2u_x u_{xt}\big)\big|_{(2.6)} = 0, \tag{2.12}
$$

where $\Gamma^{[4]}$ is the fourth-prolongation of (2.11). Equation (2.12) yields the following overdetermined system of linear partial differential equations (PDEs):

$$
\begin{aligned}
&\tau_u = 0,\ \tau_x = 0,\ \xi_u = 0,\ \xi_t = 0,\ \eta_x = 0,\ \xi_{xx} = 0,\ \eta_{tx} = 0, \\
&\eta_{uu} = 0,\ \eta_{tt} = 0,\ \tau_{tt} = 0,\ \eta_{tu} = 0,\ \eta_{xu} = 0,\ \eta_{xx} = 0, \\
&\eta_t + 2\xi_x - 2\tau_t = 0,\ \eta_u - 2\xi_x + \tau_t = 0,\ 2g\xi_x - \tau g_t = 0, \\
&\tau f_t - 4f\xi_x + 2f\tau_t = 0,\ \eta_u - 4\xi_x + 2\tau_t = 0.
\end{aligned} \tag{2.13}
$$

Solving the above system of determining equations for arbitrary f and g we find that the principal Lie algebra consists of two translation symmetries, namely

$$\Gamma_1 = \frac{\partial}{\partial x}, \; \Gamma_2 = \frac{\partial}{\partial u}.$$

2.3 Lie group classification of (2.6)

Solving the system of determining equations (2.13) also yields the following classifying relations:

$$(\alpha + \beta t)f'(t) + \gamma f(t) = 0,$$
$$(\lambda + \delta t)g'(t) + \mu g(t) = 0,$$

where $\alpha, \beta, \gamma, \lambda, \delta$ and μ are constants. Using the equivalence transformations obtained in Section 2.1, these classifying relations are invariant under the equivalence transformations (2.10) if

$$\bar{\alpha} = (\alpha + \beta a_1)e^{-a_4 - a_5}, \quad \bar{\beta} = \beta, \quad \bar{\gamma} = \gamma, \quad \bar{\lambda} = (\lambda + \delta a_1)e^{-a_4 - a_5}, \quad \bar{\delta} = \delta, \quad \bar{\mu} = \mu.$$

From these classifying relations we obtain the following six distinct cases for the functions f and g. For each case we also provide the associated symmetries, which extend the principal Lie algebra.

Case 1. $f(t)$ and $g(t)$ arbitrary but not of the form in Cases 2-6 given below.

In this case, we obtain the principal Lie algebra spanned by the operators

$$\Gamma_1 = \frac{\partial}{\partial x}, \; \Gamma_2 = \frac{\partial}{\partial u}.$$

Case 2. $f(t) = At^2$ and $g(t) = Bt^2$, where A and B are constants.

This case extends the principal Lie algebra by one scaling Lie point symmetry, namely

$$\Gamma_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Case 3. $f(t) = Ae^{2nt}$ and $g(t) = Be^{nt}$, where $A,\ B$ and n are constants.

In this case also the principal Lie algebra is extended by one Lie point symmetry, namely

$$\Gamma_3 = 2\frac{\partial}{\partial t} + nx\frac{\partial}{\partial x} + 2n(u-t)\frac{\partial}{\partial u}.$$

Case 4. $f(t) = At^{-2}$ and $g(t) = B$, where A and B are constants.

This case extends the principal Lie algebra by one Lie point symmetry, namely

$$\Gamma_3 = t\frac{\partial}{\partial t} + (2t-u)\frac{\partial}{\partial u}.$$

Case 5. $f(t) = A$ and $g(t) = Bt$, where A and B are constants.

The principal Lie algebra is extended by one Lie point symmetry

$$\Gamma_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial u}.$$

Case 6. $f(t) = A$ and $g(t) = B$, where A and B are constants.

For this case the principal Lie algebra is extended by a translation symmetry in t,

$$\Gamma_3 = \frac{\partial}{\partial t}.$$

2.4 Symmetry reductions and group invariant solutions of (2.6)

In this section we study all the cases of (2.6) for which the principal Lie algebra extends. We perform symmetry reductions and obtain exact group-invariant solutions where possible. Given any system of partial differential equations, one may construct group-invariant solutions using any group of transformations by reducing the number of variables in the system [70].

2.4.1 Case 2

$f(t) = At^2$, $g(t) = Bt^2$, where A and B are non-zero constants.

This case leads to the following form of equation (2.6):

$$u_{tt} - u_{xx} + At^2 u_{xxxx} - Bt^2 u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{2.14}$$

Its Lie point symmetry

$$\Gamma_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$$

has associated Lagrange system given by

$$\frac{dt}{t} = \frac{dx}{x} = \frac{du}{u}.$$

Solving $\dfrac{dt}{t} = \dfrac{dx}{x}$ and $\dfrac{dt}{t} = \dfrac{du}{u}$, we obtain, respectively, the two linearly independant invariants

$$J_1 = \frac{x}{t},\ J_2 = \frac{u}{t}.$$

Now, expressing J_2 as a function of J_1 we get

$$u(t,x) = t\,\phi\left(\frac{x}{t}\right). \tag{2.15}$$

Substituting this value of $u(t,x)$ into (2.14) and simplifying, we obtain a nonlinear fourth-order ODE

$$(z^2 - 2B - 1)\phi'' - 3z\phi'\phi'' + \phi\phi'' - 4Bz\phi''' + (A - Bz^2)\phi'''' = 0, \tag{2.16}$$

where $z = x/t$ and the 'prime' denotes differentiation with respect to z.

2.4.2 Case 3

$f(t) = Ae^{2nt}$, $g(t) = Be^{nt}$, where A, B and n are non-zero constants.

In this case equation (2.6) has the form

$$u_{tt} - u_{xx} + Ae^{2nt}u_{xxxx} - Be^{nt}u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0 \tag{2.17}$$

and its Lie point symmetry

$$\Gamma_3 = 2\frac{\partial}{\partial t} + nx\frac{\partial}{\partial x} + 2n(u-t)\frac{\partial}{\partial u}$$

provides us with the two invariants

$$J_1 = x^2 e^{-nt}, \; J_2 = ue^{-nt} - te^{-nt} - \frac{1}{n}e^{-nt}.$$

Expressing J_2 as a function of J_1 we get

$$u(t,x) = t + \frac{1}{n} + e^{nt}\phi(x^2 e^{-nt}). \tag{2.18}$$

The substitution of (2.18) into (2.17) yields the highly nonlinear ODE

$$\begin{aligned} &n^2\phi - n^2 z\phi' + 2n\phi\phi' - 2nz(\phi')^2 + (12A - 6Bn^2 z + n^2 z^2)\phi'' + 4nz\phi\phi'' \\ &-12nz^2\phi'\phi'' + (48Az - 14Bn^2 z^2)\phi''' + (16A - 4Bn^2 z^2)\phi'''' = 0, \end{aligned} \tag{2.19}$$

where $z = x^2 e^{-nt}$ and the 'prime' denotes differentiation with respect to z.

2.4.3 Case 4

$f(t) = At^{-2}$, $g(t) = B$, where A and B are non-zero constants.

For these values of $f(t)$ and $g(t)$ equation (2.6) takes the form

$$u_{tt} - u_{xx} + At^{-2}u_{xxxx} - Bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0 \tag{2.20}$$

and its Lie point symmetry

$$\Gamma_3 = t\frac{\partial}{\partial t} + (2t-u)\frac{\partial}{\partial u}$$

gives the two invariants

$$J_1 = x,\ \ J_2 = ut - t^2.$$

Proceeding in the same way as in the previous cases we obtain

$$u(t,x) = t + \frac{1}{t}\phi(x), \tag{2.21}$$

which leads to the nonlinear ODE

$$A\phi''''(x) - 2B\phi''(x) - \phi(x)\phi''(x) - 2\phi'^2(x) + 2\phi(x) = 0. \tag{2.22}$$

We now solve (2.22) by using the $(G'/G)-$expansion method [11].

The $(G'/G)-$expansion method assumes the solution of the ODE (2.22) to be of the form

$$\phi(x) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{2.23}$$

where $G = G(x)$ satisfies the second-order linear ordinary differential equation

$$G'' + \lambda G' + \mu G = 0 \tag{2.24}$$

with α_i, $i = 0, 1, 2, \cdots, m$, λ and μ being constants to be determined. The positive integer m is determined by considering the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in the ordinary differential equation (2.22).

In our case the balancing procedure yields $m = 2$, so the solutions of equation (2.22) are of the form

$$\phi(x) = \alpha_0 + \alpha_1(G'/G) + \alpha_2(G'/G)^2. \tag{2.25}$$

Substituting (2.25) into (2.22) and making use of the equation (2.24), collecting all terms with same powers of (G'/G) and equating each coefficient to zero leads to an overdetermined system of algebraic equations.

$$2\alpha_0 - 2\alpha_1 B\lambda\mu - 4\alpha_2 B\mu^2 + A\alpha_1\lambda^3\mu + 14A\alpha_2\lambda^2\mu^2 + 8A\alpha_1\lambda\mu^2 - \alpha_0\alpha_1\lambda\mu$$

$$
\begin{aligned}
&+16A\alpha_2\mu^3-2\alpha_1^2\mu^2-2\alpha_0\alpha_2\mu^2=0,\\
&2\alpha_1-2\alpha_1B\lambda^2-12\alpha_2B\lambda\mu-4\alpha_1B\mu+A\alpha_1\lambda^4+30A\alpha_2\lambda^3\mu+22A\alpha_1\lambda^2\mu\\
&-\alpha_0\alpha_1\lambda^2+120A\alpha_2\lambda\mu^2-5\alpha_1^2\lambda\mu-6\alpha_0\alpha_2\lambda\mu+16A\alpha_1\mu^2-10\alpha_1\alpha_2\mu^2\\
&-2\alpha_0\alpha_1\mu=0,\\
&2\alpha_2-8\alpha_2B\lambda^2-6\alpha_1B\lambda-16\alpha_2B\mu+16A\alpha_2\lambda^4+15A\alpha_1\lambda^3+232A\alpha_2\lambda^2\mu\\
&-3\alpha_1^2\lambda^2-4\alpha_0\alpha_2\lambda^2+60A\alpha_1\lambda\mu-23\alpha_1\alpha_2\lambda\mu-3\alpha_0\alpha_1\lambda-10\alpha_2^2\mu^2\\
&+136A\alpha_2\mu^2-6\alpha_1^2\mu-8\alpha_0\alpha_2\mu=0,\\
&130A\alpha_2\lambda^3-20\alpha_2B\lambda-4\alpha_1B+50A\alpha_1\lambda^2-13\alpha_1\alpha_2\lambda^2-22\alpha_2^2\lambda\mu\\
&+440A\alpha_2\lambda\mu-7\alpha_1^2\lambda-10\alpha_0\alpha_2\lambda+40A\alpha_1\mu-26\alpha_1\alpha_2\mu-2\alpha_0\alpha_1=0,\\
&240A\alpha_2\mu-12\alpha_2B-12\alpha_2^2\lambda^2+330A\alpha_2\lambda^2+60A\alpha_1\lambda-29\alpha_1\alpha_2\lambda-24\alpha_2^2\mu\\
&-4\alpha_1^2-6\alpha_0\alpha_2=0,\\
&12A\alpha_1-13\alpha_2^2\lambda+168A\alpha_2\lambda-8\alpha_1\alpha_2=0,\\
&60A\alpha_2-7\alpha_2^2=0.
\end{aligned}
$$

Solving this system of algebraic equations with the aid of Mathematica, we obtain

$$
\begin{aligned}
\alpha_0&=-\frac{60\mu}{(\lambda^2-4\mu)^2},\\
\alpha_1&=-\frac{60\lambda}{(\lambda^2-4\mu)^2},\\
\alpha_2&=-\frac{60}{(\lambda^2-4\mu)^2},\\
A&=-\frac{7}{(\lambda^2-4\mu)^2},\\
B&=-\frac{5}{2(\lambda^2-4\mu)}.
\end{aligned}
$$

Now substituting the values of these constants in (2.25) and making use of (2.21) we obtain two types of travelling wave solutions of the generalized Benney-Luke equation (2.6) given below.

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$\begin{aligned} u(t,x) &= t + \frac{1}{t}\left[\alpha_0 + \alpha_1\left(-\frac{\lambda}{2} + \delta_1 \frac{C_1 \sinh(\delta_1 x) + C_2 \cosh(\delta_1 x)}{C_1 \cosh(\delta_1 x) + C_2 \sinh(\delta_1 x)}\right)\right] \\ &+ \frac{1}{t}\left[\alpha_2\left(-\frac{\lambda}{2} + \delta_1 \frac{C_1 \sinh(\delta_1 x) + C_2 \cosh(\delta_1 x)}{C_1 \cosh(\delta_1 x) + C_2 \sinh(\delta_1 x)}\right)^2\right], \end{aligned} \tag{2.26}$$

where $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$\begin{aligned} u(t,x) &= t + \frac{1}{t}\left[\alpha_0 + \alpha_1\left(-\frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 x) + C_2 \cos(\delta_2 x)}{C_1 \cos(\delta_2 x) + C_2 \sin(\delta_2 x)}\right)\right] \\ &+ \frac{1}{t}\left[\alpha_2\left(-\frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 x) + C_2 \cos(\delta_2 x)}{C_1 \cos(\delta_2 x) + C_2 \sin(\delta_2 x)}\right)^2\right], \end{aligned} \tag{2.27}$$

where $\delta_2 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

2.4.4 Case 5

$f(t) = A,\ g(t) = Bt$, where A and B are non-zero constants.

This case yields the following form of equation (2.6):

$$u_{tt} - u_{xx} + Au_{xxxx} - Btu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{2.28}$$

Its Lie point symmetry

$$\Gamma_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial u}$$

as before gives

$$u(t,x) = t + \phi\left(\frac{x^2}{t}\right), \tag{2.29}$$

where ϕ satisfies the nonlinear ODE

$$\begin{aligned} &(2z - 4B)\phi' - 10z(\phi')^2 + (12A - 32Bz + z^2)\phi'' - 12z^2\phi'\phi'' \\ &+(48Az - 26Bz^2)\phi''' + (16Az^2 - 4Bz^3)\phi'''' = 0, \end{aligned} \tag{2.30}$$

with $z = x^2/t$ and the 'prime' denotes differentiation with respect to z.

2.4.5 Case 6

$f(t) = a$, $g(t) = b$, where a and b are nonzero constants.

This case leads us to the Benney-Luke equation (2.1). Considerable study has been done for this particular case (see, for example, [60, 64]). Here we use its Lie point symmetries to obtain travelling wave solutions. Equation (2.1) has three translation symmetries

$$\Gamma_1 = \frac{\partial}{\partial x}, \; \Gamma_2 = \frac{\partial}{\partial u}, \; \Gamma_3 = \frac{\partial}{\partial t}.$$

We seek a solution using the two symmetries Γ_1 and Γ_3 and find travelling wave solutions of (2.1). Taking the linear combination of Γ_1 and Γ_3 given by $\Gamma = k\Gamma_3 - \omega\Gamma_1$, one can solve the corresponding Lagrange system to obtain an invariant $z = kx + \omega t$, where k and ω are constants. Thus, the group-invariant solution of (2.1) is of the form

$$u = \phi(z). \tag{2.31}$$

Substitution of the above value of u into (2.1) yields the fourth-order nonlinear ODE

$$(a\omega^4 - bk^2\omega^2)\phi'''' + 3k\omega^2\phi'\phi'' + (k^2 - \omega^2)\phi'' = 0. \tag{2.32}$$

We now solve this ODE by employing Kudryashov method [18, 55, 56]. Thus we assume the exact solution of equation (2.32) can be expressed as

$$\phi(z) = \sum_{n=0}^{N} a_n \Big(G(z)\Big)^n, \tag{2.33}$$

where $G(z)$ satisfies the first-order nonlinear ODE

$$G'(z) = G^2(z) - G(z). \tag{2.34}$$

An exact solution of this equation is given by

$$G(z) = \frac{1}{1+e^z}. \tag{2.35}$$

The positive integer N in equation (2.33) is determined by the homogenous balance proceedure, which in this case yields $N = 1$. Hence the solution of (2.32) is of the form

$$\phi(z) = a_0 + a_1 G(z). \tag{2.36}$$

Substituting (2.36) into (2.32) and making use of the equation (2.34) and then equating the coefficients of the functions G^i to zero, we obtain the algebraic system of equations in terms of a_0 and a_1 given by

$$\begin{aligned}
&4\,bk^2\omega^2 a_1 - 4\,a\omega^4 a_1 - \,k\omega^2 {a_1}^2 = 0,\\
&4\,a\omega^4 a_1 - 4\,bk^2\omega^2 a_1 + \,k\omega^2 {a_1}^2 = 0,\\
&a\omega^4 a_1 - bk^2\omega^2 a_1 + k^2 a_1 - \omega^2 a_1 = 0,\\
&5\,bk^2\omega^2 a_1 - 5\,a\omega^4 a_1 - \,k\omega^2 {a_1}^2 - \,k^2 a_1 + \,\omega^2 a_1 = 0,\\
&25\,a\omega^4 a_1 - 25\,bk^2\omega^2 a_1 + 6\,k\omega^2 {a_1}^2 + \,k^2 a_1 - \,\omega^2 a_1 = 0.
\end{aligned}$$

Solving this system of algebraic equations with the aid of Maple, one possible set of values of the constants is

$$k = \pm\omega\sqrt{\frac{a\omega^2-1}{b\omega^2-1}}, \quad a_1 = \pm\frac{4\omega\,(a-b)}{\sqrt{(a\omega^2-1)(b\omega^2-1)}}.$$

As a result, a solution of the Benney-Luke equation (2.1) is given by

$$u(t,x) = a_0 \pm \frac{4w(a-b)}{(1+e^{kx+\omega t})\sqrt{(a\omega^2-1)(b\omega^2-1)}},$$

where a_0 is an arbitrary constant.

2.5 Conservation laws of (2.6)

In this section we construct conservation laws for (2.6) for two special cases, namely, Cases 4 and 6, using the multiplier method [45, 55, 71, 72]. We note that the other cases provide only the trivial conservation laws.

We first consider Case 4, that is equation (2.20).

We recall that a multiplier Q for (2.20) has the property that

$$Q(u_{tt} - u_{xx} + At^{-2}u_{xxxx} - Bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt}) = D_t T^t + D_x T^x \qquad (2.37)$$

holds identically. Here D_t and D_x are the total derivatives as given in Section 2.1. We note that the right hand side of (2.37) is a divergence expression.

The vector $T = (T^t, T^x)$ is a conserved vector of (2.20) if it satisfies

$$D_t T^t + D_x T^x|_{(2.20)} = 0. \qquad (2.38)$$

The determining equation for the first-order multipliers $Q(t, x, u, u_x, u_t)$ is given by

$$\frac{\delta}{\delta u}[Q(u_{tt} - u_{xx} + At^{-2}u_{xxxx} - Bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt})] = 0, \qquad (2.39)$$

where $\delta/\delta u$ is the Euler-Lagrange operator

$$\begin{aligned}\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_t\frac{\partial}{\partial u_t} + D_x^2\frac{\partial}{\partial u_{xx}} + D_t^2\frac{\partial}{\partial u_{tt}} + D_t D_x\frac{\partial}{\partial u_{tx}} \\ &\quad + D_x^4\frac{\partial}{\partial u_{xxxx}} + D_t^2 D_x^2\frac{\partial}{\partial u_{ttxx}}. \end{aligned} \qquad (2.40)$$

Expanding (2.39) and then splitting on derivatives of u yields an over determined system of PDEs. Solving these PDEs, we obtain

$$Q = c_1 u_x + c_2, \qquad (2.41)$$

where c_1 and c_2 are constants. For the conserved vectors of (2.20) corresponding to this first-order multiplier we solve for T^t and T^x in the equation

$$D_t T^t + D_x T^x = Q(u_{tt} - u_{xx} + At^{-2}u_{xxxx} - Bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt}). \qquad (2.42)$$

The computations of this equation are lengthy and cumbersome. However, with the help of Maple using the homotopy method we obtain the two conserved vectors

$$\begin{aligned}
T_1^t &= \frac{1}{3}uu_xu_{xx} + \frac{2}{3}{u_x}^3 + \frac{1}{2}u_{xx}Bu_{tx} - \frac{1}{2}u_{txx}u_xB - \frac{1}{2}uu_{tx} + \frac{1}{2}u_tu_x,\\
T_1^x &= -\frac{1}{2}Bu_{ttx}u_x - \frac{1}{3}uu_{tx}u_x + \frac{1}{3}u_t{u_x}^2 + \frac{1}{2}uu_{tt} - \frac{1}{2}{u_x}^2 + At^{-2}u_xu_{xxx} - \frac{1}{2}At^{-2}{u_{xx}}^2
\end{aligned}$$

and

$$\begin{aligned}
T_2^t &= \frac{1}{2}uu_{xx} + {u_x}^2 - u_{txx}B + u_t,\\
T_2^x &= -\frac{1}{2}uu_{tx} + \frac{1}{2}u_tu_x - u_x + At^{-2}u_{xxx}.
\end{aligned}$$

Secondly we consider Case 6 and find conservation laws for

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_tu_{xx} + 2u_xu_{xt} = 0. \qquad (2.43)$$

The determining equation for the first-order multipliers of (2.43), $Q(t, x, u, u_x, u_t)$ is given by

$$\frac{\delta}{\delta u}[Q(u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_tu_{xx} + 2u_xu_{xt})] = 0.$$

Proceeding in the same manner as above, the corresponding multiplier is

$$Q = c_2u_t + c_1u_x + c_3,$$

where c_1, c_2 and c_3 are constants. Consequently we obtain three conserved vectors, viz;

$$\begin{aligned}
T_1^t &= \frac{1}{3}uu_xu_{xx} + \frac{2}{3}{u_x}^3 - \frac{1}{2}uu_{tx} + \frac{1}{2}u_tu_x + \frac{1}{2}bu_{xx}u_{tx} - \frac{1}{2}bu_{txx}u_x,\\
T_1^x &= -\frac{1}{3}uu_xu_{tx} + \frac{1}{3}{u_x}^2u_t - \frac{1}{2}{u_x}^2 - \frac{1}{2}bu_xu_{ttx} + au_xu_{xxx} - \frac{1}{2}a{u_{xx}}^2 + \frac{1}{2}uu_{tt};
\end{aligned}$$

$$T_2^t = \frac{2}{3}uu_tu_{xx} + \frac{2}{3}{u_x}^2u_t + \frac{2}{3}uu_xu_{tx} + \frac{1}{2}{u_t}^2 + \frac{1}{2}bu_{xx}u_{tt} - \frac{1}{2}bu_{txx}u_t + \frac{1}{2}auu_{xxxx}$$

$$
\begin{aligned}
& -\frac{1}{2}buu_{ttxx} - \frac{1}{2}uu_{xx}, \\
T_2^x &= -\frac{2}{3}uu_tu_{tx} - \frac{2}{3}uu_{tt}u_x + \frac{1}{3}u_xu_t^{\,2} + \frac{1}{2}uu_{tx} + \frac{1}{2}buu_{tttx} - \frac{1}{2}auu_{txxx} - \frac{1}{2}u_tu_x \\
& -\frac{1}{2}bu_xu_{ttt} + \frac{1}{2}au_xu_{txx} - \frac{1}{2}au_{xx}u_{tx} + \frac{1}{2}au_{xxx}u_t
\end{aligned}
$$

and

$$
\begin{aligned}
T_3^t &= \frac{1}{2}uu_{xx} + u_x^{\,2} - bu_{txx} + u_t, \\
T_3^x &= -\frac{1}{2}uu_{tx} + \frac{1}{2}u_tu_x + au_{xxx} - u_x.
\end{aligned}
$$

2.6 Conclusion

In this chapter we studied the generalized Benney-Luke equation (2.6) with time dependent coefficients. Firstly, Lie group classification was carried out for this equation. The principal Lie algebra was determined, which consisted of two translation symmetries. The equivalence group of transformations for (2.6) was obtained and the classifying relations produced six distinct cases for the time dependent coefficients $f(t)$ and $g(t)$ for which the principal Lie algebra was extended. Lie point symmetries for these different forms of the variable coefficients admitted by the equation were obtained. Symmetry reductions were performed for all cases and exact group-invariant solutions were obtained in two cases. In both cases, travelling wave solutions were derived. Also, conservation laws were constructed for two cases by using the multiplier method.

