Chapter 7: Evaluation of MED in RLNC Networks

Contents

In this chapter we evaluate the decoding efficiency of practical decoding methods in RLNC networks through the use of the network model developed in Chapter 6. The decoding delay and decoding complexity and the number of arithmetic operations for the decoding methods will be assessed. This chapter also includes a study on the efficiency of the decoding methods in the presence of packet loss in the network.

7.1 Introduction

In the previous chapter we presented a network configuration for an RLNC network where the receiver nodes can obtain packets the global encoding vectors of which form a non-strict lower triangular generator matrix. This construction allows for the implementation of ED, as well as MED, presented in Chapter 6. In this chapter we evaluate these practical decoding methods in RLNC networks through the use of the network model developed in Chapter 6. We evaluate the efficiency of the ED and MED algorithms in terms of decoding delay and complexity and show that MED renders lower decoding delay compared to ED.

We also evaluate the performance of the two decoding methods in the presence of packet erasures. We show how MED can continue decoding with high probability after the occurrence of erased packets, where ED cannot. Lastly, we recommend further improvements to MED to allow the algorithm to decode the required source symbols successfully in the midst of lost packets.

7.2 Decoding performance in an ideal network scenario

In Chapter 6 a model was introduced to produce packets at receiver nodes that form a non-strict lower triangular generator matrix. In a network where all the max-flow paths from source to receiver nodes are approximately equal in length and are reliable, the sequential transmission of source symbols might result in receiver nodes obtaining lower triangular $\boldsymbol{G}$ matrices which contain no non-zero entries above the main diagonal. This can render a strict lower triangular matrix, which is the ideal structure because each new packet received contains only a single new source symbol so that the decoder can decode one source packet with the reception of each new encoded packet. A strict lower-triangular matrix can be characterised as a single α block, as defined in Chapter 6.

The probability that $\boldsymbol{G}$ is strictly lower triangular can be calculated by expression (6.13) and the probabilities for obtaining non-zero elements in $\boldsymbol{G}$ are listed in Table 6.1, where $P_1 = 0.94$ and $P_2 = 0.05$, as determined in Chapter 6:

$$P_{strict} = {P_1}^n \cdot \prod_{j=2}^{n} (1 - P_j)^{n-j+1} \tag{7.1}$$

The values of P_{strict} for variable values of n are illustrated in Figure 7.1.

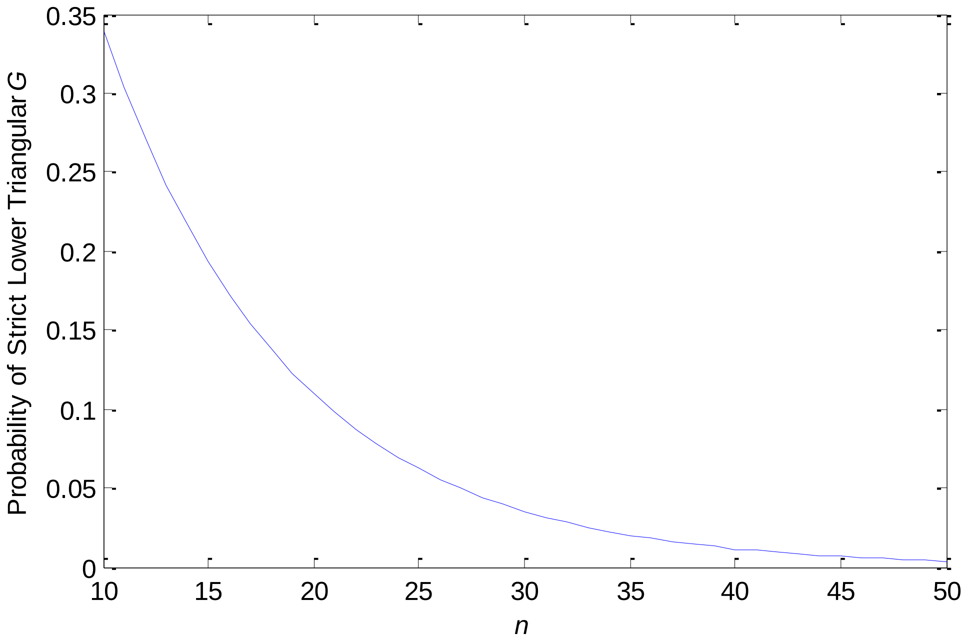

Figure 7.1: Probability of obtaining a strict lower triangular generator matrix $\boldsymbol{G}$

It can be seen that the probability of obtaining strict lower triangular matrices is not large. Although the probability of obtaining such a matrix is low, this matrix structure is evaluated as it is the ideal scenario for decoding. We evaluate the complexity of the decoding algorithms as well as the decoding delay.

7.2.1 Gaussian elimination

Owing to the fact that Gaussian elimination (GE) employs matrix inversion, decoding is successful when $\boldsymbol{G}$ is of full rank n. As described in Section 6.2.1, the GE method consists of two steps, forward elimination and backward substitution, where the forward elimination step is used to transform the matrix into a *triangular* system. As the received matrix $\boldsymbol{G}$ in this scenario is already triangular, the forward elimination step of GE can be eliminated.

Decoding delay

The decoding delay of GE equals the time a receiver node $z \in Z$ has to wait in order to collect n linearly independent packets. Thus the delay is proportional to the size of n [**8**]. The lower triangular structure of matrix $\boldsymbol{G}$ does not affect the decoding delay of GE. Thus the decoding delay of GE increases linearly with the increase in source symbols and scales linearly to the size of n.

Decoding complexity

As a result of the strict-triangular structure of $\boldsymbol{G}$, GE can only employ backward substitution as shown in Algorithm 6.2. The total number of calculations in the backward substitution step in a strict lower triangular matrix of size n is approximately [**81**]

$$\sum_{i=1}^{n-1}(n-i)=\frac{n(n-1)}{2}. \tag{7.2}$$

Hence, GE in this network scenario requires $\mathcal{O}(n^2)$ operations, which is an improvement on GE in a network where $\boldsymbol{G}$ is randomly constructed and where $\mathcal{O}(n^3)$ is predominant owing to backward substitution.

7.2.2 Earliest decoding

In an ideal network scenario where strict lower triangular matrices are received, each packet collected by the receiver is an α packet that contains a single undecoded source symbol. In a network scenario where the generator matrix is strict lower triangular, ED would perform favourably as the reception of a new encoded packet would guarantee the decoding of a source symbol. Thus the reception of a strict lower triangular generator matrix would render zero decoding delay for ED.

Decoding delay

As all the packets of $\boldsymbol{G}$ matrix only contain new information regarding a single source symbol and ED checks for decoding opportunities after every innovative packet has been received, every packet received can be decoded without delay.

Decoding complexity

As with GE, ED can also only employ the backward substitution step for decoding, as the $\boldsymbol{G}$ matrix is received in lower-triangular form. As the decoding algorithm is run at the reception of each new packet, the backward substitution algorithm for GE (Algorithm 6.2) can be adjusted for ED. Each of the n received α packets can then be decoded without delay by linearly combining all the source packets already decoded to it, as shown in Algorithm 7.1:

Algorithm 7.1: Earliest decoding: backward substitution

1: **for** $\boldsymbol{i}=\mathbf{1}\rightarrow\boldsymbol{n}$ **do**
2:   **if** $\boldsymbol{i}=\mathbf{1}$ **do**
3: $\boldsymbol{x}_i\leftarrow\boldsymbol{y}_i$
4:   **else**
5: $\boldsymbol{x}_i\leftarrow\boldsymbol{y}_i-\sum_{j=1}^{i-1}\boldsymbol{G}_{ij}\oplus\boldsymbol{x}_j$

```
    end if
end for
```

The total number of calculations performed for the decoding of the strict lower triangular $\boldsymbol{G}$ is

$$1 + 2 + 3 + \cdots + (n-1) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \qquad (7.3)$$

which is of complexity $\mathcal{O}(n^2)$, equal to that of GE.

7.2.3 Modified earliest decoding

As with ED, MED also checks for decoding opportunities with the arrival of every innovative packet.

Decoding delay

Every newly received packet is linearly combined with the decoded source symbols. With the reception of only α packets, each new packet after the linear combination will have a global encoding vector with a degree of 1, resulting in a decoded source symbol. Thus every new packet received can be decoded without delay, resulting in a zero decoding delay.

Decoding complexity

For the strict lower triangular $\boldsymbol{G}$ matrix, a simplified version of Algorithm 6.3 can be used, which is shown in Algorithm 7.2. Each newly received packet, except for the first, is linearly combined with all the decoded source symbols that it contains to render a packet that has a high probability of containing a coding vector of degree 1.

Algorithm 7.2: Simplified modified earliest decoding for strict lower triangular $\boldsymbol{G}$

Initialise $\boldsymbol{g_0} = \boldsymbol{0}, \boldsymbol{g_0} \in \boldsymbol{G}$**, and** $\boldsymbol{y_0} = \boldsymbol{0}, \boldsymbol{y_0} \in \boldsymbol{Y}$
$\boldsymbol{g_{x_1}} \leftarrow \boldsymbol{g_1}$
$\boldsymbol{x_1} \leftarrow \boldsymbol{y_1}$
for $\boldsymbol{i = 2 \rightarrow n}$ **do**
    $\boldsymbol{g_i} \leftarrow \boldsymbol{g_i} - \sum_{j=1}^{i-1} \boldsymbol{g_{ij}} \oplus \boldsymbol{x_j}$
    $\boldsymbol{d_H(g_i, g_0) = 1}$
    $\boldsymbol{g_{x_i}} \leftarrow \boldsymbol{g_i}$
    update $\boldsymbol{G, Y}$
    $\boldsymbol{x_i} \leftarrow \boldsymbol{y_i}$
end for

The total number of calculations performed for the decoding of the n source packets is

$$\sum_{i=2}^{n}(i-1)=\frac{n(n-1)}{2} \tag{7.4}$$

which is of complexity $\mathcal{O}(n^2)$.

From the above evaluations it can be seen that the MED method performs the same as ED in an ideal network scenario where instantaneous decodability is possible for both methods. GE, however, still has a decoding delay equal to n.

7.3 Decoding performance in a non-ideal network scenario

The decentralised structure of the RLNC network, variable lengths of max-flow paths, as well as other factors, influence the lower triangular structure of $\boldsymbol{G}$. This can result in the $\boldsymbol{G}$ matrix having non-zero entries above the diagonal. According to the evaluation in Chapter 6, the probability of obtaining non-zero entries above the second diagonal is ignored. Thus we assume a non-strict lower triangular matrix as only having non-zero entries on the second diagonal.

The probability $P_{non-strict}$ of obtaining a non-strict lower triangular matrix $\boldsymbol{G}$**,** with any number of non-zero entries on the second diagonal, can be calculated by expression (6.17) as determined in Chapter 6, where

$$P_{non-strict}=\prod_{j=3}^{n}(1-P_j)^{n-j+1}\cdot\sum_{\varphi=1}^{n-1}{P_1}^{n-\varphi}\cdot{P_2}^{\varphi}\cdot(1-P_2)^{n-\varphi-1}\cdot\binom{n-1}{\varphi}. \tag{7.5}$$

and which is illustrated in Figure 7.2.

