A multi-period stochastic programming approach to integrated asset and liability management of investment products with guarantees

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Thesis submitted for the degree Doctor of Philosophy in Risk Analysis at the Potchefstroom Campus of the North-West University

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Contents

List of Figures

List of Tables

Abstract

In recent years investment products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way.

This thesis presents two stochastic programming frameworks for the asset and liability management of investment products with guarantees. The asset side of these products usually contains fixed income securities. For this reason we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve) over time. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in the multi-stage stochastic programming approach. The first part of this thesis presents two methods for yield curve scenario generation. The first method uses a moment-matching approach and the second a simulation approach which takes the movement of macro-economic factors into account.

In asset and liability management under uncertainty, using stochastic programming, it is sometimes necessary to take into account flexible risk management actions, for example the reinvestment of coupons or the payment of liabilities at time steps smaller than those at which portfolio rebalancing (or restructuring, i.e. changing the portfolio composition) takes place. The yield curve scenarios at these intermediate time points have to be path dependent. Firstly this thesis proposes a moment-matching approach to construct scenario trees with path dependent intermediate discrete yield curve outcomes sufficient for the pricing of fixed income securities.

As part of the second approach we estimate an econometric model that fits the South African term structure of interest rates, using a Kalman filter approach. The proposed model includes four latent factors and three observable macro-economic factors (capacity utilisation, inflation and repo-rate). The goal is to capture the dynamic interactions between the macro-economy and the term structure. The resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate. In practice most financial institutions have views on the macro-economy. These views are produced by means of an economic scenario generator (ESG) or expert opinion. These ESG's only produce forecasts for macro-economic factors, for example the repo-rate and not a complete yield curve.

The second part of this thesis introduces and solves two asset and liability problems. The first problem is the asset and liability management of minimum liquid asset portfolios found in the banking environment and the second problem is the asset and liability management of insurance products with minimum guarantees. We discuss the formulation and implementation of these multi-stage stochastic programming models and back-test both models on real market data.

Maintaining liquid asset portfolios involves a high carry cost and is mandatory by law for some financial institutions. Taking this into account, a financial institution's aim is to manage a liquid asset portfolio in an "optimal" way, such that it keeps the minimum allowed liquid assets to comply with regulations. This thesis proposes a multi-stage dynamic stochastic programming model for liquid asset portfolio management. The model allows for portfolio rebalancing decisions over a multiperiod horizon, as well as for flexible risk management actions, such as reinvesting coupons at intermediate time steps.

The second problem is the asset and liability management of insurance products with minimum guarantees. This thesis proposes a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimise the down-side risk of these products. We investigate with-profits guarantee funds by including regular bonus payments while keeping the optimisation problem linear. The main focus is the formulation and implementation of a multi-stage stochastic programming model.

Dynamic optimization is perceived to be too difficult... It would be nice to have a generic 'sledge hammer' approach for attacking this sort of problem. [1]

A. D. Smith (1996), p. 1085

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Uittreksel

Beleggingsprodukte het in die afgelope paar jaar meer kompleks geword, deur onder andere 'n verskeidenheid waarborge en bonus opsies aan beleggers te bied. Die toename in kompleksiteit dryf die motivering vir die ondersoek na geïntegreerde bate- en lastebestuursraamwerke. Hierdie raamwerke behoort die dinamiese portefeulje toewysing realisties, op 'n risiko-beheerde wyse, aan te spreek.

Hierdie tesis bied twee stogastiese programmeringsraamwerke aan vir die bate- en lastebestuur van investeringsprodukte met waarborge. Die bate sy van hierdie produkte bevat gewoonlik vaste inkomste sekuriteite. Vir hierdie rede bestudeer ons die stogastiese evolusie van die vorm van die rentekoerstermynstruktuur. In die veld van scenario generering vir multi-periode stogasties programmering, is die literatuur van mening dat die generering van scenarios, wat die onsekerheid in die evolusie van risiko faktore oor tyd voorstel, een van die belangrikste en kritiese stappe in die multi-periode stogastiese programmeringsbenadering. Die eerste gedeelte van die tesis stel twee metodes voor vir die scenario generering van opbrengskurwes. Die eerste metode maak gebruik van 'n moment-passings benadering en die tweede metode maak gebruik van 'n simulasie benadering wat die beweging van makro-ekonomiese faktore in ag neem.

In stogastiese bate- en lastebestuur, waar van stogastiese programmering gebruik gemaak word, is dit soms nodig om van buigsame risikobestuursaksies gebruik te maak. Soos byvoorbeeld die herinvestering van koeponne of die betaling van laste, op tydstippe korter as die waar portefeuljeherbalansering of -herstrukturering plaasvind. Die opbrengskurwe scenario's by die intermediêre tydstappe moet padafhanklik wees. Die tesis stel 'n moment-passings benadering voor om scenariobome te genereer met padafhanklike intermediêre diskrete opbrengskurwe uitkomstes, wat geskik is vir die prysing van vaste inkomste sekuriteite.

As deel van die tweede benadering beraam ons 'n ekonometriese model wat die Suid-Afrikaanse rentekoers termynstruktuur pas, deur gebruik te maak van 'n *Kalman* filter benadering. Die voorgestelde model sluit vier latente faktore en drie

makro-ekonomiese faktore (kapasiteitsgebruik, inflasie en repokoers) in. Die doel is om die dinamiese interaksie tussen die makro-ekonomie en die termynstruktuur vas te vang. Die resulterende model kan gebruik word om rentekoers scenario bome te genereer wat geskik is vir die prysing van vaste inkomste sekuriteite. 'n Belangrike inset in ons scenario generator, is die investeerder se vooruitskatting van die evolusie van die repokoers. Meeste finansiële instellings het in die praktyk verwagtinge oor die makro-ekonomie. Hierdie sieninge word gewoontelik deur ekonomiese scenario generators (ESG) verskaf of deur middel van ekspert opinies. Die ESG's voorsien gewoontelik vooruitskattings van die makro-ekonomiese faktore, byvoorbeeld die repokoers, en nie die hele termyn struktuur nie.

Die tweede gedeelte van hierdie tesis bestudeer twee bate- en lastebestuurs probleme en los dit ook op. Die eerste probleem is die bate- en lastebestuur van minimum likiedebate portefeuljes wat in die bank industrie voorkom. Die tweede probleem is die bate- en lastebestuur van versekeringsprodukte met minimum waarborge. Ons bespreek die formulering en implementering van die multi-periode stogastiese programmeringsmodelle en toets albei modelle op historiese mark data.

Die instandhouding van likiedebate portefeuljes is gemoeid met 'n hoë drakoste en is volgens wet verpligtend vir sommige finansiële instellings. Dus is die doel van 'n finansiële instelling om op 'n optimale wyse hierdie portefeulje te bestuur, sodanig dat die minimum likiedebates gehou word, om aan die regulasies te voldoen. Hierdie tesis bied 'n multi-periode dinamiese stogastiese programmeringsmodel aan vir likiedebate portefeuljebestuur. Die model maak voorsiening vir multiperiode portefeuljeherbalansering, asook vir buigsame risikobestuursaksies, soos die herinvestering van koeponne, by intermediêre tydstappe.

Die tweede probleem is die bate- en lastebestuur van versekeringsprodukte met minimum waarborge. Die tesis bied 'n multi-periode dinamiese stogastiese programmeringsmodel aan, vir die geïntegreerde bate- en lastebestuur van versekeringsprodukte met waarborge. Hierdie model minimeer die negatiewe (onderkant) risiko van hierdie produkte. Ons bestudeer waarborgfondse met winste, met gereelde bonus betalings, en hou die optimerings probleem linieêr. Die hooffokus is die formulering en implementering van die multi-periode stogastiese programmeringsmodel.

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To my Lord and Saviour Jesus Christ, thank you for Your abundant love and grace.

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Chapter 1

Introduction

In recent years investment products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way.

In this thesis we present two stochastic programming frameworks for the asset and liability management of investment products with guarantees. The asset side of these products usually contains fixed income securities. For this reason we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve) over time. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in a multi-stage stochastic programming approach. The first part of this thesis presents two methods for yield curve scenario generation. The first method uses a moment-matching approach and second a simulation approach which takes the movement of macro-economic factors into account.

The second part of this thesis introduces and solves two asset and liability management problems. The first problem is the asset and liability management of minimum liquid asset portfolios, found in the banking environment, and the second problem is the asset and liability management of insurance products with minimum guarantees. We discuss the formulation and implementation of these multi-stage stochastic programming models and back-test both models on real market data.

In this introductory chapter we will discuss the background of the problem and highlight the contributions of this thesis. We also present an introduction to stochastic programming and its uses in asset and liability management and highlight the importance of scenario generation as input to these models. We will conclude with an overview of what is to follow in this thesis.

1.1 Thesis background and contribution

In this section we will discuss the background to the problem and highlight the contributions of this thesis.

1.1.1 Investment products with guarantees

Liquid asset portfolio

Banks are deposit-taking institutes and when a run on the bank occurs, due to any adverse movements of a risk factor, the bank needs to have enough liquid assets to meet public demand. The South African Banks Act (Banks Act, 94/1990) and Regulations Relating to Banks (SA, 2008) protect the public by requiring banks to keep a minimum amount in liquid assets. Liquid assets are assets which are easily redeemable for cash and are defined in Section 1 of the Banks Act as:

- Reserve Bank notes, subsidiary coins,
- Gold coin and bullion,
- Any credit balance in a clearing account with the SARB,
- Treasury bills of the RSA,
- Securities issued by virtue of section 66 of the Public Finance Management Act, 1999
- Bill issued by the Land Bank
- Securities of the SARB.

The minimum nominal amount required in liquid assets, is stipulated in Section 72 of the Banks Act (Banks Act, 94/1990) and Regulation 20 of the Regulations Relating to Banks (SA, 2008). The Banks Act (Banks Act, 94/1990) stipulates, that a bank shall hold liquid assets with respect to the value of its liabilities as may be specified by regulations. Regulation 20 of the Regulation Relating to banks (SA, 2008), requires a bank to hold over a period of one month an average daily amount of liquid assets equal to no less than 5% of its reduced liabilities. For this purpose a bank needs to keep a *statutory portfolio*, also called a *liquid asset portfolio*.

The liquid assets that are available for inclusion into the liquid asset portfolio are interest rate sensitive, low (credit) risk financial instruments. Having a low risk implies having a small return, so keeping the portfolio is mostly unprofitable. The portfolio is funded by a pool of funds with a cost equivalent to the bank's interdivisional borrowing rate. Maintaining a liquid asset portfolio involves a high carry cost, thus making it expensive to hold. However, as mentioned above, the portfolio is mandatory. Taking this and the high carry cost of the portfolio into account, the bank's aim is to manage the liquid asset portfolio in an "optimal" way, such that it keeps the minimum allowed liquid assets to comply to regulations, whilst maximising the portfolio return to cover at least the carry cost.

To manage this portfolio in an "optimal" way the bank will need to rebalance or restructure, i.e. changing the portfolio composition, on a regular basis. Changing the portfolio composition will depend on certain aspects such as expert views on risk factor movements, legislation and regulations. With these legislation and regulations to adhere to and uncertainties to consider the liquid asset portfolio management problem can be described as a multi-stage decision problem in which portfolio rebalancing actions are taken at successive future discrete time points.

This thesis will investigate the use of stochastic programming in addressing all of these aspects in a realistic way. We propose a multi-stage dynamic stochastic programming model for liquid asset portfolio management. The model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management actions, such as reinvesting coupons, at intermediate time steps, i.e. between decision times. We show how our problem closely relates to insurance products with guarantees and utilise this in the formulation. Furthermore we discuss our formulation and implementation of a multi-stage stochastic programming model that minimises the down-side risk of these portfolios and illustrate by means of back-testing, over a period of two years, rebalancing every quarter, that the proposed model addresses the risk management of these portfolios in a reasonable way.

Insurance products with guarantees

Inspired by the research of Dempster *et al.* (2006) and Consiglio *et al.* (2006), this thesis further proposes a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. As proposed in Dempster *et al.* (2006), the model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management actions, such as the reinvestment of coupons at intermediate time steps. We investigate with-profits guarantee funds as in Consiglio *et al.* (2006), by including regular bonus payments. Once these bonuses have been declared, the bonus becomes guaranteed.

Our main contribution in this area is that we keep the optimisation problem linear, by changing the way bonuses are declared. The problem is kept linear for two reasons. The first is that, by keeping the problem linear, we can model the rebalancing of the portfolio at future decision times explicitly. By doing so the dynamic stochastic programming model automatically hedge the first stage portfolio allocation against projected future uncertainties in asset returns (see Dempster *et al.*, 2003 and 2006). The second reason is that the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments. Although our bonus payments may seem unrealistic, we show that our bonus assumption mimics those proposed by Consiglio *et al.* (2006).

Furthermore we discuss the formulation and implementation of the multi-stage stochastic programming model. We also demonstrate the model's features at different levels of minimum guarantee and different levels of risk-aversion. Again we illustrate by means of back-testing, over a period of five years, that the proposed model addresses the risk management of these portfolios in a reasonable way.

1.1.2 Yield curve scenario generation

Moment-matching

One of the main sources of uncertainty in analysing the risk and return properties of investment products with guarantees is the stochastic evolution of risk factors. The asset side of these products usually contains fixed income securities. For this reason we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve) over time. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in the multi-stage stochastic programming approach (see Dupačová *et al.*, 2000; Kouwenberg, 2001).

In this thesis we apply a moment-matching procedure for the generation of yield curve scenarios. We discuss the moment-matching procedure and propose an algorithm to produce yield curve scenarios with path dependent intermediate nodes suitable for the pricing of fixed income securities. In asset and liability management under uncertainty, using stochastic programming, it is sometimes necessary to take into account flexible risk management actions, for example the reinvestment of coupons or the payment of liabilities, at time steps smaller then those at which rebalancing takes place. For this reason we propose a scenario generation algorithm that generates a balanced scenario tree with path dependent intermediate time nodes. We illustrate the stability of the scenario generation method by means of several back-tests.

Macro-economic simulation

As part of the second approach, we estimate an econometric model that fits the South African term structure of interest rates. We describe a Kalman filter state-space modelling approach for the basic three-factor yields-only model used by Diebold *et al.* (2006). The yields-only model uses only three latent factors of the yield curve and does not include macro-economic factors. One of our main contributions in this area is the model estimation for the South African term structure. Furthermore we introduce a four-factor model based on the yield curve model of Svensson (1994). The four-factor model is introduced because the Nelson and Siegel (1987) model is not flexible enough to get an acceptable cross-sectional fit to the South African term structure.

Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. We incorporate three macro-economic factors (capacity utilisation, inflation and repo-rate). According to Diebold *et al.* (2006) these three macro-economic factors are considered to be the minimum set of fundamentals needed to capture the basic macro-economic dynamics (see also Rudebusch and Svensson, 1999; Kozicki and Tinsley, 2001). For scenario generation it is not only important to capture the dynamics of the yield curve reasonably well in-sample, but it is also important to forecast the dynamics of the yield curve reasonably well out-of-sample. We show that the estimated model fits the term structure reasonably well in-sample and performs reasonably well in out-of-sample forecasting. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate. In practice most financial institutions have views on the macro-economy. These views are produced by *means of* an economic scenario generator (ESG) or expert opinion. These ESG's only produce forecasts for macro-economic factors, for example the repo-rate and not a complete yield curve. By using the Kalman filter to model the yield curve in a bidirectional approach, it is possible to close this loop and to produce a full yield curve given a set of macro-economic forecasts.

Lastly we present a parallel simulation and randomised clustering approach to generate scenario trees, which are the input for financial optimisation problems. Furthermore, we discuss the existence of arbitrage in the scenario trees and propose a method to eliminate arbitrage opportunities. We illustrate by means of several back-tests the stability of the scenarios generation method and compare the results to those of the moment-matching scenario generation method. We also discuss and compare the two proposed methods in terms of back-testing and stability.

1.2 Literature review

In this section we present an introduction to *Stochastic Programming* and its uses in asset and liability management and the importance of scenario generation as input to these models.

1.2.1 Stochastic programming

Multi-stage dynamic stochastic programming has over the past few decades become a popular tool for asset and liability management. Recognised in the 1970's, Crane (1971) presents a discrete stochastic programming model for commercial bank portfolio management. Crane (1971) shows that this model explicitly takes into account the dynamic nature of these types of problems and incorporates risk by treating future cash flows and interest rates as random variables. Bradley and Crane (1972) presents a more dynamic model for bond portfolio management, where the bond portfolio problem is viewed as a multi-staged decision problem. Kusy and Ziemba (1986) developed a multi-stage stochastic linear programming model for the asset and liability management of a bank. Their model includes the uncertainties of institutional, legal, financial, and bank-related policies. They demonstrate that the asset and liability model developed, is theoretically and operationally superior to deterministic programming models (e.g. mean variance, Markowitz, 1952). Some other notable financial planning applications can be found in Mulvey and Vladimirou (1989) and Mulvey and Vladimirou (1992).

Several authors highlighted the advantages of multi-stage dynamic stochastic programming in asset and liability modelling (see for example Mulvey *et al.*, 2003). In contrast to the usual mean-variance approach (Markowitz, 1952) with a myopic view of managing investment risk over a single period, dynamic stochastic optimisation provides the asset manager with an integrated way to model both assets and liabilities in a flexible manner that takes into account multi-period dynamic asset allocation and the valuation of liabilities under future market conditions. Using this approach the rebalancing of the asset portfolio is modelled explicitly.

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Die instandhouding van likiedebate portefeuljes is gemoeid met 'n hoë drakoste en is volgens wet verpligtend vir sommige finansiële instellings. Dus is die doel van 'n finansiële instelling om op 'n optimale wyse hierdie portefeulje te bestuur, sodanig dat die minimum likiedebates gehou word, om aan die regulasies te voldoen. Hierdie tesis bied 'n multi-periode dinamiese stogastiese programmeringsmodel aan vir likiedebate portefeuljebestuur. Die model maak voorsiening vir multiperiode portefeuljeherbalansering, asook vir buigsame risikobestuursaksies, soos die herinvestering van koeponne, by intermediêre tydstappe.

Die tweede probleem is die bate- en lastebestuur van versekeringsprodukte met minimum waarborge. Die tesis bied 'n multi-periode dinamiese stogastiese programmeringsmodel aan, vir die geïntegreerde bate- en lastebestuur van versekeringsprodukte met waarborge. Hierdie model minimeer die negatiewe (onderkant) risiko van hierdie produkte. Ons bestudeer waarborgfondse met winste, met gereelde bonus betalings, en hou die optimerings probleem linieêr. Die hooffokus is die formulering en implementering van die multi-periode stogastiese programmeringsmodel.

Dynamic optimization is perceived to be too difficult . . . It would be nice to have a generic 'sledge hammer' approach for attacking this sort of problem. [1]

A. D. Smith (1996), p. 1085

[1]Taken from Dempster *et al.* (2006) - Global asset and liability management.

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October 2009

Chapter 1

Introduction

In recent years investment products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way.

In this thesis we present two stochastic programming frameworks for the asset and liability management of investment products with guarantees. The asset side of these products usually contains fixed income securities. For this reason we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve) over time. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in a multi-stage stochastic programming approach. The first part of this thesis presents two methods for yield curve scenario generation. The first method uses a moment-matching approach and second a simulation approach which takes the movement of macro-economic factors into account.

The second part of this thesis introduces and solves two asset and liability management problems. The first problem is the asset and liability management of minimum liquid asset portfolios, found in the banking environment, and the second problem is the asset and liability management of insurance products with minimum guarantees. We discuss the formulation and implementation of these multi-stage stochastic programming models and back-test both models on real market data.

In this introductory chapter we will discuss the background of the problem and highlight the contributions of this thesis. We also present an introduction to stochastic programming and its uses in asset and liability management and highlight the importance of scenario generation as input to these models. We will conclude with an overview of what is to follow in this thesis.

1.1 Thesis background and contribution

In this section we will discuss the background to the problem and highlight the contributions of this thesis.

1.1.1 Investment products with guarantees

Liquid asset portfolio

Banks are deposit-taking institutes and when a run on the bank occurs, due to any adverse movements of a risk factor, the bank needs to have enough liquid assets to meet public demand. The South African Banks Act (Banks Act, 94/1990) and Regulations Relating to Banks (SA, 2008) protect the public by requiring banks to keep a minimum amount in liquid assets. Liquid assets are assets which are easily redeemable for cash and are defined in Section 1 of the Banks Act as:

- Reserve Bank notes, subsidiary coins,
- Gold coin and bullion,
- Any credit balance in a clearing account with the SARB,
- Treasury bills of the RSA,
- Securities issued by virtue of section 66 of the Public Finance Management Act, 1999
- Bill issued by the Land Bank
- Securities of the SARB.

The minimum nominal amount required in liquid assets, is stipulated in Section 72 of the Banks Act (Banks Act, 94/1990) and Regulation 20 of the Regulations Relating to Banks (SA, 2008). The Banks Act (Banks Act, 94/1990) stipulates, that a bank shall hold liquid assets with respect to the value of its liabilities as may be specified by regulations. Regulation 20 of the Regulation Relating to banks (SA, 2008), requires a bank to hold over a period of one month an average daily amount of liquid assets equal to no less than 5% of its reduced liabilities. For this purpose a bank needs to keep a *statutory portfolio*, also called a *liquid asset portfolio*.

The liquid assets that are available for inclusion into the liquid asset portfolio are interest rate sensitive, low (credit) risk financial instruments. Having a low risk implies having a small return, so keeping the portfolio is mostly unprofitable. The portfolio is funded by a pool of funds with a cost equivalent to the bank's interdivisional borrowing rate. Maintaining a liquid asset portfolio involves a high carry cost, thus making it expensive to hold. However, as mentioned above, the portfolio is mandatory. Taking this and the high carry cost of the portfolio into account, the bank's aim is to manage the liquid asset portfolio in an "optimal" way, such that it keeps the minimum allowed liquid assets to comply to regulations, whilst maximising the portfolio return to cover at least the carry cost.

To manage this portfolio in an "optimal" way the bank will need to rebalance or restructure, i.e. changing the portfolio composition, on a regular basis. Changing the portfolio composition will depend on certain aspects such as expert views on risk factor movements, legislation and regulations. With these legislation and regulations to adhere to and uncertainties to consider the liquid asset portfolio management problem can be described as a multi-stage decision problem in which portfolio rebalancing actions are taken at successive future discrete time points.

This thesis will investigate the use of stochastic programming in addressing all of these aspects in a realistic way. We propose a multi-stage dynamic stochastic programming model for liquid asset portfolio management. The model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management actions, such as reinvesting coupons, at intermediate time steps, i.e. between decision times. We show how our problem closely relates to insurance products with guarantees and utilise this in the formulation. Furthermore we discuss our formulation and implementation of a multi-stage stochastic programming model that minimises the down-side risk of these portfolios and illustrate by means of back-testing, over a period of two years, rebalancing every quarter, that the proposed model addresses the risk management of these portfolios in a reasonable way.

Insurance products with guarantees

Inspired by the research of Dempster *et al.* (2006) and Consiglio *et al.* (2006), this thesis further proposes a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. As proposed in Dempster *et al.* (2006), the model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management actions, such as the reinvestment of coupons at intermediate time steps. We investigate with-profits guarantee funds as in Consiglio *et al.* (2006), by including regular bonus payments. Once these bonuses have been declared, the bonus becomes guaranteed.

Our main contribution in this area is that we keep the optimisation problem linear, by changing the way bonuses are declared. The problem is kept linear for two reasons. The first is that, by keeping the problem linear, we can model the rebalancing of the portfolio at future decision times explicitly. By doing so the dynamic stochastic programming model automatically hedge the first stage portfolio allocation against projected future uncertainties in asset returns (see Dempster *et al.*, 2003 and 2006). The second reason is that the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments. Although our bonus payments may seem unrealistic, we show that our bonus assumption mimics those proposed by Consiglio *et al.* (2006).

Furthermore we discuss the formulation and implementation of the multi-stage stochastic programming model. We also demonstrate the model's features at different levels of minimum guarantee and different levels of risk-aversion. Again we illustrate by means of back-testing, over a period of five years, that the proposed model addresses the risk management of these portfolios in a reasonable way.

1.1.2 Yield curve scenario generation

Moment-matching

One of the main sources of uncertainty in analysing the risk and return properties of investment products with guarantees is the stochastic evolution of risk factors. The asset side of these products usually contains fixed income securities. For this reason we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve) over time. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in the multi-stage stochastic programming approach (see Dupačová *et al.*, 2000; Kouwenberg, 2001).

In this thesis we apply a moment-matching procedure for the generation of yield curve scenarios. We discuss the moment-matching procedure and propose an algorithm to produce yield curve scenarios with path dependent intermediate nodes suitable for the pricing of fixed income securities. In asset and liability management under uncertainty, using stochastic programming, it is sometimes necessary to take into account flexible risk management actions, for example the reinvestment of coupons or the payment of liabilities, at time steps smaller then those at which rebalancing takes place. For this reason we propose a scenario generation algorithm that generates a balanced scenario tree with path dependent intermediate time nodes. We illustrate the stability of the scenario generation method by means of several back-tests.

Macro-economic simulation

As part of the second approach, we estimate an econometric model that fits the South African term structure of interest rates. We describe a Kalman filter state-space modelling approach for the basic three-factor yields-only model used by Diebold *et al.* (2006). The yields-only model uses only three latent factors of the yield curve and does not include macro-economic factors. One of our main contributions in this area is the model estimation for the South African term structure. Furthermore we introduce a four-factor model based on the yield curve model of Svensson (1994). The four-factor model is introduced because the Nelson and Siegel (1987) model is not flexible enough to get an acceptable cross-sectional fit to the South African term structure.

Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. We incorporate three macro-economic factors (capacity utilisation, inflation and repo-rate). According to Diebold *et al.* (2006) these three macro-economic factors are considered to be the minimum set of fundamentals needed to capture the basic macro-economic dynamics (see also Rudebusch and Svensson, 1999; Kozicki and Tinsley, 2001). For scenario generation it is not only important to capture the dynamics of the yield curve reasonably well in-sample, but it is also important to forecast the dynamics of the yield curve reasonably well out-of-sample. We show that the estimated model fits the term structure reasonably well in-sample and performs reasonably well in out-of-sample forecasting. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate. In practice most financial institutions have views on the macro-economy. These views are produced by means of an economic scenario generator (ESG) or expert opinion. These ESG's only produce forecasts for macro-economic factors, for example the repo-rate and not a complete yield curve. By using the Kalman filter to model the yield curve in a bidirectional approach, it is possible to close this loop and to produce a full yield curve given a set of macro-economic forecasts.

Lastly we present a parallel simulation and randomised clustering approach to generate scenario trees, which are the input for financial optimisation problems. Furthermore, we discuss the existence of arbitrage in the scenario trees and propose a method to eliminate arbitrage opportunities. We illustrate by means of several back-tests the stability of the scenarios generation method and compare the results to those of the moment-matching scenario generation method. We also discuss and compare the two proposed methods in terms of back-testing and stability.

1.2 Literature review

In this section we present an introduction to *Stochastic Programming* and its uses in asset and liability management and the importance of scenario generation as input to these models.

1.2.1 Stochastic programming

Multi-stage dynamic stochastic programming has over the past few decades become a popular tool for asset and liability management. Recognised in the 1970's, Crane (1971) presents a discrete stochastic programming model for commercial bank portfolio management. Crane (1971) shows that this model explicitly takes into account the dynamic nature of these types of problems and incorporates risk by treating future cash flows and interest rates as random variables. Bradley and Crane (1972) presents a more dynamic model for bond portfolio management, where the bond portfolio problem is viewed as a multi-staged decision problem. Kusy and Ziemba (1986) developed a multi-stage stochastic linear programming model for the asset and liability management of a bank. Their model includes the uncertainties of institutional, legal, financial, and bank-related policies. They demonstrate that the asset and liability model developed, is theoretically and operationally superior to deterministic programming models (e.g. mean variance, Markowitz, 1952). Some other notable financial planning applications can be found in Mulvey and Vladimirou (1989) and Mulvey and Vladimirou (1992).

Several authors highlighted the advantages of multi-stage dynamic stochastic programming in asset and liability modelling (see for example Mulvey *et al.*, 2003). In contrast to the usual mean-variance approach (Markowitz, 1952) with a myopic view of managing investment risk over a single period, dynamic stochastic optimisation provides the asset manager with an integrated way to model both assets and liabilities in a flexible manner that takes into account multi-period dynamic asset allocation and the valuation of liabilities under future market conditions. Using this approach the rebalancing of the asset portfolio is modelled explicitly.

According to Zenios (2008), at each decision date the portfolio manager needs to assess the current state of the economy (i.e. interest rates and market prices), he also needs to assess future fluctuations in interest rates, market prices and cash flows at possible states of the economy at the next decision date. This information on the current state and possible future states of the economy needs to be incorporated into a the investment decision of buying and selling securities, and short-term borrowing or lending. At the next decision date the portfolio manager is faced with new information and possible future states that need to be incorporated into the new investment decision.

The stochastic programming model specifies a sequence of investment decisions at each of the discrete trading times. At each decision period in the scenario tree, the investment decision is made given the current state of the portfolio and a set of possible scenarios at successor states. Thus the current portfolio composition depends on the previous decisions and the realised scenarios in the interim. The model will determine an optimal decision at each state in the scenario tree, given the information available at that state. Given that there are a multitude of succeeding future states of the economy, the optimal decision will not depend on clairvoyance, but should anticipate the future states of the economy (Zenios, 2008).

Zenios (2008) discusses the basic stochastic programming optimisation for dynamic portfolio strategies. Zenios (2008) considers the following event tree or scenario tree. A scenario tree or event tree is a discrete approximation of the joint distribution of random factors (yield curve and stock indices). We represent the scenario tree in terms of states (nodes) $s_t^{v(t)}$, where time $t = 0, 1, 2, \ldots, T$ and $v(t) = 0, 1, 2, \ldots, N_t$ the numbers of the states at time t. The set of states at time t are denoted by $\Sigma_t = \left\{ s_t^{v(t)} | v(t) = 0, 1, \ldots, N_t \right\}$. The set of all states in the scenario tree is denoted by $\Sigma = \cup_{t=0}^{T} \Sigma_t$. Links $\varepsilon \subset \Sigma \times \Sigma$, indicate the possible transitions between states. To enforce non-anticipativity, i.e. to prevent foresight of uncertain future events, we order the elements of ε in pairs $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)} \right)$ where the dependence of the index $v(t)$ on t is explicitly indicated. The order of the states indicates that state $s_{t+1}^{v(t+1)}$ at time $t+1$ can be reached from state $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is the successor state and $s_t^{v(t)}$ the predecessor state. By using the superscript "+" to denote the successor states, and the superscript "-" to denote the predecessors, we have $s_t^{v(t)+} = s_{t+1}^{v(t+1)}$ and $s_{t+1}^{v(t+)-} = s_t^{v(t)}$. Each state $s_t^{v(t)}$ has an associated probability p_t^s, for $s \in \Sigma_t$, such that $\sum_{s \in \Sigma_t} p_t^s = 1$. Random factors in the scenario tree may be denoted by X_t^s, where time index t takes values over the times $t = 0, 1, 2, \ldots, T$, and states index s from the set $\Sigma_t = \left\{ s_t^{v(t)} | v(t) = 0, 1, \ldots, N_t \right\}$. Figure 1.2.1 gives an example of a two stage scenario tree, where the stages represents the decision times or times when branching occurs in the scenario tree $(t = 0, 1)$.

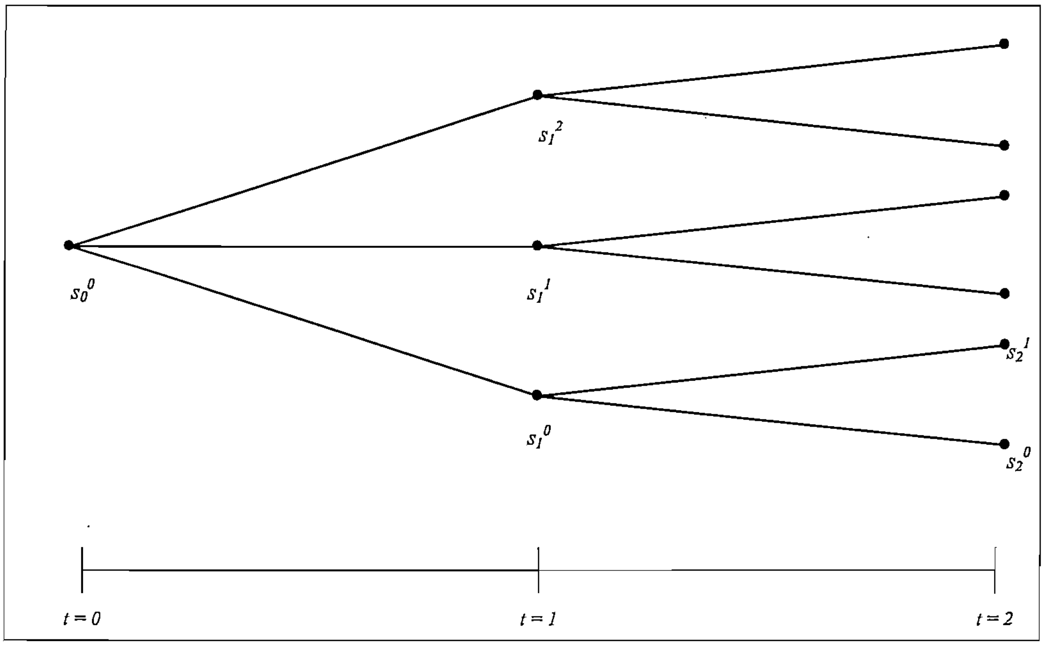

Figure 1.2.1: Graphical representation of a scenario tree

A scenario ω, is a path through the event tree or scenario tree and denoted by the sequence $\left\{s_0^{v(0)}, s_1^{v(1)}, \ldots, s_T^{v(T)}\right\}$, such that $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)}\right) \in \varepsilon$, for all $t = 0, 1, \ldots T-1$, with an associated probability $P(\omega)$. The scenarios are indexed by ω from a finite sample set Ω, and the probabilities satisfy $\sum_{\omega \in \Omega} P(\omega) = 1$. The state visited by scenario ω, at time t are denoted by $n_t(\omega)$, such that $n_t(\omega) = s_t^{v(t)}$ for all $t = 0, 1, \ldots, T$. Zenios (2008) states that by defining scenarios using an event tree, it is clear that some scenarios have common states up to a given decision time and that trading strategies will be the same for these scenarios up to this time.

This concept of non-anticipativity, meaning that decisions cannot depend on as yet observed scenarios (see Censor and Zenios, 1997), which is linked to the scenario tree structure, may be expressed in probabilistic terms as the measurability of the random factors with respect to a filtration. Following Pliska (1997) we can model the uncertainty about the interest rates and market prices with a stochastic process over the time $t = 0, 1, 2, \ldots, T$, that supports a finite probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega_1, \ldots, \omega_{N_T}\}$ is a finite sample space and P is a probability measure on Ω with $P(\omega) > 0$ for all $\omega \in \Omega$. A collection $\mathcal{F}$ of subsets of Ω is called an *algebra* on Ω if:

- $\Omega \in \mathcal{F}$
- if $F \in \mathcal{F}$ then $F^c = \Omega \backslash F \in \mathcal{F}$, and

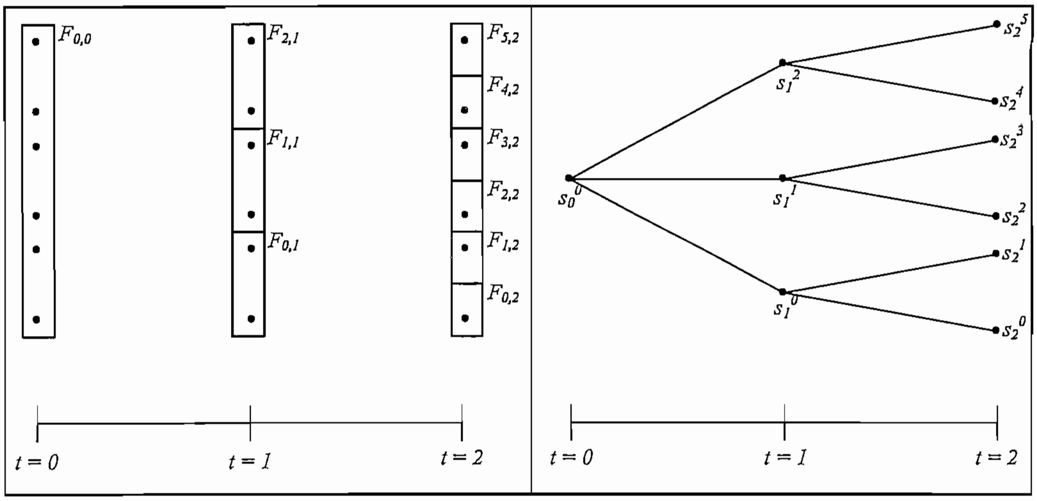

Figure 1.2.2: A finite filtration and its associated scenario tree

- if F and $G \in \mathcal{F}$ then $F \cup G \in \mathcal{F}$.

Given an algebra $\mathcal{F}_t$ on Ω, one can always find a unique collection of subsets $\{F_n\}_t$ such that

- each subset $F_n \in \mathcal{F}_t$,
- the subsets $\{F_n\}_t$ are disjoint, and
- the union of the subsets $\{F_n\}_t$ equals Ω.

Thus every algebra $\mathcal{F}_t$ corresponds to a unique partition of Ω. In the scenario tree described above their is a one-to-one mapping between the states $s \in \Sigma_t$ and the partition $\{F_n\}_t$ for each time $t = 0, 1, 2, \ldots, T$ (see Figure 1.2.2). There is also a one-to-one correspondence between the partitions of Ω and the algebras on Ω. The scenario tree can be organised as a sequence $\{\mathcal{F}_t\}$ of algebras, and the corresponding filtration is $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, 2, \ldots, T\}$.

A random variable X is said to be *measurable* with respect to a filtration $\mathbb{F}$ if the function $\omega \longrightarrow X(\omega)$ is constant on any subset in the partition corresponding to the algebra $\mathcal{F}_t$. This ensures that for each partition $\{F_n\}_t$ and consequently each corresponding state $s \in \Sigma_t$, we can associate a realisation of the random variable $X(\omega)$. In particular, if the decision-maker knows in which state the economy is, he/she will have knowledge of the current security prices, and furthermore, since the information process is made up by a nested sequence of sets, he/she will also have knowledge of past security prices (see also Consiglio and Staino, 2008).

We are now ready to state the basic asset and liability problem in mathematical terms. The following notation makes the transition to mathematical programming languages, such as SAS/OR PROC OPTMODEL, very easy. Therefore it is used throughout the thesis.

Consider the following variables and parameters for the stochastic programming model, where the time index t takes values over the times $t = 0, 1, 2, \ldots, T$, and states index s from the set $\Sigma_t = \left\{ s_t^{v(t)} | v(t) = 0, 1, \ldots, N_t \right\}$:

Time sets

$T = \{0, 1, 2, \ldots, T\}$	: set of all times considered in the stochastic program;

Index sets

$\Sigma_t = \left\{ s_t^{v(t)} \| v(t) = 0, 1, 2, \ldots, N_t \right\}$	: set of states at period t;
I	: set of all instruments;

Parameters

r_t^s	: risk free rate of return at period t in state s;
$F_{t,i}^s$	: cash flow per unit face value of asset $i \in I$ at period t in state s;
$P_{t,i}^{a,s} / P_{t,i}^{b,s}$	: ask or bid price of asset $i \in I$ at period t in state s;
f_a / f_b	: proportional transaction costs on ask or bid transactions;
p_t^s	: probability of state s at period t;
L_t^s	: liability due at period t in state s;

Decision variables

$x_t^s = \left\{ x_{t,i}^s \right\}_{i \in I}$	: face value of assets bought at period t in state s;
$y_t^s = \left\{ y_{t,i}^s \right\}_{i \in I}$	: face value of assets sold at period t in state s;
$z_t^s = \left\{ z_{t,i}^s \right\}_{i \in I}$	: face value of assets held at period t in state s;
c_t^{+s}	: cash invested in short term deposits at period t in state s;
c_t^{-s}	: cash borrowed in short term deposits at period t in state s;

Zenios (2008) mentions that there are two basic constraints in stochastic programming models for portfolio optimisation. The first considers the cash flow accounting for risk-less assets and the second is inventory balance equations for different asset classes in each state at decision times. Figure 1.2.3 illustrates the flow of cash and the inventory of assets in each state at decision times.

The formulation of the stochastic programming model is described below:

Cash balance constraints. The cash balance constraints ensure that at decision times the amount of cash that is received from selling assets, cash flows generated from holding assets and cash that was invested in the previous period are equal to

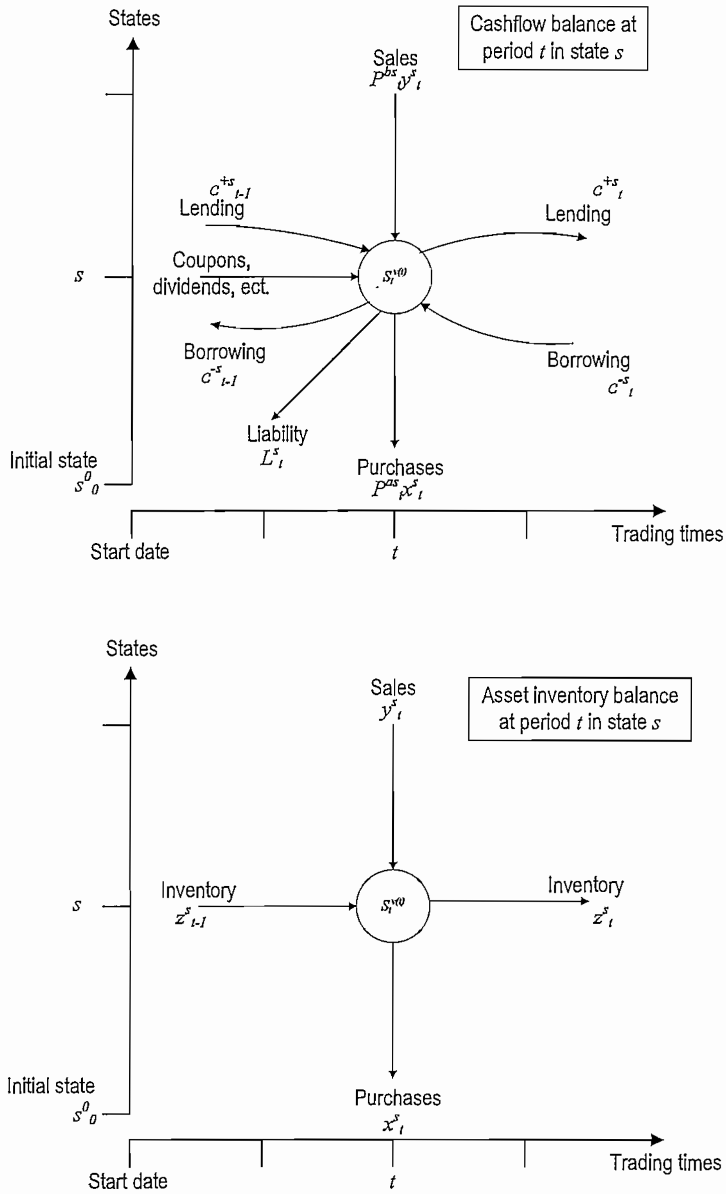

Figure 1.2.3: Cash flow balance and inventory balance at decision times

the amount of new assets bought, cash invested in the risk-less asset and liability payments:

$$\sum_{i\in I} P^{b,0}_{0,i} y^0_{0,i} + c_0 + c_0^{-0} = \sum_{i\in I} P^{a,0}_{0,i} x^0_{0,i} + c_0^{+0} + L^0_0 \text{ for } t \in \{0\} \text{ and } s \in \Sigma_0 = \{s^0_0\},$$

$$\begin{aligned} &\sum_{i\in I} P^{b,s}_{t,i} y^s_{t,i} + \sum_{i\in I} F^s_{t,i} z^{s^-}_{t-1,i} + \left(1 + r^{s^-}_{t-1}\right) c^{+s^-}_{t-1} + c^{-s}_t \\ = \ &\sum_{i\in I} P^{a,s}_{t,i} x^s_{t,i} + \left(1 + r^{s^-}_{t-1} + \delta\right) c^{s^-}_{t-1} + c_t^{+s} + L^s_t, \\ &\text{for } t \in T\backslash\{0\} \text{ and } s \in \Sigma_t. \end{aligned}$$

where δ is the spread over risk-less lending used for borrowing and c_0 denotes the initial cash.

Inventory constraints. The inventory constraints give the quantity invested in each asset in each state at each time period:

$$z^0_{0,i} = b_{0,i} + x^s_{0,i} - y^s_{0,i} \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$z^s_{t,i} = z^{s^-}_{t-1,i} + x^s_{t,i} - y^s_{t,i}, \text{ for } i \in I,\ t \in T\backslash\{0\} \text{ and } s \in \Sigma_t,$$

where $b_{0,i}$, for $i \in I$, denotes the initial portfolio.

End-of-horizon constraints. The terminal wealth of the portfolio is evaluated at the end of the planning horizon in each state $s \in \Sigma_T$ and is given by:

$$W^s_T = \sum_{i\in I} P^{b,s}_{T,i} z^{s^-}_{T-1,i} + \sum_{i\in I} F^s_{T,i} z^{s^-}_{T-1,i} + \left(1 + r^{s^-}_{T-1}\right) c^{+s^-}_{T-1} - L^s_T$$

Objective function. A utility function, $\mathcal{U}$, is used to incorporate risk-aversion in the dynamic portfolio asset allocation. The objective is to maximise the expected utility of the terminal wealth

$$\max \sum_{s\in\Sigma_T} p^s_T \mathcal{U}\left(W^s_T\right).$$

Other objective functions may be used that are more appropriate depending on the application.

In this example of the stochastic programming formulation for portfolio optimisation only two constraints where considered, namely inventory balance and cash flow constraints. Depending on the application other constraints may be modelled. Other conditions that may be included as constraints are the handling of different cash accounts, the restriction of short selling and the inclusion of portfolio constraints to limit the position in a given asset.

A wide variety of literature on the application of stochastic programming to asset and liability management problems exits. In particular, Consigli and Dempster (1998) presents the *computer-aided asset/liability management* (CALM) model for pension fund management. Klaassen (1998) considered the problem of a sequence of liability payments in the future and the construction of a portfolio to meet these liabilities. Kouwenberg (2001) and Gondzio and Kouwenberg (2001) propose a multi-stage dynamic stochastic programming model for the asset and liability management of an actual Dutch pension fund. They also present essential elements in order to tackle large-scale problems (see Gondzio and Kouwenberg, 2001). Drijver *et al.* (2001) formulate a multi-stage mixed-integer stochastic program to model the asset and liability process of a pension fund. Dempster *et al.* (2003) introduce a dynamic stochastic programming model for strategic dynamic financial analysis. Dempster *et al.* (2003) furthermore show that the dynamic stochastic programming model will automatically hedge the current portfolio allocation against future uncertainties in asset returns and costs of liabilities over the analysis horizon.