Chapter 3

Solutions and conservation laws of the symmetric regularized long wave equation

In this chapter we study the symmetric regularized long wave equation (SRLW) given by

$$u_{tt} - u_{xx} + \frac{1}{2}(u^2)_{tx} - u_{ttxx} = 0, \tag{3.1}$$

which arises in several physical applications, for example in sound waves in a plasma [73]. Exact travelling wave solutions of this equation were obtained using the (G'/G)-expansion method [73]. In the present work, Lie symmetry method along with the simplest equation method and the exp-function method are used to construct exact solutions for this equation. First the Lie point symmetries of the SRLW equation (3.1) are found using the Lie algorithm [25]. These Lie point symmetries are then used to transform (3.1) into an ordinary differential equation. The simplest equation method [14] and the exp-function method [74, 75] are then used to construct exact solutions of the ordinary differential equation, which leads to the exact solutions of the SRLW equation. Moreover, conservation laws of (3.1) are derived using the multiplier method.

The work corresponding to the exact solutions of (3.1) has been published. See [76].

3.1 Exact solutions of (3.1)

In this section we first find the Lie point symmetries of (3.1) and use them to reduce the equation to an ordinary differential equation. We then apply the simplest equation method to the ordinary differential equation and obtain exact solutions of (3.1).

3.1.1 Lie point symmetries of (3.1) and symmetry reduction

The vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of the SRLW equation (3.1) if and only if

$$X^{[4]}(u_{tt} - u_{xx} + \frac{1}{2}(u^2)_{tx} - u_{ttxx})|_{(3.1)} = 0. \tag{3.2}$$

Here $X^{[4]}$ is the fourth prolongation of X given by

$$\begin{aligned} X^{[4]} &= \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta^t\frac{\partial}{\partial u_t} \\ &\quad + \zeta^x\frac{\partial}{\partial u_x} + \zeta^{tt}\frac{\partial}{\partial u_{tt}} + \zeta^{xx}\frac{\partial}{\partial u_{xx}} + \zeta^{tx}\frac{\partial}{\partial u_{tx}} + \zeta^{ttxx}\frac{\partial}{\partial u_{ttxx}}, \end{aligned}$$

where

$$\zeta^t = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),$$

$$
\begin{aligned}
\zeta^{x} &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta^{tt} &= D_t(\zeta^t) - u_{tt} D_t(\xi^1) - u_{xt} D_t(\xi^2),\\
\zeta^{xx} &= D_x(\zeta^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2),\\
\zeta^{xt} &= D_t(\zeta^x) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2),\\
\zeta^{xxtt} &= D_x(\zeta^{ttx}) - u_{ttxx} D_x(\xi^1) - u_{xtxx} D_x(\xi^2).
\end{aligned}
$$

Expanding (3.2) and splitting on the derivatives of u, we obtain the following two translation symmetries:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x}.
\end{aligned}
$$

Let us consider the linear combination of these translation symmetries $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$, namely, the symmetry $X = X_1 + \nu X_2$, where ν is a constant. The associated Lagrange system is

$$
\frac{dt}{1} = \frac{dx}{\nu} = \frac{du}{0}
$$

and this leads to the two invariants

$$
z = x - \nu t, \quad u = F(z). \tag{3.3}
$$

Treating F as the new dependent variable and z as the new independent variable, and then substituting the value of u into the SRLW equation (3.1) transforms (3.1) to a fourth-order nonlinear ordinary differential equation

$$
(\nu^2 - 1)F''(z) - \nu F(z)F''(z) - \nu(F'(z))^2 + \nu^2 F(z)'''' = 0. \tag{3.4}
$$

3.1.2 Exact solutions of (3.1) using simplest equation method

Now the simplest equation method is used to solve (3.4) and henceforth one obtains the exact solutions of the SRLW equation (3.1). The Bernoulli and Riccati equations will be used as the simplest equations. The Bernoulli and Riccati equations are well known equations whose solutions can be expressed in terms of elementary functions. We recall from chapter one the Bernoulli equation

$$H'(z) = cH(z) + dH^2(z), \tag{3.5}$$

where c and d are constants and whose solution is

$$H(z) = c\left\{\frac{\cosh[c(z+C)] + \sinh[c(z+C)]}{1 - d\cosh[c(z+C)] - d\sinh[c(z+C)]}\right\},$$

with C a constant of integration.

We also recall the Riccati equation

$$G'(z) = cG^2(z) + dG(z) + e, \tag{3.6}$$

where c, d and e are constants with the solutions

$$H(z) = -\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(z+C)\right]$$

and

$$H(z) = -\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2c}{\theta}\sinh\left(\frac{\theta z}{2}\right)},$$

with $\theta^2 = d^2 - 4ce > 0$ and C a constant of integration.

Solutions of (3.1) using Bernoulli as the simplest equation

The solutions of the ODE (3.4) are considered to be in the form

$$F(z) = \sum_{i=0}^{M} \mathcal{A}_i (H(z))^i, \tag{3.7}$$

where $H(z)$ satisfies the Bernoulli equation, M is a positive integer that can be determined by balancing the highest order derivative term with the highest order nonlinear term, and $\mathcal{A}_i, (i = 0, 1, \cdots, M)$ are parameters to be determined.

The balancing procedure yields $M = 2$, so the solutions of (3.4) are of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1 H + \mathcal{A}_2 H^2. \tag{3.8}$$

Substituting (3.8) into (3.4) and making use of the Bernoulli equation (3.5) and then equating all coefficients of the function H^i to zero, we obtain the following algebraic system of equations in terms of $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$:

$$
\begin{aligned}
H^0 &: \ 10\nu\mathcal{A}_2^2 d^2 + 120\nu^2\mathcal{A}_2 d^4 = 0,\\
H^1 &: \ 24\nu^2\mathcal{A}_1 d^4 + 12\nu\mathcal{A}_1 d^2\mathcal{A}_2 + 336\nu^2\mathcal{A}_2 d^3 c + 18\nu\mathcal{A}_2^2 dc = 0,\\
H^2 &: \ \nu\mathcal{A}_0\mathcal{A}_1 c^2 + \mathcal{A}_1 c^2 + \nu^2\mathcal{A}_1 c^4 - \nu^2\mathcal{A}_1 c^2 = 0,\\
H^3 &: \ 8\nu\mathcal{A}_2^2 c^2 + 6\mathcal{A}_2 d^2 - 6\nu^2\mathcal{A}_2 d^2 + 21\nu\mathcal{A}_1 d\mathcal{A}_2 c + 3\nu\mathcal{A}_1^2 d^2 + 60\nu^2\mathcal{A}_1 d^3 c\\
&\quad +6\nu\mathcal{A}_0\mathcal{A}_2 d^2 + 330\nu^2\mathcal{A}_2 d^2 c^2 = 0,\\
H^4 &: \ 3\mathcal{A}_1 dc - 3\nu^2\mathcal{A}_1 dc + 4\nu\mathcal{A}_0\mathcal{A}_2 c^2 + 4\mathcal{A}_2 c^2 - 4\nu^2\mathcal{A}_2 c^2 + 3\nu\mathcal{A}_0\mathcal{A}_1 dc\\
&\quad +2\nu\mathcal{A}_1^2 c^2 + 16\nu^2\mathcal{A}_2 c^4 + 15\nu^2\mathcal{A}_1 dc^3 = 0,\\
H^5 &: \ 2\mathcal{A}_1 d^2 + 50\nu^2\mathcal{A}_1 d^2 c^2 - 10\nu^2\mathcal{A}_2 dc + 5\nu\mathcal{A}_1^2 dc + 130\nu^2\mathcal{A}_2 dc^3 - 2\nu^2\mathcal{A}_1 d^2\\
&\quad +9\nu\mathcal{A}_1 c^2\mathcal{A}_2 + 2\nu\mathcal{A}_0\mathcal{A}_1 d^2 + 10\nu\mathcal{A}_0\mathcal{A}_2 dc + 10\mathcal{A}_2 dc = 0.
\end{aligned}
$$

Solving this system, with the aid of Maple, we obtain the following values for the constants:

$$
\begin{aligned}
\mathcal{A}_0 &= \frac{-c^2\nu^2 + \nu^2 - 1}{\nu},\\
\mathcal{A}_1 &= -12cd\nu,\\
\mathcal{A}_2 &= -12\nu d^2.
\end{aligned}
$$

As a result a solution of the symmetric regularized long wave equation (3.1) using the Bernoulli equation as the simplest equation is

$$
\begin{aligned}
u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 c &\left\{\frac{\cosh[c(z+C)] + \sinh[c(z+C)]}{1 - d\cosh[c(z+C)] - d\sinh[c(z+C)]}\right\}\\
+\mathcal{A}_2 c^2 &\left\{\frac{\cosh[c(z+C)] + \sinh[c(z+C)]}{1 - d\cosh[c(z+C)] - d\sinh[c(z+C)]}\right\}^2,
\end{aligned}
$$

where $z = x - \nu t$, C is a constant of integration.

Solutions of (3.1) using Ricatti as the simplest equation

The balancing procedure yields $M=2$, so the solutions of (3.4) are of the form

$$F(z)=\mathcal{A}_0+\mathcal{A}_1G(z)+\mathcal{A}_2G^2(z). \tag{3.9}$$

Substituting (3.9) into (3.4) and making use of the Ricatti equation (3.6) and then equating all coefficients of the function G^i to zero, we obtain an algebraic system of equations in terms of $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$. Solving the resultant algebraic equations we obtain the following set of values:

$$\begin{aligned}
\mathcal{A}_0 &= -\frac{8ce\nu^2+d^2\nu^2-\nu^2+1}{\nu},\\
\mathcal{A}_1 &= -12cd\nu,\\
\mathcal{A}_2 &= -12c^2\nu.
\end{aligned}$$

It follows that the solutions for the symmetric regularized long wave equation (3.1) using the Ricatti equation as the simplest equation are

$$\begin{aligned}
u(t,x) = \mathcal{A}_0 &+ \mathcal{A}_1\left\{-\frac{d}{2c}-\frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\
&+\mathcal{A}_2\left\{-\frac{d}{2c}-\frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2
\end{aligned}$$

and

$$\begin{aligned}
u(t,x) = \mathcal{A}_0 &+ \mathcal{A}_1\left\{-\frac{d}{2c}-\frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2c}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}\\
&+\mathcal{A}_2\left\{-\frac{d}{2c}-\frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2c}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2,
\end{aligned}$$

where $z=x-\nu t$ with $\theta^2=d^2-4ce>0$ and C is a constant of integration.

3.1.3 Solution of (3.1) using the exp-function method

In this section we use the exp-function method [74, 75] to solve the symmetric regularized long wave equation (3.1). We consider solutions of (3.4) in the form

$$F(z)=\frac{\sum_{n=-b}^{c}a_ne^{nz}}{\sum_{m=-p}^{q}b_me^{mz}}, \tag{3.10}$$

where b, c, p and q are positive integers to be determined, a_n and b_m are arbitrary constants [74]. The balancing procedure of the exp-function method produces $p = b$ and $q = c$. For simplicity, we set $p = b = 1$ and $q = c = 1$ so that (3.10) is reduced to

$$F(z) = \frac{a_1 e^z + a_0 + a_{-1} e^{-z}}{b_1 e^z + b_0 + b_{-1} e^{-z}}. \tag{3.11}$$

Substituting (3.11) into (3.4) and solving the resultant ODE, with the help of Maple, one possible set of values of the constants is

$$\begin{aligned}
a_{-1} &= -\frac{b_{-1}}{\nu}, \\
a_0 &= \frac{b_0(6\nu^2 - 1)}{\nu}, \\
a_1 &= \frac{-b_0^2}{4\nu b_{-1}}, \\
b_1 &= \frac{b_0^2}{4b_{-1}}, \\
&b_0,\ b_{-1} \text{ arbitrary.}
\end{aligned}$$

As a result we obtain the solution of (3.1) as

$$u(t, x) = \frac{a_1 e^{x-\nu t} + a_0 + a_{-1} e^{-(x-\nu t)}}{b_1 e^{x-\nu t} + b_0 + b_{-1} e^{-(x-\nu t)}}.$$

3.2 Conservation laws of (3.1)

The multiplier approach is used to construct conservation laws of (3.1). The SRLW equation (3.1) can be written in expanded form as

$$E \equiv u_{tt} - u_{xx} + u_t u_x + u u_{tx} - u_{xxtt} = 0. \tag{3.12}$$

The determining equation for the zeroth-order multiplier $\Lambda(t, x, u)$ is given by

$$\frac{\delta}{\delta u}[\Lambda(u_{tt} - u_{xx} + u_t u_x + u u_{tx} - u_{xxtt})] = 0. \tag{3.13}$$

Here the Euler-Lagrange operator $\delta/\delta u$ is

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_t^2\frac{\partial}{\partial u_{tt}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_{tx}\frac{\partial}{\partial u_{tx}}+D_x^2D_t^2\frac{\partial}{\partial u_{xxtt}}$$

and the total derivatives D_t and D_x are given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+u_{ttt}\frac{\partial}{\partial u_{tt}}+u_{ttx}\frac{\partial}{\partial u_{tx}}\\
&\quad +u_{txx}\frac{\partial}{\partial u_{xx}}+\cdots+u_{xxttt}\frac{\partial}{\partial u_{xxtt}}+\cdots\\
D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+u_{xxx}\frac{\partial}{\partial u_{xx}}+u_{xxt}\frac{\partial}{\partial u_{xt}}\\
&\quad +u_{xtt}\frac{\partial}{\partial u_{tt}}+\cdots+u_{xxxtt}\frac{\partial}{\partial u_{xxtt}}+\cdots.
\end{aligned}$$

Thus we can write (3.13) as

$$\left(\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_t^2\frac{\partial}{\partial u_{tt}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_{tx}\frac{\partial}{\partial u_{tx}}+D_x^2D_t^2\frac{\partial}{\partial u_{xxtt}}\right)\times$$
$$\big(\Lambda u_{tt}-\Lambda u_{xx}+\Lambda u_tu_x+\Lambda uu_{tx}-\Lambda u_{xxtt}\big)=0.$$

Expanding and simplifying we obtain the determining equation

$$\begin{aligned}
&\Lambda_u u_{tt}-\Lambda_u u_{xx}+2\Lambda_u uu_{tx}-2\Lambda_u u_{xxtt}+\Lambda_{tt}+2\Lambda_{tu}u_t+\Lambda_{uu}u_t^2+\Lambda_u u_{tt}-\Lambda_{xx}\\
&-2\Lambda_{xu}u_x-\Lambda_{uu}u_x^2-\Lambda_u u_{xx}+\Lambda_{tx}u+\Lambda_{tu}uu_x+\Lambda_{ux}uu_t+\Lambda_{uu}uu_tu_x+\Lambda_u u_tu_x\\
&-\Lambda_{ttxx}-2\Lambda_{ttxu}u_x-\Lambda_{ttuu}u_x^2-\Lambda_{ttu}u_{xx}-2\Lambda_{tuxx}u_t-4\Lambda_{tuxu}u_tu_x-4\Lambda_{tux}u_{xt}\\
&-2\Lambda_{tuuu}u_tu_x^2-4\Lambda_{tuu}u_xu_{xt}-2\Lambda_{tuu}u_tu_{xx}-2\Lambda_{tu}u_{xxt}-\Lambda_{uuxx}u_t^2-2\Lambda_{uuxu}u_t^2u_x\\
&-4\Lambda_{uux}u_tu_{xt}-\Lambda_{uuuu}u_t^2u_x^2-\Lambda_{uuu}u_t^2u_{xx}-4\Lambda_{uuu}u_tu_xu_{xt}-2\Lambda_{uu}u_{xt}^2\\
&-2\Lambda_{uu}u_tu_{xxt}-\Lambda_{uxx}u_{tt}-2\Lambda_{uxu}u_{tt}u_x-2\Lambda_{ux}u_{xtt}-\Lambda_{uuu}u_{tt}u_x^2-\Lambda_{uu}u_{tt}u_{xx}\\
&-2\Lambda_{uu}u_xu_{xtt}=0.
\end{aligned}$$

Splitting on derivatives of u and solving the resultant system of determining equations we obtain the multiplier

$$\Lambda=\frac{1}{2}C_1(x^2+t^2)+C_2x+C_3t+C_4,$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. For the conserved vectors of (3.1) corresponding to this zeroth order multiplier we have to solve for T^1 and T^2 in the equation

$$D_t T^1 + D_x T^2 = \Lambda E.$$

The computations of this equation are lengthy and cumbersome. However, with the help of Maple we obtain that

$$\begin{aligned} T^1 &= \left\{\frac{1}{2}(t^2+x^2)C_1 + C_2 x + C_3 t + C_4\right\} u u_x - (C_1 t + C_3) u \\ &\quad + \left\{\frac{1}{2}(t^2+x^2)C_1 + C_2 x + C_3 t + C_4\right\} u_t + (C_1 t + C_3) u_{xx} \\ &\quad - \left\{\frac{1}{2}(t^2+x^2)C_1 + C_2 x + C_3 t + C_4\right\} u_{txx} \end{aligned}$$

and

$$\begin{aligned} T^2 &= -\frac{1}{2} u \left[\left\{\frac{1}{2}(t^2+x^2)C_1 + C_2 x + C_3 t + C_4\right\} u_t + (C_1 t + C_3) u\right] \\ &\quad + \frac{1}{2} u \left\{\frac{1}{2}(t^2+x^2)C_1 + C_2 x + C_3 t + C_4\right\} u_t - C_1 u_x + u(C_1 x - C_2) \\ &\quad - \left\{\frac{1}{2}(t^2+x^2)C_1 + C_2 x + C_3 t + C_4\right\} u_x. \end{aligned}$$

Hence we obtain the following four conserved vectors for (3.1):

$$\begin{aligned} T_1^1 &= \frac{1}{2}(t^2+x^2) u u_x - t u + \frac{1}{2}(t^2+x^2) u_t + t u_{xx} - \frac{1}{2}(t^2+x^2) u_{txx}, \\ T_1^2 &= -\frac{1}{2} t u^2 - \frac{1}{2}(t^2+x^2) u_x - u_x + x u, \end{aligned}$$

$$\begin{aligned} T_2^1 &= u u_{xx} + x u_t - x u_{txx}, \\ T_2^2 &= -x u_x + u, \end{aligned}$$

$$T_3^1 = t u u_x + t u_t - t u_{txx} - u + u_{xx},$$

$$T_3^2 = -\frac{1}{2}u^2 - tu_x,$$

$$T_4^1 = uu_x + u_t - u_{txx},$$
$$T_4^2 = -u_x.$$

It can be easily verified that the four conservation laws are nontrivial.

3.3 Conclusion

In this chapter exact solutions of the symmetric regularized long wave equation (3.1) were determined successfully using the simplest equation method and the exp-function method. The solutions obtained via the simplest equation method and the exp-function method are travelling wave solutions. We also constructed the conservation laws using the multiplier approach for the symmetric regularized long wave equation. All the four conserved vectors were verified and found to be nontrivial.

Chapter 4

Exact solutions for Klein-Gordon-Zakharov equations

The Klein-Gordon-Zakharov (KGZ) equations [77]

$$u_{tt} - u_{xx} + u + uv + |u|^2 u = 0, \tag{4.1a}$$

$$v_{tt} - v_{xx} - (|u|^2)_{xx} = 0, \tag{4.1b}$$

are a coupled system of nonlinear partial differential equations of two functions $u(x,t)$ and $v(x,t)$. This model describes the interaction of the Langmuir wave and the ion acoustic wave in plasma. The function $u(x,t)$ denotes the fast time scale component of electric field raised by electrons and the function $v(x,t)$ denotes the deviation of ion density from its equilibrium. Here $u(x,t)$ is a complex function and $v(x,t)$ is a real function. Note that if we remove the term $|u|^2u$, then this system reduces to the classical Klein-Gordon-Zakharov system [78]

$$u_{tt} - u_{xx} + u + uv = 0, \tag{4.2a}$$

$$v_{tt} - v_{xx} - (|u|^2)_{xx} = 0. \tag{4.2b}$$

A number of studies have been conducted for this system (4.2) in different time space [79–83]. However, for the KGZ equations (4.1), Chen [84] considered orbital stability of solitary waves, while Shi et al. [78] employed the sine-cosine method and the extended tanh method to construct exact solutions of the KGZ equations (4.1).

In this chapter we employ an entirely different approach, namely, the travelling wave variable approach along with the simplest equation method to obtain exact solutions of the KGZ equations (4.1).

This work has been published. See [85].

4.1 Solution of (4.1) using the travelling wave variable approach

The travelling wave variable approach converts the system of nonlinear partial differential equations into a system of nonlinear ordinary differential equations, which we then solve to obtain exact solutions of the system.

In order to solve the KGZ equations (4.1), we first transform it to a system of nonlinear ordinary differential equations which can then be solved in order to obtain its exact solutions.

We make the wave variable transformation

$$u = e^{i\phi}u(z), \; v = v(z), \; \phi = px + rt, \; z = kx + dt, \tag{4.3}$$

where p, r, k and d are real constants, $d \neq k$. Using this transformation, (4.1) transforms to

$$(p^2 - r^2 + 1)u + i(2rd - 2pk)u' + (d^2 - k^2)u'' + uv + u^3 = 0, \tag{4.4a}$$

$$(d^2 - k^2)v'' - (u^2)'' = 0. \tag{4.4b}$$

Integrating (4.4b) twice and taking the constants of integration to be zero we obtain

$$v = \frac{u^2}{d^2 - k^2}. \tag{4.5}$$

Now substituting (4.5) into (4.4a) we get

$$u'' = \left(\frac{r^2 - p^2 - 1}{d^2 - k^2}\right)u + \left(\frac{d^2 - k^2 + 1}{(d^2 - k^2)^2}\right)u^3, \tag{4.6}$$

which can be written in the form

$$u'' = Pu + Qu^3, \tag{4.7}$$

where

$$P = \frac{r^2 - p^2 - 1}{d^2 - k^2} \quad \text{and} \quad Q = \frac{d^2 - k^2 + 1}{(d^2 - k^2)^2}.$$

Solving (4.7), with the aid of Mathematica, we obtain the solution

$$u(z) = \pm\frac{1}{H}\, i\,\mathrm{sn}(F|\omega), \tag{4.8}$$

where $\mathrm{sn}(F|\omega)$ is a Jacobian elliptic function of the sine-amplitude [86],

$$F = \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{P^2 - 2Qc_1} - P\right)(z + c_2)^2}, \quad H = \sqrt{\frac{Q}{\sqrt{P^2 - 2Qc_1} + P}}$$

and

$$\omega = \frac{1}{Qc_1}\left\{ - Qc_1 + P\left(\sqrt{P^2 - 2Qc_1} + P\right)\right\}$$

is the modulus of the elliptic function with $0 < \omega < 1$. Here c_1 and c_2 are constants of integration. Reverting back to our original variables, we can now write the solution of our Klein-Gordon-Zakharov equations as

$$u(x,t) = \pm\frac{1}{H}\, i\,\mathrm{sn}(F|\omega), \tag{4.9}$$

where

$$F = \frac{1}{\sqrt{2}}\sqrt{\left(\sqrt{P^2 - 2Qc_1} - P\right)(kx + dt + c_2)^2},$$

ω and H are as above.

