



Constraining Interacting Dark Energy Models with Cosmological Data

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Declaration

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Abstract

In this dissertation, the background of cosmology has been investigated in the framework of the non-gravitational interaction between dark matter and dark energy models (IDE), with the latest cosmological data. Although the initial motivation for the IDE models was to alleviate the coincidence problem, recently, the models have become popular to emphasise the discrepancy between the Hubble measurements from the late- and early-cosmological measurements. We take into account the linear phenomenological models namely: $Q = \xi H \rho_{dm}$ (IDE1), $Q = \xi H \rho_{de}$ (IDE2), and $Q = 3\xi H(\rho_{dm} + \rho_{de})$ (IDE3); and non-linear models such as $Q = 3H\xi \left(\frac{\rho_{dm}\rho_{de}}{\rho_{dm} + \rho_{de}} \right)$ (IDE4) and $Q = 3H\xi \sqrt{\rho_{dm}\rho_{de}}$ (IDE5) to explore the deviation of each model from the Λ CDM model to systematically investigate the implications of alternative cosmological scenarios. Using baryonic acoustic oscillation data released from the dark energy spectroscopic instrument (*BAO DESI*), Pantheon+ Supernovae Type Ia combined with SH0ES Cepheid (*PantheonP + SH0ES*), and cosmic chronometer (*CC*) datasets, we constrain the cosmological parameters Ω_m, ξ, ω and H_0 for each model. Furthermore, we assess the viability of all models compared to Λ CDM by performing the statistical analysis through Bayesian evidence. The results indicate that one of our models, IDE4, remains a better candidate than the other IDE models. Followed by the IDE5, with potential to reduce the H_0 tension, making it the second better viable candidate.

Keywords: dark energy, dark matter, coincidence problem, Hubble tension, Λ CDM model

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Table of abbreviations

A table containing a list of abbreviations that will be used throughout the text.

IDE	Interacting Dark Energy
GR	General Relativity
ΛCDM	Lambda Cold Dark Matter
DE	Dark Energy
DM	Dark Matter
FLRW	Friedmann-Lemaître-Robertson-Walker
CMB	Cosmic Microwave Background
WL	Weak Lensing
SN Ia	Supernova Type Ia
DESI	Dark Energy Spectroscopic Instrument
BAO	Baryonic acoustic oscillation
LSS	Large-Scale Structure
CC	Cosmic Chronometer
MCMC	Markov Chain Monte Carlo

Introduction

Cosmology is the study of the origin, structure, evolution, and the eventful fate of the Universe. This study is driven by human curiosity to know more about the Universe, as they have been looking at the heavens in search of answers to deep questions about life. Questions relate to “*Where are we from? Where are we going? What are we? Does the Universe have a beginning? Will the Universe have an end? Is the Universe infinite? How did we get here? Are we special?*” and many more. Therefore, cosmology aims to answer these fundamental questions by exploring the history of the Universe, understanding its present condition, and predicting its future. In recent years, the extensive release of accurate astronomical data has opened avenues for answering questions about modern cosmology. Its level of precision enables us to evaluate cosmological models and accurately determine cosmological parameters [1, 2]. One model, namely the Λ CDM adequately explains most observations that there currently are, with the cosmological constant (Λ) explaining the observed acceleration of the Universe and cold dark matter (CDM) generating gravitational wells that align with observations on both galactic and cosmological scales. For instance, the observation of supernova type Ia (SNIa) made researchers aware in 1998 of such expansion and acceleration of the Universe after the unexpected faintness in the data [3–7]. Better yet, most independent observations, including cosmic background microwave (CMB) [8], baryon acoustic oscillations (BAO), weak lensing (WL) and many more, confirmed the fact that the Universe indeed is expanding [9–11].

The aforementioned model is said to successfully explain the observations and also provides the theoretical frameworks for modern galaxy formation [12, 13], but the model suffers from certain challenges, for example the cosmological coincidence problem, the constant problem [14, 15], the H_0 tension, and the σ_8 tension. The first refers to the fact that the measured vacuum energy density is more than 120 orders of magnitude lower than the theoretical predictions. The second alludes that the densities of dark matter and dark energy are comparable at present, but differ by orders of magnitude in the past and the predicted future [12, 16]. The third concerns the 5σ difference between the value of the Hubble constant H_0 , measured from the cosmic microwave background (CMB) and the value obtained from Type

Ia supernovae using a calibrated local distance ladder [16–18]. The last concerns the 3σ difference between the CMB measurements, the weak lensing, and the redshift studies of the parameter σ_8 , which quantifies the amplitude of fluctuations in matter and structure growth at later stages [19, 20]. These challenges open suggestions for alternative models to investigate other than the Λ CDM model.

One of the alternatives is to consider the interaction between dark energy and dark matter, which has been intensively studied in the literature [21–29]. In which the two are effectively modeled as perfect cosmological fluids, in the context of [Friedmann-Lemaître-Robertson-Walker metric](#). Given that dark energy and dark matter make up substantial portions of the Universe’s contents, it is reasonable, within the context of field theory, to examine a potential interaction between the two. Therefore, the existence of dark energy and dark matter interactions is an observational endeavour that has the potential to offer valuable insight into the nature of the dark sectors when determined. Although the two are currently unknown and the precise form of interaction cannot be derived from the principles of physics, it is wise to find the existence from phenomenological considerations [12, 16, 26, 30]. As a consequence, phenomenological interactions have been studied in the literature with different dark energy models. In particular, Dong et al. investigated two types of interactions ($Q_1 = 3\gamma\rho_{de}$ and $Q_2 = 3\gamma\rho_c$), using the latest cosmological data, which includes *CMB*, *BAO*, *LSS*, *SN Ia*, $H(z)$ and H_0 . They found that H_0 is still in tension with both the models considered and there is no significant preference for the interaction between dark matter and dark energy [31]. Furthermore, Muhsin A. et al. investigated these phenomenological interactions along with ($Q_3 = 3Hb(\rho_{de} + \rho_m)$, $Q_4 = 3Hb(\rho_{de} - \rho_m)$), $Q_5 = 3Hb\sqrt{\rho_{de}\rho_m}$, $Q_6 = 3Hb\frac{\rho_{de}\rho_m}{\rho_{de}+\rho_m}$) using only cosmic chronometer data, in which they came to the conclusion that their results are reliable, as they investigated many possible different forms of the interaction term beyond the first-order linear approximation by allowing it to depend on several combinations of the dark energy [21]. Recently, some developments considering new forms of non-gravitational and nonlinear interaction have been proposed [31, 32]; therefore, we are interested in exploring whether non-linear models can better explain the challenges faced by the Λ CDM than linear models.

The focus of this dissertation is on the observational constraints of interacting dark energy models, specifically examining how these models can account for various astrophysical phenomena and cosmological observations. We are exploring multiple specific models of interacting dark energy, such as the linear interacting dark energy models, that involve the linear interaction between dark energy and dark matter, enabling straightforward analyses of the impact on the Universe’s expansion and structure formation. Additionally, we investigate the non-linear interacting dark energy models that introduce more intricate interactions, leading to more dynamic behaviour and potentially different evolutionary patterns in the Universe. The goal

is to establish constraints on these interacting models by integrating the latest observational data, including the cosmic chronometer data [33], the dark energy spectroscopic instrument (DESI) *BAO* data [34] and the recently released supernova Pantheon+ combined with SH0ES data of 1701 light curves of 1550 distinct SNIa ranging in redshift from $z = 0.001$ to 2.26 [35]. We aim to evaluate the potential of these models to account for the accelerating expansion of the Universe and to enhance our comprehension of the fundamental components of the Universe by qualitative and quantitative comparison with observational data. Finally, check the viability of each IDE model through statistical analysis.

The outline of this dissertation is as follows: The first chapter covers the introduction and key concepts in cosmology, providing an overview of what the Universe consists of. Chapter 2 explains the challenges faced by the standard cosmological model and further outlines possible solutions. In Chapter 3, we introduce the IDE models of the interaction between dark energy and dark matter. The methodology and data are given in detail in Chapter 4. Then, in Chapter 5 we provide the observational constraints and the statistical analyses. Finally, we present our conclusions and provide plans for future research in Chapter 6.

Chapter 1

General relativity: a key to modern cosmology

General relativity (GR) is one of the most influential theories in modern physics. The theory was developed by Albert Einstein in an approximately eight-year period, in which he presented his work in 1915 to the Prussian Academy of Sciences as recognised today. The theory imposes a different view of space-time that we inhabit; instead of matter moving through a passive space-time continuity, the theory asserts that the presence of matter distorts space-time. The distortion causes the deflection of material particles and light from their classical paths. In his paper he explains the deflection of light by the Sun, which was the second test he proposed. The idea had been in discussion since the 1800s about how light may bend around a massive object under Newtonian mechanics, and a deflection angle for the Sun was calculated to be approximately 0.875 arc-seconds by Newtonian dynamics. Only few people understood the theory, Arthur Eddington being one of them, and he organised an expedition to the Island of the Principe in 1919, which photographed the star field around the Sun during the solar eclipse after Einstein predicted that starlight that passes the limb of the sun on the way to earth is deflected by measurable 1.750 arc seconds (one of Einstein's tests falsifying GR) [36, 37]. Him and the team confirmed this prediction, and their confirmation played an important role in gaining widespread acceptance of the GR theory, establishing it as a major breakthrough in physics. Since then, GR has endured numerous experimental tests with remarkable precision, securing its status as one of the foundational pillars of modern physics [38, 39].

As alluded above, GR exhibits differently in that gravitation is an effect of the curvature of space-time itself. At the core of this theory are a set of equations called the Einstein equations, the equations fundamentally relates the geometry of space-time to the energy and momentum within it. It is important to note the equivalence

principle as is the foundation of the theory before we undergo the mathematical framework, as it was one of Einstein’s quests in achieving the consistency of the theory of gravity.

1.1 The equivalence principle

GR has fundamentals, and one of them is the equivalence principle. It is one of the foundational aspects of GR and is commonly stated in three different distinct forms, varying based on the conditions and limitations applied. The most basic and least restrictive equivalence principle is appropriately named the weak equivalence principle, also called **the universality of free fall**. Performing this principle, one can say that the inertial and gravitational masses of any object are exactly equal. This is an indication that the acceleration of a test body due to gravity is independent of the mass, composition, and structure of the test body, and therefore so is the trajectory of a test body, where the test bodies are considered to be subject to gravity alone and not to any additional forces [40]. An example is the fall to Earth by Apollo 15 [41]. A key consequence of the weak equivalence principle is then that the local effects of motion of test bodies subject to a gravitational field are the same as those in a frame that is being uniformly accelerated.

We also have the other principle called the **Einstein equivalence principle** which takes the weak principle on and further states that in a laboratory that is free-falling, the results of local, non-gravitational, experiments are independent of the location and velocity of that laboratory; this means that one cannot infer the existence of a gravitational field through local, non-gravitational, experiments [42]. The Einstein equivalence principle then suggests that gravity itself should be seen as an effect of the curvature of space-time [43]. Finally, the most stringent equivalence principle is called the **strong equivalence principle**, which further broadens the scope of the laws of physics such as quantum mechanics and electromagnetism, which are the same in all freely falling reference frames, despite their location in the Universe [44, 45].

1.2 Einstein’s field equations

Up to this point we have seen how GR can appear when considering the equivalence principle and the need for a geometric explanation of gravity; let us discuss the mathematical framework required to express this physical theory of gravity. Let us recall from “*Introduction to General Relativity*” textbook [43] and [46] that a type (m,n) tensor T is a multiplier map from a collection of m dual vectors to the set of

the real numbers $\mathbf{R}$:

$$T = T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_m} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_m} dx^{\nu_1} \otimes \dots \otimes dx^{\nu_m} , \quad (1.1)$$

where $\otimes$ is the tensor product, $\partial_\mu = \partial/\partial x^\mu$ and dx^ν are the basis of the vector space and dual vector space, respectively. Coordinates transformation of a tensor T then has the following components

$$T_{\nu'_1 \dots \nu'_m}^{\mu'_1 \dots \mu'_m} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\mu'_m}}{\partial x^{\mu_m}} \dots \frac{\partial x^{\nu'_1}}{\partial x^{\nu_1}} \dots \frac{\partial x^{\nu'_m}}{\partial x^{\nu_m}} T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_m} . \quad (1.2)$$

Noting that the tensor product is symmetric [46]. The operation of tensors includes the tensor product, addition, subtraction of tensors of the same type, and contraction of tensors. Therefore, a tensor is considered symmetric in a pair of indices if it remains unchanged when those indices are swapped. Conversely, antisymmetric if the sign of the tensor changes under an exchange of indices. For instance, in a type (2,0) tensor, this translates to $T^{\mu\nu} = -T^{\nu\mu}$ for a symmetric tensor and $T^{\mu\nu} = -T^{\nu\mu}$ for an antisymmetric tensor. Here we look at $g_{\mu\nu}$ as it plays a pivotal role in GR, as seen how it replaces the flat Minkowski $\eta_{\mu\nu}$ in special relativity to now provide a metric, and thereby define a line element, for a curved space-time as;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \quad (1.3)$$

where $g_{\mu\nu}$ are the components of the metric tensor, the indices μ, ν are determined to the extent that x^0 represents the time coordinate and x^i , with $i = 1, 2, 3$, represent the spatial coordinates. The metric tensor is a type (0,2) symmetric tensor. We consider Einstein's convention in which repeated indices are summed ¹.

Remember, we are looking for the Einstein field equations. Firstly, we need to know about the curvature of a manifold, and in doing that, we need to add the most fundamental tensor, called the Riemann tensor. But to do that, we similarly need to relate vectors of tangent spaces that are close to each other, in order to get a connection. It is fundamental to define the covariant derivative, which generalises the partial derivative to a curved space-time. The connection coefficients appear in the definition of the covariant derivative, and in GR we require a torsion-free and metric compatible connection, which we say are conditions that leads to a specific connection named the Christoffel connection, with the connection coefficients given by the Christoffel symbol,

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\nu} \left[\frac{\partial g_{\alpha\mu}}{\partial x^\beta} + \frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right] , \quad (1.4)$$

where $g^{\mu\nu}$ is the inverse of the metric tensor such that $g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$, with δ_ν^μ being the Kronecker delta, defined as zero unless $\mu = \nu$, in which it is equal to 1. Note that

¹we adopt the mostly positive $(-, +, +, +)$ signature for the metric unless specified otherwise

the Christoffel symbol is not a tensor; this can be seen by applying the coordinate transformation, so for a general tensor we use the covariant derivative ∇_μ given as

$$\nabla_\alpha T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_m} = \partial_\alpha T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_m} + \sum_{k=1}^m \Gamma_{\alpha\lambda}^{\mu_k} T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \lambda \dots \mu_m} - \sum_{k=1}^n \Gamma_{\alpha\nu_k}^\lambda T_{\nu_1 \dots \lambda \dots \nu_n}^{\mu_1 \dots \mu_m} . \quad (1.5)$$

This allows us to define the parallel transport of a tensor along the path $x^\mu(\lambda)$ to be

$$\frac{dx^\alpha}{d\lambda} \nabla_\alpha T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_m} = 0 , \quad (1.6)$$

where λ is an affine parameter. Therefore, the parallel transport of the tangent vector $V^\mu = dx^\mu/d\lambda$ relates to the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 . \quad (1.7)$$

These equations can be seen as a generalisation of Newton's second law for a test particle with zero net force acting upon it. Describing how a free-falling test particle moves through curved spacetime in GR. Now we can introduce the most imperative tensor, called the Riemann, as mentioned above. The tensor can be motivated in several ways; one of the most popular is by considering the parallel transport of a vector V^μ around a closed infinitesimal loop in curved space. This leads to equation $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu = R_{\alpha\mu\nu}^\beta V^\alpha)$, where the Riemann tensor is identified as

$$R_{\mu\nu} = \frac{\partial \Gamma_{\alpha\beta}^\mu}{\partial x^\alpha} - \frac{\partial \Gamma_{\alpha\beta}^\mu}{\partial x^\beta} + \Gamma_{\alpha}^\mu \Gamma_{\mu\nu}^\nu - \Gamma_{\alpha\nu}^\mu . \quad (1.8)$$

The equation 1.8 satisfies many properties such that;

$$\begin{aligned} R_{\beta\alpha\mu\nu} &= g_{\beta\sigma} R_{\alpha\mu\nu}^\sigma , & R_{\beta\alpha\mu\nu} &= -R_{\alpha\beta\mu\nu} , \\ R_{\beta\alpha\mu\nu} &= R_{\beta\alpha\nu\mu} , & R_{\beta\alpha\mu\nu} &= R_{\mu\nu\beta\alpha} , \end{aligned} \quad (1.9)$$

therefore, we see that summing up cyclic permutations of the last three indices results in the Bianchi identity:

$$R_{\beta\alpha\mu\nu} + R_{\beta\nu\mu\alpha} + R_{\beta\mu\alpha\nu} = 0 . \quad (1.10)$$

Now, it is clear that the Riemann tensor possesses certain symmetries that significantly reduce its number of independent components. Although it initially appears to have 256 components, these symmetries include antisymmetry in its last two and first two indices. These reduces 20 independent components in four dimensions. Therefore, this identity is not the only one useful in this position because we also have the Bianchi identity, which turns out to be very useful, and it results from permutations of the covariant derivative of the Riemann tensor as

$$\nabla_\sigma R_{\beta\alpha\mu} + \nabla_\beta R_{\alpha\sigma\mu\nu} + \nabla_\alpha R_{\sigma\beta\mu\nu} . \quad (1.11)$$

Contracting equation 1.8 we obtain the Ricci tensor to be

$$R_{\mu\nu} = R_{\mu\beta\nu}^\beta . \quad (1.12)$$

The Ricci tensor has indices and is also symmetric, with 10 independent components. We then continue to find the Ricci scalar by combining the Ricci curvature and the FLRW contra-variant $g^{\mu\nu}$ and

$$R = g^{\mu\nu} R_{\mu\nu} . \quad (1.13)$$

Now, when we go on to contract equation 1.10, which is the Bianchi identity we get,

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0 , \quad (1.14)$$

this can be written in the form of Einstein tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} , \quad \nabla^\mu G_{\mu\nu} = 0 , \quad (1.15)$$

where $G_{\mu\nu}$ represents the Einstein tensor. Very close to getting the Einstein field equation but without the energy-momentum we arrive at the vacuum field equations given as

$$G_{\mu\nu} = 0 . \quad (1.16)$$

On the cosmological scales, matter, radiation and dark energy which are the constituents of the Universe can be treated as a fluid. The most general energy-momentum tensor for a fluid component $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = \rho(t) u_\mu u_\nu + p(t) (u_\mu u_\nu + g_{\mu\nu}) , \quad (1.17)$$

since the cosmological principle holds for the FLRW metric, we say it is a perfect fluid. Then the perfect fluid stress-energy tensor $T^{\mu\nu}$ can be obtained as 1.14. Since we are not focusing on perturbations in this dissertation, the fluid is assumed to be at rest in the co-moving coordinate system; therefore, only a temporal component of the four-velocity takes the form $u^\mu = (1, 0, 0, 0)$. This corresponds to that in [43], where ρ is the energy density, p is the fluid pressure, and u is the four velocity components.

Einstein initiated the development of modern cosmology [47] in his paper on the general theory of relativity. By this time, he had recognised the mutual influence of space-time and matter and realised that he had to start with the Einstein-Hilbert action in order to derive the equations of motion of the field. David Hilbert was the first to propose the action to calculate the Einstein field equations, whereas Einstein thought of the physics in bringing the equations to life. The equation takes the form:

$$A = \frac{c^4}{16\pi G} \int \sqrt{-g} \left[R - 2(\mathcal{L}'_m - \Lambda) \right] d^4x , \quad (1.18)$$

where the cosmological constant Λ is included and the Lagrangian $\mathcal{L}'_m$ describing the matter contained in the field, G is Newton's gravitational constant, and c is the speed of light. This action limits the two time-like points in space. Deriving the

Einstein field equations, taking into account the notation used in [48], equation 1.17 can be rewritten as

$$A = \int \sqrt{-g} \left[\frac{k}{2} R + (\mathcal{L}_m - k\Lambda) \right] d^4x . \quad (1.19)$$

Here, $k = \frac{c^4}{8\pi G}$, $\mathcal{L}_m = k\mathcal{L}'_m$, and the d^4x indicates that the integration takes place over all four dimensions of space-time. Therefore, the variational principle of least action with respect to the metric $g_{\mu\nu}$ becomes zero. Moreover,

$$\delta A = 0 , \quad (1.20)$$

and that yields,

$$\delta \int \sqrt{-g} \left[\frac{k}{2} R + (\mathcal{L}_m - k\Lambda) \right] d^4x = 0 . \quad (1.21)$$

Now, from the variational principle, we can obtain the Einstein equation;

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = k T_{\mu\nu} , \quad (1.22)$$

and

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_m}{\delta g^{\mu\nu}} = -2 \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_m , \quad (1.23)$$

is the stress energy momentum tensor (we followed [48] and [47] for full clarity). Equation 1.19 can be simplified as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} , \quad (1.24)$$

then, $G_{\mu\nu}$ is the Einstein tensor; therefore,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = k T_{\mu\nu} . \quad (1.25)$$

Finally, we reached finding the Einstein field equations (1.24). This gives us immunity to explore how GR can be applied to our Universe and how we arrive at the standard model of cosmology known as Λ cold-dark matter, and how the theory applies to the Universe as a whole. *“They say a scientist’s quest is a quest for truth, and so whatever diverse and elaborate theories a scientist may hold to be true must be confronted with the natural world”* [49]. To think we are still in the dark of *“whether the Universe has an end or not”*, this is a very difficult experience, but as an upcoming scientist we believe that remaining open to the new perspectives on the Universe our problems will eventually be our victories. Anyway, let us dive into how our model got here and is an eye-opener for most of us scientists.