Applications of dynamic stochastic programming in the area of fixed-income portfolios are found in Hiller and Eckstein (1993), Mulvey and Zenios (1994), Worzel *et al.* (1994), Zenios *et al.* (1998) and Consiglio and Zenios (2001). Other applications in international portfolio management are presented in Topaloglou *et al.* (2002), Topaloglou *et al.* (2004) and Topaloglou *et al.* (2008). A large variety of compilations on the application and implementation of dynamic stochastic programming in the area of finance are available in Ziemba and Mulvey (1998), Zenios and Ziemba (2006), Zenios and Ziemba (2007) and Zenios (2008), among others.

Examples of the use of dynamic stochastic programming models for asset and liability management in the insurance industry are the Russell-Yasuda Kasai model by Cariño and Ziemba (1998) and Cariño *et al.* (1998) and the Towers Perrin model by Mulvey (1996), Mulvey and Thorlacius (1998) and Mulvey *et al.* (2003). Høyland and Wallace (2001a) apply dynamic stochastic programming to analyse the implications of legal regulations in the Norwegian life insurance market.

Several applications of dynamic stochastic programming exist in the area of modelling the assets and liabilities of insurance products with guarantees, to which our two problems are closely related, such as Consiglio *et al.* (2001), Consiglio *et al.* (2003), Hochreiter *et al.* (2007) and Consiglio and De Geovanni (2008). More recent contributions specifically in the area of insurance products with minimum guarantees (related to our problem) using dynamic stochastic programming as an asset and liability management tool are Dempster *et al.* (2006) and Consiglio *et al.* (2006).

More specifically, Dempster *et al.* (2006) proposed an asset and liability management framework and gave numerical results for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay. They demonstrated the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product. Through back-testing they show that the proposed stochastic optimisation framework addresses the risk created by the guarantee in a reasonable way. Consiglio *et al.* (2006) study the same type of problem by structuring a portfolio for with-profit guarantee funds in the United Kingdom. The problem results in a non-linear optimisation problem. They demonstrated how the model can be used to analyse the alternatives to different bonus policies and reserving methods. Consiglio *et al.* (2001) investigate the asset and liability management of minimum guarantee products for the Italian Industry.

It is important to note that pricing of contingent claims and dynamic management of portfolios are two sides of the same coin. The main differences between the valuation of insurance products and dynamic portfolio management are highlighted by Consiglio *et al.* (2006). The literature on pricing products with guarantees assumes that the reference portfolio is given exogenously (e.g. equities 60% and bonds 40%), and does not address the problem of structuring this portfolio optimally. The possible upside potential is ignored. According to Dempster *et al.* (2006): *"This is where the asset manager has a potential advantage. He or she can provide the protection while still exposing the client to high-risk markets through active asset allocation to potentially higher returns".* Consiglio *et al.* (2001) have shown that the financial institution could substantially increase shareholder value by structuring the reference portfolio by viewing it as an integrated asset and liability management optimization problem. Long-term options, which forms the backbone of valuation methods, are in general only available as OTC contracts. This adds a credit risk component to the problem that is largely ignored. The replicating portfolio approach used to value these products assumes continuous rebalancing. This assumption and the other assumptions of the Black-Scholes market are unrealistic.

1.2.2 Scenario generation

One of the main sources of uncertainty in analysing the risk and return properties of investment products with guarantee is the stochastic evolution of risk factors. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in the multi-stage stochastic programming approach (see Dupačová *et al.*, 2000; Kouwenberg, 2001).

Zenios (2008) describes scenarios as the evolution of random variables over time and states that they are not forecasts of these random variables. A forecast is the prediction that the random variable will take on a specific value, whereas scenarios are possible events with certain probabilities. Zenios (2008) states further that we may not know which event will occur, but we know plausible events and the likelihood of each one.

Zenios (2008) lists the following properties for scenarios in order to be useful for financial optimisation:

Correctness: Scenarios should conform to the derived prevalent theories that model the underlying random variables. These scenarios should be derived from the *correct* theoretical models of these random variables (to the extend that these models are correct). The scenarios should capture at the same time the relevant past history and furthermore adequately depict the anticipated evolution of the underlying financial drivers and be consistent with current market conditions.

Accuracy: Scenarios should accurately approximate the theoretical model from which they are derived. As scenarios are the discrete approximations of continuous distributions, errors are inevitable. Accuracy can be ensured when first or higher order moments of the scenarios are matched with those of the underlying distribution or when a large number of scenarios are used and a fine discretisation grid.

Consistency: Scenarios that model more than one variable should ensure that the values of these variables are internally consistent.

Another requirement in financial optimisation is that these scenarios should satisfy the no-arbitrage properties. Ingersoll (1987) distinguishes between two types of arbitrage. The first type is an opportunity to construct a zero-investment portfolio that has nonnegative payoffs in all states of the world, and a strictly positive payoff in at least one state. The second type is an opportunity to construct a negative investment portfolio (i.e. providing an immediate positive cash flow) that generates a nonnegative payoff in all future states of the world (also see Klaassen, 1997 and 1998). If such an opportunity would exist an optimisation program will take advantage of it.

Various scenario generation approaches for single- and multi-period stochastic programming have been reported in literature. Zenios (2008) discusses three approaches. One approach is bootstrapping historical market data (see for example Mulvey and Vladimirou, 1989; Beltratti *et al.*, 2004). It assumes that historical observations are events that provide a representative set of likely future events. Zenios (2008) highlights that this approach is both correct and consistent, since the scenarios consist of market data that has been observed. Another advantage is the simplicity of the approach. A disadvantage of this approach is the inability to provide any causal relationship among random variables. Furthermore the approach leaves no room for expert intervention. Koskosides and Duarte (1997) address this problem by introducing forward expectation-based forecasting into the scenario generation, by adjusting the historical data with investors' future views. Another approach is the statistical analysis of historical market data where scenarios are sampled for the fitted distributions. Methods such as factor analysis (see for example Bertocchi *et al.*, 2000) and principal components (see for example Mulvey and Vladimirou, 1989 and 1992) are used to decrease the dimensionality of these distributions. Zenios (2008) highlights that this approach is correct if distributions with the correct theoretical properties are selected and accuracy can be ensured if the distribution is accurately fitted. Consistency is more difficult to achieve, and requires consistency constraints in the multivariate distribution estimation. Examples of model-based scenario generation in the insurance industry can be found in Consiglio *et al.* (2006), who use the Wilkie model (Wilkie, 1995) to generate scenarios, and Dempster *et al.* (2006), who use a Kalman filter approach to model the term structure and asset prices. The third approach is to develop discrete approximations of continuous distributions. This is done by sampling from the fitted continuous distributions of the underlying market data. Zenios (2008) states that this approach satisfies all three properties of correctness, accuracy and consistency, but errors may arise during sampling. For examples see the Towers-Perrin model by Mulvey and Thorlacius (1998) and the Yasuda-Kasai model by Cariño and Ziemba (1998).

Other scenario generation techniques described in literature is the moment-matching method and lattice structures. In the moment-matching scenario generating method, introduced by Høyland and Wallace (2001b) for multi-period problems, the decision-maker specifies the statistical properties for the random variables relevant to the optimisation problem. These properties can be specified directly or can be derived from the marginal distributions. The scenario tree is generated in such a manner that these specified statistical properties are preserved. Høyland and Wallace (2001a) implement this approach for insurance problems. The procedure was further specialised by Høyland *et al.* (2003).

Lattice structures are also commonly used in the stochastic evolution of term structures and widely used in fixed income portfolio optimisation. Examples where lattice structures are used in fixed income portfolio optimisation can be found in Worzel *et al.* (1994) and Consiglio and Zenios (2001). Dupačová *et al.* (2000) and Gülpinar *et al.* (2004) also discuss different scenario generation procedures.

1.2.3 Yield curve modelling

One of the main sources of uncertainty in analysing the risk and return properties of a portfolio of fixed income securities is the stochastic evolution of the shape of the term structure of interest rates (or yield curve, later on in the thesis the words yields and zero-rates are used interchangeably). Diebold *et al.* (2006) characterise the yield curve using three latent factors, namely level, slope and curvature. To model the dynamic interactions between the macro-economy and the yield curve, they also included observable macro-economic factors, specifically real activity, inflation and a monetary policy instrument. Other examples where a latent factor model approach is used to characterise the yield curve and that explicitly include macro-economic factors can be found in Ang and Piazzesi (2003), Hördahl *et al.* (2004) and Wu (2002). These examples, however, only consider a unidirectional linkage between the macro-economy and the yield curve. Kozicki and Tinsley (2001), Dewachter and Lyrio (2004) and Rudebusch and Wu (2003) allow for implicit feedback.

To capture the dynamics of the yield curve, Diebold *et al.* (2006) do not use a no-arbitrage factor representation such as the typically used affine no-arbitrage models (see for example Duffee, 2002; Brousseau, 2002) or canonical affine no-arbitrage models (see for example Rudebusch and Wu, 2003). Instead of using a no-arbitrage representation Diebold *et al.* (2006) suggest using a three-factor term structure model based on the yield curve model of Nelson and Siegel (1987), as used in Diebold and Li (2006), and interpret these factors as level, slope and curvature. Diebold and Li (2006) propose a two-step procedure to estimate the dynamics of the yield curve. The procedure firstly estimates the three latent factors and secondly estimates an autoregressive model for these factors. Diebold and Li (2006) use these models to forecast the term structure. Diebold *et al.* (2006) proposed a one-step approach by introducing an integrated state-space modelling approach which is preferred over the two-step Diebold-Li approach. This Kalman filter approach simultaneously fits the yield curve and estimates the underlying dynamics of these factors. The model also incorporates the estimation of the macro-economic factors and the link between the macro-economy and the latent factors driving the yield curve.

1.3 Conclusion and thesis overview

In this chapter we have presented an overview of stochastic programming and its uses in asset and liability management and the importance of scenario generation as input to these models. We also highlighted the main contributions of this thesis.

As scenario trees are the input to our portfolio optimisation problems we start off in Chapter 2 and 3 by presenting two methods for yield curve scenario generation. Since fixed income securities are usually contained in the asset side of the asset and liability management of investment products with guarantees, we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve). Chapter 2 presents a moment-matching scenario generation approach and Chapter 3 a simulation approach which includes macro-economic factors.

In Chapter 4 and 5 we will present two stochastic programming frameworks for the asset and liability management of investment products with guarantees. Chapter 4 presents a stochastic programming framework for the asset and liability management of minimum liquid asset portfolios found in the banking environment, and Chapter 5 deals with insurance products with minimum guarantees. We will discuss the formulation and the implementation of the multi-stage stochastic programming models that minimises the down-side risk of these products.

Concluding remarks are presented Chapter 6.

Chapter 2

Moment-matching yield curve scenario generation

One of the main sources of uncertainty in analysing the risk and return properties of investment products with guarantees is the stochastic evolution of risk factors. Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in the multi-stage stochastic programming approach. In this chapter[1] *we present a moment-matching approach to construct scenario trees with path dependent intermediate discrete yield curve outcomes sufficient for the pricing of fixed income securities.*

2.1 Introduction

One of the main sources of uncertainty in analysing the risk and return properties of investment products with guarantees is the stochastic evolution of risk factors. Being that fixed income securities are contained in these portfolios we are concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve). Literature in the field of scenario generation for multi-period stochastic programs has stated that the generation of a set of scenarios, which represents the uncertainty in the evolution of these risk factors over time, is one of the most important and critical steps in the multi-stage stochastic programming approach (see Dupačová *et al.*, 2000; Kouwenberg, 2001).

[1] A paper based on the work done in this chapter has been presented at the International Conference on Mathematics in Finance, South Africa, 2005.

Zenios (2008) describes scenarios as the evolution of random variables over time and states that they are not forecasts of these random variables. A forecast is the prediction that the random variable will take on a specific value, whereas scenarios are possible events with certain probabilities. Zenios (2008) states further that we may not know which event will occur, but we know plausible events and the likelihood of each one. In Chapter 1 we provide a literature review on scenario generation methods and discuss different properties for scenarios to be useful for financial optimisation.

The scenario generation approach proposed in this chapter is based on the moment-matching approach proposed by Høyland and Wallace (2001b). In asset and liability management under uncertainty, using stochastic programming, it is sometimes necessary to take into account flexible risk management actions, for example the reinvestment of coupons or the payment of liabilities, at time steps smaller then those at which rebalancing takes place. For this reason we propose a scenario generation algorithm that generates a balanced scenario tree with path dependent intermediate time nodes.

2.2 The moment-matching scenario generation method

In this section we describe the moment-matching scenario generation method introduced by Høyland and Wallace (2001b). The general idea of the method described is to generate a scenario tree with specified statistical properties. We implement this method to generate yield curve scenario trees in the subsequent sections. We start this section with a general description of the scenario structure.

2.2.1 The scenario tree structure

A typical scenario structure that is used is a fan structure, with a single common starting node (see Figure 2.2.1). In a multi-period setting where rebalancing is allowed (multiple decision times) after the first-stage, even the most carefully constructed model will have arbitrage (see Thorlacius, 2000). Since after time zero a specific scenario will realise and perfect information is revealed about the subsequent periods. Because a single path is known, knowledge of the second period information determines all subsequent information and thus by using this information a trading strategy can be designed to create arbitrage. In order to model the non-anticipativity of trading strategies, meaning that decisions cannot depend on as yet observed scenarios (see Censor and Zenios, 1997), a tree structure is used

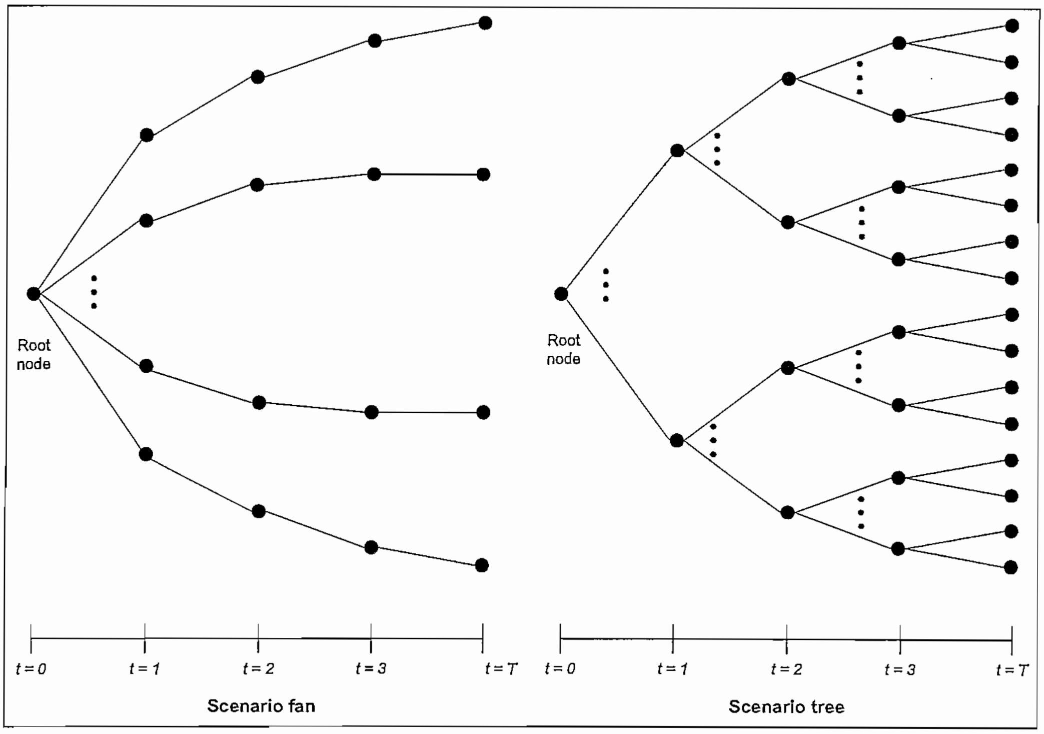

Figure 2.2.1: Graphical representation of a scenario structure

where multiple branching of the scenarios at each decision time occurs (see Figure 2.2.1). The existence of arbitrage in tree structures are discussed in Subsection 2.3.4.

A scenario tree or event tree is a discrete approximation of the joint distribution of random factors (yield curve and stock indices). We represent the scenario tree in terms of states (nodes) $s_t^{v(t)}$, where time $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$ and $v(t) = 0, 1, 2, \ldots, N_t$ the numbers of the states at time t. The set of states at time t are denoted by $\Sigma_t = \left\{ s_t^{v(t)} | v(t) = 0, 1, \ldots, N_t \right\}$. The set of all states in the scenario tree is denoted by $\Sigma = \cup_{t=0}^{T} \Sigma_t$. Links $\varepsilon \subset \Sigma \times \Sigma$, indicate the possible transitions between states. To enforce non-anticipativity, i.e. to prevent foresight of uncertain future events, we order the elements of ε in pairs $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)} \right)$ where the dependence of the index $v(t)$ on t is explicitly indicated. The order of the states indicates that state $s_{t+1}^{v(t+1)}$ at time $t+1$ can be reached from state $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is the successor state and $s_t^{v(t)}$ the predecessor state. By using the superscript "+" to denote the successor states, and the superscript "-" to denote the predecessors, we have $s_t^{v(t)+} = s_{t+1}^{v(t+1)}$ and $s_{t+1}^{v(t+1)-} = s_t^{v(t)}$. Each state $s_t^{v(t)}$ has an associated probability p_t^s, for $s \in \Sigma_t$, such that $\sum_{s \in \Sigma_t} p_t^s = 1$.

A scenario ω, is a path through the event tree or scenario tree and denoted by the sequence $\left\{s_0^{v(0)}, s_1^{v(1)}, \ldots, s_T^{v(T)}\right\}$, such that $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)}\right) \in \varepsilon$, for all $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$ with an associated probability $P(\omega)$. The scenarios are indexed by ω from a finite sample set Ω, and the probabilities satisfy $\sum_{\omega\in\Omega} P(\omega) = 1$. The state visited by scenario ω, at time t are denoted by $n_t(\omega)$, such that $n_t(\omega) = s_t^{v(t)}$ for all $t = 0, 1, \ldots, T-1$. Zenios (2008) states that by defining scenarios using an event tree, it is clear that some scenarios have common states up to a given decision time and that trading strategies will be same for these scenarios up to this time. The property of non-anticipativity may be expressed in probabilistic terms as the measurability of the random factors with respect to a filtration (see Subsection 1.2.1).

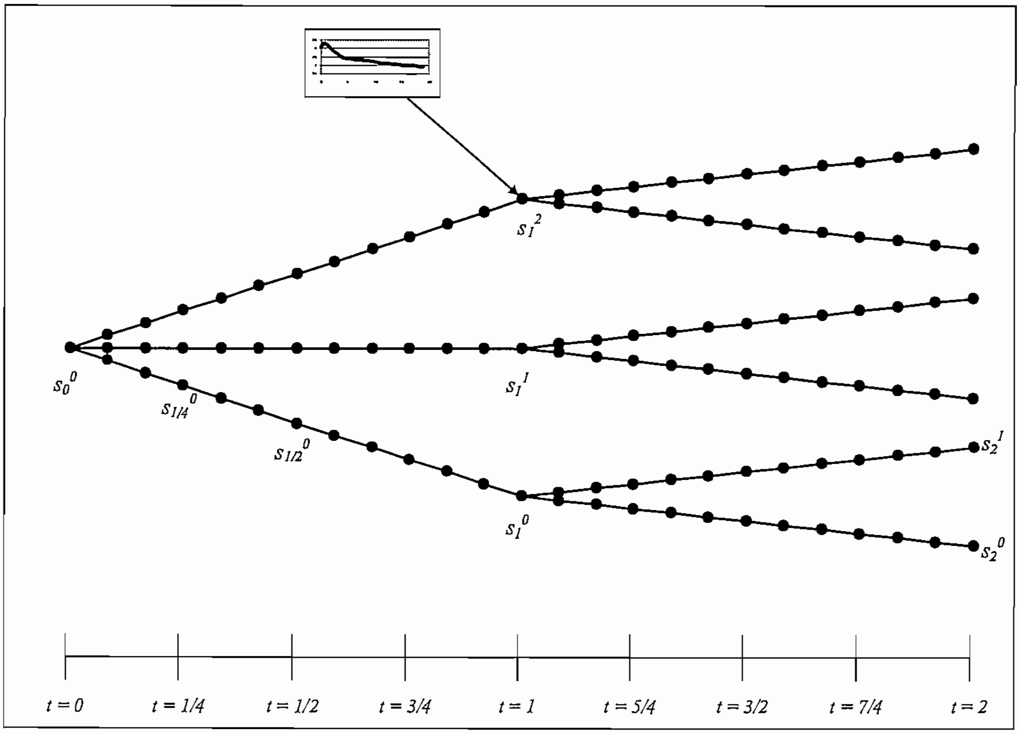

Figure 2.2.2: Graphical representation of a yield curve scenario tree

A T-period scenario tree structure is represented as a tree-string which is a string of integers specifying for each decision time $t_d = 0, 1, 2, \ldots, T-1$, i.e. the annual decision times at which the fund will trade to rebalance its portfolio, the number of branches (or branching factor) for each state in the set of states $\Sigma_{t_d} = \left\{s_{t_d}^{v(t_d)} | v(t_d) = 0, 1, \ldots, N_{t_d}\right\}$. This specification gives rise to a balanced scenario tree, in which each sub-tree in the same period has the same number of branches. Let tree-string $k = (k_0, k_1, ..., k_{t_d}, \ldots, k_{T-1})$ denote a typical tree-string, then the branching factor for decision time t_d, is given by k_{t_d}. Figure 2.2.2 gives an example of a scenario tree with a (3,2) tree-string, i.e. $k_0 = 3$ and $k_1 = 2$.

2.2.2 The scenario generation method

A moment-matching scenario generating method was introduced by Høyland and Wallace (2001b) to generate scenario trees for multi-stage decision problems. In the moment-matching scenario generating method the decision-maker specifies the statistical properties for the random variables relevant to the optimisation problem. These properties can be specified directly or can be derived from the marginal distributions. The scenario tree is generated in such a manner that these specified statistical properties are preserved. The method, based on non-linear programming, has the random variables and their probabilities in the scenario tree as decision variables. The basic idea is to minimise some measure of distance, such as the square norm, between the specified statistical properties and that of the constructed scenario tree. The general form of the algorithm generates the outcomes of the random variables simultaneously together with their probabilities.

The procedure was further extended by Høyland *et al.* (2003). In short the new procedure generates one marginal at a time and creates the joint distribution by putting the marginal distributions together. All the marginal distributions are simulated with the same number of outcomes, and the probability of the i'th outcome is the same for each marginal distribution. The i'th scenario, that is the i'th outcome of the of the joint distribution, is then created by using the i'th outcome from each marginal distribution, and given the corresponding probability. Various transforms in an iterative loop are used to fit the moments and correlations. Kaut (2003) further updates this method which leads to the improved performance of the algorithm.

Although the second method proposed by Høyland *et al.* (2003), is numerically more stable and executes much faster than the original Høyland and Wallace (2001b) method, we use the Høyland and Wallace (2001b) method in our proposed scenario generation algorithm for the following reason. In the Høyland *et al.* (2003) method it is assumed that decision-makers are able to express their expectations of the market in terms of the marginal distributions of the interest rates (or returns on other asset classes). These expectations are then converted into scenarios which can be used in stochastic programming models. The Høyland and Wallace (2001b) method also allows decision-makers to specify these expectations, but in addition allows decision-makers to include worst case outcomes to ensure extreme events. In our discussions with decision-makers (mainly in the banking sector), we have found that most decision-makers do not have expectations on the marginal distributions of the interest rates but rather have a specific view of interest rates (e.g. repo-rate), or have interest rate forecasts produced by economic scenario generators at their disposal. By using the first Høyland and Wallace (2001b) method we can include this view as one of the scenarios in the scenario generation process.

The general idea of the method described in Høyland and Wallace (2001b) is to generate a scenario tree with specified statistical properties. These specified properties can partially describe the known or unknown underlying distribution of the random variables relevant to the optimisation problem. We assume a balanced tree, where the number of branches is the same for all conditional distributions in the same period (see Figure 2.2.2). Let SP be the set of all specified statistical properties, and let SV_i be the specific value of statistical property i, where $i \in SP$. Define x to be the outcome vector of dimension $I \times k_0 + I \times k_0 \times k_1 + ... + I \times k_0 \times ... \times k_{T-1}$ and p to be the probability vector of dimension $k_0 + k_0 \times N_1 + ... + k_0 \times k_1 \times ... \times k_{T-1}$, where $k = (k_0, k_1, ..., k_{t_d}, \ldots, k_{T-1})$ is the tree-string defining the tree structure. Furthermore, let $f_i(\mathrm{x}, \mathrm{p})$ be the mathematical expression for statistical property i, $i \in SP$. Finally let w_i be the weight of statistical property i, $i \in SP$. The objective is to construct x and p such that the specified statistical properties are matched as accurately as possible. This is done by minimising a measure of distance between the statistical properties of the constructed distribution and the specified statistical properties, subject to probability constraints. Using the square norm to measure the distance, the optimisation problem can be written as

$$\begin{aligned} \min_{\mathrm{x},\mathrm{p}} \quad & \sum_{i \in SP} w_i (f_i(\mathrm{x},\mathrm{p}) - SV_i)^2 \\ s.t. \quad & \sum \mathrm{p} \cdot M = 1 \\ & \mathrm{p} \geq 0 \end{aligned} \tag{2.2.1}$$

where M is a matrix of zeros and ones, with its number of rows equal to the length of p and number of columns equal to the number of nodes in the scenario tree. M is constructed such that the each column in M extracts the conditional probabilities in the scenario tree.

The specified statistical properties that the general method uses are any central moments and co-moments in any period. In later periods, specifications can be given over all outcomes or over outcomes with a common history. For the latter, statistical properties can be either conditional on, or independent of, the outcomes of earlier periods. The statistical properties can be specified directly by the decision-maker, derived from the marginal distributions or calculated from empirical data. In addition worst case outcomes and expected outcomes can be included. In general, p is treated as a variable of the optimisation problem, but may also be treated as a parameter.

Due to the non-convexity of the non-linear optimisation problem, the solution algorithm might converge to a local optimum. But for the purpose of constructing yield curve scenarios for portfolio optimisation it is satisfactory to have a distribution equal or close to the specified distribution. An objective function value equal or close to zero indicates a perfect or good match with the specified distribution. In case of a lack there of, because of the non-convexity or inconsistent specification, the weights w_i can be used to incorporate the relative importance of some of the specified properties. In general, if the specified statistical properties are meaningful, a full or close match of the specified properties can be produced by solving the problem repeatedly from a new set of starting values until it is obtained. Kaut and Wallace (2007) state that one should not be too concerned about how well the distribution is approximated, as long as the scenario tree leads to a “good” decision.

Høyland and Wallace (2001b) discuss some pitfalls in the specification of the statistical properties. One can either *overspecify* the problem, meaning that the specified properties are too extensive relative to the size of the scenario tree, or *underspecify* the problem, meaning that the number of scenarios is large relative to the requirements of the specifications. Høyland and Wallace (2001b) suggest the idea of counting the degrees of freedom to make a guess about the size of the scenario tree, i.e. to match the number of variables in the scenario tree with the number of specifications. Furthermore Høyland and Wallace (2001b) show that the relevant statistical properties needed depend on the optimisation problem at hand. Given the optimisation problem and its objective function different scenario sets can be generated using the same statistical properties. The stability of the objective function can then be evaluated over the different set of scenarios. If the stability in the objective function is not satisfactory more statistical properties may be added. Høyland and Wallace (2001b) further suggest a sampling approach where smaller trees are generated with the same statistical properties and aggregated to one large tree, while preserving the statistical properties.

2.2.3 Generating single- and multiple-period scenario trees

A single period scenario tree can be constructed by solving Problem 2.2.1 to match the specified properties. There are two alternative ways of applying the moment-matching scenario generation method to construct a multi-period scenario tree. Constructing a multiple period tree can be done in one large overall optimization or in several smaller optimisations using a sequential approach. The sequential approach specifies statistical properties for the first period and generates first period outcomes that are consistent with these specifications. For each generated first period outcome, specify conditional distribution properties for the second period and generate second period conditional outcomes that are consistent with these specifications. This is repeated for all periods. Modelling the conditional properties, features such as mean reversion can be included. As opposed to the sequential approach, an overall approach that constructs the entire tree in one large optimisation can also be used (see Figure 2.2.3).

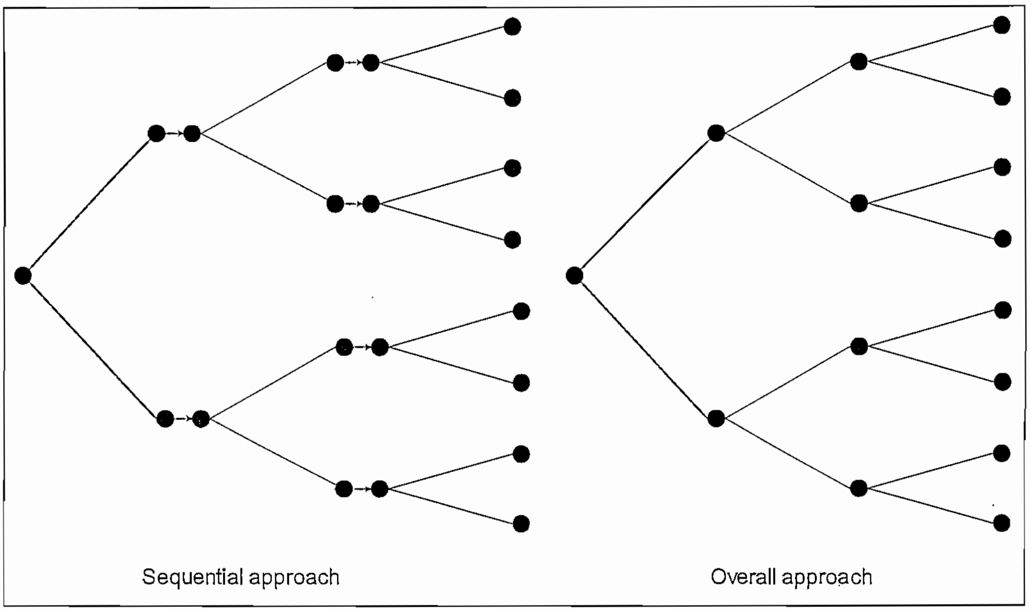

Figure 2.2.3: Sequential and overall approach to constructing a scenario tree

Although the sequential approach has a numerical advantage due to the decomposition of the problem into single period trees, it requires the distribution properties to be specified locally to each node in the scenario tree, lacking direct control of the statistical properties defined in the later periods ($t > 0$) over all the outcomes of the scenario tree. Furthermore the sequential approach involves a more rigid optimisation scheme, where the first period trees might satisfy the first period specifications, but lead to conditional second period specifications which make it impossible to obtain a perfect match. In this aspect an overall approach is more realistic than sequential approach (Høyland and Wallace, 2001b).

Høyland and Wallace (2001b) highlight that the main disadvantage of the overall approach is the degree of non-convexity and a perfect match may be difficult to compute. Furthermore, the sequential approach allows for easy updating of the statistical properties at each node of the scenario tree as input to the problem, whereas overall approach includes an updating procedure as a function in the optimisation problem. Due to the numerical advantage, we follow the sequential approach to construct scenario trees, by solving the problem from a new set of starting values until a full or close match is obtained.

2.3 Generating yield curve scenarios

In this section we apply the moment-matching scenario generation method discussed in Section 2.2 to generating yield curve scenarios with path dependent intermediate time nodes. Gülpinar *et al.* (2004) also applied this approach and model the conditional first moment with an exponential growth curve. The yield curve we are considering to generate scenarios for, has a term structure of zero-rates. The generated scenario tree must be suitable for pricing fixed income securities such as government bonds. We follow the sequential approach to construct the scenario tree.

2.3.1 Scenario generation optimisation

The scenario tree structure is described in Subsection 2.2.1. For the central moments and co-moments we assume that the first four central moments, i.e. expected value, standard deviation, skewness and kurtosis, and correlations are relevant. To generate yield curve scenarios we will generate scenarios for the log changes in the zero-rates. The objective is to generate scenarios for the log changes in the yield curve, denoted by $x_{m_i,t,j}$ with maturities m_i (in months), where $i \in I = \{1, ..., d\}$ and their associated probabilities $p_{j,t}$ for $j = 1, ..., k_t$ (k_t being the number of scenarios). Let $MOM_{m_i,n,}$, for $n = 1, 2, 3, 4$, be the first four central moments of the current conditional distribution for the log changes in the yield curve. The correlation of the log changes is denoted by R_{m_i,m_l}.

The scenarios, $x_{m_i,t,j}$, and their associated probabilities, $p_{j,t}$, are decision variables in the following non-linear optimisation problem:

$$
\begin{aligned}
\min_{x,p} \quad & \sum_{i=1}^{d}\sum_{k=1}^{4} w_{m_i,n}(mom_{m_i,t,n} - MOM_{m_i,n})^2 \\
& + \sum_{i,l \in I,\ i<l} w_{m_i,m_l}(r_{m_i,m_l,t} - R_{m_i,m_l})^2 \\
s.t. \quad & \sum_{j=1}^{k_t} p_{j,t} = 1, \\
mom_{m_i,t,1} \quad = \quad & \sum_{j=1}^{k_t} x_{m_i,t,j} p_{j,t}, \quad i \in I, \qquad (2.3.1) \\
mom_{m_i,t,n} \quad = \quad & \sum_{j=1}^{k_t} (x_{m_i,t,j} - mom_{m_i,t,1})^n p_{t,j}, \quad i \in I, \quad n = 2,3,4, \\
r_{m_i,m_l,t} \quad = \quad & \sum_{j=1}^{k_t} (x_{m_i,t,j} - mom_{m_i,t,1})(x_{m_l,t,j} - mom_{m_l,t,1}) p_{j,t}, \quad i,l \in I \text{ and } i < l \\
& p_{j,t} \geq 0, \quad j = 1, ..., k_t
\end{aligned}
$$

where $w_{m_i,n}$ and w_{m_i,m_l} are weights of the central moments and correlations respectively. Gülpinar *et al.* (2004) define the weights to be $w_{m_i,n} = \frac{w'_n}{MOM_{m_i,n}}$ and $w_{i,l} = \frac{w'_{i,l}}{R_{m_i,m_l}}$ in which w'_k and $w'_{i,l}$ are the relative importance of the central moments and correlations respectively. In the case of a lack of a perfect fit of the specified statistical properties these weights are used.

2.3.2 Scenario generation algorithm (1)

Our objective is to construct a balanced scenario tree with path dependent intermediate time nodes (see for example Figure 2.2.2). In asset and liability management under uncertainty, using stochastic programming, it is sometimes necessary to take into account flexible risk management actions, for example the reinvestment of coupons or the payment of liabilities, at time steps smaller then those at which rebalancing takes place. For this reason we propose a scenario generation algorithm that generates a balanced scenario tree with path dependent intermediate time nodes.

In order to construct the path dependent intermediate time nodes we start off by constructing a balanced branching scenario tree, using the sequential approach, that branches at intermediate time steps during the decision period. A clustering (or grouping) algorithm is then used to reduce the scenarios to single paths with intermediate time nodes. This procedure is then repeated in a sequential manner until the entire tree is constructed (see Figure 2.3.1 for an example with quarterly intermediate nodes).

Denote $Y_t = \{Y_{m_i,t}\}_{t=0,\frac{1}{12},\ldots,t}$, to be the historical yield curve data up to and including time t, where $Y_{m_i,t}$ are the historical zero-rates with maturities m_i in months, where $i \in I = \{1, ..., d\}$, d the number of zero-rates and $m_1 < ... < m_d$. Furthermore we consider $X_t = \{X_{m_1,t}\}_{t=0,\frac{1}{12},\ldots,t}$ to be the corresponding historical log changes in the yield curve up to and including time t, where $X_{m_i,t} = \left(ln \frac{Y_{t,m_i}}{Y_{t-\frac{1}{12},m_i}} \right)$. We present the scenario generation algorithm for monthly data, without loss of generality.

Denote $y^s_\tau = \left\{ y^s_{m_i,\tau} \right\}$, to be the outcomes for the zero-rates at time τ in state $s \in \Sigma_\tau = \left\{ s^{v(\tau)}_\tau = 0, 1, \ldots, k_\tau \right\}$ in the scenario tree, $x^s_{m_i,\tau}$, to be the outcomes for the log changes at time τ at state $s \in \Sigma_\tau$ in the scenario tree and p^s_τ, to be the probabilities of the outcomes at time τ in state $s \in \Sigma_\tau$ for $\tau = t + \frac{1}{12}, t + \frac{2}{12}, ..., t + T$. Furthermore denote $MOM^s_\tau = \left\{ MOM^s_{m_i,n,\tau} \right\}$, to be the specified central moments at time τ in state $s \in \Sigma_\tau$ in the scenario tree for $\tau = t, t+1, ..., t+T-1$.

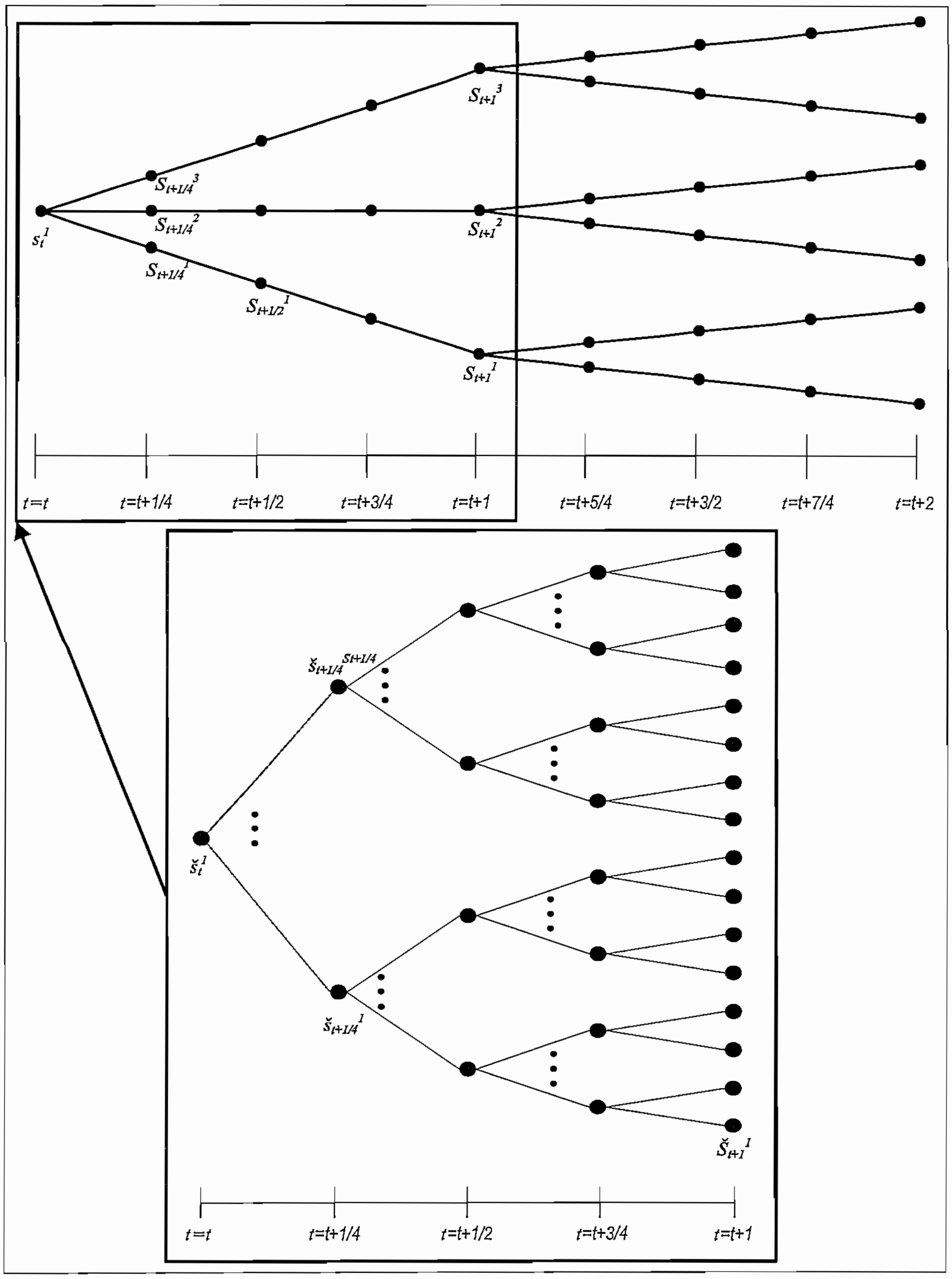

Figure 2.3.1: Scenario generation approach (with quarterly intermediate nodes)

At time t we estimate the central moments, $MOM_{m_i,n}$, for $n = 1,2,3,4$, and the correlations R_{m_i,m_l} of the historical data X_t. These statistical properties are used for the first period scenario generation. To simplify the problem we only assume the expected values to be state dependent, while the other statistical properties are assumed to be independent of the state. We fit a mean reversion process to the historical yield curve log changes in order to update the conditional expected values. The expected value $MOM^s_{m_i,1,\tau}$, in state $s \in \Sigma_\tau$, for $\tau > 0$ is given by

$$MOM^s_{m_i,1,\tau} = (1 - MRF_{m_i}) \times MRL_{m_i} + MRF_{m_i} \times x^s_{m_i,\tau}, \tag{2.3.2}$$

where MRF_{m_i} is the mean reversion factor parameter and the MRL_{m_i} the mean reversion level parameter.

The scenario generation algorithm described below is divided into three parts:

Repeat Steps 1 to 3 for each decision node in the scenario tree $s \in \Sigma_\tau$, and $\tau = t, t+1, ..., t+T-1$

Step 1: *Scenario generation.* In order to construct the path-dependent intermediate time nodes we start by constructing a balanced branching scenario sub-tree, using the sequential approach. The sub-tree branches at intermediate time nodes during the decision period. The states in the sub-tree are denoted by $\tilde{s}^{v(t')}_{t'}$ where $t' = \tau, \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + 1$ and $v(t') = 0, 1, 2, \ldots, \tilde{N}_t$ the number of states at time t', and the set of states at time t' are denoted by $\tilde{\Sigma}_t = \left\{\tilde{s}^{v(t')}_{t'} | v(t') = 0, 1, \ldots, \tilde{N}_t\right\}$ and indexed by $\tilde{s}$ (see Figure 2.3.1). Now, let $y^{\tilde{s}}_{t'} = \left\{y^{\tilde{s}}_{m_i,t'}\right\}$ be the outcomes for the zero-rates at time t' in state $\tilde{s} \in \tilde{\Sigma}_{t'}$ in the sub-tree, let $x^{\tilde{s}}_{m_i,t'}$ be the outcomes for the log changes at time t' in state $\tilde{s} \in \tilde{\Sigma}_{t'}$ in the scenario tree and let $p^{\tilde{s}}_{t'}$, be the probabilities of the outcomes at time t' in state $\tilde{s} \in \tilde{\Sigma}_{t'}$ for $t' = \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + 1$. Furthermore denote $MOM^{\tilde{s}}_{t'} = \left\{MOM^{\tilde{s}}_{m_i,n,t'}\right\}$, to be the specified central moments at time τ in state $\tilde{s} \in \tilde{\Sigma}_\tau$ in the scenario tree for $t' = \tau, \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + \frac{11}{12}$.

Repeat for each decision node in the scenario sub-tree $\tilde{s} \in \tilde{\Sigma}_{t'}$ and $t' = \tau, \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + \frac{11}{12}$.

> **Step 1.1:** Find the scenarios $x^{\tilde{s}+}_{m_i,t'}$ and probabilities $p^{\tilde{s}+}_{t'}$, for decision node $\tilde{s} \in \tilde{\Sigma}_{t'}$ by solving the optimisation problem (Problem 2.3.1).
>
> **Step 1.2:** Update the first moment for each scenario generated (Equation 2.3.2).

Step 2: *Rollup of scenarios.* In order to reduce the scenarios to single paths at intermediate time nodes a clustering (or grouping) algorithm is used to group the generated scenarios according to a distance measure. We calculate the "distance" between the discounting factors of the yield curves and that of a chosen pivot yield curve, by:

$$D_j = \sqrt{\sum_{i=1}^{d}\left(\frac{1}{\left(1+y^{\tilde{s}}_{m_i,\tau+1}\right)} - \frac{1}{(1+y^{p}_{m_i})}\right)^2}$$

with p the current pivot. Intuitively, the "distance" will give us an indication of the difference in fixed income security prices using the different scenarios. The scenarios are grouped according to the distance measure rather than taking the average over the sub-trees generated. This will ensure that extreme scenarios are grouped together, rather than averaged out over the sub-trees. The choice of distance measure does not influence the statistical properties (see discussion below).

Step 2.1: Calculate the corresponding zero-rate, $y^{\tilde{s}}_{m_i,t'}$, for each log change scenario generated, $x^{\tilde{s}}_{m_i,t'}$, where $y^{\tilde{s}}_{m_i,t'} = y^{\tilde{s}-}_{m_i,t'}\exp\left(x^{\tilde{s}}_{m_i,t'}\right)$ for $\tilde{s} \in \tilde{\Sigma}_{t'}$ and $t' = \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + 1$.

Step 2.2: From the leave nodes in the sub-tree, $\tilde{s} \in \tilde{\Sigma}_{\tau+1}$, choose an arbitrary yield curve out of the $\tilde{N}_t$ generated yield curves as the first pivot.

Step 2.3: Calculate the distance D_1 from the first pivot to the remaining yield curves (leave nodes)

Step 2.4: Choose the second pivot as the yield curve with the largest distance to the first pivot.

Step 2.5: Calculate the distances D_j of the remaining yield curves to current pivots.

Step 2.6: Assign the remaining curves to the closest pivot, forming disjoint sets (not necessarily of equal size).

Step 2.7: The next pivot is chosen as the yield curve with the largest distance to the current pivots.

Step 2.8: Repeat Steps 2.5 to 2.8, till k_τ (branching factor at time τ) disjoint sets are formed.

Step 3: *Aggregation.* The last step aggregates the grouped scenarios into single paths with intermediate time nodes and updates the expected values. The grouped scenarios are aggregated in reverse order from the leave nodes. The aggregation process will change the conditional statistical properties (see discussion below).

Step 3.1: Starting at the leave nodes, $\tilde{s} \in \tilde{\Sigma}_{\tau+1}$, for each group of scenarios, calculate the average scenario using the scenarios and their associated path probabilities. Where $y^{s}_{m_i,t'} = \sum_{\tilde{s}\in s} \frac{y^{\tilde{s}}_{m_i t'} p^{\tilde{s}}_{t'}}{p^{s}_{t'}}$, $x^{s}_{m_i,t'} = \sum_{\tilde{s}\in s} \frac{x^{\tilde{s}}_{m_i t'} p^{\tilde{s}}_{t'}}{p^{s}_{t'}}$ and $p^{s}_{t'} = \sum_{\tilde{s}\in s} p^{\tilde{s}}_{t'}$.

Step 3.2: Repeat Step 3.2 for each time step in descending order, $t' = \tau + \frac{11}{12}, \tau + \frac{10}{12}, ..., \tau + \frac{1}{12}$, by grouping the scenarios according to their successor scenarios and using the leave node path probabilities (i.e. in the intermediate nodes a single node may be used more than once).

