Conserved quantities, optimal systems and explicit solutions of certain partial differential equations

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Thesis accepted for the degree Doctor of Philosophy in Science with Applied Mathematics at the North-West University

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Graduation: May 2021

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CONSERVED QUANTITIES, OPTIMAL SYSTEMS AND EXPLICIT SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

by

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Thesis submitted for the degree of Doctor of Philosophy in Applied Mathematics at the Mafikeng Campus of the North-West University

November 2020

Supervisor: Professor C M Khalique

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Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: .....

MR INNOCENT SIMBANEFAYI

Date: 13-04-2021

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed:.....

PROF CM KHALIQUE

Date: 13 APRIL 2021

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

I. Simbanefayi, C.M. Khalique, Cnoidal and snoidal waves and conservation laws for physical space-time (3+1)-dimensional modified KdV models, Results Phys., 10 (2018) 975–979.

Chapter 3

I. Simbanefayi, C.M. Khalique, A symbolic computational approach to finding solutions and conservation laws for (3+1)-dimensional modified BBM models, Alex. Eng. J., 59 (2020) 1799–1809.

Chapter 4

I. Simbanefayi, C.M. Khalique, Conserved quantities, optimal system and explicit solutions of a (1+1)-dimensional generalised coupled mKdV-type system, J. Adv. Res., in press. https://doi.org/10.1016/j.jare.2020.10.002.

Chapter 5

I. Simbanefayi, C.M. Khalique, Group invariant solutions and conserved quantities of a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation, Mathematics, 8 (2020) 1012; doi:10.3390/math8061012.

Chapter 6

L. Zhang, I. Simbanefayi, C.M. Khalique, Travelling wave solutions and conservation laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt equation, submitted to Iran. J. Sci. Technol. Trans. A Sci.

Chapter 7

I. Simbanefayi, C.M. Khalique, An optimal system of group-invariant solutions and conserved quantities of a nonlinear fifth-order integrable equation, Open Phys., in

press.

Chapter 8

I. Simbanefayi, C.M. Khalique, Analytic solutions and conserved quantities of the coupled complex modified Korteweg-de Vries equations, submitted to Physica D.

Chapter 9

L. Zhang, I. Simbanefayi, C.M. Khalique, Conservation laws and symmetry reductions for a generalized hyperelastic-rodwave equation, submitted to Chaos Soliton. Fract.

Chapter 10

I. Simbanefayi, C.M. Khalique, Conservation laws, classical symmetries and exact solutions of a (1+1)-dimensional fifth-order integrable equation, submitted to Phys. Scr.

Dedication

For my dear wife Simiso, my lovely daughter Nomthandazo and my beloved sons Phakade & Uphakeme.

Acknowledgements

My heartfelt gratitude goes to my supervisor Professor CM Khalique for his consistent guidance and support throughout this research work. I would also like to thank Professor Lijun Zhang and Professor Maria Luz Gandarias for the collaborative work in two of the problems studied in this research. I greatly appreciate the financial support from the North-West University, Mafikeng Campus, through the postgraduate bursary scheme. Last but not least, I would like to thank my wife Mimie, my daughter Nomthie and my two boys Phakae and Phake for their patience while l slaved on without end. Finally, my deepest gratitude goes to God, my sustainer and enabler.

Abstract

In this thesis we study nine nonlinear partial differential equations (NLPDEs) from the point of view of classical Lie point symmetries and conserved quantities. These equations of choice have real world applications, primarily in the field of fluid dynamics and plasma physics. One equation models wave propagation in a hyperelastic-rod. Several (3+1)-dimensional equations are studied in detail, these are the modified Korteweg-de Vries (mKdV) and Benjamin-Bona-Mahony (BBM) equations. We also study a (3+1)-dimensional generalised Kadomtsev-Petviashvili (KP) equation. Higher dimensional equations tend to be more apt models of nonlinear interrelations between physical quantities. Moreover, we study three systems, that is, the generalised coupled mKdV, Broer-Kaup-Kupershmidt (BKK) and coupled complex mKdV systems. We also explore two fifth-order nonlinear integrable models.

Using Lie algebras, we obtain group invariant solutions, optimal systems of one dimensional subalgebras and their corresponding reductions. Techniques such as the extended Jacobi elliptic function, (G'/G)-expansion, power series solution, multiple exp-function methods are used in this work. Furthermore, we derive conservation laws for the underlying equations. Three and two dimensional renderings of selected solutions are provided. Techniques such as Noether's approach, multiplier method and Ibragimov's conservation theorem are used. In several instances we demonstrate explicitly the use of the first homotopy integral formula from variational calculus, which is sometimes attached to the multiplier method.

Introduction

Nonlinear partial differential equations (NLPDEs) have rapidly become indispensable in the quest to conceptualise the world around us. It is against this backdrop that countless research papers today explore various aspects of NLPDEs. The value attached to seeking exact solutions of NLPDEs is thus immense. However, compared to linear partial differential equations, there is no systematic way to obtain exact solutions of NLPDEs due to their complexity. Nonetheless, many ad hoc methods of finding exact solutions of NLPDEs have been developed, for example, the improved tanh method [1] the Riccati-Bernoulli sub-ODE method [2], the homogeneous balance of undetermined coefficients method [3–6], the first integral method [7], the bifurcation technique [8], the generalised unified method [9], the multiple exp-function method [10–13], dynamical system approach [14–16], simplified Hirota's method [17, 18], the (G'/G)-expansion function method [19], Kudryashov's method [20], Jacobi elliptic function expansion technique [21–24], the power series technique [25–27], and Lie group analysis approach [28–32].

Lie group theory is one of the most proficient methods for treatment of differential equations (DEs). Marius Sophus Lie (1842–99) is credited for this quintessential body of knowledge. He understood that the myriad of apparently different methods for obtaining analytic solutions of DEs were, in essence, all special cases of a broad integration approach; the theory of transformation groups. This theory is an analog of Galois theory and continues to impact mathematics and mathematical physics to

this day. It is easy to view Lie group theory as one of the most eminent techniques to derive analytic solutions of DEs.

Conservation laws are crucial for determining the extent of integrability of DEs, development of numerical schemes, reduction and solutions of partial differential equations (PDEs) amongst others. Methods for finding conservation laws include the celebrated Noether's theorem [33–35] for determining conserved vectors of systems of PDEs with a variational principle. In order to sidestep the requirement for a variational principle imposed by Noether's theorem, a modern form has been developed [36–50]. For example, the aptly named multiplier approach [45]. Another such method is the conservation theorem by Ibragimov [39,40] whose method associates every infinitesimal vector with a conservation law. Moreover, there is the partial Noether approach, which works like the Noether approach for the DEs with or without a Lagrangian [51, 52].

We give a few recent studies of NLPDEs presented in the literature. For instance, Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation was investigated in [53] and exact solutions were constructed. In [54], the (2+1)-dimensional B-type KP equation of fluid mechanics was studied and soliton molecules and some novel interaction solutions were discussed. The (2+1)-dimensional modified dispersive water-wave system was considered in [55] and variable separation solutions were obtained. The authors of [56] examined the KP-BBM equation and constructed periodic, multi wave, cross-kink wave and breather wave solutions. The Boiti-Leon-Manna-Pempinelli equation was studied using the Hirota bilinear form and rational wave solutions were obtained in [57]. Lie symmetry analysis was carried out on a generalized (2+1)-dimensional KP equation [58] and in [59], the (2+1)-dimensional dispersive long wave equation was investigated via the truncated Painlevé series method. A new method was introduced in [60] to find exact solutions for NLPDEs of mathematical physics.

This thesis is structured as follows.

In Chapter one, we provide a synopsis of pertinent concepts in Lie group theory that will be used throughout this thesis. We also outline the methods of exact solutions and conserved quantities that we will utilise.

Chapter two is a study of three (3+1)-dimensional mKdV equations. Similarity reductions along with extended Jacobi elliptic method are used to obtain exact solutions. Conserved quantities are obtained via the direct method.

In Chapter three, we explore three (3+1)-dimensional BBM equations and obtain exact solutions via symmetry reductions and simplest equation method. Noether's theorem is used to derive conservation laws.

Chapter four studies a (1+1)-dimensional mKdV system. We use elements of the optimal system of one-dimensional subalgebras to reduce the system. The power series method is used to solve the resultant ODEs. We obtain conservation laws using a homotopy integral approach.

Chapter five deals with group invariant solutions of a (3+1)-dimensional generalised KP equation. The multiplier approach and Ibragimov's conservation theorem are applied successfully to obtained local conservation laws.

Chapter six investigates the bounded travelling wave solutions of a (2+1)-dimensional BKK system using bifurcation analysis. The first homotopy integral approach is used to obtain conserved vectors.

In Chapter seven, we investigate the optimal system of group invariant solutions of a new integrable fifth-order nonlinear partial differential equation. We analyse the equation for a variational principle and in turn apply Noether's theorem and the multiplier method to get conservation laws.

In Chapter eight, we examine a (1+1)-dimensional coupled complex mKdV system.

We derive group transformations using the nine-dimensional Lie algebra of this equation. Exact solutions are obtained and conserved quantities are computed.

Chapter nine is an application of symmetry reductions on a generalized hyperelastic-rod wave equation. Using the travelling wave variable, we perform a triple reduction on the conserved vectors of the equation.

In Chapter ten, we study a (1+1)-dimensional fifth-order equation. Using the optimal systems of Lie algebras we obtain exact solutions. We also use the multiple exp-function method to obtain one and two wave solutions. Conservation are derived at the end of the chapter.

Chapter eleven provides a summary of the results in the thesis and future work is contemplated.

Bibliography is given at the end.

Chapter 1

Preliminaries

This chapter presents some preliminaries on Lie symmetry analysis and conservation laws of differential equations, which are used throughout this work. Selected methods of obtaining analytic solutions and conserved quantities of differential equations are also outlined.

1.1 One-parameter group of continuous transformations

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a),\ \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 (Lie group) A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b\, T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0\, T_a = T_a\, T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a\, T_{a^{-1}} = T_{a^{-1}}\, T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\, \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.2 Prolongations

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [31, 65].

Now let us consider a pth-order system of PDEs of n independent variables $x = (x^1, x^2, \cdots, x^n)$ and m dependent variables $u = (u^1, u^2, \cdots, u^m)$ given by

$$E_\alpha(x, u, u_{(1)}, \cdots, u_{(p)}) = 0, \quad \alpha = 1, \cdots, m. \tag{1.5}$$

Here $u_{(1)}, u_{(2)}, \ldots, u_{(p)}$ represent the collections of all first, second, ..., pth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, respectively, and the total derivative operator D_i given by (1.4).

1.2.1 Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a, about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x, u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiation in transformed variables $\bar{x}^i$. So

$$\bar{u}^\alpha_i = \bar{D}_j(u^\alpha), \qquad \bar{u}^\alpha_{ij} = \bar{D}_j(\bar{u}^\alpha_i) = \bar{D}_i(\bar{u}^\alpha_j), \cdots .$$

Now let us apply (1.6) and (1.10) as follows:

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}^\alpha_j. \end{aligned} \tag{1.11}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u^\beta_i \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}^\alpha_j = \frac{\partial \phi^\alpha}{\partial x^i} + u^\beta_i \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}^\alpha_j$ can be represented as functions of $x, u, u_{(i)}$, a for small a, i.e., (1.12) is locally invertible:

$$\bar{u}^\alpha_i = \psi^\alpha_i(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u^\alpha_i. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}^\alpha_i \approx u^\alpha_i + a\zeta^\alpha_i \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

1.2.1.1 Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}^\alpha_j) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u^\alpha_j + a\zeta^\alpha_j) &= D_i(u^\alpha + a\eta^\alpha) \end{aligned}$$

$$
\begin{aligned}
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j\text{)}. \qquad (1.15)
\end{aligned}
$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$
\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k\text{)}. \qquad (1.16)
$$

By induction (recursively)

$$
\zeta_{i_1,i_2,\ldots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{p-1}\,j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j\text{)}. \qquad (1.17)
$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$
\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha\text{)}, \\
&\vdots \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1,
\end{aligned}
$$

where

$$
X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.
$$

1.3 Group admitted by a partial differential equations

Definition 1.2 (Point symmetry) The vector field

$$
X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \qquad (1.18)
$$

is a *point symmetry* of the pth-order partial differential equations (1.5), if

$$X^{[p]}(E_\alpha) = 0 \tag{1.19}$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]}\, E_\alpha\big|_{E_\alpha=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 (Determining equation) Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.4 Infinitesimal criterion of invariance

Definition 1.5 (Invariant) A function $F(x, u)$ is called an *invariant* of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\, \phi^\alpha(x, u, a)) = F(x, u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x,u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 (Lie equations) If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u^\alpha}}{da} = \eta^\alpha(\bar{x},\bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0} = x, \quad \bar{u^\alpha}\big|_{a=0} = u\,.$$

1.5 Conservation laws

1.5.1 Fundamental operators and their relationship

Definition 1.6 (Euler-Lagrange operator) The *Euler-Lagrange operator*, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1,\ldots,m. \tag{1.25}$$

Definition 1.7 (Lie-Bäcklund operator) The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.26}$$

where $\mathcal{A}$ is the space of differential functions [31]. The generator (1.26) can be prolonged to some arbitrary order using the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.27}$$

where the prolongation coefficients are obtained by the following formulae:

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.28}$$

with *Lie characteristic function* W^α given by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{1.29}$$

The characteristic form of (1.27) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.30}$$

Definition 1.8 (Conservation law) The n-tuple vector $T = (T^1, T^2, \ldots, T^n),\ T^j \in \mathcal{A},\ \ j = 1, \ldots, n$, is a *conserved vector* of (1.5) if T^i satisfies

$$D_i T^i|_{(1.5)} = 0. \tag{1.31}$$

The equation (1.31) defines a local conservation law of system (1.5).

1.5.2 Noether Approach

In this section, we recall certain salient features of the Noether approach [33] to finding conserved quantities.

Definition 1.9 (Lagrangian) If there exists a function
$\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)})$, $s \leq p$, p being the order of equation (1.5), such that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0 \quad \alpha = 1, \cdots, m, \tag{1.32}$$

then $\mathcal{L}$ is called a Lagrangian of equation (1.5). Equation (1.32) is known as the Euler-Lagrange equation.

The vector $T = (T^1, T^2, \cdots, T^n),\ T^j \in \mathcal{A},\ j = 1, \cdots, n$, is said to be a conserved vector of (1.5) provided T^i satisfies the equation

$$D_i T^i|_{(1.5)} = 0. \tag{1.33}$$

This defines a local conservation law of (1.5).

The Noether operators associated with a Lie-Bäcklund symmetry operator (1.26) X are

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i i_1 i_2 \cdots i_s}},\ i = 1, \cdots, n, \tag{1.34}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.25) by replacing u^α with the corresponding derivatives. For example,

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s \geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 j_2 \cdots j_s}},\ i = 1, \cdots, n,\ \alpha = 1, \cdots, m, \tag{1.35}$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are linked by the operator identity [31, 32]

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.36}$$

A Lie-Bäcklund operator (1.27) X is called a Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$, if

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i), \tag{1.37}$$

where $B^i = (B^1, \cdots, B^n)$, $B^i \in \mathcal{A}$ are known as gauge functions.

Theorem 1.4 (Noether's Theorem) *For any Noether symmetry generator X connected to a Lagrangian $\mathcal{L} \in \mathcal{A}$, there correlates a vector $T = (T^1, \cdots, T^n)$, $T^i \in \mathcal{A}$, given by*

$$T^i = N^i(\mathcal{L}) - B^i, \quad i = 1, \cdots, n, \tag{1.38}$$

which is a conserved vector of the Euler-Lagrange DEs (1.32).

In using this method, it is imperative to begin by seeking for the unique Lagrangian associated with system (1.5), this is followed by determining the variational symmetries by using the determining condition (1.37). Finally, from (1.38) we obtain conserved quantities of system (1.5).

1.5.3 Ibragimov's method for finding conservation laws

The gist of Ibragimov's method [39, 40] is that every infinitesimal generator is associated with a conserved quantity, notwithstanding the absence of traditional Lagrangians which are envisaged in Noether's theorem [33]. Below we provide a synopsis of the method.

Consider the system of NLPDEs (1.5) and its adjoint equations system given by

$$E^*_\alpha(x, u, v, \cdots, u_{(p)}, v_{(p)}) = \frac{\delta}{\delta u^\alpha}(vE_\beta), \quad \alpha = 1, \cdots, m, \tag{1.39}$$

where $\delta/\delta v^\alpha$ is the Euler–Lagrange operator (1.25) and where $v = (v^1, \ldots, v^m)$ are m novel field variables .

Theorem 1.5 *Consider a system of m Equations (1.5). The adjoint system given by (1.39), inherits the symmetries of the system (1.5). Namely, if the system (1.5) admits a point transformation group with a generator $X = \xi^i \partial/\partial t + \eta \partial/\partial u^\alpha$, then the adjoint system (1.39) admits the operator X extended to the variables v^α by*

the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha} \tag{1.40}$$

with appropriately chosen $\eta^\alpha_* = \eta^\alpha_*(x, u, v)$.

The functions ξ^i and η^α are infinitiesimal generator coefficients dependent on x and u. In [39], the coefficients η^α_* in (1.40) are given by

$$\eta^\alpha_* = -\big[\lambda^\alpha_\beta + D_i(\xi^i)\big]v, \tag{1.41}$$

where λ^α_β is a constant and can be computed by utilising the equation

$$X(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{1.42}$$

We can obtain a conserved vector, for instance, for a third-order Lagrangian by applying the formula

$$\begin{aligned} C^i = {} & \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \cdots \right] \\ & + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} + \ldots \right] + D_j D_k(W^\alpha) \frac{\partial \mathcal{L}}{\partial u_{ijk}} + \cdots, \end{aligned} \tag{1.43}$$

where $\mathcal{L}$ is the Lagrangian of the system E and E^* that is defined as

$$\mathcal{L} = v^\alpha E_\alpha \tag{1.44}$$

and W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j, \; \alpha = 1, \ldots, m. \tag{1.45}$$

The reader is referred to [39,40] for a more comprehensive discussion of this method.

1.5.4 Multiplier method

The multiplier method is one of the most robust and preferred methods for deriving conserved quantities of DEs [30,43,46,47,61–63]. This method attempts to mitigate the shortcomings of Noether's theorem [33], which requires amongst other things, the existence of a variational principle or a Lagrangian before the theorem can be applied. We begin by providing a concise basis of the method.

Consider system E (1.5) of m PDEs of order p. A local conserved quantity $T^i(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(l)})$ of system (1.5) is a continuity equation (1.33) valid for the solution space ε of system (1.5).

In general, local nontrivial conserved quantities emanate from the divergence identity

$$D_{x^1}T^1 + D_{x^2}T^2 + \cdots + D_{x^n}T^n = \Lambda_\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(r)})E. \tag{1.46}$$

Here, $\Lambda_\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(r)})$ is a series of conservation law multipiers which are dependent on x, u and the derivatives of u, up to some arbitrary order $r < k$. The relationship (1.46) brings to light the pre-eminent interrelation between conserved quantities T^i and multipliers Λ_α. A determining condition to derive a set of multipliers $\Lambda_\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(r)})$ for system (1.5) is that

$$\frac{\delta}{\delta u^\alpha}(\Lambda_\alpha E) = 0, \;\; \alpha = 1, \cdots, m, \tag{1.47}$$

where $\delta/\delta u^\alpha$ is the Euler–Lagrange operator (1.25).

The condition (1.47) is requisite and adequate for Λ to be a multiplier. In this research work, we will explore the lesser known first homotopy integral formula [43]

$$\Phi = \int_0^1 \sum_{j=1}^{p} \partial_\lambda \partial^{j-1} u^\alpha_{(\lambda)} \left(\sum_{l=j}^{p} (-D)^{l-j} \cdot \left(\frac{\partial E_\alpha \Lambda^\alpha}{\partial u} \right) \Big|_{u^\alpha = u^\alpha_{(\lambda)}} \right) \mathrm{d}\lambda, \tag{1.48}$$

where $\alpha = 1, \cdots, m$, and m is the number of dependent variables. Also, $\Phi = (T, X)$ is a conserved quantity composed of conserved density T and spatial flux X. A more

rigorous and detailed treatment of the theoretical justification of the multiplier approach including proofs of the formulas utilised in this section can be found in [45].

1.6 Exact solutions

We now review some of the methods used in this work to find analytic solutions differential equations.

1.6.1 Multiple exp-function method

In order to obtain multiple wave solutions of equation (1.5) we will use the multiple exponential function method [10–13]. Here we provide a brief outline of the method. First we define the auxillary first-order PDEs

$$\theta_{i,t} = -\omega_i \theta_i, \qquad \theta_{i,x} = k_i \theta_i, \tag{1.49}$$

for some constants ω_i and k_i, where $1 \leq i \leq m$. Equations (1.49) have the exact solutions

$$\theta_i = q_i e^{\xi_i} \qquad \xi_i = k_i t - \omega_i x, \tag{1.50}$$

where q_i represents arbitrary constants. Now suppose that NLPDE (1.5) admits the solution

$$u(t,x) = \frac{p(\theta_i)}{g(\theta_i)}, \tag{1.51}$$

where

$$p = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} p_{rs,ij} \theta_r^i \theta_s^j$$

and

$$g = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} g_{rs,ij} \theta_r^i \theta_s^j.$$

We seek to determine values of the arbitrary constants $p_{rs,ij}$ and $g_{rs,ij}$. By taking all derivatives of (1.51) and substituting in equation (1.5) we obtain the determining equation

$$W(t, x, \theta_1, \theta_2, \cdots, \theta_n). \tag{1.52}$$

Collecting coefficients of θ_i in equation (1.52) results in an overdetermined system of algebraic equations. The use of computer software such as Maple or Mathematica proves indispensable in executing this algorithm, culminating in the solutions of polynomials p and g. In essence, the multiple wave solutions of equation (1.5) are given by

$$u(t, x) = \frac{p\left(q_1 e^{k_1 x - \omega_1 t}, \cdots, \cdots, q_n e^{k_n x - \omega_n t}\right)}{g\left(q_1 e^{k_1 x - \omega_1 t}, \cdots, \cdots, q_n e^{k_n x - \omega_n t}\right)}. \tag{1.53}$$

1.6.2 The extended Jacobi elliptic function method

This method was first used in [21]. It has however undergone minor modifications over time, see for example [22–24]. The space-time group invariant

$$u(t, x) = \phi(\xi), \quad \xi = x - \nu t, \tag{1.54}$$

transforms NLPDE (1.5) into a NLODE, say,

$$F(\phi, \phi', \phi'', \cdots, \phi^{(p)}). \tag{1.55}$$

Now suppose $\phi(\xi)$ can be expressed as

$$\phi(\xi) = \sum_{i=-M}^{M} A_i H(\xi)^i, \tag{1.56}$$

where M is a positive integer obtained by the balancing procedure [64]. Here $H(\xi)$ satisfies the first-order ODE

$$H'(\xi) = -\sqrt{(1 - H^2(\xi))(1 - \omega + \omega H^2(\xi))} \tag{1.57}$$

or

$$H'(\xi) = \sqrt{(1 - H^2(\xi))(1 - \omega H^2(\xi))}. \tag{1.58}$$

We recall that

$$H(\xi) = \mathrm{cn}(\xi|\omega), \tag{1.59}$$

the Jacobi cosine-amplitude function, is a solution to (1.57), whereas the Jacobi sine-amplitude function

$$H(\xi) = \mathrm{sn}(\xi|w) \tag{1.60}$$

is a solution to (1.58). Here ω is a parameter such that $0 \leq \omega \leq 1$ [66, 67].

The third step of our procedure entails substituting (1.56) subject to (1.57) or (1.58) into the ODE mentioned earlier. This produces an equation in powers of $H(\xi)$. Finally separating coefficients with respect to like powers of $H(\xi)$ yields an algebraic system of equations which we solve for potential values of A_i, $i = 0, \pm 1, \cdots \pm M$.

We note that when $\omega \to 1$, then $\mathrm{cn}(\xi|\omega) \to \mathrm{sech}(\xi)$ and $\mathrm{sn}(\xi|\omega) \to \tanh(\xi)$. Also, when $\omega \to 0$, then $\mathrm{cn}(\xi|\omega) \to \cos(\xi)$ and $\mathrm{sn}(\xi|\omega) \to \sin(\xi)$.

Delta amplitude $\mathrm{dn}(\xi|\omega)$ solutions can be obtained by using the ODE

$$H(\xi) = -\left\{(1 - H^2(\xi))(H^2(\xi) + \omega - 1)\right\}^{1/2}. \tag{1.61}$$

This would complete the copolar trio, see [66].

1.6.3 Synopsis of $(G'/G)-$expansion method

Suppose that the solution(s) of NLODE (1.55) take the following series form [19]:

$$\phi(\xi) = \sum_{i=0}^{M} A_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{1.62}$$

where $G = G(\xi)$ satisfies the second-order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0. \tag{1.63}$$

Here λ and μ take arbitrary real values. The integer M is found by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in ODE (1.55). The next step entails substituting equation (1.62) along with ODE (1.63) into equation (1.55) yielding an algebraic equation in powers of (G'/G). Collecting all terms with same order of (G'/G) together gives an overdetermined system of algebraic equations whose solutions are the desired values of A_i. Finally, ODE (1.63) has three types of well known general solutions, namely, hyperbolic, trigonometric and rational. These are substituted along with computed parameter values of A_i's into equation (1.62) to give three types of solutions of the ODE (1.55). These are by extension, solutions of NLPDE (1.5).

1.6.4 Power series solution method

The power series solution method for DEs [25–27] will also be utilised in this work. This method entails using the infinite series

$$\phi(\xi) = \sum_{z=0}^{\infty} g_z \xi^z$$

and its respective derivatives, which are then substituted into equation (1.55). This method will be outlined in full in Chapter 4.

1.7 Concluding remarks

In this chapter, we provided a concise outline of some notable aspects of Lie group analysis which provide a basis for this research work. We outlined several conserved

quantity methods which will be implemented throughout our work. Finally, we presented the methods for determining exact solutions of DEs that will be utilised in this thesis.

Chapter 2

Cnoidal and snoidal waves and conservation laws for physical space-time (3+1)-dimensional modified KdV models

2.1 Introduction

The nonlinear space-time (3+1)-dimensional modified Korteweg-de Vries (KdV) model given by

$$u_t + 6u^2 u_x + u_{xyz} = 0 \tag{2.1}$$

was studied by Wazwaz [68], which was proposed by Hereman [69,70]. A variety of exact solutions that contained soliton, kink and periodic solutions were obtained in [68]. Also, two new nonlinear (3+1) dimensional modified KdV equations, namely,

$$u_t + 6u^2 u_y + u_{xyz} = 0 \tag{2.2}$$

and

$$u_t + 6u^2 u_z + u_{xyz} = 0 \tag{2.3}$$

were introduced in [68] and soliton, kink and periodic solutions were presented along with the constraints that guarantee their existence. The importance of studying the (3+1)-dimensional equations has been given in [68]. For example, Wazwaz [71] studied an extended (3+1)-dimensional nonlinear evolution equation and found multiple soliton solutions with the aid of the simplified Hirota's method. Furthermore, the linear superposition principle was applied to a (3+1)-dimensional nonlinear evolution equation and two sets of the resonant multiple wave solutions were presented in [72].

In this chapter, we study the three nonlinear (3+1)-dimensional modified KdV equations (2.1)–(2.3). Firstly using the Lie symmetry method along with the extended Jacobi elliptic function expansion method we obtain their soliton, cnoidal and snoidal wave solutions. Furthermore, we construct conservation laws using the direct method.

The work done in this chapter has been published in [24].

2.2 Solutions and conservation laws of (2.1)

2.2.1 Lie point symmetries

We begin the solution of (2.1) by first determining its Lie point symmetries. The vector field

$$\begin{aligned} X =& \xi^1(t,x,y,z,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,z,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,z,u)\frac{\partial}{\partial y} + \xi^4(t,x,y,z,u)\frac{\partial}{\partial z} \\ &+ \eta(t,x,y,z,u)\frac{\partial}{\partial u}, \end{aligned} \tag{2.4}$$

where $\xi^i, i = 1, 2, \cdots, 4$ and η depend on t, x, y, z and u, is a Lie point symmetry of (2.1) provided

$$\mathrm{pr}^{(3)}X(u_t + 6u^2u_x + u_{xyz})|_{u_t+6u^2u_x+u_{xyz}=0} = 0. \tag{2.5}$$

Here $\mathrm{pr}^{(3)}X$ is the third prolongation [30] of X and is defined by

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xyz}\frac{\partial}{\partial u_{xyz}}, \tag{2.6}$$

where ζ_t, ζ_x, and ζ_{xyz} are determined as follows:

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_tD_t(\xi^1) - u_xD_t(\xi^2) - u_yD_t(\xi^3) - u_zD_t(\xi^4),\\
\zeta_x &= D_x(\eta) - u_tD_x(\xi^1) - u_xD_x(\xi^2) - u_yD_x(\xi^3) - u_zD_x(\xi^4),\\
\zeta_y &= D_y(\eta) - u_tD_y(\xi^1) - u_xD_y(\xi^2) - u_yD_y(\xi^3) - u_zD_y(\xi^4),\\
\zeta_{xy} &= D_x(\zeta_y) - u_{xt}D_x(\xi^1) - u_{xx}D_x(\xi^2) - u_{xy}D_x(\xi^3) - u_{xz}D_x(\xi^4),\\
\zeta_{xyz} &= D_z(\zeta_{xy}) - u_{xyt}D_z(\xi^1) - u_{xyx}D_z(\xi^2) - u_{xyy}D_z(\xi^3) - u_{xyz}D_z(\xi^4)
\end{aligned}$$

with the total derivatives D_t, D_x, D_y and D_z given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{ty}\frac{\partial}{\partial u_y} + u_{tz}\frac{\partial}{\partial u_z} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + u_{xy}\frac{\partial}{\partial u_y} + u_{xz}\frac{\partial}{\partial u_z} + \cdots,\\
D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y} + u_{yt}\frac{\partial}{\partial u_t} + u_{yx}\frac{\partial}{\partial u_x} + u_{yz}\frac{\partial}{\partial u_z} + \cdots,\\
D_z &= \frac{\partial}{\partial z} + u_z\frac{\partial}{\partial u} + u_{zz}\frac{\partial}{\partial u_z} + u_{zt}\frac{\partial}{\partial u_t} + u_{zy}\frac{\partial}{\partial u_y} + u_{zx}\frac{\partial}{\partial u_x} + \cdots.
\end{aligned} \tag{2.7}$$

Expanding (2.5) and separating the resultant determining equation with respect to the derivatives of u, yields the system of twenty two linear partial differential equations

$$\begin{aligned}
&\xi^1_x = 0,\ \xi^1_y = 0,\ \xi^1_z = 0,\ \xi^1_u = 0,\ \xi^2_y = 0,\ \xi^2_u = 0,\ \xi^2_z = 0,\ \xi^3_t = 0,\ \xi^3_z = 0,\\
&\xi^3_x = 0,\ \xi^3_u = 0,\ \xi^4_x = 0,\ \xi^4_t = 0,\ \xi^4_u = 0,\ \xi^4_y = 0,\ \eta_{yu} = 0,\ \eta_{zu} = 0,
\end{aligned}$$

$$\eta_{uu} = 0,\ \eta_{xu} = 0,\ \eta_t + 6u^2\eta_x + \eta_{xyz} = 0,\ \xi^2_x + \xi^3_y + \xi^4_z - \xi^1_t = 0,$$
$$\xi^2_t - 12u\eta - 6u^2\xi^1_t + 6u^2\xi^2_x = 0.$$

On solving the above system of PDEs for ξ^1, ξ^2, ξ^3, ξ^4 and η we obtain

$$\begin{aligned}
\xi^1 &= C_2 + C_3 t,\\
\xi^2 &= C_1 + (C_3 - C_5 - C_7)x,\\
\xi^3 &= C_4 + C_5 y,\\
\xi^4 &= C_6 + C_7 z,\\
\eta &= -\frac{1}{2}(C_5 + C_7)u,
\end{aligned}$$

where $C_i, i = 1, 2, \cdots, 7$ are arbitrary constants. The symmetry algebra is thus generated by the operators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= \frac{\partial}{\partial y},\\
X_4 &= \frac{\partial}{\partial z},\\
X_5 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\\
X_6 &= 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},\\
X_7 &= 2x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}.
\end{aligned}$$

We remark here that the groups corresponding to X_1, X_2, X_3 and X_4 demonstrate the time- and space-invariance of the equation, while the groups corresponding to X_5, X_6 and X_7 represent scaling symmetries.

2.2.2 Exact solutions by using Lie point symmetries and direct integration

In this section we find exact closed form solutions of the (3+1)-dimensional modified KdV equation (2.1). We first use the Lie symmetry method to convert the NLPDE (2.1) to a nonlinear ordinary differential equation (NLODE). We consider a linear combination of the four translation symmetries X_1, X_2, X_3 and X_4, namely

$$X = X_1 + \alpha X_2 + X_3 + X_4, \tag{2.8}$$

where α is a constant. Solving the associated Lagrange equations of (2.8), we obtain three invariants $f = \alpha t - x$, $g = x - \alpha y$ and $h = y - z$ and consequently the group invariant solution

$$u(t, x, y, z) = \theta(f, g, h) \tag{2.9}$$

of (2.1). Using the above invariants, equation (2.1) is transformed into the NLPDE

$$\alpha\theta_f - 6\theta^2\theta_f + 6\alpha\theta^2\theta_g - \alpha\theta_{hgf} + \theta_{hhf} + \alpha\theta_{hgg} - \theta_{hhg} = 0 \tag{2.10}$$

with three independent variables f, g and h. The Lie point symmetries of (2.10) are

$$\Gamma_1 = \frac{\partial}{\partial f}, \Gamma_2 = \frac{\partial}{\partial g}, \Gamma_3 = \frac{\partial}{\partial h}. \tag{2.11}$$

Again, by taking a linear combination of (2.11), i.e., $\Gamma = \Gamma_1 + \beta\Gamma_2 + \Gamma_3$, β is an arbitrary constant, we obtain two invariants $r = f - h$ and $s = g - \beta h$. Hence the group invariant $\theta(f, g, h) = \phi(r, s)$ satisfies the NLPDE

$$\begin{aligned} &\alpha\phi_r - 6\phi^2\phi_r + 6\phi^2\phi_s + \alpha\phi_{srr} + \alpha\beta\phi_{ssr} + \phi_{rrr} + 2\beta\phi_{srr} + \beta^2\phi_{ssr} \\ &- \alpha\phi_{ssr} - \alpha\beta\phi_{sss} - \alpha\beta\phi_{srr} - 2\beta\phi_{ssr} - \beta^2\phi_{ssr} = 0 \end{aligned} \tag{2.12}$$

with two independent variables r and s. Equation (2.12) has two infinitesimal symmetry generators

$$\Sigma_1 = \frac{\partial}{\partial r}, \ \Sigma_2 = \frac{\partial}{\partial s} \tag{2.13}$$

and as before, the linear combination $\Sigma_1 + \nu\Sigma_2$ (ν an arbitrary constant) yields the group invariant solution

$$\phi(r, s) = \psi(\xi), \tag{2.14}$$

where $\xi = s - \nu r$ as a solution of (2.1). Using (2.14), equation (2.12) is transformed into the third-order NLODE

$$a\psi'''(\xi) + 6b\psi^2(\xi)\psi'(\xi) - c\psi'(\xi) = 0, \tag{2.15}$$

where $a = (\nu + 1)(\beta - \nu)(\alpha + \beta - \nu)$, $b = \nu + 1$ and $c = \alpha\nu$. Integrating (2.15) twice with respect to ξ while setting the constants of integration to zero (since we are seeking a soliton solution) gives

$$\psi'(\xi)^2 = A\psi(\xi)^2 - B\psi(\xi)^4, \tag{2.16}$$

where $A = c/a$ and $B = b/a$. Basic manipulation of (2.16) gives

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\psi(\xi) = \pm\sqrt{B}\psi(\xi)\sqrt{K^2 - \psi(\xi)^2}, \tag{2.17}$$

where $K^2 = A/B$. Evidently (2.17) is a variables-separable ODE. Thus separating the variables and integrating yields

$$\operatorname{sech}^{-1}\left(\frac{\psi}{K}\right) = \pm\sqrt{B}\xi + C, \tag{2.18}$$

where C is an arbitrary constant of integration. Solving for ψ and reverting to original variables we get

$$u(t, x, y, z) = \pm\sqrt{\frac{\alpha\nu}{\nu + 1}}\operatorname{sech}\left(C \pm \sqrt{\frac{1}{(\beta - \nu)(\alpha + \beta - \nu)}}\,\xi\right), \tag{2.19}$$

where $\xi = (\nu + 1)x + (\nu - \alpha - \beta)y + (\beta - \nu)z - \alpha\nu t$, as a solution of (2.1). It should be noted that this bell shaped soliton solution (2.19) was obtained in [68] by ansatz method.

2.2.3 Exact solutions using the extended Jacobi elliptic function expansion method

In this subsection we use the extended Jacobi elliptic function expansion method [22] to obtain more closed form solutions of (2.1).

2.2.3.1 Cnoidal wave solutions

Considering the NLODE (2.15), the balancing procedure yields $M = 1$, thus (1.56) is

$$\psi(\xi) = A_{-1}H^{-1}(\xi) + A_0 + A_1H(\xi). \tag{2.20}$$

We now substitute the value of ψ from (2.20) into (2.15) and utilise (1.57) to obtain

$$\begin{aligned}
&6aA_1\omega H(\xi)^6 - 2aA_1\omega H(\xi)^4 + 2aA_{-1}\omega H(\xi)^2 + aA_1H(\xi)^4 - aA_{-1}H(\xi)^2 \\
&- 6aA_{-1}\omega + 6aA_{-1} - 6A_1^3bH(\xi)^6 - 12A_0A_1^2bH(\xi)^5 - 6A_{-1}A_1^2bH(\xi)^4 \\
&+ 6A_{-1}A_0^2bH(\xi)^2 - 6A_0^2A_1bH(\xi)^4 + 6A_{-1}^2A_1bH(\xi)^2 + 12A_{-1}^2A_0bH(\xi) \\
&+ 6A_{-1}^3b + A_1cH(\xi)^4 - A_{-1}cH(\xi)^2 = 0.
\end{aligned} \tag{2.21}$$

Equation (2.21) can be separated on like powers of $H(\xi)$ to obtain an overdetermined system of four algebraic equations

$$\begin{aligned}
&a\omega - bA_1^2 = 0 \\
&a\omega A_{-1} - aA_{-1} + bA_{-1}^3 = 0, \\
&aA_1 - 2\omega aA_1 - 6bA_1A_0^2 - 6bA_{-1}A_1^2 + cA_1 = 0, \\
&2a\omega A_{-1} - aA_{-1} + 6bA_1A_{-1}^2 + 6bA_0^2A_{-1} - cA_{-1} = 0.
\end{aligned}$$

Solving the above system gives

$$\omega = \frac{a + \alpha\nu}{2a}, \;\; A_{-1} = A_0 = 0, \;\; A_1 = \pm\sqrt{\frac{a + \alpha\nu}{2(\nu + 1)}}.$$

Therefore the solution to equation (2.1) is

$$u(t,x,y,z) = \pm\sqrt{\frac{a+\alpha\nu}{2(\nu+1)}}\ \mathrm{cn}\left(\xi\,\middle|\,\frac{a+\alpha\nu}{2a}\right),$$

where $\xi = (\nu+1)\,x + (\nu-\alpha-\beta)\,y + (\beta-\nu)\,z - \alpha\,\nu\,t$ and $a = (\nu+1)(\beta-\nu)\,(\alpha+\beta-\nu)$.

2.2.3.2 Snoidal wave solutions

In this subsection we obtain snoidal wave solutions for the equation (2.1). We recall that the balancing procedure yields $M = 1$, thus substituting the value of ψ from (2.20) into (2.15) and making use of (1.58) we obtain

$$\begin{aligned}
&6aA_1\omega H(\xi)^6 - aA_1\omega H(\xi)^4 + aA_{-1}\omega H(\xi)^2 - aA_1 H(\xi)^4 + aA_{-1}H(\xi)^2 \\
&- 6aA_{-1} + 6A_1^3 bH(\xi)^6 + 12A_0A_1^2 bH(\xi)^5 + 6A_{-1}A_1^2 bH(\xi)^4 + 6A_0^2A_1 bH(\xi)^4 \\
&- 6A_{-1}A_0^2 bH(\xi)^2 - 6A_{-1}^2A_1 bH(\xi)^2 - 12A_{-1}^2A_0 bH(\xi) - 6A_{-1}^3 b - A_1 cH(\xi)^4 \\
&+ A_{-1}cH(\xi)^2 = 0,
\end{aligned}$$

which splits into the four algebraic system of equations

$$\begin{aligned}
&a\omega + bA_1^2 = 0, \\
&aA_{-1} + bA_{-1}^3 = 0, \\
&6bA_1A_0^2 - a\omega A_1 - aA_1 + 6bA_{-1}A_1^2 - cA_1 = 0, \\
&aA_{-1}\omega + aA_{-1} - 6bA_1A_{-1}^2 - 6bA_0^2A_{-1} + cA_{-1} = 0.
\end{aligned}$$

One possible solution of the above system is

$$\begin{aligned}
&a = -\frac{\alpha\nu}{\omega + 6\sqrt{\omega}+1},\ \ A_{-1} = \sqrt{\frac{\alpha\nu(\nu+1)}{\omega+6\sqrt{\omega}+1}},\ \ A_0 = 0, \\
&A_1 = -\frac{1}{\nu+1}\sqrt{\frac{\alpha\nu(\nu+1)\omega}{\omega+6\sqrt{\omega}+1}}.
\end{aligned}$$

Consequently, the solution to (2.1) is

$$u(t,x,y,z)=\sqrt{\frac{\alpha\nu(\nu+1)}{\omega+6\sqrt{\omega}+1}}\left(\mathrm{ns}(\xi|\omega)-\frac{1}{\nu+1}\sqrt{\omega}\ \mathrm{sn}(\xi|\omega)\right),$$

where $\xi=(\nu+1)\,x+(\nu-\alpha-\beta)\,y+(\beta-\nu)\,z-\alpha\,\nu\,t$, $a=(\nu+1)(\beta-\nu)\,(\alpha+\beta-\nu)$, $\mathrm{ns}(\xi|\omega)=1/\mathrm{sn}(\xi|\omega)$ and $0\leq\omega\leq 1$.

2.2.4 Conservation laws

In this subsection we apply the direct approach to determine the conservation laws for equation (2.1). We begin by computing the zeroth-order multiplier using

$$\frac{\delta}{\delta u}\left[\Lambda(t,x,y,z,u)(u_t+6u^2u_x+u_{xyz})\right]=0, \tag{2.22}$$

where $\delta/\delta u$ is the Euler-Lagrange operator which in this case is given by

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_xD_yD_z\frac{\partial}{\partial u_{xyz}}, \tag{2.23}$$

and D_t, D_x, D_y and D_z are the total derivatives as in (2.7). Expanding (2.22) and splitting the resultant equation with respect to derivatives of u yields the twelve determining equations

$$\Lambda_{yu}=0,\ \Lambda_{xu}=0,\ \Lambda_{uu}=0,\ \Lambda_{zu}=0,\ \Lambda_{uuu}=0,\ \Lambda_{xuu}=0,\ \Lambda_{yuu}=0,$$
$$\Lambda_{zuu}=0,\Lambda_{xyu}=0,\ \Lambda_{xzu}=0,\ \Lambda_{yzu}=0,\ 6u^2\Lambda_x+\Lambda_t+\Lambda_{xyz}=0.$$

We solve the above system of equations for Λ and obtain

$$\Lambda\,(t,x,y,z,u)=C_1u+F_2\,(y,z)\,, \tag{2.24}$$

where C_1 is an arbitrary constant and F_2 is an arbitrary function of y and z. Thus the conserved vectors corresponding to (2.24) are [73, 74]

$$T_1{}^t=\frac{1}{2}u^2,$$

$$T_1{}^x = \frac{3}{2}u^4 + \frac{1}{3}uu_{yz} - \frac{1}{6}u_y u_z,$$
$$T_1{}^y = \frac{1}{3}uu_{xz} - \frac{1}{6}u_x u_z,$$
$$T_1{}^z = \frac{1}{3}uu_{xy} - \frac{1}{6}u_x u_y;$$

$$T_2{}^t = uF(y,z),$$
$$T_2{}^x = 2u^3F(y,z) + \frac{1}{3}uF_{yz}(y,z) - \frac{1}{6}u_z F_y(y,z) - \frac{1}{6}u_y F_z(y,z) + \frac{1}{3}u_{yz}F(y,z),$$
$$T_2{}^y = \frac{1}{3}u_{xz}F(y,z) - \frac{1}{6}u_x F_z(y,z),$$
$$T_2{}^z = \frac{1}{3}u_{xy}F(y,z) - \frac{1}{6}u_x F_y(y,z).$$

2.3 Exact solutions and conservation laws of (2.2)

In this section we study the second (3+1)-dimensional modified KdV equation, namely

$$u_t + 6u^2u_y + u_{xyz} = 0.$$

Using the Lie symmetry method together with direct integration, as in Section 2.2, we obtain the 1-soliton solution

$$u(t,x,y,z) = \pm\sqrt{\frac{\alpha\nu}{\nu-\alpha-\beta}}\ \operatorname{sech}\left(C \pm \sqrt{\frac{1}{(\beta-\nu)(\nu+1)}}\xi\right), \tag{2.25}$$

where $\xi = (\nu+1)x + (\nu-\alpha-\beta)y + (\beta-\nu)z - \alpha\nu t$. We have recovered the solution obtained in [68] by ansatz method. Furthermore using the extended Jacobi elliptic function expansion method we obtain the cnoidal and snoidal wave solutions of (2.2) in the form

$$u(t,x,y,z) = \pm\sqrt{\frac{a+\alpha\nu}{2(\nu-\alpha-\beta)}}\ \operatorname{cn}\left(\xi\,\middle|\,\frac{a+\alpha\nu}{2a}\right),$$
$$u(t,x,y,z) = \pm\sqrt{\frac{\alpha\nu(\nu-\alpha-\beta)}{\omega+6\sqrt{\omega}+1}}\left(\operatorname{ns}(\xi|\omega) - \frac{1}{(\nu-\alpha-\beta)}\sqrt{\omega}\ \operatorname{sn}(\xi|\omega)\right),$$

respectively. Here $\xi = (\nu+1)\,x + (\nu-\alpha-\beta)\,y + (\beta-\nu)\,z - \alpha\,\nu\,t$ and $a = (\nu+1)(\beta-\nu)\,(\alpha+\beta-\nu)$.

Following the method outlined in the previous section, we obtain the zeroth-order multiplier as

$$\Lambda\,(t,x,y,z,u) = C_1 u + F_2\,(x,z)\,, \tag{2.26}$$

where C_1 is an arbitrary constant and F_2 is an arbitrary function of x and z and the corresponding conserved vectors of (2.2) are

$$\begin{aligned}
{T_1}^t &= \frac{1}{2}u^2,\\
{T_1}^x &= -\frac{1}{6}u_z\,u_y + \frac{1}{3}\,uu_{yz},\\
{T_1}^y &= \frac{3}{2}u^4 + \frac{1}{3}uu_{xz} - \frac{1}{6}u_x u_z\\
{T_1}^z &= -\frac{1}{6}u_x u_y + \frac{1}{3}uu_{xy};
\end{aligned}$$

$$\begin{aligned}
{T_2}^t &= uF(x,z),\\
{T_2}^x &= \frac{1}{3}F(x,z)u_{yz} - \frac{1}{6}F_z\,(x,z)\,u_y,\\
{T_2}^y &= (2u^3 + \frac{1}{3}u_{xz})F\,(x,z) + \frac{1}{3}uF_{xz}\,(x,z) - \frac{1}{6}F_x\,(x,z)\,u_z - \frac{1}{6}F_z\,(x,z)\,u_x,\\
{T_2}^z &= \frac{1}{3}F(x,z)u_{xy} - \frac{1}{6}F_x\,(x,z)\,u_y.
\end{aligned}$$

2.4 Exact solutions and conservation laws of (2.3)

Finally we consider the third (3+1)-dimensional modified KdV equation

$$u_t + 6u^2 u_z + u_{xyz} = 0.$$

Proceeding as in Section 3, we obtain the soliton solution

$$u(t,x,y,z) = \pm\sqrt{\frac{\alpha\nu}{\beta-\nu}}\ \mathrm{sech}\left(C \pm \sqrt{\frac{1}{(\alpha+\beta-\nu)(\nu+1)}}\,\xi\right),$$

where $\xi = (\nu+1)\,x + (\nu-\alpha-\beta)\,y + (\beta-\nu)\,z - \alpha\,\nu\,t$ and recover the solution obtained in [68]. Moreover, we obtain the cnoidal and snoidal wave solutions of (2.3) in the form

$$u(t,x,y,z) = \pm\sqrt{\frac{a+\alpha\nu}{2b}}\ \mathrm{cn}\left(\xi\,\middle|\,\frac{a+\alpha\nu}{2a}\right),$$
$$u(t,x,y,z) = \pm\sqrt{\frac{\alpha\nu(\beta-\nu)}{\omega+6\sqrt{\omega}+1}}\left(\mathrm{ns}(\xi|\omega) + \frac{1}{\nu-\beta}\sqrt{\omega}\ \mathrm{sn}(\xi|\omega)\right),$$

respectively. Here $\xi = (\nu+1)\,x + (\nu-\alpha-\beta)\,y + (\beta-\nu)\,z - \alpha\,\nu\,t$ and $a = (\nu+1)(\beta-\nu)\,(\alpha+\beta-\nu)$.

The zeroth-order multiplier in this case is

$$\Lambda\,(t,x,y,z,u) = C_1 u + F_2\,(x,y)\,, \tag{2.27}$$

where C_1 is an arbitrary constant and F_2 is an arbitrary function of x and y and the corresponding conserved vectors of (2.3) are

$$T_1{}^t = \frac{1}{2}u^2,$$
$$T_1{}^x = -\frac{1}{6}u_z\,u_y + \frac{1}{3}\,uu_{yz},$$
$$T_1{}^y = -\frac{1}{6}u_z\,u_x + \frac{1}{3}uu_{xz},$$
$$T_1{}^z = \frac{3}{2}u^4 + \frac{1}{3}uu_{xy} - \frac{1}{6}u_x\,u_y;$$

$$T_2{}^t = uF(x,y),$$
$$T_2{}^x = \frac{1}{3}F\,(x,y)\,u_{yz} - \frac{1}{6}F_y\,(x,y)\,u_z,$$
$$T_2{}^y = \frac{1}{3}F\,(x,y)\,u_{xz} - \frac{1}{6}F_x\,(x,y)\,u_z,$$
$$T_2{}^z = 2\,F\,(x,y)\,u^3 + \frac{1}{3}uF_{xy}\,(x,y) - \frac{1}{6}F_x\,(x,y)\,u_y - \frac{1}{6}F_y\,(x,y)\,u_x + \frac{1}{3}F\,(x,y)\,u_{xy}.$$

2.5 Concluding remarks

In this chapter we studied three space-time (3+1)-dimensional modified Korteweg-de Vries equations. Such partial differential equations model many realistic problems in engineering, wave propagation, fluids, etc. The closed form exact solutions for the three equations (2.1)–(2.3) were obtained using Lie symmetry method along with the extended Jacobi elliptic expansion method. Soliton, cnoidal and snoidal waves solutions were derived. Moreover, conservation laws for the three equations (2.1)–(2.3) were obtained by the application of direct method.

Chapter 3

A symbolic computational approach to finding solutions and conservation laws for (3+1)-dimensional modified BBM models

3.1 Introduction

The regularized long-wave (RLW) equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{3.1}$$

is a (1+1)-dimensional equation and is an evolution equation. In equation (3.1), u_t represents time evolution, whereas uu_x is nonlinear term that describes the rising or falling of a wave sharply. The mixed derivative term $-u_{xxt}$ is a result of a

bounded dispersion relation. Since the (3+1)-dimensional nonlinear differential equations are considered to be more realistic compared to the (1+1) and (2+1)-dimensional equations, Wazwaz [68] recently introduced a set of three equations, viz., (3+1)-dimensional modified Benjamin-Bona-Mahony (D mBBM) equations that read

$$u_t + u_x + u^2 u_y - u_{txz} = 0, \tag{3.2}$$

$$u_t + u_z + u^2 u_x - u_{xyt} = 0 \tag{3.3}$$

and

$$u_t + u_y + u^2 u_z - u_{xxt} = 0. \tag{3.4}$$

Using a variety of ansatz, exact solutions to (3.2)–(3.4) were derived that described distinct physical structures of solutions along with some constraints. In this work, we study three (3+1)-D mBBM equations (3.2)–(3.4). By employing Noether's approach [34, 46] we construct the conservation laws of these three equations. Thereafter, we construct soliton and Jacobi elliptic function solutions of (3.2)–(3.4) by using Lie group techniques with the aid of the simplest equation technique.

The work presented in this chapter has been published in [75].