Now $v(x,t)$ can be obtained from (4.5).

It should be noted that the solution (4.9) is valid for $0 < \omega < 1$ and as ω approaches zero, the solution becomes the normal sine function, $\sin z$, and as ω approaches 1, the solution tends to the tanh function, $\tanh z$.

The profile of the solution (4.9) is given in the Figure 4.1.

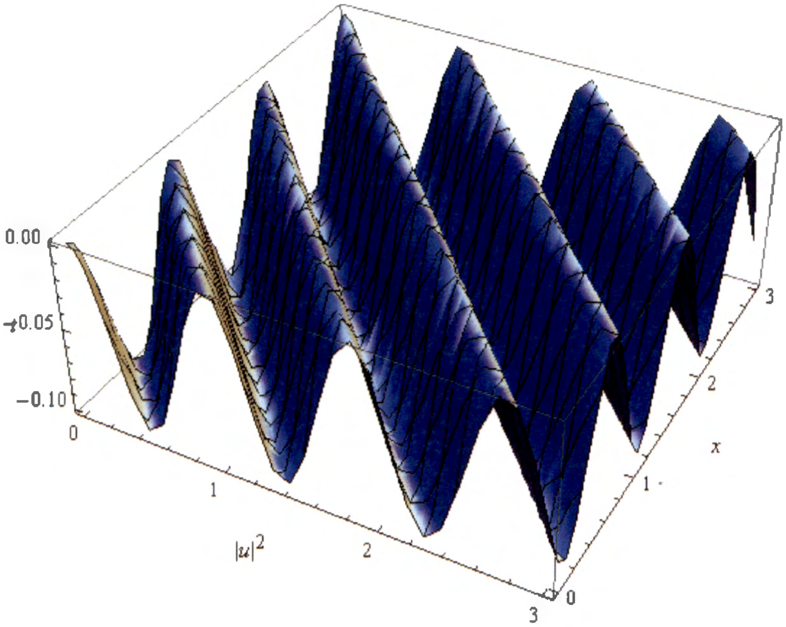

Figure 4.1: Profile of solution (4.9)

4.2 Solutions of (4.1) using the simplest equation method

We consider the solutions of (4.6) in the form

$$u(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{4.10}$$

where $G(z)$ satisfies the Bernoulli or the Riccati equation.

Solutions of (4.1) using Bernoulli as the simplest equation

We consider the Bernoulli equation

$$G'(z) = aG(z) + bG^2(z), \tag{4.11}$$

where a and b are constants.

The balancing procedure yields $M = 1$, so the solution of (4.6) is of the form

$$u(z) = A_0 + A_1 G(z). \tag{4.12}$$

Substituting (4.12) into (4.6) and making use of the Bernoulli equation (4.11) and then equating all coefficients of the function G^i to zero, we obtain the following algebraic system of equations

$$2\,A_1k^4b^2 - k^2{A_1}^3 + d^2{A_1}^3 - 4\,A_1d^2k^2b^2 + {A_1}^3 + 2\,A_1d^4b^2 = 0,$$

$$3\,A_1k^4ab + 3\,d^2A_0{A_1}^2 - 3\,k^2A_0{A_1}^2 - 6\,A_1d^2k^2ab + 3\,A_0{A_1}^2 + 3\,A_1d^4ab = 0,$$

$$p^2A_0d^2 - p^2A_0k^2 - r^2A_0d^2 + r^2A_0k^2 - A_0k^2 + d^2{A_0}^3 - k^2{A_0}^3 + A_0d^2 + {A_0}^3 = 0,$$

$$A_1d^2 + 3\,d^2{A_0}^2A_1 - p^2A_1k^2 + r^2A_1k^2 + p^2A_1d^2 + A_1d^4a^2 - 3\,k^2{A_0}^2A_1 + A_1k^4a^2$$
$$+ 3\,{A_0}^2A_1 - 2\,A_1d^2k^2a^2 - r^2A_1d^2 - A_1k^2 = 0.$$

Solving this system, with the aid of Maple, we obtain the following values for the constants

$$A_0 = \frac{a(d^2 - k^2)}{\sqrt{2(k^2 - d^2 - 1)}},$$

$$A_1 = \frac{\sqrt{2}b(d^2 - k^2)}{k^2 - d^2 - 1},$$

$$p = \sqrt{\frac{r^2d^2 - r^2k^2 + k^2 - d^2{A_0}^2 + k^2{A_0}^2 - d^2 - {A_0}^2}{d^2 - k^2}}.$$

As a result a solution of the Klein-Gordon-Zakharov equations (4.1), using the Bernoulli equation as the simplest equation, is

$$u(x,t) = e^{i(px+rt)}\Bigg[\frac{\sqrt{2}ab\,(d^2-k^2)\,(\cosh(a(kx+dt+c))+\sinh(a(kx+dt+c)))}{\sqrt{k^2-d^2-1}(1-b\cosh(a(kx+dt+c))-b\sinh(a(kx+dt+c)))} + \frac{a\,(d^2-k^2)}{\sqrt{2(k^2-d^2-1)}}\Bigg], \tag{4.13}$$

where c is a constant of integration.

Solutions of (4.1) using Riccati as the simplest equation

We use the Riccati equation given by

$$G'(z) = aG^2(z) + bG(z) + c, \tag{4.14}$$

where a, b and c are constants. The balancing procedure yields $M = 1$, so the solution of (4.6) is of the form

$$u(z) = A_0 + A_1 G(z).$$

Similar calculations yield the following set of values:

$$A_0 = \frac{b(d^2-k^2)}{\sqrt{2(k^2-d^2-1)}},$$

$$A_1 = -\frac{\sqrt{2}a(d^2-k^2)\sqrt{k^2-d^2-1}}{d^2-k^2+1},$$

$$c = -\frac{\sqrt{2(k^2-d^2-1)}(d^2b^2-k^2b^2-2-2p^2+2r^2)}{4A_1(d^2-k^2+1)}.$$

As a result the two solutions of (4.1) are

$$u(x,t) = e^{i\phi}\Bigg[A_0 + A_1\Bigg\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\Bigg\}\Bigg] \tag{4.15}$$

and

$$u(x,t) = e^{i\phi}\Bigg[A_0 + A_1\Bigg\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)$$

$$+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\Bigg\}\Bigg], \tag{4.16}$$

where $\phi = px + rt$ and $z = kx + dt$. θ is given by $\sqrt{b^2 - 4ac}$, C is a constant of integration, and A_0 and A_1 are as obtained above.

It should be noted that by substituting the above values of $u(x,t)$ into (4.5), one can now obtain the solutions for the variable $v(x,t)$.

4.3 Conclusion

In this chapter we studied the Klein-Gordon-Zakharov equations (4.1). The travelling wave hypothesis approach along with the simplest equation method were utilized to obtain new exact solutions of the Klein-Gordon-Zakharov equations. The solutions obtained were travelling wave solutions.

Chapter 5

Travelling wave solutions and conservation laws of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation

In this chapter we study the (2+1)-dimensional Burgers-Kadomtsev-Petviashvili (Burgers-KP) equation with power law nonlinearity, that is given by

$$(u_t + \alpha u^n u_x + \beta u_{xx})_x + \gamma u_{yy} = 0. \tag{5.1}$$

Here, in (5.1) α, β, γ and n are arbitrary constants. When $n = 1$ in (5.1) we obtain

$$(u_t + \alpha u u_x + \beta u_{xx})_x + \gamma u_{yy} = 0. \tag{5.2}$$

This equation (5.2) is a dissipative version of the Kadomtsev-Petviashvili (KP) equation. It is widely accepted that the Burgers-KP equation is a natural model for the propagation of the two-dimensional damped waves [87]. The solutions of (5.2) have been studied in various aspects. See for example the recent papers [88–90]. Wazwaz [88] used the tanh-coth method to obtain kink solutions and periodic solutions of (5.2). In the same paper Hirota's bilinear method was used to establish exact $N-$soliton solutions of equation (5.2). Ren and Wang [89] employed the modified direct Clarkson-Kruskal method to find new solutions of this equation. Chun and Sakthivel [90] applied the exp-function method and constructed a variety of new generalized solitonary and periodic solutions for (5.2).

In the present work we study equation (5.1) and obtain its exact solutions using Lie group analysis. Also the solution of the resultant ordinary differential equation is constructed using the $(G'/G)-$expansion method. In addition conservation laws are derived for (5.1) using the new conservation theorem by Ibragimov.

The results of this chapter have been published in [91].

5.1 Exact solutions of (5.1)

In this section we obtain exact solutions of (5.1) using Lie group analysis.

5.1.1 Exact solutions of (5.1) using Lie point symmetries

The vector field given by

$$X = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of (5.1) if

$$\mathrm{pr}^{(3)}X[(u_t + \alpha u^n u_x + \beta u_{xx})_x + \gamma u_{yy}] = 0 \tag{5.3}$$

whenever $(u_t+\alpha u^n u_x+\beta u_{xx})_x+\gamma u_{yy}=0$. Here $\mathrm{pr}^{(3)}X$ denotes the 3rd prolongation of X and is given by

$$\begin{aligned}
\mathrm{pr}^{(3)}X &= \xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}+\phi^t\frac{\partial}{\partial u_t}+\phi^x\frac{\partial}{\partial u_x}+\phi^{tx}\frac{\partial}{\partial u_{tx}}+\phi^{xx}\frac{\partial}{\partial u_{xx}} \\
&\quad +\phi^{yy}\frac{\partial}{\partial u_{yy}}+\phi^{xxx}\frac{\partial}{\partial u_{xxx}}+\cdots,
\end{aligned}$$

where

$$\begin{aligned}
\phi^t &= D_t(\eta)-u_tD_t(\xi^1)-u_xD_t(\xi^2)-u_yD_t(\xi^3), \\
\phi^x &= D_x(\eta)-u_tD_x(\xi^1)-u_xD_x(\xi^2)-u_yD_x(\xi^3), \\
\phi^{tx} &= D_x(\eta_t)-u_{tt}D_x(\xi^1)-u_{tx}D_x(\xi^2)-u_{ty}D_x(\xi^3), \\
\phi^{xx} &= D_x(\eta_x)-u_{xt}D_x(\xi^1)-u_{xx}D_x(\xi^2)-u_{xy}D_x(\xi^3), \\
\phi^{yy} &= D_y(\eta_y)-u_{yt}D_y(\xi^1)-u_{yx}D_y(\xi^2)-u_{yy}D_y(\xi^3), \\
\phi^{xxx} &= D_x(\eta_{xx})-u_{xxt}D_x(\xi^1)-u_{xxx}D_x(\xi^2)-u_{xxy}D_x(\xi^3),
\end{aligned}$$

and the total derivatives D_t, D_x and D_y are given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+\cdots \\
D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\cdots \\
D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{yt}\frac{\partial}{\partial u_t}+\cdots.
\end{aligned}$$

Expanding (5.3) we obtain the determining equation

$$\begin{aligned}
&n^2\alpha u_x^2\eta u^{n-2}-n\alpha u_x^2\eta u^{n-2}+n\alpha u_x^2\xi_t^1u^{n-1}-2n\alpha u_tu_x\xi_x^1u^{n-1}-n\alpha u_x^2\xi_x^2u^{n-1} \\
&-n\alpha u_x^2u_y\xi_u^3u^{n-1}+3n\alpha\beta u_x^2\xi_u^1u_{xx}u^{n-1}-2n\alpha u_xu_y\xi_x^3u^{n-1}+n\alpha u_x^2\eta_uu^{n-1} \\
&+2n\alpha u_x\eta_xu^{n-1}+n\alpha\eta u_{xx}u^{n-1}+3n\alpha\beta u_x^4\xi_{uu}^1u^{n-1}+6n\alpha\beta u_x^3\xi_{xu}^1u^{n-1} \\
&+3n\alpha\beta u_x^2\xi_{xx}^1u^{n-1}+3\alpha\beta\xi_u^1u_{xx}^2u^n+\alpha\xi_t^1u_{xx}u^n+\alpha u_t\xi_u^1u_{xx}u^n-\alpha u_x\xi_u^2u_{xx}u^n \\
&-\alpha\xi_x^2u_{xx}u^n-2\alpha u_x\xi_u^3u_{xy}u^n-2\alpha\xi_x^3u_{xy}u^n+2\alpha\gamma u_x\xi_u^1u_{yy}u^n+2\alpha\gamma\xi_x^1u_{yy}u^n \\
&-\alpha u_tu_x^2\xi_{uu}^1u^n+3\alpha\beta u_x^2u_{xx}\xi_{uu}^1u^n-2\alpha u_tu_x\xi_{xu}^1u^n+6\alpha\beta u_xu_{xx}\xi_{xu}^1u^n+3\alpha\beta u_{xx}\xi_{xx}^1u^n
\end{aligned}$$

$$
\begin{aligned}
&-\alpha u_t\xi^1_{xx}u^n-\alpha u_x^3\xi^2_{uu}u^n-2\alpha u_x^2\xi^2_{xu}u^n-\alpha u_x\xi^2_{xx}u^n-\alpha u_x^2u_y\xi^3_{uu}u^n+2\alpha\beta u_x\xi^1_u u_{xxx}u^n\\
&-2\alpha u_xu_y\xi^3_{xu}u^n-\alpha u_y\xi^3_{xx}u^n+\alpha u_x^2\eta_{uu}u^n+2\alpha u_x\eta_{xu}u^n+\alpha\eta_{xx}u^n+2\alpha\beta\xi^1_x u_{xxx}u^n\\
&+2\alpha^2u_x\xi^1_u u_{xx}u^{2n}+2\alpha^2\xi^1_x u_{xx}u^{2n}+2n\alpha^2u_x^3\xi^1_u u^{2n-1}+2n\alpha^2u_x^2\xi^1_x u^{2n-1}-3\beta\xi^2_u u_{xx}^2\\
&-u_x\xi^1_u u_{tt}-\xi^1_x u_{tt}-2\gamma u_y\xi^1_u u_{ty}-2\gamma\xi^1_y u_{ty}-u_x\xi^3_u u_{ty}-\xi^3_x u_{ty}-\xi^2_t u_{xx}-u_t\xi^2_u u_{xx}\\
&-2\gamma u_y\xi^2_u u_{xy}-2\gamma\xi^2_y u_{xy}-\xi^3_t u_{xy}-u_t\xi^3_u u_{xy}-3\beta\xi^3_u u_{xx}u_{xy}+\gamma\xi^1_t u_{yy}+\gamma u_t\xi^1_u u_{yy}\\
&+\gamma u_x\xi^2_u u_{yy}+\gamma\xi^2_x u_{yy}-2\gamma u_y\xi^3_u u_{yy}-2\gamma\xi^3_y u_{yy}+3\beta\gamma\xi^1_u u_{xx}u_{yy}-u_tu_x\xi^1_{tu}-u_t\xi^1_{tx}\\
&-\gamma u_tu_y^2\xi^1_{uu}-u_t^2u_x\xi^1_{uu}-3\beta u_tu_xu_{xx}\xi^1_{uu}+3\beta\gamma u_x^2u_{yy}\xi^1_{uu}-u_t^2\xi^1_{xu}-3\beta u_tu_{xx}\xi^1_{xu}\\
&+6\beta\gamma u_xu_{yy}\xi^1_{xu}+3\beta\gamma u_{yy}\xi^1_{xx}-2\gamma u_tu_y\xi^1_{yu}-\gamma u_t\xi^1_{yy}-u_x^2\xi^2_{tu}-u_x\xi^2_{tx}-u_tu_x^2\xi^2_{uu}\\
&-\gamma u_xu_y^2\xi^2_{uu}-6\beta u_x^2u_{xx}\xi^2_{uu}-u_tu_x\xi^2_{xu}-9\beta u_xu_{xx}\xi^2_{xu}-3\beta u_{xx}\xi^2_{xx}-2\gamma u_xu_y\xi^2_{yu}\\
&-\gamma u_x\xi^2_{yy}-u_xu_y\xi^3_{tu}-u_y\xi^3_{tx}-\gamma u_y^3\xi^3_{uu}-u_tu_xu_y\xi^3_{uu}-3\beta u_xu_yu_{xx}\xi^3_{uu}-3\beta u_x^2u_{xy}\xi^3_{uu}\\
&-u_tu_y\xi^3_{xu}-3\beta u_yu_{xx}\xi^3_{xu}-6\beta u_xu_{xy}\xi^3_{xu}-3\beta u_{xy}\xi^3_{xx}-2\gamma u_y^2\xi^3_{yu}-\gamma u_y\xi^3_{yy}+u_x\eta_{tu}\\
&+\eta_{tx}+\gamma u_y^2\eta_{uu}+u_tu_x\eta_{uu}+3\beta u_xu_{xx}\eta_{uu}+u_t\eta_{xu}+3\beta u_{xx}\eta_{xu}+2\gamma u_y\eta_{yu}+\gamma\eta_{yy}\\
&-3\beta u_x\xi^1_u u_{txx}-3\beta\xi^1_x u_{txx}+\beta\xi^1_t u_{xxx}+\beta u_t\xi^1_u u_{xxx}-2\beta u_x\xi^2_u u_{xxx}-2\beta\xi^2_x u_{xxx}\\
&+3\beta^2\xi^1_u u_{xx}u_{xxx}+3\beta^2u_x^2\xi^1_{uu}u_{xxx}+6\beta^2u_x\xi^1_{xu}u_{xxx}+3\beta^2\xi^1_{xx}u_{xxx}-3\beta u_x\xi^3_u u_{xxy}\\
&-3\beta\xi^3_x u_{xxy}-\beta u_tu_x^3\xi^1_{uuu}-3\beta u_tu_x^2\xi^1_{xuu}-3\beta u_tu_x\xi^1_{xxu}-\beta u_t\xi^1_{xxx}-\beta u_x^4\xi^2_{uuu}\\
&-3\beta u_x^3\xi^2_{xuu}-3\beta u_x^2\xi^2_{xxu}-\beta u_x\xi^2_{xxx}-\beta u_x^3u_y\xi^3_{uuu}-3\beta u_x^2u_y\xi^3_{xuu}-3\beta u_xu_y\xi^3_{xxu}\\
&-\beta u_y\xi^3_{xxx}+\beta u_x^3\eta_{uuu}+3\beta u_x^2\eta_{xuu}+3\beta u_x\eta_{xxu}+\beta\eta_{xxx}=0.
\end{aligned}
$$

Splitting on the derivatives of u, yields the following overdetermined system of sixteen linear partial differential equations:

$$
\begin{aligned}
&\xi^1_y=0,\ \xi^3_x=0,\ \xi^3_u=0,\ \xi^2_u=0,\ \xi^1_x=0,\ \xi^1_u=0,\ \eta_{xu}=0,\ \eta_{uu}=0,\\
&2\gamma\xi^2_y+\xi^3_t=0,\ \xi^3_{yy}-2\eta_{yu}=0,\ 3\xi^2_x-2\xi^3_y=0,\ \xi^2_x-2\xi^3_y+\xi^1_t=0,\\
&\alpha u^n\eta_{xx}+\gamma\eta_{yy}+\beta\eta_{xxx}\eta_{xt}=0,\ n\eta+u\eta_u-2u\xi^2_x+2u\xi^3_y-\eta=0,\\
&\alpha nu^n\eta-2\alpha u^nu\xi^2_x+2\alpha u^nu\xi^3_y-3\beta u\xi^2_{xx}-u\xi^2_t=0,\\
&2\alpha nu^n\eta_x-\alpha u^nu\xi^2_{xx}-\beta u\xi^2_{xxx}-\gamma u\xi^2_{yy}+u\eta_{ut}-u\xi^2_{xt}=0.
\end{aligned}
$$

Solving this system one obtains the following five Lie point symmetries:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= \frac{\partial}{\partial y}, \\
X_4 &= 2\gamma t\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \\
X_5 &= 4nt\frac{\partial}{\partial t} + 2nx\frac{\partial}{\partial x} + 3ny\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.
\end{aligned}
$$

The linear combination of the translation symmetries, $\Gamma = X_1 + X_2 + X_3$ reduces (5.1) to a partial differential equation in two independent variables. The symmetry Γ yields the following three invariants:

$$f = t - y,\ g = x - y,\ \theta = u. \tag{5.4}$$

Using the above invariants we can write the transformed equation as

$$\theta_{fg} + \alpha\theta^n\theta_{gg} + \alpha n\theta^{n-1}\theta_g^2 + \beta\theta_{ggg} + \gamma\theta_{ff} + 2\gamma\theta_{fg} + \gamma\theta_{gg} = 0, \tag{5.5}$$

which is a nonlinear PDE in two independent variables f and g. The above equation has two translation symmetries, namely

$$\Upsilon_1 = \frac{\partial}{\partial f},\ \Upsilon_2 = \frac{\partial}{\partial g}.$$

By taking a linear combination $\Upsilon_1 + c\Upsilon_2$ of the above symmetries, we see that it yields the invariants

$$z = g - cf,\ \theta = \psi.$$

Now treating ψ as new dependent variable and z as the new independent variable, (5.5) is reduced to the following third-order nonlinear ODE:

$$\alpha\psi^n\psi'' + \alpha n\psi^{n-1}(\psi')^2 + \left(\gamma\,(1-c)^2 - c\right)\psi'' + \beta\psi''' = 0, \tag{5.6}$$

where $'$ denotes the differentiation with respect to the variable z. We can integrate (5.6) twice to obtain

$$\left(\gamma\,(1-c)^2-c\right)\psi+\frac{\alpha\,\psi^{n+1}}{n+1}+\beta\,\psi'=0, \tag{5.7}$$

where the constants of integration are taken to be zero. With the aid of Maple, equation (5.7) can be integrated to give the solution

$$\psi\,(z)=\left[\frac{1}{\zeta\,(n+1)}\left\{-\alpha+C_1\,(n+1)\,\zeta \mathrm{e}^{\frac{n\zeta z}{\beta}}\right\}\right]^{-\frac{1}{n}}, \tag{5.8}$$

where $\zeta=\gamma\,(1-c)^2-c$. Reverting back to our original variables and at the same time writing the solution in terms of trigonometric functions, we have

$$u\,(t,x,y)=\left[\frac{1}{\zeta\,(n+1)}\left\{-\alpha+C_1\,(n+1)\,\zeta\left(\cosh\left(\frac{n\zeta z}{\beta}\right)+\sinh\left(\frac{n\zeta z}{\beta}\right)\right)\right\}\right]^{-\frac{1}{n}}, \tag{5.9}$$

where $\zeta=\gamma\,(1-c)^2-c,\ z=x-(1-c)y-ct$ and C_1 is a constant of integration.

By taking $n=2$, $\alpha=1$, $\beta=1$, $\gamma=1$, $C_1=1$, $c=2$ and $t=0$ in (5.9), the profile of the solution (5.9) is given in Figure 5.1.