Step 3.3: Update the expected values for each aggregate leave node $s \in \Sigma_{\tau+1}$ (Equation 2.3.2).

The rollup part of the algorithm is largely based on the interest rate sampling methods of Chueh (2002). The main differences between our application of the algorithm and that proposed by Chueh (2002) is that our scenarios are not equally probable and that we use a different measure of distance. Chueh (2002) discusses several other distance methods for interest rate sampling. Our distance method relates closely to the *relative present value distance method* discussed by Chueh (2002), in the sense that the distance measure of Chueh (2002) is based on one-year short-term interest rates and our distance measure uses zero-rates. The distance measure D_j may be seen as the sum of the squared differences between discounting factors with the different scenario yield curves. This will give us an indication of the difference in fixed income security prices using the different scenarios. Other distance measures that measure the distance between two vectors may be considered in this context. The choice of distance measure does not influence the statistical properties, but consideration should be given to aspects such as preserving extreme scenarios.

All unconditional statistical properties (variance, skewness, kurtosis and correlations) of the sub-tree will be preserved during the aggregation process. The conditional statistical properties (means) at intermediate times of the aggregated sub-tree will be altered to be conditional on the statistical properties of the previous intermediate time, i.e. $MOM_{m_i,1,t'} = (1 - MRF_{m_i}) \times MRL_{m_i} + MRF_{m_i} \times MOM_{m_i,1,t'-\frac{1}{12}}$, as the aggregation process reduces the generated sub scenario tree into single paths with intermediate time nodes (see Figure 2.3.1). This concept is also used in the next subsection. The conditional properties for the next decision period are then conditional on the final outcomes of the aggregated sub scenario tree.

The choice of the branching factor for the intermediate scenario trees, needs to be such that the number of leave nodes in the sub-tree are sufficient for the rollup procedure. For larger scenario trees, as those used in our optimisations, the choice of branching factor may be too large, which may lead to *overspecified* problems (see Høyland and Wallace, 2001b). The second drawback for larger scenario trees is the time aspect. Although solving Problem 2.3.1 for a branching factor of five may take a few milliseconds, doing this thousands of times may take several days to construct a scenario tree. Høyland *et al.* (2003) also states that the algorithm presented by Høyland and Wallace (2001b) becomes slow when the number of random variables increases. For reasons mentioned in Subsection 2.2.2 we use the Høyland and Wallace (2001b) method.

In the next subsection we suggest a shortened version of our algorithm in order to address these problems.

2.3.3 Scenario generation algorithm (2)

In this subsection we present a shortened version of the algorithm presented in the previous subsection. Instead of generating a balanced branching tree for each decision period and using a clustering algorithm to construct the path dependent sub-trees, we generate the entire sub-tree in one optimisation. This again is repeated to construct the entire scenario tree.

To incorporate the mean-reversion of the first moment in the decision sub-tree the expected values $MOM_{m_i,1,t'}$, for intermediate times $t' = \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + 1$ in the decision sub-tree is given by

$$MOM_{m_i,1,t'} = (1 - MRF_{m_i}) \times MRL_{m_i} + MRF_{m_i} \times MOM_{m_i,1,t'-\frac{1}{12}}. \tag{2.3.3}$$

Thus the statistical properties at intermediate times in the sub-tree will not dependent on a specific outcome in the previous intermediate time but only on the statistical property of the preceding intermediate time, this was also a result of the aggregation step in the pervious algorithm. In order to generate the sub-trees in one optimisation, the non-linear problem 2.3.1 is modified to be

$$\begin{aligned} \min_{x,p} \quad & \sum_{t'=\tau+\frac{1}{12}}^{\tau+1} \sum_{i=1}^{d} \sum_{n=1}^{4} w_{m_i,n}(mom_{m_i,n,t'} - MOM_{m_i,n,t'})^2 \\ & + \sum_{i,l \in I,\ i<l} w_{m_i,m_l}(r_{m_i,m_l,t'} - R_{m_i,m_l})^2 \qquad (2.3.4) \\ s.t. \quad & \sum_{s \in \Sigma_{t'}} p_{t'}^s = 1, \end{aligned}$$

$$
\begin{aligned}
mom_{m_i,1,t'} &= \sum_{s\in\Sigma_{t'}} x^s_{m_i,t'} p^s_{t'}, \;\; i \in I, \\
mom_{m_i,k,t'} &= \sum_{s\in\Sigma_{t'}} \left(x^s_{m_i,t'} - mom_{m_i,1,t'}\right)^n p^s_{t'}, \;\; i\in I, \;\; n = 2,3,4, \\
r_{m_i,m_l,t'} &= \sum_{s\in\Sigma_{t'}} \left(x^s_{m_i,t'} - mom_{m_i,1,t'}\right)\left(x^s_{m_l,t'} - mom_{m_l,1,t'}\right) p^s_{t,} \;\; i,l \in I \text{ and } i<l \\
p^s_{t'} &\geq 0, \; s \in \Sigma_{t'} \\
MOM_{m_i,n,t'} &= MOM_{m_i,n} \\
\text{for } t' &= \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + 1.
\end{aligned}
$$

The probabilities $p^s_{t'}$ are assumed to be the same for the intermediate times $t' = \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau + 1$ and $s \in \Sigma_\tau = \left\{s^{v(\tau)}_\tau = 0, 1, \ldots, k_\tau\right\}$. The first moments $MOM^s_{m_i,1,\tau}$, at state $s \in \Sigma_\tau$, for decision times $\tau = t, t+1, ..., t+T-1$ is given by

$$MOM^s_{m_i,1,\tau} = (1 - MRF_{m_i}) \times MRL_{m_i} + MRF_{m_i} \times x^s_{m_i,\tau}. \tag{2.3.5}$$

The scenario generation algorithm is described as follows:

Repeat for each decision node $s \in \Sigma_\tau = \left\{s^{v(\tau)}_\tau = 0, 1, \ldots, k_\tau\right\}$, and $\tau = t, t+1, ..., t+T-1$

> **Step 1:** Calculate the first moments $MOM_{m_i,1,t'}$ for the current decision sub-tree (Equation 2.3.3), where $MOM_{m_i,1,\tau} = MOM^s_{m_i,1,\tau}$.
>
> **Step 2:** Find the scenarios $x^s_{m_i,t'}$ and probabilities $p^s_{t'}$, where $t' = \tau + \frac{1}{12}, \tau + \frac{2}{12}, ..., \tau+1$ for decision node $s \in \Sigma_\tau$ by solving the optimisation problem (Problem 2.3.4).
>
> **Step 3:** Update the first moment for each scenario generated (Equation 2.3.5).

For scenario trees with large branching factors Høyland and Wallace (2001b) further suggest a sampling approach where smaller trees are generated with the same statistical properties and aggregated to one large tree, while preserving the statistical properties. Thus, Step 1 may be repeated several times using the same statistical properties to generate several smaller trees. These smaller scenario trees are then aggregated in order to construct the desired sub scenario tree while preserving the statistical properties.

2.3.4 Arbitrage

Klaassen (2002) illustrates that the method proposed by Høyland and Wallace (2001b) can result in arbitrage opportunities in the scenario tree if only statistical properties are imposed. Klaassen (2002) shows that arbitrage opportunities can be detected *ex post* by checking for solutions to a set of linear constraints or be excluded by including non-linear constraints in the scenario generation process.

Klaassen (2002) proposes linear constraints for two types of arbitrage. Ingersoll (1987) distinguishes these two types of arbitrage. The first type is an opportunity to construct a zero-investment portfolio that has nonnegative payoffs in all states of the world, and a strictly positive payoff in at least one state. The second type is an opportunity to construct a negative investment portfolio (i.e. providing an immediate positive cash flow) that generates a nonnegative payoff in all future states of the world.

Following the notation of Klaassen (2002), let $r_{k,t+1}^{n}$ be the return on asset class k $(k = 1, ..., K)$ between time t and $t+1$ if state n $(n = 1, ..., N)$ of the world materialises at time $t+1$. Klaassen (2002) mentions a useful result, that if the set of equations

$$\sum\nolimits_{n=1}^{N} v_n \left(1 + r_{k,t+1}^{n}\right) = 1 \text{ for all } k = 1, ..., K,$$

has a strictly positive solution v_n for all n $(n = 1, ..., N)$, then no arbitrage opportunities of the first or second type exist

(also see Ingersoll, 1987). Taking $r_{\tau,t+1}^{n}$ to be the return on a zero-coupon bond with maturity $k = \tau$, then

$$1 + r_{\tau,t+1}^{n} = \frac{P_{t+1}^{n}(\tau - 1)}{P_t(\tau)},$$

where $P_t(\tau) = \exp(-\tau y_t(\tau))$ is the price at time t of a zero-coupon bond with maturity τ and $y_t(\tau)$ the zero-rate time t with maturity τ. Thus if the set of equations

$$\sum\nolimits_{n=1}^{N} v_n \exp\left(-(\tau - 1) y_{t+1}^{n}(\tau - 1)\right) = \exp(-\tau y_t(\tau)) \text{ for all maturities } \tau,$$

has a strictly positive solution v_n for all n $(n = 1, ..., N)$, then no arbitrage opportunities of the first or second type exist in our yield curve scenarios.

Klaassen (2002) proposes that one may include these equations in the scenario generation process. This can be achieved by adding these equations as constraints to Problem 2.3.1 and 2.3.4. To avoid making the model more complex, we use these equations to check *ex post* for arbitrage in our scenario trees. If arbitrage exits we re-run the scenario generation method. Portfolio constraints in the optimisation problem, such as the restriction of short-selling and the inclusion of bid and ask spreads, will also eliminate some arbitrage opportunities.

2.3.5 Back-testing

To test our scenario generation methodology we implemented the multi-stage stochastic optimisation problem described in Dempster *et al.* (2006). Dempster *et al.* (2006) propose an asset and liability management framework and give numerical results for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay. They demonstrate the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product (see Appendix A for the model formulation). We use our scenario generation approach to generate the input scenarios for the optimisation problem. The mean reversion parameters are estimated using the market data up to an initial decision time t and scenario trees are then generated from time t to some chosen horizon $t+T$. The optimal first stage/root node decision is then implemented at time t and we measure the success of the portfolio implementation by its performance with historical data up to time $t+1$. This whole procedure is rolled forward for T trading times. At each decision time t, the parameters are re-estimated using the historical data up to and including time t.

We back-test over a period of five years, from February 2004 through to February 2009, and use different tree structures with approximately the same number of scenarios and minimise the expected average shortfall for an annual guarantee of 7% and include transaction costs. The tree structures are described in Table 2.3.1. Bonds with 5, 7, 10, 15 and 19 year maturities as well as the FTSE/JSE Top 40 index are included in the portfolio. Scenarios for the Top 40 index are generated along with the yield curve using the second scenario generation algorithm in Subsection 2.3.3.

Table 2.3.1: Tree structure for different back-tests

Year	Set1	Set2	Set3
February04	5.5.5.5.5=3125	13.4.4.4.4=332	200.2.2.2.2=3200
February05	8.8.8.8=4096	15.6.6.6=3240	400.2.2.2=3200
February06	15.15.15=3375	30.10.10=3000	400.3.3=3600
February07	56.56=3136	160.20=3200	800.4=3200
February08	3125	3328	3200

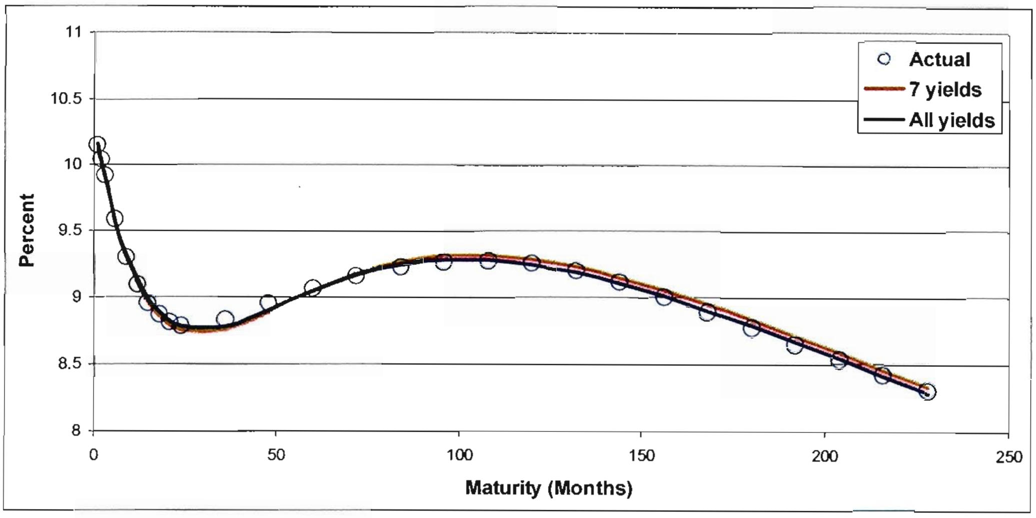

Figure 2.3.2: Svensson yield curve representation using all 27 yields and seven yields

We use the Perfect Fit Bond Curves, one of the five BEASSA Zero Coupon Yield Curve series of yield curves (see BESA, 2003a) with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months. The curves are derived from government bond data and the technical specifications are described in BESA (2003b). We use end-of-month data from August 1999 through to February 2009.

In order to decrease the dimensionality of the problem we only use the 1, 3, 12, 60, 120, 180 and 228 month yields. After scenarios have been generated for these seven yields the rest of the yield curve is interpolated using the Svensson yield curve parameterisation (see Svensson, 1994). The seven yields that are used, consisting of short, medium and long maturities, are sufficient to construct the 27 yields that we require scenarios for. In Figure 2.3.2 this is illustrated. As can be seen the interpolated yield curve, were only seven yields where used to estimate the Svensson yield curve parameters, fits the actual yield curve just as well as the yield curve where all 27 yields where used to estimate the Svensson parameters.

Descriptive statistics for the log changes of the yields and the Top 40 index (mean, standard deviation, skewness, kurtosis, minimum, maximum and autocorrelations for one month) and correlations over the entire period of back-testing are provided in Table 2.3.2 and Table 2.3.3.

Table 2.3.2: Descriptive statistics of log changes

Maturity	Mean	Std. Dev.	Skew	Kurt	Min	Max	$\hat{\rho}(1)$
1M	-0.0010	0.0393	-1.2356	3.2950	-0.1444	0.1140	0.3474
3M	-0.0014	0.0391	-1.1648	2.9384	-0.1380	0.0957	0.4101
12M	-0.0040	0.0490	-0.3146	2.0638	-0.1498	0.1383	0.3206
60M	-0.0043	0.0456	0.2221	2.4157	-0.1587	0.1709	0.1416
120M	-0.0041	0.0431	0.1667	2.2940	-0.1475	0.1505	0.0995
180M	-0.0042	0.0432	-0.1681	1.9487	-0.1515	0.1207	0.0630
228M	-0.0045	0.0441	-0.2608	1.9998	-0.1553	0.1302	0.0413
Top40	0.0086	0.0598	-0.3793	0.2190	-0.1614	0.1369	0.0753

Table 2.3.3: Historical correlations of log changes

	1M	3M	12M	60M	120M	180M	228M	Top40
1M	1.000	0.935	0.584	0.337	0.201	0.161	0.143	0.155
3M	0.935	1.000	0.796	0.454	0.264	0.211	0.185	0.183
12M	0.584	0.796	1.000	0.722	0.517	0.450	0.405	0.188
60M	0.337	0.454	0.722	1.000	0.931	0.872	0.816	0.079
120M	0.201	0.264	0.517	0.931	1.000	0.978	0.932	0.022
180M	0.161	0.211	0.450	0.872	0.978	1.000	0.984	0.012
228M	0.143	0.185	0.405	0.816	0.932	0.984	1.000	-0.010
Top40	0.155	0.183	0.188	0.079	0.022	0.012	-0.010	1.000

Furthermore, in Table 2.3.4 we provide the estimated mean reversion parameters over the back-tested period. The mean reversion parameters are fitted using least squares. With exception of the180 and 228 month zero-rates and the Top 40 index, all mean reversion level parameters are not statistically significantly from different zero. This is also apparent in Table 2.3.2 where it can be seen that the means are close to zero. This is expected, considering that we are working with the log changes in the zero-rates. Some of the mean reversion factor parameters are statistically significantly different from zero. Furthermore it is apparent from the mean reversion factor parameters, that there is a high level of mean reversion $(1-MRF)$, this can also be seen in Table 2.3.2 where there is a low level of persistence when we observe the autocorrelations.

Figure 2.3.3 illustrates back-testing results for all three scenario sets. The results are consistent with those in Dempster *et al.* (2006). Although Dempster *et al.* (2006) minimise the expected average shortfall and maximises the expected terminal wealth of the portfolio, and distinguish between them using a risk-aversion parameter, we only minimise the expected average shortfall to test the model's performance. Only shortfall is used as it plays an important role in our applications. The model performs well staying above the guarantee, although the system involves the inclusion of transaction cost which puts downward pressure on the portfolio wealth.

Table 2.3.4: Mean reversion parameter estimates

Parameters	5 Year	4 Year	3 Year	2 Year	1 Year
1 M MRL	-0.008 (0.009)	-0.007 (0.008)	-0.007 (0.006)	-0.004 (0.006)	-0.001 (0.006)
1 M MRF	**0.437** (0.125)	**0.397** (0.115)	**0.375** (0.106)	**0.395** (0.100)	**0.356** (0.093)
3 M MRL	-0.007 (0.010)	-0.007 (0.008)	-0.006 (0.007)	-0.004 (0.006)	0.000 (0.006)
3 M MRF	**0.469** (0.123)	**0.428** (0.113)	**0.396** (0.105)	**0.425** (0.098)	**0.399** (0.092)
12 M MRL	-0.006 (0.009)	-0.007 (0.008)	-0.006 (0.007)	-0.003 (0.006)	-0.001 (0.006)
12 M MRF	**0.274** (0.134)	**0.264** (0.121)	0.222 (0.112)	**0.242** (0.104)	**0.236** (0.097)
60 M MRL	-0.007 (0.007)	-0.009 (0.006)	-0.008 (0.005)	-0.006 (0.005)	-0.004 (0.005)
60 M MRF	0.185 (0.136)	0.220 (0.122)	0.132 (0.114)	0.157 (0.105)	0.157 (0.099)
120 M MRL	-0.007 (0.006)	-0.009 (0.005)	-0.008 (0.005)	-0.007 (0.005)	-0.005 (0.004)
120 M MRF	0.087 (0.138)	0.157 (0.124)	0.077 (0.114)	0.121 (0.106)	0.136 (0.100)
180 M MRL	-0.009 (0.005)	**-0.010** (0.005)	**-0.009** (0.004)	-0.007 (0.004)	-0.005 (0.004)
180 M MRF	-0.021 (0.139)	0.086 (0.125)	0.008 (0.115)	0.067 (0.106)	0.101 (0.100)
228 M MRL	**-0.010** (0.004)	**-0.011** (0.004)	**-0.009** (0.004)	-0.008 (0.004)	-0.005 (0.004)
228 M MRF	-0.129 (0.138)	0.016 (0.126)	-0.046 (0.115)	0.030 (0.107)	0.072 (0.100)
Top40 MRL	0.009 (0.009)	0.010 (0.008)	0.013 (0.007)	**0.015** (0.006)	**0.015** (0.005)
Top40 MRF	0.006 (0.139)	0.016 (0.126)	-0.007 (0.115)	-0.026 (0.107)	-0.023 (0.102)

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

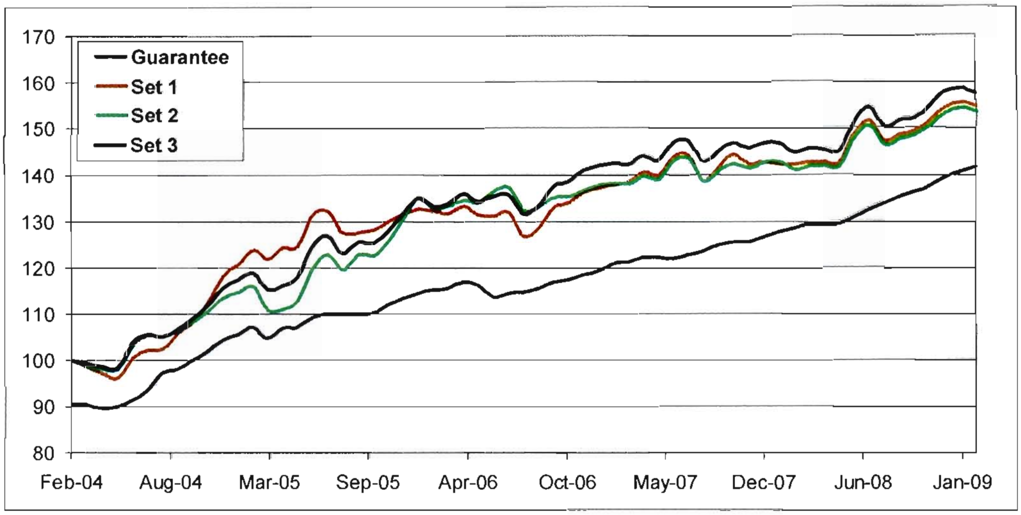

Figure 2.3.3: Moment-matching scenario back-testing results

Table 2.3.5: Moment-matching portfolio allocation stability statistics

	Objective function	Top40	5Y	7Y	10Y	15Y	19Y
Mean	-0.2634	0.0027	0.6928	0.0	0.0	0.0	0.2685
Std. Dev.	0.038	0.0028	0.0858	0.0	0.0	0.0	0.0706
Min	-0.1636	0.0000	0.2464	0.0	0.0	0.0	0.1596
Max	-0.3757	0.0105	0.8172	0.0	0.0	0.0	0.6291

Furthermore, in Table 2.3.5 we present back-testing stability statistics (Høyland and Wallace, 2001b, uses similar statistics to report stability). The model was solved for 100 different scenario sets, with a tree-string of (40.3) (120 scenarios) using all available data for model fitting. We present the mean, standard deviation, minimum and maximum of the objective function and the first stage portfolio allocations. The first stage portfolio allocation seems consistent with small standard deviation. The objective function also has a small standard deviation with no outliers when we look at the minimum and maximum, indicating the stability of the scenario generation.

The scenario generation is further tested by solving the model for 100 different scenario sets and for different number of final nodes, 120, 500, 1000 and 2000. Dempster *et al.* (2006) minimise the expected average shortfall and maximises the expected terminal wealth of the portfolio, and distinguish between them using a risk-aversion parameter (*alpha*). For each scenario set the model is solved ranging the risk-aversion parameter from 0 to 1 in steps of 0.1 (1 being the most risk-averse).

Table 2.3.6: Moment-matching efficient frontier stability statistics

	120 Scenarios		500 Scenarios		1000 Scenarios		2000 Scenarios	
Alpha	Mean	Std. Dev	Mean	Std. Dev	Mean	Std. Dev	Mean	Std. Dev
0	343.41	1.432	343.33	1.090	343.45	0.659	343.54	0.538
0.1	308.98	1.292	308.91	0.982	309.02	0.594	309.10	0.484
0.2	274.56	1.151	274.49	0.874	274.58	0.529	274.66	0.431
0.3	240.13	1.010	240.08	0.766	240.15	0.463	240.22	0.377
0.4	205.71	0.869	205.66	0.657	205.72	0.396	205.78	0.324
0.5	171.29	0.728	171.25	0.548	171.30	0.332	171.35	0.270
0.6	136.88	0.587	136.84	0.439	136.88	0.266	136.92	0.217
0.7	102.47	0.446	102.44	0.332	102.47	0.200	102.50	0.161
0.8	68.08	0.307	68.06	0.224	68.08	0.136	68.10	0.109
0.9	33.73	0.181	33.70	0.122	33.71	0.077	33.72	0.058
1	-0.26	0.038	-0.27	0.023	-0.27	0.016	-0.273	0.013

Table 2.3.6 presents the mean, standard deviation, for the different number of final nodes. In Figure 2.3.4 we display the mean frontier, by averaging the objective function values obtained over the 100 different scenario sets, and the confidence bands covering 95% of the results (Kaut *et al.*, 2007; Consiglio and Staino, 2008, uses a similar approach for scenario and model stability testing). The frontier is a decreasing function of the risk-aversion parameter *alpha*. If the value of *alpha* is closer to 1, more importance is given to the shortfall of the portfolio and less given to the expected wealth and hence a more risk-averse portfolio allocation strategy will be taken and vice versa. In the extreme case where *alpha* is 1 only the shortfall will be minimised and the expected wealth will be ignored, and where *alpha* is 0, the unconstrained case only maximises the wealth. For a 1000 final nodes the 95% region, at its maximum (alpha at 0), is 0.685% wide (a reduction of 0.4% from 500 final nodes), ensuring that the randomisation error is bounded enough (Figure 2.3.5 displays the frontier for alpha values of 0 to 0.2 for a better view). In Table 3.4.3 we also observe that the standard deviation reduces as the number of final nodes increases. The reduction is less when we increase the number of final nodes from 1000 to 2000, again ensuring that the randomisation error is bounded enough, and achieves stability.

Although back-testing assumes that the past describes the future and can in no means guarantee the success of the outcomes of these models in practice, it provides us with a way to assess the performance of the proposed algorithm. Through back-testing we see that the proposed scenario generation algorithm performs well on a portfolio optimisation problem in literature; similar results are ob-

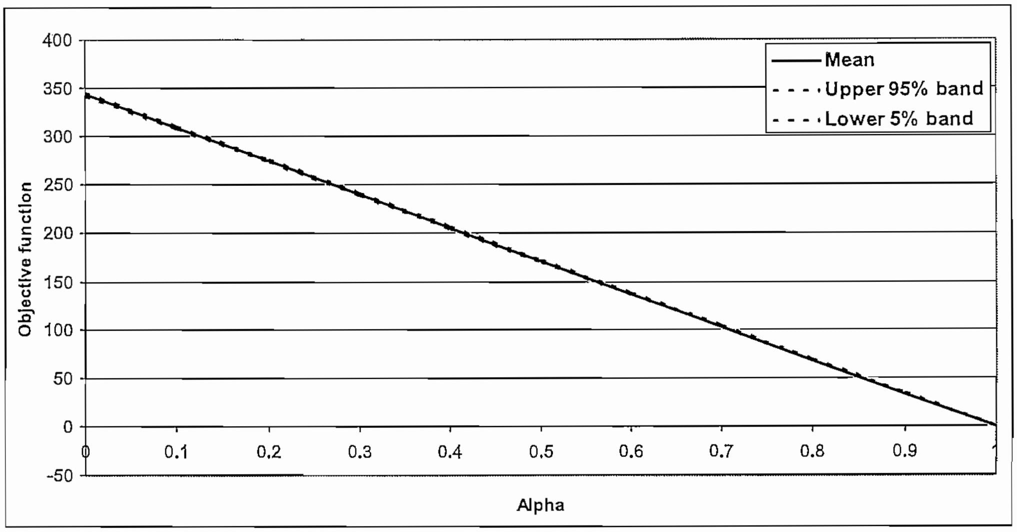

Figure 2.3.4: Average efficient frontier with 5% and 95% confidence bands.

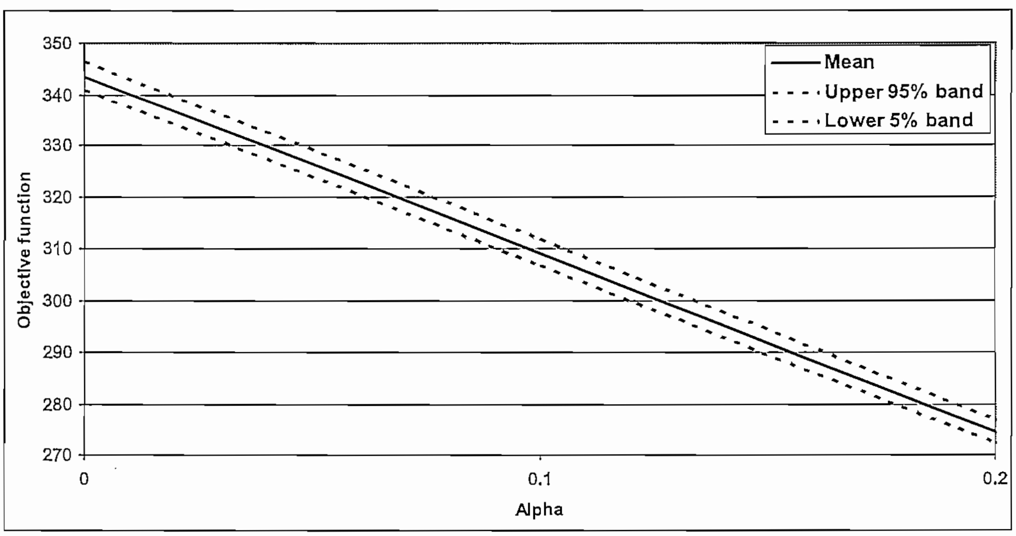

Figure 2.3.5: Extract: Average efficient frontier with 5% and 95% confidence bands.

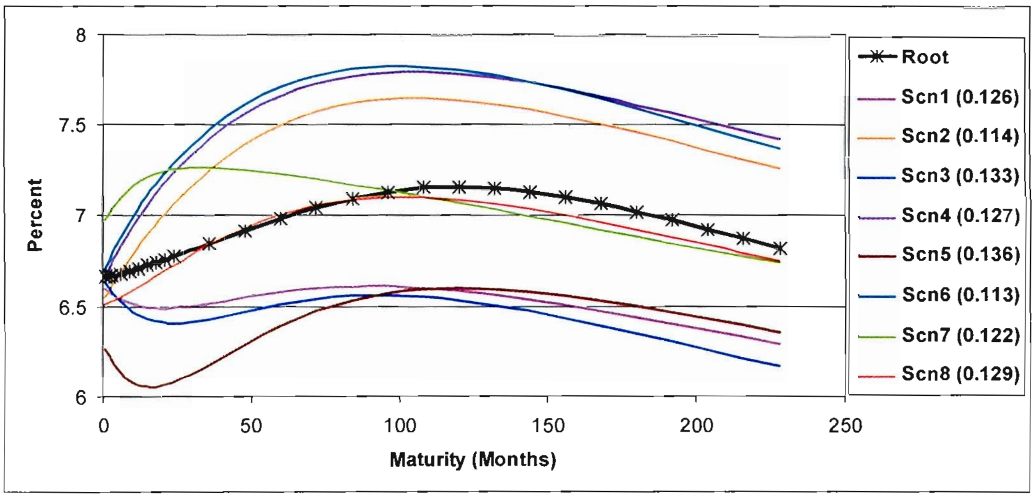

Figure 2.3.6: Moment-matching scenarios at first decision period (tree-string (8.8), monthly data with quaterly rebranching)

tained as in Dempster *et al.* (2006). We also see that stability in the objective is obtained by increasing the number of scenarios. The amount of the final number of scenarios necessary to achieve this stability may depend on the optimisation problem in question.

Figures 2.3.6 to 2.3.8 visually illustrate the scenarios generated by the proposed algorithm. Figure 2.3.6 presents the scenarios at the first decision period of a scenario tree and the root with a tree-string of (8.8) using monthly data and re-branching quarterly. The probabilities are presented in parentheses next to each scenario. Plausible scenarios are generated with no extreme scenarios and appropriate probabilities. Figure 2.3.7 presents the evolution of three scenarios (upwards, downwards and small movements) from the root to the first decision period. The evolution of the scenario from the root to the first decision period seems realistic, and shows the path dependency of the scenario generation method. Also note that the scenario generation method does not only produce parallel shifts in the level of the yield curves but also slope and curvature changes. In Figure 2.3.8 we present the scenarios at the leave nodes of the scenario tree. The scenarios are split into two groups, low probability and average probability. Some scenarios with larger movement from the root are generated with lower probability than at the first decision period, and scenarios with average movement from the root have mostly the same probabilities.

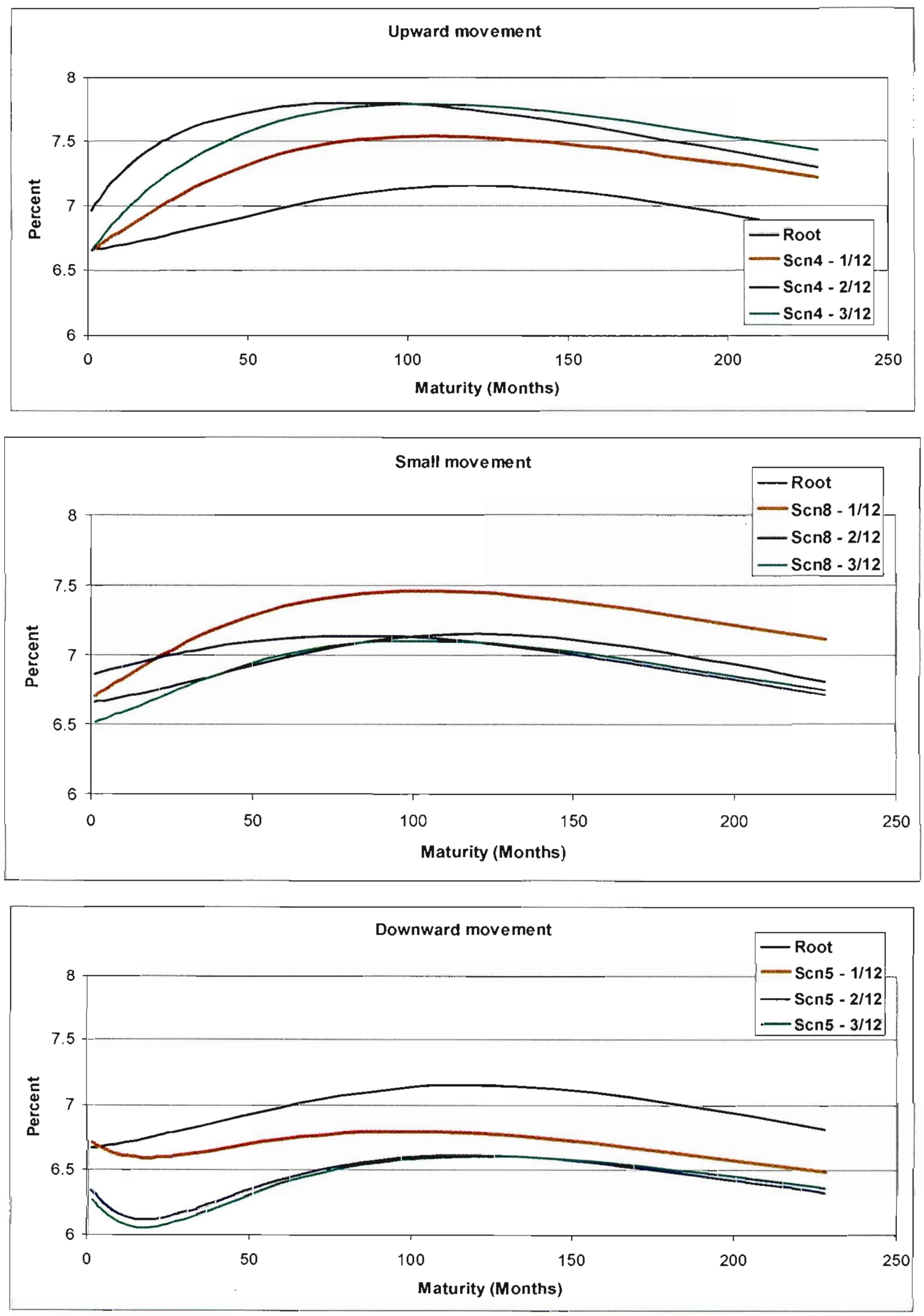

Figure 2.3.7: Evolution of different moment-matching scenarios from root to first decision period

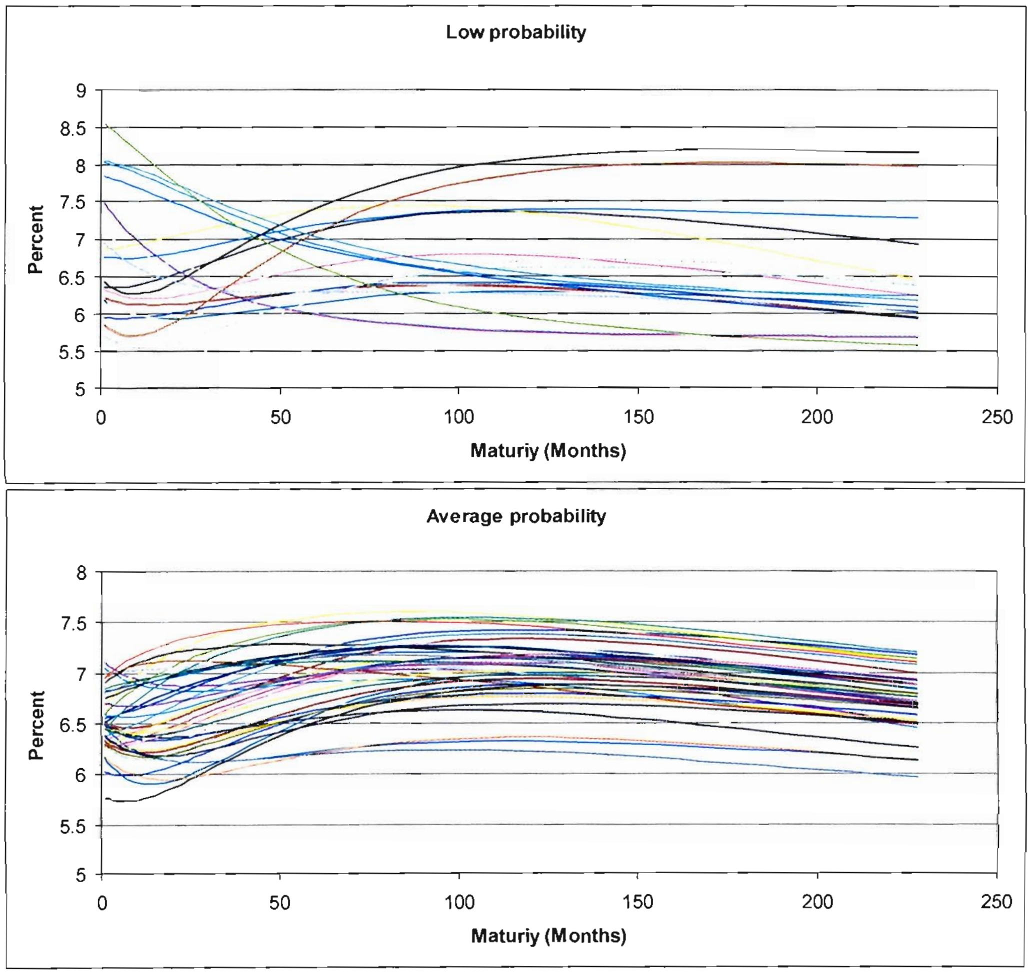

Figure 2.3.8: Moment-matching scenarios at the leave nodes of the scenario tree for low and average probabilities

2.4 Conclusion

In this chapter we have presented a moment-matching scenario generation method for generating yield curve scenarios. We proposed two methods, both generating scenarios by matching the principle moments of the underlying distributions of the log changes of the yield curve. These methods generate yield curve scenario trees with path dependent yield curves at intermediate time nodes, and where each node in the scenario tree represent the term structure of interest rates (yield curve). The second scenario generation method is preferred to the first, as it consumes less time to generate larger yield curve scenario trees.

Our proposed method uses the scenario generation method proposed by Høyland and Wallace (2001b) as apposed to the a second method proposed Høyland *et al.* (2003). The second method proposed by Høyland *et al.* (2003) is in most if not all respects better than the first method introduced by Høyland and Wallace (2001b). We used the first method for its ability to include decision-makers expected views or forecasts from economic scenario generators. Further investigations should consider making use off the Høyland *et al.* (2003) method instead of the Høyland and Wallace (2001b) method. By doing so a large reduction in computational time will be achieved.

Although back-testing assumes that the past describes the future and can in no means guarantee that success of the outcomes of these models in practice, it provides us with a way to assess the performance of the proposed algorithm. Through back-testing we have shown that the proposed scenario generation algorithm performs well on a portfolio optimisation problem in the literature. We also have shown that stability is obtained by increasing the number of scenarios. The amount of the final number of scenarios necessary to achieve this stability may depend on the optimisation problem in question.

In the next chapter we investigate the inclusion of macro-economic factors into the generation of yield curve scenarios. de Pooter *et al.* (2007) argues that models which include macro-economic factors seem more accurate in sub-periods where there is substantial uncertainty about the future path of interest rates. Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation.

Chapter 3

Macro-economic interest rate scenario generation

One of the main sources of uncertainty in analysing risk and return properties of a portfolio of fixed income securities is the stochastic evolution of the shape of the term structure of interest rates. In this chapter[1] *we estimate a model that fits the South African term structure of interest rates, using a Kalman filter approach. Our model includes four latent factors and three observable macro-economic factors (capacity utilisation, inflation and repo-rate). Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate.*

3.1 Introduction

One of the main sources of uncertainty in analysing the risk and return properties of a portfolio of fixed income securities is the stochastic evolution of the shape of the term structure of interest rates (or yield curve). Many yield curve models for example, Knez *et al.* (1994), Duffie and Kan (1996), and Dai and Singleton (2000) consider models in which unobserved factors explain the entire set of yields. These factors are often given the labels level, slope and curvature. The factors in these models are however not linked to macro-economic factors. Examples where

[1] A paper based on the work done in this chapter has been presented at the ASSA Convention, South Africa, 2008 and received the "Best First-Time Paper" award. The paper has also been accepted for publication in the South African Actuarial Journal.

a latent factor model approach is used to characterise the yield curve and that explicitly include macro-economic factors can be found in Ang and Piazzesi (2003), Hördahl *et al.* (2004) and Wu (2002). These examples, however, only consider a unidirectional linkage between the macro-economy and the yield curve. Kozicki and Tinsley (2001), Dewachter and Lyrio (2004) and Rudebusch and Wu (2003) allow for implicit feedback. Tilley (1992) provides an actuarial layman's guide to building stochastic interest rate generators.

Bernaschi *et al.* (2008) analyses and shows that besides the relation with the ECB (European central bank) official interest rate, it is extremely difficult to find, using simple linear regression analysis, a convincing relation among the parameters that describe the Italian term structure and macro-economic factors that drive the dynamics of the term structure itself. Bernaschi *et al.* (2008) concludes that a possible solution is to resort to more complex interaction models based on non-linear impulse-response functions. Ang and Piazzesi (2003) further argue the importance of describing the joint behavior of the yield curve and macro-economic factors for bond pricing, investment decisions and public policy. They state that although many term structure models use latent factors to explain term structure movements and some interpretations to what these factors mean (e.g. level slope and curvature), the factors are not given direct comparisons with macro-economic factors. These models describe the effect the latent factors have on the yield curve rather than describing the economic sources of the shocks. Ang and Piazzesi (2003) consider a unidirectional linkage between the macro-economy and the yield curve.

de Pooter *et al.* (2007) argues that models which include macro-economic factors seem more accurate in sub-periods where there is substantial uncertainty about the future path of interest rates. Furthermore models that do not include information about the macro-economy perform well in sub-periods where the term structure has a more stable pattern.

Inspired by the research of Diebold *et al.* (2006) we estimate a model that fits the South African term structure of interest rates, using a Kalman filter approach. Diebold *et al.* (2006) characterise the yield curve using three latent factors, namely level, slope and curvature. To model the dynamic interactions between the macro-economy and the yield curve, they also included observable macro-economic factors, specifically real activity, inflation and a monetary policy instrument.

To capture the dynamics of the yield curve, Diebold *et al.* (2006) do not use a no-arbitrage factor representation such as the typically used affine no-arbitrage models (see for example Duffee, 2002; Brousseau, 2002) or canonical affine no-arbitrage models (see for example Rudebusch and Wu, 2003). Instead of using a no-arbitrage representation Diebold *et al.* (2006) suggest·using a *three-factor* term structure model based on the yield curve model of Nelson and Siegel (1987), as used in Diebold and Li (2006), and interpret these factors as level, slope and curvature. Diebold and Li (2006) propose a two-step procedure to estimate the dynamics of the yield curve. The procedure firstly estimates the three latent factors and secondly estimates an autoregressive model for these factors. Diebold and Li (2006) use these models to forecast the term structure. Diebold *et al.* (2006) propose a one-step approach by introducing an integrated state-space modelling approach which is preferred over the two-step Diebold-Li approach. This Kalman filter approach allows for a bidirectional linkage between the macro-economy and the yield curve and simultaneously fits the yield curve and estimates the underlying dynamics of these factors. The model also incorporates the estimation of the macro-economic factors and the link between the macro-economy and the latent factors driving the yield curve.

In the South African yield curve context, to name a few, Maitland (2002) provides a principle component analysis approach for interpolating the South African yield curve. The methodology proposed by Maitland (2002) provides a way in which the yield curve can be interpolated from a restricted number of modelled yields, and at the same time minimises the number of yields from which to estimate the remainder of the curve. Given the first and second principal components, Maitland (2002) shows that the short rate and the long-bond yield could be used to reconstruct the South African yield curve. Stander (2000) discusses bond indices in South Africa. Using a survey, Stander (2000) establishes inadequacies in the indices as well as possible changes that should be considered. Stander (2000) further addresses criticism of the Bond Exchange-Actuaries yield curve and presents alternative empirical yield-curve models and equilibrium models. These contributions focus on the characterisation of the yield curve and do not consider forecasting or scenario generation.

In Section 3.2, we describe the Kalman filter state-space modelling approach for the basic three-factor yields-only model proposed by Diebold *et al.* (2006). Their model uses only three latent factors of the yield curve and does not include macro-economic factors. We will describe the model estimation for the South African term structure and introduce a four-factor model based on the Svensson (1994) yield curve model. We introduce the four-factor model after showing that the Nelson and Siegel (1987) model is not flexible enough to get an acceptable fit to the South African term structure.

In Section 3.3, we incorporate macro-economic factors (capacity utilisation, inflation and repo-rate) in the yields-macro model. Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. Section 3.4 describes our approach. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate. In practice these views are produced by means of an economic scenario generator (ESG) or expert opinion. We also discuss the existence of arbitrage in the scenario trees and propose a method to eliminate arbitrage opportunities.

3.2 Yields-only model

In this section we introduce the factor model representation of the yield curve. Following Diebold *et al.* (2006), we start with the yields-only model using the three-factor representation of Nelson and Siegel (1987) and use this as a benchmark for the four-factor representation of Svensson (1994). By using the more flexible four-factor model, we obtain a better cross-sectional fit. Since all the models that are described in this section are fitted using a Kalman filter approach, we start this section with an overview of the Kalman filter.

3.2.1 The Kalman filter

The Kalman filter, introduced by Kalman (1960), is a popular technique used in signal processing, control engineering and other fields. The main idea is to represent a dynamic system in terms of states (the unobserved underlying Markov process). The state equation (or transition equation) describes the dynamics of this process while the observation equation (or measurement equation) relates the observables with the unobserved states. The advantage of using a state-space representation (defined below) is that it allows the modeller to infer the properties of the unobserved yield curve drivers from the observed interest rates or yields.