1.3 Birth of standard cosmological model

The emergence of GR is widely considered the birth of modern cosmology. This means that if GR is valid on cosmological length scales, then the large-scale structure

and evolution of the Universe, the past, present, and future, are determined by [Einstein's field equations](#). Einstein applied his theory to the cosmos just 2 years after publication. In 1917 he proposed a solution to the field equations describing a closed and static Universe. The Λ added to the tensor of the Einstein equation above (equation [1.23](#)), which is the cosmological constant, by which if matter has a decelerating effect on expansion due to gravity, then Λ has an opposite 'repulsive effect', compared to Einstein's original static Universe model, the two effects cancel each other out in Einstein's solution [\[50\]](#).

According to Mash's principle, which states that the large-scale distribution of matter in the Universe determines the local frame of reference, this principle influenced Einstein in formulating GR theory. He actually believed that a new theory of gravitation incorporated this principle, particularly by adding the cosmological constant Λ , whereby no static solutions could be found in a matter-free Universe. After a short while, Willem de Sitter challenged the theory and found a well-defined solution to the field equations for a closed Universe with matter absent in the Universe, leading him to ask 'If no matter exists apart from the test body, has this inertia?' [\[51, 52\]](#). In 1922, Kornel Lanczos showed that the de Sitter, although originally presented as static, could in fact be interpreted as an expanding universe through a coordinate transformation. Then, in 1922, Alexander Friedmann published a paper in which he derived a general solution for the field equations describing an expanding curved Universe, of which the Einstein Universe and then the de Sitter Universe were special cases [\[53\]](#).

Einstein initially rejected Friedmann's solutions, believing that they contained a mathematical error; however, Friedmann corrected Einstein, showing him to have, in fact, made an error, and he then withdrew his objection publicly and realised that the error was his. A few years later, in 1927, Georges Lemaître derived Friedmann's solutions independently; he noted that there should be an approximately linear relationship between the recessional velocity and the distance of a distant galaxy to exist independently and noted that "the receding velocities of extragalactic nebulae are a cosmic effect of the expansion of the Universe.' This approximate linear relationship between the recessional velocity of a galaxy and its distance, known as Hubble's law, named after Edwin Hubble, who conducted foundational measurements of distant galaxies in 1929, unaware of the earlier theoretical work by Lemaître. He proposed that galaxies are receding from us, suggesting the expansion of the Universe. By means of observations, he established the velocity-distance relation; $v = H_0 d$. He approximated his constant as $500 \text{ kms}^{-1} \text{ Mpc}^{-1}$ after calculations [\[54\]](#). After all the efforts from scientists such as Friedmann and Lemaître to convince Einstein that the Universe is actually expanding, he continued to not believe it. However, his reluctance to accept was not rooted in personal vendetta or philosophical conviction, but was actually supported by observational data of the time. Despite the new

observational presentation to him, he insisted on not leaving his thought of a static model until Sir Eddington showed him that the static model is unstable [55], which is when he gave in and started to believe that it actually is expanding.

The Einstein-de Sitter Universe quickly became the dominant model of cosmology up to the end of the 20th century, because of its theoretical simplicity and consistency with observations at the time. This model resulted in a flat Universe composed of matter alone, meaning it assumed a universe with zero spatial curvature. The model was constructed by Einstein and de Sitter in 1932. However, this model faced a major challenge in the late 1990s, when observations on Supernovae type Ia started to be of serious doubt and demanded the reintroduction of the long-abandoned cosmological constant Λ , giving rise to the current standard model of cosmology, the Λ CDM model.

1.3.1 The Friedmann-Lemaître-Robertson-Walker metric

Recall the [Einstein field equations](#) above. Looking at the left side of the equation (1.24), this can be written as a metric $g_{\mu\nu}$. Therefore, building a cosmological model requires some assumptions. That is where human curiosity comes in with all the questions about the Universe. Copernicus took these questions into consideration and conducted research that led him to believe that there is nothing special about our location in the Universe. The publication of Newton's Principles in 1687 complemented his theory, in which he showed that the dynamics of objects on Earth and the heavens above are dictated by the same set of observable natural laws [16, 56].

Therefore, modern cosmology is based on a fundamental axiom, called the cosmological principle. This explains how tedious our position in the Universe is, noting that the Universe is homogeneous and isotropic at large scales (≥ 100 Mpc). Homogeneous simply means that the Universe has the same properties everywhere in space, and isotropic means that the Universe looks the same in every direction [43, 57, 58]. This makes cosmology concerned with the physical properties of the entire Universe. Averaging on a small scale, fortunately for the physicist, implies that only gravity needs to be taken into account. All other fundamental forces (electromagnetism, strong, weak) are limited to the scale of particle physics, or charges aligned with neutrally charged objects, and none of these effects occur with gravity. This is why cosmology applies the theoretical description of gravity, GR, to the Universe.

We assume that the constituents of the Universe present the properties of the cosmological principle. Therefore, in determining the line element of the model, the underlying symmetries are used. In this, we enforce the fundamentals of spherical

symmetry;

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \quad (1.26)$$

where t is the cosmic time followed by the spatial spherical coordinates (r, θ, ϕ) , which are not time-dependent, $a(t)$ is the scale factor responsible for the size of the Universe relative to the present day ($a(t_0) = a_0 = 1$), k denotes a constant that tells the geometry of the Universe ($k = +1, 0, -1$), when $k = 0$ the spatial section is closed, while for $k = 0$ it is flat and $k = -1$ means that the spatial section is open and r is the comoving radial distance. The metric was first derived by Friedmann in 1922 [59] and then more generally by Robertson in the late 1920s and early 1930s and Walker in 1935.

It can be seen that the metric is isotropic and homogeneous because it has only diagonal entries and does not have a different direction in space and time, and also the spatial point of the metric has the same form as a three-sphere in four-dimensional Euclidean space [43, 58]. The equation metric 1.24 will be used to evaluate the equation Einstein' field equations to derive the Friedmann equations, and we will use the natural unit conversions ($c=\hbar=k_B=1$), i.e. the speed of light, the reduced Planck constant and the Boltzmann constant, throughout this study.

With the understanding we have from all the information above about the Universe expanding, let us explore the phenomenon that is behind this expansion. A photon loses energy with the expansion of the Universe, scaling as $E \propto a^{-1}$ with the energy E , and the wavelength of a photon grows with expansion, thereby experiencing a redshift in our measures of time and distance. The relationship between the observed redshift to a photon at the present time and the scale factor at the time of emission becomes $1 + z = \frac{1}{a}$.

Returning to equation 1.23, we forget about the right side of the equation, which describes the energy-momentum content of space-time. With appropriate definitions and specifying the energy species, we will find [Friedmann's equations](#) and further find the continuity equations governing the dynamical evolution of the Universe.

1.3.2 Friedmann's equations

Using the FLRW metric equation (1.23) and the energy-momentum tensor equation (1.19) for all components with the assumption of a perfect fluid, the Einstein equations result in two independent equations. The time-time component gives the Friedman equation

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \quad (1.27)$$

and the space-space component results in

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (3p + \rho) + \frac{\Lambda}{3} . \quad (1.28)$$

The equation is called the Raychaudhuri equation. Both equations (1.21) and (1.22) are called Friedman equations relating the expansion of the Universe to the energy density ρ and the pressure p of the components of the Universe. Λ denotes the cosmological constant; the scale factor here defines the Hubble rate ($H = \frac{\dot{a}}{a}$) as $H = \frac{da}{dt} \frac{1}{a}$, where the H today is denoted as H_0 ², often parameterised in another parameter h as $H_0 = 100h \text{ kms}^{-1} \text{ Mpc}^{-1}$.

Raychaudhuri equation is related to the Friedman equation, we now need to solve 1.21, but we first have to know the relationship between energy density and pressure for a system. By doing that, we use the equation of state and with the first law of thermodynamics to the Universe as a whole given as;

$$\Delta U = \mathcal{Q} + W , \quad (1.29)$$

where it expresses the total energy in an isolated system is constant. ΔU is the change in the internal energy of the system, $\mathcal{Q}$ is the flow of heat out of the system, and W is the work done by the system [60]. The work is equal to the product of the pressure P of the system and its change in volume ΔV . Looking at the fact that we work in an expanding Universe we see that $W = -\Delta V P$. However, assuming a spherical symmetric Universe, the expansion of the Universe models is commonly modelled as an adiabatic system, which means that there is no heat exchange with the surroundings. Therefore,

$$\mathcal{Q} = 0 , \quad (1.30)$$

now, substituting W into equation 1.29, we get that

$$\Delta U + \Delta V P = 0 . \quad (1.31)$$

According to Dunning [61], this can be used to an expanding volume V of unit co-moving radius, and because in this dissertation we are considering the sphere with radius equivalent to scale factor a , it still stands. Therefore, we say the change in volume of a sphere can be written as:

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3} \pi a^3 \right) = 3VH . \quad (1.32)$$

Noting that mass (m) can be written in the form $m = \rho V$, where ρ is the energy density and V the volume, the equation becomes:

$$\dot{\rho} + 3H (\rho + p) = 0 . \quad (1.33)$$

²The subscript 0 will be used to indicate today's value of a variable throughout.

This equation describes how the energy density of the Universe evolves with time, and it is called the fluid conservation equation. It can be solved by first determining the relationship between pressure and density terms. In addition, assumptions of homogeneity made for the FLRW metric emphasises that both these terms have the same value everywhere in our Universe models $P = P(\rho)$. Therefore, introducing the equation of state (ω) for convenience [62], we get that

$$\omega = \frac{p}{\rho} . \quad (1.34)$$

After substituting 1.34 into 1.32, we get the conservation equation as;

$$\dot{\rho} + 3H\rho(1 + \omega) = 0 . \quad (1.35)$$

Equation 1.35 is a differential equation that can be solved by the separation of variables method. Moreover, integrating the equation gives us the energy density expression which tells us how the equation of state evolves with the scale factor and;

$$\rho_y = a^{-3(1+\omega_y)} \rho_{(y,0)} , \quad (1.36)$$

where the subscript denotes some fluid y , and $e^C = \rho_{(y,0)}$ is the present energy density of the fluid considered. The equation corresponds to that in [16, 48]. Here, for $\omega = 0$, this corresponds to pressure-less matter, $\omega = \frac{1}{3}$ corresponds to radiation and $\omega = -1$ corresponds to the cosmological constant (Λ).

Studying cosmology brings clarity in certain aspects regarding the Universe. Here, we explore questions of whether the Universe will continue to expand or will it eventually collapse, and whether it is finite or infinite. This takes us back to what the Universe looks like (shape), and therefore we deepen the discussions of the intrinsic geometry of the Universe. According to Einstein's field equation, mass-energy curves the fabric of space-time itself, and the geometry of the Universe is described by the energy density. Taking into account the Friedman equations, the

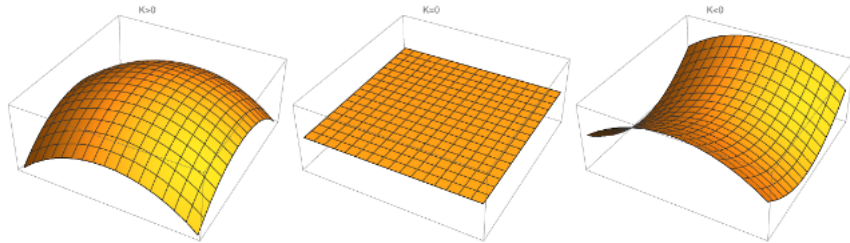


Figure 1.1: 2D examples of a closed Universe $k = 1$, an open Universe $k = -1$, and a flat Universe $k = 0$ [63].

sign of the curvature (k) determines the shape the Universe holds, and therefore negatively corresponds to a Universe that is continuously expanding, while a positive

value corresponds to a Universe where the direction of expansion is reversed and can end in a big crunch [16, 63]. The curvature can take on three possible values, positive ($k = 1$), no curvature ($k = 0$), or negative ($k = -1$) [58] as illustrated in Figure 1.1. Therefore, it can be seen that, for $\Lambda = 0$ there is a cosmological constant and for the present Hubble parameter H_0 there exists a density ρ_c such that the Universe is flat ($k = 0$) and can be obtained as

$$\rho_c = \frac{3H^2}{8\pi G}, \quad (1.37)$$

this equation is called the critical density equation, where G is Newton's gravitational constant ($= 6.67 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$) and H the Hubble constant which can be transformed into Hubble parameter $H_0 (= 67.4 \text{kms}^{-1} \text{Mpc}^{-1})$. This value corresponds to the value obtained in [14] to determine the critical density $\rho_{(c,0)}$, which can be computed with the constant values provided to be approximately $8.53 \times 10^{-27} \text{kg}$. Describing the ratio of energy density ρ to the critical density ρ_c , we introduce the fractional density parameter Ω and get that

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G \rho}{3H^2}, \quad (1.38)$$

and this leads to the relation between the density parameter Ω_y and the energy density ρ_x such that:

$$\Omega_y = \frac{8\pi G}{3H^2} \rho_y \quad \& \quad \rho_y = \frac{3H^2}{8\pi G} \Omega_y, \quad (1.39)$$

where Ω_y represents either matter (Ω_m), radiation (Ω_r) or dark energy (Ω_Λ) respectively. Furthermore, energy density ρ_y represents matter (ρ_m), radiation (ρ_r) or dark energy (ρ_Λ), respectively; therefore, substituting these equations into [Friedmann's equations](#) yields the normalised density parameters as

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda. \quad (1.40)$$

These parameters are not entirely independent, because for $\Lambda = 0$ ($\Omega_\Lambda = 0$) models, the Universe is geometrically closed, flat or open, depending on whether its total energy density, ρ , is greater than, equal to or smaller than the critical density, ρ_c . Therefore, if we know the values of two, we can find the other one, which is elucidated in the next section. We also call this equation the constraint equation. Now, using the relation between the energy density for the general fluid with the equation of state, we can write the dimensionless density parameter in terms of the evolution of the scale factor

$$\Omega_y = \frac{8\pi G}{3H^2} \rho_y = \left(\frac{H_0}{H} \right)^2 \Omega_{y,0} a^{-3(1+\omega_y)}, \quad (1.41)$$

when substituting this equation into the constraint equation, we get

$$H^2 = H_0 (\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2}), \quad (1.42)$$

giving the first Friedmann equation as is frequently seen in cosmology. As discussed at the beginning of this chapter, we noted that the Einstein-de Sitter Universe was the dominant cosmological model up to the late 20th century, which modelled the Universe as having no curvature or cosmological constant Λ . A key parameter of interest to cosmologists in addition to the expansion rate $\dot{a}$ in H , was the rate of change of the expansion rate, $\ddot{a}$, contained in dimensionless deceleration parameter q whose sign determines whether the expansion rate is increasing or decreasing and whether the Universe is expanding at an accelerating or decelerating rate. The prevailing cosmological model of the time predicted a decelerating Universe $\ddot{a} < 0$; the deceleration parameter was defined with a minus sign so that it is positive when the expansion of the Universe is decelerating. The deceleration parameter is defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2}, \quad (1.43)$$

We can therefore write the deceleration equation as

$$q_0 = \Omega_{r,0}(1+z)^4 + \frac{1}{2}\Omega_{m,0}(1+z)^3 - \Omega_{\Lambda,0}. \quad (1.44)$$

Therefore, to explain the evolution of the Universe better, we continue to plot the density parameters of the components of the Universe against Redshift ($1+z$). We can see in Figure 1.2 right panel, that the era is characterised by radiation, followed by dark matter, and currently by dark energy. This progression indicates that we are transitioning into an era where dark energy takes centre stage. Figure 1.2 left panel, observes that the Universe underwent deceleration during periods dominated by radiation and dark matter. In contrast, the current epoch is marked by acceleration, predominantly steered by the influence of dark energy.

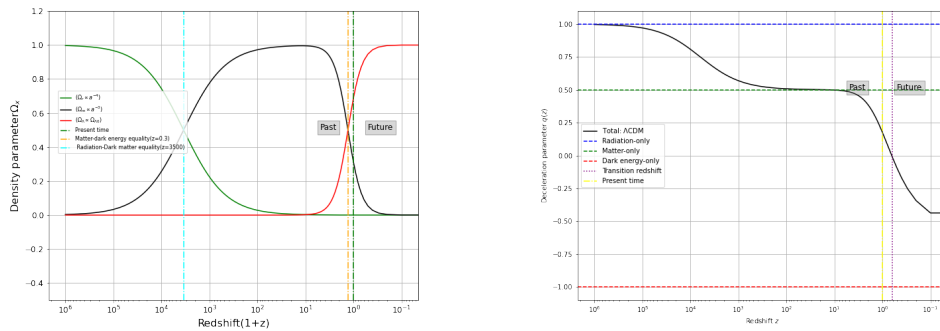


Figure 1.2: The plot on the left panel indicates how the energy components of the Universe have evolved and continue evolving. The plot on the right panel illustrates how these components have influenced and continue to influence the rate at which the Universe's expansion is accelerating, as predicted by the Λ CDM model.

1.3.3 The Λ CDM parameters

In ordinary terms, we say “*simplicity meets elegance*”, but here we say “*minimal complexity converges with optimal efficiency*”. That is, the success of the Λ CDM model lies in its simplicity. It is known for its six parameters that presently explain the cosmological data with an extremely high degree of accuracy. According to [64], the cosmic microwave background (CMB), radiation provides the most robust cosmological constraints due to its combination of very solid theoretical bases and high-quality data. Therefore, we need to understand the main process of the interaction between radiation and matter as it forms a liquid with almost uniform density that is close to thermal equilibrium. Moreover, we determine the predictions of Λ CDM using the values of cosmological parameters to tell us how the energy of the Universe is distributed among its components: radiation, matter, and the cosmological constant [15], continues to tell us the geometry of the Universe. The two

Parameters	Best fit values	Uncertainty
$\Omega_b h^2$	0.022383	0.00015
$\Omega_{cmb} h^2$	0.1198	0.0012
τ	0.0543	0.0073
$\ln(10^{10} A_s)$	3.0448	0.014
n_s	0.96605	0.0042
$100\theta_s$	1.040909	0.00031

Table 1.1: The base cosmological parameters from Planck collaboration [14].

parameters that are easily motivated are those of energy densities of matter species populating the energy-momentum content of the Universe, (see Table 1.1 for the selection of the key parameters in the base model from Planck collaboration). We were able to witness the appearance of energy densities such as Ω_r , Ω_m , Ω_Λ and Ω_k in which Ω_k is strongly constrained, and now it is assumed that the Universe is flat [14]. Therefore, Λ CDM considers a flat FLRW metric as its baseline model. In addition, the present day value of radiation, which is negligibly small, is determined by

$$\Omega_r h = \Omega_\gamma \left(1 + \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} N_{eff} \right), \quad (1.45)$$

here h represents the dimensionless reduced Hubble parameter, Ω_γ is the photon energy density determined by the CMB photon temperature at $T = 2.73K$, and N_{eff} denotes the effective number of species of neutrinos, where $N_{eff} = 3.046$ in the baseline model. Therefore, we have Ω_m to consider as a parameter and Ω_Λ which can be determined from the constraint equation. What about the four remaining parameters?

Well, these four arise when evaluating the most significant source of cosmological

observational data to date. The cosmic microwave background (CMB) radiation stands out, focusing particularly on its anisotropies. The CMB spectrum closely resembles a perfect black body spectrum and has been measured with greater precision than any other black body spectrum in the Universe as seen in reference [14]. Therefore, anisotropies of the CMB spectrum are especially responsive to cosmological models, particularly the four specific parameters that complete part of the six-parameter model. To constrain the parameters of a cosmological model using the observed CMB temperature anisotropies, researchers require a theory that can predict the theoretical spectrum based on specific cosmological parameters. This prediction can be achieved by analysing the linear perturbations in a Friedmann-Lemaître-Robertson-Walker (FLRW) Universe and the evolution of the perturbations of its energy components [65], which this study does not focus on. The latest breakthrough in detecting CMB anisotropies was launched by the Planck satellite in 2009, with the results being released in 2013. With this, the resolutions and sensitivities of the anisotropies increased substantially from 1946 to date, allowing cosmological models to be more constrained and for the Λ CDM model to further secure its place as the standard model of cosmology. Since CMB anisotropies have been measured with high resolution and sensitivity, precise parameter constraints have been imposed on cosmological models (see Figure 1.3) [14]. Therefore, it is essential to focus on the key cosmological parameters, epochs, and processes that influence the CMB anisotropy power spectrum to determine the minimal set of parameters for the concordance model of cosmology [66].

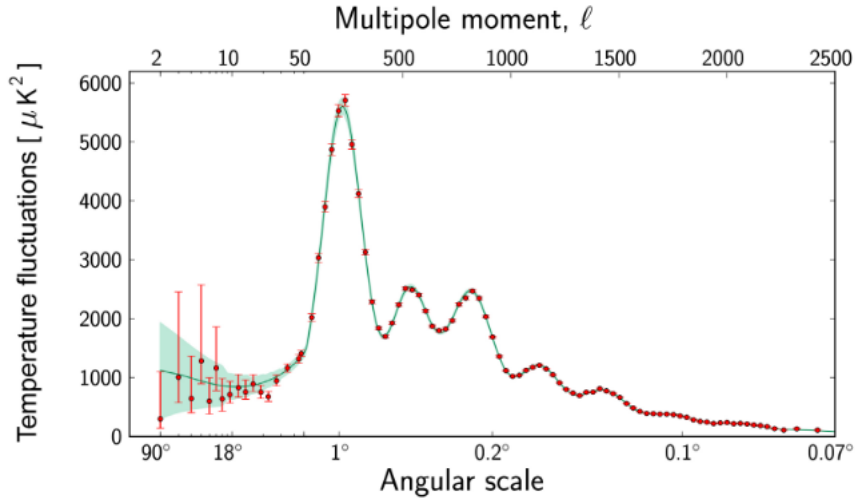


Figure 1.3: Anisotropies in the temperature of the CMB as measured by the Planck satellite [67].

