CONSERVATION LAWS AND EXACT SOLUTIONS FOR SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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Contents

Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MRS DIMPHO MILLICENT MOTHIBI

Date: ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

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PROF CM KHALIQUE

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

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Chapter 3

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Chapter 4

DM Mothibi, CM Khalique, New exact solutions of coupled Korteweg-de Vries equations. Proceedings of The 2013 International Conference on Scientific Computing (CSC'13), 2013 World Congress in Computer Science, Computer Engineering, and Applied Computing (WORLDCOMP13), 22-25 July 2013, Las Vegas, Nevada, USA, ISBN: 1-60132-238-0.

Chapter 5

DM Mothibi, CM Khalique, Exact solutions of coupled Boussinesq equations. Chapter in the book: Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science, ISBN 978-3-319-12306-6. Series: Springer Proceedings in Mathematics & Statistics, Vol. 117. Cojocaru, M., Kotsireas, I.S., Makarov, R.N., Melnik, R., Shodiev, H. (Eds.) 2015, I, 479 p. 145 illus., 102 illus. in color. Copyright Springer International Publishing Switzerland 2015.

Chapter 6

DM Mothibi, CM Khalique, Conservation laws and exact solutions of a generalized

Zakharov-Kuznetsov equation. Symmetry 2015, 7(2), 949-961.

Chapter 7

DM Mothibi, Conservation laws for Ablowitz-Kaup-Newell-Segur equation. Accepted and to appear in American Institute of Physics Conference Proceedings of 13th International Conference of Numerical Analysis and Applied Mathematics 2015.

Chapter 8

DM Mothibi, B Muatjetjeja, CM Khalique, Conservation laws and exact solutions for potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity. Submitted for publication to Journal of Computational Mathematics.

Dedication

I dedicate this work to my late aunt Dipela Cynthia Tonyane, whose memories and the support she gave kept me going. To my loving husband, kids and family. Lastly this is also dedicated to my mentor and a father figure the late Professor MT Kambule.

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Abstract

In this thesis we study some nonlinear partial differential equations which appear in several physical phenomena of the real world. Exact solutions and conservation laws are obtained for such equations using various methods. The equations which are studied in this work are a generalized coupled (2+1)-dimensional hyperbolic system, a modified Kortweg-de Vries type equation, the higher-order modified Boussinesq equation with damping term, coupled Korteweg-de Vries equations, coupled Boussinesq equations, a generalized Zakharov-Kuznetsov equation, a generalized Ablowitz-Kaup-Newell-Segur equation and a potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity.

We perform a complete Lie symmetry classification of a generalized coupled (2+1)-dimensional hyperbolic system, which models many physical phenomena in nonlinear sciences. The Lie group classification of the system provides us with eleven-dimensional equivalence Lie algebra and has several possible extensions. It is further shown that several cases arise in classifying the arbitrary parameters, the forms of which include amongst others the power and exponential functions.

We obtain exact solutions of two nonlinear evolution equations, namely, modified Kortweg-de Vries equation and higher-order modified Boussinesq equation with damping term. The $(G'/G)-$expansion method is employed to obtain the exact solutions. Travelling wave solutions of three types are obtained and these are the solitary waves, periodic and rational. In addition, the conservation laws for higher-order modified Boussinesq equation with a damping term are constructed using the multiplier approach.

The $(G'/G)-$expansion method is employed to derive the exact travelling wave solutions of coupled Korteweg-de Vries equations. The solutions obtained include the soliton solutions. Furthermore, the conservation laws for these equations are

obtained.

Travelling wave solutions of coupled Boussinesq equations are determined and conservation laws are obtained for the system using the new conservation theorem and multiplier approach.

We study a generalized Zakharov–Kuznetsov equation in three variables, which has applications in the nonlinear development of ion-acoustic waves in a magnetized plasma. Conservation laws for this equation are constructed using the new conservation theorem. Furthermore, new exact solutions are obtained by employing the Lie symmetry method along with the simplest equation method.

Conservation laws of a generalized Ablowitz-Kaup-Newell-Segur equation are constructed by using Noether theorem. The exact solutions are obtained using the Lie symmetry method together with the simplest equation method and direct integration.

Finally, a potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity, which arises in a number of significant nonlinear problems of physics and applied mathematics is studied. We carry out Noether symmetry classification on this equation. Four cases arise depending on the values of p and consequently we construct conservation laws for these cases with respect to the second-order Lagrangian. In addition, exact solutions for this equation are obtained using the Lie group analysis together with the Kudryashov method and direct integration.

List of Acronyms

KdV: Kortweg-de Vries

mKdV: modified Kortweg-de Vries

gZK:	generalized Zakharov-Kuznetsov
KP:	Kadomtsev-Petviashvili
PKPp:	potential Kadomtsev-Petviashvili with power law nonlinearity
AKNS:	Ablowitz-Kaup-Newell-Segur
gAKNS:	generalized Ablowitz-Kaup-Newell-Segur
VCPKP	variable coefficients potential Kadomtsev-Petviashvili

Introduction

A large variety of real-world physical systems are governed by nonlinear partial differential equations. Such equations are very important because they are able to describe the real features in various fields of applications, for example, fluid mechanics, gas dynamics, combustion theory, relativity, thermodynamics, biology, and many others. Nonlinear partial differential equations of real life problems are difficult to solve analytically. Finding exact solutions of the nonlinear partial differential equations is a very important task and plays an important role in nonlinear science. There has recently been much attention devoted to the search for better and more efficient solution methods for determining solutions to nonlinear partial differential equations [1–34].

In the last few decades, a variety of effective methods for finding exact solutions were discovered. These include the homogeneous balance method [3], the ansatz method [4, 5], the variable separation approach [6], the inverse scattering transform method [7], the Bäcklund transformation [8], the Darboux transformation [9], the Hirota bilinear method [10], the $(G'/G)-$expansion method [11–13], the Kudryashov method [14–24] and Lie group analysis [25–31]. Such methods were successfully applied to nonlinear partial differential equations in obtaining their exact solutions.

Lie group analysis is one of the most powerful and systematic methods to determine solutions of nonlinear differential equations. It was originally developed by Marius Sophus Lie (1842-1899). His study gave rise to the modern theory of what is now universally known as Lie groups. Ever since, a large amount of work has been published in the literature on the subject of Lie groups applied to differential equations in terms of the Lie point symmetries admitted by the equation under study. Lie point symmetry of a differential equation is a one parameter point transformation which leaves the differential equation invariant. Lie theory enables one to reduce the order of ordinary differential equations. The reduction of a partial differential equation with respect to $r-$dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

It is well-known that conservation laws play an important role in the study of differential equations. Conservation laws describe physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion [28, 35, 36]. They have been used in investigating the existence, uniqueness, and stability of solutions of nonlinear partial differential equations [37, 38]. Also, they have been used in the development and use of numerical methods [39, 40]. Recently, conservation laws were used to obtain exact solutions of some partial differential equations [41–45]. Thus, it is essential to study conservation laws of differential equations.

Sophus Lie's work had influence on many mathematicians including Emmy Noether (1882-1935). A connection between symmetries and conservation laws for differential equations is established via Noether theorem [46, 47]. In addition to Lie point symmetries, Noether symmetries are also widely studied and are associated, in particular, with those differential equations which possess Lagrangians. The Noether symmetries, which are symmetries of the Euler-Lagrange systems, have interesting applications in the study of properties of particles moving under the influence of gravitational fields.

Noether theorem [46, 47] allows construction of conservation laws systematically. However, it can only be applied to differential equations with a Lagrangian. In order to overcome this limitation, several works have been done. See for example, [48–53]. Further developments have been made in this direction and the concepts of quasi self-adjoint, weak self adjointness and nonlinear self-adjoint were introduced in [54–59].

This thesis is structured as follows:

In Chapter one, we introduce the preliminaries that are needed in our study.

In Chapter two, a complete Lie group classification is performed on a generalized coupled (2+1)-dimensional hyperbolic system. As a result, the arbitrary functions which appear in the system are specified.

Chapter three presents the travelling wave solutions of a modified Kortweg-de Vries type equation and higher-order modified Boussinesq equation with damping term using the $(G'/G)-$expansion method. Conservation laws for the latter equation are constructed using the multiplier approach.

In Chapter four, exact solution and conservation laws for the coupled Korteweg-de Vries equation are found using $(G'/G)-$expansion method and the new conservation theorem due to Ibragimov, respectively.

Chapter five studies the exact solutions and conservation laws of the coupled Boussinesq equation.

In Chapter six, the exact solutions and conservation laws of a generalized Zakharov-Kuznetsov equation are obtained using the Lie symmetry method along with the simplest equation method and the new conservation theorem due to Ibragimov, respectively.

Chapter seven deals with the exact solutions and conservation laws of a generalized Ablowitz-Kaup-Newell-Segur equation. The simplest equation method is used to obtain exact solutions and the Noether approach is employed for the construction of conservation laws.

In Chapter eight, conservation laws for a potential Kadomtsev-Petviashvili equation with power law nonlinearity equation are constructed by applying the Noether theorem. In addition, the exact solutions for this equation are obtained using Kudryashov method.

Finally, in Chapter nine, a summary of the results of the thesis is presented and future work is suggested.

Bibliography is given at the end.

Chapter 1

Preliminaries

In this chapter, we present some preliminaries on Lie symmetry analysis and conservation laws of differential equations, which are used throughout this work and are based on references [25–31, 35, 46].

1.1 One-parameter group of continuous transformations

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a), \ \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 (Lie group) A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables

x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\, \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.2 Prolongations

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)} \cdots$.

In group analysis, all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [29].

We now consider a pth-order partial differential equations, namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a, about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x,u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}^\alpha_i = \bar{D}_j(u^\alpha), \qquad \bar{u}^\alpha_{ij} = \bar{D}_j(\bar{u}^\alpha_i) = \bar{D}_i(\bar{u}^\alpha_j), \cdots.$$

Applying (1.6) and (1.10), we obtain

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}^\alpha_j, \end{aligned} \tag{1.11}$$

and so

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

Letting

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

to be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (\text{sum on } j).
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta^{\alpha}_{ij} \;=\; D_j(\eta^{\alpha}_i) - u^{\alpha}_{ik} D_j(\xi^k), \quad \text{(sum on } k\text{)}. \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} \;=\; D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}\,j} D_{i_p}(\xi^j), \quad \text{(sum on } j\text{)}. \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_i} \quad \text{(sum on } i, \alpha\text{)}, \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p} \frac{\partial}{\partial u^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}.$$

1.3 Group admitted by a partial differential equation

Definition 1.2 (Point symmetry) The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}, \tag{1.18}$$

is a *point symmetry* of the pth-order partial differential equation (1.5), if

$$X^{[p]}(E_{\alpha}) = 0 \tag{1.19}$$

whenever $E_{\alpha} = 0$. This can also be written as

$$X^{[p]}\, E_{\alpha}\big|_{E_{\alpha}=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 (Determining equation) Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.4 Infinitesimal criterion of invariance

Definition 1.5 (Invariant) A function $F(x, u)$ is called an *invariant* of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\, \phi^\alpha(x, u, a)) = F(x, u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 (Lie equations) If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u^\alpha}}{da} = \eta^\alpha(\bar{x},\bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0} = x, \quad \bar{u^\alpha}\big|_{a=0} = u\,.$$

1.5 Conservation laws

1.5.1 Fundamental operators and their relationship

Consider a pth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, given by equation (1.5).

Definition 1.6 (Euler-Lagrange operator) The *Euler-Lagrange operator*, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1,\ldots,m. \tag{1.25}$$

Definition 1.7 (Lagrangian) If there exists a function $\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)})$, $s \leq p$, p being the order of equation (1.5), such that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0 \quad \alpha = 1, \cdots, m \tag{1.26}$$

then $\mathcal{L}$ is called a Lagrangian of equation (1.5). Equation (1.26) is known as the Euler-Lagrange equation.

Definition 1.8 (Lie-Bäcklund operator) The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.27}$$

where $\mathcal{A}$ is the space of differential functions [29]. The operator (1.27) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.28}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.29}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{1.30}$$

One can write the Lie-Bäcklund operator (1.28) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.31}$$

Definition 1.9 (Conservation law) The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a *conserved vector* of (1.5) if T^i satisfies

$$D_i T^i|_{(1.5)} = 0. \tag{1.32}$$

The equation (1.32) defines a local conservation law of system (1.5).

1.5.2 Multiplier method

The multiplier approach is an effective algorithmic for finding the conservation laws for partial differential equations with any number of independent and dependent variables. Authors in [50] gave this algorithm by using the multipliers presented in [30]. A local conservation law of a given differential system arises from a linear combination formed by local multipliers (characteristics) with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the given differential equation system.

The advantage of this approach is that it does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.33}$$

hold identically, where E_α, D_i are defined by equations (1.5), (1.4) and T^i is defined in definition (1.9). The right hand side of (1.33) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.34}$$

Once the multipliers are obtained the conserved vectors are constructed by invoking the homotopy operator [50].

1.5.3 Ibragimov method for conservation laws

A new conservation theorem by Ibragimov [53] provides the procedure for computing the conserved vector associated with every symmetry of the system of pth-order differential equation (1.5).

Definition 1.10 (Adjoint equations) Consider a system of pth-order partial differential equations given by (1.5). We introduce the differential functions

$$E_\alpha^*(x, u, v, \cdots, u_{(p)}, v_{(p)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \cdots, m, \tag{1.35}$$

where $v = (v^1, \cdots, v^m)$ are new dependent variables, $v = v(x)$, and define the system of *adjoint equations* to equation (1.5) by

$$E_\alpha^*(x, u, v, \cdots, u_{(p)}, v_{(p)}) = 0, \quad \alpha = 1 \cdots, m. \tag{1.36}$$

Theorem 1.4 Any system of pth-order differential equations (1.5) considered together with its adjoint equation (1.36) has a Lagrangian. Namely, the Euler-Lagrange equations (1.26) with the Lagrangian

$$\mathcal{L} = v^\beta E_\beta^*(x, u, v, \cdots, u_{(p)}) \tag{1.37}$$

provide the simultaneous system of equations (1.5) and (1.35)−(1.36) with $2m$ dependent variables $u = u(u^1, \cdots, u^m)$ and $v = (v^1, \cdots, v^m)$.

Theorem 1.5 Consider a system of m equations (1.5). The adjoint system given by (1.36), inherits the symmetries of the system (1.5). Namely, if the system (1.5) admits a point transformation group with a generator (1.27), then the adjoint system (1.36) admits the operator (1.27) extended to the variables v^α by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha} \tag{1.38}$$

with appropriately chosen coefficients $\eta_*^\alpha = \eta_*^\alpha(x, u, v, \cdots)$.

Definition 1.11 (Nonlinearly self-adjoint) A system (1.5) is said to be nonlinearly self-adjoint if the adjoint system (1.36) is satisfied for all the solutions of (1.5) after some substitution of v^α given by

$$v^\alpha = \phi^\alpha(x, u, u_{(1)}, \cdots), \quad \alpha = 1, \cdots, m, \tag{1.39}$$

under the condition that not all ϕ^α vanish identically [56].

Theorem 1.6 (Ibragimov theorem) Any infinitesimal symmetry (Lie point,Lie-Bäcklund , nonlocal) given by (1.27) of a nonlinearly self-adjoint system (1.5) leads to a conservation law $D_i(C^i) = 0$ for the system (1.32). The components of the conserved vector are given by the formula

$$\begin{aligned} C^i &= \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \cdots \right] \\ &\quad + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] + D_j D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right], \end{aligned} \tag{1.40}$$

where W^α is is the Lie characteristic function given by (1.30)and $\mathcal{L}$ is the formal Lagrangian (1.37) [53].

1.5.4 Noether theorem

Definition 1.12 (Noether operator) The Noether operators associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{ii_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \tag{1.41}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.25) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \dots D_{j_s} \frac{\partial}{\partial u_{ij_1 j_2 \dots j_s}^\alpha}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \tag{1.42}$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity [53]

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.43}$$

Definition 1.13 (Noether symmetry) A Lie-Bäcklund operator X of the form (1.27) is called a Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$, if there exists a vector $B^i = (B^1, \cdots, B^n)$, $B^1 \in \mathcal{A}$ such that

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i) \tag{1.44}$$

if $B^i = 0$ $(i = 1, \cdots, n)$, then X is called a strict Noether symmetry corresponding to a Lagrangian $\mathcal{L} \in \mathcal{A}$.

Theorem 1.7 (Noether's Theorem) For any Noether symmetry generator X associated with a given Lagrangian $\mathcal{L} \in \mathcal{A}$, there corresponds a vector $T = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, given by

$$T^i = N^i(\mathcal{L}) - B^i, \quad i = 1, ..., n, \tag{1.45}$$

which is a conserved vector of the Euler-Lagrange differential equations (1.26).

In the Noether approach, we find the Lagrangian $\mathcal{L}$ and then equation (1.44) is used to determine the Noether symmetries. Then, equation (1.45) will yield the corresponding Noether conserved vectors.

1.6 Exact solutions

In this section we recall some methods which can be used to determine exact solutions of differential equations.

1.6.1 Description of $(G'/G)-$expansion method

The $(G'/G)-$expansion method for finding exact solutions of nonlinear differential equations was introduced in [11]. Several researchers have recently applied this method to various nonlinear differential equations. They have shown that this method provides a very effective and powerful mathematical tool for solving nonlinear equations in various fields of applied sciences (see, for example, papers [11–13].

Consider a nonlinear partial differential equation, say, in two independent variables x and t, given by

$$P(u, u_x, u_t, u_{tt}, u_{xt}, u_{xx} \cdots) = 0, \tag{1.46}$$

where $u(x,t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The essence of the $(G'/G)-$expansion method is given in the following steps.

- **Step 1.** The transformation $u(x,t) = U(z), \quad z = x - \nu t$ reduces equation (1.46) to the ordinary differential equation

$$P(U, -\nu U', U', \nu^2 U'', -\nu U'', U'' \cdots) = 0. \tag{1.47}$$

- **Step 2.** According to the $(G'/G)-$expansion method, it is assumed that the travelling wave solution of equation (1.47) can be expressed by a polynomial in (G'/G) as follows:

$$U(z) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{1.48}$$

  where $G = G(z)$ satisfies the second-order linear ordinary differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \tag{1.49}$$

  with α_i, $i = 0, 1, 2, \cdots, m$, λ and μ being constants to be determined. The positive integer m is determined by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in ordinary differential equation (1.47).

- **Step 3.** By substituting (1.48) into (1.47) and using the second-order ordinary differential equation (1.49), collecting all terms with same order of (G'/G) together, the left-hand side of (1.47) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_0, \cdots, \alpha_m \, \nu, \lambda, \mu$.

- **Step 4.** Lastly, assuming that the constants can be obtained by solving the algebraic equations in Step 3, since the general solution of (1.49) is known, then substituting the constants and the general solutions of (1.49) into (1.48) we obtain travelling wave solutions of the nonlinear partial differential equation (1.46).

1.6.2 The simplest equation method

In this subsection we recall the simplest equation method developed by Kudryashov [14,15] for finding exact solutions of nonlinear partial differential equations. Several researchers have recently applied this method to various nonlinear partial differential equations and it has been shown that this method provides a very effective and powerful mathematical tool for solving nonlinear differential equations in various fields of applied sciences (see, for example, papers [16–20]). The basic steps of the method are as follows:

Consider the nonlinear partial differential equation of the form

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{yy} \cdots) = 0. \tag{1.50}$$

Using the following transformation

$$u(t, x, y) = F(z), \quad z = k_1 t + k_2 x + k_3 y + k_4 \tag{1.51}$$

reduces equation (1.50) to an ordinary differential equation

$$E_2[F(z), k_1F'(z), k_2F'(z), k_3F'(z), k_1^2F''(z), k_2^2F''(z), k_3^2F''(z), \cdots] = 0. \quad (1.52)$$

The simplest equations that we use here are the Bernoulli equation:

$$H'(z) = aH(z) + bH^2(z), \quad (1.53)$$

and the Riccati equation:

$$G'(z) = aG^2(z) + bG(z) + c, \quad (1.54)$$

where a, b and c are constants [14, 19, 20]. We look for solutions of the nonlinear ordinary differential equation (1.52) that are of the form

$$F(z) = \sum_{i=0}^{M} A_i(G(z))^i, \quad (1.55)$$

where $G(z)$ satisfies the Bernoulli or Riccatiequation, M is a positive integer that can be determined by balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

The solution of Bernoulli Equation (1.53) we use here is given by:

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}$$

where C is a constant of integration. For the Riccati Equation (1.54), the solutions to be used are:

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right] \quad (1.56)$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \quad (1.57)$$

with $\theta = \sqrt{b^2 - 4ac}$ and C is a constant of integration.

1.6.3 Kudryashov method

In this section we present a method, due to Kudryashov, for finding exact solutions of nonlinear differential equations, which has recently appeared in [18]. It should be noted that several researchers have recently applied this method to various nonlinear differential equations and it has been shown that this method provides a very effective and powerful mathematical tool for solving nonlinear differential equations in various fields of applied sciences (see, for example, papers [21–24]).

We now recall this method and give its description. Consider a nonlinear partial differential equation, say, in two independent variables t and x, given by

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0, \tag{1.58}$$

where $u(x,t)$ is an unknown function, E is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. The algorithm of Kudryashov method consists of the following six steps:

- **Step 1.** The transformation $u(x,t) = U(z), \quad z = kx + \omega t$, where k and ω are constants, reduces equation (1.58) to the ordinary differential equation

$$E_2(U, \omega U_z, kU_z, \omega^2 U_{zz}, k^2 U_{zz}, \cdots) = 0. \tag{1.59}$$

- **Step 2.** It is assumed that the exact solution of equation (1.59) can be expressed by a polynomial in Q as follows:

$$U(z) = \sum_{n=0}^{N} a_n \Big(Q(z)\Big)^n, \tag{1.60}$$

where the coefficients a_n $(n = 0, 1, 2, \cdots, N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(z)$ is the solution of the first-order nonlinear ordinary differential equation

$$Q_z(z) = Q^2(z) - Q(z). \tag{1.61}$$

We note that the equation (1.61) has the solution given by

$$Q(z) = \frac{1}{1 + e^z}, \tag{1.62}$$

The positive integer N is determined by taking the pole order of general solution for equation (1.59). Substituting $U(z) = z^{-p}$, $p > 0$ into monomials of equation (1.59) and comparing the two or more terms with smallest powers in equation we find the value for N.

- **Step 3.** We substitute the derivatives of $U(z)$ with respect to z and the expression for $U(z)$ into equation (1.59) and as a result we obtain the equation that has the function Q, coefficients a_n $(n = 0, 1, \cdots, N)$ and parameters k, ω of equation (1.59).

- **Step 4.** The method now transforms the problem of finding the exact solution of ordinary differential equation (1.59) into the problem of looking for solution of the system of algebraic equations. Equating expressions at the different powers of Q to zero, we obtain the system of algebraic equations in the form

$$P_n(a_N, a_{N-1}, \cdots, a_0, k, \omega, \cdots) = 0, \quad (n = 0, \cdots, N). \tag{1.63}$$

- **Step 5.** Solving the system of algebraic equations, we obtain values of coefficients $a_N, a_{N-1}, \cdots, a_0$ and relations for parameters of equation (1.59). As a result, we obtain exact solutions of equation (1.59) in the form (1.60).

- **Step 6.** The presentation of solution $U(z)$ of equation (1.59) in more convenient form and checking up of solutions.

1.7 Conclusion

In this chapter we presented a brief introduction to the Lie group analysis and conservation laws of partial differential equations and gave some results which will be used throughout this thesis. We also presented algorithms of certain methods that are used to determine the exact solutions of differential equations studied in this work.

Chapter 2

Group classification of a generalized coupled (2+1)-dimensional hyperbolic system

In this chapter we perform a complete Lie group classification of the generalized coupled (2+1)-dimensional hyperbolic system

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} + f(v) = 0, \\ v_{tt} - v_{xx} - v_{yy} + g(u) = 0, \end{cases} \tag{2.1}$$

which models many physical phenomena in nonlinear sciences. Here $f(v)$ and $g(u)$ are nonzero arbitrary functions of their respective arguments. The blow up problem for positive solutions of parabolic and hyperbolic problems with reaction terms of local and nonlocal type involving a variable exponent was studied in [60]. Parabolic problems appear in many branches of applied mathematics and can be used to model, for example, chemical reactions, heat transfer and population dynamics (see [60] and references therein). Escobedo and Herrero [61] extended the work of [60] and studied the system of equations

$$\begin{cases} u_t - \Delta u = v^q, \\ v_t - \Delta v = u^p, \end{cases} \tag{2.2}$$

where p, q are arbitrary constants and investigated the boundedness and blow-up of its solutions. The uniqueness and global existence of solutions of the system (2.2) were studied in [62]. Recently, the authors of [63] considered nonlinear parabolic and hyperbolic systems with variable exponents and obtained results concerning the existence and blow-up property of solutions.

Inspired by the works done in [61–63], more recently the authors of [64] studied the coupled (2+1)-dimensional hyperbolic system

$$\begin{cases} u_{tt} - u_{xx} - u_{yy} + \alpha v^q = 0, \\ v_{tt} - v_{xx} - v_{yy} + \beta u^p = 0, \end{cases} \tag{2.3}$$

where q, p, α and β are non-zero constants. A complete Noether symmetry classification was carried out in [64] and it was shown that four main cases arose in the Noether classification with respect to the standard Lagrangian. The conservation laws were also constructed for the cases which admitted Noether point symmetries.

The work in this chapter has been submitted for publication. See [65].