Following Hamilton (Hamilton, 1994, Chap 13), let y_t denote a vector of variables (yields in our case) observed at date t that can be described in terms of f_t, a vector of unobservable states. The state-space representation of the dynamics of y is then given the following system of equations:

$$f_t = Af_{t-1} + \eta_t \text{ (Transition equation)}$$

$$y_t = Bx_t + \Lambda f_t + \varepsilon_t \text{ (Measurement equation)}$$

where the matrices A, B and Λ have appropriate dimensions, x_t is a vector of exogenous variables. The disturbances η_t and ε_t are vector white noise processes:

$$E\left(\eta_t \eta_\tau'\right) = \begin{cases} Q & \text{for } t=\tau \\ 0 & \text{otherwise} \end{cases},$$

$$E\left(\varepsilon_t \varepsilon_\tau'\right) = \begin{cases} H & \text{for } t=\tau \\ 0 & \text{otherwise} \end{cases},$$

where the matrices Q and H have appropriate dimensions. The disturbances η_t and ε_t are also assumed to be uncorrelated at all lags:

$$E\left(\eta_t \varepsilon_\tau\right) = 0, \text{for all } t \text{ and } \tau.$$

The Kalman filter is a sequential algorithm that calculates the best predictor of the unobserved states, given all previous observations. The details will be given later.

3.2.2 Factor representation

The main aim of the factor model approach is to represent the term structure (a large set of yields with various maturities) as a function of a smaller set of unobservable factors. The Nelson-Siegel representation (Nelson and Siegel, 1987) produces reliable and reasonable estimation results and has become one of the popular approaches adopted by central banks for yield curve estimation (BIS, 1999). The Nelson-Siegel model, derived from a parametric functional form for the forward rates, uses only four parameters to define a parsimonious and stable representation of the whole term structure:

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_3 \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

where $y(\tau)$ is the zero coupon yield with maturity τ and β_1, β_2, β_3 and λ are the model parameters. As demonstrated by Diebold and Li (2006), the parameters β_1, β_2 and β_3 of the Nelson-Siegel representation of the yield curve, can be interpreted as level, slope and curvature and the terms that multiply these factors are factor loadings. The parameter λ determines the shape of the curve and does not have a direct economic interpretation. To give meaning to the parameters β_1, β_2 and β_3, Diebold and Li (2006) rewrite the representation as

$$y_t(\tau) = L_t + S_t \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_t \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

where L_t, S_t and C_t are the, time-varying parameters, β_1, β_2 and β_3, and are considered unobserved factors.

Diebold *et al.* (2006) describe the state-space system as follows: The dynamics of the unobservable factors, L_t, S_t and C_t, are modelled as a vector autoregressive process of the first order which forms a state-space system. The ARMA state vector dynamics may be of any order, but the VAR(1) assumption is maintained for transparency and parsimony. The dynamics of the state vector is governed by the transition equation

$$\begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix},$$

where $t = 1, ..., T$ and μ_L, μ_S and μ_C the means of the unobservable factors, L_t, S_t and C_t.

By fixing the parameter λ (to be specified later), the measurement equation, which relates a set of N yields of the yield curve, with maturities $\tau_1, ..., \tau_N$, to the three unobserved factors, are

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \begin{pmatrix} L \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix},$$

where $t = 1, ..., T$. The state-space system can be written in matrix notation as

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t,$$

$$y_t = \Lambda f_t + \varepsilon_t,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \text{ and } f_t = \begin{pmatrix} L \\ S_t \\ C_t \end{pmatrix}.$$

The white noise disturbances in the transition and measurement equations are required to be orthogonal to each other and to the initial state for the linear least-squares optimality of the Kalman filter:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right],$$

$$E\left(f_0 \eta_t'\right) = 0,$$

$$E\left(f_0 \varepsilon_t'\right) = 0.$$

Diebold *et al.* (2006) assume that the Q matrix is non-diagonal to allow the shocks to the three term structure factors to be correlated. The H matrix is assumed to be diagonal, which implies that the deviations of the yields of various maturities from the yield curve are uncorrelated. This is quite standard and as in estimating no-arbitrage term structure models, i.i.d "measurement errors" are added to the observed yields. Given the large number of observed yields, this is also required for computational tractability (Diebold *et al.*, 2006).

3.2.3 Three-factor model estimation

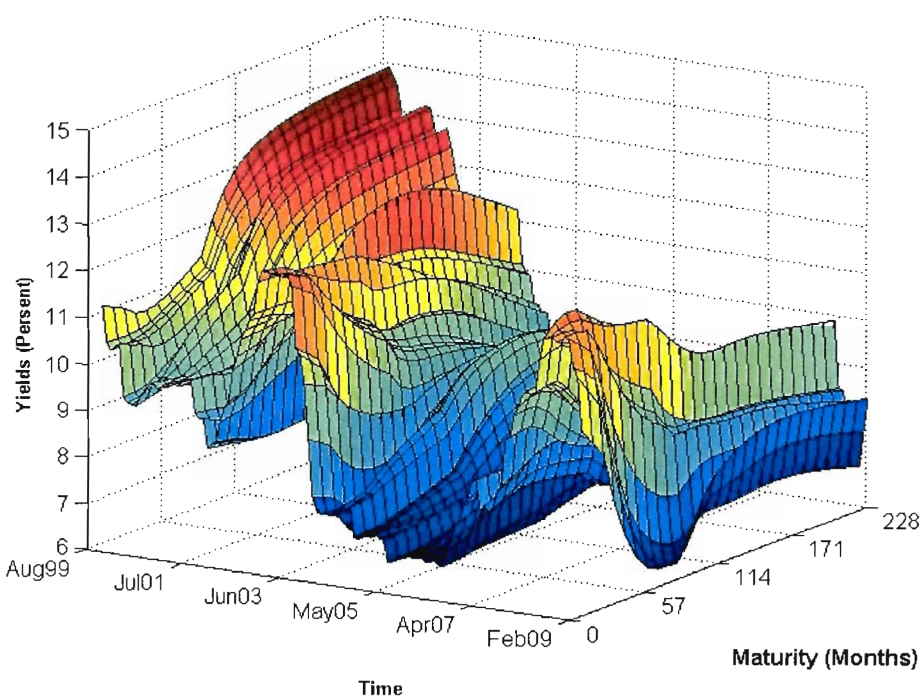

Figure 3.2.1: Yield curves, August 1999 to February 2009

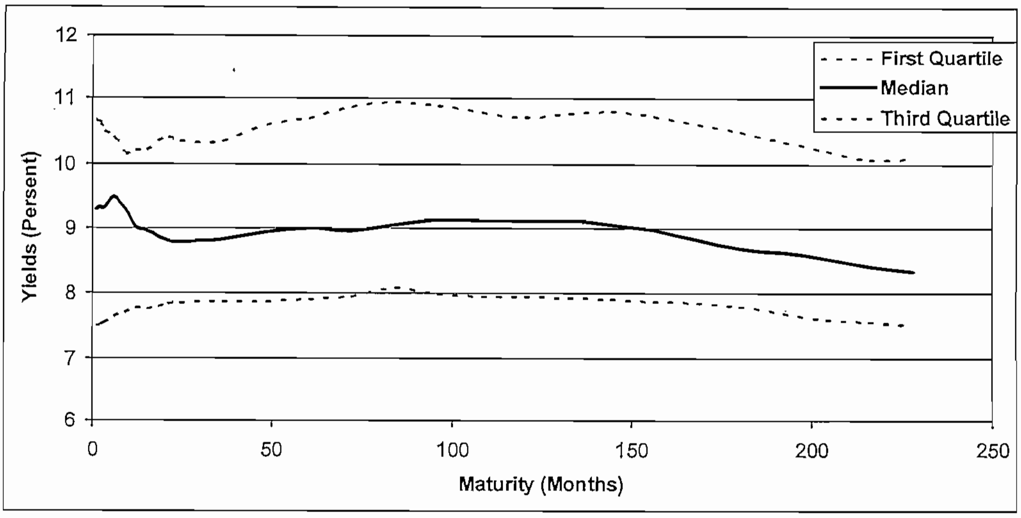

Figure 3.2.2: Median yield curve with point-wise interquartile ranges

We use the Perfect Fit Bond Curves, one of the five BEASSA Zero Coupon Yield Curve series of yield curves (see BESA, 2003a), with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months. The yields curves are zero-rates and compounded semi-annually. The yields are converted to continuous compounded yields for analysis purposes. The curves are derived from government bond data and the technical specifications are described in BESA (2003b). We use end-of-month data from August 1999 through to February 2009. Figure 3.2.1 provides a three dimensional plot of the yield curve data.

The variation in the level of the yield curve is visually apparent as is the variation in the slope and curvature of the yield curve. Descriptive statistics for the yields (mean, standard deviation, minimum, maximum and autocorrelations for one, twelve and thirty months) are provided in Table 3.2.1. It is clear that the typical yield curve is humped shape with a negative hump at about 20 months and a positive hump at about 120 months. The short rates are less volatile than the long rates but less persistent when comparing the autocorrelation with a lag of twelve months. This is the opposite compared to the U.S. term structure (see Diebold and Li, 2006). This may due to the little amount of data available. The level is persistent and varies moderately relative to its mean and the slope and the curvature are the least persistent. The slope is highly variable relative to its mean as is the curvature. In Figure 3.2.2 the median yield curve together with point-wise interquartile ranges are displayed. The humped shaped pattern, with short rates less volatile than long rate, is apparent.

Table 3.2.1: Yield curve descriptive statistics

Maturity	Mean	Std. dev.	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$
1	9.225	1.835	6.542	12.437	0.974	0.272	-0.335
2	9.215	1.806	6.554	12.383	0.975	0.271	-0.337
3	9.199	1.774	6.565	12.329	0.974	0.269	-0.334
6	9.127	1.672	6.604	12.223	0.968	0.268	-0.301
9	9.054	1.586	6.574	12.092	0.956	0.268	-0.242
12	9.003	1.533	6.531	12.058	0.945	0.272	-0.175
15	8.981	1.509	6.509	12.024	0.936	0.280	-0.113
18	8.977	1.499	6.385	11.989	0.931	0.293	-0.058
21	8.987	1.498	6.347	11.954	0.928	0.309	-0.011
24	9.005	1.502	6.295	11.918	0.927	0.327	0.030
36	9.108	1.546	6.638	12.107	0.930	0.403	0.142
48	9.219	1.606	6.912	12.615	0.937	0.463	0.194
60	9.311	1.660	6.980	12.959	0.943	0.501	0.219
72	9.381	1.702	7.026	13.190	0.946	0.524	0.232
84	9.432	1.733	7.055	13.350	0.948	0.538	0.237
96	9.466	1.756	7.072	13.466	0.949	0.548	0.236
108	9.483	1.773	7.076	13.553	0.950	0.554	0.231
120	9.481	1.789	7.069	13.621	0.950	0.560	0.225
132	9.462	1.805	7.051	13.676	0.950	0.565	0.216
144	9.427	1.823	7.023	13.721	0.951	0.570	0.207
156	9.380	1.842	6.987	13.759	0.951	0.574	0.197
168	9.325	1.863	6.945	13.792	0.952	0.579	0.186
180	9.266	1.885	6.898	13.820	0.952	0.583	0.174
192	9.204	1.907	6.847	13.845	0.953	0.588	0.162
204	9.141	1.931	6.794	13.866	0.953	0.591	0.150
216	9.077	1.956	6.698	13.886	0.954	0.594	0.138
228	9.015	1.981	6.581	13.903	0.954	0.597	0.126
Level	9.232	1.571	6.818	12.821	0.958	0.575	0.138
Slope	0.184	2.140	-3.760	4.093	0.965	0.261	-0.336
Curvature	-0.204	1.376	-5.409	2.847	0.852	0.016	-0.102

As in Diebold *et al.* (2006), the yields-only model forms a state-space system, with a VAR(1) transition equation summarising the dynamics of the vector of latent variables, and a linear measurement equation relating the observed yields to the state vector as described above. In the entire model there are 46 parameters that need to be estimated by the numerical optimisation of the relevant likelihood function. Let ψ be the vector of all parameters that need to be estimated. These parameters are the nine parameters contained in transition matrix A, the three parameters contained in the mean state vector μ, and the one parameter λ contained in the measurement matrix Λ. Furthermore the transition disturbance covariance matrix Q contains six parameters, and the measurement disturbance covariance matrix H contains 27 parameters (one variance for each of the 27 yields). Given that the matrices A and Λ are affine and assuming that the distributions of η_t, ε_t and f_0 are normal, the model is referred to as a linear Gaussian state-space model (Lemke, 2006).

It follows by assumption that the transition density $p\left(f_{t+1}|f_t\right)$ and the measurement density $p\left(y_t|f_t\right)$ are jointly normal. This implies that the prediction and filtering densities are normal,

$$f_t|\mathcal{Y}_{t-1} \sim N\left(\hat{f}_{t|t-1}, \Sigma_{t|t-1}\right),$$

$$f_t|\mathcal{Y}_t \sim N\left(\hat{f}_{t|t}, \Sigma_{t|t}\right),$$

$$y_t|\mathcal{Y}_{t-1} \sim N\left(\hat{y}_{t|t-1}, F_t\right),$$

where $\mathcal{Y}_t = \{y_1, \ldots, y_t\}$ is taken to be the sequence of observations available for estimation and $\hat{f}_{t|t-1}$, $\hat{f}_t$, $\hat{y}_{t|t-1}$ and $\Sigma_{t|t-1}$, $\Sigma_{t|t}$, F_t the sequences of conditional means, and covariance matrices respectively. These quantities can be obtained by employing the Kalman filter for a given set of parameters ψ.

The Kalman filter algorithm can be described as follows (see Lemke, 2006):

Step 1: Set $\hat{f}_{0|0} = \bar{f}_0$, $\Sigma_{0|0} = \bar{\Sigma}_0$ and set $t = 0$.

Step 2: $\hat{f}_{t-1|t-1}$ and $\Sigma_{t-1|t-1}$ are given values, but y_t has not been observed yet. Compute

$$\left(\hat{f}_{t|t-1} - \mu\right) = A\left(\hat{f}_{t-1|t-1} - \mu\right),$$

$$\Sigma_{t|t-1} = A\Sigma_{t-1|t-1}A' + Q,$$

$$\hat{y}_{t|t-1} = \Lambda\hat{f}_{t|t-1}, \text{ and}$$

$$F_t = \Lambda\Sigma_{t|t-1}\Lambda' + H.$$

Step 3: y_t has been observed. Compute

$$K_t = \Sigma_{t|t-1}\Lambda' F_t^{-1},$$

$$\hat{f}_{t|t} = \hat{f}_{t|t-1} + K_t\left(y_t - \hat{y}_{t|t-1}\right),$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - K_t\Lambda\Sigma_{t|t-1}.$$

Step 4: If $t < T$, set $t := t+1$, and go to Step 2; else, stop.

Hence the Kalman filter delivers the sequence of means and covariance matrices for the conditional distributions of interest for a given set of parameters ψ. The Kalman filter is initialised by setting $\bar{f}_0$ and $\bar{\Sigma}_0$ to the unconditional mean and unconditional covariance matrix of the state vector respectively. Under the normality assumption, the distribution of y_t conditional on $\mathcal{Y}_{t-1}$is the N-dimensional normal distribution with mean $\hat{y}_{t|t-1}$and covariance matrix F_t. The conditional density of y_t given $\mathcal{Y}_{t-1}$ and ψ can be written as (see Lemke, 2006)

$$p\left(y_t|\mathcal{Y}_{t-1};\psi\right) = \left[(2\pi)^{N/2}\sqrt{|F_t|}\right]^{-1}\cdot\exp\left[-\frac{1}{2}\left(y_t - \hat{y}_{t|t-1}\right)' F_t^{-1}\left(y_t - \hat{y}_{t|t-1}\right)\right].$$

Accordingly, the log-likelihood function becomes

$$\ln L(\psi) = -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log|F_t| - \frac{1}{2}\sum_{t=1}^{T} v_t' F_t^{-1} v_t,$$

where $v_t = \left(y_t - \hat{y}_{t|t-1}\right)$ is the vector of prediction errors.

For a given set of parameters ψ, the Kalman filter is used to compute the prediction errors v_t and their covariance matrix F_t, after which the log-likelihood function is computed. The parameters are estimated by maximising the log-likelihood function, using either the Nelder-Mead Simplex or Newton-Raphson algorithms. For more details on Kalman filtering see Harvey (1989) and Lemke (2006). Non-negativity constraints are imposed on all the variances. As in Diebold *et al.* (2006), we obtain starting parameters using the two-step Diebold-Li method and initialising the variances to 1.0. As in Diebold and Li (2006) we initialise the value of λ at 0.0609 to maximise the loading on the curvature factor at exactly 30 months, i.e. the maturity at which the hump occurs in the yield curve.

In Table 3.2.2 and Table 3.2.3 we present the estimation results for the three-factor yields-only model. In Table 3.2.2 the estimate of the A matrix indicates the highly persistent dynamics of L_t, S_t and C_t, with estimated own lag coefficients 0.945, 0.987 and 0.953 respectively. Cross factor dynamics between S_t and L_t and S_t and C_t appear to be important with statistically significant effects. The mean of

Table 3.2.2: Three-factor yields-only model estimates

	L_{t-1}	S_{t-1}	C_{t-1}	μ
L_t	**0.945** (0.026)	0.007 (0.020)	-0.022 (0.014)	**8.568** (0.886)
S_t	**0.091** (0.033)	**0.987** (0.025)	**0.110** (0.018)	0.039 (0.723)
C_t	-0.137 (0.078)	**-0.213** (0.059)	**0.953** (0.043)	-0.988 (0.597)

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

Table 3.2.3: Three-factor yields-only estimated Q matrix

	L_t	S_t	C_t
L_t	**0.161** 0.021	-0.163 0.024	0.003 0.046
S_t		**0.265** 0.035	-0.015 0.059
C_t			**1.476** 0.196

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

the level is approximately 8.5 percent, the mean of the slope and the mean of the curvature do not seem statistically significant different from zero and appear to be reasonable if compared to the mean values of the empirical estimates in Table 3.2.1. The largest eigenvalue of the A matrix is 0.96 and ensures the stationarity of the system. In Table 3.2.3 the estimates of the Q matrix indicate that transitional shock volatility increases as we move from L_t to S_t to C_t as measured by the diagonal elements. There are no significant covariance terms in the Q matrix. The estimate for λ is 0.0916 which implies that the loading on the curvature factor is maximised at a maturity of 19.85 months. This can be seen in Figure 3.2.2, where the first (inverted) hump occurs at around the maturity of 20 months.

Table 3.2.4 contains the means and standard deviations of the predicted errors (also called measurement errors, which is measured as the difference between the actual yields and the predicted model yields) for the yields-only model and the yields-macro model (macro-economic factors included, which is presented in the next section). The three-factor yields-only model fits the yield curve reasonably well in the short maturities but less so in the longer maturities, with the standard deviation also increasing for longer maturities. The results is similar to the yields-only model of Diebold *et al.* (2006).

Table 3.2.4: Summary of statistics for predicted errors of yields (percent)

Maturity	Yields-only				Yields-macro			
	Three-factor		Four-factor		Three-factor		Four-factor	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std
1	-0.052	0.377	-0.048	0.491	0.001	0.431	-0.008	0.399
2	-0.011	0.358	-0.008	0.449	0.004	0.438	0.003	0.410
3	0.017	0.359	0.020	0.430	0.010	0.317	0.021	0.311
6	0.047	0.385	0.054	0.431	-0.011	0.338	0.030	0.318
9	0.037	0.407	0.051	0.462	0.006	0.323	0.026	0.313
12	0.023	0.432	0.044	0.499	0.016	0.321	0.022	0.315
15	0.018	0.456	0.045	0.528	0.026	0.337	0.013	0.336
18	0.019	0.476	0.048	0.550	0.020	0.357	0.009	0.362
21	0.023	0.493	0.052	0.566	0.015	0.377	0.009	0.383
24	0.030	0.506	0.055	0.576	0.014	0.393	0.011	0.400
36	0.064	0.537	0.049	0.594	0.016	0.408	0.014	0.415
48	0.104	0.551	0.027	0.596	0.019	0.421	0.017	0.428
60	0.141	0.556	0.002	0.597	0.024	0.432	0.019	0.440
72	0.168	0.557	-0.018	0.604	0.049	0.466	0.023	0.475
84	0.186	0.558	-0.028	0.612	0.083	0.487	0.017	0.493
96	0.195	0.560	-0.028	0.619	0.118	0.496	0.010	0.498
108	0.192	0.560	-0.021	0.623	0.149	0.500	0.006	0.499
120	0.174	0.560	-0.013	0.627	0.170	0.501	0.007	0.499
132	0.141	0.559	-0.007	0.629	0.180	0.502	0.013	0.498
144	0.095	0.560	-0.005	0.632	0.177	0.503	0.020	0.499
156	0.038	0.565	-0.005	0.634	0.159	0.504	0.026	0.501
168	-0.025	0.574	-0.005	0.636	0.128	0.505	0.029	0.503
180	-0.091	0.589	-0.005	0.638	0.085	0.509	0.030	0.506
192	-0.160	0.608	-0.004	0.639	0.033	0.518	0.030	0.509
204	-0.229	0.633	-0.002	0.641	-0.024	0.532	0.029	0.511
216	-0.298	0.661	0.001	0.644	-0.085	0.553	0.028	0.514
228	-0.365	0.693	0.004	0.647	-0.146	0.579	0.029	0.517

We use the Kalman filter fixed-interval smoothing algorithm to obtain optimal extractions of the latent level, slope and curvature factors. The algorithm consists of a set of recursions which start with the final quantities given by the Kalman filter and work backwards (Harvey, 1989). The equations are

$$\hat{f}_{t|T} = \hat{f}_{t|t} + \Sigma_t^* \left(\hat{f}_{t+1|T} - \hat{f}_{t+1|t}\right), \text{ and}$$

$$\Sigma_{t|T} = \Sigma_{t|t} + \Sigma_t^* \left(\Sigma_{t+1|T} - \Sigma_{t+1|t}\right) \Sigma_t^{*\prime},$$

where $\Sigma_t^* = \Sigma_{t|t} A' \Sigma_{t+1|t}^{-1}$. Figure 3.2.3 plots the three smoothed estimated factors together and in Figure 3.2.4 to Figure 3.2.6 we present the three factors together with various empirical proxies and related macro-economic factors. The level factor in Figure 3.2.3 is in the neighbourhood of 8 percent and displays persistence. The slope and the curvature factors vary around zero with positive and negative values and appear less persistent. The slope factor is more persistent than the curvature factor but has a lower variance. This seems consistent if compared to the mean and autocorrelation values of the empirical estimates in Table 3.2.1.

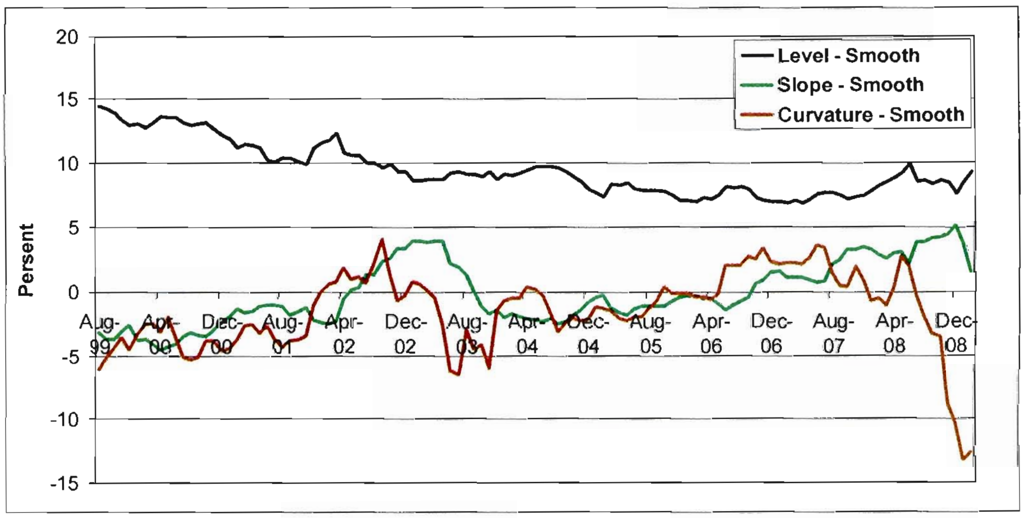

Figure 3.2.3: Estimates of the level, slope and curvature factors

Figure 3.2.4 displays the estimated level factor and two related comparison series. The first one is a commonly used empirical proxy for the level factor namely the average of the short-, medium- and long-term yields, $(y(3) + y(24) + y(228))/3$. The second is the annual percentage change in the consumer price index. There is a high correlation of 0.89 between the level factor and the empirical proxy. The correlation between the level factor and the inflation is 0.51. As stated by Diebold *et*

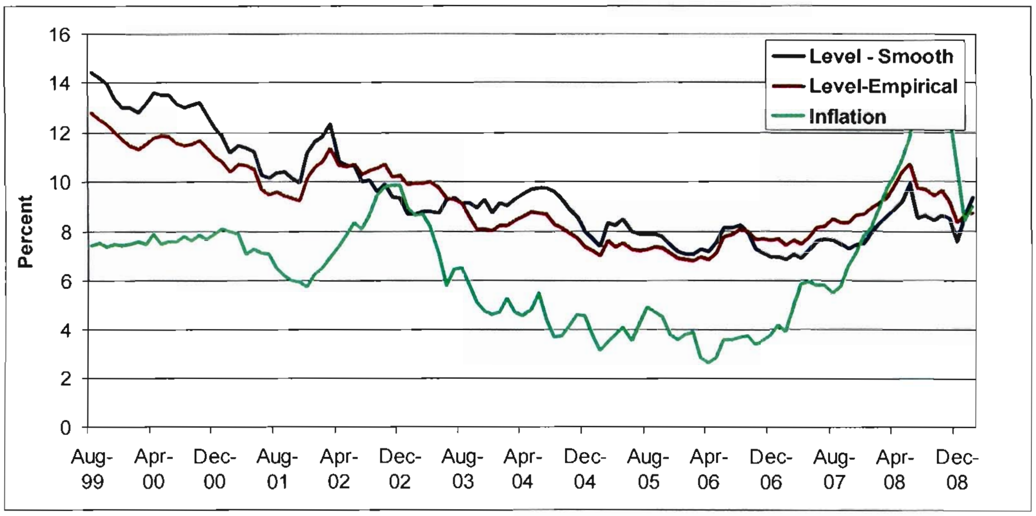

Figure 3.2.4: Three-factor yields-only model level factor and empirical estimates

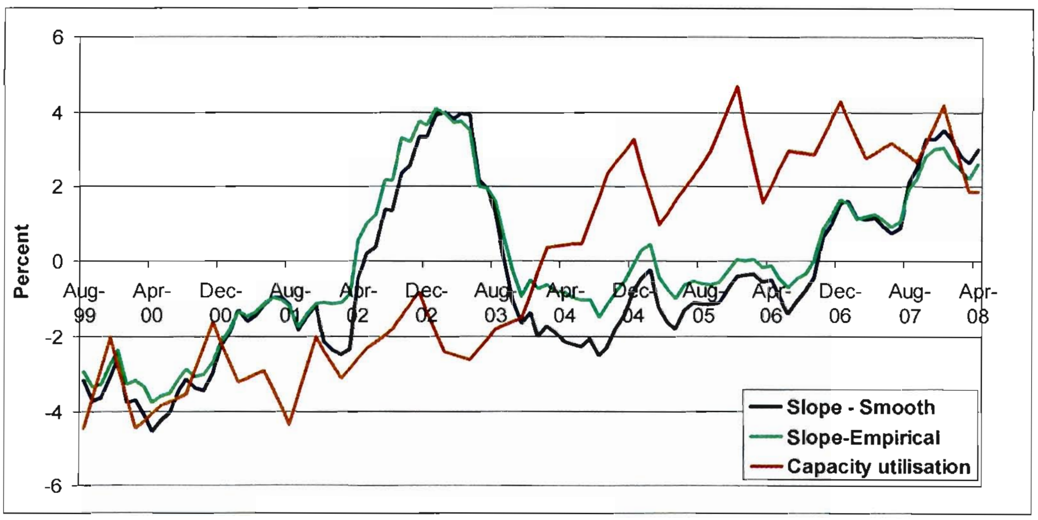

Figure 3.2.5: Three-factor yields-only model slope factor and empirical estimates

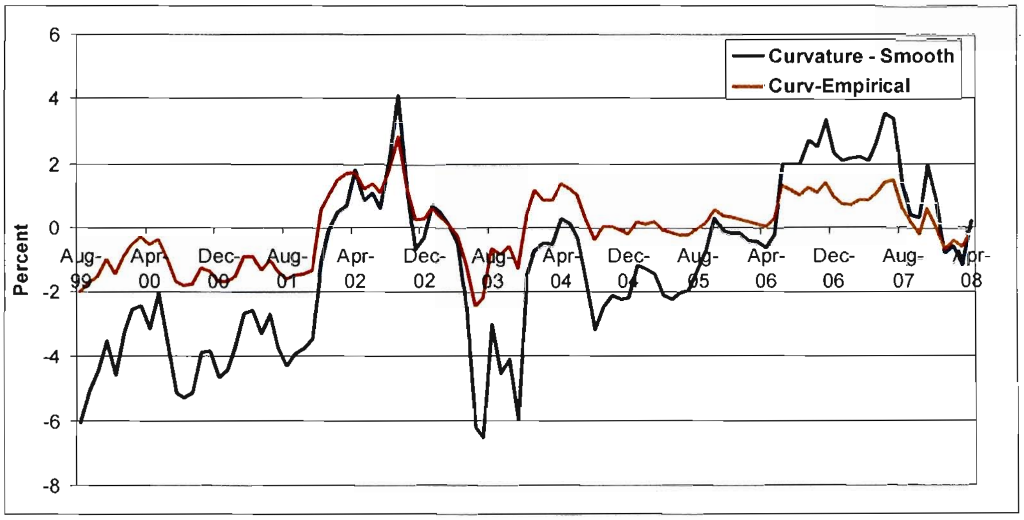

Figure 3.2.6: Three-factor yields-only model curvature factor and empirical estimates

al. (2006) this is consistent with the Fisher equation, which suggests a link between level of the yield curve and inflationary expectations.

In our estimates of the of the empirical level, slope and curvature we use the 228 month zero-rate and not the 120 month zero-rate as Diebold *et al.* (2006). The use of the 228 month zero-rate instead of the 120 month zero-rate has very little effect on the results.

Figure 3.2.5 displays the estimated slope factor and two related comparison series. The first is the empirical proxy for the slope factor namely the difference between the short- and long-term yields, $y(3) - y(228)$. The second is an indicator of macro-economic activity namely the demeaned manufacturing capacity utilisation. There is a high correlation of 0.97 between the slope factor and the empirical proxy. The correlation between the slope factor and the capacity utilisation is 0.36. Diebold *et al.* (2006) states that, as with the level factor, there is a connection between the yield curve and the cyclical dynamics of the economy.

Figure 3.2.6 displays the curvature factor and the empirical proxy for the curvature of the yield curve, which is $2y(24) - y(3) - y(228)$. There is a correlation of 0.97 between the curvature factor and the empirical proxy. Diebold *et al.* (2006) report no reliable macro-economic links to the curvature factor.

3.2.4 Four-factor model estimation

In Table 3.2.4 we have shown that the three-factor model fits the yield curve reasonably well in the short maturities but less well at the longer maturities. We extend the three-factor model to a four-factor model using the Svensson representation (Svensson, 1994) of the yield curve

$$y(\tau) = \beta_1 + \beta_2\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}\right) + \beta_3\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\right) + \beta_4\left(\frac{1-e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\right),$$

where $y(\tau)$ is the zero coupon yield with maturity τ and β_1, β_2, β_3, β_4, λ_1 and λ_2 are model parameters. Figure 3.2.7 illustrates an example fit of both the Nelson-Siegel curve and the Svensson curve on an arbitrary yield curve in our dataset. It is clearly visible that the Svensson curve is more flexible and provides a better cross-sectional fit to the South African term structure than the Nelson-Siegel curve.

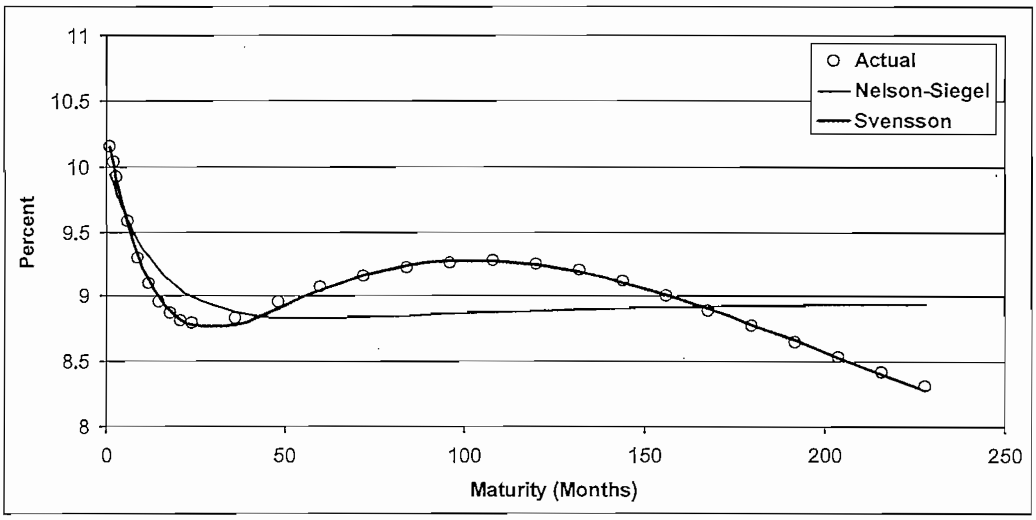

Figure 3.2.7: Nelson-Siegel fit versus Svensson fir for the yield curve

As for the Nelson-Siegel parameterisation, we rewrite the Svensson the representation as

$$y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}\right) + C_t^1\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\right) + C_t^2\left(\frac{1-e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\right),$$

where L_t, S_t, C_t^1 and C_t^2 are the, time-varying parameters, β_1, β_2, β_3 and β_4, respectively. We interpret the factors L_t, S_t, C_t^1 and C_t^2as level, slope, curvature one and curvature two. The state-space system can be extended as

$$(f_t - \mu) = A(f_t - \mu) + \eta_t,$$

$$y_t = \Lambda f_t + \varepsilon_t,$$
$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right],$$

where $f_t = \left(L_t, S_t, C_t^1, C_t^2\right)'$. The dimensions of A, μ, η_t and Q are increased as appropriate. Λ is changed to be

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda_1\tau_1}}{\lambda_1\tau_1} & \frac{1-e^{-\lambda_1\tau_1}}{\lambda_1\tau_1} - e^{-\lambda_1\tau_1} & \frac{1-e^{-\lambda_2\tau_1}}{\lambda_2\tau_1} - e^{-\lambda_2\tau_1} \\ 1 & \frac{1-e^{-\lambda_1\tau_2}}{\lambda_1\tau_2} & \frac{1-e^{-\lambda_1\tau_2}}{\lambda_1\tau_2} - e^{-\lambda_1\tau_2} & \frac{1-e^{-\lambda_2\tau_2}}{\lambda_2\tau_2} - e^{-\lambda_2\tau_2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda_1\tau_N}}{\lambda_1\tau_N} & \frac{1-e^{-\lambda_1\tau_N}}{\lambda_1\tau_N} - e^{-\lambda_1\tau_N} & \frac{1-e^{-\lambda_2\tau_N}}{\lambda_2\tau_N} - e^{-\lambda_2\tau_N} \end{pmatrix} \begin{pmatrix} L \\ S_t \\ C_t^1 \\ C_t^2 \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix}.$$

In Table 3.2.5 and Table 3.2.6 we present the estimation results for the four-factor yields-only model. In Table 3.2.5 the estimate of the A matrix indicates high persistent own dynamics of L_t, S_t, C_t^1 and C_t^2, with estimated own lag coefficients of 0.981, 1.019, 0.809 and 0.992 respectively. Some cross factor dynamics seem significantly important. The estimates indicate the persistence in L_t, C_t^1 and C_t^2 decreases and an increases in S_t as measured by the diagonal elements. The mean of the level is approximately 5.9 percent and is statistically significant different from zero. The mean of the slope is 2.612 percent, the mean of the first curvature factor is -0.529 percent, which are not statistically significant different from zero. The mean of the second curvature factor 8.009 percent and is statistically significant different from zero. The largest eigenvalue of the A matrix is 0.986 and ensures the stationarity of the system. In Table 3.2.6 the estimates indicate an increase in the transitional shock volatility as we move from L_t to S_t to C_t^1 to C_t^2. The estimate for λ_1 is 0.088 which implies that the loading on the first curvature factor is maximised at a maturity of 20.38 months and the estimate for λ_2 is 0.015 which implies that the loading on the second curvature factor is maximised at a maturity of 119.55 months. Referring to Figure 3.2.2 it can be seen that the first hump is at about 20 months and the second hump at 120 months. As shown in Table 3.2.4, the four-factor yields-only model improves on the means of the predicted errors, especially for the long maturities.

Table 3.2.5: Four-factor yields-only model estimates

	L_{t-1}	S_{t-1}	C^1_{t-1}	C^2_{t-1}	μ
L_t	**0.981** (0.044)	0.045 (0.025)	-0.026 (0.018)	0.000 (0.005)	**5.871** (1.588)
S_t	0.004 (0.037)	**1.019** (0.026)	**0.102** (0.021)	**-0.043** (0.015)	2.612 (1.336)
C^1_t	**-0.202** (0.039)	**-0.223** (0.047)	**0.809** (0.039)	**0.046** (0.012)	-0.529 (0.348)
C^2_t	**0.284** (0.055)	**0.275** (0.023)	**-0.081** (0.031)	**0.992** (0.041)	**8.009** (2.855)

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

Table 3.2.6: Four-factor yields-only estimated Q matrix

	L_t	S_t	C^1_t	C^2_t
L_t	**0.547** (0.096)	-0.057 (0.131)	0.036 (0.054)	-0.007 (0.071)
S_t		**0.490** (0.076)	-0.102 (0.057)	-0.021 (0.011)
C^1_t			**1.538** (0.084)	0.077 (0.090)
C^2_t				**4.545** (0.382)

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

Again we plot the estimated smoothed level and slope factors against empirical proxies and macro-economic factors in Figures 3.2.8 and Figure 3.2.9. We omit the curvature factors as there is no reliable macro-economic link to them. In Figure 3.2.8 we plot the estimated level factor against the empirical proxy and annual percentage change in the inflation index. There is a correlation of 0.67 between the estimated level factor and the empirical proxy. The correlation between the estimated level and the inflation is 0.28, which again suggests that inflation is linked to dynamics of the yield curve.

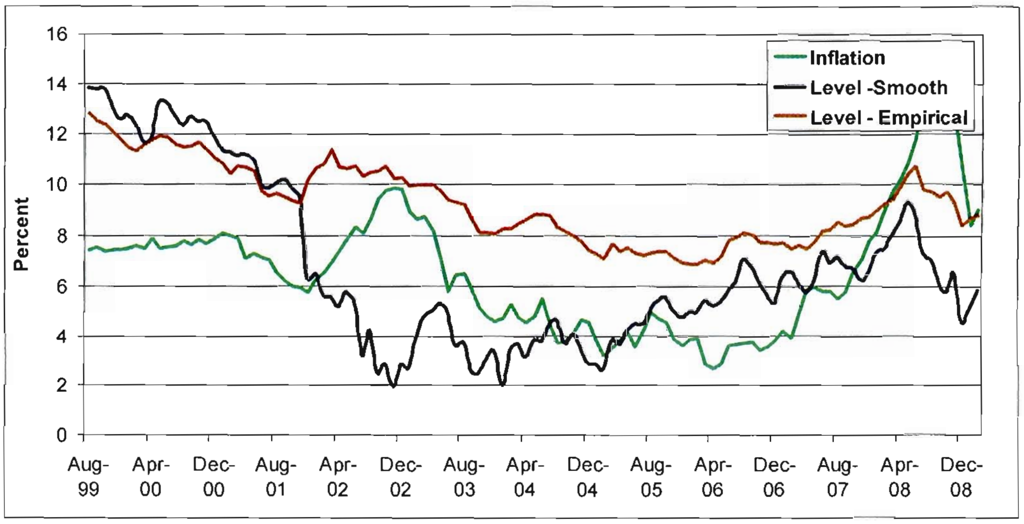

Figure 3.2.8: Four-factor yields-only model level factor and empirical estimates

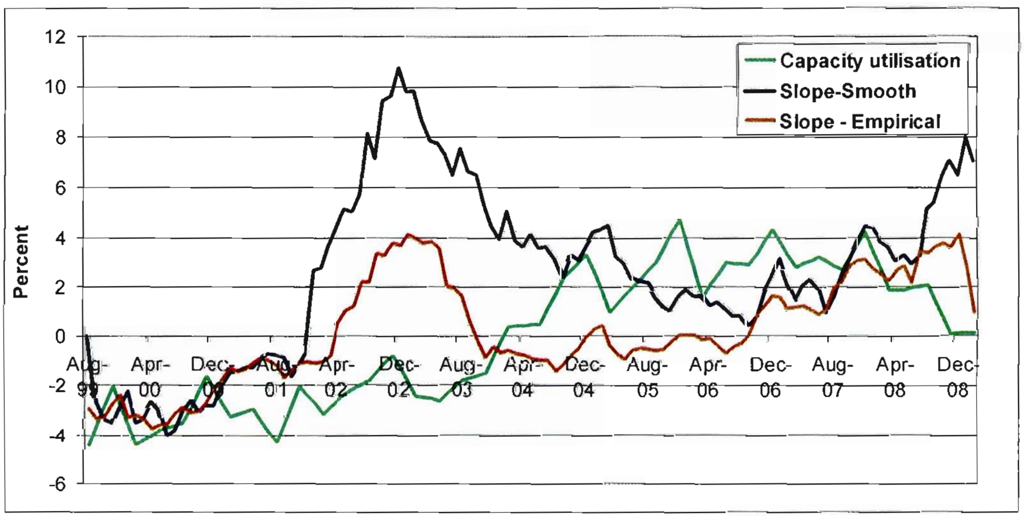

Figure 3.2.9: Four-factor yields-only model slope factor and empirical estimates

Figure 3.2.9 shows the estimated slope curve together with the empirical proxy *and* demeaned manufacturing capacity utilisation. There is a 0.84 correlation between the estimated slope factor and the empirical proxy, and a 0.30 correlation between the estimated slope factor and capacity utilisation. This also suggests a link between the capacity utilisation and the dynamics of the yield curve.

3.3 Macro-economic model

In this section we relate the four unobserved factors, level, slope and the two curvature factors, that provide a good representation of the yield curve, to the macro-economic factors. This can be done by extending the state-space model in the previous section. We also present out-of-sample forecasting results to assess how well the four-factor yields-macro model forecast the dynamics of the yield curve.

3.3.1 Yields-macro model

We include the following three macro-economic factors: manufacturing capacity utilisation (CU_t), which represents the level of real economic activity relative to potential; the annual percentage change in the inflation index (IF_t), which represent the inflation rate; and the repo-rate (RR_t), which represents the South African monetary policy instrument. According to Diebold *et al.* (2006) these three macro-economic factors are considered to be the minimum set of fundamentals needed to capture the basic macro-economic dynamics (see also Rudebusch and Svensson, 1999; Kozicki and Tinsley, 2001). We extend the four-factor yields-only model, to the four-factor yields-macro model, to incorporate the three macro-economic factors. This is done by adding the macro-economic factors to the set of state variables. The state-space system is extended as follows:

$$\begin{pmatrix} (f_t - \mu) \\ (x_t - \nu) \end{pmatrix} = \begin{pmatrix} A_{11}\,(f_{t-1} - \mu) \\ A_{21}\,(f_{t-1} - \mu) \end{pmatrix} + \begin{pmatrix} A_{12}\,(x_{t-1} - \nu) \\ A_{22}\,(x_{t-1} - \nu) \end{pmatrix} + \begin{pmatrix} \eta_t \\ \gamma_t \end{pmatrix},$$

$$y_t = \Lambda f_t + \varepsilon_t,$$

$$\begin{pmatrix} \eta_t \\ \gamma_t \\ \varepsilon_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & K & 0 \\ K' & J & 0 \\ 0 & 0 & H \end{pmatrix}\right],$$

where $f_t = \left(L_t, S_t, C_t^1, C_t^2\right)'$ and $x_t = (CU_t, IF_t, RR_t)'$. Where A_{11}, A_{12}, A_{21}, A_{22}, μ, ν, η_t, γ_t, Q, K and J have appropriate dimensions. Λ stays unchanged. This is consistent with the view that only four factors are needed to distil the information in the yield curve (Diebold *et al.*, 2006). In the four-factor yields-macro model the matrix

$$\begin{pmatrix} Q & K \\ K' & J \end{pmatrix}$$

is assumed to be non-diagonal and H is assumed to be diagonal.

As previously stated it follows by assumption that the transition density $p\left(f_{t+1}|f_t\right)$ and $p\left(x_{t+1}|x_t\right)$ and the measurement density $p\left(y_t|f_t\right)$ are jointly normal. This implies that the prediction and filtering densities are normal,

$$\left(\begin{array}{c|c} f_t & \\ x_t & \mathcal{G}_{t-1} \\ y_t & \end{array}\right) \sim N\left(\begin{pmatrix} f_t \\ x_t \\ y_t \end{pmatrix}, \begin{pmatrix} \Sigma^{ff}_{t|t-1} & \Sigma^{fx}_{t|t-1} & \Sigma^{fy}_{t|t-1} \\ \Sigma^{xf}_{t|t-1} & \Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1} \\ \Sigma^{yf}_{t|t-1} & \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1} \end{pmatrix}\right),$$

$$f_t|\mathcal{G}_t \sim N\left(\hat{f}_{t|t}, \Sigma^{ff}_{t|t}\right),$$

where $\mathcal{G}_t = \{y_1, \ldots, y_t, x_1, \ldots x_t\}$is taken to be the sequence of observations available for estimation. These quantities can be obtained by employing the Kalman filter for a given set of parameters ψ.

The Kalman filter algorithm is updated as follows:

Step 1: Set $\hat{f}_{0|0} = \bar{f}_0$, $\Sigma^{ff}_{0|0} = \bar{\Sigma}_0$ and set $t = 0$.

