



A study of a generalized potential Yu-Toda-Sasa-Fukuyama and fifth-order nonlinear partial differential equations

OAGENG LAWRENCE DITEHO

 **orcid.org/0000-0003-3857-1901**

Dissertation submitted in fulfilment of the requirements for the degree *Master of Science in Applied Mathematics* at the North-West University

Supervisor: Prof CM Khalique

Graduation: May 2023

Student number: 25391453

Contents

Declaration	iv
Dedication	vi
Acknowledgements	vii
Abstract	viii
Introduction	1
1 Preliminaries	3
1.1 Introduction	3
1.2 Differential functions	3
1.3 Group admitted by a system of PDEs	4
1.4 Euler-Lagrange operator	5
1.5 Conservation laws	5
1.5.1 Noether's approach	5
1.5.2 The multiplier approach	6
1.6 Description of Kudryashov's method	6
1.7 Conclusions	7
2 Solutions and conservation laws of BBM equation: an illustrative example	8

2.1	Introduction	8
2.2	Lie symmetry analysis of (2.1)	9
2.2.1	Lie point symmetries	9
2.2.2	Commutator relations	13
2.2.3	One-parameter group of transformations	14
2.2.4	Group-invariant solutions	14
2.3	Conservation laws of the BBM equation	18
2.3.1	Conservation laws using the multiplier method	18
2.3.2	Conservation laws using Noether's Theorem	24
2.4	Concluding remarks	27
3	Solutions and conservation laws of the (3+1)-dimensional general- ized potential Yu-Toda-Sasa-Fukuyama equation	28
3.1	Introduction	28
3.2	Solutions of the 3DgYTSF equation (3.2)	30
3.2.1	Lie point symmetries of (3.2)	30
3.2.2	One-parameter group of transformations	35
3.2.3	Invariant solutions under the symmetries $X_1, \dots, X_4$	37
3.2.4	Solution via the incomplete elliptic integral	38
3.2.5	Symmetry reductions under the symmetries X_7 and X_8	40
3.3	Conservation laws of (3.2)	43
3.3.1	Conservation laws using the multiplier approach	43
3.3.2	Conservation laws via the Noether's approach	47
3.4	Conclusion	51

4	Conserved quantities and solutions of a generalized fifth-order KdV equation	52
4.1	Introduction	52
4.2	Symmetry reductions and solutions	53
4.2.1	Point symmetries of (4.1)	53
4.2.2	Commutation table	57
4.2.3	One-parameter group of transformations	57
4.2.4	Construction of invariant solutions	58
4.3	Soliton solutions	62
4.3.1	Solitary waves	62
4.3.2	Shock waves	63
4.3.3	Singular solitary waves (Type-I)	64
4.3.4	Singular solitary waves (Type-II)	65
4.4	Conservation laws of (4.1)	66
4.4.1	Conservation laws using the multiplier approach	66
4.4.2	Conservation laws using Noether's approach	71
4.5	Conclusion	74
5	Concluding remarks and future work	75

Declaration

I OAGENG LAWRENCE DITEHO, student number 25391453, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed:

MR. OAGENG LAWRENCE DITEHO

Date:

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Master of Science degree rules and regulations have been fulfilled.

Signed:.....

PROF C.M. KHALIQUE

Date:

Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 3

C.M. Khalique, K. Plaatjie, O.L. Diteho, Symmetry solutions and conservation laws for the 3D generalized potential Yu-Toda-Sasa-Fukuyama equation of mathematical physics, *Symmetry*, (2021) 13, 2058.

Chapter 4

C.M. Khalique, O.L. Diteho, Conserved quantities and closed-form solutions of a generalized fifth-order KdV equation, to be submitted.

Dedication

To my family

Acknowledgements

Firstly, I would like to thank my supervisor Prof CM Khalique for his encouragement and guidance throughout my studies. I also thank Dr K Plaatjie for all the help he provided me with regards to my MSc studies. I would also like to thank both of them for their fruitful discussions, suggestions and comments. I am thankful to the North-West University for much needed resources. Furthermore, I acknowledge the never ending love and support from the Diteho's and Nthaudi's families. Last but not least, to my God who is my sustainer and enabler, I owe my all.

Abstract

In this work we study two nonlinear partial differential equations which arise in mathematical physics. These equations have been studied in the literature. However, we start by considering a second-order nonlinear Benjamin-Bona-Mahony equation as an illustrative example. We then focus on the (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation and the generalized fifth-order Korteweg-de Vries equation. Lie group analysis is used to construct exact solutions of these equations. Furthermore, conservation laws are derived for the underlying equations using the multiplier method and Noether's method.

Introduction

A lot of natural phenomena are modelled by nonlinear partial differential equations (NLPDEs). Several studies have been conducted for such equations in order to acquire solutions that make it easier to comprehend the real world problems. Exact solutions for these types of equations have been explored by many researchers. The development of numerous methods for computing exact solutions of NLPDEs has been archived. Some of these methods are Jacobi elliptic function expansion method [1], the homogeneous balance method [2], the Kudryashov's method [3], the ansatz method [4], the inverse scattering transform method [5], the Bäcklund transformation [6], the Darboux transformation [7], the Hirota bilinear method [8], the (G'/G) -expansion method [9] and the Lie symmetry method [10–15].

The Norwegian mathematician Marius Sophus Lie (1844-1899) invented a significant symmetry-based technique in order to solve differential equations that is known today as Lie group analysis. It was possible to acquire exact solutions of differential equations because of this method. Publications of vigorous research have been made by a lot of researchers based on Lie's work [10–15]. A procedure was introduced by a German mathematician Emmy Noether (188-1935) for emanating conservation laws for systems of differential equations that admit variational principle and this procedure is referred to as Noether's theorem [16]. Lagrangian is a requirement for a given differential equation system to have in order to admit a variational principle.

A significant role is played by conservation laws in the study of differential equations. They set out physical conserved quantities, for instance mass, energy, momentum, charge and other constants of motion. In addition, they are influential for the in-

vestigation of integrability and extraordinary solutions. See for example [16–24] and references therein.

The outline of this dissertation is as follows:

In chapter one important preliminaries are outlined with regards to Lie’s theory, Kudryashov’s method, Noether’s theorem and the multiplier approach.

In chapter two illustrations are presented of the main objectives of this research project. The one-dimensional Benjamin-Bona-Mahony equation is considered and Lie’s theory is used to obtain its group-invariant solutions. Henceforth, multiplier method is applied to extract its conservation laws.

In chapter three Lie point symmetries for the (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation are worked out and group-invariant solutions are obtained. Travelling wave solutions are constructed by applying Kudryashov’s and extended Jacobi elliptic function expansion methods. Lastly, the multiplier approach and Noether’s theorem is used to derive conservation laws.

In chapter four we study a generalized fifth-order Korteweg-de Vries equation. We compute symmetries of this equation and then utilize them to perform symmetry reductions. Also we find soliton solutions and calculate the conservation laws of the equation using two techniques, namely the multiplier method and Noether’s theorem.

In chapter five we present concluding remarks and give possibilities of some future work.

Bibliography is given at the end of the dissertation.

Chapter 1

Preliminaries

In this chapter, we present a short summary of theory of Lie symmetry analysis that is relevant to our work. This includes the procedure to calculate Lie symmetries and invariant solutions of PDEs. Moreover, we give certain methods for deriving conservation laws and constructing exact solutions of PDEs.

1.1 Introduction

Lie symmetry method is completely algorithmic and efficient technique in finding exact solutions of PDEs. It was introduced by the eminent Norwegian scholar Sophus Lie (1842-1899) in the late nineteenth century. Lie took great inspiration from the works of Galois (1811-1832), who pioneered Galois theory and group theory. Many books have appeared in the literature recently, which include [10–14].

1.2 Differential functions

Let $u = (u^1, \dots, u^m)$ be the dependent variable, which is a function of the independent variable $x = (x^1, \dots, x^n)$. The derivatives of u^α relative to x^i are

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(u_i^\alpha) = D_j D_i(u^\alpha), \quad (1.1)$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \dots, n, \quad (1.2)$$

is the total derivative operator relative to x^i . The quantities

$$x = \{x^i\}, \quad u = \{u^\alpha\}, \quad u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \quad \cdots$$

are treated as independent variables, where $u_{(r)}$ denotes set of all r th-order derivatives, $r \geq 1$. We denote by $\mathcal{A}$, the vector space of all differential functions of the form $f(x, u, u_{(1)}, \dots, u_{(r)})$ [14].

1.3 Group admitted by a system of PDEs

Contemplate a k^{th} -order system of DEs with n independent and N dependent variables

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, N. \quad (1.3)$$

System (1.3) is invariant under

$$\mathbf{X} = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (1.4)$$

the generator of a group, if and only if

$$\mathbf{X}^{[k]} E_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)})|_{(1.3)} = 0, \quad \alpha = 1, \dots, N \quad (1.5)$$

Here $\mathbf{X}^{[k]}$ denotes the k th prolongation of $\mathbf{X}$ and is given by

$$\mathbf{X}^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1 i_2 \dots i_k}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha}, \quad (1.6)$$

where $\zeta_{i_1 \dots i_s}^\alpha$ are defined by

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (1.7)$$

$$\zeta_{i_1 \dots i_s}^\alpha = D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\alpha) - u_{j_1 \dots j_{s-1}}^\alpha D_{i_s}(\xi^j), \quad s > 1. \quad (1.8)$$

The generator (1.4) of the symmetry group is called infinitesimal symmetry of (1.3). Equations (1.5) are known as determining equations of all symmetries of (1.3).

1.4 Euler-Lagrange operator

The Euler operator is specified as

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \quad (1.9)$$

whereas Lie-Bäcklund operator is

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_s}^\alpha}. \quad (1.10)$$

1.5 Conservation laws

The m -tuple $C = (C^1, C^2, \dots, C^m)$, $C^k \in \mathcal{A}$, $k = 1, 2, \dots, m$, is said to be a conserved vector of (1.3) provided

$$D_k T^k = 0 \quad (1.11)$$

holds for all solutions of (1.3). We call (1.11) a local conservation law.

1.5.1 Noether's approach

If the system (1.3) is equivalent to

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, N, \quad (1.12)$$

for some $\mathcal{L}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)}) \in \mathcal{A}$, $s \leq k$, then we say that $\mathcal{L}$ is a Lagrangian of system (1.3) and (1.12) are the Euler-Lagrange DEs.

The operator $\mathbf{X}$ in (1.10) is said to be a Noether symmetry generator corresponding to a Lagrangian $\mathcal{L}$ of Euler-Lagrange equations if

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(G^i) \quad (1.13)$$

for some gauge function $G = (G^1, G^2, \dots, G^n)$.

For each Noether symmetry N corresponding to a Lagrangian $\mathcal{L}$ there corresponds a vector $C = (C^1, C^2, \dots, C^n)$, known as Noether conserved vector, defined by

$$C^i = G^i - \xi^i \mathcal{L} - W^\alpha \frac{\delta \mathcal{L}}{\delta u_i^\alpha} - \sum_{s \geq 1} D_{i_1 \dots i_s}(W^\alpha) \frac{\delta \mathcal{L}}{\delta u_{i_1 \dots i_s}^\alpha}, \quad (1.14)$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2, \dots, N$, are characteristic functions.

Noether conserved vector C can also be written as

$$\begin{aligned} C^k = & \xi^i \mathcal{L} + (\eta - u_{x^j} \xi^j) \left\{ \frac{\partial \mathcal{L}}{\partial u_{x^k}} - \sum_{l=k}^n D_{x^i} \left(\frac{\partial \mathcal{L}}{\partial u_{x^j x^k}} \right) \right\} \\ & + \sum_{l=k}^n \{ \zeta_l - u_{x^l x^j} \xi^j \} \frac{\partial \mathcal{L}}{\partial u_{x^k x^l}} - G^i. \end{aligned} \quad (1.15)$$

1.5.2 The multiplier approach

This approach involves the variational derivative. A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ satisfies the equation [12]

$$\Lambda_\alpha E_\alpha = D_i T^i \quad (1.16)$$

identically. The multipliers Λ_α are determined using

$$\frac{\delta}{\delta u^\beta} (\Lambda_\alpha E_\alpha) = 0, \quad (1.17)$$

which holds for functions $u(x^1, x^2, \dots, x^n)$. Once we get the multipliers from (1.17), we may compute conserved vectors by invoking equation (1.16).

1.6 Description of Kudryashov's method

This technique for determining exact solutions of PDEs was introduced by the Russian mathematician N. A. Kudryashov [3].

Let us consider the NLPDE

$$\mathcal{F}(t, x, q, q_t, q_x, q_{tt}, q_{xx}, \dots) = 0. \quad (1.18)$$

We now present the narration of Kudryashov's technique:

Step 1. Let $q(x, t) = W(\zeta)$, $\zeta = kx + \omega t$, where k and ω are constants. The substitution of this in (1.18), reduces it to the ODE

$$\mathcal{E}(W, \omega W', kW', \omega^2 W'', k^2 W'', \dots) = 0. \quad (1.19)$$

Step 2. Suppose solution of (1.19) can be written as

$$W(\zeta) = \sum_{k=0}^N a_k R^k(\zeta), \quad (1.20)$$

where a_k ($k = 0, 1, \dots, N$) are unknown constants to be found, such that $a_N \neq 0$, and $R(\zeta)$ solves

$$R'(\zeta) = R^2(\zeta) - R(\zeta). \quad (1.21)$$

We recall that (1.21) has the solution

$$R(\zeta) = \frac{1}{1 + \exp(\zeta)}. \quad (1.22)$$

Step 3. Next we insert the value of $W(\zeta)$ into (1.19) and invoke (1.21) to get an equation that involves powers of R .

Step 4. Equating unlike powers of R to zero, we acquire the system of algebraic equations

$$P_k(a_N, a_{N-1}, \dots, a_0, k, \omega, \dots) = 0, \quad (k = 0, \dots, N). \quad (1.23)$$

Step 5. Finally, solution of system (1.23) yields the values of $a_0, a_1, \dots, a_{N-1}, a_N$ and connections between the parameters of (1.19). Consequently, the exact solutions of (1.19) in the form (1.20) is obtained.

1.7 Conclusions

In this chapter we presented some basic facts of Lie symmetry analysis. Furthermore, two methods for deriving conservation laws were given. These methods were Noether's theorem and the multiplier method. The description of Kudryashov's method for determining exact solutions of PDEs was recalled. The material presented in this chapter shall be employed throughout this work.

Chapter 2

Solutions and conservation laws of BBM equation: an illustrative example

In this chapter we study a one-dimensional Benjamin-Bona-Mahony equation [25] to illustrate what will be done in this research project. We start by computing its symmetries and perform symmetry reductions. Thereafter conservation laws for this equation will be derived by utilizing Noether's theorem and the multiplier method.

2.1 Introduction

Benjamin et al. [25] investigated the regularized long-wave equation, commonly known as the Benjamin-Bona-Mahony (BBM) equation, for the first time as a regularized version of the Korteweg-de-Vries (KdV) equation [26]. It was also obtained as an approximation for surface water waves in a homogeneous channel for shallow water waves. The BBM equation may be used to investigate drift waves in plasma or Rossby waves in rotating fluids in addition to shallow water waves [27]. It also gives a model of one-dimensional transmitted waves, the BBM equation is not an evolution equation in the strict sense, but it is nevertheless regarded as nonlinear partial dif-

ferential equation (PDE) of evolution type. When examining the dispersion relation for the corresponding linearized equation, the fundamental mathematical difference between KdV and BBM models is most easily observed. In this chapter we study the BBM equation

$$u_t + u_x + uu_x - u_{txx} = 0, \quad (2.1)$$

2.2 Lie symmetry analysis of (2.1)

2.2.1 Lie point symmetries

We derive all Lie point symmetries for the BBM equation (2.1). Equation (2.1) admits the Lie group of transformations with infinitesimal generator

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.2)$$

whenever

$$X^{[3]}(u_t + u_x + uu_x - u_{txx}) \Big|_{(2.1)} = 0. \quad (2.3)$$

With $X^{[3]}$ being the third extension of X and is given by

$$X^{[3]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{122} \frac{\partial}{\partial u_{txx}},$$

where the coefficients ζ s are written as

$$\zeta_1 = \eta_t + u_t \eta_u - u_t^2 \tau_u - u_t \tau_t - u_t u_x \xi_u - u_x \xi_t,$$

$$\zeta_2 = \eta_x + u_x \eta_u - u_t u_x \tau_u - u_t \tau_x - u_x \xi_x - u_x^2 \xi_u,$$

$$\begin{aligned} \zeta_{122} = & \eta_{txx} + u_{txx} \eta_u + u_{xx} \eta_{tu} + 2u_x u_{tx} \eta_{uu} + u_t u_x^2 \eta_{uuu} + u_t \eta_{xxu} + 2u_x \eta_{txu} + 2u_{tx} \eta_{xu} \\ & + u_{xx} u_t \eta_{uu} + 2u_t u_x \eta_{xuu} - u_t u_x^3 \xi_{uuu} - u_x^3 \xi_{tuu} - 3u_{tx} u_x^2 \xi_{uu} - u_{tt} u_x^2 \tau_{uu} - u_t^2 u_x^2 \tau_{uuu} \\ & + u_x^2 \eta_{tuu} - 2u_t u_x^2 \xi_{xuu} - u_t u_x^2 \tau_{tuu} - 2u_x^2 \xi_{txu} - 3u_{txx} u_x \xi_u - 2u_{ttx} u_x \tau_u - 3u_{xx} u_t u_x \xi_{uu} \\ & - 4u_t u_{tx} u_x \tau_{uu} - 4u_{tx} u_x \xi_{xu} - 2u_{tt} u_x \tau_{xu} - 2u_t^2 u_x \tau_{xuu} - u_t u_x \xi_{xxu} - 3u_{xx} u_x \xi_{tu} \\ & - 2u_{tx} u_x \tau_{tu} - 2u_t u_x \tau_{txu} - u_x \xi_{txx} - u_{xxx} u_t \xi_u - 3u_{xx} u_{tx} \xi_u - 2u_{tx}^2 \tau_u - 2u_t u_{txx} \tau_u \end{aligned}$$

$$\begin{aligned}
& -u_{xx}u_{tt}\tau_u - u_{xx}u_t^2\tau_{uu} - 2u_{txx}\xi_x - 2u_{ttx}\tau_x - 2u_{xx}u_t\xi_{xu} - 4u_tu_{tx}\tau_{xu} - u_{tx}\xi_{xx} \\
& - u_{tt}\tau_{xx} - u_t^2\tau_{xxu} - u_{xxx}\xi_t - u_{txx}\tau_t - u_{xx}u_t\tau_{tu} - 2u_{xx}\xi_{tx} - 2u_{tx}\tau_{tx} - u_t\tau_{txx}.
\end{aligned}$$

Expanding (2.3) gives

$$u_x\eta + \zeta_1 + \zeta_2 + u\zeta_2 - \zeta_{122}\Big|_{u_{txx}=u_t+u_x+uu_x} = 0. \quad (2.4)$$

Replacing the ζ s by their respective values in the above equation we obtain

$$\begin{aligned}
& \eta u_x + \eta_t + u_t\eta_u - u_t^2\tau_u - u_t\tau_t - u_tu_x\xi_u - u_x\xi_t + \eta_x + u_x\eta_u - u_tu_x\tau_u - u_t\tau_x - u_x\xi_x \\
& - u_x^2\xi_u + u\{\eta_x + u_x\eta_u - u_tu_x\tau_u - u_t\tau_x - u_x\xi_x - u_x^2\xi_u\} - \eta_{txx} - u_{txx}\eta_u - u_{xx}\eta_{tu} \\
& - 2u_xu_{tx}\eta_{uu} - u_tu_x^2\eta_{uuu} - u_t\eta_{xxu} - 2u_x\eta_{txu} - 2u_{tx}\eta_{xu} - u_{xx}u_t\eta_{uu} - 2u_tu_x\eta_{xuu} \\
& + u_x^3\xi_{tuu} + u_tu_x^3\xi_{uuu} + 3u_{tx}u_x^2\xi_{uu} + u_{tt}u_x^2\tau_{uu} + u_t^2u_x^2\tau_{uuu} - u_x^2\eta_{tuu} + 2u_tu_x^2\xi_{xuu} \\
& + u_tu_x^2\tau_{tuu} + 2u_x^2\xi_{txu} + 3u_{txx}u_x\xi_u + 2u_{ttx}u_x\tau_u + 3u_{xx}u_tu_x\xi_{uu} + 4u_tu_{tx}u_x\tau_{uu} \\
& + 4u_{tx}u_x\xi_{xu} + 2u_{tt}u_x\tau_{xu} + 2u_t^2u_x\tau_{xuu} + u_tu_x\xi_{xxu} + 3u_{xx}u_x\xi_{tu} + 2u_{tx}u_x\tau_{tu} \\
& + 2u_tu_x\tau_{txu} + u_x\xi_{txx} + u_{xxx}u_t\xi_u + 3u_{xx}u_{tx}\xi_u + 2u_t^2\tau_u + 2u_tu_{txx}\tau_u + u_{xx}u_{tt}\tau_u \\
& + u_{xx}u_t^2\tau_{uu} + 2u_{txx}\xi_x + 2u_{ttx}\tau_x + 2u_{xx}u_t\xi_{xu} + 4u_tu_{tx}\tau_{xu} + u_{tx}\xi_{xx} + u_{tt}\tau_{xx} \\
& + u_t^2\tau_{xxu} + u_{xxx}\xi_t + u_{txx}\tau_t + u_{xx}u_t\tau_{tu} + 2u_{xx}\xi_{tx} + 2u_{tx}\tau_{tx} + u_t\tau_{txx}\Big|_{u_{txx}=u_t+u_x+uu_x} = 0.
\end{aligned}$$