2.1 Equivalence transformations

An equivalence transformation (see for example [29]) of (2.1) is an invertible transformation involving the independent variables t, x, y and the dependant variables

u and v that map (2.1) into itself. The vector field

$$\begin{aligned} Y &= \xi^1(t,x,y,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v)\frac{\partial}{\partial y} \\ &\quad + \eta^1(t,x,y,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v)\frac{\partial}{\partial v} + \mu^1(t,x,y,u,v,f,g)\frac{\partial}{\partial f} \\ &\quad + \mu^2(t,x,y,u,v,f,g)\frac{\partial}{\partial g} \end{aligned} \tag{2.4}$$

is the generator of the equivalence group for (2.1) provided it is admitted by the extended system [25, 66]

$$u_{tt} - u_{xx} - u_{yy} + f(v) = 0, \;\; v_{tt} - v_{xx} - v_{yy} + g(u) = 0, \tag{2.5}$$

$$f_t = f_x = f_y = f_u = 0, \;\; g_t = g_x = g_y = g_v = 0. \tag{2.6}$$

The prolonged operator of (2.4) for the extended system (2.5)-(2.6) is given by

$$\widetilde{Y} = Y^{[2]} + \omega_t^1\frac{\partial}{\partial f_t} + \omega_x^1\frac{\partial}{\partial f_x} + \omega_y^1\frac{\partial}{\partial f_y} + \omega_u^1\frac{\partial}{\partial f_u} + \omega_t^2\frac{\partial}{\partial g_t} + \omega_x^2\frac{\partial}{\partial g_x} + \omega_y^2\frac{\partial}{\partial g_y} + \omega_v^2\frac{\partial}{\partial g_v}, \tag{2.7}$$

where $Y^{[2]}$ is the second-prolongation of (2.4) given by

$$\begin{aligned} Y^{[2]} &= Y + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_y^1\frac{\partial}{\partial u_y} + \zeta_t^2\frac{\partial}{\partial v_t} + \zeta_x^2\frac{\partial}{\partial v_x} + \zeta_y^2\frac{\partial}{\partial v_y} \\ &\quad + \zeta_{tt}^1\frac{\partial}{\partial u_{tt}} + \zeta_{xx}^1\frac{\partial}{\partial u_{xx}} + \zeta_{yy}^1\frac{\partial}{\partial u_{yy}} + \zeta_{tt}^2\frac{\partial}{\partial v_{tt}} + \zeta_{xx}^2\frac{\partial}{\partial v_{xx}} + \zeta_{yy}^2\frac{\partial}{\partial v_{yy}} + \cdots . \end{aligned}$$

Here the variables ζ's and ω's are defined by

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - u_tD_t(\xi^1) - u_xD_t(\xi^2) - u_yD_t(\xi^3), \\ \zeta_x^1 &= D_x(\eta^1) - u_tD_x(\xi^1) - u_xD_x(\xi^2) - u_yD_x(\xi^3), \\ \zeta_y^1 &= D_y(\eta^1) - u_tD_y(\xi^1) - u_xD_y(\xi^2) - u_yD_y(\xi^3), \\ \zeta_t^2 &= D_t(\eta^2) - v_tD_t(\xi^1) - v_xD_t(\xi^2) - v_yD_t(\xi^3), \\ \zeta_x^2 &= D_x(\eta^2) - v_tD_x(\xi^1) - v_xD_x(\xi^2) - v_yD_x(\xi^3), \\ \zeta_y^2 &= D_y(\eta^2) - v_tD_y(\xi^1) - v_xD_y(\xi^2) - v_yD_y(\xi^3), \end{aligned}$$

$$
\begin{aligned}
\zeta^1_{tt} &= D_t(\zeta^1_t) - u_{tt}D_t(\xi^1) - u_{tx}D_t(\xi^2) - u_{ty}D_t(\xi^3),\\
\zeta^1_{xx} &= D_x(\zeta^1_x) - u_{tx}D_x(\xi^1) - u_{xx}D_x(\xi^2) - u_{xy}D_x(\xi^3),\\
\zeta^1_{yy} &= D_y(\zeta^1_y) - u_{ty}D_y(\xi^1) - u_{xy}D_y(\xi^2) - u_{yy}D_y(\xi^3),\\
\zeta^2_{tt} &= D_t(\zeta^2_t) - v_{tt}D_t(\xi^1) - v_{tx}D_t(\xi^2) - v_{ty}D_t(\xi^3),\\
\zeta^2_{xx} &= D_x(\zeta^2_x) - v_{tx}D_x(\xi^1) - v_{xx}D_x(\xi^2) - v_{xy}D_x(\xi^3),\\
\zeta^2_{yy} &= D_y(\zeta^2_y) - v_{ty}D_y(\xi^1) - v_{xy}D_y(\xi^2) - v_{yy}D_y(\xi^3)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^1_t &= \widetilde{D}_t(\mu^1) - f_t\widetilde{D}_t(\xi^1) - f_x\widetilde{D}_t(\xi^2) - f_y\widetilde{D}_t(\xi^3) - f_u\widetilde{D}_t(\eta^1),\\
\omega^1_x &= \widetilde{D}_x(\mu^1) - f_t\widetilde{D}_x(\xi^1) - f_x\widetilde{D}_x(\xi^2) - f_y\widetilde{D}_x(\xi^3) - f_u\widetilde{D}_x(\eta^1),\\
\omega^1_y &= \widetilde{D}_y(\mu^1) - f_t\widetilde{D}_y(\xi^1) - f_x\widetilde{D}_y(\xi^2) - f_y\widetilde{D}_y(\xi^3) - f_u\widetilde{D}_y(\eta^1),\\
\omega^1_u &= \widetilde{D}_u(\mu^1) - f_t\widetilde{D}_u(\xi^1) - f_x\widetilde{D}_u(\xi^2) - f_y\widetilde{D}_u(\xi^3) - f_u\widetilde{D}_u(\eta^1),\\
\omega^2_t &= \widetilde{D}_t(\mu^2) - g_t\widetilde{D}_t(\xi^1) - g_x\widetilde{D}_t(\xi^2) - g_y\widetilde{D}_t(\xi^3) - g_v\widetilde{D}_t(\eta^2),\\
\omega^2_x &= \widetilde{D}_x(\mu^2) - g_t\widetilde{D}_x(\xi^1) - g_x\widetilde{D}_x(\xi^2) - g_y\widetilde{D}_x(\xi^3) - g_v\widetilde{D}_x(\eta^2),\\
\omega^2_y &= \widetilde{D}_y(\mu^2) - g_t\widetilde{D}_y(\xi^1) - g_x\widetilde{D}_y(\xi^2) - g_y\widetilde{D}_y(\xi^3) - g_v\widetilde{D}_y(\eta^2),\\
\omega^2_v &= \widetilde{D}_v(\mu^2) - g_t\widetilde{D}_v(\xi^1) - g_x\widetilde{D}_v(\xi^2) - g_y\widetilde{D}_v(\xi^3) - g_v\widetilde{D}_v(\eta^2),
\end{aligned}
$$

respectively, where

$$
\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + \cdots\,,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + \cdots\,,\\
D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + \cdots
\end{aligned}
$$

are the usual total differentiation operators and

$$
\widetilde{D}_t = \frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + g_t\frac{\partial}{\partial g} + \cdots\,,
$$

$$
\begin{aligned}
\widetilde{D}_x &= \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + g_x\frac{\partial}{\partial g} + \cdots\,, \\
\widetilde{D}_y &= \frac{\partial}{\partial y} + f_y\frac{\partial}{\partial f} + g_y\frac{\partial}{\partial g} + \cdots\,, \\
\widetilde{D}_u &= \frac{\partial}{\partial u} + f_u\frac{\partial}{\partial f} + g_u\frac{\partial}{\partial g} + \cdots\,, \\
\widetilde{D}_v &= \frac{\partial}{\partial v} + f_v\frac{\partial}{\partial f} + g_v\frac{\partial}{\partial g} + \cdots
\end{aligned}
$$

are the new total differentiation operators for the extended system. The application of the operator (2.7) and the invariance conditions of system (2.5)-(2.6) leads to the following overdetermined system of linear partial differential equations:

$$
\begin{aligned}
&\xi^1_u = 0,\ \xi^1_v = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \xi^3_u = 0,\ \xi^3_v = 0,\ \eta^1_{uu} = 0,\ \eta^1_{uv} = 0,\\
&\eta^1_{vv} = 0,\quad \eta^1_{vt} = 0,\ \eta^1_{vx} = 0, \eta^1_{vy} = 0,\ \xi^2_x - \xi^1_t = 0,\ \xi^3_y - \xi^1_t = 0,\ \xi^1_x - \xi^2_t = 0,\\
&\xi^1_y - \xi^3_t = 0,\ \xi^3_x + \xi^2_y = 0,\ \xi^1_{yy} + \xi^1_{xx} - \xi^1_{tt} + 2\eta^1_{ut} = 0,\ \xi^2_{yy} + \xi^2_{xx} - \xi^2_{tt} - 2\eta^1_{ux} = 0,\\
&\xi^3_{yy} + \xi^3_{xx} - \xi^3_{tt} - 2\eta^1_{uy} = 0,\ \eta^1_{tt} - \eta^1_{xx} - \eta^1_{yy} - f\eta^1_u - g\eta^1_v + 2f\xi^1_t + \mu^1 = 0,\\
&\eta^2_{uu} = 0, \eta^2_{uv} = 0,\ \eta^2_{vv} = 0,\ \eta^2_{ut} = 0,\ \eta^2_{ux} = 0, \eta^2_{uy} = 0,\ \xi^1_{yy} + \xi^1_{xx} - \xi^1_{tt} + 2\eta^2_{vt} = 0,\\
&\xi^2_{yy} + \xi^2_{xx} - \xi^2_{tt} - 2\eta^2_{vx} = 0,\ \xi^3_{yy} + \xi^3_{xx} - \xi^3_{tt} - 2\eta^2_{vy} = 0,\\
&\eta^2_{tt} - \eta^2_{xx} - \eta^2_{yy} - f\eta^2_u - g\eta^2_v + 2g\xi^1_t + \mu^2 = 0.
\end{aligned}
$$

Solving the above system, we obtain the following equivalence generators:

$$
\begin{aligned}
&Y_1 = \frac{\partial}{\partial t},\ Y_2 = \frac{\partial}{\partial x},\ Y_3 = \frac{\partial}{\partial y},\ Y_4 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},\ Y_5 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y},\\
&Y_6 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ Y_7 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f},\ Y_8 = v\frac{\partial}{\partial v} + g\frac{\partial}{\partial g},\\
&Y_9 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g},\ Y_{10} = \frac{\partial}{\partial u},\ Y_{11} = \frac{\partial}{\partial v}.
\end{aligned}
$$

Thus, the eleven-parameter equivalence group is given by

$$
\begin{aligned}
Y_1 &: \ \bar{t} = a_1 + t,\ \bar{x} = x, \bar{y} = y,\ \bar{u} = u,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,\\
Y_2 &: \ \bar{t} = t,\ \bar{x} = a_2 + x, \bar{y} = y,\ \bar{u} = u,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,\\
Y_3 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{y} = a_3 + y,\ \bar{u} = u,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,
\end{aligned}
$$

$$Y_4 \;:\; \bar{t} = a_4x + t,\ \bar{x} = a_4t + x,\ \bar{y} = y,\ \bar{u} = u,\ \bar{v} = v,\ \bar{f} = f, \bar{g} = g,$$

$$Y_5 \;:\; \bar{t} = a_5y + t,\ \bar{x} = x,\ \bar{y} = a_5t + y,\ \bar{u} = u,\ \ \bar{v} = v,\ \bar{f} = f, \bar{g} = g,$$

$$Y_6 \;:\; \bar{t} = t,\ \bar{x} = x - a_6y,\ \bar{y} = a_6x + y,\ \bar{u} = u,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,$$

$$Y_7 \;:\; \bar{t} = t,\ \bar{x} = x, \bar{y} = y,\ \bar{u} = ue^{a_7},\ \bar{v} = v,\ \bar{f} = fe^{a_7},\ \bar{g} = g,$$

$$Y_8 \;:\; \bar{t} = t,\ \bar{x} = x, \bar{y} = y,\ \bar{u} = u,\ \bar{v} = ve^{a_8},\ \bar{f} = f,\ \bar{g} = ge^{a_8},$$

$$Y_9 \;:\; \bar{t} = te^{a_9},\ \bar{x} = xe^{a_9}, \bar{y} = ye^{a_9},\ \bar{u} = u,\ \bar{v} = v,\ \bar{f} = fe^{-2a_9},\ \bar{g} = ge^{-2a_9},$$

$$Y_{10} \;:\; \bar{t} = t,\ \bar{x} = x, \bar{y} = y,\ \bar{u} = a_{10} + u,\ \bar{v} = v,\ \bar{f} = f,\ \bar{g} = g,$$

$$Y_{11} \;:\; \bar{t} = t,\ \bar{x} = x, \bar{y} = y,\ \bar{u} = u,\ \bar{v} = a_{11} + v,\ \bar{f} = f,\ \bar{g} = g$$

and their composition gives

$$\begin{aligned}
\bar{t} &= a_1 + a_4x + a_5y + te^{a_9}, \\
\bar{x} &= a_2 + a_4t - a_6y + xe^{a_9}, \\
\bar{y} &= a_3 + a_5t + a_6x + ye^{a_9}, \\
\bar{u} &= e^{a_7}(u + a_{10}), \\
\bar{v} &= e^{a_8}(v + a_{11}), \\
\bar{f} &= e^{a_7 - 2a_9}f, \\
\bar{g} &= e^{a_8 - 2a_9}g. \qquad (2.8)
\end{aligned}$$

2.2 Principal Lie algebra

According to Lie's theory the system of differential equations (2.1) is invariant under the group with generator

$$\begin{aligned}
\Gamma &= \xi^1(t,x,y,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v)\frac{\partial}{\partial y} \\
&\quad + \eta^1(t,x,y,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v)\frac{\partial}{\partial v} \qquad (2.9)
\end{aligned}$$

if and only if

$$\Gamma^{[2]}\left(u_{tt}-u_{xx}-u_{yy}+f(v)\right)\Big|_{(2.1)}=0,\quad \Gamma^{[2]}\left(v_{tt}-v_{xx}-v_{yy}+g(u)\right)\Big|_{(2.1)}=0, \tag{2.10}$$

where $\Gamma^{[2]}$ denotes the second prolongation of the generator (2.9) and the symbol $|_{(2.1)}$ means it is evaluated on system (2.1). As the ξ's and η's do not depend on any derivatives of u and v, the determining equations (2.10) split with respect to the derivatives of u and v, yielding the following overdetermined system of thirty-one linear partial differential equations:

$$\begin{aligned}
&\xi^1_u=0,\ \xi^1_v=0,\ \xi^2_u=0,\ \xi^2_v=0,\ \xi^3_u=0,\ \xi^3_v=0,\ \eta^1_{uu}=0,\ \eta^1_{uv}=0,\\
&\eta^1_{vv}=0,\quad \eta^1_{vt}=0,\ \eta^1_{vx}=0,\ \eta^1_{vy}=0,\ \xi^2_x-\xi^1_t=0,\ \xi^3_y-\xi^1_t=0,\ \xi^1_x-\xi^2_t=0,\\
&\xi^1_y-\xi^3_t=0,\ \xi^3_x+\xi^2_y=0,\ \xi^1_{yy}+\xi^1_{xx}-\xi^1_{tt}+2\eta^1_{ut}=0,\ \xi^2_{yy}+\xi^2_{xx}-\xi^2_{tt}-2\eta^1_{ux}=0,\\
&\xi^3_{yy}+\xi^3_{xx}-\xi^3_{tt}-2\eta^1_{uy}=0,\ \eta^1_{tt}-\eta^1_{xx}-\eta^1_{yy}-f\eta^1_u-g\eta^1_v+2f\xi^1_t+f'(v)\eta^2=0,\\
&\eta^2_{uu}=0,\eta^2_{uv}=0,\ \eta^2_{vv}=0,\ \eta^2_{ut}=0,\ \eta^2_{ux}=0,\ \eta^2_{uy}=0,\ \xi^1_{yy}+\xi^1_{xx}-\xi^1_{tt}+2\eta^2_{vt}=0,\\
&\xi^2_{yy}+\xi^2_{xx}-\xi^2_{tt}-2\eta^2_{vx}=0,\xi^3_{yy}+\xi^3_{xx}-\xi^3_{tt}-2\eta^2_{vy}=0,\\
&\eta^2_{tt}-\eta^2_{xx}-\eta^2_{yy}-f\eta^2_u-g\eta^2_v+2g\xi^1_t+g'(u)\eta^1=0.
\end{aligned} \tag{2.11}$$

Solving the above system for arbitrary $f(v)$ and $g(u)$, we find that the system (2.1) admits the six-dimensional Lie algebra spanned by

$$\begin{aligned}
&\textit{time translation} && \Gamma_1=\frac{\partial}{\partial t},\\
&\textit{space translation} && \Gamma_2=\frac{\partial}{\partial x},\\
&\textit{space translation} && \Gamma_3=\frac{\partial}{\partial y},\\
&\textit{Lorentz boost} && \Gamma_4=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x},\\
&\textit{Lorentz boost} && \Gamma_5=y\frac{\partial}{\partial t}+t\frac{\partial}{\partial y},\\
&\textit{Rotation} && \Gamma_6=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y},
\end{aligned}$$

which is the principal Lie algebra of the system (2.1).

2.3 Lie group classification

Solving the system (2.11), we obtain the following classifying relations:

$$(\alpha v+\beta)f'(v)+\gamma f(v)+\delta=0,$$
$$(\theta u+\lambda)g'(u)+\varphi g(u)+\omega=0,$$

where $\alpha,\beta,\gamma,\delta,\theta,\lambda,\varphi$ and ω are constants. Using the equivalence transformations obtained in Section 2.1, this classifying relation is invariant under the equivalence transformations (2.8) if

$$\bar{\alpha}=\alpha,\quad \bar{\gamma}=\gamma,\quad \bar{\beta}=e^{a_8}(\beta-a_{11}\alpha),\quad \bar{\delta}=\delta e^{a_7-2a_9},\quad \bar{\theta}=\theta,\quad \bar{\varphi}=\varphi,$$
$$\bar{\lambda}=e^{a_7}(\lambda-a_{10}\theta),\quad \bar{\omega}=\omega e^{a_8-2a_9}.$$

These classifying relations lead to the following twelve cases for the functions f and g and for each case we also provide the associated extended symmetries.

Case 1: $f(v)$ and $g(u)$ arbitrary but not of the form in Cases 2-12 given below

In this case, we obtain the principal Lie algebra

$$\Gamma_1=\frac{\partial}{\partial t},\quad \Gamma_2=\frac{\partial}{\partial x},\quad \Gamma_3=\frac{\partial}{\partial y},\quad \Gamma_4=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x},$$
$$\Gamma_5=y\frac{\partial}{\partial t}+t\frac{\partial}{\partial y},\quad \Gamma_6=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}.$$

Case 2: $f(v)=nv+\sigma$ and $g(u)=mu+\theta$, where n, σ, m and θ are constants

This case extends the principal Lie algebra by four symmetries, namely

$$\Gamma_7 = \frac{\partial}{\partial u},$$
$$\Gamma_8 = nv\frac{\partial}{\partial u}+(mu+\theta)\frac{\partial}{\partial v},$$

$$
\begin{aligned}
\Gamma_9 &= nu\frac{\partial}{\partial u} + (nv+\sigma)\frac{\partial}{\partial v}, \\
\Gamma_{10} &= nH\frac{\partial}{\partial u} + (H_{yy}+H_{xx}-H_{tt})\frac{\partial}{\partial v},
\end{aligned}
$$

where $H(t,x,y)$ is any solution of the partial differential equation

$$2H_{ttyy}+2H_{ttxx}-2H_{xxyy}-H_{yyyy}-H_{tt}-H_{xx}+mnH-mC_1-m\sigma C_2-n\theta C_3=0$$

and C_1, C_2, C_3 are arbitrary constants.

Case 3: $f(v)=\alpha v^n$ and $g(u)=\theta u^m$, where α, n, θ and m are constants

We have four subcases.

Case 3.1: $n \neq -1, m \neq -1$

The principal Lie algebra is extended by one symmetry

$$\Gamma_{11}=(mn-1)\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)-2(n+1)u\frac{\partial}{\partial u}-2(m+1)v\frac{\partial}{\partial v}.$$

Case 3.2: $n=m=-1$

This subcase extends the principal Lie algebra by two symmetries, viz.,

$$
\begin{aligned}
\Gamma_{12} &= u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \\
\Gamma_{13} &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2v\frac{\partial}{\partial v}.
\end{aligned}
$$

Case 3.3: $n=\dfrac{1}{m}$ and m is arbitrary

Here the principal Lie algebra extends by one symmetry

$$\Gamma_{14} = u\frac{\partial}{\partial u} + mv\frac{\partial}{\partial v}.$$

Case 3.4: $n=5$ and $m=5$

In this subcase the principal Lie algebra extends by the following four symmetries:

$$
\begin{aligned}
&\textit{Scaling} &\quad \Gamma_{15} &= 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \\
&\textit{Inversion} &\quad \Gamma_{16} &= 2ty\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (t^2 - x^2 + y^2)\frac{\partial}{\partial y} - uy\frac{\partial}{\partial u} - vy\frac{\partial}{\partial v}, \\
&\textit{Inversion} &\quad \Gamma_{17} &= 2xt\frac{\partial}{\partial t} + (t^2 + x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} - ux\frac{\partial}{\partial u} - vx\frac{\partial}{\partial v}, \\
&\textit{Inversion} &\quad \Gamma_{18} &= (t^2 + x^2 + y^2)\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x} + 2ty\frac{\partial}{\partial y} - ut\frac{\partial}{\partial u} - vt\frac{\partial}{\partial v}.
\end{aligned}
$$

Case 4: $n = -1$ and $g(u)$ is arbitrary

This subcase extends the principal Lie algebra by one symmetry

$$\Gamma_{19} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2v\frac{\partial}{\partial v}.$$

Case 5: $f(v)$ is arbitrary and $m = -1$

Here the principal Lie algebra extends by one symmetry

$$\Gamma_{20} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}.$$

Case 6: $f(v) = \alpha e^{nv}$ and $g(u) = \theta e^{mu}$, where α, n, θ and m are constants

This case extends the principal Lie algebra by one symmetry

$$\Gamma_{21} = mn\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2n\frac{\partial}{\partial u} - 2m\frac{\partial}{\partial v}.$$

Case 7: $f(v) = \alpha v^n$ and $g(u) = \theta e^{mu}$, where α, n, θ and m are constants

This case extends the principal Lie algebra by one symmetry

$$\Gamma_{22} = mn\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2(n+1)\frac{\partial}{\partial u} - 2mv\frac{\partial}{\partial v}.$$

Case 8: $f(v) = \alpha e^{nv}$ and $g(u) = \theta u^m$, where α, n, θ and m are constants

This case extends the principal Lie algebra by one symmetry

$$\Gamma_{23} = mn\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2nu\frac{\partial}{\partial u} - 2(m+1)\frac{\partial}{\partial v}.$$

Case 9: $f(v) = nv + \sigma$ and $g(u) = \theta u^m$, where n, σ, θ and m are constants with $m \neq n \neq 1$

In this case the principal Lie algebra extends by one symmetry

$$\begin{aligned}\Gamma_{24} &= n(m-1)\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 4nu\frac{\partial}{\partial u}\\ &\quad -2(mnv + m\sigma + nv + \sigma)\frac{\partial}{\partial v}.\end{aligned}$$

Case 10: $f(v) = \alpha v^n$ and $g(u) = mu + \theta$, where α, n, m and θ are constants with $m \neq n \neq 1$

This case extends the principal Lie algebra by one symmetry

$$\begin{aligned}\Gamma_{25} &= m(n-1)\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2(mnu + n\sigma + mu + \sigma)\frac{\partial}{\partial u}\\ &\quad -4mv\frac{\partial}{\partial v}.\end{aligned}$$

Case 11: $f(v) = nv + \sigma$ and $g(u) = \theta e^{mu}$, where n, σ, θ and m are constants

This case extends the principal Lie algebra by one symmetry

$$\Gamma_{26} = mn\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 4n\frac{\partial}{\partial u} - 2m(nv + \sigma)\frac{\partial}{\partial v}.$$

Case 12: $f(v) = \alpha e^{nv}$ and $g(u) = mu + \theta$, where α, n, m and θ are constants

This case extends the principal Lie algebra by one symmetry

$$\Gamma_{27} = mn\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2n(mu + \theta)\frac{\partial}{\partial u} - 4m\frac{\partial}{\partial v}.$$

2.4 Conclusion

In this chapter we have used the Lie group analysis to perform a complete Lie group classification of the generalized coupled (2+1)-dimensional hyperbolic system (2.1). We showed that the system admitted eleven-dimensional equivalence Lie algebra. The functional forms of the arbitrary parameters were specified via the classical method of group classification and these included the power, exponential and linear functions. The six-dimensional principal Lie algebra was also obtained and several possible extensions of it were presented.