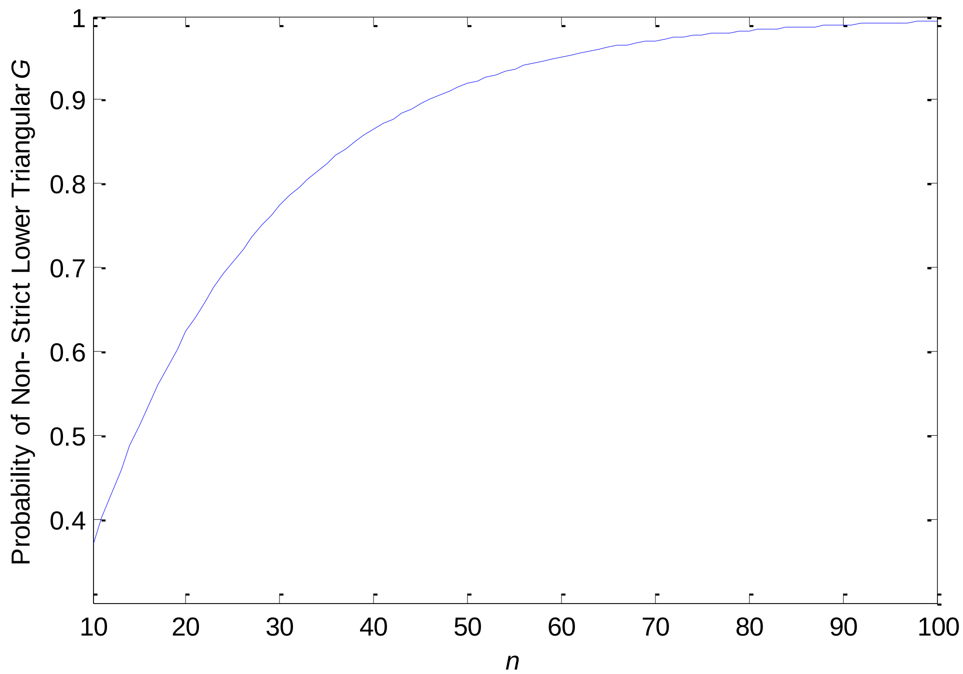

Figure 7.2: Probability of obtaining a non-strict lower triangular matrix

A non-strict lower triangular matrix can contain multiple entries on the second diagonal which can be adjacent, non-adjacent or a combination of both. Packets containing entries on the second diagonal can be characterised as β packets.

As discussed in Chapter 6, the first β packet of a β block has the potential to add two new source symbols to the receiver, which prevents a receiver from decoding these packets immediately. Subsequent packets with second diagonal 1s add new source symbols to the receiver, resulting in $(\varphi - 1)$ linearly independent packets and φ undecoded source symbols. As soon as a receiver obtains an encoded packet which does not include a second diagonal entry, this additional linearly independent packet enables decoding of the source symbols as it creates a $\varphi \times \varphi$ submatrix of full rank.

Subsequently we discuss the decoding performance of GE, ED and MED when these β packets are received in a non-strict lower triangular matrix.

7.3.1 Gaussian elimination

In the case of a strict lower triangular matrix, the forward elimination step of GE could be eliminated as the received generator matrix is already in strict lower triangular form. With the presence of non-zero entries above the diagonal, the forward elimination step must be reemployed in order to create a strict lower triangular matrix before backward substitution can be employed.

Decoding delay

As in the ideal case, the decoding delay of GE equals the time the sink has to wait in order to collect n linearly independent packets, which is proportional to the size of n [**8**].

Decoding complexity

As decoding requires the employment of forward elimination and backward substitution, shown in Algorithms 6.1 and 6.2, the total of arithmetic calculations required increases dramatically, as forward elimination in a matrix of size n requires approximately [**81**]

$$\sum_{i=1}^{n-1}(n-i)+\sum_{i=1}^{n-1}(n-i)^2 \;=\frac{n^2}{2}+\frac{2}{3}n^3 \tag{7.6}$$

calculations along with that of the backward substitution step calculated in (7.2).

From (7.2) and (7.6) is can be determined that GE requires $\mathcal{O}(n^3)$ operations for decoding via matrix inversion, which is considered computationally complex [**8**], [**39**]. In a situation of a small number of source messages, GE is an efficient decoding method, but decreases in efficiency as n becomes large.

7.3.2 Earliest decoding

Owing to the fact that decoding can commence while packets are still being collected, the decoding delay and complexity of ED can remain small. However, the decoding delay and complexity are influenced by the size of the β blocks.

As described in Chapter 6, ED determines the number of undecoded source symbols at the node (number of unknowns) as well as the number of linearly independent coding vectors (number of linear equations). When the number of linearly independent packets is equal to the number of undecoded source symbols, decoding of the subset of source symbols is possible through matrix inversion.

Decoding delay

As in the ideal case, as shown in Algorithm 7.1, each received α packet is linearly combined with the decoded source symbols, rendering a packet with a coding vector of degree 1. This results in a zero decoding delay for α packets.

When packets are received that form part of a β block with high probability, decoding can only commence once all the packets in the β block have been received, meaning that innovative packets have to be collected until the number of linearly independent packets collected is equal to the number of undecoded source symbols.

When β packets with second diagonal 1s are collected by a receiver, there are i unknowns and only $(i-1)$ equations, where $i \leq \varphi$ and φ is the size of the β block. Thus no decoding is possible. The last packet of the β block has no entry on the second diagonal, meaning that no new undecoded source symbol is added, but adds an additional innovative packet. With the addition of the last β packet, the ED algorithm is provided with φ unknowns and φ equations. When a complete β block is received, a submatrix $\boldsymbol{G_{sub}} \subseteq \boldsymbol{G}$ can be successfully inverted and the source symbols $\boldsymbol{X_{sub}} \subseteq \boldsymbol{X}$ decoded. Thus, the decoding delay created by such a β block is equal $(\varphi-1)$. An example can be seen in Figure 7.3 (a).

It may happen that the first packet with a non-zero entry on the second diagonal only introduces a single undecoded symbol, as illustrated in Figure 7.3 (b). This provides the decoder with a single equation and one unknown, which can be decoded without delay. If the second β packet also adds a single undecoded symbol, it can be decoded. Thus it is possible to receive a β block which is decodable with a decoding delay smaller than $(\varphi-1)$.

Figure 7.3: (a) β block with two undecoded symbols in first row **(b) β block with one undecoded symbol in first row**

The decoding delay of ED for a β block can be calculated. These calculations are shown in Appendix B. Assume that a β block consists of φ packets of which $\lambda = (\varphi - 1)$ adjacent packets have non-zero entries on the second diagonal. The decoding delays, obtained from the calculations shown in Appendix B, are summarised in Table 7.1 below:

Table 7.1: Decoding delay for ED for β block of φ packets

	Size of delay	Formula
Zero delay	D_0	$(1-P_1){P_x}^{\lambda-1}$
Full delay	D_λ	P_1
Other	$D_{(\lambda-i)}, i = 1,\ldots,\lambda-1$	$(1-P_1){P_x}^{i}$

The probabilities of different decoding delays in β blocks, described in the above table, are illustrated for values of $\varphi = \{3,6,9\}$ in Figure 7.4.

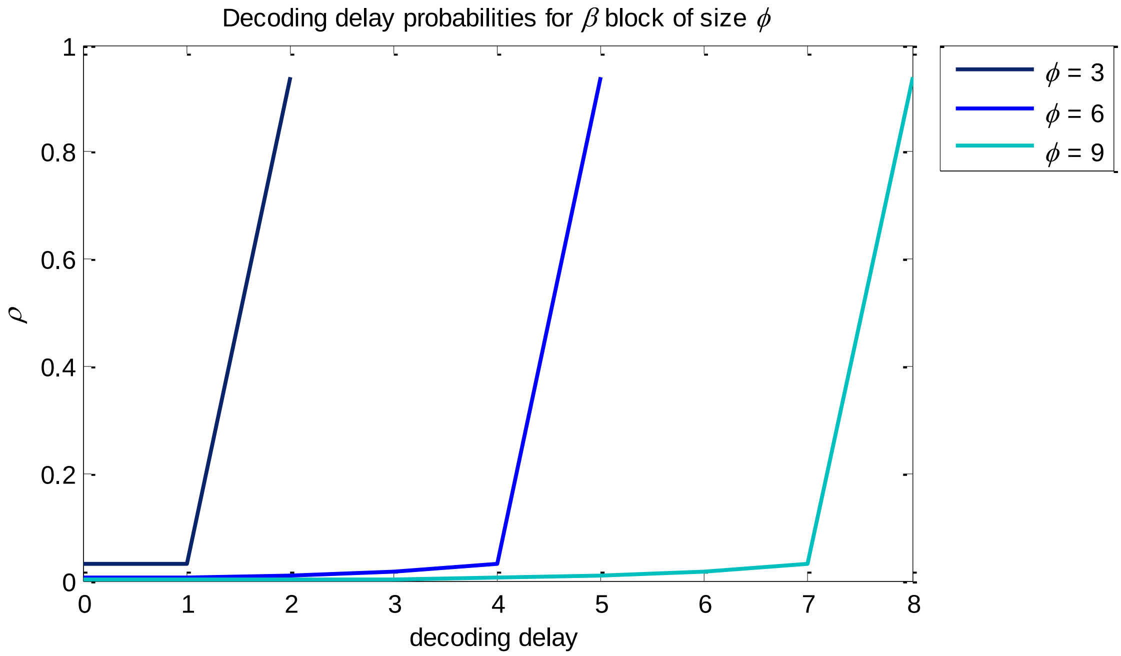

Figure 7.4: Decoding delay probabilities for ED for β blocks of various sizes

From Table 7.1 and Figure 7.4 it can be seen that when the first β packet contains a packet on the main diagonal, with probability $P_1 = 0.94$, decoding can only commence when the complete β block has been received, regardless of φ. This is due to the fact that, from $t_s = 1$, the receiver node is introduced with i unknowns and only $(i-1)$ equations which can only be changed when the last β packet is received.

There exists the possibility that ED can decode source symbols with a smaller delay than $(\varphi - 1)$, but this is only equal to $(1 - P_1) = 0.06$. In general, the decoding delay of ED for a β block can be considered equal to $(\varphi - 1)$, where φ is the size of the block. Thus ED yields a decoding delay which is scalable to φ where φ is dependent on the probability of obtaining a packet which contains a second diagonal 1. It is clear that the decoding delay of ED is independent of n, as stated in [**8**].

Decoding complexity

As in the ideal case, the decoding of α packets scales quadratic to the number of packets, as shown in Section 7.2.2. When packets are received that form a β block, decoding can only take place with high probability when all the packets of a β block have been received, as shown in the previous section. Decoding a full β block requires $\mathcal{O}(\varphi^3)$ operations due to matrix inversion, as evaluated in (7.2) and (7.6), where φ is the size of the largest β block in $\boldsymbol{G}$.

Although matrix inversion is performed on each β block, the size of φ is independent of the size of n and, as the probability of obtaining a large β block is small, the complex matrix inversion is limited to small blocks of size φ.

Note that with the reception of every new innovative packet, the packet and its global encoding vector have to be linearly combined with all the decoded source symbols to effectively remove them. This means that the last received packet can be potentially linearly combined with all $(n-1)$ decoded source symbols, influencing the decoding complexity of ED. This calculation can be deduced from Algorithm 7.2 and is shown in Algorithm 7.3.

Algorithm 7.3: Eliminating decoded source symbols from incoming packets

for $\boldsymbol{i = 2 \rightarrow n}$ **do**
    $\boldsymbol{g_i \leftarrow g_i - \sum_{j=1}^{i-1} g_{ij} \oplus x_j}$
end for

Thus, the decoding complexity of ED is not completely independent of n, as it scales linearly with the size of n.

7.3.3 Modified earliest decoding

As with ED, MED also checks for decoding opportunities with the arrival of new innovative packets. As described in Chapter 6, MED calculates the Hamming distance of the received packets' global encoding vectors to determine if the linear combinations of the received packets would render a native packet.

Decoding delay

As in the ideal case shown in Section 7.2.3, the resulting delay for the decoding of α packets is zero as each collected packet only introduces a single undecoded source symbol. The decoding delay of MED is influenced by the occurrence of β blocks in $\boldsymbol{G}$.

The MED algorithm is able to decode source symbols in a β block if their coding vectors have Hamming distances of 1. By looking at the structure of a β block, it can be seen that several subsequent packets in a β block adds a single source symbol, which may possibly lead to packets with coding vectors of Hamming distance 1. This structure allows MED to decode packets in a β block where ED cannot.

Figure 7.3 (a) is repeated in Figure 7.5, which shows the β block that results in a decoding delay scalable to the size of the β block for ED. For MED, the first packet which contains two undecoded source symbols cannot be decoded. The coding vectors of the first two packets, however, have a Hamming distance of 1. Through the linear combination of these two packets, a source packet can be decoded. The other source symbols can be decoded when the β block has been fully received.

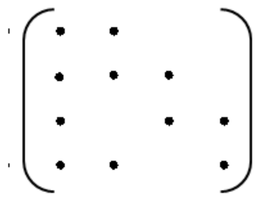

Figure 7.5: $\boldsymbol{\beta}$ block presented in Figure 7.3 (a)

Owing to the possibility of adjacent packets differing by only a single undecoded source symbol, as in the case of the matrix in Figure 7.5, the delay in decoding a β block of $\boldsymbol{G}$ can be reduced.