Substituting u_{txx} by $u_t + u_x + uu_x$ in the above equation we obtain

$$\begin{aligned}
& \eta u_x + \eta_t + u_t\eta_u - u_t^2\tau_u - u_t\tau_t - u_tu_x\xi_u - u_x\xi_t + \eta_x + u_x\eta_u - u_tu_x\tau_u - u_t\tau_x - u_x\xi_x \\
& - u_x^2\xi_u + u\{\eta_x + u_x\eta_u - u_tu_x\tau_u - u_t\tau_x - u_x\xi_x - u_x^2\xi_u\} - \eta_{txx} - \{u_t + u_x + uu_x\}\eta_u \\
& - u_{xx}\eta_{tu} - 2u_xu_{tx}\eta_{uu} - u_tu_x^2\eta_{uuu} - u_t\eta_{xxu} - 2u_x\eta_{txu} - 2u_{tx}\eta_{xu} - u_{xx}u_t\eta_{uu} - 2u_tu_x\eta_{xuu} \\
& + u_x^3\xi_{tuu} + 3u_{tx}u_x^2\xi_{uu} + u_{tt}u_x^2\tau_{uu} + u_t^2u_x^2\tau_{uuu} - u_x^2\eta_{tuu} + 2u_tu_x^2\xi_{xuu} + u_tu_x^2\tau_{tuu} + 2u_x^2\xi_{txu} \\
& + u_tu_x^3\xi_{uuu} + 3u_x\xi_u\{u_t + u_x + uu_x\} + 2u_{ttx}u_x\tau_u + 3u_{xx}u_tu_x\xi_{uu} + 4u_tu_{tx}u_x\tau_{uu} + 4u_{tx}u_x\xi_{xu} \\
& + 2u_{tt}u_x\tau_{xu} + 2u_t^2u_x\tau_{xuu} + u_tu_x\xi_{xxu} + 3u_{xx}u_x\xi_{tu} + 2u_{tx}u_x\tau_{tu} + 2u_tu_x\tau_{txu} + u_x\xi_{txx} \\
& + u_{xxx}u_t\xi_u + 3u_{xx}u_{tx}\xi_u + 2u_{tx}^2\tau_u + 2u_t\tau_u\{u_t + u_x + uu_x\} + u_{xx}u_{tt}\tau_u + u_{xx}u_t^2\tau_{uu} + 2u_t\xi_x \\
& + 2u_x\xi_x + 2uu_x\xi_x + 2u_{ttx}\tau_x + 2u_{xx}u_t\xi_{xu} + 4u_tu_{tx}\tau_{xu} + u_{tx}\xi_{xx} + u_{tt}\tau_{xx} + u_t^2\tau_{xxu} + u_{xxx}\xi_t \\
& + u_t\tau_t + u_x\tau_t + uu_x\tau_t + u_{xx}u_t\tau_{tu} + 2u_{xx}\xi_{tx} + 2u_{tx}\tau_{tx} + u_t\tau_{txx} = 0. \quad (2.5)
\end{aligned}$$

The functions (τ, ξ, η) depend on t, x and u only, thus we split (2.5) on various derivatives of u and this gives

$$\tau_x = 0, \tau_u = 0, \quad (2.6)$$

$$\xi_t = 0, \xi_u = 0, \quad (2.7)$$

$$\xi_{xxx} - 4\xi_x = 0, \quad (2.8)$$

$$\eta_{uu} = 0, \quad (2.9)$$

$$\eta_{tu} = 0, \quad (2.10)$$

$$2\eta_{xu} - \xi_{xx} = 0, \quad (2.11)$$

$$\eta + (u + 1)(\tau_t + \xi_x) = 0, \quad (2.12)$$

$$\eta_{txx} - \eta_t - (u + 1)\eta_x = 0. \quad (2.13)$$

Equations (2.6) imply that

$$\tau = A(t), \quad (2.14)$$

where $A(t)$ is an arbitrary function of t . Integrating (2.7) gives

$$\xi = B(x), \quad (2.15)$$

where $B(x)$ is an arbitrary function of x . Solving equation (2.9) we get

$$\eta = C(t, x)u + D(t, x),$$

where C and D are functions of t and x . Substituting the above value of η into equation (2.10), we obtain $C_t = 0$ and this solves to $C = C(x)$, hence

$$\eta = C(x)u + D(t, x). \quad (2.16)$$

Now substituting the values of ξ and η into equation (2.11), we get

$$2C'(x) - B''(x) = 0. \quad (2.17)$$

Solving equation (2.17) for C , we obtain

$$C(x) = \frac{1}{2}B'(x) + c_1, \quad (2.18)$$

where c_1 is a constant of integration. Thus, the value of η becomes

$$\eta = \left(\frac{1}{2} B'(x) + c_1 \right) u + D(t, x). \quad (2.19)$$

Using equation (2.13), we get

$$D_{txx} - D_t - \frac{1}{2} u^2 B''(x) - u D_x - \frac{1}{2} u B''(x) - D_x = 0. \quad (2.20)$$

Now splitting equation (2.20) on powers of u we obtain

$$u^2 : B''(x) = 0, \quad (2.21)$$

$$u : D_x + \frac{1}{2} B''(x) = 0, \quad (2.22)$$

$$u^0 : D_{txx} - D_t - D_x = 0. \quad (2.23)$$

From (2.21) we have

$$B(x) = c_2 x + c_3 \quad (2.24)$$

where c_2 and c_3 are arbitrary constants. Equations (2.22) and (2.23) simplify and gives $D = c_4$, where c_4 is an integration constant. Thus we obtain the following

$$\begin{aligned} \xi &= c_2 x + c_3, \\ \eta &= \left(\frac{1}{2} c_2 + c_1 \right) u + c_4. \end{aligned}$$

Substituting the value of ξ into equation (2.8) leads to $c_2 = 0$ and this simplifies ξ, η to

$$\xi = c_3 \quad (2.25)$$

$$\eta = c_1 u + c_4 \quad (2.26)$$

Similarly substituting the values of τ, ξ, η into equation (2.12) gives

$$c_1 u + c_4 + (u + 1) A'(t) = 0. \quad (2.27)$$

The equation (2.27) is independent of u , thus we split on powers of u and get

$$u : A'(t) + c_1 = 0, \quad (2.28)$$

$$\text{Rest} : A'(t) + c_4 = 0. \quad (2.29)$$

We note that equations (2.28) and (2.29) are only possible if $c_4 = c_1$ and hence integrating we get $A(t) = c_5 - c_1 t$. Thus we get the infinitesimals

$$\tau = c_5 - c_1 t,$$

$$\xi = c_3,$$

$$\eta = c_1 u + c_1.$$

Hence the Lie point symmetries of the BBM equation (2.1) are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= t \frac{\partial}{\partial t} - (1 + u) \frac{\partial}{\partial u}. \end{aligned} \quad (2.30)$$

2.2.2 Commutator relations

We now calculate the commutation relations for the symmetries (2.30). We start by computing $[X_1, X_3]$ using the definition of Lie bracket. We have

$$\begin{aligned} [X_1, X_3] &= X_1 X_3 - X_3 X_1 \\ &= \frac{\partial}{\partial t} \left(t \frac{\partial}{\partial t} - (1 + u) \frac{\partial}{\partial u} \right) - \left(t \frac{\partial}{\partial t} - (1 + u) \frac{\partial}{\partial u} \right) \frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial t} \\ &= X_1. \end{aligned}$$

Likewise, one can compute all the remaining commutation relations using the above procedure. The table below shows the commutation relations in table form.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	0
X_3	$-X_1$	0	0

2.2.3 One-parameter group of transformations

In this subsection we use the Lie equations [13] along with initial conditions

$$\begin{aligned}\frac{d\bar{t}}{da} &= \tau(\bar{t}, \bar{x}, \bar{u}), \quad \bar{t}|_{a=0} = t, & \frac{d\bar{x}}{da} &= \xi(\bar{t}, \bar{x}, \bar{u}), \quad \bar{x}|_{a=0} = x, \\ \frac{d\bar{u}}{da} &= \eta(\bar{t}, \bar{x}, \bar{u}), \quad \bar{u}|_{a=0} = u,\end{aligned}$$

to compute group of transformations that correspond to the Lie symmetries (2.30).

We begin by considering the symmetry X_3 , namely,

$$X_3 = t \frac{\partial}{\partial t} - (1 + u) \frac{\partial}{\partial u}.$$

Now using the above definition of Lie equations on X_3 , we have

$$\frac{d\bar{t}}{da} = \bar{t}, \quad \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = 0, \quad \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = -(1 + \bar{u}), \quad \bar{u}|_{a=0} = u.$$

Solving these equations and using their corresponding initial conditions, gives

$$\bar{t} = te^{a_3}, \quad \bar{x} = x, \quad \bar{u} = (1 + u)e^{-a_3} - 1.$$

Thus, the group of transformation T_{a_3} , corresponding to X_3 is

$$T_{a_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow \{te^{a_3}, x, (1 + u)e^{-a_3} - 1\}.$$

Proceeding in the same manner as above, we obtain group of transformation groups for the remaining operators:

$$T_{a_1} : (\bar{x}, \bar{t}, \bar{u}) \longrightarrow (t + a_1, x, u),$$

$$T_{a_2} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u),$$

$$T_{a_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{a_3}, x, (1 + u)e^{-a_3} - 1).$$

2.2.4 Group-invariant solutions

Given a point symmetry

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.31)$$

of the BBM equation (2.1), the group-invariant solutions under the one-parameter group generated by X are obtained as follows: we start by calculating two linearly independent invariants

$$J_1 = \phi(t, x), \quad J_2 = \psi(t, x, u)$$

by solving the first-order quasi-linear PDE

$$X J \equiv \tau(t, x, u) \frac{\partial J}{\partial x} + \xi(t, x, u) \frac{\partial J}{\partial t} + \eta(t, x, u) \frac{\partial J}{\partial u} = 0.$$

Then, we express one invariant as a function of the other, viz,

$$J_2 = f(J_1) \tag{2.32}$$

and solve (2.32) for the dependent variable u . Lastly, we substitute the expression of u into equation (2.1) and this gives an ODE or a PDE for the unknown function f which may or may not be solved.

We now use the above explained procedure to construct group-invariant solutions of the BBM equation (2.1).

Case X_1 .

We firstly consider the X_1 and find its group-invariant solution. This symmetry has the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

and solving for invariants one obtains

$$J_1 = x \quad \text{and} \quad J_2 = u.$$

Thus the group-invariant solution is $J_2 = f(J_1)$. This implies that $u = f(x)$, where f is an arbitrary function. Substituting this value of u into (2.1) we obtain the first-order (ODE)

$$f'(x) + f(x)f'(x) = 0,$$

whose solution is $f(x) = C_1$, where C_1 is an arbitrary constant of integration. Thus

$$u(t, x) = C_1.$$

Case X_2 .

Next the Lie symmetry X_2 , namely,

$$X_2 = \frac{\partial}{\partial x}$$

and obtain a group-invariant solution under X_2 . The characteristic equations associated with X_2 are:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}.$$

Thus, we get the two invariants

$$J_1 = t, \text{ and } J_2 = u.$$

Therefore, the invariant solution is written as $J_2 = \Psi(J_1)$, that is $u = \Psi(t)$, where Ψ is an arbitrary function. By differentiating u with respect to x and t , we get

$$u_t = \Psi'(t), \quad u_x = 0, \quad u_{xx} = 0.$$

Substituting these expressions into (2.1) we obtain the first-order ODE

$$\psi'(t) = 0,$$

which on integration gives $\psi(t) = C_2$, where C_2 is an arbitrary constant. Thus the group-invariant solution of the BBM equation (2.1) under X_2 is also

$$u(t, x) = C_2.$$

Case X_3 .

We now use the Lie point symmetry X_3 , i.e.,

$$X_3 = t \frac{\partial}{\partial t} - (1 + u) \frac{\partial}{\partial u}.$$

The characteristic equations for this symmetry are

$$\frac{dt}{t} = \frac{dx}{0} = \frac{du}{-(1+u)}$$

and they yield the invariants $J_1 = x$ and $J_2 = t(1+u)$. Consequently, the invariant solution of the BBM (2.1) under this symmetry is $J_2 = \Phi(J_1)$, where Φ is an arbitrary function. This implies that

$$u(t, x) = \frac{1}{t}\phi(x) - 1. \quad (2.33)$$

Differentiating (2.33) with respect to x and t we get

$$u_t = -\frac{1}{t^2}\phi(x), \quad u_x = \frac{1}{t}\phi'(x), \quad u_{tx} = -\frac{1}{t^2}\phi'(x), \quad u_{txx} = -\frac{1}{t^2}\phi''(x). \quad (2.34)$$

Substituting (2.34) into (2.1) yields a second-order ODE

$$\phi'' + \phi'\phi - \phi = 0. \quad (2.35)$$

Using Mathematica we obtain the solution for the second-order ODE (2.35) as

$$\int_1^{\phi(x)} \frac{1}{W\left(e^{-\frac{z^2}{2}+c_1-1}\right) + 1} dz = x + c_1, \quad (2.36)$$

where W is the ProductLog [28] and c_1, c_2 are constants of integration. Hence

$$u(t, x) = \frac{1}{t}\phi(x) - 1$$

is the solution of BBM equation (2.1) under the symmetry X_3 , where ϕ satisfies (2.36).

Case $X_1 + cX_2$. Travelling wave solutions

We now use the linear combination of the symmetries X_1 and X_2 and determine travelling wave solutions. Solving the characteristic equations associated with the symmetry $X_1 + cX_2$ we get

$$J_1 = x - ct, \quad J_2 = u$$

and the group-invariant solution $u = U(x - ct)$. Inserting this into equation (2.1) gives the nonlinear ODE

$$cU'''(\xi) + U(\xi)U'(\xi) + (1-c)U'(\xi) = 0, \quad \xi = x - ct. \quad (2.37)$$

Integrating the ODE (2.37) we obtain

$$cU'' + \frac{1}{2}U^2 + (1-c)U + C_3 = 0, \quad (2.38)$$

where C_3 is an arbitrary constant. Multiplying the ODE (2.38) by U' and integrating again gives the ODE

$$U'^2 + \frac{1}{3c}U^3 + \frac{1}{c}(1-c)U^2 + \frac{2C_3}{c}U + \frac{2C_4}{c} = 0, \quad (2.39)$$

where C_4 is an integration constant. We assume that the cubic function.

$$U^3 + 3(1-C)U^2 + 6C_3U + 6C_4 = 0, \quad (2.40)$$

has the real roots a_1, a_2, a_3 such that $a_1 > a_2 > a_3$, then (2.39) becomes

$$U'^2 = \frac{1}{3}(U - a_1)(U - a_2)(U - a_3) \quad (2.41)$$

whose general solution is [29, 30]

$$U(\xi) = a_2 + (a_1 - a_2)\text{cn}^2 \left\{ \sqrt{\frac{a_1 - a_3}{12c}} \xi | G^2 \right\}, \quad G^2 = \frac{a_1 - a_2}{a_1 - a_3}, \quad (2.42)$$

where cn is the Jacobi cosine function and $\xi = x - ct$. Thus, reverting to the original variables we get the solution of the BBM equation(2.1) as

$$u(t, x) = a_2 + (a_1 - a_2)\text{cn}^2 \left\{ \sqrt{\frac{a_1 - a_3}{12c}} (x - ct) | G^2 \right\}. \quad (2.43)$$

2.3 Conservation laws of the BBM equation

In this section we find conservation laws of the BBM equation (2.1) by using the two techniques, namely, the multiplier method and Noether's theorem.

2.3.1 Conservation laws using the multiplier method

In this section we employ the multiplier method to derive the conservation laws of the BBM equation (2.1). We look for the zeroth-order multiplier $\Lambda = \Lambda(t, x, u)$. The determining equation for the multiplier is given by

$$\frac{\delta}{\delta u} [\Lambda(t, x, u) \{u_t + u_x + uu_x - u_{txx}\}] = 0, \quad (2.44)$$

where $\delta/\delta u$ is the Euler operator given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_t D_x^2 \frac{\partial}{\partial u_{txx}} \dots \quad (2.45)$$

with D_t, D_x being the total derivatives written as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \dots \end{aligned} \quad (2.46)$$

Expanding equation (2.44) gives

$$\Lambda_u(u_t + u_x + uu_x - u_{txx}) - D_t(\Lambda) - D_x(\Lambda) - \Lambda u_x + D_x(u\Lambda) + D_t D_x^2(\Lambda) = 0. \quad (2.47)$$

Applying the total derivatives (2.46) to equation (2.47) and splitting on the derivatives of u we obtain

$$u_{xx} : \Lambda_{tu} = 0, \quad (2.48)$$

$$u_t u_{xx} : \Lambda_{uu} = 0, \quad (2.49)$$

$$u_{tx} : \Lambda_{xu} = 0, \quad (2.50)$$

$$\text{rest} : \Lambda_{txx} - u\Lambda_x - \Lambda_x - \Lambda_t = 0. \quad (2.51)$$

Integrating equation (2.49) twice gives

$$\Lambda = A(t, x)u + B(t, x) \quad (2.52)$$

where A and B are arbitrary functions of t and x . Substituting this value of Λ into (2.48) gives

$$A_t = 0$$

whose solution is

$$A = A(t).$$

Hence

$$\Lambda = A(t)u + B(t, x).$$

Inserting this value of Λ into equation (2.50), we obtain

$$A'(t) = 0$$

and this solves to

$$A(t) = c_1$$

where c_1 is an arbitrary constant. Thus

$$\Lambda = c_1 u + B(t, x).$$

Now substituting this value of Λ into equation (2.51) we get

$$B_{txx} - B_x u - B_x - B_t = 0.$$

Since A is independent of u therefore we can split the above equation on powers of u and get

$$u : B_x = 0, \tag{2.53}$$

$$u^0 : B_{txx} - B_t - B_x = 0. \tag{2.54}$$

Equation (2.53) implies that $B = B(t)$. Thus substituting this value of B into equation (2.54) we get

$$B'(t) = 0. \tag{2.55}$$

Integrating equation (2.55) gives $B(t) = c_2$, where c_2 is an arbitrary constant of integration. Thus our multipliers for the BBMe are

$$\Lambda_1 = 1, \Lambda_2 = u. \tag{2.56}$$

Multiplier Λ for BBM equation (2.1) has the property that

$$\Lambda(u_t + u_x + uu_x - u_{txx}) = D_t T^t + D_x T^x, \tag{2.57}$$

where $T^t = T^t(t, x, u, u_x)$ and $T^x = T^x(t, x, u, u_x, u_{tx})$.

We now use the multiplier the multipliers Λ_1, Λ_2 to compute conservation laws.

Case 1.

We start by using $\Lambda_1 = 1$ and equation (2.57), thus expanding we have

$$u_t + u_x + uu_x - u_{txx} = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx} + T_{u_{tx}}^x u_{txx}.$$

Splitting the above equation on the third-order derivatives of u , we obtain

$$u_{txx} : T_{u_{tx}}^x = -1, \quad (2.58)$$

$$\text{Rest} : u_t + u_x + uu_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx}. \quad (2.59)$$

Thus integrating (2.58) gives

$$T^x = -u_{tx} + A(t, x, u, u_x) \quad (2.60)$$

where A is arbitrary function of its argument. Now inserting the above value of T^x into (2.59) gives

$$u_t + u_x + uu_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx}. \quad (2.61)$$

We see that T^t and A in equation (2.61) are independent of second derivatives of u , thus we split (2.61) on second derivatives of u and get

$$u_{tx} : T_{u_x}^t = 0, \quad (2.62)$$

$$u_{xx} : A_{u_x} = 0, \quad (2.63)$$

$$\text{Rest} : u_t + u_x + uu_x = T_t^t + T_u^t u_t + T_x^x + T_u^x u_x. \quad (2.64)$$

Equation (2.63) imply that $A = A(t, x, u)$. Hence

$$T^x = -u_{tx} + A(t, x, u).$$

From equation (2.62) we deduce that

$$T^t = B(t, x, u),$$

where B is an arbitrary function of its arguments. Substituting the above values of T^t and T^x into equation (2.59) we get

$$u_t + u_x + uu_x = B_t + B_u u_t + A_x + A_u u_x \quad (2.65)$$

Splitting equation (2.65) on derivatives of u yields

$$u_t : B_u = 1, \quad (2.66)$$

$$u_x : A_u = 1 + u, \quad (2.67)$$

$$\text{rest} : B_t + A_x = 0. \quad (2.68)$$

Equation (2.66) implies that

$$B = u + C(t, x),$$

where C is an arbitrary function of t and x . Equation (2.67) gives

$$A = \frac{1}{2}u^2 + u + D(t, x),$$

where D is an arbitrary function of its arguments. Substituting the values of A and B into equation (2.68) gives $C_t + D_x = 0$, thus we take C and D as zero because they contribute to the trivial part of the conservation law. Thus the conservation law of the Burgers equation (2.1) is given by

$$\begin{aligned} T^t &= u, \\ T^x &= -u_{tx} + \frac{1}{2}u^2 + u. \end{aligned}$$

Case 2.