Chapter 3

Exact solutions of a KdV type equation and higher-order Boussinesq equation with damping term

In this chapter we study two nonlinear evolution equations, namely, the modified Kortweg-de Vries (mKdV) type equation [1]

$$uu_{xxt} - u_x u_{xt} - 4u^3 u_t + 4uu_{xxx} - 4u_x u_{xx} - 16u^3 u_x = 0 \tag{3.1}$$

and the higher-order modified Boussinesq equation with damping term [2]

$$u_{tt} + \alpha u_{txx} + \beta u_{xxxx} + \gamma[6u(u_x)^2 + 3u^2 u_{xx}] = 0. \tag{3.2}$$

It is well known that nonlinear evolution equations, such as (3.1) and (3.2), are widely used as models to describe physical phenomena in different fields of applied sciences, such as plasma waves, solid state physics, plasma physics and fluid mechanics. One of the basic physical problems for these models is to obtain their exact solutions for the better understanding of nonlinear models.

The modified Boussinesq equation

$$u_{tt} + u_{xxxx} + (u^3)_{xx} = 0, \tag{3.3}$$

arises in various physical applications and is also used to investigate the behavior of systems which are primarily linear but nonlinearity is introduced as a perturbation [2,67–69].

Yan et al. [2] obtained three types of symmetry reductions for the higher-order modified Boussinesq equation with damping term based on the direct method due to Clarkson and Kruskal and the improved direct method due to Lou. Authors in [2] also found kink-shape solitary wave solutions for equation (3.2) using direct transformation, which are of important physical significance.

Although a great deal of research work has been devoted to finding different methods to solve nonlinear evolution equations, there is no unique method. In 2007 Wang et al. [11] proposed a new method referred to as the $(G'/G)-$expansion method for finding travelling wave solutions of nonlinear evolution equations.

This work has been published. See [70].

3.1 Exact solutions of (3.1)

In this section we construct travelling wave solutions of mKdV type equation by employing the $(G'/G)-$expansion method.

As a first step we transform the mKdV type equation (3.1) to a nonlinear ordinary differential equation using the travelling wave variable

$$u(t,x) = F(z), \quad z = x - \nu t. \tag{3.4}$$

Applying the above transformation, equation (3.1) transforms to the nonlinear ordinary differential equation

$$-\nu FF''' + \nu F'F''' + 4\nu F^3F' + 4FF''' - 4F'F'' - 16F^3F' = 0, \tag{3.5}$$

which reduces to

$$(4-\nu)[FF''' - F'F'' - 4F^3F'] \;=\; 0. \tag{3.6}$$

Hence if $\nu \neq 4$, we obtain

$$FF''' - F'F'' - 4F^3F' = 0, \tag{3.7}$$

where the prime denotes the derivative with respect to z.

The $(G'/G)-$expansion method assumes the solution of equation (3.7) to be of the form given by equation (1.48). The balancing procedure yields $M = 1$, so the solution of equation (3.7) is of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1(G'/G). \tag{3.8}$$

Substituting (3.8) into (3.7), making use of the equation(1.49), collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields the following system of algebraic equations:

$$(G'/G)^0 \;:\; -\mathcal{A}_0\mathcal{A}_1\lambda^2\mu - 2\mathcal{A}_0\mathcal{A}_1\mu^2 + \mathcal{A}_1^2\lambda\mu^2 + 4\alpha\mathcal{A}_0^3\alpha_1\mu = 0 \tag{3.9}$$

$$(G'/G) \;:\; 4\mathcal{A}_0^3\mathcal{A}_1\lambda + 12\mathcal{A}_0^2\mathcal{A}_1^2\mu + \mathcal{A}_1^2\lambda^2\mu - \mathcal{A}_0\mathcal{A}_1\lambda^3 - 8\mathcal{A}_0\mathcal{A}_1\lambda\mu = 0 \tag{3.10}$$

$$\begin{aligned}(G'/G)^2 \;:\; & -7\mathcal{A}_1^2\lambda^2 - 8\mathcal{A}_1^2\mu - 7\mathcal{A}_0\mathcal{A}_1\lambda^2 - 8\mathcal{A}_0\mathcal{A}_1\mu - 2\mathcal{A}_1^2\lambda\mu \\ & +12\mathcal{A}_0\mathcal{A}_1^3\mu + 12\mathcal{A}_0^2\mathcal{A}_1^2\lambda + 4\mathcal{A}_0^3\mathcal{A}_1 = 0\end{aligned} \tag{3.11}$$

$$\begin{aligned}(G'/G)^3 \;:\; & 4\mathcal{A}_1^4\mu + 12\mathcal{A}_0\mathcal{A}_1^3\lambda + 12\mathcal{A}_0^2\mathcal{A}_1^2 + 4\mathcal{A}_1^2\lambda^2 + 4\mathcal{A}_1^2\mu \\ & -12\mathcal{A}_0\mathcal{A}_1\lambda = 0\end{aligned} \tag{3.12}$$

$$(G'/G)^4 \;:\; 4\mathcal{A}_1^4\lambda + 12\mathcal{A}_0\mathcal{A}_1^3 - 7\mathcal{A}_1^2\lambda - 6\mathcal{A}_0\mathcal{A}_1 = 0 \tag{3.13}$$

$$(G'/G)^5 \quad : \quad 4\mathcal{A}_1^4 - 4\mathcal{A}_1^2 = 0. \tag{3.14}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\mathcal{A}_0 = \frac{\lambda}{2}, \quad \mathcal{A}_1 = 1. \tag{3.15}$$

Substituting these values of $\mathcal{A}_0$, $\mathcal{A}_1$ and the corresponding solution of equation (1.49) into (3.8), we obtain three types of travelling wave solutions of equation (3.1). These are

Case 1: When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$u_1(t,x) \quad = \quad \mathcal{A}_0 + \mathcal{A}_1\bigg(- \frac{\lambda}{2} + \delta_1 \frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\bigg), \tag{3.16}$$

where $z = x - \nu t$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Case 2: When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$u_2(t,x) \quad = \quad \mathcal{A}_0 + \mathcal{A}_1\bigg(- \frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 z) + C_2 \cos(\delta_2 z)}{C_1 \cos(\delta_2 z) + C_2 \sin(\delta_2 z)}\bigg), \tag{3.17}$$

where $z = x - \nu t$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$u_3(t,x) \quad = \quad \mathcal{A}_0 + \mathcal{A}_1\bigg(- \frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\bigg), \tag{3.18}$$

where $z = x - \nu t$, C_1 and C_2 are arbitrary constants.

The solution profile for Case 1 is given below in Figure 3.1, with parameters

$C_1 = 1, \ C_2 = 0 \ \lambda = 2I \ \nu = -1.$

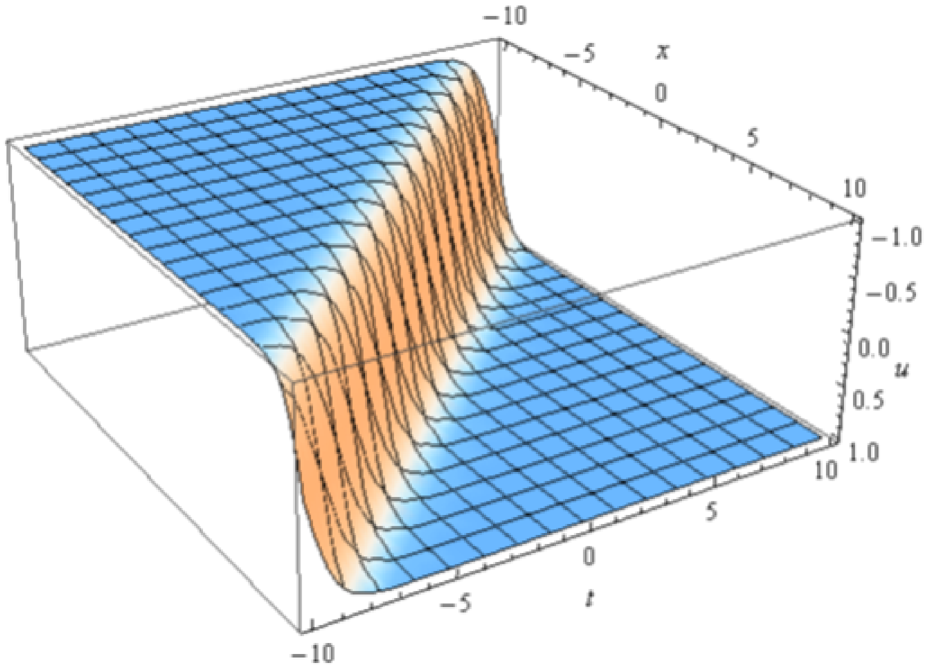

Figure 3.1: Profile of solution (3.1)

3.2 Exact solutions and conservation laws for higher-order modified Boussinesq equation with damping term

This section considers the higher-order modified Boussinesq equation (3.2) with damping term [2]. Exact solutions using $(G'/G-)$expansion method are obtained. Furthermore, conservation laws for this equation are constructed by employing the multiplier approach.

3.2.1 Exact solutions of (3.2)

Following the same procedure of the $(G'/G)-$expansion method presented in Chapter one, equation (3.2) is transformed to the ordinary differential equation

$$\nu^2 U'' - \alpha\nu U''' + \beta U'''' + \gamma[6U(U')^2 + 3U^2U''] = 0, \tag{3.19}$$

where the prime denotes the derivative with respect to z. Balancing the order of U'''' and U^2U'' in (3.19) yields $M = 1$. Hence, the solution to equation (3.19) is assumed to be of the form

$$U(z) = a_0 + a_1(G'/G). \tag{3.20}$$

Substituting (3.20) into (3.19) and making use of (1.49), we obtain the following algebraic system of equations in terms of a_0, a_1, by equating all coefficients of the functions $(G'/G)^i$ to zero.

$$\begin{aligned}
(G'/G)^0 \quad &: \quad a_1\mu\lambda\nu^2 + 2a_1\mu^2\alpha\nu + a_1\mu\lambda^2\alpha\nu + 8a_1\mu^2\lambda\beta + a_1\mu\lambda^3\beta \\
&\quad +6a_0a_1^2\mu^2\gamma + 3a_0^2a_1\mu\lambda\gamma = 0 &(3.21)\\
(G'/G) \quad &: \quad a_1\lambda^2\nu^2 + 2a_1\mu\nu^2 + 8a_1\mu\lambda\alpha\nu + a_1\lambda^3\alpha\nu + 16a_1\mu^2\beta \\
&\quad +22a_1\mu\lambda^2\beta + a_1\lambda^4\beta + 18a_0a_1^2\mu\lambda\gamma + 3a_0^2a_1\lambda^2\gamma + 6a_0^2a_1\mu\gamma \\
&\quad +6a_1^3\mu^2\gamma = 0 &(3.22)\\
(G'/G)^2 \quad &: \quad 3a_1\lambda\nu^2 + 8a_1\mu\alpha\nu + 7a_1\lambda^2\alpha\nu + 60a_1\mu\lambda\beta + 15a_1\lambda^3\beta \\
&\quad +15a_1^3\mu\lambda\gamma + 24a_0a_1^2\mu\gamma + 12a_0a_1^2\lambda^2\gamma + 9a_0^2a_1\lambda\gamma = 0 &(3.23)\\
(G'/G)^3 \quad &: \quad 2a_1\nu^2 + 12a_1\lambda\alpha\nu + 40a_1\mu\beta + 50a_1\lambda^2\beta + 9a_1^3\lambda^2\gamma \\
&\quad +18a_1^3\mu\gamma + 30a_0a_1^2\lambda\gamma + 6a_0^2a_1\gamma = 0 &(3.24)\\
(G'/G)^4 \quad &: \quad 6a_1\alpha\nu + 60a_1\lambda\beta + 21a_1^3\lambda\gamma + 18a_0a_1^2\gamma = 0 &(3.25)\\
(G'/G)^5 \quad &: \quad 24a_1\beta + 12a_1^3\gamma = 0. &(3.26)
\end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, one possible set of solution is

$$\alpha = \frac{3\lambda\sqrt{\beta}}{\sqrt{2(\lambda^2-\mu)}}, \quad \nu = -\frac{3\beta\lambda}{\alpha}, \quad a_0 = 0, \quad a_1 = \sqrt{\frac{-2\beta}{\gamma}}. \tag{3.27}$$

Substituting these values from (3.27) and the corresponding solution of equation (1.49) into (3.20), yields three types of travelling wave solutions of equation (3.19) and consequently of (3.2) as follows:

Case 1: When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution:

$$u_1(x,t) = \sqrt{\frac{-2\beta}{\gamma}}\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right)\right], \tag{3.28}$$

where $z = x - \nu t$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$ and C_1 and C_2 are arbitrary constants.

Case 2: When $\lambda^2 - 4\mu < 0$, we obtain trigonometric function solution:

$$u_2(x,t) = \sqrt{\frac{-2\beta}{\gamma}}\left[-\frac{\lambda}{2} + \delta_2\left(\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)\right], \tag{3.29}$$

where $z = x - \nu t$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$ and C_1 and C_2 are arbitrary constants.

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain the rational function solution

$$u_3(x,t) = \sqrt{\frac{-2\beta}{\gamma}}\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right), \tag{3.30}$$

where $z = x - \nu t$, and C_1 and C_2 are arbitrary constants.

3.2.2 Conservation laws of (3.2)

In this section we construct conservation laws for equation (3.2) by using the multiplier method.

We compute all multipliers of the form

$$\Lambda = \Lambda(t, x, u). \tag{3.31}$$

We consider the multiplier approach for equation (3.2),

$$\frac{\delta}{\delta u}\left[\Lambda\big(u_{tt}+\alpha u_{txx}+\beta u_{xxxx}+6\gamma u u_x^2+3\gamma u^2 u_{xx}\big)\right]=0. \tag{3.32}$$

Then

$$\Lambda\big(u_{tt}+\alpha u_{txx}+\beta u_{xxxx}+6\gamma u u_x^2+3\gamma u^2 u_{xx}\big)=D_tT^t+D_xT^x, \tag{3.33}$$

where T^t and T^x are conserved vectors. From equation (3.32), it follows that

$$(\Lambda_{tt}-\alpha\Lambda_{txx}+\beta\Lambda_{xxxx}+3\gamma u^2\Lambda_{xx})+2u_{tt}\Lambda_u=0. \tag{3.34}$$

Equation (3.34) is a polynomial identity in the variable u_{tt}. Hence equation (3.34) splits into two equations

$$\Lambda_u=0,\quad \Lambda_{tt}-\alpha\Lambda_{txx}+\beta\Lambda_{xxxx}+3\gamma u^2\Lambda_{xx}=0, \tag{3.35}$$

whose solution yields the four local conservation law multipliers

$$\Lambda_1=1,\;\; \Lambda_2=t,\;\; \Lambda_3=x,\;\; \Lambda_4=xt. \tag{3.36}$$

Consequently, we obtain the local conservation laws of equation (3.2), given by

$$\begin{aligned} T_1^t &= \alpha x u_{xx}+x u_t,\\ T_1^x &= \beta x u_{xxx}-\beta u_{xx}+3\gamma x u^2 u_x-\gamma u^3; \end{aligned} \tag{3.37}$$

$$\begin{aligned} T_2^t &= \alpha u_{xx}+u_t\\ T_2^x &= \beta u_{xxx}+3\gamma u^2 u_x; \end{aligned} \tag{3.38}$$

$$\begin{aligned} T_3^t &= \alpha x t u_{xx}+x t u_t-x u,\\ T_3^x &= \beta x t u_{xxx}-\beta t u_{xx}-\alpha u_{xx}+3\gamma x t u^2 u_x-\gamma t u^3+\alpha u; \end{aligned} \tag{3.39}$$

$$T_4^t = \alpha t u_{xx} + t u_t - u,$$
$$T_4^x = \beta t u_{xxx} + 3\gamma t u^2 u_x - \alpha u_x. \tag{3.40}$$

It is observed that the conserved vector (3.39) does not satisfy the divergence condition, viz., $D_i T^i|_{(3.2)} = 0$, as some excessive terms emerge that require some further analysis. By making a slight adjustment to these terms, it can be shown that this can be absorbed into the divergence condition.

For,

$$\begin{aligned} D_t(T_3^t) + D_x(T_3^x) &= \alpha u_{xxx} - \alpha x u_{xx} - \alpha u_x \\ &= D_x(\alpha u_{xx} - \alpha x u_x) \end{aligned} \tag{3.41}$$

hence,

$$D_t(T_3^t) + D_x(T_3^x + \alpha u_{xx} - \alpha x u_x)|_{(3.2)} = 0. \tag{3.42}$$

We now redefine the conserved vectors in the parenthesis as

$$\begin{aligned} \tilde{T_3^t} &= x(\alpha t u_{xx} + t u_t - u), \\ \tilde{T_3^x} &= \beta x t u_{xxx} - \beta t u_{xx} + 3\gamma x t u^2 u_x - \gamma t u^3 + \alpha u - \alpha x u_{xx}. \end{aligned} \tag{3.43}$$

Thus, the modified conserved vectors $\tilde{T_3^t}$ and $\tilde{T_3^x}$ satisfy the divergence condition.

3.3 Conclusion

In this chapter we studied two nonlinear partial differential equations that appear in a variety of scientific fields. These are the modified Kortweg de Vries equation and the higher-order modified Boussinesq equation with damping term. This chapter showed that $(G'/G)-$expansion method is an effective method for finding exact solutions of nonlinear evolution equations. The key ideas of the method are that the travelling wave solutions of a complicated nonlinear evolution equation can be constructed by means of various solutions of a second-order linear ordinary differential equation as presented in Chapter one. By using this method we have successfully obtained travelling wave solutions expressed in the form of hyperbolic function, trigonometric function and rational function. We have also determined the conservation laws using the multiplier approach for the higher-order modified Boussinesq equation with damping term.

Chapter 4

Solutions and conservation laws of coupled Korteweg-de Vries equations

The well-known celebrated Korteweg-de Vries (KdV) equation [71]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{4.1}$$

describes the dynamics of solitary waves. Initially, it was derived to describe shallow water waves of long wavelength and small amplitude. It is an important equation in the field of theory of integrable systems. It has infinite number of conservation laws, gives multiple-soliton solutions, and has many other physical properties. See for example [72, 73] and references therein.

The coupled Korteweg-de Vries equations, have recently been the focus of attraction for scientists, because of their many applications in scientific fields and many studies have been reported in the literature. See for example [74–77].

In this chapter we study the coupled Korteweg-de Vries equations [77],

$$\begin{cases} u_t + 6uu_x - 6vv_x + u_{xxx} = 0, \\ \\ v_t + 3uv_x + v_{xxx} = 0. \end{cases} \tag{4.2}$$

We determine the exact solutions for equation (4.2) using the $(G'/G)-$expansion method. Furthermore, conservation laws for (4.2) will be constructed by employing the new conservation theorem due to Ibragimov [53] and multiplier approach [50].

The work on the exact solutions of equation (4.2) has been published. See [78].

4.1 Solutions of (4.2)

In this section we employ the $(G'/G)-$expansion method and construct the travelling wave solutions of the coupled KdV equation (4.2) where u and v are real-valued scalar functions, t is time and x is a spatial variable.

As a first step we transform the coupled KdV equations (4.2) to a system of nonlinear ordinary differential equations using the travelling wave variable

$$u(t,x) = U(z), \;\; v(t,x) = V(z), \;\; \text{where} \;\; z = x - ct. \tag{4.3}$$

Using the above transformations, equation (4.2) transform to the nonlinear ordinary differential equations

$$\begin{cases} -cU' + 6UU' - 6VV' + U''' = 0, \\ \\ -cV' + 3UV' + V''' = 0, \end{cases} \tag{4.4}$$

where the primes denotes the derivative with respect to z.

The $(G'/G)-$expansion method assumes the solutions of equation (4.4) to be of the form given by equation (1.48).

The balancing procedure yields $M=2$, so the solutions of the ordinary differential equation (4.4) are of the form

$$U(z)=\alpha_0+\alpha_1(G'/G)+\alpha_2(G'/G)^2,\;\; V(z)=\beta_0+\beta_1(G'/G)+\beta_2(G'/G)^2. \quad (4.5)$$

Substituting (4.5) into (4.4) and making use of (1.49), and then collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields a system of algebraic equations. Solving this system of algebraic equations, using Mathematica, we obtain the following two sets of values for the constants α's and β's:

Case A

$$\alpha_0=\frac{1}{3}(c-\lambda^2-2\mu),\;\; \alpha_1=-2\lambda,\;\; \alpha_2=-2,$$
$$\beta_0=\pm\frac{\lambda}{\sqrt{6}}\sqrt{\lambda^2-4\mu-c},\;\; \beta_1=\frac{2\beta_0}{\lambda},\;\; \beta_2=0.$$

Case B

$$\alpha_0=\frac{1}{3}(c-\lambda^2-8\mu),\;\; \alpha_1=-4\lambda,\;\; \alpha_2=-4$$
$$\beta_0=\pm\frac{1}{3\sqrt{2}}\sqrt{c^2-2c\lambda^2+\lambda^4-16c\mu+64\mu^2},\;\; \beta_1=\frac{12\lambda\beta_0}{8\mu-c\lambda^2},\;\; \beta_2=\frac{\beta_1}{\lambda}.$$

Substituting the values of α's and β's from Case A and the corresponding solutions of ordinary differential equations (1.49) into (4.5), we obtain the following three types of travelling wave solutions of equation (4.2):

Case 1: When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solutions

$$\begin{aligned} u_1(t,x) &= \frac{1}{3}(c-\lambda^2-2\mu)-2\lambda\Big[-\frac{\lambda}{2}+\delta_1\Big(\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\Big)\Big] \\ &\quad -2\Big[-\frac{\lambda}{2}+\delta_1\Big(\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\Big)\Big]^2, \\ v_1(t,x) &= \pm\frac{\lambda}{\sqrt{6}}\sqrt{\lambda^2-4\mu-c}+\frac{2\beta_0}{\lambda}\Big[-\frac{\lambda}{2}+\delta_1\Big(\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\Big)\Big], \end{aligned}$$

where $z = x - ct$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Case 2: When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$\begin{aligned} u_2(t,x) &= \frac{1}{3}(c-\lambda^2-2\mu) - 2\lambda\bigg(-\frac{\lambda}{2} + \delta_2 \frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\bigg) \\ &\quad -2\bigg(-\frac{\lambda}{2} + \delta_2 \frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\bigg)^2, \\ v_2(t,x) &= \pm\frac{\lambda}{\sqrt{6}}\sqrt{\lambda^2-4\mu-c} + \frac{2\beta_0}{\lambda}\bigg(-\frac{\lambda}{2} + \delta_2 \frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\bigg), \end{aligned}$$

where $z = x - ct$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions

$$\begin{aligned} u_3(t,x) &= \frac{1}{3}(c-\lambda^2-2\mu) - 2\lambda\bigg(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\bigg) - 2\bigg(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\bigg)^2, \\ v_3(t,x) &= \pm\frac{\lambda}{\sqrt{6}}\sqrt{\lambda^2-4\mu-c} + \frac{2\beta_0}{\lambda}\bigg(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\bigg), \end{aligned}$$

where $z = x - ct$, C_1 and C_2 are arbitrary constants.

In a similar fashion, using the values of Case B we obtain three types of travelling wave solutions of equations (4.2) as follows:

Case 1: When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$\begin{aligned} u_4(t,x) &= \frac{1}{3}(c-\lambda^2-8\mu) - 4\lambda\bigg[-\frac{\lambda}{2} + \delta_1\bigg(\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\bigg)\bigg] \\ &\quad -4\bigg[-\frac{\lambda}{2} + \delta_1\bigg(\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\bigg)\bigg]^2, \\ v_4(t,x) &= \pm\frac{1}{3\sqrt{2}}\sqrt{c^2 - 2c\lambda^2 + \lambda^4 - 16c\mu + 64\mu^2} \\ &\quad + \frac{12\lambda\beta_0}{8\mu - c\lambda^2}\bigg[-\frac{\lambda}{2} + \delta_1\bigg(\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\bigg)\bigg] \\ &\quad + \frac{\beta_1}{\lambda}\bigg[-\frac{\lambda}{2} + \delta_1\bigg(\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\bigg)\bigg]^2, \end{aligned}$$

where $z = x - ct$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Case 2: When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$
\begin{aligned}
u_5(t,x) &= \frac{1}{3}(c-\lambda^2-8\mu) - 4\lambda\bigg(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\bigg) \\
&\quad -4\bigg(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\bigg)^2, \\
v_5(t,x) &= \pm\frac{1}{3\sqrt{2}}\sqrt{c^2-2c\lambda^2+\lambda^4-16c\mu+64\mu^2} \\
&\quad +\frac{12\lambda\beta_0}{8\mu-c\lambda^2}\bigg(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\bigg) \\
&\quad +\frac{\beta_1}{\lambda}\bigg(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\bigg)^2,
\end{aligned}
$$

where $z = x - ct$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions

$$
\begin{aligned}
u_6(t,x) &= \frac{1}{3}(c-\lambda^2-8\mu) - 4\lambda\bigg(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\bigg) - 4\bigg(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\bigg)^2, \\
v_6(t,x) &= \pm\frac{1}{3\sqrt{2}}\sqrt{c^2-2c\lambda^2+\lambda^4-16c\mu+64\mu^2} + \frac{12\lambda\beta_0}{8\mu-c\lambda^2}\bigg(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\bigg) \\
&\quad +\frac{\beta_1}{\lambda}\bigg(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2 z}\bigg)^2,
\end{aligned}
$$

where $z = x - ct$, C_1 and C_2 are arbitrary constants.

It should be noted that the solutions obtained in this paper by $(G'/G)-$expansion method are more general than the solutions obtained in [77].

4.2 Conservation laws of (4.2)

In this section the new conservation theorem due to Ibragimov [53] and multiplier approach [50] will be used to construct conservation laws for (4.2). To use the new conservation theorem we need to know the Lie point symmetries for equation (4.2). Thus, we first compute the symmetries for (4.2).