First, reionisation increased the abundance of free electrons, recoupling CMB photons with electrons. However, by this time, the number density of electrons had significantly decreased, preventing the photon-electron Compton scattering rate from

rising to a level comparable with the rate of expansion, thereby keeping the Universe transparent. The reionisation of electrons caused rescattering of CMB photons, leading to the suppression of anisotropies at smaller scales in the CMB temperature spectrum. Consequently, a parameter that quantifies the effect of reionisation is a valuable addition to any cosmological model utilizing the CMB anisotropy spectrum for constraints. The optical depth to reionisation, denoted as τ , effectively encapsulates the impact of reionisation given as

$$\tau = \tau(\eta_{re}) = \int_{\eta_{re}}^{\eta_0} an_e X_e \sigma_T d\eta , \quad (1.46)$$

where τ is the optical depth measuring the opaqueness of the Universe at a time $\eta(= d\eta/a(t))$ from present-day time η_0 , n_e is the number density of electrons; X_e is the ionisation fraction of electrons; and σ_T is the Thomson cross-section. Equation 1.46 illustrates the parameter τ that appears in the Λ CDM. We further continue to look at the era of recombination, where the electrons and protons were tightly coupled through Coulomb scattering to form an electron-proton fluid, which, in turn, was tightly coupled to photons through Compton scattering, thereby together forming a single oscillating baryon-photon fluid. This baryon-photon fluid then has an associated sound speed of

$$c_s = \sqrt{\frac{1}{3(1+R)}} , \quad (1.47)$$

where $R = 3\rho_b/(4\rho_\gamma)$ is a ratio of baryon to photon energy density. The sound speed is indeed sensitive to the baryonic energy density; as the baryon density increases, the sound speed of the baryon-photon fluid decreases. The comoving distance travelled by an acoustic wave by a given time η is known as the sound horizon. It can be expressed mathematically as:

$$d_s(\eta) = \int_0^\eta c_s(\eta') d\eta' , \quad (1.48)$$

where $c_s(\eta')$ is the sound speed at the conformal time η' . This integral captures the distance that sound waves can travel in the baryon-photon fluid up to that time, influencing the CMB anisotropy power spectrum. The next parameter in the Λ CDM model is the angular size of the sound horizon at decoupling. This parameter quantifies the size of the sound horizon relative to the angular scale on the sky, which is critical for understanding the CMB anisotropy power spectrum. The angular size of the sound horizon at decoupling, denoted as θ_s , can be defined as:

$$\theta_s = \frac{d_s}{D_A} , \quad (1.49)$$

where d_s is the comoving sound horizon at the time of decoupling, and D_A is the angular diameter distance to the surface of the last scattering. This parameter helps relate the physical scale of acoustic oscillations to the observed angular scale of the

CMB anisotropies, providing important information about the geometry and expansion history of the Universe Figure (1.1). The parameter θ_s effectively encapsulates dynamics from both before and after recombination, making it a powerful probe of various cosmological parameters. Since it includes a ratio of the comoving sound horizon (dependent on pre-recombination dynamics) and the angular diameter distance (dependent on post-recombination dynamics), it can provide insights $\Omega_b h^2$ (baryonic matter density) and $\Omega_{cdm} h^2$ (cold dark matter density). Ω_Λ , dominates the energy density of the Universe at late times. Lastly, the dimensionless primordial power spectrum of curvature perturbations, often denoted as $P_{\mathcal{R}}(k)$, describes the fluctuations in the density of the Universe at the time of inflation. It is a key component in cosmological models, particularly within the context of the inflationary paradigm, given as

$$P_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_0} \right)^{n_s-1}, \quad (1.50)$$

where A_s is the amplitude of the primordial power spectrum at a pivot scale k_0 , n_s is the spectral index, which characterises the scale dependence of the power spectrum, and k is the wavenumber, related to the scale of fluctuations. According to [14] the spectrum outlines the primordial fluctuations of density perturbations, which are generally believed to originate from inflation. This process established the initial conditions for the formation of large-scale structures, including stars and galaxies, in the Universe. These early fluctuations create distinct signatures in the anisotropies of the cosmic microwave background (CMB) temperature and polarisation spectra. As a result, the amplitude of the primordial power spectrum A_s and the spectral index of scalar perturbations n_s can be constrained based on CMB observations [14]. This concludes the six parameters Table 1.1 of the Λ CDM.

1.4 Conclusion

In this chapter, we introduced general relativity, well established theories of gravity to date, and explored its applications in cosmology. We continued to examine how cosmological principles give rise to the Friedmann-Lemaître-Robertson-Walker metric and explored some of the implications of this for a Universe filled by radiation, baryons, cold dark matter, curvature, and a cosmological constant. Lastly, we put this all together and, upon consideration of the CMB anisotropy spectrum, saw how the six-parameter Λ CDM model emerges. The Λ CDM model represents a major triumph in modern science, much like the standard model in particle physics, however, like the standard model of particle physics, there are some issues that cannot be overlooked and have led many cosmologists to search beyond the Λ CDM model for an explanation. In the next chapter, we will look at some of the issues of Λ CDM cosmology and see whether any can be better explained in an

extended framework.

Chapter 2

Beyond the Λ CDM model

The standard cosmological model is said to be a blessing and a curse for the study of the Universe. This model also called the Λ CDM model encompasses our understanding of the fundamental particles that compose all the material content in the Universe and the interactions between them. The idea that dark matter may exist in the Universe is very old, as Fritz Zwicky proposed that the spread in the velocities of galaxies within a galaxy cluster is so large that even their combined gravity would not allow them to coalesce into a cluster. It was also discovered that it was not sufficient in the 1930s [64]. Therefore, severe limitations in the density parameters of baryonic ($\Omega_b = 0.04 \pm 0.004$) materials resulted from the nucleosynthesis of light elements, so that the materials cannot be completely baryonic [68]. This is below the cluster mass requirement ($\Omega_m \sim 0.2$) and justifies the need for non-baryonic dark matter [64]. It was hinted by observations such as the power spectrum of fluctuations in the automated plate measurement (APM) survey of galaxies that there might be a cosmological constant component (dark energy). And now it is only fair to accept and say that the hint of the Universe being dominated by DE only gained wide approval with observations of distant supernovae in the 1990s, implying that the Universe is not only expanding, but also accelerating [69, 70]. This was completely unexpected because if it were driven by a source field (rather than a cosmological constant), that field would have a very unusual equation of state with negative pressure. Specifically, the state parameter equation $\omega = \frac{p}{\rho c^2} < -\frac{1}{3}$ applies. Moreover, Einstein's cosmological constant is equivalent to a source field with $\omega = -1$ as far as the expansion of the Universe is concerned [71]. Here we provide an overview of the most important cosmological observations of the 21st century, examining the values of various parameters. The successes and challenges faced by the standard cosmological model will be studied and further explored as possible alternative solutions to the Λ CDM model.

2.1 Λ CDM victories

We often say that Λ CDM is the cornerstone model of modern cosmology; the reason behind is its ability to simultaneously explain diverse cosmological phenomena across vastly different scales and epochs, starting from the first minutes of nucleosynthesis to the present-day accelerating expansion of the Universe. For instance

- It correctly forecasts the power spectrum and observed CMB temperature fluctuations, which are the afterglow of the Big Bang. The CMB temperature fluctuations were mapped in remarkable detail by Planck and WMAP, which revealed a pattern that matches remarkably accurately Λ CMB [14]. The measurements support a number of basic Universe-related hypotheses, and explains how large-scale structures form and change over time.
- It explains the formation and evolution of large-scale structures. The cosmic web of galaxies and voids we see in nature is replicated by complex computer simulations based on Λ CDM, such as the Millennium and Illustris projects. In galaxy surveys such as SDSS and DES, the characteristic scale of baryon acoustic oscillations, frozen sound waves from the early Universe, appears precisely where Λ CDM predicts. Even the odd movements of galaxies brought on by the gravitational pull of dark matter are exactly in line with Λ CDM predictions. These independent lines of evidence create a consistent picture of cosmic structure formation over billions of years [9, 14].
- This accounts for the observed accelerating expansion of the Universe.
- It also provides a consistent framework for understanding the composition and evolution of the Universe [69].

The Λ CDM model, with just six parameters, successfully accounts for a wide range of current observational data with remarkable precision. The model remains consistent with all cosmological observations to date, despite a few notable exceptions [14]. What we mean by consistent here is that the parameter constraints coming from independent data sets agree with one another; or put another way, the allowed parameter values inferred from differing datasets are not in tension with one another. Furthermore, statistical model selection and data analysis, in general, do not find preference for extensions to the baseline, flat, six-parameter Λ CDM model [72].

Despite its successes, the Λ CDM model faces challenges ahead. Over the last 20 years, cosmology has evolved into a precision science. This increased accuracy has revealed certain inconsistencies within this model. Although consistent with individual datasets, the results of Λ CDM fits across different data sets often contradict each other [73]. Tensions arise when comparing the predictions of the Λ CDM derived

from the early Universe versus those from the late Universe, with the Hubble tension, the vacuum catastrophe, the S_8 discrepancy, and the cosmological coincidence problem being the most prominent examples. Variations in statistical significance may be due to systematic errors, but their persistent presence strongly implies potential weaknesses in the standard cosmological scenario of the Λ CDM.

2.2 Challenges of the Λ CDM model

2.2.1 Vacuum catastrophe

The absence of static solutions for the Friedman equations led Einstein, in 1917, to revise the field equations of general relativity. Such a revision amounted to introducing a repulsive cosmological constant term in the field equations to balance the attractive pull of gravity. The vacuum catastrophe also called the cosmological constant problem, refers to the measured energy density of the vacuum being over 120 orders of magnitude smaller than the theoretical prediction. By calculating the amount of energy that a quantum fluctuation can contribute within a Planck volume, we can show how the results were obtained. In doing this, first consider the Heisenberg uncertainty principle.

$$\Delta E \Delta t \geq \frac{\hbar}{2}, \quad (2.1)$$

with ΔE and Δt being the uncertainties in energy and time, respectively. A vacuum cannot have zero energy, as this would imply a zero uncertainty principle.

However, although the cosmological constant would appear as the most obvious candidate to fit the observational data, it presents us with two deep conceptual puzzles. As discussed in the previous section, the study of the angular power spectrum of the CMB anisotropies indicates that the spatial geometry of the Universe is very close to Euclidean, and this finding constrains the total energy density to be very close to the critical density [16].

2.2.2 Cosmic coincidence problem

The “cosmic coincidence problem” (CCP) explains why the current epoch is special and why, only in this epoch, the energy density of dark energy (ρ_{de}) is of the same order of magnitude as the energy density of matter (ρ_{dm}). It has been argued that a cosmological model that includes an energy flow from dark energy to dark matter could potentially solve the late acceleration cosmic coincidence problem [28, 74], and this dynamic is explained in detail in the next chapter. On the other hand, the most

successful model (Λ CDM) does not provide any explanation in this regard; rather, the cosmological constant is fine-tuned in order of magnitude for the ratio $r = \frac{\rho_{dm}}{\rho_{de}}$ to be of order unity nowadays. The ratio for Λ CDM in this problem is given as;

$$r = \frac{\rho_{dm}}{\rho_{de}} = \frac{\rho_{dm,0}a^{-3(1+\omega_i)}}{\rho_{de,0}a^{-3(1+\omega_i)}} = r_0a^{-3(\omega_i-\omega_i)}; \quad r_{\Lambda CDM} = \frac{\rho_{dm}}{\Omega} = \frac{\rho_{dm,0}a^{-3}}{\rho_{\Omega,0}} = r_0a^{-3}, \quad (2.2)$$

where r_0 is the present ratio of dark matter to dark energy which can be referred as $\frac{\Omega_{dm,0}}{\Omega_{de,0}}$. In order to understand the magnitude of this problem, it is helpful to compute r at the Planck scale to gauge the level of fine-tuning that the Λ CDM model necessitates from its starting conditions (see e.g [16, 26] and the references therein).

2.2.3 Hubble tension

Hubble tension refers to a discrepancy between the value of the Hubble constant H_0 derived from cosmic microwave background (CMB) data from the Planck Institute of Quantum Optics and direct local distance guide measurements[29]. Consequently, there has been an apparent growth level from 2σ to 4.4σ of tension between these two values, as mentioned in the latest Planck 2018 results (Hubble constant to be approximately $H_0 = 67.4 \pm 0.5 \text{ km s}^{-1}\text{Mpc}^{-1}$) and in 2019 from the observation Magellanic Cloud Cepheid by the Hubble Space Telescope obtaining the best estimate for the Hubble constant of $H_0 = 74.03 \pm 1.42 \text{ km s}^{-1}\text{Mpc}^{-1}$ [75] in 2019 and more recently, in 2021, they obtained $H_0 = 73.0 \pm 1.4 \text{ km s}^{-1}\text{Mpc}^{-1}$. The H_0 discrepancy could potentially suggest the presence of physics that goes beyond the standard Λ CDM model, elevating this matter to one of the crucial subjects in cosmology as a whole. [76] and references therein gave a review of the realm of this tension and concluded that there is no strong case to solve this puzzle.

2.2.4 S_8 discrepancy

Although not to the same extent, discrepancies still emerge between local measurements of the strength of structure in the Universe, both our direct measurements and those obtained by other methods such as CMB analysis. The emergence of tensions between the value of the present-day Hubble parameter H_0 as inferred from early-time cosmology using cosmic microwave background (CMB) measurements, and that from local late-time cosmology using type Ia supernovae has been considered extensively in the literature as possible evidence of the existence of new physics beyond the standard cosmology scenario [26]. A similar discrepancy appears in the amplitude of the variance of the matter density field on scales of $8h^{-1} \text{ Mpc}$, $S_8 = \sigma_8(\Omega_{dm}/0.3)^{0.5}$ where Ω_m is the total matter density of the Universe). Tension S_8 , although not

statistically significant (ranging between $1.5 - 2.5\sigma$), has also received a great deal of attention. Most proposed solutions introduce some form of self-interaction in the dark sector in an attempt to erase structure in the late Universe. Other proposals include, but are not limited to, dark matter-neutrino interactions, modifications to gravity, or neutrino self-interactions [77, 78]. The apparent correlation between the two tensions indicates that one tension cannot be addressed without the other [79].

2.3 Alternative solutions to Λ CDM

Most challenges faced by the standard cosmological models are closely bound by data derived from the cosmic microwave background, indicating that our comprehension of the Universe and fundamental cosmology might still be incomplete. This creates exciting opportunities for further research and the enhancement of our theories, as new data and methodologies could help fill these gaps and deepen our knowledge of the cosmos. Based on this idea, lately observations have been important in the study of alternative theoretical frameworks and possible solutions [47, 80]. Therefore, by expanding the observational data, we can evaluate and refine these models, potentially uncovering new insights that could address existing challenges in our understanding of the Universe. Scientists have explored various methods and modifications to address these limitations. These efforts include revising existing models, introducing new interactions, and adjusting parameters in cosmological theories. By considering alternative frameworks and incorporating novel elements, researchers aim to resolve the discrepancies and enhance our understanding of the large-scale structure and evolution of the Universe. They continued to bring substantial efforts to be taken to address the inconsistencies between cosmic microwave background (CMB) data and observations of large-scale structures [10, 14, 81]. These efforts involve postulating an interaction **between dark matter and dark energy** [25, 26, 80, 82], introducing models of early dark energy [83], accounting for additional dark radiation during the recombination epoch [84], and exploring modified gravity theories along with dynamic dark energy [85]. Each of these distinct strategies seeks to resolve the discrepancies and enhance our understanding of the evolution of the Universe and the formation of cosmic structures.

In spite of the many proposed alternatives, here we consider the interacting dark energy models with interaction between dark energy and dark matter, as it has been testified that, with appropriate parameters, these models can relieve the discordances mentioned under the standard cosmological model [86]. It is intriguing because it investigates the potential for dark energy, responsible for the accelerating expansion of the Universe, to not function independently. Instead, it could interact with other components, such as dark matter, leading to an alternative evolution of the Universe

compared to the standard model. These interactions might offer explanations for certain observational anomalies that the standard Λ CDM model struggles to account for, as mentioned above. By allowing energy exchange between dark matter and dark energy, these models provide new dynamics that can be probed by current and future cosmological observations, potentially revealing new aspects of the dark sector [12].

2.4 Conclusion

In this chapter, we reviewed some of the tensions within the Λ CDM model, considering the possibility that these tensions might suggest the need to look beyond the Λ CDM framework and explored possible resolutions to these discrepancies. Lastly, we delved into the mysterious dark sector and some of the significant challenges surrounding the cosmological constant. Although the Λ CDM model remains our best cosmological model to date, it is in substantial disagreement with fundamental physics predictions regarding the vacuum energy density. These issues, along with others, have led cosmologists to seek alternative theories of dark energy. Current data allow for a potentially strong interaction between dark energy and dark matter, making it crucial to explore the possibility of interactions within the dark sector. This exploration is important to gain deeper insights into this mysterious part of dark sector's nature, which is believed to constitute approximately 95% of the Universe's energy budget, an area about which we currently have limited understanding.

Chapter 3

Review of interacting dark energy models

The behaviour of the two exotic cosmological fluids DM and DE ¹ has been one of the major issues in modern cosmology, as the former has the usual meaning of gravitationally interacting mass, but is oblivious to other standard particle physics interactions, while the latter is a new strange kind of mystery leading to the accelerated expansion of the Universe. The interaction between the two is a possible mechanism suggesting that, within particle physics or theoretical contexts, any two matter fields can influence one another. This phenomenological theory has captured the attention of the cosmology community due to its various potential implications [22]. Models that involve the interaction of DM and DE provide an equivalent framework for understanding the dark sector of the Universe. These models have been thoroughly studied in [12, 16, 22, 87–91] and offer plausible solutions to the coincidence problem and issues related to the cosmological constant, such as DE decaying into DM [12, 92–94]. Some studies proposed reasonable assumptions about the relationship between DM and DE to address the challenges associated with the Λ CDM model. Since the true nature of DE and DM remains unknown, the possibility of interaction is allowed within field theory [26].

The energy densities of DM and DE are connected in a manner that makes them dynamic and comparable, reducing the coincidence problem [95]. Several studies have already addressed the relationship between the interaction of DM and DE. Recently, a few interacting models have been investigated to help resolve certain cosmological tensions. Allowing interaction has been found to change the DE equation of state (EoS) from quintessence to phantom, suggesting that the EoS parameter exhibits an effective quintom-like nature [96]. In phantom-scalar field models, the phan-

¹DM and DE will be used throughout the dissertation in replace of dark matter and dark energy respectively.

tom crossing line leads to instabilities due to a negative kinetic correction term. Moreover, interaction theory has proven particularly useful in addressing the discrepancy between global and local measurements of the Hubble constant H_0 . Since there are no strict guidelines for selecting interaction functions, a range of linear and non-linear functions have been proposed in the literature. For example, in [96], the authors explore the physical constraints on the EoS parameter for dark-sector interactions. The motivation behind the study was for a better understanding of the mechanism that derives the late-time expansion acceleration of the Universe. They presented a thermodynamic analysis that integrates the second law of thermodynamics with observational data to derive constraints on the interaction between DM and DE. To describe the evolution of DE density, they used two parametrisations of EoS ($\omega = \omega_0 + \omega_a$) and revealed that the combination of thermodynamic limits with observational data tightens the constraints on the $(\omega_0 - \omega_a)$ parameter space. In addition, their results indicate that the assumption of minimal coupling between dark sectors leads to considerable changes in the predicted cosmic evolution. The study offers highlights on how significant it is to consider thermodynamics when dealing with cosmological models [96].

In the study [31], which is of importance to this dissertation, the authors investigate the implications of interacting dark energy models in light of the cosmological observations that were recent then. They particularly addressed the tension between local measurements of H_0 and predictions from the Λ CDM model based on Planck data. The measurement of $H_0 = 73.02 \pm 1.79$ km/s/Mpc was highlighted as higher than the Planck value of $H_0 = 67.27 \pm 0.66$ km/s/Mpc. This suggests new physics in the late-time Universe. A comprehensive data set such as CMB observations, BAO, LSS, SNIa, and $H(z)$ measurements was implemented to constrain various IDE models. Their analysis reveals no substantial evidence for the interaction between DE and DM, although they found that IDE models can moderately alleviate H_0 tension, but do not fully resolve it. *“Do we still have hope in finding a suitable model that can surpass the findings of the Λ CDM model without having to believe that one model can alleviate the coincidence problem but worsen the Hubble tension?”*

In this section, we focus on the phenomenological interaction as a linear combination of energy densities of dark sectors in the form of linear and non-linear IDE models as listed in Table (3.2). Although, it is uncommon to discuss the dynamics of interacting models with non-linear interaction functions because of their complexity, we find them fascinating to explore the dynamics of the Universe to determine whether we can glean any further information from them. We solely focus on the interaction between DM and DE, disregarding the interaction between DE and baryonic matter. Typically, a minimal coupling between DE and baryonic matter is assumed to prevent any detectable violations of the equivalence principle and to avoid potential conflicts with observational constraints regarding long-range forces. In reference

[25], it is indicated that the good agreement with the observational limits for a range of parameter combinations remains unaffected by whether baryons are included in the interaction. As a result, we assume that baryonic matter is minimally coupled in our analysis.