The decoding delay of MED for a β block can be calculated by means of the mathematical model presented in Chapter 5. The calculations are shown in Appendix B. Assume that a β block consists of φ packets of which $\lambda = (\varphi - 1)$ adjacent packets have non-zero entries on the second diagonal. The decoding delays are summarised in Table 7.2 below:

Table 7.2: Decoding delay for MED for β block of φ packets

	Size of delay	Formula
Zero delay	D_0	$(1 - P_1){P_x}^{\lambda-1}$
Full delay	D_λ	$P_1{P_x}^{\lambda-1}$
Other	$D_{(\lambda-i)}, i = 1, \dots, \lambda - 1$	$A(1 - P_1){P_x}^{\lambda-1} + BP_1{P_x}^{\lambda-1}$ $A = \binom{\lambda - 1}{\lambda - i}, B = \binom{\lambda - 1}{\lambda - i - 1}$

The probabilities of different decoding delays for MED in β blocks of sizes $\varphi = \{3,6,9\}$, described in the above table, are illustrated in Figure 7.6.

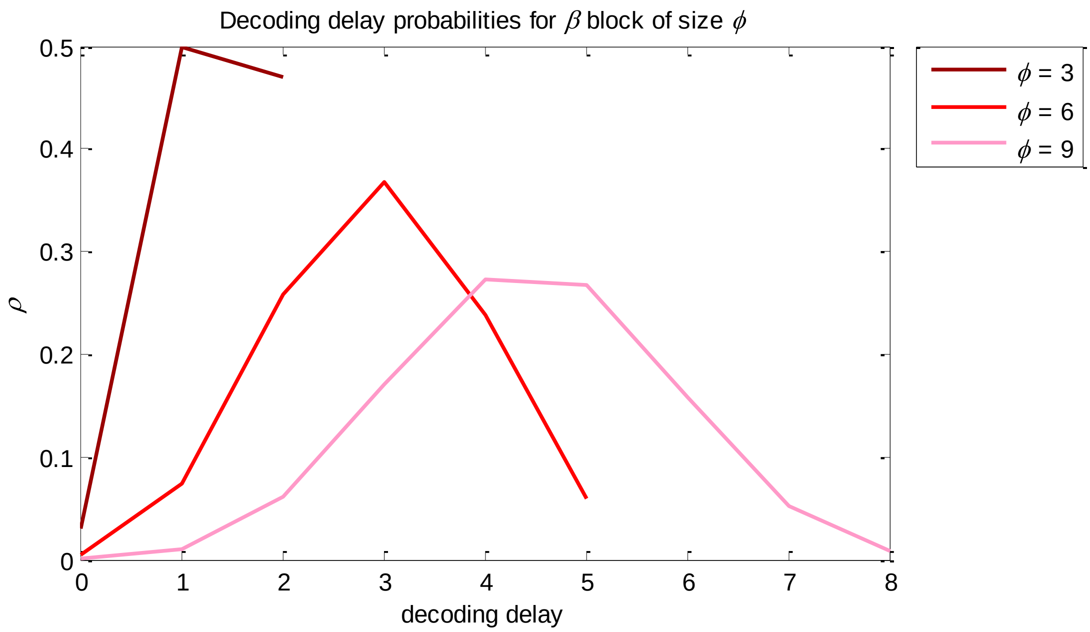

Figure 7.6: Decoding delay probabilities for MED for β blocks of various sizes

From Figure 7.6 it can be seen that as the size of φ increases, the decoding delay of MED becomes normally distributed. This means that the decoding delay of a β block can be approximated by $\varphi/2$ which is less than that of ED and independent of n. The decoding delay of MED is upper bounded by that of ED, because when no packets with $d_H = 1$ are available in a completed β block the MED algorithm reverts to ED. It can be seen though, that the probability of decoding no source symbols before the reception of the complete β block is small when m is large.

Decoding complexity

From the evaluation in Section 7.2.3 it was shown that MED performs the same as ED in an ideal network scenario where decoding complexity of α packets scales quadratic. With the reception of packets from a β block, the maximum number of packets in the β block can be seen as $\varphi + 1$, which includes the φ packets from the β block as well as the zero vector added by the MED algorithm. The $\varphi + 1$ packets are compared to each other, which can require a maximum of

$$\varphi + (\varphi - 1) + \cdots + 1 = \frac{\varphi(\varphi + 1)}{2} \tag{7.7}$$

arithmetic operations, where $\varphi \leq 1 \leq n$ is the size of β block. After these comparisons a single source symbol is decoded and eliminated from the other packets in the block, which requires a maximum of $\varphi - 1$ operations per decoded symbol. This process can be iterated a maximum of φ times before all the source symbols in the β block are decoded, which renders a decoding complexity of $\mathcal{O}(\varphi^3)$, where the size of m is determined by the largest β block in $\boldsymbol{G}$ and independent of n.

As with ED, each new innovative packet that is received and its global encoding vector must be linearly combined with all the decoded source symbols. Thus Algorithm 7.3 also forms part of the MED algorithm, as can be seen in Algorithm 6.3. Therefore, the decoding complexity of MED also scales linearly to n.

Accordingly, the order of decoding complexity of MED is equal to that of ED, meaning that it scales the same with n and φ. The decoding algorithm of MED, however, requires fewer arithmetical operations for the successful decoding of a β block when compared to ED.

7.3.4 Performance comparison of decoding methods

In Sections 7.3.2 and 7.3.3, the decoding delay of ED and MED is evaluated when β packets are received which includes non-zero entries above the main diagonal. In this section we plot the decoding delays of ED and MED, shown in Figures 7.4 and 7.6, for $\varphi = \{3,10\}$, where φ is the size of the β block. The advantage in the decoding delay of MED over ED can be seen in Figure 7.7.

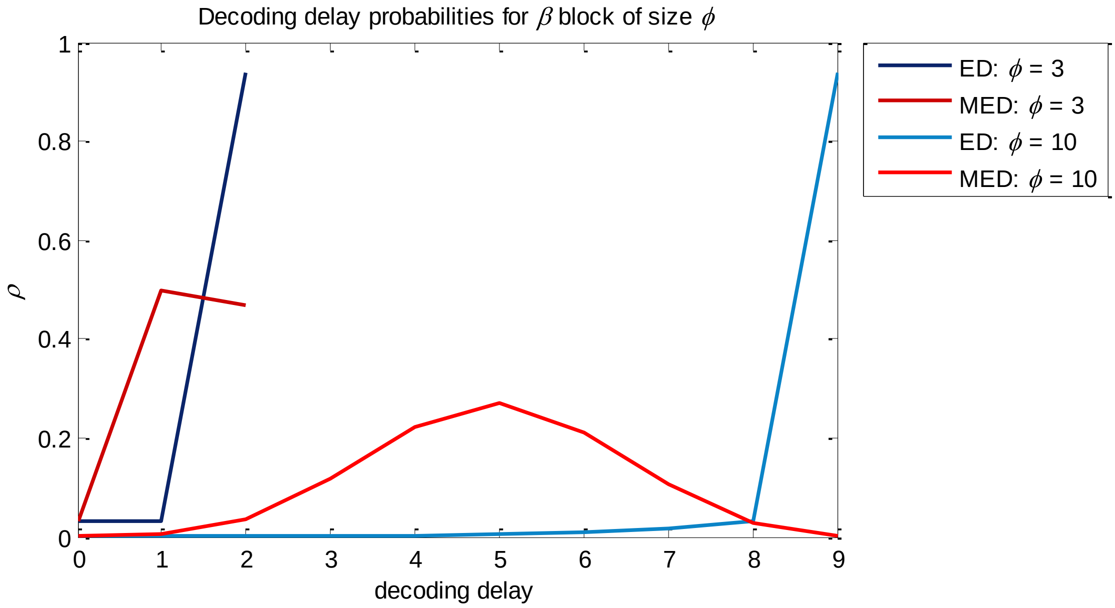

Figure 7.7: Comparison of decoding delay of ED and MED for β block

However, the probability of obtaining a β block is dependent on $P_2 = 0.05$, which was determined in Section 6.5.3. As this probability is fairly low, the probability of obtaining large β blocks is small, so the decoding delay advantage of MED shown in Figure 7.7 is not of real practical value for well-constructed networks.

It should be noted that the comparison between MED and ED using β block decoding delay provides a handy model to analyse the behaviour of the decoding methods. However, in a simulation environment, β blocks are not explicitly identified. As shown in Appendix B, the β block decoding delay always provides an upper bound on the packet decoding delay.

We illustrate the decoding delay of ED and MED by means of the network model developed in Chapter 6. Packets are randomly and independently generated with the probabilities of non-zero entries being shown in Table 6.1. One thousand generations of size n were generated and as the packets were generated, they were decoded using ED and MED. The decoding delay of ED and MED for $n = 10$ is shown in Figure 7.8, where t denotes the timesteps when a new innovative packet is collected for decoding.

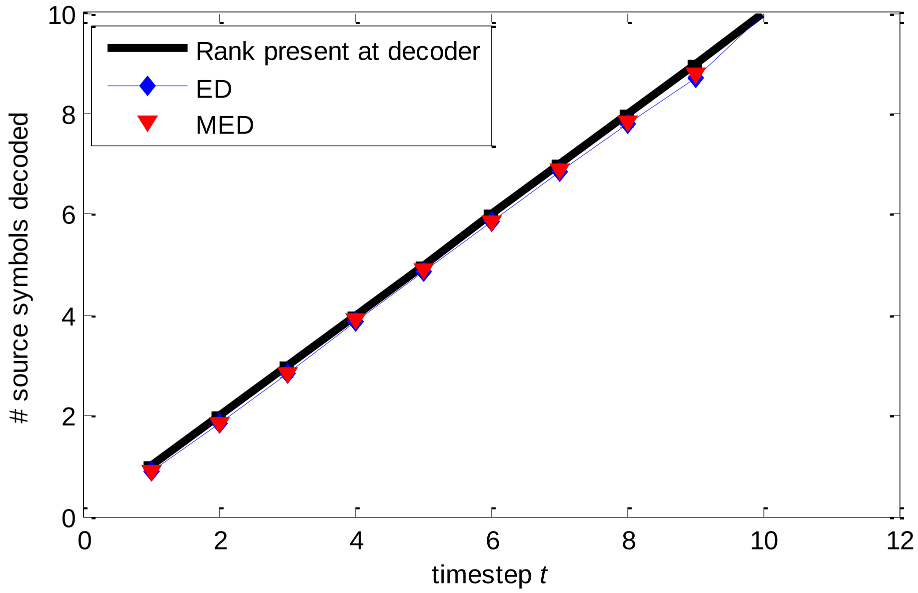

Figure 7.8: Decoding delay for ED and MED

Accordingly, the decoding delay advantage of MED over ED does not have a large practical influence, as the probability of receiving large β blocks is very small. In the following section, we consider network scenarios where packets can be lost in the network, as this influences the structure of the non-strict lower triangular matrices received.

7.4 Decoding performance in an erasure network scenario

In an RLNC network, packets can be lost with probability ρ_e due to channel interference from external devices, changes in topology, link failures or other detrimental network conditions. RLNC networks generate redundant encoded packets at the intermediate nodes which can be used for erasure control against packet loss, as described in Chapter 3. When a packet is lost within a generation, the receiver node can still obtain sufficient linearly independent packets for decoding if it waits a certain amount of time to collect enough packets. This characteristic makes RLNC robust to erasures, as successful decoding at receiver nodes does not depend on receiving packets that contain specific information, but on receiving enough linearly independent packets [**30**].

In the network environment presented in Chapter 5, however, the efficiency of decoding is improved by obtaining packets that contain specific information. These packets are collected by the receiver nodes in a systematic order which enables the construction of generator matrices of roughly lower triangular structures. This structure enables efficient decoding but can be compromised by the erasure of a packet. Although decoding would still be possible at the end of the generation if sufficient linearly independent packets have been collected, the advantage of effective decoding via ED and MED may be lost.