3.2 Conservation laws and analytic solutions

3.2.1 Conservation laws

Here conservation laws of the (3+1)-D mBBM equation (3.2) are constructed by employing Noether's approach. To utilise Noether's theorem one needs to have a Lagrangian for the corresponding Euler-Lagrange equation. Since equation (3.2) is of order three, its Lagrangian does not exist. However, to implement Noether's theorem we raise the order of (3.2) to the fourth-order by letting $u = v_y$. Thus

(3.2) becomes

$$v_{ty} + v_{xy} + v_y^2 v_{yy} - v_{txyz} = 0. \tag{3.5}$$

It can easily be verified that equation (3.5) has a Lagrangian

$$\mathcal{L} = -\frac{1}{2}v_t v_y - \frac{1}{2}v_x v_y - \frac{1}{12}v_y^4 - \frac{1}{2}v_{ty}v_{xz} \tag{3.6}$$

as $\delta\mathcal{L}/\delta v = 0$, where in our case

$$\frac{\delta\mathcal{L}}{\delta v} = -D_t\frac{\partial\mathcal{L}}{\partial v_t} - D_x\frac{\partial\mathcal{L}}{\partial v_x} - D_y\frac{\partial\mathcal{L}}{\partial v_y} + D_tD_y\frac{\partial\mathcal{L}}{\partial v_{ty}} + D_xD_z\frac{\partial\mathcal{L}}{\partial v_{xz}}.$$

Noether symmetries for (3.5) of the form

$$\begin{aligned} X =& \xi^1(t,x,y,z,v)\frac{\partial}{\partial t} + \xi^2(t,x,y,z,v)\frac{\partial}{\partial x} + \xi^3(t,x,y,z,v)\frac{\partial}{\partial y} + \xi^4(t,x,y,z,v)\frac{\partial}{\partial z} \\ &+ \eta(t,x,y,z,v)\frac{\partial}{\partial v}, \end{aligned}$$

corresponding to Lagrangian (3.6) may be obtained by solving the determining equation

$$X^{[2]}\mathcal{L} + \{D_t(\xi^1) + D_x(\xi^2) + D_y(\xi^3) + D_z(\xi^4)\}\mathcal{L} = D_t(B^1) + D_x(B^2) + D_y(B^3) + D_z(B^4), \tag{3.7}$$

where $X^{[2]}$, the second prolongation of X, is given by [32]

$$\begin{aligned} X^{[2]} =& \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \xi^4\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial v} + \zeta_t\frac{\partial}{\partial v_t} + \zeta_x\frac{\partial}{\partial v_x} + \zeta_y\frac{\partial}{\partial v_y} \\ &+ \zeta_{ty}\frac{\partial}{\partial v_{ty}} + \zeta_{xz}\frac{\partial}{\partial v_{xz}}, \end{aligned}$$

and $B^i(t,x,y,z,v)$, $(i = 1, \cdots, 4)$ are gauge functions. Now expanding (3.7), we have

$$\begin{aligned} &\frac{1}{2}v_{xz}{v_y}^2\xi^3_{tv} - \frac{1}{2}v_t v_y\xi^4_z - \frac{1}{2}v_x v_y\xi^4_z - \frac{1}{2}v_{xz}v_y\eta_{tv} - \frac{1}{2}v_{ty}v_z\eta_{xv} - \frac{1}{2}v_{xz}v_t\eta_{yv} - \frac{1}{2}v_{ty}v_x\eta_{vv} \\ &+ \frac{1}{2}v_{ty}{v_z}^2\xi^4_{xv} + \frac{1}{2}v_{xz}{v_t}^2\xi^1_{yv} + \frac{1}{2}v_{ty}{v_x}^2\xi^2_{zv} + \frac{1}{2}v_{xz}v_t\xi^t_{ty} + \frac{1}{2}v_{xz}v_x\xi^2_{ty} + \frac{1}{2}v_{xz}v_y\xi^3_{ty} \\ &+ \frac{1}{2}v_{xz}v_z\xi^4_{ty} + \frac{1}{2}v_{ty}v_t\xi^1_{xz} + \frac{1}{2}v_{ty}v_x\xi^2_{xz} + \frac{1}{2}v_{ty}v_y\xi^3_{xz} + \frac{1}{2}v_{ty}v_z\xi^4_{xz} - v_y v_x\eta_v - v_y v_t\eta_v \end{aligned}$$

$$
\begin{aligned}
&- v_{xz}v_{ty}\eta_v + \frac{1}{2}v_y{v_t}^2\xi^1_v + \frac{1}{4}{v_y}^4v_t\xi^1_v + \frac{1}{2}v_y{v_x}^2\xi^2_v + \frac{1}{4}{v_y}^4v_x\xi^2_v + \frac{1}{2}{v_y}^2v_t\xi^3_v + \frac{1}{2}{v_y}^2v_x\xi^3_v \\
&+ \frac{1}{4}{v_y}^4v_z\xi^4_v - \frac{1}{2}v_xv_y\xi^1_t + \frac{1}{2}v_yv_x\xi^2_t + \frac{1}{2}v_{xz}v_{xy}\xi^2_t + \frac{1}{2}v_{xz}v_{yy}\xi^3_t + \frac{1}{2}v_{xz}v_{yz}\xi^4_t + \frac{1}{2}v_yv_z\xi^4_t \\
&+ \frac{1}{2}v_yv_t\xi^1_x + \frac{1}{2}v_{ty}v_{tz}\xi^1_x - \frac{1}{2}v_tv_y\xi^2_x + \frac{1}{2}v_{ty}v_{yz}\xi^3_x + \frac{1}{2}v_yv_z\xi^4_x + \frac{1}{2}v_{ty}v_{zz}\xi^4_x + \frac{1}{2}v_tv_x\xi^1_y \\
&+ \frac{1}{3}v_t{v_y}^3\xi^1_y + \frac{1}{2}v_{xz}v_{tt}\xi^1_y + \frac{1}{2}v_x\xi^2_yv_t + \frac{1}{3}v_xv_y{}^3\xi^2_y + \frac{1}{2}v_{xz}v_{tx}\xi^2_y + \frac{1}{2}v_zv_t\xi^4_y + \frac{1}{2}v_zv_x\xi^4_y \\
&+ \frac{1}{3}v_z{v_y}^3\xi^4_y + \frac{1}{2}v_{xz}v_{tz}\xi^4_y + \frac{1}{2}v_{ty}v_{tx}\xi^1_z + \frac{1}{2}v_{ty}v_{xx}\xi^2_z + \frac{1}{2}v_{ty}v_{xy}\xi^3_z + \frac{1}{2}v_{xz}{v_t}^2v_y\xi^1_{vv} \\
&+ \frac{1}{2}v_{ty}{v_x}^2v_z\xi^2_{vv} + \frac{1}{2}v_{xz}{v_y}^2v_t\xi^3_{vv} + \frac{1}{2}v_{ty}{v_z}^2v_x\xi^4_{vv} + \frac{1}{2}v_{xz}v_tv_y\xi^1_{tv} + \frac{1}{2}v_{xz}v_xv_y\xi^2_{tv} \\
&+ \frac{1}{2}v_{xz}v_zv_y\xi^4_{tv} + \frac{1}{2}v_{ty}v_tv_z\xi^1_{xv} + \frac{1}{2}v_{ty}v_xv_z\xi^2_{xv} + \frac{1}{2}v_{ty}v_yv_z\xi^3_{xv} + \frac{1}{2}v_{xz}v_xv_t\xi^2_{yv} + \frac{1}{2}v_{xz}v_yv_t\xi^3_{yv} \\
&+ \frac{1}{2}v_{xz}v_zv_t\xi^4_{yv} + \frac{1}{2}v_{ty}v_tv_x\xi^1_{zv} + \frac{1}{2}v_{ty}v_yv_x\xi^3_{vz} + \frac{1}{2}v_{ty}v_zv_x\xi^4_{vz} + \frac{1}{2}v_xv_yv_t\xi^1_v + v_{xz}v_tv_{ty}\xi^1_v \\
&+ \frac{1}{2}v_{xz}v_{tt}v_y\xi^1_v + \frac{1}{2}v_{ty}v_{tz}v_x\xi^1_v + \frac{1}{2}v_{ty}v_{tx}v_z\xi^1_v + \frac{1}{2}v_tv_yv_x\xi^2_v + v_{xz}v_xv_{ty}\xi^2_v + \frac{1}{2}v_{xz}v_{xy}v_t\xi^2_v \\
&+ \frac{1}{2}v_{xz}v_{tx}v_y\xi^2_v + \frac{1}{2}v_{ty}v_{xx}v_z\xi^2_v + v_{xz}v_yv_{ty}\xi^3_v + \frac{1}{2}v_{xz}v_{yy}v_t\xi^3_v + \frac{1}{2}v_{ty}v_{xy}v_z\xi^3_v + \frac{1}{2}v_{ty}v_{yz}v_x\xi^3_v \\
&+ \frac{1}{2}v_tv_yv_z\xi^4_v + \frac{1}{2}v_xv_yv_z\xi^4_v + \frac{1}{2}v_{xz}v_{tz}v_y\xi^4_v + v_{xz}v_zv_{ty}\xi^4_v + \frac{1}{2}v_{xz}v_{yz}\xi^4_vv_t + \frac{1}{2}v_{ty}v_{zz}v_x\xi^4_v \\
&- \frac{1}{2}v_{ty}v_xv_z\eta_{vv} + \frac{1}{2}v_{ty}v_tv_xv_z\xi^1_{vv} + \frac{1}{2}v_{xz}v_xv_tv_y\xi^2_{vv} + \frac{1}{2}v_{ty}v_yv_xv_z\xi^3_{vv} + \frac{1}{2}v_{xz}v_zv_tv_y\xi^4_{vv} \\
&- \frac{1}{2}v_{xz}v_tv_y\eta_{vv} - B^2_x - B^3_y - B^4_z - B^1_t - \frac{1}{3}{v_y}^4\eta_v + \frac{1}{4}{v_y}^5\xi^3_v - \frac{1}{2}v_y\eta_t - \frac{1}{2}v_y\eta_x - \frac{1}{2}v_t\eta_y \\
&- \frac{1}{2}v_x\eta_y - \frac{1}{3}{v_y}^3\eta_y - \frac{1}{12}{v_y}^4\xi^1_t + \frac{1}{2}{v_y}^2\xi^3_t - \frac{1}{12}{v_y}^4\xi^2_x + \frac{1}{2}{v_y}^2\xi^3_x + \frac{1}{2}{v_t}^2\xi^1_y + \frac{1}{2}{v_x}^2\xi^2_y \\
&+ \frac{1}{4}{v_y}^4\xi^3_y - \frac{1}{12}{v_y}^4\xi^4_z - v_tB^1_v - v_xB^2_v - v_yB^3_v - v_zB^4_v - \frac{1}{2}v_{xz}\eta_{ty} - \frac{1}{2}v_{ty}\eta_{xz} = 0.
\end{aligned}
$$

The above equation now splits on derivatives of v yielding twenty seven homogeneous linear partial differential equations (PDEs)

$$
\begin{aligned}
&\xi^1_x = 0, \quad \xi^1_y = 0, \quad \xi^1_z = 0, \quad \xi^1_v = 0, \\
&\xi^2_t = 0, \quad \xi^2_y = 0, \quad \xi^2_z = 0, \quad \xi^2_v = 0, \\
&\xi^3_t = 0, \quad \xi^3_x = 0, \quad \xi^3_z = 0, \quad \xi^3_v = 0, \\
&\xi^4_t = 0, \quad \xi^4_x = 0, \quad \xi^4_y = 0, \quad \xi^4_v = 0, \\
&\xi^2_x + \xi^4_z = 0, \quad \xi^1_t + \xi^4_z = 0,
\end{aligned}
$$

$$\xi^1_t + \xi^2_x + \xi^4_z - 3\xi^3_y = 0, \quad \eta_{xz} = 0,$$
$$\eta_v = 0, \quad \eta_y = 0, \quad \eta_{xz} = 0,$$
$$B^1_v = 0, \quad B^2_v = 0, \quad B^4_v = 0,$$
$$B^3_v = -\frac{1}{2}\left(\eta_t + \eta_x\right),$$
$$B^1_t + B^2_x + B^3_y + B^4_z = 0.$$

Solving the above PDEs, albeit easy calculations, we obtain

$$\xi^1 = C_1 t + C_4,$$
$$\xi^2 = C_1 x + C_2,$$
$$\xi^3 = \frac{1}{3} C_1 y + C_5,$$
$$\xi^4 = -C_1 z + C_3,$$
$$\eta = F^4(t,x) + F^5(t,z),$$
$$B^1 = F^1\left(t,x,y,z\right),$$
$$B^2 = F^2\left(t,x,y,z\right),$$
$$B^3 = F^6(t,x,y,z) - \frac{1}{2}\left(F^4_t + F^4_x + F^5_t\right) v,$$
$$B^4 = F^3\left(t,x,y,z\right).$$

Thus, Noether symmetries and their corresponding gauge functions are

$$X_1 = \frac{\partial}{\partial t}, \quad B^i = 0, \ (i = 1, \cdots, 4),$$
$$X_2 = \frac{\partial}{\partial x}, \quad B^i = 0, \ (i = 1, \cdots, 4),$$
$$X_3 = \frac{\partial}{\partial y}, \quad B^i = 0, \ (i = 1, \cdots, 4),$$
$$X_4 = \frac{\partial}{\partial z}, \quad B^i = 0, \ (i = 1, \cdots, 4),$$
$$X_5 = 3t\frac{\partial}{\partial t} + 3x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 3z\frac{\partial}{\partial z}, \quad B^i = 0, \ (i = 1, \cdots, 4),$$
$$X_{F^4} = F^4(t,x)\frac{\partial}{\partial v}, \quad B^1 = B^2 = B^4 = 0, B^3 = -\frac{1}{2}(F^4_t + F^4_x)v$$

$$X_{F^5} = F^5(t,z)\frac{\partial}{\partial v}, \quad B^1 = B^2 = B^4 = 0, B^3 = -\frac{1}{2}F_t^5 v.$$

The conserved vector components corresponding to the above Noether point symmetries can be obtained by using [34]

$$T^k = \mathcal{L}\xi^k + \left(\eta^\alpha - u^\alpha_{x^j}\xi^j\right)\left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{x^l x^k}}\right)\right) + \sum_{l=k}^{n}\left(\zeta^\alpha_l - u^\alpha_{x^l x^j}\xi^j\right)\frac{\partial \mathcal{L}}{\partial u^\alpha_{x^k x^l}} - B^k. \tag{3.8}$$

Thus, in our case the conserved vectors corresponding to the Noether point symmetries $X_1, \cdots, X_5$, X_{F^4} and X_{F^5} are

$$\begin{aligned}
T_1^t &= -\frac{1}{2}v_x v_y - \frac{1}{12}v_y^4,\\
T_1^x &= \frac{1}{2}v_t v_y + \frac{1}{2}v_{tz}v_{ty},\\
T_1^y &= \frac{1}{2}v_t^2 + \frac{1}{2}v_t v_x + \frac{1}{3}v_t v_y^3 - \frac{1}{2}v_t v_{txz},\\
T_1^z &= -\frac{1}{2}v_t v_{txy};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{2}v_x v_y + \frac{1}{2}v_{xy}v_{xz},\\
T_2^x &= -\frac{1}{2}v_t v_y - \frac{1}{12}v_y^4,\\
T_2^y &= \frac{1}{2}v_x^2 + \frac{1}{2}v_t v_x + \frac{1}{3}v_x v_y^3 - \frac{1}{2}v_x v_{txz},\\
T_2^z &= -\frac{1}{2}v_x v_{txy};
\end{aligned}$$

$$\begin{aligned}
T_3^t &= \frac{1}{2}v_y^2 + \frac{1}{2}v_{yy}v_{xz},\\
T_3^x &= \frac{1}{2}v_y^2 + \frac{1}{2}v_{ty}v_{yz},\\
T_3^y &= \frac{1}{4}v_y^4 - \frac{1}{2}v_{ty}v_{xz} - \frac{1}{2}v_y v_{txz},\\
T_3^z &= -\frac{1}{2}v_y v_{txy};
\end{aligned}$$

$$
\begin{aligned}
T_4^t =& \frac{1}{2}v_y v_z + \frac{1}{2}v_{yz}v_{xz},\\
T_4^x =& \frac{1}{2}v_y v_z + \frac{1}{2}v_{ty}v_{zz},\\
T_4^y =& \frac{1}{2}v_t v_z + \frac{1}{2}v_x v_z + \frac{1}{3}v_z v_y^3 - \frac{1}{2}v_z v_{txz},\\
T_4^z =& -\frac{1}{2}v_t v_y - \frac{1}{2}v_x v_y - \frac{1}{12}v_y^4 - \frac{1}{2}v_{xz}v_{ty} - v_z v_{txy};
\end{aligned}
$$

$$
\begin{aligned}
T_5^t =& -\frac{3}{2}tv_x v_y - \frac{1}{4}tv_y^4 + \frac{3}{2}xv_x v_y + \frac{1}{2}yv_y^2 - \frac{3}{2}zv_y v_z + \frac{1}{2}v_y v_{xz} + \frac{3}{2}xv_{xy}v_{xz} + \frac{1}{2}yv_{xz}v_{yy}\\
&-\frac{3}{2}zv_{xz}u_z,\\
T_5^x =& -\frac{3}{2}xv_t v_y - \frac{1}{4}xv_y^4 + \frac{3}{2}tv_t v_y + \frac{1}{2}yv_y^2 - \frac{3}{2}zv_y v_z - \frac{3}{2}v_z v_{ty} + \frac{3}{2}tv_{ty}v_{tz} + \frac{1}{2}yv_{ty}u_z\\
&-\frac{3}{2}zv_{ty}v_{zz},\\
T_5^y =& -\frac{1}{4}yv_y^4 - \frac{1}{2}yv_{xz}v_{ty} + \frac{3}{2}tv_t^2 + \frac{3}{2}v_t v_x + v_t v_y^3 - \frac{3}{2}v_t v_{txz} + \frac{3}{2}v_t v_x + \frac{3}{2}xv_x^2 + xv_x v_y^3\\
&-\frac{3}{2}xv_x v_{txz} - \frac{1}{2}yv_y v_{txz} - \frac{3}{2}zv_z v_t - \frac{3}{2}zv_x v_z - zv_z v_y^3 + \frac{3}{2}zv_z v_{txz},\\
T_5^z =& \frac{3}{2}zv_t v_y + \frac{3}{2}zv_x v_y + \frac{1}{4}zv_y^4 + \frac{3}{2}zv_{xz}v_{ty} - \frac{3}{2}tv_t v_{txy} - \frac{3}{2}xv_x v_{txy} - \frac{1}{2}yv_y v_{txy}\\
&+\frac{3}{2}zv_z v_{txy};
\end{aligned}
$$

$$
\begin{aligned}
T_{F^4}^t =& -\frac{1}{2}v_y F^4(t,x),\\
T_{F^4}^x =& -\frac{1}{2}v_y F^4(t,x),\\
T_{F^4}^y =& \left(-\frac{1}{2}v_t - \frac{1}{2}v_x - \frac{1}{3}v_y^3 + \frac{1}{2}v_{txz}\right)F^4(t,x) + \frac{1}{2}vF_t^4 + \frac{1}{2}vF_x^4,\\
T_{F^4}^z =& \frac{1}{2}F^4(t,x)v_{txy};
\end{aligned}
$$

$$
\begin{aligned}
T_{F^5}^t =& -\frac{1}{2}v_y F^5(t,z),\\
T_{F^5}^x =& -\frac{1}{2}v_y F^5(t,z) - \frac{1}{2}v_{ty}F_z^5(t,z),
\end{aligned}
$$

$$T^y_{F^5} = -\frac{1}{2}v_t F^5(t,z) - \frac{1}{2}v_x F^5(t,z) - \frac{1}{3}v_y^3 F^5(t,z) + \frac{1}{2}v_{txz}F^5(t,z) + \frac{1}{2}vF^5_t,$$
$$T^z_{F^5} = \frac{1}{2}v_{txy}F^5(t,z).$$

Now reverting to variable u, we get nonlocal conservation laws of (3.2) as

$$T^t_1 = -\frac{1}{2}u\int u_x dy - \frac{1}{12}u^4,$$
$$T^x_1 = \frac{1}{2}u\int u_t dy + \frac{1}{2}u_t\int u_{tz}dy,$$
$$T^y_1 = \frac{1}{2}\left(\int u_t dy\right)^2 + \frac{1}{2}\int u_t dy\int u_x dy + \frac{1}{3}u^3\int u_t dy - \frac{1}{2}\int u_t dy\int u_{txz}dy,$$
$$T^z_1 = -\frac{1}{2}u_{tx}\int u_t dy;$$

$$T^t_2 = \frac{1}{2}u\int u_x dy + \frac{1}{2}u_x\int u_{xz}dy,$$
$$T^x_2 = -\frac{1}{2}u\int u_t dy - \frac{1}{12}u^4,$$
$$T^y_2 = \frac{1}{2}\left(\int u_x dy\right)^2 + \frac{1}{2}\int u_t dy\int u_x dy + \frac{1}{3}u^3\int u_x dy - \frac{1}{2}\int u_x dy\int u_{txz}dy,$$
$$T^z_2 = -\frac{1}{2}u_{tx}\int u_x dy;$$

$$T^t_3 = \frac{1}{2}u^2 + \frac{1}{2}u_y\int u_{xz}dy,$$
$$T^x_3 = \frac{1}{2}u^2 + \frac{1}{2}u_t u_z,$$
$$T^y_3 = \frac{1}{4}u^4 - \frac{1}{2}u_t\int u_{xz}dy - \frac{1}{2}u\int u_{txz}dy,$$
$$T^z_3 = -\frac{1}{2}uu_{tx};$$

$$T^t_4 = \frac{1}{2}u\int u_z dy + \frac{1}{2}u_z\int u_{xz}dy,$$
$$T^x_4 = \frac{1}{2}u\int u_z dy + \frac{1}{2}u_t\int u_{zz}dy,$$

$$
\begin{aligned}
T_4^y =& \frac{1}{2}\int u_t dy \int u_z dy + \frac{1}{2}\int u_x dy \int u_z dy + \frac{1}{3}u^3 \int u_z dy - \frac{1}{2}\int u_z dy \int u_{txz} dy,\\
T_4^z =& -\frac{1}{2}u\int u_t dy - \frac{1}{2}u\int u_x dy - \frac{1}{12}u^4 - \frac{1}{2}u_t \int u_{xz} dy - u_{tx}\int u_z dy;
\end{aligned}
$$

$$
\begin{aligned}
T_5^t =& -\frac{3}{2}tu\int u_x dy - \frac{1}{4}tu^4 + \frac{3}{2}xu\int u_x dy + \frac{1}{2}yu^2 - \frac{3}{2}zu\int u_z dy + \frac{1}{2}u\int u_{xz} dy\\
&+ \frac{3}{2}xu_x \int u_{xz} dy + \frac{1}{2}yu_y \int u_{xz} dy - \frac{3}{2}zu_z \int u_{xz} dy,\\
T_5^x =& -\frac{3}{2}xu\int u_t dy - \frac{1}{4}xu^4 + \frac{3}{2}tu\int u_t dy + \frac{1}{2}yu^2 - \frac{3}{2}zu\int u_z dy - \frac{3}{2}u_t \int u_z dy\\
&+ \frac{3}{2}tu_t \int u_{tz} dy + \frac{1}{2}yu_t u_z - \frac{3}{2}zu_t \int u_{zz} dy,\\
T_5^y =& -\frac{1}{4}yu^4 - \frac{1}{2}yu_t \int u_{xz} dy + \frac{3}{2}t\left(\int u_t dy\right)^2 + \frac{3}{2}\int u_t dy \int u_x dy + u^3 \int u_t dy\\
&- \frac{3}{2}\int u_t dy \int u_{txz} dy + \frac{3}{2}\int u_t dy \int u_x dy + \frac{3}{2}x\left(\int u_x dy\right)^2 + xu^3 \int u_x dy\\
&- \frac{3}{2}x\int u_x dy \int u_{txz} dy - \frac{1}{2}yu\int u_{txz} dy - \frac{3}{2}z\int u_z dy \int u_t dy\\
&- \frac{3}{2}z\int u_x dy \int u_z dy - zu^3 \int u_z dy + \frac{3}{2}z\int u_z dy \int u_{txz} dy,\\
T_5^z =& \frac{3}{2}zu\int u_t dy + \frac{3}{2}zu\int u_x dy + \frac{1}{4}zu^4 + \frac{3}{2}zu_t \int u_{xz} dy - \frac{3}{2}t\int u_t dy u_{tx}\\
&- \frac{3}{2}xu_{tx}\int u_x dy - \frac{1}{2}yuu_{tx} + \frac{3}{2}zu_{tx}\int u_z dy;
\end{aligned}
$$

$$
\begin{aligned}
T_{F^4}^t =& -\frac{1}{2}uF^4(t,x),\\
T_{F^4}^x =& -\frac{1}{2}uF^4(t,x),\\
T_{F^4}^y =& -\frac{1}{2}F^4(t,x)\int u_t dy - \frac{1}{2}F^4(t,x)\int u_x dy - \frac{1}{3}u^3F^4(t,x) + \frac{1}{2}F^4(t,x)\int u_{txz} dy\\
&+ \frac{1}{2}F_t^4 \int u dy + \frac{1}{2}F_x^4 \int u dy,\\
T_{F^4}^z =& \frac{1}{2}u_{tx}F^4(t,x);
\end{aligned}
$$

$$
\begin{aligned}
T^t_{F^5} &= -\frac{1}{2}uF^5(t,z),\\
T^x_{F^5} &= -\frac{1}{2}uF^5(t,z) - \frac{1}{2}u_tF^5_z,\\
T^y_{F^5} &= -\frac{1}{2}F^5(t,z)\int u_t dy - \frac{1}{2}F^5(t,z)\int u_x dy - \frac{1}{3}u^3F^5(t,z) + \frac{1}{2}F^5(t,z)\int u_{txz}dy\\
&\quad + \frac{1}{2}F^5_t\int u dy,\\
T^z_{F^5} &= \frac{1}{2}u_{tx}F^5(t,z).
\end{aligned}
$$

3.2.2 Closed form solutions

In this section, we compute soliton and Jacobi elliptic function solutions of the (3+1)-D mBBM equation (3.2) by invoking Lie point symmetry and simplest equation techniques.

Lie symmetries

The vector field (2.4) is a Lie symmetry of (3.2) if

$$X^{[3]}(u_t + u_x + u^2u_y - u_{txz})|_{(3.2)} = 0, \tag{3.9}$$

where $X^{[3]}$, the third prolongation of X, is given by [32]

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_y\frac{\partial}{\partial u_y} + \zeta_{txz}\frac{\partial}{\partial u_{txz}} \tag{3.10}$$

with $\zeta_t, \zeta_x, \zeta_x$, ζ_{tx} and ζ_{txz} being defined as

$$
\begin{aligned}
\zeta_t &= D_t(\eta) - u_tD_t(\xi^1) - u_xD_t(\xi^2) - u_yD_t(\xi^3) - u_zD_t(\xi^4),\\
\zeta_x &= D_x(\eta) - u_tD_x(\xi^1) - u_xD_x(\xi^2) - u_yD_x(\xi^3) - u_zD_x(\xi^4),\\
\zeta_y &= D_y(\eta) - u_tD_y(\xi^1) - u_xD_y(\xi^2) - u_yD_y(\xi^3) - u_zD_y(\xi^4),\\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt}D_x(\xi^1) - u_{tx}D_x(\xi^2) - u_{ty}D_x(\xi^3) - u_{tz}D_x(\xi^4),\\
\zeta_{txz} &= D_z(\zeta_{tx}) - u_{txt}D_z(\xi^1) - u_{txx}D_z(\xi^2) - u_{txy}D_z(\xi^3) - u_{txz}D_z(\xi^4).
\end{aligned}
$$

The total differential operators D_t, D_x, D_y, and D_z are given by (2.7). Equation (3.9) now splits on derivatives of v yielding twenty one homogeneous linear PDEs

$$\xi_t^4 = 0,\ \xi_t^3 = 0,\ \xi_t^2 = 0,\ \xi_{tt}^1 = 0,\ \xi_x^4 = 0,\ \xi_x^3 = 0,\ \xi_x^1 = 0,\ \xi_y^1 = 0,\ \xi_y^2 = 0,\ \xi_{yy}^3 = 0,$$
$$\xi_y^4 = 0,\ \xi_z^3 = 0,\ \xi_z^2 = 0,\ \xi_z^1 = 0,\ \xi_u^4 = 0,\ \xi_u^3 = 0,\ \xi_u^2 = 0,\ \xi_u^1 = 0,\ \xi_t^1 - \xi_x^2 = 0,$$
$$\xi_t^1 - \xi_z^4 = 0,\ \ 2\eta + u\left(\xi_t^1 - \xi_y^3\right) = 0.$$

After some simple calculations, one obtains the following values of $\xi^1, \xi^2, \xi^3, \xi^4$ and η:

$$\begin{aligned}
\xi^1 &= C_4 + C_6 t,\\
\xi^2 &= C_5 + C_6 x,\\
\xi^3 &= C_1 + C_2 y,\\
\xi^4 &= C_3 - C_6 z,\\
\eta &= \frac{1}{2}u(C_2 - C_6),
\end{aligned}$$

where C_i, $(i = 1, 2, \cdots, 6)$ are constants. Thus, Lie symmetries of (3.2) are

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial z},\ X_5 = 2y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},$$
$$X_6 = 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u},$$

which includes four translation and two scaling operators.

3.2.2.1 Soliton solution

We first perform symmetry reductions and transform the (3+1)-D mBBM equation (3.2) to a third-order nonlinear ODE. Consider the four translation symmetries X_1, X_2, X_3 and X_4. Taking a linear combination of the aforementioned generators, viz., $X = X_1 + \alpha X_2 + X_3 + X_4$, ($\alpha$ a constant) yields the invariants $f = -x + \alpha t$,

$g = -\alpha y + x$ and $h = -z + y$. Using these invariants, the invariant solution $u = \theta\,(f, g, h)$ satisfies

$$\theta_g - (1-\alpha)\theta_f + \theta^2(\theta_h - \alpha\theta_g) - \alpha(\theta_{ffh} - \theta_{fgh}) = 0. \tag{3.11}$$

Likewise, making use of the linear combination $\Gamma = \Gamma_1 + \beta\Gamma_2 + \Gamma_3$ of the Lie point symmetries

$$\Gamma_1 = \frac{\partial}{\partial f},\ \Gamma_2 = \frac{\partial}{\partial g},\ \Gamma_3 = \frac{\partial}{\partial h} \tag{3.12}$$

of (3.11) gives $r = -h + f$ and $s = -\beta h + g$ as two invariants. Hence $\theta = \phi(r, s)$ solves

$$\phi_s - (1-\alpha)\phi_r - \phi^2(\alpha\phi_s + \phi_r + \beta\phi_s) - \alpha(1-\beta)\phi_{rrs} + \alpha\phi_{rrr} - \alpha\beta\phi_{srs} = 0. \tag{3.13}$$

As before, we now utilise the two infinitesimal symmetry generators of (3.13), namely $\Sigma_1 = \partial/\partial r$ and $\Sigma_2 = \partial/\partial s$. The symmetry $\Sigma = \Sigma_1 + \nu\Sigma_2$ provides us with the invariant solution $\phi(r, s) = \psi(\xi)$ with $\xi = s - \nu r$. Thus, equation (3.13) transforms to

$$a\psi''' + b\psi^2\psi' - c\psi' = 0, \tag{3.14}$$

with $a = \nu\alpha(\nu+1)(\nu-\beta)$, $b = \alpha - \nu + \beta$ and $c = \nu - \alpha\nu + 1$. Double integration of (3.14) with integration constants zero (as we are looking for a soliton solution) yields

$$\psi'^2 = A\psi^2 - B\psi^4, \tag{3.15}$$

where $A = c/a$ and $B = b/(6a)$. Integration of equation (3.15) yields

$$\operatorname{sech}^{-1}\left(\frac{\psi}{K}\right) = \pm\sqrt{B}\xi + C^*, \tag{3.16}$$

where $K^2 = A/B$ and C^* is a constant. Equation (3.16) gives the value of ψ and hence we have

$$u = \pm\sqrt{\frac{6(\nu - \alpha\nu + 1)}{\alpha - \nu + \beta}}\operatorname{sech}\left(C^* \pm \sqrt{\frac{\alpha - \nu + \beta}{6\nu\alpha(\nu+1)(\nu-\beta)}}\,\xi\right), \tag{3.17}$$

where $\xi = -\alpha\,\nu\,t + (1+\nu)\,x - (\alpha+\beta-\nu)\,y - (\nu-\beta)\,z$. Thus, we recuperate the bell-shaped soliton solution (3.17), which was derived in [68] by using an ad hoc method. The beauty of the above method is that the form of the solution was not chosen in advance. See the graphical representation below which has been drawn using suitable parameter values of solution (3.17).

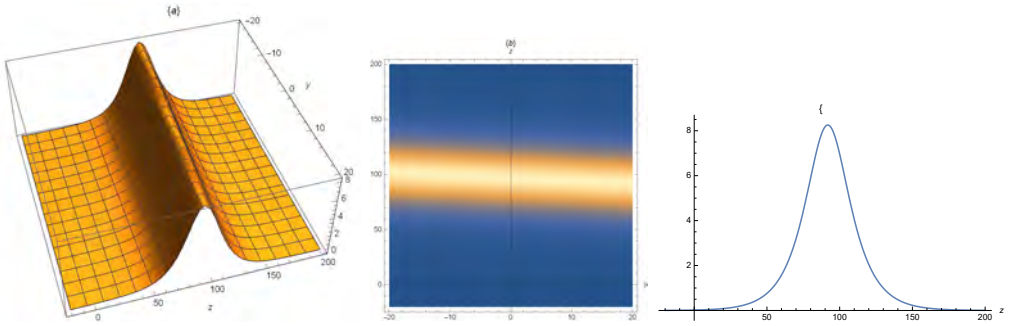


Figure 3.1: Profile of soliton solution (3.17).

3.2.2.2 Exact solutions using the simplest equation method

We now construct exact explicit solutions of the (3+1)-D mBBM equation (3.2) in terms of Jacobi elliptic functions following the method of Section 1.6.2. We apply this method to ODE (3.14).

Solutions of (3.14) using (1.57) as the simplest equation

We note that for the equation (3.14), using the balancing procedure, we get $M = 1$. Thus, (1.56) is

$$\psi(\xi) = A_{-1}H^{-1}(\xi) + A_0 + A_1H(\xi). \tag{3.18}$$

Using this value of ψ in (3.14) along with (1.57) we obtain eleven algebraic equations:

$\alpha\,A_0{A_1}^2 + \beta\,A_0{A_1}^2 - \nu\,A_0{A_1}^2 = 0,$

$\alpha\,{A_{-1}}^2A_0 - \alpha\,\omega\,{A_{-1}}^2A_0 - \beta\,\omega\,{A_{-1}}^2A_0 + \nu\,\omega\,{A_{-1}}^2A_0 + \beta\,{A_{-1}}^2A_0$

$$
\begin{aligned}
&-\nu\,{A_{-1}}^2A_0=0,\\
&2\alpha\,\omega\,{A_{-1}}^2A_0-2\nu\,\omega\,{A_{-1}}^2A_0-\alpha\,{A_{-1}}^2A_0-\beta\,{A_{-1}}^2A_0+\nu\,{A_{-1}}^2A_0=0,\\
&2\,\nu\,\omega\,A_0{A_1}^2-2\,\alpha\,\omega\,A_0{A_1}^2-2\,\beta\,\omega\,A_0{A_1}^2+\alpha\,A_0{A_1}^2+\beta\,A_0{A_1}^2-\nu\,A_0{A_1}^2=0,\\
&6\,\alpha\,\beta\,\nu^2\omega\,A_1-6\,\alpha\,\nu^3\omega\,A_1+6\,\alpha\,\beta\,\nu\,\omega\,A_1-6\,\alpha\,\nu^2\omega\,A_1+\alpha\,{A_1}^3+\beta\,{A_1}^3\\
&-\nu\,{A_1}^3=0,\\
&\alpha\,\omega\,A_0{A_1}^2-\alpha\,\omega\,{A_{-1}}^2A_0-\beta\,\omega\,{A_{-1}}^2A_0+\beta\,\omega\,A_0{A_1}^2+\nu\,\omega\,{A_{-1}}^2A_0-\nu\,\omega\,A_0{A_1}^2\\
&+\nu\,A_0{A_1}^2-\alpha\,A_0{A_1}^2-\beta\,A_0{A_1}^2=0,\\
&6\,\alpha\,\nu^3\omega^2A_{-1}-6\,\alpha\,\beta\,\nu^2\omega^2A_{-1}+12\,\alpha\,\beta\,\nu^2\omega\,A_{-1}-6\,\alpha\,\beta\,\nu\,\omega^2A_{-1}-12\,\alpha\,\nu^3\omega\,A_{-1}\\
&+6\,\alpha\,\nu^2\omega^2A_{-1}-6\,\alpha\,\beta\,\nu^2A_{-1}+12\,\alpha\,\beta\,\nu\,\omega\,A_{-1}+6\,\alpha\,\nu^3A_{-1}-12\,\alpha\,\nu^2\omega\,A_{-1}-\alpha\,\omega\,{A_{-1}}^3\\
&+\nu\,\omega\,{A_{-1}}^3-\beta\,\omega\,{A_{-1}}^3-6\,\alpha\,\beta\,\nu\,A_{-1}+6\,\alpha\,\nu^2A_{-1}+\alpha\,{A_{-1}}^3+\beta\,{A_{-1}}^3-\nu\,{A_{-1}}^3=0,\\
&14\,\alpha\,\nu^3\omega^2A_1-14\,\alpha\,\beta\,\nu^2\omega^2A_1+7\,\alpha\,\beta\,\nu^2\omega\,A_1-14\,\alpha\,\beta\,\nu\,\omega^2A_1-7\,\alpha\,\nu^3\omega\,A_1+14\,\alpha\,\nu^2\omega^2A_1\\
&+7\,\alpha\,\beta\,\nu\,\omega\,A_1-7\,\alpha\,\nu^2\omega\,A_1+\alpha\,\omega\,A_{-1}{A_1}^2+\alpha\,\omega\,{A_0}^2A_1-2\,\alpha\,\omega\,{A_1}^3+\beta\,\omega\,A_{-1}{A_1}^2\\
&+\beta\,\omega\,{A_0}^2A_1-2\,\beta\,\omega\,{A_1}^3-\nu\,\omega\,A_{-1}{A_1}^2-\nu\,\omega\,{A_0}^2A_1+2\,\nu\,\omega\,{A_1}^3+\alpha\,\nu\,\omega\,A_1+\alpha\,{A_1}^3\\
&+\beta\,{A_1}^3-\nu\,{A_1}^3-\nu\,\omega\,A_1-\omega\,A_1=0,\\
&14\,\alpha\,\beta\,\nu^2\omega^2A_{-1}-14\,\alpha\,\nu^3\omega^2A_{-1}-21\,\alpha\,\beta\,\nu^2\omega\,A_{-1}+14\,\alpha\,\beta\,\nu\,\omega^2A_{-1}\\
&+21\,\alpha\,\nu^3\omega\,A_{-1}-14\,\alpha\,\nu^2\omega^2A_{-1}+7\,\alpha\,\beta\,\nu^2A_{-1}-21\,\alpha\,\beta\,\nu\,\omega\,A_{-1}\\
&-7\,\alpha\,\nu^3A_{-1}+21\,\alpha\,\nu^2\omega\,A_{-1}+2\,\alpha\,\omega\,{A_{-1}}^3-\alpha\,\omega\,{A_{-1}}^2A_1-\alpha\,\omega\,A_{-1}{A_0}^2+2\,\beta\,\omega\,{A_{-1}}^3\\
&-\beta\,\omega\,{A_{-1}}^2A_1-\beta\,\omega\,A_{-1}{A_0}^2-2\,\nu\,\omega\,{A_{-1}}^3+\nu\,\omega\,{A_{-1}}^2A_1+\nu\,\omega\,A_{-1}{A_0}^2+7\,\alpha\,\beta\,\nu\,A_{-1}\\
&+\alpha\,A_{-1}{A_0}^2-7\,\alpha\,\nu^2A_{-1}-\alpha\,\nu\,\omega\,A_{-1}-\alpha\,{A_{-1}}^3+\alpha\,{A_{-1}}^2A_1-\beta\,{A_{-1}}^3+\beta\,{A_{-1}}^2A_1+\nu\,{A_{-1}}^3\\
&+\beta\,A_{-1}{A_0}^2-\nu\,{A_{-1}}^2A_1-\nu\,A_{-1}{A_0}^2+\alpha\,\nu\,A_{-1}+\nu\,\omega\,A_{-1}-\nu\,A_{-1}+\omega\,A_{-1}-A_{-1}=0,\\
&2\,\alpha\,\beta\,\nu^2\omega^2A_{-1}+10\,\alpha\,\beta\,\nu^2\omega^2A_1-2\,\alpha\,\nu^3\omega^2A_{-1}-10\,\alpha\,\nu^3\omega^2A_1-\alpha\,\beta\,\nu^2\omega\,A_{-1}-10\,\alpha\,\beta\,\nu^2\omega\,A_1\\
&+2\,\alpha\,\beta\,\nu\,\omega^2A_{-1}+10\,\alpha\,\beta\,\nu\,\omega^2A_1+\alpha\,\nu^3\omega\,A_{-1}+10\,\alpha\,\nu^3\omega\,A_1-2\,\alpha\,\nu^2\omega^2A_{-1}-10\,\alpha\,\nu^2\omega^2A_1\\
&+\alpha\,\beta\,\nu^2A_1-\alpha\,\beta\,\nu\,\omega\,A_{-1}-10\,\alpha\,\beta\,\nu\,\omega\,A_1-\alpha\,\nu^3A_1+\alpha\,\nu^2\omega\,A_{-1}+10\,\alpha\,\nu^2\omega\,A_1\\
&-\alpha\,\omega\,{A_{-1}}^2A_1-\alpha\,\omega\,A_{-1}{A_0}^2-2\,\alpha\,\omega\,A_{-1}{A_1}^2-2\,\alpha\,\omega\,{A_0}^2A_1+\alpha\,\omega\,{A_1}^3-\beta\,\omega\,{A_{-1}}^2A_1
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\,\omega\,A_{-1}{A_0}^2-2\,\beta\,\omega\,A_{-1}{A_1}^2-2\,\beta\,\omega\,{A_0}^2A_1+\beta\,\omega\,{A_1}^3+\nu\,\omega\,{A_{-1}}^2A_1+\nu\,\omega\,A_{-1}{A_0}^2\\
&+2\,\nu\,\omega\,A_{-1}{A_1}^2+2\,\nu\,\omega\,{A_0}^2A_1-\nu\,\omega\,{A_1}^3+\alpha\,\beta\,\nu\,A_1-\alpha\,\nu^2A_1-\alpha\,\nu\,\omega\,A_{-1}-2\,\alpha\,\nu\,\omega\,A_1\\
&+\alpha\,A_{-1}{A_1}^2+\alpha\,{A_0}^2A_1-\alpha\,{A_1}^3+\beta\,A_{-1}{A_1}^2+\beta\,{A_0}^2A_1-\beta\,{A_1}^3-\nu\,A_{-1}{A_1}^2\\
&-\nu\,{A_0}^2A_1+\nu\,{A_1}^3+\alpha\,\nu\,A_1+\nu\,\omega\,A_{-1}+2\,\nu\,\omega\,A_1-\nu\,A_1+\omega\,A_{-1}+2\,\omega\,A_1-A_1=0,\\
&-10\,\alpha\,\beta\,\nu^2\omega^2A_{-1}-2\,\alpha\,\beta\,\nu^2\omega^2A_1+10\,\alpha\,\nu^3\omega^2A_{-1}+2\,\alpha\,\nu^3\omega^2A_1+10\,\alpha\,\beta\,\nu^2\omega\,A_{-1}\\
&+3\,\alpha\,\beta\,\nu^2\omega\,A_1-10\,\alpha\,\beta\,\nu\,\omega^2A_{-1}-2\,\alpha\,\beta\,\nu\,\omega^2A_1-10\,\alpha\,\nu^3\omega\,A_{-1}-3\,\alpha\,\nu^3\omega\,A_1\\
&+10\,\alpha\,\nu^2\omega^2A_{-1}+2\,\alpha\,\nu^2\omega^2A_1-\alpha\,\beta\,\nu^2A_{-1}-\alpha\,\beta\,\nu^2A_1+10\,\alpha\,\beta\,\nu\,\omega\,A_{-1}+3\,\alpha\,\beta\,\nu\,\omega\,A_1\\
&+\alpha\,\nu^3A_{-1}+\alpha\,\nu^3A_1-10\,\alpha\,\nu^2\omega\,A_{-1}-3\,\alpha\,\nu^2\omega\,A_1-\alpha\,\omega\,{A_{-1}}^3+2\,\alpha\,\omega\,{A_{-1}}^2A_1\\
&+2\,\alpha\,\omega\,A_{-1}{A_0}^2+\alpha\,\omega\,A_{-1}{A_1}^2+\alpha\,\omega\,{A_0}^2A_1-\beta\,\omega\,{A_{-1}}^3+2\,\beta\,\omega\,{A_{-1}}^2A_1\\
&+2\,\beta\,\omega\,A_{-1}{A_0}^2+\beta\,\omega\,A_{-1}{A_1}^2+\beta\,\omega\,{A_0}^2A_1+\nu\,\omega\,{A_{-1}}^3-2\,\nu\,\omega\,{A_{-1}}^2A_1\\
&-2\,\nu\,\omega\,A_{-1}{A_0}^2-\nu\,\omega\,A_{-1}{A_1}^2-\nu\,\omega\,{A_0}^2A_1-\alpha\,\beta\,\nu\,A_{-1}-\alpha\,\beta\,\nu\,A_1+\alpha\,\nu^2A_{-1}\\
&+\alpha\,\nu^2A_1+2\,\alpha\,\nu\,\omega\,A_{-1}+\alpha\,\nu\,\omega\,A_1-\alpha\,{A_{-1}}^2A_1-\alpha\,A_{-1}{A_0}^2-\alpha\,A_{-1}{A_1}^2-\alpha\,{A_0}^2A_1\\
&-\beta\,{A_{-1}}^2A_1-\beta\,A_{-1}{A_0}^2-\beta\,A_{-1}{A_1}^2-\beta\,{A_0}^2A_1+\nu\,{A_{-1}}^2A_1+\nu\,A_{-1}{A_0}^2\\
&+\nu\,A_{-1}{A_1}^2+\nu\,{A_0}^2A_1-\alpha\,\nu\,A_{-1}-\alpha\,\nu\,A_1-2\,\nu\,\omega\,A_{-1}-\nu\,\omega\,A_1+\nu\,A_{-1}+\nu\,A_1\\
&-2\,\omega\,A_{-1}-\omega\,A_1+A_{-1}+A_1=0.
\end{aligned}
$$

Solving the above system, using Maple, we obtain

$$
\begin{aligned}
&\nu=-\frac{\alpha^2-\alpha+1+\sqrt{\alpha^4-2\,\alpha^3-\alpha^2-2\,\alpha+1}}{2\alpha^2},\\
&\beta=\frac{\nu(2\,\omega-1)+\alpha}{2\,\omega-1},\ \ A_{-1}=A_0=0,\ \ A_1=\pm\sqrt{\frac{3(\nu(1-\alpha)+1)}{\alpha}},\\
&\alpha\neq 0,\ \ \alpha\neq 1,\ \ \nu\neq\beta.
\end{aligned}
$$

Therefore the solution to (3.2) is

$$
u(t,x,y,z)=\pm\sqrt{\frac{3(\nu(1-\alpha)+1)}{\alpha}}\ \mathrm{cn}\left(\xi\,|\omega\right), \tag{3.19}
$$

where $\alpha\neq 0,\ \ \alpha\neq 1,$, $\nu\neq\beta$, and $\xi=-\alpha\,\nu\,t+(1+\nu)\,x-(\alpha+\beta-\nu)\,y-(\nu-\beta)\,z$. We note that the solution (3.17) obtained by direct integration can be derived from

solution (3.19), since $\mathrm{cn}(\xi|\omega) \to \mathrm{sech}(\xi)$ when $\omega \to 1$. However, solutions (3.19) are periodic in nature, consistent with Jacobi amplitude functions. See a 3D rendering of one of the solutions in Figure 3.2. Here we utilise suitable values for parameters α and ω.

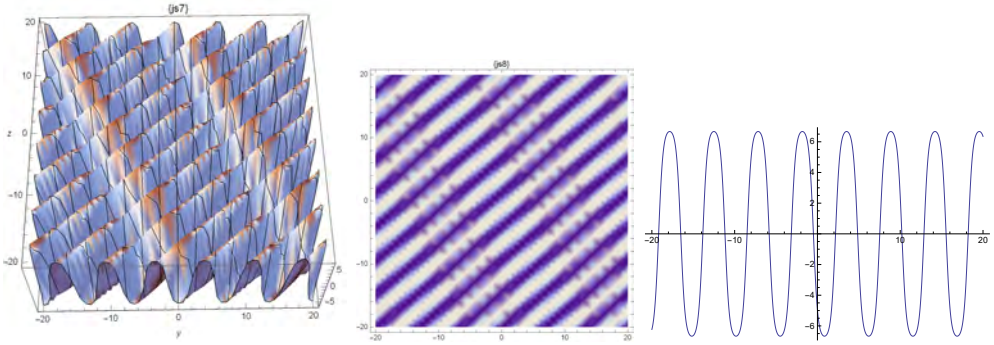


Figure 3.2: Profile of cnoidal wave solution (3.19)

Solutions of (3.14) using (1.58) as the simplest equation

Substituting (3.18) along with (1.58) into (3.14), we obtain an algebraic equation, which splits into the following eleven algebraic equations:

$$\alpha\, {A_{-1}}^2 A_0 + \beta\, {A_{-1}}^2 A_0 - \nu\, {A_{-1}}^2 A_0 = 0,$$

$$\nu\, A_0 {A_1}^2 - \alpha\, A_0 {A_1}^2 - \beta\, A_0 {A_1}^2 = 0,$$

$$\nu\, {A_{-1}}^2 A_0 - \alpha\,\omega\, {A_{-1}}^2 A_0 - \beta\,\omega\, {A_{-1}}^2 A_0 + \nu\,\omega\, {A_{-1}}^2 A_0 - \alpha\, {A_{-1}}^2 A_0$$
$$- \beta\, {A_{-1}}^2 A_0 = 0,$$

$$\alpha\,\omega\, A_0 {A_1}^2 + \beta\,\omega\, A_0 {A_1}^2 - \nu\,\omega\, A_0 {A_1}^2 + \alpha\, A_0 {A_1}^2 + \beta\, A_0 {A_1}^2 - \nu\, A_0 {A_1}^2 = 0,$$

$$\alpha\,\omega\, {A_{-1}}^2 A_0 + \beta\,\omega\, {A_{-1}}^2 A_0 - \nu\,\omega\, {A_{-1}}^2 A_0 - \alpha\, A_0 {A_1}^2 - \beta\, A_0 {A_1}^2$$
$$+ \nu\, A_0 {A_1}^2 = 0,$$

$$6\,\alpha\,\nu^3 A_{-1} - 6\,\alpha\,\beta\,\nu^2 A_{-1} - 6\,\alpha\,\beta\,\nu\, A_{-1} + 6\,\alpha\,\nu^2 A_{-1} + \alpha\, {A_{-1}}^3 + \beta\, {A_{-1}}^3 - \nu\, {A_{-1}}^3 = 0,$$

$$6\,\alpha\,\beta\,\nu^2\omega\, A_1 - 6\,\alpha\,\nu^3\omega\, A_1 + 6\,\alpha\,\beta\,\nu\,\omega\, A_1 - 6\,\alpha\,\nu^2\omega\, A_1 - \alpha\, {A_1}^3 - \beta\, {A_1}^3$$
$$+ \nu\, {A_1}^3 = 0,$$

$$7\,\alpha\,\beta\,\nu^2\omega\,A_{-1} - 7\,\alpha\,\nu^3\omega\,A_{-1} + 7\,\alpha\,\beta\,\nu^2 A_{-1} + 7\,\alpha\,\beta\,\nu\,\omega\,A_{-1} - 7\,\alpha\,\nu^3 A_{-1} - 7\,\alpha\,\nu^2\omega\,A_{-1}$$
$$- \alpha\,\omega\,{A_{-1}}^3 - \beta\,\omega\,{A_{-1}}^3 + \nu\,\omega\,{A_{-1}}^3 + 7\,\alpha\,\beta\,\nu\,A_{-1} - 7\,\alpha\,\nu^2 A_{-1} - \alpha\,{A_{-1}}^3 + \alpha\,{A_{-1}}^2 A_1$$
$$+ \alpha\,A_{-1}{A_0}^2 - \beta\,{A_{-1}}^3 + \beta\,{A_{-1}}^2 A_1 + \beta\,A_{-1}{A_0}^2 + \nu\,{A_{-1}}^3 - \nu\,{A_{-1}}^2 A_1 - \nu\,A_{-1}{A_0}^2$$
$$+ \alpha\,\nu\,A_{-1} - \nu\,A_{-1} - A_{-1} = 0,$$
$$7\,\alpha\,\nu^3\omega^2 A_1 - 7\,\alpha\,\beta\,\nu^2\omega^2 A_1 - 7\,\alpha\,\beta\,\nu^2\omega\,A_1 - 7\,\alpha\,\beta\,\nu\,\omega^2 A_1 + 7\,\alpha\,\nu^3\omega\,A_1 + 7\,\alpha\,\nu^2\omega^2 A_1$$
$$+ 7\,\alpha\,\nu^2\omega\,A_1 - 7\,\alpha\,\beta\,\nu\,\omega\,A_1 - \alpha\,\omega\,A_{-1}{A_1}^2 - \alpha\,\omega\,{A_0}^2 A_1 + \alpha\,\omega\,{A_1}^3 - \beta\,\omega\,A_{-1}{A_1}^2$$
$$+ \beta\,\omega\,{A_1}^3 - \beta\,\omega\,{A_0}^2 A_1 + \nu\,\omega\,A_{-1}{A_1}^2 + \nu\,\omega\,{A_0}^2 A_1 - \nu\,\omega\,{A_1}^3 - \alpha\,\nu\,\omega\,A_1 + \alpha\,{A_1}^3$$
$$+ \beta\,{A_1}^3 - \nu\,{A_1}^3 + \nu\,\omega\,A_1 + \omega\,A_1 = 0,$$
$$\alpha\,\nu^3\omega^2 A_{-1} - \alpha\,\beta\,\nu^2\omega^2 A_{-1} - 8\,\alpha\,\beta\,\nu^2\omega\,A_{-1} - \alpha\,\beta\,\nu^2\omega\,A_1 - \alpha\,\beta\,\nu\,\omega^2 A_{-1} + 8\,\alpha\,\nu^3\omega\,A_{-1}$$
$$+ \alpha\,\nu^3\omega\,A_1 + \alpha\,\nu^2\omega^2 A_{-1} - \alpha\,\beta\,\nu^2 A_{-1} - \alpha\,\beta\,\nu^2 A_1 - 8\,\alpha\,\beta\,\nu\,\omega\,A_{-1} - \alpha\,\beta\,\nu\,\omega\,A_1$$
$$+ \alpha\,\nu^3 A_{-1} + \alpha\,\nu^3 A_1 + 8\,\alpha\,\nu^2\omega\,A_{-1} + \alpha\,\nu^2\omega\,A_1 + \alpha\,\omega\,{A_{-1}}^3 - \alpha\,\omega\,{A_{-1}}^2 A_1 - \alpha\,\omega\,A_{-1}{A_0}^2$$
$$+ \beta\,\omega\,{A_{-1}}^3 - \beta\,\omega\,{A_{-1}}^2 A_1 - \beta\,\omega\,A_{-1}{A_0}^2 - \nu\,\omega\,{A_{-1}}^3 + \nu\,\omega\,{A_{-1}}^2 A_1 + \nu\,\omega\,A_{-1}{A_0}^2$$
$$- \alpha\,\beta\,\nu\,A_{-1} - \alpha\,\beta\,\nu\,A_1 + \alpha\,\nu^2 A_{-1} + \alpha\,\nu^2 A_1 - \alpha\,\nu\,\omega\,A_{-1} - \alpha\,{A_{-1}}^2 A_1 - \alpha\,A_{-1}{A_0}^2$$
$$- \alpha\,A_{-1}{A_1}^2 - \alpha\,{A_0}^2 A_1 - \beta\,{A_{-1}}^2 A_1 - \beta\,A_{-1}{A_0}^2 - \beta\,A_{-1}{A_1}^2 - \beta\,{A_0}^2 A_1 + \nu\,{A_{-1}}^2 A_1$$
$$+ \nu\,A_{-1}{A_0}^2 + \nu\,A_{-1}{A_1}^2 + \nu\,{A_0}^2 A_1 - \alpha\,\nu\,A_{-1} - \alpha\,\nu\,A_1 + \nu\,\omega\,A_{-1} + \nu\,A_{-1} + \nu\,A_1$$
$$+ \omega\,A_{-1} + A_{-1} + A_1 = 0,$$
$$\alpha\,\beta\,\nu^2\omega^2 A_{-1} + \alpha\,\beta\,\nu^2\omega^2 A_1 - \alpha\,\nu^3\omega^2 A_{-1} - \alpha\,\nu^3\omega^2 A_1 + \alpha\,\beta\,\nu^2\omega\,A_{-1} + 8\,\alpha\,\beta\,\nu^2\omega\,A_1$$
$$+ \alpha\,\beta\,\nu\,\omega^2 A_{-1} + \alpha\,\beta\,\nu\,\omega^2 A_1 - \alpha\,\nu^3\omega\,A_{-1} - 8\,\alpha\,\nu^3\omega\,A_1 - \alpha\,\nu^2\omega^2 A_{-1} - \alpha\,\nu^2\omega^2 A_1$$
$$+ \alpha\,\beta\,\nu^2 A_1 + \alpha\,\beta\,\nu\,\omega\,A_{-1} + 8\,\alpha\,\beta\,\nu\,\omega\,A_1 - \alpha\,\nu^3 A_1 - \alpha\,\nu^2\omega\,A_{-1} - 8\,\alpha\,\nu^2\omega\,A_1$$
$$+ \alpha\,\omega\,{A_{-1}}^2 A_1 + \alpha\,\omega\,A_{-1}{A_0}^2 + \alpha\,\omega\,A_{-1}{A_1}^2 + \alpha\,\omega\,{A_0}^2 A_1 + \beta\,\omega\,{A_{-1}}^2 A_1 + \beta\,\omega\,A_{-1}{A_0}^2$$
$$+ \beta\,\omega\,A_{-1}{A_1}^2 + \beta\,\omega\,{A_0}^2 A_1 - \nu\,\omega\,{A_{-1}}^2 A_1 - \nu\,\omega\,A_{-1}{A_0}^2 - \nu\,\omega\,A_{-1}{A_1}^2 - \nu\,\omega\,{A_0}^2 A_1$$
$$+ \alpha\,\beta\,\nu\,A_1 - \alpha\,\nu^2 A_1 + \alpha\,\nu\,\omega\,A_{-1} + \alpha\,\nu\,\omega\,A_1 + \alpha\,A_{-1}{A_1}^2 + \alpha\,{A_0}^2 A_1 - \alpha\,{A_1}^3$$
$$+ \beta\,A_{-1}{A_1}^2 + \beta\,{A_0}^2 A_1 - \beta\,{A_1}^3 - \nu\,A_{-1}{A_1}^2 - \nu\,{A_0}^2 A_1 + \nu\,{A_1}^3 + \alpha\,\nu\,A_1$$
$$- \nu\,\omega\,A_{-1} - \nu\,\omega\,A_1 - \nu\,A_1 - \omega\,A_{-1} - \omega\,A_1 - A_1 = 0.$$

A solution of the above system, using Maple, is

$$\nu = -\frac{\alpha^2 - \alpha + 1 + \sqrt{\alpha^4 - 2\,\alpha^3 - \alpha^2 - 2\,\alpha + 1}}{2\alpha^2},$$
$$\beta = \frac{\nu\,(\omega + 1) - \alpha}{\omega + 1}, \quad A_{-1} = A_0 = 0, \quad A_1 = \pm\sqrt{\frac{6(\nu(1-\alpha)+1)}{\alpha}},$$
$$\alpha \neq 0, \quad \alpha \neq 1, \quad \nu \neq \beta.$$

Therefore a solution to (3.2) becomes

$$u(t, x, y, z) = \pm\sqrt{\frac{6(\nu(1-\alpha)+1)}{\alpha}}\ \mathrm{sn}\,(\xi\,|\omega\,), \tag{3.20}$$

where $0 \leq \omega \leq 1$ and $\xi = -\alpha\,\nu\,t + (1+\nu)\,x - (\alpha + \beta - \nu)\,y - (\nu - \beta)\,z$. See the profile of solution (3.20) in Figure 3.3.

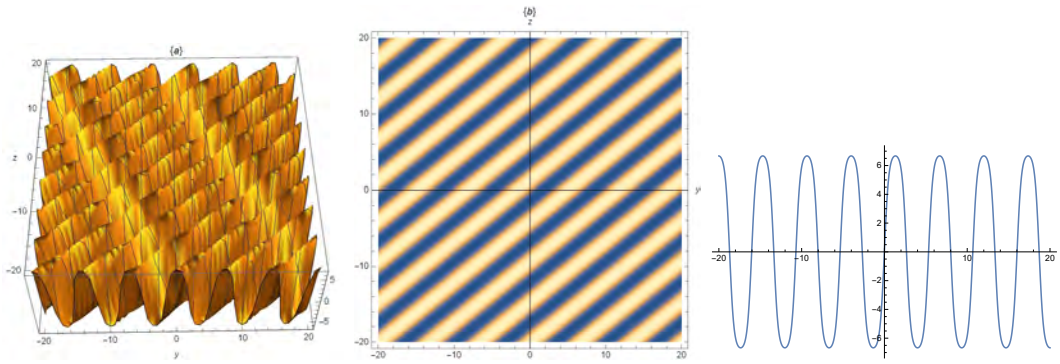


Figure 3.3: Profile of snoidal wave solution (3.20).