Step 2: $\hat{f}_{t-1|t-1}$ and $\Sigma^{ff}_{t-1|t-1}$ are given values, but y_t and x_t has not been observed yet. Compute

$$\left(\hat{f}_{t|t-1} - \mu\right) = A_{11}\left(\hat{f}_{t-1|t-1} - \mu\right) + A_{12}\left(x_{t-1} - \nu\right),$$

$$\left(\hat{x}_{t|t-1} - \mu\right) = A_{11}\left(\hat{f}_{t-1|t-1} - \mu\right) + A_{12}\left(x_{t-1} - \nu\right),$$

$$\hat{y}_{t|t-1} = \Lambda \hat{f}_{t|t-1},$$

$$\Sigma^{ff}_{t|t-1} = A_{11}\Sigma^{ff}_{t-1|t-1}A'_{11} + Q,$$

$$\Sigma^{xx}_{t|t-1} = A_{21}\Sigma^{ff}_{t-1|t-1}A'_{21} + J,$$

$$\Sigma^{yy}_{t|t-1} = \Lambda\Sigma^{ff}_{t|t-1}\Lambda' + H,$$

$$\Sigma^{fx}_{t|t-1} = A_{11}\Sigma^{ff}_{t-1|t-1}A'_{21} + K,$$

$$\Sigma^{fy}_{t|t-1} = \Sigma^{ff}_{t|t-1}\Lambda' \text{ and}$$

$$\Sigma^{xy}_{t|t-1} = \Sigma^{fx}_{t|t-1}\Lambda'.$$

Step 3: y_t and x_t has been observed. Compute

$$\hat{f}_{t|t} = \hat{f}_{t|t-1} + \begin{pmatrix} \Sigma^{fx}_{t|t-1} & \Sigma^{fy}_{t|t-1} \end{pmatrix} \begin{pmatrix} \Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1} \\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1} \end{pmatrix}^{-1} \begin{pmatrix} y_t - \hat{y}_{t|t-1} \\ x_t - \hat{x}_{t|t-1} \end{pmatrix},$$

$$\Sigma^{ff}_{t|t} = \Sigma^{ff}_{t|t-1} - \begin{pmatrix} \Sigma^{fx}_{t|t-1} & \Sigma^{fy}_{t|t-1} \end{pmatrix} \begin{pmatrix} \Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1} \\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma^{xf}_{t|t-1} \\ \Sigma^{yf}_{t|t-1} \end{pmatrix}.$$

Step 4: If $t < T$, set $t := t + 1$, and go to Step 2; else, stop.

Accordingly, the log-likelihood function becomes

$$\begin{aligned} \ln L(\psi) &= -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T} \log \left| \begin{pmatrix} \Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1} \\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1} \end{pmatrix} \right| \\ & \quad -\frac{1}{2}\sum_{t=1}^{T} v_t' \begin{pmatrix} \Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1} \\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1} \end{pmatrix}^{-1} v_t, \end{aligned}$$

where $v_t = \begin{pmatrix} y_t - \hat{y}_{t|t-1} \\ x_t - \hat{x}_{t|t-1} \end{pmatrix}$ is the vector of prediction errors.

In Table 3.3.1 and Table 3.3.2 we present the estimation results for the four-factor yields-macro model. The estimate of the A matrix again indicates high persistent own dynamics for S_t, C_t^1, C_t^2, CU_t and IF_t. Some of the cross-factor dynamics are significantly important for most factors. The estimates also indicate an increase in the transitional shock volatility as we move from L_t to S_t to C_t^1 to C_t^2 all being statistically significant different from zero, and a decrease in the transitional shock volatility as we move from CU_t to IF_t to RR_t, all being statistically significant different from zero. There are small changes in the mean of the slope and the two curvature factors. The largest eigenvalue of the A matrix is 0.98 and ensures the stationarity of the system. None of the covariance terms in the Q matrix are statistically significantly different from zero.

As shown in Table 3.2.4, the four-factor yields-macro model reduces on most of the means slightly but reduces the standard deviations of the predicted errors indicating a better fit. We also provide the means and standard deviations of the predicted errors for the three-factor yields-macro model, again the four-factor yields-macro model fits the yield curve better than the three-factor yields-macro model. The estimates for the level, slope and two curvature factors of the four-factor yields-macro model are very similar to those of the four-factor yields-only model.

Table 3.3.1: Four-factor yields-macro model estimates

	L_{t-1}	S_{t-1}	C^1_{t-1}	C^2_{t-1}	CU_t	IF_t	RR_t	μ
L_t	**0.621**	**-0.279**	-0.022	-0.008	0.065	-0.005	**0.310**	**4.769**
	(0.064)	(0.030)	(0.029)	(0.024)	(0.058)	(0.052)	(0.071)	(11.625)
S_t	0.313	**1.162**	**0.116**	0.003	-0.059	0.094	**-0.310**	2.878
	(0.164)	(0.074)	(0.039)	(0.006)	(0.089)	(0.060)	(0.123)	(12.272)
C^1_t	0.293	**0.183**	**0.960**	0.027	-0.063	**-0.175**	-0.212	0.472
	(0.206)	(0.084)	(0.049)	(0.053)	(0.112)	(0.077)	(0.178)	(8.076)
C^2_t	**1.420**	**1.406**	**0.108**	**0.877**	**-0.565**	-0.027	**-1.588**	**8.419**
	(0.139)	(0.112)	(0.014)	(0.077)	(0.155)	(0.091)	(0.271)	(19.528)
CU_t	**0.281**	**0.205**	0.023	0.019	**0.928**	-0.062	**-0.232**	**86.338**
	(0.117)	(0.077)	(0.017)	(0.012)	(0.039)	(0.033)	(0.086)	(3.465)
IF_t	**0.643**	**0.392**	**0.105**	0.048	0.055	**0.888**	**-0.303**	**5.389**
	(0.146)	(0.074)	(0.020)	(0.008)	(0.033)	(0.030)	(0.040)	(1.323)
RR_t	**0.334**	**0.187**	**0.117**	0.016	-0.022	0.023	**0.725**	**7.491**
	(0.072)	(0.008)	(0.005)	(0.009)	(0.027)	(0.020)	(0.062)	(2.563)

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

Table 3.3.2: Four-factor yields-macro estimated Q matrix

	L_{t-1}	S_{t-1}	C^1_{t-1}	C^2_{t-1}	CU_t	IF_t	RR_t
L_t	**0.559**	-0.001	0.000	0.000	0.000	0.000	0.000
	(0.078)	(0.036)	(0.087)	(0.112)	(0.077)	(0.019)	(0.042)
S_t		**0.635**	0.000	0.000	0.000	0.000	0.000
		(0.093)	(0.060)	(0.171)	(0.036)	(0.051)	(0.011)
C^1_t			**1.762**	0.000	0.000	0.000	0.000
			(0.216)	(0.167)	(0.097)	(0.198)	(0.150)
C^2_t				**4.780**	0.000	0.000	0.000
				(0.38)	(0.100)	(0.195)	(0.165)
CU_t					**0.197**	0.000	-0.001
					(0.028)	(0.021)	(0.011)
IF_t						**0.250**	0.001
						(0.040)	(0.010)
RR_t							**0.104**
							(0.013)

Note: Bold entries denote parameters estimates significant at five percent using a t-test statistic. Standard errors appear in parentheses

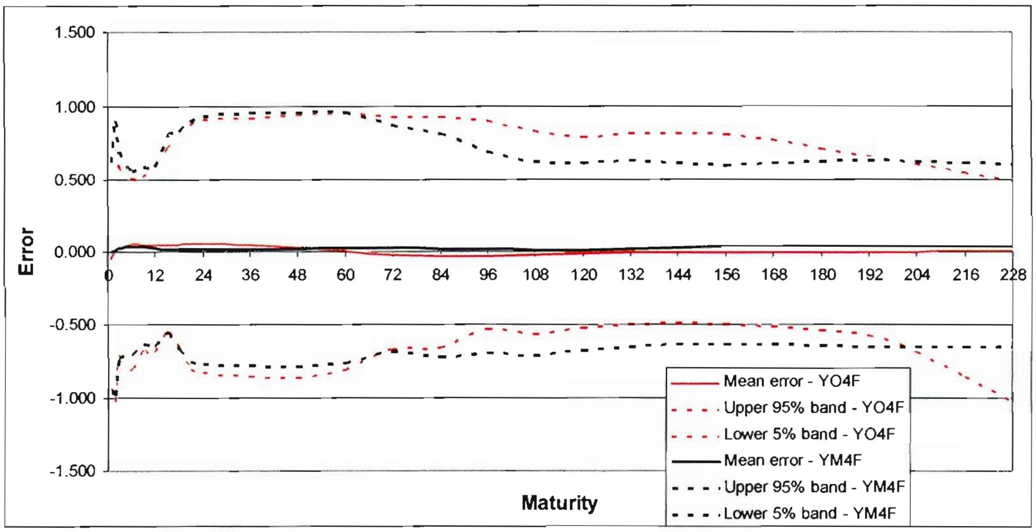

Figure 3.3.1: Mean predicted errors and confidence bands at 5% and 95%

Figure 3.3.1 presents the mean predicted error and the associated upper and lower 95% confidence bands for the four-factor yields-only (Y/O) and the four-factor yields-macro (Y/M) models. Here we can clearly see that there is little difference in the means of the two models. But as mentioned before, there is less variance in the yields-macro model especially in longer maturities, indicating a better fit. The subsequent analysis will use the four-factor yields-macro model.

3.3.2 Out-of-sample testing

For scenario generation it is not only important to capture the dynamics of the yield curve well in-sample, but it is also important to forecast the dynamics of the yield curve well out-of-sample. For this reason we estimate the four-factor yields-macro model on truncated or curtailed data sets. Using the estimated parameters we forecast the yield curve repeatedly for one, two, three and four years ahead over the period of February 2004 through to February 2009, using monthly intervals. For the purpose of asset and liability management it would be of importance to use longer periods for out-of-sample testing, but our lack of data for model fitting restricts this period. Diebold and Li (2006) model and forecast the Nelson-Siegel factors as univariate AR(1) processes for one month, six months and twelve months ahead. The model proposed by Diebold and Li (2006) outperforms other models for yield curve forecasting on all maturities. Thus we model and forecast the Svensson factors as univariate AR(1) processes in order to compare their model against our four-factor yields-macro model.

Table 3.3.3: One year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	-1.053	1.562	-1.338	2.213	-0.395	1.164
12	-0.686	1.103	-1.281	1.909	-0.021	0.742
36	-0.706	0.605	-1.666	1.417	-0.046	0.408
60	-0.929	0.593	-2.036	1.166	-0.276	0.573
120	-1.009	0.741	-2.277	0.972	-0.364	0.837
180	-0.932	0.649	-2.292	0.969	-0.291	0.752
228	-0.854	0.524	-2.274	1.022	-0.217	0.621

Table 3.3.4: Two year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	-0.881	2.004	-1.585	2.273	-0.178	1.564
12	-0.578	1.547	-1.576	1.951	0.164	1.235
36	-0.822	1.089	-2.124	1.763	-0.074	0.815
60	-1.195	1.000	-2.611	1.853	-0.459	0.638
120	-1.406	0.984	-2.947	1.951	-0.688	0.639
180	-1.289	0.954	-2.895	1.935	-0.592	0.622
228	-1.141	0.950	-2.784	1.900	-0.459	0.646

Table 3.3.5: Three year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	-0.472	1.892	-1.235	2.013	0.120	1.539
12	-0.195	1.343	-1.290	1.720	0.417	1.134
36	-0.645	1.144	-2.061	2.233	-0.056	0.668
60	-1.173	1.460	-2.703	2.835	-0.605	0.827
120	-1.506	1.710	-3.166	3.284	-0.967	1.095
180	-1.394	1.638	-3.107	3.234	-0.882	1.040
228	-1.218	1.515	-2.954	3.088	-0.725	0.936

Table 3.3.6: Four year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	0.067	1.258	-0.511	1.376	0.451	1.381
12	0.067	0.817	-0.887	1.682	0.469	1.069
36	-0.617	1.247	-1.907	3.382	-0.230	0.630
60	-1.212	2.220	-2.620	4.627	-0.842	1.555
120	-1.671	3.011	-3.212	5.663	-1.323	2.379
180	-1.573	2.860	-3.152	5.564	-1.243	2.260
228	-1.420	2.611	-3.004	5.308	-1.104	2.033

In Table 3.3.3 to Table 3.3.6 we present the out-of-sample forecasting results for maturities 3, 12, 36, 60, 120, 180 and 288 months. We define the forecast errors at time $t+h$ to be $y_{t+h}(\tau)-\hat{y}_{t+h}(\tau)$, where t is the time of parameter estimation and h the length of the period forecasted. We report the mean and standard deviation of the forecast errors. The four-factor yields-macro model outperforms the AR(1) model. The standard deviations for the AR(1) model are also larger than that of the four-factor yields-macro model. In particular the four year ahead forecast of the four-factor yields-macro model is better than that for the AR(1) model.

In practice most financial institutions have views on the macro-economy. These views are produced by means of an economic scenario generator (ESG) or expert opinion. These ESG's only produce forecasts for macro-economic factors, for example the repo-rate and not a complete yield curve. By using the Kalman filter to model the yield curve in a bidirectional approach, as mentioned in the introduction, it is possible to close this loop and to produce a full consistent yield curve given a set of macro-economic forecasts. This is done by including the macro-economic forecasts produced by such an ESG in the forecasting of the yield curve rather than the forecasted macro-economic factors of the model. Either all three macro-economic factors or only a selection thereof can be replaced. By lack of real ESG forecasts for the repo-rate we include the actual repo-rate. In Table 3.3.3 to Table 3.3.6 we present out-of-sample forecasting results where the actual repo-rate was included in the forecasting instead of the forecasted repo-rate from the model. As can be seen the forecasting error reduces, especially for the longer maturities, compared to the other models. Thus by including these forecasts a better yield curve forecast can be made.

In Figure 3.3.2 we present the quantile-quantile plots for maturities 3, 60, 120 and 228 months. We set the quantiles of the empirical distribution against the quantiles obtained by averaging over a set of scenarios generated by the four-factor yields-macro model, we also plot the 5th and 95th percentiles. The four-factor yields-

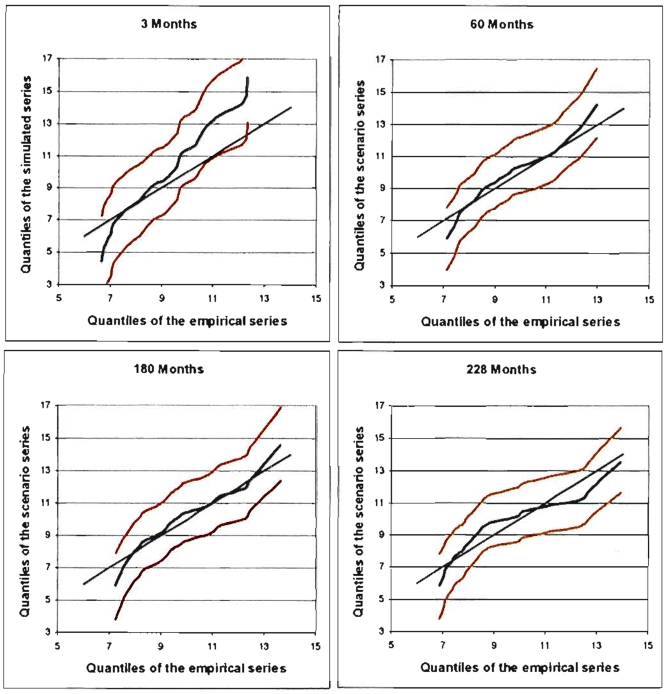

Figure 3.3.2: Quantile-quantile plots for maturities 3, 60, 120 and 228 months

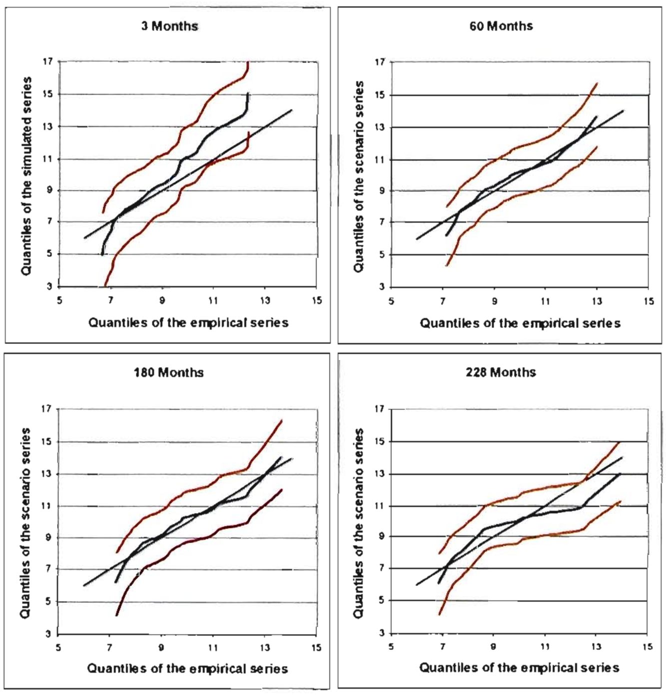

Figure 3.3.3: Quantile-quantile plots for maturities 3, 60, 120 and 228 months with sampling from errors

macro model better reproduce the empirical distribution in the medium term rates than in the short and long rates. In Figure 3.3.3 the quantile-quantile plots for 3, 60, 120 and 228 maturities are presented where the scenarios were generated by sampling from the residuals instead of using normal errors. As can be seen little improvement is gained by sampling from the residuals as apposed to using normal errors.

3.4 Scenario generation

In this section we describe the scenario generation algorithm that we use to generate yield curve scenario trees for fixed income portfolio optimisation problems. We use the four-factor yields-macro model to generate yield curve scenarios. The existence of arbitrage in the scenario trees is discussed and a method to eliminate arbitrage opportunities is proposed. We also demonstrate that the scenarios are stable by using back-testing.

3.4.1 Yield curve scenario generation

We start this section by describing a procedure based on the parallel simulation and randomised clustering approach proposed by Gülpinar *et al.* (2004) to generate a scenario tree which is the input for financial optimisation problems. The basic data structure is the scenario tree node, which contains a cluster of yield curve scenarios, one of which is designated as the centroid or representative. The final tree consists of the centroids of each node, and their branch probabilities. Gülpinar *et al.* (2004) introduced a randomised clustering algorithm. This differs from the approach proposed by Dupačová *et al.* (2000) which determines clusters that are optimal by some measure. Our approach is to group the scenarios into equal groups rather than using a clustering approach as these approaches may need a very large number of scenarios to be generated at the root node to ensure sufficient scenarios at the leaf nodes.

The specific scenario tree structure that we are interested in is a yield curve scenario tree. A T-period scenario tree structure is represented as a tree-string which is a string of integers specifying for each state $s = 1, 2, \dots, T$ the number of branches (or branching factor) for each node in that state (see Dempster *et al.*, 2006). This gives rise to balanced scenario trees, in which each sub-tree in the same period has the same number of branches. Let k_s denote the branching factor for state s, then Figure 3.4.1 gives an example of a scenario tree with a (3,2) tree-string, i.e. k_1 = 3 and k_2 = 2.

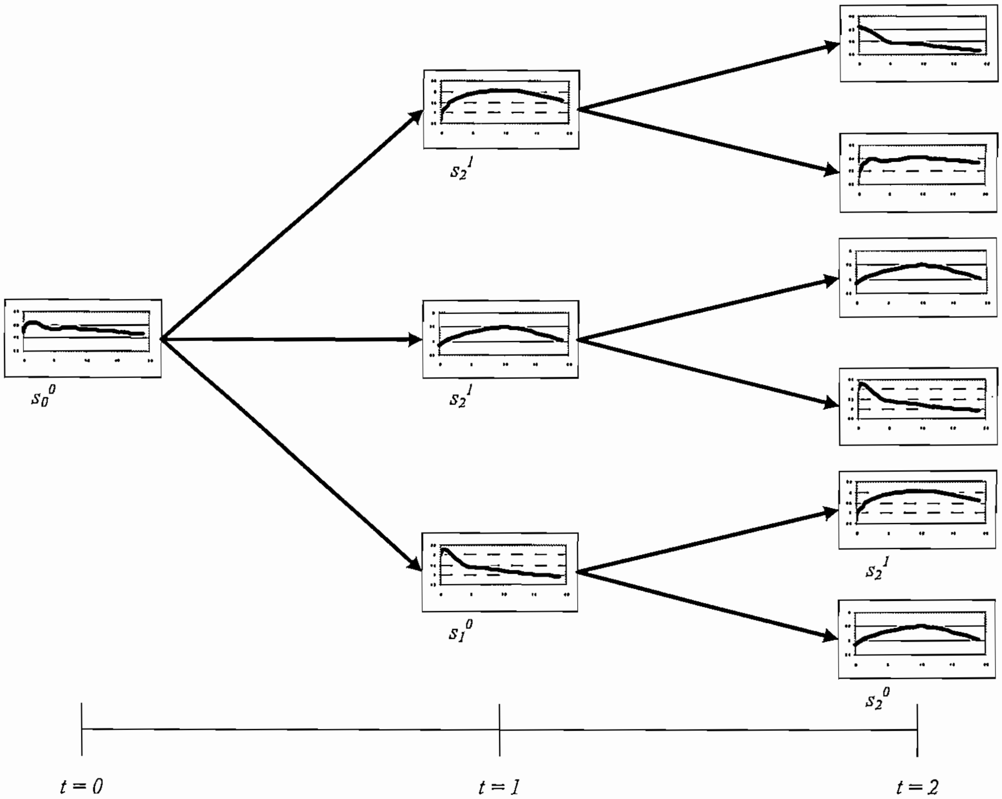

Figure 3.4.1: Graphic representation of scenarios

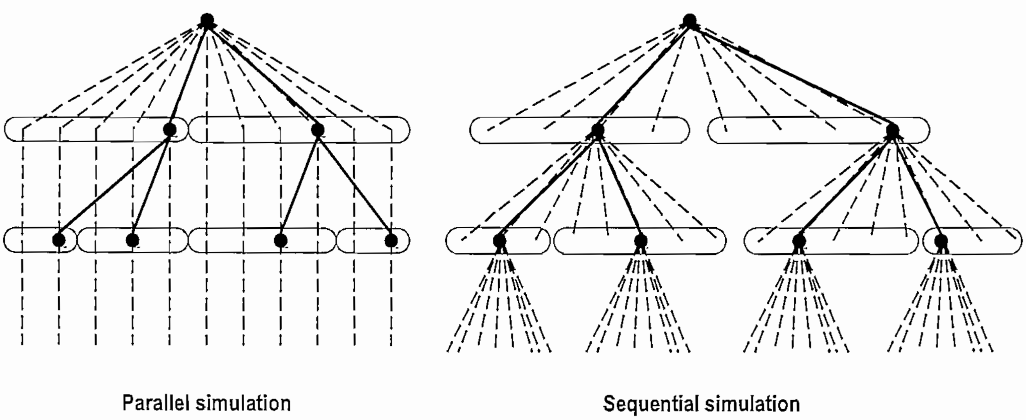

Figure 3.4.2: Two methods of simulating scenarios

It is important to point out that the generated trees are non-recombining. Given that we use four latent factors and three macro-economic factors in our yield curve model, it is very difficult to construct a recombining tree. Even having only three latent factors it is notoriously difficult to construct a recombining tree. Furthermore non-recombining trees are used in the fixed income portfolio optimisation problems as the portfolio composition is path-dependent.

Figure 3.4.2 illustrates the methods of scenario simulation, namely parallel and sequential. We use the parallel method for simulation as this method will produce more realistic extreme events in the scenario tree. The reason is that, with the number of simulations growing smaller down the tree in the parallel method, the centroids that eventually represent the scenario groups are drawn from a smaller sample size and in the sequential method, at every stage the simulated scenarios in all of the clusters are discarded, and the next simulation restarted from the centroid, which will prevent any extreme variation (Gülpinar *et al.*, 2004).

In order to group the scenarios a measure of relative position is used where we calculate the "distance" between the discounting factors of the yield curve and that of the average by:

$$D = \sum\nolimits_{\tau} \left(\frac{1}{(1 + y(\tau))^{\tau}} - \frac{1}{(1 + y^{M}(\tau))^{\tau}} \right),$$

where $y(\tau)$ is the zero-rate with maturity τ and $y^{M}(\tau)$ the average zero-rate with maturity τ. Note that the relative distance D can be negative and positive, which means that a yield curve can be positioned to the "left" or to the "right" of the average yield curve. This is to ensure realistic extreme events. Chueh (2002) discusses several other distance methods for interest rate sampling. Our relative distance method relates closely to the relative present value distance method in Chueh (2002). It is necessary to represent each group of scenarios with a single point, which becomes the data in the scenario tree. Gülpinar *et al.* (2004) argue that to prevent the scenario tree from containing scenarios that are not consistent with the simulation parameters, the centroid should not be taken to be the centre of the group, but rather the simulated scenario closest to the centre. We use the mean of the group as the notion of the centre, other notions of the centre that can be used are the median and the mode.

The main steps of our algorithm can be outlined as follow:

Step 1: At $s = 0$ create a root node group containing N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model. Each scenario is equally likely and consists of $T + 1$ sequential yield curves with the same starting point, the current yield curve (in total $(T+1)\times N$ yield curves are generated).

Step 2: Set $s := s+1$ and for each group in the previous stage, calculate the average scenario and calculate the relative position (defined above) of each scenario with respect to the average scenario.

Step 3: For each group, sort the scenarios in descending distance (relative position) order and group them into k_s equal sized groups.

Step 4: For each new group, find the scenario closest (in absolute value) to average of the group, and designate it as the centroid. Assign a probability of $\left(\prod_{i=1}^{s-1} k_i\right)^{-1}$ to each centroid.

Step 5: If $s < T$, go to Step 2, else stop.

3.4.2 Arbitrage

Filipović (1999) and other researchers such as Diebold *et al.* (2006) show that the Nelson-Siegel family of yield curve models does not impose absence of arbitrage, although these models estimate and forecast the yield curve better than arbitrage-free models (Duffee, 2002, noted that the canonical affine arbitrage-free models demonstrate disappointing out-of-sample performance). In light of this, the scenarios generated are not arbitrage free. Klaassen (2002) shows that arbitrage opportunities can be detected *ex post* by checking for solutions to a set of linear constraints or be excluded by including non-linear constraints in the scenario generation process. Christensen *et al.* (2007) derive a class of arbitrage-free affine dynamic term structure models that approximate the Nelson-Siegel yield curve specification. Christensen *et al.* (2008) extend these models to include the Svensson extension of the Nelson-Siegel yield curves.

We propose a method to reduce the presence of arbitrage *ex post*, without extending our models to the class of arbitrage-free models. We reduce the presence of arbitrage *ex post*, as apposed to excluding it by means of including non-linear constraints during the scenario generation process. This approach has no additional effect on the computational difficulty of the model estimation process and the data requirements. As the scenario generation process is a discrete approximation of the continuous evolution of the term structure, extending the models, used in the simulation process, to a class of arbitrage-free models will not ensure the exclusion of arbitrage in the generated scenarios.

Klaassen (2002) proposes linear constraints for two types of arbitrage. Ingersoll (1987) distinguishes these two types of arbitrage. The first type is an opportunity to construct a zero-investment portfolio that has nonnegative payoffs in all states of the world, and a strictly positive payoff in at least one state. The second type is an opportunity to construct a negative investment portfolio (i.e. providing an immediate positive cash flow) that generates a nonnegative payoff in all future states of the world.

Following the notation of Klaassen (2002), let $r_{k,t+1}^{n}$ be the return on asset class k $(k = 1, ..., K)$ between time t and $t+1$ if state n $(n = 1, ..., N)$ of the world materialises at time $t+1$. Klaassen (2002) mentions a useful result, that if the set of equations

$$\sum_{n=1}^{N} v_n \left(1 + r_{k,t+1}^{n}\right) = 1 \text{ for all } k = 1, ..., K,$$

has a strictly positive solution v_n for all n $(n = 1, ..., N)$, then no arbitrage opportunities of the first or second type exist (see also Ingersoll, 1987). Taking $r_{\tau,t+1}^{n}$ to be the return on a zero-coupon bond with maturity $k = \tau$, then

$$1 + r_{\tau,t+1}^{n} = \frac{P_{t+1}^{n}(\tau - 1)}{P_t(\tau)},$$

where $P_t(\tau) = e^{-\tau y_t(\tau)}$ is the price at time t of a zero-coupon bond with maturity τ. Thus if the set of equations

$$\sum_{n=1}^{N} v_n \exp\left(-(\tau - 1) y_{t+1}^{n}(\tau - 1)\right) = \exp\left(-\tau y_t(\tau)\right) \text{ for all maturities } \tau,$$

has strictly positive solution v_n for all n $(n = 1, ..., N)$, then no arbitrage opportunities of the first or second type exist in our yield curve scenarios.

The class of arbitrage-free affine dynamic term structure models that Christensen *et al.* (2007) and Christensen *et al.* (2008) derive, for the Nelson-Siegel family of yield curves, differs from the Diebold *et al.* (2006) models only in the inclusion of an additional yield-adjustment term which depends only on the maturity of the zero-coupon bond. As this term is dependent on the maturity of the bond, it can be seen as a shift in the slope of the yield curve. Now let

$$v_n = \frac{\exp\left(-y_t(1)\right)}{N} \text{ for all } n \ (n = 1, ..., N),$$

then, if we can find yield curve shifts $c_{t+1}(\tau)$ such that

$$\frac{1}{N} \sum_{n=1}^{N} \exp\left(-(\tau - 1)\left(y_{t+1}^{n}(\tau - 1) + c_{t+1}(\tau)\right)\right) = \frac{\exp\left(-\tau y_t(\tau)\right)}{\exp\left(-y_t(1)\right)} \text{ for all maturities } \tau,$$

no arbitrage opportunities exist in the yield curve scenarios. Thus, if for a zero coupon bond with maturity τ, the mean of the calculated present values, using different scenarios, equals the current price, for all maturities τ, no arbitrage opportunities exist in the yield curve scenarios (this is consistent with no-arbitrage literature).

Given the small size of branching factors of the scenario trees generated it may not be possible to find realistic solutions to the yield curve shifts $c_{t+1}(\tau)$. Thus to eliminate most of the arbitrage opportunities in the scenario trees we propose the following algorithm:

Step 1: At the root node create a group of N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model (as for the scenario tree). Each scenario is equally likely and consists of T sequential yield curves.

Step 2: At each branching time of the scenario tree calculate the average of the N generated scenarios (at the root node the current yield curve is used).

Step 3: Then for each average yield curve and the corresponding one-period ahead scenarios solve

$$\frac{1}{N}\sum_{n=1}^{N} \exp\left(-(\tau-1)\left(y_{t+1}^{n}(\tau-1)+c_{t+1}(\tau)\right)\right)=\frac{\exp\left(-\tau y_t(\tau)\right)}{\exp\left(-y_t(1)\right)}$$

for all maturities, to obtain the yield curve shifts $c_{t+1}(\tau)$.

Step 4: Add the amount $c_{t+1}(\tau)$ to the original scenario tree yield curves.

The described method removes most of the arbitrage opportunities in the scenario tree with a few opportunities left in sub-trees. For scenario trees with a short horizon all opportunities may be removed. We judge this reduction of arbitrage opportunities as sufficient since portfolio constraints in optimisation problems, such as the restriction of short-selling and the inclusion of bid and ask spreads, will eliminate the remaining arbitrage opportunities.

3.4.3 Back-testing

To test our scenario generation methodology we implemented the multi-stage stochastic optimisation problem described in Dempster *et al.* (2006). Dempster *et al.* (2006) propose an asset and liability management framework and give numerical results for a simple example of a closed-end guaranteed fund where no contri-

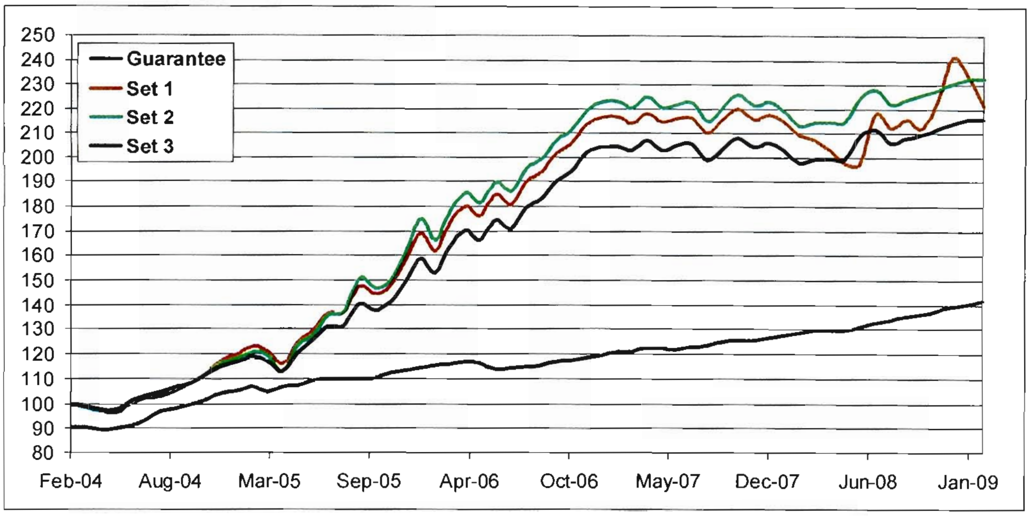

Figure 3.4.3: Macro-economic scenario back-testing results

Table 3.4.2: Macro-economic portfolio allocation stability statistics

	Objective function	Top40	5Y	7Y	10Y	15Y	19Y
Mean	-0.325	0.0135	0.9619	0.0018	0.0	0.0	0.0051
Std. Dev.	0.221	0.0035	0.0176	0.0093	0.0	0.0	0.0063
Min	-0.1929	0.0057	0.8945	0.0000	0.0	0.0	0.0000
Max	-0.4283	0.0248	0.9990	0.9808	0.0	0.0	0.0228

minimum and maximum of the objective function and the first stage portfolio allocations. The first stage portfolio allocation seems consistent with small standard deviation. The objective function also have small standard deviations with no outliers when we look at the minimum and maximum, furthermore indicating the stability of the scenario generation.

The scenario generation is further tested by again solving the model for 100 different scenario sets and for different number of final nodes, 120, 500, 1000 and 2000. Dempster *et al.* (2006) minimise the expected average shortfall and maximises the expected terminal wealth of the portfolio, and distinguish between them using a risk-aversion parameter (*alpha*). For each scenario set the model is solved ranging the risk-aversion parameter from 0 to 1 in steps of 0.1 (1 being the most risk-averse). Table 3.4.3 presents the mean, standard deviation, for the different number of final nodes. In Figure 3.4.4 we display the mean frontier, by averaging the objective function values obtained over the 100 different scenario sets, and the confidence bands covering 95% of the results (Kaut *et al.*, 2007; Consiglio and Staino, 2008, uses a similar approach for scenario and model stability testing).

butions are allowed after the initial cash outlay. They demonstrate the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product (see Appendix A for the model formulation). We use our scenario generation approach to generate the input scenarios for the optimisation problem. The four-factor yields-macro model is fitted to market data up to an initial decision time t and scenario trees are generated from time t to some chosen horizon $t+T$. The optimal first stage/root node decision is then implemented at time t and we measure the success of the portfolio implementation by its performance with historical data up to time $t+1$. This whole procedure is rolled forward for T trading times. At each decision time t, the parameters of the four-factor yields-macro model are re-estimated using the historical data up to and including time t.

We back-test over a period of five years, from February 2004 through to February 2009, and use different tree structures with approximately the same number of scenarios. The tree structures are described in Table 3.4.1. Bonds with 5, 7, 10, 15 and 19 year maturities as well as the FTSE/JSE Top 40 index are included in the portfolio (Dempster *et al.*, 2006, include bonds with different maturities and an equity index). In order to generate scenarios for the Top 40 index, the index is modelled using a simple linear regression model incorporating the three macro-economic factors. We minimise the expected average shortfall for an annual guarantee of 9% and include transaction costs.

Table 3.4.1: Tree structure for different back-tests

Year	Set1	Set2	Set3
February04	5.5.5.5.5=3125	13.4.4.4.4=332	200.2.2.2.2=3200
February05	8.8.8.8=4096	15.6.6.6=3240	400.2.2.2=3200
February06	15.15.15=3375	30.10.10=3000	400.3.3=3600
February07	56.56=3136	160.20=3200	800.4=3200
February08	3125	3328	3200

Figure 3.4.3 illustrates the back-testing portfolio values and the minimum guarantee for all three scenario sets. The results are consistent with those in Dempster *et al.* (2006). Although Dempster *et al.* (2006) minimise the expected average shortfall and maximises the expected terminal wealth of the portfolio, and distinguish between them using a risk-aversion parameter, we only minimise the expected average shortfall to test the model's performance. Only shortfall is used as it plays an important role in our applications. The model performs well staying above the guarantee at all times, achieving a high terminal wealth.

In Table 3.4.2 we present back-testing stability statistics. The model was solved for 100 different scenario sets, with a tree-string of 40.3 (120 scenarios) using all available data for model fitting. Table 3.4.2 presents the mean, standard deviation,

Table 3.4.3: Macro-economic efficient frontier stability statistics

	120 Scenarios		500 Scenarios		1000 Scenarios		2000 Scenarios	
Alpha	Mean	Std. Dev	Mean	Std. Dev	Mean	Std. Dev	Mean	Std. Dev
0	377.92	2.848	376.75	2.754	377.27	2.319	377.17	2.347
0.1	339.80	2.711	338.75	2.621	339.22	2.209	339.13	2.234
0.2	301.69	2.567	300.74	2.482	301.17	2.093	301.09	2.116
0.3	263.58	2.414	262.73	2.178	263.12	1.970	263.06	1.991
0.4	225.47	2.252	224.73	2.009	225.08	1.839	225.02	1.857
0.5	187.36	2.077	186.72	1.970	187.03	1.698	186.98	1.714
0.6	149.26	1.887	148.72	1.825	148.99	1.546	148.95	1.557
0.7	111.16	1.676	110.72	1.622	110.95	1.377	110.92	1.384
0.8	73.08	1.433	72.73	1.394	72.91	1.186	72.89	1.187
0.9	35.21	1.055	34.90	1.051	34.97	0.916	34.95	0.893
1	-0.325	0.221	-0.37	0.228	-0.37	0.192	-0.39	0.186

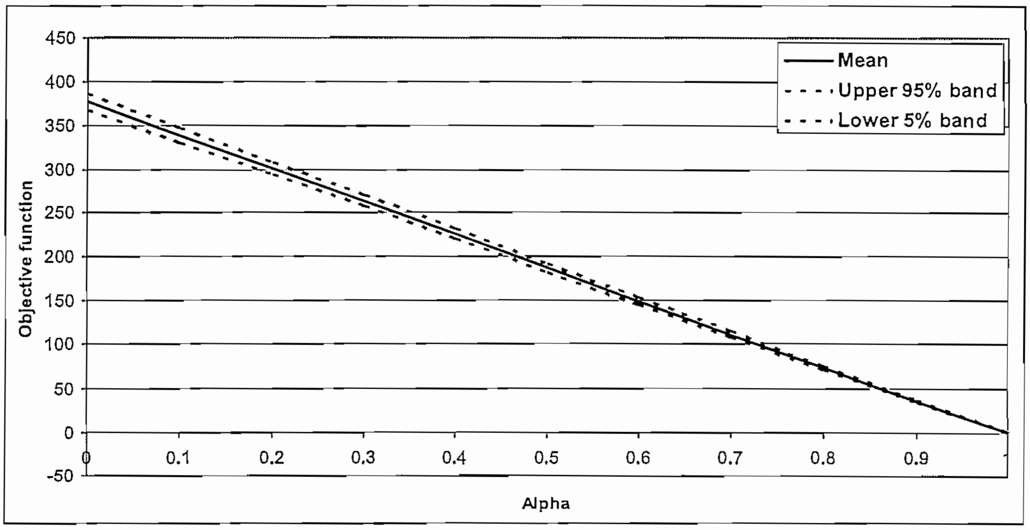

Figure 3.4.4: Average efficient frontier with 5% and 95% confidence bands.

The frontier is a decreasing function of the risk-aversion parameter *alpha*. If the value of *alpha* is closer to 1, more importance is given to the shortfall of the portfolio and less given to the expected wealth and hence a more risk-averse portfolio allocation strategy will be taken and vice versa. In the extreme case where *alpha* is 1 only the shortfall will be minimised and the expected wealth will be ignored, and where *alpha* is 0, the unconstrained case only maximises the wealth. For a 1000 final nodes the 95% region, at its maximum (alpha at 0), is 4.9% wide (a reduction of 2% from 500 final nodes), ensuring that the randomisation error is bounded enough. In Table 3.4.3 we also observe that the standard deviation decreases as the number of final nodes increases. The reduction is less or none at all when we increase the number of final nodes from 1000 to 2000, again ensuring that the randomisation error is bounded enough, and achieves stability.

Although back-testing assumes that the past describes the future and can in no means guarantee that success of the outcomes of these models in practice, it provides us with a way to assess the performance of the proposed algorithm. Through back-testing we see that the proposed scenario generation algorithm performs well on a portfolio optimisation problem in literature, similar results are obtained as in Dempster *et al.* (2006). We also see that stability in the objective is obtained by increasing the number of scenarios. The amount of the final number of scenarios necessary to achieve this stability may depend on the optimisation problem in question.

3.4.4 Moment-matching versus macro-economic scenario generation

In this section we compare the moment-matching scenario generation method described in Chapter 2 with the macro-economic scenario generation method presented in this chapter.

In Figure 3.4.5 we present the back-tested portfolio results for the moment-matching scenario generation method and the macro-economic scenario generation method, using scenario set 1 (see Table 3.4.1). Also included in Figure 3.4.5 are the portfolio values if we where to invest all the funds only in one of the assets, namely 5, 7, 10, 15, 19 year bonds and the Top 40 Index. From Figure 3.4.5 it is clear that, with both scenario generation methods, the optimisation model performs well with no shortfall. It is also clear that when the macro-economic scenarios are used the model allocates more funds to the risky asset, achieving a higher terminal wealth. The model mainly invests in bonds when the moment-matching scenarios are used. The model clearly performs better, in terms of the portfolio value, when the macro-economic scenarios are used. This difference in the allocation of fewer funds to

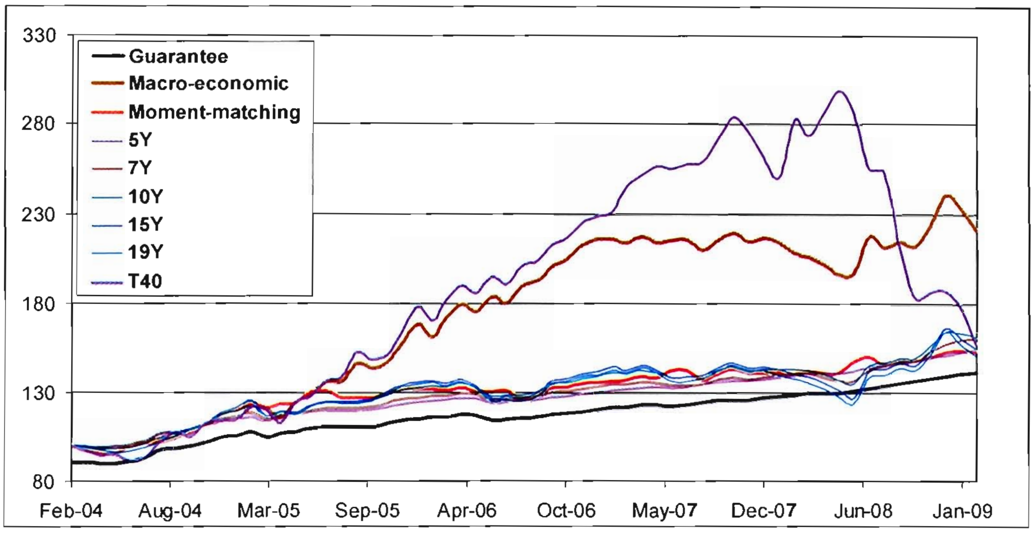

Figure 3.4.5: Moment-matching versus macro-economic scenario back-testing results

the risky asset, when using the moment-matching scenarios, may be due to the moment-matching scenario generation method producing realistic extreme scenarios for the risky asset, which the model will take into account (remember that we are looking at the most risk-averse case). We further see that when the macro-economic scenarios are used, the model shifts the portfolio allocations from the risky asset to bonds after the third year. It may be that the macro-economic scenario generation method produces more realistic extreme scenarios for the risky asset from this point onwards. A reasonable explanation for this may be the small amount of data available in order to fit the macro-economic models, which improves as time increases. Also our main focus is on the scenario generation of the yield curve and not the risky asset.

In Table 3.4.4 we present back-testing stability statistics for the moment-matching and macro-economic scenario generation methods. The model was solved for 100 different scenario sets for each scenario generation method, with a tree-string of (40.3) (120 scenarios) using all available data for model fitting. We present the mean, standard deviation of the objective function and the first stage portfolio allocations. In Tables 3.4.4 we observe stability in the both the objective function and the first stage portfolio allocations, for both methods. The optimisation model allocates more funds to the risky asset when the macro-economic scenarios are used, as mentioned previously.

Furthermore, we compare the moment-matching and macro-economic scenario generation methods in terms of the stability of the efficient frontier. The model

Table 3.4.4: Portfolio allocation stability statistics

	Objective function	Top40	5Y	7Y	10Y	15Y	19Y
Moment-matching							
Mean	-0.2643	0.0027	0.6928	0.0000	0.0	0.0	0.2685
Std. Dev.	0.0015	0.0028	0.0858	0.0000	0.0	0.0	0.0706
Macro-economic							
Min	-0.3246	0.0135	0.9619	0.0018	0.0	0.0	0.0051
Max	0.0024	0.0035	0.0176	0.0093	0.0	0.0	0.0063

is solved for 100 different scenario sets, where for each set the model is solved ranging the risk-aversion parameter from 0 to 1 in steps of 0.1 (1 being the most risk-averse). Table 3.4.5 presents the mean and standard deviation, for a 1000 final nodes, for the moment-matching and macro-economic scenario generation methods. Both the moment-matching and macro-economic scenario generation methods displayed sufficient stability when the final number of nodes where increased to 1000. In Table 3.4.5 we observe that model produces a lower efficient frontier when using the moment-matching scenarios. This may be due to the model allocating less funds to the risky asset when the moment-matching scenarios are used. It is also clear that the model displays less variance in the efficient frontier when the moment-matching scenarios are used and indicates more stability. This can also be seen in Figure 3.4.6 where we display the mean frontier, by averaging the objective function values obtained over the 100 different scenario sets, and the confidence bands covering 95% of the results, for both scenario generation methods. To get a better view we only display the results for a risk-aversion parameter of between 0 to 0.4. The 95% region, at its maximum (alpha at 0), is 0.685% wide for the moment-matching scenarios and 4.9% wide for the macro-economic scenarios, showing that the randomisation error is bounded more when using the moment-matching scenarios.

From the results it is clear that both scenario generation methods display stability in the objective function and the portfolio allocations. We observe that the objective function is more stable when the moment-matching scenarios are used. In terms of back-testing the macro-economic scenarios performs better achieving a higher terminal wealth. Gülpinar *et al.* (2004) investigates different scenario generation methods. They state that although the moment-matching approach is arguably the most theoretically sound way to generate scenario trees, the effort (in terms of time spent) yielded no perceptible gains in back-testing over the faster simulation approach. Gülpinar *et al.* (2004) only use the Høyland and Wallace (2001b) method in their investigation. By using the Høyland *et al.* (2003) method, which is more stable and executes faster, one might come to a different conclusion.