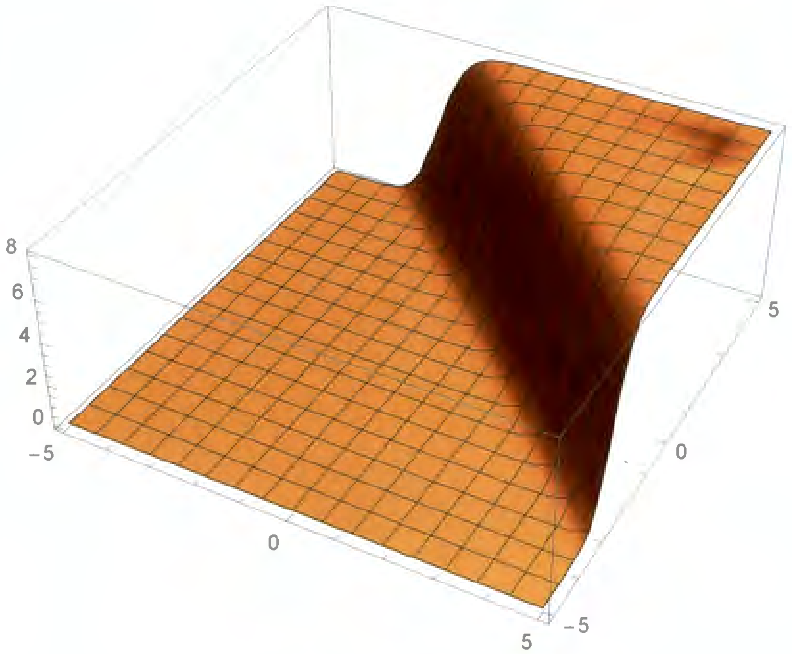

Figure 5.1: Profile of solution (5.9)

5.1.2 Exact solutions of (5.1) using the (G'/G)-expansion method

In this section we use the $(G'/G)-$expansion method [11,92], to obtain a few exact solutions of the Burgers-KP equation (5.1).

By introducing the transformation

$$\psi(z) = F(z)^{\frac{1}{n}}, \tag{5.10}$$

equation (5.6) transforms to the third-order nonlinear ordinary differential equation for F, viz.,

$$\begin{aligned}&\alpha n F(z)^2 F'(z)^2 + \alpha n^2 F(z)^3 F''(z) + n(1-n)\zeta F(z) F'(z)^2 \\ &+ n^2 \zeta F(z)^2 F''(z) + (2n^2 - 3n + 1)\zeta F'(z)^3\end{aligned}$$

$$+3n(1-n)\beta F(z)F'(z)F''(z)+\beta n^2F(z)^2F'''(z)=0. \qquad (5.11)$$

where $\zeta=\gamma\,(1-c)^2-c$.

Let us consider the solutions of (5.11) in the form

$$F(z)=\sum_{i=0}^{M}\mathcal{A}_i\left(\frac{G'(z)}{G(z)}\right)^i, \qquad (5.12)$$

where $G(z)$ satisfies

$$G''+\lambda G'+\mu G=0 \qquad (5.13)$$

and λ and μ are constants. The homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (5.11) determines the value of M and $\mathcal{A}_0,\cdots,\mathcal{A}_M$, are constants to be determined.

Application of the balancing procedure to the ODE (5.11) yields $M=1$, so the solutions of (5.11) are of the form

$$F(z)=\mathcal{A}_0+\mathcal{A}_1\left(\frac{G'(z)}{G(z)}\right). \qquad (5.14)$$

Substituting (5.14) into (5.11) and using (5.13) leads to an overdetermined system of algebraic equations. Solving this system of algèbraic equations with the aid of Maple, we obtain

$$c=\frac{2\gamma n+n-\sqrt{4\gamma n^2+n^2-4\beta\gamma n\sqrt{\lambda^2-4\mu}}}{2\gamma n},$$
$$\mathcal{A}_0=\frac{(n+1)(\beta\lambda-\gamma n\,(c-1)^2+cn)}{2\alpha n},$$
$$\mathcal{A}_1=\frac{\beta(n+1)}{\alpha n}.$$

Now using the general solution of (5.13) in (5.14), we have the following three types of travelling wave solutions of the Burgers-KP equation (5.1):

When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solution

$$u(x,y,t)=\left[\mathcal{A}_0+\mathcal{A}_1\left(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\right)\right]^{1/n},$$

where $z = x - (1-c)y - ct$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$u(x,y,t) = \left[\mathcal{A}_0 + \mathcal{A}_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)\right]^{1/n},$$

where $z = x - (1-c)y - ct$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u(x,y,t) = \left[\mathcal{A}_0 + \mathcal{A}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)\right]^{1/n},$$

where $z = x - (1-c)y - ct$, C_1 and C_2 are arbitrary constants.

5.2 Conservation laws of (5.1)

In this section we construct conservation laws for (5.1) using the conservation theorem due to Ibragimov [48]. First we find the adjoint equation for (5.1), which is given by

$$\frac{\delta}{\delta u}[v((u_t + \alpha u^n u_x + \beta u_{xx})_x + \gamma u_{yy})] = 0, \tag{5.15}$$

where $v = v(t,x,y)$ is the new variable. The operator $\delta/\delta u$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} + D_t D_x\frac{\partial}{\partial u_{tx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_x^3\frac{\partial}{\partial u_{xxx}} + \cdots \tag{5.16}$$

and the total derivatives D_t, D_x and D_y are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots \\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y} + u_{yt}\frac{\partial}{\partial u_t} + \cdots . \end{aligned} \tag{5.17}$$

Now expanding equation (5.15) using (5.16) and (5.17) we obtain the adjoint equation as

$$v_{tx} + \alpha u^n v_{xx} - \beta v_{xxx} + \gamma v_{yy} = 0. \tag{5.18}$$

We now consider the system formed by the equation (5.1) and the adjoint equation (5.18), i.e.,

$$\begin{aligned} &(u_t + \alpha u^n u_x + \beta u_{xx})_x + \gamma u_{yy} = 0, \\ &v_{tx} + \alpha u^n v_{xx} - \beta v_{xxx} + \gamma v_{yy} = 0. \end{aligned} \tag{5.19}$$

The third-order Lagrangian for this system is given by

$$L = v[(u_t + \alpha u^n u_x + \beta u_{xx})_x + \gamma u_{yy}]. \tag{5.20}$$

Applying Ibragimov's theorem [48] we deduce that the conserved quantities of system (5.19) associated with a symmetry of (5.1) are given by

$$\begin{aligned} T^i =& \xi^i L + W\left[\frac{\delta L}{\delta u_i} - D_j\left(\frac{\delta L}{\delta u_{ij}}\right) + D_j D_k\left(\frac{\delta L}{\delta u_{ijk}}\right)\right] + D_j W\left[\frac{\delta L}{\delta u_{ij}} - D_k\left(\frac{\delta L}{\delta u_{ijk}}\right)\right] \\ &+ D_j D_k(W)\frac{\delta L}{\delta u_{ijk}} + \cdots, \end{aligned}$$

where $W = \eta - \xi^i u_i$ is the Lie characteristic function. We have five cases.

(i) Considering the Lie point symmetry $X_1 = \partial/\partial t$, the Lie characteristic function is $W = -u_t$. Hence the conserved vector corresponding to the Lie point symmetry X_1 is

$$\begin{aligned} T_1^t =& v\left\{\alpha u^{n-1}\left(u u_{xx} + n u_x^2\right) + \frac{u_{tx}}{2} + \beta u_{xxx} + \gamma u_{yy}\right\} + \frac{1}{2}u_t v_x, \\ T_1^x =& -u_t\left\{\alpha u^{n-1}\left(n v u_x - u v_x\right) - \frac{v_t}{2} + \beta v_{xx}\right\} - u_{tx}\left(\alpha v u^n - \beta v_x\right) \\ &- \beta v u_{txx} - \frac{1}{2} v u_{tt}, \\ T_1^y =& \gamma\left(u_t v_y - v u_{ty}\right). \end{aligned}$$

(ii) For the Lie point symmetry $X_2 = \partial/\partial x$, the Lie characteristic function is $W = -u_x$ and hence the conserved vector is

$$\begin{aligned}
T_2^t &= \frac{1}{2}\left(u_x v_x - v u_{xx}\right), \\
T_2^x &= \alpha u_x v_x u^n + \frac{1}{2} v\left(u_{tx} + 2\gamma u_{yy}\right) + \frac{1}{2} u_x\left(v_t - 2\beta v_{xx}\right) + \beta u_{xx} v_x, \\
T_2^y &= \gamma\left(u_x v_y - v u_{xy}\right).
\end{aligned}$$

(iii) For $X_3 = \partial/\partial y$, the Lie characteristic function is $W = -u_y$, the conserved vector is

$$\begin{aligned}
T_3^t &= \frac{1}{2}\left(u_y v_x - v u_{xy}\right), \\
T_3^x &= -u_{xy}\left(\alpha v u^n - \beta v_x\right) - u_y\left\{\alpha u^{n-1}\left(n v u_x - u v_x\right) - \frac{v_t}{2} + \beta v_{xx}\right\} \\
&\quad -\beta v u_{xxy} - \frac{1}{2} v u_{ty}, \\
T_3^y &= v\left\{\alpha u^{n-1}\left(u u_{xx} + n u_x^2\right) + u_{tx} + \beta u_{xxx}\right\} + \gamma u_y v_y.
\end{aligned}$$

(iv) For the Lie point symmetry $X_4 = 2\gamma t\partial/\partial y - y\partial/\partial x$, the Lie characteristic function is $W = y u_x - 2\gamma t u_y$ and the conserved vector is

$$\begin{aligned}
T_4^t &= \frac{1}{2}\left\{v\left(y u_{xx} - 2\gamma t u_{xy}\right) + v_x\left(2\gamma t u_y - y u_x\right)\right\}, \\
T_4^x &= -\frac{1}{2u}\left[2\alpha u^n\left\{2\gamma t v\left(u_{xy} u + n u_x u_y\right) + v_x u\left(y u_x - 2\gamma t u_y\right)\right\}\right. \\
&\quad + u\left\{v\left(2\gamma\left(2\beta t u_{xxy} + t u_{ty} + u_y + y u_{yy}\right) + y u_{tx}\right) + 2\beta v_x\left(y u_{xx} - 2\gamma t u_{xy}\right)\right. \\
&\quad \left.\left. + \left(2\beta v_{xx} - v_t\right)\left(2\gamma t u_y - y u_x\right)\right\}\right], \\
T_4^y &= \gamma\left[v\left\{2t\left(\alpha u_{xx} u^n + u_{tx} + \beta u_{xxx}\right) + 2\alpha n t u_x^2 u^{n-1} + y u_{xy} + u_x\right\}\right. \\
&\quad \left. + 2\gamma t u_y v_y - y u_x v_y\right].
\end{aligned}$$

(v) Finally taking the symmetry $X_5 = 4nt\partial/\partial t + 2nx\partial/\partial x + 3ny\partial/\partial y - 2u\partial/\partial u$, we have $W = -2u - 4ntu_t - 2nxu_x - 3nyu_y$. The corresponding conserved vector is

$$T_5^t = \frac{1}{2}\left[-8ntv\left\{\alpha u^{n-1}\left(u_{xx} u + n u_x^2\right) + u_{tx} + \beta u_{xxx} + \gamma u_{yy}\right\}\right.$$

$$
\begin{aligned}
&+ v\left\{n\left(4tu_{tx}+3yu_{xy}+2xu_{xx}\right)+2(n+1)u_{x}\right\}\\
&-v_{x}\left\{2u+n\left(4tu_{t}+2xu_{x}+3yu_{y}\right)\right\}\big],\\
T_{5}^{x} =& -2nxv\left\{\alpha u^{n-1}\left(u_{xx}u+nu_{x}^{2}\right)+u_{tx}+\beta u_{xxx}+\gamma u_{yy}\right\}\\
&+\left\{2u+n\left(4tu_{t}+2xu_{x}+3yu_{y}\right)\right\}\left\{\alpha u^{n-1}\left(nu_{x}v-v_{x}u\right)-\frac{v_{t}}{2}+\beta v_{xx}\right\}\\
&+\left\{n\left(4tu_{tx}+3yu_{xy}+2xu_{xx}\right)+2(n+1)u_{x}\right\}\left(\alpha vu^{n}-\beta v_{x}\right)\\
&+\beta v\left\{n\left(4tu_{txx}+3yu_{xxy}+2xu_{xxx}\right)+(4n+2)u_{xx}\right\}\\
&+\frac{1}{2}v\left\{n\left(2xu_{tx}+3yu_{ty}+4tu_{tt}\right)+(4n+2)u_{t}\right\},\\
T_{5}^{y} =& \, v\left[n\left\{-3\alpha yu^{n-1}\left(u_{xx}u+nu_{x}^{2}\right)-3y\left(u_{tx}+\beta u_{xxx}\right)+2\gamma\left(2tu_{ty}+xu_{xy}\right)\right\}\right.\\
&\left.+\gamma(3n+2)u_{y}\right]-\gamma v_{y}\left\{2u+n\left(4tu_{t}+2xu_{x}+3yu_{y}\right)\right\}.
\end{aligned}
$$

Notice that these conserved vectors involve v, the solution of the adjoint equation, and hence gives an infinite number of conservation laws.

5.3 Conclusion

In this chapter we obtained exact solutions of the generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation (5.1) by employing the Lie symmetry analysis to reduce the partial differential equation to an ordinary differential equation then integrating the resultant equation. The $(G'/G)-$expansion method was also employed to obtain travelling wave solutions. Moreover, the conservation laws for the underlying equation were derived by using the new conservation theorem due to Ibragimov. We obtained five conservation laws corresponding to the five Lie point symmetries of the equation.

Chapter 6

Traveling wave solutions and Conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity

In this chapter we study the Korteweg-de Vries-Burgers equation with power law nonlinearity (KdVBp) given by

$$u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx} = 0, \tag{6.1}$$

where α, β and γ are constants. When $n = 1$, this equation becomes the Korteweg-de Vries-Burgers equation, which was derived for a wide class of nonlinear system in weak nonlinearity and long wavelength approximation [93]. In this study we employ Lie symmetry analysis along with the Kudryashov approach to find exact solutions of (6.1). Furthermore, we construct conservation laws for (6.1) using the new conservation theorem due to Ibragimov.

The work contained in this chapter has been published in [56].

6.1 Similarity reduction and exact solutions

In this section we first compute Lie point symmetries of (6.1) and then use them along with the Kudryashov method to find exact solutions of (6.1).

6.1.1 Similarity reduction of (6.1)

Firstly we calculate the Lie point symmetries of (6.1) using the Lie algorithm. These Lie point symmetries are then used to transform (6.1) into an ordinary differential equation. Kudryashov approach [18] is then applied to the ordinary differential equation and as a result we obtain the exact solutions of (6.1).

The symmetries of (6.1) will be generated by the vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}.$$

The application of the third prolongation, $\mathrm{pr}^{(3)}X$ to (6.1) leads to the overdetermined system of linear partial differential equations

$$\begin{aligned}
&\xi^1_x = 0,\ \xi^1_u = 0,\ \xi^2_{uuu} = 0,\ \beta\xi^2_{uu} - 3\gamma\xi^2_{xuu} + \gamma\eta_{uuu} = 0,\\
&2u^n\alpha\gamma\xi^2_{uu} - \beta^2\xi^2_{uu} + 3\beta\gamma\xi^2_{xuu} - \beta\gamma\eta_{uuu} = 0,\\
&4u^n\alpha\gamma\xi^2_{uu} - \beta^2\xi^2_{uu} + 3\beta\gamma\xi^2_{xuu} - \beta\gamma\eta_{uuu} = 0,\\
&6u^n\alpha\gamma\xi^2_{uu} - \beta^2\xi^2_{uu} + 3\beta\gamma\xi^2_{xuu} - \beta\gamma\eta_{uuu} = 0,\\
&2\beta\xi^2_{xu} - \beta\eta_{uu} - 3\gamma\xi^2_{xxu} + 3\gamma\eta_{xuu} = 0,\\
&\alpha u^n\eta_x - \beta\eta_{xx} + \gamma\eta_{xxx} + \eta_t = 0,\\
&6\gamma\eta_{xuu} - 6\gamma\xi^2_{xxu} + 4\beta\xi^2_{xu} - 2\beta\eta_{uu} + 3\alpha u^n\xi^2_u = 0,
\end{aligned}$$

$3\alpha u^n\xi_u^2 + 2\beta\xi_{xu}^2 - 3\gamma\xi_{xxu}^2 + 3\gamma\eta_{xuu} - \beta\eta_{uu} = 0,$

$5\alpha\beta u^n\xi_u^2 - 9\alpha\gamma u^n\xi_{xu}^2 + 3\alpha\gamma u^n\eta_{uu} + 4\beta^2\xi_{xu}^2 - 2\beta^2\eta_{uu} - 6\beta\gamma\xi_{xxu}^2$

$+6\beta\gamma\eta_{xuu} = 0,$

$u\xi_t^2 - 2\alpha uu^n\xi_x^2 - \alpha nu^n\eta - 3\gamma u\eta_{xxu} - \beta u\xi_{xx}^2 + 2\beta u\eta_{xu} + \gamma u\xi_{xxx}^2 = 0,$

$2\alpha\beta u^n\xi_u^2 - 9\alpha\gamma u^n\xi_{xu}^2 + 4\beta^2\xi_{xu}^2 + 3\alpha\gamma u^n\eta_{uu} - 2\beta^2\eta_{uu} - 6\beta\gamma\xi_{xxu}^2$

$+6\beta\gamma\eta_{xuu} = 0,$

$3\alpha^2\gamma u^{2n}\xi_u^2 - 2\alpha\beta^2u^n\xi_u^2 + 9\alpha\beta\gamma u^n\xi_{xu}^2 - 3\alpha\beta\gamma u^n\eta_{uu} - 2\beta^3\xi_{xu}^2 + \beta^3\eta_{uu}$

$+3\beta^2\gamma\xi_{xxu}^2 - 3\beta^2\gamma\eta_{xuu} = 0,$

$\alpha nu^n\eta - u\xi_x^2 - \gamma u\xi_{xxx}^2 + \alpha nu^n\xi_t^1 + \beta u\xi_{xx}^2 - 2\beta u\eta_{xu} - \alpha uu^n\xi_x^2$

$+3\gamma u\eta_{xxu} = 0,$

$3\alpha\gamma uu^n\xi_{xx}^2 - \alpha\beta nu^n\eta + \beta u\xi_t^2 - \alpha\beta uu^n\xi_x^2 - \beta^2u\xi_{xx}^2 - 3\alpha\gamma uu^n\eta_{xu} + 2\beta^2u\eta_{xu}$

$+\beta\gamma u\xi_{xxx}^2 - 3\beta\gamma\eta_{xxu} = 0.$

Solving this system we obtain two translation symmetries

$$X_1 = \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x}.$$

Considering the linear combination symmetry $X = kX_1 - \omega X_2$, one can easily solve the corresponding Lagrange system to obtain an invariant $z = kx + \omega t$, where k and ω are constants, and so the group-invariant solution of (6.1) is of the form

$$u = F(z). \tag{6.2}$$

Substituting the values of (6.2) into (6.1) we obtain the third-order nonlinear ordinary differential equation

$$\gamma k^3F'''(z) - \beta k^2F''(z) + \alpha kF(z)^nF'(z) + \omega F'(z) = 0. \tag{6.3}$$

Integrating (6.3) once with respect to the variable z and taking the constant of integration to be zero reduces (6.3) to the second-order nonlinear ordinary differential equation

$$\gamma k^3 F'''(z) - \beta k^2 F'(z) + \frac{\alpha k F(z)^{n+1}}{n+1} + \omega F(z) = 0. \tag{6.4}$$

This ordinary differential equation can not be integrated directly so we introduce the transformation

$$F(z) = G(z)^{\frac{1}{n}} \tag{6.5}$$

which then transforms (6.4) to the nonlinear ordinary differential equation for G, namely,

$$\begin{aligned} &n^2\omega G(z)^2 + n^3\omega G(z)^2 + \alpha\, kn^2 G(z)^3 - \gamma\, k^3(n^2-1)G'(z)^2 \\ &+k^2 n(n+1)G(z)\left(k\gamma G''(z) - \beta G'(z)\right) = 0. \end{aligned} \tag{6.6}$$

6.1.2 Exact solutions of (6.6) using Kudryashov approach

In this subsection we employ Kudryashov method [18] to find exact solutions of the nonlinear ordinary differential equation (6.6). We consider the solutions of (6.6) in the form

$$G(z) = \sum_{i=0}^{N} a_i Q(z)^i, \tag{6.7}$$

where the function $Q(z)$ satisfies the first-order ODE

$$Q'(z) = Q(z)^2 - Q(z), \tag{6.8}$$

whose solution can be written as

$$Q(z) = \frac{1}{1+e^z}. \tag{6.9}$$

The balancing procedure yields $N = 2$, and so the solution of (6.6) is of the form

$$G(z) = a_0 + a_1 Q(z) + a_2 Q(z)^2. \tag{6.10}$$

Substituting (6.10) into (6.6) and making use of the equation (6.8) and then equating the coefficients of Q^i to zero, we obtain an algebraic system of equations in terms of a_i $(i = 0, 1, 2)$.

Solving the resultant system of algebraic equations with the aid of Maple, one possible set of values of the constants is

$$
\begin{aligned}
a_0 &= -\frac{2\gamma k^2(n+1)(n+2)}{\alpha n^2},\\
a_1 &= \frac{4\gamma k^2(n+1)(n+2)}{\alpha n^2},\\
a_2 &= -\frac{2\gamma k^2(n+1)(n+2)}{\alpha n^2},\\
\beta &= \frac{\gamma k(n+4)}{n},\\
\omega &= -\frac{2\gamma k^2}{n^2}\Big(4n - 5kn - 18k + 16\Big).
\end{aligned}
$$

As a result, a solution of the Korteweg-de Vries Burgers equation with power law nonlinearity (6.1) is of the form

$$
u(t,x) = \left\{a_0 + a_1\left(\frac{1}{1+e^{kx+\omega t}}\right) + a_2\left(\frac{1}{1+e^{kx+\omega t}}\right)^2\right\}^{1/n}, \tag{6.11}
$$

where a_0, a_1 and a_2 are as given above.

6.2 Construction of conservation laws for (6.1)

In this section we construct conservation laws for the KdVBp equation (6.1). The new conservation theorem due to Ibragimov will be used [48]. First we find the adjoint of equation (6.1) given by

$$
\frac{\delta}{\delta u}[v(u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx})] = 0, \tag{6.12}
$$

where $\delta/\delta u$ is the Euler-Lagrange operator

$$
\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}} \tag{6.13}
$$

and the total derivatives D_t and D_x are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots . \end{aligned} \tag{6.14}$$

Now expanding equation (6.12) with the help of (6.13) and (6.14) we obtain the adjoint equation

$$v_t + \alpha u^n v_x + \beta v_{xx} + \gamma v_{xxx} = 0. \tag{6.15}$$

In this case the system formed by the KdVBp equation (6.1) together with its adjoint equation is given by

$$u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx} = 0, \tag{6.16a}$$

$$v_t + \alpha u^n v_x + \beta v_{xx} + \gamma v_{xxx} = 0. \tag{6.16b}$$

The third-order Lagrangian for this system of equations (6.16) is

$$L = v(u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx}). \tag{6.17}$$

Now applying the conservation theorem due to Ibragimov [48], the conserved vector components are obtained using

$$T^i = \xi L + W\frac{\delta L}{\delta u_i} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W)\frac{\delta L}{\delta u_{i_1\ldots i_s}}, \tag{6.18}$$

where L is the Lagrangian given in (6.17) and $W = \eta - \xi^1 u_t - \xi^2 u_x$ is the Lie characteristic function.