3.3 Conservation laws and exact solutions of (3.3)

Following the procedure of the previous section, we have the following nonlocal conservation laws of the (3+1)-D mBBM equation (3.3):

$$T_1^t = -\frac{1}{2}u\int u_z dx - \frac{1}{12}u^4,$$
$$T_1^x = \frac{1}{2}\left(\int u_t dx\right)^2 + \frac{1}{2}\int u_t dx \int u_z dx + \frac{1}{3}u^3\int u_t dx - \frac{1}{2}u_{ty}\int u_t dx + \frac{1}{2}u_t\int u_{ty} dx,$$

$$T_1^y = -\frac{1}{2}u_{tx}\int u_t dx,$$
$$T_1^z = \frac{1}{2}u\int u_t dx;$$

$$T_2^t = \frac{1}{2}u^2 + \frac{1}{2}u_x u_y,$$
$$T_2^x = \frac{1}{4}u^4 - \frac{1}{2}uu_{ty},$$
$$T_2^y = -\frac{1}{2}u_{tx},$$
$$T_2^z = -\frac{1}{2}u_{ty}\int u_y dx;$$

$$T_3^t = \frac{1}{2}u\int u_y dx + \frac{1}{2}u_x\int u_{yy}dx,$$
$$T_3^x = \frac{1}{2}\int u_t dx\int u_y dx + \frac{1}{2}\int u_y dx\int u_z dx + \frac{1}{3}u^3\int u_y dx - \frac{1}{2}u_{ty}\int u_y dx + \frac{1}{2}u_y\int u_{ty}dx,$$
$$T_3^y = -\frac{1}{2}u\int u_t dx - \frac{1}{2}u\int u_z dx - \frac{1}{12}u^4 - \frac{1}{2}u_x\int u_{ty}dx - \frac{1}{2}u_{tx}\int u_y dx,$$
$$T_3^z = \frac{1}{2}u\int u_y dx;$$

$$T_4^t = \frac{1}{2}u\int u_z dx + \frac{1}{2}u_x\int u_{yz}dx,$$
$$T_4^x = \frac{1}{2}\int u_t dx\int u_z dx + \frac{1}{2}\left(\int u_z dx\right)^2 + \frac{1}{3}u^3\int u_z dx - \frac{1}{2}u_{ty}\int u_z dx + \frac{1}{2}u_z\int u_{ty}dx,$$
$$T_4^y = -\frac{1}{2}u_{tx}\int u_z dx,$$
$$T_4^z = -\frac{1}{12}u^4 - \frac{1}{2}u\int u_t dx - \frac{1}{2}u_x\int u_{ty}dx;$$

$$
\begin{aligned}
T_5^t =& \frac{1}{2}tu\int u_z dx - \frac{1}{2}zu\int u_z dx - \frac{1}{12}zu^4 + \frac{1}{12}tu^4 + \frac{1}{2}yu\int u_y dx + \frac{1}{2}u_x\int u_y dx \\
&+ \frac{1}{2}yu_x\int u_{yy}dx, \\
T_5^x =& \frac{1}{2}\int udx\int u_t dx + \frac{1}{2}\int udx\int u_z dx + \frac{1}{3}u^3\int udx - \frac{1}{2}u_{ty}\int udx \\
&+ t\left(\int u_t dx\right)^2 + t\int u_t dx\int u_z dx + \frac{2}{3}tu^3\int u_t dx - tu_{ty}\int u_t dx \\
&+ z\int u_t dx\int u_z dx + z\left(\int u_z dx\right)^2 + \frac{2}{3}zu^3\int u_z dx - zu_{ty}\int u_z dx \\
&+ \frac{1}{2}u\int u_{ty}dx + tu_t\int u_{ty}dx + zu_z\int u_{ty}dx, \\
T_5^y =& -\frac{1}{2}u_{tx}\int udx - tu_{tx}\int u_t dx - zu_{tx}\int u_z dx, \\
T_5^z =& u\int u_t dx - zu\int u_t dx - \frac{1}{6}zu^4 - zu_x\int u_{ty}dx + \frac{1}{2}u\int udx;
\end{aligned}
$$

$$
\begin{aligned}
T_{F^1}^t =& -\frac{1}{2}uF^1(y,z) - \frac{1}{2}u_xF_y^1, \\
T_{F^1}^x =& -\frac{1}{2}F^1(y,z)\int u_t dx - \frac{1}{2}F^1(y,z)\int u_z dx - \frac{1}{3}u^3F^1(y,z) + \frac{1}{2}u_{ty}F^1(y,z) \\
&+ \frac{1}{2}F_z^1\int udx, \\
T_{F^1}^y =& \frac{1}{2}u_{tx}F^1(y,z), \\
T_{F^1}^z =& -\frac{1}{2}uF^1(y,z);
\end{aligned}
$$

$$
\begin{aligned}
T_{F^2}^t =& -\frac{1}{2}uF^2(t,z), \\
T_{F^2}^x =& -\frac{1}{2}F^2(t,z)\int u_t dx - \frac{1}{2}F^2(t,z)\int u_z dx - \frac{1}{3}u^3F^2(t,z) + \frac{1}{2}u_{ty}F^2(t,z) \\
&+ \frac{1}{2}F_t^2\int udx + \frac{1}{2}F_z^2\int udx, \\
T_{F^2}^y =& -\frac{1}{2}uF^2(t,z) - \frac{1}{2}F_z^2\int u_{ty}dx, \\
T_{F^2}^z =& -\frac{1}{2}uF^2(t,z).
\end{aligned}
$$

The Lie symmetry technique and direct integration yields the soliton solution

$$u = \pm\sqrt{\frac{6(\alpha\nu+\nu-\beta)}{\nu+1}}\,\mathrm{sech}\left(C \pm \sqrt{\frac{\nu+1}{6\alpha\nu(\nu-\alpha+\beta)(\nu+1)}}\,\xi\right)$$

of (3.3), where $\xi = -\alpha\,\nu\,t+(1+\nu)\,x-(\alpha+\beta-\nu)\,y-(\nu-\beta)\,z$ and C is a constant. As before, we have retrieved the bell shaped soliton solution of [68]. Moreover, the cnoidal and snoidal wave solutions of (3.3) are

$$u(t,x,y,z) = \pm\sqrt{3\,\alpha}\,\mathrm{cn}\,(\xi\,|\omega)\,,\ \ \alpha>0. \tag{3.21}$$

$$u(t,x,y,z) = \pm\sqrt{6\,\alpha}\,\mathrm{sn}\,(\xi\,|\omega)\,,\ \ \alpha>0, \tag{3.22}$$

respectively, with $\xi = -\alpha\,\nu\,t+(1+\nu)\,x-(\alpha+\beta-\nu)\,y-(\nu-\beta)\,z$.

3.4 Conservation laws and exact solutions of (3.4)

For the third (3+1)-D mBBM equation (3.4), we have the following nonlocal conservation laws:

$$\begin{aligned}
T_1^t &= -\frac{1}{2}u\int u_y dz - \frac{1}{12}u^4,\\
T_1^x &= -\frac{1}{2}u_{tx}\int u_t dz + \frac{1}{2}u_t\int u_{tx}dz,\\
T_1^y &= -\frac{1}{2}u\int u_t dz,\\
T_1^z &= \frac{1}{2}\left(\int u_t dz\right)^2 + \frac{1}{2}\int u_t dz\int u_y dz + \frac{1}{2}u^3\int u_t dz - \frac{1}{2}\int u_t dz\int u_{txx}dz;
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{2}u\int u_x dz + \frac{1}{2}u_x\int u_{xx}dz,\\
T_2^x &= -\frac{1}{2}u\int u_t dz - \frac{1}{2}u\int u_y dz - \frac{1}{12}u^4 - \frac{1}{2}u_{tx}\int u_x dz,\\
T_2^y &= \frac{1}{2}u\int u_x dz,
\end{aligned}$$

$$T_2^z = \frac{1}{2}\int u_t dz \int u_x dz + \frac{1}{2}\int u_x dz \int u_y dz + \frac{1}{2}u^3 \int u_x dz - \frac{1}{2}\int u_x dz \int u_{txx} dz;$$

$$\begin{aligned}
T_3^t &= \frac{1}{2}u\int u_y dz + \frac{1}{2}u_y \int u_{xx} dz,\\
T_3^x &= -\frac{1}{2}u_{tx}\int u_y dz + \frac{1}{2}u_t \int u_{xy} dz,\\
T_3^y &= -\frac{1}{2}u\int u_t dz - \frac{1}{12}u^4 - \frac{1}{2}u_t \int u_{xx} dz,\\
T_3^z &= \frac{1}{2}\int u_t dz \int u_y dz + \frac{1}{2}\left(\int u_y dz\right)^2 + \frac{1}{3}u^3 \int u_y dz - \frac{1}{2}\int u_y dz \int u_{txx} dz;
\end{aligned}$$

$$\begin{aligned}
T_4^t &= \frac{1}{2}u^2 + \frac{1}{2}u_z \int u_{xx} dz,\\
T_4^x &= -\frac{1}{2}uu_{tx} + \frac{1}{2}u_t u_x,\\
T_4^y &= \frac{1}{2}u^2,\\
T_4^z &= \frac{1}{4}u^4 - \frac{1}{2}u_t \int u_{xx} dz - \frac{1}{2}u\int u_{txx} dz;
\end{aligned}$$

$$\begin{aligned}
T_5^t &= -tu\int u_y dz - \frac{1}{6}tu^4 + \frac{1}{2}u\int u dz + yu\int u_y dz + \frac{1}{2}uz\int u_{xx} dz + yu_y \int u_{xx} dz,\\
T_5^x &= -\frac{1}{2}u_{tx}\int u dz - tu_{tx}\int u_t dz - yu_{tx}\int u_y dz + \frac{1}{2}u_t \int u_x dz + tu_t \int u_{tx} dz\\
&\quad + yu_t \int u_{xy} dz,\\
T_5^y &= -yu\int u_t dz - \frac{1}{6}yu^4 - yu_t \int u_{xx} dz + \frac{1}{2}u\int u dz + tu\int u_t dz,\\
T_5^z &= \frac{1}{2}\int u dz \int u_t dz + \frac{1}{2}\int u dz \int u_y dz + \frac{1}{3}u^3 \int u dz - \frac{1}{2}\int u dz \int u_{txx} dz\\
&\quad + t\left(\int u_t dz\right)^2 + t\int u_t dz \int u_y dz + \frac{2}{3}tu^3 \int u_t dz - t\int u_t dz \int u_{txx} dz\\
&\quad + y\int u_t dz \int u_y dz + y\left(\int u_y dz\right)^2 + \frac{2}{3}yu^3 \int u_y dz - y\int u_y dz \int u_{txx} dz;
\end{aligned}$$

$$
\begin{aligned}
T_6^t =& -6yu\int u_y dz + 6tu\int u_y dz - yu^4 + tu^4 - \frac{3}{2}u\int u dz + zu^2 - \frac{1}{2}u\int u_{xx}dz \\
&+ 3x\int u_{xx}dzu_x + zu_z\int u_{xx}dz, \\
T_6^x =& -3xu\int u_t dz - 3xu\int u_y dz - \frac{1}{2}xu^4 + \frac{3}{2}u_{tx}\int u dz - 6yu_{tx}\int u_t dz \\
&+ 6tu_{tx}\int u_t dz - 3xu_{tx}\int u_x dz - zuu_{tx} + \frac{3}{2}u_t\int u_x dz + 6yu_t\int u_{tx}dz \\
&- 6tu_t\int u_{tx}dz + zu_tu_x, \\
T_6^y =& 6yu\int u_t dz - \frac{3}{2}u\int u dz - 6tu\int u_t dz + 3xu\int u_x dz\int u dz + zu^2, \\
T_6^z =& \frac{1}{2}\int u dz\int u_t dz + \frac{1}{2}\int u dz\int u_y dz + \frac{1}{3}u^3\int u dz - \frac{1}{2}\int u dz\int u_{txx}dz \\
&+ t\left(\int u_t dz\right)^2 + t\int u_t dz\int u_y dz + \frac{2}{3}tu^3\int u_t dz - t\int u_t dz\int u_{txx}dz \\
&+ y\int u_t dz\int u_y dz + y\left(\int u_y dz\right)^2 + \frac{2}{3}yu^3\int u_y dz - y\int u_y dz\int u_{txx}dz;
\end{aligned}
$$

$$
\begin{aligned}
T_{F^1}^t =& -\frac{1}{2}xuF^1(t,y), \\
T_{F^1}^x =& \frac{1}{2}xu_{tx}F^1(t,y) - \frac{1}{2}u_tF^1(t,y), \\
T_{F^1}^y =& -\frac{1}{2}xuF^1(t,y), \\
T_{F^1}^z =& -\frac{1}{2}xF^1(t,y)\int u_t dz - \frac{1}{2}xF^1(t,y)\int u_y dz - \frac{1}{3}xu^3F^1(t,y) \\
&+ \frac{1}{2}xF^1(t,y)\int u_{txx}dz + \frac{1}{2}F_t^1\int u dz + \frac{1}{2}F_x^1\int u dz;
\end{aligned}
$$

$$
\begin{aligned}
T_{F^2}^t =& -\frac{1}{2}uF^2(t,y), \\
T_{F^2}^x =& \frac{1}{2}u_{tx}F^2(t,y), \\
T_{F^2}^y =& -\frac{1}{2}uF^2(t,y), \\
T_{F^2}^z =& -\frac{1}{2}F^2(t,y)\int u_t dz - \frac{1}{2}F^2(t,y)\int u_y dz - \frac{1}{3}u^3F^2(t,y) + \frac{1}{2}F^2(t,y)\int u_{txx}dz
\end{aligned}
$$

$$+\frac{1}{2}F_t^2\int u dz+\frac{1}{2}F_y^2\int u dz.$$

The one-soliton solution of (3.4) is

$$u=\pm\sqrt{\frac{6(\alpha+\beta+\alpha\nu-\nu)}{\beta-\nu}}\operatorname{sech}\left(C\pm\sqrt{\frac{\beta-\nu}{6\alpha\nu(\nu+1)^2}}\,\xi\right), \tag{3.23}$$

$$\alpha\neq 0,\ \nu\neq 0,\ \beta\neq\nu,\ \nu\neq 1, \tag{3.24}$$

and the cnoidal and snoidal wave solutions of (3.4) are

$$u=\pm\frac{\sqrt{6\nu\,\omega\ (\nu+1)\,(2\,\nu^2\omega-\nu^2+2\,\nu\,\omega-\nu-1)}}{2\,\nu^2\omega-\nu^2+2\,\nu\,\omega-\nu-1}\,\mathrm{cn}\,(\xi\,|\omega)\,, \tag{3.25}$$

$$u=\pm\sqrt{6\,\alpha}\,\mathrm{cn}\,(\xi\,|\omega)\,,\ \ \alpha>0, \tag{3.26}$$

respectively, with $\xi=-\alpha\,\nu\,t+(1+\nu)\,x-(\alpha+\beta-\nu)\,y-(\nu-\beta)\,z$.

3.5 Concluding remarks

In this chapter, we studied the three (3+1)-D mBBM equations, which were introduced in [68]. We obtained soliton solutions for each equation without any foregoing assumptions about the form of the solutions. Furthermore, we constructed Jacobi elliptic function solutions of the three equations with the aid of the simplest equation method. Jacobi elliptic solutions are more general as they contain trigonometric and hyperbolic functions depending on the behaviour of the parameter ω. The obtained solutions are presented graphically. Finally, we successfully derived nonlocal conservation laws for each (3+1)-D mBBM equation by enlisting the famous Noether's theorem.

Chapter 4

Conserved quantities, optimal system and explicit solutions of a (1+1)-dimensional generalised coupled mKdV-type system

4.1 Introduction

In [76], Foursov performed a classification of coupled potential KdV-type and modified KdV-type equations that possess higher symmetries, and eleven new systems were presented for which Hamiltonian and bi-Hamiltonian formulations were provided for some of the equations. His work focused on coupled and symmetric systems of type $u_t = F[u, v]$ and $v_t = F[v, u]$. It is against this backdrop that the following previously unknown coupled system was obtained in [76]:

$$\begin{aligned} u_t - u^2 v_x - 3uvu_x - u_{xxx} = 0, \\ v_t - v^2 u_x - 3uvv_x - v_{xxx} = 0. \end{aligned} \tag{4.1}$$

In this chapter, however, we investigate the generalised coupled mKdV-type system

$$\begin{aligned} E_1 &\equiv u_t - u^2 v_x - \alpha u v u_x - u_{xxx} = 0, \\ E_2 &\equiv v_t - v^2 u_x - \alpha u v v_x - v_{xxx} = 0 \end{aligned} \tag{4.2}$$

with α a constant. System (4.1) is interesting in that it is non-decouplable and possesses higher generalised symmetries. To the best of our knowledge, the above system (4.2) is studied extensively for the first time in this chapter and the results are therefore new. With the application of Lie symmetry analysis we seek to derive conserved quantities and exact solutions for the system (4.2). For this we shall utilise multiplier approach via the homotopy integral and Lie group analysis along with power series solution method.

The work presented in this chapter has been published in [77].

4.2 Conserved quantities

For the generalised coupled mKdV-type system (4.2) with multipliers Λ^1 and Λ^2 we determine the corresponding conserved quantities by using the homotopy integral method, but first we find the multipliers.

To compute all multipliers of (4.2), we use the determining condition

$$\frac{\delta}{\delta u}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] = 0, \;\; \frac{\delta}{\delta v}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] = 0. \tag{4.3}$$

Here the operators $\delta/\delta u$ and $\delta/\delta v$ are the well known Euler operators (1.25), which in this instance are specified as

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_x^2 \frac{\partial}{\partial v_{xx}} - D_x^3 \frac{\partial}{\partial v_{xxx}} \end{aligned} \tag{4.4}$$

with D_t and D_x being the total derivatives (1.4) explicitly given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xt} \frac{\partial}{\partial u_t} + v_{xt} \frac{\partial}{\partial v_t} + \cdots .$$

Here we seek to compute second order multipliers

$$\Lambda^m = \Lambda^m(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}), \quad m = 1, 2.$$

Expanding (4.3) and comparing powers of derivatives of u, v, we acquire a system of thirty six PDEs. The solution to this system of PDEs is akin to the algorithm for obtaining Lie point symmetries. Solving this system of PDEs gives

$$\begin{aligned}
\Lambda^1 =& C_1 \left[\left\{ (\alpha+1) t u v^2 + x v + 3\, t v_{xx} \right\} \right] \\
&+ \frac{1}{\alpha+1} [C_2 \{ (\alpha+1)\, u v^2 + 3\, v_{xx} \} + C_3\, (\alpha+1)\, v], \ \alpha \neq -1, \\
\Lambda^2 =& C_1 \left[x u + \left\{ (\alpha+1)\, v u^2 + 3\, u_{xx} \right\} t \right] \\
&+ \frac{1}{\alpha+1} [C_2 \{ \left((\alpha+1) u^2 v + 3\, u_{xx} \right) \}] + C_3\, u, \ \alpha \neq -1
\end{aligned} \tag{4.5}$$

with C_1, C_2 and C_3 constants. Thus, we obtain three multipliers given by

$$\begin{aligned}
&\Lambda_1^1 = (\alpha+1)\, t u v^2 + x v + 3\, t v_{xx}, \ \Lambda_1^2 = x u + \left((\alpha+1)\, v u^2 + 3\, u_{xx} \right) t; \\
&\Lambda_2^1 = \frac{1}{\alpha+1} \left\{ (\alpha+1)\, u v^2 + 3\, v_{xx} \right\}, \ \Lambda_2^2 = \frac{1}{\alpha+1} \left\{ (\alpha+1) u^2 v + 3\, u_{xx} \right\}; \\
&\Lambda_3^1 = v, \ \Lambda_3^2 = u.
\end{aligned} \tag{4.6}$$

From these three sets of multipliers, conserved densities T^i and spatial fluxes X^i for $i = 1, 2, 3$ can be obtained in several ways [43]. In this work, we use the first

homotopy integral (1.48), here given explicitly by

$$
\begin{aligned}
T &= \int_0^1 \left(u \left(\frac{\partial E_1 \Lambda_1}{\partial u_t} \right) \Big|_{u=u_{(\lambda)}} + v \frac{\partial E_2 \Lambda_2}{\partial v_t} \Big|_{v=v_{(\lambda)}} \right) \mathrm{d}\lambda, \\
X &= \int_0^1 \Bigg[u \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial u_x} \right) \Big|_{u=u_{(\lambda)}} - D_x \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xx}} \right) \Big|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxx}} \right) \Big|_{u=u_{(\lambda)}} \right\} \\
&\quad + u_x \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xx}} \right) \Big|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xx}} \right) \Big|_{u=u_{(\lambda)}} \right\} + u_{xx} \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxx}} \right) \Big|_{u=u_{(\lambda)}} \\
&\quad + u \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial u_x} \right) \Big|_{u=u_{(\lambda)}} - D_x \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xx}} \right) \Big|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xxx}} \right) \Big|_{u=u_{(\lambda)}} \right\} \\
&\quad + u_x \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xx}} \right) \Big|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xx}} \right) \Big|_{u=u_{(\lambda)}} \right\} + u_{xx} \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xxx}} \right) \Big|_{u=u_{(\lambda)}} \\
&\quad + v \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial v_x} \right) \Big|_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xxx}} \right) \Big|_{v=v_{(\lambda)}} \right\} \\
&\quad + v_x \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} \right\} + v_{xx} \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xxx}} \right) \Big|_{v=v_{(\lambda)}} \\
&\quad v \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial v_x} \right) \Big|_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xxx}} \right) \Big|_{v=v_{(\lambda)}} \right\} \\
&\quad + v_x \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} \right\} + v_{xx} \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xxx}} \right) \Big|_{v=v_{(\lambda)}} \Bigg] \mathrm{d}\lambda.
\end{aligned}
\tag{4.7}
$$

Choosing the homotopy $u_{(\lambda)} = \lambda u$, for Λ_1^1 and Λ_1^2 we have

$$
\begin{aligned}
T^1 &= \int_0^1 \left(2\, u^2 \lambda^3 t v^2 \alpha + 2\, u^2 \lambda^3 v^2 t + 2\, \lambda\, uvx + 3\lambda u v_{xx} t + 3 t v \lambda u_{xx} \right) \mathrm{d}\lambda \\
&= \frac{1}{2} \alpha t v^2 u^2 + \frac{1}{2} t v^2 u^2 + xvu + \frac{3}{2} t u v_{xx} + \frac{3}{2} t v u_{xx}
\end{aligned}
$$

and

$$
\begin{aligned}
X^1 = \int_0^1 \big(& -v_x \lambda\, u - \lambda\, v u_x - 4\alpha\, \lambda^3 t v u v_x u_x - 2\alpha x u^2 \lambda^3 v^2 - 2\lambda^5 \alpha^2 t v^3 u^3 - 4\lambda^5 \alpha t u^3 v^3 \\
& - 6\lambda t v_{xx} u_{xx} - 2\lambda x u_{xx} v - 3\, \lambda t v u_{xt} - 2\lambda^5 t u^3 v^3 - 3\, \lambda t\, u v_{tx} - 2\lambda\, u x v_{xx} - 4\lambda^3 t v^2 u_x^2 \\
& - 4\lambda^3 t v_x^2 u^2 + 2\lambda x v_x u_x + 3\lambda t u_x v_t + 3\lambda t u_t v_x + 2\lambda^3 \alpha\, t u^2 v_x^2 - 4\, v^2 t \lambda^3 u u_{xx} \\
& + 2\alpha\, \lambda^3 t v^2 u_x^2 - 4\, v_{xx} t \lambda^3 v\, (u)^2 - 2\, u^2 \lambda^3 v^2 x + 8\, v \lambda^3 u v_x u_x t - 4\, v^2 \alpha\, \lambda^3 t u u_{xx} \\
& - 4\lambda^3 \alpha t v u^2 v_{xx} \big)\, \mathrm{d}\lambda \\
= & -\frac{1}{2}\, \alpha\, x u^2 v^2 + \frac{1}{2}\, \alpha\, t u_x^2 v^2 - \frac{1}{3}\, \alpha^2 t u^3 v^3 + \frac{1}{2}\, \alpha\, t u^2 v_x^2 - \frac{2}{3} \alpha\, t \alpha v^3 u^3 - t u_{xx} u v^2
\end{aligned}
$$

$$
\begin{aligned}
&- t\,vu^2v_{xx} - \alpha\,tuu_x\,vv_x - \alpha\,tuu_{xx}\,v^2 + 2\,tuu_x\,vv_x - \alpha\,tu^2vv_{xx} - \frac{1}{2}\,vu_x - \frac{1}{2}\,uv_x \\
&- \frac{3}{2}\,t\,vu_{tx} - tu_x^2v^2 - \frac{1}{2}\,xu^2v^2 - \frac{1}{3}t\,u^3v^3 + \frac{3}{2}\,tu_x\,v_t - xuv_{xx} + xu_x\,v_x - xvu_{xx} \\
&- 3\,tu_{xx}\,v_{xx} + \frac{3}{2}\,tu_t\,v_x - tu^2v_x^2 - \frac{3}{2}\,tuv_{tx}.
\end{aligned}
$$

In a similar manner, we obtain two more conserved quantities of (4.2) associated with the two multipliers $(\Lambda_2^1, \Lambda_2^2)$ and $(\Lambda_3^1, \Lambda_3^2)$. These are

$$
\begin{aligned}
T^2 =& \frac{1}{2(\alpha+1)}\left\{\alpha\,u^2v^2 + v^2u^2 + 3\,uv_{xx} + 3\,u_{xx}\,v\right\}, \\
X^2 =& -\frac{1}{6(\alpha+1)}\left\{2\,\alpha^2u^3v^3 + 4\,\alpha\,u^3v^3 + 2\,u^3v^3 + 6\,\alpha\,u^2vv_{xx} - 3\,\alpha\,u^2v_x^2 + 6\,\alpha\,uu_x\,vv_x\right. \\
&+6\,\alpha\,uu_{xx}\,v^2 - 3\,\alpha\,{u_x}^2v^2 + 6\,v_{xx}\,vu^2 + 6\,u^2v_x^2 - 12\,uu_x\,vv_x + 6\,uu_{xx}\,v^2 + 6\,{u_x}^2v^2 \\
&\left.+9\,uv_{tx} - 9\,u_t\,v_x + 9\,u_{tx}\,v - 9\,u_x\,v_t + 18\,u_{xx}\,v_{xx}\right\};
\end{aligned}
$$

$$
\begin{aligned}
T^3 =& uv, \\
X^3 =& v_x\,u_x - uv_{xx} - u_{xx}\,v - \frac{1}{2}\,(\alpha+1)\,u^2v^2.
\end{aligned}
$$

4.3 analytic solutions of (4.2)

Here we utilise point symmetries of the generalised coupled mKdV-type system (4.2) and build up an optimal system of one-dimensional subalgebras in order to achieve symmetry reductions as well as symmetry invariant solutions.

4.3.1 Optimal system of one-dimensional subalgebras for (4.2)

The symmetry group of (4.2) can be obtained using Maple and it consists of

$$
X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ X_3 = u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v},\ X_4 = 3\,t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} - 2u\,\frac{\partial}{\partial u},
$$

which are space and time translations and scaling symmetries, respectively. We now seek to exploit one elementary facet of Lie algebras, that is, the bilinear product property. The commutators of these Lie symmetries are tabulated in Table 4.3.1, where the entry at the intersection of ith row with jth column is a reckoning of Lie bracket $[X_i, X_j]$ [30, 31].

Table 4.31. Commutation relations of 4-dimensional Lie algebra of (4.2)

$[\ ,\ ]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	0	$3X_2$
X_3	0	0	0	0
X_4	$-X_1$	$-3X_2$	0	0

We now enlist Lie series [30]

$$\mathrm{Ad}(\exp(\varepsilon \mathrm{X_i}))\mathrm{X_j} = \sum_{\mathrm{n}=0}^{\infty} \frac{\varepsilon^{\mathrm{n}}}{\mathrm{n!}} (\mathrm{ad}\, \mathrm{X_i})^{\mathrm{n}}(\mathrm{X_j}),$$

together with bilinear products in Table 4.31 to obtain adjoint representations. The results are tabulated in Table 4.32.

Table 4.32. Adjoint table of Lie algebra of (4.2)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 - \varepsilon X_1$
X_2	X_1	X_2	X_3	$X_4 - 3\varepsilon X_2$
X_3	X_1	X_2	X_3	X_4
X_4	$e^{\varepsilon} X_1$	$e^{3\varepsilon} X_2$	X_3	X_4

With the aid of Table 4.31 and Table 4.32 and by prudently applying adjoint maps, see for instance [30, 42, 78] we see that the optimal system of 1-dimensional subalgebras is spanned by

$$\begin{aligned} X_1 =& \frac{\partial}{\partial x}, \\ X_2 =& \frac{\partial}{\partial t}, \\ X_1 + X_2 =& \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \\ X_2 - X_1 =& \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \\ cX_3 + X_4 =& x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + (c-2)u\frac{\partial}{\partial u} - cv\frac{\partial}{\partial v}, \\ aX_1 + bX_2 + X_3 =& a\frac{\partial}{\partial x} + b\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \end{aligned}$$

with a, b, c constants. The symmetry $(x, t, u, v) \mapsto (-x, t, u, v)$ is discrete and maps $X_2 - X_1$ onto $X_1 + X_2$, thus reducing our optimal system to five inequivalent subgroups, viz.,

$$\{X_1, X_2, X_1 + X_2, cX_3 + X_4, aX_1 + bX_2 + cX_3\}. \tag{4.8}$$

4.3.2 Symmetry reductions and explicit solutions of (4.2)

We now present symmetry reductions and some explicit solutions of (4.2) according to the optimal system (4.8).

4.3.2.1 Symmetry reductions

We proceed to compute the invariant solutions of each of the five cases presented in (4.8). Firstly we transform the NLPDE system (4.2) into several systems of nonlinear ordinary differential equations (NLODEs). Furthermore, we present the solutions for (4.2).

Case 1. X_1

For vector field X_1, we obtain the invariant $\xi = x$ and the group invariant solutions $u(t,x) = U(\xi)$ and $v(t,x) = V(\xi)$. Substitution of these values of u, v in (4.2) gives the NLODEs

$$\begin{aligned} U''' + 3UVU' + V'U^2 = 0, \\ V''' + 3VUV' + V^2U' = 0. \end{aligned} \tag{4.9}$$

Case 2. X_2

Without giving detail, it can be readily seen that this case leads to the obvious constant solutions

$$U = k_1, \;\; V = k_2 \tag{4.10}$$

with k_1, k_2 constants.

Case 3. $X_1 + X_2$

The third member of (4.8), namely $X_1 + X_2$ provides us with the invariant $\xi = x - t$ and group invariant solutions $u = U(\xi)$ and $v = V(\xi)$. Consequently, U and V satisfy

$$\begin{aligned} U''' + \alpha UU'V + V'U^2 + U' = 0, \\ V''' + \alpha VUV' + U'V^2 + V' = 0. \end{aligned} \tag{4.11}$$

Case 4. $cX_3 + X_4$

In this case, the invariants $\xi = xt^{1/3}$, $u = t^{(c-2)/3}U(\xi)$ and $v = t^{c/3}V(\xi)$ are apparent, with functions U, V conforming to the system

$$\begin{aligned} 3\,U''' + 3\alpha UVU' + 3\,U^2V' + \xi U' - (c-2)U = 0, \\ 3\,V''' + 3\alpha UVV' + 3\,V^2U' + \xi V' + cV = 0. \end{aligned} \tag{4.12}$$

Case 5. $aX_1 + bX_2 + X_3$

In this final instance, we obtain the invariants $\xi = bx - at$, $u = e^{t/b}U(\xi)$ and $v = e^{-t/b}V(\xi)$. By substituting these invariant solutions into system (4.2) we find

the following system of nonlinear ODEs:

$$\begin{aligned} &b^2U^2V' + \alpha\, b^2UVU' + b^4U''' + abU' - U = 0, \\ &b^2V^2U' + \alpha\, b^2UVV' + b^4V''' + abV' + V = 0. \end{aligned} \tag{4.13}$$

4.3.2.2 Explicit solutions of (4.2)

In this subsection, we determine exact power series solutions [25–27] for the ODEs (4.9) and (4.11–4.13). For the ODE (4.9), we have the power series form

$$U(\xi) = \sum_{j=0}^{\infty} p_j \xi^j \quad \text{and} \quad V(\xi) = \sum_{j=0}^{\infty} q_j \xi^j, \tag{4.14}$$

where p_j and q_j for $j = 1, 2, \cdots$ are undetermined constants. From (4.14), we obtain

$$\begin{aligned} U'(\xi) &= \sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j, \quad V'(\xi) = \sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j, \\ U''(\xi) &= \sum_{j=0}^{\infty}(j+1)(j+2)p_{j+2}\xi^j, \quad V''(\xi) = \sum_{j=0}^{\infty}(j+1)(j+2)q_{j+2}\xi^j, \\ U'''(\xi) &= \sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j, \\ V'''(\xi) &= \sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j. \end{aligned} \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.9) we have

$$\begin{aligned} &\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j\right) \\ &+ \left(\sum_{j=0}^{\infty}p_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) = 0, \\ &\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) \end{aligned}$$

$$+\left(\sum_{j=0}^{\infty} q_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j\right)=0,$$

which simplifies to

$$\begin{aligned}
&\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j+3\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\xi^j\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\xi^j=0,\\
&\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j+3\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\xi^j\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\xi^j=0.
\end{aligned}\tag{4.16}$$

From (4.16) the following recursion formulae are understood:

$$\begin{aligned}
p_{j+3}&=-\frac{1}{(j+1)(j+2)(j+3)}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\right.\\
&\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\right),\\
q_{j+3}&=-\frac{1}{(j+1)(j+2)(j+3)}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\right.\\
&\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\right),
\end{aligned}\tag{4.17}$$

for $j=0,1,2,\cdots$. Thus, by choosing the constants p_m and q_m, $m=0,1,2$, successive coefficients can be obtained, uniquely, from (4.17). We have, for instance,

$$\begin{aligned}
p_3&=-\frac{1}{6}p_0^2q_1-\frac{1}{2}p_0q_0p_1,\\
q_3&=-\frac{1}{6}q_0^2p_1-\frac{1}{2}p_0q_0q_1;\\
p_4&=-\frac{1}{12}p_0^2q_2-\frac{1}{4}p_2p_0q_0-\frac{5}{24}p_1p_0q_1-\frac{1}{8}p_1^2q_0,\\
q_4&=-\frac{1}{12}p_2q_0^2-\frac{1}{4}p_0q_2q_0-\frac{5}{24}p_1q_1q_0-\frac{1}{8}p_0q_1^2;
\end{aligned}$$

$$p_5 = \frac{1}{20}p_0^3 q_0 q_1 + \frac{1}{12}p_1 p_0^2 q_0^2 - \frac{2}{15}p_2 p_0 q_1 - \frac{7}{60}p_1 p_0 q_2 - \frac{3}{20}p_1 p_2 q_0 - \frac{1}{15}p_1^2 q_1,$$
$$q_5 = \frac{1}{20}p_0 p_1 q_0^3 + \frac{1}{12}p_0^2 q_1 q_0^2 - \frac{7}{60}p_2 q_1 q_0 - \frac{2}{15}p_1 q_2 q_0 - \frac{1}{15}p_1 q_1^2 - \frac{3}{20}p_0 q_1 q_2;$$

and so on. Consequently, the exact power series solution of (4.9) or in fact (4.2), is given by

$$\begin{aligned} u(t,x) =& p_0 + p_1\xi + p_2\xi^2 - \left(\frac{1}{6}p_0^2 q_1 + \frac{1}{2}p_0 q_0 p_1\right)\xi^3 - \left(\frac{1}{12}p_0^2 q_2 + \frac{1}{4}p_2 p_0 q_0\right. \\ &\left. + \frac{5}{24}p_1 p_0 q_1 + \frac{1}{8}p_1^2 q_0\right)\xi^4 + \left(\frac{1}{20}p_0^3 q_0 q_1 + \frac{1}{12}p_1 p_0^2 q_0^2 - \frac{2}{15}p_2 p_0 q_1 - \frac{7}{60}p_1 p_0 q_2\right. \\ &\left. - \frac{3}{20}p_1 p_2 q_0 - \frac{1}{15}p_1^2 q_1\right)\xi^5 - \sum_{j=3}^{\infty}\frac{j!}{(j-3)!}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_i q_{k-i} p_{j-k+1}\right. \\ &\left. + \sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_i p_{k-i} q_{j-k+1}\right)\xi^{j+3}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} v(t,x) =& q_0 + q_1\xi + q_2\xi^2 - \left(\frac{1}{6}q_0^2 p_1 + \frac{1}{2}p_0 q_0 q_1\right)\xi^3 - \left(\frac{1}{12}p_2 q_0^2 + \frac{1}{4}p_0 q_2 q_0 + \frac{5}{24}p_1 q_1 q_0\right. \\ &\left. + \frac{1}{8}p_0 q_1^2\right)\xi^4 + \left(\frac{1}{20}p_0 p_1 q_0^3 + \frac{1}{12}p_0^2 q_1 q_0^2 - \frac{7}{60}p_2 q_1 q_0 - \frac{2}{15}p_1 q_2 q_0 - \frac{1}{15}p_1 q_1^2\right. \\ &\left. - \frac{3}{20}p_0 q_1 q_2\right)\xi^5 - \sum_{j=3}^{\infty}\frac{j!}{(j-3)!}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_i q_{k-i} q_{j-k+1}\right. \\ &\left. + \sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_i q_{k-i} p_{j-k+1}\right)\xi^{j+3}. \end{aligned} \tag{4.19}$$

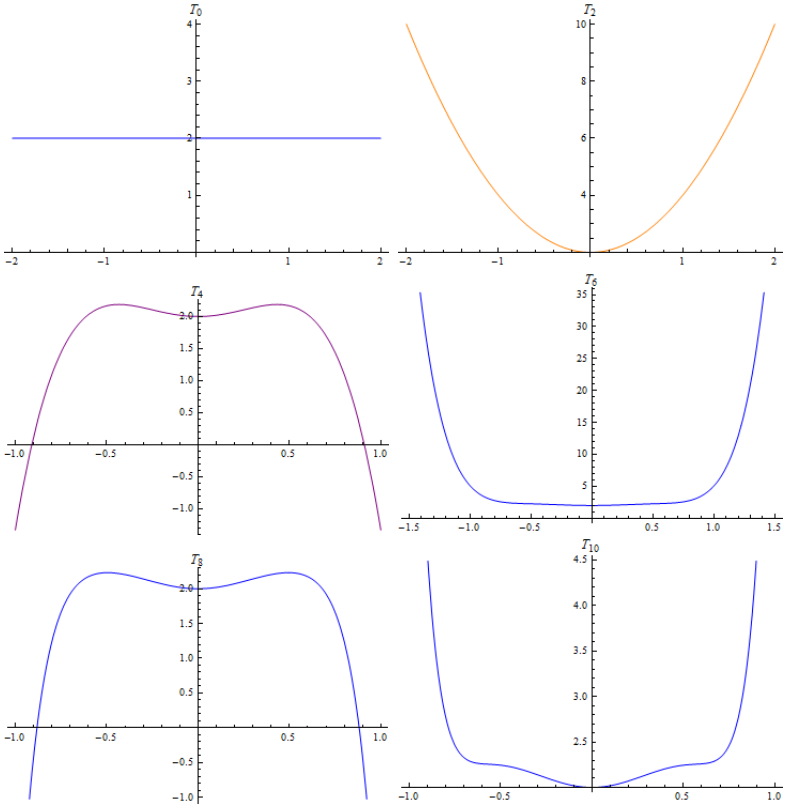


Figure 4.1: Profiles of even partial sums for solution (4.18).

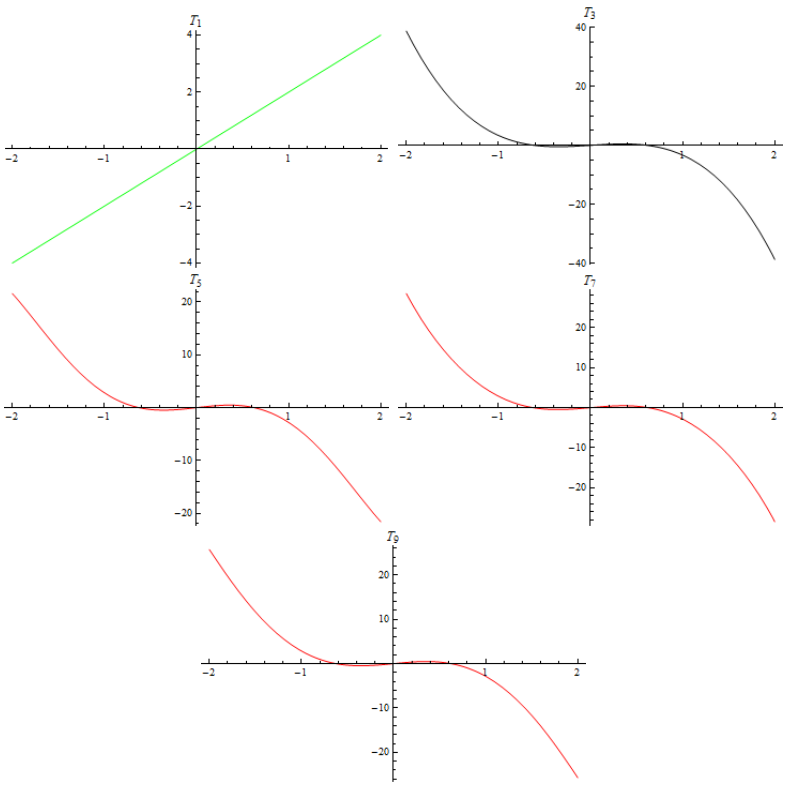


Figure 4.2: Profiles of odd partial sums for solution (4.18).

Indications from Figure 4.1 and Figure 4.2 are that solution (4.18) is convergent. Figure 4.3 is a consolidation of the above solution profiles.

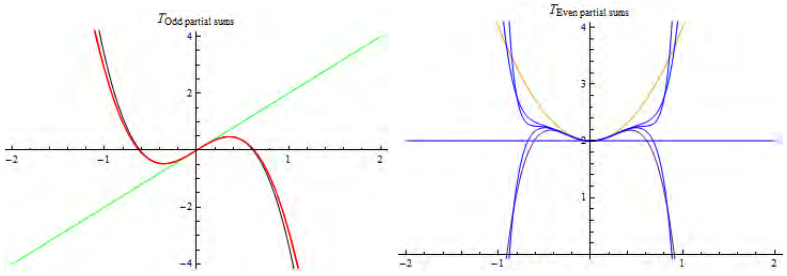


Figure 4.3: Consolidated profiles of partial sums $T_0 \cdots T_{10}$ for solution (4.18).

We now employ the same procedure to solve system (4.12). Substituting (4.15)

into (4.12), we have

$$
\begin{aligned}
&3\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\alpha\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j\right)\\
&+3\left(\sum_{j=0}^{\infty}p_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right)+\xi\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right)-(c-2)\left(\sum_{j=0}^{\infty}p_j\xi^j\right)=0,\\
&3\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\alpha\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right)\\
&+3\left(\sum_{j=0}^{\infty}q_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)p_j\xi^j\right)+\xi\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right)+c\left(\sum_{j=0}^{\infty}q_j\xi^j\right)=0.
\end{aligned}
$$

Simplifying the above system we now have

$$
\begin{aligned}
&3\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\xi^j\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\xi^j+\sum_{j=0}^{\infty}(j-c+2)p_j\xi^j=0,\\
&3\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\xi^j\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\xi^j+\sum_{j=0}^{\infty}(j+c)q_j\xi^j=0.
\end{aligned}
\tag{4.20}
$$

The system (4.20) is true if the coefficients of ξ^j, $j\in\mathbb{Z}^+$ are equal to 0:

$$
\begin{aligned}
p_{j+3} =& -\frac{1}{3(j+1)(j+2)(j+3)}\left(3\alpha\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\right.\\
&\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}+j-c+2\right)\\
q_{j+3} =& -\frac{1}{3(j+1)(j+2)(j+3)}\left(3\alpha\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\right.\\
&\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}+j+c\right)
\end{aligned}
\tag{4.21}
$$

for $j = 0, 1, 2, \cdots$. Thus, for arbitrary constants p_m and q_m, $m = 0, 1, 2$, we have

$$
\begin{aligned}
p_3 =& -\frac{1}{6}\,\alpha\, p_0\, q_0\, p_1 - \frac{1}{18}\,{p_0}^2 q_1 + \frac{1}{18}c - \frac{1}{9},\\
q_3 =& -\frac{1}{6}\,\alpha\, p_0\, q_0\, q_1 - \frac{1}{18}\,{q_0}^2 p_1 - \frac{1}{18}c;
\end{aligned}
$$

$$
\begin{aligned}
p_4 =& -\frac{1}{12}\,\alpha\, p_0\, q_0\, p_2 - \frac{1}{36}\,{p_0}^2 q_2 - \frac{1}{72}(3\,\alpha + 2)\, p_0\, q_1\, p_1 - \frac{1}{24}\,\alpha\,{p_1}^2 q_0 + \frac{1}{72}c - \frac{1}{24},\\
q_4 =& -\frac{1}{12}\,\alpha\, p_0\, q_0\, q_2 - \frac{1}{36}\,{q_0}^2 p_2 - \frac{1}{72}(3\,\alpha + 2)\, q_0\, q_1\, p_1 - \frac{1}{24}\,\alpha\,{q_1}^2 p_0 - \frac{1}{72}c - \frac{1}{72};
\end{aligned}
$$

$$
\begin{aligned}
p_5 =& \frac{1}{120}\alpha^2 {p_0}^2 {q_0}^2 p_1 + \frac{1}{180}\alpha\,{p_0}^3 q_0\, q_1 - \frac{1}{360}\alpha c\, p_0\, q_0 + \frac{1}{180}\alpha\, p_0\, q_0 + \frac{1}{1080}{p_0}^2 {q_0}^2 p_1\\
&+ \frac{1}{1080}{p_0}^2 c - \frac{1}{30}\, p_0\, q_1\, p_2\,\alpha - \frac{1}{90} p_0\, q_1\, p_2 - \frac{1}{60} p_0\, p_1\, q_2\,\alpha - \frac{1}{45}\, p_0\, p_1\, q_2\\
&- \frac{1}{20}\,\alpha\, p_1\, q_0\, p_2 - \frac{1}{60}\alpha {p_1}^2 q_1 - \frac{1}{180}{p_1}^2 q_1 + \frac{1}{180}c - \frac{1}{45},\\
q_5 =& \frac{1}{120}\alpha^2 {p_0}^2 {q_0}^2 q_1 + \frac{1}{180}\alpha\, p_0\,{q_0}^3 p_1 + \frac{1}{360}\alpha\, c p_0\, q_0 + \frac{1}{1080}{q_0}^2 {p_0}^2 q_1 - \frac{1}{1080}c{q_0}^2\\
&+ \frac{1}{540}{q_0}^2 - \frac{1}{30}\, q_0\, p_1\, q_2\,\alpha - \frac{1}{90} q_0\, p_1\, q_2 - \frac{1}{60} q_0\, q_1\, p_2\,\alpha - \frac{1}{45}\, q_0\, q_1\, p_2\\
&- \frac{1}{20}\,\alpha\, p_0\, q_1\, q_2 - \frac{1}{60}{q_1}^2 p_1\,\alpha - \frac{1}{180}{q_1}^2 p_1 + \frac{1}{180}c - \frac{1}{90}.
\end{aligned}
\tag{4.22}
$$

Hence the exact power series solution to (4.12) is

$$
\begin{aligned}
u(t,x) =& t^{(c-2)/3}\left\{ p_0 + p_1\xi + p_2\xi^2 - \left(\frac{1}{6}\,\alpha\, p_0\, q_0\, p_1 + \frac{1}{18}\,{p_0}^2 q_1 - \frac{1}{18}c + \frac{1}{9}\right)\xi^3\right.\\
&- \left(\frac{1}{12}\,\alpha\, p_0\, q_0\, p_2 + \frac{1}{36}\,{p_0}^2 q_2 + \frac{1}{72}(3\,\alpha + 2)\, p_0\, q_1\, p_1 + \frac{1}{24}\,\alpha\,{p_1}^2 q_0\right.\\
&\left.-\frac{1}{72}c + \frac{1}{24}\right)\xi^4 + \left(\frac{1}{120}\alpha^2 {p_0}^2 {q_0}^2 p_1 + \frac{1}{180}\alpha\,{p_0}^3 q_0\, q_1 - \frac{1}{360}\alpha c\, p_0\, q_0\right.\\
&+\frac{1}{180}\alpha\, p_0\, q_0 + \frac{1}{1080}{p_0}^2 {q_0}^2 p_1 + \frac{1}{1080}c{p_0}^2 - \frac{1}{30}\, p_0\, q_1\, p_2\,\alpha - \frac{1}{90} p_0\, q_1\, p_2\\
&-\frac{1}{60} p_0\, p_1\, q_2\,\alpha - \frac{1}{45}\, p_0\, p_1\, q_2 - \frac{1}{20}\,\alpha\, p_1\, q_0\, p_2 - \frac{1}{60}\alpha {p_1}^2 q_1 - \frac{1}{180}{p_1}^2 q_1\\
&\left.+\frac{1}{180}c - \frac{1}{45}\right)\xi^5 + \sum_{n=3}^{\infty}\frac{3n!}{(n-3)!}\left(3\alpha\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)p_i q_{k-i} p_{n-k+1}\right.
\end{aligned}
$$

$$+\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)p_i p_{k-i} q_{n-k+1}+n-c+2\Bigg)\xi^{n+3}\Bigg\}, \tag{4.23}$$

$$\begin{aligned}
v(t,x) = t^{c/3}\Bigg\{ & q_0+q_1\xi+q_2\xi^2-\left(\frac{1}{6}\,\alpha\, p_0\, q_0\, q_1+\frac{1}{18}\,{q_0}^2 p_1+\frac{1}{18}c\right)\xi^3-\left(\frac{1}{12}\,\alpha\, p_0\, q_0\, q_2\right. \\
& \left.+\frac{1}{36}\,{q_0}^2 p_2+\frac{1}{72}(3\,\alpha+2)\,q_0\, q_1\, p_1+\frac{1}{24}\,\alpha\,{q_1}^2 p_0+\frac{1}{72}c+\frac{1}{72}\right)\xi^4 \\
& +\left(\frac{1}{120}\alpha^2 {p_0}^2 {q_0}^2 q_1+\frac{1}{180}\alpha\, p_0\, {q_0}^3 p_1+\frac{1}{360}\alpha\, c p_0\, q_0+\frac{1}{1080}{q_0}^2 {p_0}^2 q_1\right. \\
& -\frac{1}{1080}c{q_0}^2+\frac{1}{540}{q_0}^2-\frac{1}{30}\,q_0\, p_1\, q_2\,\alpha-\frac{1}{90}q_0\, p_1\, q_2-\frac{1}{60}q_0\, q_1\, p_2\,\alpha \\
& \left.-\frac{1}{45}\,q_0\, q_1\, p_2-\frac{1}{20}\,\alpha\, p_0\, q_1\, q_2-\frac{1}{60}{q_1}^2 p_1\,\alpha-\frac{1}{180}{q_1}^2 p_1+\frac{1}{180}c-\frac{1}{90}\right)\xi^5 \\
& -\sum_{n=3}^{\infty}\frac{3n!}{(n-3)!}\left(3\alpha\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)p_i q_{k-i} q_{n-k+1}\right. \\
& \left.+\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)q_i q_{k-i} p_{n-k+1}+n+c\right)\xi^{n+3}\Bigg\}.
\end{aligned} \tag{4.24}$$

We now provide $3D$ renderings of (4.23) and (4.24) for $n=0,1,2$ and for arbitrary values of $p_0, q_0, p_1, q_1, p_2, q_2$ and c in Figures 4.4, 4.5 and 4.6.

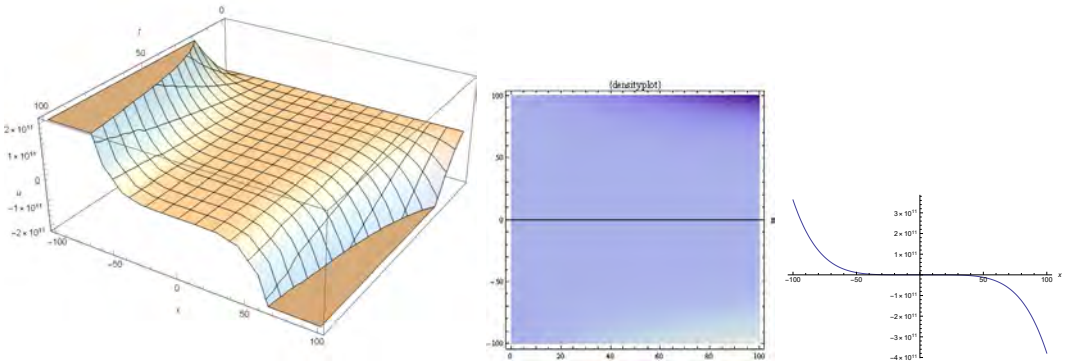


Figure 4.4: Profiles of solution (4.23)

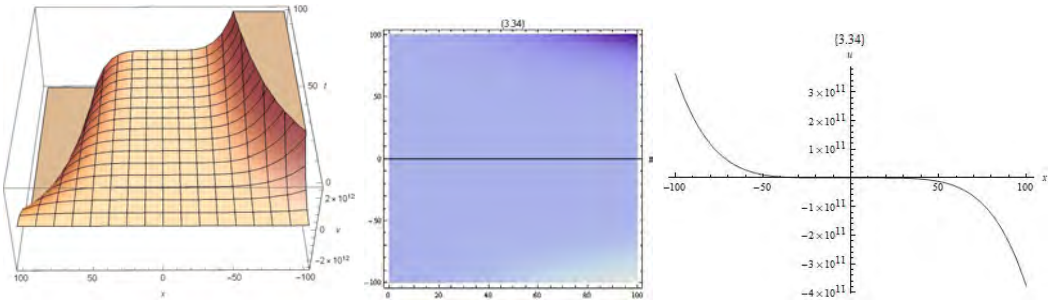


Figure 4.5: Profiles of solution (4.24)

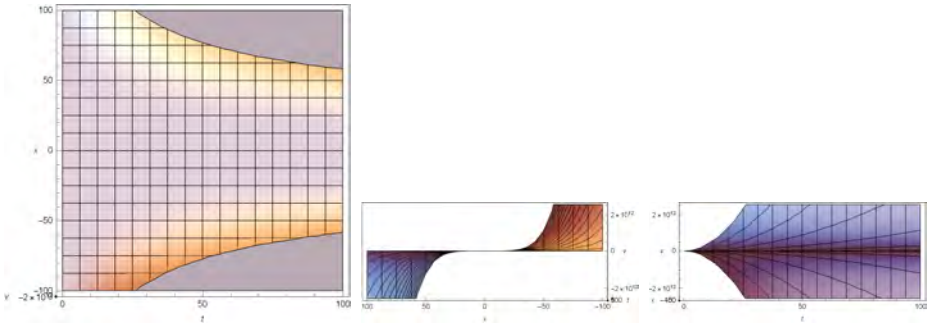


Figure 4.6: Profiles of (4.24) from different vantage points

4.4 Concluding remarks

In this chapter, we considered a (1+1)-dimensional generalised coupled mKdV-type system. It has no variational principle and as such, we employed the multiplier method and used a homotopy integral formula to derive the conserved quantities to which it conforms. As already seen, the homotopy integral approach is relatively simple, concise and elegant compared to other traditional avenues of computing conserved quantities. In this work, we provided a step-by-step illustrative example of this algorithm with the aid of a system which has not been previously studied. Again we sought the optimal system of one-dimensional algebras for this system by invoking its four-dimensional Lie algebras. This enabled us to transform the

system into several systems of NLODEs. With the aid of the power series solution method, we solved some of the systems of NLODEs and obtained its exact solutions which are by extension also solutions of system (4.2).

Chapter 5

Group invariant solutions and conserved quantities of a (3+1)-dimensional generalized Kadomtsev–Petviashvili equation

5.1 Introduction

The (3+1)-dimensional Kadomtsev–Petviashvili (KP) equations [79]

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0 \tag{5.1}$$

have their origins in the classic 1970 work of two Soviet physicists Kadomtsev and Petviashvili [80]. It models waves in scenarios where the ratio between the depth of water and the wavelength is very small coupled with insubstantial nonlinear restoring forces. It is also a generalisation of the Korteweg–de Vries (KdV) equation, named after the Dutch mathematicians Korteweg and de Vries [81]. The highest order term u_{xxx} caters for weak dispersion, while the term uu_x describes the

nonlinearity aspect of a wave and the last two terms of (5.1) describe diffractive divergence [82]. The sign "$\pm$" in the last two terms corresponds to either positive or negative dispersion. Since then, several researchers have studied equations of the form (5.1), see for instance [83–88]. Investigations into (5.1) have ranged from establishing the stability of their solitons [85], determining their integrability properties [86], Painlevé analysis [87] and deriving their exact multiple wave solutions [88]. The (3+1)-dimensional generalised KP equation

$$u_{tx} + u_{ty} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxxy} - u_{zz} = 0 \tag{5.2}$$

to the best of our knowledge was introduced in [89], where the Plücker relation for determinants was used to obtain one Wronskian solution. Furthermore, in the same work, the Jacobi identity for determinants was used to establish a Grammian solution for (5.2). Multiple solitons and multiple singular solitons for Equation (5.2) were derived in [90] using a simplified form of Hirota's method. In their work, the researchers in [90] contrast their results with those obtained in [89] in terms of the spatial variable z. Later in [91], an extra term u_{tz}, was added to (5.2), thus introducing a new form of the (3+1)-dimensional generalised KP equation, which we denote here by (3+1)-D gKPe, and it reads

$$u_{tx} + u_{ty} + u_{tz} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxxy} - u_{zz} = 0. \tag{5.3}$$

In their work [91], the authors showed that addition of the new term affects the dispersion relations significantly. Furthermore, they applied Hirota's direct method to determine the multiple soliton solutions of (5.3). In [92], the Equation (5.3) was studied and exact solutions were obtained through the use of Hirota's bilinear method and an extended homoclinic test approach.

In this chapter, we seek to derive the exact solutions of the (3+1)-D gKPe (5.3) by making use of its Lie point symmetries and direct integration. We will not employ

ad hoc methods as often is the case when treating NLPDEs of this nature. Furthermore, we will for the first time derive the conserved quantities of Equation (5.3) by using two approaches.

We want to reiterate here that a study of higher dimensional NLPDEs is critical to our understanding of the world around us since they are more realistic models of natural and man-made phenomena [68].

The work presented in this chapter has been published in [93].