4.2.1 Lie point symmetries of (4.2)

The vector field

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + +\eta^2(t,x,u,v)\frac{\partial}{\partial v}$$

is a Lie point symmetry of (4.2) if

$$X^{[3]}\,(u_t + 6uu_x - 6vv_x + u_{xxx})|_{(4.2)} = 0,\;\; X^{[3]}\,(v_t + 3uv_x + v_{xxx})|_{(4.2)} = 0. \quad (4.6)$$

Expanding (4.6) and then splitting on the derivatives of u and v, we obtain the following overdetermined system of linear partial differential equations:

$$\xi^1_x = 0,\;\; \xi^1_u = 0,\;\; \xi^1_v = 0,\;\; \xi^2_u,\;\; \xi^2_v = 0,\;\; \eta^2_u = 0,\;\; \eta^2_{vv} = 0,$$

$$\eta^1_{uu} = 0,\;\; \eta^1_{uv} = 0,\;\; \eta^1_{vv} = 0,\;\; \eta^1_{xv} = 0,\;\; \eta^1_{xu} - \xi^2_{xx} = 0,\;\; \eta^2_{xv} - \xi^2_{xx} = 0,$$

$$3\xi^2_x - \xi^1_t = 0,\;\; \xi^1_t - 3\xi^2_x = 0,\;\; \eta^2_t + \eta^2_{xxx} + 3u\eta^2_x = 0,\; \eta^1_{xxx} + 6u\eta^1_x + \eta^1_t - 6v\eta^1_x = 0,$$

$$3\eta^1 - \xi^2_t - \xi^2_{xxx} + 3\eta^2_{xxv} + 6u\xi^2_x = 0,,\;\; 2v\xi^2_x - 2v\eta^2_v - 2v\xi^1_t + 2v\eta^1_u - 2\eta^2 + u\eta^1_v = 0,$$

$$6u\xi^1_t + 6\eta^1 - \xi^1_t - \xi^2_{xxx} + 3\eta^1_{xxu} - 6u\xi^2_x = 0.$$

Solving the above system of partial differential equations, one obtains the following three Lie point symmetries:

$$\textit{time translation} \qquad X_1 = \frac{\partial}{\partial t},$$

$$\textit{space translation} \qquad X_2 = \frac{\partial}{\partial x},$$

$$\textit{scaling} \qquad X_3 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}.$$

In the next section we will construct conservation laws corresponding to each of these symmetries.

4.2.2 Application of the new conservation theorem

This section constructs conservation laws corresponding to each of the above two translational symmetries X_1, X_2 and scaling symmetry X_3 for the system (4.2).

Using equation (1.37), a third-order Lagrangian for the system (4.2)

$$\begin{cases} F_1 \equiv u_t + 6uu_x - 6vv_x + u_{xxx} = 0, \\ F_2 \equiv v_t + 3uv_x + v_{xxx} = 0. \end{cases}$$

is given by

$$L \equiv p(u_t + 6uu_x - 6vv_x + u_{xxx}) + q(v_t + 3uv_x + v_{xxx}), \tag{4.7}$$

where p and q are two new dependent variables. Taking into account equation (4.7), the adjoint system for the coupled KdV equations (4.2) obtained from equation (1.36) is

$$\begin{cases} F_1^* \equiv 3qv_x - p_{xxx} - 6up_x - p_t = 0, \\ F_2^* \equiv 6vp_x - q_{xxx} - 3uq_x - 3u_xq - q_t = 0. \end{cases} \tag{4.8}$$

According to [53], system (4.2) is said to be nonlinearly self-adjoint if the adjoint system (4.8) obeys the condition

$$\begin{cases} F_1^*|_{p=\phi(t,x,u,v),q=\psi(t,x,u,v)} = \lambda_{11}F_1 + \lambda_{12}F_2, \\ F_2^*|_{p=\phi(t,x,u,v),q=\psi(t,x,u,v)} = \lambda_{21}F_1 + \lambda_{22}F_2 \end{cases} \tag{4.9}$$

with regular undetermined coefficients λ_{ij} $(i, j = 1, 2)$. Following the substitution

$$p = \phi(t, x, u, v), \;\; q = \psi(t, x, u, v) \tag{4.10}$$

with $\phi(t, x, u, v) \neq 0$ or $\psi(t, x, u, v) \neq 0$ and since ϕ and ψ do not depend on the derivatives $u_t, v_t, u_{xx}, \cdots$, equation (4.9) split into the following equations:

$$\lambda_{11} = -\phi_u, \;\; \lambda_{12} = -\phi_v, \;\; \lambda_{21} = -\psi_u, \;\; \lambda_{22} = -\psi_v,$$

$$\phi_{uu} = 0, \;\; \phi_{vv} = 0, \;\; \phi_{uv} = 0, \;\; \phi_{xu} = 0, \;\; \phi_{xv} = 0,$$

$$\psi_{uu} = 0, \;\; \psi_{vv} = 0, \;\; \psi_{uv} = 0, \;\; \psi_{xu} = 0, \;\; \psi_{xv} = 0,$$

$$6v\phi_u + 3u\phi_v - 3\psi = 0,\ \phi_{xxx} + 6u\phi_x + \phi_t = 0,$$
$$6v\phi_u + 3u\psi_u - 3\psi = 0,\ 6v\phi_v - 6v\psi_u = 0,$$
$$\psi_{xxx} - 6v\phi_x + 3u\psi_x + \psi_t = 0.$$

Solving the above system, we get

$$p = \phi(t, x, u, v) = -c_1 u + c_2,$$
$$q = \psi(t, x, u, v) = -2c_1 v, \tag{4.11}$$

where c_1, c_2 are arbitrary constants. The adjoint equation (4.8) becomes equivalent to the original system (4.2) after substituting the values of p and q. This shows that the system (4.2) is nonlinearly self-adjoint when p and q take the values given by (4.11).

We now apply Theorem 1.6 to the nonlinearly self-adjoint equations (4.2) to find conservation laws for the following three cases:

(i) We first consider the Lie point symmetry $X_1 = \partial/\partial t$ of (4.2). Corresponding to this symmetry the Lie characteristic functions are $W^1 = -u_t$ and $W^2 = -v_t$. Thus, by using the new conservation theorem due to Ibragimov [53], the components of the conserved vector associated with the symmetry X_1 are given by

$$\begin{aligned} C_1^t &= c_3 u_{xxx} + 6c_3 u u_x - 6c_3 v v_x - c_2 u u_{xxx} - 2c_2 v v_{xxx} - 6c_2 u^2 u_x, \\ C_1^x &= 6c_2 u^2 u_t - 6c_3 u u_t + c_2 u_t u_{xx} + c_2^2 u u_{tx} - c_2 c_3 u_{tx} - c_2^2 u u_{txx} + c_2 c_3 u_{txx} \\ &\quad + 6c_3 v v_t + 2c_2 v_t v_{xx} + 4c_2^2 v_x v_{tx} + 2c_2 v v_{txx}. \end{aligned}$$

It is observed that the above conserved vectors do not satisfy the divergence condition, viz., $D_i C^i|_{(4.2)} = 0$, as some excessive terms emerge that require some further analysis. By making a slight adjustment to these terms, it can be shown that these can be absorbed into the divergence condition.

For,

$$D_t(C_1^t) + D_x(C_1^x) = c_3 u_{txxx} - c_2 u u_{txxx} + c_2 u_{tx} u_{xx} c_2^2 u_x u_{tx} + c_2^2 u u_{txx}$$

$$
\begin{aligned}
&-c_2c_3u_{txx} - c_2^2u_xu_{txx} - c_2^2uu_{txxx} + c_2c_3u_{txxx} + 2c_2v_{tx}v_{xx} \\
&+4c_2^2v_{tx}v_{xx} + c_2^2v_xv_{txx} + 2c_2v_xv_{txx} \\
= \;& D_t(c_3u_{xxx}) + D_x(c_2c_3u_{txx}) + D_x(c_2u_tu_{xx}) - D_t(c_2uu_{xxx}) \\
&+D_x(c_2^2uu_{tx}) - D_x(c_2^2uu_{txx}) - D_x(c_2c_3u_{tx}) + D_x(2c_2v_xv_{tx}) \\
&+D_x(4c_2^2v_xv_{tx}) \qquad (4.12)
\end{aligned}
$$

hence,

$$
\begin{aligned}
&D_t(6c_3uu_x - 6c_3vv_x - 2c_2vv_{xxx} - 6c_2u^2u_x) + D_x(6c_2u^2u_t - 6c_3uu_t + c_2u_tu_{xx} \\
&+6c_3vv_t + 2c_2v_tv_{xx} + 2c_2vv_{txx} - c_2u_tu_{xx} - 2c_2v_xv_{tx})\,|_{(4.2)} = 0 \qquad (4.13)
\end{aligned}
$$

We now redefine the conserved vectors in the parenthesis as:

$$
\begin{aligned}
\tilde{C}_1^t &= 6c_3uu_x - 6c_3vv_x - 2c_2vv_{xxx} - 6c_2u^2u_x \\
\tilde{C}_1^x &= 6c_2u^2u_t - 6c_3uu_t + c_2u_tu_{xx} + 6c_3vv_t + 2c_2v_tv_{xx} + 2c_2vv_{txx} \\
&\quad -c_2u_tu_{xx} - 2c_2v_xv_{tx}. \qquad (4.14)
\end{aligned}
$$

Thus, the modified conserved vectors $\tilde{C}_1^t$, and $\tilde{C}_1^x$, satisfy the divergence condition and they represent the energy conservation law.

(ii) Likewise, the Lie point symmetry $X_2 = \partial/\partial x$ has the Lie characteristic functions $W^1 = -u_x$ and $W^2 = -v_x$. Invoking Ibragimov theorem we obtain the conserved vector whose components are:

$$
\begin{aligned}
C_2^t &= c_2uu_x - c_3u_x + 2c_2vv_x, \\
C_2^x &= c_3u_t - c_2uu_t - 2c_2vv_t,
\end{aligned}
$$

which represents the linear momentum conservation law.

(iii) Finally, the Lie point symmetry $X_3 = 3t\partial/\partial t + x\partial/\partial x - 2u\partial/\partial u - 2v\partial/\partial v$ gives $W^1 = -(3tu_t + xu_x + 2u)$ and $W^2 = -(3tv_t + xv_x + 2v)$ and so the associated conserved vector has components

$$
C_3^t = 18c_3tuu_x - 18c_2tu^2u_x - 18c_3tvv_x - 3c_2tuu_{xxx} + 3c_3tu_{xxx} - 6c_2tvv_{xxx}
$$

$$
\begin{aligned}
& +2c_2u^2 - 2c_3u + c_2xuu_x - c_3xu_x + 4c_2v^2 + 2c_2xvv_x, \\
C_3^x &= c_3xu_t - c_2xuu_t - 2c_2xvv_t + 12c_2u^3 - 12c_3u^2 + 18c_2tu^2u_t - 18c_3tuu_t \\
& +6c_2uu_{xx} + 3c_2tu_tu_{xx} - 3c_2u_x^2 - 3c_2tu_xu_{tx} - 4c_3u_{xx} + 3c_2tuu_{txx} \\
& -3c_3tu_{txx} - 12c_2uv^2 + 12c_3v^2 + 18c_3tvv_t + 12c_2uv^2 + 12c_2vv_{xx} \\
& +6c_2tv_tv_{xx} - 4c_2v_x^2 - 6c_2tv_xv_{tx} - 2c_2v_x^2 + 6c_2tvv_{txx},
\end{aligned}
$$

which represents the boost momentum conservation law.

4.2.3 Conservation laws for (4.2) using multiplier approach

We now use the multiplier method to find conservation laws of (4.2). The second-order multipliers $\Lambda(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ for (4.2) are given by

$$
\Lambda_1 = \frac{1}{2}C_1v^2 + \frac{1}{2}C_2u + C_3v + C_4, \tag{4.15}
$$

$$
\Lambda_2 = C_1(uv + v_{xx}) + C_2v + C_3u, \tag{4.16}
$$

where C_1, C_2, C_3 and C_4 are constants [79]. Corresponding to the above multipliers we have the following four conserved vectors:

$$
\begin{aligned}
C_1^t &= \frac{1}{2}(uv^2 + vv_{xx}), \\
C_1^x &= \frac{1}{2}(u_{xx}v^2 + v_{xx}^2 + v_tv_x - vv_{tx} + 3u^2v^2) + uvv_{xx} + uv_x^2 - u_xvv_x - \frac{3}{4}v^4;
\end{aligned} \tag{4.17}
$$

$$
\begin{aligned}
C_2^t &= \frac{1}{4}u^2 + \frac{1}{2}v^2, \\
C_2^x &= u^3 + vv_{xx} + \frac{1}{2}(uu_{xx} - v_x^2) - \frac{1}{4}u_x^2;
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
C_3^t &= uv, \\
C_3^x &= uv_{xx} + u_{xx}v - u_xv_x + 3u^2v - 2v^3;
\end{aligned} \tag{4.19}
$$

$$C_4^t = u,$$
$$C_4^x = u_{xx} + 3u^2 - 3v^2. \tag{4.20}$$

4.3 Conclusion

In this chapter we studied coupled Korteweg-de Vries equation (4.2) that appear in many scientific fields. The $(G'/G)-$expansion method was used to derive exact travelling wave solutions of the coupled KdV equations. The solutions obtained were expressed in the form of hyperbolic function, trigonometric function and rational solutions. Furthermore, conservation laws were derived by using the new conservation theorem and the multiplier approach.

Chapter 5

Exact solutions and conservation laws of coupled Boussinesq equations

In this chapter we study the coupled Boussinesq equations [80]

$$\begin{cases} u_t + uu_x + v_x + au_{xxt} = 0, \\ v_t + (uv)_x + bu_{xxx} = 0, \end{cases} \tag{5.1}$$

where u and v are real-valued scalar functions, t is time and x is a spatial variable. We determine the travelling wave solutions of (5.1) by using the $(G'/G)-$expansion method. In addition, the conservation laws for system (5.1) are constructed using both the new conservation theorem [53] and the multiplier approach [50].

Exact solutions presented in this chapter have been published. See [81].

5.1 Exact solutions of coupled Boussinesq equations

In this section we employ the $(G'/G)-$expansion method and construct the travelling wave solutions of the coupled Boussinesq equations (5.1).

We first transform the coupled Boussinesq equations (5.1) to nonlinear ordinary differential equations using the traveling wave variable

$$u(t,x) = U(\xi),\ \ v(t,x) = V(\xi),\ \ \text{where}\ \ \xi = x - ct. \tag{5.2}$$

Using the above transformations, equation (5.1) transform to the nonlinear ordinary differential equations

$$\begin{cases} acU''' + cU' - UU' - V' = 0, \\ bU''' - cV' + VU' + UV' = 0, \end{cases} \tag{5.3}$$

where the primes denotes the derivative with respect to ξ.

We assume that the solutions of (5.3) are of the form given by (1.48). The balancing procedure yields $M = 2$, so the solutions of equation (5.3) are of the form

$$U(\xi) = \alpha_0 + \alpha_1(G'/G) + \alpha_2(G'/G)^2,\ \ V(\xi) = \beta_0 + \beta_1(G'/G) + \beta_2(G'/G)^2. \tag{5.4}$$

Substituting (5.4) into (5.3), making use of the equation (1.49), collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields the following system of algebraic equations:

$$\begin{aligned}
(G'/G)^0 \ &: \ \alpha_0\alpha_1\mu - a\alpha_1c\lambda^2\mu - 6a\alpha_2c\lambda\mu^2 - 2a\alpha_1c\mu^2 + \beta_1\mu - \alpha_1c\mu = 0,\\
(G'/G) \ &: \ \alpha_0\alpha_1\lambda - a\alpha_1c\lambda^3 - 14a\alpha_2c\lambda^2\mu - 8a\alpha_1c\lambda\mu - 16a\alpha_2c\mu^2 + \alpha_1^2\mu\\
&\quad +2\alpha_0\alpha_2\mu + \beta_1\lambda + 2\beta_2\mu - \alpha_1c\lambda - 2\alpha_2c\mu = 0,\\
(G'/G)^2 \ &: \ 3\alpha_1\alpha_2\mu - 8a\alpha_2c\lambda^3 - 7a\alpha_1c\lambda^2 - 52a\alpha_2c\lambda\mu - 8a\alpha_1c\mu + \alpha_1^2\lambda
\end{aligned}$$

$$
\begin{aligned}
& && +2\alpha_0\alpha_2\lambda+\alpha_0\alpha_1+2\beta_2\lambda+\beta_1-2\alpha_2c\lambda-\alpha_1c=0,\\
(G'/G)^3 &:&& 3\alpha_1\alpha_2\lambda-38a\alpha_2c\lambda^2-12a\alpha_1c\lambda-40a\alpha_2c\mu+2\alpha_2^2\mu+\alpha_1^2\\
& && +2\alpha_0\alpha_2+2\beta_2-2\alpha_2c=0,\\
(G'/G)^4 &:&& 3\alpha_1\alpha_2-54a\alpha_2c\lambda-6a\alpha_1c+2\alpha_2^2\lambda=0,\\
(G'/G)^5 &:&& 24a\alpha_2c+2\alpha_2^2=0,\\
(G'/G)^6 &:&& \beta_1c\mu-\alpha_1\beta_0\mu-\alpha_0\beta_1\mu+\alpha_1(-b)\lambda^2\mu-6\alpha_2b\lambda\mu^2-2\alpha_1b\mu^2=0,\\
(G'/G)^7 &:&& \beta_1c\lambda-\alpha_1\beta_0\lambda-\alpha_0\beta_1\lambda-2\alpha_2\beta_0\mu-2\alpha_1\beta_1\mu-2\alpha_0\beta_2\mu+\alpha_1(-b)\lambda^3\\
& && -14\alpha_2b\lambda^2\mu-8\alpha_1b\lambda\mu-16\alpha_2b\mu^2+2\beta_2c\mu=0,\\
(G'/G)^8 &:&& 2\beta_2c\lambda-2\alpha_2\beta_0\lambda-2\alpha_1\beta_1\lambda-2\alpha_0\beta_2\lambda-3\alpha_2\beta_1\mu-3\alpha_1\beta_2\mu-\alpha_1\beta_0-\alpha_0\beta_1\\
& && -8\alpha_2b\lambda^3-7\alpha_1b\lambda^2-52\alpha_2b\lambda\mu-8\alpha_1b\mu+\beta_1c=0,\\
(G'/G)^9 &:&& 2\beta_2c-3\alpha_2\beta_1\lambda-3\alpha_1\beta_2\lambda-4\alpha_2\beta_2\mu-2\alpha_2\beta_0-2\alpha_1\beta_1-2\alpha_0\beta_2-38\alpha_2b\lambda^2\\
& && -12\alpha_1b\lambda-40\alpha_2b\mu=0,\\
(G'/G)^{10} &:&& -4\alpha_2\beta_2\lambda-3\alpha_1\beta_2-3\alpha_2\beta_1-54\alpha_2b\lambda-6\alpha_1b=0,\\
(G'/G)^{11} &:&& -4\alpha_2\beta_2-24\alpha_2b=0.
\end{aligned}
$$

With the aid of Mathematica, we obtain the following set of values for the constants α's and β's:

$$\alpha_0=\frac{2ac^2(a\lambda^2+8a\mu+1)+b}{2ac},\ \alpha_1=12ac\lambda,\ \alpha_2=12ac,$$

$$\beta_0=\frac{b^2-2a^2bc^2\lambda^2-16a^2bc^2\mu}{4a^2c^2},\ \beta_1=-6b\lambda,\ \beta_2=-6b.$$

Substituting the values of α's and β's and the corresponding solutions of second-order ordinary differential equation (1.49) into (5.4), we obtain the following three types of travelling wave solutions of equations (5.1):

Case 1: When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$\begin{aligned}
u_1(t,x) &= \frac{2ac^2(a\lambda^2 + 8a\mu + 1) + b}{2ac} \\
&\quad + 12ac\lambda\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1\sinh(\delta_1\xi) + C_2\cosh(\delta_1\xi)}{C_1\cosh(\delta_1\xi) + C_2\sinh(\delta_1\xi)}\right)\right] \\
&\quad + 12ac\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1\sinh(\delta_1\xi) + C_2\cosh(\delta_1\xi)}{C_1\cosh(\delta_1\xi) + C_2\sinh(\delta_1\xi)}\right)\right]^2, \\
v_1(t,x) &= \frac{b^2 - 2a^2bc^2\lambda^2 - 16a^2bc^2\mu}{4a^2c^2} \\
&\quad - 6b\lambda\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1\sinh(\delta_1\xi) + C_2\cosh(\delta_1\xi)}{C_1\cosh(\delta_1\xi) + C_2\sinh(\delta_1\xi)}\right)\right], \\
&\quad - 6b\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1\sinh(\delta_1\xi) + C_2\cosh(\delta_1\xi)}{C_1\cosh(\delta_1\xi) + C_2\sinh(\delta_1\xi)}\right)\right]^2,
\end{aligned}$$

where $\xi = x - ct$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Case 2: When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$\begin{aligned}
u_2(t,x) &= \frac{2ac^2(a\lambda^2 + 8a\mu + 1) + b}{2ac} \\
&\quad + 12ac\lambda\left[-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2\xi) + C_2\cos(\delta_2\xi)}{C_1\cos(\delta_2\xi) + C_2\sin(\delta_2\xi)}\right] \\
&\quad + 12ac\left[-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2\xi) + C_2\cos(\delta_2\xi)}{C_1\cos(\delta_2\xi) + C_2\sin(\delta_2\xi)}\right]^2, \\
v_2(t,x) &= \frac{b^2 - 2a^2bc^2\lambda^2 - 16a^2bc^2\mu}{4a^2c^2} \\
&\quad - 6b\lambda\left[-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2\xi) + C_2\cos(\delta_2\xi)}{C_1\cos(\delta_2\xi) + C_2\sin(\delta_2\xi)}\right] \\
&\quad - 6b\left[-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2\xi) + C_2\cos(\delta_2\xi)}{C_1\cos(\delta_2\xi) + C_2\sin(\delta_2\xi)}\right]^2,
\end{aligned}$$

where $\xi = x - ct$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions

$$\begin{aligned} u_3(t,x) &= \frac{2ac^2(a\lambda^2 + 8a\mu + 1) + b}{2ac} + 12ac\lambda\left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right] \\ &\quad +12ac\left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right]^2, \end{aligned}$$

$$\begin{aligned} v_3(t,x) &= \frac{b^2 - 2a^2bc^2\lambda^2 - 16a^2bc^2\mu}{4a^2c^2} - 6b\lambda\left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right] \\ &\quad -6b\left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right]^2, \end{aligned}$$

where $\xi = x - ct$, C_1 and C_2 are arbitrary constants.

It should be noted that the solutions obtained in this paper by $(G'/G)-$expansion method are more general than the solutions obtained in [80].

5.2 Conservation laws of (5.1)

In this section we construct conservation laws for (5.1) by using both the new conservation theorem and multiplier approach.

5.2.1 Lie point symmetries of (5.1)

The vector field

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + +\eta^2(t,x,u,v)\frac{\partial}{\partial v}$$

is a Lie point symmetry of (5.1) if

$$X^{[3]}\left(u_t + uu_x + v_x + au_{xxt}\right)\Big|_{(5.1)} = 0, \qquad X^{[3]}\left(v_t + (uv)_x + bu_{xxx}\right)\Big|_{(5.1)} = 0. \tag{5.5}$$

Expanding (5.5) and then splitting on the derivatives of u and v, we obtain the following overdetermined system of linear partial differential equations:

$$
\begin{aligned}
&\xi^1_x = 0,\ \xi^1_u = 0,\ \xi^1_v = 0,\ \xi^2_t = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \eta^2_u = 0,\ \eta^1_v = 0,\ \eta^1_{ut} = 0,\\
&\eta^1_{uu} = 0,\ a\eta^1_{xxu} + 2\xi^2_x = 0,\ \eta^1_{xu} - \xi^2_{xx} = 0,\ 2\eta^1_{xu} - \xi^2_{xx} = 0,\\
&a\eta^1_{txx} + u\eta^1_x + \eta^2_x + \eta^1_t = 0,\ \eta^1 + 2u\xi^2_x - u\eta^1_u + u\eta^2_v = 0,\\
&b\eta^1_{xxx} + v\eta^1_x + \eta^2_t + u\eta^2_x = 0,\ 2v\xi^2_x - b\xi^2_{xxx} + 3b\eta^1_{xxu} + \eta^2 = 0,\\
&u\xi^1_t - au\eta^1_{xxu} - u\xi^2_x + \eta^1 = 0,\ 3\xi^2_x - \xi^1_t - \eta^1_u + \eta^2_v = 0,\\
&\xi^1_t - a\eta^1_{xxu} - \eta^1_u + \eta^2_v - \xi^2_x = 0.
\end{aligned}
$$

The solution of the above system of partial differential equations yields the following two translational Lie point symmetries:

$$
\begin{aligned}
&\textit{time translation} \qquad X_1 = \frac{\partial}{\partial t},\\
&\textit{space translation} \qquad X_2 = \frac{\partial}{\partial x}.
\end{aligned}
$$

5.2.2 Application of the new conservation theorem

We consider the coupled Boussinesq equation (5.1)

$$
\begin{cases}
F_1 \equiv u_t + uu_x + v_x + au_{xxt} = 0,\\[2ex]
F_1 \equiv u_t + uu_x + v_x + au_{xxt} = 0,
\end{cases}
$$

and use equation (1.37) to obtain the third-order Lagrangian. This is given by

$$
\mathcal{L} \equiv v^1(u_t + uu_x + v_x + au_{xxt}) + v^2(v_t + (uv)_x + bu_{xxx}), \tag{5.6}
$$

where v^1 and v^2 are two new dependent variables. The relation (1.36) yields the following adjoint equations for the coupled Boussinesq equations:

$$\begin{cases} F_1^* \equiv -(v_t^1 + uv_x^1 + vv_x^2 + bv_{xxx}^2 + av_{txx}^1) = 0, \\ F_2^* \equiv -(v_t^2 + v_x^1 + uv_x^2) = 0. \end{cases} \tag{5.7}$$

We now carry out a nonlinear self-adjoint classification of system (5.1). The system (5.1) is said to be nonlinearly self-adjoint if the adjoint system (5.7) satisfies the following equations:

$$\begin{cases} F_1^*|_{v^1=\phi(t,x,u,v),v^2=\psi(t,x,u,v)} = \lambda_{11}F_1 + \lambda_{12}F_2, \\ F_2^*|_{v^1=\phi(t,x,u,v),v^2=\psi(t,x,u,v)} = \lambda_{21}F_1 + \lambda_{22}F_2 \end{cases} \tag{5.8}$$

with regular undetermined coefficients $\lambda_{ij}\,(i,j=1,2)$ and following the substitution

$$v^1 = \phi(t,x,u,v), \;\; v^2 = \psi(t,x,u,v) \tag{5.9}$$

with $\phi(t,x,u,v) \neq 0$ or $\psi(t,x,u,v) \neq 0$. Since ϕ and ψ do not depend on the derivatives $u_t, v_t, u_{xx}, \cdots$, equations (5.8) split into the following system of equations:

$$\begin{aligned} &\lambda_{11} = -\phi_u, \;\; \lambda_{12} = \lambda_{21} = -\psi_u, \;\; \lambda_{22} = -\psi_v, \\ &\phi_u = \phi_v = \psi_u = \psi_v = 0, \\ &b\psi_{xxx} + a\phi_{txx} + u\phi_u + v\psi_x + \psi_t = 0, \\ &u\psi_x + \psi_t + \phi_x = 0. \end{aligned}$$

Solving the above system, we get

$$\begin{aligned} v^1 &= \phi(t,x,u,v) = c_1, \\ v^2 &= \psi(t,x,u,v) = c_2, \end{aligned} \tag{5.10}$$

where c_1 and c_2 are arbitrary constants. The equations (5.7) become identical to the coupled Boussinesq equation (5.1)) after the substitution of equations (5.10). Hence the system (5.1) is a nonlinearly self-adjoint system.