Table 3.4.5: Moment-matching and macro-economic efficient frontier stability statistics

	Moment-matching		Macro-economic	
Alpha	Mean	Std. Dev	Mean	Std. Dev
0	343.450	0.659	377.269	2.319
0.1	309.015	0.594	339.220	2.209
0.2	274.582	0.529	301.171	2.093
0.3	240.152	0.463	263.123	1.970
0.4	205.724	0.396	225.076	1.839
0.5	171.300	0.332	187.030	1.698
0.6	136.882	0.266	148.986	1.546
0.7	102.472	0.200	110.946	1.377
0.8	68.075	0.136	72.913	1.186
0.9	33.707	0.077	34.966	0.916
1	-0.274	0.016	-0.374	0.192

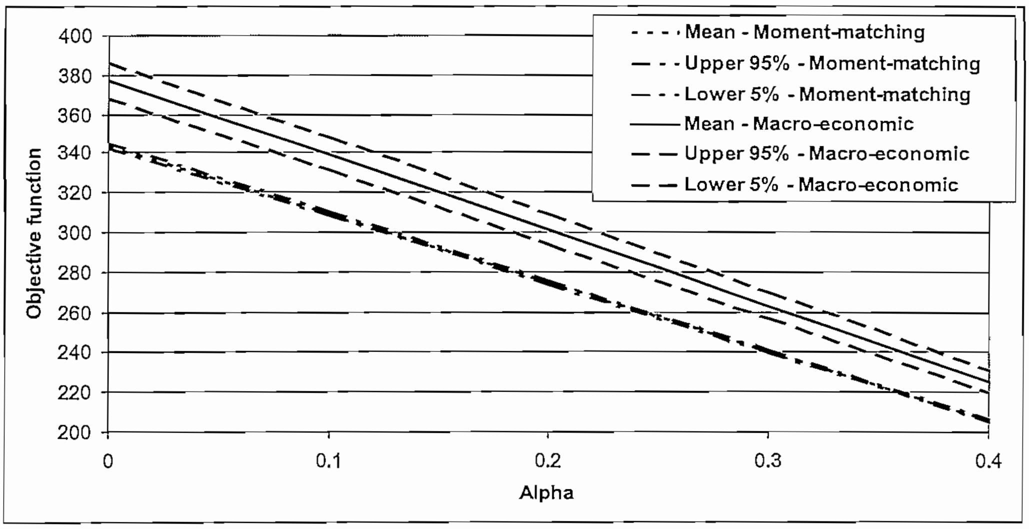

Figure 3.4.6: Moment-matching and macro-economic average efficient frontier with 5% and 95% confidence bands.

Both methods further enable the decision-maker to specify their view on the future movement of the yield curve in some way. In the first moment-matching approach this can be done by keeping the view on the interest rates fix in the optimisation and in the second macro-economic approach it is done by including the view on the macro-economic factors. The moment-matching method may also be used instead of the simulation method in the macro-economic approach. This can be done by using the Kalman filter approach in order to estimate the statistical specifications of the distribution and using these specifications in the moment-matching scenario generation approach. Gülpinar *et al.* (2004) also proposes a *hybrid* approach where the scenarios are generated using the simulation approach. The probabilities are then determined using the moment-matching approach. It is the purpose this chapter and Chapter 2 to propose suitable methods for yield curve scenario generation and further investigation on the different approaches and the mixing thereof, might be fruitful.

3.5 Conclusion

This chapter considered the estimation and characterisation of the South African term structure with respect to macro-economic factors and its use in scenario generation for fixed income portfolio optimisation. We have estimated a yield curve model that incorporates four yield curve factors (level, slope and two curvature factors) and three macro-economic factors (real activity, inflation and the stance of monetary policy). The estimated model fits the term structure reasonably well in-sample as shown in the results. We also test the model in out-of-sample forecasting for horizons up to four years. For the purpose of asset and liability management it would be of importance to use longer periods for out-of-sample testing, but the lack of data for model fitting restricted this period. The model also performs reasonably well in out-of-sample forecasting. We have shown that better performance can be realised by including forecasts for the macro-economic factors generated by an economic scenario generator. By lack of forecast data we used the actual repo-rate.

We also proposed a parallel simulation approach for yield curve scenario tree generation. The procedure was tested and the performance was measured by out-of-sample back-testing in terms of the value of a fixed income portfolio optimization problem described in the literature. Although back-testing assumes that the past describes the future and can in no means guarantee that success of the outcomes of these models in practice, it provides us with a way to assess the performance of the proposed algorithm. Through back-testing we have shown that the proposed scenario generation algorithm performs well on a portfolio optimisation problem in literature. We also have shown that stability is obtained by increasing the number of scenarios. The amount of the final number of scenarios necessary to achieve this stability may depend on the optimisation problem in question.

The existence of arbitrage in the scenario trees was discussed and a method to eliminate arbitrage opportunities *ex post* was proposed. Future consideration may be given to other methods in order to exclude arbitrage opportunities either during simulation or removing the arbitrage opportunities *ex post*.

Furthermore we compared the moment-matching scenario generation method to the macro-economic scenario generation method in terms of back-testing and stability. From the results it is clear that both scenario generation methods display stability in the objective function and the portfolio allocations. The objective function is more stable when the moment-matching scenarios are used. In terms of back-testing the macro-economic scenarios performs better achieving a higher terminal wealth. It was the purpose this chapter and Chapter 2 to propose suitable methods for yield curve scenario generation and further investigation on the different approaches and the mixing thereof, might be fruitful.

In this chapter and the previous chapter we have presented two methods for yield curve scenario generation. In the next two chapters we will use these methods as input to our multi-stage stochastic programming models. To illustrate the suitability of both methods, to portfolio optimisation with fixed income instruments, we will use the moment-matching approach in Chapter 4 and the macro-economic approach in Chapter 5. The moment-matching approach is used in Chapter 4, as the liquid asset portfolio has a the horizon between one and two years. The macro-economic approach is used in Chapter 5, as the guarantee fund has a much longer horizon.

Chapter 4

Liquid asset portfolio

Maintaining liquid asset portfolios involves a high carry cost and is mandatory by law for most financial institutions. Taking this into account a financial institution's aim is to manage a liquid asset portfolio in an "optimal" way, such that it keeps the minimum required liquid assets to comply with regulations. In this chapter[1] we propose a multi-stage dynamic stochastic programming model for liquid asset portfolio management. The model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management decisions, such as reinvesting coupons, at intermediate time steps. We show how our problem closely relates to insurance products with guarantees and utilise this in the formulation. We model the uncertainty in terms of scenario trees by using the moment-matching techniques introduced in Chapter 2.

We will discuss our formulation and implementation of a multi-stage stochastic programming model that minimises the down-side risk of these portfolios. The model is back-tested on real market data over a period of two years

4.1 Introduction

In a publication by the Basel Committee on Banking Supervision (BCBS, 2000) liquidity is defined as the ability to fund increases in assets and meet obligations as they come due, without incurring unacceptable losses. Managing this liquidity is one of the most important activities in a bank. This also entails a bank to not

[1]Papers based on the work done in this chapter has been presented at the SPXI Conference, Austria, 2007 and the MMEI Conference, Czech Republic, 2009. A paper on this chapter has also been accepted for publication in a special issue of the journal Kybernetika containing selected papers from the MMEI Conference, 2009.

only measure its liquidity position on an ongoing basis, but to assess the evolution of funding requirements under different scenarios and adverse conditions. BCBS (2008a) reviews the *Sound practices for managing liquidity risk in banking organisations* (see BCBS, 2000). One of the findings in BCBS (2008a) is that many banks considered severe and prolonged liquidity disruptions as improbable and did not conduct stress tests, which takes into account the probability of market wide strain or the severity and the duration of these disruptions. BCBS (2008b) expands the guidance of BCBS (2000) in a number of key areas. One area is the maintenance of an adequate level of liquidity, *including a cushion of liquid assets*. BCBS (2008b) provides guidance around seventeen principles for managing liquidity risk. Two of these principles are as follow:

Principle 1:

A bank is responsible for the sound management of liquidity risk. A bank should establish a robust liquidity risk management framework that ensures it maintains sufficient liquidity, including a cushion of unencumbered, high quality liquid assets, to withstand a range of stress events, including those involving the loss or impairment of both unsecured and secured funding sources. Supervisors should assess the adequacy of both a bank's liquidity risk management framework and its liquidity position and should take prompt action if a bank is deficient in either area in order to protect depositors and to limit potential damage to the financial system.

Principle 12:

A bank should maintain a cushion of unencumbered, high quality liquid assets to be held as insurance against a range of liquidity stress scenarios, including those that involve the loss or impairment of unsecured and typically available secured funding sources. There should be no legal, regulatory or operational impediment to using these assets to obtain funding.

Principle 12 in BCBS (2008b) states further that the continuous availability of an adequate cushion of unencumbered, high quality liquid assets, that can be sold or pledged to obtain funds in a range of stress scenarios, is a critical element in the bank's resilience to liquidity stress. BCBS (2008b) also provides guidance on the size of the cushion of liquid assets. To guard it self against the most severe stress scenarios, the bank should hold a portfolio of the most reliably liquid assets, such as cash and high quality government bonds or similar instruments. There should furthermore be no legal, regulatory or operational impediment to the use of these assets to obtain funding, and these assets should be used in the event of severe liquidity stress. Thus, this cushion of liquid assets serves as insurance in the event the no other funding can be obtained for liquidity purposes.

In South Africa the South African Banks Act (Banks Act, 94/1990) and Regulations Relating to Banks (SA, 2008) requires banks to keep a minimum amount in liquid assets. Liquid assets are assets which are easily redeemable for cash, and are defined in Section 1 of the Banks Act as:

- Reserve Bank notes, subsidiary coins,
- Gold coin and bullion,
- Any credit balance in a clearing account with the SARB,
- Treasury bills of the RSA,
- Securities issued by virtue of section 66 of the Public Finance Management Act, 1999
- Bill issued by the Land Bank
- Securities of the SARB.

The minimum nominal amount that is required, in liquid assets, is stipulated in Section 72 of the Banks Act (Banks Act, 94/1990) and Regulation 20 of the Regulations Relating to Banks (SA, 2008). The Banks Act (Banks Act, 94/1990) stipulate, that a bank shall hold liquid assets with respect to the value of its liabilities as may be specified by regulations. Regulation 20 of the Regulation Relating to banks (SA, 2008), requires a bank to hold over a period of one month an average daily amount of liquid assets equal to no less than 5% of its reduced liabilities. The reduced liabilities are defined as follow:

$$
\begin{aligned}
& \textit{Liabilities (on balance sheet)} \\
& - \textit{amount of funding received from head office and other branches} \\
& \qquad \textit{in the same group} \\
& - \textit{amount owing by banks, branches and mutual banks in the RSA} \\
= \; & \textit{reduced liabilities}
\end{aligned}
$$

Regulation 20 of the Regulation Relating to banks (SA, 2008) further states that a bank shall not pledge or otherwise encumber any portion of the liquid assets held by it in compliance with this provision. For this purpose a bank needs to keep a *statutory portfolio*, also called a *liquid asset portfolio*. Since a bank may not pledge or otherwise encumber any portion of the liquid asset portfolio, we analyse the portfolio separately from the bank's other portfolios, i.e. as a stand-alone portfolio.

From a management perspective, with respect to the calculation of the weighted average cost of funds, it may also be necessary to keep the portfolio separate in order to determine the costs of the fund. Given the regulatory background and the liquid asset requirement our main objective in this chapter is the management of the liquid asset portfolio and not the adequacy of the liquid asset requirement.

The liquid assets that are available for inclusion into the liquid asset portfolio are interest rate sensitive, low (credit) risk financial instruments. Having a low risk implies having a small return, so keeping the portfolio is mostly unprofitable. The portfolio is funded by a pool of funds with a cost equivalent to the bank's interdivisional borrowing rate. Maintaining a liquid asset portfolio involves a high carry cost, thus making it expensive to hold. However, as mentioned above, the portfolio is mandatory. Taking this into account and the high carry cost of the portfolio the bank's aim is to manage the liquid asset portfolio in an "optimal" way, such that it keeps the minimum required liquid assets to comply to regulations, whilst maximising the portfolio return to cover at least the carry cost.

To manage this portfolio in an "optimal" way the bank will need to rebalance or change the composition of the portfolio on a regular basis. Changing the portfolio composition will depend on certain aspects such as expert views on risk factors movements, legislation and regulations. With these legislation and regulations to adhere to and uncertainties to consider, the liquid asset portfolio management problem can be described as a multi-stage decision problem in which portfolio rebalancing actions are taken at successive future discrete time points. At each decision period it needs to be decided which assets to buy, which to sell and which to hold.

The aim of the manager of the liquid asset portfolio, is to identify and minimise risks by analysing the market, legislation, portfolio data, and many other factors. This Chapter will investigate the use of stochastic programming in addressing all of these aspects in a realistic way. In Chapter 1 we provide a literature review on *Stochastic Programming* and discuss the basic stochastic programming optimisation for dynamic portfolio strategies. In Section 4.2 we discuss the formulation and implementation of the multi-stage stochastic programming model. Section 4.3 presents back-testing results. The back-tests are done on real market data over a period of two years.

4.2 Scenario optimisation framework

In this section we discuss the formulation of a multi-stage stochastic programming model that minimizes the down-side risk of liquid asset portfolios. As mentioned in the introduction the liquid assets that are available for inclusion into the liquid asset portfolio are interest rate sensitive, low (credit) risk financial instruments. Having a low risk implies having a small return, so keeping the portfolio is mostly unprofitable. The portfolio is funded by a pool of funds with a cost equivalent to the bank's interdivisional borrowing rate, i.e. the portfolio is funded internally by the bank. This can be seen as the department responsible for keeping the liquid asset portfolio, borrowing funds form other divisions in the bank in order to fund the portfolio. Thus capital that may have been invested in other securities can now only be invested in liquid assets. Thus maintaining this portfolio involves a high carry cost, and makes it an expensive portfolio to hold.

The value of the liquid asset portfolio is bounded from below by the liquid asset requirement. The liquid asset requirement as mentioned in the introduction depends on the liabilities of the bank. This requirement grows over time as the bank's business grows, i.e. as the bank takes on more clients with deposit facilities, the amount of liabilities increases and thus the liquid asset requirement increases. Thus the funds invested by the bank for the purpose of meeting the liquid asset requirement, needs to have a growth rate of at least the rate at which the bank's liabilities are growing.

In this light our liquid asset problem can be seen as an minimum guarantee problem. Where the bank can be seen as a client investing an up-front amount of money, equal to the liquid asset requirement, into a guarantee fund, where the guaranteed rate of return is equal to the growth rate of the liabilities of the bank. Dempster *et al.* (2006) propose an asset and liability management framework for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay, which can be adapted to address our problem. They demonstrate the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product. Through back-testing they show that the proposed stochastic optimisation framework addresses the risk created by the guarantee in a reasonable way. We propose a similar framework for the asset and liability management of the liquid asset portfolio problem. Other than Dempster *et al.* (2006) we will assume that any shortfall in the portfolio will be funded by the bank, and that the funds provided by the bank for the purpose of the liquid asset requirement will carry some sort of liability payment equal to the interdivisional borrowing rate.

In the next subsections we will discuss the model features, the model variables and parameters and its dynamics together with its constraints. We will also discuss the objective function.

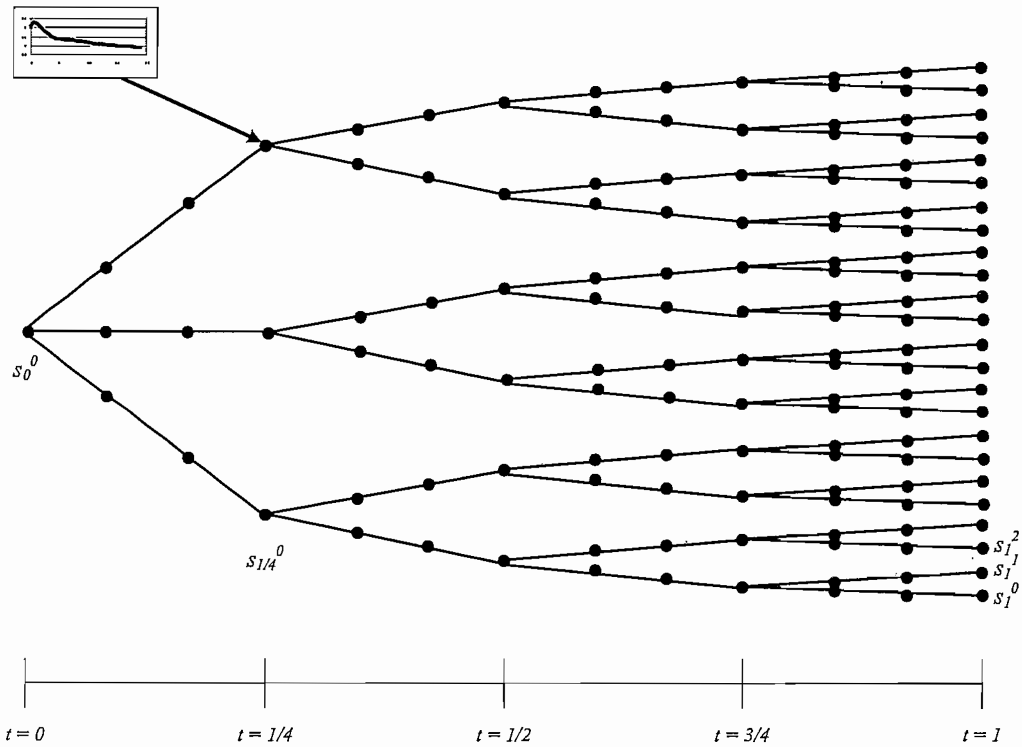

Figure 4.2.1: Graphical representation of a yield curve scenario tree

4.2.1 Model features

We investigate the optimal asset allocation of a bank's liquid asset portfolio. It is assumed that the portfolio will be funded up-front by the bank and this will be assumed to be a liability for the portfolio. This liability will grow at the maximum of the one month zero-rate, $r^s_{t,\frac{1}{12}}$ (assumed to be the interdivisional borrowing rate) and the minimum liquid asset growth rate g^s_t, throughout the lifetime of the portfolio and is payable at the end of the horizon. By minimising the shortfall of the portfolio we will ensure that the assets will cover at least the maximum of the minimum liquid requirement or the liabilities at the cost of the interdivisional borrowing rate. The portfolio will be rebalanced quarterly. Any shortfall in the portfolio, at rebalancing, will be funded by the bank. This is to ensure that the minimum liquid requirement is achieved. The shortfall payments will accrue to a shortfall fund separate from the liabilities, which will grow at the one month zero-rate. The time horizon of the portfolio is T years and we will use three asset classes as liquid assets namely, (semi-annual) coupon bearing government bonds, gold and three and six month treasury bills.

To represent uncertainty, we generate the future yield curves and construct a scenario tree. A scenario tree is a discrete approximation of the joint distribution of random factors (yield curve and gold prices). To facilitate the mathematical formulation of the optimisation problem, we represent the scenario tree in terms of states (nodes) $s_t^{v(t)}$, where time $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$ and $v(t) = 0, 1, 2, \ldots, N_t$ the numbers of the states at time t. The set of states at time t are denoted by $\Sigma_t = \left\{ s_t^{v(t)} | v(t) = 0, 1, \ldots, N_t \right\}$. The set of all states in the scenario tree is denoted by $\Sigma = \cup_{t=0}^{T} \Sigma_t$. Links $\varepsilon \in \Sigma \times \Sigma$, indicate the possible transitions between states. To enforce non-anticipativity, i.e. to prevent foresight of uncertain future events, we order the elements of ε in pairs $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)} \right)$ where the dependence of the index $v(t)$ on t is explicitly indicated. The order of the states indicates that state $s_{t+1}^{v(t+1)}$ at time $t+1$ can be reached from state $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is the successor state and $s_t^{v(t)}$ the predecessor state. By using the superscript "+" to denote the successor states, and the superscript "-" to denote the predecessors, we have $s_t^{v(t)+} = s_{t+1}^{v(t+1)}$ and $s_{t+1}^{v(t+1)-} = s_t^{v(t)}$. Each state $s_t^{v(t)}$ has an associated probability p_t^s, for $s \in \Sigma_t$, such that $\sum_{s \in \Sigma_t} p_t^s = 1$.

The quarterly decision times $t_d = 0, \frac{1}{4}, \frac{1}{2}, \ldots, T - \frac{1}{4}$, are the times which the fund will trade to rebalance its portfolio. We represent the branching of the tree structure with a tree-string, which is a string of integers specifying for each decision time t_d the number of branches for each node in state Σ_d. This specification gives rise to a balanced scenario tree where each sub-tree in the same period has the same number of branches. Figure 4.2.1 gives an example of a scenario tree with a (3,2,2,2) tree-string, giving a total of 24 scenarios.

4.2.2 Model variables and parameters

The following notation will be used for variables and parameters of the model, where the time index t takes values over the times $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$, and states index s takes values from the set $\Sigma_t = \{ s_t^v | v = 1, 2, \ldots, N_t \}$ (The following notation easy translates to mathematical programming languages, such as SAS/OR PROC OPTMODEL, and are therefore used):

Time sets

$T^{total} = \left\{ 0, \frac{1}{12}, \frac{2}{12}, \ldots, T \right\}$: set of all times considered in the stochastic program;

$T^d = \left\{ 0, \frac{1}{4}, \frac{1}{2}, \ldots, T - \frac{1}{4} \right\}$: set of decision times;

$T^i = T^{total} \backslash T^d$: set of intermediate times;

Index sets

$\Sigma_t = \{s_t^v \| v = 1, 2, \ldots, N_t\}$	: set of states at time t;
$B = \{B_\tau\}$	: set of government bonds with maturity denoted by τ;
GZ	: set of gold prices;
$M3$	: set of 3 month treasury bills;
$M6 = \{M6_{3M}, M6_{6M}\}$	: set of 6 month treasury bills with time to maturity;
$I = B \cup GZ \cup M3 \cup M6$	: set of all instruments;

Parameters

δ_{B_τ}	: coupon rate of a government bond with maturity τ;
F_{B_τ}	: face value of a government bond with maturity τ;
D^s_{t,B_T}	: coupon payment date of a government bond with maturity τ at time t in state s;
$F_{M3 \cup M6}$	: face value of a treasury bills;

$r^s_{t,\tau}$	: zero-rate with maturity τ at time t in state s;
g^s_t	: minimum liquid asset growth rate at time t in state s;
$mg^s_t = \max\left(g^s_t, r^s_{t,\frac{1}{12}}\right)$	: liability growth rate at time t in state s;
$P^{a,s}_{t,i} / P^{b,s}_{t,i}$	: ask or bid price of asset $i \in I$ at time t in state s;
f_a / f_b	: proportional transaction costs on ask or bid transactions;
p^s_t	: probability of state s at time t;
L_0	: initial liability at the root node;
$z_0 = \left\{z_{0,i}\right\}_{i \in I}$	: initial quantities of assets at the root node;
MLA_0	: initial minimum liquid asset requirement;

Decision variables

$x^s_t = \left\{x^s_{t,i}\right\}_{i \in I}$	: quantities of assets *bought* at time t in state s;
$y^s_t = \left\{y^s_{t,i}\right\}_{i \in I}$	: quantities of assets *sold* at time t in state s;
$z^s_t = \left\{z^s_{t,i}\right\}_{i \in I}$	: quantities of assets *held* at time t in state s, from time t to $t + \frac{1}{12}$;
W^s_t	: value of portfolio wealth at time t in state s;
L^s_t	: value of liability account at time t in state s;
EF^s_t	: value of the shortfall fund at time t in state s;;
c^s_t	: amount of extra cash provided at time t in state s;
SF^s_t	: amount of shortfall at time t in state s;

4.2.3 Instrument pricing

We assume all bonds to pay semi-annual coupons at rate δ_{B_τ} and derive bid and ask prices by adding a spread, sp, to the zero-rates. Let $P^{a,s}_{t,B_\tau}$ denote the ask price of a coupon bearing bond with maturity τ at time t:

$$\begin{aligned} P^{a,s}_{t,B_\tau} &= F_{B_\tau} \exp\left(-\left(\tau - t\right)\left(r^s_{t,\tau-t} + sp\right)\right) \\ &\quad + \sum\nolimits_{m=\left\{\frac{\lfloor 2t\rfloor}{2}+\frac{1}{2},\frac{\lfloor 2t\rfloor}{2}+1,\ldots,\frac{\lfloor 2t\rfloor}{2}+\tau\right\}} \frac{1}{2}\delta_{B_\tau} F_{B_\tau} \exp\left(-\left(m-t\right)\left(r^s_{t,(m-t)} + sp\right)\right), \end{aligned}$$

$$\text{for } t \in T^{total}, \text{ and } s \in \Sigma_t,$$

where the principal amount is discounted in the first term and the coupon payment stream in the second term. The integral part is denoted by $\lfloor . \rfloor$. Let $P^{b,s}_{t,B_\tau}$ be the bid price of the bond with maturity τ at time t:

$$\begin{aligned} P^{b,s}_{t,B_\tau} &= F_{B_\tau} \exp\left(-\left(\tau - t\right)\left(r^s_{t,\tau-t} - sp\right)\right) \\ &\quad + \sum\nolimits_{m=\left\{\frac{\lfloor 2t\rfloor}{2}+\frac{1}{2},\frac{\lfloor 2t\rfloor}{2}+1,\ldots,\frac{\lfloor 2t\rfloor}{2}+\tau\right\}} \frac{1}{2}\delta_{B_\tau} F_{B_\tau} \exp\left(-\left(m-t\right)\left(r^s_{t,(m-t)} - sp\right)\right), \end{aligned}$$

$$\text{for } t \in T^{total}, \text{ and } s \in \Sigma_t.$$

The use of a bid or ask price when including transaction costs are not always necessary in this context as transaction costs are sufficient to prevent the simultaneous buying and selling of assets in the optimisation model. However transaction costs on bonds are relatively low compared to the bid/ask spread. In South Africa the typical bid/ask spread on bonds is 0.5% and transaction costs are 0.012%.

Treasury bills, with maturities of three and six months, are assumed to be sold at a discount. Furthermore we assume treasury bills to be held until maturity and then sold at the face value. Let $P^{a,s}_{t,M3}$ denote the ask price of a three month treasury bill at time t:

$$P^{a,s}_{t,M3} = F_{M3} \exp\left(-\tau_1 r^s_{t,\tau_1}\right), \tau_1 = \frac{3}{12} + \frac{3}{13}\lfloor\frac{12t}{3}\rfloor - t \text{ for } t \in T^{total}, \text{ and } s \in \Sigma_t,$$

and $P^{b,s}_{t,M3}$ denote the bid price of a three month treasury bill at time t:

$$P^{b,s}_{t,M3} = F_{M3} \exp\left(-\tau_2 r^s_{t,\tau_2}\right), \tau_2 = \tau_1 - \frac{3}{12}\lfloor\frac{12\tau_1}{3}\rfloor \text{ for } t \in T^{total}, \text{ and } s \in \Sigma_t.$$

Six month treasury bills are split into two sets. The first set contains six month treasury bills which have time to maturity of six months on the previous decision time and the second set contains six month treasury bills which have time to maturity of three months on the previous decision time. Let $P^{a,s}_{t,M6_{6M}}$ denote the ask price of a six month treasury bill at time t, with a time to maturity of six months on the previous decision time:

$$P^{a,s}_{t,M6_{6M}} = F_{M6}\exp\left(-\tau_1 r^s_{t,\tau_1}\right), \tau_1 = \frac{3}{12} + \frac{3}{13}\lfloor\frac{12t}{3}\rfloor - t \text{ for } t \in T^{total}, \text{ and } s \in \Sigma_t,$$

and $P^{b,s}_{t,M6_{6M}}$ denote the bid price of a six month treasury bill at time t, with a time to maturity of six months on the previous decision time:

$$P^{b,s}_{t,M6_{6M}} = F_{M6}\exp\left(-\tau_2 r^s_{t,\tau_1}\right), \tau_2 = \tau_1 - \frac{3}{12}\lfloor\frac{12\tau_1}{3}\rfloor \text{ for } t \in T^{total}, \text{ and } s \in \Sigma_t.$$

Furthermore let $P^{a,s}_{t,M6_{3M}}$ denote the ask price of a six month treasury bill at time t, with a time to maturity of three months on the previous decision time:

$$P^{a,s}_{t,M6_{3M}} = F_{M6}\exp\left(-\tau_1 r^s_{t,\tau_1}\right), \; \tau_1 = \frac{6}{12} + \frac{3}{13}\lfloor\frac{12t}{3}\rfloor - t \text{for } t \in T^{total}, \text{ and } s \in \Sigma_t,$$

and $P^{b,s}_{t,M6_{3M}}$ denote the bid price of a six month treasury bill at time t, with a time to maturity of three months on the previous decision time:

$$P^{b,s}_{t,M6_{3M}} = F_{M6}\exp\left(-\tau r^s_{t,\tau}\right), \tau_2 = \tau_1 - \frac{3}{12}\lfloor\frac{12\tau_1}{3}\rfloor \text{ for } t \in T^{total}, \text{ and } s \in \Sigma_t.$$

4.2.4 Variable dynamics and constraints

To simplify our notation, we assume all face values to be 100. We further assume all bonds to pay semi-annual coupons at rate δ_{B_τ}. At coupon payment times t, $D^s_{t,B_t} = 1$ if a coupon is due at time t in state s and $D^s_{B_t} = 0$ otherwise. The bond cash flows, $CF^s_{t,i}$, per unit face value are then calculated as:

$$CF^s_{t,i} = D^s_{t,i}\frac{1}{2}\delta_i F_i \text{, for } i \in B,\, t \in T^{total} \text{ and } s \in \Sigma_t.$$

The variable dynamics and constraints for the minimum liquid asset problem are:

Cash balance constraints. The cash balance constraints ensure that the amount of cash that is received, from selling assets, coupon payments at decision times and extra cash supplied for shortfall, is equal to the amount of assets bought:

$$\sum\nolimits_{i\in I} P^{b,s}_{t,i} y^s_{t,i}\left(1 - f_b\right) + \sum\nolimits_{i\in B} CF^s_{t,i_\tau} z_{0,i} + c^s_t = \sum\nolimits_{i\in I} P^{a,s}_{t,i} x^s_{t,i}\left(1 + f_a\right),$$

$$\text{for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$\sum_{i\in I} P^{b,s}_{t,i} y^s_{t,i} (1-f_b) + \sum_{i\in B} CF^s_{t,i} z^{s-}_{t-\frac{1}{12},i} + c^s_t = \sum_{i\in I} P^{a,s}_{t,i} x^s_{t,i} (1+f_a),$$

$$\text{for } t \in T^d \setminus \{0\} \text{ and } s \in \Sigma_t.$$

Short sale constraints. The short sale constraints eliminate the possibility of short-selling assets in each state at each time period:

$$x^s_{t,i} \geq 0, \text{ for all } i \in I,\ t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t,$$

$$y^s_{t,i} \geq 0, \text{ for all } i \in I,\ t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t,$$

$$z^s_{t,i} \geq 0, \text{ for all } i \in I,\ t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t.$$

Inventory constraints. The inventory constraints give the quantity invested in each asset in each state at each time period. The inventory constraints for bonds, gold and the three month treasury bills are straight forward, as the quantity of assets that are held for the next time period equals the quantity that was held in the previous time period plus the quantity bought minus the quantity sold:

$$z^s_{t,i} = z_{0,i} + x^s_{t,i} - y^s_{t,i}, \text{ for } i \in I \setminus \{M6\},\ t \in \{0\} \text{ and } s \in \Sigma_t$$

$$z^s_{t,i} = z^{s-}_{t-\frac{1}{12},i} + x^s_{t,i} - y^s_{t,i}, \text{ for } i \in I \setminus \{M6\},\ t \in T^{total} \setminus \{0\} \text{ and } s \in \Sigma_t.$$

As mentioned previously treasury bills are held to maturity. Due to the quarterly rebalancing of the portfolio, the six month treasury bills are split into two sets. The first set contains six month treasury bills which have time to maturity of six months on the previous decision time and the second set contains six month treasury bills which have time to maturity of three months on the previous decision time.

The inventory constraints for the six month treasury bills with time to maturity of three months are:

$$z^s_{t,i} = z_{0,j}, \text{ for } i \in \{M6_{3M}\},\ j \in \{M6_{6M}\},\ t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$z^s_{t,i} = z^{s-}_{t-\frac{1}{12},j}, \text{ for } i \in \{M6_{3M}\},\ j \in \{M6_{6M}\},\ t \in T^d \setminus \{0\} \text{ and } s \in \Sigma_t.$$

Furthermore the inventory constraints for the six month treasury bills with time to maturity of six months are:

$$z_{t,i}^{s}=x_{t,i}^{s},\ \text{for } i\in\{M6_{6M}\},\ t\in\{0\}\ \text{and } s\in\Sigma_t,$$

$$z_{t,i}^{s}=x_{t-\frac{1}{12},i}^{s-},\ \text{for } i\in\{M6_{6M}\},\ t\in T^d\backslash\{0\}\ \text{and } s\in\Sigma_t.$$

This enables us to distinguish between six month treasury bills that mature at the current decision time an those that only mature at the next decision time. For intermediate time periods the inventory constraints are the same as for the rest of the instruments:

$$z_{t,i}^{s}=z_{t-\frac{1}{12},i}^{s-}+x_{t,i}^{s}-y_{t,i}^{s},\ \text{for } i\in M6,\ t\in T^i\ \text{and } s\in\Sigma_t.$$

Information constraints. As the portfolio is only rebalanced at decision times, the information constraints ensure that the portfolio can not be changed between decision times:

$$x_{t,i}^{s}=y_{t,i}^{s}=0\ \text{for } i\in I\backslash\{B\},\ t\in T^i\ \text{and } s\in\Sigma_t.$$

The information constraints for bonds are included in the coupon reinvestment constraints.

Coupon reinvestment constraints. The coupon reinvestment constraints ensure that the coupons that are paid at the coupon times are reinvested in the same coupon bearing bonds:

$$x_{t,i}^{s}=\frac{CF_{t,i}^{s}z_{t-\frac{1}{12},i}^{s-}}{P_{t,i}^{a,s}(1+f_a)},\ \text{for } i\in\{B\},\ t\in T^i\ \text{and } s\in\Sigma_t,$$

$$y_{t,i}^{s}=0,\ \text{for } i\in\{B\},\ t\in T^i\ \text{and } s\in\Sigma_t,$$

Rollover constraints. Rollover constraints ensure that treasury bills are sold at maturity.

$$y_{t,i}^{s}=z_{0,i},\ \text{for } i\in\{M3,M6_{3m}\},\ t\in\{0\}\ \text{and } s\in\Sigma_t,$$

$$y_{t,i}^{s}=z_{t-\frac{1}{12},i}^{s-},\ \text{for } i\in\{M3,M6_{3m}\},\ t\in T^d\backslash\{0\}\ \text{and } s\in\Sigma_t,$$

$$x_{t,i}^{s}=0,\ \text{for } i\in\{M6_{3M}\},\ t\in T^d\ \text{and } s\in\Sigma_t,$$

$$y_{t,i}^{s}=0,\ \text{for } i\in\{M6_{6M}\},\ t\in T^d\ \text{and } s\in\Sigma_t.$$

Portfolio wealth constraints. The portfolio wealth constraints determine the value of the liquid asset portfolio in each state at each time period. The value of the portfolio wealth is determined after rebalancing, i.e. any extra cash c_t^s that has been provided to fund shortfalls below the minimum liquid asset requirement, is taken into account by the cash balance constraints:

$$W_t^s = \sum_{i\in I} P_{t,i}^{a,s} z_{t,i}^s (1+f_a), \text{ for } t \in T^{total} \backslash T \text{ and } s \in \Sigma_t.$$

The terminal portfolio wealth is given by:

$$W_T^s = \sum_{i\in I} P_{T,i}^{b,s} z_{T-\frac{1}{12},i}^{s-} (1-f_b) + \sum_{i\in\{B\}} CF_{t,i}^s z_{T-\frac{1}{12},i}^{s-}, \text{ for } s \in \Sigma_T$$

Liability account constraints. The liability account constraints determine the value of the liability account in each state at each time period. The liabilities are assumed to grow at liability growth rate, $mg_t^s = \max\left(g_t^s, r_{t,\frac{1}{12}}^s\right)$, the maximum of the one month zero-rate and the minimum liquid asset growth rate:

$$L_t^s = L_0, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$L_t^s = L_{t-\frac{1}{12}}^{s-} \exp\left(\frac{1}{12} mg_{t-\frac{1}{12}t}^{s-}\right), \text{ for } t \in T^{total} \backslash \{0\} \text{ and } s \in \Sigma_t.$$

Shortfall fund constraints. The shortfall fund constraints determine the value of the shortfall fund in each state at each time period. The shortfall fund is assumed to grow at the one month zero-rate plus any extra cash c_t^s that has been provided to fund shortfalls below the liability:

$$EF_t^s = c_t^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$EF_t^s = EF_{t-\frac{1}{12}}^{s-} \exp\left(\frac{1}{12} r_{t-\frac{1}{12}t}^{s-}\right) + c_t^s, \text{ for } t \in T^{total} \backslash \{T\} \text{ and } s \in \Sigma_t.$$

Shortfall constraints. The shortfall constraints determine the shortfall of the portfolio with respect to the liabilities and the shortfall fund in each state at each time period. By including the shortfall fund, when calculating the shortfall, and by minimising the shortfall we will encourage growth in portfolio wealth to cover not only the liabilities but also extra costs. The shortfall is calculated by using the value of the portfolio wealth after transaction:

$$SF_t^s + W_t^s \geq MLA_t^s + EF_t^s, \text{ for } t \in T^{total} \text{ and } s \in \Sigma_t,$$

where $SF_t^s \geq 0$ for $t \in T^{total}$ and $s \in \Sigma_t$.

Shortfall funding constraints. The shortfall funding constraints determine the amount of extra cash needed at decision times to ensure no shortfall below the liabilities.

The amount of extra cash is calculated by using the value of the portfolio wealth before transaction:

$$c_t^s + W_t^{bs} \geq L_t^s, \text{ for } t \in T^{total} \text{ and } s \in \Sigma_t,$$

where

$$W_t^{bs} = \sum\nolimits_{i \in I} P_{t,i}^{b,s} z_{0,i} (1 - f_b) + \sum\nolimits_{i \in \{B\}} CF_{t,i}^s F_i z_{0,i},$$

$$\text{with } c_t^s \geq 0 \text{ for } t \in \{0\}, \text{ and } s \in \Sigma_t,$$

and

$$W_t^{bs} = \sum\nolimits_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-} (1 - f_b) + \sum\nolimits_{i \in \{B\}} CF_{t,i}^s z_{t-\frac{1}{12},i}^{s-},$$

$$\text{with } c_t^s \geq 0 \text{ for } t \in T^d \backslash \{0\}, \text{ and } s \in \Sigma_t,$$

is the value of the portfolio wealth before transactions.

Portfolio composition constraints. Portfolio composition constraints can be introduced in order to reduce concentration risk. The following portfolio constraints are taken into account:

$$\frac{\sum_{i \in GZ} P_{t,i}^{a,s} z_{t,i}^s}{W_t^s} < 0.5, \text{ for } t \in T^d \text{ and } s \in \Sigma_t,$$

$$\frac{\sum_{i \in M3 \cup M6} P_{t,i}^{a,s} z_{t,i}^s}{W_t^s} < 0.7, \ t \in T^d \text{ and } s \in \Sigma_t,$$

$$\frac{\sum_{i \in M3 \cup M6} P_{t,i}^{a,s} z_{t,i}^s}{W_t^s} > 0.3, \ t \in T^d \text{ and } s \in \Sigma_t,$$

the first constraint ensures that the amount held in gold is restricted to be no more than 50% of the total portfolio wealth and the remaining two ensure that the amount held in treasury bills is between 30% and 70%.

4.2.5 Objective function

When managing the minimum liquid asset portfolio there are two main goals to take into account. The first aim is the management of the investment strategies of the fund in order to comply with the minimum liquid asset requirement and the liability. To manage the risk of underperforming, the shortfall was defined to be amount by which the portfolio's wealth falls below the liabilities (accruing at the maximum of the minimum liquid asset growth rate and the interdivisional borrowing rate) and the shortfall funding account. To quantify this risk we consider the expected average shortfall over all times (see Dempster *et al.*, 2006). Another measure of risk that may be considered is the *conditional value-at-risk* (CVaR), which in it self is also a expected shortfall measure. The second aim is to minimise the extra cost necessary to stay above the minimum liquid asset requirement, i.e. the extra cash that needs to be borrowed. The objective we consider is the minimum average expected shortfall over all periods and the average expected extra cash that is needed for shortfall funding. Dempster *et al.* (2006) have shown that monitoring shortfall at intermediate nodes improve results. The objective function is given as:

$$\max_{\left\{\begin{array}{c} x_{t,i}^s, y_{t,i}^s, z_{t,i}^s \\ i \in I, t \in T^d \cup \{T\}, s \in \Sigma_t \end{array}\right\}} \left\{ \begin{array}{c} \alpha \sum_{t \in T^{total}} \sum_{s \in \Sigma_t} \frac{SF_t^s}{|T^{total}|} p_t^s \\ + (1-\alpha) \sum_{t \in T^d} \sum_{s \in \Sigma_t} \frac{c_t^s}{|T^d|} p_t^s \end{array} \right\}$$

where the value of $0 \leq \alpha < 1$ sets the level of importance of the expected average shortfall relative to the cost. If the value of α is closer to 1, more importance is given to the shortfall and less given to the extra cost of the portfolio and visa versa. The value of α may not be equal to 1, as this will result in the problem to be infeasible, because there is no restriction on the amount of extra cash.

4.3 Results

In this section we discuss the performance of the model. The first part explains the data and instruments that are used to generate scenario trees, which is the input to the mathematical optimisation problem. In the second part we present back-tested results for the model for different levels of the minimum liquid asset requirements and different levels of alpha.

4.3.1 Data and instruments

We use eight different assets, namely, (semi-annual) coupon bearing bonds with maturities 5, 7, 10, 15 and 19 years, gold and three and six month treasury bills. We use the Perfect Fit Bond Curves, one of the five BEASSA Zero Coupon Yield Curve series of yield curves (see BESA, 2003a), with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months. The curves are derived from government bond data and the technical specifications are described in BESA (2003b). We decrease the dimensionality of the problem by using the 1, 3, 12, 60, 120, 180 and 228 months

Table 4.3.1: Tree structures used for back-testing the minimum liquid asset portfolio

Year	Tree-string
February07	9.3.3.3.2.2.2.2=3888
May07	18.3.3.3.2.2.2=3888
August07	36.3.3.3.2.2=3888
Novemeber07	72.3.3.3.2=3888
February08	144.3.3.3=3888
May08	432.3.3=3888
August08	432.9=3888
November08	3888

yields. After scenarios have been generated for these seven yields the rest of the yield curve is interpolated using the Svensson yield curve parameterisation (see Svensson, 1994). The gold returns are generated together with the yield returns. We use end-of-month data from August 1999 through to February 2009. The tree structures used in back-testing are displayed in Table 4.3.1.

We use the moment-matching scenario generation approach, described in Subsection 2.3.3 of Chapter 2, to generate the input scenarios for the optimisation problem. The mean reversion parameters are fitted to the market data up to an initial decision time t and scenario trees are generated from time t to time T. The optimal first stage/root node decisions are then implemented at time t. The success of the portfolio strategy is measured by its performance with historical data up to time $t + \frac{3}{12}$. This whole procedure is rolled forward. At each decision time t, the mean reversion parameters are re-estimated using the historical data up to and including time t.

4.3.2 Back-testing results

We perform back-tests over a period of two years, from February 2007 through to February 2009. We assume the growth rate for the minimum liquid asset requirement to be constant over the entire period, $g_t^s = g$. We back-test for different levels of minimum liquid asset requirement growth rate and for different levels of alpha. For each of these back-tests, at different levels, we report the expected average shortfall (EAShf), taken to be

$$\sum_{t \in \mathcal{T}^{total}} \sum_{s \in \Sigma_t} \frac{SF_t^s}{|\mathcal{T}^{total}|} p_t^s$$

at each decision time and the actual average shortfall, taken to be

$$\sum_{t=1}^{T} \frac{SF_t}{T}$$

We also report the expected cost of the portfolio taken to be the expected present value of the final shortfall fund

$$\sum_{s\in\Sigma_T} \left(\frac{EF_T^s}{\prod_{t=1}^{T} \exp\left(\frac{1}{12} r_{t,t+\frac{1}{12}}^{s_t^{v(t)}}\right)} \right) p_t^s,$$

where $\left(s_t^{v(t)}, s_{t+\frac{1}{12}}^{v(t+\frac{1}{12})}\right) \in \varepsilon$ and the actual cost of the portfolio, taken to be

$$\left(\frac{EF_T}{\prod_{t=1}^{T} \left(\frac{1}{12} r_{t,t+\frac{1}{12}}\right)} \right).$$

The following portfolio composition constraints were included, the amount of funds invested in gold and treasury bills were restricted to be no more than 50% and between 30% and 70% respectively.

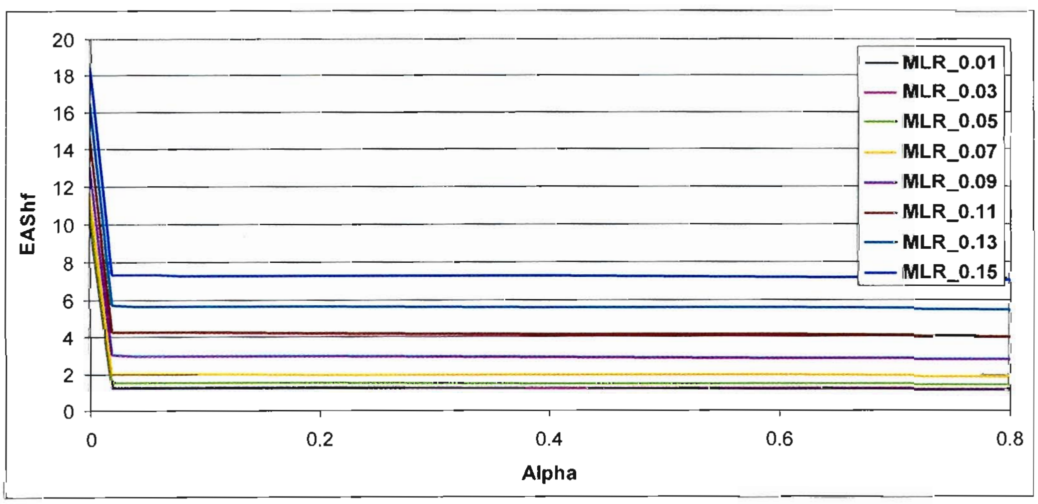

Figure 4.3.1: Expected average shortfall for different levels of alpha and minimum liquid asset requirement growth rate (MLR), at February 2007

In Figure 4.3.1 we present the expected average shortfall for different levels of alpha and minimum liquid asset requirement growth rates (MLR) at February 2007. As expected the expected average shortfall increases as the requirement increases and decreases as alpha increases and more importance is given to the minimum shortfall in the objective. Also note that the expected average shortfall is very similar

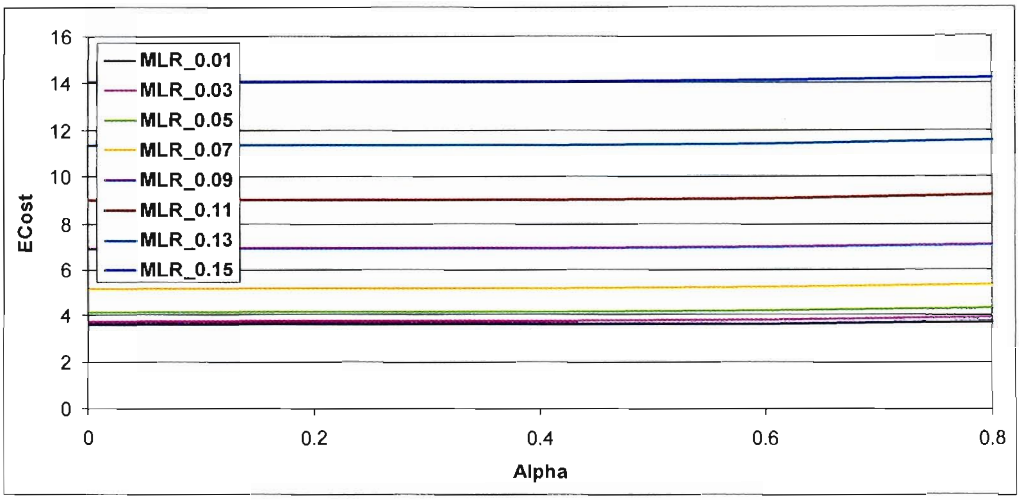

Figure 4.3.2: Expected cost for different levels of alpha and minimum liquid asset requirement growth rate (MLR), at February 2007

for minimum liquid asset requirement growth rates of 1% to 7%. This is due to the one month zero-rate, which is above 7% resulting in the liability growth rate to be above 7%. Also in Figure 4.3.1 it is clear that the expected average shortfall decreases as alpha increases and more importance is given to minimum shortfall. Figure 4.3.2 presents the expected cost for different levels of alpha and minimum liquid asset requirement growth rates at February 2007. It is clear the expected costs increase as the requirement increases and as alpha increases and more importance is given to minimum shortfall.