5.2 Exact Solutions of the (3+1)-D gKPe

In this section, we determine the exact solutions of the (3+1)-D gKPe (5.3) by utilising its Lie point symmetries and direct integration. Roughly speaking, a Lie point symmetry of a PDE is a local group of transformations acting on the independent and dependent variables of the PDE that maps every solution of the PDE to another solution of the same PDE. That is to say, it maps the solution set of the PDE to itself. It can be seen, using for example, MathLie package [94], that equation (5.3) has a finite-dimensional Lie subalgebra L_6, which is generated by the vector fields

$$\begin{aligned} X_1 =& \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial z}, \\ X_5 =& -t\frac{\partial}{\partial t} + z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (2t+z)\frac{\partial}{\partial z}, \\ X_6 =& 15t\frac{\partial}{\partial t} + 3(4t+x)\frac{\partial}{\partial x} + 3(3y-2z)\frac{\partial}{\partial y} - 3(4t-z)\frac{\partial}{\partial z} \\ &+ (4x+4y-3u)\frac{\partial}{\partial u}, \end{aligned} \tag{5.4}$$

and the two infinite-dimensional subalgebras generated by

$$X_{F_1} = F_1(t)\frac{\partial}{\partial u},\ X_{F_2} = F_2(t+z)\frac{\partial}{\partial u}, \tag{5.5}$$

where F_1 and F_2 are arbitrary functions of their arguments.

5.2.1 Invariant Solutions under the Symmetries $X_1, \cdots, X_4$

Firstly, we utilise the translation symmetries and reduce the (3+1)-D gKPe (5.3) to a fourth-order ODE. Thus, using the four translation symmetries X_1, X_2, X_3 and X_4 of equation (5.3), we can obtain the following invariant solution:

$$u(t,x,y,z) = U(p), \;\; p = \gamma x + \alpha y + \beta z - \nu t, \tag{5.6}$$

where γ, α, β and ν are constants. This reduces equation (5.3) to

$$\alpha\gamma^3 U'''' + 6\alpha\gamma^2 U' U'' - (\alpha\nu + \beta^2 + \beta\nu + \gamma\nu)U'' = 0. \tag{5.7}$$

Integrating (5.7) once with respect to p yields

$$\alpha\gamma^3 U''' + 3\alpha\gamma^2 U'^2 - (\alpha\nu + \beta^2 + \beta\nu + \gamma\nu)U' + K_0 = 0 \tag{5.8}$$

with K_0 a constant. Now letting

$$U'(p) = \gamma\,\phi(p), \;\; \omega = -\frac{\alpha\nu + \beta^2 + \beta\nu + \gamma\nu}{\alpha\gamma^3} \;\text{ and }\; K_1 = \frac{K_0}{\alpha\gamma^4},$$

Equation (5.8) becomes

$$\phi'' + 3\phi^2 - \omega\phi + K_1 = 0. \tag{5.9}$$

This ODE can be integrated easily. Multiplying (5.9) by ϕ' and integrating once with respect to p gives

$$\phi'^2 = -(2\phi^3 - \omega\phi^2 + 2K_1\phi + 2K_2) \tag{5.10}$$

with K_2 an integration constant. If the algebraic equation $\phi^3 - \frac{1}{2}\omega\phi^2 + K_1\phi + K_2 = 0$ has the roots $\lambda_1 \geq \lambda_2 \geq \lambda_3$, then we have

$$\phi'^2 = -2(\phi - \lambda_1)(\phi - \lambda_2)(\phi - \lambda_3). \tag{5.11}$$

Equation (5.11) has the well-known solution expressed in terms of the Jacobi elliptic function, that is,

$$\phi(p) = \lambda_2 + (\lambda_1 - \lambda_2)\mathrm{cn}^2\left(\sqrt{\frac{\lambda_1 - \lambda_3}{2}}\, p \,|M^2\right), \quad M^2 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}. \tag{5.12}$$

Here $\mathrm{cn}(p|M^2)$ is the Jacobi elliptic cosine function. Furthermore, pertinent to note is that when $M^2 \to 1$, $\mathrm{cn}(p|M^2) \to \mathrm{sech}\,(p)$ and when $M^2 \to 0$, $\mathrm{cn}(p|M^2) \to \cos(p)$ [66]. A comparison of Equations (5.9) and (5.10) reveals that

$$\omega = 2(\lambda_1 + \lambda_2 + \lambda_3), \quad K_1 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \quad K_2 = -\lambda_1\lambda_2\lambda_3.$$

Reverting to original variables, the solution of the (3+1)-D gKPe (5.3) is thus

$$u(t,x,y,z) = \frac{\{\lambda_2 - (1-M^2)\lambda_1\}\gamma p}{M^2} + \frac{(\lambda_1 - \lambda_2)\,\gamma \mathrm{dn}\,(A\,p|M^2)\,E\,(\mathrm{am}\,(A\,p|M^2)\,|M^2)}{AM^2\sqrt{\mathrm{dn}\,(A\,p|M^2)^{\,2}}}, \tag{5.13}$$

where $\mathrm{E}(p|M^2)$ is the elliptic integral of the second kind, $\mathrm{am}(p|M^2)$ is the amplitude function, $\mathrm{dn}(p|M^2)$ is the delta amplitude function, $\mathrm{cn}(p|M^2)$ is the Jacobi cosine function and $A = \sqrt{(\lambda_1 - \lambda_3)/2}$ [66, 95]. Figure 5.1 depicts the profile of solution (5.13) for suitable parameter values.

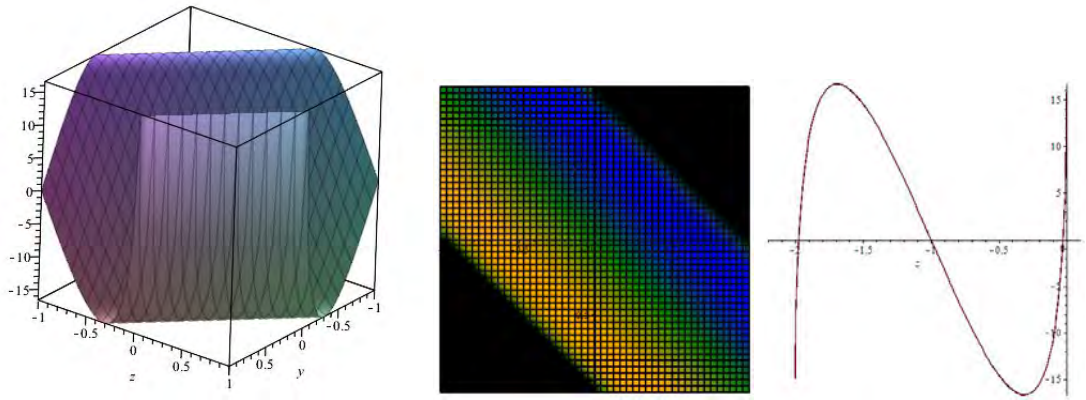


Figure 5.1: Profiles of solution (5.13).

Figure 5.1 depicts the coexistence of bright and dark solitons. It is well-known that bright soliton profile are characterized by hyperbolic secant. The bright soliton

solution usually takes a bell-shaped figure and propagates undistorted without any change in shape for arbitrarily long distances. However, dark soliton solutions, configured also as topological optical solitons, are given by hyperbolic tangent.

Important to note is that Equation (5.10) is reminiscent of the ODE obtained in the quintessential work of Korteweg and de Vries in [96]. This ODE is associated with long waves propagating along a rectangular canal. The ODE (5.10) describes stationary waves and by imposing certain constraints such as having the fluid undisturbed at infinity, Korteweg and de Vries obtained negative and positive solitary waves as well as cnoidal wave solutions [96, 97].

5.2.2 Invariant Solution under the Symmetry X_5

We now turn our attention to the point symmetry X_5. The usual computations yield the group invariant solution $u(t, x, y, z) = U(\xi)$, where $\xi = t(t + z)$. This, substituted into (5.3) gives the ODE

$$\xi U'' + U' = 0, \tag{5.14}$$

whose solution is $U(\xi) = C_1 + C_2 \ln(\xi)$, and hence the group invariant solution of the (3+1)-D gKPe (5.3) under the symmetry X_5 is

$$u(t, x, y, z) = C_1 + C_2 \ln\{t(t + z)\} \tag{5.15}$$

with C_1, C_2 integration constants. The corresponding graphical representation of solution (5.15) is given in Figure 5.2.

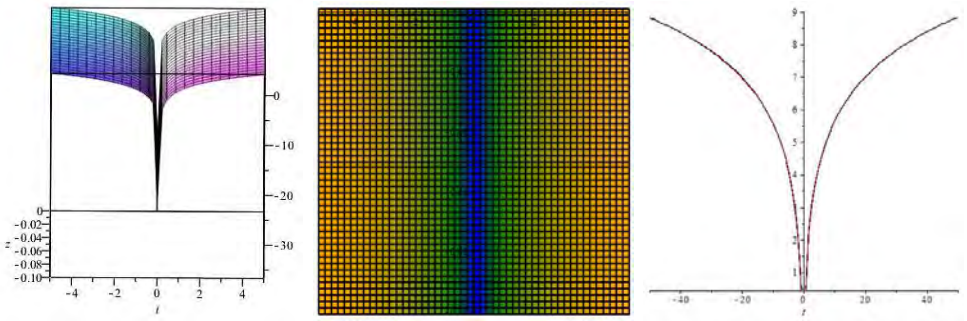


Figure 5.2: Profiles of solution (5.15).

The profiles portrayed in Figure 5.2 are typical of a grey singular soliton solution, more so the distinctive asymptotic form observable for $\lim_{t\to 0} u(t,x,y,z) = -\infty$.

5.2.3 Invariant Solution under the Symmetry X_6

Finally, we compute the group invariant solution under X_6. By following the usual procedure we obtain the group invariant solution

$$u(t,x,y,z) = t^{-\frac{1}{5}}U(\xi) - \frac{2}{9}t + \frac{2}{3}x + \frac{1}{3}y + \frac{1}{3}z, \tag{5.16}$$

where $\xi = t^{-1/5}(x-t)$. Substituting (5.16) into (5.3) and simplifying ultimately yields the second-order ODE

$$\xi U'' + 2U' = 0. \tag{5.17}$$

Equation (5.17) can be solved much like Equation (5.14), which in turn gives the invariant solution of the (3+1)-D gKPe (5.3) under the group generated by X_6:

$$u(t,x,y,z) = C_1 + C_2\, t^{1/5}(x-t)^{-1} \tag{5.18}$$

with C_1, C_2 integration constants. In Figure 5.3, solution (5.18) is depicted.

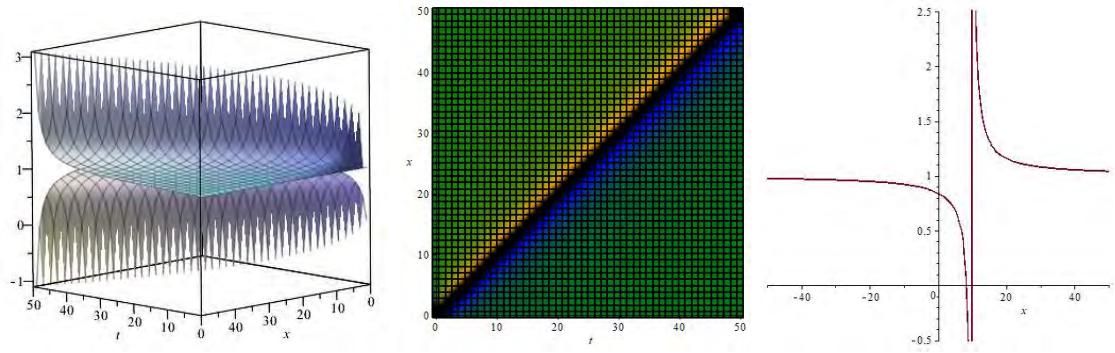


Figure 5.3: Profiles of solution (5.18).

The density plot coupled with the two dimensional profile in Figure 5.3 depicts a singular soliton-like wave with a singularity in the spatial domain at $0 \leq x \leq 20$.

5.3 Conserved Quantities of the (3+1)-D gKPe

In this section, we compute the conserved quantities of the (3+1)-D gKPe (5.3). In our work we use the multiplier approach and the conservation theorem due to Ibragimov to derive the conserved quantities of (5.3).

5.3.1 Application of the Method

In order to determine conserved quantities of Equation (5.3), we begin by computing the first order multipliers

$$\Lambda = \Lambda(t, x, u, u_x, u_y, u_z),$$

by utilising condition (1.47), that is,

$$\frac{\delta}{\delta u}(\Lambda E) = 0, \tag{5.19}$$

where

$$E \equiv u_{tx} + u_{ty} + u_{tz} + 3u_x u_{xy} + 3u_{xx} u_y + u_{xxxy} - u_{zz} = 0.$$

The Euler operator is given by (1.25).

$$\begin{aligned}\frac{\delta}{\delta u} =& \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_t D_y \frac{\partial}{\partial u_{ty}} + D_t D_z \frac{\partial}{\partial u_{tz}} \\ &+ D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_z^2 \frac{\partial}{\partial u_{zz}} + D_x^3 D_y \frac{\partial}{\partial u_{xxxy}}\end{aligned}$$

and D_t, D_x, D_y and D_z are total derivative operators (1.4). Expanding (5.19) and splitting on derivatives of u, we obtain the following system of nine multiplier determining equations:

$$\begin{aligned}&\Lambda_{ty} + \Lambda_{tz} - \Lambda_{zz} = 0,\ \Lambda_{tu_x} = 0,\ \Lambda_{yu_x} = 0,\ \Lambda_{zu_x} = 0,\ \Lambda_{u_x u_x} = 0, \\ &\Lambda_x = 0,\ \Lambda_u = 0,\ \Lambda_{u_y} = 0,\ \Lambda_{u_z} = 0. \end{aligned} \tag{5.20}$$

The solution algorithm of system (5.20) is similar to that of obtaining Lie point symmetries. However, with the aid of Gem, a Maple based package [73], this can be expedited. We thus have the multiplier

$$\Lambda = C u_x \tag{5.21}$$

with C an integration constant. The conserved quantities of (5.3) are obtained via the divergence identity

$$D_t T^t + D_x T^x + D_y T^y + D_z T^z = E\Lambda,$$

where T^t is a conserved density and T^x, T^y, T^z are spatial fluxes [43]. Thus, after some calculations, we obtain the following conservation law:

$$\begin{aligned}T^t =& \frac{1}{2} {u_x}^2 + \frac{1}{2} u_y u_x + \frac{1}{2} u_z u_x, \\ T^x =& u u_x u_{xy} + 2 {u_x}^2 u_y + \frac{1}{2} u_y u_{xxx} - \frac{1}{2} u_{xy} u_{xx} + \frac{1}{2} u_{xxy} u_x + \frac{1}{2} u u_{ty} + \frac{1}{2} u u_{tz} \\ &+ \frac{1}{2} u u_{xxxy} - \frac{1}{2} u u_{zz},\end{aligned}$$

$$T^y = -uu_x\,u_{xx} - \frac{1}{2}\,uu_{tx} - \frac{1}{2}\,uu_{xxxx},$$
$$T^z = -\frac{1}{2}\,uu_{tx} + \frac{1}{2}\,uu_{xz} - \frac{1}{2}\,uz\,u_x.$$

Since the derivatives of u in the multiplier (5.21) are of a lower order than both leading derivatives of equation (5.3), that is, u_{tx} and u_{xxxy}, the conserved quantities derived here are low-order conservation laws [43].

5.3.2 Ibragimov's Approach

We now determine the conservation laws of (5.3) by applying a theorem due to Ibragimov [39, 40]. As in the case of multiplier method, this theorem does not demand availability of a Lagrangian and is established on a concept of an adjoint equation. Thus, it applies to an arbitrary differential equation, irrespective of whether or not it comes from a variational principle.

5.3.2.1 Application of the Method

To begin, we define the adjoint equation of (5.3) as [39]

$$\begin{aligned}
F^* =& \frac{\delta}{\delta u}\left\{v(u_{tx} + u_{ty} + u_{tz} + 3u_xu_{xy} + 3u_{xx}u_y + u_{xxxy} - u_{zz})\right\} \\
=& \left(\frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_tD_x\frac{\partial}{\partial u_{tx}} + D_tD_y\frac{\partial}{\partial u_{ty}} + D_tD_z\frac{\partial}{\partial u_{tz}}\right. \\
&\left. + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_y\frac{\partial}{\partial u_{xy}} + D_z^2\frac{\partial}{\partial u_{zz}} + D_x^3D_y\frac{\partial}{\partial u_{xxxy}}\right)\left\{v\,(u_{tx} + u_{ty}\right. \\
&\left. + u_{tz} + 3u_xu_{xy} + 3u_{xx}u_y + u_{xxxy} - u_{zz})\right\} \\
=& v_{tx} + v_{ty} + v_{tz} + 6v_xu_{xy} + 3u_xv_{xy} + 3u_yv_{xx} + v_{xxxy} - v_{zz} = 0. \qquad (5.22)
\end{aligned}$$

We have introduced a new variable $v = v(t, x, y, z)$. According to [39], the Equation (5.3) considered together with its adjoint (5.22) has a Lagrangian $\mathcal{L}$, given

by

$$\mathcal{L} = vF = v(u_{tx} + u_{ty} + u_{tz} + 3u_x u_{xy} + 3u_{xx}u_y + u_{xxxy} - u_{zz}). \tag{5.23}$$

Notice how $\delta\mathcal{L}/\delta u = F^*$ and $\delta\mathcal{L}/\delta v = F$. The Lagrangian (5.23) is equivalent to the second order Lagrangian

$$\mathcal{L} = v(u_{tx} + u_{ty} + u_{tz} + 3u_x u_{xy} + 3u_{xx}u_y - u_{zz}) + v_{xx}u_{xy}. \tag{5.24}$$

Now, equation (5.22) admits all the symmetries (5.4) and (5.5) of (5.3) extended to the new variable $v(t, x, y, z)$. That is, the generators (5.4) and (5.5) become

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u} + \eta^* \frac{\partial}{\partial v} \tag{5.25}$$

with

$$\eta^* = \eta^*(t, x, y, z, u, v) = -\left\{\lambda + D_t(\xi^1) + D_x(\xi^2) + D_y(\xi^3) + D_z(\xi^4)\right\} v. \tag{5.26}$$

The parameter λ is determined by using

$$X'(F) = \lambda F, \tag{5.27}$$

where X' is the generator (5.4) prolonged to all the derivatives in (5.3), that is,

$$\begin{aligned} X' = X &+ \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{tz} \frac{\partial}{\partial u_{tz}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} \\ &+ \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{zz} \frac{\partial}{\partial u_{zz}} + \zeta_{xxxy} \frac{\partial}{\partial u_{xxxy}}. \end{aligned} \tag{5.28}$$

Here $X = \xi^1 \partial/\partial t + \xi^2 \partial/\partial x + \xi^3 \partial/\partial y + \xi^4 \partial/\partial z + \eta \partial/\partial u$ and $\xi^1, \cdots, \xi^4$, and η are functions of (t, x, y, z, u). Furthermore, ζ_x, ζ_y, ζ_{tx}, ζ_{ty}, ζ_{tz}, ζ_{xx}, ζ_{xy}, ζ_{zz} and ζ_{xxxy} are coefficient functions, each generally given by (1.17) and explicitly by the following formulas:

$$
\begin{aligned}
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) - u_z D_x(\xi^4),\\
\zeta_y &= D_y(\eta) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3) - u_z D_y(\xi^4),\\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2) - u_{ty} D_x(\xi^3) - u_{tz} D_x(\xi^4),\\
\zeta_{ty} &= D_y(\zeta_t) - u_{tt} D_y(\xi^1) - u_{tx} D_y(\xi^2) - u_{ty} D_y(\xi^3) - u_{tz} D_y(\xi^4),\\
\zeta_{tz} &= D_z(\zeta_t) - u_{tt} D_z(\xi^1) - u_{tx} D_z(\xi^2) - u_{ty} D_z(\xi^3) - u_{tz} D_z(\xi^4),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3) - u_{xz} D_x(\xi^4),\\
\zeta_{xy} &= D_y(\zeta_x) - u_{xt} D_y(\xi^1) - u_{xx} D_y(\xi^2) - u_{xy} D_y(\xi^3) - u_{xz} D_y(\xi^4),\\
\zeta_{zz} &= D_z(\zeta_x) - u_{zt} D_z(\xi^1) - u_{zx} D_z(\xi^2) - u_{zy} D_z(\xi^3) - u_{zz} D_z(\xi^4),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2) - u_{xxy} D_z(\xi^3) - u_{xxz} D_x(\xi^4),\\
\zeta_{xxxy} &= D_y(\zeta_{xxx}) - u_{xxxt} D_y(\xi^1) - u_{xxxx} D_y(\xi^2) - u_{xxxy} D_y(\xi^3) - u_{xxxz} D_y(\xi^4).
\end{aligned}
\tag{5.29}
$$

Now using Equations (5.3), (5.27) and (5.28) we compute the values of the parameter λ corresponding to each of the vector fields in (5.4) and (5.5).

Cases $X_1, \cdots, X_4$

For the time translation symmetry X_1, we have $\xi^1 = 1$ and $\xi^2 = \xi^3 = \xi^4 = \eta = 0$. Thus, it is easy to see that

$$\zeta_x = \zeta_y = \zeta_{tx} = \zeta_{ty} = \zeta_{tz} = \zeta_{xx} = \zeta_{xy} = \zeta_{zz} = \zeta_{xxxy} = 0.$$

Consequently, we have $X_1'(F) = 0F$, that is $\lambda = 0$. From (5.26), we obtain $\eta^* = 0$ and the new generator (5.4) retains the form of X_1, i.e., $Y_1 = \partial/\partial t$. Since the generator coefficients are all constants, the translation symmetries will retain their form and this leads us to conclude that $Y_2 = \partial/\partial x$, $Y_3 = \partial/\partial y$ and $Y_4 = \partial/\partial z$.

Case X_5

Here, we have the generator coefficients $\xi^1 = -t$, $\xi^2 = x$, $\xi^3 = z$, $\xi^4 = 2t + z$ and

$\eta = 0$ from which we can establish the following:

$$\zeta_x = 0, \quad \zeta_y = 0, \quad \zeta_{tx} = u_{tx} - 2u_{xz}, \quad \zeta_{ty} = u_{ty} - 2u_{yz}, \quad \zeta_{tz} = -2u_{zz} - u_{tx} - u_{ty},$$
$$\zeta_{xx} = 0, \quad \zeta_{xy} = 0, \quad \zeta_{zz} = -2u_{xz} - 2u_{yz} - 2u_{zz}, \quad \zeta_{xxxy} = 0.$$

Consequently, we can verify that $X_5'(F) = 0F$, that is, $\lambda = 0$. From (5.26), we can further establish that $\eta^* = 0$. The generator for the adjoint Equation (5.22) is thus $Y_5 = -t\partial/\partial t + z\partial/\partial x + z\partial/\partial y + (2t + z)\partial/\partial z$.

Case X_6

The vector field X_6 has the coefficients $\xi^1 = 15t$, $\xi^2 = 12t + 3x$, $\xi^3 = 9y - 6z$, $\xi^4 = -12t + 3z$ and $\eta = 4x + 4y - 3u$. The reckoning of the coefficient functions (5.29) yields

$$\zeta_x = 4 - 6u_x, \quad \zeta_y = 4 - 12u_y, \ \zeta_{tx} = -21u_{tx} - 12u_{xx} + 12u_{xz},$$
$$\zeta_{ty} = -27u_{ty} - 12u_{xy} + 12u_{yz}, \ \zeta_{tz} = -21u_{tz} - 12u_{xz} + 12u_{zz} + 6u_{ty},$$
$$\zeta_{xx} = -9u_{xx}, \ \zeta_{xy} = -15u_{xy}, \ \zeta_{zz} = -9u_{zz} + 12u_{yz}, \ \zeta_{xxxy} = -21u_{xxxy}.$$

Now, from Equations (5.27), (5.28) and (5.29), we have

$$\begin{aligned} X_6'(F) =& -21u_{tx} - 21u_{ty} - 21u_{tz} - 63u_xu_{xy} - 63u_{xx}u_y - 21u_{xxxy} + 21u_{zz} \\ =& -21(u_{tx} + u_{ty} + u_{tz} + 3u_xu_{xy} + 3u_{xx}u_y + u_{xxxy} - u_{zz}) \\ =& -21F. \end{aligned}$$

We can thus see that $\lambda = -21$. Consequently,

$$\begin{aligned} \eta^* =& -(-21 + D_t(15t) + D_x(12t + 3x) + D_y(9y - 6z) + D_z(-12t + 3z))\, v \\ =& -7v. \end{aligned}$$

We now have the vector field:
$Y_6 = 15t\partial t + (12t + 3x)\partial x + (9y - 6z)\partial y + (3z - 12t)\partial z + (4x + 4y - 3u)\partial u - 9v\partial v$, corresponding to (5.22).

Cases X_7, X_8

Finally, considering the infinite-dimensional vector fields X_7 and X_8 we have, for (5.22), $Y_7 = F(t)\partial u$ and $Y_8 = F(t+z)\partial u$, respectively.

To compute the conservation laws of (5.3), we utilise the formula [39]

$$C^i = \xi^i \mathcal{L} + W^\alpha \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_k \frac{\partial \mathcal{L}}{\partial u^\alpha_{ik}} \right) + D_k(W^\alpha) \frac{\partial \mathcal{L}}{\partial u^\alpha_{ik}}, \tag{5.30}$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u^\alpha_j$. The reckoning of (5.30) gives the following conserved quantities:

$$\begin{aligned}
T_1 =& -vu_{zz} + 3vu_xu_{xy} + 3vu_yu_{xx} + vu_{xxxy} + \frac{1}{2}v_zu_t + \frac{1}{2}v_yu_t + \frac{1}{2}v_xu_t + \frac{1}{2}vu_{tz} \\
&+ \frac{1}{2}vu_{ty} + \frac{1}{2}vu_{tx}, \\
X_1 =& \frac{3}{2}v_yu_xu_t + 3u_yv_xu_t + \frac{3}{2}vu_{xy}u_t + \frac{3}{4}v_{xxy}u_t + \frac{1}{2}v_tu_t - \frac{3}{2}vu_xu_{ty} - \frac{1}{4}v_{xx}u_{ty} \\
&- 3vu_yu_{tx} - \frac{1}{2}v_{xy}u_{tx} + \frac{1}{2}v_xu_{txy} + \frac{1}{4}v_yu_{txx} - \frac{3}{4}vu_{txxy} - \frac{1}{2}vu_{tt}, \\
Y_1 =& \frac{3}{2}u_xv_xu_t - \frac{3}{2}vu_{xx}u_t + \frac{1}{4}v_{xxx}u_t + \frac{1}{2}v_tu_t - \frac{3}{2}vu_xu_{tx} - \frac{1}{4}v_{xx}u_{tx} + \frac{1}{4}v_xu_{txx} \\
&- \frac{1}{4}vu_{txxx} - \frac{1}{2}vu_{tt}, \\
Z_1 =& -v_zu_t + \frac{1}{2}v_tu_t + vu_{tz} - \frac{1}{2}vu_{tt};
\end{aligned}$$

$$\begin{aligned}
T_2 =& -\frac{1}{2}u_{xz}v - \frac{1}{2}u_{xy}v - \frac{1}{2}u_{xx}v + \frac{1}{2}u_xv_y + \frac{1}{2}u_xv_z + \frac{1}{2}u_xv_x, \\
X_2 =& 3u_xu_{xy}v - u_{zz}v + \frac{1}{4}u_{xxxy}v + u_{tz}v + u_{ty}v + \frac{1}{2}u_{tx}v + \frac{1}{2}v_tu_x + \frac{3}{2}u_x^2v_y + 3u_xu_yv_x \\
&+ \frac{3}{4}u_xv_{xxy} - \frac{1}{2}u_{xx}v_{xy} - \frac{1}{4}v_{xx}u_{xy} + \frac{1}{2}v_xu_{xxy} + \frac{1}{4}u_{xxx}v_y, \\
Y_2 =& -3u_{xx}u_xv - \frac{1}{4}u_{xxxx}v - \frac{1}{2}u_{tx}v + \frac{1}{2}v_tu_x + \frac{3}{2}u_x^2v_x + \frac{1}{4}u_xv_{xxx} - \frac{1}{4}u_{xx}v_{xx} \\
&+ \frac{1}{4}u_{xxx}v_x, \\
Z_2 =& u_{xz}v - \frac{1}{2}u_{tx}v + \frac{1}{2}v_tu_x - u_xv_z;
\end{aligned}$$

$$
\begin{aligned}
T_3 =& -\frac{1}{2}u_{yz}v-\frac{1}{2}u_{yy}v-\frac{1}{2}u_{xy}v+\frac{1}{2}u_yv_x+\frac{1}{2}u_yv_z+\frac{1}{2}u_yv_y,\\
X_3 =& -\frac{3}{2}u_yu_{xy}v-\frac{3}{2}u_xu_{yy}v-\frac{3}{4}u_{xxyy}v-\frac{1}{2}u_{ty}v+\frac{1}{2}v_tu_y+3u_y^2v_x+\frac{3}{2}u_xu_yv_y+\frac{3}{4}u_yv_{xxy}\\
&-\frac{1}{2}u_{xy}v_{xy}+\frac{1}{2}v_xu_{xyy}-\frac{1}{4}u_{yy}v_{xx}+\frac{1}{4}v_yu_{xxy},\\
Y_3 =& -u_{zz}v+\frac{3}{2}u_xu_{xy}v+\frac{3}{2}u_{xx}u_yv+\frac{3}{4}u_{xxxy}v+u_{tz}v+\frac{1}{2}u_{ty}v+u_{tx}v+\frac{1}{2}v_tu_y+\frac{3}{2}u_xu_yv_x\\
&-\frac{1}{4}v_{xx}u_{xy}+\frac{1}{4}v_xu_{xxy}+\frac{1}{4}u_yv_{xxx},\\
Z_3 =& u_{yz}v-\frac{1}{2}u_{ty}v+\frac{1}{2}v_tu_y-u_yv_z;
\end{aligned}
$$

$$
\begin{aligned}
T_4 =& \frac{1}{2}u_zv_z-\frac{1}{2}vu_{zz}+\frac{1}{2}u_zv_y-\frac{1}{2}vu_{yz}+\frac{1}{2}u_zv_x-\frac{1}{2}vu_{xz},\\
X_4 =& \frac{3}{2}u_zv_yu_x-\frac{3}{2}vu_{yz}u_x+3u_zu_yv_x-3vu_yu_{xz}+\frac{3}{2}vu_zu_{xy}-\frac{1}{2}u_{xz}v_{xy}+\frac{1}{2}v_xu_{xyz}-\frac{1}{4}u_{yz}v_{xx}\\
&+\frac{1}{4}v_yu_{xxz}+\frac{3}{4}u_zv_{xxy}-\frac{3}{4}vu_{xxyz}+\frac{1}{2}u_zv_t-\frac{1}{2}vu_{tz},\\
Y_4 =& \frac{3}{2}u_zu_xv_x+\frac{1}{4}u_{xxz}v_x-\frac{3}{2}vu_xu_{xz}-\frac{3}{2}vu_zu_{xx}-\frac{1}{4}u_{xz}v_{xx}+\frac{1}{4}u_zv_{xxx}-\frac{1}{4}vu_{xxxz}+\frac{1}{2}u_zv_t\\
&-\frac{1}{2}vu_{tz},\\
Z_4 =& -u_zv_z+3vu_xu_{xy}+3vu_yu_{xx}+vu_{xxxy}+\frac{1}{2}u_zv_t+\frac{1}{2}vu_{tz}+vu_{ty}+vu_{tx};
\end{aligned}
$$

$$
\begin{aligned}
T_5 =& -\frac{1}{2}vu_z+tv_zu_z+\frac{1}{2}zv_zu_z+tv_yu_z+\frac{1}{2}zv_yu_z+tv_xu_z+\frac{1}{2}zv_xu_z-\frac{1}{2}zvu_{zz}-\frac{1}{2}vu_y\\
&+\frac{1}{2}zv_zu_y+\frac{1}{2}zu_yv_y-tvu_{yz}-zvu_{yz}-\frac{1}{2}zvu_{yy}-\frac{1}{2}vu_x+\frac{1}{2}zv_zu_x+\frac{1}{2}zv_yu_x+\frac{1}{2}zu_yv_x\\
&+\frac{1}{2}zu_xv_x-tvu_{xz}-zvu_{xz}-zvu_{xy}-3tvu_xu_{xy}-\frac{1}{2}zvu_{xx}-3tvu_yu_{xx}-tvu_{xxxy}\\
&-\frac{1}{2}tv_zu_t-\frac{1}{2}tv_yu_t-\frac{1}{2}tv_xu_t-\frac{1}{2}tvu_{tz}-\frac{1}{2}tvu_{ty}-\frac{1}{2}tvu_{tx},\\
X_5 =& 3zv_xu_y^2+\frac{3}{2}zv_yu_xu_y+6tu_zv_xu_y+3zu_zv_xu_y+3zu_xv_xu_y-6tvu_{xz}u_y-3zvu_{xz}u_y\\
&-\frac{3}{2}zvu_{xy}u_y+\frac{3}{4}zv_{xxy}u_y-3tv_xu_tu_y+\frac{1}{2}zv_tu_y+3tvu_{tx}u_y+\frac{3}{2}zv_yu_x^2-vu_z-zvu_{zz}\\
&+3tu_zv_yu_x+\frac{3}{2}zu_zv_yu_x-3tvu_{yz}u_x-\frac{3}{2}zvu_{yz}u_x-\frac{3}{2}zvu_{yy}u_x+3tvu_zu_{xy}+\frac{3}{2}zvu_zu_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&+3zvu_xu_{xy}-tu_{xz}v_{xy}-\frac{1}{2}zu_{xz}v_{xy}-\frac{1}{2}zu_{xy}v_{xy}+tv_xu_{xyz}+\frac{1}{2}zv_xu_{xyz}+\frac{1}{2}zv_xu_{xyy}\\
&-\frac{1}{2}zv_{xy}u_{xx}-\frac{1}{2}tu_{yz}v_{xx}-\frac{1}{4}zu_{yz}v_{xx}-\frac{1}{4}zu_{yy}v_{xx}-\frac{1}{4}zu_{xy}v_{xx}+\frac{1}{2}tv_yu_{xxz}\\
&+\frac{1}{4}zv_yu_{xxz}+\frac{1}{4}zv_yu_{xxy}+\frac{1}{2}zv_xu_{xxy}+\frac{3}{2}tu_zv_{xxy}+\frac{3}{4}zu_zv_{xxy}+\frac{3}{4}zu_xv_{xxy}-\frac{3}{2}tvu_{xxyz}\\
&-\frac{3}{4}zvu_{xxyz}-\frac{3}{4}zvu_{xxyy}+\frac{1}{4}zv_yu_{xxx}+\frac{1}{4}zvu_{xxxy}+\frac{1}{2}vu_t-\frac{3}{2}tv_yu_xu_t-\frac{3}{2}tvu_{xy}u_t\\
&-\frac{3}{4}tv_{xxy}u_t+tu_zv_t+\frac{1}{2}zu_zv_t+\frac{1}{2}zu_xv_t-\frac{1}{2}tu_tv_t-tvu_{tz}+\frac{1}{2}zvu_{tz}+\frac{1}{2}zvu_{ty}\\
&+\frac{3}{2}tvu_xu_{ty}+\frac{1}{4}tv_{xx}u_{ty}+\frac{1}{2}zvu_{tx}+\frac{1}{2}tv_{xy}u_{tx}-\frac{1}{2}tv_xu_{txy}-\frac{1}{4}tv_yu_{txx}+\frac{3}{4}tvu_{txxy}+\frac{1}{2}tvu_{tt};
\end{aligned}
$$

$$
\begin{aligned}
Y_5=&\frac{3}{2}zv_xu_x^2+3tu_zv_xu_x+\frac{3}{2}zu_zv_xu_x+\frac{3}{2}zu_yv_xu_x-3tvu_{xz}u_x-\frac{3}{2}zvu_{xz}u_x\\
&+\frac{3}{2}zvu_{xy}u_x-3zvu_{xx}u_x+\frac{1}{4}zv_{xxx}u_x-\frac{3}{2}tv_xu_tu_x+\frac{1}{2}zv_tu_x+\frac{3}{2}tvu_{tx}u_x\\
&-vu_z-zvu_{zz}-3tvu_zu_{xx}-\frac{3}{2}zvu_zu_{xx}+\frac{3}{2}zvu_yu_{xx}-\frac{1}{2}tu_{xz}v_{xx}-\frac{1}{4}zu_{xz}v_{xx}\\
&-\frac{1}{4}zu_{xy}v_{xx}-\frac{1}{4}zu_{xx}v_{xx}+\frac{1}{2}tv_xu_{xxz}+\frac{1}{4}zv_xu_{xxz}+\frac{1}{4}zv_xu_{xxy}+\frac{1}{4}zv_xu_{xxx}\\
&+\frac{1}{2}tu_zv_{xxx}+\frac{1}{4}zu_zv_{xxx}+\frac{1}{4}zu_yv_{xxx}-\frac{1}{2}tvu_{xxxz}-\frac{1}{4}zvu_{xxxz}+\frac{3}{4}zvu_{xxxy}\\
&-\frac{1}{4}zvu_{xxxx}+\frac{1}{2}vu_t+\frac{3}{2}tvu_{xx}u_t-\frac{1}{4}tv_{xxx}u_t+tu_zv_t+\frac{1}{2}zu_zv_t+\frac{1}{2}zu_yv_t\\
&-\frac{1}{2}tu_tv_t-tvu_{tz}+\frac{1}{2}zvu_{tz}+\frac{1}{2}zvu_{ty}+\frac{1}{2}zvu_{tx}+\frac{1}{4}tv_{xx}u_{tx}-\frac{1}{4}tv_xu_{txx}\\
&+\frac{1}{4}tvu_{txxx}+\frac{1}{2}tvu_{tt},\\
Z_5=&-2tu_zv_z-zu_zv_z-zu_yv_z-zu_xv_z+tu_tv_z+vu_y+zvu_{yz}+vu_x+zvu_{xz}\\
&+6tvu_xu_{xy}+3zvu_xu_{xy}+6tvu_yu_{xx}+3zvu_yu_{xx}+2tvu_{xxxy}+zvu_{xxxy}\\
&+\frac{1}{2}vu_t+tu_zv_t+\frac{1}{2}zu_zv_t+\frac{1}{2}zu_yv_t+\frac{1}{2}zu_xv_t-\frac{1}{2}tu_tv_t+\frac{1}{2}zvu_{tz}+2tvu_{ty}\\
&+\frac{1}{2}zvu_{ty}+2tvu_{tx}+\frac{1}{2}zvu_{tx}+\frac{1}{2}tvu_{tt};
\end{aligned}
$$

$$
T_6=-3vu_z-9tu_{zz}v-\frac{3}{2}zu_{zz}v-3u_yv+6tu_{yz}v-\frac{9}{2}yu_{yz}v+\frac{3}{2}zu_{yz}v-\frac{9}{2}yu_{yy}v
$$

$$
\begin{aligned}
&+3zvu_{yy}-3u_xv-\frac{3}{2}xu_{xz}v-\frac{3}{2}zu_{xz}v-6tu_{xy}v-\frac{3}{2}xu_{xy}v-\frac{9}{2}yu_{xy}v+3zu_{xy}v\\
&+45tu_xu_{xy}v-6tu_{xx}v-\frac{3}{2}xu_{xx}v+45tu_yu_{xx}v+15tu_{xxxy}v+\frac{15}{2}tu_{tz}v+\frac{15}{2}tu_{ty}v\\
&+\frac{15}{2}tvu_{tx}+4v-2xv_z-2yv_z+\frac{3}{2}uv_z-6tu_zv_z+\frac{3}{2}zu_zv_z+\frac{9}{2}yv_zu_y-3zv_zu_y\\
&-2xv_y-2yv_y+\frac{3}{2}uv_y-6tu_zv_y+\frac{3}{2}zu_zv_y+\frac{9}{2}yu_yv_y-3zu_yv_y+6tv_zu_x\\
&+\frac{3}{2}xv_zu_x+6tv_yu_x+\frac{3}{2}xv_yu_x-2xv_x-2yv_x+\frac{3}{2}uv_x-6tu_zv_x+\frac{3}{2}zu_zv_x\\
&+\frac{9}{2}yu_yv_x-3zu_yv_x+6tu_xv_x+\frac{3}{2}xu_xv_x+\frac{15}{2}tv_zu_t+\frac{15}{2}tv_yu_t+\frac{15}{2}tv_xu_t,
\end{aligned}
$$

$$
\begin{aligned}
X_6=&\,27yv_xu_y^2-18zv_xu_y^2+12vu_y-36vu_xu_y+\frac{27}{2}yv_yu_xu_y-9zv_yu_xu_y-12xv_xu_y\\
&-12yv_xu_y+9uv_xu_y-36tu_zv_xu_y+9zu_zv_xu_y+36tu_xv_xu_y+9xu_xv_xu_y\\
&+36tvu_{xz}u_y-9zvu_{xz}u_y-\frac{27}{2}yvu_{xy}u_y+9zvu_{xy}u_y-3v_{xx}u_y+\frac{27}{4}yv_{xxy}u_y\\
&-\frac{9}{2}zv_{xxy}u_y+45tv_xu_tu_y+\frac{9}{2}yv_tu_y-3zv_tu_y-45tvu_{tx}u_y+18tv_yu_x^2+\frac{9}{2}xv_yu_x^2\\
&+6vu_z-12tvu_{zz}-3xvu_{zz}-6xv_yu_x-6yv_yu_x+\frac{9}{2}uv_yu_x-18tu_zv_yu_x\\
&+\frac{9}{2}zu_zv_yu_x+18tvu_{yz}u_x-\frac{9}{2}zvu_{yz}u_x-\frac{27}{2}yvu_{yy}u_x+9zvu_{yy}u_x-6xvu_{xy}\\
&-6yvu_{xy}+\frac{9}{2}uvu_{xy}-18tvu_zu_{xy}+\frac{9}{2}zvu_zu_{xy}+36tvu_xu_{xy}+9xvu_xu_{xy}\\
&+\frac{15}{2}v_xu_{xy}-3u_xv_{xy}+6tu_{xz}v_{xy}-\frac{3}{2}zu_{xz}v_{xy}-\frac{9}{2}yu_{xy}v_{xy}+3zu_{xy}v_{xy}+2v_{xy}\\
&-6tv_xu_{xyz}+\frac{3}{2}zv_xu_{xyz}+\frac{9}{2}yv_xu_{xyy}-3zv_xu_{xyy}+\frac{9}{4}v_yu_{xx}-6tv_{xy}u_{xx}-\frac{3}{2}xv_{xy}u_{xx}\\
&+3tu_{yz}v_{xx}-\frac{3}{4}zu_{yz}v_{xx}-\frac{9}{4}yu_{yy}v_{xx}+\frac{3}{2}zu_{yy}v_{xx}-3tu_{xy}v_{xx}-\frac{3}{4}xu_{xy}v_{xx}+v_{xx}\\
&-3tv_yu_{xxz}+\frac{3}{4}zv_yu_{xxz}-\frac{27}{2}vu_{xxy}+\frac{9}{4}yv_yu_{xxy}-\frac{3}{2}zv_yu_{xxy}+6tv_xu_{xxy}\\
&+\frac{3}{2}xv_xu_{xxy}-3xv_{xxy}-3yv_{xxy}+\frac{9}{4}uv_{xxy}-9tu_zv_{xxy}+\frac{9}{4}zu_zv_{xxy}+9tu_xv_{xxy}\\
&+\frac{9}{4}xu_xv_{xxy}+9tvu_{xxyz}-\frac{9}{4}zvu_{xxyz}-\frac{27}{4}yvu_{xxyy}+\frac{9}{2}zvu_{xxyy}+3tv_yu_{xxx}\\
&+\frac{3}{4}xv_yu_{xxx}+3tvu_{xxxy}+\frac{3}{4}xvu_{xxxy}-9vu_t+\frac{45}{2}tv_yu_xu_t+\frac{45}{2}tvu_{xy}u_t+\frac{45}{4}tv_{xxy}u_t\\
&-2xv_t-2yv_t+\frac{3}{2}uv_t-6tu_zv_t+\frac{3}{2}zu_zv_t+6tu_xv_t+\frac{3}{2}xu_xv_t+\frac{15}{2}tu_tv_t+18tvu_{tz}
\end{aligned}
$$

$$
\begin{aligned}
&+ 3xvu_{tz} - \frac{3}{2}zvu_{tz} + 12tvu_{ty} + 3xvu_{ty} - \frac{9}{2}yvu_{ty} + 3zvu_{ty} - \frac{45}{2}tvu_xu_{ty} \\
&- \frac{15}{4}tv_{xx}u_{ty} + 6tvu_{tx} + \frac{3}{2}xvu_{tx} - \frac{15}{2}tv_{xy}u_{tx} + \frac{15}{2}tv_xu_{txy} + \frac{15}{4}tv_yu_{txx} \\
&- \frac{45}{4}tvu_{txxy} - \frac{15}{2}tvu_{tt},
\end{aligned}
$$

$$
\begin{aligned}
Y_6 = &- 9vu_x^2 + 18tv_xu_x^2 + \frac{9}{2}xv_xu_x^2 - 6xv_xu_x - 6yv_xu_x + \frac{9}{2}uv_xu_x - 18tu_zv_xu_x \\
&+ \frac{9}{2}zu_zv_xu_x + \frac{27}{2}yu_yv_xu_x - 9zu_yv_xu_x + 18tvu_{xz}u_x - \frac{9}{2}zvu_{xz}u_x + \frac{27}{2}yvu_{xy}u_x \\
&- 9zvu_{xy}u_x - 36tvu_{xx}u_x - 9xvu_{xx}u_x - \frac{3}{2}v_{xx}u_x + 3tv_{xxx}u_x + \frac{3}{4}xv_{xxx}u_x \\
&+ \frac{45}{2}tv_xu_tu_x + 6tv_tu_x + \frac{3}{2}xv_tu_x - \frac{45}{2}tvu_{tx}u_x + 6vu_z - 9yvu_{zz} + 6zvu_{zz} \\
&+ 6xvu_{xx} + 6yvu_{xx} - \frac{9}{2}uvu_{xx} + 18tvu_zu_{xx} - \frac{9}{2}zvu_zu_{xx} + \frac{27}{2}yvu_yu_{xx} \\
&- 9zvu_yu_{xx} + \frac{9}{4}v_xu_{xx} + 3tu_{xz}v_{xx} - \frac{3}{4}zu_{xz}v_{xx} - \frac{9}{4}yu_{xy}v_{xx} + \frac{3}{2}zu_{xy}v_{xx} \\
&- 3tu_{xx}v_{xx} - \frac{3}{4}xu_{xx}v_{xx} + v_{xx} - 3tv_xu_{xxz} + \frac{3}{4}zv_xu_{xxz} + \frac{9}{4}yv_xu_{xxy} - \frac{3}{2}zv_xu_{xxy} \\
&- 3vu_{xxx} + 3tv_xu_{xxx} + \frac{3}{4}xv_xu_{xxx} - xv_{xxx} - yv_{xxx} + \frac{3}{4}uv_{xxx} - 3tu_zv_{xxx} \\
&+ \frac{3}{4}zu_zv_{xxx} + \frac{9}{4}yu_yv_{xxx} - \frac{3}{2}zu_yv_{xxx} + 3tvu_{xxxz} - \frac{3}{4}zvu_{xxxz} + \frac{27}{4}yvu_{xxxy} \\
&- \frac{9}{2}zvu_{xxxy} - 3tvu_{xxxx} - \frac{3}{4}xvu_{xxxx} - 9vu_t - \frac{45}{2}tvu_{xx}u_t + \frac{15}{4}tv_{xxx}u_t - 2xv_t \\
&- 2yv_t + \frac{3}{2}uv_t - 6tu_zv_t + \frac{3}{2}zu_zv_t + \frac{9}{2}yu_yv_t - 3zu_yv_t + \frac{15}{2}tu_tv_t + 6tvu_{tz} \\
&+ 9yvu_{tz} - \frac{15}{2}zvu_{tz} + \frac{9}{2}yvu_{ty} - 3zvu_{ty} - 6tvu_{tx} - \frac{3}{2}xvu_{tx} + 9yvu_{tx} \\
&- 6zvu_{tx} - \frac{15}{4}tv_{xx}u_{tx} + \frac{15}{4}tv_xu_{txx} \\
&- \frac{15}{4}tvu_{txxx} - \frac{15}{2}tvu_{tt}, \\
Z_6 = &\, 12vu_z + 12tv_zu_z - 3zv_zu_z - 6tv_tu_z + \frac{3}{2}zv_tu_z + 4xv_z + 4yv_z - 3uv_z \\
&- 6vu_y - 9yv_zu_y + 6zv_zu_y + 9yvu_{yz} - 6zvu_{yz} - 6vu_x - 12tv_zu_x \\
&- 3xv_zu_x + 12tvu_{xz} + 3xvu_{xz} - 36tvu_xu_{xy} + 9zvu_xu_{xy} - 36tvu_yu_{xx}
\end{aligned}
$$

$$
\begin{aligned}
&+ 9zvu_yu_{xx} - 12tvu_{xxxy} + 3zvu_{xxxy} - 9vu_t - 15tv_zu_t - 2xv_t - 2yv_t \\
&+ \frac{3}{2}uv_t + \frac{9}{2}yu_yv_t - 3zu_yv_t + 6tu_xv_t + \frac{3}{2}xu_xv_t + \frac{15}{2}tu_tv_t + 9tvu_{tz} + \frac{3}{2}zvu_{tz} \\
&- 12tvu_{ty} - \frac{9}{2}yvu_{ty} + 6zvu_{ty} - 18tvu_{tx} - \frac{3}{2}xvu_{tx} + 3zvu_{tx} - \frac{15}{2}tvu_{tt};
\end{aligned}
$$

$$
\begin{aligned}
T_{F_1} =& \frac{1}{2}F_1(t)v_x - \frac{1}{2}F_1(t)v_y - \frac{1}{2}F_1(t)v_z, \\
X_{F_1} =& -\frac{3}{2}F_1(t)u_{xy}v + \frac{1}{2}F_1'v - \frac{3}{2}F_1(t)u_xv_y - 3F_1(t)u_yv_x - \frac{3}{4}F_1(t)v_{xxy} - \frac{1}{2}F_1(t)v_t, \\
Y_{F_1} =& \frac{3}{2}F_1(t)u_{xx}v + \frac{1}{2}F_1'v - \frac{3}{2}F_1(t)u_xv_x - \frac{1}{4}F_1(t)v_{xxx} - \frac{1}{2}F_1(t)v_t, \\
Z_{F_1} =& \frac{1}{2}F_1'v + F_1(t)v_z - \frac{1}{2}F_1(t)v_t;
\end{aligned}
$$

$$
\begin{aligned}
T_{F_2} =& \frac{1}{2}F_2'v - \frac{1}{2}v_xF_2(t+z) - \frac{1}{2}v_yF_2(t+z) - \frac{1}{2}v_zF_2(t+z), \\
X_{F_2} =& -\frac{3}{2}F_2(t+z)u_{xy}v + \frac{1}{2}F_2'v - \frac{3}{2}u_xv_yF_2(t+z) - 3u_yv_xF_2(t+z) \\
&- \frac{3}{4}F_2(t+z)v_{xxy} - \frac{1}{2}v_tF_2(t+z), \\
Y_{F_2} =& \frac{3}{2}u_{xx}F_2(t+z)v + \frac{1}{2}F_2'v - \frac{3}{2}u_xv_xF_2(t+z) - \frac{1}{4}v_{xxx}F_2(t+z) - \frac{1}{2}v_tF_2(t+z), \\
Z_{F_2} =& -\frac{1}{2}F_2'v + v_zF_2(t+z) - \frac{1}{2}v_tF_2(t+z).
\end{aligned}
$$

5.4 Concluding remarks

In this chapter, we used Lie symmetry methods to obtain analytic solutions of the (3+1)-dimensional generalised KP equation (5.3). The first solution contains an elliptic integral of the second kind, an amplitude function, a delta amplitude function and a Jacobi cosine function. It is common knowledge that these functions degenerate to trigonometric or hyperbolic functions depending on the behaviour of the parameter M^2. Thus, the solution (5.13) is to our knowledge the most

general solution of (5.3) that has been obtained to date. Some of the solutions obtained were presented graphically. Furthermore, we obtained logarithmic and rational solutions. Finally, we computed the conserved quantities of equation (5.3) using the multiplier method as well as Ibragimov's conservation theorem. The former method yielded a local low-order conserved quantity, while the later method yielded eight conservation laws with each conserved vector corresponding to an infinitesimal generator.

Chapter 6

Travelling wave solutions and conservation laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt equations

6.1 Introduction

The Broer-Kaup-Kupershmidt (BKK) system of equations

$$\begin{aligned} E_1 &\equiv u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} = 0, \\ E_2 &\equiv v_t + v_{xx} + 2(uv)_x = 0. \end{aligned} \tag{6.1}$$

is one of the most popular system of NLPDEs to emerge in the past few decades. This is evidenced by the vast array of scholars who have researched various aspects of the system. It has applications in fluid dynamics where it models dispersive shallow water waves travelling in equal depth. By using a convenient scaling trans-

formation, it has been shown in [98, 99] that the (2+1)-dimensional asymmetric Davey-Stewartson system [100]

$$\bar{q}_t + \frac{\bar{q}_{xx}}{2} + 2\bar{q}\partial_y^{-1}(\bar{q}\bar{r})_x = 0, \quad \bar{r}_t + \frac{\bar{r}_{xx}}{2} + 2\bar{r}\partial_y^{-1}(\bar{q}\bar{r})_x = 0,$$

transforms into (6.1) under the transformation

$$\bar{q} = \exp\left(-\int^x u\mathrm{dx}\right), \quad \bar{r} = -v\exp\left(\int^x u\mathrm{dx}\right).$$

Solitoff and dromion solutions are obtained in the same work [98]. Also, the Kadomtsev-Petviashvili equation transforms into the BKK equation under a symmetry constraint, see for example, [100, 101]. In [101] the modified extended Fan sub-equation method is used to obtain soliton-like and Jacobi elliptic wave function-like solutions of (6.1). In [102], Bäcklund transformation and variable separation approach was used to obtain dromions, lumps and peakons through the introduction of an arbitrary function. Again, in [103, 104] an auxiliary equation method was utilised to obtain its exact travelling wave solutions. He's semi-inverse method was applied in [105] to establish a variational principle of the BKK system. Several other researchers have utilised different ad hoc methods to establish different solutions of (6.1). Kassem and Rashed [106] came up with closed form solutions of (6.1) by using hidden symmetries of its Lie optimal systems. The most recent work on the BKK system was by Tang et al. [107] who presented the double Wronskian solutions by using Hirota's method and binary bell polynomials.

Due to its undeniably vast applicability, continued study of the BKK system remains necessary. In this chapter, we further explore the (2+1)-dimensional BKK system (6.1). Unlike most of the previous research on this system, we do not employ ad hoc methods to obtain its analytic solutions, but utilise a standard Lie based integration method [30]. Here we provide a detailed outline of the derivation of the bounded travelling wave solutions, which include kink and anti-kink profiles. We also outline how periodic solutions of a snoidal nature are obtained. Moreover,

the homotopy integral approach to finding conservation laws is explored in detail. To the best of our knowledge, the literature is devoid of explicit applications of this approach, more so for NLPDEs with mixed derivatives, our work is novel in this regard.

The work presented in this chapter has been submitted for possible publication. See [108].

6.2 Bounded travelling wave solutions of the BKK equations (1)

Let $\xi = x + by + ct$, then the BKK equations (6.1) becomes

$$\begin{aligned} &bcu'' - bu''' + 2b(uu')' + 2v'' = 0, \\ &cv' + v'' + 2(uv)' = 0. \end{aligned} \tag{6.2}$$

Integrating (6.2) once with respect to ξ and letting the constants of integration be zero, one gets

$$\begin{aligned} &bcu' - bu'' + 2buu' + 2v' = 0, \\ &(2u + c)v + v' = 0. \end{aligned} \tag{6.3}$$

Integrating the first equation of (6.3) once again and then solving for v gives

$$v = \frac{1}{2}(bu' - bu^2 - bcu + g), \tag{6.4}$$

where g is a constant of integration. Substituting the expression of v from (6.4) into the second equation of system (6.3), we have for the case when $b \neq 0$, that

$$u'' = (2u + c)\left(u^2 + cu - \frac{g}{b}\right) + \frac{g_2}{b}. \tag{6.5}$$

Let $u' = y$, then equation (6.5) is equivalent to the following planar dynamical system:

$$u' = y, \quad y' = (2u + c)\left(u^2 + cu - \frac{g}{b}\right). \tag{6.6}$$

Suppose $u_0 > -\frac{c}{2}$ satisfies $g = bu_0(c + u_0)$, then system (6.6) has three equilibrium points $E_1(-c - u_0, 0)$, $E_2(-\frac{c}{2}, 0)$ and $E_3(u_0, 0)$, where $-c - u_0 < -\frac{c}{2} < u_0$. By determining the Jacobian determinant, we know that E_1 and E_3 are saddle points and E_2 is a center. Clearly, system (6.6) is a Hamiltonian system with Hamiltonian

$$H(u, y) = \frac{y^2}{2} - \frac{1}{2}u^4 - cu^3 - \frac{1}{2}\left(c^2 - 2\frac{g}{b}\right)u^2 + \frac{gc}{b}u. \tag{6.7}$$

Hence $H(E_2) = h_2$ and $H(E_1) = H(E_3) = h_0$, where $h_1 = \frac{1}{2}u_0^2(u_0 + c)^2$ and $h_2 = -\frac{1}{32}c^2(c^2 + 8cu_0 + 8u_0^2)$. Therefore, we have the phase portrait of (6.6) (see Figure 1). It shows that there are two heteroclinic orbits connecting the two saddle points E_1 and E_3 which consist of the boundary of a family of periodic orbits. As is well-known, the two heteroclinic orbits of (6.6) which are determined by $H(u, y) = h_1$ correspond to a kink and an anti-kink for u of (6.1) respectively and the periodic orbits given by $H(u, y) = h$ with $h_2 < h < h_1$ correspond to periodic travelling wave solutions. Hence, one sees easily from Figure 1 that there are a family of periodic wave solutions and a kink and an anti-kink for u of (6.1). Note that v is determined by (6.4) which is equivalent to

$$v = \frac{b}{2}(u' - u^2 - cu + u_0(c + u_0)), \tag{6.8}$$

so (u, v) is a periodic travelling wave solution of (6.1) if u is periodic.