We now employ Theorem 1.6 to construct conservation laws for system (5.1).

Case 1: Translation of time $X_1 = \dfrac{\partial}{\partial t}$

The Lie characteristic functions for X_1 are $W^1 = -u_t$, $W^2 = -v_t$. Invoking equation (1.40) the components of the conserved vector associated with the symmetry X_1 are given by

$$
\begin{aligned}
C_t^1 &= c_1uu_x + c_1v_x + c_2u_xv + c_2uv_x + bc_2u_{xxx}, \\
C_x^1 &= -(c_1uu_t + c_2u_tv + bc_2u_{txx} + ac_1u_{ttx} + c_1v_t + c_2uv_t),
\end{aligned}
$$

which represents the energy conservation law.

Case 2: Translation in x direction $X_2 = \dfrac{\partial}{\partial x}$

Likewise, the Lie point symmetry X_2 has the Lie characteristic functions $W^1 = -u_x$ and $W^2 = -v_x$, and we obtain the conserved vector whose components are

$$
\begin{aligned}
C_2^t &= -(c_1u_x + ac_1u_{txx} - c_2v_x), \\
C_2^x &= c_1u_t + c_2v_t.
\end{aligned}
$$

The above conserved vectors do not satisfy the divergence condition. Following the technique presented in the previous chapter it can be shown that these vectors can be absorbed into the divergence condition.

For X_1 we have

$$
\begin{aligned}
D_t(C_1^t) + D_x(C_1^x) &= ac_1u_{ttxx} \\
&= D_t(ac_1u_{txx})
\end{aligned}
\tag{5.11}
$$

hence,

$$
D_t(C_1^t - ac_1u_{txx}) + D_x(C_1^x) \mid_{(5.1)} = 0. \tag{5.12}
$$

We now redefine the conserved vectors as

$$\begin{aligned}
\tilde{C}_1^t &= c_1uu_x + c_1v_x + c_2u_xv + c_2uv_x + bc_2u_{xxx} - ac_1u_{txx} \\
\tilde{C}_1^x &= -(c_1uu_t + c_2u_tv + bc_2u_{txx} + ac_1u_{ttx} + c_1v_t + c_2uv_t). \qquad (5.13)
\end{aligned}$$

For X_2 we have

$$\begin{aligned}
D_t(C_2^t) + D_x(C_2^x) &= ac_1u_{ttxx} \\
&= D_t(ac_1u_{txx}) \qquad (5.14)
\end{aligned}$$

hence,

$$D_t(C_2^t - ac_1u_{txx}) + D_x(C_2^x) \mid_{(5.1)} = 0. \qquad (5.15)$$

We now redefine the conserved vectors as

$$\begin{aligned}
\tilde{C}_2^t &= -(c_1u_x + c_2v_x), \\
\tilde{C}_2^x &= c_1u_t + c_2v_t. \qquad (5.16)
\end{aligned}$$

Thus, the modified conserved vectors (5.13) and (5.16) satisfy the divergence condition and they represent the linear momentum conservation law.

5.2.3 Conservation laws for (5.1) using multiplier approach

We now construct the conservation laws for equation (5.1) by employing the multiplier method. The zero-order multipliers $\Lambda(t, x, u, v)$ for (5.1) are given by [79]

$$\Lambda_1 = C_2, \quad \Lambda_2 = C_1,$$

where C_1 and C_2 are constants. Corresponding to the above multipliers, we have the following two conserved vectors:

$$C_t^1 = v,$$

$$C_x^1 = uv + bu_{xx};$$

and

$$C_t^2 = u + au_{xx},$$
$$C_x^2 = \frac{1}{2}u^2 + v.$$

5.3 Conclusion

In this chapter we computed exact travelling wave solutions of coupled Boussinesq equations using $(G'/G)-$expansion method. The solutions obtained were expressed in the form of hyperbolic functions, trigonometric functions and rational solutions. We further constructed conservation laws by employing the new conservation theorem and multiplier approach.

Chapter 6

Conservation laws and exact solutions of a generalized Zakharov-Kuznetsov equation

This chapter aims to study the generalized Zakharov-Kuznetsov (gZK) equation [82, 83]

$$u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0, \tag{6.1}$$

where α, β and n are nonzero arbitrary constants and $u = u(t, x, y)$.

Many important phenomena and dynamic processes in physics, applied mathematics and engineering can be described by higher-dimensional extensions of the Kortweg-de Vries (KdV) equation. Zakharov and Kuznetsov successfully proposed one such model [84]. The Zakharov-Kuznetsov (ZK) equation given by

$$u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0 \tag{6.2}$$

is one of the known two dimensional generalizations of the KdV equation studied in the literature. The ZK equation governs the behaviour of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [84–86]. In [82,87], Wazwaz used the extended tanh method and the sine-cosine ansatz to find solitons and periodic solutions for (6.2). Moussa [88] also obtained some similarity solutions by using symmetry group method. Equation (6.2) was also studied amongst others by Peng [89] and by using extended mapping method abundant periodic wave solutions were obtained.

In [82] the extended tanh method was employed and solitons and periodic solutions were derived for (6.1), which may be helpful to describe waves features in plasma physics. The Cole-Hopf transformation and the first integral technique were used in [83] to obtain complex solutions for equation (6.1).

The work presented in this chapter has been published in [90].

6.1 Conservation laws

In this section the new conservation theorem due to Ibragimov [53] will be used to construct conservation laws for (6.1). For this purpose we need to first find Lie symmetries of (6.1).

6.1.1 Lie point symmetries of (6.1)

The vector field

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u} \qquad (6.3)$$

is a Lie point symmetry of (6.1) if

$$X^{[3]}\left[u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x\right]\Big|_{(6.1)} = 0. \qquad (6.4)$$

Expanding (6.4) and then splitting on the derivatives of $u(t,x,y)$, we obtain the following overdetermined system of linear partial differential equations:

$$\xi^3_t = 0,\ \ \xi^2_y = 0,\ \ \xi^3_x = 0,\ \ \xi^3_u = 0,\ \ \xi^2_u = 0,\ \ \xi^1_x = 0,$$
$$\xi^1_y = 0,\ \ \xi^1_u = 0,\ \ \xi^2_{xx} = 0,\ \ \eta_{xu} = 0,\ \ \eta_{uu} = 0,\ \ \xi^3_{yy} - 2\eta_{yu} = 0,$$
$$\beta u\eta_{yyu} - u\xi^2_t + 2\alpha u^{n+1}\xi^2_x + n\alpha u^n\eta = 0,$$
$$\beta u\eta_{yyu} - u\xi^2_t + 2\alpha u^{n+1}\xi^3_y + n\alpha u^n\eta = 0,$$
$$\beta\eta_{xxx} + \beta\eta_{xyy} + \alpha u^n\eta_x + \eta_t = 0,$$
$$\beta u\eta_{yyu} - u\xi^2_t + n\alpha u^n\eta + \alpha u^{n+1}\xi^1_t - \alpha u^{n+1}\xi^2_x = 0.$$

Solving the above system of partial differential equations one obtains the following four Lie point symmetries:

$$\begin{aligned}
&\textit{translation in time} && X_1 = \frac{\partial}{\partial t},\\
&\textit{spatial translational} && X_2 = \frac{\partial}{\partial x},\\
&\textit{spatial translational} && X_3 = \frac{\partial}{\partial y},\\
&\textit{Scaling} && X_4 = 3nt\frac{\partial}{\partial t} + nx\frac{\partial}{\partial x} + ny\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.
\end{aligned}$$

6.1.2 Application of the new conservation theorem

The gZK equation together with its adjoint equation are given by

$$\begin{cases} E \equiv u_t + \alpha u^n u_x + \beta u_{xxx} + \beta u_{xyy} = 0, \\ E^* \equiv v_t + \alpha v_x u^n + \beta v_{xxx} + \beta v_{xyy} = 0. \end{cases} \tag{6.5}$$

The third-order Lagrangian for equation (6.5) is given by

$$L = v(u_t + \alpha u^n u_x + \beta u_{xxx} + \beta u_{xyy}), \tag{6.6}$$

which can be reduced to the second-order Lagrangian

$$L = v(u_t + \alpha u^n u_x) - \beta v_x u_{xx} - \beta v_x u_{yy}. \tag{6.7}$$

We have the following four cases:

Case 1: $X_1 = \dfrac{\partial}{\partial t}$

Corresponding to this symmetry the Lie characteristic functions are $W^1 = -u_t$ and $W^2 = -v_t$. Thus, by using Theorem 1.6, the components of the conserved vector associated with the symmetry X_1 are given by

$$\begin{aligned}
C_1^t &= \alpha v u^n u_x - \beta v_x(u_{xx} + u_{yy}),\\
C_1^x &= \beta v_x u_{tx} + \beta v_t(u_{xx} + u_{yy}) - \alpha v u^n u_t - \beta u_t v_{xx},\\
C_1^y &= \beta v_x u_{ty} - \beta u_t v_{xy}
\end{aligned}$$

and this represents the energy conservation law.

Case 2: $X_2 = \dfrac{\partial}{\partial x}$

The Lie point symmetry X_2 has the Lie characteristic functions $W^1 = -u_x$ and $W^2 = -v_x$. Invoking Theorem 1.6 we obtain the conserved vector whose components are:

$$\begin{aligned}
C_2^t &= -v u_x,\\
C_2^x &= \beta v_x u_{xx} - \beta u_x v_{xx} + v u_t,\\
C_2^y &= \beta v_x u_{xy} - \beta u_x v_{xy},
\end{aligned}$$

which represents the linear momentum conservation law.

Case 3: $X_3 = \dfrac{\partial}{\partial y}$

The Lie characteristic functions for X_3 are $W^1 = -u_y$ and $W^2 = -v_y$. Application of Theorem 1.6 yields the following conserved vectors:

$$C_3^t = -v u_y,$$

$$C_3^x = \beta v_x u_{xy} + \beta v_y(u_{xx} + u_{yy}) - \beta u_y v_{xx} - \alpha v u_n u_y,$$
$$C_3^y = v(u_t + \alpha u^n u_x) - \beta v_x u_{xx} - \beta u_y v_{xy},$$

which represents the linear momentum conservation law.

Case 4: $X_4 = 3nt\dfrac{\partial}{\partial t} + nx\dfrac{\partial}{\partial x} + ny\dfrac{\partial}{\partial y} - 2u\dfrac{\partial}{\partial u}$

Finally, the Lie point symmetry X_4 gives $W^1 = -(2u + 3ntu_t + nxu_x + nyu_y)$ and $W^2 = (2-2n)v - 3ntv_t - nxv_x - nyv_y$ and so the associated conserved vector has components:

$$\begin{aligned}
C_4^t &= v(3\alpha ntu^n u_x - 2u - nxu_x - nyu_y) - 3\beta ntv_x(u_{xx} + u_{yy}),\\
C_4^x &= v(nxu_t - 2\alpha u^{n+1} - 3\alpha ntu^n u_t - \alpha nyu^n u_y) + \beta nyv_y(u_{xx} + u_{yy})\\
&\quad +3\beta ntv_t(u_{xx} + u_{yy}) + \beta v_x(2u_x + 3ntu_{tx} + nu_x + nxu_{xx} + nyu_{xy})\\
&\quad -\beta v_{xx}(2u + 3ntu_t + nxu_x + nyu_y) - 2\beta v(u_{xx} + u_{yy}) + 2\beta nv(u_{xx} + u_{yy}),\\
C_4^y &= \beta v_x(2u_y + 3ntu_{ty} + nxu_{xy} + nu_y - nyu_{xx}) - \beta v_{xy}(2u + 3ntu_t + nxu_x + nyu_y)\\
&\quad +nyv(u_t + \alpha u^n u_x),
\end{aligned}$$

which represents the boost momentum conservation law.

6.2 Exact solutions of (6.1)

In this section we obtain exact solutions of (6.1) using firstly its Lie point symmetries and secondly by employing the simplest equation method.

6.2.1 Exact solutions of (6.1) using its Lie point symmetries

First of all we utilize the linear combination of the three translation symmetries, namely $X = X_1 + \nu X_2 + X_3$ and reduce the gZK equation (6.1) to a partial differential equation in two independent variables. The associated Lagrange system is

$$\frac{dt}{1} = \frac{dx}{\nu} = \frac{dy}{1} = \frac{du}{0},$$

which yields the following three invariants:

$$f = t - y, \quad g = x - \nu y, \quad \theta = u. \tag{6.8}$$

By considering θ as the new dependent variable and f and g as new independent variables, the gZK equation (6.1) transforms to

$$\theta_f + \alpha\theta^n\theta_g + \beta(\nu^2 + 1)\theta_{ggg} + 2\beta\nu\theta_{fgg} + \beta\theta_{ffg} = 0, \tag{6.9}$$

which is a nonlinear partial differential equation in two independent variables. Further symmetry reduction of (6.9) can be done by using its symmetries. The equation (6.9) has the two translational symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}.$$

The combination $\Gamma_1 + k\Gamma_2$, of the two symmetries Γ_1 and Γ_2, for an arbitrary constant k, yields the two invariants

$$z = g - kf \quad \text{and} \quad W = \theta,$$

which gives rise to a group invariant solution $W = W(z)$. Consequently using these invariants, (6.9) is transformed into the third-order nonlinear ordinary differential equation

$$\beta(1 + (\nu - k)^2)W''' + \alpha W^n W' - kW' = 0. \tag{6.10}$$

The integration of (6.10) yields

$$\beta(1 + (\nu - k)^2)W'' + \frac{\alpha}{n+1}W^{n+1} - kW = 0, \tag{6.11}$$

where the constant of integration has been taken to be zero, because we are looking for soliton solutions. Equation (6.11) can be integrated easily by first multiplying it by W'. We then obtain the first-order variables separable equation

$$\frac{\beta(1+(\nu-k)^2)}{2}W'^2 + \frac{\alpha}{(n+1)(n+2)}W^{n+2} - \frac{k}{2}W^2 = 0, \qquad (6.12)$$

which can be integrated easily. After integrating and reverting back to the original variables, we obtain the following group-invariant solution of the gZK equation (6.1) for arbitrary values of n in the form

$$u(t,x,y) = \left(\frac{k(n+1)(n+2)}{2\alpha}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}(R), \qquad (6.13)$$

where

$$R = \frac{n\sqrt{k}(C_1+z)}{2\sqrt{\beta(1+(\nu-k)^2)}},$$

$$z = x - kt - (\nu-k)y.$$

Note that (6.13) represents a non-topological soliton solution. A sketch of the solution (6.13) with $n=2$, $\alpha=2$, $k=5$, $\nu=1$, $\beta=1$, $t=0$ and $C_1=1$ is given in Figure 6.1.

6.2.2 Exact solutions of (6.1) using simplest equation method

In this subsection we use the simplest equation method described in Section 1.6.2, to solve the nonlinear third-order ordinary differential equation (6.10) for $n=1,2$.

Solutions of (6.1) using the Bernoulli equation as the simplest equation

n=1

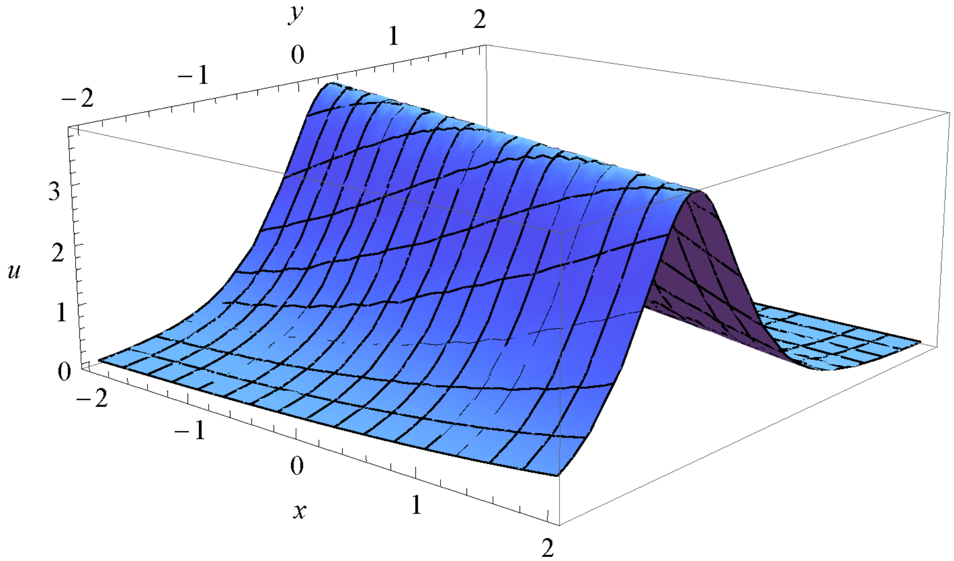

Figure 6.1: Profile of solution (6.13)

In this case the balancing procedure yields $M = 2$ and solutions of (6.10) are of the form

$$W(z) = A_0 + A_1 G + A_2 G^2. \qquad (6.14)$$

We insert this value of $W(z)$ in (6.10). Then using the Bernoulli equation (1.53) and, thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of five equations in terms of A_0, A_1, A_2, namely

$$\begin{aligned}
24\,b^3\beta\,k^2A_2 - 48\,b^3\beta\,k\nu\,A_2 + 24\,b^3\beta\,\nu^2A_2 + 24\,b^3\beta\,A_2 + 2\,\alpha\,bA_2{}^2 &= 0,\\
a^3\beta\,k^2A_1 - 2\,a^3\beta\,k\nu\,A_1 + a^3\beta\,\nu^2A_1 + a^3\beta\,A_1 + a\alpha\,A_0A_1 - akA_1 &= 0,\\
54\,ab^2\beta\,k^2A_2 - 108\,ab^2\beta\,k\nu\,A_2 + 54\,ab^2\beta\,\nu^2A_2 + 6\,b^3\beta\,k^2A_1 - 12\,b^3\beta\,k\nu\,A_1 &\\
+6\,b^3\beta\,\nu^2A_1 + 54\,ab^2\beta\,A_2 + 6\,b^3\beta\,A_1 + 2\,a\alpha\,A_2{}^2 + 3\,\alpha\,bA_1A_2 &= 0,\\
38\,a^2b\beta\,k^2A_2 - 76\,a^2b\beta\,k\nu\,A_2 + 38\,a^2b\beta\,\nu^2A_2 + 12\,ab^2\beta\,k^2A_1 - 24\,ab^2\beta\,k\nu\,A_1 &\\
+12\,ab^2\beta\,\nu^2A_1 + 38\,a^2b\beta\,A_2 + 12\,ab^2\beta\,A_1 + 3\,a\alpha\,A_1A_2 + 2\,\alpha\,bA_0A_2 &\\
+\alpha\,bA_1{}^2 - 2\,bkA_2 &= 0,\\
8\,a^3\beta\,k^2A_2 - 16\,a^3\beta\,k\nu\,A_2 + 8\,a^3\beta\,\nu^2A_2 + 7\,a^2b\beta\,k^2A_1 - 14\,a^2b\beta\,k\nu\,A_1 &
\end{aligned}$$

$$+7\,a^2b\beta\,\nu^2A_1+8\,a^3\beta\,A_2+7\,a^2b\beta\,A_1+2\,a\alpha\,A_0A_2+a\alpha\,{A_1}^2$$
$$+\alpha\,bA_0A_1-2\,akA_2-bkA_1\;=\;0.$$

With the aid of Maple, we solve the above system and obtain

$$\begin{aligned}
A_0 &= \frac{1}{\alpha}\{2a^2\beta k\nu-a^2\beta\nu^2-a^2\beta-a^2\beta k^2+k\},\\
A_1 &= \frac{1}{\alpha}\{12ab\beta\left(2k\nu-\nu^2-k^2-1\right)\},\\
A_2 &= \frac{1}{\alpha}\{12b^2\beta\left(2k\nu-\nu^2-k^2-1\right)\}.
\end{aligned}$$

Therefore the solution of (6.1), for $n=1$ is given by

$$\begin{aligned}
u(t,x,y)=&A_0+aA_1\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\\
&+A_2a^2\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}^2,
\end{aligned}\tag{6.15}$$

where $z=x-kt-(\nu-k)y$ and C is an arbitrary constant of integration.

n = 2

The balancing procedure yields $M=1$ so the solutions of (6.10) take the form

$$W(z)=A_0+A_1G.\tag{6.16}$$

As before, substituting (6.16) into (6.10), we obtain the algebraic system of equations

$$\begin{aligned}
6\,b^3\beta\,k^2A_1-12\,b^3\beta\,k\nu\,A_1+6\,b^3\beta\,\nu^2A_1+\alpha\,b{A_1}^3+6\,b^3\beta\,A_1 &= 0,\\
a^3\beta\,k^2A_1-2\,a^3\beta\,k\nu\,A_1+a^3\beta\,\nu^2A_1+a^3\beta\,A_1+a\alpha\,{A_0}^2A_1-akA_1 &= 0,\\
12\,ab^2\beta\,k^2A_1-24\,ab^2\beta\,k\nu\,A_1+12\,ab^2\beta\,\nu^2A_1+a\alpha\,{A_1}^3+12\,ab^2\beta\,A_1 &\\
+2\,\alpha\,bA_0{A_1}^2 &= 0,\\
7\,a^2b\beta\,k^2A_1-14\,a^2b\beta\,k\nu\,A_1+7\,a^2b\beta\,\nu^2A_1+7\,a^2b\beta\,A_1+2\,a\alpha\,A_0{A_1}^2 &
\end{aligned}$$

$$+\alpha\, bA_0{}^2A_1 - bkA_1 \;=\; 0$$

whose solution is

$$A_0 = \pm\sqrt{\frac{3k}{\alpha}},\;\; A_1 = \pm\frac{2b}{a}\sqrt{\frac{3k}{\alpha}},\;\; \beta = \frac{2k}{a^2\left(2k\nu - \nu^2 - k^2 - 1\right)}.$$

Therefore, the solution of (6.1), for $n = 2$ is given by

$$u(t,x,y) = A_0 + aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}, \tag{6.17}$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration. A sketch of the solution (6.17) is given in Figure 6.2.