3.1 Background evolution

Recalling that DE and DM evolve independently in the Λ CDM model, their energy densities are managed separately [12, 23, 97]. In contrast, all the energy densities of the dark sector can be found in IDE models because of other postulated interactions. Furthermore, the energy momentum tensor of the dark sector, $T_{(i)}^{\mu\nu}$, satisfied as; $\sum_i \nabla_\mu T_{(i)}^{\mu\nu} \equiv 0$, where i represents dark energy (de) or dark matter (dm) respectively, and Einstein's summation convention has been adopted, as in [47, 48]. We consider the flat Universe, and therefore, we deduce the conservation using the energy momentum tensor of the perfect fluid using the FLRW metric (1.23), as

$$3H^2 = 8\pi G(\rho_{de} + \rho_{dm} + \rho_b + \rho_r) , \quad (3.1)$$

where ρ_{de} , ρ_{dm} , ρ_b , and ρ_r represent energy density de, dm, baryonic matter and radiation, respectively. Therefore, since we are only focusing on the interaction between DE and DM, we neglect the contributions of (uncoupled) baryon(ρ_b) and radiation(ρ_r) components, therefore,

$$3H^2 = 8\pi G(\rho_{de} + \rho_{dm}) . \quad (3.2)$$

The conservation of the dark sector can also be expressed mathematically from the conservation equations for both fluids as

$$\dot{\rho}_{dm} + \dot{\rho}_{de} + 3H(\rho_{dm} + \rho_{de} + P_{dm} + P_{de}) = 0 , \quad (3.3)$$

with $H = \frac{\dot{a}}{a}$ being the Hubble constant. This equation (3.3) provides evidence of interaction among the components admitting the mutual exchange of energy and momentum. We also describe the dimensionless Hubble parameter $H(z)$ satisfying the relation:

$$H(z)^2 = H_0^2 \left(\Omega_{de,0} \frac{\rho_{de}}{\rho_{de,0}} + \Omega_{dm,0} \frac{\rho_{dm}}{\rho_{dm,0}} \right) , \quad (3.4)$$

where $\Omega_{de,0}$ and $\Omega_{dm,0}$ represent today's energy-density fractions of dark energy and dark matter. We further present the corresponding solutions for the dimensionless Hubble parameter ($H(z)$), which will be used to constrain each model. We assume a flat FLRW Universe with $\Omega_{dm} + \Omega_{de} = 1$ ² and a constant equation of state (EoS) ω_{de} of DE, in which the Friedmann equation under such assumptions is equation (3.3)

²Note Ω_{de} is a derived parameter.

above. Now, by assuming an arbitrary interaction (Q), equation 3.3 can be separated into DM and DE equations, creating a mechanism whereby the components can exchange energy momentum in a non-gravitational way [12, 16]. Therefore, the dynamic equations of DM and DE have the following separate conservation equations:

$$\dot{\rho}_{dm} + 3H\rho_{dm} = Q, \quad (3.5)$$

$$\dot{\rho}_{de} + 3H(1 + w_{de})\rho_{de} = -Q, \quad (3.6)$$

where the dot denotes the derivative, ρ_{dm} and ρ_{de} are the energy densities for DM and DE, respectively, $\omega = \frac{P_{de}}{\rho_{de}}$ is the equation of state of DE and Q^3 is the interacting term and determines the energy transfer rate between dark energy and dark matter. So, if $Q > 0$, we say that the energy is transferred from DE to DM and when $Q < 0$ the energy is transferred from DM to DE, as in [16, 29, 98]. The analysis in [92] showed that both decaying routes are possible as,

$$Q \begin{cases} > 0 & \text{DE to DM} , \\ = 0 & \text{No interaction} , \\ < 0 & \text{DM to DE} . \end{cases} \quad (3.7)$$

We can further rewrite equations (3.5) and (3.6) in relations to redshift ($z = a^{-1} - 1$) and $\frac{d}{dt} = -H(1+z)\frac{d}{dz}$;

$$(1+z)\frac{d\rho_{dm}}{dz} - 3\rho_{dm} = -\frac{Q}{H} \quad ; \quad (1+z)\frac{d\rho_{de}}{dz} - 3(1+w_{de})\rho_{de} = \frac{Q}{H}. \quad (3.8)$$

Therefore, we say that the energy pressure relation can depend on the redshift (time), which was originally from equations (3.5) and (3.6). The EoS component describes the relationship between pressure P and energy density ρ in standard cosmology. However, in the IDE models, the effective EoS describes the total energy budget and expansion behaviour of the Universe, accounting for both DE and DM, as well as the interaction between them. Therefore, effective background EoS for the Universe is defined as

$$\omega_{eff} = \frac{p_{tot}}{\rho_{tot}} = \frac{p_{dm} + p_{de}}{\rho_{dm} + \rho_{de}}. \quad (3.9)$$

Describing the equivalent uncoupled models in the background as given by equations (3.5) and (3.6), this equation becomes

$$\dot{\rho}_{dm} + 3H\rho_{dm}(1 + \omega_{eff}) = 0, \quad (3.10)$$

$$\dot{\rho}_{de} + 3H\rho_{de}(1 + \omega_{eff}) = 0. \quad (3.11)$$

Now, incorporating equations (3.5), (3.6), (3.10) and (3.11), we see that interaction between the effective background equations of state for the DM and DE change to:

$$\omega_{dm}^{eff} = -\frac{Q}{3H\rho_{dm}} \quad ; \quad \omega_{de}^{eff} = \omega_{de} + \frac{Q}{3H\rho_{de}}, \quad (3.12)$$

³IDE models will be labelled as $Q_{1,2,3,4,5}$ to avoid confusion with different expressions expressed with the same variable

and it follows that;

$$Q > 0 \rightarrow \begin{cases} \omega_{dm}^{eff} > 0 & \text{DM Redshift is faster than the scale factor } a^{-3}, \\ \omega_{de}^{eff} < \omega_{de} & \text{DE has more accelerating power,} \end{cases} \quad (3.13)$$

$$Q < 0 \rightarrow \begin{cases} \omega_{dm}^{eff} > 0 & \text{DM Redshift is slower than the scale factor } a^{-3}, \\ \omega_{de}^{eff} < \omega_{de} & \text{DE has less accelerating power.} \end{cases} \quad (3.14)$$

If $\omega_{de} = -1$ and $Q < 0$ or $Q > 0$, DE can behave as quintessence $\omega^{eff} > -1$ or uncoupled $\omega^{eff} < -1$ DE, respectively. If there is no interaction between DM and DE ($Q = 0$), the effective equations of state reduce back to the uncoupled model, such that $\omega_{dm}^{eff} = \omega_m = 0$ and $\omega_{de}^{eff} = \omega_{de}$ [24, 99]. Analysing these equations enables us to understand the properties and aspects of DE and its effect on the expansion of the cosmos. Additionally, it helps in distinguishing the separate impacts of DM and DE, offering a deeper insight into their functions in the development of the Universe (see [97] and references therein). The effective equation of state enables us to predict the implications of a dark coupling (see [97]) as they were introduced to the cosmic coincidence problem at first.

3.2 Consequences of IDE models

As the saying goes “It never rains but it pours”, as much as we believe that IDE models address Λ CDM challenges, there are far-reaching cosmological consequences that this model faces. They encounter both theoretical and observational implications. They are known for challenging the assumption that dark energy is simply a cosmological constant with a fixed energy density, but is often modelled as a scalar field. They may motivate modified gravity theories, such as scalar-tensor or vector-tensor theories, which naturally predict couplings between dark sectors. From the observational implications, they can reconcile the discrepancy between the Hubble constant measured locally and inferred from the CMB by modifying late-time expansion history.

In [97], the author indicates that the implications hold only if the IDE model is viable and “any cosmological model may be considered unviable due to theoretical concerns, such as internal inconsistencies, instabilities, negative densities and so forth” [16]. The study continues to indicate, in Table 3.1, the problems that only a model free of them can be deemed viable if it meets observational constraints, such as predicting an expansion history that coincides with cosmological data [16]. But let us look at how that can come about. The table alludes to the fact that when $\omega_{dm}^{eff} < 0$ DM is slower, resulting in fewer DM in the past and delaying the radiation-matter equality. In contrast, when $\omega_{dm}^{eff} > \omega_{de}$, it means that DE redshifts

more rapidly, leading to a higher amount of DE than in the past. This combination of decreased DM and increased DE results in both cosmic jerk and matter-dark energy equality that occurred earlier in cosmic history [100, 101]. This results in lower values of the Hubble parameters at late times, causing the tension that concerns late-time observations. Because the Hubble parameter and therefore the expansion rate are lower during much of the Universe’s expansion, it implies that the Universe took longer to reach its current size. Consequently, the Universe should also be older. All this happens when the flow of energy takes place from DE to DM. The opposite happens when the flow of energy is from DM to dark energy. The concerns raised

	$Q > 0$	$Q < 0$
Energy flow	$DE \rightarrow DM$	$DM \rightarrow DE$
Effective equation of state	$\omega_{dm}^{eff} < \omega_{dm} ; \omega_{de}^{eff} > \omega_{de}$	$\omega_{dm}^{eff} > \omega_{dm} ; \omega_{de}^{eff} < \omega_{de}$
Coincidence problem	Alleviates	Worsens
Age of Universe	Older	Younger
Radiation-matter equality	Later	Earlier
Cosmic jerk	Earlier	Later
Matter-DE equality	Earlier	Later

Table 3.1: Summarised consequences of interacting DE models (adapted from [16]).

by these models are that they often mimic each other by varying other cosmological parameters, leading to parameter degeneracies. They lack a strong theoretical underpinning from particle physics, and therefore the phenomenological interactions. “Again, we end up asking whether a suitable cosmological model that will allow to solve all the challenges can be found.”

3.3 Phenomenological IDE models

In this dissertation, we consider IDE models labelled in Table (3.2) with interactions between DM and DE, focusing on coupling constant models, with dimensionless coupling parameter ξ . Since the form of the interaction term Q , as explained in (3.7) is still an open question, it is advised to assume a certain possible route to study the challenges of interaction between DM and DE as has been done in several studies [27, 31, 102]. Here, we solely focus on the case where $\omega = -1$ to avoid introducing additional parameters. We present several mathematical expressions for the coupling Q between DE and DM, and further give the expressions corresponding to $H(z)$, which will be used to constrain each model.

Mark	Interaction	References
IDE1	$\xi H \rho_{dm}$	[16, 21, 26]
IDE2	$\xi H \rho_{de}$	[16, 21, 24, 69]
IDE3	$\xi H (\rho_{dm} + \rho_{de})$	[12, 21, 87, 91]
IDE4	$3\xi H (\frac{\rho_{dm}\rho_{de}}{\rho_{dm}+\rho_{de}})$	[21, 103]
IDE5	$3H\xi\sqrt{\rho_{dm}\rho_{de}}$	[103, 104]

Table 3.2: List of linear interactions (IDE1-IDE3) and non-linear interactions (IDE4 & IDE5) considered.

3.3.1 Linear models

The literature extensively examines interaction models in which the energy transfer rate follows linear dependencies, specifically in cases where Q scales proportionally with either dark matter density $Q \propto \rho_{dm}$ or dark energy density $Q \propto \rho_{de}$. These canonical linear parametrisations have become foundational in coupled dark energy studies [21, 25, 26, 38, 39, 97, 104]. Therefore, here we focus on the three linear models, Q proportional to the DM density, the second proportional to the DE, and a general case being a linear combination of DE and DM energy densities, which we often say is an asymmetric model due to the sign change.

$$\textbf{IDE1: } Q_1 = \xi H \rho_{dm}$$

In this model DM and DE background densities evolve as two fluids, coupled with a strength linearly dependent on the DM density present at any given time of the cosmic evolution, and equations (3.5) and (3.6) become:

$$\dot{\rho}_{dm} + 3H\rho_{dm} = \xi H \rho_{dm} , \quad (3.15)$$

$$\dot{\rho}_{de} + 3H\rho_{de}(1 + \omega_{de}) = -\xi H \rho_{dm} . \quad (3.16)$$

These two correspond to two fluids with effective equations of state given as

$$\omega_{dm}^{eff} = -\frac{\xi}{3} , \quad (3.17)$$

and,

$$\omega_{de}^{eff} = \omega_{de} + \frac{\xi}{3} , \quad (3.18)$$

respectively. These equations match those in [16, 24]. Now, to solve the equations (3.15) and (3.16), we obtain analytical solutions for DM (ρ_{dm}) and DE (ρ_{de}) densities as

$$\rho_{dm} = a^{\xi-3} \rho_{(dm,0)} \quad ; \quad \rho_{de} = (1+z)^{3-\xi} \rho_{dm,0} , \quad (3.19)$$

$$\begin{aligned}\rho_{de} &= \left[\rho_{de,0} + \rho_{dm,0} A (1 - a^{\xi+3\omega}) \right] a^{-3(1+\omega)}, \\ \rho_{de}(z) &= \left[\rho_{de,0} + \rho_{dm,0} A (1 - (1+z)^{-\xi-3\omega}) \right] (1+z)^{3(1+\omega)}.\end{aligned}\tag{3.20}$$

where $A = \left(\frac{\xi}{\xi+3\omega} \right)$, $\rho_{dm,0}$ and $\rho_{de,0}$ are the DM and DE energy densities today. We continue to calculate the Hubble parameter using the assumed Friedman equation 3.3 with the dimensionless components as

$$\begin{aligned}H^2(z) &= \frac{8\pi G}{3} (\rho_{dm}(z) + \rho_{de}(z)) \\ &= H_0 \sqrt{(1+z)^{3-\xi} \Omega_{dm,0} + [1 - \Omega_{dm,0} + \Omega_{dm,0} A (1 - (1+z)^{-\xi-3\omega})] (1+z)^{3(1+\omega)}}.\end{aligned}\tag{3.21}$$

The equation is consistent to the dimensional hubble rate found in [21, 97, 105].

$$\textbf{IDE2: } Q_2 = \xi H \rho_{de}$$

In a similar manner as in IDE1, DM and DE background densities evolve as two fluids and equations (3.5) and (3.6) become:

$$\dot{\rho}_{dm} + 3H\rho_{dm} = \xi H \rho_{de}, \tag{3.22}$$

$$\dot{\rho}_{de} + 3H\rho_{de}(1 + \omega_{de}) = -\xi H \rho_{de}. \tag{3.23}$$

To obtain analytical solutions for how the DM ρ_{dm} and DE ρ_{de} densities evolve, we solve the conservation equations with the interacting term $Q = \xi H \rho_{de}$, where we get that

$$\rho_{dm} = a^{-3} \left[\rho_{dm,0} + \rho_{de,0} \frac{\xi}{\xi + 3\omega} (1 - a^{-(\xi+3\omega)}) \right], \tag{3.24}$$

for the DM density and then for ρ_{de} through use of the first order differential equation we reach that

$$\rho_{de} = a^{-\xi-3(1+\omega)} \rho_{(de,0)}. \tag{3.25}$$

The solutions match with the energy densities found in [16, 106]. Equations (3.24) and (3.25) also reduce back to the Λ CDM model when $\xi = 0$ and $\omega = -1$ as:

$$\rho_{dm} = \frac{3H_0^2}{8\pi G} (1+z)^3 \left[\Omega_{(dm,0)} + \Omega_{(de,0)} A \left[1 - (1+z)^{(\xi+3\omega)} \right] \right], \tag{3.26}$$

$$\rho_{de} = \frac{3H_0^2}{8\pi G} (1+z)^{\xi+3(1+\omega)} \Omega_{(de,0)}. \tag{3.27}$$

Therefore, we find that the effective equations of state for the models are

$$\omega_{dm}^{eff} = -\frac{\xi}{3} \frac{\rho_{de}}{\rho_{dm}}, \tag{3.28}$$

and

$$\omega_{de}^{eff} = \omega_{de} + \frac{\xi}{3}, \tag{3.29}$$

respectively. It can be seen that ω_{dm}^{eff} is constant throughout cosmic evolution, while ω_{de}^{eff} is dynamic with a dependence on the coincidence problem ratio $r = \rho_{dm}/\rho_{de}$. We can continue to calculate the Hubble parameter using the assumed Friedman equation (3.3) with the dimensionless components as;

$$\begin{aligned} H^2(z) &= \frac{8\pi G}{3}(\rho_{dm}(z) + \rho_{de}(z)) \\ &= H_0 \sqrt{[\Omega_{(dm,0)} + (1 - \Omega_{(dm,0)})A(1 - (1+z)^{(\xi+3\omega)})] + (1+z)^{\xi+3(1+\omega)}(1 - \Omega_{(dm,0)})} . \end{aligned} \quad (3.30)$$

Similarly to IDE1, the equation is consistent to the dimensional Hubble rate found in [21, 97, 105].

$$\textbf{IDE3: } Q_3 = H\xi(\rho_{dm} + \rho_{de})$$

We can also write the interaction between DE and DM in proportion to the total energy densities of DE and DM, so that the continuity equations are

$$\dot{\rho}_{dm} + 3H\rho_{dm} = \xi H(\rho_{de} + \rho_{de}) , \quad (3.31)$$

$$\rho_{de} + 3H\rho_{de}(1 + \omega) = -\xi H(\rho_{dm} + \rho_{de}) , \quad (3.32)$$

And we can convert this equations to

$$\frac{d\rho_{de}}{dz} - \frac{3}{1+z}\rho_{dm} + \frac{\xi}{1+z}(\rho_{de} + \rho_{dm}) = 0 , \quad (3.33)$$

$$\frac{d\rho_{de}}{dz} - \frac{3}{1+z}(1 + \omega)\rho_{de} - \frac{\xi}{1+z}(\rho_{de} + \rho_{dm}) = 0 . \quad (3.34)$$

Since it is not easy to write out analytic solutions of ρ_{de} and ρ_{dm} when we substitute the parameterisations of EoS, we completely rely on numerical calculations to investigate the time evolution of the DE and DM in these cases.

3.3.2 Non-linear models

These are rarely studied IDE models in the literature. The two were considered in [21] using cosmic chronometer data and the Gaussian process in studying the expansion history of the Universe. We are adding the two to our list in order to offer a well understood conclusion without biases of the most studied linear IDE models.

$$\textbf{IDE4: } Q_4 = 3H\xi\left(\frac{\rho_{dm}\rho_{de}}{\rho_{dm} + \rho_{de}}\right)$$

One interesting model, we say its non-linear as its proportionality to the interacting term comes with the twos'. We use numerical ODEs to solve the conservation

equations as

$$-(1+z)H\frac{d\rho_{dm}}{dz} + 3H\rho_{dm} = Q , \quad (3.35)$$

$$-(1+z)H\frac{d\rho_{de}}{dz} + 3H\rho_{de}(1+\omega_{de}) = -Q , \quad (3.36)$$

using the redshift relation $\frac{d}{dt} = -H(1+z)\frac{d}{dz}$. Therefore,

$$\frac{d\rho_{dm}}{dz} = \frac{3\rho_{dm} - 3\xi\left(\frac{\rho_{dm}\rho_{de}}{\rho_{dm}+\rho_{de}}\right)}{1+z} , \quad (3.37)$$

$$\frac{d\rho_{de}}{dz} = \frac{3(1+\omega_{de})\rho_{de} + 3\xi\left(\frac{\rho_{dm}\rho_{de}}{\rho_{dm}+\rho_{de}}\right)}{1+z} . \quad (3.38)$$

Since we are only considering the dark sectors, the Friedmann equation as mentioned above

$$H^2 = H_0^2(\rho_{dm} + \rho_{de}) , \quad (3.39)$$

We use the iterative Runge-Kutta numerical differential equations solver to integrate conservation equations and Friedmann equations. The set up shows that DM deviates from $(1+z)^3$ due to the interacting model and DE grows slower than $(1+z)^{3(1+\omega_{de})}$.

$$\mathbf{IDE5:} \quad Q_5 = 3H\xi\sqrt{\rho_{dm}\rho_{de}}$$

Same as the one IDE4, is non-linear as its proportionality to the interacting term comes with the twos'. By two's we mean $Q \propto \rho_{dm}\rho_{de}$. In the same manner we use numerical ODEs to solve the conservation equations as

$$-(1+z)H\frac{d\rho_{dm}}{dz} + 3H\rho_{dm} = Q , \quad (3.40)$$

$$-(1+z)H\frac{d\rho_{de}}{dz} + 3H\rho_{de}(1+\omega_{de}) = -Q , \quad (3.41)$$

using the redshift relation $\frac{d}{dt} = -H(1+z)\frac{d}{dz}$. Therefore,

$$\frac{d\rho_{dm}}{dz} = \frac{3\rho_{dm} - 3\xi\left(\sqrt{\rho_{dm}\rho_{de}}\right)}{1+z} , \quad (3.42)$$

$$\frac{d\rho_{de}}{dz} = \frac{3\rho_{de}(1+\omega_{de}) + 3\xi\left(\sqrt{\rho_{dm}\rho_{de}}\right)}{1+z} . \quad (3.43)$$

Since only considering the dark sectors, the Friedmann equation as mentioned above is

$$H^2 = H_0^2(\rho_{dm} + \rho_{de}) , \quad (3.44)$$

We use the iterative Runge-Kutta numerical differential equation solver to integrate conservation equations and Friedmann equations.

3.4 Conclusion

In this chapter, we delved into the dynamics of coupled DE and DM interactions, focusing on phenomenological interaction terms motivated by cosmological observations. We derived the conservation equations for dark sectors in the presence of our interacting term Q . Five different models, with an additional metastable model were considered. Out of all, we managed to obtain exact analytical solutions for certain models and explicit numerical solutions for the others. Here, we were just laying the groundwork for understanding how energy transfer between dark sectors alters cosmic evolution. The models and methods developed here will be faced with observational data in the next chapters to test their viability.

Chapter 4

Methodology and observational data

In [Chapter 3](#), we presented various DE interaction models and discussed their behaviour in calculations. Our aim now is to validate these models using existing observational data to refine the cosmological parameters. In account of this, we outline the methods used to derive the constraints on the parameters of our models, focusing on the efficient sampling technique provided by MCMC. In addition, we detail the datasets and observational measurements employed in the analysis. Lastly, we will present the discussion and results obtained in [Chapter 5](#).

4.1 Methodology

Python package EMCEE is utilised to maximise the likelihood functions, thereby determining the constraints on the free parameters. Given that, we employ Kosmulator¹-a Python package that utilises the Hammer package (EMCEE) to run the Markov Chain Monte Carlo (MCMC), generated from the original version (see [\[48\]](#) for the initial code) to study modified gravity and alternatively the cosmological models. Therefore, we ensure that the expansion of the Universe has been accounted for by choosing the priors on the cosmological parameter as flat as we are not restricting or rather studying one direction of our energy densities and they range as $0 \leq H_0 \leq 100$, $-1.5 \leq \omega \leq 0.2$, $0.01 \leq \Omega_m \leq 0.4$, and $-5 \leq \xi \leq 5$. These priors are used for Λ CDM, and all models that are considered, with the applicable set of parameters for each model. The reasoning behind the relaxed priors is to avoid imposing the fixed interaction term. In that we are avoiding biases of energy travelling towards the quintessence or phantom behaviour. Rather, freeing the inter-

¹Kosmulator package is publicly available at: <https://github.com/renierht/Kosmulator/>

acting term allows one not to obscure the nature of dark coupling. That allows for a robust identification of where new physics lie, whether in early-time or late-time energy exchange.