Now using the multiplier $\Lambda = u$ and the determining equation for conservation laws (2.57), we have

$$uu_t + uu_x + u^2u_x - uu_{txx} = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx} + T_{u_{tx}}^x u_{txx}.$$

Splitting the above equation on derivatives of u we obtain

$$u_{txx} : T_{u_{tx}}^x = -u, \quad (2.69)$$

$$\text{Rest} : uu_t + uu_x + u^2u_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx}. \quad (2.70)$$

Equation (2.69) implies that

$$T^x = -uu_{tx} + A(t, x, u, u_x),$$

where A is arbitrary function of its argument. Substituting the above value of T^t into (2.70) gives

$$uu_t + uu_x + u^2u_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + A_x - u_x u_{tx} + A_u u_x + A_{u_x} u_{xx}. \quad (2.71)$$

Separating the above on second derivatives of u gives

$$u_{xx} : A_{u_x} = 0, \quad (2.72)$$

$$u_{tx} : T_{u_x}^t = u_x, \quad (2.73)$$

$$\text{Rest} : uu_t + uu_x + u^2u_x = T_t^t + T_u^t u_t + A_x + A_u u_x. \quad (2.74)$$

Solving equation (2.72) we get

$$A = A(t, x, u),$$

thus

$$T^x = -uu_{tx} + A(t, x, u),$$

where A is an arbitrary function of its arguments. Equation (2.73) implies that

$$T^t = \frac{1}{2}u_x^2 + B(t, x, u),$$

where B is an arbitrary function of its arguments. Substituting the above values of T^t and T^x into equation (2.59) we get

$$uu_t + uu_x + u^2u_x = B_t + B_u u_t + A_x + A_u u_x \quad (2.75)$$

Splitting equation (2.75) on derivatives of u yields

$$u_t : B_u = u, \quad (2.76)$$

$$u_x : A_u = u + u^2, \quad (2.77)$$

$$\text{Rest} : B_t + A_x = 0. \quad (2.78)$$

Solving (2.77) we get

$$A = \frac{1}{2}u^2 + \frac{1}{3}u^3 + C(t, x) \quad (2.79)$$

where C is an arbitrary function of t and x . Integrating equation (2.76) gives

$$B = \frac{1}{2}u^2 + D(t, x) \quad (2.80)$$

where D is an arbitrary function of its argument. Substituting the values of A and B into equation (2.78) gives $D_t + C_x = 0$, thus we take B and C as zero because they contribute to the trivial part of the conservation law. Thus the conservation law of the BBM equation (2.1) is given by

$$\begin{aligned} T^t &= \frac{1}{2}u^2 + \frac{1}{2}u_x^2, \\ T^x &= -uu_{tx} + \frac{1}{3}u^3 + \frac{1}{2}u^2. \end{aligned}$$

2.3.2 Conservation laws using Noether's Theorem

We utilize Noether's approach to derive conservation laws for (2.1). This equation is of third order and as such does not have a Lagrangian. To make it variational we increase its order by allowing $u = v_x$. Thus we have

$$v_{tx} + v_{xx} + v_x v_{xx} - v_{txx} = 0. \quad (2.81)$$

It can be verified that equation (2.81) has a second-order Lagrangian

$$\mathcal{L} = -\frac{1}{2}v_t v_x - \frac{1}{2}v_x^2 - \frac{1}{6}v_x^3 - \frac{1}{2}v_{tx} v_{xx},$$

as

$$\frac{\delta \mathcal{L}}{\delta v} = 0$$

on equation (2.81). Here $\delta/\delta v$ is the Euler operator and is defined as

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t D_x \frac{\partial}{\partial v_{tx}} + D_x^2 \frac{\partial}{\partial v_{xx}} + \dots$$

The determining equation for Noether point symmetries is given by

$$X^{[2]}(\mathcal{L}) + \mathcal{L} \{D_t(\tau) + D_x(\xi)\} - D_t(B^1) - D_x(B^2) = 0, \quad (2.82)$$

with B^1 and B^2 being gauge terms depending on t, x, v , $X^{[2]}$ is the second extension for the infinitesimal generator

$$X = \tau(t, x, v) \frac{\partial}{\partial t} + \xi(t, x, v) \frac{\partial}{\partial x} + \eta(t, x, v) \frac{\partial}{\partial v}.$$

Expanding equation (2.82), gives

$$\begin{aligned}
& -\frac{1}{2}v_x \{ \eta_t + v_t \eta_v - v_t \tau_t - v_t^2 \tau_v - v_x \xi_t - v_t v_x \xi_v \} - \frac{1}{2}v_t \{ \eta_x + v_x \eta_v - v_t \tau_x - v_t v_x \tau_v \\
& - v_x \xi_x - v_x^2 \xi_v \} - v_x \{ \eta_x + v_x \eta_v - v_t \tau_x - v_t v_x \tau_v - v_x \xi_x - v_x^2 \xi_v \} - \frac{1}{2}v_x^2 \{ \eta_x + v_x \eta_v \\
& - v_t \tau_x - v_t v_x \tau_v - v_x \xi_x - v_x^2 \xi_v \} - \frac{1}{2}v_{xx} \{ \eta_{tx} + v_x \eta_{tv} + v_t \eta_{xv} + v_{xt} \eta_v + v_t v_x \eta_{vv} \\
& - v_{xt} \tau_{tv} - v_{xt} \xi_{xv} - v_t^2 \tau_{xv} - 2v_t v_{xt} \tau_v - v_x v_{tt} \tau_v - v_t^2 v_x \tau_{vv} - v_x \xi_{xt} - v_{xx} \xi_t - v_x^2 \xi_{tv} \\
& - 2v_x v_{xt} \xi_v - v_t v_{xx} \xi_v - v_t v_x^2 \xi_{vv} \} - \frac{1}{2}v_{xt} \{ \eta_{xx} + 2v_x \eta_{xv} + v_{xx} \eta_v + v_x^2 \eta_{vv} - 2v_{xx} \xi_x \\
& - v_x \xi_{xx} - 2v_x^2 \xi_{xv} - 3v_x v_{xx} \xi_v - v_x^3 \xi_{vv} - 2v_{xt} \tau_x - v_t \tau_{xx} - 2v_t v_x \tau_{xv} - v_t v_{xx} \tau_v - 2v_x v_{xt} \tau_v \\
& - v_t v_x^2 \tau_{vv} \} - B_t^1 - v_t B_v^1 - B_x^2 - v_x B_v^2 = 0.
\end{aligned}$$

Splitting the above equation on derivatives of v yields

$$\tau_t = 0, \tau_x = 0, \tau_v = 0, \quad (2.83)$$

$$\xi_x = 0, \xi_t = 0, \xi_v = 0, \quad (2.84)$$

$$\eta_x = 0, \eta_v = 0, \quad (2.85)$$

$$\eta_x + 2B_v^1 = 0, \quad (2.86)$$

$$\eta_t + 2B_v^2 = 0, \quad (2.87)$$

$$B_x^1 + B_t^2 = 0. \quad (2.88)$$

Equations (2.83) imply that

$$\tau = C_1,$$

where C_1 is an arbitrary constant of integration. Integrating equations (2.84) gives

$$\xi = C_2,$$

where C_2 is an arbitrary constant of integration. Equations (2.85) gives

$$\eta = f(t), \quad (2.89)$$

where $f(t)$ is an arbitrary function of t . Using equation (2.87), we obtain

$$B^2 = -\frac{1}{2}f'(t)v + E(t, x), \quad (2.90)$$

where $E(t, x)$ is an arbitrary function of t and x . From (2.86), we obtain

$$B^1 = G(t, x),$$

where $G(t, x)$ is an arbitrary function of t and x . Equation (2.88) gives $E_t + G_x = 0$, thus we take E and G to be zero as they contribute to the trivial part of the conserved vector. Thus Noether point symmetries and their gauge functions are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & B^1 &= 0, & B^2 &= 0, \\ X_2 &= \frac{\partial}{\partial x}, & B^1 &= 0, & B^2 &= 0, \\ X_f &= f(t) \frac{\partial}{\partial v}, & B^1 &= 0, & B^2 &= -\frac{1}{2} v f'(t). \end{aligned}$$

We now construct the conservation laws corresponding to each Noether point symmetry using [31]

$$\begin{aligned} T^t &= \mathcal{L}\tau + (\eta - v_t\tau - v_x\xi) \frac{\partial \mathcal{L}}{\partial v_t} + (\zeta_2 - v_{tx}\tau - v_{xx}\xi) \frac{\partial \mathcal{L}}{\partial v_{tx}} - B^1, \\ T^x &= \mathcal{L}\xi + (\eta - v_t\tau - v_x\xi) \left\{ \frac{\partial \mathcal{L}}{\partial v_x} - D_t \left(\frac{\partial \mathcal{L}}{\partial v_{tx}} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial v_{xx}} \right) \right\} \\ &\quad + (\zeta_2 - v_{tx}\tau - v_{xx}\xi) \frac{\partial \mathcal{L}}{\partial v_{xx}} - B^2. \end{aligned}$$

Case 1. The first Noether symmetry $X_1 = \partial/\partial t$ has the conserved vector

$$\begin{aligned} T_1^t &= -\frac{1}{4} v_t v_{xxx} - \frac{1}{4} v_{xx} v_{tx} - \frac{v_x^3}{6} - \frac{v_x^2}{2}, \\ T_1^x &= \frac{1}{2} v_t v_x^2 + v_t v_x + \frac{v_{tx}^2}{2} - \frac{3}{4} v_t v_{txx} + \frac{1}{4} v_{tt} v_{xx} + \frac{v_t^2}{2}. \end{aligned}$$

Case 2. The second operator $X_2 = \partial/\partial x$ yields the consevation law

$$\begin{aligned} T_2^t &= \frac{v_x^2}{2} - \frac{1}{4} v_{xxx} v_x + \frac{v_{xx}^2}{4}, \\ T_2^x &= -\frac{3}{4} v_x v_{txx} + \frac{1}{4} v_{xx} v_{tx} + \frac{v_x^3}{3} + \frac{v_x^2}{2}. \end{aligned}$$

Case 3. Finally for the operator $X_f = f(t)\partial/\partial v$, we get the conserved vector

$$\begin{aligned} T_f^t &= \left(\frac{1}{4} v_{xxx} - \frac{1}{2} v_x \right) f(t), \\ T_f^x &= \left(\frac{3}{4} v_{txx} - \frac{1}{2} v_x^2 - v_x - \frac{1}{2} v_t \right) f(t) + \left(\frac{1}{2} v - \frac{1}{4} v_{xx} \right) f'(t). \end{aligned}$$

Now going to original variable u , we obtain conservation laws for BBM equation (2.1) as

$$\begin{aligned}
T_1^t &= -\frac{1}{4} \left(\int u_t dx \right) u_{xx} - \frac{1}{4} u_x u_t - \frac{1}{6} u^3 - \frac{1}{2} u^2, \\
T_1^x &= \frac{1}{2} \left(\int u_t dx \right) u^2 + \left(\int u_t dx \right) u + \frac{1}{2} u_t^2 - \frac{3}{4} \left(\int u_t dx \right) u_{tx} + \frac{1}{4} \left(\int u_{tt} dx \right) u_x \\
&\quad + \frac{1}{2} \left(\int u_t dx \right)^2, \\
T_2^t &= \frac{1}{2} u^2 - \frac{1}{4} u u_{xx} + \frac{1}{4} u_x^2, \\
T_2^x &= -\frac{3}{4} u u_{tx} + \frac{1}{4} u_t u_x + \frac{1}{3} u^3 + \frac{1}{2} u^2, \\
T_f^t &= \left(\frac{1}{4} u_{xx} - \frac{1}{2} u \right) f(t), \\
T_f^x &= \left\{ \frac{3}{4} u_{tx} - \frac{1}{2} u^2 - u - \frac{1}{2} \left(\int u_t dx \right) \right\} f(t) + \left\{ \frac{1}{2} \left(\int u dx \right) - \frac{1}{4} u_x \right\} f'(t).
\end{aligned}$$

2.4 Concluding remarks

In this chapter we presented an illustration of what this research project will be about by constructing Lie point symmetries of the third-order BBM equation (2.1). We then used the obtained symmetries to perform symmetry reductions. Finally conservation laws for this equation were derived using the multiplier approach and Noether's theorem.

Chapter 3

Solutions and conservation laws of the (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation

In this chapter we study a fourth-order (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation by first computing its Lie point symmetries and then performing symmetry reductions. Exact group invariant solutions are obtained including the solution in terms of incomplete elliptic integral. Furthermore, conservation laws for this equation are derived using both the multiplier and Noether's methods.

3.1 Introduction

The (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama (3DYTSF) equation

$$u_{xxxx} - 4u_{tx} + 3\alpha u_{yy} + 4u_x u_{xz} + 2u_z u_{xx} = 0 \quad (3.1)$$

was first studied in [32] using strong symmetry and its travelling wave solutions were obtained. Yan [33] investigated (3.1) and gained auto-Bäcklund transformations. Using auto-Bäcklund transformations some closed-form solutions of (3.1) were determined. These included the non-travelling wave solutions, soliton-like solutions and rational solutions. In [34], the authors investigated (3.1) using homoclinic and extended homoclinic test techniques, the two-soliton method together with bilinear form method and gained some new wave solutions that included periodic kink-wave, periodic soliton, cross kink wave and doubly periodic wave solutions. The exp-function method was employed in [35] and new solitary and periodic solutions with free parameters were found. Applying a modification of extended homoclinic test method, the authors of [36] determined analytic solutions of (3.1).

The authors of [37], using 1D subalgebras, found group invariant solutions of (3.1) that contained arbitrary functions. Some of the obtained solutions were sketched. Closed-form solutions including lump solutions and interaction solutions of (3.1) were determined by invoking the generalized Hirota bilinear method [38]. Analytical solutions and conservation laws for the 2D form of (3.1) were presented in [39] and moreover, 2D and 3D graphical representations of solutions were given. N-soliton solutions were constructed for (3.1) by applying bilinear transformation that included period soliton, line soliton, lump soliton and their interaction. Furthermore, for certain solutions their images were sketched and their dynamic behaviour were discussed [40]. In [41], the authors applied the extended homoclinic test and Hirota bilinear method and presented a class of lump solutions of (3.1). Moreover, periodic lump-type solutions were constructed. In [42], (3.1) was transformed to the 2D potential YTSE equation (see ref [39]). General lump solutions of this equation were found and its propagation path was presented. By letting $u = w_x$ the authors of [43] raised the order of 3DYTSF equation to five and invoked Lie symmetry method and established dark, bright, topological, Peregrine and multi-soliton.

In this chapter we work with the three-dimensional generalized potential Yu-Toda-Sasa-Fukuyama (3DgYTSE) equation, viz.,

$$u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx} = 0, \quad (3.2)$$

where α and β are real constants. We determine its exact solutions by invoking symmetry analysis together with various other methods. Furthermore, conserved quantities of (3.2) are determined using two approaches: multiplier approach and Noether's approach.

The content of this chapter has been published in [44].

3.2 Solutions of the 3DgYTSF equation (3.2)

3.2.1 Lie point symmetries of (3.2)

In this subsection we construct Lie point symmetries of the 3DgYTSF (3.2). The vector field for equation (3.2) is written as

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \quad (3.3)$$

where τ, ξ, ϕ, ψ and η are functions of t, x, y, z and u . We recall that (3.3) is a Lie point symmetry of (3.2) if

$$X^{[4]}E|_{E=0} = 0, \quad (3.4)$$

where $E \equiv u_{xxxx} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx}$. In this case $X^{[4]}$ is the fourth prolongation [12] of (3.3) and is defined by

$$\begin{aligned} X^{[4]} = X &+ \zeta_2 \frac{\partial}{\partial u_x} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{33} \frac{\partial}{\partial u_{yy}} + \zeta_{24} \frac{\partial}{\partial u_{xz}} \\ &+ \zeta_{2224} \frac{\partial}{\partial u_{xxxz}}, \end{aligned} \quad (3.5)$$

where the ζ_s are determined as follows:

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) - u_y D_t(\phi) - u_z D_t(\psi), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\phi) - u_z D_x(\psi), \\ \zeta_3 &= D_y(\eta) - u_t D_y(\tau) - u_x D_y(\xi) - u_y D_y(\phi) - u_z D_y(\psi), \\ \zeta_4 &= D_z(\eta) - u_t D_z(\tau) - u_x D_z(\xi) - u_y D_z(\phi) - u_z D_z(\psi), \\ \zeta_{12} &= D_z(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi) - u_{ty} D_x(\phi) - u_{tz} D_x(\psi), \end{aligned}$$

$$\begin{aligned}
\zeta_{22} &= D_x(\zeta_2) - u_{tx}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\phi) - u_{xz}D_x(\psi), \\
\zeta_{33} &= D_y(\zeta_3) - u_{ty}D_y(\tau) - u_{xy}D_y(\xi) - u_{yy}D_y(\phi) - u_{yz}D_y(\psi), \\
\zeta_{24} &= D_z(\zeta_2) - u_{tx}D_z(\tau) - u_{xx}D_z(\xi) - u_{xy}D_z(\phi) - u_{xz}D_z(\psi), \\
\zeta_{2224} &= D_z(\zeta_{222}) - u_{txxx}D_z(\tau) - u_{xxxx}D_z(\xi) - u_{xxxxy}D_z(\phi) - u_{xxxz}D_z(\psi). \quad (3.6)
\end{aligned}$$

Here D_t , D_x , D_y and D_z are the total derivatives and are defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z} \cdots, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{xz} \frac{\partial}{\partial u_z} \cdots, \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{ty} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + u_{yz} \frac{\partial}{\partial u_z} \cdots, \\
D_z &= \frac{\partial}{\partial z} + u_z \frac{\partial}{\partial u} + u_{tz} \frac{\partial}{\partial u_t} + u_{xz} \frac{\partial}{\partial u_x} + u_{yz} \frac{\partial}{\partial u_y} + u_{zz} \frac{\partial}{\partial u_z} \cdots. \quad (3.7)
\end{aligned}$$

From equation (3.4) we get

$$\left\{ X + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{33} \frac{\partial}{\partial u_{yy}} + \zeta_{24} \frac{\partial}{\partial u_{xz}} + \zeta_{2224} \frac{\partial}{\partial u_{xxxz}} \right\} \left(u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx} \right) \Big|_{(3.2)} = 0. \quad (3.8)$$

After expanding the above equation we obtain

$$2\alpha\zeta_2 u_{xz} + \alpha\zeta_4 u_{xx} - 2\alpha\zeta_{12} + \alpha\zeta_{22} u_z + \beta\zeta_{33} + 2\alpha\zeta_{24} u_x + \zeta_{2224} \Big|_{(3.2)} = 0. \quad (3.9)$$

Substituting values of ζ 's from (3.6) into equation (3.9), replacing u_{xxxz} by $2\alpha u_{xt} - \beta u_{yy} - 2\alpha u_x u_{xz} - \alpha u_{xx} u_z$ and splitting on various derivatives of u , we get

$$\tau_x = 0, \tau_z = 0, \tau_u = 0, \tau_y = 0, \quad (3.10)$$

$$\tau_{tt} = 0, \quad (3.11)$$

$$\xi_u = 0, \xi_z = 0, \quad (3.12)$$

$$\phi_z = 0, \phi_x = 0, \phi_u = 0, \quad (3.13)$$

$$\psi_x = 0, \psi_y = 0, \psi_u = 0, \quad (3.14)$$

$$\phi_{ty} = 0, \quad (3.15)$$

$$\phi_{yy} = 0, \quad (3.16)$$

$$\xi_x + \tau_t - 2\phi_y = 0, \quad (3.17)$$

$$\beta\xi_y - \alpha\phi_t = 0, \quad (3.18)$$

$$\psi_z + 4\phi_y - 3\tau_t = 0, \quad (3.19)$$

$$\eta_x + \psi_t = 0, \quad (3.20)$$

$$\eta_z + 2\xi_t = 0, \quad (3.21)$$

$$\eta_u - \tau_t + 2\phi_y = 0, \quad (3.22)$$

$$\beta\eta_{yy} + 2\alpha\psi_{tt} = 0. \quad (3.23)$$

Equations (3.10) imply that

$$\tau = A(t), \quad (3.24)$$

where A is an arbitrary function of t . Equation (3.11) gives

$$A''(t) = 0. \quad (3.25)$$

Integrating the above gives

$$A(t) = C_1 t + C_2, \quad (3.26)$$

where C_1 and C_2 are arbitrary constants. Thus

$$\tau = C_1 t + C_2. \quad (3.27)$$

From equations (3.13) we get

$$\phi = B(t, y), \quad (3.28)$$

where B is an arbitrary function of t and y . Equations (3.14) yield

$$\psi = C(t, z), \quad (3.29)$$

where C is an arbitrary function of t and z . From equation (3.16), we get

$$B_{yy} = 0. \quad (3.30)$$

Integrating equation (3.30) twice gives

$$B = D(t)y + F_1(t), \quad (3.31)$$

where D and F_1 are arbitrary functions of t . Using equation (3.15), we obtain

$$D'(t) = 0,$$

which solves to $D(t) = C_3$, where C_3 is an arbitrary constant. Thus

$$\phi = C_3 y + F_1(t). \quad (3.32)$$

From equation (3.19) we obtain

$$C_z = 3C_1 - 4C_3 \quad (3.33)$$

and integrating this, gives

$$C = 3zC_1 - 4C_3z + F_2(t). \quad (3.34)$$

F_2 is an arbitrary function of t . Thus

$$\psi = 3zC_1 - 4C_3z + F_2(t) \quad (3.35)$$

From equations (3.12) we obtain

$$\xi = E(t, x, y), \quad (3.36)$$

where E is an arbitrary function of its argument. Substituting the above value of ξ into (3.18) we get

$$E_y = \frac{\alpha}{\beta} F_1'(t), \quad (3.37)$$

which solves to

$$E = \frac{\alpha}{\beta} y F_1'(t) + G(t, x). \quad (3.38)$$

where G is an arbitrary function of t and x . Thus

$$\xi = \frac{\alpha}{\beta} y F_1'(t) + G(t, x). \quad (3.39)$$

From equation (3.17), we obtain

$$G_x = 2C_3 - C_1 \quad (3.40)$$

and integrating the above with respect to x we obtain

$$G = 2xC_3x - C_1 + F_3(t), \quad (3.41)$$

where F_3 is an arbitrary function of t . Thus

$$\xi = 2xC_3x - C_1 + F_3(t) \quad (3.42)$$

From equation (3.22), we obtain

$$\eta_u = -2C_3 + C_1, \quad (3.43)$$

which solves to

$$\eta = uC_1 - 2uC_3 + F(t, x, y, z), \quad (3.44)$$

where F is an arbitrary function of its argument. Equation (3.21) gives

$$F_z = -2 \left(\frac{\alpha}{\beta} y F_1''(t) + F_3'(t) \right), \quad (3.45)$$

whose solution is

$$F = -2z \left(\frac{\alpha}{\beta} y F_1''(t) + F_3'(t) \right) + K(t, x, y), \quad (3.46)$$

where K is an arbitrary function of its argument. Equation (3.20) gives

$$K_x = -F_2'(t), \quad (3.47)$$

which solves to

$$K = -xF_2'(t) + L(t, y), \quad (3.48)$$

where L is an arbitrary function of t and y . From equation (3.23), we get

$$L_{yy} = -\frac{2\alpha}{\beta} F_2''(t). \quad (3.49)$$

Integrating the above twice with respect to y gives

$$L = -\frac{\alpha}{\beta} y^2 F_2''(t) + yF_5(t) + F_4(t). \quad (3.50)$$

Where F_4 and F_5 are arbitrary functions of t . Thus

$$\eta = uC_1 - 2uC_3 - \frac{2\alpha}{\beta}yzF_1''(t) - \frac{\alpha}{\beta}y^2F_2''(t) - xF_2'(t) - 2zF_3'(t) + yF_5(t) + F_4(t). \quad (3.51)$$