Solutions of (6.1) using the Riccati equation as the simplest equation

n=1

For this case the balancing procedure gives $M = 2$ and so (1.56) becomes

$$W(z) = A_0 + A_1G + A_2G^2. \tag{6.18}$$

The insertion of this value of $W(z)$ into (6.10) and making use of the Riccati equation (1.54) yields the following algebraic system of equations in terms of A_0, A_1, A_2:

$$\begin{aligned}
24\,a^3\beta\,k^2A_2 - 48\,a^3\beta\,k\nu\,A_2 + 24\,a^3\beta\,\nu^2A_2 + 24\,a^3\beta\,A_2 + 2\,a\alpha\,A_2{}^2 &= 0,\\
6\,a^3\beta\,k^2A_1 - 12\,a^3\beta\,k\nu\,A_1 + 6\,a^3\beta\,\nu^2A_1 + 54\,a^2b\beta\,k^2A_2 - 108\,a^2b\beta\,k\nu\,A_2 &\\
+54\,a^2b\beta\,\nu^2A_2 + 6\,a^3\beta\,A_1 + 54\,a^2b\beta\,A_2 + 3\,a\alpha\,A_1A_2 + 2\,\alpha\,bA_2{}^2 &= 0,\\
2\,a\beta\,c^2k^2A_1 - 4\,a\beta\,c^2k\nu\,A_1 + 2\,a\beta\,c^2\nu^2A_1 + b^2\beta\,ck^2A_1 - 2\,b^2\beta\,ck\nu\,A_1 &\\
+b^2\beta\,c\nu^2A_1 + 6\,b\beta\,c^2k^2A_2 - 12\,b\beta\,c^2k\nu\,A_2 + 6\,b\beta\,c^2\nu^2A_2 + 2\,a\beta\,c^2A_1 &\\
+b^2\beta\,cA_1 + 6\,b\beta\,c^2A_2 + \alpha\,cA_0A_1 - ckA_1 &= 0,\\
12\,a^2b\beta\,k^2A_1 - 24\,a^2b\beta\,k\nu\,A_1 + 12\,a^2b\beta\,\nu^2A_1 + 40\,a^2\beta\,ck^2A_2 - 80\,a^2\beta\,ck\nu\,A_2 &\\
+40\,a^2\beta\,c\nu^2A_2 + 38\,ab^2\beta\,k^2A_2 - 76\,ab^2\beta\,k\nu\,A_2 + 38\,ab^2\beta\,\nu^2A_2 + 12\,a^2b\beta\,A_1 &
\end{aligned}$$

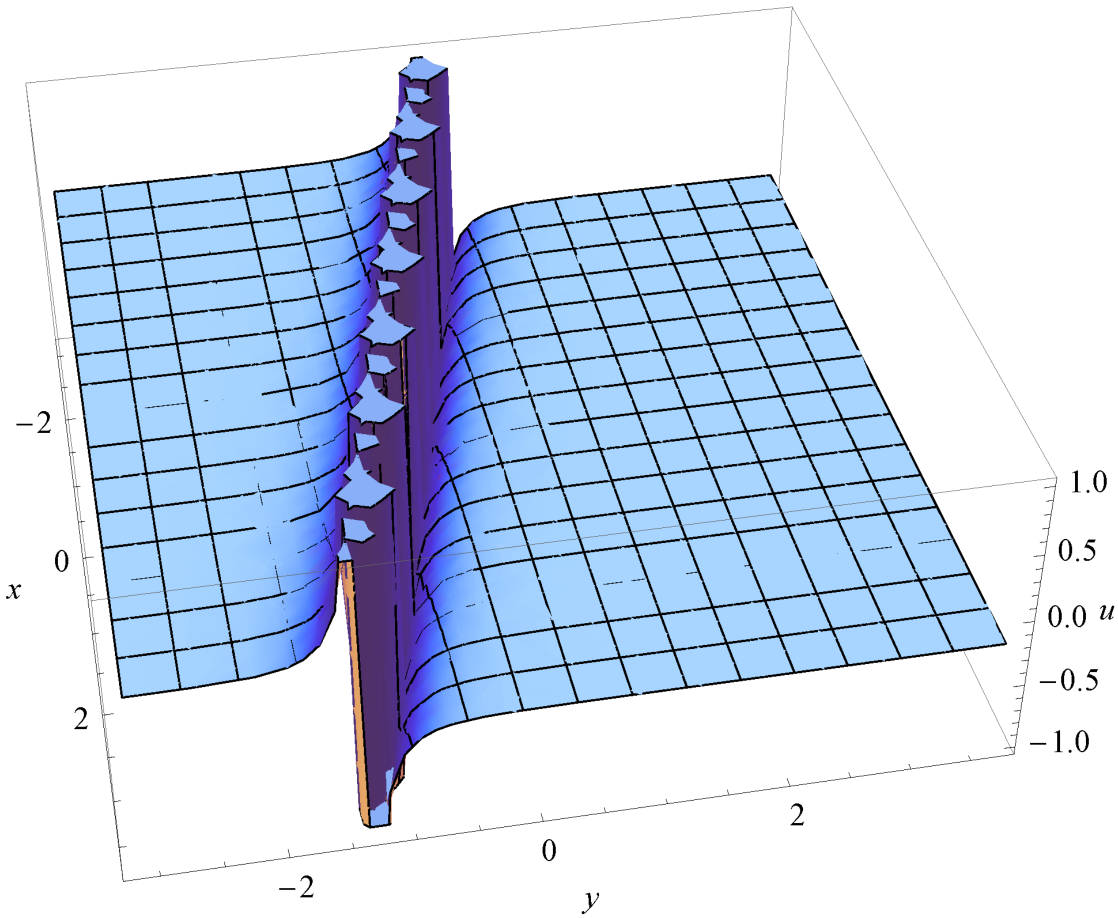

Figure 6.2: Profile of solution (6.17)

$$
\begin{aligned}
+40\,a^2\beta\,cA_2 + 38\,ab^2\beta\,A_2 + 2\,a\alpha\,A_0A_2 + a\alpha\,{A_1}^2 + 3\,\alpha\,bA_1A_2 & \\
+2\,\alpha\,c{A_2}^2 - 2\,akA_2 &= 0, \\
8\,ab\beta\,ck^2A_1 - 16\,ab\beta\,ck\nu\,A_1 + 8\,ab\beta\,c\nu^2A_1 + 16\,a\beta\,c^2k^2A_2 - 32\,a\beta\,c^2k\nu\,A_2 & \\
+16\,a\beta\,c^2\nu^2A_2 + b^3\beta\,k^2A_1 - 2\,b^3\beta\,k\nu\,A_1 + b^3\beta\,\nu^2A_1 + 14\,b^2\beta\,ck^2A_2 & \\
-28\,b^2\beta\,ck\nu\,A_2 + 14\,b^2\beta\,c\nu^2A_2 + 8\,ab\beta\,cA_1 + 16\,a\beta\,c^2A_2 + b^3\beta\,A_1 & \\
+14\,b^2\beta\,cA_2 + \alpha\,bA_0A_1 + 2\,\alpha\,cA_0A_2 + \alpha\,c{A_1}^2 - bkA_1 - 2\,ckA_2 &= 0, \\
8\,a^2\beta\,ck^2A_1 - 16\,a^2\beta\,ck\nu\,A_1 + 8\,a^2\beta\,c\nu^2A_1 + 7\,ab^2\beta\,k^2A_1 - 14\,ab^2\beta\,k\nu\,A_1 & \\
+7\,ab^2\beta\,\nu^2A_1 + 52\,ab\beta\,ck^2A_2 - 104\,ab\beta\,ck\nu\,A_2 + 52\,ab\beta\,c\nu^2A_2 + 8\,b^3\beta\,k^2A_2 &
\end{aligned}
$$

$$-16\,b^3\beta\,k\nu\,A_2 + 8\,b^3\beta\,\nu^2 A_2 + 8\,a^2\beta\,cA_1 + 7\,ab^2\beta\,A_1 + 52\,ab\beta\,cA_2 + 8\,b^3\beta\,A_2$$
$$+a\alpha\,A_0A_1 + 2\,\alpha\,bA_0A_2 + \alpha\,b{A_1}^2 + 3\,\alpha\,cA_1A_2 - akA_1 - 2\,bkA_2 \quad = \quad 0.$$

The solution of the above system using Maple gives

$$A_0 = \frac{1}{\alpha}\{16a\beta ck\nu - 8a\beta c\nu^2 - 8a\beta c - 8a\beta ck^2 - \beta b^2\nu^2 - \beta b^2 - \beta b^2k^2 + 2\beta b^2k\nu + k\},$$
$$A_1 = \frac{1}{\alpha}\{12ab\beta\left(2k\nu - \nu^2 - k^2 - 1\right)\},\ A_2 = \frac{1}{\alpha}\{12a^2\beta\left(2k\nu - \nu^2 - k^2 - 1\right)\}.$$

Consequently, the solutions of (6.1) are

$$\begin{aligned} u(t,x,y) =&A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\} \\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\}^2 \end{aligned} \tag{6.19}$$

and

$$\begin{aligned} u(t,x,y) =&A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \end{aligned} \tag{6.20}$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration.

n = 2

The balancing procedure yields $M = 1$, so the solutions of (6.10) are of the form:

$$W(z) = A_0 + A_1G \tag{6.21}$$

Substituting (6.21) into (6.10) and using the Riccati equation, we obtain the following algebraic system of equations:

$$-6bA_1c^3\nu + 3aA_1^3c = 0,$$

$$6aA_1^2A_0c + 3aA_1^3d - 12bA_1c^2d\nu = 0,$$

$$3aA_1A_0^2e - 2bA_1ce^2\nu - A_1e\nu - bA_1d^2e\nu = 0,$$

$$-8bA_1cde\nu - A_1d\nu + 6aA_1^2A_0e - bA_1d^3\nu + 3aA_1A_0^2d = 0,$$

$$-A_1c\nu + 3aA_1^3e - 7bA_1cd^2\nu + 3aA_1A_0^2c + 6aA_1^2A_0d - 8bA_1c^2e\nu = 0.$$

Solving the above algebraic equations, one obtains

$$A_0 = \pm\frac{b}{2\sqrt{\alpha}}\sqrt{6\beta(2k\nu - \nu^2 - k^2 - 1)},\ A_1 = \pm\frac{a}{\sqrt{\alpha}}\sqrt{6\beta(2k\nu - \nu^2 - k^2 - 1)},$$
$$a = \frac{b^2\beta\, k^2 - 2\, b^2\beta\, k\nu + b^2\beta\,\nu^2 + b^2\beta + 2\, k}{4\,\beta\, c\,(k^2 - 2\,\nu\, k + \nu^2 + 1)}.$$

Hence, we have the following solutions of (6.1) for $n = 2$

$$u(x, y, t) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z + C)\right)\right\} \tag{6.22}$$

and

$$u(t, x, y) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}, \tag{6.23}$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration. A sketch of the solution (6.23) is given in Figure 6.3.

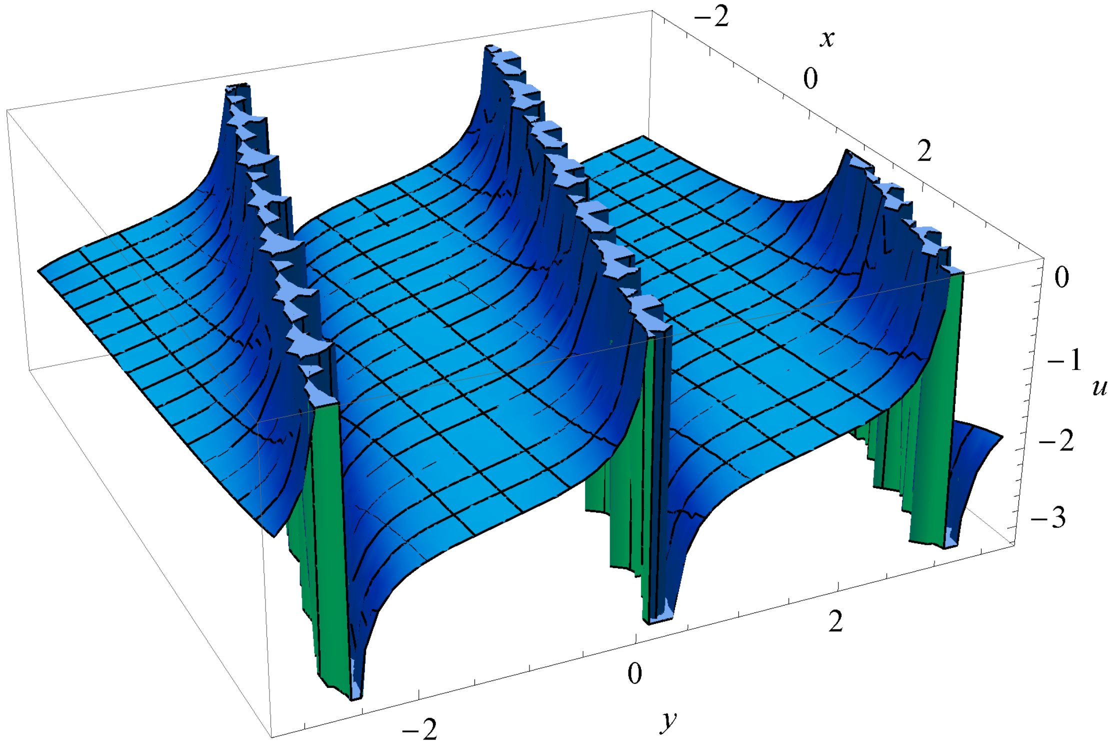

Figure 6.3: Profile of the solution of (6.23).

6.3 Concluding remarks

In this chapter we studied the generalized Zakharov–Kuznetsov equation (6.1). We derived the conservation laws of this equation using the new conservation theorem by Ibragimov. In addition, the Lie point symmetries of the gZK equation were obtained and used in conjunction with the simplest equation method to obtain exact solutions of the gZK equation. The solutions obtained here are new and more general than the ones obtained before in [82] and [83].

Chapter 7

Conservation laws and exact solutions for a generalized Ablowitz-Kaup-Newell-Segur equation

In this chapter we study the generalized Ablowitz-Kaup-Newell-Segur (gAKNS) equation, which has many applications in several physical phenomena. The gAKNS equation is given by

$$u_{xxxy} + 2\beta u_x u_{xy} + \beta u_y u_{xx} + \gamma u_{xt} + \alpha u_{xx} = 0 \tag{7.1}$$

with constant coefficients α, γ and β. We perform the Noether symmetry analysis for (7.1). Thereafter, we construct the conservation laws for those cases which admit the Noether operators. Furthermore, we compute exact solutions for gAKNS equation using the simplest equation method.

We note that when $\beta = \gamma = 4$, equation (7.1) becomes the Ablowitz-Kaup-Newell-

Segur (AKNS) equation

$$u_{xxxy} + 8u_x u_{xy} + 4u_y u_{xx} + 4u_{xt} + \alpha u_{xx} = 0, \tag{7.2}$$

which describes the wave phenomena observed in fluid dynamics, plasma and elastic media. Cheng and Hao in [91] obtained the periodic wave solutions of (7.2) using bilinear Backlund transformation whereas Wazwaz [92] obtained the soliton solutions for equation (7.2) using the simplified form of the bilinear method.

Part of this chapter has been accepted for publication. See [93].

7.1 Conservation laws of equation (7.1)

The vector field

$$X = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u} \tag{7.3}$$

is called a Noether point symmetry corresponding to a second-order Lagrangian $\mathcal{L}$ of (7.1) if

$$X^{[2]}(\mathcal{L}) + \{D_t(\xi^1) + D_x(\xi^2) + D_y(\xi^3)\}\mathcal{L} = D_t(B^1) + D_x(B^2) + D_y(B^3) \tag{7.4}$$

for some gauge functions $B^1(t,x,y,u)$, $B^2(t,x,y,u)$ and $B^3(t,x,y,u)$. Here $X^{[2]}$ is the second-order prolongation defined in Chapter one. For any Noether point symmetry X corresponding to a given Lagrangian $\mathcal{L}$, there corresponds a conserved vector $T = (T^i, \ldots, T^n)$, $i = 1\ldots, n$, if T^i satisfies

$$D_i T^i|_{(7.1)} = 0. \tag{7.5}$$

The equation (7.5) defines a local conservation law of the equation (7.1). It can be verified that the second-order Lagrangian of the gAKNS equation (7.1) is

$$\mathcal{L} = \frac{1}{2}u_{xx}u_{xy} - \frac{1}{2}\beta u_x^2 u_y - \frac{1}{2}\gamma u_t u_x - \frac{1}{2}\alpha u_x^2. \tag{7.6}$$

The insertion of $\mathcal{L}$ from (7.6) into equation (7.4) and splitting with respect to derivatives of $u(t,x,y)$ yields linear overdetermined system of partial differential equations:

$$
\begin{aligned}
\xi^1_x = 0,\ \xi^1_y = 0,\ \xi^1_u = 0,\ \xi^2_{xx} = 0,\ \xi^2_y = 0,\ \xi^2_u = 0,\ \xi^3_x = 0,\ \xi^3_u = 0,\\
\eta_{uu} = 0,\ \eta_{xx} = 0,\ \eta_{xu} = 0,\ \eta_{uy} = 0,\ \eta_{xy} = 0,\ 2\beta\eta_x - \gamma\xi^3_t = 0,\\
2\gamma\eta_u + \gamma\xi^3_y = 0,\ 2\eta_u + \xi^1_t - 2\xi^2_x = 0,\ 2\eta_u + \xi^1_t - 2\xi^2_x = 0,\\
3\beta\eta_u - \beta\xi^2_x + \beta\xi^1_t = 0,\ 2\alpha\eta_u + \beta\eta_y - \alpha\xi^2_x + \alpha\xi^1_t - \gamma\xi^2_t + \alpha\xi^3_y = 0,\\
B^1_u + \frac{1}{2}\gamma\eta_x = 0,\ B^2_u + \alpha\eta_x + \frac{1}{2}\gamma\eta_t = 0,\ B^3_u = 0,\ B^1_t + B^2_x + B^3_y = 0.
\end{aligned}
\tag{7.7}
$$

After some tedious and lengthy calculations, the above equations yield the following six Noether point symmetries together with their corresponding gauge functions:

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad B^1 = B^2 = B^3 = 0,\\
&X_2 = \frac{\partial}{\partial y}, \quad B^1 = B^2 = B^3 = 0,\\
&X_3 = \frac{2\beta}{\gamma} t\frac{\partial}{\partial y} + x\frac{\partial}{\partial u}, \quad B^1 = -\frac{1}{2}\gamma u,\ B^2 = -\alpha u,\ B^3 = 0,\\
&X_4 = -4t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} + \left(\frac{3\alpha}{\beta y} + u\right)\frac{\partial}{\partial u}, \quad B^1 = B^2 = B^3 = 0,\\
&X_5 = e(t)\frac{\partial}{\partial x} + \frac{\gamma}{\beta} y e'(t)\frac{\partial}{\partial u}, \quad B^1 = B^3 = 0, \quad B^2 = -\frac{\gamma^2}{2\beta} u y e''(t),\\
&X_6 = r(t)\frac{\partial}{\partial u}, \quad B^1 = B^3 = 0, \quad B^2 = -\frac{1}{2}\gamma u r'(t).
\end{aligned}
$$

Invoking Noether theorem 1.7, the six conserved vectors associated with these six Noether point symmetries are, respectively,

$$
\begin{aligned}
&T^1_1 = \frac{1}{2}u_{xx}u_{xy} - \frac{1}{2}\beta u_x^2 u_y - \frac{1}{2}\alpha u_x^2,\\
&T^2_1 = \beta u_t u_x u_y + \frac{1}{2}\gamma u_t^2 + \alpha u_t u_x + u_t u_{xxy} - \frac{1}{2}u_{tx}u_{xy} - \frac{1}{2}u_{ty}u_{xx},\\
&T^3_1 = \frac{1}{2}\beta u_t u_x^2 + \frac{1}{2}u_t u_{xxx} - \frac{1}{2}u_{tx}u_{xx};
\end{aligned}
\tag{7.8}
$$

$$
\begin{aligned}
T_2^1 &= \frac{1}{2}\gamma u_x u_y,\\
T_2^2 &= \beta u_x u_y^2 + \frac{1}{2}\gamma u_t u_y + \alpha u_x u_y + u_y u_{xxy} - \frac{1}{2}u_{xy}^2 - \frac{1}{2}u_{xx}u_{yy},\\
T_2^3 &= \frac{1}{2}u_y u_{xxx} - \frac{1}{2}\gamma u_t u_x - \frac{1}{2}\alpha u_x^2;
\end{aligned} \tag{7.9}
$$

$$
\begin{aligned}
T_3^1 &= \beta t u_x u_y - \frac{1}{2}\gamma x u_x + \frac{1}{2}\gamma u,\\
T_3^2 &= \beta t u_t u_y - \beta x u_x u_y - \frac{1}{2}\gamma x u_t - \alpha x u_x - x u_{xxy} + \frac{2\beta^2}{\gamma}t u_x u_y^2 + \frac{2\alpha\beta}{\gamma}t u_x u_y\\
&\quad + \frac{2\beta}{\gamma}t u_y u_{xxy} + \frac{1}{2}u_{xy} - \frac{\beta}{\gamma}t u_{xy}^2 - \frac{\beta}{\gamma}t u_{xx}u_{yy} + \alpha u,\\
T_3^3 &= \frac{\beta}{\gamma}t u_y u_{xxx} - \frac{1}{2}x u_{xxx} - \beta t u_t u_x - \frac{\alpha\beta}{\gamma}t u_x^2 - \frac{1}{2}\beta x u_x^2 + \frac{1}{2}u_{xx};
\end{aligned} \tag{7.10}
$$

$$
\begin{aligned}
T_4^1 &= 2\beta t u_x^2 u_y - 2t u_{xx}u_{xy} + 2\alpha t u_x^2 - \frac{3\alpha\gamma}{2\beta}y u_x - \frac{1}{2}\gamma u u_x - \frac{1}{2}\gamma x u_x^2 - \gamma y u_x u_y,\\
T_4^2 &= u_x u_{xy} - \frac{1}{2}\alpha x u_x^2 - \frac{1}{2}\beta x u_x^2 u_y - 5\alpha y u_x u_y - \beta u u_x u_y - 4\beta t u_t u_x u_y - 2\beta y u_x u_y^2\\
&\quad - \frac{3\alpha\gamma}{2\beta}y u_t - \frac{1}{2}\gamma u u_t - 2\gamma t u_t^2 - \gamma y u_t u_y - \frac{3\alpha^2}{\beta}y u_x - \alpha u u_x - 4\alpha t u_t u_x - \frac{3\alpha}{\beta}y u_{xxy}\\
&\quad - u u_{xxy} - 4t u_t u_{xxy} - x u_x u_{xxy} - 2y u_y u_{xxy} + 2t u_{tx}u_{xy} + \frac{1}{2}x u_{xx}u_{xy} + y u_{xy}^2 + \frac{3\alpha}{2\beta}u_{xx}\\
&\quad + \frac{3}{2}u_y u_{xx} + 2t u_{ty}u_{xx} + y u_{xx}u_{yy};\\
T_4^3 &= \gamma y u_t u_x - \frac{1}{2}\alpha y u_x^2 - \frac{3\alpha}{2\beta}y u_{xxx} - \frac{1}{2}\beta u u_x^2 - \frac{1}{2}u u_{xxx} - 2\beta t u_t u_x^2 - 2t u_t u_{xxx}\\
&\quad - \frac{1}{2}\beta x u_x^3 - \frac{1}{2}x u_x u_{xxx} - y u_y u_{xxx} + u_x u_{xx} + \frac{1}{2}x u_{xx}^2 + 2t u_{tx}u_{xx};
\end{aligned} \tag{7.11}
$$

$$
\begin{aligned}
T_5^1 &= \frac{1}{2}\gamma e(t)u_x^2 - \frac{\gamma^2}{2\beta}y e'(t)u_x,\\
T_5^2 &= \frac{1}{2}\beta e(t)u_x^2 u_y + \frac{1}{2}\alpha e(t)u_x^2 - \gamma y e(t)u_x u_y - \frac{\gamma^2}{2\beta}y e'(t)u_t - \frac{\alpha\gamma}{\beta}y e'(t)u_x\\
&\quad - \frac{\gamma}{\beta}y e'(t)u_{xxy} + e(t)u_x u_{xxy} + \frac{\gamma}{2\beta}e'(t)u_{xx} + \frac{\gamma^2}{2\beta}u y e''(t),
\end{aligned}
$$

$$T_5^3 = \frac{1}{2}\beta e(t)u_x^3 - \frac{1}{2}\gamma y e'(t)u_x^2 - \frac{\gamma}{2\beta}e'(t)u_{xxx} - \frac{1}{2}e(t)u_{xx}^2; \tag{7.12}$$

$$\begin{aligned}
T_6^1 &= -\frac{1}{2}\gamma r(t)u_x,\\
T_6^2 &= \frac{1}{2}\gamma u r'(t) - \beta r(t)u_x u_y - \frac{1}{2}\gamma r(t)u_t - \alpha r(t)u_x - r(t)u_{xxy},\\
T_6^3 &= -\frac{1}{2}\beta r(t)u_x^2 - \frac{1}{2}r(t)u_{xxx}.
\end{aligned} \tag{7.13}$$

The above conserved vectors do not satisfy the divergence condition (7.5), as some excessive terms emerge that require some further analysis. By making a slight adjustment to these terms, it can be shown that these can be absorbed into the divergence condition. Further analysis of the conserved vector (7.8) yields

$$\begin{aligned}
D_t(T_1^1) + D_x(T_1^2) + D_y(T_1^3) &= -\frac{1}{2}u_{xx}u_{txy} + \frac{1}{2}u_t u_{xxxy}\\
&= D_x\left(-\frac{1}{2}u_{xx}u_{ty}\right) + D_y\left(\frac{1}{2}u_t u_{xxx}\right)
\end{aligned} \tag{7.14}$$

and hence

$$D_t(T_1^1) + D_x\left(T_1^2 + \frac{1}{2}u_{xx}u_{ty}\right) + D_y\left(T_1^3 - \frac{1}{2}u_t u_{xxx}\right) = 0. \tag{7.15}$$