Figures 4.3.3 and 4.3.4 present similar results for the expected average shortfall and expected cost for different levels of alpha and minimum liquid asset requirement growth rates at February 2008. It is also apparent that the expected average shortfall decreases considerably as soon as shortfall is being minimised in the objective function, showing the advantage of minimising shortfall and not only the extra cash that is needed for shortfall funding. Also remember that shortfall was defined to be the amount that the wealth portfolio is below the liabilities and the shortfall fund in each state at each time period and that the extra cash needed for shortfall is calculated by only using the shortfall below the liabilities. By minimising only the extra cash needed for shortfall below liabilities will result in a larger shortfall below liabilities when including the shortfall fund. As more importance is given to the minimum shortfall in the objective the expected average shortfall decreases less. There is also a slow increase in the expected cost as alpha increases. Also note that expected average shortfall increases over time and the expected cost decreases over time.

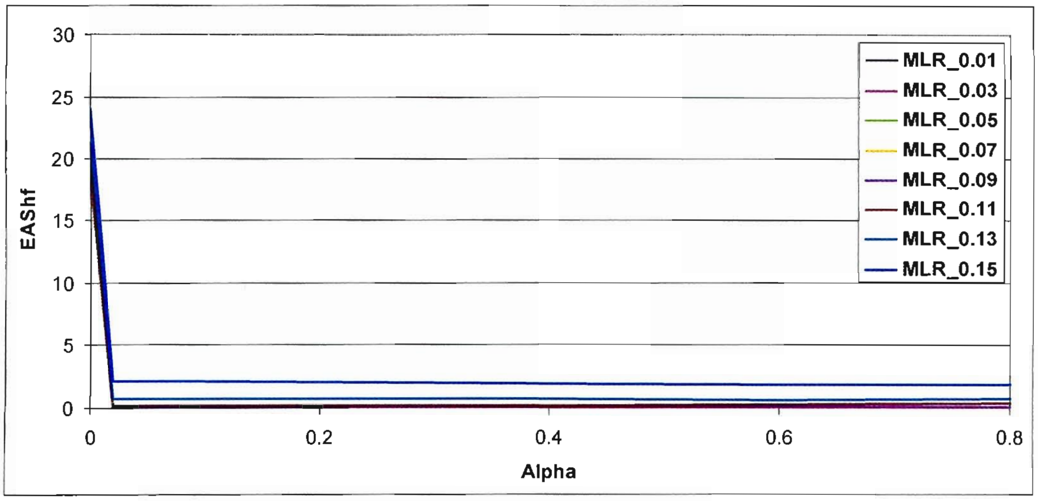

Figure 4.3.3: Expected average shortfall for different levels of alpha and minimum liquid asset growth rate (MLR), at February 2008

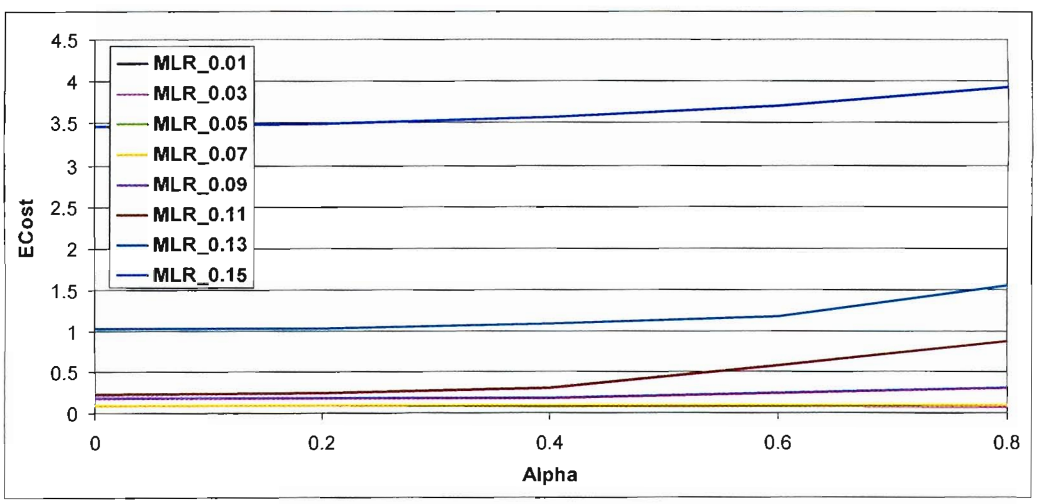

Figure 4.3.4: Expected cost for different levels of alpha and minimum liquid asset requirement growth rate (MLR), at February 2008

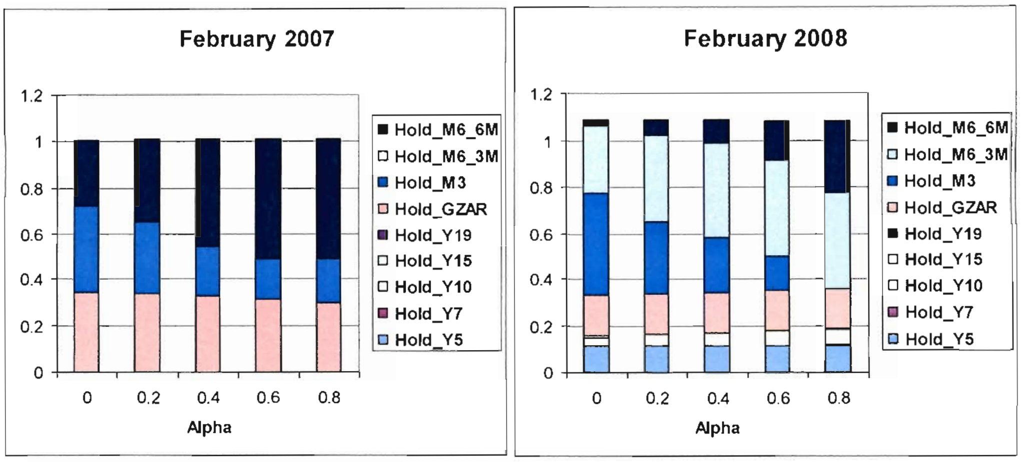

Figure 4.3.5: Asset allocation for different levels of alpha

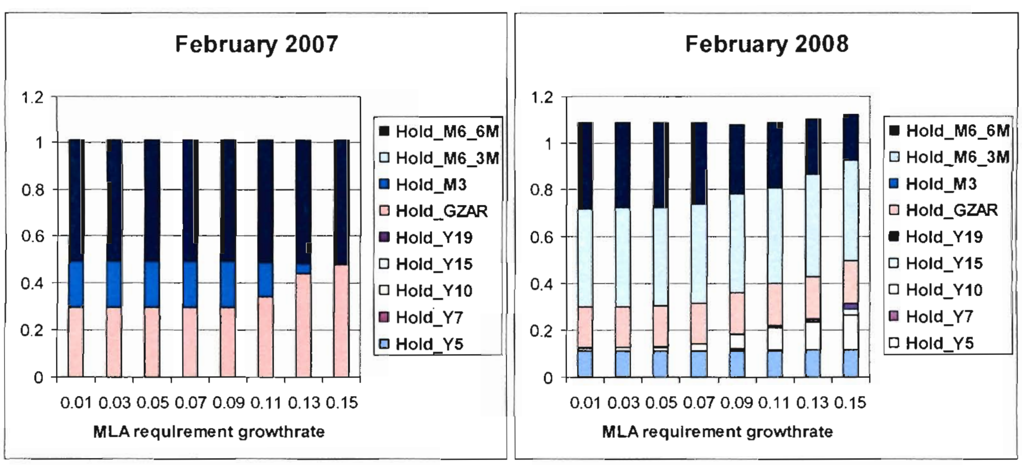

Figure 4.3.6: Asset allocation for different levels of minimum liquid asset requirement growth rate (MLR)

In Figure 4.3.5 we present the asset allocation for different levels of alpha at a minimum liquid asset requirement growth rate of 9% at February 2007 and February 2008. As expected less funds are allocated to gold (which is a more risky asset) as alpha increases and more importance is given to the minimum shortfall. Figure 4.3.6 presents the asset allocation for different levels of minimum liquid asset requirement growth rates, at alpha of 0.6, at February 2007 and February 2008. More funds are allocated to gold as the minimum liquid asset requirement growth rate increases. At the offset of the portfolio funds are allocated only to gold and treasury bills, where over time more funds are allocated to bonds. In Figure 4.3.5 we see that as alpha increases less funds are allocated to gold and three month treasury bills and more funds are allocated to six month treasury bill and five and ten year bonds.

In Figure 4.3.6 we see that as the minimum liquid asset requirement growth rate increases more funds are allocated to gold and five and ten year bonds and less are allocated to treasury bills.

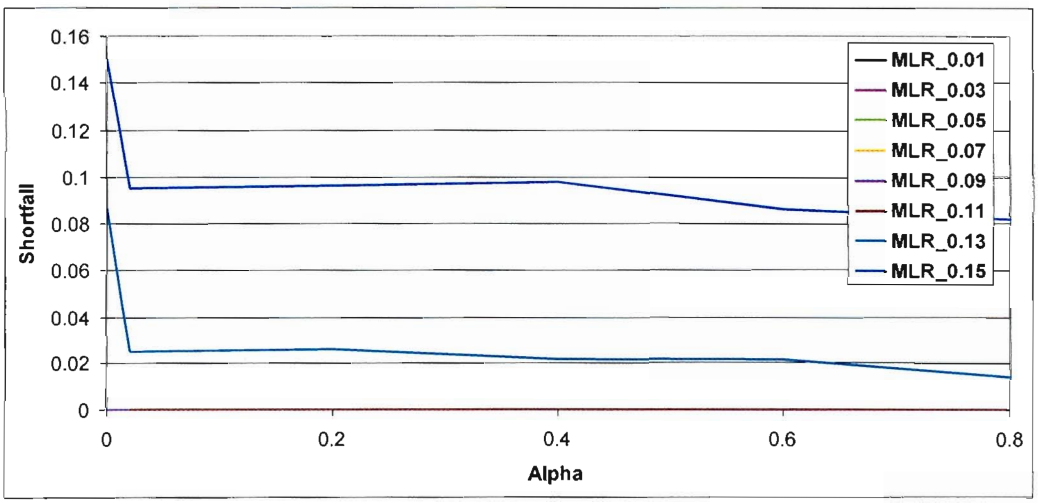

Figure 4.3.7: Actual average shortfall for different levels of alpha and minimum liquid asset requirement growth rate (MLR)

Figures 4.3.7 and 4.3.8 present the actual average shortfall and cost for different levels of alpha and minimum liquid asset requirement growth rates. The average shortfall decreases as alpha increases and more importance is given to minimum shortfall. For minimum liquid asset requirements of 11% and below no shortfall is reported and the shortfall increases as the requirement increases from 11%. The cost of the portfolio increases as alpha and the minimum liquid asset requirement increases. From minimum liquid asset requirements of 9% and above, extra costs are needed to stay above the requirement.

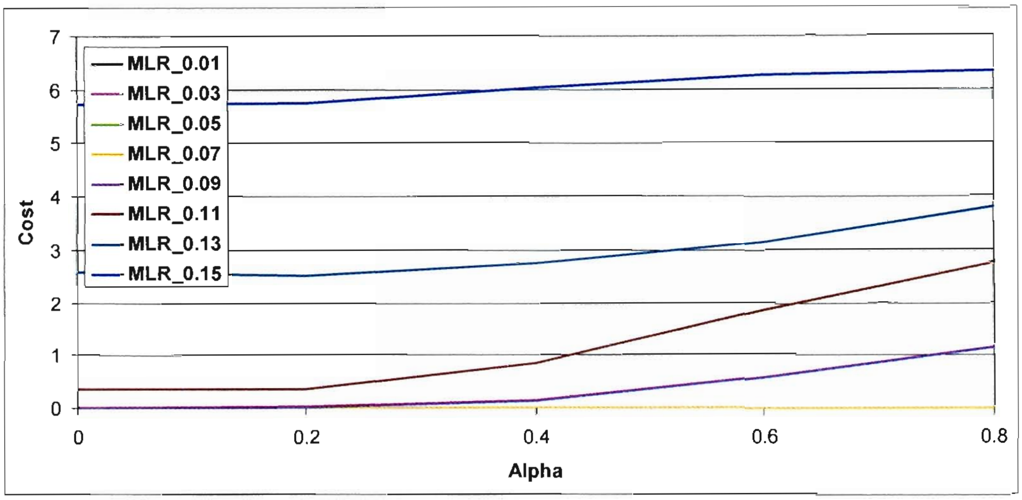

Figure 4.3.8: Actual cost for different levels of alpha and minimum liquid asset requirement growth rate

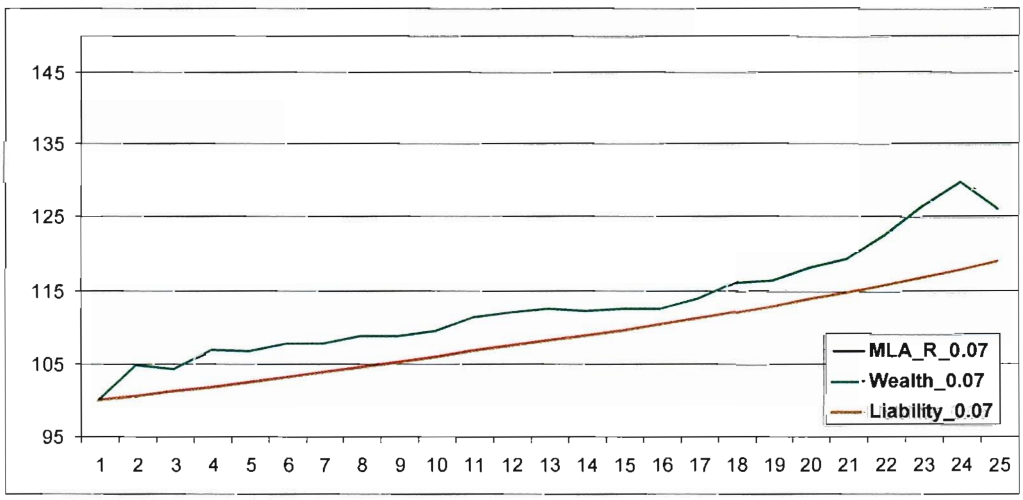

Figure 4.3.9: Wealth and liability accounts at 7% minimum liquid asset growth rate

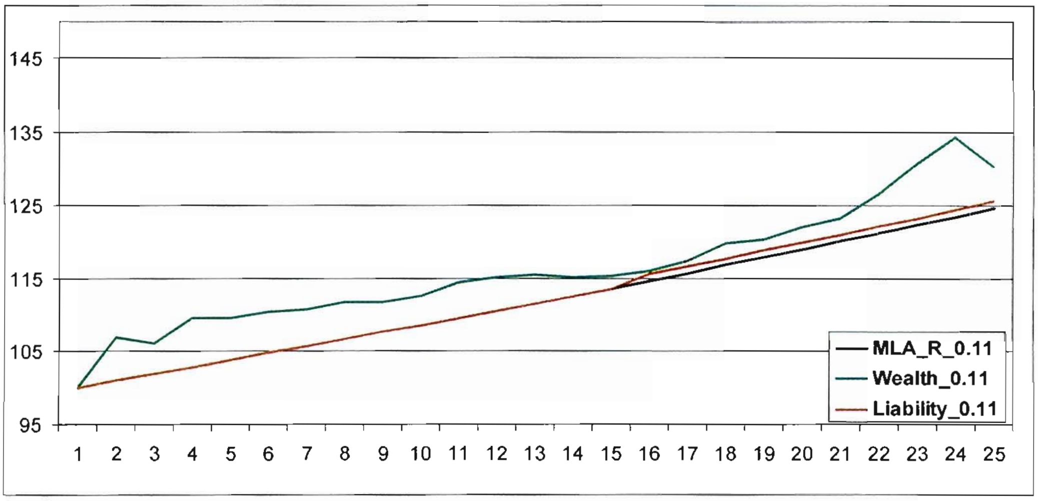

Figure 4.3.10: Wealth and liability accounts at 11% minimum liquid asset growth rate

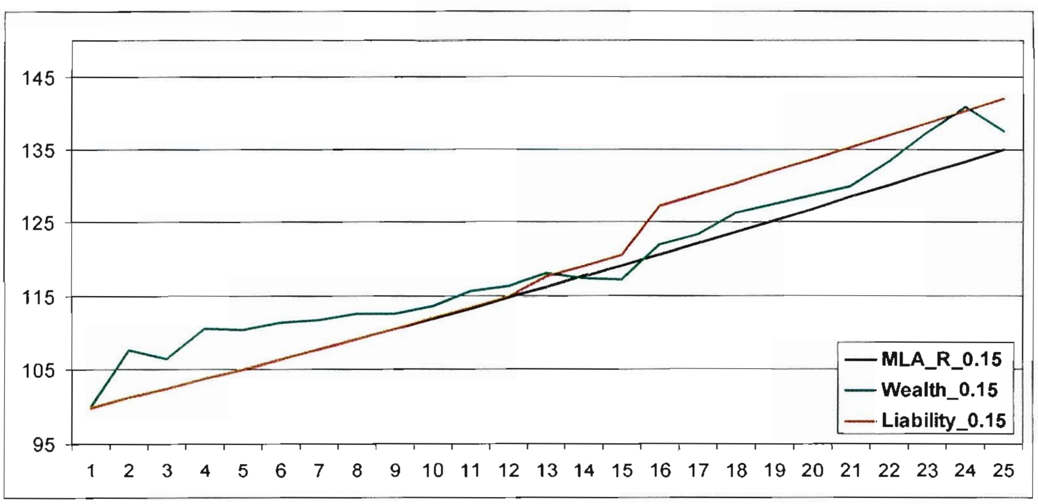

Figure 4.3.11: Wealth and liability accounts at 15% minimum liquid asset growth rate

In Figures 4.3.9, 4.3.10 and 4.3.11 we present the wealth account, liability account (including the liability account and shortfall fund account) and minimum liquid requirement account for 7%, 11% and 15% where alpha is taken to be 0.5. As mentioned above no shortfall is reported for requirements of below 11% and extra cash is required to stay above the requirement at 11% and 15%. At 15% shortfall below the minimum liquid requirement is funded. It is also not possible to ensure no-shortfall below the requirement and the shortfall fund.

4.4 Conclusion

This chapter presented a multi-stage dynamic stochastic programming model for the integrated asset and liability management of minimum liquid asset portfolios. The model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management decisions, such as reinvesting coupons, at intermediate time steps. We have shown that our problem is related to insurance products with guarantees and utilised this in the formulation.

We have shown the model features at different levels of alpha (importance of minimum expected average shortfall) and minimum liquid asset requirement growth rates. The model performs as expected with average shortfall decreasing and cost increasing as alpha increases. Also the average shortfall and cost increase as the minimum liquid asset requirement growth rates increase.

This model can also be used when analysing the investment decision made by the financial institution and may play an important role in liquidity management, when concerning different levels of liability growth rates.

In the next chapter we present a stochastic programming framework for the asset and liability management of insurance products with guarantees.

Chapter 5

Insurance products with guarantees

In recent years insurance products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way.

In this chapter[1] *we propose a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. We investigate with-profits guarantee funds by including regular bonus payments while keeping the optimisation problem linear.*

The uncertainty is represented in terms of scenario trees using a four-factor term structure model that includes macro-economic factors (inflation, capacity utilisation and repo-rate). We construct scenario trees with path dependent intermediate discrete yield curve outcomes suitable for the pricing of fixed income securities. The main focus of this Chapter is the formulation and implementation of a multi-stage stochastic programming model. The model is back-tested on real market data over a period of five years.

[1]A paper based on the work done in this chapter has been presented at the ASSA Convention, South Africa, 2008. The paper has also been accepted for publication in the South African Actuarial Journal.

5.1 Introduction

In recent years multi-stage dynamic stochastic programming models have become a popular tool for integrated asset and liability modelling. In contrast to the usual mean-variance approach (Markowitz, 1952) with a myopic view of managing investment risk over a single period, dynamic stochastic optimisation provides the asset manager with an integrated way to model both assets and liabilities in a flexible manner that takes into account multi-period dynamic asset allocation and the valuation of liabilities under future market conditions. Using this approach the rebalancing of the asset portfolio is modelled explicitly. Examples of the use of dynamic stochastic programming models in asset and liability management can be found in Kouwenberg (2001) and Mulvey *et al.* (2003). Dempster *et al.* (2003) show that the dynamic stochastic programming model will automatically hedge the current portfolio allocation against future uncertainties in asset returns and costs of liabilities over the analysis horizon. These models are also flexible enough to take into account multi-period horizons, portfolio constraints such as no short-selling, transaction costs and the investor's level of risk-aversion and utility.

Insurance products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way. Examples of the use of dynamic portfolio optimisation models for asset and liability management in the insurance industry are the Russell-Yasuda Kasai model by Cariño and Ziemba (1998), the Towers Perrin model by Mulvey and Thorlacius (1998) and the CALM model of Consigli and Dempster (1998). More recent contributions specifically in the area of insurance products with minimum guarantees using dynamic stochastic programming as an asset and liability management tool are Dempster *et al.* (2006) and Consiglio *et al.* (2006).

Dempster *et al.* (2006) propose an asset and liability management framework and give numerical results for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay. They demonstrate the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product. Through back-testing they show that the proposed stochastic optimisation framework address the risk created by the guarantee in a reasonable way.

Consiglio *et al.* (2006) study the same type of problem by structuring a portfolio for with-profit guarantee funds in the United Kingdom. The optimisation problem results in a non-linear optimisation problem. They demonstrate how the model can be used to analyse the alternatives to different bonus policies and reserving methods. Consiglio *et al.* (2001) investigate the asset and liability management of minimum guarantee products for the Italian Industry.

Inspired by the research of Dempster *et al.* (2006) and Consiglio *et al.* (2006), we propose a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees, that minimises the down-side risk of these products. As proposed in Dempster *et al.* (2006), our model also allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management decisions, such as the reinvestment of coupons at intermediate time steps. We investigate with-profits guarantee funds as in Consiglio *et al.* (2006), by including regular bonus payments. Once these bonuses have been declared, the bonuses become guaranteed. To keep the optimisation problem linear, we change the way bonuses are declared. We keep the problem linear, for two reasons. The first is that, by keeping the problem linear, we can model the rebalancing of the portfolio at future decision times. By doing so the dynamic stochastic programming model automatically hedges the first stage portfolio allocation against projected future uncertainties in asset returns (see Dempster *et al.*, 2003 and 2006). The second reason is that the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments.

For the South African insurance market Katz and Rosenberg (2005) use a sample smoothed bonus annuity contract to illustrate the weaknesses of traditional pricing, valuation and risk management tools used by life offices particularly in a low interest rate environment. Furthermore Katz and Rosenberg (2005) describe, illustrate and argue the merits of a coherent pricing, valuation and risk management framework for managing smoothed bonus contracts.

Pricing of contingent claims and dynamic management of portfolios are two sides of the same coin. The main differences between the valuation of insurance products and dynamic portfolio management are highlighted by Consiglio *et al.* (2006). The literature on pricing products with guarantees assumes that the reference portfolio is given exogenously (e.g. equities 60% and bonds 40%), and does not address the problem of structuring this portfolio optimally. The possible upside potential is ignored. According to Dempster *et al.* (2006): *"This is where the asset manager has a potential advantage. He or she can provide the protection while still exposing the client to high-risk markets through active asset allocation to potentially higher returns"*. Consiglio *et al.* (2001) have shown that the financial institution could substantially increase shareholder value by structuring the reference portfolio. This can be done by viewing it as an integrated asset and liability management optimization problem. Long-term options, which form the backbone of valuation methods, are in general only available as OTC contracts. This adds a credit risk component to the problem that is largely ignored. The replicating portfolio approach used to value these products assumes continuous rebalancing. This assumption and the other assumptions of the Black-Scholes market are unrealistic.

Foroughi *et al.* (2003) explore the risks faced by South African life insurance companies arising from the provision of investment guarantees in these products. Furthermore Foroughi *et al.* (2003) examine various forms of investment guarantees available in South Africa and the business issues created by writing these products. They compare existing methods used to value these products and discuss practical issues around the building of such asset and liability models. Foroughi *et al.* (2003) identified, non-profit immediate annuities, participating immediate annuities, unit-linked saving products with a maturity guarantee and smoothed-bonus savings products with a maturity guarantee as the main four products with investment guarantees sold in South Africa.

We represent the uncertainty in terms of scenario trees by using a four-factor term structure model that includes macro economic factors (inflation, capacity utilisation and repo-rate). We construct scenario trees with path dependent intermediate discrete yield curve outcomes suitable for the pricing of fixed income securities (see Chapter 3).

The rest of this chapter is structured as follows. In Section 5.2 we discuss the formulation and implementation of the multi-stage stochastic programming model. Section 5.3 presents back-testing results. The back-tests are done on real market data over a period of five years.

5.2 Scenario optimisation framework

In this section we propose a linear multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees, that minimises the down-side risk of these products.

5.2.1 Model features

As in Consiglio *et al.* (2006) we investigate the optimal asset allocation of with-profits guarantee funds, by including regular bonuses. The fund is operated by a proprietary company on a 90/10 basis, where policyholder benefits in 90% of the growth in asset share and the share holders 10%. It is assumed that there is a cohort of policyholders, paying a single up-front premium and that no contributions are allowed thereafter. At maturity there is an underlying guarantee to pay a minimum rate of return of g on the initial premium. In addition to receiving a guaranteed rate of return on the initial premium, policyholders also receive several bonuses. Bonuses are meant to reflect the overall performance of the firm's portfolio, and to correspond to "Policyholders' Reasonable Expectations". Two types of bonuses are received by the policyholder, namely regular bonuses (declared annually) and terminal bonuses (awarded upon maturity). Regular bonuses are "vesting", in other words they are guaranteed once declared and cannot be reduced (Consiglio *et al.*, 2006). The time horizon of the fund is T years. We use two asset classes, namely, (semi-annual) coupon bearing bonds and equities modelled using indices.

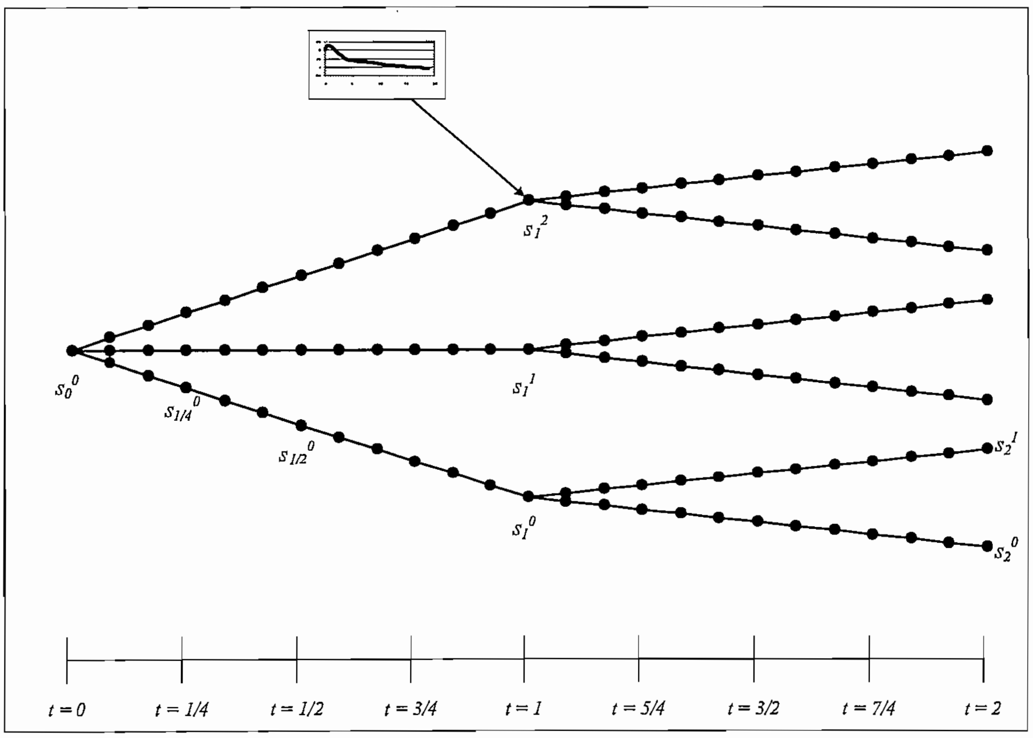

Figure 5.2.1: Graphical representation of a yield curve scenario tree

To represent uncertainty, we simulate the future yield curves to construct a scenario tree. A scenario tree is a discrete approximation of the joint distribution of random factors (yield curve and stock indices). To facilitate the mathematical formulation of the optimisation problem, we represent the scenario tree in terms of states (nodes) $s_t^{v(t)}$, where time $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$ and $v(t) = 0, 1, 2, \ldots, N_t$ the numbers of the states at time t. The set of states at time t are denoted by $\Sigma_t = \left\{ s_t^{v(t)} | v(t) = 0, 1, \ldots, N_t \right\}$. The set of all states in the scenario tree is denoted by

$\Sigma = \cup_{t=0}^{T}\Sigma_t$. Links $\varepsilon \in \Sigma \times \Sigma$, indicate the possible transitions between states. To enforce non-anticipativity, i.e. to prevent foresight of uncertain future events, we order the elements of ε in pairs $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)}\right)$ where the dependence of the index $v(t)$ on t is explicitly indicated. The order of the states indicates that state $s_{t+1}^{v(t+1)}$ at time $t+1$ can be reached from state $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is the successor state and $s_t^{v(t)}$ the predecessor state. By using the superscript "+" to denote the successor states, and the superscript "-" to denote the predecessors, we have $s_t^{v(t)+} = s_{t+1}^{v(t+1)}$ and $s_{t+1}^{v(t+1)-} = s_t^{v(t)}$. Each state $s_t^{v(t)}$ has an associated probability p_t^s, for $s \in \Sigma_t$, such that $\sum_{s\in\Sigma_t} p_t^s = 1$.

The annual decision times $t_d = 0, 1, 2, \ldots, T-1$, are the times which the fund will trade to rebalance its portfolio. We represent the branching of the tree structure with a tree-string, which is a string of integers specifying for each decision time t_d the number of branches for each state $s \in \Sigma_t$. This specification gives rise to a balanced scenario tree where each sub-tree in the same period has the same number of branches. Figure 5.2.1 gives an example of a scenario tree with a (3,2) tree-string, giving a total of 6 scenarios.

5.2.2 Model variables and parameters

The following notation will be used for variables and parameters of the model, where the time index t takes values over the times $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, T$, and states index s takes values from the set $\Sigma_t = \{s_t^v | v = 1, 2, \ldots, N_t\}$ (The following notation easy translates to mathematical programming languages, such as SAS/OR PROC OPTMODEL, and are therefore used):

Time sets

$T^{total} = \{0, \frac{1}{12}, \frac{2}{12}, \ldots, T\}$	: set of all times considered in the stochastic program;
$T^d = \{0, 1, 2, \ldots, T-1\}$	: set of decision times;
$T^i = T^{total} \backslash T^d$	: set of intermediate times;
$T^c = \{\frac{1}{2}, \frac{3}{2}, \ldots, T-\frac{1}{2}\}$	: set of coupon payment times between decision times;

Index sets

$\Sigma_t = \{s_t^v | v = 1, 2, \ldots, N_t\}$	: set of states at time t;
SI	: set of stock indices;
$B = \{B_\tau\}$	: set of government bonds with maturity τ;
$I = SI \cup B$	: set of all instruments;

Parameters

δ_{B_τ}	: coupon rate of a government bond with maturity τ;
F_{B_τ}	: face value of a government bond with maturity τ;
$r^s_{t,\tau}$	: zero-rate with maturity τ at time t in state s;
g	: minimum guaranteed rate of return;
ρ	: regulatory equity to debt ratio;
$r^s_{t,b}$	: benchmark rate at time t in state s;
γ	: policyholders' rate of participation in the profits of the firm;
β	: target terminal bonus;
$P^{a,s}_{t,i}/P^{b,s}_{t,i}$	: ask or bid price of asset $i \in I$ at time t in state s;
f_a/f_b	: proportional transaction costs on ask or bid transactions;
p^s_t	: probability of state s at time t;

Decision variables

$x^s_t = \left\{x^s_{t,i}\right\}_{i \in I}$	: quantities of assets *bought* at time t in state s;
$y^s_t = \left\{y^s_{t,i}\right\}_{i \in I}$	: quantities of assets *sold* at time t in state s;
$z^s_t = \left\{z^s_{t,i}\right\}_{i \in I}$	: quantities of assets *held* at time t in state s, from time t to $t + \frac{1}{12}$;
A^s_t	: value of assets account at time t in state s;
L^s_t	: value of liability account at time t in state s;
E^s_t	: value of equity account at time t in state s;
c^s_t	: amount of equity provided by shareholders at time t in state s;
SF^s_t	: amount of shortfall at time t in state s;
RB^s_t	: regular bonus payment declared at time t in state s;
TB^s_T	: policyholders' terminal bonus at time T in state s;

5.2.3 Bond pricing

We assume all bonds to pay semi-annual coupons of $\delta_{B_\tau}$and derive bid and ask prices by adding a spread, sp, to the zero-rates. Let $P^{a,s}_{t,B_\tau}$ denote the ask price of a coupon bearing bond with maturity τ at time t:

$$\begin{aligned} P^{a,s}_{t,B_\tau} &= F_{B_\tau} \exp\left(-\left(\tau - t\right)\left(r^s_{t,\tau-t} + sp\right)\right) \\ &\quad + \sum\nolimits_{m=\left\{\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \dots, \frac{\lfloor 2t \rfloor}{2}+\tau\right\}} \frac{1}{2}\delta_{B_\tau} F_{B_\tau} \exp\left(-\left(m - t\right)\left(r^s_{t,(m-t)} + sp\right)\right), \end{aligned}$$

for $t \in T^{total}$, and $s \in \Sigma_t$, where the principal amount is discounted in the first term and the coupon payment stream in the second term. The integral part is denoted by $\lfloor . \rfloor$. Let $P^{b,s}_{t,B_\tau}$ be the bid price of the bond with maturity τ at time t:

$$\begin{aligned} P^{b,s}_{t,B_\tau} &= F_{B_\tau} \exp\left(-(\tau - t)\left(r^s_{t,\tau-t} - sp\right)\right) \\ &+ \sum\nolimits_{m=\left\{\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \ldots, \frac{\lfloor 2t \rfloor}{2}+\tau\right\}} \frac{1}{2}\delta_{B_\tau} F_{B_\tau} \exp\left(-(m-t)\left(r^s_{t,(m-t)} - sp\right)\right), \end{aligned}$$

$$\text{for } t \in T^{total}, \text{and } s \in \Sigma_t.$$

The use of a bid or ask price when including transaction costs are not always necessary in this context as transaction costs are sufficient to prevent the simultaneous buying and selling of assets in the optimisation model. However transaction costs on bonds are relatively low compared to the bid/ask spread. In South Africa the typical bid/ask spread on bonds is 0.5% and transaction costs are 0.012%.

5.2.4 Variable dynamics and constraints

To simplify our notation, we assume all face values to be 100. We further assume all bonds to pay semi-annual coupons at rate δ_{B_τ} at yearly decision times and six months in between. The bond cash flows, $CF^s_{t,i}$, per unit face value are then calculated as:

$$CF^s_{t,i} = \frac{1}{2}\delta_i F_i, \text{ for } i \in B,\ t \in T^d \cup T^c \text{ and } s \in \Sigma_t.$$

The variable dynamics and constraints for the minimum guarantee problem are:

Cash balance constraints. The cash balance constraints ensure that the amount of cash that is received from selling assets, coupon payments at decision times and equity supplied for shortfall, is equal to the amount of assets bought:

$$\sum\nolimits_{i\in I} P^{a,s}_{0,i} x^s_{0,i} (1 + f_a) = A^s_0 \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$\sum\nolimits_{i\in I} P^{b,s}_{t,i} y^s_{t,i} (1 - f_b) + \sum\nolimits_{i\in I\setminus\{SI\}} CF^s_{t,i} z^{s-}_{t-\frac{1}{12},i} + c^s_t = \sum\nolimits_{i\in I} P^{a,s}_{t,i} x^s_{t,i} (1 + f_a),$$

$$\text{for } t \in T^d \setminus \{0\} \text{ and } s \in \Sigma_t.$$

Short sale constraints. The short sale constraints eliminate the possibility of short-selling assets in each state at each time period:

$$x^s_{t,i} \geq 0, \text{ for all } i \in I,\ t \in T^{total} \backslash \{T\} \text{ and } s \in \Sigma_t,$$

$$y^s_{t,i} \geq 0, \text{ for all } i \in I,\ t \in T^{total} \backslash \{0\} \text{ and } s \in \Sigma_t,$$

$$z^s_{t,i} \geq 0, \text{ for all } i \in I,\ t \in T^{total} \backslash \{T\} \text{ and } s \in \Sigma_t.$$

Inventory constraints. The inventory constraints give the quantity invested in each asset in each state at each time period:

$$z^s_{0,i} = x^s_{0,i} \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$z^s_{t,i} = z^{s-}_{t-\frac{1}{12},i} + x^s_{t,i} - y^s_{t,i}, \text{ for } i \in I,\ t \in T^{total} \backslash \{0\} \text{ and } s \in \Sigma_t.$$

Information constraints. As the portfolio is only rebalanced at decision times, the information constraints ensure that portfolio can not be changed between decision times,

$$x^s_{t,i} = y^s_{t,i} = 0 \text{ for } i \in I,\ t \in T^i \backslash T^c \text{ and } s \in \Sigma_t.$$

Coupon reinvestment constraints. The coupon reinvestment constraints ensure that the coupons that are paid at the coupon times are reinvested in the same coupon bearing bonds:

$$x^s_{t,i} = \frac{CF^s_{t,i} z^{s-}_{t-\frac{1}{12},i}}{P^{a,s}_{t,i}(1+f_a)}, \text{ for } i \in I \backslash \{SI\},\ t \in T^c \text{ and } s \in \Sigma_t,$$

$$y^s_{t,i} = 0, \text{ for } i \in I \backslash \{SI\},\ t \in T^c \text{ and } s \in \Sigma_t,$$

$$x^s_{t,SI} = 0, y^s_{t,SI} = 0,\ t \in T^c \text{ and } s \in \Sigma_t$$

Asset account constraints. The asset account constraints determine the value of the asset account in each state at each time period. The value of the asset account is determined after rebalancing, i.e. any equity c^s_t that has been provided by shareholders to fund shortfalls, is taken into account by the cash balance constraints:

$$A^s_0 = L_0 + E^s_0, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t$$

$$A^s_t = \sum\nolimits_{i \in I} P^{a,s}_{t,i} z^s_{t,i} (1+f_a), \text{ for } t \in T^{total} \backslash \{T\} \text{ and } s \in \Sigma_t$$

$$A_T^s = \sum_{i \in I} P_{T,i}^{b,s} z_{T-\frac{1}{12},i}^{s-} (1 - f_b) + \sum_{i \in I \setminus \{SI\}} CF_{t,i}^s z_{T-\frac{1}{12},i}^{s-}, \text{ for } s \in \Sigma_T$$

Liability account constraints. The liability account constraints determine the value of the liability account in each state at each time period. The liability grows at the guaranteed rate of return plus any regular bonus payments that are declared:

$$L_0^s = L_0, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$L_t^s = L_{t-\frac{1}{12}}^{s-} \exp\left(\frac{1}{12} g\right) + RB_t^s, \text{ for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t.$$

Equity account constraints. The equity account constraints determine the value of the equity account in each state at each time period. The equity grows at the one month zero-rate. The shortfall is funded by the shareholders by the infusion of additional equity:

$$E_0^s = c_0^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t$$

$$E_t^s = E_{t-\frac{1}{12}}^{s-} \exp\left(\frac{1}{12} r_{t-\frac{1}{12} t}^{s-}\right) + c_t^s, \text{ for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t.$$

Regular bonus constraints. The regular bonus constraints determine the amount of the regular bonus payment in each state for each decision time. To determine the amount of the regular bonus we follow the approach described by Consiglio *et al.* (2006) which is based on that of Ross (1989) where the regular bonuses are determined by aiming for a target terminal bonus, i.e. the firm wishes the policyholders' terminal benefit to be a fixed portion of the total benefit received. Regular bonuses are assumed to be declared at decision times only (i.e. annually).

It is assumed that the asset account will grow constant at the current benchmark rate, $r_{t,b}^s$, up to termination, giving the terminal asset value as:

$$A_T^s = A_t^{bs} \exp\left(r_{t,b}^s (T - t)\right),$$

where $A_t^{bs} = \sum_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-} (1 - f_b) + \sum_{i \in I \setminus \{SI\}} CF_{t,i}^s z_{t-\frac{1}{12},i}^{s-}$ is the value of the asset account before transactions. It is further assumed that the liabilities will grow at the minimum growth rate, g, up to termination. Furthermore, it is assumed that the regular bonus payment, RB_t^s, that is declared at time t will stay constant throughout the remainder of the term and will be invested at the minimum guarantee, g. Thus the terminal liability value is:

$$L_T^s = L_{t-\frac{1}{12}}^{s-} \exp\left(g(T - t + \frac{1}{12})\right) + RB_t^s \left(\frac{\exp(g(T-t)) - 1}{\exp(g) - 1} + \exp(g(T-t))\right),$$

where

$$\left(\frac{\exp\left(g\left(T-t\right)\right)-1}{\exp\left(g\right)-1}+\exp\left(g\left(T-t\right)\right)\right),$$

is the accumulated value of a constant annuity with payment of one cash unit from time t to T invested at the minimum guarantee g.

The terminal bonus, $TB_T^s = \gamma\left(A_T^s - L_T^s\right)$, received by the policyholders need to constitute β% of the total amount received by the policyholders:

$$\frac{TB_T^s}{TB_T^s + L_T^s} = \beta.$$

Solving for RB_t^s yields

$$RB_t^s = \frac{\gamma\left(1-\beta\right)A_t^{b,s}\exp\left(r_{t,b}^s\left(T-t\right)\right)-\left(\beta+\gamma\left(1-\beta\right)\right)L_{t-\frac{1}{12}}^{s-}\exp\left(g\left(T-t+\frac{1}{12}\right)\right)}{\left(\beta+\gamma\left(1-\beta\right)\right)\left(\frac{\exp\left(g\left(T-t\right)\right)-1}{\exp\left(g\right)-1}+\exp\left(g\left(T-t\right)\right)\right)}.$$

When the expected terminal asset amount exceeds the expected terminal liability amount regular bonuses will increase. Conversely, when the expected terminal liability amount exceeds the expected terminal asset amount the regular bonus will be negative. As this will be unfair towards policyholders to declare negative bonuses the following regular bonus constraint is introduced:

$$RB_t^s >= \frac{\gamma\left(1-\beta\right)A_t^{b,s}\exp\left(r_{t,b}^s\left(T-t\right)\right)-\left(\beta+\gamma\left(1-\beta\right)\right)L_{t-\frac{1}{12}}^{s-}\exp\left(g\left(T-t+\frac{1}{12}\right)\right)}{\left(\beta+\gamma\left(1-\beta\right)\right)\left(\frac{\exp\left(g\left(T-t\right)\right)-1}{\exp\left(g\right)-1}+\exp\left(g\left(T-t\right)\right)\right)},$$

$$\text{for } t \in \left(T^d \setminus \{0\}\right) \cup \{T\}, \text{ and } s \in \Sigma_t$$

where $RB_t^s >= 0$ and $RB_t^s = 0$ for $t \in \left(T^i \cup \{0\}\right) \setminus \{T\}$, and $s \in \Sigma_t$. By enforcing the regular bonus constraints the optimisation will determine the regular bonus amount RB_t^s at each decision period.

Consiglio *et al.* (2006) also consider the working party approach based on Chadburn (1997) which is based on work done by the Institute of Actuaries Working Party. This approach declares regular bonuses (in return form) to reflect the benchmark return subject to the liability account remaining lower than the value of the reduced asset account, where the reduced assets accumulate at 75% of the return on assets. Consiglio *et al.* (2006) test their model with both these features and find that bonus policies based on aiming for a target terminal bonus outperforms bonus polices based on the working party approach.

Shortfall constraints. The shortfall constraints determine the regulatory shortfall of the portfolio in each state at each time period. The shortfall is calculated by using the value of the asset account before transaction:

$$SF_0^s + L_0 >= (1+\rho)\, L_0^s, \text{ for } t \in \{0\} \text{ and } s \in \Sigma_t,$$

$$SF_t^s + A_t^{bs} >= (1+\rho)\, L_{t-\frac{1}{12}}^s e^{\frac{1}{12}g}, \text{ for } t \in T^{total} \setminus \{0\} \text{ and } s \in \Sigma_t$$

where $A_t^{bs} = \sum_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-} (1 - f_b) + \sum_{i \in I \setminus \{SI\}} CF_{t,i}^s z_{t-\frac{1}{12},i}^{s-}$ is the value of the asset account before transactions and $SF_t^s >= 0$ for $t \in T^{total}$, and $s \in \Sigma_t$. The shortfall SF_t^s at decision periods are funded by the shareholders equity payment, c_t^s, thus $c_t^s = SF_t^s$ for $t \in T^d \cup \{T\}$ and $s \in \Sigma_t$, and zero at intermediate nodes, $c_t^s = 0$ for $t \in T^i \setminus \{T\}$, and $s \in \Sigma_t$. By enforcing the shortfall constraints the optimisation will determine the amount of equity c_t^s to be provided by the shareholders at each decision period.