To derive the kink and anti-kink, we integrate along the heteroclinic orbits $H(u, y) = h_1$, that is,

$$y_\pm = \pm(u - u_0)(u + c + u_0).$$

Substituting the above equation into the first equation of (6.6) and then integrating

the resulting differential equation yields

$$u_{\pm}(\xi) = -\frac{c}{2} \pm \frac{1}{2}(c+2u_0)\tanh\left(\frac{1}{2}(c+2u_0)(\xi-\xi_0)\right), \tag{6.9}$$

where ξ_0 is a constant of integration. It implies that for arbitrary constants ξ_0, c and $u_0 > -\frac{c}{2}$, (6.9) determine a kink and an anti-kink for u of (6.1) respectively. Taking account of (6.8), we have the corresponding

$$v_+(\xi) = \frac{b}{4}(c+2x_0)^2\mathrm{sech}^2\left(\frac{1}{2}(c+2u_0)(\xi-\xi_0)\right) \tag{6.10}$$

and

$$v_-(\xi) = 0,$$

that is to say, $(u_+(\xi), v_+(\xi))$ and $(u_-(\xi), v_-(\xi))$ are two travelling wave solutions to (6.1).

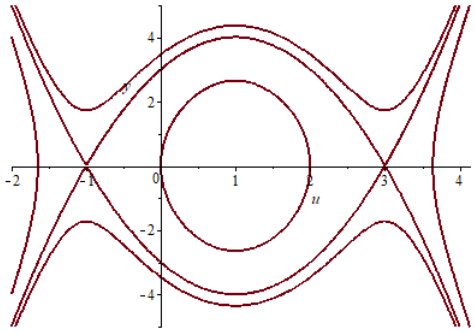


Figure 6.1: Phase orbit of (6.6) and the corresponding bounded travelling wave solutions.

For the periodic orbits determined by $H(u,y) = h$ with $h_2 < h < h_1$, we know from (6.7) that the periodic orbit passing through $(u_2, 0)$, for arbitrary $u_2 \in (-c-u_0, -\frac{c}{2})$, is determined by

$$y = \pm\sqrt{(u-u_1)(u-u_2)(u_3-u)(u_4-u)} \quad (u_2 \le u \le u_3), \tag{6.11}$$

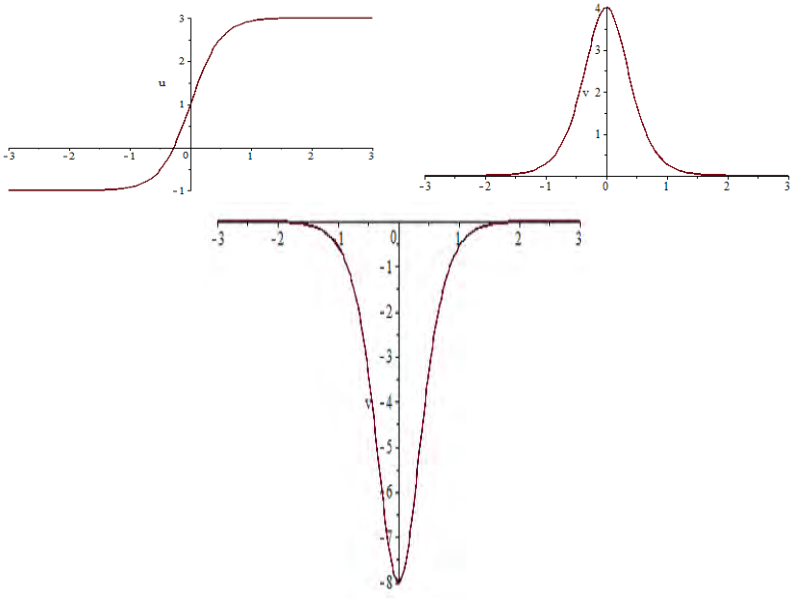
where u_1, u_3 and u_4 satisfying $u_1 < -c-u_0$ and $-\frac{c}{2} < u_3 < u_0 < u_4$ are three real roots of the following cubic algebraic equation:

$$u^3+(2c+u_2)u^2+\left((c+u_2)^2-2u_0(c+u_0)\right)u+u_2(c+u_2)^2-2u_0(c+u_0)(c+u_2) = 0.$$

Inserting equation (6.11) into the first equation of (6.6) and then integrating the resulting differential equation yields

$$u(\xi) = u_1 + \frac{(u_2 - u_1)(u_3 - u_1)}{(u_3 - u_1) - (u_3 - u_2)sn^2(\Omega\xi, q)}, \tag{6.12}$$

where $\Omega = \frac{1}{2}\sqrt{(u_4 - u_2)(u_3 - u_1)}$ and $q = \sqrt{\frac{(u_3-u_2)(u_4-u_1)}{(u_3-u_1)(u_4-u_2)}}$. Therefore, we derive a family of periodic travelling wave solutions $u(\xi), v(\xi)$ to (6.1), where $u(\xi)$ is given by (6.12) and the associated $v(\xi)$ is derived by inserting (6.12) into (6.8) for arbitrary $u_2 \in (-c - u_0, -\frac{c}{2})$.

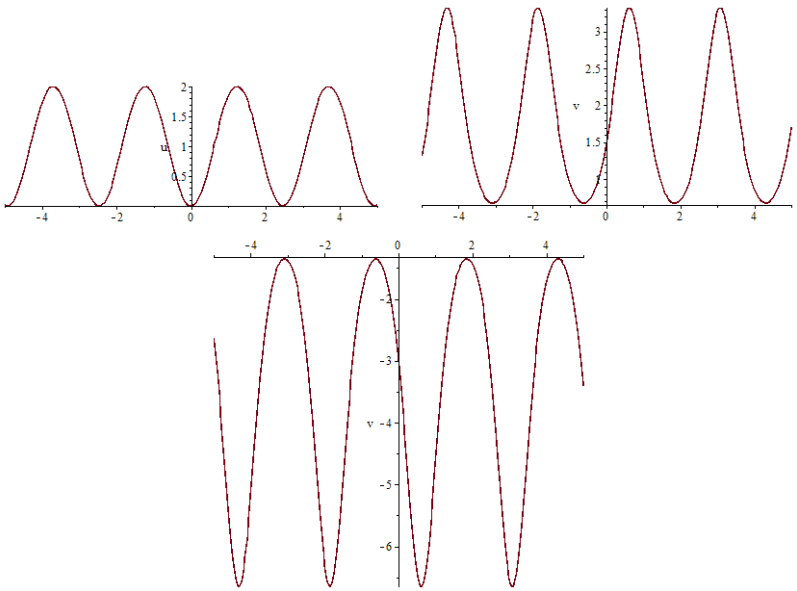


(1) $u = u_+(\xi)$ (2) $v = v_+(\xi)$ with $b = 1$ (3) $v = v_+(\xi)$ with $b = -2$

Figure 6.2: Bounded travelling wave solutions with $c = -2$ and $u_0 = 3$.

Theorem 6.1 *For arbitrary real numbers b, c and $u_0 > -\frac{c}{2}$, let $\xi = x + by + ct$. Then the (2+1)-dimensional BKK system (6.1) has the following bounded travelling wave solutions:*

(1) two families of travelling wave solutions $(u_+(\xi), v_+(\xi))$ and $(u_-(\xi), v_-(\xi))$, where $u_+(\xi)$ is of kink shape but the associated $v_+(\xi)$ is of solitary shape whose



(1) $u = u(\xi)$ (2) $v = v(\xi)$ with $b = 1$ (3) $v = v(\xi)$ with $b = -2$

Figure 6.3: Periodic travelling wave solutions with $c = -2$, $u_0 = 3$ and $u_2 = 0$.

amplitude is determined by b (See Figure 6.2). However, $u_-(\xi)$ *is of anti-kink shape and* $u_-(\xi) = 0$*;*

(2) a family of periodic travelling wave solutions $(u(\xi), v(\xi))$ *given by (6.12) and the associated* $v(\xi)$ *is determined by (6.8) for arbitrary* $u_2 \in (-c - u_0, -\frac{c}{2})$ *(See Figure 6.3).*

6.3 Conservation laws of the BKK equations (1.1)

A conservation law is a divergence expression $D_t T + D_x X + D_y Y = 0$ subject to solutions of the BKK system (6.1). The conserved density T and the spatial flux (X, Y) are functions of t, x, y, u, v and the derivatives of u and v. The BKK system (6.1) does not possess a variational principle as it does not satisfy the Helmholtz conditions [43]. However, conserved quantities may be obtained using a more

direct approach which does not require the existence of a variational principle. In this work we employ the multiplier approach to compute conservation laws. This approach capitalises on the well known correspondence between multipliers and conservation laws [43]. Furthermore, we will employ the first homotopy integral formula to compute conserved quantities. To date, the authors have not come across an explicit application of this formula to a system with mixed derivatives. This work will thus serve as an illustrative example, amongst other things. A determining condition to find multipliers of the BKK system (6.1) is

$$\begin{aligned} \frac{\delta}{\delta u}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] &= 0, \\ \frac{\delta}{\delta v}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] &= 0, \end{aligned} \tag{6.13}$$

where $\delta/\delta u$ and $\delta/\delta v$ are Euler-Lagrange operators (1.25), here given by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_tD_y\frac{\partial}{\partial u_{ty}} + D_xD_y\frac{\partial}{\partial u_{xy}} - D_x^2D_y\frac{\partial}{\partial u_{xxy}}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}}. \end{aligned} \tag{6.14}$$

Furthermore, D_t, D_x and D_y are total derivatives (1.4) given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t} + \cdots, \\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + u_{yt}\frac{\partial}{\partial u_t} + v_{yt}\frac{\partial}{\partial v_t} + \cdots. \end{aligned}$$

Here we seek to compute first order conservation law multipliers

$$\Lambda^\alpha = \Lambda^\alpha(t, x, u, v, u_x, v_x, u_y, v_y), \quad \alpha = 1, 2.$$

Expanding (6.13) and splitting on derivatives of u and v, we obtain the following system of multiplier determining equations:

$$\Lambda^1_{ty} = 0,\ \ \Lambda^2_{ty} = 0,\ \ 2\Lambda^1_{xx} - \Lambda^2_t = 0,\ \ \Lambda^1_{xy} = 0,\ \ \Lambda^2_x = 0,\ \ \Lambda^1_u = 0,\ \ \Lambda^2_u = 0,\ \ \Lambda^2_{u_x} = 0,$$

$$\Lambda^1_v = 0,\ \ \Lambda^2_v = 0,\ \ \Lambda^2_v = 0,\ \ \Lambda^1_{u_x} = 0,\ \ \Lambda^2_{u_x} = 0,$$
$$\Lambda^1_{v_x} = 0,\ \ \Lambda^2_{v_x} = 0,\ \ \Lambda^1_{u_y} = 0,\ \ \Lambda^2_{u_y} = 0.$$

Solving the above system for Λ^1 and Λ^2, we obtain

$$\begin{aligned} \Lambda^1 &= \frac{1}{4}F_2'(t)\,x^2 + F_3(t)\,x + F_4(y) + F_5(t), \\ \Lambda^2 &= F_1(y) + F_2(t), \end{aligned} \tag{6.15}$$

where F_i, $i = 1, \cdots, 5$ are arbitrary functions of their respective arguments. We thus have the following five multipliers

$$\begin{aligned} &\Lambda^1_1 = 0, && \Lambda^2_1 = F_1(y); \\ &\Lambda^1_2 = \frac{1}{4}F_2'(t)x^2, && \Lambda^2_2 = F_2(t); \\ &\Lambda^1_3 = F_3(t)x, && \Lambda^2_3 = 0; \\ &\Lambda^1_4 = F_4(y), && \Lambda^2_4 = 0; \\ &\Lambda^1_5 = F_5(t), && \Lambda^2_5 = 0. \end{aligned} \tag{6.16}$$

The homotopy integral formula [43] is a revolutionary approach for computing conserved vectors and is given by (1.48). In accordance with system (6.1) and

multipliers (6.15) we give formula (1.48) in explicit form as

$$T = \int_0^1 \left(uD_y \left(\frac{\partial E_1 \Lambda^1}{\partial u_{ty}} \right) \Big|_{u=u_{(\lambda)}} + v \left(\frac{\partial E_2 \Lambda^2}{\partial v_t} \right) \Big|_{v=v_{(\lambda)}} \right) \mathrm{d}\lambda,$$

$$\begin{aligned} X = \int_0^1 \Bigg\{ & u \Bigg\{ \left(\frac{\partial E_1 \Lambda^1}{\partial u_x} \right) \Big|_{u=u_{(\lambda)}} + \left(\frac{\partial E_2 \Lambda^2}{\partial u_x} \right) \Big|_{u=u_{(\lambda)}} - D_y \left(\frac{\partial E_1 \Lambda^1}{\partial u_{xy}} \right) \Big|_{u=u_{(\lambda)}} \\ & + D_x D_y \left(\frac{\partial E_1 \Lambda^1}{\partial u_{xxy}} \right) \Big|_{u=u_{(\lambda)}} \Bigg\} - u_x D_y \left(\frac{\partial E_1 \Lambda^1}{\partial u_{xxy}} \right) \Big|_{u=u_{(\lambda)}} \\ & + v \Bigg\{ \left(\frac{\partial E_2 \Lambda^2}{\partial v_x} \right) \Big|_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_1 \Lambda^1}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_2 \Lambda^2}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} \Bigg\} \\ & + v_x \left(\frac{\partial E_2 \Lambda^2}{\partial v_{xx}} \right) \Big|_{v=v_{(\lambda)}} \Bigg\} \mathrm{d}\lambda, \end{aligned}$$

$$\begin{aligned} Y = \int_0^1 \Bigg\{ & u \left(\frac{\partial E_1 \Lambda^1}{\partial u_y} \right) \Big|_{u=u_{(\lambda)}} + u_t \left(\frac{\partial E_1 \Lambda^1}{\partial u_{ty}} \right) \Big|_{u=u_{(\lambda)}} + u_x \left(\frac{\partial E_1 \Lambda^1}{\partial u_{xy}} \right) \Big|_{u=u_{(\lambda)}} \\ & + u_{xx} \left(\frac{\partial E_1 \Lambda^1}{\partial u_{xxy}} \right) \Big|_{u=u_{(\lambda)}} \Bigg\} \mathrm{d}\lambda. \end{aligned} \tag{6.17}$$

Choosing the homotopy $u_{(\lambda)} = \lambda u$, and $v_{(\lambda)} = \lambda v$ as is usually the case, we have for Λ_1^1 and Λ_1^2

$$\begin{aligned} T_1 &= \int_0^1 v F_1 \mathrm{d}\lambda \\ &= v F_1, \\ X_1 &= \int_0^1 \left(4\lambda u v F_1 + v_x F_1 \right) \mathrm{d}\lambda \\ &= 2uv F_1 + v_x F_1, \\ Y_1 &= 0. \end{aligned}$$

$T_1 = vF_1$ is mass density, and $X_1 = 2uv + v_x F_1$ is mass flux or momentum.

Similarly, the complete set of flux densities and spatial fluxes are derived to obtain

$$T_2 = \frac{1}{4} x^2 u_y F_2' + v F_2,$$

$$X_2 = \frac{1}{2}\, uu_y F_2' x^2 + 2\, uvF_2 + \frac{1}{2}\, xu_y F_2' - \frac{1}{4} x^2 u_{xy} F_2' - xvF_2' + \frac{1}{2} x^2 v_x F_2' + v_x F_2,$$
$$Y_2 = -\frac{1}{4}\, u\left(x^2 F_2'' + 2xuF_2' + 2\, F_2'\right);$$

$$T_3 = \ = xu_y\, F_3,$$
$$X_3 = 2xuu_y F_3 + u_y\, F_3 - u_{xy} F_3 x - 2\, vF_3 + 2xv_x\, F_3,$$
$$Y_3 = \ - xuF_3' + uF_3;$$

$$T_4 = u_y\, F_4,$$
$$X_4 = \ = 2\, uu_y F_4 - u_{xy} F_4 + 2\, v_x\, F_4,$$
$$Y_4 = 0;$$

$$T_5 = u_y\, F_5,$$
$$X_5 = 2\, uF_5 u_y - u_{xy}\, F_5 + 2\, v_x\, F_5,$$
$$Y_5 = \ - uF_5'.$$

The conservation laws obtained here are local and infinitely many because of the presence of arbitrary functions. Due to the order of the dependencies of the adjoint symmetries being of a lower order than the leading derivatives of the underlying system, the conserved quantities are low-order.

6.4 Concluding remarks

In this chapter we carried out a bifurcation analysis of the (2+1)-dimensional BKK system (6.1). We obtained bounded travelling wave solutions in the form of kink, bright and dark solitons as well as periodic waves. We also provided the parametric conditions for which these solutions hold. Finally, using the first homotopy integral

approach we successfully derived conservation laws for the BKK system. Due to the presence of arbitrary functions in the multipliers, we obtained infinitely many conserved vectors.

Chapter 7

An optimal system of group-invariant solutions and conserved quantities of a nonlinear fifth-order integrable equation

7.1 Introduction

The most general form of a fifth-order KdV equation reads [109]

$$u_t + \alpha u^2 u_x + \nu u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0 \tag{7.1}$$

with arbitrary real constants α, ν and γ. Equation (7.1) models long waves propagating in shallow water and in the presence of gravity. Several famous fifth-order KdV equations emanate from equation (7.1), for instance, the fifth-order Sawada-Kotera equation [110]

$$u_t + 45u^2 u_x + 15u_x u_{xx} + 15u u_{xxx} + u_{xxxxx} = 0$$

occurs when $\alpha = 3\nu$, $\nu = \gamma$ and $\gamma = 15$. On the other hand, when $\alpha = 5/2\nu$, $\nu = 5/4\gamma$ and $\gamma = 20$, we have the nonlinear fifth-order Kaup-Kupershmidt equation [111]

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0.$$

In this chapter, we study a newly formulated integrable fifth-order equation

$$F \equiv \beta u_{xxx} - 4u_tu_{xxx} - 12u_{xx}u_{tx} - 8u_xu_{txx} - u_{txxxx} + u_{ttt} = 0,\ \beta \neq 0. \tag{7.2}$$

Equation (7.2) was developed in [112], where the researchers performed a Painlevé test to prove its integrability. The authors went further to find different types of soliton solutions of (7.2) using Hirota's method. In a sense, equation (7.2) is a second generation hybrid between the fifth-order Kawahara equation [113]

$$u_t + \alpha uu_x + \nu u_{xxx} + \gamma u_{xxxxx} \tag{7.3}$$

and another fifth-order equation [114]

$$u_{xxx} - 4u_tu_{xxx} - 12u_{xx}u_{tx} - 8u_xu_{txx} - u_{txxxx} + u_{ttt} = 0. \tag{7.4}$$

In other words, equation (7.2) is a progression from equation (7.4) due to the addition of a second dispersive term (u_{xxx}), by lending credence from the Kawahara equation, which also has the two dispersive terms (u_{xxx}) and (u_{txxxx}). Notwithstanding this backdrop, integrable models are of great interest to mathematicans and physicists for the plethora of information they possess. The integrability of equation (7.2), which has been proven in [112], makes it a significant model and worth studying. In this chapter, we seek to obtain a more comprehensive set of solutions with the aid of an optimal system of one-dimensional subalgebras. Also, we will derive a group of transformations corresponding to the symmetry algebra of (7.2).

Although the solutions given in [112] are extensive, it will be seen that our solutions subsume those in [112] since we obtain hyperbolic, parabolic, elliptic and power

series solutions. We will go a step further to determine the conserved vectors attributable to this equation.

The work presented in this chapter has been published in [115].

7.2 Lie group analysis of (7.2)

7.2.1 Infinitesimal generators

For a one-parameter group of transformations

$$\bar{t} \to t + a\xi^1(t,x,u),\ \bar{x} \to x + a\xi^2(t,x,u),\ \bar{u} \to u + a\eta(t,x,u)$$

with a small parameter a, we have the corresponding infinitesimal generator

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{7.5}$$

The vector field (7.5) in conjunction with (7.2) satisfies the invariance condition

$$\mathrm{pr}^{(5)}X(\beta u_{xxx} - 4u_t u_{xxx} - 12u_{xx}u_{tx} - 8u_x u_{txx} - u_{txxxx} + u_{ttt})|_{F=0} = 0. \tag{7.6}$$

Here, $\mathrm{pr}^{(5)}X$ is the fifth-order extension of (7.5) given by [30]

$$\begin{aligned}\mathrm{pr}^{(5)}X = X &+ \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{ttt}\frac{\partial}{\partial u_{ttt}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}} \\ &+ \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{txxxx}\frac{\partial}{\partial u_{txxxx}}\end{aligned}$$

and ζ_t, ζ_x, ζ_{tx}, ζ_{xx}, ζ_{ttt}, ζ_{txx}, ζ_{xxx}, ζ_{txxxx} are prolongation coefficient functions which can be obtained from (1.16) and (1.17). From (7.6), we obtain the following system of twelve linear homogeneous PDEs:

$$\begin{aligned}&{\xi^1}_x = 0,\ {\xi^1}_u = 0,\ {\xi^1}_{tt} = 0,\ {\xi^1}_{tx} = 0,\ {\xi^1}_{tu} = 0,\ {\xi^2}_t = 0,\ {\xi^2}_u = 0, \\ &2{\xi^2}_x - {\xi^1}_t = 0,\ \eta_{ttt} = 0,\ \eta_x = 0,\ \xi^1_t + 2\eta_u = 0,\ 8\eta_t - 3\beta{\xi^1}_t = 0.\end{aligned} \tag{7.7}$$

Solving system (7.7) yields the generator coefficients

$$\xi^1 = \frac{8C_2}{3\beta}t + C_1,\ \ \xi^2 = \frac{4C_2}{3\beta}x + C_4,\ \ \eta = -\frac{4C_2}{3\beta}u + C_2 t + C_3 \tag{7.8}$$

and ultimately we obtain a four-dimensional Lie algebra L_4, spanned by the generators

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = 8t\frac{\partial}{\partial t} + 4x\frac{\partial}{\partial x} + (3\beta t - 4u)\frac{\partial}{\partial u},\ X_4 = \frac{\partial}{\partial u}. \tag{7.9}$$

7.2.2 Group transformations of known solutions

The one-parameter groups G_i of transformations generated by the four-dimensional Lie algebra (7.9) can be obtained by solving the Lie equations [31] and these are

$$\begin{aligned}
&G_1 : (t, x, u) \to (t + a, x, u),\\
&G_2 : (t, x, u) \to (t, x + a, u),\\
&G_3 : (t, x, u) \to \left(te^{8a}, xe^{4a}, ue^{-4a} + \frac{1}{4}\beta t e^{-4a}\left(e^{12a} - 1\right)\right),\\
&G_4 : (t, x, u) \to (t, x, u + a).
\end{aligned}$$

Thus, we have the following theorem.

Theorem 7.1 *If $u = m(t, x)$ is a solution of the PDE (7.2), then so are the functions*

$$\begin{aligned}
&G_1 : u^{(1)} = m(t - a, x),\\
&G_2 : u^{(2)} = m(t, x - a),\\
&G_3 : u^{(3)} = e^{-4a}m(te^{-8a}, xe^{-4a}) + \frac{1}{4}\beta t\left(1 - e^{-12a}\right),\\
&G_4 : u^{(4)} = m(t, x) + a.
\end{aligned}$$

This means that from any known solution of (7.2) one can obtain infinitely many new exact solutions of (7.2) by the repeated use of the above family of solutions.

7.2.3 Optimal system of one-parameter group invariant solutions

We now utilise the method given in [30] to find an optimal system of one-dimensional subalgebras corresponding to the Lie algebra (7.9). Accordingly, we begin by determining the commutation relations, $[X_i, X_j] = X_i(X_j) - X_j(X_i)$, between vector fields (7.9). This leads to the commutator table given in Table 7.1.

Table 7.1 Commutation relations of 4-dimensional Lie algebra (7.9)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$8X_1 + 3\beta X_4$	0
X_2	0	0	$4X_2$	0
X_3	$-8X_1 - 3\beta X_4$	$-4X_2$	0	$4X_4$
X_4	0	0	$-4X_4$	0

The Lie series

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{Ad}\, X_i)^n(X_j)$$

along with the results of Table 7.1 yield adjoint representations, which we present in Table 7.2.

Table 7.2 Adjoint table of Lie algebra (7.9)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$-8\epsilon X_1 + X_3 - 3\beta\epsilon X_4$	X_4
X_2	X_1	X_2	$-4\epsilon X_2 + X_3$	X_4
X_3	$e^{8\epsilon}X_1 + \frac{1}{4}\beta e^{-4\epsilon}\left(e^{12\epsilon} - 1\right)$	$e^{4\epsilon}X_2$	X_3	$e^{-4\epsilon}X_4$
X_4	X_1	X_2	$X_3 + 4\epsilon X_4$	X_4

Using the Tables 7.1 and 7.2, and the procedure given in [30,78], we obtain an optimal system of one-dimensional subalgebras spanned by $X_1, X_2, X_3, X_1 \pm X_2, X_2 \pm X_4$. However, we note that the vector field $X_1 - X_2$ can be mapped to the vector field $X_1 + X_2$ by the discrete symmetry $(t, x, u) \longmapsto (t, -x, u)$. This leaves us with the following optimal system of one-dimensional subalgebras of (7.2) that is spanned by

$$X_1, X_2, X_3, X_1 + X_2, X_2 \pm X_4. \tag{7.10}$$

We now determine an optimal system of group invariant solutions corresponding to (7.10). This will give us closed-form solutions of (7.2).

7.2.3.1 Cases X_1, X_2 and $X_2 \pm X_4$

From the generators X_1, X_2 and $X_2 \pm X_4$, we obtain the group invariant solutions:

$$u(t, x) = \phi_1(\xi),\ \xi = x, \tag{7.11}$$

$$u(t, x) = \phi_2(\xi),\ \xi = t, \tag{7.12}$$

$$u(t, x) = \phi_3(\xi) \pm x,\ \xi = t, \tag{7.13}$$

respectively. Each of the group invariant solutions in (7.11)–(7.13) transform equation (7.2) into the ODE $\phi_j'''(\xi) = 0$, $j = 1, 2, 3$. This is easily solved to give the

solution

$$\phi_j(\xi) = \frac{1}{2}C_1\xi^2 + C_2\xi + C_3,\ j = 1, 2, 3, \tag{7.14}$$

with C_1, C_2, C_3 integration constants. Thus,

$$u_1 = \frac{1}{2}C_1x^2 + C_2x + C_3, \tag{7.15}$$

$$u_2 = \frac{1}{2}C_1t^2 + C_2t + C_3, \tag{7.16}$$

$$u_3 = \frac{1}{2}C_1t^2 + C_2t + C_3 \pm x \tag{7.17}$$

are the group invariant solutions of (7.2) under the optimal system elements X_1, X_2 and $X_2 \pm X_4$, respectively.

7.2.3.2 Case $X_1 + X_2$

We now focus on the optimal system element $X_1 + X_2$. Using the usual Lie theory, we obtain the group invariant solution $u(t, x) = \phi(\xi)$, $\xi = x - t$, which transforms equation (7.2) into the nonlinear ordinary differential equation (NLODE)

$$\phi'''' + (\beta - 1)\phi''' + 12(\phi'''\phi' + \phi''^2) = 0. \tag{7.18}$$

(a) Solutions of (7.2) by direct integration of (7.18)

Integrating (7.18) twice with respect to ξ gives

$$\phi''' + 6\phi'^2 + (\beta - 1)\phi' + C_1\xi + C_2 = 0 \tag{7.19}$$

with integration constants C_1 and C_2. Equation (7.19) cannot be solved to produce solutions in terms of classical functions, see for example [116–118]. At this point we are compelled to explore two interesting cases which lead to elliptic and hyperbolic solutions.

(i) **The Jacobi elliptic function solutions of (7.2)**

Consider the NLODE (7.19) with $C_1 = 0$ and let $\phi'(\xi) = (1/2)\,\Theta(\xi)$. Then equation (7.19) becomes

$$\Theta'' + 3\Theta^2 - \rho\Theta + C_4 = 0, \tag{7.20}$$

where $\rho = 1 - \beta$ and $C_4 = 2C_2$. The solution for equation (7.2) will thus take the form $u(t,x) = 1/2 \int \theta(\xi)\mathrm{d}\xi$. The NLODE (7.20) is a principal equation and arises from many famous NLPDEs. See for example [116]. Multiplying (7.20) by Θ' and integrating with respect to ξ yields

$$\Theta'^2 + 2\Theta^3 - \rho\Theta^2 + 2C_4\Theta + 2C_5 = 0. \tag{7.21}$$

Equation (7.21) leads us to the well-known Jacobi elliptic cosine function solution of (7.20), namely [66, 95]

$$\Theta(\xi) = \mu_2 + (\mu_1 - \mu_2)\,\mathrm{cn}^2\left\{\sqrt{\frac{\mu_1 - \mu_3}{2}}\xi, M^2\right\},\ M^2 = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3}. \tag{7.22}$$

Here μ_1, μ_2 and μ_3 (with $\mu_1 \geq \mu_2 \geq \mu_3$, $0 \leq M^2 \leq 1$) are roots of the algebraic equation

$$\Theta^3 - \frac{1}{2}\rho\Theta^2 + C_4\Theta + C_5 = 0$$

and thus satisfy the differential equation

$$\Theta'^2 = -2(\Theta - \mu_1)(\Theta - \mu_2)(\Theta - \mu_3).$$

Moreover, $C_4 = 2C_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3$ and $C_5 = \mu_1\mu_2\mu_3$. Finally, we recuperate the variable u to obtain the solution of (7.2) as

$$u(t,x) = \frac{(\mu_1 - \mu_2)\,\mathrm{sn}\,(A\,\xi|M^2)\cos^{-1}\{\mathrm{dn}(A\,\xi|M^2)\}}{2A\sqrt{1 - \mathrm{dn}\,(A\,\xi|M^2)^2}} + 1/2\mu_2\xi, \tag{7.23}$$

where $A = \sqrt{\mu_1 - \mu_3}/\sqrt{2}$, $\beta = 1 - 2(\mu_1 + \mu_2 + \mu_3)$ and $\xi = x - t$.

The analytic solution (7.23) is periodic in nature, and its profile is sketched in Figure 7.1.

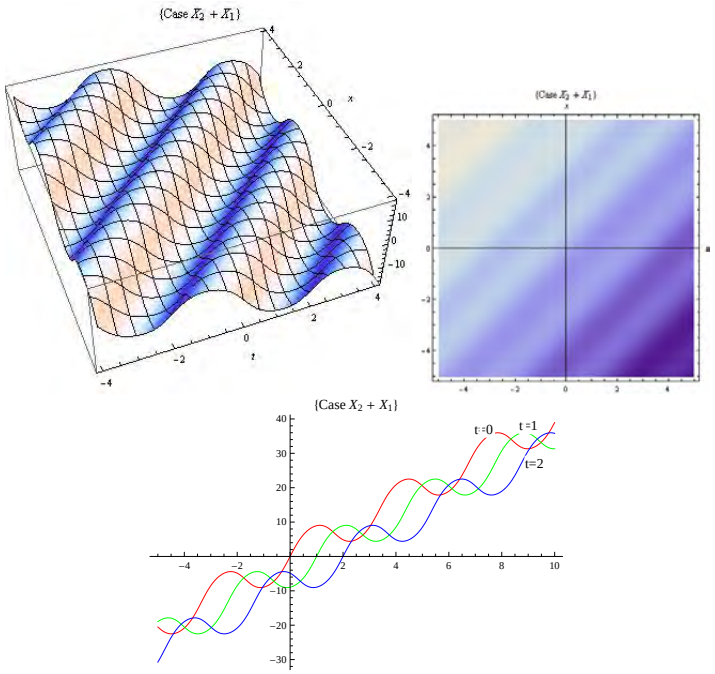


Figure 7.1: Profile of periodic solution (7.23).

(ii) **The hyperbolic solutions of (7.2)**

By letting the integration constants C_1 and C_2 equal to zero and $\phi'(\xi) = \psi(\xi)$ in (7.19), equation (7.19) becomes

$$\psi'' + (\beta - 1)\psi + 6\psi^2 = 0. \tag{7.24}$$

Multiplying by ψ' and integrating with respect to ξ leads us to

$$\psi'^2 + 4\psi^3 + (\beta - 1)\psi^2 = 0, \tag{7.25}$$

where once again, the integration constant is taken as zero. The solution of equation (7.25) is

$$\psi(\xi) = -\frac{1}{4}(\beta - 1)\left[1 - \tanh^2\left\{\frac{1}{2}\left(\pm\sqrt{1-\beta}C_3 - \sqrt{1-\beta}\xi\right)\right\}\right] \tag{7.26}$$

with C_3 an integration constant. In order to recover the variable u, we integrate (7.26) with respect to ξ to obtain the topological kink soliton solution

$$u(t,x) = \frac{1-\beta}{2\sqrt{1-\beta}} \tanh\left(\frac{1}{2}\sqrt{1-\beta}\,(\xi \pm C_3)\right) \tag{7.27}$$

with its profile shown in Figure 7.2.

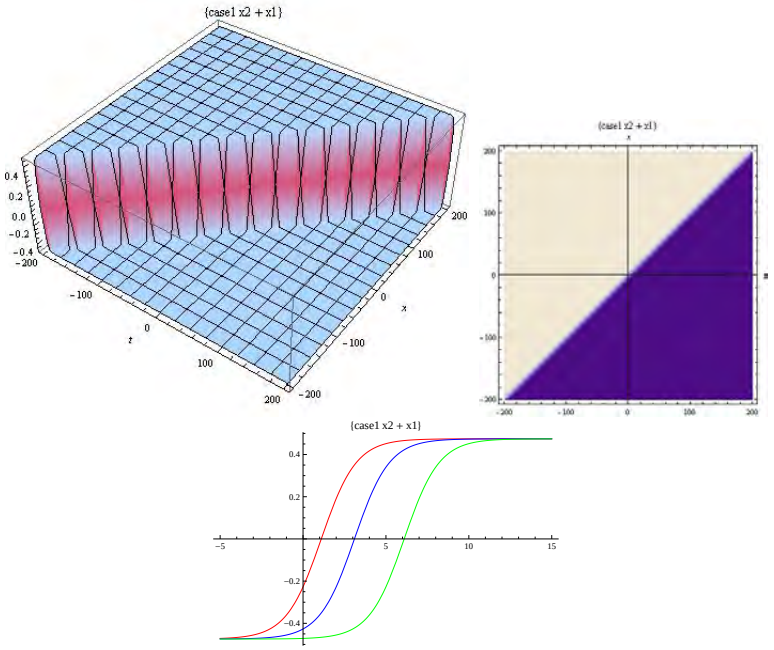


Figure 7.2: Profile of topological kink soliton (7.27).

Remark. Travelling wave solutions of (7.2) can be obtained by taking the linear combination $X_1 + \nu X_2$, where ν is the wave velocity. The corresponding group invariant solution for this combination of vector fields is $u(t,x) = \phi(\xi)$, $\xi = x - \nu t$, which consequently yields the travelling wave solution of (7.2) as

$$u(t,x) = \frac{(\mu_1 - \mu_2)\,\mathrm{sn}\,(A\,\xi|M^2)\cos^{-1}\{\mathrm{dn}(A\,\xi|M^2)\}}{A\sqrt{1-\mathrm{dn}\,(A\,\xi|M^2)^2}} + \mu_2\xi, \tag{7.28}$$

where $A = \sqrt{\mu_1 - \mu_3}/\sqrt{2}$, $\beta = \nu\{\nu^2 - 2(\mu_1 + \mu_2 + \mu_3)\}$ and $\xi = x - \nu t$.

(b) Solutions of (7.2) via the $(G'/G)-$**expansion method** Using the $(G'/G)-$expansion technique [19], we now solve the NLODE (7.18). Let equation (7.18) have the formal solution (1.62). We aim to find the values of the coefficients $A_0, A_1, \cdots A_m$. In the case of the NLODE (7.18), we obtain $m = 1$. See, for example [64]. Thus, from (1.62) we have

$$\phi(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \tag{7.29}$$

Substituting (7.29) into (7.18) and making use of the linear ODE (1.63) we obtain an algebraic equation in A_0 and A_1, whose solution is

$$A_0 = A_0, \;\; A_1 = 1, \;\; \beta = 1 - (\lambda^2 - 4\mu).$$

Thus, we have the following three types of solutions for (7.2):

(i) When $1 - \beta > 0$, we have

$$u(t,x) = A_0 + \Omega_1 \left(\frac{B_1 \cosh(\Omega_1 \xi) + B_2 \sinh(\Omega_1 \xi)}{B_1 \sinh(\Omega_1 \xi) + B_2 \cosh(\Omega_1 \xi)} \right) - \frac{\lambda}{2}, \tag{7.30}$$

where $\Omega_1 = \sqrt{1-\beta}/2$, $\xi = x - t$ and A_0, B_1 and B_2 are arbitrary constants. The solution profile of (7.30) is presented in Figure 7.3.

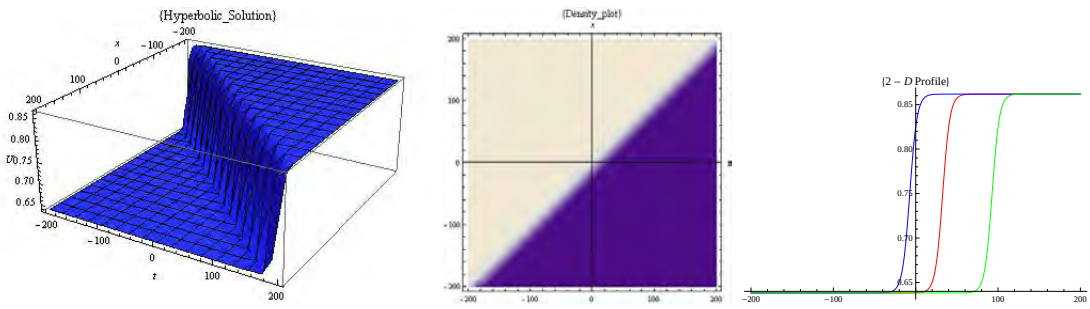


Figure 7.3: Topological kink soliton profile of (7.30).

(ii) When $1 - \beta < 0$, we get

$$u(t,x) = A_0 + \Omega_2 \left(\frac{B_1 \sin(\Omega_2 \xi) - B_2 \cos(\Omega_2 \xi)}{B_1 \cos(\Omega_2 \xi) + B_2 \sin(\Omega_2 \xi)} \right) - \frac{\lambda}{2}, \tag{7.31}$$

where $\Omega_2 = \sqrt{\beta - 1}/2$, $\xi = x - t$ and A_0, B_1 and B_2 are arbitrary constants. The solution profile of (7.31) is sketched in Figure 7.4.

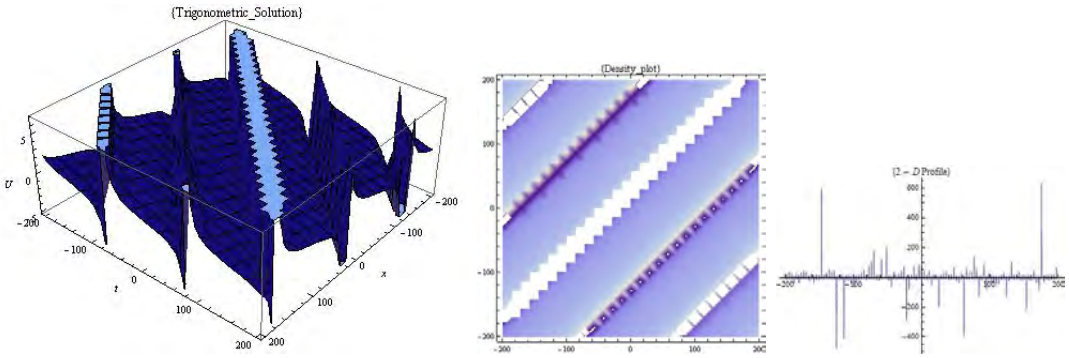


Figure 7.4: Periodic behaviour with singularities for (7.31).

(iii) When $\beta = 1$, we obtain

$$u(t,x) = A_0 + \frac{B_2}{B_2\xi + B_1} - \frac{\lambda}{2}, \tag{7.32}$$

where $\xi = x - t$ and A_0, B_1 and B_2 are arbitrary constants. The profile of solution (7.32) is given in Figure 7.5.

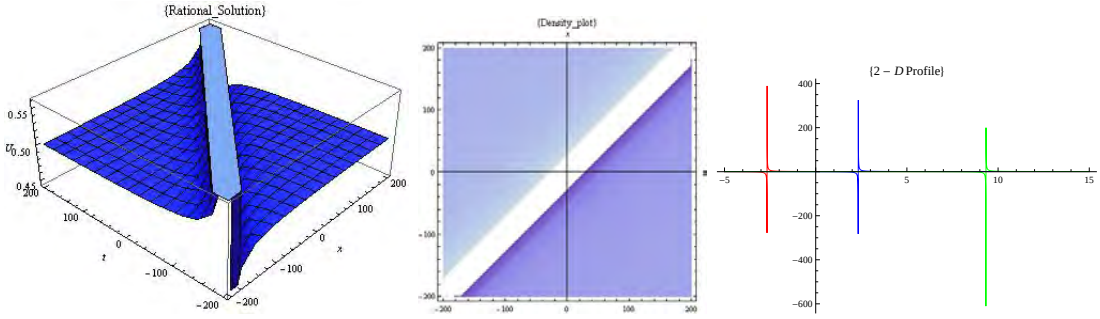


Figure 7.5: Bright and dark soliton (7.32) with a singularity.

7.2.3.3 Case X_3

Finally, we consider the optimal system element X_3, which culminates in the group invariant solution $u(t,x) = (1/\sqrt{t})\phi(\xi) + 1/4\,\beta t$, $\xi = x/\sqrt{t}$. Thus, equation (7.2)

is transformed into the NLODE

$$4\xi\phi^{(5)} + 20\phi^{(4)} + 48\xi\phi'\phi''' - \xi^3\phi''' + 96\xi\phi'^2 + 96\xi\phi\phi'' + 16\phi\phi''' + 192\phi'\phi''$$
$$- 33\xi\phi' - 12\xi^2\phi'' - 15\phi = 0. \tag{7.33}$$

The power series solution method is ideal for solving such complicated nonlinear differential equations (7.33). See, for example [25–27]. To begin, let the solution of (7.33) take the form

$$\phi(\xi) = \sum_{z=0}^{\infty} g_z \xi^z \tag{7.34}$$

with constants g_z, $z = 0, 1, 2, \cdots$, to be determined. The various derivatives of (7.34) are thus

$$\begin{aligned}
\phi'(\xi) &= \sum_{z=0}^{\infty}(z+1)g_{z+1}\xi^z, \\
\phi''(\xi) &= \sum_{z=0}^{\infty}(z+1)(z+2)g_{z+2}\xi^z, \\
\phi'''(\xi) &= \sum_{z=0}^{\infty}(z+1)(z+2)(z+3)g_{z+3}\xi^z, \\
\phi^{(4)}(\xi) &= \sum_{z=0}^{\infty}(z+1)(z+2)(z+3)(z+4)g_{z+4}\xi^z, \\
\phi^{(5)}(\xi) &= \sum_{z=0}^{\infty}(z+1)(z+2)(z+3)(z+4)(z+5)g_{z+5}\xi^z.
\end{aligned} \tag{7.35}$$

Substituting the results of (7.35) into (7.33) gives

$$4\sum_{z=1}^{\infty} z(z+1)(z+2)(z+3)(z+4)g_{z+4}\xi^z + 20\sum_{z=0}^{\infty}(z+1)(z+2)(z+3)$$
$$(z+4)g_{z+4}\xi^z + 48\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}(j+1)(z-j)(z+1-j)(z+2-j)g_{j+1}g_{z+2-j}\xi^z$$
$$- \sum_{z=3}^{\infty}(z-2)(z-1)zg_z\xi^z + 96\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}(j+1)(z-j)g_{j+1}g_{z-j}\xi^z$$
$$+ 96\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}(z-j)(z+1-j)g_j g_{z+1-j}\xi^z + 16\sum_{z=0}^{\infty}\sum_{j=0}^{z}(z+1-j) \tag{7.36}$$

$$(z+2-j)(z+3-j)g_j g_{z+3-j}\xi^z + 192\sum_{z=0}^{\infty}\sum_{j=0}^{z}(j+1)(z+1-j)$$
$$(z+2-j)g_{j+1}g_{z+2-j}\xi^z - 33\sum_{z=1}^{\infty} z g_z\xi^z - 12\sum_{z=2}^{\infty}(z-1)z g_z\xi^z$$
$$-15\sum_{z=0}^{\infty} g_z\xi^z = 0.$$

Equation (7.36) leads to

$$480g_5\xi + 2880g_6\xi^2 + 4\sum_{z=3}^{\infty} z(z+1)(z+2)(z+3)(z+4)g_{z+4}\xi^z + 480g_4$$
$$+ 2400g_5\xi + 7200g_6\xi^2 + 20\sum_{z=3}^{\infty}(z+1)(z+2)(z+3)(z+4)g_{z+4}\xi^z$$
$$+ 288g_1g_3\xi + 1152g_1g_4\xi^2 + 48\sum_{z=3}^{\infty}\sum_{j=0}^{z-1}(j+1)(z-j)(z+1-j)(z+2-j)$$
$$g_{j+1}g_{z+2-j}\xi^z - \sum_{z=3}^{\infty}(z-2)(z-1)z g_z\xi^z + 96g_1^2\xi + 192g_1g_2\xi^2$$
$$+ 96\sum_{z=3}^{\infty}\sum_{j=0}^{z-1}(j+1)(z-j)g_{j+1}g_{z-j}\xi^z + 192g_0g_2\xi + 576g_0g_3\xi^2 \qquad (7.37)$$
$$+ 96\sum_{z=3}^{\infty}\sum_{j=0}^{z-1}(z-j)(z+1-j)g_j g_{z+1-j}\xi^z + 96g_0g_3 + 384g_0g_4\xi$$
$$+ 960g_0g_5\xi^2 + 16\sum_{z=3}^{\infty}\sum_{j=0}^{z}(z+1-j)(z+2-j)(z+3-j)g_j g_{z+3-j}\xi^z$$
$$+ 384g_1g_2 + 1152g_1g_3\xi + 2304g_1g_4\xi^2 + 192\sum_{z=3}^{\infty}\sum_{j=0}^{z}(j+1)(z+1-j)$$
$$(z+2-j)g_{j+1}g_{z+2-j}\xi^z - 33g_1\xi - 66g_2\xi^2 - 33\sum_{z=3}^{\infty} z g_z\xi^z - 12g_2\xi^2$$
$$-12\sum_{z=3}^{\infty}(z-1)z g_z\xi^z - 15g_0 - 15g_1\xi - 15g_2\xi^2 - 15\sum_{z=3}^{\infty} g_z\xi^z = 0.$$

Now comparing coefficients of ξ in equation (7.37) we have, for $z = 0, 1, 2$:

$$96g_0g_3 + 384g_1g_2 - 15g_0 + 480g_4 = 0, \qquad (7.38)$$

$$192g_0g_2 + 384g_0g_4 + 96{g_1}^2 + 1440g_1g_3 - 48g_1 + 2880g_5 = 0, \tag{7.39}$$

$$576g_0g_3 + 960g_0g_5 + 192g_1g_2 + 3456g_1g_4 - 93g_2 + 10080g_6 = 0 \tag{7.40}$$

and generally for $z \geq 3$ we have the recurrence relation

$$\begin{aligned} g_{z+4} = &- \frac{1}{(4z+20)(z+1)(z+2)(z+3)(z+4)} \left\{ 48 \sum_{j=0}^{z-1} (j+1)(z-j) \right. \\ &(z+1-j)(z+2-j)g_{j+1}g_{z+2-j} - 33zg_z - 12(z-1)zg_z - 15g_m \\ &+96 \sum_{j=0}^{z-1} (j+1)(z-j)g_{j+1}g_{z-j} + 96 \sum_{j=0}^{z-1} (z-j)(z+1-j)g_jg_{z+1-j} \\ &+16 \sum_{j=0}^{m} (z+1-j)(z+2-j)(z+3-j)g_jg_{z+3-j} - (z-2)(z-1)zg_z \\ &\left. +192 \sum_{j=0}^{m} (j+1)(z+1-j)(z+2-j)g_{j+1}g_{z+2-j} \right\}. \end{aligned} \tag{7.41}$$

From equations (7.38)–(7.40) and for arbitrary $g_0, \cdots, g_3$, we have the following:

$$\begin{aligned} g_4 =& \frac{1}{32}g_0 - \frac{1}{5}\, g_0\, g_3 - \frac{4}{5}\, g_1\, g_2, \\ g_5 =& \frac{2}{75}{g_0}^2 g_3 - \frac{1}{15}\, g_0\, g_2 + \frac{8}{75}\, g_0\, g_1\, g_2 - \frac{1}{240}{g_0}^2 - \frac{1}{30}\, {g_1}^2 - \frac{1}{2}\, g_1\, g_3 + \frac{1}{60}g_1, \\ g_6 =& \frac{1}{2520}g_0^3 - \frac{4}{1575}g_3g_0^3 - \frac{16}{1575}g_1g_2g_0^2 + \frac{2}{315}g_2g_0^2 + \frac{1}{315}g_1^2g_0 - \frac{31}{2520}g_1g_0 \\ &+ \frac{61}{525}g_1g_3g_0 - \frac{2}{35}g_3g_0 + \frac{48}{175}g_1^2g_2 - \frac{2}{105}g_1g_2 + \frac{31}{3360}g_2. \end{aligned} \tag{7.42}$$

Using the recursion formula (7.41), successive terms g_m, $m = 7, 8, \cdots$, can be

determined uniquely. Thus, the power series solution of (7.33) can be written as

$$\begin{aligned}\phi(\xi) =& g_0 + g_1\xi + g_2\xi^2 + g_3\xi^3 + \left(\frac{1}{32}g_0 - \frac{1}{5}\,g_0\,g_3 - \frac{4}{5}\,g_1\,g_2\right)\xi^4 + \left(\frac{2}{75}{g_0}^2 g_3\right.\\ &-\frac{1}{15}\,g_0\,g_2 + \frac{8}{75}\,g_0\,g_1\,g_2 - \frac{1}{240}{g_0}^2 - \frac{1}{30}\,{g_1}^2 - \frac{1}{2}\,g_1\,g_3 + \left.\frac{1}{60}g_1\right)\xi^5\\ &+\left(\frac{1}{2520}g_0^3 - \frac{4}{1575}g_3g_0^3 - \frac{16}{1575}g_1g_2g_0^2 + \frac{2}{315}g_2g_0^2 + \frac{1}{315}g_1^2g_0 - \frac{31}{2520}g_1g_0\right.\\ &+\left.\frac{61}{525}g_1g_3g_0 - \frac{2}{35}g_3g_0 + \frac{48}{175}g_1^2g_2 - \frac{2}{105}g_1g_2 + \frac{31}{3360}g_2\right)\xi^6 + \sum_{z=3}^{\infty} g_{z+4}\xi^{z+4}.\end{aligned} \tag{7.43}$$

Finally, the solution of (7.2) is

$$\begin{aligned}u(t,x) =& g_0 + g_1\frac{x}{\sqrt{t}} + g_2\frac{x^2}{t} + g_3\frac{x^3}{t^{3/2}} + \left(\frac{1}{32}g_0 - \frac{1}{5}\,g_0\,g_3 - \frac{4}{5}\,g_1\,g_2\right)\frac{x^4}{t^2} + \left(\frac{2}{75}{g_0}^2 g_3\right.\\ &-\frac{1}{15}\,g_0\,g_2 + \frac{8}{75}\,g_0\,g_1\,g_2 - \frac{1}{240}{g_0}^2 - \frac{1}{30}\,{g_1}^2 - \frac{1}{2}\,g_1\,g_3 + \left.\frac{1}{60}g_1\right)\frac{x^5}{t^{5/2}}\\ &+\left(\frac{1}{2520}g_0^3 - \frac{4}{1575}g_3g_0^3 - \frac{16}{1575}g_1g_2g_0^2 + \frac{2}{315}g_2g_0^2 + \frac{1}{315}g_1^2g_0 - \frac{31}{2520}g_1g_0\right.\\ &+\left.\frac{61}{525}g_1g_3g_0 - \frac{2}{35}g_3g_0 + \frac{48}{175}g_1^2g_2 - \frac{2}{105}g_1g_2 + \frac{31}{3360}g_2\right)\frac{x^6}{t^3}\\ &-\sum_{z=3}^{\infty}\frac{(z+4)!}{(4z+20)z!}\left(48\sum_{j=0}^{z-1}(j+1)(z-j)(z+1-j)(z+2-j)\right.\\ &g_{j+1}g_{z+2-j} - 33zg_z - 12(z-1)zg_z - 15g_z + 96\sum_{j=0}^{z-1}(j+1)(z-j)\\ &g_{j+1}g_{z-j} + 96\sum_{j=0}^{z-1}(z-j)(z+1-j)g_jg_{z+1-j} + 16\sum_{j=0}^{z}(z+1-j)\\ &(z+2-j)(z+3-j)g_jg_{z+3-j} - (z-2)(z-1)zg_z\\ &\left.+192\sum_{j=0}^{z}(j+1)(z+1-j)(z+2-j)g_{j+1}g_{z+2-j}\right)\left(\frac{x}{\sqrt{t}}\right)^{z+4}.\end{aligned} \tag{7.44}$$

The graphical depiction of the behavioural pattern of solution (7.44) is given in Figure 7.6.

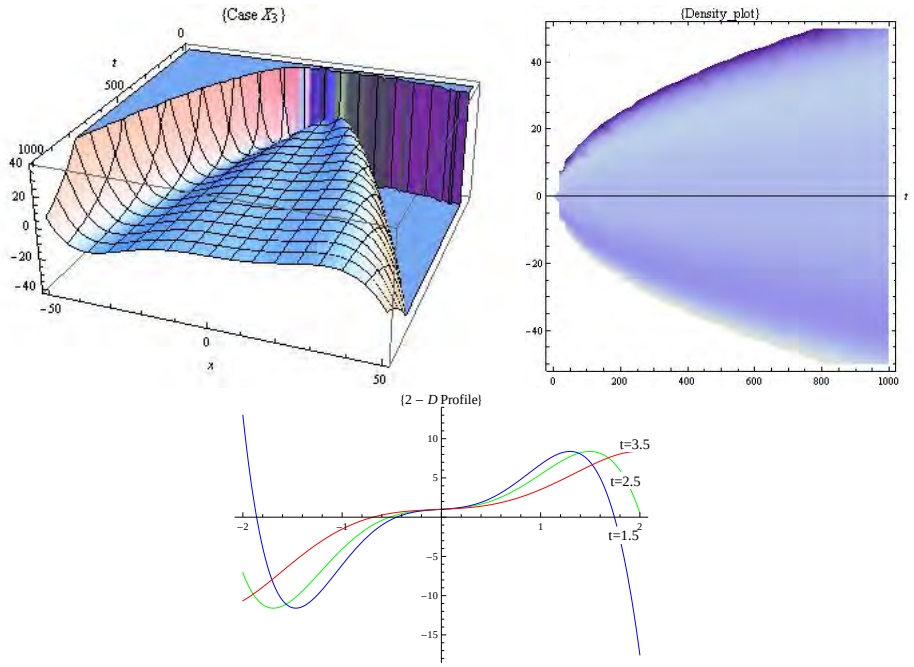


Figure 7.6: Depiction of cumulative partial sums up to T_6 for solution (7.44).

7.3 Conserved quantities of (7.2)

7.3.1 Noether's approach

We begin by applying the classical Noether's approach [33] to determine conserved quantities of (7.2). The Helmholtz conditions dictate that for a NLPDE to have a variational principle, it must, amongst other things have an even-order [43]. Equation (7.2) is of order five and thus cannot have a variational principle. To remedy this we introduce $u = v_x$ to transform (7.2) into the sixth-order NLPDE

$$F^* \equiv \beta v_{xxxx} - 4v_{tx}v_{xxxx} - 12v_{xxx}v_{txx} - 8v_{xx}v_{txxx} - v_{txxxxx} + v_{tttx} = 0. \qquad (7.45)$$

For the transformed equation (7.45) to have a variational principle, it must indeed satisfy the Helmholtz conditions

$$\frac{\partial F^*}{\partial(\partial^q v)} = (-1)^q (E_v^q(F^*))^t. \tag{7.46}$$

Here $q = 1, \cdots, Q$, where Q is the order of equation (7.45). Due to the presence of the transpose t, generally the number of field variables v should equal the number of components in system F^*, else the Helmholtz conditions do not hold and extension F^* would not possess a variational principle. Also, E_v^q represents higher Euler operators, which are given by

$$E_v^{(r)}(F^*) = \frac{\partial F^*}{\partial(\partial^r v)} - \binom{r+1}{r} D \cdot \frac{\partial F^*}{\partial(\partial^{r+1} v)} + \cdots + \binom{q}{r} (-D)^{q-r} \cdot \frac{\partial F^*}{\partial(\partial^r v)} \tag{7.47}$$

with $r = 1 \cdots q$ and D represents the total derivatives $D = (D_t, D_x)$. It can be shown with relative ease that a reckoning of (7.46) will confirm that equation (7.45) indeed has a variational principle. A Lagrangian can thus be recovered for (7.45) as

$$\begin{aligned} \mathcal{L} =& \frac{8}{3} v_{txx} \left(v v_{xxx} + v_x v_{xx} \right) + \frac{4}{3} v_{xxx} \left(v v_{txx} + v_x v_{tx} \right) + \frac{1}{2} \beta {v_{xx}}^2 \\ &+ \frac{1}{2} v_{xxx} v_{txx} + \frac{1}{2} v_{tt} v_{tx} - 4 v_{xxx} v v_{txx}. \end{aligned} \tag{7.48}$$

The Lagrangian (7.48) conforms with condition $\delta\mathcal{L}/\delta v = F^* = 0$, as expected.