We therefore see that the 3DgYTSF equation (3.2) has the infinitesimals

$$\begin{aligned} \tau &= C_1t + C_2, \\ \xi &= -C_1x + 2C_3x + \frac{\alpha F_1'(t)y}{\beta} + F_3(t), \\ \phi &= C_3y + F_1(t), \\ \psi &= 3C_1z - 4C_3z + F_2(t), \\ \eta &= C_1u - 2C_3u - \frac{2\alpha F_1''(t)yz}{\beta} - \frac{\alpha F_2''(t)y^2}{\beta} - F_2'(t)x - 2F_3'(t)z + F_5(t)y + F_4(t). \end{aligned}$$

Hence the Lie point symmetries of the 3DgYTSF equation (3.2) are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial y}, \\ X_4 &= \frac{\partial}{\partial z}, \\ X_5 &= \frac{\partial}{\partial u}, \\ X_6 &= y \frac{\partial}{\partial u}, \\ X_7 &= t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + 3z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}, \\ X_8 &= 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - 4z \frac{\partial}{\partial z} - 2u \frac{\partial}{\partial u}. \end{aligned}$$

3.2.2 One-parameter group of transformations

In this subsection we compute the one-parameter groups corresponding to the symmetries $X_1, \dots, X_8$. We solve the following Lie equations with initial conditions:

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}), \quad \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}), \quad \bar{x}|_{a=0} = x,$$

$$\begin{aligned}\frac{d\bar{y}}{da} &= \phi(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}), \quad \bar{y}|_{a=0} = y, \quad \frac{d\bar{z}}{da} = \psi(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}), \quad \bar{z}|_{a=0} = z, \\ \frac{d\bar{u}}{da} &= \eta(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}), \quad \bar{u}|_{a=0} = u,\end{aligned}$$

to compute the one-parameter group of transformations. For each X_i , let T_{a_i} be the corresponding group. We first compute one-parameter group corresponding to infinitesimal generator

$$X_7 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + 3z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da_7} = \bar{t}, \quad \bar{t}|_{a_7=0} = t, \quad \frac{d\bar{x}}{da_7} = -\bar{x}, \quad \bar{x}|_{a_7=0} = x, \quad \frac{d\bar{z}}{da_7} = 3\bar{z}, \quad \bar{z}|_{a_7=0} = z, \quad \frac{d\bar{u}}{da_7} = \bar{u}, \quad \bar{u}|_{a_7=0} = u.$$

Solving the above equations we obtain

$$\bar{t} = te^{a_7}, \quad \bar{x} = xe^{-a_7}, \quad \bar{y} = y, \quad \bar{z} = ze^{3a_7}, \quad \bar{u} = ue^{a_7}.$$

Thus the one-parameter group T_{a_7} corresponding to the operator X_7 is given by

$$T_{a_7} : (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (te^{a_7}, xe^{-a_7}, y, ze^{3a_7}, ue^{a_7}).$$

If we continue in the same manner as above, we get the following one-parameter groups for $X_i (i = 1, \dots, 8)$, respectively

$$\begin{aligned}T_{a_1} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t + a_1, x, y, z, u), \\ T_{a_2} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t, x + a_2, y, z, u), \\ T_{a_3} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t, x, y + a_3, z, u), \\ T_{a_4} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t, x, y, z + a_4, u), \\ T_{a_5} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t, x, y, z, u + a_5), \\ T_{a_6} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t, x, y, z, u + a_6 y), \\ T_{a_7} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (te^{a_7}, xe^{-a_7}, y, ze^{3a_7}, ue^{a_7}), \\ T_{a_8} &: (\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \longrightarrow (t, xe^{2a_8}, ye^{a_8}, ze^{-4a_8}, ue^{-2a_8}).\end{aligned}$$

Consequently, if $u = M(t, x, y, z)$ is a known solution of equation (3.2), then by using the symmetry groups $T_{a_i} (i = 1, 2, 3, \dots, 8)$, so are the functions

$$u_1 = M(t + a_1, x, y, z),$$

$$\begin{aligned}
u_2 &= M(t, x + a_2, y, z), \\
u_3 &= M(t, x, y + a_3, z), \\
u_4 &= M(t, x, y, z + a_4), \\
u_5 &= M(t, x, y, z) + a_5, \\
u_6 &= M(t, x, y, z) - a_6 y, \\
u_7 &= M(te^{a_7}, xe^{-a_7}, y, ze^{3a_7})e^{-a_7}, \\
u_8 &= M(t, xe^{2a_8}, ye^{a_8}, ze^{-4a_8})e^{2a_8}.
\end{aligned}$$

3.2.3 Invariant solutions under the symmetries $X_1, \dots, X_4$

We now utilize the four translation symmetries and perform symmetry reduction of (3.2). We consider the symmetry $X = X_1 + aX_2 + bX_3 + cX_4$, where a, b, c are arbitrary constants. Solving the characteristics equation of the PDE ($XI = 0$) we obtain the invariants

$$f = x - at, \quad g = y - bt, \quad h = z - ct, \quad u = \theta(f, g, h).$$

When using the above invariant solution, equation (3.2) reduces to

$$\theta_{fffh} + 2\alpha(a\theta_{ff} + b\theta_{fg} + c\theta_{fh}) + \beta\theta_{gg} + 2\alpha\theta_f\theta_{fh} + \alpha\theta_h\theta_{ff} = 0. \quad (3.52)$$

The reduced PDE (3.52) has the following as its Lie point symmetries:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}, \quad \Gamma_3 = \frac{\partial}{\partial h}, \quad \Gamma_4 = \frac{\partial}{\partial \theta}, \quad \Gamma_5 = g \frac{\partial}{\partial \theta}, \\
\Gamma_6 &= \left(\frac{2}{3}f + \frac{\alpha bg}{3\beta} \right) \frac{\partial}{\partial f} + g \frac{\partial}{\partial g} + \left(\frac{1}{3} \frac{(-4ah - 4cf - 2\theta)\beta + 2\alpha b^2 h}{\beta} \right) \frac{\partial}{\partial \theta}, \\
\Gamma_7 &= \left(\frac{\alpha bg}{3\beta} - \frac{1}{3}f \right) \frac{\partial}{\partial f} + h \frac{\partial}{\partial h} + \left(\frac{1}{3} \frac{(-4ah + 2cf + \theta)\beta + 2\alpha b^2 h}{\beta} \right) \frac{\partial}{\partial \theta}.
\end{aligned}$$

By considering the symmetry $\Gamma_1 + d\Gamma_2 + e\Gamma_3$, where d and e are constants, we see that it provides us with three invariants, namely

$$r = g - df, \quad s = h - ef, \quad \theta = \phi(r, s).$$

As above, utilization of these invariants, reduces one independent variable of (3.52) and we obtain a reduced PDE

$$\begin{aligned}
& -d^3\phi_{rrrs} - 3d^2e\phi_{rrss} - 3de^2\phi_{rsss} - e^3\phi_{ssss} + 2\alpha ad^2\phi_{rr} + 4\alpha ade\phi_{rs} + 2\alpha ae^2\phi_{ss} \\
& - 2\alpha bd\phi_{rr} - 2\alpha be\phi_{rs} - 2\alpha cd\phi_{rs} - 2\alpha ce\phi_{ss} + \beta\phi_{rr} + 2\alpha d^2\phi_r\phi_{rs} + 2\alpha de\phi_r\phi_{ss} \\
& + 4\alpha de\phi_s\phi_{rs} + 3\alpha e^2\phi_s\phi_{ss} + \alpha d^2\phi_s\phi_{rr} = 0.
\end{aligned} \tag{3.53}$$

Lie symmetries of equation (3.53) are

$$\begin{aligned}
\Sigma_1 &= \frac{\partial}{\partial r}, \Sigma_2 = \frac{\partial}{\partial s}, \Sigma_3 = \frac{\partial}{\partial \phi}, \Sigma_4 = r \frac{\partial}{\partial r} \\
&+ \left(3s - 2 - \frac{2er}{d} \right) \frac{\partial}{\partial s} + \left(\frac{6\beta er}{\alpha d^3} - f - 8as + \frac{8bs}{d} - \frac{4\beta s}{\alpha d^2} + \frac{8aer}{d} - \frac{10ber}{d^2} + \frac{2cr}{d} \right) \frac{\partial}{\partial \phi}.
\end{aligned}$$

By considering the symmetry $\Sigma_1 + \omega\Sigma_2$, where ω is a constant, we see that it yields two invariants, namely

$$\xi = s - \omega r, \quad \phi = \psi(\xi).$$

Using these invariants, (3.53) reduces to the ODE

$$\begin{aligned}
& (2\alpha ad^2\omega^2 + 2\alpha ae^2 + 2\alpha bew + 2\alpha cd\omega + \beta\omega^2 - 4\alpha adew - 2\alpha bd\omega^2 - 2\alpha ce) \psi'' \\
& - (6\alpha dew - 3\alpha d^2\omega^2 - 3\alpha e^2) \psi'\psi'' + (e^3 - \omega^3d^3 - 3de^2\omega + 3d^2e\omega^2) \psi'''' = 0.
\end{aligned} \tag{3.54}$$

We rewrite equation (3.54) as

$$A\psi'' - B\psi'\psi'' + C\psi'''' = 0, \tag{3.55}$$

where the A, B, C are given by

$$\begin{aligned}
A &= 2\alpha ad^2\omega^2 + 2\alpha ae^2 + 2\alpha bew + 2\alpha cd\omega + \beta\omega^2 - 4\alpha adew - 2\alpha bd\omega^2 - 2\alpha ce, \\
B &= 6\alpha dew - 3\alpha d^2\omega^2 - 3\alpha e^2, \quad C = e^3 - \omega^3d^3 - 3de^2\omega + 3d^2e\omega^2.
\end{aligned}$$

3.2.4 Solution via the incomplete elliptic integral

We aim to present the solution of equation (3.2) in terms of the incomplete elliptic integral solution. Firstly we need to convert this partial differential equation into

the ordinary differential equation which we did and obtained (3.55). We then turn our attention to the NODE (3.55) and integrating it twice to get

$$C\psi''' - \frac{1}{3}B\psi'^3 + A\psi'^2 + 2c_1\psi' + 2c_2 = 0, \quad (3.56)$$

where c_1, c_2 are integration constants. Now allowing $\psi'(\xi) = \Psi(\xi)$, equation (3.56) transforms to the ODE

$$\Psi'^2 - \frac{B}{3C}\Psi^3 + \frac{A}{C}\Psi^2 + \frac{2c_1}{C}\Psi + \frac{2c_2}{C} = 0. \quad (3.57)$$

Now to get the solution for (3.57), we start by assuming that γ_1, γ_2 and γ_3 are real roots of the cubic polynomial

$$\Psi^3 - \frac{3A}{B}\Psi^2 - \frac{6c_1}{B}\Psi - \frac{6c_2}{B} = 0, \quad (3.58)$$

such that they satisfy the condition $\gamma_1 > \gamma_2 > \gamma_3$ then general solution for equation (3.57) is [30, 45]

$$\Psi(\xi) = \gamma_2 + (\gamma_1 - \gamma_2)\text{cn}^2 \left\{ \sqrt{\frac{\alpha}{4}(\gamma_1 - \gamma_3)} \xi, X^2 \right\}, \quad X^2 = \frac{\gamma_1 - \gamma_2}{\gamma_1 - \gamma_3}, \quad (3.59)$$

where (cn) is the Jacobi cosine function.

Integration of equation (3.59) gives $\psi(\xi)$ and thus reverting to the variables t, x, y, z, u , we acquire the solution of 3DgYTSF equation (3.2) as

$$u(t, x, y, z) = \sqrt{\frac{12C(\gamma_1 - \gamma_2)^2}{B(\gamma_1 - \gamma_3)X^8}} \left[\text{EllipticE} \left\{ \text{sn} \left(\sqrt{\frac{B(\gamma_1 - \gamma_3)}{12C}} \xi, X^2 \right), X^2 \right\} \right] + \left\{ \gamma_2 - (\gamma_1 - \gamma_2) \frac{1 - X^4}{X^4} \right\} \xi + k_0, \quad (3.60)$$

where k_0 is a constant, $\xi = (ae - c + b\omega - a\omega d)t - (e + \omega d)x - \omega y + z$ and $\text{EllipticE}[g, k]$ being the incomplete elliptic integral given as [?, 45]

$$\text{EllipticE}[v, m] = \int_0^v \sqrt{\frac{1 - m^2 s^2}{1 - s^2}} ds.$$

The wave profile of the periodic solution (3.60), for parametric values $\gamma_1 = 102, \gamma_2 = 53, \gamma_3 = -58, C = -0.7, B = 0.25, c_1 = 1.2, z = 0, e = 0.5, d = 0.25, \omega = -1.01, a = -4.5, c = -0.6$ at $t = -15$ is given in Figure 1.

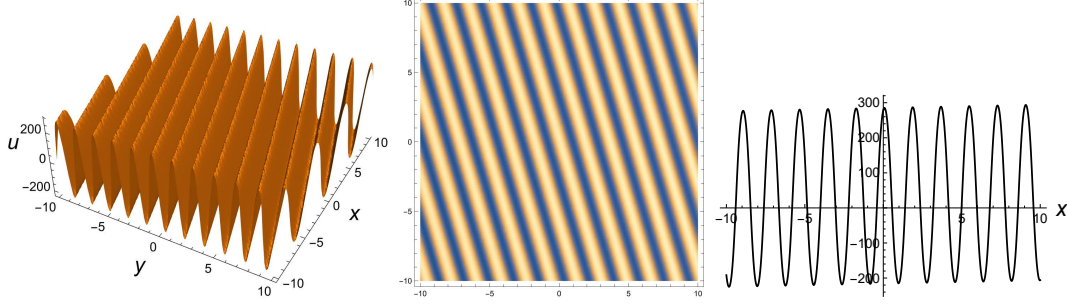


Figure 3.1: The 3D and 2D solution profile of (3.60)

3.2.5 Symmetry reductions under the symmetries X_7 and X_8

In this subsection we perform symmetry reductions under the scaling symmetries X_7 and X_8 with the aim of obtaining solutions where possible.

Group invariant solution under X_7

Here we consider the Lie symmetry X_7 , i.e.,

$$X_7 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + 3z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}. \quad (3.61)$$

Upon solving characteristic equations for this operator we obtain the invariants

$$f = tx, g = y, h = \frac{z}{t^3}, u = t \Phi(f, g, h). \quad (3.62)$$

This invariant solution reduces equation (3.2) to a reduced PDE of the form

$$\Phi_{fffh} - 4\alpha\Phi_f - 2\alpha f\Phi_{ff} + 6\alpha h\Phi_{fh} + \beta\Phi_{gg} + 2\alpha\Phi_f\Phi_{fh} + \alpha\Phi_h\Phi_{ff} = 0. \quad (3.63)$$

Equation (3.63) has the following six symmetries:

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial g}, \Gamma_2 = g \frac{\partial}{\partial \Phi}, \Gamma_3 = \frac{\partial}{\partial \Phi}, \Gamma_4 = \frac{\partial}{\partial f} + 2h \frac{\partial}{\partial \Phi}, \\ \Gamma_5 &= 2f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g} - 4h \frac{\partial}{\partial h} - 2\phi \frac{\partial}{\partial \Phi}, \Gamma_6 = \frac{\partial}{\partial h} - \left(3f + \frac{6\alpha g^2}{\beta} \right) \frac{\partial}{\partial \Phi}. \end{aligned} \quad (3.64)$$

The symmetry Γ_4 gives three invariants $g, h, U = \Phi - 2fh$ and consequently, the group-invariant solution is

$$\Phi(f, g, h) = U(g, h) + 2fh,$$

which reduces equation (3.63) to

$$U_{gg} + \frac{12\alpha}{\beta}h = 0,$$

whose solution is given by

$$U = -\frac{6\alpha}{\beta}g^2h + gM(h) + K(h),$$

where M and K are functions of h . Thus, the invariant solution of the 3DYTSF equation (3.2) under the symmetry X_7 is

$$u(t, x, y, z) = \frac{2xz}{t} - \frac{6\alpha y^2 z}{\beta t^2} + tyM\left(\frac{z}{t^3}\right) + tK\left(\frac{z}{t^3}\right). \quad (3.65)$$

In Figure 3.2, we depict the solution (3.65) with $M(z/t^3) = \cos(z/t^3)$, $K(z/t^3) = \text{sech}(z/t^3)$ and parametric values $\beta = 2$, $\alpha = 2$, $x = 0$, $t = 1$, $z = 15$.

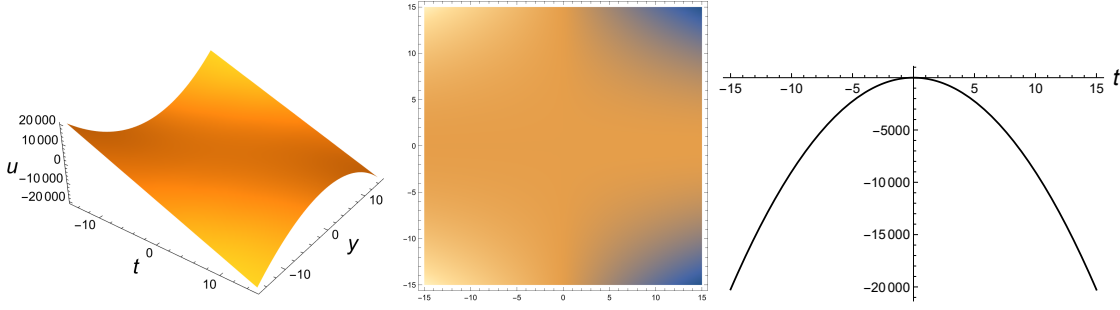


Figure 3.2: The 3D and 2D solution profile of (3.65) with $M = \cos(z/t^3)$, $K = \text{sech}(z/t^3)$

In Figure 3.3, we demonstrate the solution (3.65) with different choices of arbitrary functions, i.e., $M(z/t^3) = \text{sech}(z/t^3)$, $K(z/t^3) = \cos(z/t^3)$ and the parameters $\beta = 2$, $\alpha = 2$, $x = 0$, $t = 1$, $z = 15$.

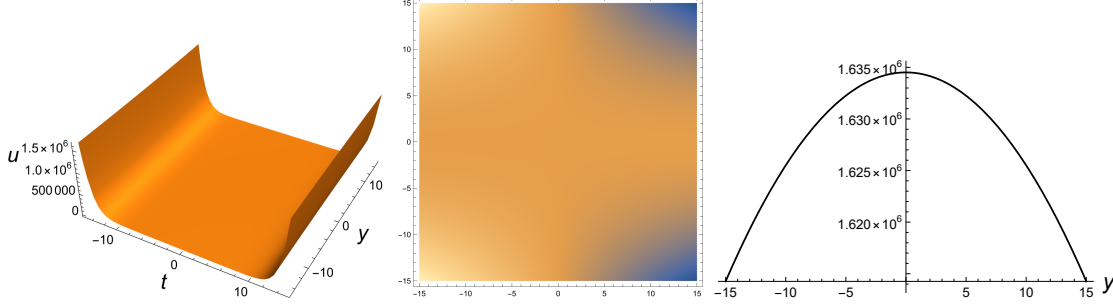


Figure 3.3: The 3D and 2D solution profile of (3.65) with $M = \text{sech}(z/t^3)$, $K = \cos(z/t^3)$

Group invariant solution under X_8

For the symmetry X_8 we get the group-invariant solution as

$$u(t, x, y, z) = \frac{1}{x} F(f, g, h), \quad (3.66)$$

where f, g, h, F are the invariants given by

$$f = t, g = \frac{y}{\sqrt{x}}, h = x^2 z, F = xu.$$

Substituting (3.66) into the 3DgYTSF equation (3.2) gives the NODE

$$\begin{aligned} & 36ghF_{ghh} - 12g^2hF_{gghh} + 48gh^2F_{ghhh} - 8\alpha gF_{fg} + 32\alpha hFF_{hh} + g^3F_{gggh} - 64h^3F_{hhhh} \\ & - 16\alpha F_f + 32\alpha hF_{fh} - 16\alpha hF_h^2 - 192h^2F_{hhh} + 3g^2F_{ggh} - 3gF_{gh} - 8\beta F_{gg} - 48hF_{hh} \\ & - 8\alpha gFF_{gh} - 6\alpha gF_gF_h - 4\alpha g^2F_gF_{gh} - 96\alpha h^2F_hF_{hh} - 2\alpha g^2F_hF_{gg} + 16\alpha ghF_gF_{hh} \\ & + 32\alpha ghF_hF_{gh} = 0 \end{aligned} \quad (3.67)$$

whose symmetry includes $\Gamma_1 = \partial/\partial f$. Using the symmetry Γ_1 , we get the invariants $j_1 = g, j_2 = h, F = \Psi$, which reduce equation (3.67) to the NPDE

$$\begin{aligned} & 36gh\Psi_{ghh} - 12g^2h\Psi_{gghh} + 48gh^2\Psi_{ghhh} + 32\alpha h\Psi\Psi_{hh} + g^3\Psi_{gggh} - 64h^3\Psi_{hhhh} - 16\alpha\Psi_h^2 \\ & - 192h^2\Psi_{hhh} + 3g^2\Psi_{ggh} - 3g\Psi_{gh} - 8\beta\Psi_{gg} - 48h\Psi_{hh} - 8\alpha g\Psi\Psi_{gh} - 6\alpha g\Psi_g\Psi_h \\ & - 4\alpha g^2\Psi_g\Psi_{gh} - 96\alpha h^2\Psi_h\Psi_{hh} - 2\alpha g^2\Psi_h\Psi_{gg} + 16\alpha gh\Psi_g\Psi_{hh} + 32\alpha gh\Psi_h\Psi_{gh} = 0 \end{aligned}$$

with two independent variables. Thus, we have reduced the number of independent variables of 3DgYTSF equation (3.2) by two.