In this section we consider the scenario where the sink node does not obtain all the systematically encoded packets. We evaluate the decoding performances of GE, ED and MED in light of an erased packet and show that MED is more resilient to packet erasures than ED and GE.

7.4.1 Influence of erasure on lower triangular $\boldsymbol{G}$ matrix

In our network environment where received packets form a non-strict lower triangular matrix of size n, we label the packets received before the erasure with indices $1, \ldots, \kappa$. Thus the lost packet would be labelled $(\kappa + 1)$ and the packets received afterward labelled $(\kappa + 2), \ldots, n$. Assume that ρ_e is the probability of the erasure of an innovative packet occurring in the transmission channel of a specific sink node. The number of successful innovative packet receptions that can be expected until the first erasure is:

$$\begin{aligned} E &= \sum_{\kappa=1}^{\infty} (1-\rho_e)^{\kappa} \rho_e \cdot \kappa \\ &= \rho_e \sum_{\kappa=0}^{\infty} (1-\rho_e)^{\kappa} \cdot \kappa \\ &= \frac{1-\rho_e}{\rho_e} \end{aligned} \tag{7.8}$$

As the first κ packets obtained by the receiver nodes include a single undecoded source symbol with high probability, packet $y(e_{\kappa+1})$, when not received due to an erasure, possibly carried a new undecoded source packet which is not included in the generator matrix of the receiver $\boldsymbol{G}$. Thus the following packet, $y(e_{\kappa+2})$, can potentially introduce two new undecoded source packets to a sink node. If packets subsequent to $y(e_{\kappa+1})$ continue to introduce a single undecoded source symbol each, $\boldsymbol{G}$ would consist of n source symbols and only $(n-1)$ linearly independent packets. Owing to the redundancy introduced in the RLNC network, innovative packet $y(e_{n+1})$ can be obtained which would render $\boldsymbol{G}$ as being of full rank.

The loss of a packet can be seen as the main diagonal of the $\boldsymbol{G}$ matrix shifting to the right, as illustrated in Figure 7.9.

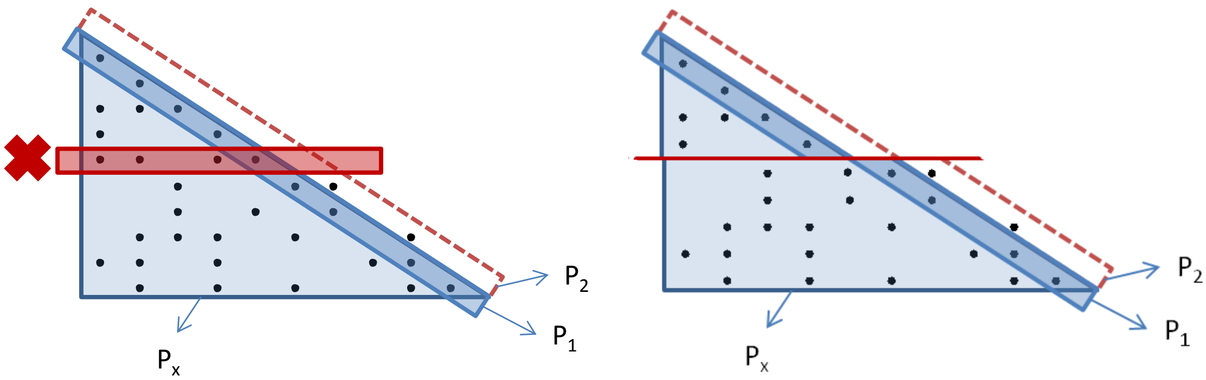

Figure 7.9: (a) Non-strict lower triangular matrix $\boldsymbol{G}$ (b) Matrix $\boldsymbol{G}$ with a single packet erasure

It can be seen from Figure 7.9 that the first κ received packets are still packets with probabilities of non-zero entries on the main and second diagonals as shown in Table 6.1 and repeated in Table 7.3 (a). After a packet is erased, it can be seen from Figure 7.9 (b) that the probabilities of the non-zero entries on the different diagonals have shifted to the right. These probabilities are shown in Table 7.3 (b).

Table 7.3: (a) Probabilities of non-zero elements in G before erased packet

	Diagonal	Probability
P_1	1 (main)	0.94
P_2	2	0.05
$P_i, i \geq 3$:	≥ 3	$< 0.02 \approx 0$
P_x	below main	0.5

(b) Probabilities of non-zero elements in G after erased packet

	Diagonal	Probability
P_1	2	0.94
P_2	3	0.05
$P_i, i \geq 3$:	≥ 4	0
P_x	main and below	0.5

Figure 7.9 and Table 7.3 show that the probability of packet $y(e_{\kappa+2})$ introducing at least two undecoded source symbols to the receiver is high. The probability of having a non-zero entry on the main and second diagonal has changed to $P_x = 0.5$ and $P_1 = 0.94$ respectively. Packet $y(e_{\kappa+3})$ can introduce a new source symbol with probability $P_1 = 0.94$ as this now is the probability of a non-zero entry on the second diagonal. In addition, packet $y(e_{\kappa+2})$ can now introduce three undecoded source symbols to the receiver as it may contain a non-zero entry on the third diagonal with probability $P_2 = 0.05$.

7.4.2 Matrix characterisation

After an erasure, most packets have non-zero entries on the second diagonal with possibility $P_1 = 0.94$. These packets can be viewed as packets that form a β block as discussed in the previous section, but with different probabilities for entries on the main and second diagonal. An example of such a β block is illustrated in Figure 7.10.

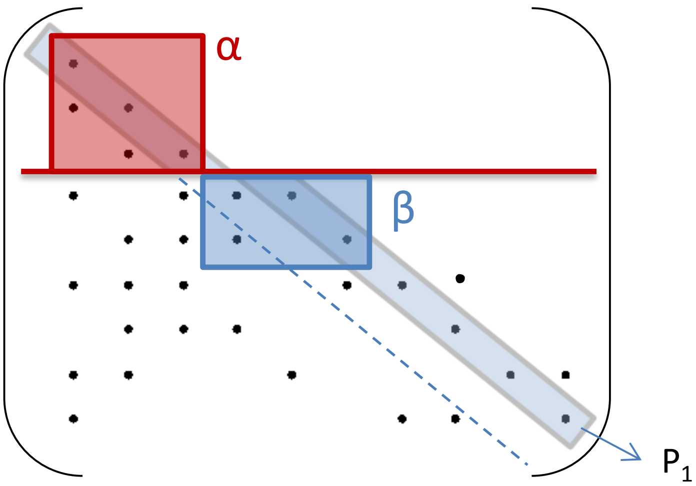

Figure 7.10: Characterisation of α and β blocks after erasure

Consider a β block described in Section 6.4.3: without an erasure the receiver is guaranteed to receive a β block where the number of unknowns matches the number of equations. Thus it is guaranteed that the β packets with 1s on the second diagonal would always be succeeded by an innovative packet without a 1 on the second diagonal. After an erasure, as shown in Figure 7.10, the reception of an innovative packet at the end of the block, which does not contain any new source symbols, is not guaranteed. Thus, it is possible in a β block that the number of undecoded source symbols is more than the number of linearly independent packets. In addition, the probability of collecting the additional innovative packet after the reception of the β packets is not guaranteed.

Following the process in Section 6.4.2, we can determine the probability of obtaining a β block, consisting of φ packets with adjacent 1s on the second diagonal, after a packet erasure. Assume that κ packets were successfully received and decoded before the packet erasure. Thus, the remaining generation is of size $\ell = (n - \kappa - 1)$.

Obtaining a β block with φ packets with adjacent 1s on the second diagonal, where $1 \leq \varphi \leq (\ell - 1)$, requires the following:

- φ entries on the second diagonal must be 1, with probability ${P_1}^{\varphi}$
- $\ell - \varphi - 1$ entries on the second diagonal must be 0, with probability $(1 - P_1)^{\ell-\varphi-1}$
- the φ non-zero entries can be placed in $\ell - \varphi$ ways on the second diagonal
- the φ entries must not have entries on the third diagonal, with probability $(1 - P_2)^{\varphi}$

Thus, the probability of collecting φ adjacent packets with 1s on the second diagonal from ℓ packets is

$$P_{\beta\_\varphi} = (\ell - \varphi) \cdot {P_1}^{\varphi} \cdot (1 - P_1)^{\ell-\varphi-1} \cdot (1 - P_2)^{\varphi} \tag{7.10}$$

Such a β block can be seen as a submatrix of size φ which forms a lower Hessenberg matrix with non-zero entries on the second diagonal.

From (7.10) we can calculate the probability, P_β, of obtaining a β block in the generator matrix after an erasure with any number of non-zero entries on the second diagonal. To obtain P_β we sum the probabilities of P_{β_φ} for $\varphi = \{1,2, \dots, (\ell - 1)\}$

$$P_\beta = \sum_{\varphi=1}^{\ell-1} (\ell - \varphi) \cdot {P_1}^{\varphi} \cdot (1 - P_1)^{\ell-\varphi-1} \cdot (1 - P_2)^{\varphi}. \tag{7.11}$$

The values of P_β for variable values of ℓ are illustrated in Figure 7.11.

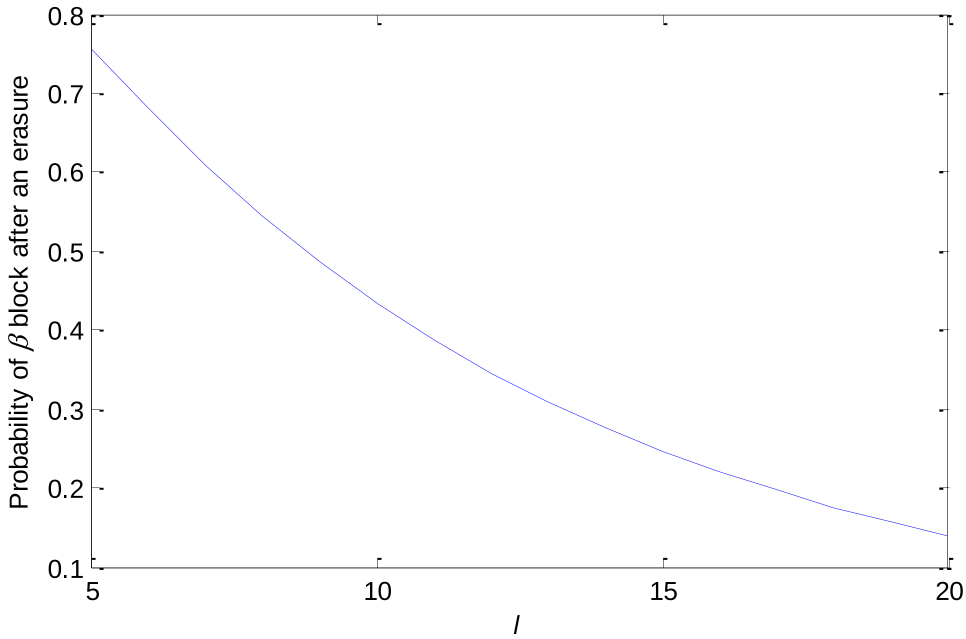

Figure 7.11: Probability of obtaining a β block after a packet erasure

In addition, a receiver node can now obtain a packet that can potentially introduce three undecoded source symbols to the receiver, as it may contain a non-zero entry on the third diagonal with probability P_2. An example can be seen in Figure 7.9 (b) where the first packet after the erasure has a non-zero entry on the third diagonal. Another example can be seen in Figure 7.10 where the packet received after the β block has a 1 on the third diagonal.

We characterise a γ block as a submatrix containing adjacent packets with 1s on the third diagonal plus all the following packets without third diagonal entries as well as the preceding undecoded packets. An example of a γ block is illustrated in Figure 7.12. Figure 7.12 is the equivalent matrix shown in Figure 7.10, but with the new matrix characterisation.