In order to obtain the variational symmetries $Y = \xi^1 \partial/\partial t + \xi^2 \partial/\partial x + \eta \partial/\partial v$ corresponding to Lagrangian (7.48), we shall use the invariance condition

$$\mathrm{pr}^{(3)} Y(\mathcal{L}) + \mathcal{L} D_t \xi^1 + D_x \xi^2 = D_t B^1 + D_x B^2, \tag{7.49}$$

where $\mathrm{pr}^{(3)} Y$ is the third prolongation of Y obtainable from

$$\mathrm{pr}^{(3)} Y = Y + \zeta_x \frac{\partial}{\partial v_x} + \zeta_{tt} \frac{\partial}{\partial v_{tt}} + \zeta_{tx} \frac{\partial}{\partial v_{tx}} + \zeta_{xx} \frac{\partial}{\partial v_{xx}} + \zeta_{txx} \frac{\partial}{\partial v_{txx}} + \zeta_{xxx} \frac{\partial}{\partial v_{xxx}}.$$

Note that in (7.49), ξ^1, ξ^2, η, B^1 and B^2 are functions of (t, x, v). The invariance condition (7.49) yields the following system of twelve linear PDEs:

$$\begin{aligned} &\xi^1_t = 0,\ \xi^1_x = 0,\ \xi^1_v = 0,\ \xi^2_t = 0,\ \xi^2_x = 0,\ \xi^2_v = 0, \\ &\eta_{tt} = 0,\ \eta_x = 0,\ \eta_v = 0,\ B^1_v = 0,\ B^2_v = 0,\ B^1_t + B^2_x = 0. \end{aligned} \tag{7.50}$$

From system (7.50) we can readily infer without any tedious calculations that

$$\xi^1 = C_1,\ \xi^2 = C_2,\ \eta = C_3 t + C_4,\ B^1 = F(t, x),\ B^2 = H(t, x) + P(t)$$

with $H(t, x) = -\int F_t(t, x)dx$. The above generator coefficients lead us to the variational symmetries

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t},\ B^1 = B^2 = 0, \\ Y_2 &= \frac{\partial}{\partial x},\ B^1 = B^2 = 0, \\ Y_3 &= t\frac{\partial}{\partial v},\ B^1 = B^2 = 0, \\ Y_4 &= \frac{\partial}{\partial v},\ B^1 = B^2 = 0. \end{aligned} \tag{7.51}$$

Following the algorithm outlined in Section 1.5.2 we obtain the following nonlocal conserved quantities of equation (7.2):

$$\begin{aligned} T^t_1 = &\frac{1}{36}\left\{2\left(16u_t + 9\beta\right)u_x^2 + 8\left(8uu_{tx} - 9u_{xx}\left(\int u_t dx\right)\right)u_x - 8\left(u_{xxx}\int u_t dx\right.\right. \\ &\left.\left. -7u_t u_{xx}\right)u + 3\left(4u_{xx}u_{tx} + \left(2u_{xxx} - 3\int u_{tt}dx\right)u_t + \left(9u_{tt} - 2u_{xxx}\right)\int u_t dx\right)\right\}, \\ T^x_1 = &\frac{1}{36}\left\{4\left(\left(9\beta - 34u_t\right)u_{xx}\int u_t dx + 2u\int u_{tt}dx - 3u_{tt}\right) + 32u_x^2\int u_{tt}dx - 18{u_{tx}}^2\right. \\ &-4\left(9\beta u_t - 20{u_t}^2 + 56u_{tx}\int u_t dx + 16uu_{tt}\right)u_x - 64uu_t u_{tx} + 8uu_{txx}\int u_t dx \\ &+24u_t u_{txx} - 30u_{txxx}\int u_t dx + 6u_{xxx}\int u_{tt}dx - 9\left(\int u_{tt}dx\right)^2 \\ &\left.+9\left(\int u_t dx\right)\left(\int u_{ttt}dx\right)\right\}; \end{aligned}$$

$$
\begin{aligned}
T_2^t =& \frac{1}{36}\left\{\left(6u_{xxx} - 9\int u_{tt}dx - 96uu_{xx}\right)u_x - 18{u_t}^2 + 32{u_x}^3 - 6{u_{xx}}^2\right.\\
&\left. + \left(27u_{tt} - 8uu_{xxx} - 6u_{xxxx}\right)u\right\},\\
T_2^x =& \frac{1}{36}\left\{\left(4u_{xx}\left(9\beta - 32u_t\right) - 30u_{txxx} + 9\int u_{ttt}dx\right)u + 2\left(56u_t - 9\beta\right)u_x^2\right.\\
&\left. +8u^2u_{txx} - 12u_{xx}u_{tx} + 8u_x\left(3u_{txx} - 26uu_{tx}\right) + 3u_t\left(3\int u_{tt}dx + 2u_{xxx}\right)\right\};
\end{aligned}
$$

$$
\begin{aligned}
T_3^t =& \frac{1}{2}u_t + \frac{1}{36}t\left\{72u_xu_{xx} - 27u_{tt} + 8uu_{xxx} + 6u_{xxxx}\right\},\\
T_3^x =& \frac{1}{36}\left\{224tu_xu_{tx} - 4\left(2u + 9\beta t - 34tu_t\right)u_{xx} - 8tuu_{txx} + 30tu_{txxx}\right.\\
&\left. +9\int u_{tt}dx - 9t\int u_{ttt}dx - 32u_x^2 - 6u_{xxx}\right\};
\end{aligned}
$$

$$
\begin{aligned}
T_4^t =& 2u_xu_{xx} - \frac{3}{4}u_{tt} + \frac{2}{9}uu_{xxx} + \frac{1}{6}u_{xxxx},\\
T_4^x =& \frac{1}{36}\left\{4\left(34u_t - 9\beta\right)u_{xx} + 224u_xu_{tx} - 8uu_{txx} + 30u_{txxx} - 9\int u_{ttt}dx\right\}.
\end{aligned}
$$

7.3.2 Multiplier approach

One advantage of the multiplier approach is that it does not require the existence of a variational principle in order to obtain conserved quantities of differential equations [30]. The determining condition for a multiplier (1.47) will give us multipliers $\Lambda(t, x, u)$. The Euler operator (1.25), in this case, is given by

$$
\begin{aligned}
\frac{\delta}{\delta u} =& \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_tD_x\frac{\partial}{\partial u_{tx}} + D_x^2\frac{\partial}{\partial u_{xx}} - D_t^3\frac{\partial}{\partial u_{ttt}}\\
&- D_tD_x^2\frac{\partial}{\partial u_{txx}} - D_x^3\frac{\partial}{\partial u_{xxx}} - D_tD_x^4\frac{\partial}{\partial u_{txxxx}}
\end{aligned}
$$

and D_t, D_x are given by (1.4). From equation (1.47) and by following the procedure similar to the usual Lie symmetry algorithm, six multiplier determining equations

are realised, which are

$$\Lambda_{uu} = 0,\ \Lambda_{xu} = 0,\ \Lambda_{xx} = 0,\ \Lambda_{tu} = 0,\ \Lambda_{tx} = 0,\ \Lambda_{ttt} = 0. \tag{7.52}$$

Solving equations (7.52) one obtains the multipliers corresponding to equation (7.2) as

$$\Lambda = C_1 u + C_2 x + \frac{1}{2}C_3 t^2 + C_4 t + C_5, \tag{7.53}$$

where C_1,. . . ,C_5 are constants. We now employ the first homotopy integral formula (1.48) with equation (7.2) along with multipliers (7.53). We obtain

$$\begin{aligned} T =& C_2 u u_{tt} - 2C_1 u u_{xx} - 4C_2 u u_x u_{xx} + C_3 u - \frac{1}{2}C_2 u u_{xxxx} - \frac{1}{2}C_2 u_t^2 - C_3 t u_t - u_t C_4 \\ &+ C_1 x u_{tt} + \frac{1}{2}C_3 t^2 u_{tt} + C_4 t u_{tt} + C_5 u_{tt} \end{aligned}$$

and

$$\begin{aligned} X =& C_5\beta u_{xx} - C_1 x u_{txxx} - C_4 t u_{txxx} - C_1\beta u_x - \frac{1}{2}C_3 t^2 u_{txxx} + C_1\beta x u_{xx} + \frac{1}{2}C_3\beta t^2 u_{xx} \\ &+ C_4\beta t u_{xx} - \frac{1}{2}C_2 u u_{txxx} - 8C_5 u_x u_{tx} + 6C_1 u_t u_x - \frac{1}{2}C_2 u_{tx} u_{xx} - 4C_5 u_t u_{xx} \\ &+ \frac{1}{2}u_{txx} C_2 u_x + 2u u_{tx} C_1 + 4u_t C_2 u_x^2 + \frac{1}{2}C_2 u_t u_{xxx} - \frac{1}{2}C_2\beta u_x^2 - C_5 u_{txxx} + C_1 u_{txx} \\ &- 4C_2 u u_{xx} u_t - 4C_1 x u_t u_{xx} - 2C_3 t^2 u_t u_{xx} - 4C_4 t u_t u_{xx} - 8C_1 x u_x u_{tx} - 4C_3 t^2 u_x u_{tx} \\ &+ C_2\beta u u_{xx} - 8C_4 t u_x u_{tx} - 4C_2 u u_{tx} u_x \end{aligned}$$

and consequently, we have the following five conserved quantities of (7.2):

$$\begin{aligned} T_1^t =& x u_{tt} - 2u u_{xx}, \\ T_1^x =& u_{txx} + \beta x u_{xx} - 8x u_x u_{tx} - 4x u_t u_{xx} + 2u u_{tx} - \beta u_x + 6u_t u_x - x u_{txxx}; \end{aligned}$$

$$\begin{aligned} T_2^t =& u u_{tt} - \frac{1}{2}u u_{xxxx} - 4u u_x u_{xx} - \frac{1}{2}u_t^2, \\ T_2^x =& 4u_t {u_x}^2 - \frac{1}{2}u u_{txxx} + \frac{1}{2}u_x u_{txx} - \frac{1}{2}u_{xx} u_{tx} + \frac{1}{2}u_t u_{xxx} - 4u u_x u_{tx} + \beta u u_{xx} - 4u u_t u_{xx} \\ &- \frac{1}{2}\beta {u_x}^2; \end{aligned}$$

$$T_3^t = \frac{1}{2}t^2 u_{tt} + u - tu_t,$$
$$T_3^x = \frac{1}{2}\beta t^2 u_{xx} - \frac{1}{2}t^2 u_{txxx} - 2t^2 u_t u_{xx} - 4t^2 u_x u_{tx};$$

$$T_4^t = tu_{tt} - u_t,$$
$$T_4^x = \beta t u_{xx} - 8tu_x u_{tx} - 4tu_t u_{xx} - tu_{txxx};$$

$$T_5^t = u_{tt},$$
$$T_5^x = \beta u_{xx} - 4u_t u_{xx} - 8u_x u_{tx} - u_{txxx}.$$

Remarks As can be seen, the conserved vectors obtained here are first integrals of equation (7.2).

7.4 Concluding remarks

In this chapter, we performed an extensive study of a fifth-order integrable NLPDE (7.2), which was lately established in the literature and consisted of two dispersive terms. We obtained group transformations under which the equation (its solutions) remained invariant. Furthermore, we deduced an optimal system of one-dimensional subalgebras culminating in several group invariant solutions. This resulted in parabolic, trigonometric, hyperbolic, elliptic and power series solutions. The corresponding solution profiles depict topological kink soliton and periodic behaviour. Moreover, we investigated the existence of a variational principle in relation to Helmholtz conditions and went on to derive nonlocal conserved vectors corresponding to the variational principle obtained. Local and low-order conserved quantities were computed using a homotopy integral formula and multipliers.

Chapter 8

Analytic solutions and conserved quantities of the coupled complex modified Korteweg-de Vries equations of plasma physics

8.1 Introduction

The complex modified Korteweg-de Vries (cmKdV) equation [118] is given by

$$u_t + 6\epsilon|u|^2 u_x + u_{xxx} = 0,\ \epsilon \pm 1. \tag{8.1}$$

This equation is a model for weakly dispersive shallow water waves. Equation (8.1) is integrable and belongs to a hierarchy of nonlinear Schrödinger (NLS) equations. Several researchers have explored various aspects of the cmKdV equation. For instance, in [119] researchers managed to show that equation (8.1) is gauge equivalent to a spin equation by utilising gauge transformations. They also found exact soliton and periodic solutions via Darboux transformation. In [120], the existence

and orbital stability of travelling wave solutions of (8.1) was determined. Furthermore, breathers and localised solitons of the cmKdV equation were found in [121]. Of note however is that the cmKdV equation (8.1) was extended to the coupled cmKdV system (ccmKdV)

$$\begin{aligned} u_t + u_{xxx} + 6(|u|^2 + |v|^2)u_x = 0, \\ v_t + v_{xxx} + 6(|u|^2 + |v|^2)v_x = 0, \end{aligned} \tag{8.2}$$

which was seen for the first time in [122]. System (8.2) has been put forward as a nonlinear model for the propagation of plasma waves as well as transverse waves in a molecular chain model. The authors of [122] gave a binary Darboux transformation for the ccmKdV system (8.2). Furthermore, the authors manage to obtain several types of solutions which are in terms of trigonometric, hyperbolic, exponential and rational functions. The solution profiles, as expected are different forms of solitons. By introducing transformations $u(t,x) = p(t,x) + iq(t,x)$, $v(t,x) = r(t,x) + is(t,x)$ and a parameter generalisation, system (8.2) becomes a (1+1)-dimensional generalised coupled mKdV ((1+1)-dgcmKdV) system

$$\begin{aligned} E_1 &\equiv p_t + p_{xxx} + \alpha(p^2 + q^2 + r^2 + s^2)p_x = 0, \\ E_2 &\equiv q_t + q_{xxx} + \alpha(p^2 + q^2 + r^2 + s^2)q_x = 0, \\ E_3 &\equiv r_t + r_{xxx} + \alpha(p^2 + q^2 + r^2 + s^2)r_x = 0, \\ E_4 &\equiv s_t + s_{xxx} + \alpha(p^2 + q^2 + r^2 + s^2)s_x = 0, \end{aligned} \tag{8.3}$$

where α is a non-zero real constant. In this chapter, we study the (1+1)-dgcmKdV system (8.3). Firstly, using Lie symmetry analysis, we establish a one-parameter group of transformations of the solutions of this system. Moreover, we find group invariant solutions and utilise a simplest equation approach along with three differential equations, each satisfied by the copolar trio of elliptic functions cn, sn and dn to obtain analytic solutions. Furthermore, using the power series solution method we secure an extensive and unique set of solutions of the system under

study. Finally, we determine conserved quantities to which system (8.3) conforms by using the multiplier approach and provide an explicit form of the first homotopy integral formula.

The work presented in this chapter has been submitted for possible publication [123].

8.2 Lie algebra and one-parameter group transformations

The infinitesimal transformations corresponding to system (8.3) are given by [25]

$$\begin{aligned}
&\bar{t} \approx t + a\xi^1(t,x,p,q,r,s), \;\; \bar{x} \approx x + a\xi^2(t,x,p,q,r,s),\\
&\bar{p} \approx p + a\eta^1(t,x,p,q,r,s), \;\; \bar{q} \approx q + a\eta^2(t,x,p,q,r,s),\\
&\bar{r} \approx r + a\eta^3(t,x,p,q,r,s), \;\; \bar{s} \approx s + a\eta^4(t,x,p,q,r,s)
\end{aligned} \tag{8.4}$$

with $a \ll 1$.

An infinitesimal vector field of transformations (8.4) is given by

$$\begin{aligned}
X =& \xi^1(t,x,p,q,r,s)\frac{\partial}{\partial t} + \xi^2(t,x,p,q,r,s)\frac{\partial}{\partial x} + \eta^1(t,x,p,q,r,s)\frac{\partial}{\partial p}\\
&+ \eta^2(t,x,p,q,r,s)\frac{\partial}{\partial q} + \eta^3(t,x,p,q,r,s)\frac{\partial}{\partial r} + \eta^4(t,x,p,q,r,s)\frac{\partial}{\partial s}.
\end{aligned} \tag{8.5}$$

The third prolongation of vector field (8.5)

$$\begin{aligned}
\mathrm{pr}^{(3)}X =& X + \zeta^1_t\frac{\partial}{\partial p_t} + \zeta^1_x\frac{\partial}{\partial p_x} + \zeta^1_{xxx}\frac{\partial}{\partial p_{xxx}} + \zeta^2_t\frac{\partial}{\partial q_t} + \zeta^2_x\frac{\partial}{\partial q_x} + \zeta^2_{xxx}\frac{\partial}{\partial q_{xxx}}\\
&+ \zeta^3_t\frac{\partial}{\partial r_t} + \zeta^3_x\frac{\partial}{\partial r_x} + \zeta^3_{xxx}\frac{\partial}{\partial r_{xxx}} + \zeta^4_t\frac{\partial}{\partial s_t} + \zeta^4_x\frac{\partial}{\partial s_x} + \zeta^4_{xxx}\frac{\partial}{\partial s_{xxx}}
\end{aligned} \tag{8.6}$$

with prolongation coefficients (1.16) and (1.17). Equation (8.6) gives the determining equation for obtaining an r-dimensional Lie algebra associated with system

(8.3). Using Maple the following nine-dimensional Lie algebra L_9 is obtained:

$$\begin{aligned}
X_1 =& \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} - r\frac{\partial}{\partial r} - s\frac{\partial}{\partial s},\\
X_4 =& p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p},\ X_5 = q\frac{\partial}{\partial s} - s\frac{\partial}{\partial q},\ X_6 = p\frac{\partial}{\partial s} - s\frac{\partial}{\partial p},\\
X_7 =& q\frac{\partial}{\partial r} - r\frac{\partial}{\partial q},\ X_8 = p\frac{\partial}{\partial r} - r\frac{\partial}{\partial p},\ X_9 = r\frac{\partial}{\partial s} - s\frac{\partial}{\partial r}.
\end{aligned} \tag{8.7}$$

We now construct one-parameter group of transformations $G_1, \ldots, G_9$ generated by L_9 (8.7) by solving the Lie equations

$$\begin{aligned}
\frac{d\bar{t}}{da} =& \xi^1(\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}),\ \bar{t}|_{a=0} = t,\quad \frac{d\bar{x}}{da} = \xi^2(\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}),\ \bar{x}|_{a=0} = x,\\
\frac{d\bar{p}}{da} =& \eta^1(\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}),\ \bar{p}|_{a=0} = p,\quad \frac{d\bar{q}}{da} = \eta^2(\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}),\ \bar{q}|_{a=0} = q,\\
\frac{d\bar{r}}{da} =& \eta^3(\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}),\ \bar{r}|_{a=0} = r,\quad \frac{d\bar{s}}{da} = \eta^4(\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}),\ \bar{s}|_{a=0} = s.
\end{aligned} \tag{8.8}$$

Thus, we obtain

$$\begin{aligned}
&G_1 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t + a_1, x, p, q, r, s),\\
&G_2 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x + a_2, p, q, r, s),\\
&G_3 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (te^{3a_3}, xe^{a_3}, pe^{-a_3}, qe^{-a_3}, re^{-a_3}, se^{-a_3}),\\
&G_4 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x, p\cos a_4 - q\sin a_4, p\sin a_4 + q\cos a_4, r, s),\\
&G_5 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x, p, q\cos a_5 - s\sin a_5, r, q\sin a_5 + s\cos a_5),\\
&G_6 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x, p\cos a_6 - s\sin a_6, q, r, p\sin a_6 + s\cos a_6),\\
&G_7 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x, p, q\cos a_7 - r\sin a_7, q\sin a_7 + r\cos a_7, s),\\
&G_8 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x, p\cos a_8 - r\sin a_8, q, p\sin a_8 + r\cos a_8, s),\\
&G_9 : (\bar{t}, \bar{x}, \bar{p}, \bar{q}, \bar{r}, \bar{s}) \to (t, x, p, q, r\cos a_9 - s\sin a_9, r\sin a_9 + s\cos a_9).
\end{aligned}$$

Using the above results, we can now state the following theorem:

Theorem 8.1 *If $p = f(t,x), q = g(t,x), r = h(t,x)$ and $s = w(t,x)$ is a solution of (8.3), then the one-parameter family of solutions generated by group $G_1, \ldots, G_9$*

are:

$$G_{a_1} : p = f(t + a_1, x), q = g(t + a_1, x), r = h(t + a_1, x), s = w(t + a_1, x);$$

$$G_{a_2} : p = f(t, x + a_2), q = g(t, x + a_2), r = h(t, x + a_2), s = w(t, x + a_2);$$

$$G_{a_3} : p = f(te^{3a_3}, xe^{a_3})e^{a_3}, q = g(te^{3a_3}, xe^{a_3})e^{a_3},$$

$$G_{a_3} : r = h(te^{3a_3}, xe^{a_3})e^{a_3}, s) = w(te^{3a_3}, xe^{a_3})e^{a_3};$$

$$G_{a_4} : p = f(t, x)\sec a_4 + g(t, x)\tan a_4, q = g(t, x)\sec a_4 - f(t, x)\tan a_4,$$

$$G_{a_4} : r = h(t, x), s = w(t, x);$$

$$G_{a_5} : p = f(t, x), q(t, x) = g(t, x)\sec a_5 + w(t, x)\tan a_5,$$

$$G_{a_5} : r = h(t, x), s(t, x) = w(t, x)\sec a_5 - g(t, x)\tan a_5;$$

$$G_{a_6} : p = f(t, x)\sec a_6 + w(t, x)\tan a_6, q = g(t, x),$$

$$G_{a_6} : r = h(t, x), s = w(t, x)\sec a_6 - f(t, x)\tan a_6;$$

$$G_{a_7} : p = f(t, x), q = g(t, x)\sec a_7 + h(t, x)\tan a_7,$$

$$G_{a_7} : r = h(t, x)\sec a_7 - g(t, x)\tan a_7, s = w(t, x);$$

$$G_{a_8} : p = f(t, x)\sec a_8 + h(t, x)\tan a_8, q = g(t, x),$$

$$G_{a_8} : r = h(t, x)\sec a_8 - f(t, x)\tan a_8, s = w(t, x);$$

$$G_{a_9} : p = f(t, x), q = g(t, x),$$

$$G_{a_9} : r = h(t, x)\sec a_9 + s(t, x)\tan a_9, s = s(t, x)\sec a_9 - h(t, x)\tan a_9.$$

8.3 Analytic solutions

We now seek to derive exact solutions of system (8.3) by making use of the time translation symmetry X_1, space translation symmetry X_2 and scaling symmetry X_3. Group invariant solutions [93] for the rotation symmetries $X_4, \ldots, X_9$ cannot be recovered unless perhaps by taking linear combinations of X_1 and/or X_2 with any of $X_4, \ldots, X_9$. In this work, we will only explore exact solutions emanating

from symmetries X_1, X_2 and X_3 as already alluded to.

8.3.1 Case $X_1 + cX_2$

Using Lie theory this linear combination gives the group invariant (travelling wave) solutions

$$p = w(\xi),\ q = h(\xi),\ r = g(\xi),\ s = f(\xi),\ \xi = x - ct, \tag{8.9}$$

where c is the speed of the wave. With (8.9) we transform system (8.3) into the system of nonlinear ordinary differential equations (NLODEs)

$$\begin{aligned} &w''' - cw' + \alpha(w^2 + h^2 + g^2 + f^2)w' = 0,\\ &h''' - ch' + \alpha(w^2 + h^2 + g^2 + f^2)h' = 0,\\ &g''' - cg' + \alpha(w^2 + h^2 + g^2 + f^2)g' = 0,\\ &f''' - cf' + \alpha(w^2 + h^2 + g^2 + f^2)f' = 0. \end{aligned} \tag{8.10}$$

We seek to obtain solutions of system (8.10) in terms of the main copolar trio of Jacobian elliptic functions, that is, elliptic cosine $\mathrm{cn}(\xi|\omega)$, elliptic sine $\mathrm{sn}(\xi|\omega)$ and delta amplitude $\mathrm{dn}(\xi|\omega)$. Here ω represents a parameter such that $0 \leq \omega \leq 1$. All remaining nine Jacobian functions can be written in terms of at least one of the elliptic functions in this copolar trio. See for example [66, 67]. What makes Jacobi elliptic functions interesting is that they degenerate to trigonometric, hyperbolic and by extension exponential functions depending on the value of parameter ω, see Table 8.3.1.

Table 8.3.1. Copolar trio for $\omega = 0$ and $\omega = 1$

	$\omega = 0$	$\omega = 1$
$\mathrm{sn}(\xi\|\omega)$	$\sin\omega$	$\tanh\omega$
$\mathrm{cn}(\xi\|\omega)$	$\cos\omega$	$\mathrm{sech}\,\omega$
$\mathrm{dn}(\xi\|\omega)$	1	$\mathrm{sech}\,\omega$

The elliptic functions $\mathrm{cn}(\xi|\omega)$, $\mathrm{sn}(\xi|\omega)$ and $\mathrm{dn}(\xi|\omega)$ are solutions of the following NLODEs:

$$\phi(\xi) = -\left\{(1-\phi^2(\xi))(1-\omega+\omega\phi^2(\xi))\right\}^{1/2}, \tag{8.11}$$

$$\phi(\xi) = \left\{(1-\phi^2(\xi))(1-\omega\phi^2(\xi))\right\}^{1/2}, \tag{8.12}$$

$$\phi(\xi) = -\left\{(1-\phi^2(\xi))(\phi^2(\xi)+\omega-1)\right\}^{1/2}, \tag{8.13}$$

respectively. Suppose system (8.3) has solutions of the form

$$f(\xi) = \sum_{i=0}^{M} A_i\phi(\xi),\ g(\xi) = \sum_{i=0}^{M} B_i\phi(\xi),\ h(\xi) = \sum_{i=0}^{M} C_i\phi(\xi),\ w(\xi) = \sum_{i=0}^{M} D_i\phi(\xi). \tag{8.14}$$

Here A_i, B_i, C_i and D_i are constants to be determined. For system (8.10), $M = 1$ and is determined by the balancing procedure. Thus, (8.14) becomes

$$\begin{aligned} f(\xi) =& A_0 + A_1\phi(\xi),\ g(\xi) = B_0 + B_1\phi(\xi), \\ h(\xi) =& C_0 + C_1\phi(\xi),\ w(\xi) = D_0 + D_1\phi(\xi). \end{aligned} \tag{8.15}$$

8.3.1.1 Elliptic cosine solutions

Periodic solutions

Substituting (8.15) along with (8.11) into NLODE system (8.10), we get a system

of four algebraic equations in terms of $\phi(\xi)$. We can further split this system by collecting coefficients of powers of $\phi(\xi)$ leading to a system of twenty one equations in terms of parameters A_i, B_i, C_i D_i, c, α and ω. Using Maple we get the following solution

$$\begin{aligned}
c =& \frac{1}{3{A_0}^2}\left(3\,\alpha\,{A_0}^4 + 3\,\alpha\,{B_0}^2{A_0}^2 + \alpha\,{B_1}^2{A_0}^2 + 3\,\alpha\,{C_0}^2{A_0}^2 + \alpha\,{C_1}^2{A_0}^2 + 3\,\alpha\,{D_0}^2{A_0}^2\right.\\
&+\alpha\,{D_1}^2{A_0}^2 + \alpha\,{B_0}^2{B_1}^2 + 2\,\alpha\,B_0B_1C_0C_1 + 2\,\alpha\,B_0B_1D_0D_1 + \alpha\,{C_0}^2{C_1}^2 - 3\,{A_0}^2\\
&\left.+2\,\alpha\,C_0C_1D_0D_1 + \alpha\,{D_0}^2{D_1}^2\right),\\
\omega =& \frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right),\ A_1 = -\frac{1}{A_0}\left(B_0B_1 + C_0C_1 + D_0D_1\right),\\
\alpha =& \alpha,\ A_0 = A_0,\ B_0 = B_0,\ B_1 = B_1,\ C_0 = C_0,\ C_1 = C_1,\ D_0 = D_0,\ D_1 = D_1.
\end{aligned}$$

Thus for $\phi(\xi) = \mathrm{cn}(\xi|\omega)$ we have the solution

$$p(t,x) = D_0 + D_1\mathrm{cn}\left(x - c\,t\,\middle|\,\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right), \tag{8.16}$$

$$q(t,x) = C_0 + C_1\mathrm{cn}\left(x - c\,t\,\middle|\,\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right), \tag{8.17}$$

$$r(t,x) = B_0 + B_1\mathrm{cn}\left(x - c\,t\,\middle|\,\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right), \tag{8.18}$$

$$s(t,x) = \frac{1}{A_0}\left\{A_0^2 - \left(B_0B_1 + C_0C_1 + D_0D_1\right)\mathrm{cn}\left(x - c\,t\,\middle|\,\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right)\right\}, \tag{8.19}$$

For brevity we will only pay attention to the graphical depiction of solution (8.19), which is provided in Figure 8.1.

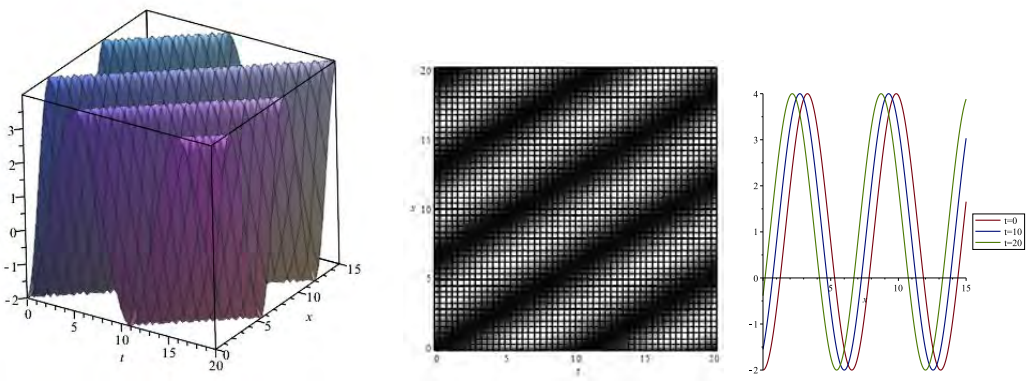


Figure 8.1: Periodic wave, a depiction of solution (8.19) with parameter values $A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, \alpha = 0.2, D_0 = 1, D_1 = 1$.

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The periodic nature of solution (8.19) is aptly depicted in Figure 8.1 and is typical behaviour of an elliptic cosine function. In this instance, the resultant value of ω corresponding to the selected parameter values is $\omega = 0.4$. As evidenced by the two-dimensional profile in Figure 8.1, taken at $t = 0$, $t = 10$ and $t = 20$, the periodic transverse wave propagates with constant amplitude and frequency.

Bell-shaped dark solitons

Maintaining the parameter values in Figure 8.1 but replacing $\alpha = 0.2$ with $\alpha = 0.5$ leads to $\omega = 1$ and solutions (8.16)–(8.19) degenerate to

$$p(t,x) = D_0 + D_1 \text{sech}(x - ct), \tag{8.20}$$

$$q(t,x) = C_0 + C_1 \text{sech}(x - ct), \tag{8.21}$$

$$r(t,x) = B_0 + B_1 \text{sech}(x - ct), \tag{8.22}$$

$$s(t,x) = \frac{1}{A_0}\left\{A_0^2 - (B_0B_1 + C_0C_1 + D_0D_1)\,\text{sech}(x - ct)\right\}. \tag{8.23}$$

A bell-shaped dark soliton profile of solution (8.23), typical of the hyperbolic secant is observed in Figure 8.2.

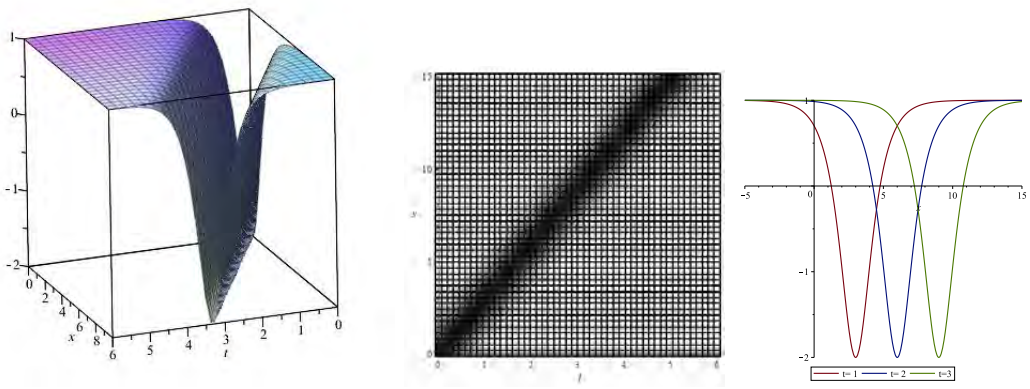

Figure 8.2: A depiction of dark soliton solution (8.23) with parameter values $A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, \alpha = 0.5, D_0 = 1, D_1 = 1$.

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Periodic sine wave solutions

Although $\alpha \neq 0$, it should be noted that as $\alpha \to 0$ then $\omega \to 0$ and solutions (8.16)–(8.19) begin to exhibit an increasingly sinusoidal behaviour, that is,

$$p(t,x) = D_0 + D_1 \sin(x - ct), \tag{8.24}$$

$$q(t,x) = C_0 + C_1 \sin(x - ct), \tag{8.25}$$

$$r(t,x) = B_0 + B_1 \sin(x - ct), \tag{8.26}$$

$$s(t,x) = \frac{1}{A_0}\left\{A_0^2 - (B_0B_1 + C_0C_1 + D_0D_1)\sin(x - ct)\right\}, \tag{8.27}$$

with a typical sine wave-like deportment as shown in Figure 8.3.

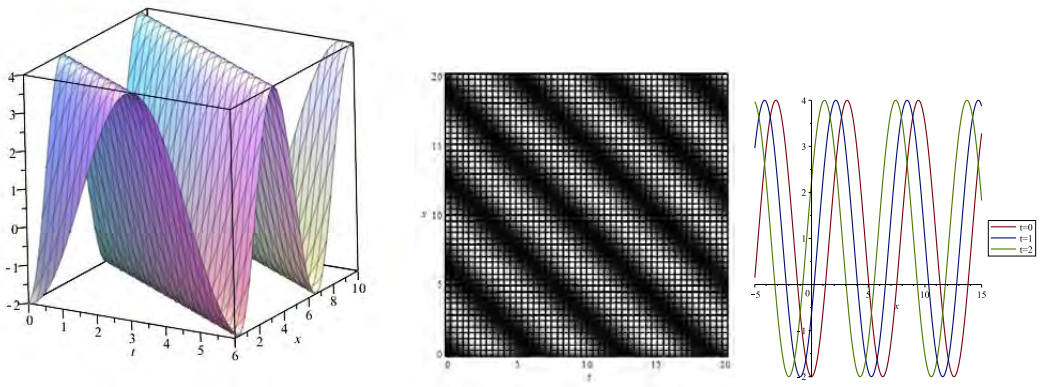

Figure 8.3: Periodic wave solution (8.27) with parameter values
$A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, \alpha \to 0, D_0 = 1, D_1 = 1.$

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8.3.1.2 Elliptic sine solutions

Periodic wave solutions

Elliptic sine solutions can be obtained by substituting equations (8.15) along with NLODE (8.12) into (8.10). The resultant parameter values are

$$\begin{aligned} c =& \frac{1}{6A_0^2}\left(6\,\alpha\,{A_0}^4 + 6\,\alpha\,{A_0}^2{B_0}^2 + \alpha\,{A_0}^2{B_1}^2 + 6\,\alpha\,{A_0}^2{C_0}^2 + \alpha\,{A_0}^2{C_1}^2 + 6\,\alpha\,{A_0}^2{D_0}^2\right. \\ & +\alpha\,{A_0}^2{D_1}^2 + \alpha\,{B_0}^2{B_1}^2 + 2\,\alpha\,B_0B_1C_0C_1 + 2\,\alpha\,B_0B_1D_0D_1 + \alpha\,{C_0}^2{C_1}^2 \\ & \left.+2\,\alpha\,C_0C_1D_0D_1 + \alpha\,{D_0}^2{D_1}^2 - 6\,{A_0}^2\right), \\ \omega =& -\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right),\ A_1 = -\frac{1}{A_0}\left(B_0B_1 + C_0C_1 + D_0D_1\right), \\ \alpha =& \alpha,\ A_0 = A_0,\ B_0 = B_0,\ B_1 = B_1,\ C_0 = C_0,\ C_1 = C_1,\ D_0 = D_0,\ D_1 = D_1, \end{aligned}$$

and consequently the set of solutions satisfying system (8.3) are given by

$$p(t,x) = D_0 + D_1\mathrm{sn}\left(x - c\,t\,\middle|-\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right), \tag{8.28}$$

$$q(t,x) = C_0 + C_1\mathrm{sn}\left(x - c\,t\,\middle|-\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right), \tag{8.29}$$

$$r(t,x) = B_0 + B_1 \mathrm{sn}\left(x - c\,t \,\middle|\, -\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right), \tag{8.30}$$

$$s(t,x) = \frac{1}{A_0}\left\{A_0^2 - (B_0B_1 + C_0C_1 + D_0D_1)\right.$$
$$\left.\times \,\mathrm{sn}\left(x - c\,t \,\middle|\, -\frac{1}{6}\,\alpha\left({A_1}^2 + {B_1}^2 + {C_1}^2 + {D_1}^2\right)\right)\right\}. \tag{8.31}$$

Obviously the restriction $\alpha < 0$ is necessary for solutions (8.28)–(8.31) to hold. For brevity we will only explore various aspects of (8.31).

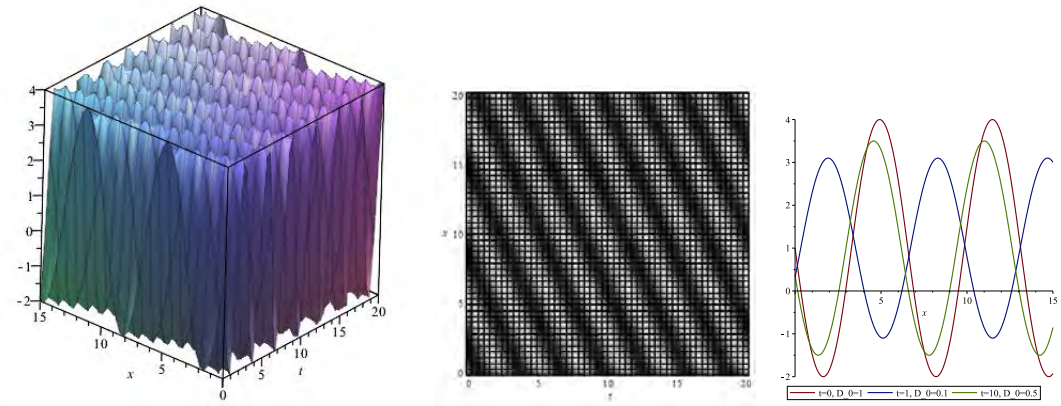

Figure 8.4: Periodic wave solution (8.31) with parameter values
$A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, \alpha = -0.2, D_0 = 1, D_1 = 1.$

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Figure 8.4 depicts a periodic profile of the elliptic sine solution (8.31). Variation of parameter D_0 for $0 \leq D_0 \leq 1$ directly leads to vertical translation as seen in the two-dimensional profile of Figure 10.4. In this instance the value of parameter $\omega = 0.4$.

Topological kink and anti-kink solitons

Hyperbolic tangent soliton solutions are recovered when $\alpha = -0.5$ since this leads to $\omega = 1$ provided the parameter values given in Figure 10.4 remain unchanged. Thus, solutions (8.28) –(8.31) become

$$p(t,x) = D_0 + D_1 \tanh(x - c\,t), \tag{8.32}$$

$$q(t,x) = C_0 + C_1 \tanh(x - ct), \tag{8.33}$$

$$r(t,x) = B_0 + B_1 \tanh(x - ct), \tag{8.34}$$

$$s(t,x) = \frac{1}{A_0}\left\{A_0^2 - (B_0B_1 + C_0C_1 + D_0D_1)\tanh(x - ct)\right\}. \tag{8.35}$$

The corresponding solution profiles for (8.35) are provided in Figure 8.5.

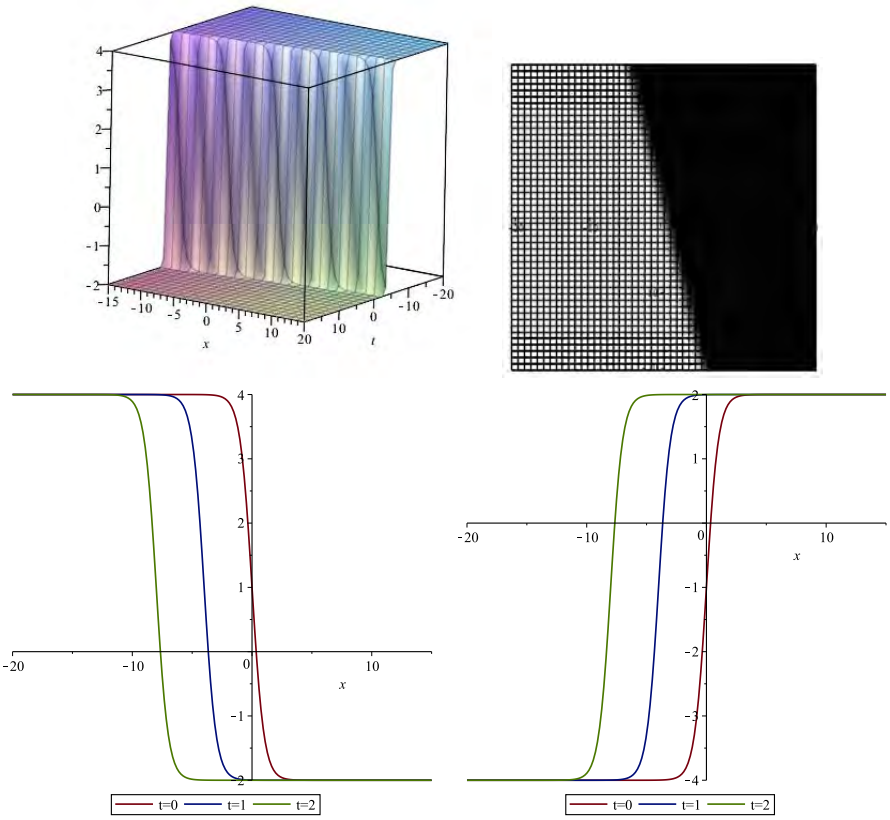

Figure 8.5: Topological kink and anti-kink soliton profiles of (8.31) with parameter values $A_0 = \pm 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, \alpha = -0.5, D_0 = 1, D_1 = 1$.

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A topological kink soliton profile is observable, for instance, when $A_0 = 1$ while an anti-kink profile can be seen when $A_0 = -1$.

Periodic cosine solutions

Periodic cosine solutions can be recovered by considering the case $\alpha \to 0$ resulting in the trigonometric solution

$$p(t,x) = D_0 + D_1 \cos(x - c\,t), \tag{8.36}$$

$$q(t,x) = C_0 + C_1 \cos(x - c\,t), \tag{8.37}$$

$$r(t,x) = B_0 + B_1 \cos(x - c\,t), \tag{8.38}$$

$$s(t,x) = \frac{1}{A_0}\left\{A_0^2 - (B_0 B_1 + C_0 C_1 + D_0 D_1)\cos(x - c\,t)\right\}. \tag{8.39}$$

The resultant solution profile is depicted by the Figure 8.6.

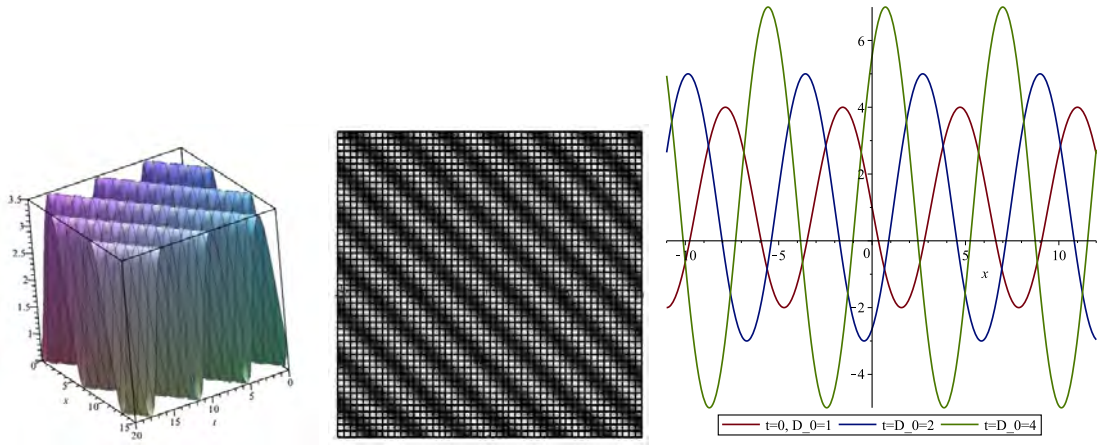

Figure 8.6: Periodic wave profiles of (8.35) with parameter values $A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, \alpha = -0.5, D_0 \in (1,2,4), D_1 = 1$.

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As can be seen in Figure 8.6, an increase in at least one of the parameters A_i, ..., D_i leads to a commensurate increase in wave amplitude, while period and frequency remain unchanged.

8.3.1.3 Delta amplitude solutions

Periodic wave solutions

Likewise, we obtain delta amplitude solutions of system (8.3) by utilising (8.15) together with (8.13) and obtain parameter values

$$\begin{aligned}
\omega =&\alpha\left(A_0^2+B_0^2+C_0^2+D_0^2\right)-c+2,\\
\alpha =&\alpha,\ c=c,\ A_0=A_0,\ B_0=B_0,\ C_0=C_0,\ C_1=C_1,\ D_0=D_0,\ D_1=D_1,\\
A_1 =&-\frac{\alpha(A_0^2+B_0^2)(C_0C_1+D_0D_1)-B_0(\alpha B_0C_0C_1+\alpha B_0D_0D_1-\sqrt{\Delta})}{\alpha A_0(A_0^2+B_0^2)},\\
B_1 =&-\frac{B_0}{\alpha A_0(A_0^2+B_0^2)}\left(\alpha B_0C_0C_1+\alpha B_0D_0D_1-\sqrt{\Delta}\right),\\
\Delta =&\,6\,\alpha {A_0}^2\left({A_0}^2+{B_0}^2\right)-\alpha^2{A_0}^4{C_1}^2-\alpha^2{A_0}^4{D_1}^2-\alpha^2{A_0}^2{B_0}^2{C_1}^2-\alpha^2{A_0}^2{B_0}^2{D_1}^2\\
&-\alpha^2{A_0}^2{C_0}^2{C_1}^2-2\,\alpha^2{A_0}^2C_0C_1D_0D_1-\alpha^2{A_0}^2{D_0}^2{D_1}^2.
\end{aligned}$$

For certain α such that $0<\alpha<1$, we can ensure that the parameter values hold. Consequently, we have the delta amplitude solutions

$$p(t,x)=D_0+D_1\mathrm{dn}\left(x-c\,t\,\middle|\alpha\left(A_0^2+B_0^2+C_0^2+D_0^2\right)-c+2\right), \tag{8.40}$$

$$q(t,x)=C_0+C_1\mathrm{dn}\left(x-c\,t\,\middle|\alpha\left(A_0^2+B_0^2+C_0^2+D_0^2\right)-c+2\right), \tag{8.41}$$

$$\begin{aligned}
r(t,x)=&B_0\left\{1-\frac{1}{\alpha A_0(A_0^2+B_0^2)}\left(\alpha B_0C_0C_1+\alpha B_0D_0D_1-\sqrt{\Delta}\right)\right.\\
&\left.\times\mathrm{dn}\left(x-c\,t\,\middle|\alpha\left(A_0^2+B_0^2+C_0^2+D_0^2\right)-c+2\right)\right\},
\end{aligned} \tag{8.42}$$

$$\begin{aligned}
s(t,x)=&\frac{1}{A_0}\left\{A_0^2-\left\{\frac{\alpha\left(A_0^2+B_0^2\right)\left(C_0C_1+D_0D_1\right)-B_0\left(\alpha B_0C_0C_1+\alpha B_0D_0D_1\right)}{\alpha(A_0^2+B_0^2)}\right.\right.\\
&\left.\left.+\frac{B_0}{\alpha(A_0^2+B_0^2)}\sqrt{\Delta}\right\}\times\ \mathrm{dn}\left(x-c\,t\,\middle|\alpha\left(A_0^2+B_0^2+C_0^2+D_0^2\right)-c+2\right)\right\}.
\end{aligned} \tag{8.43}$$

Not surprisingly, solutions (8.40)–(8.43) are periodic in nature. Let us explore, say, solution (8.42). For example, when $\alpha=0.5, c=3.5, A_0=1, B_0=1, B_1=1, C_0=1, C_1=1, D_0=1, D_1=1$, we have the periodic wave profile shown in Figure 8.7.

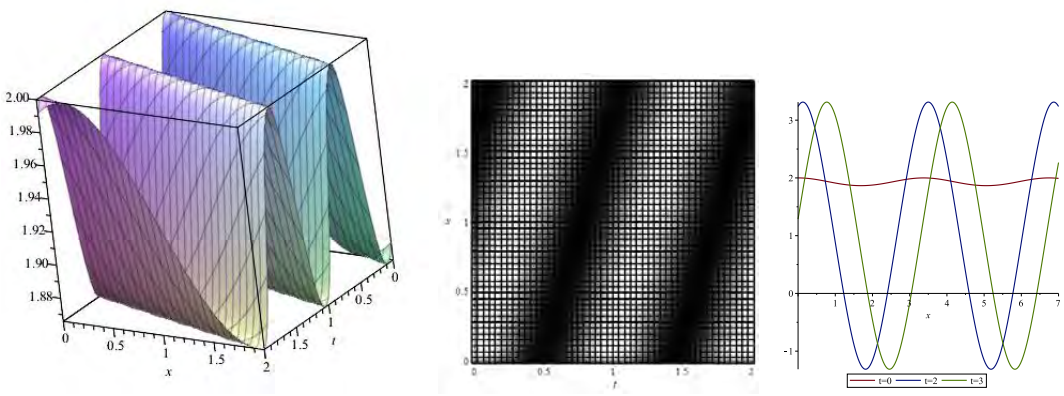

Figure 8.7: Periodic wave profiles of (8.43) with parameter values $\alpha = 0.5, c = 3.5, A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, D_0 = 1, D_1 = 1$.

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Bell-shaped bright solitons

If we take the wave speed, say $c = 3$, we have hyperbolic secant solutions akin to (8.40)–(8.43). For example, solution (8.42) becomes

$$r(t,x) = B_0 \left\{ 1 - \frac{1}{\alpha A_0 (A_0^2 + B_0^2)} \left(\alpha B_0 C_0 C_1 + \alpha B_0 D_0 D_1 - \sqrt{\Delta} \right) \operatorname{sech}(x - c\,t) \right\} \tag{8.44}$$

and is depicted in Figure 8.8.

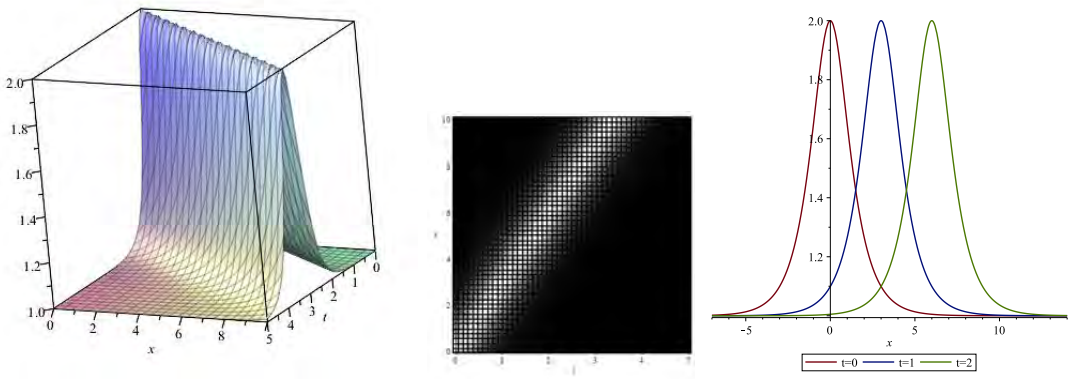

Figure 8.8: Bell-shaped bright soliton profile of (8.44) with parameter values $\alpha = 0.5, c = 3, A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, D_0 = 1, D_1 = 1$.

Constant solution

A constant is often part of the set of solutions of a system of NLPDEs and system (8.3) is not an exception. Of interest, however, is that solutions (8.40)–(8.43) degenerate to arbitrary constants for the non-unique instance $c = 4$ and $\alpha = 0.5, c = 4, A_0 = 1, B_0 = 1, B_1 = 1, C_0 = 1, C_1 = 1, D_0 = 1, D_1 = 1$, which gives $\omega = 0$. See Table 8.3.1.