The KdVBp equation (6.1) has two translation symmetries, namely, $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$. The Lie characteristic function corresponding to the Lie point symmetry $X_1 = \partial/\partial t$ is $W = -u_t$. Applying the new conservation theorem, the components of the conserved vector associated to this Lie point symmetry are given by

$$T_1^t = \alpha v u^n u_x - \beta v u_{xx} + \gamma v u_{xxx},$$

$$T_1^x = -\alpha v u^n u_t - \beta v_x u_t - \gamma v_{xx} u_t + \beta v u_{xt} + \gamma v_x u_{xt} - \gamma v u_{xxt},$$

which gives the conservation of energy. In the same manner, the Lie characteristic function corresponding to the Lie point symmetry $X_2 = \partial/\partial x$ is $W = -u_x$ and hence the corresponding conserved vector is

$$\begin{aligned} T_2^t &= -v u_x, \\ T_2^x &= v u_t - \beta v_x u_x - \gamma v_{xx} u_x + \gamma v_x u_{xx}, \end{aligned}$$

which gives the conservation of momentum.

6.3 Conclusion

In this chapter we studied the Korteweg-de Vries-Burgers equation with power law nonlinearity (6.1). Exact solutions of (6.1) were obtained using Lie symmetry analysis along with the Kudryashov method. The solutions obtained were travelling wave solutions. Furthermore we derived two conservation laws corresponding to the two translation symmetries using the new conservation theorem due to Ibragimov.

Chapter 7

Noether symmetries, conservation laws and exact solutions of a four parameter Boussinesq system

The main aim of this chapter is to derive conservation laws using Noether theorem and obtain exact travelling wave solutions for the third-order Boussinesq type system [94]

$$
\begin{aligned}
&u_t + v_x + vu_x + uv_x + av_{xxx} - bu_{txx} = 0, \\
&v_t + u_x + vv_x + cu_{xxx} - dv_{txx} = 0,
\end{aligned}
\tag{7.1}
$$

which models bi-directional propagation of nonlinear dispersive long surface waves with small amplitudes in a channel. Here x denotes the distance along the channel scaled by still water of depth h_0 and t is the time scaled by $\sqrt{h_0/g}$, with the gravitational acceleration g. The dependent variable $u = u(x,t)$ is proportional to the derivation of the liquid's free surface from its rest position at the location x at time t and $v = v(x,t)$ is proportional to the horizontal velocity at the height θh_0 where $\theta \in [0,1]$. Boussinesq was the first to explain the existence of Scott Russel's solitary wave mathematically. He employed a variety of asymptotically equivalent equations to describe water waves in the small-amplitude, long-wave regime. This four-parameter class of model equations was derived by Bona et al. [95], where the parameters a, b, c and d are real constants satisfying the restrictions

$$a = \frac{1}{2}(\theta^2 - \frac{1}{3})\lambda,\ b = \frac{1}{2}(\theta^2 - \frac{1}{3})(1-\lambda),$$
$$c = \frac{1}{2}(1-\theta^2)\mu,\ d = \frac{1}{2}(1-\theta^2)(1-\mu),$$

where $\lambda,\ \mu \in \Re$.

In [94] the authors used the collocation method based on exponential cubic B-spline functions to solve the one dimensional Boussinesq system numerically. Naz et al. [96] obtained conservation laws for a third-order variant Boussinesq system of the form

$$u_t + vu_x + uv_x + v_{xxx} = 0,\ v_t + u_x + vv_x = 0 \tag{7.2}$$

by first increasing the order of the system to guarantee a Lagrangian and then employed Noether theorem to construct the conservation laws. Yaşar and Giresunlu [97] also worked on (7.2) to obtain exact solutions using the simplest equation method and derived conservation laws using the multiplier method.

Dougalis et al. [98] made a numerical study of some aspects of stability of solitary-wave solutions of the Bona-Smith family of Boussinesq systems of equations (7.1). In that study, long time stability and possible blow-up solutions were examined under small and large perturbations covering perturbation of amplitudes, system coefficients, etc. In [99], the Galerkin finite element periodic B-spline method was used to approximate the cnoidal and solitary wave solutions of periodic initial value problem for a four variable Boussinesq system.

The results of this chapter have been submitted to [100].

7.1 Conservation laws of (7.1) using Noether approach

The four parameter Boussinesq system is a third-order system of two partial differential equations and does not possess a Lagrangian. Using the transformations $u = U_x$, $v = V_x$, we can increase the order of (7.1) by one, and at the same time taking $d = b$, system (7.1) becomes

$$\begin{aligned} U_{tx} + V_{xx} + V_x U_{xx} + U_x V_{xx} + aV_{xxxx} - bU_{txxx} &= 0, \\ V_{tx} + U_{xx} + V_x V_{xx} + cU_{xxxx} - bV_{txxx} &= 0. \end{aligned} \tag{7.3}$$

It can easily be verified that a Lagrangian of system (7.3) is

$$L = \frac{1}{2}(aV_{xx}^2 + cU_{xx}^2 - V_x^2 - U_x^2 - U_x V_x^2 - V_t U_x - U_t V_x - bV_{tx}U_{xx} - bU_{tx}V_{xx}) \tag{7.4}$$

as it satisfies $\delta L/\delta U = 0$ and $\delta L/\delta V = 0$. Here the Euler-Lagrange operators $\delta/\delta U$ and $\delta/\delta V$ are defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_t^2 \frac{\partial}{\partial U_{tt}} + D_x^2 \frac{\partial}{\partial U_{xx}} + D_x D_t \frac{\partial}{\partial U_{tx}} - \cdots, \tag{7.5}$$

and

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} + D_x^2 \frac{\partial}{\partial V_{xx}} + D_x D_t \frac{\partial}{\partial V_{tx}} - \cdots, \tag{7.6}$$

where

$$D_t = \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + U_{tt} \frac{\partial}{\partial U_t} + V_{tt} \frac{\partial}{\partial V_t} + U_{tx} \frac{\partial}{\partial U_x} + V_{tx} \frac{\partial}{\partial V_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} + V_{xx} \frac{\partial}{\partial V_x} + U_{tx} \frac{\partial}{\partial U_t} + V_{tx} \frac{\partial}{\partial V_t} + \cdots,$$

are the operators of total differentiation. The Lie-Bäcklund operator given by,

$$X = \xi^1(t,x,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,U,V)\frac{\partial}{\partial x} + \eta^1(t,x,U,V)\frac{\partial}{\partial U}$$

$$+\eta^2(t,x,U,V)\frac{\partial}{\partial V}+\zeta_t^1\frac{\partial}{\partial U_t}+\zeta_t^2\frac{\partial}{\partial V_t}+\zeta_x^1\frac{\partial}{\partial U_x}+\zeta_x^2\frac{\partial}{\partial V_x}$$
$$+\zeta_{tx}^1\frac{\partial}{\partial U_{tx}}+\zeta_{tx}^2\frac{\partial}{\partial V_{tx}}+\zeta_{xx}^1\frac{\partial}{\partial U_{xx}}+\zeta_{xx}^2\frac{\partial}{\partial V_{xx}}+\cdots \tag{7.7}$$

is a Noether operator corresponding to the Lagrangian (7.4) if it satisfies

$$X(L)+L[D_t(\xi^1)+D_x(\xi^2)]=D_t(B^1)+D_x(B^2), \tag{7.8}$$

where $B^1(t,x,U,V)$, $B^2(t,x,U,V)$ are the gauge terms. The expansion of equation (7.8) with the Lagrangian (7.4) yields the following overdetermined system of partial differential equations:

$$\begin{aligned}
&\xi_x^1=0,\ \xi_u^1=0,\ \xi_v^1=0,\ \xi_t^2=0,\ \xi_u^2=0,\ \xi_v^2=0,\\
&\eta_x^1=0,\ \eta_v^1=0,\ \eta_x^2=0,\ \eta_u^2=0,\ B_u^1=0,\ B_v^1=0,\\
&\frac{a\xi_t^1}{2}-\frac{3a\xi_x^2}{2}+a\eta_v^2=0,\ \frac{c\xi_t^1}{2}-\frac{3c\xi_x^2}{2}+c\eta_u^1=0,\\
&b\xi_x^2-\frac{b\eta_u^1}{2}-\frac{b\eta_v^2}{2}=0,\ \xi_x^2-\frac{\xi_t^1}{2}-\frac{\eta_u^1}{2}-\eta_v^2=0,\\
&B_v^2+\frac{\eta_t^1}{2}=0,\ \frac{1}{2}b\eta_{tv}^2=0,\ \frac{1}{2}b\eta_{vv}^2=0,\ B_u^2+\frac{\eta_t^2}{2}=0,\\
&\frac{1}{2}b\eta_{tu}^1=0,\ c\xi_{xx}^2=0,\ \frac{1}{2}b\xi_{xx}^2=0,\ \frac{\eta_u^1}{2}+\frac{\eta_v^2}{2}=0,\ \frac{1}{2}b\eta_{uu}^1=0,\\
&B_t^1+B_x^2+\frac{1}{2}u^2\xi_t^1+\frac{1}{2}v^2\xi_t^1+\frac{1}{2}u^2\xi_x^2+u\eta^1+\frac{1}{2}v^2\xi_x^2+v\eta^2=0.
\end{aligned}$$

The solution of the above system yields the following Noether point symmetries and gauge terms

$$\begin{aligned}
\xi^1&=C_1,\\
\xi^2&=C_2,\\
\eta^1&=E(t),\\
\eta^2&=F(t),\\
B^1&=\alpha(t,x),\\
B^2&=-\frac{1}{2}VE'(t)-\frac{1}{2}UF'(t)+\beta(t,x),
\end{aligned}$$

$$\alpha_t + \beta_t = 0. \tag{7.9}$$

The above results will now be used to find the components of the conserved vectors associated with the Lagrangian (7.4). Here we set $\alpha(t,x) = 0$ and $\beta(t,x) = 0$ as they contribute to the trivial part of the conserved vector. The formula for conserved vectors due to Noether theorem for the second-order Lagrangian (7.4) is defined by

$$\begin{aligned} T^t = & -B^1 + \xi^1 L + W^1\left[\frac{\partial L}{\partial U_t} - D_t\frac{\partial L}{\partial U_{tt}} - D_x\frac{\partial L}{\partial U_{tx}}\right] \\ & + W^2\left[\frac{\partial L}{\partial V_t} - D_t\frac{\partial L}{\partial V_{tt}} - D_x\frac{\partial L}{\partial V_{tx}}\right] + D_t(W^1)\frac{\partial L}{\partial U_{tt}} + D_t(W^2)\frac{\partial L}{\partial V_{tt}} \\ & + D_x(W^1)\frac{\partial L}{\partial U_{tx}} + D_x(W^2)\frac{\partial L}{\partial V_{tx}}, \end{aligned} \tag{7.10}$$

$$\begin{aligned} T^x = & -B^2 + \xi^2 L + W^1\left[\frac{\partial L}{\partial U_x} - D_t\frac{\partial L}{\partial U_{xt}} - D_x\frac{\partial L}{\partial U_{xx}}\right] \\ & + W^2\left[\frac{\partial L}{\partial V_x} - D_t\frac{\partial L}{\partial V_{xt}} - D_x\frac{\partial L}{\partial V_{xx}}\right] + D_t(W^1)\frac{\partial L}{\partial U_{xt}} + D_t(W^2)\frac{\partial L}{\partial V_{xt}} \\ & + D_x(W^1)\frac{\partial L}{\partial U_{xx}} + D_x(W^2)\frac{\partial L}{\partial V_{xx}}, \end{aligned} \tag{7.11}$$

where $W^1 = \eta^1 - U_t\xi^1 - U_x\xi^2$ and $W^2 = \eta^2 - V_t\xi^1 - V_x\xi^2$ are the characteristic functions. Using equations (7.10) and (7.11) together with (7.9) yields the independent conserved vectors for system (7.3). Setting $C_1 = 1$ and $C_2 = 0$ we have $\xi^1 = 1$, $\xi^2 = 0$, $\eta^1 = 0$, $\eta^2 = 0$, $B^1 = 0$, $B^2 = 0$, $W^1 = -U_t$ and $W^2 = -V_t$. Substituting these into equation (7.10) and equation (7.11) gives

$$\begin{aligned} T_1^t = & \ \frac{1}{2}aV_{xx}^2 - \frac{1}{4}bU_tV_{xxx} - \frac{1}{4}bV_tU_{xxx} - \frac{1}{4}bV_{xx}U_{tx} - \frac{1}{4}bU_{xx}V_{tx} \\ & + \frac{1}{2}cU_{xx}^2 - \frac{1}{2}U_xV_x^2 - \frac{U_x^2}{2} - \frac{V_x^2}{2}, \\ T_1^x = & \ aV_tV_{xxx} - aV_{xx}V_{tx} + bU_{tx}V_{tx} - \frac{3}{4}bV_tU_{txx} - \frac{3}{4}bU_tV_{txx} + \frac{1}{4}bU_{tt}V_{xx} \\ & + \frac{1}{4}bV_{tt}U_{xx} + cU_tU_{xxx} - cU_{xx}U_{tx} + \frac{1}{2}U_tV_x^2 + V_tU_xV_x + U_tV_t \\ & + U_tU_x + V_tV_x. \end{aligned} \tag{7.12}$$

Using the transformation $u = U_x, v = V_x$ and reverting back to the original variables we have

$$\begin{aligned}
T_1^t &= \frac{1}{2}av_x^2 - \frac{1}{4}bv_{xx}\int u_t dx - \frac{1}{4}bu_{xx}\int v_t dx - \frac{1}{4}bv_x u_t - \frac{1}{4}bu_x v_t \\
&+\frac{1}{2}cu_x^2 - \frac{1}{2}uv^2 - \frac{u^2}{2} - \frac{v^2}{2}, \\
T_1^x &= (av_{xx} - \frac{3}{4}bu_{tx} + uv + v)\int v_t dx + \frac{1}{4}bv_x\int u_{tt}dx + \frac{1}{4}bu_x\int v_{tt}dx \\
&+(cu_{xx} - \frac{3}{4}bv_{tx} + \frac{1}{2}u^2 + u)\int u_t dx + \int u_t dx\int v_t dx - av_x v_t \\
&+bu_t v_t - cu_x u_t.
\end{aligned}$$

Now setting $C_2 = 1$, $C_1 = 0$, we have $\xi^1 = 0$, $\xi^2 = 1$, $\eta^1 = 0$, $\eta^2 = 0$, $B^1 = 0$, $B^2 = 0$, $W^1 = -U_x$ and $W^2 = -V_x$ and substituting these results into equations (7.10) and (7.11) and thereafter, using the transformation $u \to U_x, v \to V_x$ and reverting back to the original variables, we obtain

$$\begin{aligned}
T_2^t &= U_x V_x - \frac{1}{4}bU_{xxx}V_x - \frac{1}{4}bU_x V_{xxx} + \frac{1}{2}bU_{xx}V_{xx} \\
&= uv - \frac{1}{4}bvu_{xx} - \frac{1}{4}buv_{xx} + \frac{1}{2}bu_x v_x, \\
T_2^x &= aV_x V_{xxx} - \frac{1}{2}aV_{xx}^2 - \frac{3}{4}bU_x V_{txx} + \frac{1}{4}bV_{xx}U_{tx} + \frac{1}{4}bU_{xx}V_{tx} \\
&-\frac{3}{4}bV_x U_{txx} + cU_{xxx}U_x - \frac{1}{2}cU_{xx}^2 + U_x V_x^2 + \frac{U_x^2}{2} + \frac{V_x^2}{2} \\
&= avv_{xx} - \frac{1}{2}av_x^2 - \frac{3}{4}buv_{tx} + \frac{1}{4}bv_x u_t + \frac{1}{4}bu_x v_t \\
&-\frac{3}{4}bvu_{tx} + cuu_{xx} - \frac{1}{2}cu_x^2 + uv^2 + \frac{u^2}{2} + \frac{v^2}{2}.
\end{aligned}$$

In the same way setting $\eta^1 = E(t)$, $\eta^2 = 0$, $C_1 = 0$, $C_2 = 0$, $\xi^1 = 0$, $\xi^2 = 0$, $B^1 = 0$, $B^2 = -\frac{1}{2}VE'(t)$, we get $W^1 = E(t)$ and $W^2 = 0$. Substituting these into equations (7.10) and (7.11) and then reverting back to the original variables we get

$$\begin{aligned}
T_{(E)}^t &= \frac{1}{4}bE(t)V_{xxx} - \frac{1}{2}E(t)V_x \\
&= \left(\frac{1}{4}bv_{xx} - \frac{1}{2}v\right)E(t),
\end{aligned}$$

$$
\begin{aligned}
T^x_{(E)} &= \left(\frac{1}{2}V - \frac{1}{4}bV_{xx}\right)E'(t) + \left(\frac{3}{4}bV_{txx} - cU_{xxx} - U_x - \frac{1}{2}V_x^2 - \frac{1}{2}V_t\right)E(t) \\
&= \left(\frac{1}{2}\int vdx - \frac{1}{4}bv_x\right)E'(t) + \left(\frac{3}{4}bv_{tx} - cu_{xx} - u - \frac{1}{2}v^2 - \frac{1}{2}\int v_t dx\right)E(t).
\end{aligned}
$$

Setting $E(t) = 1$ we obtain the nonlocal conservation law

$$
\begin{aligned}
T^t_3 &= \frac{1}{4}bv_{xx} - \frac{1}{2}v, \\
T^x_3 &= \frac{3}{4}bv_{tx} - cu_{xx} - u - \frac{1}{2}v^2 - \frac{1}{2}\int v_t dx.
\end{aligned}
$$

Similarly for when $\eta^2 = F(t)$, $\eta^1 = 0$, $C_1 = 0$, $C_2 = 0$, $\xi^1 = 0$, $\xi^2 = 0$, $B^1 = 0$ and $B^2 = -\frac{1}{2}UF'(t)$, we get $W^1 = 0$ and $W^2 = F(t)$. The same procedure yields the conserved quantities

$$
\begin{aligned}
T^t_{(F)} &= \frac{1}{4}bF(t)U_{xxx} - \frac{1}{2}F(t)U_x \\
&= \frac{1}{4}bF(t)u_{xx} - \frac{1}{2}uF(t), \\
T^x_{(F)} &= \left(\frac{1}{2}U - \frac{1}{4}bU_{xx}\right)F'(t) + \left(\frac{3}{4}bU_{txx} - aV_{xxx} - U_xV_x - \frac{1}{2}U_t - V_x\right)F(t) \\
&= \left(\frac{1}{2}\int udx - \frac{1}{4}bu_x\right)F'(t) + \left(\frac{3}{4}bu_{tx} - av_{xx} - uv - \frac{1}{2}\int u_t dx - v\right)F(t).
\end{aligned}
$$

Also setting $F(t) = 1$ we obtain the nonlocal conservation law

$$
\begin{aligned}
T^t_4 &= \frac{1}{4}bu_{xx} - \frac{1}{2}u, \\
T^x_4 &= -av_{xx} + \frac{3}{4}bu_{tx} - uv - \frac{1}{2}\int u_t dx - v.
\end{aligned}
$$

For arbitrary values of $E(t)$ and $F(t)$ infinitely many nonlocal conservation laws exist for system (7.1).

7.2 Exact solutions of (7.1) using Kudryashov method

In this section we employ Kudryashov method to find exact solutions of the four parameter Boussinesq system (7.1). First we transform the PDE system (7.1) to a system of ordinary differential equations using the ansatz

$$u(t,x) = \Theta(\xi), \;\; v(t,x) = \Phi(\xi), \;\; \xi = kx + \omega t. \tag{7.13}$$

Substitution of (7.13) into (7.1) we obtain the system of ordinary differential equations

$$\begin{aligned} &ak^3\Phi_{\xi\xi\xi} - bk^2\omega\Theta_{\xi\xi\xi} + k\Phi_\xi + k\Phi\Theta_\xi + k\Theta\Phi_\xi + \omega\Theta_\xi = 0, \\ &ck^3\Theta_{\xi\xi\xi} - dk^2\omega\Phi_{\xi\xi\xi} + k\Phi\Phi_\xi + k\Theta_\xi + \omega\Phi_\xi = 0. \end{aligned} \tag{7.14}$$

We consider the solutions of (7.14) to be of the forms

$$\Theta(\xi) = \sum_{i=0}^{M} A_i Q(\xi)^i, \;\; \Phi(\xi) = \sum_{i=0}^{M} B_i Q(\xi)^i \tag{7.15}$$

where the function $Q(\xi)$ satisfies the first-order ODE

$$Q'(\xi) = Q(\xi)^2 - Q(\xi) \tag{7.16}$$

whose solution can be written in terms of the hyperbolic functions as

$$Q(\xi) = \frac{1}{1 + \cosh(\xi) + \sinh(\xi)}. \tag{7.17}$$

In (7.15) M is a positive integer that can be determined by the balancing procedure and A_i, B_i are constants to be determined. For (7.14) the balancing procedure yields $M = 2$, and so the solutions of (7.14) are of the form

$$\Theta(\xi) = A_0 + A_1 Q(z) + A_2 Q(z)^2,$$

$$\Phi(\xi) = B_0 + B_1 Q(z) + B_2 Q(z)^2. \tag{7.18}$$

The substitution of (7.18) into (7.14) and making use of (7.16) yields the following system of algebraic equations in terms of A_i and B_i, $i = 0, 1, 2$:

$$\begin{aligned}
&A_1bk^2\omega - aB_1k^3 - A_1B_0k - A_0B_1k - A_1\omega - B_1k = 0,\\
&7aB_1k^3 - 8aB_2k^3 - 7A_1bk^2\omega + 8A_2bk^2\omega + A_1B_0k - 2A_2B_0k\\
&\ + A_0B_1k - 2A_1B_1k - 2A_0B_2k + A_1\omega - 2A_2\omega + B_1k - 2B_2k = 0,\\
&38aB_2k^3 - 12aB_1k^3 + 12A_1bk^2\omega - 38A_2bk^2\omega + 2A_2B_0k + 2A_1B_1k - 3A_2B_1k\\
&\ + 2A_0B_2k - 3A_1B_2k + 2A_2\omega + 2B_2k = 0,\\
&6aB_1k^3 - 54aB_2k^3 - 6A_1bk^2\omega + 54A_2bk^2\omega + 3A_2B_1k + 3A_1B_2k - 4A_2B_2k = 0,\\
&24aB_2k^3 - 24A_2bk^2\omega + 4A_2B_2k = 0,\\
&B_1dk^2\omega - A_1ck^3 - A_1k - B_0B_1k - B_1\omega = 0,\\
&7A_1ck^3 - 8A_2ck^3 + A_1k - 2A_2k - 7B_1dk^2\omega + 8B_2dk^2\omega - B_1^2k\\
&\ + B_0B_1k - 2B_0B_2k + B_1\omega - 2B_2\omega = 0,\\
&38A_2ck^3 - 12A_1ck^3 + 2A_2k + 12B_1dk^2\omega - 38B_2dk^2\omega + B_1^2k + 2B_0B_2k\\
&\ + 2B_2\omega - 3B_1B_2k = 0,\\
&6A_1ck^3 - 54A_2ck^3 - 6B_1dk^2\omega + 54B_2dk^2\omega - 2B_2^2k + 3B_1B_2k = 0,\\
&24A_2ck^3 - 24B_2dk^2\omega + 2B_2^2k = 0.
\end{aligned}$$

Solving this system of algebraic equations with the aid of Mathematica, one possible set of values of the constants is

$$\begin{aligned}
A_0 &= \frac{1}{324}\left(10B_2^2 + 9B_1B_2 - 324\right),\\
A_1 &= \frac{1}{36}\left(18B_1^2 + 34B_2B_1 + 15B_2^2\right),\\
A_2 &= \frac{1}{36}\left(-18B_1^2 - 36B_2B_1 - 17B_2^2\right),\\
B_0 &= -\frac{18\omega + 9kB_1 + 8kB_2}{18k},
\end{aligned}$$

$$
\begin{aligned}
B_1 &= \frac{3}{c\left(ck^2-9\right)^2}\left\{\pm\sqrt{6cd^2\omega^2\left(ck^2-9\right)^2\left(ck^2-5\right)}+6cdk\omega\left(ck^2-9\right)\right\},\\
B_2 &= \frac{18dk\omega}{9-ck^2},\\
a &= \frac{24B_2dk\omega+54B_1^2+108B_2B_1+49B_2^2}{216k^2},\\
b &= \frac{2\left(3d-cdk^2\right)}{ck^2-9}.
\end{aligned}
$$

As a result we obtain the solutions of (7.1) as

$$
u(t,x) = A_0 + A_1\left(\frac{1}{1+\cosh(\xi)+\sinh(\xi)}\right) + A_2\left(\frac{1}{1+\cosh(\xi)+\sinh(\xi)}\right)^2, \tag{7.19}
$$

$$
v(t,x) = B_0 + B_1\left(\frac{1}{1+\cosh(\xi)+\sinh(\xi)}\right) + B_2\left(\frac{1}{1+\cosh(\xi)+\sinh(\xi)}\right)^2 \tag{7.20}
$$

where $\xi = kx + \omega t$.