MCMC simulation

MCMC is a widely used method to sample complex distributions due to its minimal knowledge requirements about the target distribution. Generally, it suffices to evaluate the unknown probability density to sample from the desired distribution. It has found applications in various fields, including computer vision, machine learning, and computational biology [107]. By providing the distance coefficient for any model, this simulation identifies the best-fit value of the luminosity distance function for various combinations of cosmological parameters, followed by a corresponding statistical analysis of these outcomes. Consequently, one only needs to define the distance modulus equations for the IDE models, modify the MCMC code, and interpret the results obtained from the simulation. Initially, this MCMC simulation was used for the Λ CDM model to establish a benchmark to compare IDE models. Once the most suitable cosmological parameters for the interacting dark energy model have been determined, the results will be statistically compared with those from the Λ CDM model. Therefore, MCMC can be used to calculate the optimal values of the free parameters present in the models of this study. We follow the approach used in [108] where they used the three-way steps to explain the stringency of the parameters. The first alludes to the well-constrained parameter, showing that it follows a Gaussian probability distribution within the designated search space. The second is a neutrally constrained parameter, meaning that only one side of the Gaussian distribution has been identified in the search space. This suggests that there may be a more desirable value, as the average is slightly shifted away from the peak of the distribution. Finally, the unconstrained parameter, where a uniform distribution is obtained for that specific parameter.

The approach involves fitting the *BAO*, *CC* and *PantheonP + SH0ES* separately for each IDE model and the Λ CDM model, after we combine the datasets as *BAO + CC*, *BAO + PantheonP + SH0ES*, and *BAO + CC + PantheonP + SH0ES*. This thorough method aims to identify the parameter values that are most likely indicative of the underlying physical reality for each particular cosmological model studied. In order to statistically compare the different cosmological models and determine which model has the greatest support for it in light of the data, we compute the Bayes factor for each IDE model relative to the Λ CDM model. Therefore, the Bayes factor gives a measure of the level of support for one model over another and intrinsically accounts for additional degrees of freedom, thereby making it a useful tool in statistical model comparison. Therefore, Bayes factor of a model i relative

to the Λ CDM model is given by

$$B_{i,\Lambda\text{CDM}} = \frac{P(D|i)}{P(D|\Lambda\text{CDM})}, \quad (4.1)$$

where D is a given data set, and $P(D|i)$ and $P(D|\Lambda\text{CDM})$ are the so-called evidences for models i and ΛCDM respectively. Consequently, the greater the evidence for model i relative to ΛCDM , the greater the Bayes factor. We therefore translate the numerical Bayes factor into a qualitative statement on the strength of evidence against ΛCDM , calculating $\ln B_{i,\Lambda\text{CDM}}$. We note that when $\ln B_{i,\Lambda\text{CDM}} < 0$ for a model, there is no evidence or support for that model over ΛCDM .

Note²-DESI *BAO* is regarded as incompatible with certain datasets that include the SNIa data and degeneracies that it brings on H_0 and r_d . It is clear from the first year *BAO* data released by DESI [34] that the degeneracies are persistent in the sense that H_0 could not be constrained alone. Therefore, in their defence, [109] presented a paper constraining the two (H_0 and r_d) combined using a PantheonP compilation of 1550 supernovae and found the compatibility among the two. In a study published last year after the release of *BAO* data [110], in confirmation of such degeneracies they found that their findings suggest that IDE models with $Q \propto \rho_{de}$ support the presence of an interaction where dark energy decays into dark matter, whereas the existence of the $Q \propto \rho_{dm}$ interaction depends on the form of the proportionality coefficient. Now, let us continue by providing a brief background of the data set considered.

4.2 Data

In this section, we present the datasets used to put constraints on the models considered. While there are numerous cosmological datasets available, such as large-scale structure constraints from early investigations like the CMB [111], Baryon acoustic oscillations (*BAO*) [112, 113] from the SDSS and Union3, our focus here is on a selection of datasets that provide the most relevant and robust constraints for our analysis such as the recently published the *PantheonP + SH0ES* compilation data [35], 32 data points of the cosmic chronometer [33] and the *-BAO* data from the dark energy spectroscopic Instrument[34]. Using these data sets, we aim to determine the most suitable cosmological parameters for the IDE models considered in this dissertation within practical limits. Cosmological parameters include H_0, ω, Ω_m and ξ . Subsequently, we performed statistical analyses to assess whether the predicted expansion history of the IDE models is in agreement with observations from the data. Finally, compare the results with those obtained from the MCMC analysis of the free parameters in the ΛCDM model using the same datasets.

²Disclaimer!!! See the sub-section of DESI-*BAO* below

4.2.1 DESI measurements of *BAO*

*BAO*³ provides a standard ruler for a typical length scale to measure the angular diameter distance and the Hubble parameter. It can be measured by analysing the spatial distribution of galaxies and consequently provides an independent way to measure the expansion rate of the Universe and can impose significant constraints on the cosmological parameters. The DESI *BAO* data includes tracers of the bright galaxy sample (BGS), luminous red galaxies (LRG), emission line galaxies (ELG), quasars (QSO), and the forest in the redshift range $0.1 \leq z \leq 4.2$. These tracers are described through the transverse co-moving distance, the angle-averaged distance, and the Hubble horizon, where is the co-moving sound horizon at the drag epoch, according to [34]. Table 4.1 below illustrates the statistical values for the DESI sample used for the DESI *BAO* measurements. The Hubble distance $D_H(z)$, co-

Tracers	z	z_{eff}	D_m/r_d	D_H/r_d	D_v/r_d
BGS	0.1-0.4	0.295	-	-	7.93 ± 0.15
LRG1	0.4-0.6	0.590	13.62 ± 0.25	20.98 ± 0.61	-0.445
LRG2	0.6-0.8	0.706	16.85 ± 0.32	20.08 ± 0.60	-0.420
LRG3+ELGI	0.8-1.1	0.930	21.71 ± 0.28	17.88 ± 0.35	-0.389
ELG2	1.1-1.6	1.317	27.79 ± 0.69	13.82 ± 0.42	-0.444
QSO	0.8-2.1	1.491	-	-	26.07 ± 0.67
LyaQSO	1.77-4.16	2.330	39.71 ± 0.94	8.52 ± 0.17	-0.477

Table 4.1: Statistics of the DESI samples utilised in the DESI *BAO* measurements [34].

moving angular diameter distance $D_M(z)$, and volume-average distance $D_V(z)$ are computed as follows:

$$D_H(z) = \frac{c}{H(z)}, \quad (4.2)$$

$$D_m(z) = (1+z)D_A(z), \quad (4.3)$$

$$D_v(z) = \left(\frac{(1+z)^2 D_A(z)^2 c z}{H(z)} \right)^{1/3}, \quad (4.4)$$

where $D_A(z) = (1+z)^{-2} D_L(z)$ is the distance of the angular diameter, related to the distance of luminosity $D_L(z)$ through a redshift-dependent scaling. Therefore, *BAO* can be measured by analysing the spatial distribution of galaxies and consequently provides an independent way to measure the expansion rate of the Universe and can impose significant constraints on the cosmological parameters. The χ^2 likelihood function of each ratio can be written as

$$\chi_{BAO}^2 = \sum_i^N \left\{ \left[\frac{X_i - X_{th}}{\sigma_{X_i}} \right]^2 + \ln(2\pi\sigma_{X_i}^2) \right\}. \quad (4.5)$$

³*BAO* data used is the DESI-*BAO* and will be labelled *BAO* throughout the dissertation

DESI-*BAO* limitations

In cosmology, physics requires that different observations align consistently. In short, a late-time accelerated expansion is only plausible if CMB, *BAO*, SNIa, and many other observational measurements agree on its existence. Recently, DESI released the first round of cosmological constraints based on *BAO*, providing clues of evolution in the dark sector [34]. Therefore, there have been noticeable limitations with the dataset, in that we say it is a suggestive dataset but not inconclusive, as it needs some cross-checks with other probes to rule out systematics or statistical noise [114]. With noise we mean the inability of *BAO* to constrain H_0 and r_d alone (see Table 3 in [34] for the constraint H_0). Therefore, in consideration of the $\omega_0\omega_a$ CDM model with a redshift-dependent DE equation of state, where the assumption of $\omega_0 = -1$ and $\omega_a = 0$ recovers the standard model, DESI is preferred for $\omega_0 = -1$, driven by the Luminous Red Galaxy (LRG)(see Table 4.1) sample with effective redshift $z_{eff} = 0.51$. They combined DESI with Planck CMB, PantheonPlus, Union3 and DES data; therefore, the preferences increased to 2.5σ , 3.5σ and 3.9σ over Λ CMB, respectively. Our takeaway from this is that DESI cannot resolve the H_0 tension without external probes. But how sure can we be if we do not incorporate the very data into our IDE models because scientific knowledge is built on verifiable, reproducible evidence?

Here, we integrate DESI-*BAO* into the IDE models examined throughout this dissertation. The main aspect of our analysis includes careful accountability for the degeneracies that arise when combining *BAO* with other measurements. These degeneracies, which have been extensively studied in the recent literature, can significantly influence constraints on dark energy dynamics and the interaction between dark sectors [115–117].

4.2.2 SNIa *PantheonP* + *SH0ES* data

Type Ia supernovae were the first observation to reveal evidence of the accelerating expansion of the Universe as mentioned in Chapter 1. Due to their unique abilities to trace the expansion history of the Universe over the past 13 billion years, we often say that it is fundamental to the concordance model [5, 69]. These supernovae, occurring in binary star systems, are important for cosmology because of their uniform intrinsic brightness, which makes them ideal for measuring distances to distant galaxies. Therefore, the need arises for the distance modulus to compare the predictions of the cosmological models considered.

In this dissertation, we use the *PantheonP* + *SH0ES* SNIa dataset, which includes more data and a wider redshift range than the original Pantheon compilation. The

latter used a compilation of 1048 SNIa samples to study the expansion history of the Universe with a redshift range of $z \geq 0.01$, and the *PantheonP + SH0ES* analysis includes an even larger set of 1701 SNIa samples [118] with an extended redshift range of $0.01 < z < 2.50$. The compilation provides improved constraints on the cosmological parameters. Therefore, the key advancement in the *PantheonP + SH0ES* analysis is the inclusion of SH0ES Cepheid host distance anchors in the likelihood function. This addition helps to break the degeneracy between the absolute magnitude M and the Hubble constant H_0 , which is a limitation when using SNIa data alone. Furthermore, extended redshift coverage improves the handling of systematic uncertainties, thereby allowing more precise and reliable cosmological inferences [118, 119]. We follow the formalism used in the [118, 119], in which cosmological parameters are constrained by minimising the χ^2 likelihood as;

$$-2 \ln(L) = \chi^2 = \Delta D^T C_{stat+syst}^{-1} \Delta D, \quad (4.6)$$

where D is the vector of the 1701 SNIa distance modulus residuals which are computed using

$$\Delta D_i = \mu_i - \mu_Q(z_i), \quad (4.7)$$

here, Q is the interacting term(models). Therefore, the model distance is defined as;

$$\mu_Q(z_i) = 5 \log(d_L(z_i)/10\text{pc}), \quad (4.8)$$

where d_L is the model-based luminosity distance that includes the parameters describing the expansion history. Herein, the luminosity distance for flat Universe is described as

$$d_L(z) = (1+z)c \int_0^z \frac{dz'}{H(z')}. \quad (4.9)$$

The parameters M and H_0 are mentioned to be degenerate when analysing the Pantheon compilation alone. However, they presented constraints that include the recently released SH0ES Cepheid host distance anchors [72] in the likelihood that facilitate constraints on M and H_0 . Therefore, when using the SH0ES Cepheid host distances, we then modify the residuals as

$$\Delta D'_i = \begin{cases} \mu_i - \mu_i^{\text{Cepheid}} & i \in \text{Cepheid hosts} \\ \mu_i - \mu_Q(z_i) & \text{otherwise,} \end{cases} \quad (4.10)$$

where μ_i^{Cepheid} is the Cepheid calibrated host-galaxy distance provided by SH0ES, and $\mu_i - \mu_i^{\text{Cepheid}}$ is sensitive to the parameters M and H_0 and is largely insensitive to Ω_m or ω [118].

4.2.3 Cosmic chronometers (CC)

Here, we use 32 cosmic chronometer data points, published in the study by Michele et al. study [33]. This dataset is based on the differential ages of passive galaxies, which

include the full covariance matrix accounting for systematic uncertainty stemming from the initial mass function, stellar library, and metallicity. Generally, *CC* is known for its simplest technique for measuring the Hubble parameter as a redshift function, independent of the cosmological model or the Cepheid distance ladder. Instead, it relies on estimating stellar ages using well-established stellar population synthesis techniques. By comparing the age difference between the two passively evolving galaxies at different redshifts, we can determine the observable quantity $\frac{\Delta z}{\Delta t}$. This, in turn, allows for the calculation of the Hubble parameter through the relation

$$H(z) = -\frac{1}{1+z} \frac{dz}{dt}, \quad (4.11)$$

where dt/dz is the look-back time differential change with redshift. The cosmic chronometers have been shown to be more reliable than other methods based on absolute galaxy age determination. Moreover, direct measurement of $H(z)$ can be employed to calibrate the distance ladder, since the energy scale of the Universe is determined independently. Consequently, here we will use these data in comparison with the calibrated Cepheid employed by SH0ES on the direct distance ladder. They span the redshift range $0.07 < z < 1.965$ and constitute the most updated data set on *CC* in the literature. In recent years, important efforts have been dedicated to building the error budget of *CC* data, as in [33]. They use full covariance matrix data, computed as

$$C_{xy} = C_{xy}^{stats} + C_{xy}^{sys}, \quad (4.12)$$

where C_{xy}^{stats} contains the statistical errors and C_{xy}^{sys} accounts for several effects related to the estimate of physical properties of the galaxies. The corresponding χ_H^2 estimator is given by:

$$\chi_H^2 = \sum_{i=1}^{32} \frac{(H(z_i, Q) - H_{\text{obs}}(z_i))^2}{\sigma_H^2(z_i)}, \quad (4.13)$$

where $H(z_i, Q)$ are the theoretical Hubble parameter values at redshift z_i with model parameters Q , and $H_{\text{obs}}(z_i)$ are the observed Hubble data values at z_i , with observational error $\sigma_H(z_i)$ [33].

4.2.4 Joint datasets

Here, we employ the total likelihood function $L \propto \chi^2/2$ to obtain combined constraints for the model parameters, utilising data from DESI-*BAO*, *CC*, and *PantheonP* + *SH0ES* samples.

- *BAO+CC*: The combination is considered in constraining the cosmological models in this dissertation. As mentioned above, we are not imposing a fixed term in order to avoid biases. In the same way, we explore all possibilities

of the observations. The combination is believed to break the degeneracies in the estimation of cosmological parameters; therefore, in [34], the combination yielded tight constraints for Λ CDM. However, this reinforced the 3σ discrepancy for the late Universe [10], due to the slightly higher H_0 estimate and not solving the tension.

- *BAO + PantheonP + SH0ES* In the literature, the combination of these two is highly avoided, understandably so, because the two use different measures to constrain different conditions. In which DESI-BAO constrains r_d the sound horizon and the angular distance d_A , while Pantheon+ focuses on providing the luminosity distance. However, when calibrated with SH0ES, they should break H_0 degeneracies through the absolute scale.
- *CC + PantheonP + SH0ES* This combination is said to improve the crisis faced by cosmology. It is believed to provide critical cosmological tension from the late Universe, specifically H_0 tension. Planck collaborations [10] recorded $H_0 = 67.4 \pm 0.6$ for the late Universe and [118] recorded $H_0 = 73.1 \pm 0.9$, which slightly reduces the model independence that inclines to which Moresco [33] recorded for *CC* alone.
- *BAO + CC + PantheonP + SH0ES* For this combination, robust and comprehensive analysis of parameters is expected as the three harnesses on cross-checking the consistency across early and late Universe, enabling tighter cosmological parameter constraints and providing critical insight into the ongoing H_0 tension.

Therefore, the relevant chi-square functions for the combined analysis are given as;

$$\chi_{joint}^2 = \chi_{BAO}^2 + \chi_{CC}^2 , \quad (4.14)$$

$$\chi_{joint}^2 = \chi_{BAO}^2 + \chi_{PantheonP + SH0ES}^2 , \quad (4.15)$$

$$\chi_{joint}^2 = \chi_{CC}^2 + \chi_{PantheonP + SH0ES}^2 , \quad (4.16)$$

$$\chi_{joint}^2 = \chi_{BAO}^2 + \chi_{CC}^2 + \chi_{PantheonP + SH0ES}^2 . \quad (4.17)$$

By minimising the χ^2 total, we obtain the best-fit cosmological parameters. Each data set contributes independently to the total χ^2 , ensuring a robust statistical analysis in different observational probes.

4.3 Conclusion

In this chapter we provided a brief description of the methods and techniques used to acquire the observational results in Chapter 5. In addition, we explain the three

datasets utilized throughout this dissertation. Taking into consideration the risk we took in assessing the discrepancies that DESI-*BAO* data brings, as per recent literature, we further added *PantheonP + SH0ES*, which has the artefacts of increasing the Hubble tension when combined with *BAO*. We explain the expected results of the combined dataset as it is an unusual combination of datasets. Furthermore, MCMC simulation was explained and how we are going to use the EMCEE Python package to constrain parameters. Upon getting the observational constraints, MCMC is used to plot the corner plots and gives us the distribution across the models. Consequently, the corner plots in the next chapter visually display the results of the MCMC simulation used to determine the free parameters of the models. The symbol $\pm$ indicates uncertainties, while the best-fit value for each parameter appears without the symbol $\pm$.

Chapter 5

Results and discussion

In this chapter, we discuss the results obtained using methods and different data sets mentioned in Chapter 4. We present inferences of the confidence level parameters 68% and 95% for the models considered in Table 5.1. We compare the results of each model to those of the constraints of the parameters Λ CDM. Among these models, the Λ CDM model has the least cosmological parameters $\{\Omega_m, H_0\}$ compared to the IDE models $\{\Omega_m, H_0, \omega, \xi\}$.

5.1 Observational constraints

We start our analysis with the Λ CDM model, integrating the cosmological constraints from the data sets mentioned in Chapter 4. Figure 5.1 presents the marginalised posterior distribution in 1D and the joint confidence regions in 2D at the confidence level 1σ and 2σ for the main cosmological parameters.

- Here, the red shaded regions correspond to constraints derived from the *BAO + CC* and we obtain the following best fit values with 1σ uncertainties: $H_0 = 69.149^{+1.721}_{-0.704}$ km/s/Mpc, $\Omega_m = 0.296 \pm 0.014$ and $r_d = 147.239^{+3.489}_{-3.348}$ Mpc. The results show agreement with the Planck 2018 CMB observations [14].
- For *CC + PantheonP + SH0ES*, which is the green shaded region in Figure 5.1 we obtained the following best fit $\Omega_m = 0.313^{+0.016}_{-0.015}$, $H_0 = 72.222^{+0.889}_{-0.882}$ km/s/Mpc.
- Then, the purple shaded regions show the combination between BAO and *PantheonP + SH0ES*, and the combination yields $H_0 = 73.726^{+1.027}_{-1.003}$ km/s/Mpc, $\Omega_m = 0.310^{+0.012}_{-0.012}$ and $r_d = 136.660^{+2.240}_{-2.230}$ Mpc uncertainties, which we assume that H_0 overestimates and show inconsistency between the two datasets.

- Finally, we combined the three data sets of *BAO + CC + PantheonP + SH0ES* obtaining the following uncertainties: $H_0 = 0.303^{+0.011}_{-0.011}$ km/s/Mpc, $\Omega_m = 0.303^{+0.011}_{-0.011}$ and $r_d = 139.922^{+1.918}_{-1.879}$ Mpc.

Figure 5.1 shows tightly constrained parameters Ω_m and H_0 , meaning repeated runs of the MCMC algorithm produce posterior distribution with values consistently confined within the 1σ and 2σ levels indicating high statistical stability and robustness in their estimation, thereby reinforcing the precision and reliability of the inferred cosmological parameters. As mentioned above, we are carefully accounting for the

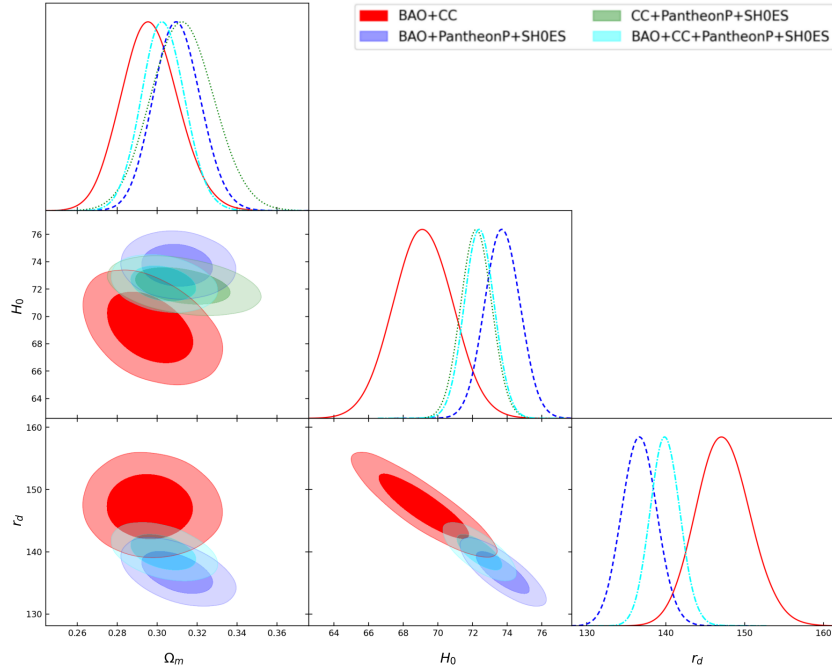


Figure 5.1: Two-dimensional plots illustrating 1D marginalised distributions with 1σ and 2σ contours of cosmological parameters for the Λ CDM model from the joint datasets.

inconsistencies and discrepancies of *BAO* mentioned in the recent literature [117, 120]. Figure A.1 left panel in the appendix shows the two-dimensional plots and 1D marginalized distributions with 1σ and 2σ contours of cosmological parameters for the Λ CDM model from only *CC*, *BAO* and *PantheonP + SH0ES*. The plot is included to show the discrepancies found in the dataset, as it can be seen that results do not show any consistencies when constraining the H_0 alone with *BAO*, meaning the H_0 tension worsens. However, oddly, Ω_m is very much consistent with the measurements found in the results of [34, 109].