8.3.2 Case X_3

From the symmetry generator X_3, we have the group invariant solution

$$p = t^{-1/3}w(\xi),\ q = t^{-1/3}h(\xi),\ r = t^{-1/3}g(\xi),\ s = t^{-1/3}f(\xi),\ \xi = \frac{x}{t^{1/3}}, \tag{8.45}$$

which transforms system (8.3) into

$$\begin{aligned}
&3\alpha f^2 f' + 3\alpha w^2 f' + 3\alpha h^2 f' + 3\alpha g^2 f' - f - \xi f' + 3f''' = 0,\\
&3\alpha f^2 g' + 3\alpha w^2 g' + 3\alpha h^2 g' + 3\alpha g^2 g' - g - \xi g' + 3g''' = 0,\\
&3\alpha f^2 h' + 3\alpha w^2 h' + 3\alpha h^2 h' + 3\alpha g^2 h' - h - \xi h' + 3h''' = 0,\\
&3\alpha f^2 w' + 3\alpha w^2 w' + 3\alpha h^2 w' + 3\alpha g^2 w' - w - \xi w' + 3w''' = 0.
\end{aligned} \tag{8.46}$$

We now use the power series method [77] to obtain analytic solutions of NLODE system (8.46). Let the solutions of system (8.46) take the form

$$\begin{aligned} f(\xi) &= \sum_{z=0}^{\infty} a_z \xi^z, \ g(\xi) = \sum_{z=0}^{\infty} b_z \xi^z, \\ h(\xi) &= \sum_{z=0}^{\infty} c_z \xi^z, \ w(\xi) = \sum_{z=0}^{\infty} d_z \xi^z \end{aligned} \tag{8.47}$$

with constants $a_z, b_z, c_z, d_z,\ z = 0, 1, 2, \ldots$ to be determined. Substituting (8.47) into (8.46), we have

$$\begin{aligned} &3\alpha(a_0^2 a_1 + b_0^2 a_1 + c_0^2 a_1 + d_0^2 a_1) - a_0 + 18a_3 + 3\sum_{z=1}^{\infty}(z+1)(z+2)(z+3)b_{z+3}\xi^z \\ &+ 3\alpha\left(\sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_i a_{k-i} a_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_i b_{k-i} a_{z-k+1}\right. \\ &\left.+ \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_i c_{k-i} a_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_i d_{k-i} a_{z-k+1}\right)\xi^z \\ &- \sum_{z=1}^{\infty} z a_z \xi^z - \sum_{z=1}^{\infty} a_z \xi^z = 0, \end{aligned} \tag{8.48}$$

$$\begin{aligned} &3\alpha(a_0^2 b_1 + b_0^2 b_1 + c_0^2 b_1 + d_0^2 d_1) - b_0 + 18b_3 + 3\sum_{z=1}^{\infty}(z+1)(z+2)(z+3)b_{z+3}\xi^z \\ &+ 3\alpha\left(\sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_i a_{k-i} b_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_i b_{k-i} b_{z-k+1}\right. \\ &\left.+ \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_i c_{k-i} b_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_i d_{k-i} b_{z-k+1}\right)\xi^z \\ &- \sum_{z=1}^{\infty} z b_z \xi^z - \sum_{z=1}^{\infty} b_z \xi^z = 0, \end{aligned} \tag{8.49}$$

$$\begin{aligned} &3\alpha(a_0^2 c_1 + b_0^2 c_1 + c_0^2 c_1 + d_0^2 c_1) - c_0 + 18c_3 + 3\sum_{z=1}^{\infty}(z+1)(z+2)(z+3)c_{z+3}\xi^z \\ &+ 3\alpha\left(\sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_i a_{k-i} c_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_i b_{k-i} c_{z-k+1}\right. \\ &\left.+ \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_i c_{k-i} c_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_i d_{k-i} c_{z-k+1}\right)\xi^z \end{aligned}$$

$$-\sum_{z=1}^{\infty} z c_z \xi^z - \sum_{z=1}^{\infty} c_z \xi^z = 0, \tag{8.50}$$

$$3\alpha(a_0^2 d_1 + b_0^2 d_1 + c_0^2 d_1 + d_0^2 d_1) - d_0 + 18 d_3 + 3\sum_{z=1}^{\infty}(z+1)(z+2)(z+3) d_{z+3}\xi^z$$
$$+ 3\alpha\left(\sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1) a_i a_{k-i} d_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1) b_i b_{k-i} d_{z-k+1}\right.$$
$$\left.+\sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1) c_i c_{k-i} d_{z-k+1} + \sum_{z=1}^{\infty}\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1) d_i d_{k-i} d_{z-k+1}\right)\xi^z$$
$$-\sum_{z=1}^{\infty} z d_z \xi^z - \sum_{z=1}^{\infty} d_z \xi^z = 0. \tag{8.51}$$

From equations (8.48)–(8.51) and for arbitrary a_i, b_i, c_i, d_i, $i = 0, 1, 2$, we have

$$\begin{aligned}
a_3 =& \frac{1}{18}\left(a_0 - 3\alpha(a_0^2 a_1 + b_0^2 a_1 + c_0^2 a_1 + d_0^2 a_1)\right), \\
b_3 =& \frac{1}{18}\left(b_0 - 3\alpha(a_0^2 b_1 + b_0^2 b_1 + c_0^2 b_1 + d_0^2 b_1)\right), \\
c_3 =& \frac{1}{18}\left(c_0 - 3\alpha(a_0^2 c_1 + b_0^2 c_1 + c_0^2 c_1 + d_0^2 c_1)\right), \\
d_3 =& \frac{1}{18}\left(d_0 - 3\alpha(a_0^2 d_1 + b_0^2 d_1 + c_0^2 d_1 + d_0^2 d_1)\right),
\end{aligned}$$

respectively, and the following recursion formulae:

$$a_{z+3} = \frac{1}{3(z+1)(z+2)(z+3)} \left\{ a_z(z+1) - 3\alpha \left(\sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) a_i a_{k-i} a_{z-k+1} \right.\right.$$
$$+ \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) b_i b_{k-i} a_{z-k+1} + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) c_i c_{k-i} a_{z-k+1}$$
$$\left.\left. + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) d_i d_{k-i} a_{z-k+1} \right) \right\},$$
$$b_{z+3} = \frac{1}{3(z+1)(z+2)(z+3)} \left\{ b_z(z+1) - 3\alpha \left(\sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) a_i a_{k-i} b_{z-k+1} \right.\right.$$
$$+ \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) b_i b_{k-i} b_{z-k+1} + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) c_i c_{k-i} b_{z-k+1}$$
$$\left.\left. + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) d_i d_{k-i} b_{z-k+1} \right) \right\},$$
$$c_{z+3} = \frac{1}{3(z+1)(z+2)(z+3)} \left\{ c_z(z+1) - 3\alpha \left(\sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) a_i a_{k-i} c_{z-k+1} \right.\right.$$
$$+ \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) b_i b_{k-i} c_{z-k+1} + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) c_i c_{k-i} c_{z-k+1}$$
$$\left.\left. + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) d_i d_{k-i} c_{z-k+1} \right) \right\},$$
$$d_{z+3} = \frac{1}{3(z+1)(z+2)(z+3)} \left\{ d_z(z+1) - 3\alpha \left(\sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) a_i a_{k-i} d_{z-k+1} \right.\right.$$
$$+ \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) b_i b_{k-i} d_{z-k+1} + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) c_i c_{k-i} d_{z-k+1}$$
$$\left.\left. + \sum_{k=0}^{z} \sum_{i=0}^{k} (z-k+1) d_i d_{k-i} d_{z-k+1} \right) \right\},$$

for $z = 1, 2, \ldots$. Therefore, the explicit power series solution of system (8.56) under the action of infinitesimal vector field X_3 is

$$p(t,x) = a_0 + a_1 \left(\frac{x}{t^{1/3}} \right) + a_2 \left(\frac{x^2}{t^{2/3}} \right) + \frac{1}{18} \left(a_0 - 3\alpha (a_0^2 a_1 + b_0^2 a_1 + c_0^2 a_1 + d_0^2 a_1) \right) \left(\frac{x^3}{t} \right)$$

$$+\frac{1}{3}\sum_{z=1}^{\infty}\frac{z!}{(z+3)!}\left\{a_z(z+1)-3\alpha\left(\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_ia_{k-i}a_{z-k+1}\right.\right.$$
$$+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_ib_{k-i}a_{z-k+1}+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_ic_{k-i}a_{z-k+1}$$
$$\left.\left.+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_id_{k-i}a_{z-k+1}\right)\right\}\left(\frac{x}{t^{1/3}}\right)^{z+3}, \tag{8.52}$$

$$q(t,x)=b_0+b_1\left(\frac{x}{t^{1/3}}\right)+b_2\left(\frac{x^2}{t^{2/3}}\right)+\frac{1}{18}\left(b_0-3\alpha(a_0^2b_1+b_0^2b_1+c_0^2b_1+d_0^2b_1)\right)\left(\frac{x^3}{t}\right)$$
$$+\frac{1}{3}\sum_{z=1}^{\infty}\frac{z!}{(z+3)!}\left\{b_z(z+1)-3\alpha\left(\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_ia_{k-i}b_{z-k+1}\right.\right.$$
$$+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_ib_{k-i}b_{z-k+1}+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_ic_{k-i}b_{z-k+1}$$
$$\left.\left.+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_id_{k-i}b_{z-k+1}\right)\right\}\left(\frac{x}{t^{1/3}}\right)^{z+3} \tag{8.53}$$

$$r(t,x)=c_0+c_1\left(\frac{x}{t^{1/3}}\right)+c_2\left(\frac{x^2}{t^{2/3}}\right)+\frac{1}{18}\left(c_0-3\alpha(a_0^2c_1+b_0^2c_1+c_0^2c_1+d_0^2c_1)\right)\left(\frac{x^3}{t}\right)$$
$$+\frac{1}{3}\sum_{z=1}^{\infty}\frac{z!}{(z+3)!}\left\{c_z(z+1)-3\alpha\left(\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_ia_{k-i}c_{z-k+1}\right.\right.$$
$$+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_ib_{k-i}c_{z-k+1}+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_ic_{k-i}c_{z-k+1}$$
$$\left.\left.+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_id_{k-i}c_{z-k+1}\right)\right\}\left(\frac{x}{t^{1/3}}\right)^{z+3} \tag{8.54}$$

$$s(t,x)=d_0+d_1\left(\frac{x}{t^{1/3}}\right)+d_2\left(\frac{x^2}{t^{2/3}}\right)+\frac{1}{18}\left(d_0-3\alpha(a_0^2d_1+b_0^2d_1+c_0^2d_1+d_0^2d_1)\right)\left(\frac{x^3}{t}\right)$$
$$+\frac{1}{3}\sum_{z=1}^{\infty}\frac{z!}{(z+3)!}\left\{d_z(z+1)-3\alpha\left(\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)a_ia_{k-i}d_{z-k+1}\right.\right.$$
$$+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)b_ib_{k-i}d_{z-k+1}+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)c_ic_{k-i}d_{z-k+1}$$
$$\left.\left.+\sum_{k=0}^{z}\sum_{i=0}^{k}(z-k+1)d_id_{k-i}d_{z-k+1}\right)\right\}\left(\frac{x}{t^{1/3}}\right)^{z+3}. \tag{8.55}$$

Figure 8.9 is a depiction of solution (8.52) and solutions (8.53)–(8.55) have analogous profiles.

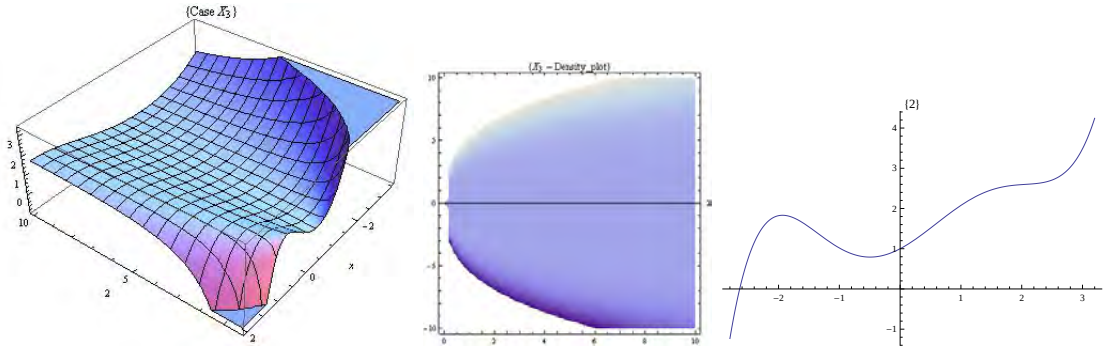

Figure 8.9: Cumulative partial sums of solution (8.52) up to T_5 with parameter values $a_0 = 1, a_1 = 1, a_2 = 0.5, b_0 = 2, b_1 = 2, b_2 = 2, c_0 = 2, c_1 = 1, c_2 = 1, d_0 = 1, d_1 = 0.9, d_2 = 3, \alpha = 0.29$.

8.4 Conserved currents

In this Section, we employ the first homotopy integral formula [43] explicitly to obtain nine local and low-order conserved quantities of the (1+1)-dgcmKdV system (8.3). The reckoning of

$$\begin{aligned}
\frac{\delta}{\delta p}\left[\Lambda^1 E_1 + \Lambda^2 E_2 + \Lambda^3 E_3 + \Lambda^4 E_4\right] &= 0,\\
\frac{\delta}{\delta q}\left[\Lambda^1 E_1 + \Lambda^2 E_2 + \Lambda^3 E_3 + \Lambda^4 E_4\right] &= 0,\\
\frac{\delta}{\delta r}\left[\Lambda^1 E_1 + \Lambda^2 E_2 + \Lambda^3 E_3 + \Lambda^4 E_4\right] &= 0,\\
\frac{\delta}{\delta s}\left[\Lambda^1 E_1 + \Lambda^2 E_2 + \Lambda^3 E_3 + \Lambda^4 E_4\right] &= 0,
\end{aligned} \tag{8.56}$$

indeed yields all multipliers of system (8.3). The well-known Euler operator $\delta/\delta p$ is given by [45]

$$\frac{\delta}{\delta p} = \frac{\partial}{\partial p} - D_t\frac{\partial}{\partial p_t} - D_x\frac{\partial}{\partial p_x} + D_x^2\frac{\partial}{\partial p_{xx}} - D_x^3\frac{\partial}{\partial p_{xxx}} \tag{8.57}$$

and analogously for each field variable in system (8.3). The total differential operators D_t and D_x are, respectively given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + p_t\frac{\partial}{\partial p} + q_t\frac{\partial}{\partial q} + r_t\frac{\partial}{\partial r} + s_t\frac{\partial}{\partial s} + p_{tt}\frac{\partial}{\partial p_t} + q_{tt}\frac{\partial}{\partial q_t} + r_{tt}\frac{\partial}{\partial r_t} + s_{tt}\frac{\partial}{\partial s_t} \\
&\quad + p_{tx}\frac{\partial}{\partial p_x} + q_{tx}\frac{\partial}{\partial q_x} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + p_x\frac{\partial}{\partial p} + q_x\frac{\partial}{\partial q} + r_x\frac{\partial}{\partial r} + s_x\frac{\partial}{\partial s} + p_{xx}\frac{\partial}{\partial p_x} + q_{xx}\frac{\partial}{\partial q_x} + r_{xx}\frac{\partial}{\partial r_x} + s_{xx}\frac{\partial}{\partial s_x} \\
&\quad + p_{xt}\frac{\partial}{\partial p_t} + q_{xt}\frac{\partial}{\partial q_t} + \cdots.
\end{aligned} \tag{8.58}$$

Here we seek to compute second-order multipliers with dependencies

$$\Lambda^m = \Lambda^m(t, x, p, q, r, s, p_x, q_x, r_x, s_x, p_{xx}, q_{xx}, r_{xx}, s_{xx}), \quad m = 1, \ldots, 4. \tag{8.59}$$

The choice of multiplier dependencies stems from the fact that all the leading derivatives of system (8.3) can be recovered by deriving the dependencies p, q, r, $s, \ldots,$ s_{xx} in (8.57) with respect to t or x. Due to the voluminous nature of computations involved in (8.56), we can use software such as Maple or Mathematica to solve the resultant system of partial differential equations. The following set of nine conserved quantity multipliers represent solutions of determining condition (8.56):

$$\begin{aligned}
\Lambda^1 &= \left(C_1x + C_6 - \frac{1}{3}\alpha\left(3C_1t - C_4\right)\left(q^2 + r^2 + s^2\right)\right)p - \frac{1}{3}\alpha\left(3C_1t - C_4\right)p^3 \\
&\quad - 3C_1tp_{xx} - C_5s_x + C_4p_{xx} - C_8r_x - C_9q_x, \\
\Lambda^2 &= \left(C_1x + C_6 - \frac{1}{3}\alpha\left(3C_1t - C_4\right)\left(p^2 + r^2 + s^2\right)\right)q - \frac{1}{3}\alpha\left(3C_1t - C_4\right)q^3 \\
&\quad - 3C_1tq_{xx} - C_2s_x + C_4q_{xx} + C_7r_x + C_9p_x, \\
\Lambda^3 &= \left(C_1x + C_6 - \frac{1}{3}\alpha\left(3C_1t - C_4\right)\left(p^2 + q^2 + s^2\right)\right)r - \frac{1}{3}\alpha\left(3C_1t - C_4\right)r^3 \\
&\quad - 3C_1tr_{xx} + C_4r_{xx} + C_7q_x + C_8p_x, \\
\Lambda^4 &= \left(C_1x + C_6 - \frac{1}{3}\alpha\left(3C_1t - C_4\right)\left(p^2 + q^2 + r^2\right)\right)s - \frac{1}{3}\alpha\left(3C_1t - C_4\right)r^3 \\
&\quad - 3C_1ts_{xx} + C_2q_x + C_3r_x + C_4s_{xx} + C_5p_x
\end{aligned} \tag{8.60}$$

with $C_1, \ldots, C_9$ as free constants. Conserved densities T^i and spatial fluxes X^i for $i = 1, \ldots, 9$ can be obtained by using the first homotopy integral formula (1.48), see also [77]. Thus, for (8.3) and (8.60), formula (1.48) is explicitly given by

$$
\begin{aligned}
T = & \int_0^1 \left(p \left(\frac{\partial E_m \Lambda^m}{\partial p_t} \right) |_{p=p\lambda} + q \left(\frac{\partial E_m \Lambda^m}{\partial q_t} \right) |_{q=q\lambda} + r \left(\frac{\partial E_m \Lambda^m}{\partial r_t} \right) |_{r=r\lambda} \right. \\
& \left. + s \left(\frac{\partial E_m \Lambda^m}{\partial s_t} \right) |_{s=s\lambda} \right) \mathrm{d}\lambda, \\
X = & \int_0^1 \left\{ p \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial p_x} \right) |_{p=p\lambda} - D_x \left(\frac{\partial E_m \Lambda^m}{\partial p_{xx}} \right) |_{p=p\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial p_{xxx}} \right) |_{p=p\lambda} \right\} \right. \\
& + p_x \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial p_{xx}} \right) |_{p=p\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial p_{xx}} \right) |_{p=p\lambda} \right\} + p_{xx} \left(\frac{\partial E_m \Lambda^m}{\partial p_{xxx}} \right) |_{p=p\lambda} \\
& + q \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial q_x} \right) |_{q=q\lambda} - D_x \left(\frac{\partial E_m \Lambda^m}{\partial q_{xx}} \right) |_{q=q\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial q_{xxx}} \right) |_{q=q\lambda} \right\} \\
& + q_x \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial q_{xx}} \right) |_{q=q\lambda} + D_x^2 \left(\frac{E_m \Lambda^m}{\partial q_{xx}} \right) |_{q=q\lambda} \right\} + q_{xx} \left(\frac{\partial E_m \Lambda^m}{\partial q_{xxx}} \right) |_{q=q\lambda} \\
& r \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial r_x} \right) |_{r=r\lambda} - D_x \left(\frac{\partial E_m \Lambda^m}{\partial r_{xx}} \right) |_{r=r\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial r_{xxx}} \right) |_{r=r\lambda} \right\} \\
& + r_x \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial r_{xx}} \right) |_{r=r\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial r_{xx}} \right) |_{r=r\lambda} \right\} + r_{xx} \left(\frac{\partial E_2 \Lambda_2}{\partial r_{xxx}} \right) |_{r=r\lambda} \\
& + s \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial s_x} \right) |_{s=s\lambda} - D_x \left(\frac{\partial E_m \Lambda^m}{\partial s_{xx}} \right) |_{s=s\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial s_{xxx}} \right) |_{s=s\lambda} \right\} \\
& \left. + s_x \left\{ \left(\frac{\partial E_m \Lambda^m}{\partial s_{xx}} \right) |_{s=s\lambda} + D_x^2 \left(\frac{\partial E_m \Lambda^m}{\partial s_{xx}} \right) |_{s=s\lambda} \right\} + s_{xx} \left(\frac{\partial E_m \Lambda^m}{\partial s_{xxx}} \right) |_{s=s\lambda} \right\} \mathrm{d}\lambda.
\end{aligned}
\tag{8.61}
$$

Now considering the case

$$
\begin{aligned}
\Lambda_1^1 &= xp - \alpha\, tp^3 - \alpha tpq^2 - \alpha tpr^2 - \alpha tps^2 - 3tp_{xx}, \\
\Lambda_1^2 &= xq - \alpha tp^2 q - \alpha tq^3 - \alpha tqr^2 - \alpha tqs^2 - 3tq_{xx}, \\
\Lambda_1^3 &= xr - \alpha tp^2 r - \alpha tq^2 r - \alpha tr^3 - \alpha trs^2 - 3tr_{xx}, \\
\Lambda_1^4 &= xs - \alpha tp^2 s - \alpha tq^2 s - \alpha tr^2 s - \alpha ts^3 - 3ts_{xx},
\end{aligned}
$$

we have

$$
T^1 = \int_0^1 \left(\lambda xp^2 - \alpha t\lambda^3 p^4 - 2\alpha t\lambda^3 p^2 q^2 - 2\alpha t\lambda^3 p^2 r^2 - 2\alpha t\lambda^3 p^2 s^2 - 3\lambda tpp_{xx} \right.
$$

$$
\begin{aligned}
&-\alpha t\lambda^3 q^4 - 2\lambda^3\alpha t q^2 r^2 - 2\lambda^3\alpha t q^2 s^2 + \lambda x q^2 - 3\lambda t q q_{xx} - \alpha t\lambda^3 r^4 \\
&-2\lambda^3\alpha t r^2 s^2 + \lambda x r^2 - 3\lambda t r r_{xx} - \lambda^3\alpha t s^4 + \lambda x s^2 - 3\lambda t s s_{xx}\big)\,\mathrm{d}\lambda \\
=&\frac{1}{2}xp^2 - \frac{1}{4}\alpha t p^4 - \frac{1}{2}\alpha t p^2 q^2 - \frac{1}{2}\alpha t p^2 r^2 - \frac{1}{2}\alpha t p^2 s^2 - \frac{3}{2}t p p_{xx} \\
&-\frac{1}{4}\alpha t q^4 - \frac{1}{2}\alpha t q^2 r^2 - \frac{1}{2}\alpha t q^2 s^2 + \frac{1}{2}x q^2 - \frac{3}{2}t q q_{xx} - \frac{1}{4}\alpha t r^4 \\
&-\frac{1}{2}\alpha t r^2 s^2 + \frac{1}{2}x r^2 - \frac{3}{2}t r r_{xx} - \frac{1}{4}\alpha t s^4 + \frac{1}{2}x s^2 - \frac{3}{2}t s s_{xx}
\end{aligned}
$$

and

$$
\begin{aligned}
X^1 = \int_0^1 \big(&2\lambda x q q_{xx} - 3\lambda t q_x q_t - \lambda x q_x^2 - 3\lambda t q_{xx}^2 + 3\lambda t q_{tx} - \lambda x p_x^2 + 2\lambda x p p_{xx} \\
&-3\lambda t p_{xx}^2 + 3\lambda t p p_{tx} - 3\lambda t p_t p_x + 3\lambda t s s_{tx} + 2\lambda x s s_{xx} - 3\lambda t s_x s_t - \lambda x s_x^2 \\
&-3\lambda t s_{xx}^2 + 3\lambda t r r_{tx} + 2\lambda x r r_{xx} - 3\lambda t r_t r_x - \lambda x r_x^2 - 3\lambda t r_{xx}^2 - 6\lambda^5\alpha^2 t p^2 q^2 s^2 \\
&-6\lambda^5\alpha^2 t p^2 r^2 s^2 - 6\lambda^5\alpha^2 t s^2 q^2 r^2 + 8\lambda^3\alpha t p s s_x p_x + 8\lambda^3\alpha t q s s_x q_x + 8\lambda^3\alpha t r s s_x r_x \\
&+8\lambda^3\alpha t p p_x q q_x + 8\alpha\lambda^3 t p r p_x r_x - 6\lambda^5\alpha^2 t p^2 q^2 r^2 + 8\lambda^3\alpha t r q q_x r_x + \lambda^3\alpha x p^4 \\
&+\lambda^3\alpha x s^4 - \lambda^5\alpha^2 t s^6 - 4\lambda^3 t s^3\alpha s_{xx} - \lambda^5\alpha^2 t r^6 + \lambda^3\alpha x r^4 - 4\alpha\lambda^3 t r^3 r_{xx} + \lambda^3\alpha x q^4 \\
&-4\lambda^3\alpha t q^3 q_{xx} - \lambda^5\alpha^2 t q^6 - \lambda^5\alpha^2 t p^6 - 4\lambda^3\alpha t p^3 p_{xx} - 4\lambda^3\alpha t s^2 p_x^2 - 4\lambda^3\alpha t s^2 q_x^2 \\
&-4\lambda^3\alpha t p^2 s_x^2 - 4\lambda^3\alpha t q^2 s_x^2 - 4\lambda^3\alpha t r^2 s_x^2 - 4\lambda^3\alpha t s p^2 s_{xx} - 4\lambda^3\alpha t s q^2 s_{xx} \\
&-4\lambda^3\alpha t s r^2 s_{xx} - 4\alpha\lambda^3 t s^2 r_x^2 + 2\alpha\lambda^3 x s^2 p^2 + 2\lambda^3\alpha x s^2 q^2 + 2\alpha\lambda^3 r^2 x - 3\lambda^5\alpha^2 t s^2 p^4 \\
&-3\lambda^5\alpha^2 t s^2 s^4 q^2 - 3\lambda^5\alpha^2 t q^4 s^2 - 3\lambda^5 r^2 s^4\alpha^2 t - 3\lambda^5\alpha^2 t r^4 s^2 - 3\lambda^5\alpha^2 t s^4 p^2 \\
&+2\lambda^3\alpha x r^2 q^2 - 4\lambda^3\alpha r q^2 r_{xx} t - 4\lambda^3\alpha r s^2 r_{xx} t - 4\lambda^3\alpha r^2 p_x^2 t - 4\lambda^3\alpha r^2 q_x^2 t \\
&-4\lambda^3\alpha t p^2 r_x^2 - 4\lambda^3\alpha t q^2 r_x^2 - 4\lambda^3 r\alpha t p^2 r_{xx} - 3\lambda^5\alpha^2 t r^4 p^2 - 3\lambda^5\alpha^2 t p^4 r^2 \\
&-3\lambda^5\alpha^2 t q^2 r^4 - 3\lambda^5\alpha^2 t q^4 r^2 + 2\alpha\lambda^3 x p^2 r^2 + 2\alpha\lambda^3 x p^2 q^2 - 4\lambda^3\alpha q p^2 t q_{xx} \\
&-4\lambda^3\alpha t q r^2 q_{xx} - 4\alpha\lambda^3 t q s^2 q_{xx} - 4\lambda^3\alpha t q^2 p_x^2 - 4\lambda^3\alpha t q_x^2 p^2 - 4\lambda^3 p\alpha t r^2 p_{xx} \\
&-4\lambda^3\alpha t s^2 p p_{xx} - 3\lambda^5\alpha^2 t p^2 q^4 - 3q^2\lambda^5\alpha^2 t p^4 - 4\lambda^3\alpha t p q^2 p_{xx} + \lambda p p_x + \lambda s s_x \\
&+\lambda r r_x + \lambda q q_x\big)\,\mathrm{d}\lambda \\
=&x q q_{xx} - \frac{3}{2}t q_x q_t - \frac{1}{2}x q_x^2 - \frac{3}{2}t q_{xx}^2 + \frac{3}{2}t q_{tx} - \frac{1}{2}x p_x^2 + x p p_{xx}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{2}tp_{xx}^2+\frac{3}{2}tpp_{tx}-\frac{3}{2}tp_tp_x+\frac{3}{2}tss_{tx}+xss_{xx}-\frac{3}{2}ts_xs_t-\frac{1}{2}xs_x^2\\
&-\frac{3}{2}ts_{xx}^2+\frac{3}{2}trr_{tx}+xrr_{xx}-\frac{3}{2}tr_tr_x-\frac{1}{2}xr_x^2-\frac{3}{2}tr_{xx}^2-\alpha^2tp^2q^2s^2\\
&-\alpha^2tp^2r^2s^2-\alpha^2ts^2q^2r^2+2\alpha tpss_xp_x+2\alpha tqss_xq_x+2\alpha trss_xr_x\\
&+2\alpha tpp_xqq_x+2\alpha tprp_xr_x-\alpha^2tp^2q^2r^2+2\alpha trqq_xr_x+\frac{1}{4}\alpha xp^4\\
&+\frac{1}{4}\alpha xs^4-\frac{1}{6}\alpha^2ts^6-ts^3\alpha s_{xx}-\frac{1}{6}\alpha^2tr^6+\frac{1}{4}\alpha xr^4-\alpha tr^3r_{xx}+\frac{1}{4}\alpha xq^4\\
&-\alpha tq^3q_{xx}-\frac{1}{6}\alpha^2tq^6-\frac{1}{6}\alpha^2tp^6-\alpha tp^3p_{xx}-\alpha ts^2p_x^2-\alpha ts^2q_x^2\\
&-\alpha tp^2s_x^2-\alpha tq^2s_x^2-\alpha tr^2s_x^2-\alpha tsp^2s_{xx}-\alpha tsq^2s_{xx}\\
&-\alpha tsr^2s_{xx}-\alpha ts^2r_x^2+\frac{1}{2}\alpha xs^2p^2+\frac{1}{2}x\alpha s^2q^2+\frac{1}{2}\alpha xr^2-\frac{1}{2}\alpha^2ts^2p^4\\
&-\frac{1}{2}\alpha^2ts^2s^4q^2-\frac{1}{2}\alpha^2tq^4s^2-\frac{1}{2}r^2s^4\alpha^2t-\frac{1}{2}\alpha^2tr^4s^2-\frac{1}{2}\alpha^2ts^4p^2\\
&+\frac{1}{2}\alpha xr^2q^2-\alpha rq^2r_{xx}t-\alpha rs^2r_{xx}t-\alpha r^2p_x^2t-\alpha r^2q_x^2t\\
&-\alpha tp^2r_x^2-\alpha tq^2r_x^2-r\alpha tp^2r_{xx}-\frac{1}{2}\alpha^2tr^4p^2-\frac{1}{2}\alpha^2tp^4r^2\\
&-\frac{1}{2}\alpha^2tq^2r^4-\frac{1}{2}\alpha^2tq^4r^2+\frac{1}{2}\alpha xp^2r^2+\frac{1}{2}\alpha xp^2q^2-\alpha qp^2tq_{xx}\\
&-\alpha tqr^2q_{xx}-\alpha tqs^2q_{xx}-\alpha tq^2p_x^2-\alpha tq_x^2p^2-p\alpha tr^2p_{xx}\\
&-\alpha ts^2pp_{xx}-\frac{1}{2}\alpha^2tp^2q^4-\frac{1}{2}q^2\alpha^2tp^4-\alpha tpq^2p_{xx}+\frac{1}{2}pp_x+\frac{1}{2}ss_x\\
&+\frac{1}{2}rr_x+\frac{1}{2}qq_x.
\end{aligned}
$$

Likewise, conserved quantities corresponding to the different sets of multipliers $\Lambda_2^1\ldots\Lambda_2^4$; $\Lambda_3^1\ldots\Lambda_3^4$; $\ldots$; $\Lambda_9^1\ldots\Lambda_9^4$ are computed to give the following conservation laws:

$$
\begin{aligned}
T^2=&\frac{1}{2}sq_x-\frac{1}{2}qs_x,\\
X^2=&\frac{1}{2}qs_t+q_xs_{xx}-s_xq_{xx}-\frac{1}{2}sq_t;
\end{aligned}
$$

$$
T^3=\frac{1}{2}sr_x-\frac{1}{2}rs_x,
$$

$$X^3 = \frac{1}{2} r s_t + r_x s_{xx} - s_x r_{xx} - \frac{1}{2} s r_t;$$

$$\begin{aligned}
T^4 =& \frac{1}{12}\alpha p^4 + \frac{1}{6}\alpha p^2 q^2 + \frac{1}{6}\alpha p^2 r^2 + \frac{1}{6}\alpha p^2 s^2 + \frac{1}{2} p p_{xx} + \frac{1}{12}\alpha q^4 + \frac{1}{6}\alpha q^2 r^2 + \frac{1}{6}\alpha q^2 s^2 \\
&+ \frac{1}{2} q q_{xx} + \frac{1}{12}\alpha r^4 + \frac{1}{6}\alpha r^2 s^2 + \frac{1}{2} r r_{xx} + \frac{1}{12}\alpha s^4 + \frac{1}{2} s s_{xx}, \\
X^4 =& \frac{1}{18} q^6 \alpha^2 - \frac{1}{2} q q_{tx} - \frac{1}{2} p \frac{\partial^2}{\partial x \partial t} p + \frac{1}{18} p^6 \alpha^2 + \frac{1}{2} s_x s_t + \frac{1}{2} p_x p_t + \frac{1}{18} r^6 \alpha^2 - \frac{1}{2} r r_{tx} \\
&+ \frac{1}{2} q_t q_x + \frac{1}{18} s^6 \alpha^2 - \frac{1}{2} s s_{tx} + \frac{1}{2} r_x r_t + \frac{1}{3}\alpha p^2 s_x^2 + \frac{1}{6}\alpha^2 p^2 s^4 + \frac{1}{6}\alpha^2 p^4 s^2 + \frac{1}{3}\alpha r^2 s_x^2 \\
&+ \frac{1}{3}\alpha q_x^2 p^2 + \frac{1}{3} q_x^2 \alpha r^2 + \frac{1}{3}\alpha q^2 s_x^2 + \frac{1}{6}\alpha^2 q^4 s^2 + \frac{1}{6} q^2 r^4 \alpha^2 + \frac{1}{6}\alpha^2 q^2 s^4 + \frac{1}{3}\alpha s^2 q_x^2 \\
&+ \frac{1}{3}\alpha s^3 s_{xx} + \frac{1}{3}\alpha r^3 r_{xx} + \frac{1}{3}\alpha q^3 q_{xx} + \frac{1}{3}\alpha p^3 p_{xx} + \frac{1}{3}\alpha r^2 p_x^2 + \frac{1}{6}\alpha^2 r^4 p^2 + \frac{1}{6}\alpha^2 r^2 p^4 \\
&+ \frac{1}{6}\alpha^2 r^2 q^4 + \frac{1}{6}\alpha^2 r^4 s^2 + \frac{1}{6}\alpha^2 r^2 s^4 + \frac{1}{3}\alpha r_x^2 p^2 + \frac{1}{3}\alpha r_x^2 q^2 + \frac{1}{3}\alpha r_x^2 s^2 + \frac{1}{6}\alpha^2 q^4 p^2 \\
&+ \frac{1}{6}\alpha^2 q^2 p^4 + \frac{1}{3}\alpha q^2 p_x^2 + \frac{1}{3}\alpha s^2 p_x^2 - \frac{2}{3}\alpha s_x s p p_x - \frac{2}{3}\alpha s_x s q q_t - \frac{2}{3}\alpha s r r_x s_x \\
&- \frac{2}{3}\alpha r p r_x p_x - \frac{2}{3}\alpha r q q_t r_x - \frac{2}{3}\alpha p q q_x p_x + \frac{1}{3}\alpha q p^2 q_{xx} + \frac{1}{3}\alpha p^2 r r_{xx} + \frac{1}{3}\alpha p^2 s s_{xx} \\
&+ \frac{1}{3}\alpha s^2 q q_{xx} + \frac{1}{3}\alpha s^2 r r_{xx} + \frac{1}{3}\alpha^2 s^2 p^2 r^2 + \frac{1}{3}\alpha^2 s^2 q^2 r^2 + \frac{1}{3}\alpha^2 s^2 p^2 q^2 + \frac{1}{3}\alpha^2 r^2 p^2 q^2 \\
&+ \frac{1}{3}\alpha r^2 p p_{xx} + \frac{1}{3}\alpha r^2 q q_{xx} + \frac{1}{3}\alpha r^2 s s_{xx} + \frac{1}{3}\alpha q^2 p p_{xx} + \frac{1}{3}\alpha q^2 r r_{xx} + \frac{1}{3} q^2 \alpha s s_{xx} \\
&+ \frac{1}{3}\alpha p s^2 p_{xx} + \frac{1}{2} s_{xx}^2 + \frac{1}{2} q_{xx}^2 + \frac{1}{2} r_{xx} + \frac{1}{2} p_{xx}^2;
\end{aligned}$$

$$\begin{aligned}
T^5 =& \frac{1}{2} s p_x - \frac{1}{2} p s_x, \\
X^5 =& \frac{1}{2} p s_t + p_x s_{xx} - s_x p_{xx} - \frac{1}{2} s p_t;
\end{aligned}$$

$$\begin{aligned}
T^6 =& \frac{1}{2} p^2 + \frac{1}{2} q^2 + \frac{1}{2} r^2 + \frac{1}{2} s^2, \\
X^6 =& \frac{1}{4}\alpha s^4 + \frac{1}{4}\alpha r^4 + \frac{1}{4}\alpha q^4 + \frac{1}{4}\alpha p^4 + \frac{1}{2}\alpha q^2 s^2 + \frac{1}{2} p^2 \alpha s^2 + \frac{1}{2}\alpha p^2 q^2 + \frac{1}{2}\alpha p^2 r^2 \\
&+ \frac{1}{2}\alpha s^2 r^2 + \frac{1}{2}\alpha r^2 q^2 - \frac{1}{2} r_x^2 - \frac{1}{2} s^2 - \frac{1}{2} p_x^2 + p p_{xx} + q q_{xx} - \frac{1}{2} q_x^2 + s s_{xx} + r r_{xx};
\end{aligned}$$

$$T^7 = \frac{1}{2}qr_t - \frac{1}{2}qr_x,$$
$$X^7 = \frac{1}{2}qr_t + q_x r_{xx} - r_x q_{xx} - \frac{1}{2}rq_t;$$

$$T^8 = \frac{1}{2}rp_x - \frac{1}{2}pr_x,$$
$$X^8 = \frac{1}{2}pr_t + p_x r_{xx} - r_x p_{xx} - \frac{1}{2}rp_t;$$

$$T^9 = \frac{1}{2}qp_x - \frac{1}{2}pq_x,$$
$$X^9 = \frac{1}{2}pq_t + p_x q_{xx} - q_x p_{xx} - \frac{1}{2}qp_t.$$

A casual glance at these results reveals that the conserved quantities are local and low-order. They are also non-trivial first integrals of system (8.3). The multiplier approach via the first homotopy integral formula proves to be a reliable method for determining local conservation laws of a system of differential equations. System (8.3) does not possess a variational principle. This is where the classical Noether's approach falters as the conservation laws for this system cannot be retrieved. However, with the multiplier method, we could obtain local non-trivial conserved vectors with relative ease.

8.5 Concluding remarks

In this work, we studied the (1+1)-dimensional generalised coupled mKdV system (8.3) of four equations, which was derived from the extended complex coupled cmKdV system (8.2), using the transformations $u(t,x) = p(t,x) + iq(t,x)$, $v(t,x) = r(t,x) + is(t,x)$. We obtained a nine-dimensional Lie algebra for system (8.3). A one-parameter group of transformations for the solutions of (8.3)

were determined. These were the most general solutions of system (8.3). Using infinitesimal vector fields X_1, X_2 and X_3 we obtained group invariant solutions. With the aid of the main copolar trio elliptic functions along with simplest equation method, we obtained elliptic cosine and sine solutions as well as delta amplitude function solutions. Also, we gave parameter conditions suitable for degeneration of the elliptic functions into trigonometric, hyperbolic and constant solutions. As evidenced in the graphical depictions, solutions of system (8.3) obtained here are periodic, bell-shaped bright and dark solitons, topological kink and anti-kink solitons. Furthermore, by using the multiplier method coupled with the first homotopy integral formula, we computed nine local and low-order conserved quantities. The multiplier approach coupled with the first homotopy integral formula is one of the most effective, efficient and reliable methods for determining conserved quantities.

Chapter 9

Conservation laws and symmetry reductions for a generalized hyperelastic-rod wave equation

9.1 Introduction

The generalized hyperelastic-rod wave (gHRW) equation

$$E \equiv u_t - u_{xxt} + \frac{1}{2}g(u)_x - \alpha(2u_x u_{xx} + u u_{xxx}) = 0,\ g(u)' \neq 0, \tag{9.1}$$

was seen for the first time in [124], where the authors proved the existence of a strongly continuous semigroup of global weak solutions for $t > 0$ and $x \in \Re$. This equation describes dispersive waves propagating in hyperelastic-rods. By using the transformation $g(u) = 2\kappa u + 3u^2$, equation (9.1) becomes the well known Camassa-Holm equation [125]

$$u_t + 2\kappa u_x - u_{xxt} + 3u u_x - 2u_x u_{xx} - u u_{xxx} = 0. \tag{9.2}$$

For $\kappa > 0$, equation (9.2) represents the one directional motion of shallow waves. The time and spatial dimensions are represented by t and x, while the wave velocity is given by $u(t, x)$. For the case $g(u) = 3u^2$ we obtain the equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{9.3}$$

which made its debut appearance in [126], where the authors found its unbounded travelling wave solutions. It is in the work of Dai [126] that we find the name hyperelastic-rod wave equation. Equation (9.3) can be obtained from the Camassa-Holm equation by taking $\kappa = 0$. While the hyperelastic-rod equation appears to be a variation of the Camassa-Holm equation (9.2), its physical significance is different because it was derived independently of equation (9.2) by Dai, see for example [126, 127].

The gHRW equation (9.1) along with other nonlinear dispersive equations made another appearance in [128] where their geometric integrators were presented. In this present work we seek to explore the gHRW equation further. We derive the low-order conservation laws of the gHRW equation. We will make use of conservation laws to obtain a multi-reduction of the gHRW equation to a first-order ordinary differential equation (ODE), without having to do successive single reductions. Furthermore, we study several plausible cases of the arbitrary function $g(u)$ with the aim of utilising Lie group analysis to obtain symmetry reductions to first or second-order ODEs and where possible, to obtain exact solutions.

The work presented in this chapter has been submitted for possible publication [129].

9.2 Conserved quantities

In this section we obtain low-order conservation laws [43] of the gHRW equation which are invariant under the time and space translation group. We use these conserved quantities to obtain a set of first integrals of the gHRW equation. Since each first integral yields a further reduction of the ODE, we directly obtain a *triple-reduction* of the PDE under the symmetry, to a first-order ODE [130].

Given a PDE

$$G(t, x, u, u_t, u_x, \ldots) = 0,$$

a (local) conservation law is a continuity equation

$$D_t T + D_x \Phi = 0 \tag{9.4}$$

holding on the space of solutions $u(x, t)$ of the PDE, where T and Φ are the conserved density and the spatial flux, which are functions of t, x, u, and derivatives of u. Here D_t, D_x are total derivatives, and (T, Φ) is the conserved current. Every non-trivial conservation law, modulo a trivial conservation law, arises from a multiplier Q which is a function of t, x, u, and derivatives of u, such that

$$D_t T + D_x \Phi = QG$$

.

For the gHRW equation (9.1) we will consider the low-order multipliers

$$Q(t, x, u, u_t, u_x, u_{tx}, u_{xx}), \tag{9.5}$$

for which the determining equation (1.47) splits into an overdetermined linear system which is straightforward to solve for Q and $g'(u) \neq 0$.

Invariance of a PDE $G(t, x, u, u_t, u_x, \ldots) = 0$ under the translation symmetry

$$\mathcal{X} = \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial x}, \tag{9.6}$$

gives rise to travelling wave solutions, with $u = U$ and $\xi = x - \nu t$ being the invariants.

Proposition 9.1 *All low-order multipliers* (9.5) *admitted by the gHRW equation (9.1) with* $g'(u) \neq 0$*, give rise to conservation laws which are invariant under the translation group and they are given by*
(i) $g(u)$ *arbitrary:*

$$Q_{1a} = 1, \tag{9.7}$$

$$Q_{1b} = u, \tag{9.8}$$

$$Q_{1c} = \frac{1}{2\alpha}(\alpha(2uu_{xx} + u_x^2) + 2u_{tx} - g(u)). \tag{9.9}$$

(ii) $g(u) = 3\alpha u^2 + au + b$*:*

$$Q_2 = -\frac{1}{2\alpha\sqrt{4\alpha(u_{xx} - u) - a}}. \tag{9.10}$$

These multipliers yield non-trivial conservation laws of low-order, summarized as follows.

Theorem 9.1 *(i) The conservation laws admitted by the gRHW equation (9.1) for* $g'(u) \neq 0$ *are given by:*

Case 1a: $g(u)$ *arbitrary*

$$\begin{aligned} T_{1a} =& u, \\ X_{1a} =& -\alpha u u_{xx} - u_{tx} - \frac{1}{2}\alpha u_x^2 + \frac{1}{2}g(u). \end{aligned} \tag{9.11}$$

Case 1b: $g(u)$ *arbitrary*

$$\begin{aligned} T_{1b} =& \frac{1}{2}u_x^2 + \frac{1}{2}u^2, \\ X_{1b} =& -\alpha u^2 u_{xx} - u u_{tx} + \frac{1}{2}\int u g(u) du. \end{aligned} \tag{9.12}$$

Case 1c: $g(u)$ arbitrary

$$\begin{aligned} T_{1c} =& -\frac{1}{2}uu_x^2 - \frac{1}{2}\alpha\int g du, \\ X_{1c} =& \frac{1}{2\alpha}u_t^2 + uu_tu_x - \frac{1}{2}\alpha u^2u_{xx}^2 \\ &+ \frac{1}{2}u_{xx}(ug(u) - \alpha u_x^2 - 2u_{tx}) - \frac{1}{8\alpha}(g(u)^2 - \alpha u_x^2 - 2u_{tx}). \end{aligned} \tag{9.13}$$

Case 2: $g(u) = 3\alpha u^2 + c_1u + a$

$$\begin{aligned} T_2 =& \frac{1}{4\alpha^2}\sqrt{4\alpha u_{xx} - 4\alpha u - a}, \\ X_2 =& \frac{1}{4\alpha^2}\sqrt{4\alpha u_{xx} - 4\alpha u - a}u. \end{aligned} \tag{9.14}$$

Substituting the group invariant

$$u(t,x) = U(x - \nu t) \tag{9.15}$$

into equation (9.1), with $g(u) = u^p$ yields the nonlinear third-order ODE

$$2(\alpha U - \nu)U''' - (pU^{p-1} - 4\alpha U'' - 2\nu)U' = 0. \tag{9.16}$$

First integrals of the travelling wave ODE (9.16) can be found by symmetry reductions of all conservation laws that are invariant under the travelling wave symmetry (9.6) of the PDE. The reduction can be understood through considering the action of (9.6) on a conservation law expressed in terms of the canonical variables.

On travelling wave solutions $u = U(\xi)$, a symmetry-invariant conservation law $D_tT + D_x\phi = 0$ reduces to the form $D_\xi(\phi - \nu T) = 0$. This yields a first integral $\psi = \phi - \nu T = C(const.)$ of the travelling wave ODE [130].

By using the conservation laws (9.11), (9.12), (9.13) we get three first integrals, that is,

$$\Psi_1 = (\alpha U - \nu)U'' + \nu U + 1/2(\alpha U'^2 - U^p) = C_1, \tag{9.17}$$

$$\Psi_2 = (\alpha U^2 - \nu U)U'' + \frac{\nu}{2}(U'^2 + U^2) - \frac{p}{2(p+1)}U^{p+1} = C_2, \tag{9.18}$$

$$\begin{aligned}\Psi_3 = &\left(\frac{\alpha}{2}U^2 - \nu U + \frac{\nu^2}{2a}\right)U''^2 + \left(\frac{1}{2}(\alpha U - \nu)U'^2 - \frac{1}{2\alpha}(\nu U^p - \alpha U^{p+1})\right)U'' \\ &+ \frac{\alpha}{8}U'^4 + \left(\frac{\nu}{2}U - \frac{1}{4\alpha}(\alpha U^p + 2\nu^2)\right)U'^2 \\ &- \frac{1}{8\alpha(p+1)}(4\nu U^{p+1} - (p+1)U^{2p}) = C_3.\end{aligned} \tag{9.19}$$

By combining the two first integrals (9.17) and (9.18) we obtain a triple reduction to a first-order autonomous equation

$$U'^2 + \frac{1}{(p+1)(\alpha U - \nu)}(U^{p+1} - \nu(p+1)U^2 + 2(p+1)C_1U - 2(p+1))C_2 = 0. \tag{9.20}$$

The third first integral is not functionally independent. The remaining conservation laws are also invariant under the translation group yielding first integrals.

9.3 Exact solutions

In this section we perform a symmetry classification of equation (9.1). An infinitesimal vector field is given by

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{9.21}$$

for some generator coefficient functions ξ^1, ξ^2 and η to be determined. The invariance condition

$$\mathrm{pr}^{(3)}X(\Delta)|_{\Delta=0} = 0 \tag{9.22}$$

yields the determining equation for the generator coefficients. Here, $\mathrm{pr}^{(3)}X$ is a third order extension of generator (9.21). Taking into consideration determining condition (9.22) we now consider different cases for the coefficient function $g(u)$.

9.3.1 Quadratic function case

In this subsection we consider the case

$$g(u) = 3\alpha u^2 + au + b \tag{9.23}$$

where a and b are free constants. Thus equation (9.1) is now

$$u_t - u_{xxt} + 3\alpha u u_x + \frac{1}{2}au_x - \alpha(2u_x u_{xx} + uu_{xxx}) = 0 \tag{9.24}$$

and admits the following 3-dimensional Lie algebra:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 4\alpha t\frac{\partial}{\partial t} - a\alpha t\frac{\partial}{\partial x} - (a + 4\alpha u)\frac{\partial}{\partial u}. \tag{9.25}$$

(i) **Travelling wave reduction**

The linear combination $X_1+\nu X_2$ gives the group invariant $u(t, x) = \phi(\xi)$ with $\xi = x - \nu t$ which transforms nonlinear partial differential equation (NLPDE) (9.24) into the nonlinear ordinary differential equation (NLODE)

$$\nu\phi''' - \alpha\phi\phi''' - 2\alpha\phi'\phi'' + 3\alpha\phi\phi' + \frac{1}{2}(a - \nu)\phi' = 0. \tag{9.26}$$

Integrating equation (9.26) once with respect to ξ gives

$$\nu\phi'' - \alpha\phi\phi'' + \frac{3}{2}\alpha\phi^2 - \frac{1}{2}\alpha\phi'^2 + \frac{1}{2}a\phi - a\phi'^2 - \nu\phi + C_1 = 0 \tag{9.27}$$

with integration constant C_1. Equation (9.27) admits the Lie point symmetry $X = \partial/\partial\xi$ with first-order invariants $z = \phi(\xi)$ and $\psi(z) = \phi'$ which reduce NLODE (9.27) into the first-order Bernoulli differential equation (DE)

$$\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{\alpha}{2(\alpha z - \nu)}\psi = \frac{3\alpha z^2 - (2\nu - a)z + 2C_1}{2(\alpha z - \nu)}\psi^{-1}, \tag{9.28}$$

Letting $w(z) = \psi^2$ we have a first-order linear differential equation given by

$$w' + \frac{\alpha}{\alpha z - \nu} = \frac{3\alpha z^2 + az - 2\nu z + 2C_1}{\alpha z - \nu}. \tag{9.29}$$

Equation (9.29) has solution

$$w\left(z\right)=\frac{2\alpha z^{3}+z^{2}a-2\nu z^{2}+4C_{1}z+2C_{2}}{2(\alpha\,z-\nu)},\tag{9.30}$$

where C_2 is a constant of integration. Reverting to original variables we have

$$u(t,x)=\phi=\pm\frac{\sqrt{2}}{2}\int\sqrt{\frac{2\alpha\phi^{3}+(a-2\nu)\phi^{2}+4C_{1}\phi+2C_{2}}{\alpha\phi-\nu}}\mathrm{d}\xi+C_{3},\tag{9.31}$$

where C_3 is an integration constant.

(ii) **Symmetry reduction using vector field X_3**

From infinitesimal generator X_3 in (9.25) we have the group invariant solution

$$u\left(t,x\right)=\frac{4\alpha\phi(\xi)-at}{4\alpha t}\quad\text{with}\quad\xi=\frac{1}{4}at+x,$$

such that equation (9.24) is transformed into the NLODE

$$\alpha\phi\phi'''+2\alpha\phi'\phi''-\phi''-3\alpha\phi\phi'+\phi=0.\tag{9.32}$$

Equation (9.32) has the symmetry $X=\partial/\partial\xi$ with differential invariants $z=\phi(\xi)$ and $f(z)=\phi'$. Using these invariants, equation (9.32) is reduced to the second-order NLODE

$$\alpha\,zf^{2}f''+\alpha\,zff'^{2}+2\alpha f^{2}f'-3\alpha zf-ff'+z=0.\tag{9.33}$$

We have thus obtained a reduced form of equation (9.24).

(iii) **Special case $g(u)=3\alpha u^2$**

By taking $a=b=0$ in (9.23), equation (9.24) becomes

$$u_{t}-u_{xxt}+3\alpha uu_{x}-\alpha(2u_{x}u_{xx}+uu_{xxx})=0.\tag{9.34}$$

Equation (9.34) admits the Lie algebra

$$X_{1}=\frac{\partial}{\partial t},\qquad X_{2}=\frac{\partial}{\partial x},\qquad X_{3}=t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}.\tag{9.35}$$

Travelling wave reduction and solutions

As before, a linear combination of Lie point symmetries X_1 and X_2 transforms equation (9.35) into the NLODE

$$\nu\phi''' - \alpha\phi\phi''' - 2\alpha\phi'\phi'' + 3\alpha\phi\phi' - \nu\phi' = 0. \tag{9.36}$$

Integrating (9.36) with respect to ξ gives the second order ODE

$$\nu\phi'' - \alpha\phi\phi'' + \frac{3}{2}\alpha\phi^2 - \frac{1}{2}\alpha\phi'^2 - \nu\phi + C_1 = 0, \tag{9.37}$$

where C_1 is an integration constant. Equation (9.37) can be reduced to the first-order DE

$$\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{\alpha}{2(\alpha z - \nu)}\psi = \frac{3\alpha z^2 - 2\nu z + 2C_1}{2(\alpha z - \nu)}\psi^{-1}, \tag{9.38}$$

by using generator $X = \partial/\partial\xi$. Solving (9.38) and reverting to original variables gives

$$\phi'^2 = \frac{\alpha\phi^3 - \nu\,\phi^2 + 2C_1\phi + C_2}{\alpha\phi - \nu}. \tag{9.39}$$

Thus for arbitrary α and ν the integral equation

$$u(t,x) = \phi(\xi) = \pm \int \sqrt{\phi^2 + \frac{2C_1\phi + C_2}{\alpha\phi - \nu}}\mathrm{d}\xi + C_3. \tag{9.40}$$

is understood. For the special instance $\alpha = 2C_1$ and $\nu = -C_2$ we have the solutions

$$u(t,x) = \pm\sinh(C_4 + \xi) \tag{9.41}$$

with the following profile:

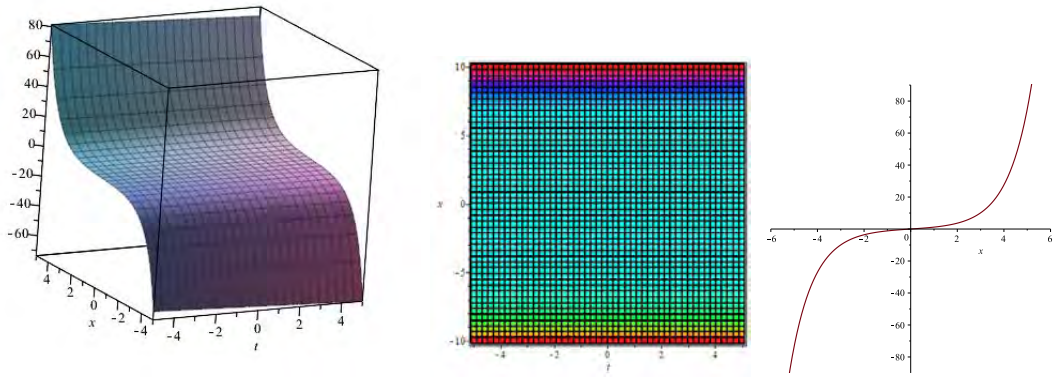

Figure 9.1: Depiction of solution (9.41) with parameter values $C_4 = -0.05, \nu = 0.08$.

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Alternatively, taking the constants of integration $C_1 = C_2 = 0$ we have the solution

$$u(t, x) = C_5 \mathrm{e}^{\pm\xi}, \tag{9.42}$$

Symmetry reduction using generator X_3

Using the usual Lie theory, vector field X_3 yields the group invariant solution $u(t, x) = \phi(\xi)/t$, $\xi = x$. Accordingly, equation (9.34) is transformed into the NLODE

$$\alpha\phi\phi''' + 2\alpha\phi'\phi'' - \phi'' - 3\alpha\phi\phi' + \phi = 0, \tag{9.43}$$

which is identical to (9.32) and thus the reduced equation (9.33) follows.

9.3.2 Power law nonlinearity case

For the case $g(u) = u^p$, we have NLPDE

$$u_t - u_{xxt} + \frac{1}{2}pu^{p-1}u_x - \alpha(2u_xu_{xx} + uu_{xxx}) = 0, \tag{9.44}$$

which admits time and space translation symmetries $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$ respectively. Hence (9.44) becomes

$$\nu\phi''' - \alpha\phi\phi''' + \frac{1}{2}p\phi^{p-1}\phi' - 2\alpha\phi'\phi'' - \nu\phi' = 0. \tag{9.45}$$

Integrating (9.45) once with respect to ξ we get NLODE

$$\nu\phi'' - \alpha\phi\phi'' - \frac{1}{2}\alpha\phi'^2 + \frac{1}{2}\phi^p - \nu\phi + C_1 = 0, \tag{9.46}$$

which admits the Lie point symmetry $X = \partial/\partial\xi$ with differential invariants $\psi(z) = \phi'(\xi)$ and $z = \phi(\xi)$. Using these invariants, the differential order of equation (9.46) is reduced by one to give the first-order DE

$$\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{\alpha}{2(\alpha z - \nu)}\psi = \frac{z^p - 2\nu z + 2C_1}{2(\alpha z - \nu)}\psi^{-1}. \tag{9.47}$$

Solving (9.47) and reintroducing the native variables we have

$$\phi'^2 - \frac{\phi^{p+1} - (p+1)(\nu\phi^2 - 2C_1\phi - C_2)}{(\alpha\phi - \nu)(p+1)} = 0$$

with constant of integration C_2. Thus we have managed to perform a double symmetry reduction of equation (9.44). Interesting to note is the convergence of two different approaches for reduction of order of differential equations, that is, the reduced equation (9.20) obtained by making use of conservation laws is similar to the equation obtained via symmetry reduction.

9.3.3 Exponential function case.

Suppose $g(u) = A\mathrm{e}^{Bu}$, then equation (9.1) becomes

$$u_t - u_{xxt} + \frac{1}{2}ABu_x\mathrm{e}^{Bu} - \alpha(2u_xu_{xx} + uu_{xxx}) = 0 \tag{9.48}$$

with free constants A and B. Equation (9.48) only admits translation symmetries, that is, $X_1 = \partial/\partial t$ and $X_1 = \partial/\partial x$. Thus symmetry reduction via the travelling

wave variable is appropriate as in the previous sections. This gives the third-order NLODE

$$\nu\phi''' - \alpha\phi\phi''' + \frac{1}{2}AB\phi' \mathrm{e}^{B\phi} - 2\alpha\phi'\phi'' - \nu\phi' = 0. \tag{9.49}$$

The second-order DE

$$\nu\phi'' - \alpha\phi\phi'' + \frac{1}{2}A\mathrm{e}^{B\phi} - \frac{1}{2}\alpha\phi'^2 - \nu\phi + C_1 = 0, \tag{9.50}$$

is a first integral of (9.49) with respect to ξ. As in previous sections, equation (9.50) can be written in terms of the differential invariants $z = \phi(\xi)$ and $\psi(z) = \phi'(\xi)$ to get

$$\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{\alpha}{2(\alpha z - \nu)}\psi = \frac{A\mathrm{e}^{Bz} - 2\nu z + 2C_1}{2(\alpha z - \nu)\psi}. \tag{9.51}$$

Making the substitution $w(z) = \psi^2$, equation (9.51) is transformed into

$$w' + \frac{\alpha}{\alpha\, z - \nu}w = \frac{A\mathrm{e}^{Bz} - 2\,\nu\, z + 2\,C_1}{\alpha\, z - \nu} \tag{9.52}$$

which is a linear DE with integrating factor

$$\begin{aligned}\mu(z) =& \exp\left(\int \frac{\alpha}{\alpha\, z - \nu}dz\right)\\ =& \alpha z - \nu.\end{aligned}$$

Multiplying (9.52) with $\mu(z)$ and integrating the resultant variables separable DE we finally have

$$\begin{aligned}u(t,x) = \phi(\xi) = \pm\frac{1}{B}\int \{&A\left(B\alpha\phi - (B\nu + \alpha)\right)\mathrm{e}^{B\phi} - B^2\left(2/3\nu\alpha\phi^3 - \left(C_1\alpha + \nu^2\right)\phi^2\right.\\ &\left.+2\nu C_1\phi - C_2\right)\}^{1/2}\,\mathrm{d}\xi + C_3\end{aligned}$$

where C_3 is a constant of integration.