Portfolio composition constraints. Portfolio composition constraints can be introduced in order to reduce concentration risk and to consider the policyholder expectations on the underlying asset mix. The following constraints are taken into account:

$$\frac{\sum_{i \in \tilde{I}} P_{t,i}^{a,s} z_{t,i}^s}{W_t^s} < \pi, \text{ for } t \in T^d, \tilde{I} \subset I \text{ and } s \in \Sigma_t,$$

where $\tilde{I}$ may be some subset of I and π the upper limit for the proportion of asset share invested in the subset of assets $\dot{I}$.

5.2.5 Objective function

When managing a minimum guarantee fund there are two main goals to take into account. The first aim is the management of the investment strategies of the fund. The second is to maximise the shareholder value taking into account the minimum guarantee given to policyholders. The shareholders' final wealth is given as $(1-\gamma)((A_T - E_T) - L_T) + E_T$ where $(1-\gamma)((A_T - E_T) - L_T)$ is the excess amount they receive after the liability and the equity has been paid. The objective to consider is the maximum expected excess wealth of the shareholders and the minimum average expected shortfall over all periods in order to ensure the minimum guarantee to policyholders. Dempster *et al.* (2006) have shown that monitoring shortfall at intermediate nodes improve results. The objective function is given as:

$$\max_{\left\{\begin{array}{c} x_{t,i}^s, y_{t,i}^s, z_{t,i}^s \\ i \in I, t \in T^d \cup \{T\}, s \in \Sigma_t \end{array}\right\}} \left\{ \begin{array}{c} (1-\alpha) \sum_{s \in \Sigma_T} (1-\gamma)\left((A_T^s - L_T^s) - E_T^s\right) \\ -\alpha \sum_{t \in T^{total}} \sum_{s \in \Sigma_t} p_t^s \frac{SF_t^s}{|T^{total}|} \end{array} \right\}$$

where the value of $0 \leq \alpha \leq 1$ sets the level of risk-aversion and can be chosen freely. If the value of α is to closer 1, more importance is given to the shortfall of the portfolio and less given to the expected excess wealth of the shareholders and hence a more risk-averse portfolio allocation strategy will be taken and vice versa. In the extreme case where $\alpha = 1$ only the shortfall will be minimised and the expected excess wealth will be ignored, and where $\alpha = 0$, the unconstrained case only maximises the expected excess wealth of the shareholders.

5.3 Results

In this section we discuss the performance of the model. The first part explains the data and instruments we use to generate scenario trees, which is the input to our mathematical optimisation problem. In the second part we present back-tested results for the model for different levels of the guarantee rate and different levels of risk-aversion.

5.3.1 Data and instruments

We use six different assets, namely, (semi-annual) coupon bearing bonds with maturities 5, 7, 10, 15 and 19 years and the FTSE/JSE Top 40 equity index. Scenarios for the equity index are generated along with the yield curve by modelling the FTSE/JSE Top 40 index with respect to the three macro-economic factors. We use the Perfect Fit Bond Curves, one of the five BEASSA Zero Coupon Yield Curve series of yield curves (see BESA, 2003a), with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months. The curves are derived from government bond data and the technical specifications are described in BESA (2003b). We use end-of-month data from August 1999 through to February 2009 and a tree structure with approximately the same number of scenarios. The tree structure used in back-testing is displayed in Table 5.3.1.

Table 5.3.1: Tree structure used for back-testing

Year	Tree-string
February04	5.5.5.5.5=3125
February05	8.8.8.8=4096
February06	15.15.15=3375
February07	56.56=3136
February08	3125

We use the scenario generation approach, described in Section 3.4 of Chapter 3, to generate the input scenarios for the optimisation problem. We estimate the yield curve dynamics with the four-factor yield curve representation of Svensson (1994). The four unobserved factors, level, slope and the two curvature factors, which provide a good representation of the yield curve, are linked to the macro-economic factors by means of a state-space model. We include the following three variables as measures of the state of the economy: manufacturing capacity utilisation, which represents the level of real economic activity relative to potential; the annual percentage change in the inflation index, which represents the inflation rate; and the repo-rate, which represents the monetary policy instrument. According to Diebold *et al.* (2006) these three macro-economic factors are considered to be the minimum set of fundamentals needed to capture the basic macro-economic dynamics. The model parameters are estimated using a Kalman filter approach. For a complete description of the model and the calibration of the model parameters see Chapter 3. The four-factor yields-macro model is fitted to market data up to an initial decision time t and scenario trees are generated from time t to some chosen horizon $t+T$. The optimal first stage/root node decisions are then implemented at time t. The success of the portfolio strategy is measured by its performance with historical data up to time $t+1$. This whole procedure is rolled forward for T trading times. At each decision time t, the parameters of the four-factor yields-macro model are re-estimated using the historical data up to and including time t.

5.3.2 Back-testing results

We perform back-tests over a period of five years, from February 2004 through to February 2009, for different levels of minimum guarantee and for different levels of risk-aversion. For each of these back-tests, at different levels of minimum guarantee and for different levels of risk-aversion, we report the annual expected excess return on equity (ExROE), taken to be

$$\sqrt[T]{\sum_{s\in\Sigma_T}\left(\frac{(1-\gamma)\left(A_T^s-L_T^s\right)+\gamma E_T^s}{E_T^s}\right)p_T^s}-1,$$

and the annual actual excess return on equity, taken to be

$$\sqrt[T]{\frac{(1-\gamma)\left(A_T-L_T\right)+\gamma E_T}{E_T}}-1.$$

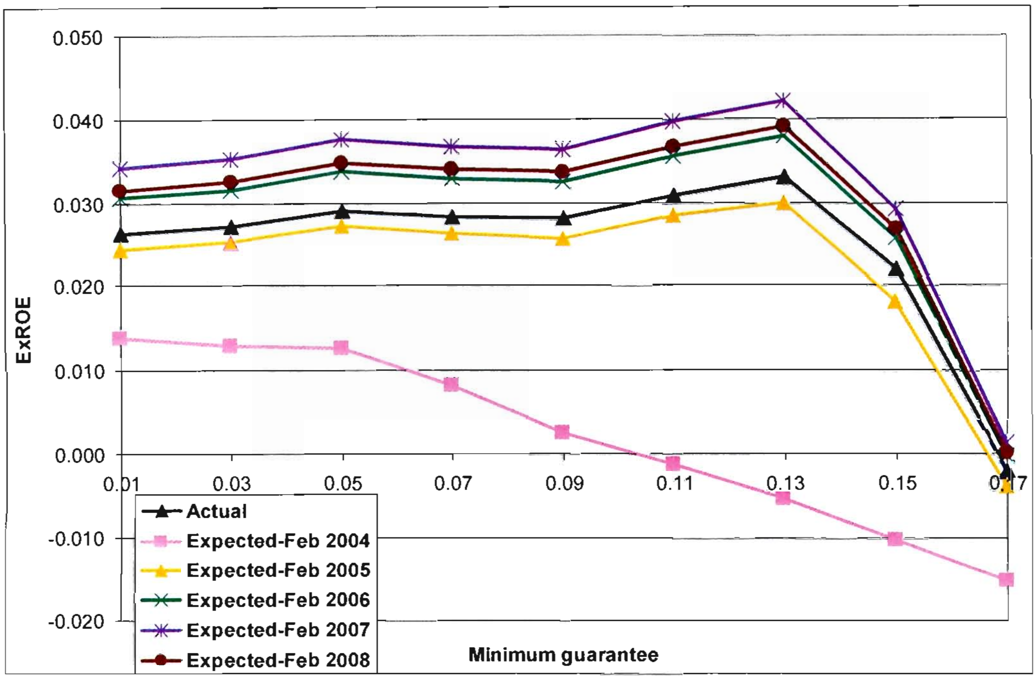

Figure 5.3.1: Shareholders annual excess return on equity for different levels of minimum guarantee at $\alpha = 0.5$

We also report the expected cost of the guarantee taken to be the expected present value of the final equity deducting the regulatory equity or equity at the start

$$\sum_{s \in \Sigma_T} \left(\frac{E_T^s}{\prod_{t=1}^T \exp\left(\frac{1}{12} r^{s_t^{v(t)}}_{t,t+\frac{1}{12}}\right)} - E_0 \right) p_T^s,$$

where $\left(s_t^{v(t)}, s_{t+\frac{1}{12}}^{v\left(t+\frac{1}{12}\right)}\right) \in \varepsilon$ and the actual cost of the guarantee, taken to be

$$\left(\frac{E_T}{\prod_{t=1}^T \exp\left(\frac{1}{12} r_{t,t+\frac{1}{12}}\right)} - E_0 \right).$$

In Figure 5.3.1 we present the expected ExROE at decision times and the actual ExROE for different levels of the minimum guarantee. The model underestimates the ExROE, the expected ExROE improves as more data becomes available, for model estimation, after the first decision time. The actual ExROE increases as the minimum guarantee increases up to 13%, after 13% the ExROE decreases as the minimum guarantee increases. In Figure 5.3.2 we present the expected cost

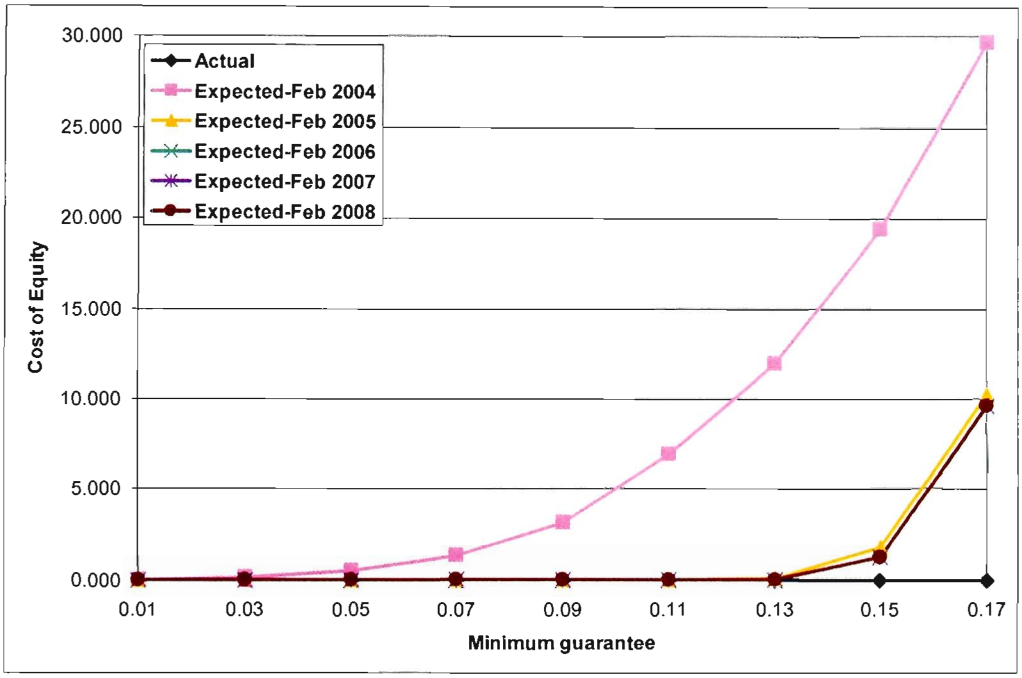

Figure 5.3.2: Cost of equity for different levels of minimum guarantee at $\alpha = 0.5$

of the guarantee at the decision times and the actual cost of the guarantee. The model firstly overestimates the cost of the guarantee and as more data becomes available for model estimation, after the first decision time, the expected cost of the guarantee improves. For a minimum guarantee of less than 9% the model requires no additional equity, as the minimum guarantee increases above 13% the amount of equity required increases.

It is expected that the ExROE will decrease as the minimum guarantee increases which is in contrast to our results. The increase in the ExROE if the minimum guarantee increases up to 13% can be explained from the zero cost of the guarantee for minimum guarantees of less than 13%. Therefore a higher excess return on equity can be achieved for higher levels of minimum guarantee if this comes at no extra cost to the shareholders. If a cost is incurred to ensure a certain level of minimum guarantee, the ExROE will decrease if the minimum guarantee increases.

Figure 5.3.3 presents the performance of the asset account and the liability account at 1%, 9% and 15% minimum guarantee. The asset level stays above the liability level over the entire period. Regular bonuses are paid up to a minimum guarantee of 15%, and that regular bonuses decrease as the level of minimum guarantee increases. This is due to the average benchmark rate, the rate used to determine the regular bonus payments, being around 8%. Thus for lower levels of minimum

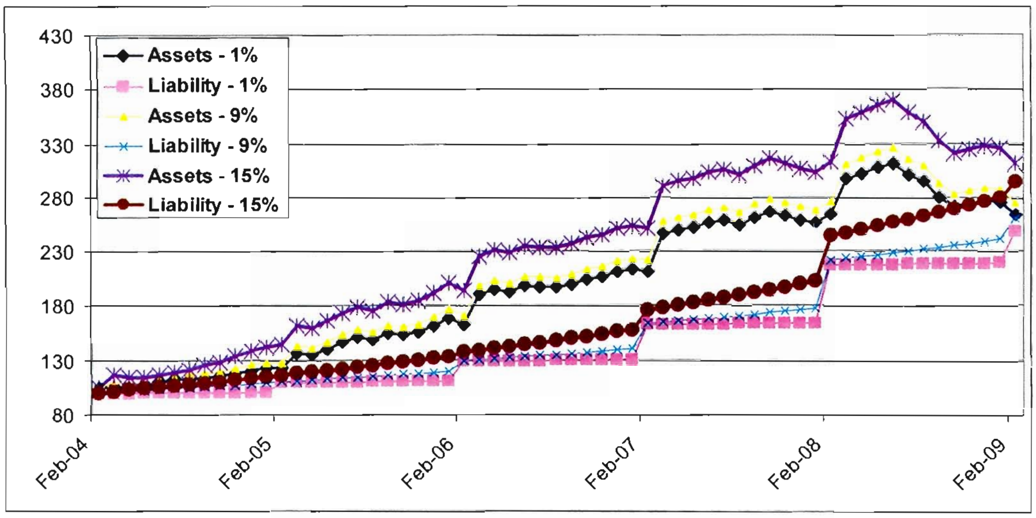

Figure 5.3.3: Asset and liability account at 1%, 9% and 15% minimum guarantee at $\alpha = 0.5$

guarantee the amount of regular bonus payments declared will be higher.

Consiglio *et al.* (2006) specify regular bonuses in return form, which is more realistic than our formulation of discrete annual payments which we define in order to keep the problem linear. Consiglio *et al.* (2006) assume that the bonus return, RB_t^s, that is declared at time t will stay constant throughout the remainder of the term giving the terminal liability value as:

$$L_T^s = L_t^s e^{g(T-t)} \exp\left(RB_t^s (T-t)\right).$$

With all other assumptions staying constant the regular bonus yields:

$$RB_t^s = \max\left[\frac{1}{(T-t)} \ln\left(\frac{\gamma(1-\beta) A_t^{b,s} \exp\left(r_{t,b}^s (T-t)\right)}{(\beta + \gamma(1-\beta)) L_t^s \exp\left(g(T-t)\right)}\right), 0\right].$$

We have also implemented the liability process proposed by Consiglio *et al.* (2006) and included it in our back-testing performance results. Figure 5.3.4 shows that our discrete approximation of bonuses mimics the more realistic approach of Consiglio *et al.* (2006). Recall that our approach has the added advantage of keeping the overall problem linear which allows us to include more realistic portfolio management constraints.

Figure 5.3.5 shows the first stage optimal asset allocation at the (forward rolling) rebalancing times for different levels of the minimum guarantee. The February 2004

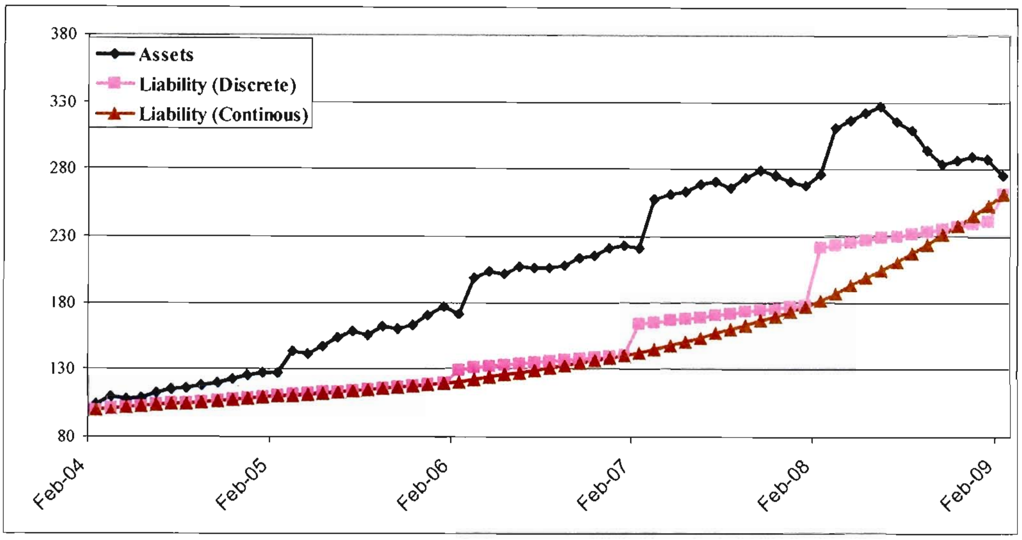

Figure 5.3.4: Liabilities with different bonus options at 1% minimum guarantee

asset allocation seems consistent. At reasonable levels of minimum guarantee the portfolio is less aggressive and allocates lower proportion of asset share to the risky asset. As the level of minimum guarantee increases more asset share is allocated to the risky asset up to a proportion of 30% of the portfolio wealth. This is a result of the portfolio composition constraints, which is set to restrict the proportion of assets share invested in the equity index to 30%. Also at higher levels of guarantee more asset share is allocated to long-term maturity bonds. After the first stage the portfolio invests more asset share in the risky asset for lower levels of minimum guarantee, this is due to the higher benchmark rate, and is necessary in order to pay bonuses. Again the proportion of asset share invested in the risky asset is restricted to 30%. If this restriction is lifted more proportions of asset share will be allocated to the risky asset. The asset allocation also does not change dramatically from one year to the next.

In Figure 5.3.6 we present the expected ExROE at rebalancing times and the actual ExROE for different levels of risk-aversion at a minimum guarantee of 9% and 15%. The model underestimates the ExROE as previously and the expected ExROE improves as more data becomes available. The ExROE decreases as the level of risk-aversion increases. For a minimum guarantee of 9% the ExROE remains constant as the risk-aversion level moves from 0 to 0.8 and then suddenly drop in the most risk-averse case. For a minimum guarantee of 15% the ExROE remains constant as the risk-aversion level moves from 0 to 0.4 and then decreases.

Figure 5.3.7 presents the expected cost of the guarantee at the decision times

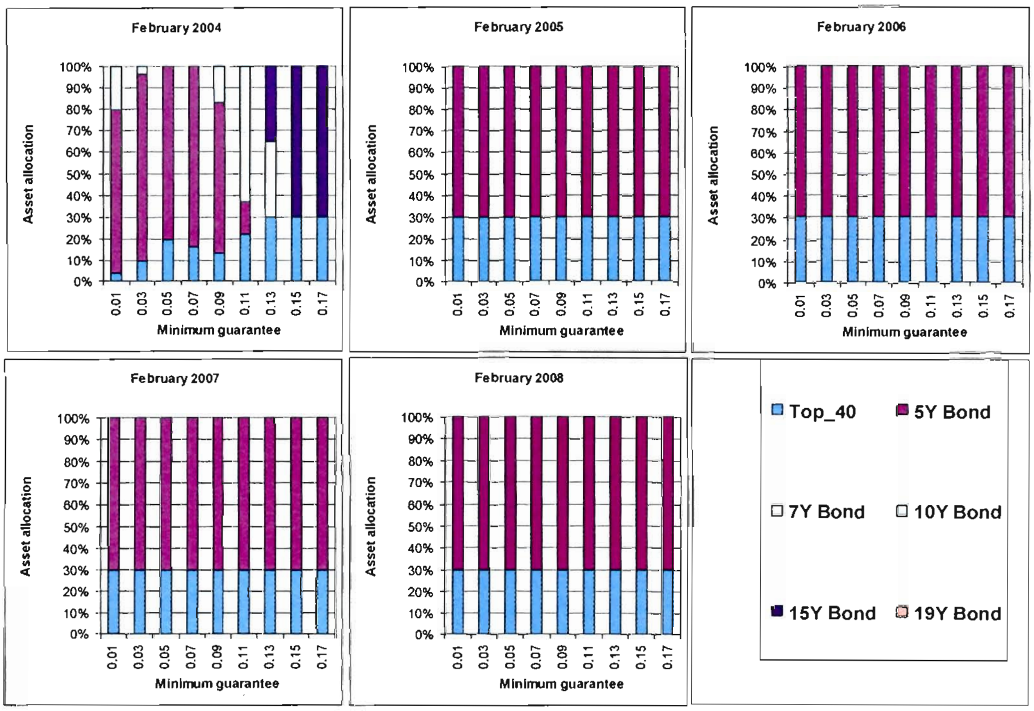

Figure 5.3.5: Asset allocation for different levels of minimum guarantee at $\alpha = 0.5$

and the actual cost of the guarantee. The model firstly overestimates the cost of the guarantee and as more data becomes available the expected cost of the guarantee improves. The expected cost of the guarantee decreases as the level of risk-aversion increases for a minimum guarantee of 9%. For a minimum guarantee of 15%, the expected cost of the guarantee increases as the level of risk-aversion increases from 0.4 to 1.

Because extra equity is now required to achieve a minimum guarantee of 9% the ExROE will remain constant if the expected cost of the guarantee is included on the objective function (i.e. $\alpha < 1$). If only the shortfall is minimised (i.e. $\alpha = 1$) a lower ExROE will be achieved. This results from the actual cost of the guarantee being zero for all levels of risk-aversion at a minimum guarantee of 9%. As a cost is incurred for a minimum of guarantee 15% if the level of risk-aversion increases for 0.4 to 1, the ExROE will decrease. A more risk-averse portfolio at much higher levels of minimum guarantee will require extra equity, as more importance is given to the shortfall of the portfolio.

Figure 5.3.8 presents the performance of the asset account and the liability account at 0, 0.6 and 1 level of risk-aversion for a minimum guarantee of 9% and 15%. The asset level stays above the liability level over the entire period. At levels 0 and 0.6 of

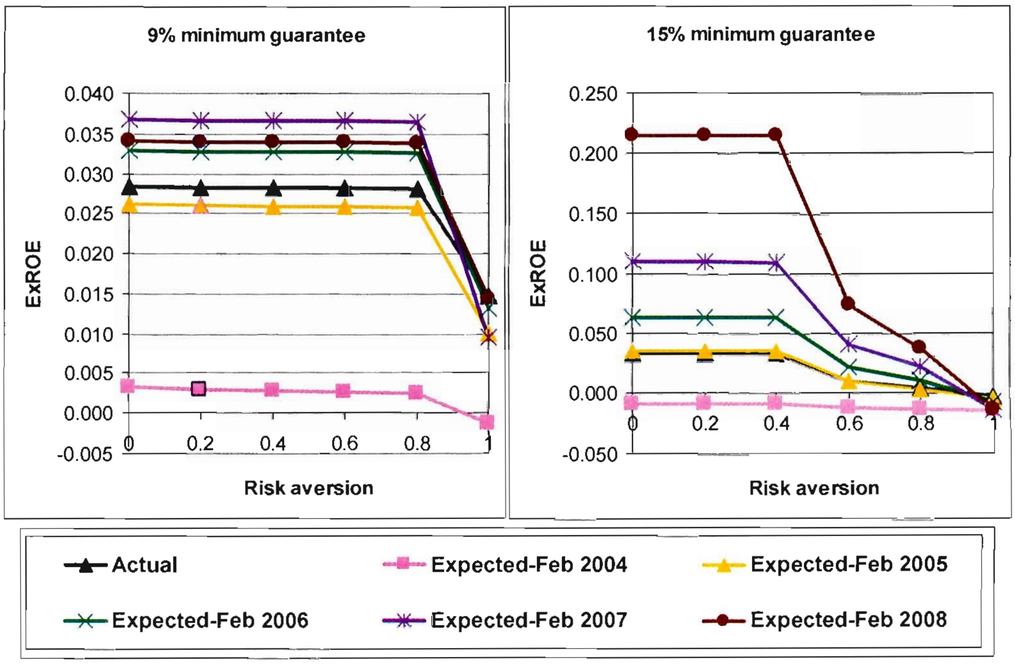

Figure 5.3.6: Shareholders annual excess return on equity for different levels of risk-aversion

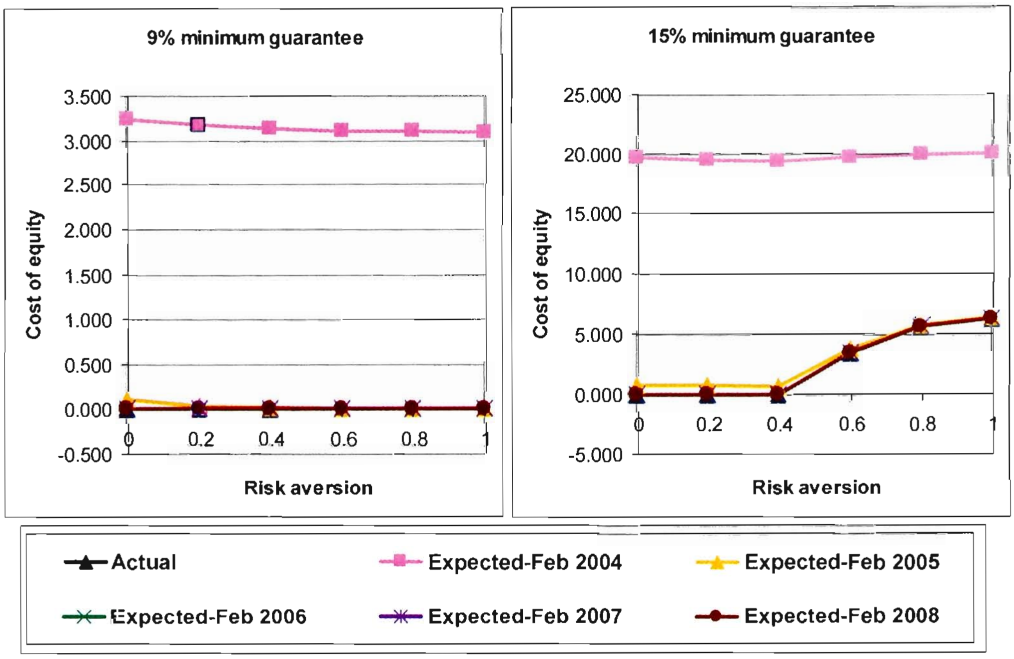

Figure 5.3.7: Cost of equity for different levels of risk-aversion

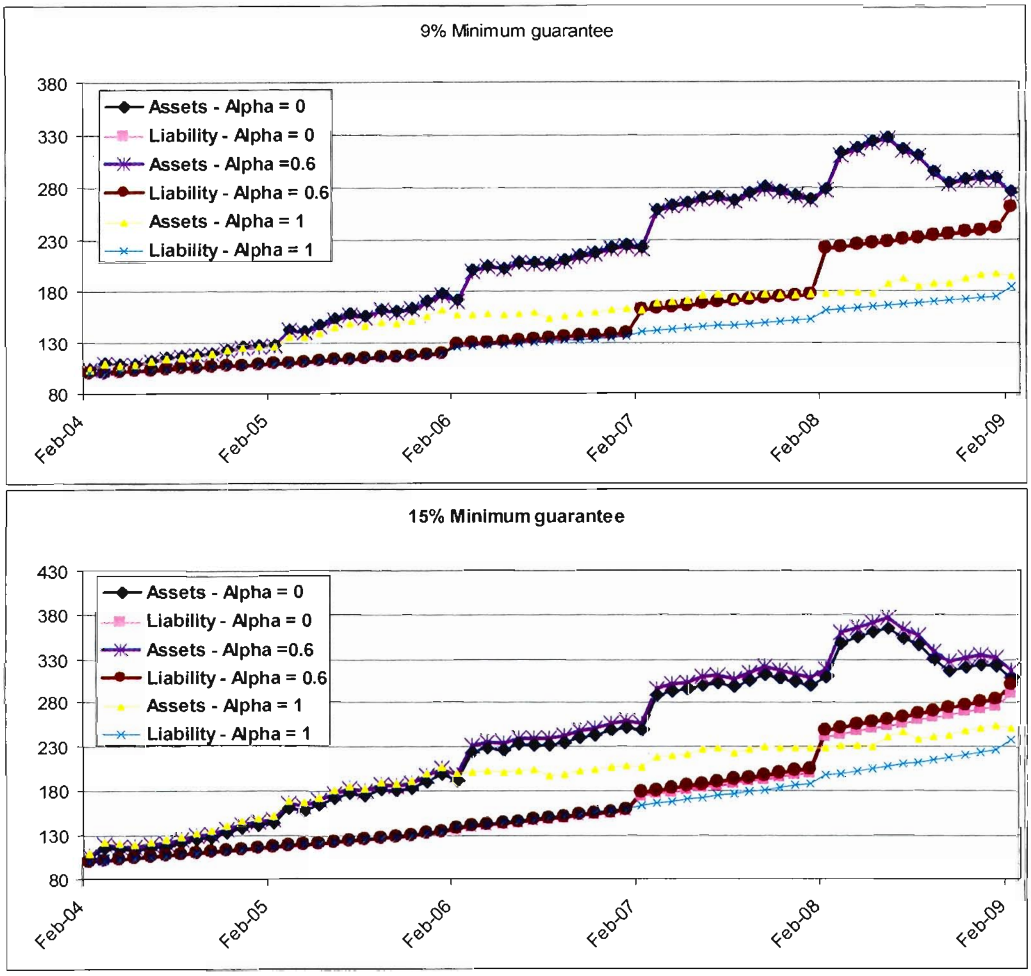

Figure 5.3.8: Asset and liability account at different levels of risk-aversion

risk-aversion, the model tends to be more aggressive and at level 1 of risk-aversion the model is more conservative. As mentioned above, for a minimum guarantee of 9%, a zero cost of the guarantee is achieved for all levels of risk-aversion. The performance of the asset account and liability account, as well as the payment of bonuses, are constant for levels of risk-aversion less than 1, due to the zero cost of the guarantee. If only the shortfall is minimised (i.e. $\alpha = 1$), the portfolio pays less bonuses and achieves a lower level of assets at end horizon. The portfolio however still achieves the minimum guarantee of 9% at no extra cost for the guarantee. For a minimum guarantee of 15% extra equity is needed for higher levels of risk-aversion and a lower ExROE is achieved. This is seen in Figure 5.3.8 where we observe that the model tends to be more conservative from a level of 0.6 of risk-aversion.

Figure 5.3.9 shows the first stage optimal asset allocation at decision times for

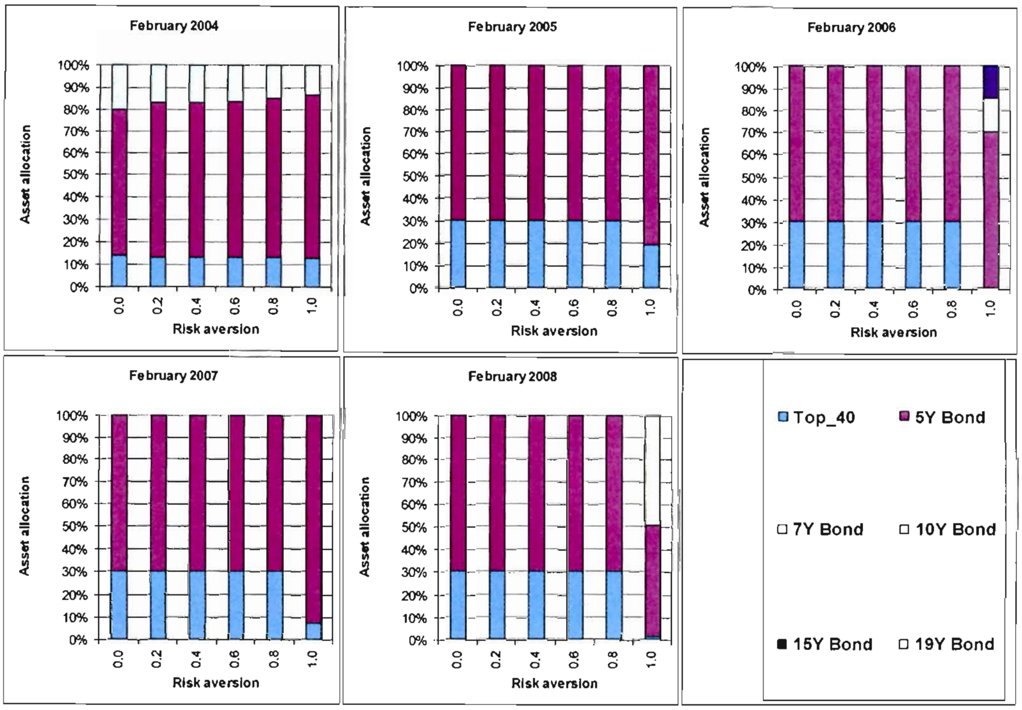

Figure 5.3.9: Asset allocation different levels of risk-aversion at 9% minimum guarantee

different levels of risk-aversion for a 9% minimum guarantee. As the level of risk-aversion increases the portfolio is more conservative and allocates a lower proportion of asset share to the risky asset, this is also apparent form the above discussion. In the most risk-averse situation, where only the shortfall is minimised (i.e. $\alpha = 1$), an even lower proportion of asset share is allocated to the risky asset. The asset allocation further does not change dramatically from one year to the next. The asset allocation for a minimum guarantee of 15% does not differ much form the asset allocations for a minimum guarantee of 9% except that more asset share is invested in longer-term bonds in the first stage (see Figure 5.3.5).

5.4 Conclusion

We have presented a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. We have included regular bonus payments and kept the optimisation problem linear, which enables us to model the rebalancing of the portfolio at future decision times. Also, by keeping the optimisation problem linear, the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments. We have also shown that our bonus assumption mimics the more realistic bonus payments proposed by Consiglio *et al.* (2006).

Furthermore, we have shown the model features at different levels of minimum guarantee and different levels of risk-aversion. The back-testing results show that the proposed stochastic optimisation framework successfully considers the risks created by the guarantee and declaration of bonus payments. As Consiglio *et al.* (2006) have shown, the model can also be used for analysing the investment decision made by the insurance firm.

The with-profits guarantee funds discussed in this chapter is operated on a 90/10 basis, where policyholder benefits in 90% of the growth in asset share and the share holders 10%. These products do however occur to a lesser extend in the South African insurance market. The main goal of this chapter was to extend on the work of Dempster *et al.* (2006) and Consiglio *et al.* (2006). This however provides wide scope for further research of these products in the South African insurance market.

Chapter 6

Summary and conclusion

In this chapter we summarise by presenting concluding remarks and end by suggesting further research directions.

6.1 Summary

This thesis presented two stochastic programming frameworks for the asset and liability management of investment products with guarantees. The first part of this thesis presented two methods for yield curve scenario generation. The first method employs a moment-matching approach and second a simulation approach which takes the movement of macro-economic factors into account.

The second part of this thesis introduced and solved two asset and liability problems. The first problem was the asset and liability management of minimum liquid asset portfolios, found in the banking environment, and the second problem was the asset and liability management of insurance products with minimum guarantees. We discussed the formulation and implementation of these multi-stage stochastic programming models and back-tested both models on real market data.

As scenario trees are the input to our portfolio optimisation problems we have started off in Chapter 2 and 3 by presenting two methods for yield curve scenario generation. Since fixed income securities are usually contained in the asset side of the asset and liability management of investment products with guarantees, we were concerned with the stochastic evolution of the shape of the term structure of interest rates (or yield curve). Chapter 2 presented a moment-matching scenario generation approach for generating yield curve scenarios. We proposed two methods, both generating scenarios by matching the principal moments of the underlying distributions of the log changes of the yield curve. These methods generate yield curve scenario trees with path dependent yield curves at intermediate time nodes, where each node in the scenario tree represents the term structure of interest rates (yield curve). The second scenario generation method was preferred to the first, as it consumes less time to generate larger yield curve scenario trees. The scenario generation method was tested and the performance was measured by out-of-sample back-testing in terms of the value of a fixed income portfolio optimisation problem described in the literature. The results demonstrated a reasonably sound way to generate stable yield curve scenario trees for fixed income portfolio optimisation.

Chapter 3 presented a simulation approach which includes macro-economic factors. The chapter considered the estimation and characterisation of the South African term structure with respect to macro-economic factors and its use in scenario generation for fixed income portfolios. We have estimated a yield curve model that incorporates four yield curve factors (level, slope and two curvature factors) and three macro-economic factors (real activity, inflation and the stance of monetary policy). The estimated model fits the term structure reasonably well in-sample as shown in the results. The model also performs reasonably well in out-of-sample forecasting. We have shown that better performance can be realised by including forecasts for the macro-economic factors generated by an economic scenario generator. We also proposed a parallel simulation approach for generating yield curve scenario trees. The procedure was tested and the performance was measured by out-of-sample back-testing in terms of the value of a fixed income portfolio optimisation problem described in the literature. The results demonstrated a reasonably sound way to generate stable yield curve scenario trees. We also discussed the existence of arbitrage in the scenario trees and proposed a method to eliminate arbitrage opportunities.

Furthermore we compared the moment-matching scenario generation method to the macro-economic scenario generation method in terms of back-testing and stability in Chapter 3. From the results it is clear that both scenario generation methods display stability in the objective function and the portfolio allocations. The objective function is more stable when the moment-matching scenarios are used. In terms of back-testing the macro-economic scenarios performs better achieving a higher terminal wealth.

In Chapter 4 and 5 we have presented two stochastic programming frameworks for the asset and liability management of investment products with guarantees. Chapter 4 presented a stochastic programming framework for the asset and liability management of minimum liquid asset portfolios found in the banking environment.

The model allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management actions, such as reinvesting coupons, at intermediate time steps. We have shown that our problem is related to insurance products with guarantees and utilised this in the formulation. We have shown the model features at different levels of importance of the minimum expected average shortfall and minimum liquid asset requirement growth rates. The model performed as expected. This model can also be used when analysing the investment decision made by the financial institution and may play an important role in liquidity management, when concerning different levels of liability growth rates.

Chapter 5 presented a stochastic programming framework for the asset and liability management of insurance products with minimum guarantees, that minimises the down-side risk of these products. We included regular bonus payments and kept the optimisation problem linear, which enabled us to model the rebalancing of the portfolio at future decision times. Also, by keeping the optimisation problem linear, the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments. We have shown that our bonus assumption mimics the bonus payments proposed by Consiglio *et al.* (2006). Furthermore we have shown the model features at different levels of minimum guarantee and different levels of risk-aversion. As Consiglio *et al.* (2006) have shown, the model can also be used for analysing the investment decision made by the insurance firm.

6.2 Future directions

In Chapters 2 and 3 we have presented two methods for the generation of yield curve scenario trees. In both instances we mentioned the existence of arbitrage in the scenarios. For the first moment-matching approach the problem is dealt with easily by either including extra constraints in the scenario generation optimisation approach or by checking *ex post* for arbitrage in the scenarios. If any arbitrage exists, the scenarios are re-generated. For the simulation approach we have presented a method to eliminate the arbitrage opportunities *ex post*. Consideration may be given to other methods in order to exclude arbitrage opportunities either during simulation or removing arbitrage opportunities *ex post*.

Our proposed method in Chapter 2 uses the scenario generation method proposed by Høyland and Wallace (2001b) as apposed to the a second method proposed Høyland *et al.* (2003). The second method proposed by Høyland *et al.* (2003) is in most if not all respects better than the first method introduced by Høyland and

Wallace (2001b). We used the first method for its ability to include decision-makers expected views on the yield curve. Consideration may be given to the Høyland *et al.* (2003) method and possible approaches may be investigated to include the decision-makers' expected views on the yield curve.

In Chapter 3 we have shown that better performance can be realised by including forecasts for the macro-economic factors generated by an economic scenario generator. By lack of forecast data we use the actual repo-rate. Further investigation should be given to this idea and the possible advantages of including such forecasts should be tested on real data.

It was the purpose of Chapter 2 and Chapter 3 to propose suitable methods for yield curve scenario generation and further investigation may be necessary in the different approaches and the mixing thereof.

Chapter 4 presented a stochastic programming framework for the asset and liability management of minimum liquid asset portfolios found in the banking environment. Gold was included as one of the liquid assets in the portfolio. As gold prices are quoted in dollars, the inclusion of gold in the portfolio creates currency risk. Consideration may be given to the inclusion of hedging instruments in order to address this problem. Other hedging instruments may also be considered, such as interest rate caps and floors to deal with interest rate risk. Note however that the inclusion of these derivatives may introduce counterparty credit risk.

In Chapter 5 we have presented a stochastic programming framework for the asset and liability management for insurance products with minimum guarantees. Consideration may be given to the inclusion of other minimum guarantee policy features such as the attrition of policyholders and the inclusion of other types of bonus features whilst still keeping the problem linear.

The with-profits guarantee funds discussed in Chapter 5 is operated by on a 90/10 basis, where policyholder benefits in 90% of the growth in asset share and the share holders 10%. These products do however occur to a lesser extend in the South African insurance market. The main goal of this chapter was to extend on the work of Dempster *et al.* (2006) and Consiglio *et al.* (2006). This however provides wide scope for further research of these products in the South African insurance market.

Appendix A

Minimum guarantee - model formulation

In this appendix we present the multi-period stochastic programming model of Dempster *et al.* (2006) for minimum guarantees. Given a set of scenarios the stochastic program results in a large linear program.

A.1 Variable parameters of the model

The following notation will be used for variables and parameters of the model:

Time sets

$\mathcal{T}^{total} = \left\{0, \frac{1}{12}, \frac{2}{12}, \ldots, T\right\}$	: set of all times considered in the stochastic program;
$\mathcal{T}^{d} = \{0, 1, 2, \ldots, T-1\}$	: set of decision times;
$\mathcal{T}^{i} = \mathcal{T}^{total} \backslash \mathcal{T}^{d}$	: set of intermediate times;
$\mathcal{T}^{c} = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, T-\frac{1}{2}\right\}$	: set of coupon payment time between decision times;

Instruments

$S_t(\omega)$	: Stock index level at time t in scenario ω;
$B_t^T(\omega)$	: Treasury security with maturity T at time t in scenario ω;
$\delta_t^{B^T}(\omega)$	: coupon rate of Treasury security with maturity T at time t in scenario ω;
F^{B^T}	: face value of Treasury security with maturity T;
$Z_t(\omega)$	: zero-coupon Treasury security price at time t in scenario ω;

Risk management barrier

$y_{t,T}(\omega)$	: zero-coupon Treasury yield with maturity T at time t in scenario ω;
G	: annual guaranteed return;
$L_t(\omega)$	: barrier at time t in scenario ω;

Portfolio evolution

A	: set of all assets;
$P_{t,a}^{buy}(\omega)/P_{t,a}^{sell}(\omega)$	: buy/sell price of asset $a \in A$ at time t in scenario ω;
f/g	: transaction costs of buying and selling;
$x_{t,a}(\omega)$	: quantity held of asset $a \in A$ between time t and $t+1$ in scenario ω;
$x_{t,a}^{+}(\omega)/x_{t,a}^{-}(\omega)$	: quantity bought/sold of asset $a \in A$ at time t in scenario ω;
W_t^s	: portfolio wealth at time $t \in T^{total}$ in scenario ω;
$h_t(\omega) :=$ $\max(0, L_t(\omega) - W_t(\omega))$	: shortfall at time t in scenario ω;

A.2 Model formulation

The formulation of the multi-period stochastic programming model of Dempster *et al.* (2006) for minimum guarantees are as follow:

$$\max_{\left\{\substack{x_{t,a}(\omega),x_{t,a}^{+}(\omega),x_{t,a}^{-}(\omega): \\ a\in A,\omega\in\Omega,t\in T^d\cup\{T\}}\right\}} \left\{(1-\alpha)\left(\sum_{\omega\in\Omega} p(\omega) \sum_{t\in T^d\cup\{T\}} W_t(\omega)\right) - \alpha\left(\sum_{\omega\in\Omega} p(\omega) \sum_{t\in T^d\cup\{T\}} \frac{h_t(\omega)}{|T^{total}|}\right)\right\}$$

subjected to:

$\sum_{a\in A} fP_{0,a}^{buy}(\omega)\, x_{0,a}^{+}(\omega) = W_0(\omega)$ for $\omega \in \Omega$,

$\sum_{a\in A\backslash\{S\}} \frac{1}{2}\delta_{t-1}^{a}(\omega)\, F^a x_{t,a}^{-}(\omega) + \sum_{a\in A} gP_{t,a}^{sell}(\omega)\, x_{t,a}^{-}(\omega) = \sum_{a\in A} fP_{t,a}^{buy}(\omega)\, x_{0,a}^{+}(\omega)$ for $\omega \in \Omega, t \in T^d \backslash \{0\}$

$W_t(\omega) = \sum_{a\in A} fP_{t,a}^{buy}(\omega)\, x_{t,a}(\omega)$ for $\omega \in \Omega, t \in T^{total} \backslash \{T\}$

$W_T(\omega) = \sum_{a\in A} gP_{T,a}^{sell}(\omega)\, x_{T-\frac{1}{12},a}(\omega) + \sum_{a\in A\backslash\{S\}} \frac{1}{2}\delta_{T-1}^{a} F^a x_{T-\frac{1}{12},a}(\omega)$ for $\omega \in \Omega$

$x_{0,a}(\omega) = x_{0,a}^{+}(\omega)$ for $a \in A, \omega \in \Omega$

$x_{t,a}(\omega) = x_{t-\frac{1}{12},a}(\omega) + x_{t,a}^{+}(\omega) + x_{t,a}^{-}(\omega)$ for $a \in A, \omega \in \Omega, t \in T^{total} \backslash \{0\}$

$x_{t,a}^{+}(\omega) = x_{t,a}^{-}(\omega) = 0$ for $a \in A, \omega \in \Omega, t \in T^i \backslash T^c$

$x^{+}_{t,a}(\omega) = \frac{\frac{1}{2}\delta^{a}_{t}(\omega)F^{a}x_{t-\frac{1}{12},a}(\omega)}{fP^{buy}_{t,a}(\omega)}$ for $a \in A\backslash\{S\}, \omega \in \Omega, t \in T^{c}$

$x^{-}_{t,a}(\omega) = 0$ for $a \in A\backslash\{S\}, \omega \in \Omega, t \in T^{c}$

$x^{+}_{t,S}(\omega) = x^{-}_{t,S}(\omega) = 0$ for $a \in A\backslash\{S\}, \omega \in \Omega, t \in T^{c}$

$h_t(\omega) + W_t(\omega) \geq L_t(\omega)$ for $\omega \in \Omega, t \in T^{total}$

$h_t(\omega) \geq 0$ for $\omega \in \Omega, t \in T^{total}$

$$L_t(\omega) = W_0(1+G)^T Z_t(\omega)$$

$$= W_0(1+G)^T e^{-y_{t,T}(\omega)(T-t)} \text{ for } \omega \in \Omega, t \in T^{total}$$

$0 \leq \alpha \leq 1$

Non-anticipatively constraints

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