We show in Table 5.1 the summarised numerical constraints and parameter correlations for the Λ CDM model. Our analysis yields $H_0 = 73.575^{+0.114}_{-0.471}$ km/s/Mpc from the *PantheonP + SH0ES* dataset. This value demonstrates consistency with the

previous findings in [118], where systematic uncertainties of 1σ contributed. Significantly, these uncertainties are insufficient in magnitude to alleviate or resolve the Hubble tension, as they account for only a minor fraction of the observed discrepancy between early and late universe H_0 measurements. DESI *BAO* on the other hand, registers $H_0 = 69.328^{+7.181}_{-6.556}$ km/s/Mpc to favour the lowest correlation with high error bars, which makes it statistically comparable, but this value agrees with that reported in Planck’s early universe estimate $H_0 = 67.425 \pm 0.53$ km/s/Mpc rather than to the *PantheonP* + *SH0ES* value. DESI *BAO* alone has the implication of not being able to resolve the Hubble tension due to its low precision, which is adequately 10% less than Planck’s. The H_0 value for *CC* alone falls between the values reported in the early universe estimate from Planck’s and *PantheonP* + *SH0ES* in the late universe with the mean uncertainty of the ≈ 2.6 universe. This makes it difficult to conclude whether this dataset alone can resolve the H_0 tension since it is just neutral. However, when combined with CMB Λ CDM shows minimal evidence of deviations (see [14]). The combination of all datasets *BAO* + *CC* + *PantheonP* + *SH0ES* increases the H_0 tension with $H_0 = 72.392^{+0.859}_{-0.855}$ km/s/Mpc to close 6σ . The systematics of the results rule out our Λ CDM.

The joint data set *BAO* + *CC* yields $H_0 = 69.149^{+1.721}_{-1.704}$ km/s/Mpc, which is very close to *CC* alone if they are not in agreement with each other. In that we can conclude and say that this value lies between the estimated value of the early universe and the late universe. *CC* alone is similar, but shows a significant precision of $\approx 1.7\sigma$, as a result combining the two gives the highlights but does not resolve the tension but alleviates it. In contrast, *BAO* + *PantheonP* + *SH0ES* increases the value H_0 due to the disagreement between the data sets and the early estimation of the universe. In hindsight, the value is very close to the one reported to the late universe estimation [118]. *CC* + *PantheonP* + *SH0ES* gives consistency with the late universe estimate but a bit lower at 0.7σ [118], this could be due to the independence leverage of a model. However, the model disagrees with the early universe with a higher precision of 5.1σ [11].

After complete observation of Figure 5.1 and Table 5.1, we can conclude and account for the risk we took to measure the parameters considered with the DESI *BAO* data, which was reported to have degeneracies in [34], especially when H_0 is limited alone, as the data do not measure absolute distances directly. Rather, they measure the ratios of distances to the sound scale which is r_d [34]. When constrained together, this tests for deviations from Λ CDM. We then worsened the analysis by joining the calibrated *PantheonP* + *SH0ES*, this combination exacerbates the tension H_0 to more than 5σ . This contrasts with the sound-horizon predictions of Λ CDM, because this is not caused by some sort of underlying physics but by the incompatibility of the two datasets. Unlike the combination of all datasets, *CC* + *PantheonP* + *SH0ES* maintains the H_0 tension at 5σ , reducing the calibrated *Pan-*

theonP + *SH0ES*. Despite the degeneracies that lie within the *BAO* and *PantheonP* + *SH0ES*, we cannot rule out Λ CDM before cross-checking other available datasets. In that we still believe that this model remains successful and fits most of the observations. But remember that we are still in search of a model that can change the perspectives of us believing in Λ CDM, so we continue to do the analogy of other cosmological models in comparison to Λ CDM.

Moving on to the considered IDE models, as mentioned in Chapter 3, we considered both linear and non-linear models. We summarize the constraints of the cosmological parameters in Table 5.1 for our IDE models.

5.1.1 IDE1

The observational constraints for this model are shown in Figure 5.4 and Table 5.1, where we show the marginalized 1D and 2D posterior distributions for the constrained parameters H_0, Ω_m, ω and ξ based on the joint dataset *BAO* + *CC*, *BAO* + *PantheonP* + *SH0ES*, *CC* + *PantheonP* + *SH0ES* and *BAO* + *CC* + *PantheonP* + *SH0ES*. Also in Figure A.1 left panel, we show the parameter constraints measured with only DESI *BAO* alone, *CC* alone, and *PantheonP* + *SH0ES* alone. Upon these measurements, all standalone datasets produce a negative interacting term ξ , which means $\xi \neq 0$, in that it is allowed at the confidence level 68% (*BAO* alone: $\xi = 0.087^{+0.160}_{-0.129}$, *CC* alone: $\xi = 0.040^{+0.181}_{-0.215}$, *PantheonP* + *SH0ES* alone: $\xi = 0.381^{+0.396}_{-0.406}$). The model shows similar constraints compared to Λ CDM, as can be seen in the joint *BAO* + *PantheonP* + *SH0ES* show an overestimation of H_0 . This shows consistency with the H_0 value of Λ CDM. Taking this into account, we confirm that there are in fact degeneracies that occur when combining the two datasets [34, 117], in which Λ CDM also experiences it. It is clear that this model fails to tightly constrain the cosmological parameters compared to the Λ CDM model, causing a significant deviation between the two. However, the constraints of *BAO* + *PantheonP* + *SH0ES* do not show a noticeable difference; this stems from the revealed degeneracies between the two datasets when combined. When looking at Ω_m estimations, specifically the $\{\Omega_m, \xi\}$ contour in Figure 5.4 shows that the inverse correlation is found with H_0 , which makes this parameter behave accordingly with increasing or decreasing Hubble constant. Interestingly, when looking at the constraints in Figure 5.4 of all the joint datasets, we can see that the tilt of the contour toward the vertical position shows minimal or no correlation between them. Consequently, the individual data sets in Figure A.1 reveal that a strong correlation is deduced. We summarize the constraints for this model in Table 5.1. The best-fit values of the cosmological parameters are $H_0 = 68.890^{+2.589}_{-2.483}$ km/s/Mpc, $\Omega_m = 0.106^{+0.020}_{-0.016}$, $\omega = -1.061^{+0.173}_{-0.223}$, $\xi = 0.097^{+0.161}_{-0.134}$ from the joint *BAO* + *CC* measurements, $H_0 = 68.028^{+0.772}_{-0.817}$ km/s/Mpc, $\Omega_m = 0.201^{+0.114}_{-0.075}$, $\omega = -0.578^{+0.210}_{-0.345}$,

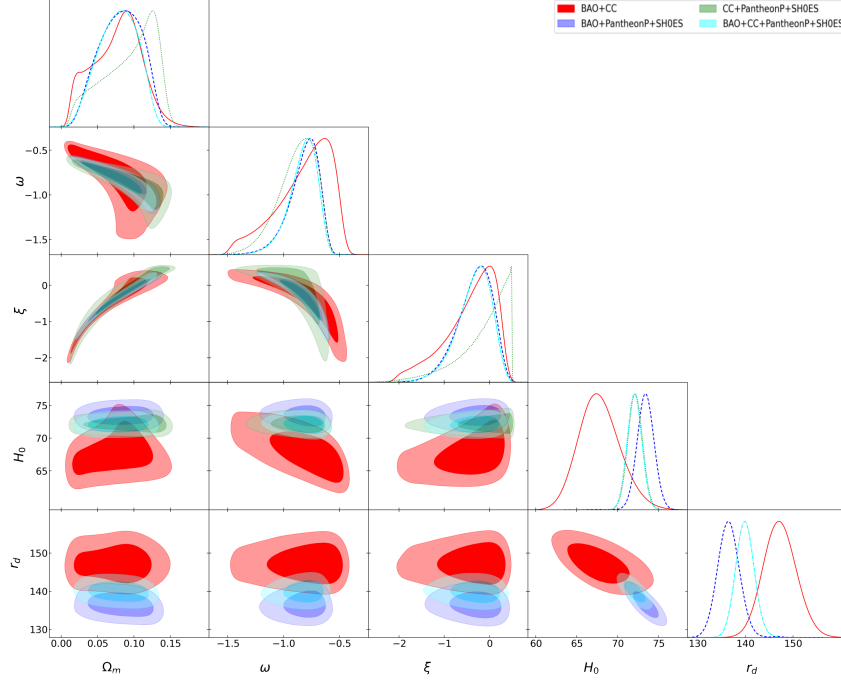


Figure 5.2: Two-dimensional plots illustrating 1D marginalized distributions with 1σ and 2σ contours of cosmological parameters for the IDE1 model from the combination of all.

$\xi = 0.114^{+0.076}_{-0.077}$ from $CC + PantheonP + SH0ES$, and $H_0 = 72.063^{+0.887}_{-0.881}$ km/s/Mpc, $\Omega_m = 0.303^{+0.326}_{-0.427}$, $\omega = -0.813^{+0.107}_{-0.142}$, $\xi = 0.079^{+0.027}_{-0.029}$ from $BAO + CC + PantheonP + SH0ES$. It can be seen that the parameter ξ for the joint datasets favours the quintessence in which DE decays into DM, contributing to the accelerating expansion of the universe at present. This combination provides strong constraints on all parameters, but $BAO + CC$, $BAO + PantheonP + SH0ES$, and $CC + PantheonP + SH0ES$ weakly constrain the parameters except for H_0 . Thus, the constraint estimate value H_0 from $CC + PantheonP + SH0ES$ shows alignment with the value found in [10] when combined with SNIa and BAO. This model does not reconcile the tension H_0 , as can be seen from the inconsistencies in the values. In summary, this interacting model shows mild interaction in the dark sector and shows consistency with the non-interacting cosmology.

5.1.2 IDE2

The observational constraints for the IDE2 model are shown in Figure 5.3 and Table 5.1. In the same way as in IDE1, the marginalized 1D and 2D posterior distributions for parameters $\{H_0, \Omega, \omega, \xi\}$ based on the joint datasets $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, $BAO + CC + PantheonP + SH0ES$ are presented. The values for H_0 are recorded as $67.360^{+2.643}_{-2.291}$ km/s/Mpc,

$73.350^{+1.084}_{-1.114}$ km/s/Mpc, $72.105^{+0.885}_{-0.827}$ km/s/Mpc, $71.874^{+0.987}_{-1.192}$ km/s/Mpc, respectively. The estimate values for all the combined data show consistency with the 1σ precision of the early universe [14] and the late universe [72], in that the $BAO + CC$ estimate is consistent with the recorded Planck collaboration $\{H_0 = 67.4 \pm 0.5\}$ km/s/Mpc in the early universe, which is in alignment with Λ CDM. Despite being close to the value recorded in the early Universe it is still lower than the recently recorded estimate for the late universe, in that this combination does not resolve the H_0 tension. The joint $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, $BAO + CC + PantheonP + SH0ES$ shows consistency with the late universe $\{H_0 = 73.0 \pm 1.0\}$ km/s/Mpc recorded by Riess [72] which is also higher than the estimate recorded from the early Universe. This combination shows potential in alleviating tension H_0 , but we cannot confirm before conducting the statistical analysis discussed in the next section.

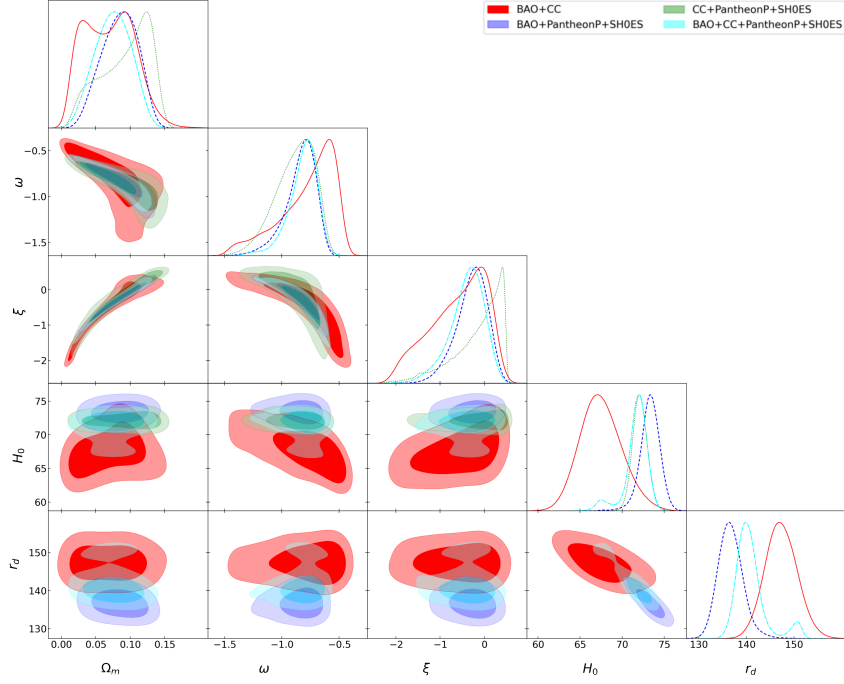


Figure 5.3: Two-dimensional plots illustrating 1D marginalized distributions with 1σ and 2σ contours of cosmological parameters for the IDE2 model from the combination of all.

Concerning Ω_m , the estimated values for all joint datasets are lower than the one recorded for Λ CDM in which CMB and LSS prefer the higher $\Omega_m \approx 0.31$ at least. However, we record $\Omega_m = 0.151^{+0.037}_{-0.042}$, $\Omega_m = 0.149^{+0.027}_{-0.029}$, $\Omega_m = 0.100^{+0.030}_{-0.050}$, $\Omega_m = 0.098^{+0.028}_{-0.029}$ for the combined datasets $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, $BAO + CC + PantheonP + SH0ES$, respectively. This reveals the dominance of DE and less DM in the universe due to the noticeable shift from the parameter. In this regard, Ω_m implicates the H_0 tension and other

challenges faced by Λ CDM generally, but the large uncertainties make it marginally consistent with Λ CDM. Now, looking at our interacting term ξ estimate values confirms the low estimate for Ω_m as it shows energy transfers from DM to DE which quantifies the interaction between the two. The recorded estimate values are $\xi = -0.473^{+0.553}_{-0.832}$, $\xi = -0.245^{+0.312}_{-0.394}$, $\xi = 0.012^{+0.363}_{-0.726}$, $\xi = -0.344^{+0.333}_{-0.432}$ for the combined datasets $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, $BAO + CC + PantheonP + SH0ES$, respectively. Added on is ω with $\omega = -0.713^{+0.171}_{-0.321}$ for $BAO + CC$, $\omega = -0.816^{+0.107}_{-0.153}$ for $BAO + PantheonP + SH0ES$, $\omega = -0.861^{+0.153}_{-0.201}$ for $CC + PantheonP + SH0ES$ and for $CC + BAO + PantheonP + SH0ES$ $\omega = -0.797^{+0.100}_{-0.139}$. The estimate of all the joint data hints at the phantom dark energy, which is slightly less than the cosmological constant $\omega = -1$.

In summary $BAO + CC$ reinforce the tension of H_0 , even though there is not much deviation between this model and the model of Λ CDM. $BAO + PantheonP + SH0ES$ on the other end reduces the tension H_0 , but Λ CDM remains intact due to Ω_m and ω moving toward Planck values that show no strong evidence for the cosmological constant $\omega \neq -1$ and $\xi \neq 0$, these are consistent with the results recorded in [121]. Lastly, the full combination $BAO + CC + PantheonP + SH0ES$ does not solve H_0 , but reduces it as its value falls between the early and late Universe. Therefore, the IDE2 model does not fully resolve the H_0 , despite reducing the uncertainties considering the all-joint combination.

5.1.3 IDE3

The observational constraints for this model are also shown in Table 5.1. We continue to focus on the cosmological models at hand H_0, Ω_m, ω and ξ . This model shows observational constraints similar to those of IDE1. Similarly to IDE1, with a slight difference, this model shows a mild interaction cosmology. Thus, $\xi = -1.963^{+0.037}_{-0.026}$ for the joint $BAO + CC$ datasets and $\xi = -2.060^{+0.035}_{-0.040}$ for all combined $BAO + CC + PantheonP + SH0ES$. So, in terms of the estimation parameter ξ , we can safely say that a mild interaction between dark sectors is allowed for this model. Regarding Ω_m for $BAO + CC$ is much lower than that of Λ CDM, in which the value of H_0 also seems to be underestimating those found in [33] and [34]. When adding $PantheonP + SH0ES$ the value of H_0 increases in alignment to that of Λ CDM, however, Ω_m overestimates reaching $0.398^{+0.001}_{-0.003}$, which also does not align with any of [118]. The estimate of H_0 for the combined $BAO + CC + PantheonP + SH0ES$ agrees with the estimate value in [122] with predictions of ξ greater than zero, the constraints for the cosmological parameters are $H_0 = 72.063^{+0.887}_{-0.881}$, $\Omega_m = 0.179^{+0.027}_{-0.029}$, $\omega = -0.813^{+0.107}_{-0.142}$, $\xi = -0.303^{+0.326}_{-0.427}$. Their results showed consistency with 2σ precision with the late estimated value H_0 [118]. However, compared to the Λ CDM model, the tension

is neither eased nor solved when looking at the results obtained in [75]. Rather, the energy travels from DM to DE considering the lowest recorded value of Ω_m with a negative interacting term ξ . We also show the marginalized 2D and 1D

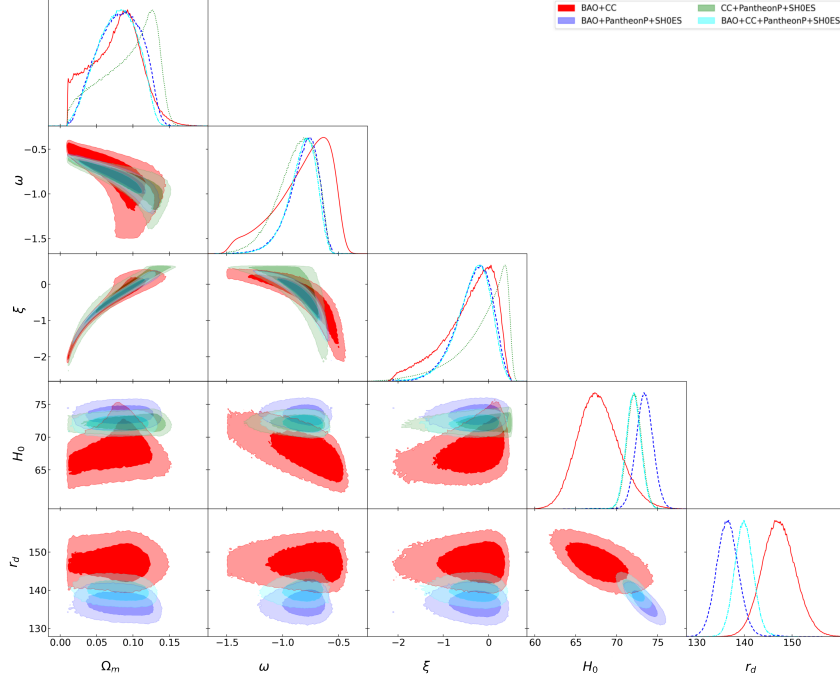


Figure 5.4: Two-dimensional plots illustrating 1D marginalized distributions with 1σ and 2σ contours of cosmological parameters for the IDE3 model from the combination of all.

distributions with the contours of 1σ and 2σ of the cosmological parameters in Figure 5.4, since it can be seen that this model does not show tight constraints for the cosmological parameters, except for the H_0 parameter. The model also has a slight tilt on the contours $\{\Omega_m, \xi\}$ and $\{\Omega_m, \omega\}$, the tilt means that the parameters are not independently constrained showing the degeneracy effect on our measurements. Thus, the IDE3 model does not alleviate the H_0 tension and shows little to no evidence of an interaction in the dark sector.