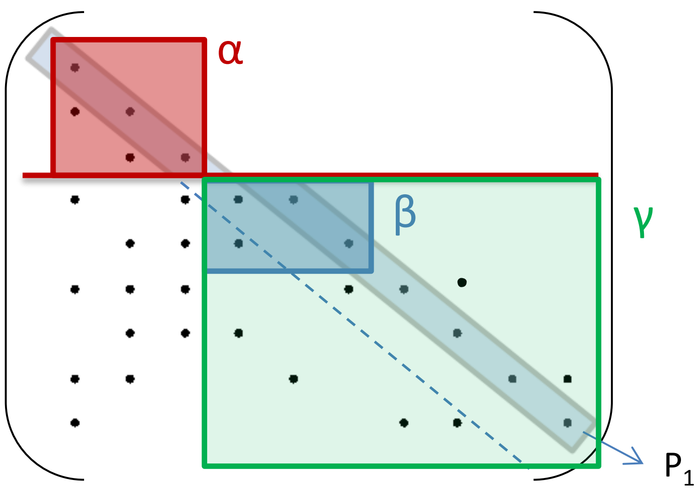

Figure 7.12: Characterisation of γ blocks

It can be seen from Figure 7.12 that the γ block includes the packets from the preceding β block, as they cannot be decoded. The last packet from a γ block is a packet without a third diagonal entry. It can be seen from Figure 7.12 that the γ block contains seven undecoded source symbols and only six linearly independent global encoding vectors. Thus it is possible that a γ block can contain more undecoded source symbols than linearly independent packets.

Following the process in Section 6.4.2, we determine the probability of obtaining a γ block of size θ. The γ block consists of $\lambda = (\theta - 1)$ adjacent packets with 1s on the third diagonal plus a packet without a third diagonal entry. Assume that κ packets were successfully received and decoded before the packet erasure. Thus the remaining generation is of size $\ell = (n - \kappa - 1)$.

Obtaining a γ block with $\lambda = (\theta - 1)$ adjacent packets with entries on the second diagonal, where $1 \leq \lambda \leq (\ell - 2)$, requires the following:

- λ entries on the third diagonal must be 1, with probability ${P_2}^{\lambda}$
- $\ell - \lambda - 2$ entries on the third diagonal must be 0, with probability $(1 - P_2)^{\ell-\lambda-2}$
- λ entries on the second diagonal can be either 0 or 1, with probability 1.
- $\ell - \lambda - 1$ entries on the second diagonal must be 1, with probability ${P_1}^{\ell-\lambda-1}$
- the λ non-zero adjacent entries can be placed in $(\ell - 1 - \lambda)$ ways in the third diagonal

Thus the probability of collecting λ adjacent packets with 1s in the third diagonal from ℓ packets to form a γ block of size θ is

$$P_{\gamma\_\theta} = (\ell - 1 - \lambda) \cdot {P_1}^{\ell-\lambda-1} \cdot {P_2}^{\lambda} \cdot (1 - P_2)^{\ell-\lambda-2} \tag{7.12}$$

From (7.12) we can calculate the probability, P_{γ}, of obtaining a γ block in the generator matrix after an erasure, with any number of non-zero entries on the third diagonal. To obtain P_{γ} we sum the probabilities of P_{γ_θ} for $\lambda = \{1,2, \dots, (\ell - 2)\}$

$$P_{\lambda} = \sum_{\lambda=1}^{\ell-2} (\ell - 1 - \lambda) \cdot {P_1}^{\ell-\lambda-1} \cdot {P_2}^{\lambda} \cdot (1 - P_2)^{\ell-\lambda-2}. \tag{7.13}$$

The values of P_{λ} for variable values of ℓ are illustrated in Figure 7.13.

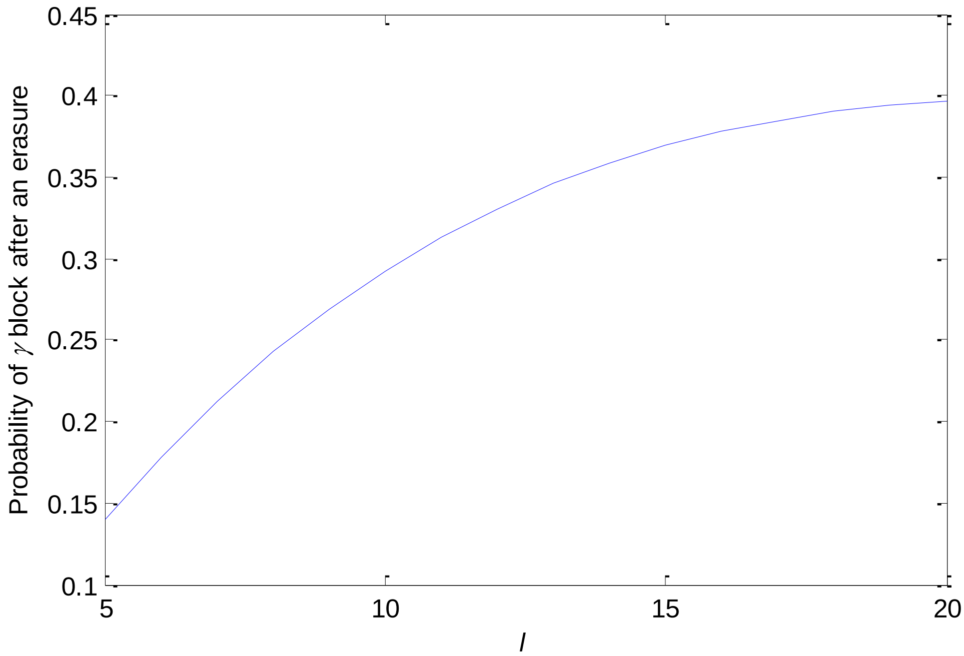

Figure 7.13: Probability of obtaining a γ block after a packet erasure

It can be seen from the above evaluations that β and γ blocks can be constructed by the receiver node, but may be of insufficient rank to successfully decode. In the following sections the decoding performance of GE, ED and MED will be evaluated in a scenario where a receiver node does not obtain all the systematically encoded packets. We determine the probability that the decoding methods can continue decoding when a packet is lost. In Section 7.3, the decoding delay was calculated as we were guaranteed sufficient linearly independent packets for decoding. In the case of packet loss, the possibility of decoding is more important as we wish to determine the efficiency of the decoding methods in the light of packet erasures.

7.4.2 Gaussian elimination

Owing to the fact that RLNC generates additional encoded packets, the receiver node can still obtain sufficient linearly independent packets for decoding if it waits a certain amount of time to collect enough packets. As GE is performed only when sufficient linearly independent packets are collected, it can decode the data without affecting the decoding complexity, as decoding is always performed on the whole set of encoded packets from a generation. However, the decoding delay will be influenced as the receiver nodes now have to wait longer in order to obtain the additional innovative packet required for matrix inversion.

7.4.3 Earliest decoding

The ED method has the advantage that even when a packet is lost, the packets received before the erasure can be decoded with high probability, as discussed in the previous section. However, as soon as one packet is erased, the probability of decoding more source symbols declines. For the sake of simplicity we assume that all the κ packets before the erasure were successfully decoded. Thus, we start our evaluation with the reception of packet $y(e_{\kappa+2})$. This evaluation will be extended to scenarios where an erasure occurs when a subset of the κ source symbols present at the receiver is not decoded before the packet loss.

As described in Chapter 6, ED determines the number of undecoded source symbols at the receiver node (number of unknowns), as well as the number of linearly independent coding vectors (number of linear equations). When the number of linearly independent packets is equal to the number of undecoded source symbols, decoding of the subset of source symbols is possible through matrix inversion. Thus, the only way that ED can decode source symbols before the end of the generation, is if it obtains i linearly independent packets which contain i undecoded source symbols.

β blocks

First, we determine the probability of decoding for ED when the φ packets collected by a receiver node form a β block. We assume that the receiver has obtained the β block with probability P_β, shown in (7.11) and now determine the possibility of decoding source symbols from the φ packets with 1s on the second diagonal. The first packet received in the β block is labelled $y(e_{\kappa+2})$ as we assume the first κ packets before the erasure were decoded.

With ED, the only possibility of decoding the first packet $y(e_{\kappa+2})$ without delay, is if the packet contains a single new source symbol only. This implies that the main diagonal contains a 0, with probability P_x. This provides the decoder with a single equation and a single unknown which can be decoded without delay. This probability is shown in the first row in Table 7.4. Note that the probability for the 1 on the second diagonal is included in P_β.

The next β packet, $y(e_{\kappa+3})$, with a 1 in the second diagonal can also be decoded without delay if it contains just a single undecoded symbol, with probability P_x. However, $y(e_{\kappa+3})$ can only be decoded if $y(e_{\kappa+2})$ has been decoded, as it would once again provide the decoder with a single equation and one unknown. The same logic follows for all the adjacent packets with a 1 on the second diagonal. This probability is shown in the second row in Table 7.4. When all the packets from the β block are decoded without delay, the packet following the β block, $y\left(e_{\kappa+2+\varphi}\right)$ would be decodable when it contains only a single source symbol, which can be a 1 on any diagonal of the matrix: main, second or third, with the probability depicted in the third row of Table 7.4. This packet does not form part of the β block but may assist in decoding packets from the β block, as discussed subsequently.

If one of the φ packets contains more than a single undecoded source symbol, the receiver will have $(i+1)$ unknowns and only i equations, where $1 \leq i \leq \varphi$ and decoding is not possible. Decoding can only continue if the receiver obtains an additional innovative packet $y\left(e_{\kappa+2+\varphi}\right)$ without a 1 on the second diagonal. Note that the probability of obtaining such an additional packet is not included in P_β, as the packet reception is not guaranteed. If such a packet is received, decoding can commence once again. The probability of obtaining such a packet is depicted in the fourth row of Table 7.4

Examples of the above scenarios are illustrated in Figure 7.14. Figure 7.14 (a) shows the decoding of source symbols without delay, while Figure 7.14 (b) illustrates a scenario where the decoding of the source symbols of the β block is dependent on the reception of the additional innovative packet without a 1 on the second diagonal.

Figure 7.14: (a) Example of a β block decodable without delay **(b) Example of a β block decodable with delay**

Table 7.4 summarises the probabilities of decoding source symbol/s at the reception of packets of a β block. The calculations for determining these probabilities are provided in Appendix B.

Table 7.4: Probability of decoding packets in a β block with ED

	Probability of decoding	Probability of not decoding	symbols present
First β packets	P_x	$1 - P_x$	2
Adjacent β packet	P_x (dependent on success of previous packet)	$1 - P_x$	2
Additional packet with 0 in second diagonal	$\frac{P_x \cdot (1 - P_1 P_2)}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$	$\frac{P_x \cdot (P_1 + P_2)}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$	3
	$\frac{P_x \cdot (1 - P_1) \cdot (1 - P_2)}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$	$\frac{P_1 + (1 - P_1) P_2}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$	≥ 4

From the values in Table 7.4 it can be deduced that the possibility of ED successfully decoding the source symbols in a β block relies mainly on the reception of the additional packet without a second diagonal 1. It is clear from Table 7.4 that the possibility is not large. This makes sense as the RLNC network has been developed to ensure that the received packets contain a 1 on the main diagonal with high probability. After the erasure, however, the diagonal shifted, causing the main diagonal entry to become a second diagonal entry.

The probabilities shown in Table 7.4 are used to calculate the probability of decoding source symbols from a β block for ED. In our calculation we included the possibility of obtaining an additional innovative packet without a 1 on the second diagonal, thus a maximum of $(\varphi + 1)$ source symbols can be decoded. The decoding probabilities for ED for β blocks of variable size φ are shown in Figure 7.15.