3.3 Conservation laws of (3.2)

In this section we construct conservation laws of the 3DgYTSF equation (3.2) by using two different approaches, namely, the multiplier method and Noether's approach.

3.3.1 Conservation laws using the multiplier approach

We seek first-order multiplier $Q = Q(t, x, y, z, u, u_t, u_x, u_y)$ by applying the determining equation for the multipliers

$$\frac{\delta}{\delta u} \{Q(u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx})\} = 0, \quad (3.68)$$

where $\delta/\delta u$ is the Euler operator

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_z \frac{\partial}{\partial u_z} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_y^2 \frac{\partial}{\partial u_{yy}} \\ & + D_x D_z \frac{\partial}{\partial u_{xz}} + D_x^3 D_z \frac{\partial}{\partial u_{xxxz}} + \dots, \end{aligned} \quad (3.69)$$

with D_t, D_x, D_y, D_z being total derivatives as in (3.7). Expanding equation (3.68) yields

$$\begin{aligned} & u_{xxxz} Q_u - 2\alpha u_{tx} Q_u + \beta u_{yy} Q_u + 2\alpha u_x u_{xz} Q_u + \alpha u_{xx} u_z Q_u - D_t (u_{xxxz} Q_{u_t} - 2\alpha u_{tx} Q_{u_t} \\ & + \beta u_{yy} Q_{u_t} + 2\alpha u_x u_{xz} Q_{u_t} + \alpha u_{xx} u_z Q_{u_t}) - D_x (u_{xxxz} Q_{u_x} - 2\alpha u_{tx} Q_{u_x} + \beta u_{yy} Q_{u_x} \\ & + 2\alpha u_x u_{xz} Q_{u_x} + \alpha u_{xx} u_z Q_{u_x}) - D_y (u_{xxxz} Q_{u_y} - 2\alpha u_{tx} Q_{u_y} + \beta u_{yy} Q_{u_y} + 2\alpha u_x u_{xz} Q_{u_y} \\ & + \alpha u_{xx} u_z Q_{u_y}) - D_x (2\alpha u_{xz} Q) - D_t D_x (-2\alpha Q) - D_y^2 (\beta Q) - D_x D_z (2\alpha u_x Q + D_x^2 (\alpha u_z Q) \\ & - D_z (\alpha u_{xx} Q) - D_x^3 D_z (Q)) = 0, \end{aligned}$$

which on applying the total derivatives (3.7) and splitting on various derivatives of u gives

$$\begin{aligned} & Q_x = 0, Q_u = 0, Q_{yy} = 0, Q_{zz} = 0, Q_{tu_t} = 0, Q_{yu_t} = 0, Q_{yu_y} = 0, Q_{zu_t} = 0, \\ & Q_{zu_y} = 0, Q_{u_t u_t} = 0, Q_{u_t u_x} = 0, Q_{u_t u_y} = 0, 2Q_{tu_x} - Q_z = 0, Q_{u_x u_x} = 0, Q_{u_y u_y} = 0, \\ & \alpha Q_{tu_y} - \beta Q_{yu_x} = 0, Q_{u_x u_y} = 0, Q_{zu_x} = 0. \end{aligned} \quad (3.70)$$

Solving equations (3.70), we obtain the multipliers

$$Q = yzF_5''(t) + \frac{1}{2}yu_x F_5'(t) + yu_x C_1 + yF_7(t) + F_{10}'(t)z + C_2u_t + \frac{1}{2}u_x F_{10}(t) + u_x C_3 \\ + \frac{\beta u_y}{2\alpha}F_5(t) + \frac{\beta t u_y}{\alpha}C_1 + u_y C_4 + F_{16}(t),$$

where $F_5(t), F_7(t), F_{10}(t)$ and $F_{16}(t)$ are functions of t whereas C_1, C_2, C_3 and C_4 are arbitrary constants. The conservation laws are then constructed using the divergence identity

$$D_t T^t + D_x T^x + D_y T^y + D_z T^z = Q(u_{xxxz} - 2\alpha u_{tx} + \beta u_{yy} + 2\alpha u_x u_{xz} + \alpha u_z u_{xx}),$$

where T^t is the conserved density, and T^x, T^y, T^z are spatial fluxes. Thus, after some calculations, conserved vectors corresponding to the eight multipliers are given below.

Case 1. For the first multiplier $Q_1 = \frac{1}{\alpha}(\alpha y u_x + \beta t u_y)$, the corresponding conserved vector $(T_1^t, T_1^x, T_1^y, T_1^z)$ is given by

$$T_1^t = \frac{1}{2}\alpha y u u_{xx} - \frac{1}{2}\alpha y u_x^2 + \frac{1}{2}\beta t u u_{xy} - \frac{1}{2}\beta t u_x u_y, \\ T_1^x = \frac{1}{3}\alpha y u u_x u_{xz} - \frac{1}{3}\beta t u u_x u_{yz} - \frac{1}{3}\beta t u u_z u_{xy} + \frac{2}{3}\alpha y u_x^2 u_z + \frac{2}{3}\beta t u_x u_y u_z - \frac{1}{2}\alpha y u u_{tx} \\ + \frac{1}{2}\beta t u u_{ty} + \frac{1}{2}\beta y u u_{yy} - \frac{1}{2}\alpha y u_t u_x - \frac{1}{2}\beta t u_t u_y + \frac{1}{8}y u u_{xxxz} - \frac{3}{8\alpha}\beta t u u_{xxyy} \\ + \frac{1}{2}\beta u u_y + \frac{5}{8}y u_x u_{xxz} + \frac{1}{4\alpha}\beta t u_x u_{xyz} - \frac{3}{8}y u_{xx} u_{xz} - \frac{1}{8\alpha}\beta t u_{xx} u_{yz} + \frac{1}{8}y u_z u_{xxx} \\ + \frac{1}{8\alpha}\beta t u_z u_{xxy} + \frac{3}{8\alpha}\beta t u_y u_{xxz} - \frac{1}{4\alpha}\beta t u_{xy} u_{xz}, \\ T_1^y = \frac{2}{3}\beta u_t u_x u_{xz} + \frac{1}{3}\beta u_t u_z u_{xx} - \beta t u u_{tx} - \frac{1}{2}\beta y u u_{xy} - \frac{1}{2}\beta u u_x + \frac{1}{2}\beta y u_x u_y + \frac{1}{2\alpha}\beta^2 t u_y^2 \\ + \frac{1}{2\alpha}\beta t u u_{xxxz}, \\ T_1^z = -\frac{1}{3}\alpha y u u_x u_{xx} - \frac{1}{3}\beta t u u_x u_{xy} + \frac{1}{3}\alpha y u_x^3 + \frac{1}{3}\beta t u_x^2 u_y - \frac{1}{8}y u u_{xxxx} - \frac{1}{8\alpha}\beta t u u_{xxyy} \\ + \frac{1}{4}y u_x u_{xxx} + \frac{1}{8\alpha}\beta t u_x u_{xxy} - \frac{1}{8}y u_{xx}^2 - \frac{1}{8\alpha}\beta t u_{xx} u_{xy} + \frac{1}{8\alpha}\beta t u_y u_{xxx}.$$

Case 2. For the second multiplier $Q_2 = u_t$, we obtain the corresponding conservation law as

$$T_2^t = \frac{2}{3}\alpha u u_x u_{xz} + \frac{1}{3}\alpha u u_z u_{xx} - \frac{1}{2}\alpha u u_{tx} + \frac{1}{2}\beta u u_{yy} + \frac{1}{2}u u_{xxxz} - \frac{1}{2}\alpha u_t u_x,$$

$$\begin{aligned}
T_2^x &= \frac{1}{3}\alpha uu_z u_{tx} - \frac{1}{3}\alpha uu_x u_{tz} + \frac{2}{3}\alpha u_t u_x u_z + \frac{1}{2}\alpha uu_{tt} - \frac{1}{2}\alpha u_t^2 - \frac{3}{8}uu_{txxz} + \frac{1}{8}u_z u_{txx} \\
&\quad + \frac{1}{4}u_x u_{txz} + \frac{3}{8}u_t u_{xxz} - \frac{1}{4}u_{tx} u_{xz} - \frac{1}{8}u_{tz} u_{xx}, \\
T_2^y &= -\frac{1}{2}\beta uu_{ty} + \frac{1}{2}\beta u_t u_y, \\
T_2^z &= \frac{1}{3}\alpha u_t u_x^2 - \frac{1}{3}\alpha uu_{tx} + \frac{1}{8}u_x u_{txx} - \frac{1}{8}uu_{txxx} + \frac{1}{8}u_t u_{xxx} - \frac{1}{8}u_{tx} u_{xx}.
\end{aligned}$$

Case 3. For the second multiplier $Q_3 = u_x$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_3^t &= \frac{1}{2}\alpha uu_{xy} - \frac{1}{2}\alpha u_x u_y, \\
T_3^x &= \frac{1}{3}\alpha uu_x u_{xz} + \frac{2}{3}\alpha u_x^2 u_z - \frac{1}{2}\alpha uu_{tx} - \frac{1}{2}\alpha u_t u_x + \frac{5}{8}u_x u_{xxz} + \frac{1}{2}\beta uu_{yy} + \frac{1}{8}uu_{xxxz} \\
&\quad + \frac{1}{8}u_z u_{xxx} - \frac{3}{8}u_{xx} u_{xz}, \\
T_3^y &= -\frac{1}{2}\beta uu_{xy} + \frac{1}{2}\beta u_x u_y, \\
T_3^z &= \frac{1}{3}\alpha u_x^3 - \frac{1}{3}\alpha uu_x u_{xx} + \frac{1}{4}u_x u_{xxx} - \frac{1}{8}uu_{xxxx} - \frac{1}{8}u_{xx}^2.
\end{aligned}$$

Case 4. For the multiplier $Q_4 = u_y$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_4^t &= \frac{1}{2}\alpha uu_{xy} - \frac{1}{2}\alpha u_x u_y, \\
T_4^x &= -\frac{1}{3}\alpha uu_x u_{yz} - \frac{1}{3}\alpha uu_z u_{xy} + \frac{2}{3}\alpha u_x u_y u_z + \frac{1}{2}\alpha uu_{ty} - \frac{1}{2}\alpha u_t u_y - \frac{3}{8}uu_{xxyz} \\
&\quad - \frac{1}{8}u_{xx} u_{yz} + \frac{1}{8}u_z u_{xxy} + \frac{3}{8}u_y u_{xxz} + \frac{1}{4}u_x u_{xyz} - \frac{1}{4}u_{xy} u_{xz}, \\
T_4^y &= \frac{2}{3}\alpha uu_x u_{xz} + \frac{1}{3}\alpha uu_z u_{xx} - \alpha uu_{tx} + \frac{1}{2}uu_{xxxz} + \frac{1}{2}\beta u_y^2, \\
T_4^z &= \frac{1}{3}\alpha u_x^2 u_y - \frac{1}{3}\alpha uu_x u_{xy} + \frac{1}{8}u_x u_{xxy} - \frac{1}{8}uu_{xxxy} + \frac{1}{8}u_y u_{xxx} - \frac{1}{8}u_{xx} u_{xy}.
\end{aligned}$$

Case 5. For multiplier $Q_5 = \left(F''(t)z + \frac{1}{2}F'(t)\right)y + \frac{1}{2\alpha}\beta F(t)u_y$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_5^t &= -\alpha F''(t)yzu_x + \frac{1}{4}\alpha F(t)yu u_{xx} - \frac{1}{4}\alpha F'(t)yu_x^2 + \frac{1}{4}\beta F(t)uu_{xy} - \frac{1}{4}\beta F(t)u_x u_y, \\
T_5^x &= \frac{1}{4}F'(t)\beta yuu_{yy} - \frac{1}{6}\beta F(t)uu_z u_{xy} - \frac{1}{6}\beta F(t)uu_x u_{yz} + \frac{1}{3}\beta F(t)u_x u_y u_z \\
&\quad - \frac{3}{16\alpha}\beta F(t)uu_{xxyz} - \frac{1}{16\alpha}\beta F(t)u_{xx} u_{yz} - \frac{1}{8\alpha}\beta F(t)u_{xy} u_{xz} + \frac{1}{16\alpha}\beta F(t)u_z u_{xxy}
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{16\alpha}\beta F(t)u_y u_{xxz} + \frac{1}{8\alpha}\beta F(t)u_x u_{xyz} + \alpha F'''(t)yzu - \frac{1}{4}\alpha F''(t)yu u_x - \alpha F''(t)yzu_t \\
& - \frac{1}{4}\alpha F'(t)yu u_{tx} + \frac{1}{3}\alpha F'(t)yu_x^2 u_z - \frac{1}{4}\alpha F'(t)yu_t u_x - \frac{1}{4}F''(t)yu_{xx} + \alpha F''(t)yzu_x u_z \\
& + \frac{1}{6}\alpha F'(t)yu u_x u_{xz} + \frac{1}{16}F'(t)yu_{xxxz} - \frac{3}{16}F'(t)yu_{xx} u_{xz} + \frac{1}{16}F'(t)yu_z u_{xxx} \\
& + \frac{5}{16}F'(t)yu_x u_{xxz} + \frac{1}{4}\beta F'(t)uu_y + \frac{1}{4}\beta F(t)uu_{ty} - \frac{1}{4}\beta F(t)u_t u_y + \frac{3}{4}F''(t)yzu_{xxz}, \\
T_5^y &= \beta F''(t)yzu_y - \beta F''(t)zu - \frac{1}{4}\beta F'(t)yu u_{xy} + \frac{1}{4}\beta F'(t)yu_x u_y - \frac{1}{4}\beta F'(t)uu_x \\
& + \frac{1}{3}\beta F(t)uu_x u_{xz} + \frac{1}{6}\beta F(t)uu_z u_{xx} - \frac{1}{2}\beta F(t)uu_{tx} + \frac{1}{4\alpha}\beta^2 F(t)u_y^2 + \frac{1}{4\alpha}\beta F(t)uu_{xxxz}, \\
T_5^z &= \frac{1}{2}\alpha F''(t)yzu_x^2 + \frac{1}{4}F''(t)yzu_{xxx} - \frac{1}{6}\alpha F'(t)yu u_x u_{xx} + \frac{1}{6}\alpha F'(t)yu_x^3 - \frac{1}{6}F'(t)yu u_{xxxx} \\
& + \frac{1}{8}F'(t)yu_x u_{xxx} - \frac{1}{16}F'(t)yu_x^2 u_{xx} - \frac{1}{6}\beta F(t)uu_x u_{xy} + \frac{1}{6}\beta F(t)u_x^2 u_y - \frac{1}{16\alpha}\beta F(t)uu_{xxxy} \\
& + \frac{1}{16\alpha}\beta F(t)u_x u_{xxy} + \frac{1}{16\alpha}\beta F(t)u_y u_{xxx} - \frac{1}{16\alpha}\beta F(t)u_{xx} u_{xy}.
\end{aligned}$$

Case 6. For the multiplier $Q_6 = F(t)y$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_6^t &= -\alpha F(t)yu_x, \\
T_6^x &= \alpha F'(t)yu + \alpha F(t)yu_x u_z - \alpha F(t)yu_t + \frac{3}{4}F(t)yu_{xxz}, \\
T_6^y &= \beta(F'(t)zu_y - \frac{1}{4}\beta F(t)uu_{xy} + \frac{1}{4}\beta F(t)u_x u_y), \\
T_6^z &= \alpha F'(t)yu + \alpha F(t)yu_x u_z - \alpha F(t)yu_t + \frac{3}{4}F(t)yu_{xxz}.
\end{aligned}$$

Case 7. For the multiplier $Q_7 = F'(t)z + \frac{1}{2}F(t)u_x$, we obtain the corresponding conservation law as

$$\begin{aligned}
T_7^t &= -\alpha zu_x F'(t) + \left(\frac{1}{4}\alpha uu_{xx} - \frac{1}{4}\alpha u_x^2 \right) F(t), \\
T_7^x &= \alpha zu F''(t) + \left(\alpha zu_x u_z - \frac{1}{4}\alpha uu_x - \alpha zu_t + \frac{3}{4}zu_{xxz} - \frac{1}{4}u_{xx} \right) F'(t) + \left(\frac{1}{3}\alpha u_x^2 u_z \right. \\
& + \frac{1}{6}\alpha uu_x u_{xz} - \frac{1}{4}\alpha u_t u_x - \frac{1}{4}\alpha uu_{tx} + \frac{5}{16}u_x u_{xxz} + \frac{1}{4}\beta uu_{yy} + \frac{1}{16}uu_{xxxz} + \frac{1}{16}u_z u_{xxx} \\
& \left. - \frac{3}{16}u_{xx} u_{xz} \right) F(t), \\
T_7^y &= \beta F'(t)zu_y + \left(\frac{1}{4}\beta u_x u_y - \frac{1}{4}\beta uu_{xy} \right) F(t), \\
T_7^z &= \left(\frac{1}{2}\alpha zu_x^2 + \frac{1}{4}zu_{xxx} \right) F'(t) + \left(\frac{\alpha}{6}u_x^3 - \frac{\alpha}{6}u_x u_{xx} + \frac{1}{8}u_x u_{xxx} - \frac{1}{6}uu_{xxxx} - \frac{1}{6}u_x^2 \right) F(t)
\end{aligned}$$

Case 8. For the last multiplier $Q_8 = F(t)$, we obtain the corresponding conservation law as

$$\begin{aligned} T_8^t &= -\alpha u_x F(t), \\ T_8^x &= \alpha F'(t)u + \left(\alpha u_x u_z - \alpha u_t + \frac{3}{4} u_{xxz} \right) F(t), \\ T_8^y &= \beta u_y F(t), \\ T_8^z &= \left(\frac{1}{2} \alpha u_x^2 + \frac{1}{4} u_{xxx} \right) F(t). \end{aligned}$$

3.3.2 Conservation laws via the Noether's approach

In this subsection we utilize Noether's approach to derive conservation laws for the 3DgYTSF equation (3.2). This equation is of fourth-order and it has a Lagrangian. It can be verified that equation (3.2) has a second-order Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{2} (u_{xx} u_{xz} - \beta u_y^2 - \alpha u_x^2 u_z) + \alpha u_t u_x$$

as $\delta \mathcal{L} / \delta u = 0$ on the equation (3.2). Here the Euler-Lagrange operator $\delta / \delta u$ is given as in (3.69). The determining equation for Noether point symmetries is

$$\begin{aligned} X^{[2]}(\mathcal{L}) + \mathcal{L} \{ D_t(\tau) + D_x(\xi) + D_y(\phi) + D_z(\psi) \} - D_t(B^1) - D_x(B^2) \\ - D_y(B^3) - D_z(B^4) = 0, \end{aligned} \quad (3.71)$$

where gauge terms B^1, B^2, B^3 and B^4 depend on t, x, y, z and u . Here $X^{[2]}$ is the second prolongation of the infinitesimal generator X and is defined by

$$X^{[2]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_3 \frac{\partial}{\partial u_y} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{24} \frac{\partial}{\partial u_{xz}}.$$

Hence expanding equation (3.71) with the Lagrangian $\mathcal{L}$ gives

$$\begin{aligned} & \alpha u_x \{ \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u - u_y \phi_t - u_t u_y \phi_u - u_z \psi_t - u_t u_z \psi_u \} \\ & + \alpha u_t \{ \eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u - u_y \phi_x - u_x u_y \phi_u - u_z \psi_x - u_x u_z \psi_u \} \\ & - \alpha u_x u_z \{ \eta_x + u_x \eta_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u - u_y \phi_x - u_x u_y \phi_u - u_z \psi_x - u_x u_z \psi_u \} \end{aligned}$$

$$\begin{aligned}
& -\beta u_y \{ \eta_y + u_y \eta_u - u_t \tau_y - u_t u_y \tau_u - u_x \xi_y - u_x u_y \xi_u - u_y \phi_y - u_y^2 \phi_u - u_z \psi_y - u_y u_z \psi_u \} \\
& - \frac{1}{2} \alpha u_x^2 \{ \eta_z + u_z \eta_u - u_t \tau_z - u_t u_z \tau_u - u_x \xi_z - u_x u_z \xi_u - u_y \phi_z - u_y u_z \phi_u - u_z \psi_z - u_z^2 \psi_u \} \\
& + \frac{1}{2} u_{xz} \{ \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \tau_u - u_x^2 \tau_{uu} - 2u_{xx} \xi_x - 2u_x^2 \xi_{xu} - 3u_x u_{xx} \xi_u - u_x^3 \xi_{uu} - 2u_{tx} \tau_x \\
& - u_t \tau_{xx} - 2u_t u_x \tau_{xu} - u_t u_{xx} \tau_u - 2u_x u_{tx} \tau_u - u_t u_x^2 \tau_{uu} - u_{xy} \phi_x - u_x u_{xy} \phi_u - u_{xz} \psi_x \\
& - u_x u_{xz} \psi_u \} + \frac{1}{2} u_{xx} \{ \eta_{xz} + u_z \eta_{xu} + u_x \eta_{zu} + u_x u_z \eta_{uu} - u_t \tau_{xz} - u_t u_z \tau_{xu} - u_t u_x \tau_{zu} \\
& - u_t u_x u_z \tau_{uu} - u_x \xi_{xz} - u_x u_z \xi_{xu} - u_x^2 \xi_{zu} - u_x^2 u_z \xi_{uu} - u_y \phi_{xz} - u_y u_z \phi_{xu} - u_x u_y \phi_{zu} \\
& - u_x u_y u_z \phi_{uu} - u_z \psi_{xz} - u_z^2 \psi_{xu} - u_x u_z \psi_{zu} - u_x u_z^2 \psi_{uu} + u_{xz} \eta_u - u_{tz} \tau_x - u_x u_{tz} \tau_u - u_t u_{xz} \tau_u \\
& - u_{xz} \xi_x - 2u_x u_{xz} \xi_u - u_{yz} \phi_x - u_x u_{yz} \phi_u - u_y u_{xz} \phi_u - u_{zz} \psi_x - u_z u_{xz} \psi_u - u_x u_{zz} \psi_u \} \\
& - B_t^1 - u_t B_u^1 - B_x^2 - u_x B_u^2 - B_y^3 - u_y B_u^3 - B_z^4 - u_z B_u^4 = 0.
\end{aligned}$$