5.1.4 IDE4

We finally consider the nonlinear models, the least studied models in the field of cosmology, here we present the ratio-based coupling model. Similarly, we present the observational constraints in Figure 5.6 and the best-fit values in Table 5.1. The observational constraints with joint datasets slightly favour the higher values of H_0 and lower values of Ω_m compared to the values of Λ CDM. Thus, when BAO is combined with CC , $(H_0 = 68.364^{+2.710}_{-2.444})$ km/s/Mpc suggests being lower than the value found in the early universe [10]. In spite of that, moving closer to the late universe probes, we have

great uncertainties. Whereas, $BAO + PantheonP + SH0ES$ ($H_0 = 73.620^{+1.025}_{-1.013}$) km/s/Mpc shows consistency with the recorded value of $H_0 = 73.05 \pm 0.1$ km/s/Mpc from SH0ES [118] but shows tension from the recorded value in the early universe. Combining $CC + PantheonP + SH0ES$ ($H_0 = 72.087^{+0.890}_{-0.871}$) km/s/Mpc on the other end shows slight possibilities to reduce tension H_0 , as its value lies between the early and late universe probes [11, 118]. Continuing to combine all probes $BAO + CC + PantheonP + SH0ES$ ($H_0 = 72.083^{+0.880}_{-0.880}$) km/s/Mpc, the estimate value suggests robustness, resulting in high possibilities of reducing tension. Now, concerning the constraints on the parameter ξ , we see that for $BAO+CC$ ($\xi = -0.044^{+0.085}_{-0.125}$) showing consistency within the confidence level 68%. $BAO + PantheonP + SH0ES$ $\xi = -0.074^{+0.087}_{-0.104}$ at a confidence level greater than 68%. We also record the $CC + PantheonP + SH0ES$ as $\xi = 0.021^{+0.097}_{-0.153}$ which shows consistency with the confidence level 68%, and finally combining all the datasets $BAO + CC + PantheonP + SH0ES$ with $\xi = -0.074^{+0.086}_{-0.110}$ at more than 68% as well. Therefore, the data sets show compatibility with nonzero values of the ξ parameter, indicating that the possibility of an interaction in the dark sector through this coupling function is expected. Now, the observational constraints on the parameters as shown in Figure 5.6

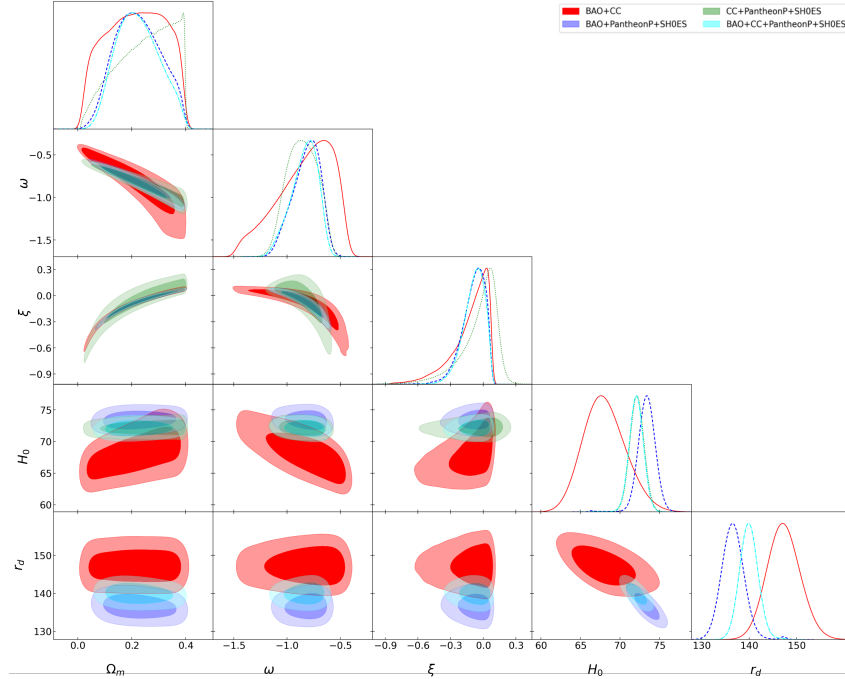


Figure 5.5: Two-dimensional plots illustrating 1D marginalized distributions with 1σ and 2σ contours of cosmological parameters for the IDE4 model from the combination of all.

are weakly constrained, apart from the H_0 parameter showing compatibility among the combined datasets with a smooth Gaussian distribution. Let us look at the contours of $\{\Omega_m, \omega\}$, $\{\Omega_m, \xi\}$ and $\{\omega, \xi\}$ they have a tilted form showing the degeneracy between the parameters. This comes from the set priors for each parameter.

Despite the degeneracies shown from the contours, this model reveals weak observational support for the interaction between DM and DE. Moreover, the model can help alleviate the H_0 tension due to the recorded uncertainties.

5.1.5 IDE5

Now the geometric mean coupling model. It is mentioned that this model gives a persistent modification to standard cosmology [103, 104] and is also one of the rarely studied in the literature. We present the observational constraint in Figure 5.3 for the joint datasets: $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, and $BAO + CC + PantheonP + SH0ES$, respectively. Similarly, we present them all together with the standalone datasets: BAO , CC , and $PantheonP + SH0ES$ in Table 5.1. One would say that freeing priors for the parameters was a mistake; here, we take it as broadening the understanding of hidden physics behind every model. This model is a perfect example of “Let free”, the reason being the constraints it records. Looking at Ω_m constraints for $BAO + CC$ ($\Omega_m = 0.151^{+0.037}_{-0.042}$), $BAO + PantheonP + SH0ES$ ($\Omega_m = 0.149^{+0.027}_{-0.029}$), $CC + PantheonP + SH0ES$ ($\Omega_m = 0.100^{+0.030}_{-0.050}$), $BAO + CC + PantheonP + SH0ES$ ($\Omega_m = 0.098^{+0.039}_{-0.044}$), see [103] where the study recorded the estimate value of Ω_m as $0.3233^{+0.0081}_{-0.0084}$ using the combined datasets SN+CMB+BAO+ H_0 . We can say the scientific physics behind this estimate is valid and justifiable, at the same time we question the results we got and want to rule out the model immediately, but doing that would need extended cross checks with other observational data in which our horizon of understanding levels up. It is also clear that the estimate values for H_0 favour the higher values, especially those on $BAO + PantheonP + SH0ES$ at $H_0 = 76.809^{+1.888}_{-2.060}$, the $CC + PantheonP + SH0ES$ at $H_0 = 77.891^{+1.602}_{-3.309}$ and the $BAO + CC + PantheonP + SH0ES$ at $H_0 = 75.589^{+1.886}_{-2.119}$ which to us is very unusual values to have ever been recorded, these values are hard to reconcile with those recorded from the early and late universe. But for $BAO + CC$ constraints on $H_0 = 70.955^{+4.213}_{-3.748}$ which falls between the early and late universe records of $H_0 = 73.04 \pm 1.04$ for the latter and $H_0 = 67.4 \pm 0.5$ for the first. This model completely increases the Hubble tension compared to all of the models above.

Regarding the observational constraints on the coupling parameter ξ , we can see that for the combined data sets $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, and $BAO + CC + PantheonP + SH0ES$, results in $\xi = -0.157^{+0.252}_{-0.508}$, $\xi = -0.097^{+0.183}_{-0.405}$, $\xi = 0.007^{+0.299}_{-0.442}$, and $\xi = -0.091^{+0.171}_{-0.397}$, respectively. Nevertheless, data sets allow for non-zero values of the interacting term ξ ; therefore, the possibility of an interaction between DM and DE through this coupling function is equally probable. This possibility only hints at the direction of energy transfer in which it favours the DE to DM regime. In summary, this model can alleviate the Hubble

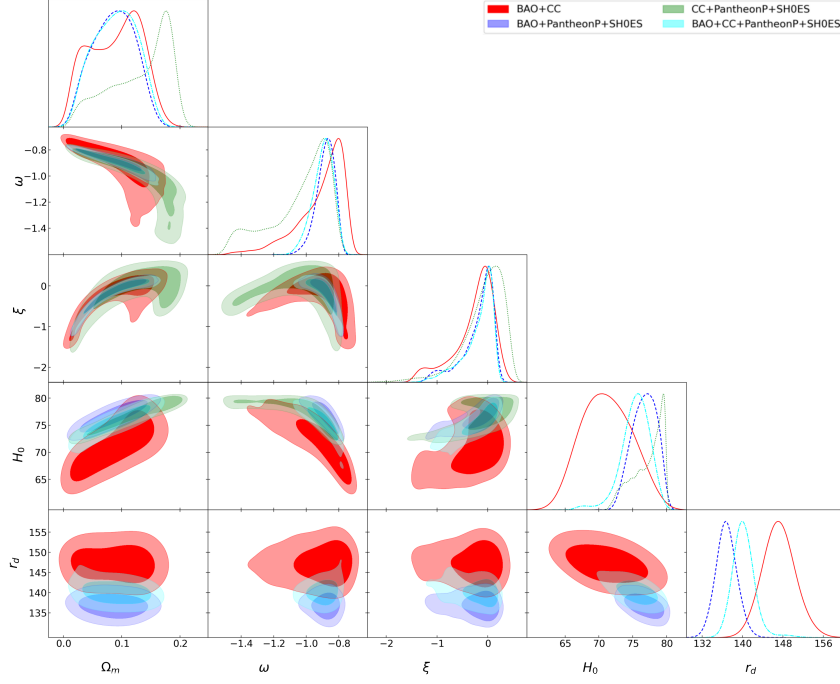


Figure 5.6: Two-dimensional plots illustrating 1D marginalized distributions with 1σ and 2σ contours of cosmological parameters for the IDE5 model from the combination of all.

tension as it records the locally (late universe) appealing H_0 estimate for $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$, and $BAO + CC + PantheonP + SH0ES$ combined datasets, at the same time it exacerbates the tension with the early universe records. However, $BAO + CC$ is in alignment with the early universe.

5.2 Statistical analysis

For comparing different models, a proper analysis method must be chosen. Therefore, we choose to use the widely used χ^2 comparison method because it is only fair since we are considering the same number of parameters for our models. We further perform a posterior likelihood analysis with flat priors for the parameters. To do that, we employ the analytical formula in computing the theoretical calculations, implementing the Akaike Information Criteria (AIC) & Bayesian Information Criteria (BIC) from [123–125], using Λ CDM as a benchmark model to compare the viability of the IDE models. These model selection criterion help to determine whether the model is accepted or rejected relative to Λ CDM. The two can be defined as

$$AIC = \chi^2 + 2k, \quad (5.1)$$

$$BIC = \chi^2 + k \ln N, \quad (5.2)$$

Data Compilation	H_0	Ω_m	ω	ξ	M_{abs}	r_d
Best-fit values for Λ CDM						
<i>BAO</i>	$69.328^{+7.181}_{-6.556}$	$0.295^{+0.015}_{-0.014}$	-	-	-	$147.063^{+15.300}_{-13.806}$
<i>CC</i>	$68.252^{+2.803}_{-2.477}$	$0.317^{+0.048}_{-0.050}$	-	-	-	-
<i>PantheonP + SH0ES</i>	$73.575^{+0.114}_{-0.471}$	0.331 ± 0.018	-	-	$-19.246^{+0.009}_{-0.013}$	-
<i>BAO + CC</i>	$69.149^{+1.721}_{-1.704}$	$0.296^{+0.014}_{-0.014}$	-	-	-	$147.234^{+3.489}_{-3.348}$
<i>BAO + PantheonP + SH0ES</i>	$73.716^{+1.041}_{-1.009}$	$0.310^{+0.012}_{-0.012}$	-	-	-19.251 ± 0.030	$136.675^{+2.288}_{-2.210}$
<i>CC + PantheonP + SH0ES</i>	$72.222^{+0.889}_{-0.882}$	$0.313^{+0.016}_{-0.015}$	-	-	-19.294 ± 0.025	-
<i>BAO + CC + PantheonP + SH0ES</i>	$72.392^{+0.859}_{-0.855}$	0.303 ± 0.011	-	-	-19.288 ± 0.025	$139.922^{+1.918}_{-1.879}$
Best-fit values for IDE1						
<i>BAO</i>	$69.073^{+7.301}_{-6.259}$	$0.106^{+0.020}_{-0.016}$	$-1.047^{+0.175}_{-0.224}$	$0.087^{+0.160}_{-0.129}$	-	$146.960^{+14.868}_{-14.177}$
<i>CC</i>	$67.717^{+3.125}_{-3.136}$	$0.202^{+0.120}_{-0.076}$	$-0.553^{+0.120}_{-0.076}$	$0.040^{+0.181}_{-0.215}$	-	-
<i>PantheonP + SH0ES</i>	$73.356^{+1.016}_{-1.018}$	$0.205^{+0.118}_{-0.074}$	$-0.695^{+0.274}_{-0.403}$	$0.381^{+0.396}_{-0.406}$	-19.248 ± 0.029	-
<i>BAO + CC</i>	$68.890^{+2.589}_{-2.483}$	$0.106^{+0.020}_{-0.016}$	$-1.061^{+0.173}_{-0.223}$	$0.097^{+0.161}_{-0.134}$	-	$147.338^{+3.468}_{-3.437}$
<i>BAO + PantheonP + SH0ES</i>	$73.451^{+1.029}_{-1.029}$	$0.201^{+0.114}_{-0.075}$	$-0.578^{+0.210}_{-0.345}$	$0.114^{+0.076}_{-0.077}$	$-19.249^{+0.030}_{-0.030}$	$136.318^{+2.307}_{-2.304}$
<i>CC + PantheonP + SH0ES</i>	$68.028^{+0.772}_{-0.817}$	$0.201^{+0.114}_{-0.075}$	$-0.578^{+0.210}_{-0.345}$	$0.114^{+0.076}_{-0.077}$	$-19.312^{+0.024}_{-0.026}$	-
<i>BAO + CC + PantheonP + SH0ES</i>	$72.063^{+0.887}_{-0.881}$	$0.079^{+0.027}_{-0.029}$	$-0.813^{+0.107}_{-0.142}$	$-0.303^{+0.326}_{-0.427}$	$-19.356^{+0.165}_{-0.065}$	-
Best-fit values for IDE2						
<i>BAO</i>	$67.605^{+5.111}_{-4.193}$	$0.294^{+0.017}_{-0.015}$	$-0.749^{+0.557}_{-0.535}$	$-0.215^{+3.461}_{-3.150}$	-	$150.774^{+9.686}_{-10.539}$
<i>CC</i>	$72.865^{+0.432}_{-0.563}$	$0.138^{+0.091}_{-0.035}$	$-1.055^{+0.285}_{-0.353}$	$-1.290^{+0.224}_{-0.322}$	-	-
<i>PantheonP + SH0ES</i>	$73.422^{+0.895}_{-0.916}$	$0.335^{+0.018}_{-0.018}$	$-0.831^{+0.018}_{-0.449}$	$0.001^{+0.070}_{-0.070}$	$-19.249^{+0.025}_{-0.028}$	-
<i>BAO + CC</i>	$67.360^{+2.643}_{-2.291}$	$0.151^{+0.037}_{-0.042}$	$-0.713^{+0.171}_{-0.321}$	$-0.473^{+0.553}_{-0.832}$	-	$147.189^{+3.463}_{-3.163}$
<i>BAO + PantheonP + SH0ES</i>	$73.350^{+1.084}_{-1.114}$	$0.149^{+0.027}_{-0.027}$	$-0.816^{+0.107}_{-0.153}$	$-0.245^{+0.312}_{-0.394}$	$-19.250^{+0.031}_{-0.031}$	$136.563^{+2.616}_{-2.386}$
<i>CC + PantheonP + SH0ES</i>	$72.105^{+0.885}_{-0.827}$	$0.100^{+0.030}_{-0.050}$	$-0.861^{+0.153}_{-0.201}$	$0.012^{+0.363}_{-0.726}$	$-19.312^{+0.030}_{-0.045}$	-
<i>BAO+CC + PantheonP + SH0ES</i>	$71.874^{+0.987}_{-1.192}$	$0.098^{+0.028}_{-0.029}$	$-0.797^{+0.100}_{-0.139}$	$-0.344^{+0.333}_{-0.432}$	$-19.249^{+0.030}_{-0.030}$	$140.325^{+2.751}_{-2.088}$
Best-fit values for IDE3						
<i>BAO</i>	$69.294^{+7.116}_{-6.450}$	$0.106^{+0.020}_{-0.016}$	$-0.857^{+0.130}_{-0.151}$	$-0.420^{+0.249}_{-0.232}$	-	$147.207^{+14.627}_{-14.356}$
<i>CC</i>	$64.086^{+4.191}_{-2.736}$	$0.202^{+0.120}_{-0.076}$	$-1.125^{+0.295}_{-0.252}$	$-0.677^{+0.427}_{-0.333}$	-	-
<i>PantheonP + SH0ES</i>	$78.548^{+0.988}_{-1.245}$	$0.205^{+0.118}_{-0.074}$	$-1.94^{+0.262}_{-0.203}$	$-1.372^{+0.113}_{-0.122}$	-19.257 ± 0.027	-
<i>BAO + CC</i>	$61.141^{+1.182}_{-0.793}$	$0.106^{+0.020}_{-0.016}$	$-1.487^{+0.013}_{-0.009}$	$-1.963^{+0.037}_{-0.026}$	-	$146.812^{+3.586}_{-3.615}$
<i>BAO + PantheonP + SH0ES</i>	$70.891^{+0.968}_{-0.962}$	$0.104^{+0.001}_{-0.003}$	$-1.229^{+0.011}_{-0.005}$	$-2.0366^{+0.036}_{-0.042}$	-	$129.352^{+2.028}_{-2.016}$
<i>CC + PantheonP + SH0ES</i>	$72.132^{+0.875}_{-0.875}$	$0.289^{+0.002}_{-0.002}$	$-0.66^{+0.245}_{-0.395}$	$0.393^{+0.261}_{-0.260}$	-19.287 ± 0.025	-
<i>BAO+CC + PantheonP + SH0ES</i>	$68.203^{+0.783}_{-0.781}$	$0.398^{+0.001}_{-0.003}$	$-1.494^{+0.010}_{-0.005}$	$-2.060^{+0.035}_{-0.040}$	$-19.257^{+1.257}_{-1.025}$	$130.384^{+1.586}_{-1.956}$
Best-fit values for IDE4						
<i>BAO</i>	$69.160^{+7.207}_{-6.412}$	$0.232^{+0.099}_{-0.089}$	$0.830^{+0.178}_{-0.283}$	$-0.060^{+0.093}_{-0.121}$	-	$145.682^{+1.040}_{-13.991}$
<i>CC</i>	$66.093^{+5.670}_{-3.926}$	$0.262^{+0.0093}_{-0.103}$	$-0.710^{+0.295}_{-0.485}$	$0.031^{+0.539}_{-0.171}$	-	-
<i>PantheonP + SH0ES</i>	$73.313^{+1.024}_{-1.192}$	$0.268^{+0.094}_{-0.109}$	$-0.820^{+0.127}_{-0.152}$	$0.009^{+0.165}_{-0.192}$	-19.247 ± 0.029	-
<i>BAO + CC</i>	$68.364^{+2.710}_{-2.444}$	$0.248^{+0.094}_{-0.098}$	$-0.858^{+0.197}_{-0.290}$	$-0.044^{+0.085}_{-0.125}$	-	$147.212^{+3.501}_{-3.352}$
<i>BAO + PantheonP + SH0ES</i>	$73.620^{+1.025}_{-1.013}$	$0.221^{+0.092}_{-0.074}$	$-0.802^{+0.102}_{-0.144}$	$-0.074^{+0.087}_{-0.104}$	-19.248 ± 0.030	$136.592^{+2.290}_{-2.235}$
<i>CC + PantheonP + SH0ES</i>	$72.087^{+0.890}_{-0.871}$	$0.270^{+0.091}_{-0.106}$	$-0.860^{+0.126}_{-0.153}$	$0.021^{+0.097}_{-0.153}$	-19.287 ± 0.025	-
<i>BAO + CC + PantheonP + SH0ES</i>	$72.083^{+0.880}_{-0.880}$	$0.219^{+0.088}_{-0.077}$	$-0.824^{+0.111}_{-0.144}$	$-0.074^{+0.086}_{-0.110}$	$-19.287^{+0.025}_{-0.065}$	$139.936^{+1.940}_{-1.886}$
Best-fit values for IDE5						
<i>BAO</i>	$68.108^{+4.067}_{-3.862}$	$0.232^{+0.099}_{-0.089}$	$0.745^{+0.178}_{-0.283}$	$-0.102^{+0.097}_{-0.158}$	-	$145.682^{+1.040}_{-13.991}$
<i>CC</i>	$67.133^{+6.159}_{-4.426}$	$0.206^{+0.0093}_{-0.103}$	$-0.710^{+0.295}_{-0.485}$	$0.031^{+0.539}_{-0.171}$	-	-
<i>PantheonP + SH0ES</i>	$73.313^{+1.024}_{-1.192}$	$0.268^{+0.094}_{-0.109}$	$-0.820^{+0.127}_{-0.152}$	$0.009^{+0.165}_{-0.192}$	-19.247 ± 0.029	-
<i>BAO + CC</i>	$70.330^{+4.326}_{-3.934}$	$0.107^{+0.053}_{-0.041}$	$-0.846^{+0.081}_{-0.167}$	$-0.157^{+0.252}_{-0.508}$	-	$146.977^{+3.536}_{-3.324}$
<i>BAO + PantheonP + SH0ES</i>	$76.658^{+2.658}_{-2.692}$	$0.187^{+0.041}_{-0.041}$	$-0.875^{+0.052}_{-0.062}$	$-0.097^{+0.183}_{-0.405}$	$-19.224^{+0.030}_{-0.032}$	$135.124^{+2.146}_{-2.162}$
<i>CC + PantheonP + SH0ES</i>	$78.235^{+1.250}_{-3.129}$	$0.149^{+0.032}_{-0.070}$	$-0.999^{+0.126}_{-0.317}$	$-0.011^{+0.293}_{-0.463}$	-19.287 ± 0.025	-
<i>BAO + CC + PantheonP + SH0ES</i>	$75.198^{+2.123}_{-2.753}$	$0.098^{+0.039}_{-0.044}$	$-0.890^{+0.052}_{-0.074}$	$-0.091^{+0.171}_{-0.397}$	-	$140.102^{+2.092}_{-1.953}$

Table 5.1: Observational constraints at 68% and 95% confidence levels on the Λ CDM and the IDE Models are obtained using from the *BAO*, *CC*, *Pantheon+*, *BAO + CC*, *CC + PantheonP + SH0ES* and *BAO + CC + PantheonP + SH0ES* data. Note that H_0 is expressed in units of $\text{km s}^{-1} \text{Mpc}^{-1}$.

where, k is the number of parameters, and N denotes the number of data points used in the fit. Not forgetting that χ^2 represents the maximum likelihood of the model. BIC imposes a stronger penalty on model complexity compared to the AIC, particularly for larger datasets. Models with smaller values ΔAIC or ΔBIC are said to be statistically favoured by the data. Therefore, to clearly highlight the distinctions between the Λ CDM model and alternative dark energy models, we normalize ΔAIC and ΔBIC so that the Λ CDM model equals zero. According to [123] the

interpretation of Bayes values are as follows;

- when $\Delta\text{AIC} \leq 2$ means that the model holds a strong observational support for the fitted data and is statistically competitive,
- $4 \leq \Delta\text{AIC} \leq 7$ means the model has moderate observational support, and not too competitive statistically, and finally
- $\Delta\text{AIC} \geq 10$ means the model does not have observational support.

At the same time ΔBIC is interpreted as;

- when $0 \leq \Delta\text{BIC} \leq 2$ shows low evidence on the model,
- here when $4 \leq \Delta\text{BIC} \leq 6$ the model is moderately favoured showing positive evidence.
- $6 \leq \Delta\text{BIC} \leq 10$ the model here is preferred, which gives strong evidence.
- Lastly, when the model is $\Delta\text{BIC} \geq 10$ it shows no evidence, which means that the model is disfavoured.