We now redefine the conserved vectors in the parenthesis as

$$\begin{aligned}
\tilde{T}_1^1 &= \frac{1}{2}u_{xx}u_{xy} - \frac{1}{2}\beta u_x^2 u_y - \frac{1}{2}\alpha u_x^2,\\
\tilde{T}_1^2 &= \beta u_t u_x u_y + \frac{1}{2}\gamma u_t^2 + \alpha u_t u_x + u_t u_{xxy} - \frac{1}{2}u_{tx}u_{xy},\\
\tilde{T}_1^3 &= \frac{1}{2}\beta u_t u_x^2 - \frac{1}{2}u_{tx}u_{xx}.
\end{aligned} \tag{7.16}$$

Thus, the modified conserved vectors $\tilde{T}_1^1$, $\tilde{T}_1^2$, and $\tilde{T}_1^3$ satisfy the divergence condition. Likewise, we can then construct the conserved quantities associated with equations (7.9) to (7.13) as

$$\tilde{T}_2^1 = \frac{1}{2}\gamma u_x u_y,$$

$$\tilde{T}_2^2 = \beta u_x u_y^2 + \frac{1}{2}\gamma u_t u_y + \alpha u_x u_y + \frac{1}{2}u_y u_{xxy} - \frac{1}{2}u_{xy}^2 - \frac{1}{2}u_{xx}u_{yy},$$

$$\tilde{T}_2^3 = \frac{1}{2}u_y u_{xxx} - \frac{1}{2}\gamma u_t u_x - \frac{1}{2}\alpha u_x^2; \tag{7.17}$$

$$\tilde{T}_3^1 = \beta t u_x u_y - \frac{1}{2}\gamma x u_x + \frac{1}{2}\gamma u$$

$$\tilde{T}_3^2 = \beta t u_t u_y - \beta x u_x u_y - \frac{1}{2}\gamma x u_t - \alpha x u_x - x u_{xxy} + \frac{2\beta^2}{\gamma}t u_x u_y^2 + \frac{2\alpha\beta}{\gamma}t u_x u_y$$
$$+ \frac{\beta}{\gamma}t u_y u_{xxy} + \frac{1}{2}u_{xy} - \frac{\beta}{\gamma}t u_{xy}^2 - \frac{\beta}{\gamma}t u_{xx}u_{yy} + \alpha u,$$

$$\tilde{T}_3^3 = \frac{\beta}{\gamma}t u_y u_{xxx} - \beta t u_t u_x - \frac{\alpha\beta}{\gamma}t u_x^2 - \frac{1}{2}\beta x u_x^2 + \frac{1}{2}u_{xx} + \frac{\beta}{\gamma}t u_{xx}u_{xy}; \tag{7.18}$$

$$\tilde{T}_4^1 = 2\beta t u_x^2 u_y - 2t u_{xx}u_{xy} + 2\alpha t u_x^2 - \frac{3\alpha\gamma}{2\beta}y u_x - \frac{1}{2}\gamma u u_x - \frac{1}{2}\gamma x u_x^2 - \gamma y u_x u_y,$$

$$\tilde{T}_4^2 = u_y u_{xx} - \frac{1}{2}\alpha x u_x^2 - \frac{1}{2}\beta x u_x^2 u_y - 5\alpha y u_x u_y - \beta u u_x u_y - 4\beta t u_t u_x u_y - 2\beta y u_x u_y^2$$
$$- \frac{3\alpha\gamma}{2\beta}y u_t - \frac{1}{2}\gamma u u_t - 2\gamma t u_t^2 - \gamma y u_t u_y - \frac{3\alpha^2}{\beta}y u_x - \alpha u u_x - 4\alpha t u_t u_x$$
$$- \frac{3\alpha}{2\beta}y u_{xxy} - u u_{xxy} - 2t u_t u_{xxy} - x u_x u_{xxy} - y u_y u_{xxy} + u_x u_{xy}$$
$$+ 2t u_{tx}u_{xy} + y u_{xy}^2 + \frac{3\alpha}{2\beta}u_{xx} + 2t u_{ty}u_{xx} + y u_{xx}u_{yy},$$

$$\tilde{T}_4^3 = \gamma y u_t u_x - \frac{1}{2}\alpha y u_x^2 - \frac{3\alpha}{2\beta}y u_{xxx} - \frac{1}{2}\beta u u_x^2 - 2\beta t u_t u_x^2 - 2t u_t u_{xxx}$$
$$- \frac{1}{2}\beta x u_x^3 - y u_y u_{xxx} + u_x u_{xx} + \frac{1}{2}x u_{xx}^2 - y u_{xx}u_{xy}; \tag{7.19}$$

$$\tilde{T}_5^1 = \frac{1}{2}\gamma e(t) u_x^2 - \frac{\gamma^2}{2\beta}y e'(t) u_x,$$

$$\tilde{T}_5^2 = \frac{1}{2}\beta e(t) u_x^2 u_y + \frac{1}{2}\alpha e(t) u_x^2 - \gamma y e(t) u_x u_y - \frac{\gamma^2}{2\beta}y e'(t) u_t - \frac{\alpha\gamma}{\beta}y e'(t) u_x$$
$$- \frac{\gamma}{2\beta}y e'(t) u_{xxy} + e(t) u_x u_{xxy} + \frac{\gamma}{2\beta}e'(t) u_{xx} + \frac{\gamma^2}{2\beta}u y e''(t),$$

$$\tilde{T}_5^3 = \frac{1}{2}\beta e(t) u_x^3 - \frac{1}{2}\gamma y e'(t) u_x^2 - \frac{\gamma}{2\beta}e'(t) u_{xxx} - \frac{1}{2}e(t) u_{xx}^2; \tag{7.20}$$

$$\tilde{T}_6^1 = -\frac{1}{2}\gamma r(t) u_x,$$
$$\tilde{T}_6^2 = \frac{1}{2}\gamma u r'(t) - \beta r(t) u_x u_y - \frac{1}{2}\gamma r(t) u_t - \alpha r(t) u_x - r(t) u_{xxy},$$
$$\tilde{T}_6^3 = -\frac{1}{2}\beta r(t) u_x^2. \tag{7.21}$$

Therefore the gAKNS equation possess the conserved vectors (7.16) to (7.21) which satisfy the divergence condition (7.5).

7.2 Exact solutions of equation (7.1)

In this section we obtain exact solutions of (7.1) using the direct integration and the simplest equation method.

7.2.1 Exact solutions of (7.1) using direct integration

As a first step we transform the gAKNS equation (7.1) to a nonlinear ordinary differential equation using the travelling wave variable

$$u(t, x, y) = F(z), \;\; z = x + y - ct. \tag{7.22}$$

Applying the above transformation, equation (7.1) transforms to the nonlinear ordinary differential equation

$$F''''(z) + 3\beta F'(z) F''(z) + (\alpha - c\gamma) F''(z) = 0. \tag{7.23}$$

Integrating equation (7.23) once with respect to z and letting the constant of integration to be zero, we obtain

$$F'''(z) + \frac{3}{2}\beta F'(z)^2 + (\alpha - c\gamma) F'(z) = 0. \tag{7.24}$$

Multiplying equation (7.24) by $F''(z)$ yields

$$F'''(z)F''(z) + \frac{3}{2}\beta F'(z)^2 F''(z) + (\alpha - c\gamma)F'(z)F''(z) = 0. \tag{7.25}$$

Integrating equation (7.25) and taking the constant of integration to be zero, we obtain the second-order ordinary differential equation

$$F''(z)^2 + \beta F'(z)^3 + (\alpha - c\gamma)F'(z)^2 = 0.$$

Solving the above equation, we obtain

$$F(z) = \frac{-2\sqrt{\alpha - c\gamma}}{\beta} \tan\left\{\frac{1}{2}\sqrt{\alpha - c\gamma}\,(z + C_1)\right\} + C_2,$$

where $z = x + y - ct$ and C_1 and C_2 are constants of integration. Hence a solution of (7.1) is given by

$$u(t, x, y) = \frac{-2\sqrt{\alpha - c\gamma}}{\beta} \tan\left\{\frac{1}{2}\sqrt{\alpha - c\gamma}\,(x + y - ct + C_1)\right\} + C_2.$$

7.2.2 Solution of (7.1) using the Riccati equation as the simplest equation

We now use the simplest equation method, described is Section 1.6.2, to determine more exact solutions of the gAKNS equation (7.1).

The simplest equation that we use here is Riccati equation (1.54). We look for solutions of the nonlinear ordinary differential equation (7.23) that are of the form given by (1.55).

The balancing procedure yields $M = 1$, so the solutions of (7.23) are of the form

$$W(z) = A_0 + A_1 G. \tag{7.26}$$

Substituting (7.26) into (7.23) and using the Riccati equation (1.54) and then equating all coefficients of the powers of function $G(z)$ to zero, we obtain the following algebraic system of equations in terms of A_0 and A_1:

$$6a^3\beta A_1^2 + 24a^4 A_1 = 0,$$

$$15a^2 b\beta A_1^2 + 60a^3 bA_1 = 0,$$

$$3b\beta c^2 A_1^2 + 8abc^2 A_1 + b^3 cA_1 - bc^2\gamma A_1 + \alpha bcA_1 = 0,$$

$$18abc\beta A_1^2 + 3b^3\beta A_1^2 + 60a^2 bcA_1 + 15ab^3 A_1 - 3abc\gamma A_1 + 3ab\alpha A_1 = 0,$$

$$12a^2 c\beta A_1^2 + 12ab^2\beta A_1^2 + 40a^3 cA_1 + 50a^2 b^2 A_1 - 2a^2 c\gamma A_1 + 2a^2\alpha A_1 = 0,$$

$$6ac^2\beta A_1^2 + 6b^2 c\beta A_1^2 + 16a^2 c^2 A_1 + 22ab^2 cA_1 - 2ac^2\gamma A_1 + b^4 A_1 - b^2 c\gamma A_1$$
$$+2ac\alpha A_1 + \alpha b^2 A_1 = 0.$$

The solution of the above system using Maple gives

$$a = -\frac{1}{4c}\left(c\gamma - b^2 - \alpha\right), \quad A_1 = \frac{c\gamma - b^2 - \alpha}{c\beta}.$$

Consequently, the solutions of (7.1) are

$$u(x, y, t) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z + C)\right)\right\} \tag{7.27}$$

and

$$u(t, x, y) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}, \tag{7.28}$$

where $z = x + y - ct$, A_0 and C are arbitrary constants.

7.3 Conclusion

In this chapter we studied the gAKNS equation (7.1). We obtained six Noether symmetries for the gAKNS equation. Thereafter, Noether theorem was employed to construct six non-trivial conserved vectors associated with the Noether symmetries. Furthermore, we obtained exact solutions for (7.1) using direct integration and the simplest equation method.

Chapter 8

Conservation laws and exact solutions for a potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity

This chapter aims to study a potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity (PKPp), which arises in a number of significant nonlinear problems of physics and applied mathematics.

Recall that the celebrated Korteweg-de Vries (KdV) equation, which was first introduced by Boussinesq [96], governs the dynamics of solitary waves. Originally it was derived to describe shallow water waves of long wavelength and small amplitude. Kadomtsev and Petviashvili generalized the KdV equation to two spatial dimensions and produced an equation, which is named after their names Kadomtsev-Petviashvili (KP) equation [97]

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{8.1}$$

where $u = u(t, x, y)$ is a scalar function, x and y are respectively the longitudinal and transverse spatial coordinates and $\sigma^2 = \pm 1$. The KP equation is a nonlinear partial differential equation which describes long waves of small amplitude with slow dependence on the transverse coordinate [97–99].

The (2+1)-dimensional potential Kadomtsev-Petviashvili (PKP) equation

$$u_{tx} + \frac{3}{2}u_x u_{xx} + \frac{1}{4}u_{xxxx} + \frac{3}{4}u_{yy} = 0 \tag{8.2}$$

has been studied by various scientists [100–104] and a substantial amount of research has been done on finding exact solutions using various methods and techniques. In [100] new travelling wave solutions for the PKP equation using the homogeneous balance method were obtained. Kaya and El-Sayed [101] investigated the PKP equation and obtained the numerical soliton-like solutions by employing Adomian decomposition method. Li and Zhang [102] improved the key steps of homogeneous balance method to obtain soliton, multisoliton and rational-type solutions of the PKP equation.

The generalization of equation (8.2) is the so called potential Kadomtsev-Petviashvili equation with $p-$power nonlinearity (PKPp) and is given by

$$u_{tx} + \alpha u_x^p u_{xx} + \beta u_{xxxx} - \gamma u_{yy} = 0, \tag{8.3}$$

where α, β, γ and p are non-zero constants. The authors of [105], studied equation (8.3) only for two special cases of p, namely for $p = 1$ and $p = 2$ and obtained non-travelling wave solutions. In fact, the PKPp equation was reduced to a (1+1)-dimensional partial differential equation via classical Lie group method and the reduced equations were further studied to obtain certain exact solutions of PKPp equation.

Gupta and Bansal [106] studied the (2+1)-dimensional variable coefficients potential Kadomtsev-Petviashvili (VCPKP) equation

$$u_{tx} + \alpha(t)u_x u_{xx} + \beta(t)u_{xxxx} + \delta(t)u_{yy} = 0, \tag{8.4}$$

where $\alpha(t)$, $\beta(t)$ and $\delta(t)$ are arbitrary functions, for its integrability and for exact solutions. The methods and techniques used in [105] were employed and certain general solutions of VCPKP equation were obtained.

In this chapter we construct conservation laws for equation (8.3) using Noether theorem and thereafter we obtain its exact solutions using the Kudryashov method.

The work presented in this chapter has been submitted for publication. See [107].

8.1 Conservation laws of equation (8.3)

The vector field

$$X = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u} \quad (8.5)$$

is called a Noether point symmetry corresponding to a second-order Lagrangian L of (8.3) if

$$X^{[2]}(L) + \{D_t(\xi^1) + D_x(\xi^2) + D_y(\xi^3)\}L = D_t(B^1) + D_x(B^2) + D_y(B^3) \quad (8.6)$$

for some gauge functions $B^1(t,x,y,u)$, $B^2(t,x,y,u)$ and $B^3(t,x,y,u)$.
Here $X^{[2]}$ is the second-order prolongation given by

$$\begin{aligned} X^{[2]} &= \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \\ &\quad \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \cdots, \end{aligned}$$

where the expressions for $\zeta_t, \zeta_x, \zeta_{tx}, \zeta_{tt}$ and ζ_{xx} are given in [44].

The total differential operators are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots, \end{aligned}$$

$$D_y = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots .$$

We notice that to determine the Noether operators for the PKPp equation (8.3), we need to consider four cases separately.

Case 1: p is arbitrary but $p \neq 1, -1, -2$

In this case a Lagrangian of the PKPp equation (8.3) is given by

$$L = -\frac{1}{2}u_t u_x - \frac{\alpha}{(p+1)(p+2)}u_x^{p+2} + \frac{\beta}{2}u_{xx}^2 + \frac{\gamma}{2}u_y^2. \tag{8.7}$$

The substitution of this value of L into (8.6) yields an overdetermined system of nineteen linear partial differential equations. These are:

$$\begin{aligned}
&\xi^1_x = 0, \quad \xi^1_y = 0, \quad \xi^1_u = 0, \quad \xi^2_t = 0, \quad \xi^2_{xx} = 0, \quad \xi^2_u = 0, \quad \xi^3_x = 0, \quad \xi^3_u = 0,\\
&\eta_x = 0, \quad \eta_{uu} = 0, \quad \eta_u + \frac{1}{2}\xi^3_y = 0, \quad \frac{1}{2}\xi^3_t - \gamma\xi^2_y = 0, \quad \gamma\eta_y - B^3_u = 0,\\
&\frac{1}{2}\eta_t + B^2_u = 0, \quad \gamma\eta_u - \frac{1}{2}\gamma\xi^3_y + \frac{1}{2}\gamma\xi^2_x + \frac{1}{2}\gamma\xi^1_t = 0,\\
&\beta\eta_u + \frac{1}{2}\beta\xi^3_y + \frac{1}{2}\beta\xi^1_t - \frac{3}{2}\beta\xi^2_x = 0, \quad B^1_u = 0, \quad B^1_t + B^2_x + B^3_y = 0,\\
&\frac{\alpha}{p+1}\eta_u + \frac{\alpha}{(p+1)(p+2)}\xi^1_t - \frac{\alpha}{p+2}\xi^2_x + \frac{\alpha}{(p+1)(p+2)}\xi^3_y = 0.
\end{aligned}$$

Solving the above system of partial differential equations, we obtain

$$\begin{aligned}
\xi^1 &= C_1,\\
\xi^2 &= C_2 y + C_3,\\
\xi^3 &= 2\gamma t C_2 + C_4,\\
\eta &= yR(t) + H(t),\\
B^1 &= Q(t,x,y),\\
B^2 &= M(t,x,y) - \frac{1}{2}u(yR'(t) + H'(t)),\\
B^3 &= N(t,x,y) + \gamma u R(t),
\end{aligned} \tag{8.8}$$

where C_1, C_2, C_3, C_4 are constants and $R(t),\ H(t),\ Q(t,x,y),\ M(t,x,y)$ and $N(t,x,y)$ are arbitrary functions of their arguments. The corresponding generic vectors of the infinitesimal transformations which leave equation (8.3) invariant together with their gauge functions are:

$$X_1 = \frac{\partial}{\partial t}, \quad B^1 = B^2 = B^3 = 0, \tag{8.9}$$

$$X_2 = \frac{\partial}{\partial x}, \quad B^1 = B^2 = B^3 = 0, \tag{8.10}$$

$$X_3 = \frac{\partial}{\partial y}, \quad B^1 = B^2 = B^3 = 0, \tag{8.11}$$

$$X_4 = y\frac{\partial}{\partial x} + 2\gamma t\frac{\partial}{\partial y}, \quad B^1 = B^2 = B^3 = 0, \tag{8.12}$$

$$X_5 = yR(t)\frac{\partial}{\partial u}, \quad B^1 = 0,\ B^2 = -\frac{1}{2}uyR'(t),\ B^3 = \gamma uR(t), \tag{8.13}$$

$$X_6 = H(t)\frac{\partial}{\partial u}, \quad B^1 = B^3 = 0,\ B^2 = -\frac{1}{2}uH'(t). \tag{8.14}$$

Applying Theorem 1.7, we obtain the six nontrivial conserved vectors associated with the above Noether point symmetries, given by

$$T_1^1 = \frac{1}{2}\beta u_{xx}^2 - \frac{\alpha}{(p+1)(p+2)}u_x^{p+2} + \frac{1}{2}\gamma u_y^2,$$
$$T_1^2 = \beta u_t u_{xxx} + \frac{\alpha}{p+1}u_t u_x^{p+1} + \frac{1}{2}u_t^2 - \beta u_{tx}u_{xx},$$
$$T_1^3 = -\gamma u_t u_y; \tag{8.15}$$

$$T_2^1 = \frac{1}{2}u_x^2,$$
$$T_2^2 = \beta u_x u_{xxx} - \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma u_y^2 + \frac{\alpha}{p+2}u_x^{p+2},$$
$$T_2^3 = -\gamma u_x u_y; \tag{8.16}$$

$$T_3^1 = \frac{1}{2}u_x u_y,$$
$$T_3^2 = \frac{1}{2}u_t u_y + \frac{\alpha}{p+1}u_x^{p+1}u_y + \beta u_y u_{xxx} - \beta u_{xy}u_{xx},$$
$$T_3^3 = -\frac{1}{2}u_t u_x - \frac{\alpha}{(p+1)(p+2)}u_x^{p+2} + \frac{1}{2}\beta u_{xx}^2 - \frac{1}{2}\gamma u_y^2; \tag{8.17}$$

$$
\begin{aligned}
T_4^1 &= \frac{1}{2}yu_x^2 + \gamma t u_x u_y, \\
T_4^2 &= \beta y u_x u_{xxx} + 2\gamma\beta t u_y u_{xxx} + \frac{1}{2}\gamma y u_y^2 + \gamma t u_t u_y - \frac{1}{2}\beta y u_{xx}^2 \\
&\quad + \frac{\alpha}{p+1} y u_x^{p+2} + \frac{2\alpha\gamma}{p+1} t u_y u_x^{p+1} - 2\beta\gamma t u_{xy} u_{xx}, \\
T_4^3 &= \beta\gamma t u_{xx}^2 - \gamma^2 t u_y^2 - \gamma y u_x u_y - \gamma t u_t u_x - \frac{2\alpha\gamma}{(p+1)(p+2)} t u_x^{p+2};
\end{aligned} \tag{8.18}
$$

$$
\begin{aligned}
T_5^1 &= -\frac{1}{2} y R(t) u_x, \\
T_5^2 &= -\frac{1}{2} y R(t) u_t - \frac{\alpha}{p+1} y R(t) u_x^{p+1} - \beta y R(t) u_{xxx} + \frac{1}{2} u y R'(t), \\
T_5^3 &= \gamma y R(t) u_y - \lambda u R(t);
\end{aligned} \tag{8.19}
$$

$$
\begin{aligned}
T_6^1 &= -\frac{1}{2} u_x H(t), \\
T_6^2 &= -\frac{1}{2} u_t H(t) - \frac{\alpha}{p+1} u_x^{p+1} H(t) - \beta u_{xxx} H(t) + \frac{1}{2} u H'(t), \\
T_6^3 &= \gamma u_y H(t).
\end{aligned} \tag{8.20}
$$

Case 2: $p = 1$

When $p = 1$, the PKPp equation (8.3) becomes

$$
u_{tx} + \alpha u_x u_{xx} + \beta u_{xxxx} - \gamma u_{yy} = 0. \tag{8.21}
$$

We observe that equation (8.21) has a Lagrangian formulation with the Lagrangian

$$
L = -\frac{1}{2} u_t u_x - \frac{1}{6}\alpha u_x^3 + \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma u_y^2. \tag{8.22}
$$

Following the same procedure as in Case 1, we obtain ten Noether symmetries; X_1, X_2, X_3 and X_4 are given, by (8.9)-(8.12) and the other six are given below:

$$
X_5 = y\frac{\partial}{\partial u}, \quad B^1 = B^2 = 0, \quad B^3 = \gamma u, \tag{8.23}
$$

$$
X_6 = \alpha t\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \quad B^1 = -\frac{1}{2}u, \quad B^2 = B^3 = 0, \tag{8.24}
$$

$$X_7 = \frac{1}{2}\alpha t^2\frac{\partial}{\partial x} + (xt + \frac{1}{2\gamma}y^2)\frac{\partial}{\partial u}, \quad B^1 = -\frac{1}{2}ut;$$
$$B^2 = -\frac{1}{2}ux; \quad B^3 = yu, \tag{8.25}$$
$$X_8 = \alpha yt\frac{\partial}{\partial x} + \alpha\gamma t^2\frac{\partial}{\partial y} + xy\frac{\partial}{\partial u}, \quad B^1 = -\frac{1}{2}uy;$$
$$B^2 = 0; \quad B^3 = \gamma ux, \tag{8.26}$$
$$X_9 = \frac{1}{2}\alpha yt^2\frac{\partial}{\partial x} + \frac{1}{3}\alpha\gamma t^3\frac{\partial}{\partial y} + (xty + \frac{1}{6\gamma}y^3)\frac{\partial}{\partial u},$$
$$B^1 = -\frac{1}{2}uty; \quad B^2 = -\frac{1}{2}uxy; \quad B^3 = \gamma uxt + \frac{1}{2}uy^2, \tag{8.27}$$
$$X_{10} = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \quad B^1 = B^2 = B^3 = 0. \tag{8.28}$$