Splitting the above equation on derivatives of u yields the following system of over-determined PDE's:

$$\begin{aligned}
& \tau_u = 0, \xi_u = 0, \phi_u = 0, \psi_u = 0, \psi_y = 0, B_u^4 = 0, \eta_{xx} = 0, \eta_{xz} = 0, \tau_x = 0, \phi_x = 0, \\
& \phi_z = 0, \psi_x = 0, \tau_z = 0, \xi_z = 0, \psi_x = 0, \tau_y = 0, 3\alpha \xi_y - 2\phi_t = 0, B_u^3 + 3\alpha \eta_y = 0, \eta_{uu}, \\
& \xi_{xx} = 0, \eta_{xu} = 0, \eta_{zu} = 0, B_u^2 - 2\eta_t = 0, \eta_x + \psi_t = 0, B_u^1 - 2\eta_x = 0, \eta_z + 2\xi_t = 0, \\
& 2\eta_u + \phi_y + \psi_z = 0, B_t^1 + B_x^2 + B_y^3 + B_z^4 = 0, 3\eta_u + \tau_t - \xi_x + \phi_y = 0, 2\eta_u + \tau_t + 2\xi_x, \\
& + \phi_y + 2\eta_u + \tau_t + \xi_x - \phi_y + \psi_z = 0
\end{aligned} \tag{3.72}$$

Solving the above overdetermined system of PDEs, we obtain the Noether symmetries and their gauge functions as follows:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, B^t = 0, B^x = 0, B^y = 0, B^z = 0, \\
X_2 &= \frac{\partial}{\partial x}, B^t = 0, B^x = 0, B^y = 0, B^z = 0, \\
X_3 &= \frac{\partial}{\partial y}, B^t = 0, B^x = 0, B^y = 0, B^z = 0, \\
X_4 &= \frac{\partial}{\partial z}, B^t = 0, B^x = 0, B^y = 0, B^z = 0, \\
X_5 &= \frac{\partial}{\partial u}, B^t = 0, B^x = 0, B^y = 0, B^z = 0, \\
X_6 &= y \frac{\partial}{\partial u}, B^t = 0, B^x = 0, B^y = 0, B^z = 0,
\end{aligned}$$

$$X_7 = 7t \frac{\partial}{\partial t} + 3x \frac{\partial}{\partial x} + 5y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - 3u \frac{\partial}{\partial u}, \quad B^t = 0, \quad B^x = 0, \quad B^y = 0, \quad B^z = 0.$$

We now determine the conservation laws corresponding to each Noether point symmetry using [31]

$$\begin{aligned} T^t &= \mathcal{L}\tau + K_1 \frac{\partial \mathcal{L}}{\partial u_t} - B^t, \\ T^x &= \mathcal{L}\xi + K_1 \left\{ \frac{\partial \mathcal{L}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right\} + K_2 \frac{\partial \mathcal{L}}{\partial u_{xx}} + K_3 \frac{\partial \mathcal{L}}{\partial u_{xz}} - B^x, \\ T^y &= \mathcal{L}\phi + K_1 \frac{\partial \mathcal{L}}{\partial u_y} - B^y, \\ T^z &= \mathcal{L}\psi + K_1 \left\{ \frac{\partial \mathcal{L}}{\partial u_z} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xz}} \right) \right\} - B^z, \end{aligned} \tag{3.73}$$

where $K_1 = \eta - u_t\tau - u_x\xi - u_y\phi - u_z\psi$, $K_2 = \zeta_2 - u_{xt}\tau - u_{xx}\xi - u_{xy}\phi - u_{xz}\psi$ and $K_3 = \zeta_4 - u_{tz}\tau - u_{xz}\xi - u_{yz}\phi - u_{zz}\psi$. Hence by using formula (3.73), we obtain conservation laws whose components are

$$\begin{aligned} T_1^t &= \frac{1}{2}u_{xx}u_{xz} - \frac{3}{2}\alpha u_y^2 - u_x^2u_z, \\ T_1^x &= 2u_tu_xu_z + \frac{3}{4}u_tu_{xxz} - \frac{1}{4}u_{xx}u_{tz} - \frac{1}{2}u_{tx}u_{xz} - 2u_t^2, \\ T_1^y &= 3\alpha u_tu_y, \\ T_1^z &= u_tu_x^2 + \frac{1}{4}u_tu_{xxx} - \frac{1}{4}u_{xx}u_{tx}; \end{aligned}$$

$$\begin{aligned} T_2^t &= -2u_x^2, \\ T_2^x &= u_x^2u_z - \frac{1}{4}u_{xx}u_{xz} + \frac{3}{4}u_xu_{xxz} - \frac{3}{2}\alpha u_y^2, \\ T_2^y &= 3\alpha u_xu_y, \\ T_2^z &= u_x^3 + \frac{1}{4}u_{xxx}u_x - \frac{1}{4}u_{xx}^2; \end{aligned}$$

$$\begin{aligned} T_3^t &= -2u_xu_y, \\ T_3^x &= -2u_tu_y + 2u_xu_yu_z - \frac{1}{2}u_{xy}u_{xz} - \frac{1}{4}u_{xx}u_{yz} + \frac{3}{4}u_yu_{xxz}, \\ T_3^y &= 2u_tu_x - u_x^2u_z + \frac{1}{2}u_{xx}u_{xz} + \frac{3}{2}\alpha u_y^2, \\ T_3^z &= u_x^2u_y - \frac{1}{4}u_{xx}u_{xy} + \frac{1}{4}u_{xxx}u_y; \end{aligned}$$

$$T_4^t = -2u_x u_z,$$

$$T_4^x = -2u_t u_z + 2u_x u_z^2 + \frac{3}{4}u_z u_{xxz} - \frac{1}{2}u_{xz}^2 - \frac{1}{4}u_{xx} u_{zz},$$

$$T_4^y = 3\alpha u_y u_z,$$

$$T_4^z = 2u_t u_x + \frac{1}{4}u_{xx} u_{xz} + \frac{1}{4}u_{xxx} u_z - \frac{3}{2}\alpha u_y^2;$$

$$T_5^t = 2u_x,$$

$$T_5^x = 2u_t - 2u_x u_z - \frac{3}{4}u_{xxz},$$

$$T_5^y = -3\alpha u_y,$$

$$T_5^z = -u_x^2 - \frac{1}{4}u_{xxx};$$

$$T_6^t = 2yu_x,$$

$$T_6^x = 2yu_t - 2yu_x u_z - \frac{3}{4}yu_{xxz},$$

$$T_6^y = -3\alpha y u_y,$$

$$T_6^z = -yu_x^2 - \frac{1}{4}yu_{xxx};$$

$$T_7^t = \frac{7}{2}tu_{xx}u_{xz} - \frac{21}{2}\alpha tu_y^2 - 7tu_x^2 u - 6uu_x - 6xu_x^2 - 10yu_x u_y - 2zu_x u_z,$$

$$\begin{aligned} T_7^x = & -\frac{9}{2}\alpha xu_y^2 + 3xu_x^2 u_z - 6uu_t - 14tu_t^2 - 10yu_t u_y - 2zu_t u_z + 6uu_t u_z + 14tu_t u_x u_z \\ & + 10yu_x u_y u_z + 2zu_x u_z^2 + \frac{3}{2}uu_{xxxz} + \frac{7}{2}tu_t u_{xxz} + \frac{3}{2}xu_x u_{xxz} + \frac{5}{2}yu_y u_{xxz} \\ & + \frac{1}{2}zu_z u_{xxz} - 3u_x u_{xz} - \frac{7}{2}tu_{tx} u_{xz} - \frac{5}{2}yu_{xy} u_{xz} - 2u_{xx} u_z - \frac{7}{2}tu_{tz} u_{xx} - \frac{3}{2}xu_{xx} u_{xz} \\ & - \frac{5}{2}u_{xx} u[yz] - \frac{1}{2}zu_{xx} u_{zz}, \end{aligned}$$

$$\begin{aligned} T_7^y = & 10yu_t u_x + \frac{5}{2}yu_{xx} u_{xz} + \frac{15}{2}\alpha y u_y^2 - 5yu_x^2 u_z + 9\alpha u u_y + 21\alpha t u_t u_y + 9\alpha x u_x u_y \\ & + 3\alpha z u_y u_z, \end{aligned}$$

$$T_7^z = 2zu_t u_x + \frac{1}{2}zu_{xx} u_{xz} - \frac{3}{2}\alpha z u_y^2 + 3uu_x^2 + 7tu_t u_x^2 + 3xu_x^3 + 5yu_x^2 u_y + \frac{3}{2}uu_{xxx}$$

$$+ \frac{7}{2}tu_tu_{xxx} + \frac{3}{2}xu_xu_{xxx} + \frac{5}{2}yu_yu_{xxx} + \frac{1}{2}zu_zu_{xxx}.$$

3.4 Conclusion

In this chapter we presented the Lie point symmetries, symmetry reductions and exact solutions of the fourth-order (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation (3.2). The group invariant solutions obtained included solution in terms of incomplete elliptic integral. It should be noted that the obtained solutions are new and different from the ones presented in the literature. We then derived the conservation laws for this equation by using the multiplier approach which produced eight multipliers and consequently, eight conservation laws. Finally, we employed the celebrated Noether's approach to construct conservation laws for the underlying equation.

Chapter 4

Conserved quantities and solutions of a generalized fifth-order KdV equation

In this chapter we study a generalized fifth-order Korteweg-de Vries (KdV) equation. Firstly, we compute point symmetries of this model and then utilize them to perform symmetry reductions. Also we find soliton solutions and calculate the conservation laws of this equation via the two techniques, namely the multiplier method and Noether's theorem.

4.1 Introduction

The physical problems that arise from different fields of mathematics and science are modelled by the nonlinear partial differential equations (PDEs). The problems incorporate fluid dynamics, chemical physics, hydrodynamics, fluid mechanics, heat and mass transfer, solid-state physics, chemical kinematics, plasma physics, etc.

The KdV equation [26] is a nonlinear PDE that has a wide range of applications. Magneto-acoustic waves, ion-acoustic waves, nonlinear LC circuit waves, and shallow

water waves in plasmas are some of the examples. Several extensions and modifications have resulted in a variety of KdV equations that appear in three, five, seven, or higher order forms [46–51]. In [46], the authors used the group method and the multiplier approach to study the generalized fifth-order KdV equation that reads

$$u_t + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} = 0, \quad (4.1)$$

where α , β and γ are the nonzero arbitrary constants.

In this chapter we aim to find all the point symmetry of (4.1) and use them to perform symmetry reductions. In addition, we find groups of transformations corresponding to each symmetry of (4.1). Moreover, conservation laws using the multiplier approach are obtained for (4.1). Furthermore, we derive conservation laws for (4.1) when $\beta = 2\alpha$ by invoking Noether's theorem.

The contents of this chapter are being prepared as a paper which is yet to be submitted [52].

4.2 Symmetry reductions and solutions

4.2.1 Point symmetries of (4.1)

We establish all Lie point symmetries of (4.1). We consider a vector field Γ written as

$$\Gamma = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \Phi \frac{\partial}{\partial u}, \quad (4.2)$$

with the coefficients (τ, ξ, Φ) depending on t, x, u . We recall that Γ is a point symmetry of (4.1) if it satisfies

$$\Gamma^{[5]}G|_{(4.1)} = 0, \quad (4.3)$$

where

$$G \equiv u_t + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx}.$$

We note that $\Gamma^{[5]}$ is the fifth extension of (4.2) and is defined as

$$\Gamma^{[5]} = \Gamma + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{222} \frac{\partial}{\partial u_{xxx}} + \zeta_{2222} \frac{\partial}{\partial u_{xxxx}}, \quad (4.4)$$

where the coefficients ζ s are

$$\begin{aligned} \zeta_1 &= D_t(\Phi) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_2 &= D_x(\Phi) - u_t D_x(\tau) - u_x D_x(\xi), \\ \zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), \\ \zeta_{222} &= D_x(\zeta_{22}) - u_{txx} D_x(\tau) - u_{xxx} D_x(\xi), \\ \zeta_{2222} &= D_x(\zeta_{222}) - u_{txxx} D_x(\tau) - u_{xxxx} D_x(\xi), \end{aligned}$$

with D_t, D_x being total derivatives given as

$$\begin{aligned} D_x &= \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{tx} \partial_{u_t} + \dots, \\ D_t &= \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \dots. \end{aligned} \quad (4.5)$$

Expansion of (4.3) leads to

$$\alpha \Phi u_{xxx} + 2\gamma \Phi u u_x + \zeta_1 + \beta u_{xx} \zeta_2 + \gamma u^2 \zeta_2 + \beta u_x \zeta_{22} + \alpha u \zeta_{222} + \zeta_{2222} \Big|_{(4.1)} = 0. \quad (4.6)$$

Substituting the values of ζ s into (4.6), replacing u_{xxxxx} by $-u_t - \alpha u u_{xxx} - \beta u_x u_{xx} - \gamma u^2 u_x$ and decomposing the resultant equation on various derivatives of u , we get

$$\tau_x = 0, \tau_u = 0, \quad (4.7)$$

$$\xi_u = 0, \quad (4.8)$$

$$\xi_{xx} = 0, \quad (4.9)$$

$$5\xi_x - \tau_t = 0, \quad (4.10)$$

$$\Phi_t = 0, \quad (4.11)$$

$$\Phi_{uu} = 0, \quad (4.12)$$

$$\Phi_u - 3\xi_x + \tau_t = 0, \quad (4.13)$$

$$5\alpha\Phi + 2\alpha u \tau_t = 0, \quad (4.14)$$

$$10\gamma u \Phi + 4\gamma u^2 \tau_t - 5\xi_t = 0. \quad (4.15)$$

From (4.7), we obtain

$$\tau = A(t), \quad (4.16)$$

where A is a function of t . Integration of (4.8) gives

$$\xi = B(t, x), \quad (4.17)$$

where B is a function of x and t . Substituting the values of τ and ξ into (4.10), gives $5B_x - A'(t) = 0$ and this solves to

$$B = \frac{1}{5}xA'(t) + C(t),$$

where C is a function of t . Hence

$$\xi = \frac{1}{5}xA'(t) + C(t). \quad (4.18)$$

The value of ξ from (4.18) satisfies equation (4.9). Equation (4.12) solves to

$$\Phi = uD(t, x) + E(t, x), \quad (4.19)$$

where D and E are functions of t and x . Substituting this value of Φ into equation (4.11) gives

$$uD_t + E_t = 0.$$

Splitting the above equation on various powers of u , we obtain

$$u : D_t = 0, \quad (4.20)$$

$$\text{Rest} : E_t = 0. \quad (4.21)$$

Solving equations (4.20) and (4.21) respectively gives $D = D(x)$ and $E = E(x)$. Hence

$$\Phi = uD(x) + E(x).$$

Substituting the values of Φ , τ , and ξ into equation (4.13) gives

$$D(x) + \frac{2}{5}A'(t) = 0. \quad (4.22)$$

Thus, solving (4.22) we get the values

$$D(x) = c_1, \quad A(t) = -\frac{5}{2}c_1t + c_2, \quad (4.23)$$

where c_1 and c_2 are integration constants. Hence

$$\begin{aligned} \tau &= -\frac{5}{2}c_1t + c_2, \\ \xi &= -\frac{1}{2}c_1x + C(t), \\ \Phi &= c_1u + E(x). \end{aligned}$$

Using equation (4.14), we get

$$E(x) = 0, \quad (4.24)$$

which implies that

$$\Phi = c_1u. \quad (4.25)$$

Substituting the values of τ , ξ , Φ into equation (4.15) gives

$$C'(t) = 0, \quad (4.26)$$

and this solves to $C(t) = c_3$, where c_3 is a constant. Therefore the infinitesimals of equation(4.1) are

$$\tau = -\frac{5}{2}c_1t + c_2, \quad (4.27)$$

$$\xi = -\frac{1}{2}c_1x + c_3, \quad (4.28)$$

$$\Phi = c_1u. \quad (4.29)$$

Hence the Lie point symmetries of the fifth-order KdV equation (4.1) are

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial t}, \\ \Gamma_2 &= \frac{\partial}{\partial x}, \\ \Gamma_3 &= 5t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \end{aligned}$$

with Γ_1 representing time translation, Γ_2 representing space translation and Γ_3 representing scaling symmetry.

4.2.2 Commutation table

We seek the commutation relations for the symmetries $(\Gamma_1, \Gamma_2, \Gamma_3)$. We calculate $[\Gamma_1, \Gamma_3]$ using the definition of Lie bracket. Thus we have

$$\begin{aligned}
[\Gamma_1, \Gamma_3] &= \Gamma_1\Gamma_3 - \Gamma_3\Gamma_1 \\
&= \frac{\partial}{\partial t} \left(5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} \right) - \left(5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} \right) \frac{\partial}{\partial t} \\
&= 5 \frac{\partial}{\partial t} \\
&= 5\Gamma_1.
\end{aligned}$$

Similarly, the preceding approach can be used to compute all of the remaining commutation relations. The commutation relations are shown in Table 1.

Table 1. Lie brackets of (4.1)

$[\Gamma_k, \Gamma_i]$	Γ_1	Γ_2	Γ_3
Γ_1	0	0	$5\Gamma_1$
Γ_2	0	0	Γ_2
Γ_3	$-5\Gamma_1$	$-\Gamma_2$	0

4.2.3 One-parameter group of transformations

Solving the following Lie equations with initial conditions

$$\begin{aligned}
\frac{d\bar{t}}{da} &= \tau(\bar{t}, \bar{x}, \bar{u}), \quad \bar{t}|_{a=0} = t, \\
\frac{d\bar{x}}{da} &= \xi(\bar{t}, \bar{x}, \bar{u}), \quad \bar{x}|_{a=0} = x, \\
\frac{d\bar{u}}{da} &= \Phi(\bar{t}, \bar{x}, \bar{u}), \quad \bar{u}|_{a=0} = u,
\end{aligned}$$

we obtain the one-parameter group of transformations. For each Γ_i , let T_{a_i} be the corresponding group. We first compute one-parameter group corresponding to infinitesimal generator Γ_3 , namely

$$\Gamma_3 = 5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 5\bar{t}, \quad \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = \bar{x}, \quad \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = -2\bar{u}, \quad \bar{u}|_{a=0} = u.$$

Solving the above equations we obtain

$$\bar{t} = te^{5a_3}, \quad \bar{x} = xe^{a_3}, \quad \bar{u} = ue^{-2a_3}.$$

Thus the one-parameter group T_{a_3} corresponding to the operator Γ_3 is given by

$$T_{a_3} : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{5a_3}, xe^{a_3}, ue^{-2a_3}).$$

If we continue in the same manner as above, we get the following one-parameter groups for the remaining operators. Thus we obtain the group transformations generated by the infinitesimal generator Γ_i ($i = 1, 2, 3$),

$$T_1 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u),$$

$$T_2 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u),$$

$$T_3 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (te^{5a_3}, xe^{a_3}, ue^{-2a_3}).$$

Theorem. If $u = N(t, x)$ is a known solution of equation (4.1), then by using the symmetry groups T_i ($i = 1, 2, 3$), so are the functions

$$u_1 = N(t + a, x),$$

$$u_2 = N(t, x + a),$$

$$u_3 = e^{-2a} N(e^{-5a}t, e^{-a}x).$$

4.2.4 Construction of invariant solutions

The fifth-order KdV equation (4.1) has three point symmetries so we consider three cases of invariant solutions. Also travelling wave solutions are derived as case 4.

Case 1. First, we look at the symmetry operator Γ_1 , namely.,

$$\Gamma_1 = \frac{\partial}{\partial t}.$$

This point symmetry has its associated characteristic equations given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}$$

and solving these give the invariants

$$J_1 = x \text{ and } J_2 = u. \quad (4.30)$$

Thus the invariant solution related to (4.30) is

$$u = F(x). \quad (4.31)$$

Inserting this value of u into (4.1), we obtain the nonlinear ODE

$$\alpha F F''' + \beta F' F'' + \gamma F^2 F' + F'''' = 0. \quad (4.32)$$

In case when $\beta = \alpha$, the nonlinear ODE (4.32) becomes

$$\alpha F F''' + \alpha F' F'' + \gamma F^2 F' + F'''' = 0$$

and can be integrated to fourth-order nonlinear ODE

$$\alpha F F'' + \gamma \frac{1}{3} F^3 + F'''' = C_1,$$

where C_1 is a constant of integration. We have managed to reduce the fifth-order PDE (4.1) to a fourth order nonlinear ODE for the special case when $\beta = \alpha$.