The interest is on the the relative value between different models, so to do the comparison we write the formulation for the information criteria as

$$\Delta\text{AIC} = \Delta\chi^2 + 2\Delta k , \quad (5.3)$$

$$\Delta\text{BIC} = \Delta\chi^2 + \Delta k \ln N . \quad (5.4)$$

In this dissertation, we chose to investigate five different IDE models with an additional metastable model. For convenience, in the following, the model with $Q_1 = 3\xi H \rho_{dm}$ is denoted as IDE1, the model with $Q_2 = 3\xi H \rho_{de}$ is denoted as IDE2, the model with $Q_3 = 3\xi H (\rho_{dm} + \rho_{de})$ is denoted as IDE3, the model with $Q_4 = 3\xi H \frac{\rho_{dm}\rho_{de}}{\rho_{dm}+\rho_{de}}$ is denoted as IDE4, and the model with $Q_5 = 3\xi H \sqrt{\rho_{dm}\rho_{de}}$ is denoted as IDE5.

Model Comparison

In the previous section, we presented the observational constraints for the considered IDE models. So, here we compare these models based on their observational implications and extend by conducting a Bayesian evidence analysis in order to test the observational viability of the models.

ΛCDM as shown above still has the upper hand when it comes to fitting with observational data. This model shows consistency with the measurements despite the

inconsistencies DESI-*BAO* and SNIa (*PantheonP* + *SH0ES* compilation) datasets bring (see [117]), one would say it compensates with the measurements. We use this model as a true model to compare the IDE models explained above. First is the IDE1 which is $\xi H\rho_{dm}$ - here, four cosmological parameters were constrained against the joint data *BAO* + *CC*, *BAO* + *PantheonP* + *SH0ES*, *CC* + *PantheonP* + *SH0ES*, and *BAO* + *CC* + *PantheonP* + *SH0ES* respectively, from the results shown in Figure 5.1 this model shows mostly a Gaussian distribution across the parameters, except for the distribution on Ω_m and ξ from *CC* + *PantheonP* + *SH0ES*. However, the consistency in distribution compared to Λ CDM does not exempt the model from completely fitting the data. This model shows consistency with both the early and late universe H_0 estimate values, indicating potential in alleviating the H_0 tension. Although, Ω_m results underestimated values compared to Λ CDM revealing possible new physics, and that gets confirmed by positive ξ values recorded revealing the direction of the energy transfer from DM to DE. In conclusion, this model shows minimal to no interaction between dark sectors and cannot solve the Hubble tension. The second model being $\xi H\rho_{de}$, the results on this model show slight similarities to the ones of IDE1, especially on the recorded low values of Ω_m . However, negative ξ is resulted revealing opposite direction DE $\rightarrow$ DM, and the direction confirms the expansion of the Universe. The model also does present the Gaussian distribution in Figure 5.3, which means parameters are well constrained but the same as in IDE1, parameters Ω_m and ξ are not well constrained. This model compared to Λ CDM shows there is interaction between DE and DM, but does not resolve the H_0 tension. Moreover, IDE2 shows better results compared to IDE1 in solving H_0 tension, with the cross-check against other observational probes to break degeneracies in *BAO*. In summary, this model does not resolve the H_0 tension. IDE3- $3H\xi(\rho_{dm} + \rho_{de})$ being the third model considered, this model also does not resolve the H_0 tension like IDE1 model the value of Ω_m underestimating the values of Λ CDM. Now the fourth model, IDE4- $3H\xi\left(\frac{\rho_{dm}\rho_{de}}{\rho_{dm}+\rho_{de}}\right)$ which draws to the DE $\rightarrow$ DM direction. Meaning a slight underestimation values of Ω_m with negatively curated ξ values. This model shows possibilities in relieving the H_0 tension better compared to the rest of the model. However, compared to Λ CDM model their results are close each, though IDE4 records slightly low Ω_m estimates. With more cross-checks from other observational probes IDE4 would resolve the tension. Moving on to the fifth considered model IDE5- $3H\xi\sqrt{\rho_{dm}\rho_{de}}$, this model favours the highest H_0 estimate values, meaning it favours the late universe estimate but has tension with the early universe. In that this model cannot resolve the H_0 tension. In summary, none of these models are ruled out by the considered data, despite the inconsistencies and the degeneracies brought by combining them. In our defence, sailing without compass was not ideal, meaning “in the absence of curiosity, research loses its exploratory essence, reducing the scientific process to routine data collection rather than a quest for understanding”. In that case, we perform statistical analysis

on our models to clarify and verify the viability of these models.

In Table 5.2 we present the statistical results for Λ CDM, IDE1, IDE2, IDE3, IDE4, and IDE5, in which the likelihood distribution, χ^2 , χ^2_{red} , AIC, BIC, Δ AIC and Δ BIC for each model are recorded. Beginning with the Λ CDM model, we deduce that the results recorded for Λ CDM are favored by the observational data over all the IDE models and are comparable to those of most of the literature, as seen in Table 5.2. Therefore, IDE4 is relatively close to Λ CDM compared to IDE1, IDE2, IDE3, and IDE5. IDE4 records the Δ AIC results as 3.304, 0.446, 2.860, 1.325 for $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$ and $BAO + CC + PantheonP + SH0ES$, respectively. These results show that the model is substantial and observationally supported according to this analysis. However, the very model resulted in Δ BIC for $BAO + CC$, $BAO + PantheonP + SH0ES$, $CC + PantheonP + SH0ES$ and $BAO + CC + PantheonP + SH0ES$ of 4.827, 6.286, 5.725, 7.201, respectively; these results show positive and strong support against Λ CDM. This model is considered viable, considering the support from Bayes's criterion. Followed by IDE5 with strong and positive support from Δ BIC but a combination of substantially favoured by the observational support. Models IDE1, IDE2 and IDE3 show the strongest support from the Δ BIC observationally, but before it could reach Δ BIC, Δ AIC had them ruled out as they presented weak to less support from observational data. In conclusion, linear models struggle observationally, whereas the non-linear models show potential in resolving the cosmological problem at hand H_0 tension.

Data	$\mathcal{L}(\phi \text{data})$	χ^2	χ^2_{red}	AIC	BIC	ΔAIC	ΔBIC
Λ CDM							
<i>CC</i>	-7.2599	14.5198	0.5007	18.520	21.388	-	-
<i>BAO</i>	-6.3740	12.7481	1.14165	18.748	20.203	-	-
<i>PantheonP + SH0ES</i>	-726.4244	1452.8489	0.8784	1458.849	1475.087	-	-
<i>BAO + CC</i>	-13.7293	27.4587	0.6865	33.459	38.742	-	-
<i>BAO + PantheonP + SH0ES</i>	-733.9593	1467.9186	0.8816	1475.919	1497.598	-	-
<i>CC + PantheonP + SH0ES</i>	-738.1128	1476.2255	0.8761	1482.226	1498.519	-	-
<i>BAO + CC + PantheonP + SH0ES</i>	-744.8523	1489.7045	0.8784	1497.705	1519.458	-	-
TDE1							
Data	$\mathcal{L}(\phi \text{data})$	χ^2	χ^2_{red}	AIC	BIC	ΔAIC	ΔBIC
<i>CC</i>	-10.1924	20.3849	0.7550	28.385	34.121	9.865	12.733
<i>BAO</i>	-7.1107	14.2213	2.0316	24.221	26.646	5.473	6.443
<i>PantheonP + SH0ES</i>	-727.7852	1455.5704	0.8811	1465.570	1492.634	6.721	17.547
<i>BAO + CC</i>	-20.8869	41.7739	1.0993	51.774	60.580	18.315	21.838
<i>BAO + PantheonP + SH0ES</i>	-732.1674	1464.3349	0.8805	1476.335	1508.855	0.406	11.246
<i>CC + PantheonP + SH0ES</i>	-737.5261	1475.0521	0.8764	1485.052	1512.209	2.87	13.689
<i>BAO + CC + PantheonP + SH0ES</i>	-743.5065	1487.0130	0.8778	1499.013	1531.643	1.308	12.185
TDE2							
Data	$\mathcal{L}(\phi \text{data})$	χ^2	χ^2_{red}	AIC	BIC	ΔAIC	ΔBIC
<i>CC</i>	-8.4231	16.8461	0.6239	24.846	30.582	9.865	11.321
<i>BAO</i>	-10.1314	20.2628	2.8947	30.263	32.687	5.473	7.243
<i>PantheonP + SH0ES</i>	-755.5968	1511.1936	0.9148	1521.194	1548.257	6.721	17.547
<i>BAO + CC</i>	-13.9757	27.9514	0.7356	37.951	46.757	18.425	21.948
<i>BAO + PantheonP + SH0ES</i>	-732.2718	1464.5435	0.8807	1476.544	1509.063	14.730	15.570
<i>CC + PantheonP + SH0ES</i>	-737.5311	1475.0622	0.8764	1485.062	1512.219	15.058	20.920
<i>BAO + CC + PantheonP + SH0ES</i>	-743.4845	1486.9691	0.8778	1498.969	1531.599	165.901	17.778
TDE3							
Data	$\mathcal{L}(\phi \text{data})$	χ^2	χ^2_{red}	AIC	BIC	ΔAIC	ΔBIC
<i>CC</i>	-8.4231	16.8461	0.6239	24.846	30.582	9.865	12.733
<i>BAO</i>	-10.1314	20.2628	2.8947	30.263	32.687	5.473	6.443
<i>PantheonP + SH0ES</i>	-755.5968	1511.1936	0.9148	1521.194	1548.257	6.721	17.547
<i>BAO + CC</i>	-15.3239	41.8766	1.1020	51.877	60.683	18.425	19.948
<i>BAO + PantheonP + SH0ES</i>	-804.3294	1608.6589	0.9673	1620.659	1653.179	0.865	13.920
<i>CC + PantheonP + SH0ES</i>	-737.5311	1475.0622	0.8764	1485.062	1584.440	75.058	85.927
<i>BAO + CC + PantheonP + SH0ES</i>	-825.8027	1651.6053	0.9750	1663.605	1696.236	1.901	12.778
TDE4							
Data	$\mathcal{L}(\phi \text{data})$	χ^2	χ^2_{red}	AIC	BIC	ΔAIC	ΔBIC
<i>CC</i>	-7.8542	15.7084	0.5818	23.708	29.444	5.190	8.032
<i>BAO</i>	-6.0551	12.1101	1.7300	22.110	24.535	3.361	4.212
<i>PantheonP + SH0ES</i>	-762.3928	1524.7855	0.8990	1534.786	1561.980	5.797	16.675
<i>BAO + CC</i>	-13.3778	26.7555	0.7041	36.756	45.562	3.304	4.827
<i>BAO + PantheonP + SH0ES</i>	-732.1876	1464.3753	0.8806	1476.375	1508.895	0.446	6.286
<i>CC + PantheonP + SH0ES</i>	-737.5430	1475.0859	0.8765	1485.086	1512.242	2.860	5.725
<i>BAO + CC + PantheonP + SH0ES</i>	-743.5146	1487.0291	0.8778	1499.029	1531.659	1.325	7.201
TDE5							
Data	$\mathcal{L}(\phi \text{data})$	χ^2	χ^2_{red}	AIC	BIC	ΔAIC	ΔBIC
<i>CC</i>	-7.2538	14.7439	0.6895	22.258	28.3654	4.859	7.105
<i>BAO</i>	-7.0551	11.2569	1.8964	21.453	24.9856	3.475	4.325
<i>PantheonP + SH0ES</i>	-812.3584	1532.3687	1.2402	1574.2543	1589.6331	4.258	10.624
<i>BAO + CC</i>	-16.2335	27.2581	0.7236	32.2369	44.2584	3.568	5.751
<i>BAO + PantheonP + SH0ES</i>	-751.6298	1464.3753	0.9556	1479.2689	1501.893	1.548	7.594
<i>CC + PantheonP + SH0ES</i>	-725.5430	14.2536	1.0257	1490.0256	1512.242	2.864	10.759
<i>BAO + CC + PantheonP + SH0ES</i>	-715.2546	1456.0451	0.9854	1531.2362	1612.3684	1.691	11.562

Table 5.2: Summary of the statistical analysis for each model.

Chapter 6

Conclusion and future plans

A brief discussion of the history of cosmology, from the fundamental theory of general relativity through to its cosmological applications by virtue of the Friedmann-Lemaître-Robertson-Walker metric has been highlighted. After outlining the different components of the matter content of the universe, we examined the groundbreaking detection of the Cosmic Microwave Background and how these elements collectively shaped the six-parameter model Λ CDM. Although Λ CDM has achieved remarkable success, it is presented with several significant challenges that require serious consideration in [Chapter 2](#). Challenges include the cosmological problem, the Hubble tension, vacuum catastrophe, and σ_8 tension. Upon explaining the challenges brought about by the standard model, we explored possible alternative models proposed in an effort to resolve the tensions within Λ CDM. One of the alternatives that stood out was the Interacting dark energy model, not to say that the other alternatives are not plausible to be studied, but here we chose to explore and give IDE the chance to prove that one of the models can help alleviate challenges in the Λ CDM model. The study of interacting dark-energy models is an advancing field in cosmology that deepens our understanding of the expansion of the universe and the nature of its essential components by integrating theoretical frameworks with observational data. So, we introduced five phenomenological IDE models in [Chapter 3](#) and derived the corresponding expression of the Hubble parameter for each model. In [Chapter 4](#), we demonstrate the observational data considered to constrain the IDE models and the methodology used to evaluate the viability of theoretically plausible models. So, we incorporated three different datasets mentioned and acknowledged certain limitations in DESI-BAO and the necessary considerations in its application. The reason for acknowledging such limitations is that there has been a noticeable trend of preferential use of specific measurements that align with established theoretical frameworks while excluding datasets that exhibit tensions or have been contested by even a limited number of studies in the cosmological literature. We indulged in the use of MCMC simulation and explained how corner plots provide

a powerful diagnostic tool for assessing the constraining power of our data, enabling a clear classification of parameters.

After the release of DESI *BAO* datasets together with Pantheon Plus + SH0ES, this dissertation mainly focused on the exploration of the constraining of the observational parameters for each of the IDE models, including Λ CDM as presented in Chapter 5. In here, we have set our flat priors for our parameters as $0 \leq H_0 \leq 100$, $-1.5 \leq \omega \leq 0.2$, $0.01 \leq \Omega_m \leq 0.4$, and $-5 \leq \xi \leq 5$, and we have set these flat priors for all parameters, deliberately avoiding the imposition of restrictive theoretical regimes. In that, we commit to minimal prior influence, ensuring that posterior constraints are driven dominantly by the observational data rather than subjective assumptions. We considered linear and nonlinear IDE models, while linear interactions are analytically simpler and widely studied in earlier works, tend to impose tighter priors on the coupling parameter ξ and often fail to fully reconcile the Hubble tension, especially when local H_0 measurements from SH0ES are mentioned in the literature (see [118]). Nonlinear models, on the other hand, show greater flexibility and are more adept at alleviating the H_0 discrepancy without strongly deviating from standard cosmological constraints, supported by recent studies [122].

Confronting both linear and non-linear interacting dark energy models with current observational datasets such as *BAO + CC*, *BAO + PantheonP + SH0ES*, *CC + PantheonP + SH0ES*, and the full combination *BAO + CC + PantheonP + SH0ES* brings about a change in the studies conducted in the literature. We combined datasets that are not commonly combined due to their differing observational methodologies and potential systematic inconsistencies; despite the reported inconsistencies, we went ahead and constrained the parameters on the IDE models and had a whole lot of a roller coaster bombardment of Hubble values and demised values. But, combining these datasets enabled a more comprehensive exploration of cosmological models, enhanced parameter constraints, and provided critical insight into current tensions. To ensure an unbiased parameter space, we chose flat priors for all model parameters. The idea here was to avoid imposing any prior assumption regarding the direction of energy transfer between DM and DE. This choice allowed observational data to drive the constraints on cosmological parameters without favouring specific scenarios such as that entailed in equation (3.7). We maintained a non-informative prior distribution, ensuring that our inferences reflect the true constraining power of the observational data rather than prior theoretical biases.

In conclusion, exploring all possible observational data available will help cosmology answer the commonly asked questions about the Universe. We investigated IDE models using the combination of the cosmological datasets: DESI *BAO*, *PantheonP + SH0ES*, and *CC* data. Taking responsibility for the known parameter degeneracies

in the literature, particularly those arising from the joint use of *DESI-BAO* and *PantheonP + SH0ES* data. By constraining a set of linear and non-linear models within this framework, we evaluated their viability compared to the standard Λ CDM model.

Our results indicate that IDE4, which is a nonlinear model, not only fits current observational data better than other IDE models, but also is statistically viable, a key challenge in contemporary cosmology. This model demonstrated competitive performance against Λ CDM and performed better than the IDE1, IDE2, IDE3 and IDE5 in terms of statistical consistency and physical plausibility. These findings support the viability of nonlinear interactions in the dark sector as a promising extension of the standard cosmological model. In our future research, we could benefit from extending this analysis to include additional cosmological probes such as gravitational lensing, redshift-space distortions, and full CMB power spectra. Furthermore, a deeper exploration of the theoretical foundations (Perturbations) and stability conditions of nonlinear IDE models would provide important insights into the physical viability of the model.

Appendix A

Degeneracies of measurements: MCMC results

Taking note of the reported tensions and inconsistencies between the DESI *BAO* data and the *PantheonP + SH0ES* compilation, we continue to analyse the dataset independently in order to enhance our understanding of the impact of degeneracies in parameter estimation. Using our MCMC methods, we put constraints on our parameters in Figure A.1, Figure A.3, Figure A.2. Results show that the degeneracies between ω and ξ vary significantly across datasets and models, with joint dataset combinations providing tighter bounds and helping to partially alleviate parameter degeneracies. Notably, standalone constraints from *PantheonP + SH0ES* tend to favour higher values of H_0 , while BAO data alone suggests a lower H_0 , consistent with previous literature. This comparative analysis enhances our understanding of how dataset-specific tensions influence parameter estimation in IDE scenarios and guides the interpretation of interacting dark sector physics.

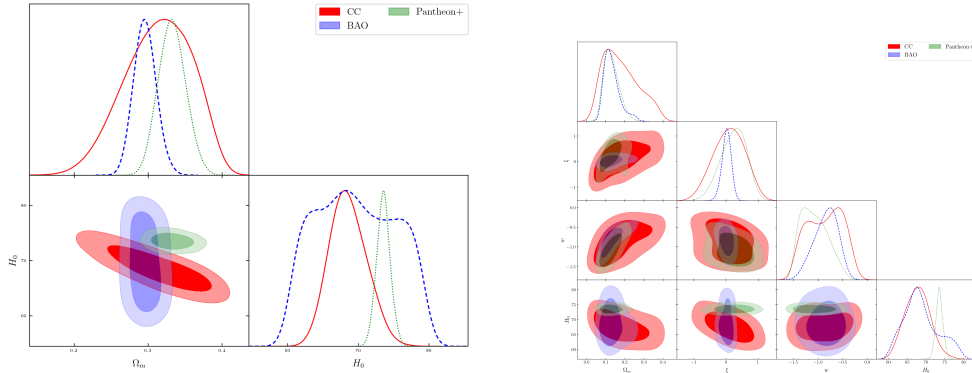


Figure A.1: Two-dimensional plot and 1D marginalized distribution with 1σ and 2σ contours of cosmological parameters for Λ CDM and IDE1 models.

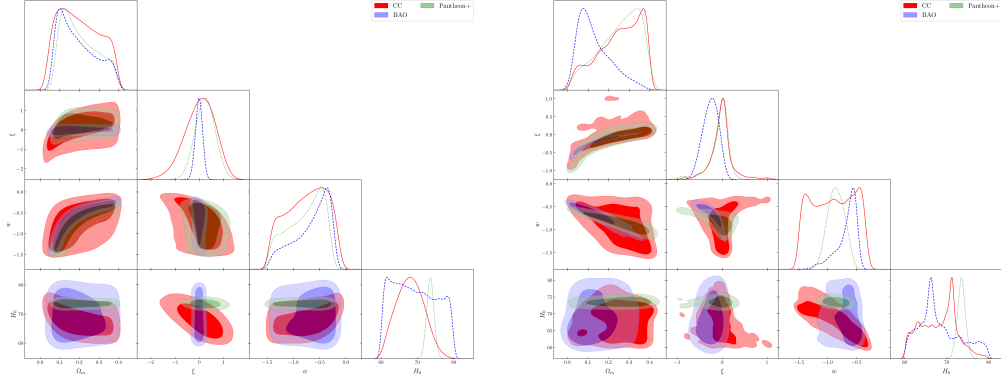


Figure A.2: Two-dimensional plot and 1D marginalized distribution with 1σ and 2σ contours of cosmological parameters for IDE2 and IDE3 models.

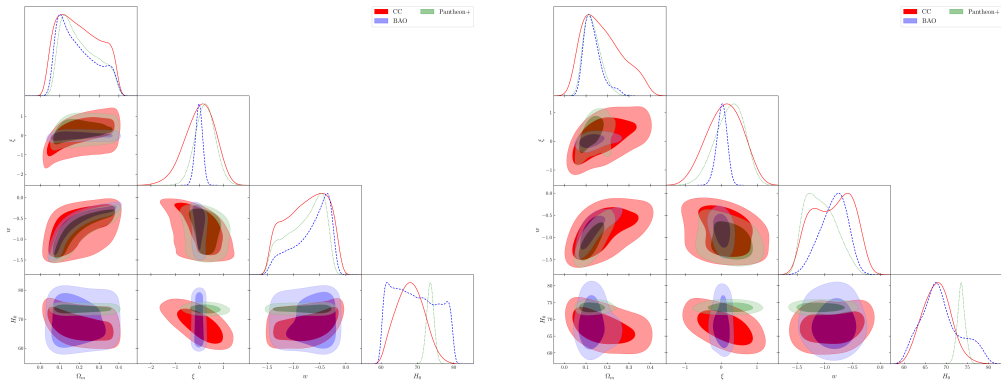


Figure A.3: Two-dimensional plot and 1D marginalized distribution with 1σ and 2σ contours of cosmological parameters for IDE4 and IDE5 models.

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