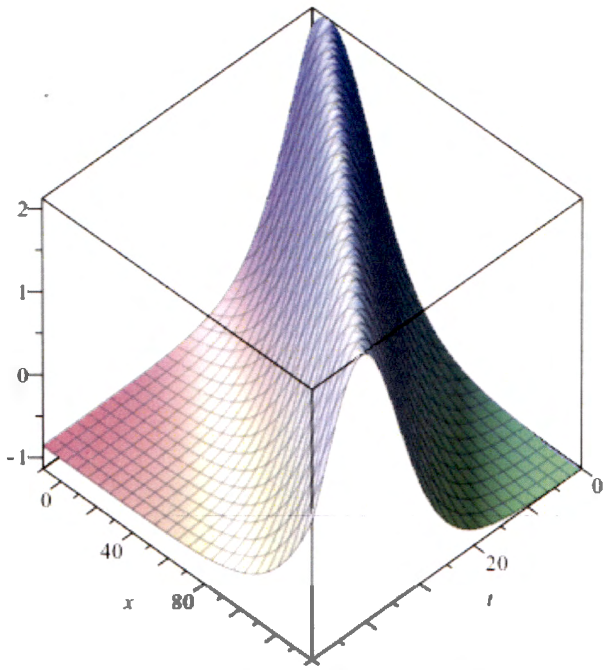

Figure 7.1: Profile of solution (7.19)

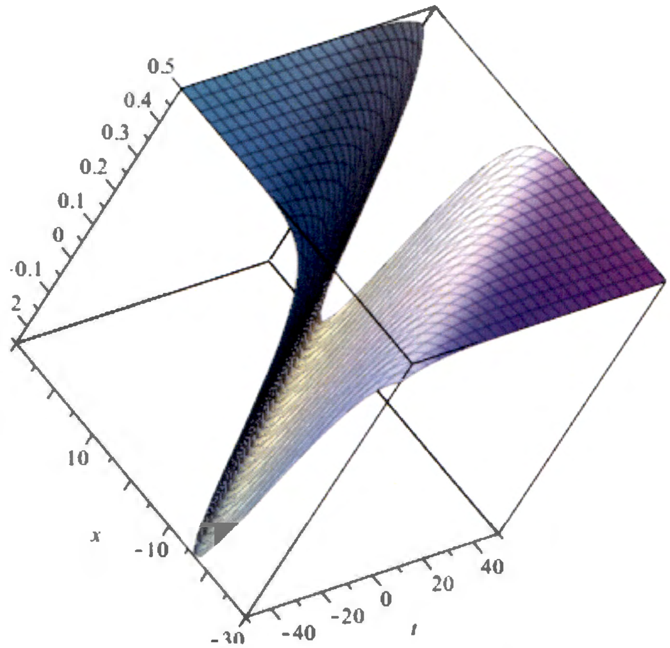

Figure 7.2: Profile of solution (7.20)

7.3 Conclusion

In this chapter we have studied the four parameter Boussinesq system (7.1). In order to apply Noether approach, the transformations $u = U_x$, $v = V_x$ were utilized. The system was then transformed to the fourth-order system in U and V variables which admits a Lagrangian. Noether symmetries and conservation laws were obtained for the fourth order system. The inverse transformations were then used to obtain the conservation laws for the original third-order Boussinesq system. Travelling wave solutions were also obtained using the Kudryashov method for the four parameter third-order Boussinesq system (7.1).

Chapter 8

Exact solutions and conservation laws of an integrable coupling with Korteweg-de Vries equation

The Korteweg-de Vries (KdV) equation [101]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{8.1}$$

is an important equation in the scientific field and in the theory of integrable systems. It describes the unidirectional propagation of long waves of small amplitude and has a lot of applications in a number of physical contexts such as hydromagnetic waves, stratified internal waves, ion-acoustic waves, plasma physics and lattice dynamics [102]. Equation (8.1) has multiple-soliton solutions and an infinite number of conservation laws and many other physical properties. See for example [103, 104] and references therein.

Recently focus has shifted to the study of coupled Korteweg-de Vries equations because of their many applications in scientific fields. See for example [105–108].

In this chapter we study the (2+1)-dimensional integrable coupling system with the Korteweg-de Vries equation [102]

$$u_t - 6uu_x - u_{xxx} = 0, \tag{8.2a}$$

$$v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx} = 0. \tag{8.2b}$$

We determine the exact solutions for the system (8.2) using the $(G'/G)-$expansion method. Furthermore, conservation laws for (8.2) will be constructed by employing the new conservation theorem due to Ibragimov [48] and multiplier approach [45].

This work has been submitted for publication. See [109].

8.1 Lie point symmetries and symmetry reductions of (8.2)

The vector field

$$\begin{aligned}\Gamma =& \xi^1(t,x,y,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v)\frac{\partial}{\partial y} \\ &+ \eta^1(t,x,y,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v)\frac{\partial}{\partial v}\end{aligned} \tag{8.3}$$

is used to generate the symmetry group of (8.2). The system of equations (8.2) is invariant under the generator (8.3) if and only if

$$\Gamma^{[3]}\left(u_t - 6uu_x - u_{xxx}\right)|_{(8.2)} = 0, \tag{8.4a}$$

$$\Gamma^{[3]}\left(v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx}\right)|_{(8.2)} = 0, \tag{8.4b}$$

where $\Gamma^{[3]}$ is the third prolongation of (8.3) given by

$$\begin{aligned}\Gamma^{[3]} &= \Gamma + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_3\frac{\partial}{\partial u_y} + \zeta_1\frac{\partial}{\partial v_t} + \zeta_2\frac{\partial}{\partial v_x} \\ &\quad + \zeta_{222}\frac{\partial}{\partial u_{xxx}} + \zeta_{223}\frac{\partial}{\partial u_{xxy}} + \zeta_{222}\frac{\partial}{\partial v_{xxx}} + \cdots .\end{aligned}$$

Expanding (8.4) and then splitting on the derivatives of u and v, we obtain the following overdetermined system of linear partial differential equations:

$$
\begin{aligned}
&\xi^1_x = 0,\ \xi^1_y = 0,\ \xi^1_u = 0,\ \xi^1_v = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \xi^2_y = 0,\\
&\xi^3_t = 0,\ \xi^3_x = 0,\ \xi^3_v = 0,\ \eta^1_v = 0,\ \ \eta^1_{uu} = 0,\ \eta^2_{uu} = 0,\ \ \eta^2_{vv} = 0,\ \ \eta^2_{uv} = 0,\\
&\eta^1_{xu} - \xi^2_{xx} = 0,\ \ 2\eta^1_{xu} - \xi^2_{xx} = 0,\ \eta^2_{xv} - \xi^2_{xx} = 0,\ \ \eta^2_{xu} + \eta^1_{yu} = 0,\\
&\ -3\xi^2_x + \xi^1_t = 0,\ \ 4u\xi^2_x + 2\eta^1 + \eta^1_{xxu} = 0,\\
&\eta^1_{xxx} - \eta^1_t + 6u\eta^1_x = 0,\ -\eta^1_u + \xi^3_y + \eta^2_v - \xi^2_x = 0,\\
&2v\xi^2_x + 2v\xi^3_y + 2\eta^2 + \eta^2_{xxu} + 2\eta^1_{xyu} = 0,\\
&12u\xi^2_x + 6\eta^1 - \xi^2_{xxx} + 3\eta^1_{xxu} + \xi^2_t = 0,\ \ -2\xi^2_x - \xi^3_y - \eta^2_v + \xi^1_t + \eta^1_u = 0,\\
&6u\eta^1_y + 6v\eta^1_x - \eta^2_t + 3\eta^1_{yxx} + \eta^2_{xxx} + 6u\eta^2_x = 0,\\
&6u\xi^2_x + 6u\xi^3_y - 6u\eta^1_u + 6\eta^1 - \xi^2_{xxx} + 3\eta^2_{xxv} + \xi^2_t + 6u\eta^2_v = 0.
\end{aligned}
$$

Solving the above system of partial differential equations, one obtains the following four Lie point symmetries:

$$
\begin{aligned}
&\textit{time translation} && \Gamma_1 = \frac{\partial}{\partial t},\\
&\textit{space translation} && \Gamma_2 = \frac{\partial}{\partial x},\\
&\textit{space translation} && \Gamma_3 = \frac{\partial}{\partial y},\\
&\textit{scaling} && \Gamma_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.
\end{aligned}
$$

Taking the symmetry given by $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ and solving the resultant Lagrange system we obtain four invariants

$$f = t - y,\ g = x - y,\ u = \theta,\ v = \psi.$$

Applying these invariants, system (8.2) is reduced to a system of partial differential equations of two functions θ and ψ in two independent variables f and g:

$$\theta_f - 6\theta\theta_g - \theta_{ggg} = 0, \tag{8.5a}$$

$$\psi_f - 6(\theta\psi)_g + 6\theta\theta_f + 6\theta\theta_g + 3\theta_{ggf} + 3\theta_{ggg} - \psi_{ggg} = 0. \tag{8.5b}$$

Equations (8.5) has the following three Lie point symmetries:

$$X_1 = \frac{\partial}{\partial f},$$
$$X_2 = \frac{\partial}{\partial g},$$
$$X_3 = 6f\frac{\partial}{\partial f} + (2g+3f)\frac{\partial}{\partial g} - (4\theta + \frac{1}{2})\frac{\partial}{\partial \theta} - (8\psi + 2\theta + \frac{1}{2})\frac{\partial}{\partial \psi},$$

Considering the symmetry $X = X_1 + \alpha X_2$ given by the linear combination of X_1 and X_2 we get the invariants

$$z = g - \alpha f,\ \theta = H,\ \psi = J.$$

This further reduces (8.2) to a coupled system of third-order ordinary differential equations in two functions $H(z)$ and $J(z)$, i.e.,

$$\alpha H' + 6HH' + H''' = 0, \tag{8.6a}$$
$$\alpha J' + 6(HJ)' - 6(1-\alpha)HH' - 3(1-\alpha)H''' + J''' = 0, \tag{8.6b}$$

where the prime denotes derivative with respect to z.

8.2 Exact solutions of (8.2) using the $(G'/G)-$expansion method

In this section we employ the $(G'/G)-$expansion method to construct travelling wave solutions of the system of third-order ODEs (8.6). The $(G'/G)-$expansion method assumes the solutions of equation (8.6) to be of the form

$$H(z) = \sum_{i=0}^{M} \mathcal{A}_i \left(\frac{G'(z)}{G(z)}\right)^i, \qquad J(z) = \sum_{i=0}^{M} \mathcal{B}_i \left(\frac{G'(z)}{G(z)}\right)^i, \tag{8.7}$$

where $G(z)$ satisfies the second-order ODE given by

$$G'' + \lambda G' + \mu G = 0, \tag{8.8}$$

where λ and μ arbitrary constants. The value of M is determined by using the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (8.6). The $\mathcal{A}_i$ and $\mathcal{B}_i$ ($i = 0, 1, \cdots, M$) are parameters to be determined. In our case the balancing procedure yields $M = 2$, so the solutions of the ordinary differential equation (8.6) are of the form

$$H(z) = \mathcal{A}_0 + \mathcal{A}_1(G'/G) + \mathcal{A}_2(G'/G)^2, \tag{8.9a}$$

$$J(z) = \mathcal{B}_0 + \mathcal{B}_1(G'/G) + \mathcal{B}_2(G'/G)^2. \tag{8.9b}$$

Substituting (8.9) into (8.6) and making use of (8.8), and then collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields a system of algebraic equations in $\mathcal{A}_i$ and $\mathcal{B}_i$, $i = 0, 1, 2$. Solving this system with the aid of Mathematica gives the following set of values:

$$\begin{aligned}
\mathcal{A}_0 &= -\frac{1}{6}\left(\alpha + \lambda^2 + 8\mu\right), \ \mathcal{A}_1 = -2\lambda, \ \mathcal{A}_2 = -2, \\
\mathcal{B}_0 &= \frac{1}{6}(\alpha - 1)\left(\alpha - 2\lambda^2 - 16\mu\right), \ \mathcal{B}_1 = 4\lambda(1 - \alpha), \ \mathcal{B}_2 = 4(1 - \alpha).
\end{aligned}$$

Hence we obtain the following three types of travelling wave solutions for the system (8.2):

Case 1.

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$\begin{aligned}
u_1(t, x, y) &= -\frac{1}{6}\left(\alpha + \lambda^2 + 8\mu\right) + 2\lambda\left[\frac{\lambda}{2} - \delta_1\left(\frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\right)\right] \\
&\quad -2\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\right)\right]^2, \\
v_1(t, x, y) &= \frac{1}{6}(\alpha - 1)\left(\alpha - 2\lambda^2 - 16\mu\right)
\end{aligned}$$

$$
\begin{aligned}
&-4(-\lambda+\alpha\lambda)\left[-\frac{\lambda}{2}+\delta_1\left(\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\right)\right] \\
&-4(-1+\alpha)\left[-\frac{\lambda}{2}+\delta_1\left(\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\right)\right]^2,
\end{aligned}
$$

where $z = x-(1-\alpha)y-\alpha t$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2-4\mu}$, C_1 and C_2 are arbitrary constants.

Case 2.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution

$$
\begin{aligned}
u_2(t,x,y) &= \frac{1}{6}\left(-\alpha-\lambda^2-8\mu\right)-2\lambda\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\right) \\
&\quad -2\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\right)^2, \\
v_2(t,x,y) &= \frac{1}{6}(\alpha-1)\left(\alpha-2\lambda^2-16\mu\right) \\
&\quad -4(-\lambda+\alpha\lambda)\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\right) \\
&\quad -4(-1+\alpha)\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\right)^2.
\end{aligned}
$$

where $z = x-(1-\alpha)y-\alpha t$, $\delta_2 = \frac{1}{2}\sqrt{4\mu-\lambda^2}$, C_1 and C_2 are arbitrary constants.

Case 3.

When $\lambda^2 - 4\mu = 0$, we obtain the rational solution

$$
\begin{aligned}
u_3(t,x,y) &= \frac{1}{6}\left(-\alpha-\lambda^2-8\mu\right)-2\lambda\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\right) \\
&\quad -2\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\right)^2, \\
v_3(t,x,y) &= \frac{1}{6}(\alpha-1)\left(\alpha-2\lambda^2-16\mu\right)-4(-\lambda+\alpha\lambda)\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\right) \\
&\quad -4(-1+\alpha)\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\right)^2.
\end{aligned}
$$

where $z = x-(1-\alpha)y-\alpha t$, C_1 and C_2 are arbitrary constants.

8.3 Conservation laws of (8.2) using the multiplier approach

In this section we construct conservation laws for system (8.2) using the multiplier approach. The determining equations for the multipliers are given by

$$\frac{\delta}{\delta u}[Q_1(u_t - 6uu_x - u_{xxx})] = 0, \tag{8.10a}$$

$$\frac{\delta}{\delta u}[Q_2(v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx})] = 0, \tag{8.10b}$$

where Q_1 and Q_2 are the second-order multipliers. Now expanding equations (8.10) and solving for Q_1 and Q_2 we obtain the second-order multipliers

$$\begin{aligned}
Q_1 = & -x\left(\frac{5}{2}u^3 + uu_{xx} - \frac{1}{4}(u_x^2 - u_{tx})\right) f_2'(y) - \left(6tux + x^2\right) f_1'(y) + v f_4(y) \\
& - x\left(3u^2 + u_{xx}\right) f_3'(y) - xuf_4'(y) + 6tuf_5'(y) + \frac{15}{2}vu^2 + 6tvf_1(y) \\
& + \frac{1}{4}\left((8u_{xy} + 4v_{xx})u + 4u_{xx}v + (-2u_y - 2v_x)u_x + u_{ty} + v_{tx}\right) f_2(y) \\
& + \frac{1}{4}\left(10u^3 + 4uu_{xx} - {u_x}^2 + u_{tx}\right) f_6(y) + \left(6uv + 2u_{xy} + v_{xx}\right) f_3(y) \\
& + \left(3u^2 + u_{xx}\right) f_7(y) + (6tu + x) f_8(y) + uf_9(y) + f_{10}(y),
\end{aligned}$$

$$\begin{aligned}
Q_2 = & \frac{1}{4}\left(10u^3 + 4uu_{xx} - {u_x}^2 + u_{tx}\right) f_2(y) + (6tu + x) f_1(y) \\
& + \left(3u^2 + u_{xx}\right) f_3(y) + f_4(y)u + f_5(y),
\end{aligned}$$

where f_i, $i = 1, \cdots, 10$ are arbitrary functions of the variable y. As a result we have the following ten conserved vectors:

$$\begin{aligned}
T_1^t = & \, 6uvf(y)t - 3xf'(y)tu^2 - ux^2f'(y) + vf(y)x, \\
T_1^x = & \, 3u^2x^2f'(y) - xu_xf'(y) + u_{xx}x^2f'(y) - 2u_{xy}f(y)x - v_{xx}f(y)x \\
& - 12uu_{xy}f(y)t - 6uv_{xx}f(y)t - 6vu_{xx}f(y)t + 6v_xu_xf(y)t \\
& - 6vuf(y)x - 3{u_x}^2xf'(y)t + 6u_xu_yf(y)t + 6uu_{xx}xf'(y)t
\end{aligned}$$

$$+ f(y)v_x + u_y\, f(y) - 36\, v f(y)u^2 t + 12\, x f'(y) t u^3,$$

$$T_1^y = f(y)u_x - 6\, u u_{xx}\, f(y)t - u_{xx}\, f(y)x - 12\, u^3 f(y)t + 3\, {u_x}^2 f(y)t$$
$$- 3\, u^2 f(y)x;$$

$$\begin{aligned}
T_2^t = {} & \frac{1}{16} f(y) v u_{xxxx} + \frac{1}{16} f(y) u u_{ty} + \frac{7}{6} f(y) u^2 u_{xy} + \frac{1}{4} f(y) u u_{xxxy} \\
& + \frac{1}{16}\, f(y) u v_{tx} + \frac{7}{12} f(y) u^2 v_{xx} + \frac{1}{16}\, f(y) u v_{xxxx} - \frac{1}{12}\, f(y) {u_x}^2 v \\
& - \frac{5}{8}\, x f'(y) u^4 + \frac{5}{2}\, v f(y) u^3 - \frac{1}{16}\, f\,(y)\, u_{xxx}\, v_x + \frac{1}{16}\, u_y\, f(y) u_t \\
& - \frac{1}{16}\, u_y\, f(y) u_{xxx} - \frac{3}{16}\, u_x\, f(y) u_{xxy} + \frac{1}{16}\, u_x\, f(y) v_t - \frac{1}{16}\, u_x\, f(y) v_{xxx} \\
& + \frac{1}{16}\, f(y) u_t\, v_x - \frac{7}{12} f'(y) u^2 u_{xx}\, x + \frac{1}{12}\, u {u_x}^2 f'(y) x - \frac{1}{6}\, u f(y) u_x\, u_y \\
& - \frac{1}{16}\, u_x\, f'(y) u_t\, x + \frac{1}{16}\, u_x\, f'(y) u_{xxx}\, x - \frac{1}{16}\, u u_{tx}\, f'(y) x \\
& - \frac{1}{16}\, u u_{xxxx}\, f'(y) x + \frac{7}{6}\, v f\,(y)\, u u_{xx} - \frac{1}{6}\, u f(y) u_x\, v_x + \frac{1}{16}\, f(y) v u_{tx}, \\
T_2^x = {} & -\frac{3}{16}\, f(y) u u_{txxy} - \frac{1}{2}\, v {u_{xx}}^2 f\,(y) - \frac{1}{8}\, u_{xx}\, f(y) u_{ty} - \frac{1}{8}\, u_{xx}\, f(y) v_{tx} \\
& - \frac{1}{8}\, v_{xx}\, f(y) u_{tx} - \frac{1}{4}\, u_{xy}\, f(y) u_{tx} - \frac{1}{16}\, f\,(y)\, u_{xxx}\, v_t + \frac{1}{8}\, u_y\, f(y) u_{txx} \\
& - \frac{1}{16}\, f'(y) {u_t}^2 x - \frac{3}{16}\, u_t\, f(y) u_{xxy} + \frac{1}{8}\, u_t\, f(y) v_t - \frac{1}{16}\, u_t\, f\,(y)\, v_{xxx} \\
& - \frac{1}{16}\, f(y) v u_{txxx} - \frac{1}{16}\, f(y) v u_{tt} - \frac{7}{12} f(y) u^2 u_{ty} - \frac{1}{16}\, f(y) u v_{tt} \\
& - \frac{7}{12} f(y) u^2 v_{tx} - \frac{1}{16}\, f(y) u v_{txxx} + \frac{1}{6}\, u_x f'\,(y)\, u u_t\, x - 5\, u^3 u_{xy}\, f\,(y) \\
& + \frac{1}{8}\, f(y) u_x\, v_{txx} - 15\, v f\,(y)\, u^4 + \frac{1}{4}\, f(y) u_x\, u_{txy} + \frac{1}{4}\, v_{xx}\, f(y) {u_x}^2 \\
& - \frac{5}{2}\, u^3 v_{xx}\, f(y) + \frac{1}{8}\, v_x\, f(y) u_{txx} + 3\, x f'(y) u^5 + \frac{1}{2}\, u_{xy}\, f(y) {u_x}^2 \\
& + \frac{1}{16}\, u_t f'(y) u_{xxx}\, x - \frac{1}{6}\, u f(y) u_x\, v_t - \frac{1}{6}\, f(y) u_x\, v u_t - \frac{7}{6}\, v f(y) u u_{tx} \\
& + \frac{5}{2}\, u^3 u_{xx} f'(y) x - \frac{3}{4}\, {u_x}^2\, (f'(y))\, x u^2 + \frac{3}{2}\, u_x\, u_y\, f\,(y)\, u^2 \\
& + \left(\frac{3}{2}\, v_x\, u_x\, u^2 - \frac{7}{2}\, u^2 u_{xx}\, v + \frac{3}{2}\, v u {u_x}^2 - 2\, u u_{xx} u_{xy}\right) f(y)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{4}\,u_{xx}\,{u_x}^2 f'\left(y\right)x+\frac{1}{2}\,u_{xx}\,u_x\,f(y)u_y+\frac{7}{12}u^2u_{tx}\,f'\left(y\right)x \\
& +\frac{1}{2}\,u{u_{xx}}^2\left(f'(y)\right)x-uu_{xx}\,f(y)v_{xx}+\frac{1}{2}\,v_x\,f(y)u_x\,u_{xx}-\frac{1}{6}\,u_y\,f(y)uu_t \\
& -\frac{1}{6}\,u_t\,f(y)uv_x+\frac{1}{8}\,u_{xx}\,f'\left(y\right)xu_{tx}-\frac{1}{8}\,u_{txx}\,\left(f'(y)\right)u_x\,x \\
& +\frac{1}{16}\,uu_{tt}\,\left(f'\left(y\right)\right)x+\frac{1}{16}\,uu_{txxx}\,f'(y)x-2\,uu_{xx}\,f(y)u_{xy}, \\
T_2^y = & -\frac{7}{12}f(y)u^2u_{tx}-\frac{1}{16}\,f\left(y\right)uu_{tt}+\frac{1}{8}\,f(y)u_x\,u_{txx}-\frac{1}{16}\,f(y)uu_{txxx} \\
& -\frac{5}{2}\,u^3f(y)u_{xx}+\frac{3}{4}\,u^2f(y){u_x}^2+\frac{1}{4}\,{u_x}^2f(y)u_{xx}-\frac{1}{2}\,uf(y){u_{xx}}^2 \\
& -\frac{1}{16}\,u_t\,f(y)u_{xxx}-\frac{1}{8}\,u_{xx}\,f(y)u_{tx}-3\,u^5f(y)+\frac{1}{16}\,{u_t}^2f(y) \\
& -\frac{1}{6}\,uf(y)u_x\,u_t;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t = & -xf'(y)u^3+3\,f(y)u^2v+f(y)uu_{xy}+\frac{1}{2}\,f(y)uv_{xx} \\
& +\frac{1}{2}\,f(y)u_{xx}\,v-\frac{1}{2}\,xf'(y)uu_{xx}, \\
T_3^x = & \,\frac{1}{2}f'(y){u_{xx}}^2x-2\,u_{xx}\,f(y)u_{xy}-u_{xx}\,f(y)v_{xx}-\frac{1}{2}\,f(y)uu_{ty} \\
& -\left(6\,u^2u_{xy}+\frac{1}{2}uv_{tx}+3u^2v_{xx}+\frac{1}{2}vu_{tx}\right)f(y)+\frac{9}{2}\,xf'\left(y\right)u^4 \\
& -18\,vf(y)u^3+\frac{1}{2}\,u_y\,f(y)u_t+\frac{1}{2}\,u_x\,f(y)v_t+\frac{1}{2}\,f(y)u_t\,v_x \\
& +3f'\left(y\right)u^2u_{xx}\,x-\frac{1}{2}u_xf'(y)u_t\,x+\frac{1}{2}\,uu_{tx}f'\left(y\right)x-6\,vf(y)uu_{xx}, \\
T_3^y = & -3\,u^2f(y)u_{xx}+\frac{1}{2}\,u_x\,f\left(y\right)u_t-\frac{1}{2}\,uf(y)u_{tx}-\frac{9}{2}\,u^4f(y) \\
& -\frac{1}{2}\,{u_{xx}}^2f(y);
\end{aligned}
$$