Case 2. Next, we consider the Lie symmetry

$$\Gamma_2 = \frac{\partial}{\partial x} \quad (4.33)$$

and proceed to find invariant solution. For this symmetry we have

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}$$

as its characteristic equations. Thus, they give invariants

$$J_1 = t, \text{ and } J_2 = u.$$

Hence the group-invariant solution is

$$u = \Psi(t)$$

and inserting this value into (4.1), we obtain

$$\Psi'(t) = 0,$$

which on integration gives $\Psi(t) = c_1$, where c_1 is an arbitrary constant. Thus the solution for (4.1) under Γ_2 is

$$u(t, x) = c_1.$$

Case 3. We now consider the Lie point symmetry

$$\Gamma_3 = 5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}. \quad (4.34)$$

For this symmetry, continuing in the same manner as above we obtain invariants

$$C_1 = xt^{-\frac{1}{5}}, \quad C_2 = ut^{\frac{2}{5}}. \quad (4.35)$$

Consequently, the invariant solution under Γ_3 is written as

$$u(t, x) = t^{-\frac{2}{5}} \phi(z), \quad z = xt^{-\frac{1}{5}}. \quad (4.36)$$

Substituting (4.36) into (4.1) yields a nonlinear ODE

$$-2\phi - z\phi' + 5\alpha\phi\phi''' + 5\beta\phi'\phi'' + 5\gamma\phi^2\phi' + 5\phi'''' = 0.$$

Case 4. (Travelling wave solution) For the symmetry $\Gamma_1 + c\Gamma_2$, we obtain two invariants

$$\xi = x - ct, \quad u = U, \quad (4.37)$$

which result in invariant solution $u = U(\xi)$. Using this, equation (4.1) is transformed to the nonlinear ODE

$$-cU' + \alpha UU''' + \beta U'U'' + \gamma U^2U' + U'''' = 0. \quad (4.38)$$

To solve equation (4.38), we utilize the power series method. With this technique, we suppose that (4.38) has the solution of the form

$$U(\xi) = \sum_{n=0}^{\infty} \mathbf{c}_n \xi^n, \quad (4.39)$$

where $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are constants to be determined. Inserting (4.39) into the nonlinear ODE (4.38) gives

$$\begin{aligned}
& 120c_5 + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\xi^n + \gamma c_0^2 c_1 \\
& + \gamma \sum_{n=1}^{\infty} \sum_{k=0}^n \sum_{j=0}^k (n+1-k)c_j c_{k-j} c_{n+1-k} \xi^n + 2\beta c_1 c_2 + \beta \sum_{n=1}^{\infty} \sum_{k=0}^n (k+1) \\
& \times (n+1-k)(n+2-k)c_{k+1} c_{n+2-k} \xi^n + 6\alpha c_0 c_3 + \alpha \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) \\
& (n+2-k) \times (n+3-k)c_k c_{n+3-k} c_k c_{n+3-k} \xi^n - \frac{1}{5} \sum_{n=1}^{\infty} n c_n \xi^n \\
& - \left(\frac{2}{5} c_0 + \frac{2}{5} \sum_{n=1}^{\infty} c_n \xi^n \right) = 0. \tag{4.40}
\end{aligned}$$

Inspecting the coefficients of $n = 0$ in (4.40), we obtain

$$\mathbf{c}_5 = \frac{\frac{2}{5}\mathbf{c}_0 - \gamma\mathbf{c}_0^2 c_1 - 2\beta\mathbf{c}_1 \mathbf{c}_2 - 6\gamma\mathbf{c}_0 \mathbf{c}_3}{120}, \tag{4.41}$$

whereas when $n \geq 1$ we have

$$\begin{aligned}
\mathbf{c}_{n+5} = & \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \times \left(\gamma \sum_{k=1}^n \sum_{j=0}^k (n+1-k) \mathbf{c}_j \mathbf{c}_{k-j} \mathbf{c}_{n+1-k} \right. \\
& + \beta \sum_{k=0}^n (k+1)(n+1-k)(n+2-k) \mathbf{c}_{k+1} \mathbf{c}_{n+2-k} + \alpha (n+1-k)(n+2-k) \\
& \left. (n+3-k) \mathbf{c}_k \mathbf{c}_{n+3-k} - \frac{1}{5} n \mathbf{c}_n - \frac{2}{5} \mathbf{c}_n \right). \tag{4.42}
\end{aligned}$$

Thus, the power series solution of (4.42) is

$$U(\xi) = \mathbf{c}_0 + \mathbf{c}_1 \xi + \mathbf{c}_2 \xi^2 + \mathbf{c}_3 \xi^3 + \mathbf{c}_4 \xi^4 + \mathbf{c}_5 \xi^5 + \sum_{n=1}^{\infty} \mathbf{c}_{n+5} \xi^{n+5}.$$

Consequently the power series solution of (4.1) is

$$\begin{aligned}
u(t, x) = & \mathbf{c}_0 + \mathbf{c}_1(x - ct) + \mathbf{c}_2(x - ct)^2 + \mathbf{c}_3(x - ct)^3 + \mathbf{c}_4(x - ct)^4 \\
& + \frac{\frac{2}{5}\mathbf{c}_0 - \gamma\mathbf{c}_0^2 c_1 - 2\beta\mathbf{c}_1 \mathbf{c}_2 - 6\gamma\mathbf{c}_0 \mathbf{c}_3}{120} (x - ct)^5 + \mathbf{c}_{n+5} (x - ct)^{n+5}, \tag{4.43}
\end{aligned}$$

where the coefficient of $\mathbf{c}_{n+5}$ for all values such that $n \geq 1$ are given by (4.42).

4.3 Soliton solutions

We now attempt to obtain solitary wave solutions to the model equation (4.1). The procedure of undetermined coefficients will be adopted to retrieve these solutions. We construct four types of solutions for (4.1).

4.3.1 Solitary waves

To obtain solitary waves we assume

$$u(t, x) = A \operatorname{sech}^p z, \quad z = B(x - vt), \quad (4.44)$$

where A represents the amplitude of the soliton, B denotes the inverse width of the soliton whereas v is the speed. The positive parameter p will be obtained using the balancing principle. Inserting (4.44) into (4.1) gives

$$\begin{aligned} & (v - p^4 \delta b^4) \operatorname{sech}^{p+1} \tau - p^2 (\beta + \alpha) A B^2 \operatorname{sech}^{2p+1} \tau - \gamma A^2 \operatorname{sech}^{(2+1)p+1} \tau + (1 + p)(2\alpha \\ & + p(\beta + \alpha)) A B^2 \operatorname{sech}^{2p+3} \tau + 2(1 + p)(2 + p)[2 + p(2 + p)] \delta B^4 \operatorname{sech}^{p+3} \tau - (1 + p) \\ & (2 + p)(3 + p)(4 + p) \delta B^4 \operatorname{sech}^{p+5} \tau = 0. \end{aligned} \quad (4.45)$$

Equating the exponents using balancing principle $(2 + 1)p + 1$ with $p + 5$ yields $p = 2$. The system (4.45) gives

$$\begin{aligned} & (v - 16\delta B^4) \operatorname{sech}^3 \tau - 4[(\beta + \alpha)A - 60\delta B^2] B^2 \operatorname{sech}^5 \tau - [360\delta B^4 + \gamma A^2 - 6(2\alpha + \beta) \\ & A B^2] \operatorname{sech}^7 \tau = 0. \end{aligned} \quad (4.46)$$

Thus equating the coefficient of the linearly independent function $\operatorname{sech}^j \tau$ for $j = 3, 5, 7$ to zero we obtain

$$v = 16\delta B^4, \quad (4.47)$$

$$A = \frac{60\delta B^2}{\beta + \alpha}, \quad (4.48)$$

constrained by

$$\beta + \alpha \neq 0 \quad (4.49)$$

and the identity

$$(2\alpha + \beta)(\alpha + \beta) - (\beta + \alpha)^2 - 10\gamma\delta = 0. \quad (4.50)$$

Thus, the solitary wave solution of (4.1) is

$$u(t, x) = A \operatorname{sech}^2[B(x - vt)] \quad (4.51)$$

where speed v is provided in (4.47) and the amplitude A is given in (4.48). Therefore, for solution to exist the identity (4.50) has to be satisfied along with the condition (4.49).

4.3.2 Shock waves

We assume

$$u(t, x) = A \tanh^p \tau. \quad (4.52)$$

Substituting (4.52) into (4.1) gives

$$\begin{aligned} & [v - 2p^2(5 + 3p^2)\delta B^4] \tanh^{p-1} \tau - \gamma A^2 \tanh^{(2+1)p-2} \tau + 4(p-1)(p-2)[2 + p(p-2)] \delta \\ & B^4 \tanh^{p-3} \tau - (p-1)(p-2)(p-3)(p-4) \delta B^4 \tanh^{p-5} \tau + 2p^2(\beta + \alpha) AB^2 \tanh^{2p-1} \tau \\ & + (p-1) \times [2\alpha - p(\beta + \alpha)] AB^2 \tanh^{2p-3} \tau + 4(p+1)(p+2)[2 + p(2+p)] \delta B^4 \\ & \tanh^{p+1} \tau - (p+1)(p+2)(p+3)(p+4) \delta B^4 \tanh^{p+3} \tau - (p+1)[2\alpha + p(\beta + \alpha)] \\ & AB^2 \tanh^{2p+1} \tau = 0, \end{aligned} \quad (4.53)$$

where the meaning of A, p and τ is as in the previous subsection. Equation (4.53) can be written as

$$\begin{aligned} & [v - 136\delta B^4 - 2\beta AB^2] \tanh \tau + 8[(\beta + \alpha)A + 60\delta B^2] B^2 \tanh^3 \tau - [6(\beta + 2\alpha) AB^2 \\ & + 360\delta B^4 + \gamma A^2] \tanh^5 \tau = 0. \end{aligned} \quad (4.54)$$

Equating to zero the coefficients of the linearly independent functions $\tanh^j \tau$ for $j = 1, 3, 5$ take us to

$$v = 2(\beta A + 68\delta B^2) B^2 \quad (4.55)$$

and the amplitude is

$$A = -\frac{60\delta B^2}{\beta + \alpha} \quad (4.56)$$

along with the condition (4.49). The identity

$$(\beta + \alpha)^2 - (2\alpha + \beta)(\alpha + \beta) + 10\gamma\delta = 0 \quad (4.57)$$

has to be satisfied for the shock waves to exist. Hence the shock wave to (4.1) is given by

$$u(t, x) = A \tanh^2[B(x - vt)], \quad (4.58)$$

where the amplitude is given in (4.56) with the speed to be (4.55) along with the corresponding conditions.

4.3.3 Singular solitary waves (Type-I)

The starting hypothesis we take is

$$u(t, x) = A \operatorname{csch}^p \tau. \quad (4.59)$$

The substitution of (4.59) into (4.1) yields

$$\begin{aligned} & (v - p^4 \delta B^4) \operatorname{csch}^{p+1} \tau - p^2 (\beta + \alpha) A B^2 \operatorname{csch}^{2p+1} \tau - \gamma A^n \operatorname{csch}^{(n+1)p+1} \tau - 2(1 + p) \\ & (2 + p) \times (2 + p(2 + p)) \delta B^4 \operatorname{csch}^{p+3} \tau - (1 + p) [2\alpha + p(\beta + \alpha)] A B^2 \operatorname{csch}^{2p+3} \tau \\ & - (1 + p)(2 + p)(3 + p)(4 + p) \delta B^4 \operatorname{csch}^{p+5} \tau = 0, \end{aligned} \quad (4.60)$$

where the meaning of A, p and τ are as in the previous subsection. Equation (4.60) becomes

$$\begin{aligned} & (v - 16\delta B^4) \operatorname{csch}^3 \tau - 4[(\beta + \alpha)A + 60\delta B^2] B^2 \operatorname{csch}^5 \tau - [360\delta B^4 + \gamma A^2 + 6(2\alpha + \beta) \\ & A B^2] \operatorname{csch}^7 \tau = 0. \end{aligned} \quad (4.61)$$

After equating the coefficients of linearly independent functions $\operatorname{csch}^j \tau$ for $j = 3, 5, 7$ to zero one gets the wave speed as in (4.47), the amplitude as in (4.56), and the

identity turns out to be the same as in (4.57). Thus the type-I singular solitary wave solution to equation (4.1) is given by

$$u(t, x) = A \operatorname{Acsch}^2[B(x - vt)], \quad (4.62)$$

where the speed and amplitude are given in (4.47) and (4.56) respectively, while the identity (4.57) has to be satisfied in order to preserve the existence of a solitary wave.

4.3.4 Singular solitary waves (Type-II)

To get type-II singular solitary wave solutions to (4.1) we assume

$$u(t, x) = A \coth^p \tau, \quad (4.63)$$

where the meaning of A, p and τ are as in the previous subsection. Inserting (4.63) into (4.1) we get

$$\begin{aligned} & [v - 2p^2(5 + 3p^2)\delta B^4] \coth^{p-1} \tau - \gamma A^2 \coth^{(2+1)p-1} \tau + 4(p-1)(p-2)[2 + p(p-2)]\delta \\ & B^4 \coth^{p-3} \tau - (p-1)(p-2)(p-3)(p-4)\delta B^4 \coth^{p-5} \tau + 2p^2(\beta + \alpha)AB^2 \coth^{2p-1} \tau \\ & + (p-1) \times [2\alpha - p(\beta + \alpha)]AB^2 \coth^{2p-3} \tau + 4(p+1)(p+2)[2 + p(2+p)]\delta B^4 \\ & \coth^{p+1} \tau - (p+1)(p+2)(p+3)(p+4)\delta B^4 \coth^{p+3} \tau - (p+1)[2\alpha + p(\beta + \alpha)]AB^2 \\ & \coth^{2p+1} \tau = 0. \end{aligned} \quad (4.64)$$

After replacing p by 2, equation (4.64) reduces to

$$\begin{aligned} & [v - 136\delta B^4 - 2\beta AB^2] \coth \tau + 8[(\beta + \alpha)A + 60\delta B^2]B^2 \coth^3 \tau - [6(\beta + 2\alpha)AB^2 \\ & + 360\delta B^4 + \gamma A^2] \coth^5 \tau = 0. \end{aligned} \quad (4.65)$$

The results (4.55) to (4.57) reappear. Thus the type-II singular solitary wave solution to (4.1) is

$$u(t, x) = A \coth^2[B(x - vt)], \quad (4.66)$$

where the amplitude is given (4.56) while the speed is (4.55) along with the corresponding conditions.

4.4 Conservation laws of (4.1)

In this section we construct conservation laws for the fifth-order KdV equation (4.1).

4.4.1 Conservation laws using the multiplier approach

The multiplier technique is used in this section to calculate conserved vectors for the fifth-order KdV equation (4.1). We shall consider zeroth-order multipliers, viz., $\Lambda = \Lambda(t, x, u)$, and use these multipliers to establish the conservation laws of (4.1). Recall that this equation's zeroth-order multipliers Λ are determined from

$$\frac{\delta}{\delta u} \left\{ \Lambda (u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxx}) \right\} = 0. \quad (4.67)$$

Expansion of (4.67) results in

$$\begin{aligned} & \Lambda_u (u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxx}) + \alpha u_{xxx} \Lambda + 2\gamma u u_x \Lambda - D_t(\Lambda) \\ & - D_x(\beta u_{xx} \Lambda) - D_x(\gamma u^2 \Lambda) - D_x^2(\beta u_x \Lambda) - D_x^3(\alpha u \Lambda) - D_x^5(\Lambda) = 0. \end{aligned} \quad (4.68)$$

Using the total derivatives, viz.,

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots \end{aligned} \quad (4.69)$$

in (4.68), we get

$$\begin{aligned} & u_t \Lambda_u + \alpha u u_{xxx} \Lambda_u + \beta u_x u_{xx} \Lambda_u + \gamma u^2 u_x \Lambda_u + u_{xxxx} \Lambda_u + \alpha u_{xxx} \Lambda + 2\gamma u u_x \Lambda - \Lambda_t + u_t \Lambda_u \\ & - \beta u_{xx} \Lambda_x - \beta u_x u_{xx} \Lambda_u - \beta u_{xxx} \Lambda - \gamma u^2 \Lambda_x + \gamma u^2 u_x \Lambda_u - 2\gamma u u_x \Lambda - \beta u_x \Lambda_{xx} - 2\beta u_x^2 \Lambda_{xu} \\ & - 2\beta u_{xx} \Lambda_x - 3\beta u_x u_{xx} \Lambda_u - \beta u_x^3 \Lambda_{uu} - \beta u_{xxx} \Lambda - \alpha u \Lambda_{xxx} - 3\alpha u u_x \Lambda_{xxu} - 3\alpha u u_x^2 \Lambda_{xuu} \\ & - 6\alpha u_x^2 \Lambda_{xu} - 2\alpha u u_{xx} \Lambda_{xu} - 3\alpha u_x \Lambda_{xx} - 3\alpha u_{xx} \Lambda_x - \alpha u u_x^3 \Lambda_{uuu} - 2\alpha u u_{xx} u_x \Lambda_{xu} - 3\alpha u_x^3 \Lambda_{uu} \\ & - 3\alpha u_x u_{xx} \Lambda_u - \alpha u_{xxx} \Lambda - \Lambda_{xxxx} + 5u_x \Lambda_{xxxu} + 10u_{xx} \Lambda_{xxu} + 30u_x u_{xx} \Lambda_{xuu} + 10u_{xxx} \Lambda_{xuu} \\ & + 10u_x^2 \Lambda_{xxxu} + 10u_x^3 \Lambda_{xxuu} + 5u_x^4 \Lambda_{xu} + 30u_x^2 u_{xx} \Lambda_{xuu} + 15u_{xx}^2 \Lambda_{xuu} + 20u_x u_{xxx} \Lambda_{xuu} \\ & + 5u_{xxx} \Lambda_{xu} + u_x^5 \Lambda_{uuuu} + 10u_x^3 u_{xx} \Lambda_{uuu} + 15u_x u_{xx}^2 \Lambda_{uuu} + 10u_x^2 u_{xxx} \Lambda_{uuu} + 10u_{xx} u_{xxx} \Lambda_{uu} \\ & + 5u_x u_{xxx} \Lambda_{uu} + u_{xxxx} \Lambda_u = 0. \end{aligned} \quad (4.70)$$

The following system of PDEs emerge from splitting equation (4.70) on the derivatives of u :

$$\Lambda_u = 0, \quad (4.71)$$

$$\Lambda_x = 0, \quad (4.72)$$

$$\Lambda_t + \gamma u^2 \Lambda_x = 0. \quad (4.73)$$

From (4.71) and (4.72), we get

$$\Lambda = A(t), \quad (4.74)$$

where A is a function depending on t . Plugging (4.74) in (4.73), we get

$$A'(t) = 0, \quad (4.75)$$

whose solution gives

$$A(t) = C_1, \quad (4.76)$$

where C_1 is an integration constant. Thus, equation (4.1) has the multiplier

$$\Lambda(t, x, u) = C_1, \quad (4.77)$$

which means the equation is in conserved form itself. The multiplier (4.77) of equation (4.1) holds for

$$\Lambda(u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx}) = D_t(T^t) + D_x(T^x), \quad (4.78)$$

where $T^t = T^t(t, x, u, u_x)$ and $T^x = T^x(t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$. We take $C_1 = 1$ and proceed to find the values of T^t, T^x .

$$\begin{aligned} u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} &= T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x + T_{u_x}^x u_{xx} \\ &+ T_{u_{xx}}^x u_{xxx} + T_{u_{xxx}}^x u_{xxxx} + T_{u_{xxxx}}^x u_{xxxxx}. \end{aligned} \quad (4.79)$$

We see that T^x is independent of u_{xxxxx} , therefore we split (4.79) on powers of this derivative and this yields

$$u_{xxxxx} : T_{u_{xxxxx}}^x = 1, \quad (4.80)$$

$$\begin{aligned}
\text{Rest : } u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x &= T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + T_x^x + T_u^x u_x \\
&+ T_{u_x}^x u_{xx} + T_{u_{xx}}^x u_{xxx} + T_{u_{xxx}}^x u_{xxxx}.
\end{aligned} \tag{4.81}$$

The solution of (4.80) is

$$T^x = u_{xxxx} + A(t, x, u, u_x, u_{xx}, u_{xxx}), \tag{4.82}$$

where A depends on its argument. Replacing T^x into (4.81) gives

$$\begin{aligned}
u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x &= T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx} \\
&+ A_{u_{xx}} u_{xxx} + A_{u_{xxx}} u_{xxxx},
\end{aligned} \tag{4.83}$$

and this splits on u_{xxxx} to yield

$$u_{xxxx} : A_{u_{xxx}} = 0, \tag{4.84}$$

$$\begin{aligned}
\text{Rest : } u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x &= T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + A_x + A_u u_x \\
&+ A_{u_x} u_{xx} + A_{u_{xx}} u_{xxx}.
\end{aligned} \tag{4.85}$$

Equation (4.84) implies that

$$A = A(t, x, u, u_x, u_{xx}). \tag{4.86}$$

Thus, from (4.86) we see that we may further split (4.85) on powers of u_{xxx} and this results in

$$u_{xxx} : A_{u_{xx}} = \alpha u, \tag{4.87}$$

$$\text{Rest : } u_t + \beta u_x u_{xx} + \gamma u^2 u_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx}. \tag{4.88}$$

The solution of (4.87) gives

$$A = \alpha u u_{xx} + C(t, x, u, u_x), \tag{4.89}$$

where C is an arbitrary function of its argument. Now substituting the value of A into equation (4.88) we get

$$u_t + \beta u_x u_{xx} + \gamma u^2 u_x = T_t^t + T_u^t u_t + T_{u_x}^t u_{tx} + C_x + \alpha u_x u_{xx} + C_u u_x + C_{u_x} u_{xx}.$$

Splitting the above equation on the second-order derivatives of u yields

$$u_{tx} : T_{u_x}^t = 0, \quad (4.90)$$

$$u_{xx} : \alpha u_x + C_{u_x} = \beta u_x, \quad (4.91)$$

$$Rest : u_t + \gamma u^2 u_x = T_t^t + T_u^t u_t + C_x + C_u u_x. \quad (4.92)$$

From (4.90), we see that

$$T^t = T^t(t, x, u). \quad (4.93)$$

Integration of (4.91) and solving for C gives

$$C = \frac{1}{2}\beta u_x^2 - \frac{1}{2}\alpha u_x^2 + D(t, x, u), \quad (4.94)$$

where D is a function of its argument. Substituting value of C in (4.92), we obtain

$$u_t + \gamma u^2 u_x = T_t^t + T_u^t u_t + D_x + D_u u_x. \quad (4.95)$$

Splitting (4.95) on the derivatives of u gives

$$u_t : 1 = T_u^t, \quad (4.96)$$

$$u_x : D_u = \gamma u^2, \quad (4.97)$$

$$Rest : T_t^t + D_x = 0. \quad (4.98)$$

Equation (4.96) imply that

$$T^t = u + F(t, x), \quad (4.99)$$

where F is a function of its argument. Integrating (4.97) gives

$$D = \frac{1}{3}\gamma u^3 + E(t, x), \quad (4.100)$$

where E is a function of t and x . Substituting the values of D and T^t into equation (4.98) we get

$$E_x + F_t = 0. \quad (4.101)$$

The functions F and E contribute to the trivial part of the conservation law thus we take them to be zero. Thus the components of the conservation law are

$$\begin{aligned} T^t &= u, \\ T^x &= u_{xxxx} + \alpha u u_x + \frac{1}{2} \beta u_x^2 - \frac{1}{2} \alpha u_x^2 + \frac{1}{3} \gamma u^3. \end{aligned}$$

A special case.