9.3.4 Logarithmic function case.

Finally, for the case $g(u) = \ln(u)$, we have

$$u_t - u_{xxt} + \frac{1}{2}u^{-1}u_x - \alpha(2u_xu_{xx} + uu_{xxx}) = 0, \tag{9.53}$$

and following the method of the previous section we finally have

$$u(t,x) = \phi(\xi) = \pm \frac{1}{6} \int \big\{ 18(\alpha - 2\nu)\phi \ln \phi - 24\alpha\nu\phi^3 - 9(\alpha - 4C_1\alpha - 4\nu^2)\phi^2 \\ +36\nu(1 - 2C_1)\phi + 36C_2 \big\}^{1/2} \, \mathrm{d}\xi + C_3$$

9.4 Concluding remarks

In this paper we reduced the third-order gHRW equation into a first-order autonomous ODE. By making use of the time and space translation group, conservation laws were reduced into first integrals, which in turn led to the reduced form of the gHRW equation . We went further to perform several symmetry reductions of the gHRW equation for different cases of the arbitrary function $g(u)$. For the case $g(u) = u^p$, we applied the travelling wave variable directly onto our equation (9.1) and managed to recover the reduced form of the gHRW equation obtained via triple reduction using conservation laws. Lie point symmetries are an efficient way of transforming a NLPDE into an ODE of a lower-order. For special values of the free constants we managed to obtain hyperbolic sine and exponential solutions for the instance $g(u) = 3\alpha u^2$.

Chapter 10

Lie symmetry analysis and conserved quantities of a (1+1)-dimensional fifth-order integrable equation

10.1 Introduction

In this chapter, we study a newly formulated integrable fifth-order equation

$$E \equiv u_{ttt} - au_{txxxx} - bu_{txx} - c(2u_{tx}u_{xx} + u_t u_{xxx} + u_x u_{txx}) - d(u_{tx}u_{xx} + u_x u_{txx}) = 0, \tag{10.1}$$

where a, b, c, d are nonzero constants. Equation (10.1) can be used as a model for bidirectional water waves propagating in a shallow medium. This is a new equation first seen in [131] where its integrability was demonstrated by using a singularity manifold analysis. Several methods, including the simplified Hirota approach and a form of Riccati expansion method, are used to obtain various types of solutions

of equation (10.1). Some solutions obtained include multiple solitons, kink and periodic waves. By taking $a = 1$, $b = 0, c = d = 4$, equation (10.1) becomes the fifth-order integrable equation

$$F \equiv \beta u_{xxx} - 4u_t u_{xxx} - 12u_{xx}u_{tx} - 8u_x u_{txx} - u_{txxxx} + u_{ttt} = 0,\ \beta \neq 0, \qquad (10.2)$$

which has been studied extensively in [112, 115].

In our work we seek to use the classical Lie Group approach to derive analytic solutions of equation (10.1). We begin by deriving elements of a one dimensional optimal system of subalgebras and obtain the corresponding similarity reductions. As such, the group invariant solutions take rational, hyperbolic, Jacobian elliptic and elliptic integral of the second kind forms. Furthermore, power series solutions are obtained for the highly nonlinear transformed equation in one of the cases. One and two wave solutions are also obtained via a multiple exponential function method. Finally, five low-order conserved quantities are obtained via the multiplier approach.

The work presented in this chapter has been submitted for possible publication [132].

10.2 Lie group analysis

We shall now perform Lie group analysis on the underlying equation (10.1).

10.2.1 Classical symmetries

The one-parameter group of transformations

$$\bar{t} \to t + a\xi^1(t, x, u),\ \bar{x} \to x + a\xi^2(t, x, u),\ \bar{u} \to u + a\eta(t, x, u)$$

with a small parameter a, correspond to vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{10.3}$$

The vector field (10.3) in conjunction with (10.1) conforms to condition

$$\mathrm{pr}^{(5)}X(u_{ttt} - au_{txxxx} - bu_{txx} - c(2u_{tx}u_{xx} + u_t u_{xxx} + u_x u_{txx}) - d(u_{tx}u_{xx} + u_x u_{txx}))|_{E=0} = 0. \tag{10.4}$$

Here, $\mathrm{pr}^{(5)}X$ is a fifth-order extension of (10.3), see for example [30] and the references therein. Following the usual Lie theory and using Maple it is easy to see that equation (10.1) has a four-dimensional Lie algebra L_4, spanned by the generators

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \left(\frac{2bx}{c+d} + u\right)\frac{\partial}{\partial u},\ X_4 = \frac{\partial}{\partial u}. \tag{10.5}$$

10.2.2 Optimal system of one dimensional subalgebras

The optimal system of one-dimensional subalgebras for the Lie algebra (10.5) can be obtained by using the commutation relations and the adjoint representations given in Tables 10.1 and 10.2, respectively [30, 78, 133].

Table 10.1. Commutation relations of 4-dimensional Lie algebra (10.5)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$2X_1$	0
X_2	0	0	$X_2 - \frac{2b}{c+d}X_4$	0
X_3	$-2X_1$	$-X_2 + \frac{2b}{c+d}X_4$	0	X_4
X_4	0	0	$-X_4$	0

Table 10.2. Adjoint table of Lie algebra (10.5)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$-2\epsilon X_1 + X_3$	X_4
X_2	X_1	X_2	$-\epsilon X_2 + X_3 + \frac{2b}{c+d}\epsilon X_4$	X_4
X_3	$e^{2\epsilon}X_1$	$e^{\epsilon}X_2 - \frac{b}{c+d}e^{-\epsilon}(-1+e^{2\epsilon})$	X_3	$e^{-\epsilon}X_4$
X_4	X_1	X_2	$X_3 + \epsilon X_4$	X_4

Using Table 10.2 and following the method detailed in, for example [30, 78, 133], we obtain the following set of elements of the optimal system:

$$\{X_2,\ X_3,\ X_1 + X_2,\ X_1 + X_4\}\,. \tag{10.6}$$

10.3 Group Invariant solutions

Using results of the previous section we will now proceed to construct an optimal system of group invariant solutions through symmetry reductions [30, 133].

10.3.1 Case $X_1 + kX_2$

In this section we seek to obtain group invariant solutions corresponding to the travelling wave variable. The linear combination $X_1 + kX_2$, with k a constant, gives the invariant solution $u = U(\xi)$, $\xi = x - kt$ leading to the nonlinear ordinary differential equation (NLODE)

$$aU'''' + (b - k^2)U'' + (2c + d)U'U'' + (2c + d)U''^2 = 0. \tag{10.7}$$

Integrating (10.7) twice with respect to ξ we obtain the third-order NLODE

$$aU'' + \frac{1}{2}(2c + d)U'^2 + (b - k^2)U' + C_1\xi + C_2 = 0 \tag{10.8}$$

for some arbitrary constants of integration C_1 and C_2.

10.3.1.1 Elliptic integral solutions

Suppose we take $C_1 = 0$ and let

$$U' = \frac{6a}{2c+d}V(\xi),$$

then from equation (10.8) we have

$$V'' + 3V^2 - \omega V + C_3 = 0. \tag{10.9}$$

Here, $\omega = 6(k^2 - b)/(2c + d)$ and $C_3 = C_2(2c + d)/6$. The solution of NLODE (10.7) is thus given by

$$U(\xi) = \frac{6a}{2c+d}\int V(\xi)\mathrm{d}\xi. \tag{10.10}$$

Many NLPDEs can be reduced to equation (10.9), see for example [20, 93]. Multiplying equation (10.9) by V' and integrating with respect to ξ we have the first-order ODE

$$V'^2 + 2V^3 - \omega V^2 + 2C_3V + 2C_4 = 0, \tag{10.11}$$

which we can write as

$$V'^2 = -2(V - \alpha_1)(V - \alpha_2)(V - \alpha_3). \tag{10.12}$$

Here α_1, α_2 and α_3 are some roots of the equation $V^3 - \omega V^2/2 + C_3V + C_4 = 0$, with $\alpha_1 > \alpha_2 > \alpha_3$. According to [95] and taking into account equation (10.10), we have the elliptic solution

$$u(t,x) = \frac{6a\left\{(\alpha_1-\alpha_3)^2\,\mathrm{E}\left(\mathrm{sn}\left(\Delta\xi, M^2\right), M^2\right) - \Delta\left(\left(\alpha_2 - 2\,\alpha_3\right)\alpha_1 + {\alpha_3}^2\right)\xi\right\}}{\Delta\,(2c+d)\,(\alpha_1-\alpha_2)} \tag{10.13}$$

with

$$M^2 = \frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3},\ \Delta = \sqrt{\frac{\alpha_1-\alpha_3}{2}}.$$

Here, E is the incomplete elliptic integral of the second kind, which is given by

$$E(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{1-t^2}\mathrm{dt},$$

whereas (sn) is the Jacobi elliptic sine function. Figure 10.1 is a depiction of solution (10.13).

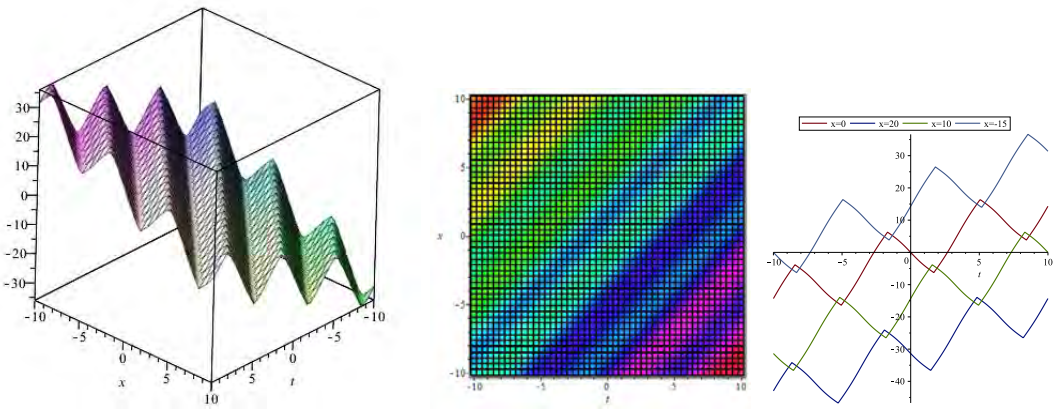

Figure 10.1: Profile of elliptic solution (10.13) for $a = c = k = \alpha_3 = 1, b = d = \alpha_2 = 2$, $\alpha_1 = 3$.

10.3.1.2 Rational and hyperbolic solutions

Considering the NLODE (10.8) again, multiplying by U'' and integrating with respect to ξ we have the second-order ODE

$$3akU''^2 + 2ckU'^3 + dkU'^3 + 3k(b - k^2)U'^2 + 6(C_1\xi + C_2)U' - 6C_1U + 6C_3 = 0. \tag{10.14}$$

In order to obtain solutions in terms of elementary functions, we take the arbitrary constants $C_i = 0$, $i = 1, \ldots, 3$. Equation (10.14) with $C_1 = C_2 = C_3 = 0$ will thus admit the Lie point symmetries

$$X_1 = \partial/\partial\xi, X_2 = \partial/\partial U. \tag{10.15}$$

From generator X_1 we can reduce equation (10.14) into the first-order ODE

$$3aG'^2 + 2cG^3(z) + G^3(z)d - 3k^2G^2(z) + 3bG^2(z) = 0. \tag{10.16}$$

The reader should note the use of appropriate invariants $z = \xi$ and $G(z) = U'$ culminating in (10.16). Solving (10.16) with the aid of Maple and reverting to native variables we have the following rational and hyperbolic solutions respectively:

$$u(t,x) = C_4 + \frac{3\,(k^2-b)}{2\,c+d}(x-kt) \tag{10.17}$$

$$u(t,x) = C_5 + \frac{6a\,(k^2-b)}{(2\,c+d)\,\sqrt{a\,(k^2-b)}}\tanh\left[\frac{\sqrt{a\,(k^2-b)}\,(x-kt-C_4)}{2a}\right], \tag{10.18}$$

where C_4 and C_5 are integration constants. Profile of solution (10.18) is given in Figure 10.2.

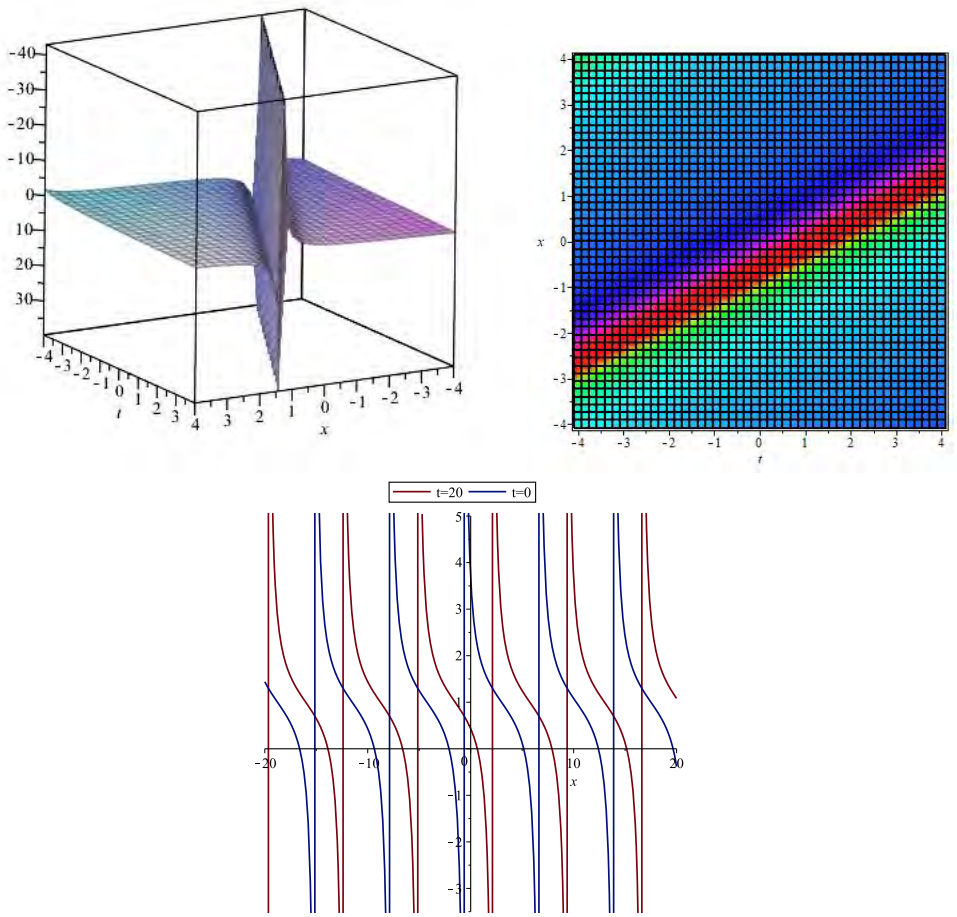

Figure 10.2: Profile of singular soliton solution (10.18) for $a = b = c = C_5 = 1, d = 5, k = 0.5, C_4 = 3$.

Likewise, using symmetry generator X_2, equation (10.14) is reduced to the first-order ODE

$$3aG' + (2c + d)G(z) + 3(b - k^2) = 0. \tag{10.19}$$

From (10.19) we recover the rational solution (10.17) as is, and the hyperbolic solution

$$u(t, x) = \frac{6\sqrt{a(k^2 - b)}}{2\,c + d} \tanh\left[\frac{\sqrt{a\,(k^2 - b)}\,(C_7 + x - kt)}{2a}\right] + C_6 \tag{10.20}$$

with integration constants C_6 and C_7. Taking the same parameter values as in Figure 10.2, the one soliton profile of solution (10.20) is given in Figure 10.3.

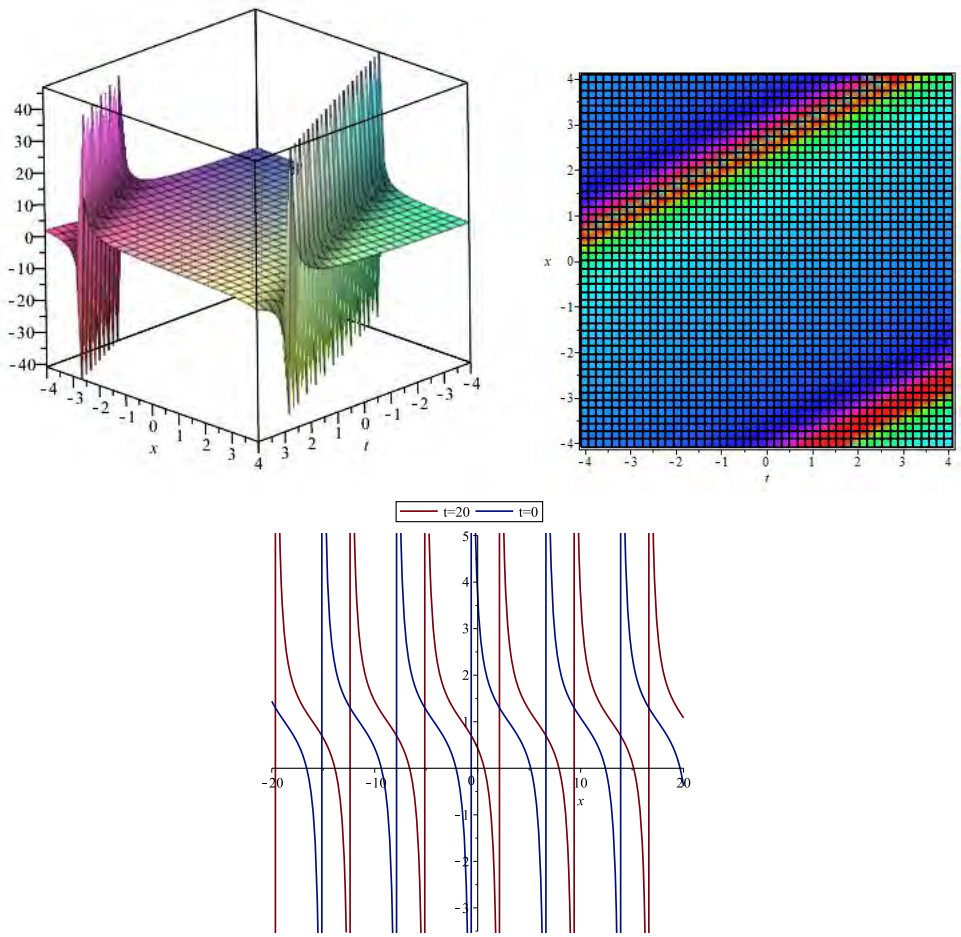

Figure 10.3: Profile of singular soliton solution (10.20) for $a = b = c = C_5 = 1, d = 5, k = 0.5, C_4 = 3$.

10.3.2 Case X_2

Using optimal system element X_2 we have the invariant $u = U(t)$, which in turn gives us the ODE $U''' = 0$ and thus we obtain

$$u(t,x) = \frac{1}{2}C_1 t^2 + C_2 t + C_3$$

as a solution of (10.1), where C_1, C_2 and C_3 are arbitrary constants.

10.3.3 Case $X_1 + X_4$

The optimal system element $X_1 + X_4$ provides the invariant solution $u = t + U(x)$ that leads us to the solution

$$u(t,x) = \frac{1}{2}C_1 x^2 + C_2 x + C_3$$

of (10.1). Here C_1, C_2 and C_3 are arbitrary constants.

10.3.4 Case X_3

The group invariant solution corresponding to vector X_3 is $u(t,x) = 1/t^{1/2}U(\xi) - b/(c+d)x$, where $\xi = x/t^{1/2}$. Accordingly, the NLPDE (10.1) is transformed into the fifth-order highly nonlinear ODE

$$\begin{aligned} &4a\xi U^{(5)} + 20aU^{(4)} + 4\xi(2c+d)U'U''' + 4cUU''' - \xi^3 U''' - 12\xi^2 U'' \\ &+ 4(7c+5d)U'U'' + 4\xi(d+2c)U''^2 - 33\xi U' - 15U = 0. \end{aligned} \tag{10.21}$$

Equation (10.21) can be integrated once with respect to ξ, giving

$$\begin{aligned} &4aU^{(4)} + 16aU''' + 4cUU'' - \xi^3 U'' + 4\xi(d+2c)U'U'' - 9\xi^2 U' + 8(d+c)U'^2 \\ &- 15\xi U + C_1 = 0, \end{aligned} \tag{10.22}$$

where C_1 is a constant of integration. We will use the power series solution method [25, 27, 77] to solve equation (10.22). If we let

$$U(\xi) = \sum_{z=0}^{\infty} g_z \xi^z, \tag{10.23}$$

where g_z, $z = 0, 1, \ldots$, are constants to be determined, then the various derivatives of (10.22) are

$$\begin{aligned}
U'(\xi) &= \sum_{z=0}^{\infty} (z+1) g_{z+1} \xi^z, \\
U''(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2) g_{z+2} \xi^z, \\
U'''(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2)(z+3) g_{z+3} \xi^z, \\
U^{(4)}(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2)(z+3)(z+4) g_{z+4} \xi^z.
\end{aligned} \tag{10.24}$$

By taking into account the derivatives (10.24), equation (10.22) is transformed into

$$\begin{aligned}
&96ag_4 + 480ag_5\xi + 1440ag_6\xi^2 + 4a \sum_{z=3}^{\infty} (z+1)(z+2)(z+3)(z+4) g_{z+4} \xi^z \\
&96ag_3 + 384ag_4\xi + 960ag_5\xi^2 + 16a \sum_{z=3}^{\infty} (z+1)(z+2)(z+3) g_{z+3} \xi^z \\
&8cg_0g_2 + 24cg_0g_3\xi + 48cg_0g_4\xi^2 + 4c \sum_{z=3}^{\infty} \sum_{j=0}^{z} (z+1-j)(z+2-j) g_j g_{z+2-j} \xi^z \\
&- \sum_{z=3}^{\infty} (z-2)(z-1) g_{z-1} \xi^z + 4(d+2c) \left(2g_1g_2\xi + 6g_1g_3\xi^2\right) + 4(d+2c) \\
&\times \sum_{z=3}^{\infty} \sum_{j=0}^{z-1} (j+1)(z-j)(z+1-j) g_{j+1} g_{z+1-j} \xi^z - 9\left(g_1 + g_2\xi + g_3\xi^2\right) \\
&- 9 \sum_{z=3}^{\infty} (z+1) g_{z+1} \xi^z + 8(d+c) g_1^2 \xi^2 + 8(d+c) \sum_{z=3}^{\infty} \sum_{j=0}^{z-2} (j+1)(z-1-j) \\
&\times g_{j+1} g_{z-1-j} \xi^z - 15g_1\xi - 15g_2\xi^2 - 15 \sum_{z=3}^{\infty} g_z \xi^z + C_1 = 0.
\end{aligned} \tag{10.25}$$

For more details, see [25, 77] and the references therein. Now splitting equation (10.25) with respect to powers of ξ, we have

$$\xi^0 \ : \ 8\,cg_0\,g_2 + 96\,ag_3 + 96\,ag_4 + C_1 - 9\,g_1 = 0, \tag{10.26}$$

$$\xi^1 \ : \ 24\,cg_0\,g_3 + 16\,cg_1\,g_2 + 8\,dg_1\,g_2 + 384\,ag_4 + 480\,ag_5 - 15\,g_1 - 9\,g_2 = 0, \tag{10.27}$$

$$\begin{aligned}\xi^2 \ : \ & 48\,cg_0\,g_4 + 8\,c{g_1}^2 + 48\,cg_1\,g_3 + 8\,d{g_1}^2 + 24\,dg_1\,g_3 + 960\,ag_5 + 1440\,ag_6 \\ & - 15\,g_2 - 9\,g_3 = 0,\end{aligned} \tag{10.28}$$

and generally for $z \geq 3$ we have the recurrence relation

$$\begin{aligned}g_{z+4} = & -\frac{1}{4a(z+1)(z+2)(z+3)(z+4)}\left\{4c\sum_{j=0}^{z}(z+1-j)(z+2-j)g_j g_{z+2-j}\right. \\ & -(z-2)(z-1)g_{z-1} + 16a(z+1)(z+2)(z+3)g_{z+3} \\ & +4(d+2c)\sum_{j=0}^{z-1}(j+1)(z-j)(z+1-j)g_{j+1}g_{z+1-j} - 9(z+1)g_{z+1} \\ & \left.+8(d+c)\sum_{j=0}^{z-2}(j+1)(z-1-j)g_{j+1}g_{z-1-j} - 15g_z\right\}.\end{aligned} \tag{10.29}$$

Taking g_i, $i = 0, \ldots, 3$ arbitrary, we can determine explicit values for g_4, g_5 and g_6. From equations (10.26)–(10.28), we have the following:

$$\begin{aligned}g_4 = & \frac{1}{96a}\left(9g_1 - 8cg_0g_2 - 96ag_3 - C_1\right), \\ g_5 = & \frac{1}{480a}\left\{\left(\left(-16c - 8d\right)g_1 + 32cg_0 + 9\right)g_2 - 24cg_0g_3 + 384ag_3 + 4C_1 - 21g_1\right\}, \\ g_6 = & \frac{1}{2880a^2}\left(-1536a^2g_3 + \left(\left(-16c - 16d\right){g_1}^2 + \left(\left(64g_2 - 96g_3\right)c + 32dg_2\right.\right.\right. \\ & -48dg_3 + 84\big)g_1 - 128g_0\left(g_2 - 3/2g_3\right)c - 16C_1 - 6g_2 + 18g_3\big)a \\ & \left.\left.+cg_0\left(8cg_0g_2 + C_1 - 9g_1\right)\right)\right).\end{aligned} \tag{10.30}$$

Likewise, from (10.29) we can deduce exact values of successive terms g_m, $m = 7, 8, \ldots$. Thus, the power series solution of (10.1) can be written as

$$\begin{aligned}
u(t,x) = & g_0 + g_1\xi + g_2\xi^2 + g_3\xi^3 + \frac{1}{96a}\left(9g_1 - 8cg_0g_2 - 96ag_3 - C_1\right)\xi^4 \\
& \frac{1}{480a}\left\{\left(\left(-16c - 8d\right)g_1 + 32cg_0 + 9\right)g_2 - 24cg_0g_3 + 384ag_3 + 4C_1 - 21g_1\right\}\xi^5 \\
& + \frac{1}{2880a^2}\left(-1536a^2g_3 + \left(\left(-16c - 16d\right){g_1}^2 + \left(\left(64g_2 - 96g_3\right)c + 32dg_2\right.\right.\right. \\
& \left.\left.\left. -48dg_3 + 84\right)g_1 - 128g_0\left(g_2 - 3/2g_3\right)c - 16C_1 - 6g_2 + 18g_3\right)a \right. \\
& \left. + cg_0\left(8cg_0g_2 + C_1 - 9g_1\right)\right)\xi^6 \\
& - \frac{1}{4a}\sum_{z=3}^{\infty}\frac{z!}{(z+4)!}\left\{4c\sum_{j=0}^{z}(z+1-j)(z+2-j)g_jg_{z+2-j}\right. \\
& -(z-2)(z-1)g_{z-1} + 16a(z+1)(z+2)(z+3)g_{z+3} \\
& +4(d+2c)\sum_{j=0}^{z-1}(j+1)(z-j)(z+1-j)g_{j+1}g_{z+1-j} - 9(z+1)g_{z+1} \\
& \left. +8(d+c)\sum_{j=0}^{z-2}(j+1)(z-1-j)g_{j+1}g_{z-1-j} - 15g_z\right\}\xi^{z+4}.
\end{aligned}$$

10.4 One and two-wave solutions

In this section we seek multiple-wave solutions of equation (10.1) using the multiple exponential function method [10].

10.4.1 One-soliton solutions

In order to obtain one-wave solutions we take $p(\theta_1) = A_0 + A_1\theta_1$ and $g(\theta_1) = B_0 + B_1\theta_1$. Following the steps alluded to in Section 1.6.1, we obtain the following two sets of dispersion relations:

(i) Case 1.

$$A_1 = \frac{(12\,B_0\,ak_1 + 2\,A_0\,c + A_0\,d)\,B_1}{B_0\;(2\,c+d)},\;\; \omega_1 = \sqrt{ak_1{}^2 + b}k_1. \tag{10.31}$$

(ii) Case 2.

$$A_1 = \frac{A_0\,B_1}{B_0},\;\; \omega_1 = \omega_1. \tag{10.32}$$

In both cases A_0, B_0 and B_1 are arbitrary constants. Thus, the one-wave solutions corresponding to (10.31) and (10.32) are given by

$$u(t,x) = \frac{A_0 + A_1 e^{k_1 x - \omega_1 t}}{B_0 + B_1 e^{k_1 x - \omega_1 t}} \tag{10.33}$$

and the one-soliton profile of (10.32) is depicted in Figure 10.4.

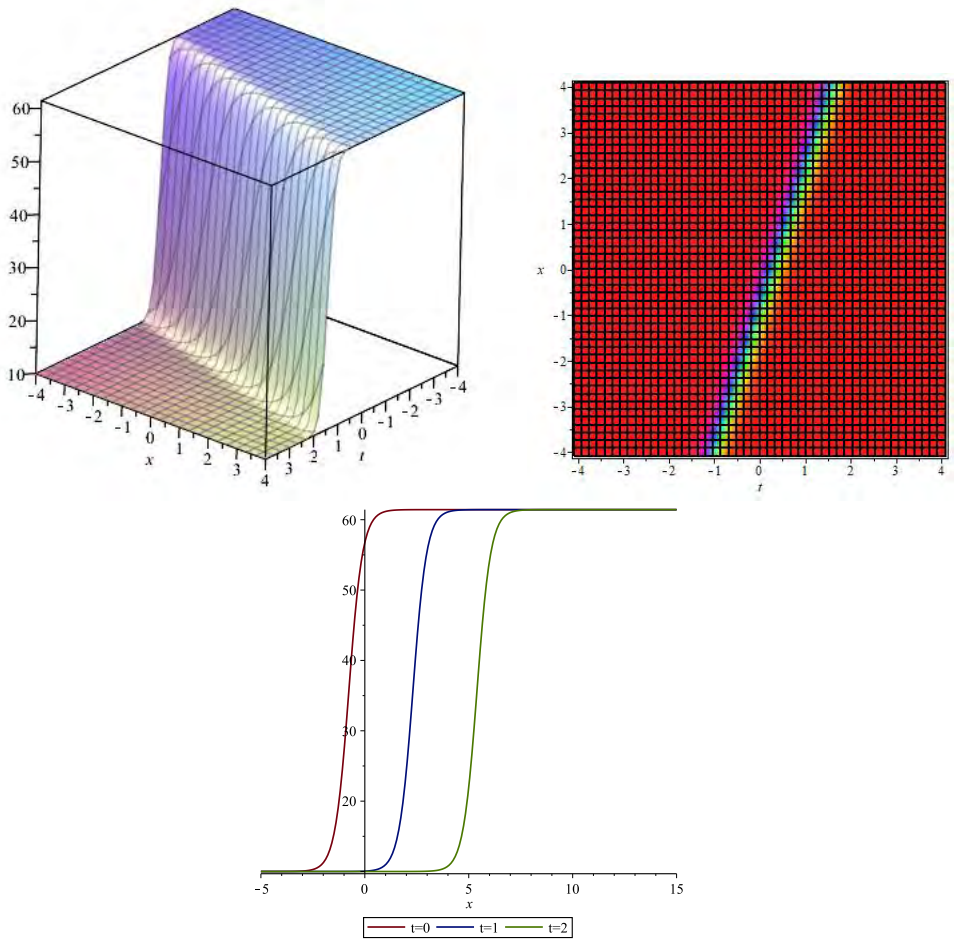

Figure 10.4: Profile of kink soliton (10.20) for $a = A_0 = B_1 = 1$, $c = B_0 = 0.1$, $k_1 = 3$ $b = d = 0.5$.

10.4.2 Two-soliton solutions

As outlined in Section 1.6.1, two-wave solutions are obtained by taking $u = p/g$, where

$$p(\theta_1, \theta_2) = 2(k_1\theta_1 + k_2\theta_2 + A_{12}(k_1 + k_2)\theta_1\theta_2), \quad g(\theta_1, \theta_2) = (1 + \theta_1 + \theta_2 + A_{12}\theta_1\theta_2)$$

and $\theta_i = e^{k_i x - \omega_i t}$. Also, A_{12} is a constant to be determined. Consequently, we have the following parameter relations:

(i) Case 1.

$$\omega_1 = \frac{\left(a{k_1}^3 + bk_1 + \sqrt{\left(a{k_2}^2 + b\right)\left(a{k_1}^2 + b\right)}k_2\right)k_1}{\sqrt{a{k_1}^4 + a{k_2}^4 + b{k_1}^2 + b{k_2}^2 + 2\,k_1\,k_2\sqrt{\left(a{k_2}^2 + b\right)\left(a{k_1}^2 + b\right)}}},$$

$$\omega_2 = -\frac{\left(a{k_2}^3 + bk_2 + \sqrt{\left(a{k_2}^2 + b\right)\left(a{k_1}^2 + b\right)}k_1\right)k_2}{\sqrt{a{k_1}^4 + a{k_2}^4 + b{k_1}^2 + b{k_2}^2 + 2\,k_1\,k_2\sqrt{\left(a{k_2}^2 + b\right)\left(a{k_1}^2 + b\right)}}},$$

$$A_{12} = \frac{a{k_1}^4 + a{k_2}^4 + b{k_1}^2 + b{k_2}^2 + 2\sqrt{{k_1}^2{k_2}^2\left(a{k_2}^2 + b\right)\left(a{k_1}^2 + b\right)}}{\left(k_1 + k_2\right)^2\left(a{k_1}^2 + a{k_2}^2 + b\right)}, \tag{10.34}$$

$$c = d = 2a.$$

(ii) Case 2.

$$\omega_1 = \frac{k_1\,k_2\left({k_1}^2 - 2\,{k_2}^2\right)\sqrt{a\left(k_1 - 2\,k_2\right)\left(2\,{k_2}^3 - {k_1}^3 + {k_2}^2 k_1\right)}}{\left(k_1 - 2\,k_2\right)\left({k_1}^3 - k_1\,{k_2}^2 - 2\,{k_2}^3\right)},$$

$$\omega_2 = \frac{k_1\,k_2\,\left({k_1}^2 - k_1\,k_2 - {k_2}^2\right)\sqrt{a\left(k_1 - 2\,k_2\right)\left(2\,{k_2}^3 - {k_1}^3 + {k_2}^2 k_1\right)}}{\left(k_1 - 2\,k_2\right)\left({k_1}^3 - k_1\,{k_2}^2 - 2\,{k_2}^3\right)}, \tag{10.35}$$

$$A_{12} = \frac{2\,k_1 - 3\,k_2}{k_2},\quad b = -\frac{a\left({k_1}^6 - 2\,{k_1}^5 k_2 + 4\,{k_2}^6\right)}{\left(k_1 - 2\,k_2\right)\left({k_1}^3 - k_1\,{k_2}^2 - 2\,{k_2}^3\right)},\quad c = d = 2a.$$

(iii) Case 3.

$$\omega_1 = \omega_2 = \sqrt{a{k_1}^2 + b}k_1,\quad A_{12} = \frac{(2a + d){k_1}^2 + 2\,b}{2(a{k_1}^2 + b)},$$
$$c = 3a - \frac{1}{2}d,\quad k_2 = -k_1. \tag{10.36}$$

The profiles of solutions (10.34)–(10.36) are given in Figures 10.5–10.7.

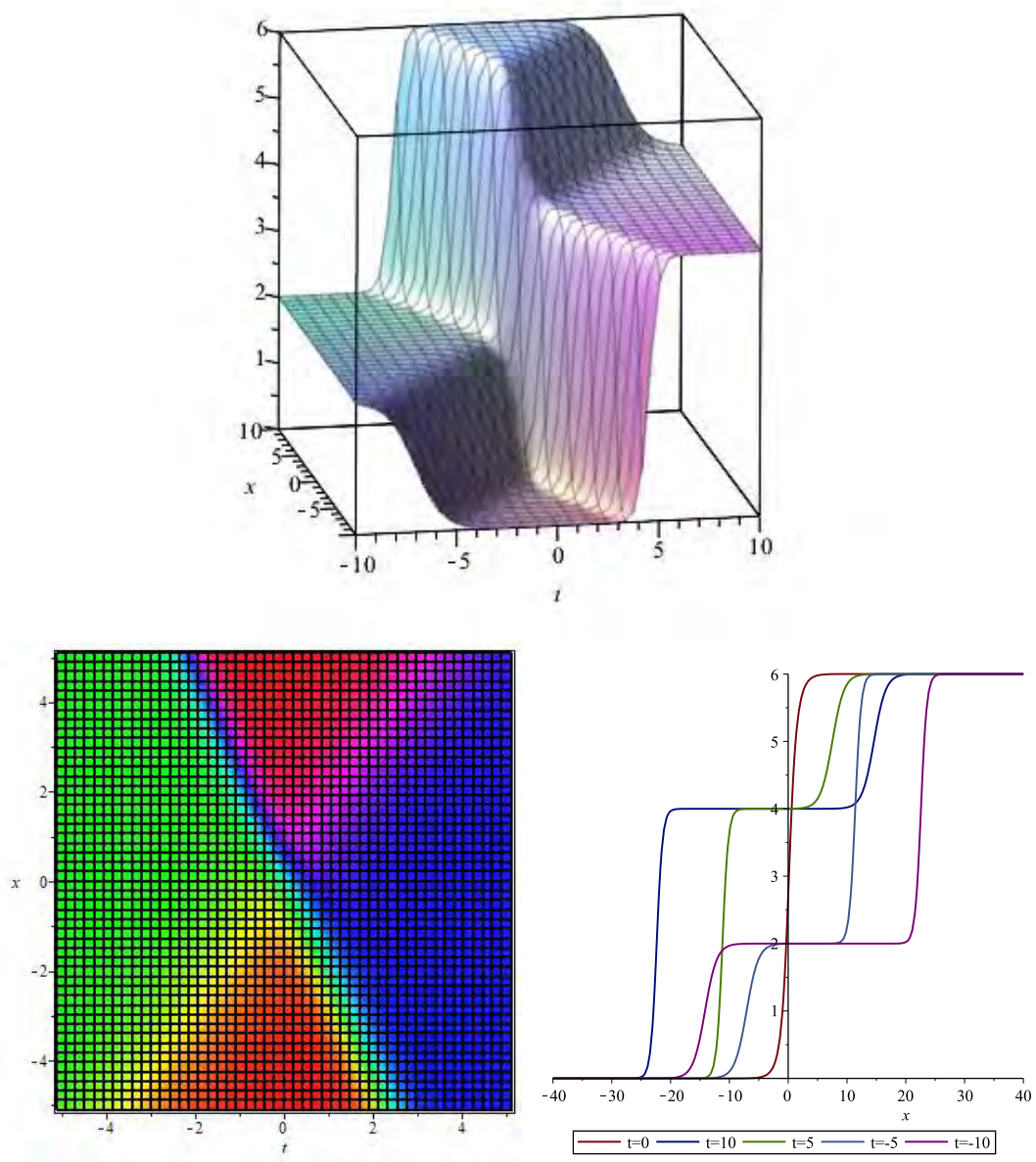

Figure 10.5: Interaction of kink and anti-kink solitons. Solution (10.34) for $a = b = k_1 = 1$ and $k_2 = 2$.

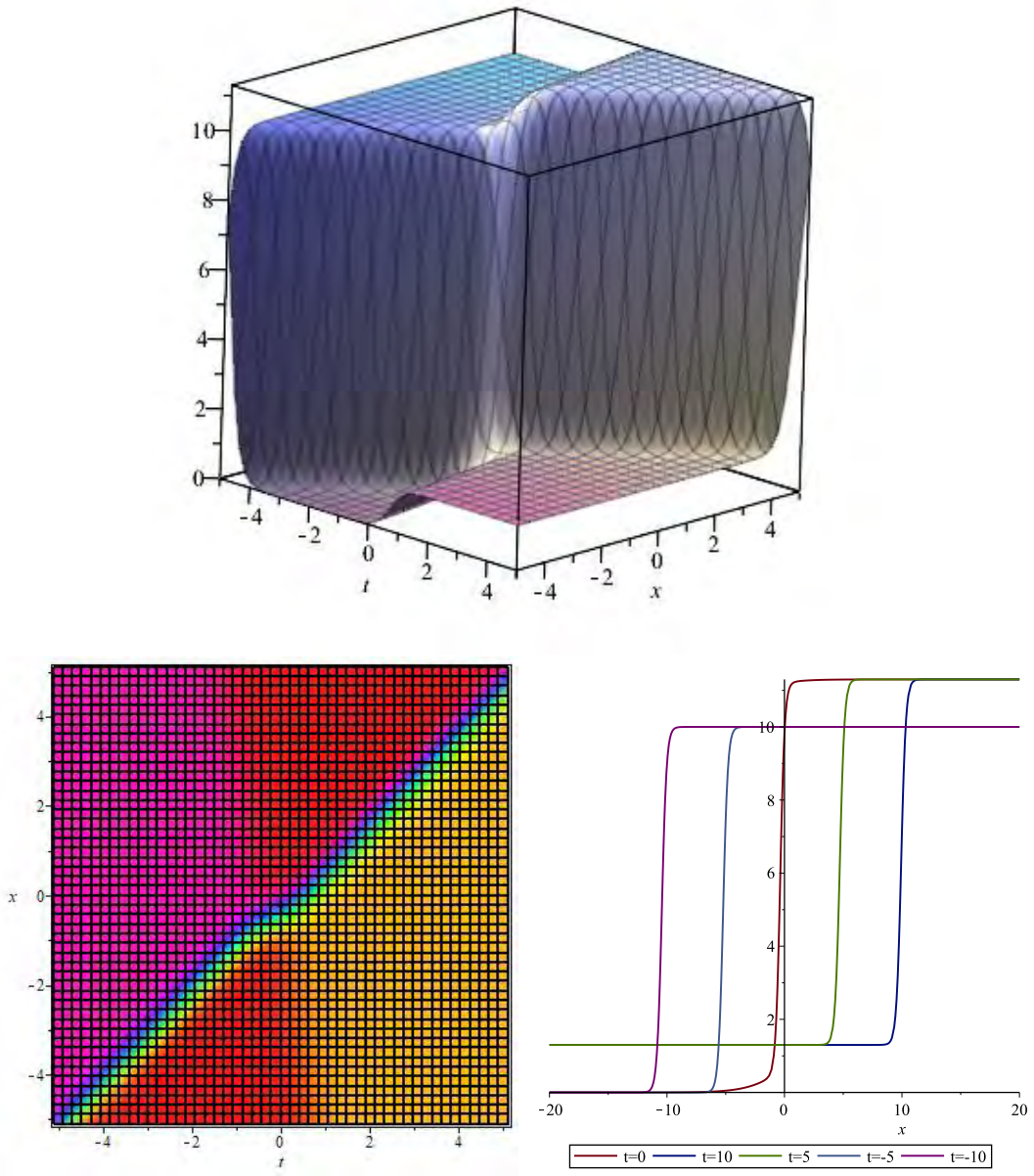

Figure 10.6: Interaction of two kink solitons with different vertical displacements. Solution (10.35) for $a = -2$, k_1=5, and $k_2 = 0.65$.

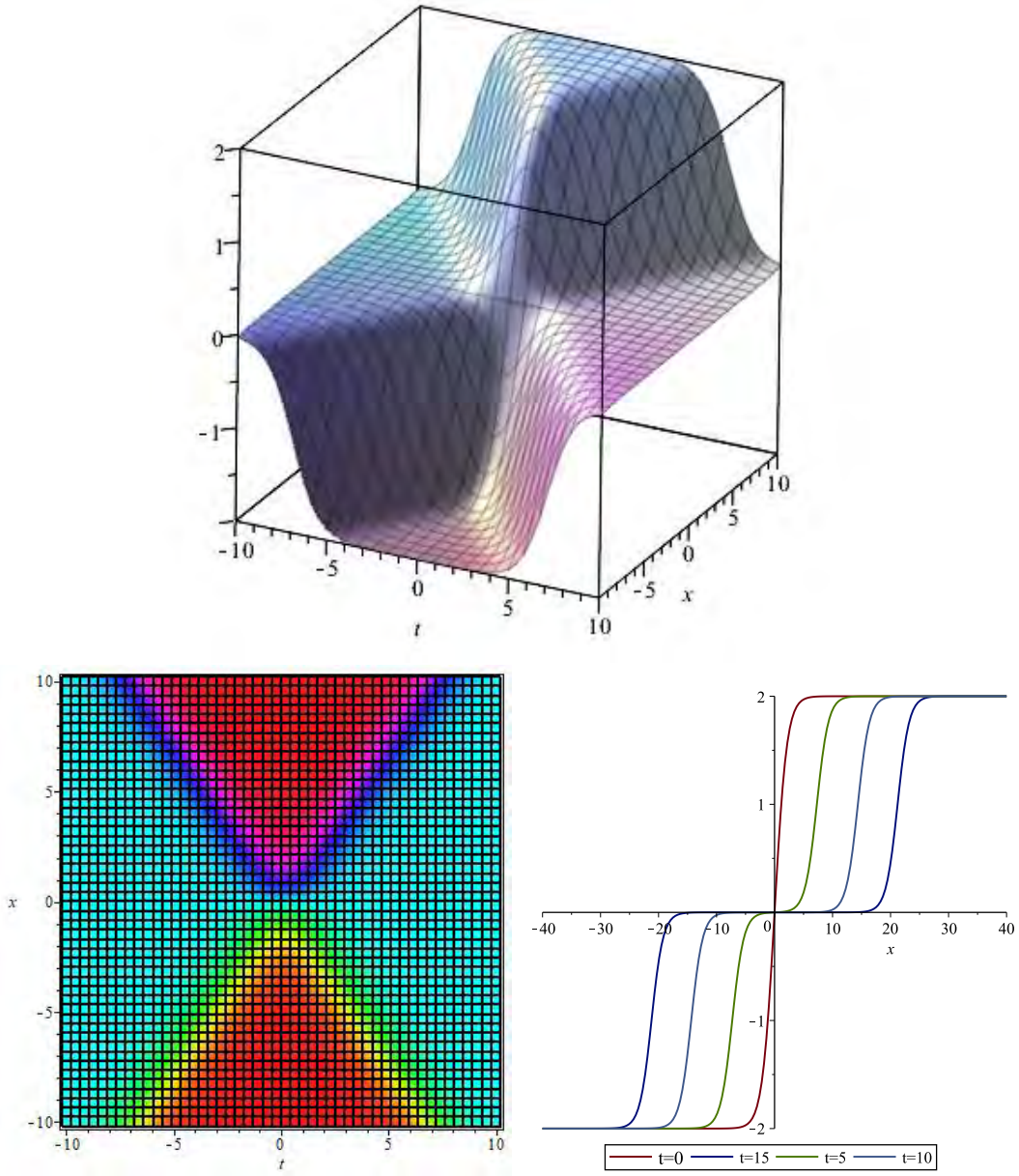

Figure 10.7: Interaction of dark and bright solitons. Solution (10.36) for $a = b = d = k_1 = 1$.

10.5 Conservation laws of (10.1)

Using the multiplier approach [45, 77, 134], we determine local low-order conservation laws of equation (10.1). This method is a modern form of the classical Noether's approach, see for example [43].

Multipliers (Λ) can be determined via the determining condition

$$\frac{\delta}{\delta u}(\Lambda E) = 0 \tag{10.37}$$

with E given by equation (10.1) and where $\delta/\delta u$ is the Euler operator as given in [115]. Multiplier determining condition (10.37) gives the following six linear PDEs:

$$\Lambda_{uu} = 0,\ \Lambda_{xu} = 0,\ \Lambda_{xx} = 0,\ \Lambda_{tu} = 0,\ \Lambda_{tx} = 0,\ \Lambda_{ttt} = 0. \tag{10.38}$$

The solution algorithm of (10.38) is analogous to that of obtaining Lie point symmetries. In this instance we obtain the zeroth-order multipliers $\Lambda(t, x, u)$

$$\Lambda = C_1 x + C_2 u + \frac{1}{2}C_3 t^2 + C_4 t + C_5, \tag{10.39}$$

where C_1,...,C_5 are constants. For the case (10.1) and (10.39), formula (1.48) takes the form

$$
\begin{aligned}
T=\int_0^1 \Bigg\{ & u\left(\left(\frac{\partial E\Lambda}{\partial u_t}\right)\Big|_{u=u_{(\lambda)}} - D_x\left(\frac{\partial E\Lambda}{\partial u_{tx}}\right)\Big|_{u=u_{(\lambda)}} + D_t^2\left(\frac{\partial E\Lambda}{\partial u_{ttt}}\right)\Big|_{u=u_{(\lambda)}}\right. \\
& \left. + D_x^2\left(\frac{\partial E\Lambda}{\partial u_{txx}}\right)\Big|_{u=u_{(\lambda)}} + D_x^4\left(\frac{\partial E\Lambda}{\partial u_{txxxx}}\right)\Big|_{u=u_{(\lambda)}}\right) + u_t\left(-D_t\left(\frac{\partial E\Lambda}{\partial u_{ttt}}\right)\Big|_{u=u_{(\lambda)}}\right) \\
& + u_{tt}\left(\frac{\partial E\Lambda}{\partial u_{ttt}}\right)\Big|_{u=u_{(\lambda)}}\Bigg\}\,\mathrm{d}\lambda,
\end{aligned}
$$

$$
\begin{aligned}
X=\int_0^1 \Bigg\{ & u\left(\left(\frac{\partial E\Lambda}{\partial u_x}\right)\Big|_{u=u_{(\lambda)}} - D_x\left(\frac{\partial E\Lambda}{\partial u_{xx}}\right)\Big|_{u=u_{(\lambda)}} + D_x^2\left(\frac{\partial E\Lambda}{\partial u_{xxx}}\right)\Big|_{u=u_{(\lambda)}}\right) \\
& + u_t\left(\left(\frac{\partial E\Lambda}{\partial u_{tx}}\right)\Big|_{u=u_{(\lambda)}} - D_x\left(\frac{\partial E\Lambda}{\partial u_{txx}}\right)\Big|_{u=u_{(\lambda)}} - D_x^3\left(\frac{\partial E\Lambda}{\partial u_{txxxx}}\right)\Big|_{u=u_{(\lambda)}}\right) \\
& + u_x\left(\left(\frac{\partial E\Lambda}{\partial u_{xx}}\right)\Big|_{u=u_{(\lambda)}} - D_x\left(\frac{\partial E\Lambda}{\partial u_{xxx}}\right)\Big|_{u=u_{(\lambda)}}\right) + u_{tx}\left(\left(\frac{\partial E\Lambda}{\partial u_{txx}}\right)\Big|_{u=u_{(\lambda)}}\right. \\
& \left. + D_x^2\left(\frac{\partial E\Lambda}{\partial u_{txxxx}}\right)\Big|_{u=u_{(\lambda)}}\right) + u_{xx}\left(\left(\frac{\partial E\Lambda}{\partial u_{xxx}}\right)\Big|_{u=u_{(\lambda)}}\right) \\
& + u_{txx}\left(-D_x\left(\frac{\partial E\Lambda}{\partial u_{txxxx}}\right)\Big|_{u=u_{(\lambda)}}\right) + u_{txxx}\left(\left(\frac{\partial E\Lambda}{\partial u_{txxxx}}\right)\Big|_{u=u_{(\lambda)}}\right)\Bigg\}\,\mathrm{d}\lambda.
\end{aligned}
\tag{10.40}
$$

Thus the conserved quantities of equation (10.1) are

$$
\begin{aligned}
T_1^t =& -\frac{1}{2}cxuu_{xxx} - \frac{3}{2}cxu_xu_{xx} - dxu_xu_{xx} + xu_{tt} - bxu_{xx} - axu_{xxxx}, \\
T_1^x =& \frac{1}{2}cxuu_{txx} - \frac{1}{2}cuu_{tx} + \frac{1}{2}cu_tu_x - \frac{1}{2}cxu_{xx}u_t;
\end{aligned}
$$

$$
\begin{aligned}
T_2^t =& -\frac{1}{3}cu^2u_{xxx} - cuu_xu_{xx} - \frac{2}{3}duu_xu_{xx} + uu_{tt} - \frac{1}{2}{u_t}^2 - \frac{1}{2}buu_{xx} - \frac{1}{2}auu_{xxxx}, \\
T_2^x =& \frac{1}{3}cu^2u_{txx} + \frac{2}{3}cu_t{u_x}^2 + \frac{1}{3}du_t{u_x}^2 + \frac{1}{2}bu_xu_t - \frac{1}{2}buu_{tx} - \frac{1}{2}auu_{txxx} + \frac{1}{2}au_xu_{txx} \\
& - \frac{1}{2}au_{xx}u_{tx} + \frac{1}{2}au_{xxx}u_t - \frac{2}{3}cuu_{tx}u_x - \frac{1}{3}duu_{tx}u_x - \frac{1}{3}cu_{xx}uu_t;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t =& -\frac{1}{4}cut^2u_{xxx} - \frac{3}{4}u_xu_{xx}t^2 - \frac{1}{2}dt^2u_xu_{xx} + u - tu_t + \frac{1}{2}t^2u_{tt} - \frac{1}{2}bt^2u_{xx} - \frac{1}{2}at^2u_{xxxx}, \\
T_3^x =& \frac{1}{4}cut^2u_{txx} + \frac{1}{2}ctuu_{xx} + \frac{1}{2}ctu_x^2 + \frac{1}{2}dtu_x{}^2 - \frac{1}{4}ct^2u_tu_{xx} + btu_x + atu_{xxx};
\end{aligned}
$$

$$T_4^t = -\frac{1}{2}ctuu_{xxx} - \frac{3}{2}ctu_xu_{xx} - dtu_xu_{xx} - u_t + tu_{tt} - btu_{xx} - atu_{xxxx},$$
$$T_4^x = \frac{1}{2}ctuu_{txx} + \frac{1}{2}cuu_{xx} + \frac{1}{2}cu_x^2 + \frac{1}{2}du_x{}^2 - \frac{1}{2}ctu_{xx}u_t + bu_x + au_{xxx};$$

$$T_5^t = -\frac{1}{2}cuu_{xxx} - \frac{3}{2}cu_xu_{xx} - du_xu_{xx} + u_{tt} - bu_{xx} - au_{xxxx},$$
$$T_5^x = \frac{1}{2}cuu_{txx} - \frac{1}{2}cu_tu_{xx}.$$

Remark: These conserved vectors are local and low-order conserved vectors.

10.6 Concluding remarks

In this chapter we studied the new fifth-order integrable equation (10.1) from the point of view of Lie symmetries. We obtained an optimal system of group invariant solutions which culminated in various types of exact solutions, namely, rational, quadratic, hyperbolic, incomplete elliptic integral of the second kind solutions. The solution profiles depicted one singular soliton and periodic waves. Furthermore, with the aid of the multiple exponential method we constructed one and two soliton solutions. Kink, anti-kink, dark and bright soliton were observed. Five low-order conservation laws were obtained by utilising the multiplier method.

Chapter 11

Conclusions and future work

The aim of this work was to derive closed form solutions of selected nonlinear partial differential equations (NLPDEs) and to determine their conservation laws. We did this from the point of view of Lie group analysis.

In Chapter one we gave a brief theoretical justification of ideas that are pertinent to this thesis. This, amongst other things included definitions and theorems of certain concepts in Lie symmetry analysis, conserved quantities and other selected ad hoc methods for obtaining analytic solutions of NLPDEs.

Chapter two presented local conservation laws of three mKdV models. Through similarity reductions and direct integration we also obtained one-soliton solutions of each equation. Periodic solutions were obtained via the extended Jacobi elliptic function method.

In Chapter three we derived the variational symmetries of three (3+1)-D mBBM equations by using the Noether condition. Thereafter, nonlocal conserved vectors for the underlying equations were obtained. By taking a linear combination of the space-time Lie point symmetries we performed successive similarity reductions culminating in a bright one-soliton solution and periodic solutions from the Jacobi

elliptic function method.

Chapter four outlined an in-depth study of the four-dimensional Lie algebras of a coupled mKdV system. Five inequivalent subgroups were constructed and were used to obtain systems of ODEs. The power series solution method was used to solve some of these ODEs. We also introduced and utilised the first homotopy integral formula here to obtain three local conservation laws.

In Chapter five we obtained an eight-dimensional Lie algebra for (3+1)-dimensional generalised KP equation. Two of these were infinite dimensional symmetries. We thus obtained group invariant solutions from the remaining six vector fields. We obtained elliptic, logarithmic and rational solutions. Conserved quantities of this equation were computed using the multiplier method as well as Ibragimov's conservation theorem.

In Chapter six we performed a bifurcation analysis of a (2+1)-dimensional BKK system. We managed to decouple the system and consequently obtained a second-order ODE which coincided with a Hamiltonian planar dynamical system with three arbitrary constants. Ultimately, bounded travelling wave solutions were obtained and these included kink, anti-kink and periodic waves. By employing a homotopy integral approach, we derived five local conserved quantities of the BKK system.

Chapter seven is an extensive study of a fifth-order integrable NLPDE. A one parameter group of transformations corresponding to the four-dimensional Lie algebra was derived. An optimal system of group invariant solutions were obtained. Using Helmholtz conditions, the fifth-order integrable was proven to have a variational principle after increasing the order by one, thus we derived nonlocal conserved vectors using Noether's theorem. Finally, local and low-order conserved quantities were computed using a homotopy integral formula.

In Chapter eight we presented group transformations of a nine-dimensional Lie algebra for the (1+1)-dimensional generalised coupled mKdV system of four equations. One-parameter group of transformations for the nine-dimensional Lie algebra was derived. Using the copolar trio along with the simplest equation method, we obtained various kinds of analytic solutions. Nine local conservation laws of the system were also derived.

Chapter nine gave a triple-reduction of a third-order generalized hyperelastic-rod wave equation into a first-order autonomous ODE. Using the transformation $u(t, x) = U(x - ct)$, conserved quantities were reduced into first integrals. We also performed symmetry reductions of the equation for various possible cases of the arbitrary coefficient function.

In Chapter ten we investigated various types of solutions of a fifth-order NLPDE. Using an optimal system we obtained several varieties of group invariant solutions. We also computed one and two-wave solutions with the aid of the multiple exp-function method. Kink, anti-kink, interactions of dark and bright soliton were observed. Five low-order conservation laws were obtained by utilising the multiplier method.

For future work we plan to use the conservation laws derived here to obtain exact solutions of the nonlinear partial differential equations studied in this thesis.

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