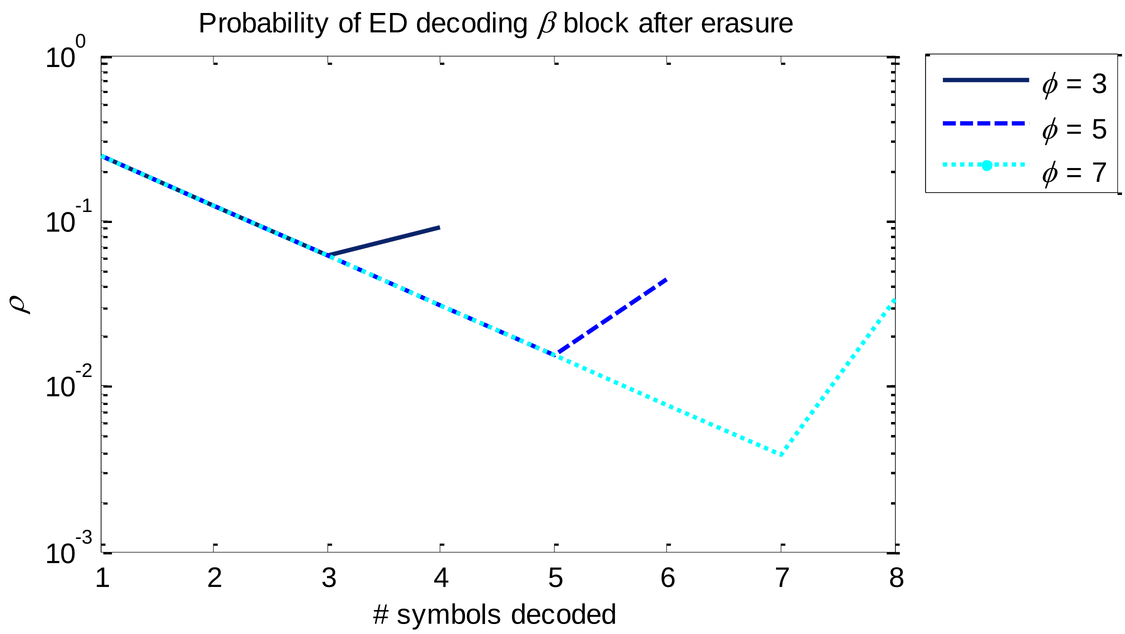

Figure 7.15: Probability of decoding source symbols in β block with ED

The form of the curves seen in Figure 7.15 can be explained. The decoding of the φ packets with a 1 on the second diagonal is dependent on the decoding of the previous packet. Thus, the probability of decoding decreases as φ increases. The possibility of decoding all the $(\varphi + 1)$ symbols from the β block exists when the additional innovative packet is received, which is indicated by the increase in decoding probability at $(\varphi + 1)$. It can be seen, however, that the possibility of receiving such a packet falls below 10% for $\varphi > 1$.

Next we determine the decoding probability of ED when a γ block is received.

γ blocks

In Section 7.4.2 the γ block was presented such that the first packet received in the γ block is a packet which contains a 1 on its third diagonal. We determine the probability of decoding for ED when $\lambda = (\theta - 1)$ adjacent packets with a 1 on the third diagonal, as well as a packet without a 1 on the third diagonal, are collected by a receiver node to form a γ block. We assume that the receiver has obtained the λ block with probability P_γ, as shown in (7.13). We now determine the possibility of decoding source symbols included in the γ block. The first packet received in the γ block is labelled $y(e_{\kappa+2})$, as we assume that the first κ packets before the erasure were decoded.

As with a β block, ED can only decode the first packet $y(e_{\kappa+2})$ without delay if it contains a single new source symbol. Thus the packet must only not contain entries on the main and second diagonals, with probability $P_x(1 - P_1)$. This provides the decoder with a single equation and a single unknown which can be decoded without delay. This probability is shown in the first row in Table 7.5.

When $y(e_{\kappa+2})$ is decoded, the following packet $y(e_{\kappa+3})$ with a 1 in the third diagonal can also be decoded without delay, if it contains only a single undecoded source symbol with probability ${P_x}^2$.

As with a β block, $y(e_{\kappa+3})$ can only be decoded if $y(e_{\kappa+2})$ was decoded, as it would once again provide the decoder with a single equation and one unknown. The same logic follows for all the adjacent packets with a 1 on the third diagonal. This probability is shown in the second row in Table 7.5. When all the $\lambda = (\theta - 1)$ packets have been successfully decoded, the last packet of the γ block can be decoded if it only adds a single source symbol. The probability is depicted in the third row of Table 7.5.

As soon as one of the λ packets with a 1 in the third diagonal contains more than a single undecoded source packet, the receiver can potentially have $(i + 2)$ unknowns and only i equations, where $1 \leq i \leq \gamma$, and decoding is not possible. When the receiver has $(i + 2)$ unknowns and only i equations, the last packet of the γ block cannot enable decoding, as it would provide the receiver with $(i + 2)$ unknowns and only $(i + 1)$ equations.

Examples of the above scenarios are illustrated in Figure 7.16. Figure 7.16 (a) shows how all the received packets can be decoded without delay, as each packet introduces only a single undecoded source symbol to the receiver. Figure 7.16 (b) illustrates a γ block where there are $(\theta + 1)$ unknowns and only θ equations.

Figure 7.16: (a) Example of a γ block decodable without delay (b) Example of a γ block that is not decodable

Regardless of the abovementioned decoding probabilities, it can be seen from Figure 7.16 that the only way in which a decoder can decode most of the source symbols in a γ block is when a single column is left empty. This can be explained as follows: The λ received packets with third diagonal 1s can lead to a decoder containing $(\theta + 1)$ unknowns and only $(\theta - 1)$ equations. The last packet of the block without a third diagonal 1 can add an innovative equation without another source symbol, leading to $(\theta + 1)$ unknowns and θ equations. Subsequent packets without third diagonal 1s add a single source symbol, where the number of equations still remains fewer than the number of unknowns.

If a single column of the γ submatrix remains empty, meaning that only θ source symbols are included in the $\lambda = (\theta - 1)$ packets with a 1 in the third diagonal, the block can be decoded when the last packet of the γ block is collected. The erased packet was supposed to add an additional equation in order to decode the extra source symbol. As this equation is now lost, the presence of zeros in a column of the matrix would erase a source symbol and allow decoding. The probability of θ innovative packets containing a zero entry in a specific column so that θ source symbols can be decoded can be approximated by

$$P_x{}^{\lambda} \cdot \frac{P_x{}^2}{1 - P_x{}^2} \tag{7.14}$$

where $\frac{P_x{}^2}{1-P_x{}^2}$ is the probability that the last packet from the γ block contains a zero entry in the specific column, which is shown in the fourth row of Table 7.5. Note that θ source symbols are the maximum number that can be decoded in a γ block as a result of the rank deficiency.

Table 7.5 summarises the probabilities of decoding source symbol/s at the reception of packets of a γ block. The process of obtaining these values is described in Appendix B.

Table 7.5: Probability of decoding in γ block for ED

	Probability of decoding	Probability of not decoding	symbols present
γ packets	$P_x(1-P_1)$	$1-[P_x(1-P_1)]$	2
Adjacent γ packet	$P_x{}^2$ (dependent on success in previous packet)	$1-P_x{}^2$	2
Packet without 1 in third diagonal	$\frac{P_x}{1-P_x{}^2}$ (dependent on empty column, with probability $P_x{}^\lambda$)	$\frac{P_x{}^2}{1-P_x{}^2}$	2
	$\frac{P_x{}^2}{1-P_x{}^2}$ (dependent on empty column, with probability $P_x{}^\lambda$)	$\frac{P_x}{1-P_x{}^2}$	≥ 3

From the values in Table 7.5 it can be deduced that the possibility of ED successfully decoding the source symbols in a γ block mainly relies on the last packet without a third diagonal 1. From Table 7.5, the probability of decoding source symbols in a γ block of variable size θ can be determined and is shown in Figure 7.17.

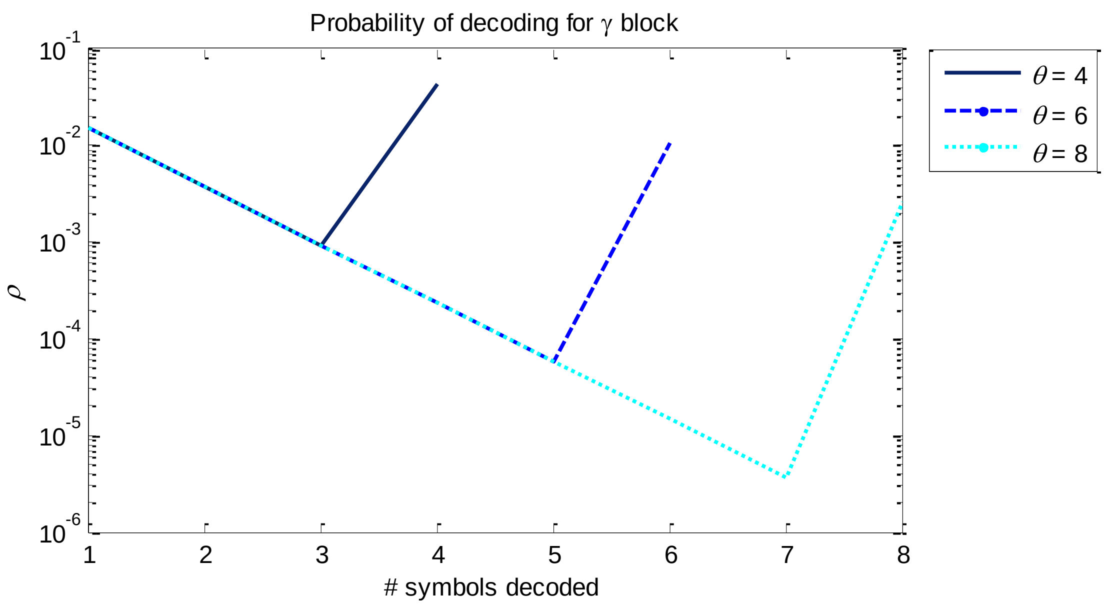

Figure 7.17: Probability of decoding in γ block with ED

The form of the curves seen in Figure 7.17 can be explained as follows: The decoding of the $\lambda = (\theta - 1)$ packets with a 1 on the third diagonal is dependent on the decoding of the previous packet. Thus, the probability of decoding decreases as λ increase. The possibility of decoding θ symbols from the γ block exists when all the packets received from the γ block have a zero entry in the same column, with the probability shown in (7.14). This possibility is the reason for the increase in decoding probability at θ. Although there is an increase in decoding possibility, the probability falls below 5% for all sizes of θ. Thus it is clear to see that a single erasure at a receiver node can cause ED to effectively stop decoding.

From the above discussion it is clear that the practical performance of ED is dependent on the probability of an erased packet, ρ_e. The decoding efficiency of ED is greatly influenced by the position of the erased packet in the generation. The decoding delay of the κ packets successfully received and decoded before the erasure is approximately linear as calculated in Section 7.3.2. After an erasure it is highly probable that the ED method would be unlikely to decode further source symbols.

As RLNC inherently generated additional encoded packets, these packets can be used to counter the rank deficiency in $\boldsymbol{G}$ at the receiver nodes. The reception of an additional randomly encoded packet at the end of a generation would allow ED to obtain a submatrix of full rank which can be successfully inverted. The receiver nodes, however, have to wait until the end of the generation to decode the source symbols, which can lead to a decoding delay scalable to $(n - \kappa)$, which has a negative influence on delay sensitive networks. Also, it requires matrix inversion of complexity $\mathcal{O}[(n - \kappa)^3]$, which can lead to an increase in computational resource usage.

7.4.4 Modified earliest decoding

From the above analyses it is clear that ED is an effective decoding technique to use in a network scenario where packets are received in a non-strict lower triangular form. Its performance is, however, severely affected by erasures in the channel and requires the transmission of redundant packets over the network to ensure successful decoding. From the previous section it can be seen that the performance of MED is similar to that of ED in lossless environments. MED, however, is more robust to packet erasures as it was developed to maintain a lower decoding delay in an erasure channel. MED continues the decoding process when a packet erasure has occurred as it eliminates the mutual information between packets for decoding that are not directly affected by the rank deficiency of the submatrix.

The MED algorithm is able to decode source symbols in β or γ blocks if their coding vectors have Hamming distances of 1, regardless of the rank of the block. Although the first packet of the β or γ block has the potential to add up to three new source symbols to the receiver, several subsequent packets in the block add only a single source symbol. Thus there exists a large possibility that subsequent packets have coding vectors of Hamming distance 1, which allows MED to decode packets where ED cannot.

β blocks

First, we determine the probability of decoding for MED when φ successive packets collected by a receiver node form a β block. We assume that the receiver has obtained the β block with probability P_β, presented in (7.11), and now determine the possibility of decoding packets from the block. The first packet received in the β block is labelled $y(e_{\kappa+2})$ and we assume that the first κ packets received have been successfully decoded.