We consider a special case of equation (4.1) when $\beta = 2\alpha$. Following the above procedure, we obtain the fourth-order multiplier as

$$\Lambda(t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}) = \frac{1}{6} (2\gamma u^3 + 6\alpha u u_{xx} + 3\alpha u_x^2 + 6u_{xxxx}) C_1 + C_2 u + C_3,$$

where C_1, C_2 and C_3 are arbitrary constants. The conserved vectors corresponding to the three multipliers are given below.

Case 1. For the first multiplier $\Lambda_1 = \frac{1}{3}\gamma u^3 + \alpha u u_{xx} + \frac{1}{2}\alpha u_x^2 + u_{xxxx}$, the corresponding conserved vector is given by

$$\begin{aligned} T_1^t &= \frac{1}{2} u (\gamma u^3 + 4\alpha u u_{xx} + 2\alpha u_x^2 + 6u_{xxxx}), \\ T_1^x &= \frac{1}{2} u_{xxxx}^2 + \frac{1}{8} \alpha^2 u_x^4 + \frac{1}{3} \alpha \gamma u^4 u_{xx} + \frac{1}{6} \alpha \gamma u^3 u_x^2 + \frac{1}{2} \alpha^2 u u_x^2 u_{xx} - \frac{1}{2} u_{tx} u_{xx} - \frac{1}{2} u u_{txxx} \\ &\quad + \frac{1}{2} u_x u_{txx} + \frac{1}{2} \alpha u_x^2 u_{xxxx} - \frac{1}{3} \alpha u^2 u_{tx} + \frac{1}{2} u_t u_{xxx} + \frac{1}{18} \gamma^2 u^6 + \frac{1}{2} \alpha^2 u^2 u_{xx}^2 + \frac{1}{3} \gamma u^3 u_{xxxx} \\ &\quad + \frac{1}{3} \alpha u u_t u_x + \alpha u u_{xx} u_{xxxx}. \end{aligned}$$

Case 2. For the first multiplier $\Lambda_2 = u$, the corresponding conserved vector is

$$\begin{aligned} T_1^t &= \frac{1}{2} u^2, \\ T_1^x &= \alpha u^2 u_{xx} - u_x u_{xxx} + \frac{1}{2} u_{xx}^2 + u u_{xxxx} + \frac{1}{4} \gamma u^4. \end{aligned}$$

Case 3. For the first multiplier $\Lambda_3 = 1$, the corresponding conserved vector is given by

$$\begin{aligned} T_1^t &= u, \\ T_1^x &= \frac{1}{3} \gamma u^3 + \alpha u u_{xx} + \frac{1}{2} \alpha u_x^2 + u_{xxxx}. \end{aligned}$$

4.4.2 Conservation laws using Noether's approach

We use Noether's technique to derive conservation laws for the fifth-order KdV equation (4.1) with $\beta = 2\alpha$ in this subsection. Since equation is of fifth-order it does not have a Lagrangian. However, to overcome this limitation, we take $u = v_x$. Thus, we have

$$v_{tx} + \alpha v_x v_{xxxx} + 2\alpha v_{xx} v_{xxx} + \gamma v_x^2 v_{xx} + v_{xxxxx} = 0. \quad (4.102)$$

It can be easily verified that equation (4.102) has a third-order Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{2}v_t v_x + \frac{1}{2}\alpha v_x v_{xx}^2 - \frac{1}{12}\gamma v_x^4 - \frac{1}{2}v_{xxx}^2,$$

because $\delta\mathcal{L}/\delta v = 0$ on (4.102). The Euler operator $\delta/\delta v$ in our case is

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_x^2 \frac{\partial}{\partial v_{xx}} - D_x^3 \frac{\partial}{\partial v_{xxx}} + \dots$$

We compute the Noether point symmetries of (4.102) using the determining equation

$$X^{[3]}(\mathcal{L}) + \mathcal{L} \{D_t(\tau) + D_x(\xi)\} - D_t(B^1) - D_x(B^2) = 0, \quad (4.103)$$

with $B^1 = B^1(t, x, v)$ and $B^2 = B^2(t, x, v)$ being the gauge functions and $X^{[3]}$ is the third extension of the infinitesimal generator

$$X = \tau(t, x, v) \frac{\partial}{\partial t} + \xi(t, x, v) \frac{\partial}{\partial x} + \Phi(t, x, v) \frac{\partial}{\partial v}.$$

As a result of expanding equation (4.103) with the Lagrangian $\mathcal{L}$, we obtain

$$\begin{aligned} & -\frac{1}{2}v_x \{ \Phi_t + v_t \Phi_v - v_t \tau_t - v_t^2 \tau_v - v_x \xi_t - v_t v_x \xi_v \} - \frac{1}{2}v_t \{ \Phi_x + v_x \Phi_v - v_t \tau_x - v_t v_x \tau_v \\ & - v_x \xi_x - v_x^2 \xi_v \} + \frac{1}{2}\alpha v_{xx}^3 \{ \Phi_x + v_x \Phi_v - v_t \tau_x - v_t v_x \tau_v - v_x \xi_x - v_x^2 \xi_v \} - \frac{1}{3}\gamma v_x^3 v_{xx} \{ \Phi_x \\ & + v_x \Phi_v - v_t \tau_x - v_t v_x \tau_v - v_x \xi_x - v_x^2 \xi_v \} + \alpha v_x v_{xx} v_{xxx} \{ \Phi_{xx} + 2v_x \Phi_{xv} + v_{xx} \Phi_v + v_x^2 \Phi_{vv} \\ & - 2v_{xx} \xi_x - v_x \xi_{xx} - 2v_x^2 \xi_{xv} - 3v_x v_{xx} \xi_v - v_x^3 \xi_{vv} - 2v_{xt} \tau_x - v_t \tau_{xx} - 2v_t v_x \tau_{xv} - v_t v_{xx} \tau_v \\ & - 2v_x v_{xt} \tau_v - v_t v_x^2 \tau_{vv} \} - v_{xxxx} \zeta_{222} - \frac{1}{2}v_t v_x \tau_t + \frac{1}{2}\alpha v_x v_{xx}^2 \tau_t - \frac{1}{12}\gamma v_x^4 \tau_t - \frac{1}{2}v_{xxx}^2 \tau_t \\ & - \frac{1}{2}v_t^2 v_x \tau_u + \frac{1}{2}\alpha v_t v_x v_{xx}^2 \tau_u - \frac{1}{12}\gamma v_t v_x^4 \tau_u - \frac{1}{2}v_t v_{xxx}^2 \tau_u - \frac{1}{2}v_t v_x \xi_x + \frac{1}{2}\alpha v_x v_{xx}^2 \xi_x \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{12}\gamma v_x^4 \xi_x - \frac{1}{2}v_{xxx}^2 \xi_x - \frac{1}{2}v_t v_x^2 \xi_u + \frac{1}{2}\alpha v_x^2 v_{xx}^2 \xi_u - \frac{1}{12}\gamma v_x^5 \xi_u - \frac{1}{2}v_x v_{xxx}^2 \xi_u - B_t^1 - v_t B_v^1 \\
& - B_x^2 - v_x B_v^2 = 0.
\end{aligned}$$

The following is the result of decomposing the above equation on various derivatives of v :

$$\tau_t = 0, \quad \tau_x = 0, \quad \tau_v = 0, \quad (4.104)$$

$$\xi_t = 0, \quad \xi_x = 0, \quad \xi_v = 0, \quad (4.105)$$

$$\Phi_x = 0, \quad \Phi_v = 0, \quad (4.106)$$

$$2\Phi_v + \tau_t = 0, \quad (4.107)$$

$$2B_v^2 + \Phi_t = 0, \quad (4.108)$$

$$B_t^1 + B_x^2 = 0, \quad (4.109)$$

$$B_v^1 = 0. \quad (4.110)$$

Equations (4.104) imply that

$$\tau = C_1, \quad (4.111)$$

where C_1 is an integration constant. Equations (4.105) give

$$\xi = C_2, \quad (4.112)$$

where C_2 is a constant. Solving equations (4.106) we get

$$\Phi = f(t), \quad (4.113)$$

where f is a function of t . From equation (4.107), we get

$$B^2 = -\frac{1}{2}v f'(t) + A(t, x), \quad (4.114)$$

where A is an arbitrary functions of its argument. Equation (4.110) gives

$$B^1 = C(t, x), \quad (4.115)$$

where C is a arbitrary function of its argument. Substituting the values of B^1 and B^2 into (4.109), we get

$$A_x + C_t = 0, \quad (4.116)$$

thus, we take $A = C = 0$ as they contribute to the trivial part of the conserved vector and this gives Noether point symmetries and their gauge functions as:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0, \\ X_2 &= \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0 \\ X_f &= f(t) \frac{\partial}{\partial v}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2} u f'(t). \end{aligned}$$

Thus, using the formulae for conservation laws [31]

$$C^k = \mathcal{L} \xi^k + (\Phi - u_j \xi^j) \left\{ \frac{\partial \mathcal{L}}{\partial u_{x^k}} - \sum_{l=1}^k D_{x^l} \left(\frac{\partial \mathcal{L}}{\partial u_{x^l x^k}} \right) \right\} + \sum_{l=k}^n (\zeta_j - u_{x^l x^j} \xi^j) \frac{\partial \mathcal{L}}{u_{x^k x^l}} - B^k,$$

we obtain the following conserved vectors:

$$\begin{aligned} C_1^t &= \frac{1}{2} \alpha v_{xx}^2 v_x - \frac{1}{12} \gamma v_x^4 - \frac{1}{2} v_{xxx}^2, \\ C_1^x &= \frac{1}{2} v_t^2 + \frac{1}{2} \alpha v_t v_{xx}^2 + \frac{1}{3} \gamma v_t v_x^3 + \alpha v_t v_{xxx} v_x - \alpha v_{xx} v_x v_{tx} + v_t v_{xxxx} \\ &\quad - v_{xxx} v_{tx} + v_{xxx} v_{txx}; \end{aligned}$$

$$\begin{aligned} C_2^t &= \frac{1}{2} v_x^2, \\ C_2^x &= \alpha v_{xxx} v_x^2 + \frac{1}{4} \gamma v_x^4 + v_{xxxx} v_x + \frac{1}{2} v_{xxx}^2 - v_{xx} v_{xxxx}; \end{aligned}$$

$$\begin{aligned} C_f^t &= -\frac{1}{2} v_x f(t), \\ C_f^x &= \frac{1}{2} u f'(t) - \left(\frac{1}{3} \gamma v_x^3 + \frac{1}{2} v_t + \frac{1}{2} \alpha v_{xx}^2 + \alpha v_{xxx} v_x + v_{xxxx} \right) f(t). \end{aligned}$$

Thus, reverting to the original variables, we obtain

$$\begin{aligned} C_1^t &= \frac{1}{2} \alpha u_x^2 u - \frac{1}{12} \gamma u^4 - \frac{1}{2} u_{xx}^2, \\ C_1^x &= \int u_t dx \left(\frac{1}{2} \int u_t dx + \frac{1}{2} \alpha u_x^2 + \frac{1}{3} \gamma u^3 + \alpha u_{xx} u + u_{xxxx} \right) \\ &\quad - \alpha u_x u u_t - u_{xxx} u_t + u_{xx} u_{tx}; \end{aligned}$$

$$C_2^t = \frac{1}{2} u^2,$$

$$C_2^x = \alpha u_{xx} u^2 + \frac{1}{4} \gamma u^4 + u_{xxxx} u + \frac{1}{2} u_{xx}^2 - u_x u_{xxx};$$

$$C_f^t = -\frac{1}{2} u f(t),$$

$$C_f^x = \frac{1}{2} f'(t) \int u dx - \left(\frac{1}{3} \gamma u^3 + \frac{1}{2} \int u_t dx + \frac{1}{2} \alpha u_x^2 + \alpha u_{xx} u + u_{xxxx} \right) f(t).$$

4.5 Conclusion

The Lie point symmetries, symmetry reductions and exact solutions of the fifth-order KdV equation (4.1) were presented in this chapter. Also some soliton solutions were presented. We then derived the conservation laws for the equation by using the multiplier approach. We obtained one multiplier and that was equal to a constant. This meant that the equation (4.1) is already in the conserved form. However, by considering a special case of equation (4.1) when $\beta = 2\alpha$ we acquired three multipliers and consequently, three conserved vectors. Moreover, for this special case of the equation we invoked Noether's theorem and derived three conservation laws that included one local and two nonlocal conservation laws.

Chapter 5

Concluding remarks and future work

It is critical to study nonlinear partial differential equations (NLPDEs) as they model real world problems. Thus, it is important to find their exact solutions and conservation laws. In this dissertation we considered three NLPDEs, namely, Benjamin-Bona-Mahony (BBM) equation, the (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation and the generalized fifth-order KdV equation.

In chapter one we gave basic literature on Lie's theory which was used in this dissertation. Techniques for obtaining the exact solutions of partial differential equations were described and two procedures for deriving conservation laws were also discussed.

In chapter two we studied 1-dimensional BBM equation as an illustrative example. We started by computing its symmetries, performed symmetry reductions and constructed exact solutions. Thereafter, conservation laws for the equation were derived by utilizing the Noether's theorem and the multiplier method.

Chapter three studied a fourth-order (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation by first computing its Lie point symmetries and then performing symmetry reductions. Exact solutions were obtained using direct integration. Furthermore, conservation laws for this equation were derived using both

the multiplier and Noether's methods.

In chapter four we presented Lie point symmetries and used them to find exact solutions of the generalized fifth-order KdV equation. Also some soliton solutions were constructed. Moreover, we derived conservation laws by employing the multiplier method and Noether's theorem.

For future work we intend to explore the possibility of using the obtained conservation laws to perform double reductions for equations studied in this work.

Bibliography

- [1] Z. Zhang, Jacobi elliptic function expansion method for the modified Korteweg-de Vries-Zakharov-Kuznetsov and the Hirota equations, *Phys. Lett. A.*, 289 (2001) 69–74
- [2] M. Wang, Y. Zhou, Z. Li, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, *Phys. Lett. A.*, 216 (1996) 67–75
- [3] N.A. Kudryashov, One method for finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simulat.*, 17 (2012) 2248–2253
- [4] J. Hu, H. Zhang, A new method for finding exact traveling wave solutions to nonlinear partial differential equations, *Phys. Lett. A.*, 286 (2001) 175–179
- [5] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991
- [6] C.H. Gu, *Soliton Theory and Its Application*, Zhejiang Science and Technology Press, Zhejiang, 1990
- [7] V.B. Matveev, M.A. Salle, *Darboux Transformations and Solitons*, Springer, New York, 1991
- [8] R. Hirota, *The direct method in soliton theory*, Cambridge University Press, Cambridge, 2004

- [9] M. Wang, X. Li, J. Zhang, The (G'/G) -expansion method and travelling wave solutions for linear evolution equations in mathematical physics, *Phys. Lett. A.*, 372 (2008) 417–423
- [10] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982
- [11] G.W. Bluman, S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989
- [12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, second ed., Springer-Verlag, Berlin, 1993
- [13] N.H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, Vols 1–3, CRC Press, Boca Raton, Florida, 1994–1996
- [14] N.H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley & Sons, Chichester, NY, 1999
- [15] I. Simbanefayi, C.M. Khalique, Travelling wave solutions and conservation laws for the Korteweg-de Vries-Bejamin-Bona-Mahony equation, *Results in Physics*, 8 (2018) 57–63
- [16] E. Noether, Invariante Variationsprobleme, *Nachr. Konig. Gesell. Wissen., Göttingen, Math.-Phys. Kl. Heft 2* (1918) 235–257 (English translation in *Transport Theory Statist. Phys.*, 1 (3) (1971) 186–207)
- [17] R.J. Leveque, *Numerical Methods for Conservation Laws*, 2nd Edition, Birkhäuser-Verlag, Basel, 1992
- [18] A.V. Mikhailov, A.B. Shabat, R.I. Yamilov, On an extension of the module of invertible transformations, *Dokl. Akad. Nauk SSSR*, 295 (1987) 288–291
- [19] A.V. Mikhailov, A.B. Shabat, R.I. Yamilov, Extension of the module of invertible transformations and classification of integrable systems, *Commun. Math. Phys.*, 115 (1988) 1–19

- [20] A. Sjöberg, Double reduction of PDEs from the association of symmetries with conservation laws with applications, *Appl. Math. Comput.*, 84 (2007) 608–616
- [21] A. Sjöberg, On double reductions from symmetries and conservation laws, *Non-linear Anal. Real World Appl.*, 10 (2009) 3472–3477
- [22] G.W. Bluman, A.F. Cheviakov, S.C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2010
- [23] E. Yasar, T. Özer, On symmetries, conservations laws and similarity solutions of foam drainage equation, *Internat. J. Non-Linear Mech.*, 46 (2011) 357–362
- [24] G.G. Polat, Ö. Orhan, T. Özer, On new conservation laws of fin equation, *Adv. Math. Phys.*, Volume 2014, Article ID 695408, 16 pages
- [25] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equations for long waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, (1972) 272, 4778
- [26] D.J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, *Philos. Mag.* 39(1895) 1838-1843.
- [27] J. Meiss, W. Horton, Fluctuation spectra of drift wave soliton gas. *Phys. Fluids* 25(1982) 1838-1843.
- [28] Wolfram Research (1996), ProductLog, Wolfram Language function, <https://reference.wolfram.com/language/ref/ProductLog.html> 20 May 2022.
- [29] N.A. Kudryashov, *Analytical theory of nonlinear differential equations*. Moskow-Igevsck: Institute of computer investigations; 2004.
- [30] M. Abramowitz, I. Stegun, *Handbook of mathematical functions*. New York: Dover; 1972.

- [31] W. Sarlet, Comment on Conservation laws of higher order nonlinear PDEs and the variational conservation laws in the class with mixed derivatives, J. Phys. A: Math. Theor. 43 (2010) 458001.
- [32] S.J. Yu, K. Toda, N. Sasa, T. Fukuyama, N soliton solutions to the Bogoyavlenskii-Schiff equation and a quest for the soliton solution in (3+1) dimensions, J. Phys. A: Math. Gen. 31 (1998) 3337–3347.
- [33] Z. Yan, New families of nontravelling wave solutions to a new (3+1)-dimensional potential-YTSF equation, Physics Letters A, 318 (2003) 78–83.
- [34] X. Zeng, Z. Dai, D. Li, New periodic soliton solutions for the (3+1)-dimensional potential-YTSF equation, Chaos, Solitons and Fractals, 42 (2009) 657–661.
- [35] Y.P. Wang, Solving the (3+1)-dimensional potential-YTSF equation with Exp-function method, 2007 ISND J. Phys. Conf. Ser., 96 (2008) 012186.
- [36] M.T. Darvishi, M. Najafi, A modification of extended homoclinic test approach to solve the (3+1)-dimensional potential-YTSF equation, Chinese Phys. Lett., 28 (2011) 040202.
- [37] R. Cimpoeasu, Multiple invariant solutions of the 3D potential Yu-Toda-Sasa-Fukuyama equation via symmetry technique, International Journal of Modern Physics B, 34 (2020) 2050188.
- [38] L. Huang, J. Manafian, G. Singh, K.S. Nisar, M.K.M. Nasution, New lump and interaction soliton, N-soliton solutions and the LSP for the (3+1)-D potential-YTSF-like equation, Results in Physics 29 (2021) 104713.
- [39] K. Plaattje, C.M. Khalique, On the solutions and conservation laws of the Yu-TodaSasaFukuyama equation of plasma physics, Results in Physics 29 (2021) 104706.
- [40] H. Ma, Q. Cheng, A. Deng, N-soliton solutions and localized wave interaction solutions of a (3+1)-dimensional potential-YuTodaSasaFukuyama equation Modern Physics Letters B, 35 (2021) 2150277.

- [41] M.J. Dong, S.F. Tian, X.B. Wang, T.T. Zhang, Lump-type solutions and interaction solutions in the (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama equation, *Analysis and Mathematical Physics*, 9 (2019) 1511–1523.
- [42] Z.Q. Li, S.F. Tian, H. Wang, J.J. Yang, T.T. Zhang, Characteristics of the lump, lumpoff and rogue wave solutions in a (3+1)-dimensional generalized potential Yu-Toda-Sasa-Fukuyama equation, *Modern Physics Letters B*, 33 (2019) 1950291.
- [43] V. Jadaun, N.R. Singh, Soliton solutions of generalized (3+1)-dimensional Yu-Toda-Sasa-Fukuyama equation using Lie symmetry analysis, *Analysis and Mathematical Physics*, 10 (2020) 42.
- [44] C.M. Khalique, K. Plaattje, O.L. Diteho, Symmetry solutions and conservation laws for the 3D generalized potential Yu-Toda-Sasa-Fukuyama equation of mathematical physics, *Symmetry*, (2021) 13, 2058
- [45] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integration, Series, and Products*, 7th ed.; Academic Press: New York, NY, USA, 2007
- [46] G. Wang, A.H. Kara, K. Fakhar, J. Vege-Guzman, A. Biswas, Group analysis, exact solutions and conservation laws of a generalized fifth order KdV equation, *Chaos, Solitons and Fractals*, 86 (2016) 8–15
- [47] H. Ahmad, T.A. Khan, S. Yao, An efficient approach for the numerical solution of fifth-order Kdv equations, *Open Maths.*, 18 (2020) 738–748.
- [48] K. Sawada, T. Kotera, A method for finding N-soliton solutions of the KdV equation and KdV-like equation, *Prog. Theor. Phys.* 51 (1974) 1355–1367.
- [49] P. Caudrey, R. Dodd, J. Gibbon, A new hierarchy of Korteweg-de Vries equations, *Proc. R. Soc. Lond. A Math. Phys. Sci.* 351 (1976) 1666, 407–422.
- [50] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.* 21 (1968) 467–490

- [51] A.M. Wazwaz, Solitons and periodic solutions for the fifth-order KdV equation, Appl. Math. Lett. 19 (2006) 1162–1167.
- [52] C.M. Khalique, O.L. Diteho, Conserved quantities and closed-form solutions of a generalized fifth-order KdV equation, to be submitted.