$$
\begin{aligned}
T_4^t = & \,\frac{1}{2}\,u\left(-f'(y)xu+2\,f(y)v\right), \\
T_4^x = & \,2\,xf'(y)u^3-6\,f(y)u^2v-\frac{1}{2}\,xf'(y){u_x}^2-2\,f(y)uu_{xy} \\
& -f(y)uv_{xx}+f(y)u_x\,u_y+f(y)u_x\,v_x-f(y)u_{xx}\,v+xf'(y)uu_{xx},
\end{aligned}
$$

$$T_4^y = -2\,u^3 f(y) + \frac{1}{2}\,{u_x}^2 f(y) - uu_{xx}\,f(y);$$

$$T_5^t = f(y)v + 3\,tu^2 f'(y)\,,$$

$$T_5^x = -6\,uu_{xx}\,tf'(y) + u_x\,f'(y) - 2\,f(y)u_{xy} - f(y)v_{xx} - 6\,f(y)uv$$
$$- 12\,tu^3 f'(y) + 3\,{u_x}^2 t f'(y),$$

$$T_5^y = -3\,u^2 f(y) - u_{xx}\,f(y);$$

$$T_6^t = -\frac{1}{12}\,uf(y){u_x}^2 + \frac{1}{16}\,uu_{xxxx}\,f(y) - \frac{1}{16}\,u_x\,f(y)u_{xxx} + \frac{7}{12}u^2 f(y)u_{xx}$$
$$+ \frac{1}{16}\,u_x\,f(y)u_t + \frac{1}{16}\,uf(y)u_{tx} + \frac{5}{8}\,u^4 f(y),$$

$$T_6^x = -\frac{7}{12}f(y)u^2 u_{tx} - \frac{1}{16}\,f\,(y)\,uu_{tt} + \frac{1}{8}\,f(y)u_x\,u_{txx} - \frac{1}{16}\,f(y)uu_{txxx}$$
$$- \frac{5}{2}\,u^3 f(y)u_{xx} + \frac{3}{4}\,u^2 f(y){u_x}^2 + \frac{1}{4}\,{u_x}^2 f(y)u_{xx} - \frac{1}{2}\,uf(y){u_{xx}}^2$$
$$- \frac{1}{16}\,u_t\,f(y)u_{xxx} - \frac{1}{8}\,u_{xx}\,f(y)u_{tx} - 3\,u^5 f(y) + \frac{1}{16}\,{u_t}^2 f(y)$$
$$- \frac{1}{6}\,uf(y)u_x\,u_t,$$

$$T_6^y = 0;$$

$$T_7^t = u^3 f(y) + \frac{1}{2}\,uu_{xx}\,f(y),$$

$$T_7^x = -3\,u^2 f(y)u_{xx} + \frac{1}{2}\,u_x\,f\,(y)\,u_t - \frac{1}{2}\,uf(y)u_{tx} - \frac{9}{2}\,u^4 f(y)$$
$$- \frac{1}{2}\,{u_{xx}}^2 f(y),$$

$$T_7^y = 0;$$

$$T_8^t = 3\,tu^2 f(y) + uxf(y),$$

$$T_8^x = f(y)u_x - 6\,uu_{xx}\,f(y)t - u_{xx}\,f(y)x - 12\,u^3 f(y)t + 3\,{u_x}^2 f(y)t$$
$$- 3\,u^2 f(y)x,$$

$$T_8^y = 0;$$

$$\begin{aligned} T_9^t &= \frac{1}{2}u^2 f(y), \\ T_9^x &= -2u^3 f(y) + \frac{1}{2}{u_x}^2 f(y) - u u_{xx} f(y), \\ T_9^y &= 0; \end{aligned}$$

$$\begin{aligned} T_{10}^t &= f(y)u, \\ T_{10}^x &= -3u^2 f(y) - u_{xx} f(y), \\ T_{10}^y &= 0. \end{aligned}$$

8.4 Conclusion

In this chapter we studied the (2+1)-dimensional coupling system with the KdV equation (8.2). Lie symmetry analysis along with the $(G'/G)-$expansion method was used to derive exact travelling wave solutions of (8.2). The solutions obtained were expressed in the form of hyperbolic function, trigonometric function and rational solutions. Furthermore, ten conservation laws were derived by using the multiplier approach.

Chapter 9

Exact solutions and conservation laws for the coupled Benjamin-Bona-Mahony equations

In this chapter we study the third-order coupled Benjamin-Bona-Mahony (BBM) equations [110]

$$\begin{aligned} &u_t + v_x + vu_x + uv_x - u_{txx} = 0, \\ &v_t + u_x + v_x v + uu_x - v_{txx} = 0, \end{aligned} \tag{9.1}$$

which are used to model small-amplitude long waves on the surface of water in a channel. In [110], travelling wave solutions of the coupled BBM equations were obtained using the $(G'/G)-$expansion method. Cui and Zhao [111], worked on the existence and orbital stability of bell-shaped solitary waves with zero and nonzero asymptotic value. In this study, exact solutions are obtained using Kudryashov method and the multiplier method is used to construct conservation laws.

The contents of this chapter have been accepted for publication. See [112].

9.1 Exact solutions of (9.1) using Kudryashov method

In this section we employ Kudryashov method to find exact solutions of the coupled Benjamin-Bona -Mahony equations (9.1). However, first we transform the PDE system (9.1) into a system of ordinary differential equations using the transformations

$$u(t,x) = \phi(z), \quad v(t,x) = \psi(z), \quad z = kx + \omega t. \tag{9.2}$$

Substituting of (9.2) into (9.1) we obtain the system of ordinary differential equations

$$\begin{aligned} &\omega\phi' + k\psi\phi' + k\psi + k\phi\psi' - k^2\omega\phi''' = 0, \\ &k\phi' + \phi\phi' + \omega\psi' + k\psi\psi' - k^2\omega\psi''' = 0. \end{aligned} \tag{9.3}$$

We consider the solutions of (9.3) in the form

$$\phi(z) = \sum_{i=0}^{M} A_i Q(z)^i, \quad \psi(z) = \sum_{i=0}^{N} B_i Q(z)^i, \tag{9.4}$$

where the function $Q(z)$ satisfies

$$Q'(z) = Q(z)^2 - Q(z) \tag{9.5}$$

whose solution can be written in terms of elementary functions as

$$Q(z) = \frac{1}{1 + \cosh(z) + \sinh(z)}. \tag{9.6}$$

In equation (9.4) M and N are positive integers. In our case the balancing procedure gives $M = 2$ and $N = 2$, and so the solutions of (9.3) are of the form

$$\phi(z) = A_0 + A_1 Q(z) + A_2 Q(z)^2$$

$$\psi(z) = \quad B_0 + B_1Q(z) + B_2Q(z)^2. \tag{9.7}$$

The substitution of (9.7) into (9.3) and making use of (9.5) yields the following overdetermined system of algebraic equations in terms of A_i and B_i, $i = 0, 1, 2$:

$$
\begin{aligned}
& A_1k^2\omega - A_1B_0k - A_0B_1k - A_1\omega - B_1k = 0, \\
& A_1B_0k - 2A_2B_0k + A_0B_1k - 2A_1B_1k - 2A_0B_2k - 7A_1k^2\omega + 8A_2k^2\omega \\
& + A_1\omega - 2A_2\omega + B_1k - 2B_2k = 0, \\
& 2A_2B_0k + 2A_1B_1k - 3A_2B_1k + 2A_0B_2k - 3A_1B_2k + 12A_1k^2\omega - 38A_2k^2\omega \\
& + 2A_2\omega + 2B_2k = 0, \\
& 3A_2B_1k + 3A_1B_2k - 4A_2B_2k - 6A_1k^2\omega + 54A_2k^2\omega = 0, \\
& 4A_2B_2k - 24A_2k^2\omega = 0, \\
& B_1k^2\omega - A_0A_1k - A_1k - B_0B_1k - B_1\omega = 0 \\
& A_1k - A_1^2k + A_0A_1k - 2A_0A_2k - 2A_2k - 7B_1k^2\omega + 8B_2k^2\omega - B_1^2k + B_0B_1k \\
& - 2B_0B_2k + B_1\omega - 2B_2\omega = 0, \\
& A_1^2k + 2A_0A_2k - 3A_1A_2k + 2A_2k + 12B_1k^2\omega - 38B_2k^2\omega + B_1^2k + 2B_0B_2k \\
& - 3B_1B_2k + 2B_2\omega = 0, \\
& B_1^2k - 38B_2k^2\omega + 2B_0B_2k - 3B_1B_2k + 2B_2\omega = 0, \\
& 3A_1A_2k - 2A_2^2k - 6B_1k^2\omega + 54B_2k^2\omega - 2B_2^2k + 3B_1B_2k = 0, \\
& 2A_2^2k - 24B_2k^2\omega + 2B_2^2k = 0.
\end{aligned}
$$

Solving this system of algebraic equations with the aid of Mathematica, one possible set of values of the constants is

$$
\begin{aligned}
& A_0 \quad \text{arbitrary}, \\
& A_1 = 6k\omega, \\
& A_2 = -6k\omega, \\
& B_0 = A_0 + k\omega - \frac{\omega}{k} + 1,
\end{aligned}
$$

$$B_1 = -\,6k\omega,$$
$$B_2 = 6k\omega.$$

As a result we obtain the solution of (9.1) as

$$u(t,x) = A_0 + A_1\left(\frac{1}{1+\cosh(z)+\sinh(z)}\right) + A_2\left(\frac{1}{1+\cosh(z)+\sinh(z)}\right)^2,$$
$$v(t,x) = B_0 + B_1\left(\frac{1}{1+\cosh(z)+\sinh(z)}\right) + B_2\left(\frac{1}{1+\cosh(z)+\sinh(z)}\right)^2,$$

where $z = kx + \omega t$ and the values of A_i and B_i, $i = 0, 1, 2$ are as given above.

9.2 Conservation laws of (9.1)

In this section we use the multiplier method to find conservation laws of the coupled BBM system (9.1), namely,

$$E_1 \equiv u_t + v_x + vu_x + uv_x - u_{txx} = 0,$$
$$E_2 \equiv v_t + u_x + v_x v + uu_x - v_{txx} = 0 \tag{9.8}$$

A conservation law of the system (9.8) is a space-time divergence such that

$$D_t T^t + D_x T^x = 0 \tag{9.9}$$

holds for all solutions $(u(t,x), v(t,x))$ of the system (9.8). The vector $T = (T^t, T^x)$ is called the conserved vector of the system (9.8). We look for zero-order multipliers Λ_1 and Λ_2, that is, Λ_1 and Λ_2 depend on t, x, u, v. The multipliers Λ_1 and Λ_2 of the system (9.8) have the property that

$$\Lambda_1 E_1 + \Lambda_2 E_2 = D_t T^t + D_x T^x \tag{9.10}$$

for all functions $u(t,x)$ and $v(t,x)$ and D_t, D_x are the usual operators of total differentiation. The determining equations for the multipliers are obtained by solving the system

$$\frac{\delta}{\delta u}[\Lambda_1 E_1 + \Lambda_2 E_2] = 0, \tag{9.11a}$$
$$\frac{\delta}{\delta v}[\Lambda_1 E_1 + \Lambda_2 E_2] = 0, \tag{9.11b}$$

where $\delta/\delta u$ and $\delta/\delta v$ are the standard Euler-Lagrange operators.

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_x^2 D_t \frac{\partial}{\partial u_{xxt}} + \cdots \tag{9.12}$$

and

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} - D_x^2 D_t \frac{\partial}{\partial v_{xxt}} + \cdots, \tag{9.13}$$

respectively. Expanding system (9.11) yields an overdetermined system of partial differential equations. After solving this with the help of Maple, we obtain the two multipliers

$$\begin{aligned} \Lambda_1(t,x,u,v) &= C_1 v + C_2 u + C_4 \\ \Lambda_2(t,x,u,v) &= C_1 u + C_2 v + C_3 \end{aligned} \tag{9.14}$$

where C_1, C_2, C_3 and C_4 are constants. Hence we obtain the following four conserved vectors:

$$\begin{aligned} T_1^t =& 3uv - uv_{xx} + u_x v_x - vu_{xx}, \\ T_1^x =& 2u^3 + 3u^2 + 3v^2 - 4v_{tx}u + 2u_t v_x - 4vu_{tx} + 2v_t u_x + 6uv^2, \end{aligned}$$

$$\begin{aligned} T_2^t =& v_x^2 + u_x^2 - 2uu_{xx} - 2vv_{xx} + 3u^2 + 3v^2, \\ T_2^x =& 2v^3 - 4u_{tx} - 4v_{tx}v + 6u^2 v + 6vu + 2u_t u_x + 3v_t v_x, \end{aligned}$$

$$\begin{aligned} T_3^t =& 3v - v_{xx}, \\ T_3^x =& 6u + 3u^2 + 3v^2 - 4v_{tx}, \end{aligned}$$

$$
\begin{aligned}
T_4^t &= 3u - u_{xx},\\
T_4^x &= 3vu + 3v - 2u_{tx}.
\end{aligned}
$$

9.3 Conclusion

In this chapter we obtained the exact solutions of the coupled Benjamin-Bona-Mahony system (9.1) using Kudryashov method. The solutions obtained are travelling wave solutions. The conservation laws for the coupled BBM system (9.1) were also derived by using the multiplier method, which resulted into four conservation laws.

Chapter 10

Conservation laws and exact solutions of the convective Cahn-Hilliard Equation

In this chapter we obtain conservation laws as well as travelling wave solutions of the fourth-order convective Cahn-Hilliard (CCH) equation [113]

$$u_t + (u - u^3 + \varepsilon^2 u_{xx})_{xx} = 0. \tag{10.1}$$

Equation (10.1) is relevant to the study of phase transitions in alloys, glasses and polymers. The constant ε represents the width of the interfaces. Stationary solutions of this equation were obtained by an extension of the method of matched asymptotic expansions that retains exponentially small terms [113]. Several aspects in various forms of this equation including phase transitions, thermodynamics, natural boundary conditions, equilibrium solutions and their stability have been considered [114, 115].

The results of this chapter have been submitted for publication. See [116].

10.1 Conservation laws of (10.1)

In this section we construct conservation laws of the CCH equation (10.1) using the multiplier approach.

The CCH equation (10.1) can be written in expanded form as

$$E \equiv u_t + u_{xx} - 3u^2 u_{xx} - 6uu_x^2 + \varepsilon^2 u_{xxxx} = 0. \tag{10.2}$$

Now the determining equation for the zeroth-order multiplier $\Lambda(t, x, u)$ is

$$\frac{\delta}{\delta u}(\Lambda E) = 0, \tag{10.3}$$

where $\delta/\delta u$ is the usual Euler-Lagrange operator.

Expanding equation (10.3) and then splitting on derivatives of u leads to an overdetermined system of twelve determining equations. Solving the resultant system we obtain the multiplier

$$\Lambda .= C_1 x + C_2, \tag{10.4}$$

where C_1 and C_2 are arbitrary constants.

For the conserved vectors of (10.1) corresponding to this multiplier we have to solve for T^1 and T^2 in the equation

$$D_t T^t + D_x T^x = \Lambda E. \tag{10.5}$$

We obtain

$$T^t = (C_1 x + C_2) u \tag{10.6}$$

and

$$T^x = -5u_x (C_1 x + C_2) u^2 - \frac{1}{3} u \left(-3 C_1 u^2 - 6 (C_1 x + C_2) u u_x\right) - C_1 u$$

$$+ \quad u_x\,(C_1\,x + C_2) - \varepsilon^2 u_{xx}\,C_1 + \varepsilon^2 u_{xxx}\,(C_1\,x + C_2)\,, \tag{10.7}$$

where C_1 and C_2 are arbitrary constants. Thus corresponding to the above multiplier we obtain the following two local conserved vectors for (10.1);

$$\begin{aligned} T_1^1 &= x\,u, \\ T_1^2 &= -3\,x\,u^2 u_x + \varepsilon^2\,x\,u_{xxx} + u^3 - \varepsilon^2 u_{xx} + x\,u_x - u, \end{aligned}$$

$$\begin{aligned} T_2^1 &= u, \\ T_2^2 &= -3\,u^2 u_x + \varepsilon^2 u_{xxx} + u_x. \end{aligned}$$

10.2 Travelling wave solutions of (10.1)

First the Lie point symmetries of the CCH equation (10.1) are found using the Lie algorithm [24]. These Lie point symmetries are then used to transform (10.1) into an ordinary differential equation. The simplest equation method is then used to obtain solutions of the resultant ODE, and whence the exact solutions of the CCH equation.

10.2.1 Lie point symmetries and symmetry reduction of (10.1)

The symmetry group of the CCH equation (10.1) is generated by the vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{10.8}$$

Applying the fourth prolongation $X^{[4]}$ to (10.1) and solving the resultant overdetermined system of linear partial differential equations, we obtain the following two translation symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$

The linear combination of X_1 and X_2, namely, $X = X_1 + \nu X_2$, where ν is a constant, yields the following two invariants:

$$z = x - \nu t, \quad u = F(z). \tag{10.9}$$

Treating F as the new dependent variable and z as the new independent variable, the convective Cahn-Hilliard equation (10.1) transforms to a fourth-order nonlinear ODE

$$\varepsilon^2 F''''(z) + F''(z) - 3F(z)^2 F''(z) - 6F(z)(F'(z))^2 - \nu F'(z) = 0. \tag{10.10}$$

10.2.2 Exact solutions of (10.10) using simplest equation method

We now use the simplest equation method to solve the ODE (10.10).

The simplest equations that will be used here are the Bernoulli and Riccati equations. These are well-known nonlinear ordinary differential equations whose solutions can be expressed in terms of elementary functions.

The Bernoulli equation we consider here is

$$H'(z) = aH(z) + bH^2(z), \tag{10.11}$$

where a and b are constants. Its solution can be written as

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}.$$

The Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \tag{10.12}$$

where a, b and c are constants, allows the following two solutions:

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)},$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

Let us consider the solutions of (10.10) in the form

$$F(z) = \sum_{i=0}^{M} A_i(H(z))^i, \tag{10.13}$$

where $H(z)$ satisfies the Bernoulli or Riccati equations, M is a positive integer that can be determined by the balancing procedure and A_i, $(i = 0, 1, \cdots, M)$, are parameters to be determined.

Solutions of (10.1) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 1$, and so the solution of (10.10) is of the form

$$F(z) = A_0 + A_1 H. \tag{10.14}$$

Substituting (10.14) into (10.10) and making use of the Bernoulli equation (10.11) and then equating the coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_i, $(i = 0, 1)$. Solving this system with the aid of Maple, one possible set of values of the constants is

$$\begin{aligned} A_0 &= \frac{10}{9}mk^2 - \frac{7}{18}m + \frac{1}{18},\ A_1 = \frac{ma}{3b},\ \varepsilon = \frac{km}{3b}, \\ \nu &= -\frac{b}{36}\left(400k^4m^2 - 180k^2m^2 + 40k^2m - 7m^2 - 18m + 1\right), \end{aligned}$$

where m is any solution of the equation $3m^2 + 1 = 0$, and k is any solution of the equation $2k^2 - 1 = 0$. As a result a solution of the convective Cahn-Hilliard equation (10.1) using the Bernoulli equation as the simplest equation is

$$u(t,x) = A_0 + A_1 c \left\{ \frac{\cosh[c(z+C)] + \sinh[c(z+C)]}{1 - d\cosh[c(z+C)] - d\sinh[c(z+C)]} \right\}, \tag{10.15}$$

where $z = x - \nu t$, C is a constant of integration, A_0 and A_1 are as obtained above.