With MED, source symbols can be decoded when the Hamming distance of two packets' global encoding vectors is 1, which does not require the number of unknowns to match the number of equations. To decode a source symbol immediately on receipt, packet $y(e_{\kappa+2})$ must only contain a single new source symbol. This implies that the main diagonal contains a 0, with probability P_x. Note that the probability of a 1 on the second diagonal is included in P_β. This probability is shown in the first row in Table 7.6.

With the reception of the following packet, $y(e_{\kappa+3})$, with a 1 on the second diagonal, a source symbol can be decoded with probability P_x. The successful decoding is not dependent on the successful decoding of the previous packet. This is due to the fact that the Hamming distance of packets $y(e_{\kappa+2})$ and $y(e_{\kappa+3})$ may be equal to 1 or the packet may contain a single undecoded source symbol. This holds true for all the φ packets with a 1 on the second diagonal. This probability is shown in the second row in Table 7.6.

As with the discussion in Section 7.4.3, an innovative packet, $y(e_{\kappa+2+\varphi})$, without a 1 in the second diagonal, can be received following the β block. The probability of obtaining such an additional packet is not included in P_β and not guaranteed. If such a packet is received, the packet can enable the decoding of all the $(\varphi+1)$ source symbols in the β block or a single source symbol. The probability of obtaining such a packet is depicted in the last three rows of Table 7.4, as the probability depends on the number of source symbols still undecoded.

An example of MED in a β block is illustrated in Figure 7.18. The first packet can be decoded without delay, where the second packet introduces two undecoded source symbols and cannot be decoded. MED enables decoding of a source symbol when the third packet has been received. The rest of the source symbols are decoded with the reception of the additional innovative packet without a 1 on the second diagonal. It can be seen from the example that two source symbols can be decoded in the β block without the reception of the additional packet.

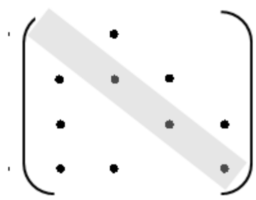

Figure 7.18: Example of a β block of size φ plus additional innovative packet

Table 7.6 summarises the probabilities of decoding source symbol/s on reception of the packets of a β block through the use of MED. The process of obtaining these values is described in Appendix B.

Table 7.6: Probability of decoding packets in a β block with MED

	Probability of decoding	Symbols present	Symbols decoded
β packets	P_x	2	1
Adjacent β packet	P_x	2	1
Additional packet without 1 in second diagonal	$\frac{P_x \cdot (1 - P_1 P_2)}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$	3	1
	$\frac{P_x \cdot (1 - P_1) \cdot (1 - P_2)}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$	≥ 4	$(\varphi + 1)$
	$\frac{P_x[P_1(1 - P_2) + (1 - P_1)P_2]}{1 - [P_x \cdot (1 - P_1) \cdot (1 - P_2)]}$		1

The values in Table 7.6 correspond with the values for ED in Table 7.4. The difference, however, is the fact that the successful decoding of a packet with MED is not dependent on the successful decoding of the previous received packet and that MED is able to decode a single source symbol with the reception of the additional packet. These values are used to calculate the probability of decoding for MED in β blocks with variable size φ, shown in Figure 7.19. In our calculation we included the possibility of obtaining an additional innovative packet without a 1 on the second diagonal, thus a maximum of $(\varphi + 1)$ source symbols can be decoded.

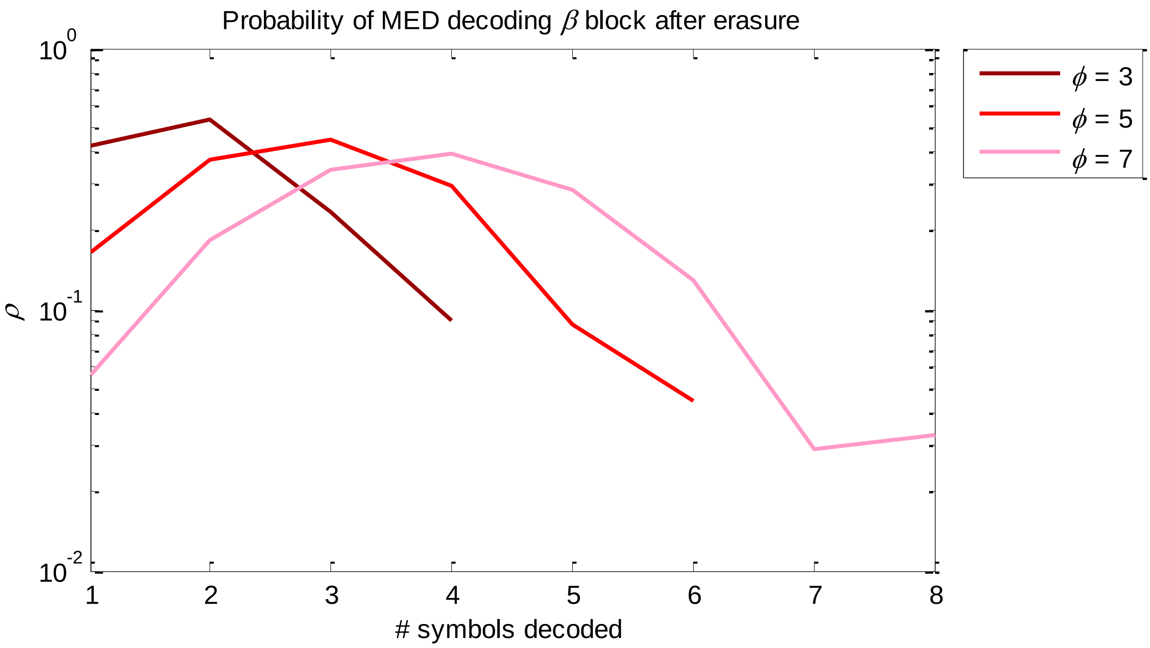

Figure 7.19: Probability of decoding source symbols in β block with MED

From Figure 7.19 it can be seen that the probability that MED will decode packets from a β block after a packet erasure has occurred takes the form of a normal distribution up to the inflection point as φ increases. This can be explained as follows: The probability of decoding a source symbol with the reception of any of the φ packets is $P_x = 0.5$. When only one source symbol is decoded, it may be due to the reception of any one of the φ packets or the additional packet. When two source symbols are decoded, it can be as a result of the reception of any two of the φ packets, where the packet can be received in ${}_{\varphi}C_2$ ways. Also it can be as a result of the reception of any one of the φ packets and the additional packet which can be received in ${}_{\varphi}C_1$ ways. Thus, when $1 \leq i \leq \varphi$ source symbols are decoded, it can be as a result of the reception of any i of the φ packets, where the packet can be received in ${}_{\varphi}C_i$ ways; or any $(i-1)$ of the φ packets and the additional packet, which can be received in ${}_{\varphi}C_{(i-1)}$ ways. These possible combinations of packets received form a bell curve that can be observed in Figure 7.19. The possibility exists that the additional innovative packet enables the decoding of all the $(\varphi + 1)$ symbols from the β block. This is indicated by the increase in decoding probability at $(\varphi + 1)$. It can be seen that the maximum decoding probability of MED is approximately 45%.

Next we determine the decoding probability of MED when a γ block is received.

γ blocks

In this section we determine the probability of decoding with MED when a γ block is obtained by a receiver node with probability P_γ, as shown in (7.13). As discussed in Section 7.4.2, a γ block contains $\lambda = (\theta - 1)$ adjacent packets with a 1 on the third diagonal, as well as a packet without a 1 on the third diagonal. The first packet received in the γ block is labelled $y(e_{\kappa+2})$ as we assume the first κ packets before the erasure were decoded.

As with a β block, MED can decode source symbols when the Hamming distance of two packets is 1 or when any packet only contains one undecoded source symbol which does not require the number of unknowns to match the number of equations. To decode a source symbol with the reception of packet $y(e_{\kappa+2})$, it must not contain entries on the main and second diagonals, with probability $P_x(1 - P_1)$. This probability is shown in the first row in Table 7.7.

With the reception of the following packet, $y(e_{\kappa+3})$, with a 1 in the third diagonal, a source symbol can be decoded with probability ${P_x}^2$. This is due to the fact that the Hamming distance of packets $y(e_{\kappa+2})$ and $y(e_{\kappa+3})$ may be equal to 1, or that $y(e_{\kappa+3})$ contains only a single undecoded source symbol. This holds true for all the θ packets with a 1 in the third diagonal, where the reception of a packet can enable the decoding of a source symbol regardless of the previous decoding success. This probability is shown in the second row in Table 7.7.

The last packet of the γ block does not contain a 1 on the third diagonal. The collection of this packet can have different results.

1. This packet can lead to the decoding of a single source symbol, with the probability dependent on the number of unknown source symbols in the γ block. The probabilities can be approximated by the values shown in Table 7.7.
2. When a single column of the γ block is empty, with probability ${P_x}^{\lambda}$, as described in Section 7.4.3, this packet can cause the decoding of θ source symbols. The probability is shown in the fifth column in Table 7.7. An example is illustrated in Figure 7.20 (a).
3. This packet can cause the decoding of $(\theta - 1)$ source symbols through the iterative process of MED. The last column in Table 7.7 shows the approximate probability for the decoding of $(\theta - 1)$ source packet when the number of undecoded source symbols is more than three. An example is illustrated in Figure 7.20 (b).

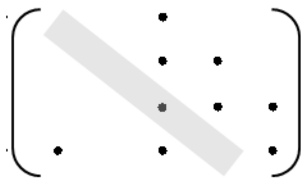
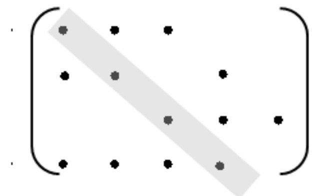

Figure 7.20: (a) Example of a γ block where θ source symbols can be decoded **(b) Example of a γ block where $(\theta - 1)$ source symbols can be decoded**

It is clear that MED can decode source symbols in a γ block that is not of full rank. Note, however, that it is impossible to decode all the source packets as the γ block is rank deficient with high probability. When a single column of the γ block is open, the block can be of full rank with θ unknowns and θ equations, as seen in Figure 7.20 (a).

We summarise the probabilities of decoding a source symbol in a γ block when a packet is received in Table 7.7. The calculations performed in obtaining these probabilities are included in Appendix B. Note that these decoding probabilities are not dependent on the decoding success of the previous packets.

Table 7.7: Probability of decoding in γ block for MED

	Probability of decoding	Symbols present	Symbols decoded
γ packets	$P_x(1-P_1)$	2	1
Adjacent γ packets	${P_x}^2$	$i \geq 2$	1
Packet with 0 in 3rd diagonal	$\frac{P_x}{1-{P_x}^2}$	2	1
Packet with 0 in third diagonal (dependent on empty column, with probability ${P_x}^{\lambda}$)	$\frac{{P_x}^2}{1-{P_x}^2}$	$i \geq 3$	1
	$\frac{{P_x}^2}{1-{P_x}^2}$		θ
Packet with 0 in third diagonal (no empty column, with probability $(1-{P_x}^{\lambda})$)	$\frac{3{P_x}^{i-1}}{1-{P_x}^2}$	$i \geq 3$	1
Packet with 0 in third diagonal (no empty column, with probability $(1-{P_x}^{\lambda})$)	$\frac{{P_x}^2}{1-{P_x}^2}$	$i > 3$	$(\theta - 1)$

From Table 7.7, the probability of decoding source symbols in a γ block of variable size θ by MED can be determined and is shown in Figure 7.21.