The application of Noether theorem [46] yields ten conserved quantities. The first four are obtained from the operators (8.9)-(8.12), given by equations (8.15)-(8.18) with $p = 1$, respectively, while the other six conserved vectors associated with the operators (8.23)-(8.28) are, respectively, given by

$$T_5^1 = \frac{1}{2}yu_x,$$
$$T_5^2 = \frac{1}{2}yu_t + \frac{1}{2}\alpha yu_x^2 + \beta yu_{xxx},$$
$$T_5^3 = \gamma u - \gamma yu_y; \tag{8.29}$$

$$T_6^1 = \frac{1}{2}\alpha tu_x^2 - \frac{1}{2}xu_x + \frac{1}{2}u,$$
$$T_6^2 = \alpha\beta tu_xu_{xxx} - \beta xu_{xxx} + \beta u_{xx} + \frac{1}{3}\alpha^2tu_x^3 - \frac{1}{2}\alpha\beta tu_{xx}^2$$
$$+ \frac{1}{2}\alpha\gamma yu_y^2 - \frac{1}{2}xu_t - \frac{1}{2}\alpha xu_x^2,$$
$$T_6^3 = \gamma xu_y - \alpha\gamma tu_xu_y; \tag{8.30}$$

$$T_7^1 = \frac{1}{4}\alpha t^2u_x^2 - \frac{1}{2}xtu_x - \frac{1}{4\gamma}y^2u_x + \frac{1}{2}ut,$$
$$T_7^2 = \frac{1}{6}\alpha^2t^2u_x^3 - \frac{1}{4}\alpha\beta t^2u_{xx}^2 + \frac{1}{4}\alpha\gamma t^2u_y^2 - \frac{1}{2}xtu_t - \frac{1}{2}\alpha xtu_x^2$$

$$
\begin{aligned}
&- \beta xtu_{xxx} - \frac{1}{4\gamma}y^2u_t - \frac{1}{4\gamma}\alpha y^2u_x^2 - \frac{1}{2\gamma}\beta y^2u_{xxx} + \frac{1}{2}\alpha\beta t^2u_{xxx} \\
&+ \beta tu_{xx} + \frac{1}{2}ux, \\
T_7^3 &= \gamma xtu_y + \frac{1}{2}y^2u_y - \frac{1}{2}\alpha\gamma t^2u_xu_y - yu; \qquad (8.31)
\end{aligned}
$$

$$
\begin{aligned}
T_8^1 &= \frac{1}{2}\alpha\gamma t^2u_xu_y + \frac{1}{2}\alpha ytu_x^2 - \frac{1}{2}xyu_x + \frac{1}{2}yu, \\
T_8^2 &= \frac{1}{3}\alpha^2ytu_x^3 - \frac{1}{2}\alpha\beta ytu_{xx}^2 + \frac{1}{2}\alpha\gamma ytu_y^2 - \frac{1}{2}xyu_t - \frac{1}{2}\alpha xyu_x^2 \\
&- \beta xyu_{xxx} + \alpha\beta yyu_xu_{xxx} + \frac{1}{2\alpha\gamma}t^2u_tu_y + \frac{1}{2}\alpha^2\gamma t^2u_x^2u_y \\
&+ \alpha\beta\gamma t^2u_yu_{xxx} + \beta yu_{xx} - \alpha\beta\gamma t^2u_{xx}u_{xy}, \\
T_8^3 &= \frac{1}{2}\alpha\beta\gamma t^2u_{xx}^2 - \frac{1}{2}\alpha\gamma t^2u_tu_x - \frac{1}{6}\alpha^2\gamma t^2u_x^3 - \frac{1}{2}\alpha\gamma^2t^2u_y^2 \\
&+ \gamma xyu_y - \alpha\gamma ytu_xu_y - \gamma xu; \qquad (8.32)
\end{aligned}
$$

$$
\begin{aligned}
T_9^1 &= \frac{1}{6}\alpha\gamma t^3u_xu_y + \frac{1}{4}\alpha yt^2u_x^2 - \frac{1}{12\gamma}y^3u_x - \frac{1}{2}xyt + \frac{1}{2}ytu, \\
T_9^2 &= \frac{1}{6}\alpha^2yt^2u_x^3 - \frac{1}{4}\alpha\beta yt^2u_{xx}^2 + \frac{1}{4}\alpha\gamma yt^2u_y^2 - \frac{1}{2}xyu_t - \frac{1}{2}\alpha xytu_x^2 \\
&- \beta xytu_{xxx} - \frac{1}{12\gamma}y^3u_t - \frac{1}{12\gamma}\alpha y^3u_x^2 - \frac{1}{6\gamma}\beta y^3u_{xxx} + \frac{1}{2}\alpha\beta yt^2u_xu_{xxx} \\
&+ \frac{1}{6}\alpha\gamma t^3u_tu_y + \frac{1}{6}\alpha^2\gamma t^3u_x^2u_y + \frac{1}{3}\alpha\beta\gamma t^3u_yu_{xxx} + \beta tyu_{xx} \\
&- \frac{1}{3}\alpha\beta\gamma t^3u_{xx}u_{xy} + \frac{1}{2}xyu, \\
T_9^3 &= \frac{1}{6}\alpha\beta\gamma t^3u_{xx}^2 - \frac{1}{6}\alpha\gamma t^3u_tu_x - \frac{1}{18}\alpha^2\gamma t^3u_x^3 - \frac{1}{6}\alpha\gamma^2t^3u_y^2 \\
&+ \gamma txyu_y + \frac{1}{6}y^3u_y - \frac{1}{2}\alpha\gamma yt^2u_xu_y - \gamma txu - \frac{1}{2}y^2u; \qquad (8.33)
\end{aligned}
$$

$$
\begin{aligned}
T_{10}^1 &= \frac{3}{2}\beta tu_{xx} - \frac{1}{2}\alpha tu_x^3 + \frac{3}{2}\gamma tu_y^2 + \frac{1}{2}uu_x + \frac{1}{2}xu_x^2 + yu_xu_y, \\
T_{10}^2 &= \frac{1}{3}\alpha xu_x^3 - \frac{1}{2}\beta xu_{xx}^2 + \frac{1}{2}\gamma xu_y^2 + \frac{1}{2}uu_t + \frac{1}{2}\alpha uu_x^2 + \beta uu_{xxx} \\
&+ \frac{3}{2}tu_t^2 + \frac{3}{2}\alpha tu_tu_x^2 + 3\beta tu_tu_{xxx} + \beta xu_xu_{xxx} + yu_tu_y + \alpha yu_x^2u_y
\end{aligned}
$$

$$+ 2\beta y u_y u_{xxx} - \beta u_x u_{xx} - 3\beta t u_{tx} u_{xx} - \beta u_x u_{xx} - 2\beta y u_{xx} u_{xy},$$

$$T_{10}^3 = \beta y u_{xx}^2 - y u_t u_x - \frac{1}{3}\alpha y u_x^3 - \gamma y u_y^2 - \gamma u u_y - 3\gamma t u_t u_y - \gamma x u_x u_y. \tag{8.34}$$

Case 3. $p = -1$

We now study the case when $p = -1$. Here equation (8.3) becomes

$$u_{tx} + \alpha u_x^{-1} u_{xx} + \beta u_{xxxx} - \gamma u_{yy} = 0. \tag{8.35}$$

It can easily be seen that equation (8.35) has a Lagrangian

$$L = -\frac{1}{2} u_t u_x - \alpha u_x \ln(u_x) + \frac{1}{2}\beta u_{xx}^2 + \frac{\gamma}{2} u_y^2. \tag{8.36}$$

Following the above procedure, one obtains six Noether point symmetries which are the same as in Case 1. Thus, the associated conserved vectors are:

$$\begin{aligned} T_1^1 &= \frac{1}{2}\gamma u_y^2 + \frac{1}{2}\beta u_{xx}^2, \\ T_1^2 &= \frac{1}{2} u_t^2 + \alpha u_t \ln(u_x) + \alpha u_t + \beta u_t u_{xxx} - \beta u_{tx} u_{xx}, \\ T_1^3 &= -\gamma u_t u_y; \end{aligned} \tag{8.37}$$

$$\begin{aligned} T_2^1 &= \frac{1}{2} u_x^2, \\ T_2^2 &= \beta u_x u_{xxx} - \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma u_y^2 + \alpha u_x, \\ T_2^3 &= -\gamma u_x u_y; \end{aligned} \tag{8.38}$$

$$\begin{aligned} T_3^1 &= \frac{1}{2} u_x u_y, \\ T_3^2 &= \frac{1}{2} u_t u_y + \alpha u_y \ln(u_x) + \alpha u_y + \beta u_y u_{xxx} - \beta u_{xy} u_{xx}, \\ T_3^3 &= \frac{1}{2}\beta u_{xx}^2 - \frac{1}{2} u_t u_x - \alpha u_x \ln(u_x) - \frac{1}{2}\gamma u_y^2; \end{aligned} \tag{8.39}$$

$$
\begin{aligned}
T_4^1 &= \frac{1}{2}yu_x^2 + \gamma t u_x u_y,\\
T_4^2 &= \beta y u_x u_{xxx} + 2\gamma\beta t u_y u_{xxx} + \frac{1}{2}\gamma y u_y^2 + \gamma t u_t u_y - \frac{1}{2}\beta y u_{xx}^2\\
&\alpha y u_x + 2\alpha\gamma t u_y \ln(u_x) + 2\alpha\gamma t u_y - 2\beta\gamma t u_{xy} u_{xx},\\
T_4^3 &= \beta\gamma t u_{xx}^2 - \gamma^2 t u_y^2 - \gamma y u_x u_y - \gamma t u_t u_x - 2\alpha\gamma t u_x \ln(u_x);
\end{aligned}
\tag{8.40}
$$

$$
\begin{aligned}
T_5^1 &= -\frac{1}{2}R(t) y u_x,\\
T_5^2 &= \frac{1}{2}R'(t) y u - R(t) y \left(\frac{1}{2}u_t - \alpha \ln(u_x) - \alpha - \beta u_{xxx}\right),\\
T_5^3 &= \gamma R(t) y \left(u_y - u\right);
\end{aligned}
\tag{8.41}
$$

$$
\begin{aligned}
T_6^1 &= -\frac{1}{2}H(t) u_x,\\
T_6^2 &= \frac{1}{2}H'(t) u - H(t) \left(\frac{1}{2}u_t - \alpha \ln(u_x) - \alpha - \beta u_{xxx}\right),\\
T_6^3 &= \gamma H(t) y u_y.
\end{aligned}
\tag{8.42}
$$

Case 4: $p = -2$

In this case the PKPp equation (8.3) becomes

$$u_{tx} + \alpha u_x^{-2} u_{xx} + \beta u_{xxxx} - \gamma u_{yy} = 0 \tag{8.43}$$

and it admits the Lagrangian

$$L = -\frac{1}{2}u_t u_x + \alpha \ln(u_x) + \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma u_y^2. \tag{8.44}$$

Following the above procedure, in this case we obtain the same six Noether point symmetries as in Case 1. Thus, the corresponding conserved vectors are:

$$
\begin{aligned}
T_1^1 &= \alpha \ln(u_x) + \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma u_y^2,\\
T_1^2 &= \frac{1}{2}u_t^2 - \alpha u_t u_x^{-1} + \beta u_t u_{xxx} - \beta u_{tx} u_{xx},
\end{aligned}
$$

$$T_1^3 = -\gamma u_t u_y; \tag{8.45}$$

$$\begin{aligned}
T_2^1 &= \frac{1}{2}u_x^2, \\
T_2^2 &= \beta u_x u_{xxx} - \frac{1}{2}\beta u_{xx}^2 + \frac{1}{2}\gamma u_y^2 - \alpha + \alpha \ln(u_x), \\
T_2^3 &= -\gamma u_x u_y;
\end{aligned} \tag{8.46}$$

$$\begin{aligned}
T_3^1 &= \frac{1}{2}u_x u_y, \\
T_3^2 &= \frac{1}{2}u_t u_y - \alpha u_y u_x^{-1} + \beta u_y u_{xxx} - \beta u_{xy} u_{xx}, \\
T_3^3 &= \frac{1}{2}\beta u_{xx}^2 - \frac{1}{2}u_t u_x + \alpha \ln(u_x) - \frac{1}{2}\gamma u_y^2;
\end{aligned} \tag{8.47}$$

$$\begin{aligned}
T_4^1 &= \frac{1}{2}y u_x^2 + \gamma t u_x u_y, \\
T_4^2 &= \beta y u_x u_{xxx} + 2\gamma\beta t u_y u_{xxx} + \frac{1}{2}\gamma y u_y^2 + \gamma t u_t u_y - \frac{1}{2}\beta y u_{xx}^2 \\
&\quad + \alpha y \ln(u_x) - 2\alpha\gamma t u_y u_x^{-1} - \alpha y - 2\beta\gamma t u_{xy} u_{xx}, \\
T_4^3 &= \beta\gamma t u_{xx}^2 + 2\alpha\gamma t \ln(u_x) - \gamma^2 t u_y^2 - \gamma y u_x u_y - \gamma t u_t u_x;
\end{aligned} \tag{8.48}$$

$$\begin{aligned}
T_5^1 &= -\frac{1}{2}y u_x R(t), \\
T_5^2 &= y R(t)\left(\alpha u_x^{-1} - \frac{1}{2}u_t - \beta u_{xxx}\right) + \frac{1}{2}u R'(t), \\
T_5^3 &= \gamma R(t)\left(y u_y - u\right);
\end{aligned} \tag{8.49}$$

$$\begin{aligned}
T_6^1 &= -\frac{1}{2}u_x H(t), \\
T_6^2 &= \frac{1}{2}u H'(t) - H(t)\left(\frac{1}{2}u_t - \alpha u_x^{-1} + \beta u_{xxx}\right), \\
T_6^3 &= \gamma u_y H(t).
\end{aligned} \tag{8.50}$$

8.2 Exact solutions of (8.3)

In this section we firstly construct exact solutions of PKPp equation (8.3) with $p = 1$ by employing the Lie symmetry method together with the Kudryashov method. Secondly, we use direct integration to find solutions of PKPp equation (8.3) for $p = 2$.

8.2.1 Solutions of (8.3) with $p = 1$ using Kudryashov method

When $p = 1$, equation (8.3) takes the form

$$u_{tx} + \alpha u_x u_{xx} + \beta u_{xxxx} - \gamma u_{yy} = 0. \tag{8.51}$$

The Lie point symmetries of equation (8.51) are [108]

$$\begin{aligned}
X_1 &= 72\alpha\gamma^2 F_1(t)\frac{\partial}{\partial t} + \left(12\alpha\gamma y^2 F_1''(t) + 24\alpha\gamma^2 x F_1'(t)\right)\frac{\partial}{\partial x} + 48\alpha\gamma^2 y F_1'(t)\frac{\partial}{\partial y} \\
&\quad + \left(12\gamma^2 x^2 F_1''(t) - 24\alpha\gamma^2 u F_1'(t) + 12\gamma x y^2 F_1'''(t) + y^4 F_1^{(4)}(t)\right)\frac{\partial}{\partial u},
\end{aligned}$$

$$\begin{aligned}
X_2 &= 2\alpha\gamma F_2(t)\frac{\partial}{\partial x} + \left(2\gamma x F_2'(t) + y^2 F_2''(t)\right)\frac{\partial}{\partial u}, \\
X_3 &= 12\alpha\gamma^2 F_3(t)\frac{\partial}{\partial y} + 6\alpha\gamma y F_3'(t)\frac{\partial}{\partial x} + \left(6\gamma x y F_3''(t) + y^3 F_3'''(t)\right)\frac{\partial}{\partial u}, \\
X_4 &= y F_4(t)\frac{\partial}{\partial u}, \\
X_5 &= F_5(t)\frac{\partial}{\partial u}.
\end{aligned}$$

When $F_1(t) = F_2(t) = F_3(t) = 1$, we get three translational symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}. \tag{8.52}$$

We now make use of the linear combination of these three translation symmetries, namely $X = X_1 + \nu X_2 + X_3$ and transform equation (8.51) to a partial differential equation in two variables. The symmetry X has three invariants

$$f = t - y,\ g = x - \nu y,\ \theta = u.$$

Considering θ as the new dependent variable and f and g, as new independent variables, equation (8.51) transforms to

$$\beta\theta_{gggg} + \alpha\theta_g\theta_{gg} + (1 - 2\gamma\nu)\theta_{fg} - \gamma\theta_{ff} - \gamma\nu^2\theta_{gg} = 0, \tag{8.53}$$

which is a nonlinear partial differential equation in two independent variables f and g.

We now further reduce equation (8.53) to an ordinary differential equation by first finding Lie point symmetries of this equation. The symmetry group of equation (8.53) is spanned by the following vector fields [108]:

$$\Gamma_1 = \frac{\partial}{\partial f},\ \Gamma_2 = \frac{\partial}{\partial g},\ \Gamma_3 = -\alpha f\frac{\partial}{\partial g},$$
$$\Gamma_4 = 6\nu\gamma f\frac{\partial}{\partial f} + (2\nu^2\gamma f + 2\nu\gamma g - g)\frac{\partial}{\partial g}.$$

The linear combination $\Gamma = \Gamma_1 + c\Gamma_2$, where c is a constant, yields the two invariants

$$z = g - cf,\ W = \theta.$$

By considering W as a dependent variable and z as an independent variable, equation (8.53) transforms to a nonlinear fourth-order ordinary differential equation

$$\beta W''''(z) + \alpha W'(z)W''(z) + (2c\nu\gamma - c^2\gamma - \nu^2\gamma - c)W''(z) = 0. \tag{8.54}$$

Integrating equation (8.54) with respect to z and letting the constant of integration to be zero, we obtain

$$\beta W''' + \frac{\alpha}{2}W'^2 + (2c\nu\gamma - c^2\gamma - \nu^2\gamma - c)W' = 0. \tag{8.55}$$

The Kudryashov method assumes the solution of equation (8.55) to be of the form given by equation (1.60). Using Step 2 of the Kudryashov method, we obtain $N = 1$. Therefore, the solution of (8.55) is given in the form

$$W(z) = \mathcal{A}_0 + \mathcal{A}_1 Q. \tag{8.56}$$

Substituting the derivatives of $W(z)$ into equation (8.55) we obtain the equation that has the function Q and coefficients $\mathcal{A}_n$ $(n = 0, 1)$. Collecting all terms with same powers of Q and equating each coefficient to zero, yields the following system of algebraic equations:

$$\begin{aligned}
Q &: \; c^2\gamma\mathcal{A}_1 - 2c\gamma\nu\mathcal{A}_1 + \gamma\nu^2\mathcal{A}_1 - \beta\mathcal{A}_1 + c\mathcal{A}_1 = 0\\
Q^2 &: \; 7\beta\mathcal{A}_1 - c\mathcal{A}_1 - \gamma\nu^2\mathcal{A}_1 + 2\gamma\nu\mathcal{A}_1 + \frac{1}{2}\alpha\mathcal{A}_1^2 - c^2\gamma\mathcal{A}_1 = 0\\
Q^3 &: \; \alpha\mathcal{A}_1^2 + 12\beta\mathcal{A}_1 = 0\\
Q^4 &: \; \frac{1}{2}\alpha\mathcal{A}_1^2 + 6\beta\mathcal{A}_1 = 0.
\end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\mathcal{A}_1 = \frac{12}{\alpha}\left(2c\gamma\nu - c^2\gamma - \gamma\nu^2 - c\right), \quad \beta = c^2\gamma - 2c\gamma + \gamma\nu^2 + c \tag{8.57}$$

and so we obtain exact solution of equation (8.55) in the form

$$U(z) = \mathcal{A}_0 + \frac{12}{\alpha}\left(2c\gamma\nu - c^2\gamma - \gamma\nu^2 - c\right)Q(z), \quad Q(z) = \frac{1}{1+e^z}. \tag{8.58}$$

Reverting back to the original variables, we obtain the solution of (8.51) as

$$u(t,x,y) = C + \frac{12}{\alpha}\left(2c\gamma\nu - c^2\gamma - \gamma\nu^2 - c\right)(1 + e^{x+(c-\nu)y-ct})^{-1}, \tag{8.59}$$

where C is an arbitrary constant.

The profile of the solution (8.59) is given in Figure 8.1, with parameters $\beta = 2, \alpha = 2, \gamma = 1, c = 2, \nu = 1, A_1 = -18, t = 0, C = 0$.

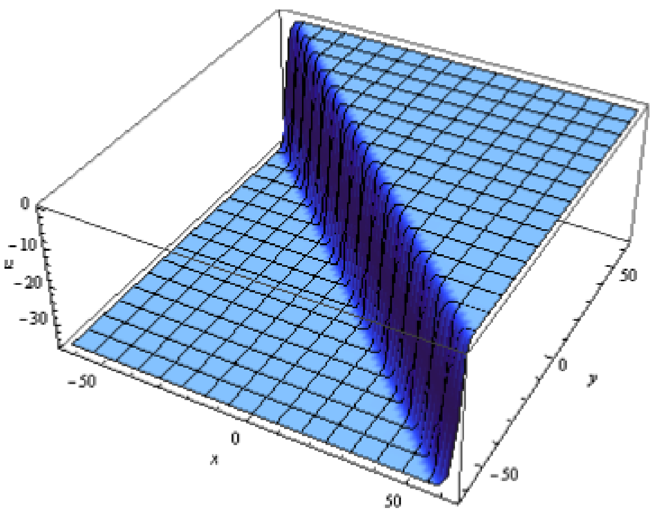

Figure 8.1: Profile of solution (8.59)

8.2.2 Solutions of (8.3) with $p = 2$ using direct integration

When $p = 2$, equation (8.3) becomes

$$u_{tx} + \alpha u_x^2 u_{xx} + \beta u_{xxxx} - \gamma u_{yy} = 0 \tag{8.60}$$

and its Lie point symmetries are [108]

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = y\frac{\partial}{\partial x} + 2\gamma t\frac{\partial}{\partial y},$$
$$\Gamma_5 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \quad \Gamma_6 = yF_1(t)\frac{\partial}{\partial u}, \quad \Gamma_7 = F_2(t)\frac{\partial}{\partial u}.$$

As before we now make use of $\Gamma = \Gamma_1 + \nu\Gamma_2 + \Gamma_3$ and transform equation (8.60) to a partial differential equation in two variables. The symmetry Γ has three invariants

$$f = t - y, \; g = x - \nu y, \; \phi = u$$

and considering ϕ as the new dependent variable and f and g as new independent variables, equation (8.60) transforms to

$$\beta\phi_{gggg} + \alpha\phi_g^2\phi_{gg} + (1 - 2\gamma\nu)\phi_{fg} - \gamma\phi_{ff} - \gamma\nu^2\phi_{gg} = 0. \tag{8.61}$$

The symmetry group of (8.61) is spanned by [108]

$$X_1 = \frac{\partial}{\partial f}, \quad X_2 = \frac{\partial}{\partial g}, \quad X_3 = f\frac{\partial}{\partial \phi}, \quad X_4 = \frac{\partial}{\partial \phi}.$$

The linear combination $X = cX_1 + X_2$, where c is a constant, yields the two invariants

$$z = f - cg, \ W = \phi.$$

By considering W as a dependent variable and z as an independent variable, equation (8.61) transforms to a nonlinear fourth-order ordinary differential equation

$$\beta c^4 W''''(z) + \alpha c^4 W'(z)^2 W''(z) + (2c\nu\gamma - c^2\nu^2\gamma - c - \gamma)W''(z) = 0. \qquad (8.62)$$

Integrating the above equation two times and taking the constants of integration to be zero, we obtain

$$\beta c^4 W''(z)^2 + \frac{1}{6}\alpha c^4 W'(z)^4 + (2c\nu\gamma - c^2\nu^2\gamma - c - \gamma)W'(z)^2 = 0. \qquad (8.63)$$

This equation can be integrated twice and gives us two solutions of (8.60) in the original variables as

$$u_1(x,y,t) = \frac{2}{\sqrt{\mathcal{A}}}\tan^{-1}\left[\sqrt{\mathcal{AB}}\ \exp\{\sqrt{\mathcal{B}}\,(t - cx - (1-c\nu)y + C_1)\}\right] + C_2$$

and

$$u_2(x,y,t) = \frac{2}{\sqrt{\mathcal{A}}}\tan^{-1}\left[\frac{1}{\sqrt{\mathcal{AB}}}\ \exp\{\sqrt{\mathcal{B}}\,(t - cx - (1-c\nu)y - C_3)\}\right] + C_4,$$

where C_1, C_2, C_3, C_4 are arbitrary constants of integration and

$$\mathcal{A} = \frac{\alpha}{6\beta}, \ \ \mathcal{B} = \frac{c^2\gamma\nu^2 - 2c\gamma\nu + c + \gamma}{\beta c^4}.$$

8.3 Concluding remarks

We constructed several conservation laws with respect to the parameter p for the PKPp equation using Noether theroem. Furthermore, the symmetry reduction method was employed to reduce the PKPp equation to a third-order ordinary differential equation. Thereafter, the Kudryashov method was successfully applied to obtain the new exact solutions for the PKPp equation when $p = 1$. Solutions when $p = 2$ were obtained by employing direct integration. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equations (8.21) and (8.60).

Chapter 9

Conclusions

The exact solutions for nonlinear partial differential equations are of important significance for the explanation of some practical physical problems. The aim of this work was to obtain exact solutions and conservation laws of some nonlinear partial differential equations by using various methods.

Chapter one provided relevant background, definitions and theorems of the important concepts that were used to carry out the calculations in this work.

In Chapter two a complete Lie group classification was performed on a generalized coupled (2+1)-dimensional hyperbolic system. Lie group classification dealt with the use of the equivalence transformation to simplify the forms of the symmetry operators. The functional forms of the arbitrary parameters were specified via the classical method of group classification, and these included the combination of power law, exponential, logarithm and linear forms. This system admitted 11-dimensional equivalence Lie algebra. The principal Lie algebra was also obtained and several possible extensions of the principal Lie algebra were presented.

In Chapter three we computed exact solutions for the modified Kortweg-de Vries equation and higher-order modified Boussinesq equation with damping term by employing the $(G'/G)-$expansion method. We obtained three types of solutions, namely, hyperbolic function solutions, trigonometric function solutions and rational function solutions. The conservation laws for the higher-order modified Boussinesq equation with damping term were derived using the multiplier method.

Chapter four dealt with coupled Korteweg-de Vries equations. New exact solutions using $(G'/G)-$expansion method were obtained. In addition, conservation laws were constructed using the new conservation theorem and multiplier method.

In Chapter five we analysed coupled Boussinesq equations that appeared in many scientific fields. The $(G'/G)-$expansion method was effectively used to derive exact travelling wave solutions. Furthermore, conservation laws were constructed using two different approaches; the new conservation theorem and the multiplier approach.

In Chapter six we studied the generalized Zakharov–Kuznetsov equation. We derived the conservation laws of this equation using the new conservation theorem. Furthermore, the Lie point symmetries of the underlying equation were obtained and used in conjunction with the simplest equation method to obtain exact solutions.

The generalized Ablowitz-Kaup-Newell-Segur equation was studied in Chapter seven. We obtained six Noether symmetries and used them to construct six non-trivial conserved vectors associated with these symmetries. The exact solutions were obtained using the Lie point symmetries together with the simplest equation method and direct integration.

In Chapter eight we derived the conservation laws for PKPp equation using Noether approach. We considered four cases that arose from different values of p. For each case different Lagrangian was obtained and conserved vectors were constructed. Moreover, the Lie symmetry method together with Kudryashov method was used to obtain solutions for PKPp equation with $p = 1$. Lastly, Lie symmetry method was used to transform the PKPp equation with $p = 2$ to an ordinary differential equation, which was solved by direct integration.

In future, conservation laws obtained in this study could be used to find exact solutions for the nonlinear partial differential equations.

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