Solutions of (10.1) using Riccati equation as the simplest equation

The balancing procedure yields $M = 1$, and so the solution of (10.10) is of the form

$$F(z) = A_0 + A_1 H. \tag{10.16}$$

Proceeding in the same manner as above we obtain

$$\begin{aligned}
A_0 &= \frac{1}{18}\left(60kb\varepsilon - \frac{21b\varepsilon}{k} + 1\right), \\
A_1 &= \frac{a\varepsilon}{k}, \\
c &= \frac{k}{108\varepsilon(20k^2-9)}\left(3600k^2b^2\varepsilon^2 + 180b^2\varepsilon^2 + 120kb\varepsilon - \frac{963^2\varepsilon^2}{k^2} - \frac{60b\varepsilon}{k} + 1\right), \\
\nu &= -\frac{k}{162\varepsilon(20k^2-9)}\left(432000k^3b^3\varepsilon^3 - 356400kb^3\varepsilon^3 + 10800k^2b^2\varepsilon^2\right. \\
&\quad \left. + \frac{27540b^3\varepsilon^3}{k} - 14040b^2\varepsilon^2 + \frac{21330b^3\varepsilon^3}{k^3} + \frac{4320b^2\varepsilon^2}{k^2} - 1\right),
\end{aligned}$$

where k is any solution of the equation $2k^2 - 1 = 0$.

It follows that the solutions for the convective Cahn-Hilliard equation (10.1) using the Ricatti equation as the simplest equation are

$$u(t,x) = A_0 + A_1\left\{ -\frac{d}{2c} - \frac{\theta}{2c}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$$

and

$$u(t,x) = A_0 + A_1\left\{ -\frac{d}{2c} - \frac{\theta}{2c}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2c}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$$

where $z = x - \nu t$ with $\theta^2 = d^2 - 4ce$ and C is a constant of integration. The values of A_0 and A_1 are as given above.

10.3 Conclusion

In this chapter we studied the convective Cahn-Hilliard equation (10.1). The multiplier method was employed to construct the conservation laws, which resulted into two conservation laws. Lie Symmetry analysis along with the simplest equation method were used to find solutions of this equation. The solutions obtained are travelling wave solutions.

Chapter 11

Concluding remarks

The main objective of this work was to obtain exact solutions and conservation laws of some nonlinear partial differential equations of mathematical physics by using various methods. The exact solutions for these nonlinear partial differential equations play a vital role in the explanation of their physical phenomena.

In Chapter one, we provided the background to the study. This included definitions, theorems and the structures of methods that were used to carry out the various calculations in this work.

Chapter two presented a study of the generalized Benney-Luke equation with time dependent coefficients. A complete Lie group classification was performed on this equation. Equivalence transformations were used to simplify the forms of the symmetry operators. The principal Lie algebra consisting of two translation symmetries was found. The Lie group classification produced six different cases for the time dependent coefficients. Possible extensions of the principal Lie algebra were presented for each of the six cases. Exact group-invariant solutions in the two possible cases were calculated. In both cases, travelling wave solutions were obtained. Also, conservation laws were constructed for these two cases.

In Chapter three exact travelling wave solutions of the symmetric regularized long wave equation were computed using the simplest equation method and the exp-function method. Solutions obtained were travelling wave solutions. Conservation laws for this equation were also constructed using the multiplier approach.

A system formed by the Klein-Gordon-Zakharov equations was studied in Chapter four. Exact solutions for this system were found using the travelling wave variable approach and the simplest equation method. A profile of the solution produced via the travelling wave variable approach was also presented graphically.

Chapter five studied the exact solutions and conservation laws of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation. Exact solutions were obtained using direct integration and also by the $(G'/G)-$expansion method. The new conservation theorem due to Ibragimov was used to find conservation laws.

In Chapter six, the exact solutions and conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity were obtained using the Lie symmetry method along with Kudryashov method and the new conservation theorem due to Ibragimov, respectively.

Chapter seven studied a four parameter Boussinesq system. Noether approach was used to find the conservation laws of the system whereas the Kudryashov method was employed to obtain exact solutions of the system.

In chapter eight the exact solutions and conservation laws of an integrable coupling with Korteweg-de Vries equation were obtained. The $(G'/G)-$expansion method and multiplier approach were employed for finding the exact solutions and for the construction of conservation laws respectively.

Chapter nine presented the exact solutions and conservation laws for coupled Benjamin-Bona-Mahony equations. The Kudryashov method was used to obtain the exact solutions and the multiplier method was applied to derive the conservation laws of the system.

In Chapter ten, conservation laws for a convective Cahn-Hillard equation were constructed by applying the multiplier approach and exact solutions were obtained using the simplest equation method.

Future studies could focus on the stability of the exact solutions obtained in this research work. Furthermore, conservation laws obtained for the nonlinear differential equations could be used to construct more exact solutions.

Bibliography

[1] A.M. Wazwaz, A modified KdV-type equation that admits a variety of travelling wave solutions: kinks, solitons, peakons and cuspons, Phys. Scr. 86 (2012) 1–6.

[2] Z. Yan, F. Xie, H. Zhang, Symmetry reductions, integrability and solitary wave solutions to higher-order modified Boussinesq equations with damping term, Commun. Theor. Phys. 36 (2001) 1–6.

[3] M.L. Wang, Y.B. Zhou, Z.B. Li, Applications of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A 216 (1996) 67–75.

[4] J.L. Hu, Explicit solutions to three nonlinear physical models, Phys. Lett. A 287 (2001) 81–89.

[5] J.L. Hu, H. Zhang, A new method for finding exact traveling wave solutions to nonlinear partial differential equations, Phys. Lett. A 286 (2001) 175–179.

[6] S. Lou, J. Lu, Special solutions from variable separation approach: Davey-Stewartson equation, J. Phys. A: Math. Gen. 29 (1996) 4209–4215.

[7] M.J. Ablowitz, P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.

[8] C.H. Gu, Soliton Theory and Its Application, Zhejiang Science and Technology Press, Zhejiang, 1990.

[9] V.B. Matveev, M.A. Salle, Darboux Transformation and Soliton, Springer, Berlin, 1991.

[10] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.

[11] M.L. Wang, Z.B. Li, J.L. Zhang, The $(G'/G)-$expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008) 417–423.

[12] A. Bekir, E. Aksoy, Exact solutions of shallow water wave equation by using the $(G'/G)-$expansion method, Wave Random Complex 22 (2012) 1–15.

[13] L.X. Li, M.L. Wang, The $(G'/G)-$expansion method and travelling wave solutions for a higher-order nonlinear Schrödinger equation, Appl. Math. Comput. 208 (2009) 440–445.

[14] N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos Soliton Fract. 24 (2005) 1217–1231.

[15] N.A. Kudryashov, Exact solitary waves of the Fisher equation, Phys. Lett. A 342 (2005) 99–106.

[16] N.K. Vitanov, Application of simplest equations of Bernoulli and Riccati kind for obtaining exact travelling-wave solutions for a class of PDEs with polynomial nonlinearity, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010) 2050–2060.

[17] N.K. Vitanov, Z.I. Dimitrova, Application of the method of simplest equation for obtaining exact traveling-wave solutions for two classes of model PDEs

from ecology and population dynamics, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010) 2836–2845.

[18] N.A. Kudryashov, One method for finding exact solutions of nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012) 2248–2253.

[19] C.M. Khalique, On the solutions and conservation laws of a coupled Kadomtsev-Petviashvili equation, J. Appl. Math. 2013 (2013) 1–7.

[20] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.

[21] N.H. Ibragimov, Transformation Groups Applied to Mathematical Physics, Reidel, Dordrecht, 1985.

[22] H. Stephani, Differential Equations: Their Solutions using Symmetries, Cambridge University Press, Cambridge, 1989.

[23] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[24] N.H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 1–3, CRC Press, Boca Raton, Florida, 1994–1996.

[25] P.J. Olver, Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics, 2nd Edition, Springer-Verlag, Berlin, 1993.

[26] N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, John Wiley & Sons, Chichester, 1999.

[27] C. Qu, Classification and reduction of some systems of quasilinear partial differential equations, Nonlinear Anal. 42 (2000) 301–327.

[28] C. Qu, Symmetries and solutions to the thin film equations, J. Math. Anal. Appl. 317 (2006) 381–397.

[29] C. Qu, Y. Fub, Y. Liu, Well-posedness, wave breaking and peakons for a modified μ-Camassa-Holm equation, J. Funct. Anal. 266 (2014) 433–477.

[30] G.W. Bluman, A.F. Cheviakov, S. Anco, Applications of Symmetry Methods to Partial Differential Equations, Springer, New York, 2010.

[31] B. Muatjetjeja, C.M. Khalique, Conservation laws for a variable coefficient variant Boussinesq system, Abstr. Appl. Anal. 2014 (2014) 1–5.

[32] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. 21 (1968) 467–490.

[33] T.B. Benjamin, The stability of solitary waves, Proc. R. Soc. Lond. A 328 (1972) 153–183.

[34] R.J. LeVeque, Numerical Methods for Conservation Laws, 2nd Edition, Birkhäuser-Verlag, Basel, 1992.

[35] E. Godlewski, P. Raviart, Numerical Approximation of Hyperbolic Systems of Conservation Laws, Springer, Berlin, 1996.

[36] A. Sjöberg, Double reduction of PDEs from the association of symmetries with conservation laws with applications, Appl. Math. Comput. 84 (2007) 608–616.

[37] A.H. Bokhari, A.J. Al-Dweik, A.H. Kara, F.M. Mahomed, F.D. Zaman, Double reduction of a nonlinear (2 + 1) wave equation via conservation laws, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 1244–1253.

[38] G.L. Caraffini, M. Galvani, Symmetries and exact solutions via conservation laws for some partial differential equations of mathematical physics, Appl. Math. Comput. 219 (2012) 1474–1484.

[39] R. Naz, Z. Ali, I. Naeem, Reductions and new exact solutions of ZK, Gardner KP, and modified KP equations via generalized double reduction theorem, Abstr. Appl. Anal. 2013 (2013) 1–11.

[40] R. Tracina, M.S. Bruzón, M.L. Grandarias, M. Torrisi, Nonlinear self-adjointness, conservation laws, exact solutions of a system of dispersive evolution equations, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 3036–3043.

[41] E. Noether, Invariante variationsprobleme, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse 2 (1918) 235–257.

[42] E. Bessel-Hagen, Über die erhaltungssätze der elektrodynamik, Math. Ann. 84 (1921) 258–276.

[43] H. Steudel, Über die zuordnung zwischen invarianzeigenschaften und erhaltungssatzen, Zeit Naturforsch 17A (1962) 129–132.

[44] S. Anco, G.W. Bluman, Direct construction of conservation laws from field equations, Phys. Rev. Lett. 78 (1997) 2869–2873.

[45] S.C. Anco, G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications, European J. Appl. Math. 13 (2002) 545–566.

[46] S.C. Anco, G.W. Bluman, Direct construction method for conservation laws for partial differential equations. Part II: General treatment, European J. Appl. Math. 13 (2002) 567–585.

[47] A.H. Kara, F.M. Mahomed, Noether-type symmetries and conservation laws via partial Lagrangians, Nonlinear Dynam. 45 (2006) 367–383.

[48] N.H. Ibragimov, A new conservation theorem, J. Math. Anal. Appl. 333 (2007) 311–328.

[49] N.H. Ibragimov, Quasi-self-adjoint differential equations, Arch ALGA 4 (2007) 55–60.

[50] M.L. Gandarias, Weak self-adjoint differential equations, J. Phys. A: Math. Theor. 44 (2011) 262001.

[51] N.H. Ibragimov, Nonlinear self-adjointness and conservation laws, J. Phys. A: Math. Theor. 44 (2011) 432002.

[52] N.H. Ibragimov, E.D. Avdonina, Nonlinear self-adjointness, conservation laws and the construction of solutions of partial differential equations using conservation laws, Russian Math. Surveys 68 (2013) 889–921.

[53] M.L. Gandarias, C.M. Khalique, Symmetries, solutions and conservation laws of a class of nonlinear dispersive wave equations, Commun. Nonlinear Sci. Numer. Simulat. 32 (2016) 114–121.

[54] R. Tracinà, M.S. Bruzón, M.L. Gandarias, On the nonlinear self-adjointness of a class of fourth-order evolution equations, Appl. Math. Comput. 275 (2016) 299–304.

[55] T. Motsepa, C.M. Khalique, Conservation laws and solutions of a generalized coupled (2+1)-dimensional burgers system, Comput. Math. Appl. 74 (2017) 1333–1339.

[56] I.E. Mhlanga, C.M. Khalique, Travelling wave solutions and conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity, Malays. J. Math. Sci. 11 (S) (2017) 1–8.

[57] A.N. González, The Cauchy problem for Benney-Luke and generalized Benney-Luke equations, Differential Integral Equations 20 (2007) 1341–1362.

[58] T. Mizumachi, R.L. Pego, J.R. Quintero, Asymptotic stability of solitary waves in the Benney-Luke model of water waves, arXiv:1202.0450 (2012).

[59] O.F. Gözükızıl, Ş. Akçağıl, Travelling wave solutions to the Benney-Luke and the higher-order improved Boussinesq equations of Sobolev type, Abstr. Appl. Anal. 2012 (2012) 10, article ID 890574.

[60] J. Akter, A.M. Akbar, Exact solutions to the Benney-Luke equation and the Phi-4 equations by using modified simple equation method, Results Phys. 5 (2015) 125–130.

[61] J.R. Quintero, Existence and analyticity of lump solutions for generalized Benney-Luke equations, Revista Colombiana de Matemáticas 36 (2002) 71–95.

[62] J.R. Quintero, J.C.M. noz Grajales, Instability of solitary waves for a generalized Benney-Luke equation, Nonlinear Anal. 68 (2008) 3009–3033.

[63] J.J. Wang, L.T. Wang, K.D. Yang, Some new exact traveling wave solutions for the generalized Benney-Luke (gBL) equation with any order, Applied Mathematics 3 (4) (2012) 309–314.

[64] M.S. Bruzón, Classical and nonclassical symmetries of a generalized Benney-Luke Equation, Int. J. Mod. Phys. B 30 (2016) 11.

[65] J.R. Quintero, Nonlinear stability of solitary waves for a 2-d Benney-Luke equation, Discrete Contin. Dyn. Syst. 13 (2005) 203–218.

[66] J.R. Quintero, A remark on the Cauchy problem for the generalized Benney-Luke equation, Differential Integral Equations 21 (2008) 859–890.

[67] J. Li, Exact traveling wave solutions to 2D-generalized Benney-Luke equation, Appl. Math. Mech. (English Ed.) 29 (11) (2008) :1391–1398.

[68] S. Wang, G. Xu, G. Chen, Cauchy problem for the generalized Benney-Luke equation, J. Math. Phys. 48 (7) (2007) 073521.

[69] I.E. Mhlanga, C.M. Khalique, A study of a generalized BenneyLuke equation with time-dependent coefficients, Nonlinear Dyn. (2017) 1–10.
URL `https://doi.org/10.1007/s11071-017-3745-1`

[70] P.J. Olver, P. Rosenau, Group-invariant solutions of differential equations, SIAM J. Appl. Math. 47 (2) (1987) 263–278.

[71] T. Motsepa, C.M. Khalique, M.L. Gandarias, Symmetry analysis and conservation laws of the Zoomeron equation, Symmetry 9 (2017) 27.

[72] A.F. Cheviakov, GeM software package for computation of symmetries and conservation laws of differential equations, Comput. Phys. Comm. 176 (1) (2007) 48–61.

[73] F. Chand, A.K. Malik, Exact traveling wave solutions of some nonlinear equations using $(G'/G)-$expansion method, Int. J. Nonlinear Sci. 14 (2012) 416–424.

[74] J.H. He, X.H. Wu, Exp-function method for nonlinear wave equations, Chaos Soliton Fract. 30 (2006) 700–708.

[75] J.H. He, X.H. Wu, Exp-function method and its application to nonlinear equations, Chaos Soliton Fract. 38 (2008) 903–910.

[76] I.E. Mhlanga, C.M. Khalique, Exact solutions of the symmetric regularized long wave equation and the Klein-Gordon-Zakharov equations, Abstr. Appl. Anal. 2014 (2014) 1–7.

[77] T. Wang, J. Chen, L. Zhang, Conservative difference methods for the Klein-Gordon-Zakharov equations, Comput. Appl. Math. 205 (2007) 430–452.

[78] Q. Shi, Q. Xiao, X. Liu, Extended wave solutions for a nonlinear Klein-Gordon-Zakharov system, Appl. Math. Comput. 218 (2012) 9922–9929.

[79] B.L. Guo, G.W. Yuan, Global smooth solution for the Klein-Gordon-Zakharov equations, J. Math. Phys. 36 (8) (1995) 4119–4124.

[80] T. Ozawa, K. Tsutaya, Y. Tsutsumi, Normal form and global solutions for the Klein-Gordon-Zakharov equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (4) (1995) 459–503.

[81] K. Tsutaya, Global existence of small amplitude solutions for the Klein-Gordon-Zakharov equations, Nonlinear Anal. 27 (12) (1996) 1373–1380.

[82] G. Adomian, Non-perturbative solution of the Klein-Gordon-Zakharov equation, Appl. Math. Comput. 81 (1) (1997) 89–92.

[83] Y. Shang, Y. Huang, W. Yuan, New exact traveling wave solutions for the Klein-Gordon-Zakharov equations, Comput. Math. Appl. 56 (2008) 1441–1450.

[84] L. Chen, Orbital stability of solitary waves for the Klein-Gordon-Zakharov equations, Acta. Math. Appl Sin. (English Ser.) 15 (1) (1999) 54–64.

[85] I.E. Mhlanga, C.M. Khalique, On the exact solutions of the Klein-Gordon-Zakharov equations, in: M. Cojocaru, I.S. Kotsireas, R.N. Makarov, R. Melnik, H. Shodiev (Eds.), Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science, Vol. 117, Springer International Publishing, Switzerland, 2015, pp. 301–307.

[86] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, 7th Edition, Academic Press, New York, 2007.

[87] B. Kojok, Sharp well-posedness for Kadomtsev-Petviashvili-Burgers (KPBII) equation in R2, J. Differential Equations 242 (2007) 211–247.

[88] A.M. Wazwaz, Multiple-front solutions for the Burgers-Kadomtsev-Petviashvili equation, Appl. Math. Comput. 200 (2008) 437–443.

[89] B. Ren, J.Y. Wang, Full symmetry groups and exact solutions to BKP and GKP equations, Physics Research International 2014 (2014) 4, article ID 786965.

[90] C. Chun, R. Sakthivel, New soliton and periodic solutions for two nonlinear physical models, Z. Naturforsch. 65a (2010) 1049–1054.

[91] I.E. Mhlanga, C.M. Khalique, Travelling wave solutions and conservation laws of a generalized (2+1)-dimensional Burgers-Kadomtsev-Petviashvili equation, in: T. Simos, C. Tsitouras (Eds.), Proceedings of the International Conference on Numerical Analysis and Applied Mathematics 2016 (ICNAAM-2016), Vol. 1863, AIP Conference Proceedings, 2017, p. 280006.

[92] K.R. Adem, C.M. Khalique, Conservation laws and traveling wave solutions of a generalized nonlinear ZK-BBM equation, Abstr. Appl. Anal. 2014 (2014) 5, article ID 139513.

[93] T.E. Sayed, A.E. Danaf, Numerical solution of the Korteweg-de Vries-Burgers equation by using quintic spline method, Stud. Univ. Babeş-Bolyai Math. XLVII (2) (2002) 41–54.

[94] O. Ersoy, I. Dağ, A. Korkmaz, Solitary wave simulations of the Boussinesq systems, arXiv:1605.04901 (2016).

[95] J.L. Bona, M. Chen, J.C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory, J. Nonlinear Sci. 12 (2002) 283–318.

[96] R. Naz, F.M. Mahomed, T. Hayat, Conservation laws for third-order variant Boussinesq system, Appl. Math. Lett. 23 (2010) 883–886.

[97] E. Yaşar, I.B. Giresunlu, Lie symmetry reductions, exact solutions and conservation laws of the third order variant Boussinesq system, Acta Phys. Polon. A 128 (3) (2015) 243–252.

[98] V.A. Dougalis, A. Durán, M.A. López-Marcos, D.E. Mitsotakis, A numerical study of the stability of solitary waves of the Bona-Smith family of Boussinesq systems, J. Nonlinear Sci. 17 (2007) 569–607.

[99] D.C. Antonopoulos, D.C. Dougalis, D.E. Mitsotakis, Galerkin approximations of periodic solutions of Boussinesq systems, Bull. Greek Math. Soc. 57 (2010) 13–30.

[100] I.E. Mhlanga, C.M. Khalique, Noether symmetries, conservation laws and exact solutions of a four parameter Boussinesq system, Submitted for publication in Results Phys.

[101] D.J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, Phil. Mag. 39 (1895) 422–443.

[102] W.X. Ma, B. Fuchssteiner, The bi-Hamiltonian structure of the perturbation equations of the KdV hierarchy, Phys. Lett. A 213 (1996) 49–55.

[103] A.M. Wazwaz, Integrability of coupled KdV equations, Cent. Eur. J. Phy. 9 (2011) 835–840.

[104] A.M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Springer, New York, 2009.

[105] D.S. Wang, Integrability of a coupled KdV system: Painlevé property, Lax pair and Bäcklund transformation, Appl. Math. Comput. 216 (2010) 1349–1354.

[106] C.X. Li, A hierarchy of coupled Korteweg-de Vries equations and the corresponding finite-dimensional integrable system, J . Phys. Soc. Jpn. 73 (2004) 327–331.

[107] Z. Qin, A finite-dimensional integrable system related to a new coupled KdV hierarchy, Phys. Lett. A 355 (2006) 452–459.

[108] A.M. Wazwaz, The tanh and sine-cosine method for compact and noncompact solutions of nonlinear Klein Gordon equation, Appl. Math. Comput. 167 (2005) 1179–1195.

[109] I.E. Mhlanga, C.M. Khalique, Exact solutions and conservation laws of an integrable coupling with Korteweg-de Vries equation, Submitted for publication in Open Phys.

[110] Y. Zhao, Q. Xu, New exact travelling wave solutions for some coupled BBM equations, Applied Mathematics 5 (2015) 1508–1515.

[111] L.W. Cui, Y. Zhao, Orbital stabilityof solitary waves for coupled BBM equations, Adv. Math. (China) 41 (2012) 341–346.

[112] I.E. Mhlanga, C.M. Khalique, Exact solutions and conservation laws for the coupled Benjamin-Bona -Mahony equations, in: T. Simos, C. Tsitouras (Eds.), Proceedings of the International Conference on Numerical Analysis and Applied Mathematics 2017 (ICNAAM-2017), Accepted and to appear in AIP Conference Proceedings.

[113] M.D. Korzec, P. Evans, A. Munch, B. Wagner, Stationary solutions of driven fourth- and sixth-order Cahn-Hilliard-type equations, J. Appl. Math. 69 (2008) 348–374.

[114] S.R. Choudhury, General similarity reductions of a family of Cahn-Hilliard equations, Nonlinear Anal. 24 (1995) 131–146.

[115] A. Novick-Cohen, L.A. Segel, Nonlinear aspects of the Cahn-Hilliard equation, Physica D 10 (1984) 277–298.

[116] I.E. Mhlanga, C.M. Khalique, Conservation laws and exact solutions of the convective Cahn-Hilliard Equation, Submitted for publication in Comput. Math. Appl.