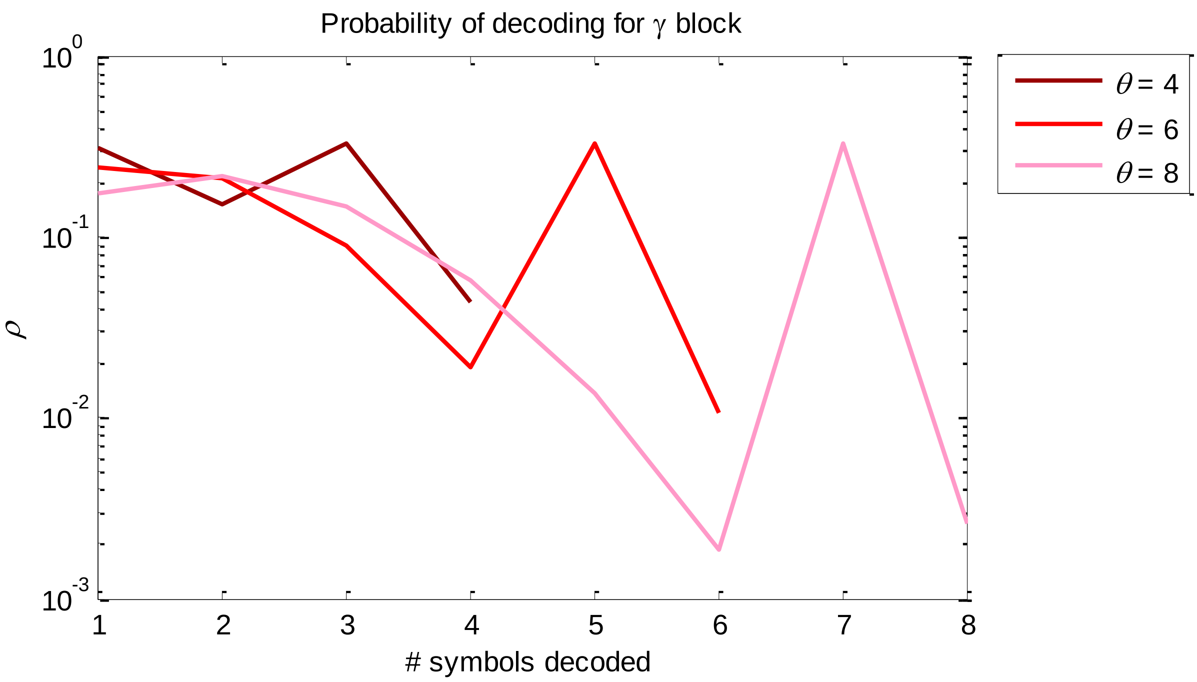

Figure 7.21: Probability of decoding in γ block with MED

From Figure 7.21 it can be seen that the probability for MED to decode packets from a γ block takes the form of a normal distribution as θ increases. This can be explained as follows: The $\lambda = (\theta - 1)$ packets with third diagonal 1s can potentially enable the decoding of up to $1 \leq i \leq (\theta - 1)$ source symbols. Thus, when $1 \leq i \leq (\theta - 1)$ source symbols are decoded, it can be as a result of the reception of any i of the λ packets, where the packet can be received in ${}_{(\theta-1)}C_i$ ways. These possible combinations of packets received form the bell curve that can be observed in Figure 7.21. However, it can be seen that the curve slopes only until $(\theta - 2)$ and not $(\theta - 1)$. This is due to the last packet of the γ block, which can potentially decode $(\theta - 1)$ or θ source symbols, as explained previously.

As can be seen from the above figure, MED is able to continue decoding after the loss of a packet. In practical networks, the size of the γ blocks obtained at the receiver nodes would be small, as a result of P_2 being small, thus MED would be able to continue decoding with a good probability where ED would fail.

7.5 Comparison of decoding probabilities

When Table 7.5 and Table 7.7 are compared, the following can be deduced: As a result of the high probability of adjacent packets differing by only a single undecoded source symbol, the probability of decoding source symbols in a γ block for MED is much larger than for ED. ED is able only to decode θ source symbols with the reception of the last packet when a single column of the γ block is left open, thus with probability

$$P_x{}^{\lambda} \cdot \frac{P_x{}^2}{1 - P_x{}^2}. \tag{7.15}$$

MED is able to decode θ source symbols with the reception of the last packet when a single column of the γ block is left open, with probability the same as ED, (7.15). MED is able, however, to decode $(\theta - 1)$ source symbols with the reception of the last packet without the need for a column of the γ block to be left open, with probability

$$(1 - P_x{}^{\lambda}) \cdot \frac{P_x{}^2}{1 - P_x{}^2}. \tag{7.16}$$

when $P_x = (1 - P_x)$ and the number of undecoded source symbols are larger than 3.

Thus MED is able to decode at least $(\theta - 1)$ source symbols with probability no less than

$$(1 - P_x{}^{\lambda}) \cdot \frac{P_x{}^2}{1 - P_x{}^2} + P_x{}^{\lambda} \cdot \frac{P_x{}^2}{1 - P_x{}^2} \tag{7.17}$$

which reduce to

$$\frac{{P_x}^2}{1-{P_x}^2}. \tag{7.18}$$

This leads us to conclude that for large γ blocks the probability, ${P_x}^\lambda, \lambda = (\theta - 1)$, diminishes significantly with each added row in the γ block, making decoding of θ source symbols, using ED, improbable. However, MED is independent of ${P_x}^\lambda$ and has a probability no less than $\frac{{P_x}^2}{1-{P_x}^2} = 0.333$, where $P_x = (1 - P_x)$, of decoding at least $(\theta - 1)$ source symbols.

In Figure 7.22 we plot the decoding probabilities of ED and MED, shown in Figures 7.17 and 7.21 for $\theta = 4$, where θ is the size of the γ block. The advantage in the decoding probability of MED over ED can be clearly seen.

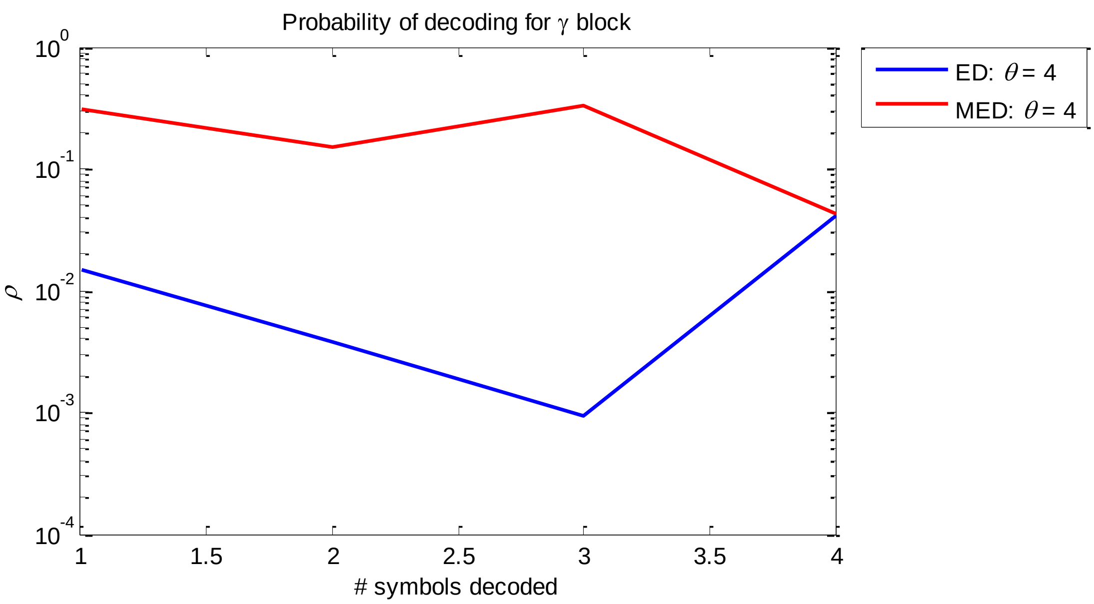

Figure 7.22: Probability of decoding in γ block for ED and MED

7.6 Decoding performance in a practical network

Next we evaluate the decoding performance of ED and MED in an RLNC network with the network configuration presented in Chapter 6. One thousand generations of size $n = 100$ were generated with different seeds. Packets are randomly and independently generated with the probabilities of non-zero entries shown in Table 6.1. A single packet erasure is guaranteed in each generation. As these packets are generated, they are presented to an MED and ED decoder. The median of the decoding delay of ED and MED is shown in Figure 7.23, where t_s denotes the timesteps when a new innovative packet is collected for decoding.

An RLNC network generates redundant packets which would render the decoding of all the source symbols highly probable at the end. We are, however, interested in the decoding performance when packets of the lower triangular structure are still received. Thus the number of decoded packets are only calculated until timestep $t_s = 99$.

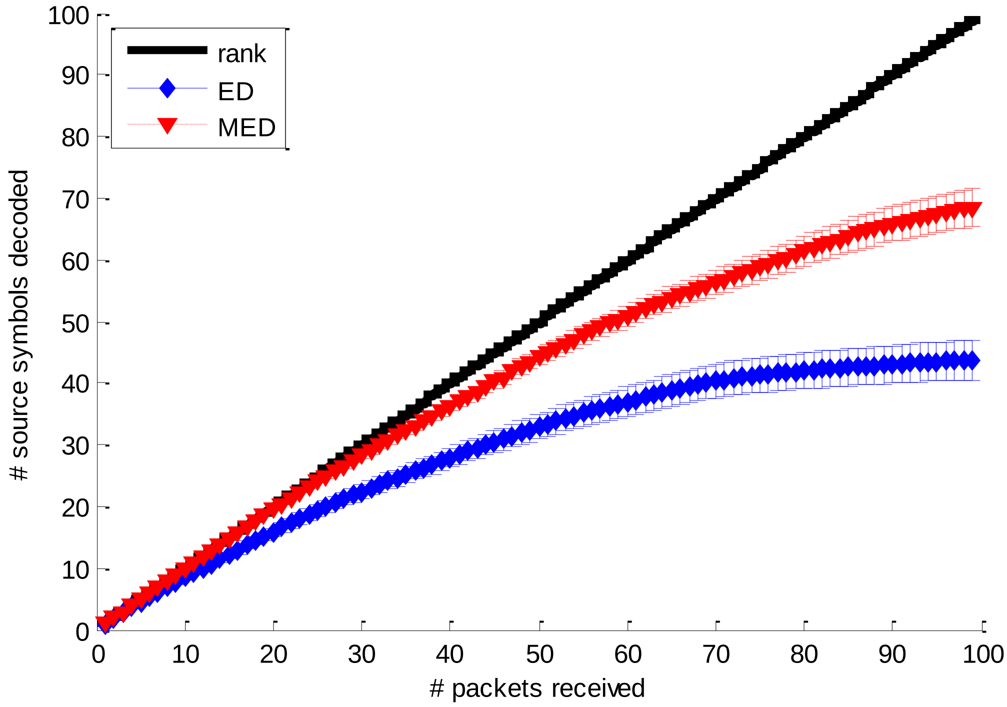

Figure 7.23: Decoding delay for ED and MED in the presence of packet loss

The first curve shows the rank of the generator matrix at the receiver at timestep t_s. The red and blue curves depict the decoding performance of MED and ED, respectively. It can be seen from the above figure that MED is more robust to packet loss than ED. Clearly, when a packet is erased, ED cannot continue to decode packets with low decoding delay. By contrast, the MED algorithm can decode a large portion of source symbols with low decoding delay in the presence of packet erasure without erasure protection.

In Figure 7.23 it can be seen that the number of source symbols decoded by MED is higher than the number of source symbols decoded by ED for the same packets received. MED decoded approximately 70% of the transmitted packets in the presence of an erased packet, where ED decoded approximately 40%. As with ED, the receiver node can decode the other source symbols at the end of the generation when an additional innovative packet can be obtained with high probability.

Network coding was developed specifically for multicast networks. Such networks are widely employed for content streaming applications on both the internet and private networks. These applications are often delay sensitive. From the evaluation of MED it can be seen that its can provide a noticeable advantage in the decoding delay of the data as opposed to GE and ED.

7.7 Conclusion

This chapter evaluated and discussed the decoding performance of three decoding methods used in RLNC networks: GE, ED and MED, introduced in Chapter 6.

In lossless networks, the practical performance of ED and MED are approximately equal. In the case of a single packet loss, the lower triangular structure of the global encoding vectors of the received packets are compromised which affects the decoding performance of both methods. The decoding probabilities for both methods were determined and it was shown that MED has more resilience to packet loss in this network scenario and can decode packets as they are collected, producing a lower decoding delay than ED.

We showed that MED can continue decoding with high probability after the occurrence of an erased packet, where ED cannot. A packet erasure causes the structure of the generator matrix to change as the main diagonal of the matrix shifts to the right, compromising the lower triangular structure of the matrix. This resilience to the change in the structure of the generator matrix at the receiver allows the implementation of an erasure code as an outer code at the source of the network. An outer erasure code with a sufficient rate would enable the receiver to use MED to obtain the source message with minimal delay.